Reheating in Early Universe Cosmology

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ABSTRACT

Inflationary cosmology is the standard paradigm of early universe physics. Inflation leaves the universe in a non-thermal state, very cold and effectively empty of matter. Hence any successful inflationary model must explain how the inflationary phase is connected to the high temperatures at Big Bang Nucleosynthesis as well as explaining the production of the Standard Model particles. The *Reheating mechanism* has been introduced to inflationary cosmology in order to explain transfer of energy which is stored in the inflaton field to other dynamical degrees of freedom in the universe and render the universe hot. Therefore, reheating is an integral part of inflationary cosmology. Studying particle production in models with a concrete background in high energy physics and the evolution of cosmological perturbations during reheating are the two main directions of the current thesis. Particle production via non-gravitational channels for G-inflation, Axion Monodromy inflation and an Asymptotically Safe Quantum Field Theory have been studied. These channels proved to be efficient. A study of the evolution of cosmological perturbations for a massless inflation toy model and for the asymptotically safe field theory showed that perturbations are highly excited even on large cosmological scales during reheating, while it was shown than in axion monodromy inflation no such significant evolution on large scales occur.

ABRÉGÉ

L'inflation cosmique est le paradigme standard de la physique de l'univers primordial. L'inflation laisse l'univers dans un état non thermique, très froid et vide de matière. Par conséquent, il est impératif que tout modèle inflationniste explique la relation entre la phase d'inflation et les températures élevées de la nucléosynthèse du Big Bang et détaille la production des particules du modèle standard. Le mécanisme de réchauffage et de refroidissement a été introduit dans la cosmologie inflationniste afin d'expliquer le transfert de l'énergie conservée dans le champ d'inflation à divers auters degrés de liberté dynamiques dans l'univers, et contribue à l'échauffement de l'univers. Donc, le mécanism de réchauffage est une partie intégrante de la cosmologie inflationniste. L'étude de la production des particules, dans le contexte de la physique des hautes énergies, et de l'évolution des perturbations cosmologiques pendant le réchauffement, est l'objectif de la thèse actuelle. La production des particules par des voies non gravitationnelles pour l'inflation G, l'inflation d'axion monodromique, ainsi qu'une théorie quantique des champs asymptotiquement sûre a été étudiée et s'est révélée efficace. L'étude de l'évolution des perturbations cosmologiques pour un modèle jouet de d'inflationon sans masse et de la théorie de champ asymptotiquement sûre a prouvé les excitations de perturbations même fur de grandes échelles cosmologiques pendant le réchauffage, alors que l'inflation d'axion monodromique ne présente pas une évolution aussi considérable fur de grandes échelles.

DEDICATION

To my family

ACKNOWLEDGEMENTS

I owe my deepest gratitude to my supervisor, Robert Brandenberger. During the course of my PhD, his patience and his sincerity with his wide knowledge of physics nurtured me towards learning cosmology and doing research. For his generosity with his time, for his continual advice and supervision, for his support whenever I needed, and for his valuable suggestions during preparation of this thesis, I am really grateful.
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Also, I wish to thank my collaborators, Yi-Fu Cai, Elisa Ferreira, Evan McDonough, Ryo Namba, Ole Svendsen and Jun'ichi Yokoyama. It was a great pleasure to discuss and collaborate with them.
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PREFACE

Satement of Originality

- **Chapters 2 and 3** are new literature reviews that I wrote to introduce the reader to *inflationary cosmology*, *cosmological perturbation theory* and *reheating mechanisms* after infaltion.
- **Chapter 4** is based on [1], a paper in collaboration with Robert Brandenberger and Jun'ichi Yokoyama. Myself and Robert Brandenberger contributed equally in the calculations. Jun'ichi Yokoyama contributed to the discussions.
- **Chapter 5** is based on [2, 3, 4, 5].
  - [2] is a paper in collaboration with Robert Brandenberger, Yi-Fu Cai and Elisa Ferreira. All the authors contributed to the discussions. I led the project and performed most of the calculations. Robert Brandenberger also contributed in writing the manuscript.
  - In [3], a paper in collaboration with Robert Brandenberger, I led the project and performed all the calculations. Robert Brandenberger supervised the project and contributed to the writing.

– [4] is a paper in collaboration with Evan McDonough and Robert Brandenberger, Evan McDonough was the leading author in the project. I led the discussions in sections VI and VII. I also reprodeuced most of calculations in other sections. Robert Brandenberger contributed to the discussions and to the writing of the manuscript.

– [5] is a paper in collaboration with Ole Svendsen and Robert Brandenberger. Myself and Ole Svendsen contributed equally in the calculations. Robert Brandenberger also contributed to discussions and to the writing of the manuscript.

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INTRODUCTION

In Gamow's classical Big Bang cosmology, the Big Bang was just a hypothetical explosion in the beginning of the Universe out of a singularity. In modern inflationary cosmology, which solves the horizon, the flatness, and the origin-of-structure problems simultaneously, the Big Bang is physically defined as the moment when the energy density of the inflaton field is converted to matter fields and the Universe has first become radiation dominated. The study of these processes namely, the reheating mechanism, is therefore of utmost importance to clarify the evolution of our Universe, and has an important impact on the evolution of inhomogeneities in the energy density of the universe. In this thesis, after reviewing the basics of early universe cosmology and of reheating after inflation, we discuss particle production in some inflationary models which have a concrete background in high energy physics and the gravitational effects in reheating mechanism. Reheating typically begins first with a linear explosive particle production (i.e. preheating). The linear regime of the resonance terminates by backreaction effects, though resonance might be continuing in a non-linear regime. In later stages, reheating may continue with the perturbative channel of the single-particle decay of Quantum Field Theory (QFT). Finally, a thermalization process is needed to produce a state in which the particles have a thermal distribution.

Also, due to the coupling between matter and gravity sectors, one can expect that the explosive particle production during preheating is accompanied by a dramatic evolution of the cosmological perturbations. This is another aspect of reheating physics that we are going to discuss and analyze in the thesis.

In this thesis, first we introduce the basics of inflationary cosmology and cosmological perturbation theory. Then in chapter 3, we review the basics of the reheating mechanism by discussing the perturbative approach and parametric resonance.

We continue with a discussion on the backreaction effects which shut off the linear stage of the resonance. In chapter 4, we introduce a class of G-inflation models. We investigate direct particle production channels and compared the efficiency with respect to the well-established gravitational particle production mechanism for the model. We then provide the conditions for which the direct channels will eventually dominate the particle production. In chapter 5, we include gravitational effects during the reheating analysis and study parametric resonance of cosmological perturbations. We introduce *massless preheating*, *axion monodromy*, and inflation in a *asymptotically safe QFT*. After discussing particle production in these models, we study the growth of entropy perturbations and the constraints that the induced curvature perturbation will put on the models. Finally we conclude the thesis in chapter 6.

Chapter 2

BASICS

2.1 Introduction

In order to review the essential basics of early universe physics, in the following we will introduce *inflationary cosmology*, the standard paradigm of early universe. Then we will discuss the theory of *cosmological perturbations* which is essential to understand structure formation in the universe. The two are the basic knowledge needed to follow the discussions in the subsequent chapters. Reader can refer to [6] for a recent review.

2.2 Inflationary Cosmology

The paradigm that the universe has underwent an exponential expansion at early times has become the standard paradigm of early universe cosmology [7, 8, 6, 9]. The inflationary paradigm not only solves some important problems of the standard Big Bang cosmology, but also provides a causal mechanism for the generation of primordial density perturbations [10]. In this section we briefly review the inflationary scenario which will be used throughout the thesis.

2.2.1 Friedmann-Robertson-Walker Cosmology and Decelerating Universe

Modern cosmology rests on two pillars: the cosmological principle and Einstein's theory of relativity. The cosmological principle states that the universe is spatially homogeneous and isotropic on large scales ($> 100Mpc$) and evolves with time. We can formulate this principle in general relativity with the maximally symmetric Friedmann-Robertson-Walker (FRW) metric

$$ds^2 = -dt^2 + a^2(t)dx^2, \tag{2.1}$$

where the so called *scale factor* $a(t)$ is a dimensionless function of time. The scale factor is a measure of the size of spatial hypersurfaces at each moment of time and we work in natural unites ($\hbar = c = 1$).

Assuming a FRW metric and considering the case of perfect fluid matter, we can work out the Einstein field equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{2.2}$$

The $0-0$ component is the first Friedmann equation

$$H^2 + \frac{k^2}{a^2} = \frac{\rho}{3M_p^2}, \tag{2.3}$$

where ρ is the energy density, $H \equiv \frac{\dot{a}(t)}{a(t)}$ is the Hubble parameter, k is the spatial cyrvature and the Planck mass is $M_P = (8\pi G)^{-1/2}$. The Bianchi identity applied to the Einstein field equations, gives conservation of the energy momentum tensor which can be written as

$$\dot{\rho} = -3H(\rho + P), \tag{2.4}$$

where P is the pressure and " $\dot{}$ " is the derivative with respect to time. Then the difference between the $0-0$ component and the $i-i$ one gives an equation for the acceleration of the expansion which is the second Friedmann equation

$$\ddot{a} = -\frac{a}{6M_P^2}(\rho + 3P). \tag{2.5}$$

Ordinary matter satisfies the strong energy condition, $\rho + 3P > 0$ and hence for ordinary matter the acceleration is always negative. As is the case in the standard Big Bang cosmology, for a universe in which ordinary matter always dominates, there will be serious problems concerning initial conditions.

2.2.2 Initial Condition Problems

Assuming that ordinary matter always dominates, we recover a decelerating universe for all times in the evolutionary history of the universe. Having said that, there are observations for which there is no explanation in the context of a decelerating universe. A few of the problems are listed below.

Flatness Problem

The *flatness problem* is related to the fine-tuning of the spatial curvature of universe to be consistent with the current observations of the energy density of the universe. Current observations are consistent with $k \simeq 0$ (see recent result from the Planck satellite [11]). Defining $3M_P^2H^2 \equiv \rho_c$, where ρ_c is the energy density in a spatially flat universe, we can rewrite the Friedmann equation as follows

$$\frac{\rho}{\rho_c} = 1 + \frac{k^2}{(aH)^2}, \tag{2.6}$$

where $(aH)^{-1}$ is the comoving Hubble radius[1] . Taking time derivatives of both sides of the above equation and assuming that the strong energy condition satisfies, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\rho}{\rho_c}) > 0, \tag{2.7}$$

which means $\frac{d}{dt}(aH)^{-1} > 0$. Hence the ratio $\frac{\rho}{\rho_c}$ continues to deviate from unity. On the other hand, current observations shows that we live in a universe with almost no spatial curvature which is consistent with $\frac{\rho}{\rho_c} \simeq 1$. But we just found the ratio of $\frac{\rho}{\rho_c}$ is a growing function of time. Therefore, considering the age of the universe, we find that the initial value of the ratio was extremely close to unity which is the case for the flat universe ($k = 0$ case). So the problem is how to justify such a highly fine-tuned flatness for the universe at initial times.

Horizon Problem

The standard Big Bang cosmology is unable to explain the observed homogeneity and isotropy of the Cosmic Microwave Background Radiation (CMB). The CMB is a temperature map of the universe which shows that all regions of the universe had almost the same temperature (to roughly one part in 10^5) at the time of photon decoupling [12]. Considering photons reaching us from opposite directions in the sky, the ratio between the time of photon decoupling and the present time indicates that the sources of the photons at the time of photon decoupling were in a spatial

[1] The comoving Hubble radius is the scale below which microphysics dominates over gravity and above which gravity dominates.

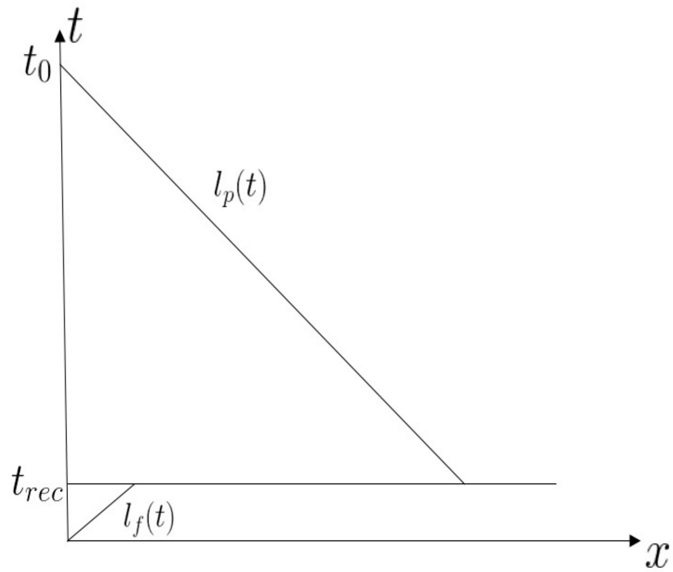

Figure 2–1: A sketch of space-time diagram that illustrates the horizon problem: The past light cone l_p at the last scattering surface is much larger than the forward light cone $l_f(t)$ at the same time.[13].

distance greater than the particle horizon at that time. In other words, the past light cone over which the CMB is observed at photon decoupling is much larger than the comoving forward light cone at the same time, as sketched in Figure 2-1.

Therefore there is no causal mechanism that can explain such homogeneity and isotropy.

Primordial Fluctuations

Observations of the CMB also show the existence of large scale primordial energy density fluctuations as in Figure 2-2 (temperature fluctuations which are associated to energy density fluctuations). Again in the context of Standard Big Bang cosmology, there is no causal mechanism to generate such fluctuations of very early times since the wavelength of the perturbations was greater that the horizon at those early times, $\lambda > H^{-1}$, as can be seen in Figure 2-3.

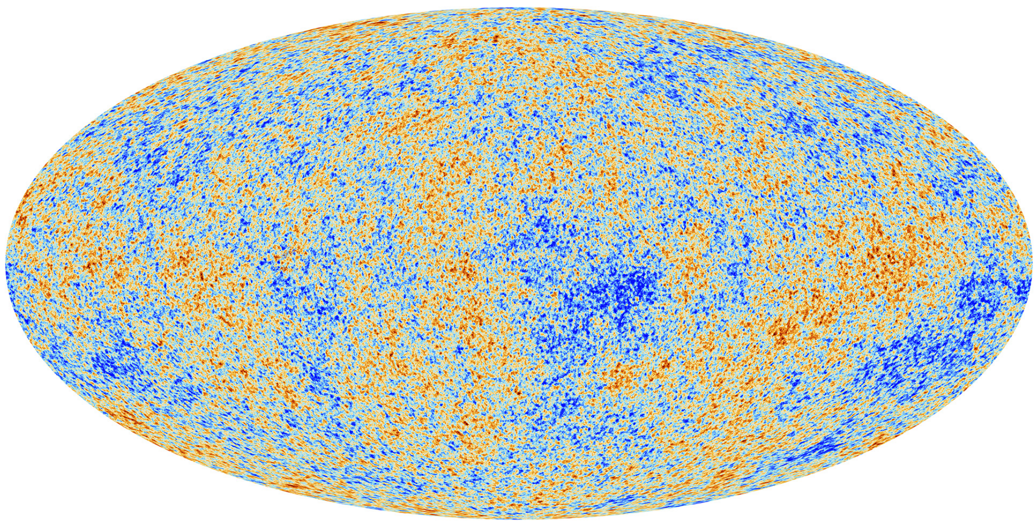

Figure 2–2: CMB thermal fluctuations as measured by Planck 2015. Red spots correspond under dense regions while the blue spots correspond to over dense regions. Perturbations are roughly one part in 10^5 [12].

We can add more problems to the list [13], but as we saw above Standard Big Bang cosmology is unable to give any explanation for them.

2.2.3 Inflation; A Solution

The idea of inflationary cosmology is basically to add an early phase of accelerated expansion of space to the standard Big Bang cosmology. In that sense, inflation is not a fundamental theory rather it is a "paradigm" that can successfully give a reasonable explanation for the initial conditions of Standard Big Bang cosmology. It also gave some important predictions for the growth of perturbations which were later verified by observations of the CMB and the large scale structure (for recent reviews see [8, 6, 9, 13, 14]). From Eq. (2.5) it is clear that by violating the strong energy condition, $\rho + 3P > 0$, one gets an accelerated expansion ($\ddot{a} > 0$). Let us see how inflation will solve the initial condition problems.

For the case of the *flatness problem*, we should note that if we violate the strong energy condition, the time derivative of Friedmann equation gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\rho}{\rho_c}) < 0, \tag{2.8}$$

which in turn means $\frac{d}{dt}(aH)^{-1} < 0$. In other words, the comoving Hubble radius shrinks during inflation and the ratio $\frac{\rho}{\rho_c}$ evolves towards unity. Therefore if inflation lasts long enough, $\frac{\rho}{\rho_c} = 1$ will be the attractor for any arbitrary initial ”non-flat” universe and hence no fine-tuning is needed.

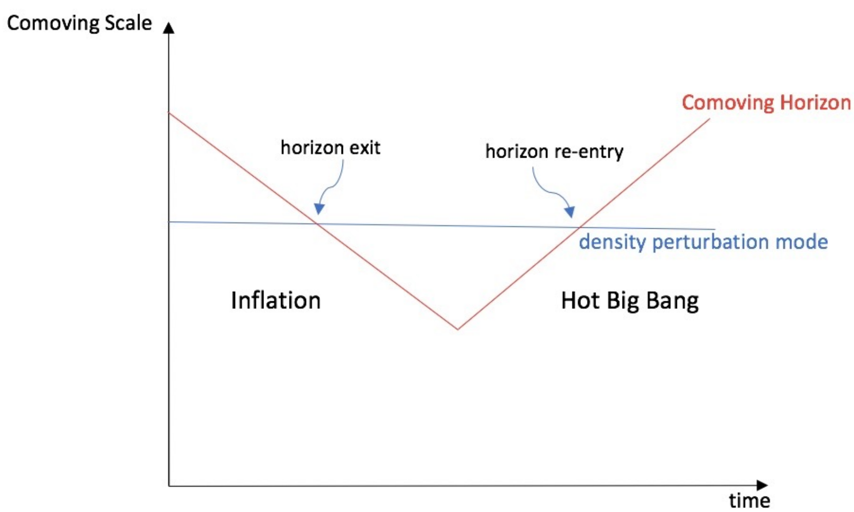

Figure 2–3: While comoving length scales k^{-1} remain constant, the comoving Hubble radius $(aH)^{-1}$ shrinks. [9].

For the same reason, the shrinking of the Hubble radius during inflation will provide the possibility for generation of the large scale *primordial perturbations*, see Figure 2-3. Since the scales of cosmological perturbations are sub-Hubble during early phases of inflation, a causal mechanism can generate small quantum fluctuations which of later stages of inflation become classical and develop into cosmological perturbations.

For the *horizon problem*, we note that in Standard Big Bang cosmology the particle horizon is the same as the Hubble radius, and therefore there is no physical explanation for the observed homogeneity and isotropy of the CMB. To address the problem, all we need is to provide that the homogeneity scale in the universe originates at sub-Hubble scales and become much bigger than the Hubble radius at some early times. Again, we can easily see that the shrinking of the Hubble radius during inflation provides that. A sketch of the inflationary space-time is given in Figure 2-4.

2.2.4 How to Obtain Inflation

As we said before, the key feature we need for inflation is to violate the strong energy condition and have $\rho + 3P < 0$. It means that if the dominant component of the universe has the equation of state $P = w\rho$ with $w < -\frac{1}{3}$, the universe is in the accelerated expansion phase. This criterion on the equation of state is incompatible with the standard description of matter as a classical ideal gas and none of the familiar types of matter have that equation of state (radiation has $w = \frac{1}{3}$, dust has $w = 0$). The breakdown of the ideal gas picture in the early universe is no surprise as on such high energy scales one can reasonably expect this to happen. Instead, matter needs to be described by Quantum Field Theory (QFT). Now let us see if QFT can introduce matter that can provide the required equation of state. The

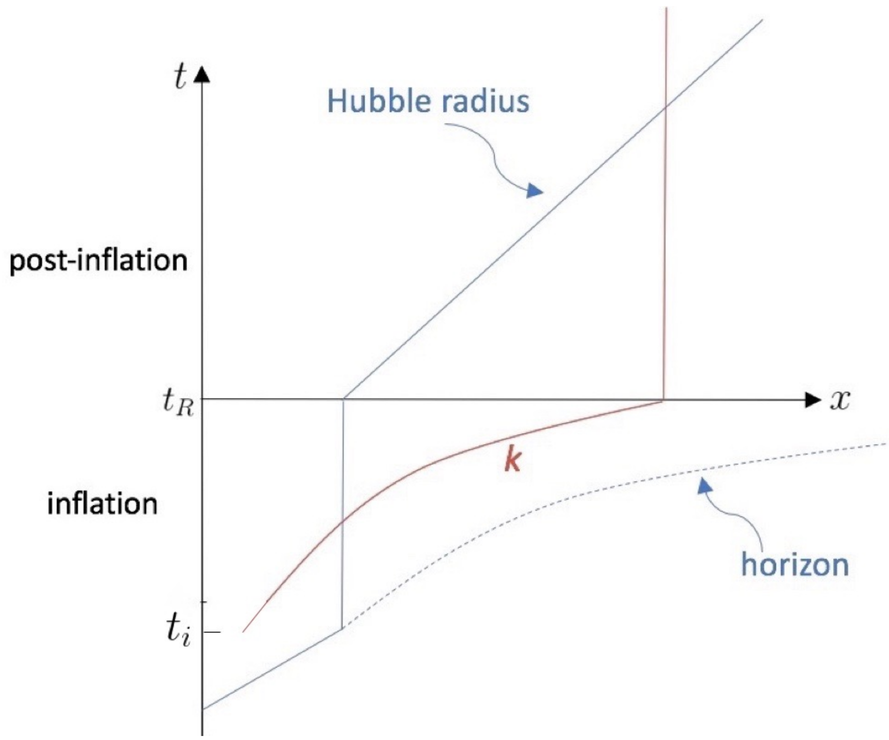

Figure 2–4: A sketch of the inflationary space-time. The horizontal axis corresponds to physical distances and the vertical axis is physical time [6].

simplest possibility is to consider a spin 0 bosonic field, a so called “scalar field". For the moment let us consider the case of a single homogeneous scalar field, ϕ, with the Lagrangian

$$\mathcal{L}_m = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{2.9}$$

Since we are considering the case of a homogeneous scalar field

$$\phi = \phi(t)$$

where “t“ is the physical time

$$\frac{1}{2}\partial_\mu\phi\partial^\mu\phi = \frac{1}{2}\dot{\phi}^2,$$

By varying the action for matter content with respect to the metric we can read the energy momentum tensor for the free scalar field. The energy density and pressure are

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{2.10}$$

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{2.11}$$

Then the equation of state for the scalar field is

$$w = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}. \tag{2.12}$$

Now it can easily be seen that if the kinetic energy is subdominant for some period of time, $\dot{\phi}^2 \ll V(\phi)$, we obtain the equation of state

$$w \simeq -1 < -\frac{1}{3},$$

which is required for accelerated expansion to happen. The condition of having kinetic energy to be subdominant is so the called “slow-roll" condition. This is the key feature behind the potential-driven methods of obtaining inflation. In this chapter we only focus on potential-driven scenarios of inflation but later in chapter 4 when we discuss G-inflation we will introduce an alternative method of obtaining inflation. However, to solve the initial condition problems, we need not only inflation but also long enough inflation. We need the kinetic energy to be subdominant for a relatively long time. Using the dynamics of the scale factor one can translate this condition into

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \ll 1, \tag{2.13}$$

$$\eta \equiv \frac{\dot{\epsilon}}{H\epsilon} \ll 1. \tag{2.14}$$

The former is the first slow-roll condition which guarantees inflation while the latter is the second slow-roll condition and provides a long enough duration for inflation.

2.2.5 Dynamical Attractor

There is a good question that the reader can ask at this point. It seems that in order to solve the initial condition problems, we have introduced a phase of evolution to the history of the universe where we have imposed another set of initial conditions or fine-tuning. One possible approach to answer this reasonable question is to argue for the “naturalness" of the slow-roll condition [14]. Since the pre-inflation physics is unknown, discussing naturalness may not be possible. However, there is an intuitive way to see that the inflationary solution is an attractor which in some sense might be seen as a natural solution.

To see the attractor solution, we note that if we start off with a potential energy dominated initial condition then the slow-roll condition is already satisfied. If instead we start off with a kinetic energy dominated state and small potential energy, from the continuity equation, Eq. (2.4), one can see $\dot{\rho} \simeq -6H\rho$ which in turn means that the kinetic energy redshifts as a^{-6} and very quickly becomes subdominant, hence leading to a slow-roll state. In this sense, slow-roll condition can be considered as a dynamical attractor.

Up to now we only considered the homogeneous case. In next section we will move on and consider inhomogeneities in both matter content and metric.

2.3 Cosmological Perturbations

Homogeneous and isotropic spacetime has been very successful to describe our universe on very large scales. However, we know that on smaller scales the content of the universe (galaxies, stars, planets and ...) has been distributed very inhomogeneously. Also, small fluctuations from CMB map, as in Fig. 2-2, indicate that the inhomogeneities in energy density of the universe was small at the time of CMB. Hence to obtain a consistent description of the universe, the inhomogeneities observed on small scales, can be treated as small perturbations above the homogeneous and isotropic background spacetime.

Cosmological perturbation theory (linearized gravity in an expanding universe) is a cornerstone of modern cosmology [10, 15]. In the context of inflationary cosmology the idea is that the cosmological structures originated from quantum fluctuations of the primordial fields which amplified and became classical perturbations during inflation and through gravitational instability made the structures in the universe. In this section first we discuss a coordinate-based approach to study cosmological perturbations, and then at the end of the chapter in a more geometrical approach we will introduce the covariant formalism for the perturbations.

2.3.1 Perturbations in Matter and Metric

Due to the coupling between matter and spacetime, perturbations in matter will induce perturbations in metric. The starting point of cosmological theory is to consider linear fluctuations around a FRW background. The perturbed Einstein equations take the form

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu}, \tag{2.15}$$

where $\delta G_{\mu\nu}$ is the perturbed Einstein tensor and $\delta T_{\mu\nu}$ is the perturbations in energy momentum tensor. If the matter content of the universe is a single scalar field, then for the perturbations in the matter one can consider

$$\phi(x,t) = \bar{\phi}(t) + \delta\phi(x,t),$$

where the first term on the right hand side is the homogeneous background field and the second term describes perturbations. For the metric perturbations we can consider

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}.$$

The first term on the right hand side is the metric for the FRW background spacetime and the second term is the perturbations. From Eq.(2.1) we know the background metric but how about the perturbation tensor? Since the full metric is a symmetric tensor, therefore there are 10 degrees of freedom for $\delta g_{\mu\nu}$ which need to be determined. Based on the differences in transformation properties under spatial rotations, we can classify perturbations and decompose them into three types, namely four scalar modes, four vector modes and two tensor modes. These modes are decoupled at the linear level and can be studied separately. Throughout most of the thesis we only consider linear perturbations and neglect higher orders.

Vector perturbations decay in an expanding spacetime. Tensor modes which describe the gravitational waves are decoupled from matter perturbations at linear order. Therefore from now on we only focus on scalar modes and investigate the dynamics and their evolution.

2.3.2 Gauge freedom

For scalar perturbations the line element can be written as

$$ds^2 = -(1+2A)dt^2 + 2a(\partial_i B)dtdx^i + a^2[(1-2\psi)\delta_{ij} + \partial_i\partial_j E]dx^i dx^j. \tag{2.16}$$

As we said there are four degrees of freedom in the metric for scalar perturbations. If we consider a single scalar field as the matter content, this will add one more degree of freedom to the dynamics and one may naively think there are five degrees of freedom in total while physically there is only one. Using two constraint equations in the Einstein equations will remove two degree of freedoms but there are still two gauge artifacts. To elaborate more on the issue, let us remember the diffeormorphism invariance (or general covariance) of General Relativity. Meaning that although the physics remains unchanged under coordinate transformations, such coordinate transformations lead to fictitious perturbations which might be interpreted mistakenly as real perturbations while they are only gauge artifacts.

There are two ways to resolve the gauge-dependent ambiguity. Fixing the gauge and working with a specific choice of coordinates is one possibility which reduces the original four scalar degrees of freedom to only two. Alternatively one may define a set of “gauge-invariant" variables which do not change under coordinate transformations. The two methods are equivalent and depending on the problem, working with one might be more convenient than the other one. Here we prefer to work with the latter for the purpose of future applications in the next chapters.

Knowing the behaviour of both the metric and matter perturbations under coordinate transformations, the gauge-invariant variables have been chosen as combinations of the metric and matter perturbations which remain unchanged under such transformations. One famous variable of this kind is the "*curvature perturbation*"

$$-\zeta \equiv \psi + \frac{H}{\dot{\rho}}\delta\rho, \tag{2.17}$$

which in the uniform density gauge ($\delta\rho = 0$) is equal to ψ [2] . Knowing the transformation rules for both ψ and $\delta\rho$ under the coordinate transformations

$$\psi \to \psi + H\delta t,$$

and

$$\delta\rho \to \delta\rho - \dot{\rho}\delta t,$$

then one can easily prove that ζ is indeed a gauge-invariant variable.

2.3.3 Power Spectrum of Curvature Perturbations

In order to find the evolution of curvature perturbations, the most straightforward way is to expand the Einstein-Hilbert action to second order in ζ and find the equation of motion for it. For the moment we work in the case of a single scalar field as the matter content. Using the second order constraint equations and doing some integrations by part one can find a simple form for the second order action for

[2] The spatial scalar curvature (Ricci scalar) at first order in perturbations is given by $R \propto \nabla^2\psi$ hence the name curvature perturbation for the variable.

curvature perturbations [10]:

$$S^{(2)} = \int dtd^3xa^3\frac{\dot{\phi}^2}{2H^2}[\dot{\zeta}^2 - \frac{1}{a^2}(\partial_i\zeta)^2]. \tag{2.18}$$

Defining $z \equiv \frac{a\dot{\phi}}{H}$ and working with conformal time $ad\tau = dt$ one can rewrite this action in terms of the *Sasaki-Mukhanov* variable $v \equiv z\zeta$:

$$S^{(2)} = \frac{1}{2}\int d\tau d^3x[(v')^2 - (\partial_i v)^2 + \frac{z''}{z}v^2]. \tag{2.19}$$

Then we can read off the equation of motion for the Sasaki-Mukhanov variable

$$v'' - \nabla^2 v - \frac{z''}{z}v = 0. \tag{2.20}$$

Going to Fourier space and defining

$$v(\tau, k) = \frac{1}{(2\pi)^{3/2}V^{1/2}}\int d^3x\mathrm{e}^{-ik.x}v(x,\tau),$$

where V is a cutoff volume. We can rewrite Eq.(2.20) for mode functions v_k

$$v_k'' + (k^2 - \frac{z''}{z})v_k = 0. \tag{2.21}$$

It is obvious that if $k^2 > \frac{z''}{z}$ then the mode function v_k has a oscillatory solution while if $k^2 < \frac{z''}{z}$ then v_k has a growing solution. Similar to the *Jean's length* in the Newtonian theory [15], in this equation $\frac{z''}{z}$ defines a length scale below which modes oscillate with constant amplitudes while above that length scale, modes have a growing solution ($v_k \sim z$).

Now to solve the equation of motion for the Sasaki-Mukhanov variable, let us rewrite the mass term $\frac{z''}{z}$ in terms of the slow-roll parameters Eq.(2.14-15). From the definition of z one can write

$$z^2 = 2a^2\epsilon,$$

where we have used the Friedmann equation and the continuity equation to obtain the right hand side. Therefore

$$\frac{z'}{z} = (aH)[1 + \frac{1}{2}\eta], \tag{2.22}$$

$$\frac{z''}{z} = (aH)^2[2 - \epsilon + \frac{3}{2}\eta - \frac{1}{2}\epsilon\eta + \frac{1}{4}\eta^2 + \frac{\dot{\eta}}{H}]. \tag{2.23}$$

The above equations are exact. We also know that the slow-roll parameters are very small during inflation, and therefore we can find an approximate expression for the mass term to first order in the slow-roll parameters. From the definition of ϵ, Eq.(2.14), we can write

$$aH \simeq -\frac{1}{\tau}(1 + \epsilon), \tag{2.24}$$

where we used "$\simeq$" instead of equality since the expression is valid only to first order in slow-roll parameters. From now on we only consider linear order in slow-roll parameters and simply use the equality sign. Using Eq.(2.25) we will find the expression for the mass term

$$\frac{z''}{z} = \frac{1}{\tau^2}[2 + 3(\epsilon + \frac{1}{2}\eta)]. \tag{2.25}$$

Defining $\nu = \frac{3}{2} + \epsilon + \frac{1}{2}\eta$ as in ref.[8], and putting Eq.(2.26) back into the Sasaki-Mukhanov equation we get

$$v_k'' + (k^2 - \frac{\nu^2 - \frac{1}{4}}{\tau^2})v_k = 0. \tag{2.26}$$

This equation has an exact solution in terms of Hankel functions:

$$v_k = \frac{\sqrt{-\pi\tau}}{2} e^{i(1+2\nu)\pi/4}[c_1 H_\nu^{(1)}(-k\tau) + c_2 H_\nu^{(2)}(-k\tau)]. \tag{2.27}$$

Note that $\tau < 0$ during inflation. In order to determine the constants c_1 and c_2 we impose the Minkowski vacuum initial condition ($k\tau \to -\infty$, one can see this as a condition on very small scales where geometry is flat)

$$v_k \to \frac{e^{-ik\tau}}{\sqrt{2k}}, \tag{2.28}$$

which corresponds to $c_1 = 1$ and $c_2 = 0$. This gives the solution of the Sasaki-Mukhanov equation

$$v_k(\tau) = \frac{\sqrt{-\pi\tau}}{2} e^{i(1+2\nu)\pi/4} H_\nu^{(1)}(-k\tau). \tag{2.29}$$

Using the asymptotic expansion for small arguments of the Hankel function (large scale perturbations as $k \ll aH$ [3]) we find

$$v_k = i\sqrt{\frac{-\tau}{2\pi}}\Gamma(\nu)(\frac{-k\tau}{2})^{-\nu}, \tag{2.30}$$

where $\Gamma(\nu)$ is the gamma function. For large arguments (small scale perturbations as $k \gg aH$) we find

$$v_k = \frac{e^{-ik\tau}}{\sqrt{2k}}(1 - \frac{i}{k\tau}). \tag{2.31}$$

[3] aH is the length scale which determines if a mode is a large scale mode or a small scale mode. We can find this from the Sasaki-Mukhanov equation and noting $\frac{z''}{z}$ is almost $(aH)^2$ to leading order. We will refer to large scale modes as the *super-Hubble* modes and to small scale modes as the *sub-Hubble* modes.

Note that since we are going to calculate the power spectrum of the fluctuations, the phase in Eq.(2.30) is not important and we dropped it in the above equations.

Having found the expression for the Sasaki-Mukhanov variable for small and large scales, we can get the power spectrum of curvature perturbations accordingly. By definition the power spectrum of ζ is given by

$$\mathcal{P}_\zeta \equiv \frac{k^3}{2\pi^2}\left|\frac{v_k}{z}\right|^2. \tag{2.32}$$

For the case of super-Hubble modes we get

$$\mathcal{P}_\zeta \simeq \frac{k^3}{4\pi^3}\Gamma^2(\nu)(\frac{k}{2})^{-2\nu} \equiv A_\zeta k^{n_s-1}, \tag{2.33}$$

where A_ζ is the amplitude of the power spectrum or the so called *COBE normalization*. From observations of the cosmic microwave background radiation [12] we know that $A_\zeta \sim 10^{-10}$. Also n_s is the spectral tilt of the power spectrum and for our case

$$n_s - 1 = 3 - 2\nu.$$

If $n_s - 1 > 0$ the spectrum is so called *blue-tilted* which means more power is on smaller scales while $n_s - 1 < 0$ corresponds to a *red-tilted* spectrum and has more power on large scales. The case of $n_s = 1$ is the so called *scale-invariant* spectrum which means all scales have the same power in the spectrum. From recent observations [12], we know $n_s - 1 = 0.032 \pm 0.006$, meaning that the power spectrum of curvature perturbations is almost scale-invariant with a tiny deviation which makes it slightly red-tilted. One of the great successes of the inflationary paradigm was the prediction of the spectrum of perturbations [16] which was later verified by different observations.

2.3.4 Multi-field Inflation and Entropy Modes on Large Scales

In what we discussed previously we only considered the case of single field inflation. Now we wish to generalize the idea to the case of multiple scalar fields as the matter content. The setup is as follows: we will consider N scalar fields which all have canonical kinetic energy and are minimally coupled to gravity.

One of the fields plays the role of the inflaton field and drives exponential expansion of space and the other fields do not contribute to inflation [17]. The Lagrangian is

$$\mathcal{L} = \frac{1}{2}\sum_{I}^{N} g^{\mu\nu}\partial_\mu\phi_I\partial_\nu\phi_I - V(\phi_I, ..., \phi_N). \tag{2.34}$$

From this Lagrangian one can easily read off the equations of motion for the background fields

$$\ddot{\phi}_I + 3H\dot{\phi}_I + V_{,\phi_I} = 0, \tag{2.35}$$

where $V_{,\phi_I}$ is the derivative of the potential with respect to ϕ_I. The Friedmann equation becomes

$$H^2 = \frac{\frac{1}{2}\sum_I \dot{\phi}_I^2 + V(\phi_I, ..., \phi_N)}{3M_P^2}. \tag{2.36}$$

To study perturbations, instead of considering fluctuations in each field, here we will work with the perturbations of energy density and pressure. The evolution of energy density perturbations is given by energy momentum conservation (the Bianchi identity, $\nabla_\mu T^{\mu\nu} = 0$) [8]:

$$\dot{\delta}\rho + 3H(\delta\rho + \delta P) = \frac{k^2}{a^2}\delta q + (\rho + P)[3\dot{\psi} + k^2(\dot{E} + \frac{B}{a})], \tag{2.37}$$

where

$$\delta\rho = \sum_I [\dot{\phi}_I(\dot{\delta}\phi_I - \dot{\phi}_I A) + \frac{\partial V}{\partial \phi_I}\delta\phi_I], \tag{2.38}$$

$$\delta P = \sum_I [\dot{\phi}_I(\dot{\delta}\phi_I - \dot{\phi}_I A) - \frac{\partial V}{\partial \phi_I}\delta\phi_I], \tag{2.39}$$

and the momentum perturbations are

$$\delta q = -\sum_I \phi_I \partial_i \delta\phi_I. \tag{2.40}$$

We can rewrite the pressure perturbations in terms of an adiabatic piece plus a non-adiabatic piece. The adiabatic piece is $\delta P_{ad} = \frac{\dot{P}}{\dot{\rho}}\delta\rho$ hence non-adiabatic contribution can be written as

$$\delta P_{nad} = \delta P - \frac{\dot{P}}{\dot{\rho}}\delta\rho. \tag{2.41}$$

As we discussed before, we will work with gauge-invariant variables. Rewriting Eq.(2.38) will give us the evolution of curvature perturbations. Also, scales of cosmological interest are the large scale ones ($k \to 0$), Therefore the evolution of curvature perturbations on large scales is given by

$$\dot{\zeta} = -H\frac{\delta P_{nad}}{\rho + P}. \tag{2.42}$$

For the case of single field inflation, since the pressure is totally determined by the energy density, the non-adiabatic pressure perturbation on large scales is zero and curvature perturbations freeze out on super-Hubble scales. However in the case of multi-field inflation the non-adiabatic pressure is in general non-zero, and hence curvature perturbations evolve even on super-Hubble scales. Now let us investigate the connection between non-adiabatic pressure perturbations and entropy perturbations. By definition, the gauge-invariant entropy perturbations are [17]

$$\mathcal{S} = H(\frac{\delta P}{\dot{P}} - \frac{\delta \rho}{\dot{\rho}}). \tag{2.43}$$

Then one can easily verify that on super-Hubble scales [4]

$$\mathcal{S} = \frac{H}{\dot{P}} \delta P_{nad}.$$

Therefore we can rewrite Eq.(2.43) in terms of entropy perturbations

$$\dot{\zeta} = -\frac{\dot{P}}{(\rho + P)} \mathcal{S}, \tag{2.44}$$

and using the continuity equation we get

$$\dot{\zeta} = 3Hc_s^2 \mathcal{S}, \tag{2.45}$$

where we have used the definition of the speed of sound $c_s^2 = \frac{\dot{P}}{\dot{\rho}}$. From Eq.(2.46) we can see on super-Hubble scales, entropy perturbations source curvature perturbations and make ζ evolve. This result is very important and we will use it when we study the evolution of curvature perturbations on super-Hubble scales in the next chapters.

[4] Note that entropy perturbations are non-zero even in the case of a single field case. But they are negligible on super-Hubble scales. See [18] for further discussions.

2.3.5 Covariant Formalism for Cosmological Perturbations

As we promised in the beginning of the section, now we are going to discuss perturbations in a geometrical approach, namely the *covariant formalism* [19, 20, 21, 22, 23, 24]. The key feature of this “non-perturbative" formalism is that the variables which are defined as perturbations vanish at the background level. After introducing the formalism, in order to make the comparison between this approach and the coordinate-based approach we discussed in previous subsections possible, we will use two approximations: 1) linear order in perturbation and 2) large scale limit. In what follows, first we introduce the basics of the formalism in the case of a perfect fluid, and then we will consider the case of two coupled scalar fields (which indeed is the situation which we study in chapter 5).

A Perfect Fluid

We consider space-time as a manifold with a preferred flow direction which is characterized by a four-velocity u^a which satisfies the normalization condition $u^a u_a = -1$.

The energy-momentum tensor for the perfect fluid is

$$T^a_b = (\rho + P)u^a u_b + P g^a_b, \tag{2.46}$$

where ρ and P are energy density and pressure, respectively. The spatial projection tensor orthogonal to the fluid four-velocity is

$$h_{ab} \equiv g_{ab} + u_a u_b\,, \tag{2.47}$$

and satisfies the relations

$$\begin{aligned} h^a_{\ b} h^b_{\ c} &= h^a_{\ c} \text{ and} \\ h_a^{\ b} u_b &= 0\,. \end{aligned} \tag{2.48}$$

The expansion parameter of space is given by

$$\Theta \equiv \nabla_a u^a \,, \tag{2.49}$$

where ∇_a is the covariant derivative. The acceleration, $\dot{u^a}$, is defined through the projected covariant derivative along the four-velocity. To be more precise, let us define the time evolution of any quantity in the covariant formalism by the Lie derivative with respect to the flow direction in the manifold. For a one form Y_a the Lie derivative is defined by

$$\dot{Y}_a \equiv \mathcal{L}_u Y_a = u^c \nabla_c Y_a + Y_c \nabla_a u^c \,. \tag{2.50}$$

For a scalar f only the first term arises, i.e. $\dot{f} = u^a \nabla_a f$. Therefore the acceleration is defined by

$$\dot{u^a} = \mathcal{L}_u u^a \,. \tag{2.51}$$

For each comoving observer, we can define the logarithm α of the local scale factor by integrating Θ along the fluid world lines:

$$\int d\tau \Theta \equiv 3\alpha \,. \tag{2.52}$$

Key to the covariant approach is to make use of variables which vanish on the unperturbed space-time. Following [19, 20, 24] one can define the “projected covariant derivative" operator

$$D_a \equiv h_a^b \nabla_b \,. \tag{2.53}$$

It is the projection onto the hypersurface perpendicular to the vector field tangent to the flow lines. Next, one can introduce the spatially projected covariant derivative (projected gradient) of the energy density

$$D_a\rho \equiv h_a^{\ b}\nabla_b\rho = \partial_a\rho + u_a\dot{\rho} \,, \tag{2.54}$$

of the pressure

$$D_aP \equiv h_a^{\ b}\nabla_bP = \partial_aP + u_a\dot{P} \,, \tag{2.55}$$

and of the expansion parameter

$$D_a\Theta \equiv h_a^{\ b}\nabla_b\Theta = \partial_a\Theta + u_a\dot{\Theta} \,. \tag{2.56}$$

As these quantities vanish in FRW space-time, they yield a fully geometrical and non-perturbative characterization of perturbations.

Knowing that, let us work out the generalized curvature and non-adiabatic pressure perturbation in a geometrical way. Starting point are the conservation equations for the energy-momentum tensor whose first component yields the continuity equation

$$\dot{\rho} + \Theta(\rho + P) = 0 \,. \tag{2.57}$$

Using the projected gradient of this equation one can define curvature covector as below [21]

$$\zeta_a \equiv D_a\alpha - \frac{\dot{\alpha}}{\dot{\rho}} D_a\rho \,. \tag{2.58}$$

Then the time evolution of this quantity is

$$\dot{\zeta}_a \;=\; \mathcal{L}_u \zeta_a \;=\; -\frac{\Theta}{3(\rho+P)}(D_a P - c_s^2 D_a \rho), \tag{2.59}$$

where $c_s^2 \equiv \frac{\dot{P}}{\dot{\rho}}$ is the generalized speed of sound as defined in [21]. Comparing this equation with the familiar equation of motion in linear theory, Eq. (2.43), where the right-hand side of the equation is the non-adiabatic pressure perturbation, we define the non-adiabatic pressure covector as

$$P_a^{(nad)} \;\equiv\; D_a P - \frac{\dot{P}}{\dot{\rho}} D_a \rho\,. \tag{2.60}$$

Making use of the definitions in equations (2.52, 53, 55) one can rewrite the curvature and the non-adiabatic pressure covectors in terms of ordinary gradients

$$\zeta_a \;=\; \partial_a \alpha - \frac{\dot{\alpha}}{\dot{\rho}} \partial_a \rho, \tag{2.61}$$

$$P_a^{(nad)} \;=\; \partial_a P - \frac{\dot{P}}{\dot{\rho}} \partial_a \rho\,. \tag{2.62}$$

For the case of a single scalar field one can show that non-adiabatic pressure covector is

$$P_a^{(nad)} \;=\; 2\frac{\dot{\phi}}{\dot{\rho}} V_{,\phi} D_a \rho, \tag{2.63}$$

which vanishes in the long wavelength approximation. Then considering $\dot{\rho} = -\Theta\dot{\phi}^2$ in equation (2.59), leads to the equation

$$\dot{\zeta}_a \;=\; \frac{2}{3}\frac{V_{,\phi}}{\dot{\phi}^3} D_a \rho, \tag{2.64}$$

for the time evolution of the curvature covector for the case of a single scalar field. The right-hand side of this equation vanishes in the long wavelength approximation. This yields the conclusion that in the case of a single perfect fluid the curvature fluctuation ζ is conserved on super-Hubble scales at arbitrary order in perturbation theory.

Two Scalar Fields

The extension to the case of two scalar fields was given in [21]. We introduce it here and then we will use it in chapter 5 when we apply the covariant formalism to study the case of massless preheating. The first step is to identify the adiabatic and the entropy components of the fluctuations in this two field system. To do this we use the formalism developed in [17] in which we are given two scalar fields ϕ and χ which both have non-vanishing backgrounds which are evolving in time. The adiabatic field σ is tangent to the field trajectory, the entropy field s is orthogonal to it. We can introduce the corresponding unit vectors in two-dimensional field space via

$$e_\sigma^I \equiv \frac{1}{\sqrt{\dot\phi^2+\dot\chi^2}}\left(\dot\phi,\dot\chi\right), \tag{2.65}$$

$$e_s^I \equiv \frac{1}{\sqrt{\dot\phi^2+\dot\chi^2}}\left(-\dot\chi,\dot\phi\right), \tag{2.66}$$

where I is the field space index. The angle θ of the trajectory in field space is then given (in the small angle approximation) by

$$\theta = \frac{\dot{\chi}}{\dot{\phi}}. \tag{2.67}$$

Using the above definitions, we can set up the adiabatic and entropy field covectors by taking the respective projective covariant derivatives of the basis fields ϕ and χ:

$$\sigma_a \equiv e^I_\sigma \nabla_a \varphi_I = \cos\theta \nabla_a \phi + \sin\theta \nabla_a \chi, \tag{2.68}$$

$$s_a \equiv e^I_s \nabla_a \varphi_I = -\sin\theta \nabla_a \phi + \cos\theta \nabla_a \chi\,. \tag{2.69}$$

Note that s_a is orthogonal to u^a and we have $u^a s_a = 0$, but this is not the case for the adiabatic covector since $u^a \sigma_a = \dot{\sigma}$.

The geometrical variables which describe the field perturbations are obtained by taking the spatially projected version of the above equations

$$\sigma^\perp_a \equiv e^I_\sigma D_a \varphi_I = \sigma_a + \dot{\sigma} u_a\,, \tag{2.70}$$

$$s^\perp_a \equiv e^I_s D_a \varphi_I = s_a\,. \tag{2.71}$$

Note that these fluctuations are well defined non-perturbatively.

From the Klein-Gordon equations for the ϕ and χ fields, we can find the adiabatic and entropy components of the Klein-Gordon equations. Using these equations we can find the evolution equation for the adiabatic σ_a and entropy s_a covectors. The resulting equation for the adiabatic component is [21]

$$
\begin{aligned}
(\ddot{\sigma}_a)^{\perp} &+ \Theta(\dot{\sigma})^{\perp}_a + \dot{\sigma} D_a \Theta + \big(V_{,\sigma\sigma} + \dot{\theta}\frac{V_{,s}}{\dot{\sigma}}\big)\sigma^{\perp}_a - D_a(\nabla^c \sigma^{\perp}_c), \\
&= \big(\dot{\theta} - \frac{V_{,s}}{\dot{\sigma}}\big) s_a)^{\cdot} + \big(\ddot{\theta} - V_{,s\sigma} + \Theta\dot{\theta}\big) s_a - D_a Y_{(s)}\,,
\end{aligned}
\tag{2.72}
$$

where

$$
Y_{(s)} = \frac{1}{\dot{\sigma}}\big(\dot{s}_a + \dot{\theta}\sigma^{\perp}_a\big) s^a\,. \tag{2.73}
$$

The equation for the entropy component is [21]

$$
\begin{aligned}
\ddot{s}_a &+ \big(\Theta - \frac{1}{\dot{\sigma}}(\nabla^c \sigma^{\perp}_c - Y_{(s)})\big)\dot{s}_a + \big(V_{,ss} + \dot{\theta}^2 - 2\dot{\theta}\frac{V_{,s}}{\dot{\sigma}}\big) s_a \\
&\quad - D_a(\nabla_c s^c) \\
&= \frac{\dot{\theta}}{\dot{\sigma}}\big(D_a \Pi - \frac{\dot{\Pi}}{\dot{\sigma}}\sigma^{\perp}_a - 2\epsilon_a\big) - \frac{1}{\dot{\sigma}}\big(D_c s^c + Y_{(\sigma)}\big)^{\cdot}\sigma^{\perp}_a \\
&\quad + D_a Y_{(\sigma)}\,,
\end{aligned}
\tag{2.74}
$$

where ϵ_a is the covector associated with the comoving energy density perturbation and

$$
Y_{(\sigma)} \equiv \frac{1}{\dot{\sigma}}\big(\dot{s}_a + \dot{\theta}\sigma^{\perp}_a\big)\sigma^{\perp,a}\,. \tag{2.75}
$$

The first approximation we make is to linearize these equations (the expansion parameter is the amplitude of the fluctuations, which in our case is proportional to $\hbar$.) This yields greatly simplified equations

$$
\begin{aligned}
(\ddot{\sigma}_a)^{\perp} &+ 3H(\dot{\sigma})^{\perp}_a + \dot{\sigma} D_a \Theta + \big(V_{,\sigma\sigma} - \dot{\theta}^2\big)\sigma^{\perp}_a - D_a(D^c \sigma^{\perp}_c) \\
&\simeq 2(\dot{\theta} s_a)^{\cdot} - 2\frac{V_{,s}}{\dot{\sigma}}\dot{\theta} s_a\,,
\end{aligned}
\tag{2.76}
$$

and

$$\ddot{s}_a + 3H\dot{s}_a + \left(V_{,ss} + 3\dot{\theta}^2\right)s_a - D_a(D_c s^c) \simeq -2\frac{\dot{\theta}}{\dot{\sigma}}\epsilon_a \,. \tag{2.77}$$

The final approximation we make is to focus on long wavelengths, i.e. we work in the leading order gradient expansion in which also the comoving energy density fluctuation vanishes. This yields our final equations

$$(\ddot{\sigma}_a)^{\perp} + 3H(\dot{\sigma})^{\perp}_a + \dot{\sigma}D_a\Theta + \left(V_{,\sigma\sigma} - \dot{\theta}^2\right)\sigma^{\perp}_a \simeq 2(\dot{\theta}s_a)^{\cdot} - 2\dot{\theta}\frac{V_{,\sigma}}{\dot{\sigma}}s_a, \tag{2.78}$$

and

$$\ddot{s}_a + 3H\dot{s}_a + \left(V_{,ss} + 3\dot{\theta}^2\right)s_a \simeq 0\,. \tag{2.79}$$

As is well known in the linear theory of cosmological perturbations (see e.g. [10] for an overview and [25] for an introduction), the entropy fluctuations are not affected by the amplitude of the adiabatic perturbation. On the other hand, entropy fluctuations induce a growing adiabatic mode.

2.4 Conclusion

In this chapter we have reviewed *Inflation* as the standard paradigm of early universe physics and also the theory of *cosmological perturbations*. First we introduced the Big Bang cosmology and the initial condition problems and discussed inflationary cosmology as a solution to the problems. Then we introduced perturbations in the matter and gravity sectors in the context of Einstein's theory of General Relativity.

We then solved for the perturbed field equations. We introduced gauge-invariant variables and discussed curvature perturbations in detail. Then we extended our discussion to multi-field cases and introduced entropy perturbations. Finally as a geometrical approach to cosmological perturbations, we introduced the covariant formalism and discussed the non-linear evolution of perturbations in this formalism.

Chapter 3

REHEATING OF THE UNIVERSE

3.1 Introduction

Inflation leaves the cosmos in a non-thermal state, very cold and effectively empty of matter. Hence any successful inflationary model must explain how the inflationary phase is connected to the high temperatures at Big Bang Nucleosynthesis as well as explaining the production of the Standard Model particles and Baryogenesis. The *reheating* mechanism has been introduced to inflationary cosmology in order to explain transfer of energy which is stored in the inflaton field to other dynamical degrees of freedom in the universe and render the universe hot (for recent reviews see [26, 27]). Therefore reheating is an integral part of any inflationary model, without which inflation would not be a viable early universe scenario. In this chapter we present a review of different reheating mechanisms with more emphasis on *parametric resonance.*

3.2 Perturbative Reheating

Reheating was initially discussed using perturbation theory in Quantum Field Theory using perturbative single-particle decay [28]. In this scenario, the inflaton field which oscillates around its minimum after inflation is considered as a collection of scalar particles which each decay to Standard Model particles and radiation. Let us study a toy model to have a better understanding of the process. To make sure that the energy transfer from the inflaton field eventually completes, we consider the trilinear interaction of the form

$$\mathcal{L}_{int} = -\sigma\phi\chi^2,$$

where χ is the scalar decay product and the coupling constant σ has dimensions of mass. Then the decay rate is

$$\Gamma_{\phi\to\chi\chi} = \frac{\sigma^2}{8\pi m_\phi}. \tag{3.1}$$

Without considering the energy loss due to the decay, the equation of motion for the inflaton field is

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0. \tag{3.2}$$

One may take into account the energy loss due to the decay, by inserting a damping term in the equation of motion[1]

$$\ddot{\phi} + 3H\dot{\phi} + \Gamma\dot{\phi} + V_{,\phi} = 0. \tag{3.3}$$

As long as the decay rate is smaller than the Hubble parameter, the decay products are redshifted very fast and particle production is not efficient. But when the Hubble parameter drops to the value $H \sim \Gamma$ and below, production becomes efficient and the inflaton field will decay. It is assumed that the decay products will eventually thermalize. To find the temperature at the end of reheating, as an estimate we consider all the energy density of the universe (or at least the dominant part of that) to be in radiation. Therefore

$$\rho = \frac{g_*\pi^2}{30}T^4 = 3M_P^2H^2, \tag{3.4}$$

[1] Note that the effects of fluctuations are not taken into account this way and this is one of the problems of the perturbative reheating.

where $g_* \sim 10^2$ is the number of relativistic degrees of freedom for the Standard Model at high energies. As an upper bound one can use $\Gamma \sim H$ and find an expression for the reheating temperature

$$T_{reh} \simeq \frac{1}{2}\sqrt{M_P\Gamma}. \tag{3.5}$$

3.2.1 Problems with the Perturbative Analysis

One of the issues with the perturbative approach is that it does not take into account the coherent nature of the inflaton oscillations at the end of inflation. In the language of quantum mechanics, the inflaton field is in a coherent state and we can treat it classically. In the next section we will show how dramatically the reheating scenario will change by taking into account the coherency of the inflaton field after inflation.

Another issue which is related to above mentioned problem concerns application of S-matrix theory in QFT which is the building block for perturbative reheating. S-matrix theory is developed in QFT to study scattering processes and connect the initial state of the system to the final state. In S-matrix theory, we assume a "small number" of particles which have a "space-like separation" from each other in both initial and final state, and then through a unitary transformation (S-matrix) we connect the two states. In our case, the initial state is the state of the inflaton field at the end of inflation and clearly we can see that the coherent state is not a state with a small number of particles with space-like separation. Instead it is a condensate of a homogeneous field which coherently oscillates and can be treated classically.

Therefore the initial state does not satisfy the requirements of S-matrix theory.

3.3 Parametric Resonance and Preheating

As was initially discussed [29], the coherent nature of the inflaton field oscillations drives *parametric resonance* which is very efficient in the energy transfer from the inflaton field. This period which is happening right after inflation is named *preheating* [30]. It can be considered as the first (linear) stage of reheating which is followed by non-linear dynamics and the thermalization process. In what follows we will discuss in detail how coherent oscillations of the inflaton field induce instabilities in the equation of motion for the fluctuations. Note that in this chapter we only consider matter perturbations and neglect the gravitational effects. We will include gravitational effects in our analysis in chapter 5.

3.3.1 A First Look at Parametric Resonance

We present the preheating mechanism with a simple toy model with Lagrangian

$$V(\phi,\chi) = \frac{1}{2}m_\phi^2\phi^2 + \frac{1}{2}m_\chi^2\chi^2 + \frac{1}{2}g^2\chi^2\phi^2, \tag{3.6}$$

where ϕ is the inflaton field which is coupled to a subdominant scalar field χ via the interaction term $g^2\chi^2\phi^2$. We also assume that the homogeneous value of the χ field is negligible. As we discussed in the previous section, the inflaton field can be treated as a classical field at the end of inflation. Thus, in the setup of Lagrangian Eq. (3.6) the classical field ϕ acts as an external force (time-dependent effective mass) in the evolution of the *quantum field* χ. Therefore we are going to study quantum production of χ particles in the external classical background of the inflaton field. The amplitude of inflaton field oscillations is in principle decreasing with time due to the expansion of the universe. However, in this subsection we are going to neglect the expansion of space and treat the problem as in a Minkowski background. Since the time period of preheating is very short compared to the Hubble time (H^{-1}) typically, this is a good approximation. We will consider effects of the expanding background in the next subsection.

In a Minkowski background the inflaton field oscillates as

$$\phi(t) = \Phi \sin(m_\phi t), \tag{3.7}$$

where Φ is the amplitude of oscillations. Since (3.6) is quadratic in χ, the Fourier modes of χ evolve independently. We use the Fourier expansion in the form

$$\chi(x,t) = \int \frac{d^3k}{(2\pi)^{3/2}V^{1/2}} \chi_k(t) e^{ikx}, \tag{3.8}$$

where V is a normalization volume. Then the modes χ_k satisfy the equation[2]

$$\ddot{\chi}_k + (k^2 + m_\chi^2 + g^2\Phi^2 \sin^2(m_\phi t))\chi_k = 0. \tag{3.9}$$

From the mathematical point of view this is the equation of a harmonic oscillator with a time-varying mass. If the mass term is a periodic function of time, Eq.(3.9) is known as *Hill's equation*. Working with the dimensionless time variable $z \equiv m_\phi t$,

[2] Note that we neglect the expansion of the universe, and we normalize the scale factor at the end of inflation to be $a_0 = 1$.

we can rewrite this equation in the conventional form of

$$\chi_k'' + (A_k - 2q\cos(2z))\chi_k = 0, \tag{3.10}$$

where $A_k \equiv \frac{2q(k^2+m_\chi^2)}{m_\phi^2}$ and $q \equiv \frac{g^2\Phi^2}{4m_\phi^2}$. If the mass term is harmonic and not only periodic, as in Eq. (3.10), then the Hill equation is known as the *Mathieu equation*. The *Floquet theorem* says that the Mathieu equation has a solution of the form [31]

$$\chi_k = e^{\mu(k)z}P_1(z) + e^{-\mu(k)z}P_2(z), \tag{3.11}$$

where $\mu(k)$ is so the called *Floquet exponent*. In general it is a complex valued function of the momentum k which is determined by parameters A_k and q. The functions $P_{1,2}$ are periodic functions of z. If $\mu(k)$ is real then the dominant mode of Eq.(3.11) is exponentially growing (unstable) while if it is pure imaginary the mode functions are just oscillating (stable). As we said, $\mu(k)$ is determined by A_k and q which depend on momentum, amplitude of the inflaton oscillations, the inflaton and preheat field's masses and the coupling constant. Therefore in parameter space there are different “stable" and “unstable" regions. Figure 3-1 displays stable and unstable regions in the parameter space of the theory of Eq.(3.6)

Exponential growth of the mode functions χ_k means efficient particle production and growth of the number density of created particles n_k. Knowing the energy of each mode $E_k = \frac{1}{2}(|\dot{\chi}|^2 + \omega_k^2|\chi_k|^2)$, one can write the expression for the number density

$$n_k = \frac{\omega_k}{2}(\frac{|\dot{\chi}|^2}{\omega_k^2} + |\chi_k|^2) - \frac{1}{2}. \tag{3.12}$$

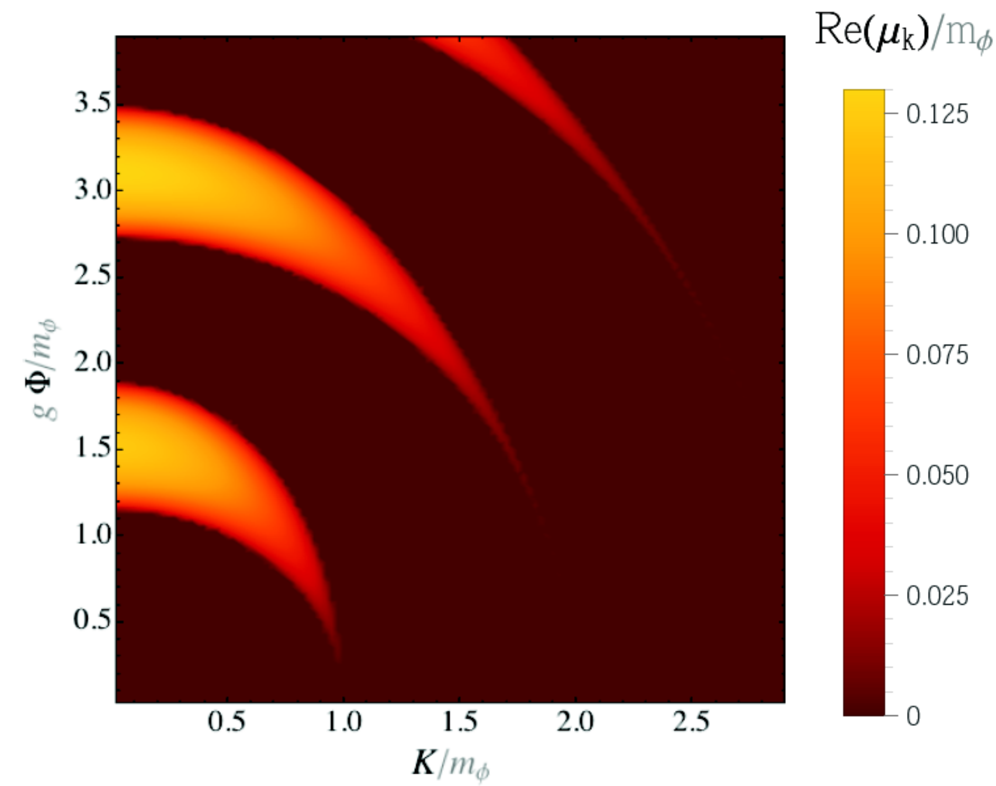

Figure 3–1: Structure of instability bands for the theory of Eq.(3.6). The horizontal axis is the rescaled momentum where $K = \sqrt{(k^2 + m_\chi^2)}$. The real part of $\mu(k)$ has been rescaled by the mass of the inflaton as well as the inflaton amplitude [26].

Considering $\chi_k \propto e^{\mu z}$, then for modes in the instability bands the number density is $n_k \propto e^{2\mu z}$. It is clear that the efficient particle production we discussed only happens for certain momenta, and therefore it leads to a highly non-thermal state and will be followed by a thermalization process in later stages.

Depending on the parameters, if only one narrow instability band contributes to the resonance then particle production is in the so called *narrow resonance* regime. While if broad ranges of momenta contribute, it is in the *broad resonance* regime.

Broad resonance happens when $q \gg 1$ whereas for narrow resonance usually $q \leqslant 1$. Although mathematically the above-mentioned instability structure of resonance works for both narrow and broad resonances, it has been shown [30] that the perturbative expansion of the solution does not converge in the case of broad resonance. Instead for the large values of q parameter, we should focus on the violation of the adiabaticity condition $\frac{\dot{\omega}_k}{\omega_k^2} \gg 1$ for particle production[3] . Remembering $q = \frac{g^2\Phi^2}{4m_\phi^2}$, in our case to obtain broad resonance, we need either a large amplitude of oscillations or a small mass for the inflaton field.

In the broad resonance regime, then during most of the period of oscillation, the effective mass of the χ field ($m_\chi^2(t) = m_\chi^2 + g^2\phi^2(t)$)is much greater than the inflaton field mass m_ϕ, and the system is in the adiabatic regime. Considering a hierarchy of $m_\phi \gg m_\chi$ for the bare masses of the fields, we find that only for a short period of time in the vicinity of $\phi = 0$ a violation of the adiabaticity condition occurs and significant particle production happens. Then again as the inflaton field evolves, the system returns to the adiabatic regime and particle number become invariant. This evolution happens in each oscillation until resonance gets shut off by backreaction effects. The evolution of the resonance is contrasted in the narrow and broad regimes in Figure 3-2.

[3] From the WKB approximation we know if the frequency $\omega_k = (k^2 + m_x^2 + g^2\Phi^2 \sin^2(m_\phi t))^{1/2}$ is a slowly varying function of time, we are in the adiabatic regime ($\frac{\dot{\omega}_k}{\omega_k^2} \ll 1$). In the adiabatic regime the particle number is an adiabatic invariant, meaning there is no particle production. Instead in the non-adiabatic regime ($\frac{\dot{\omega}_k}{\omega_k^2} \gg 1$), significant particle production is expected [30].

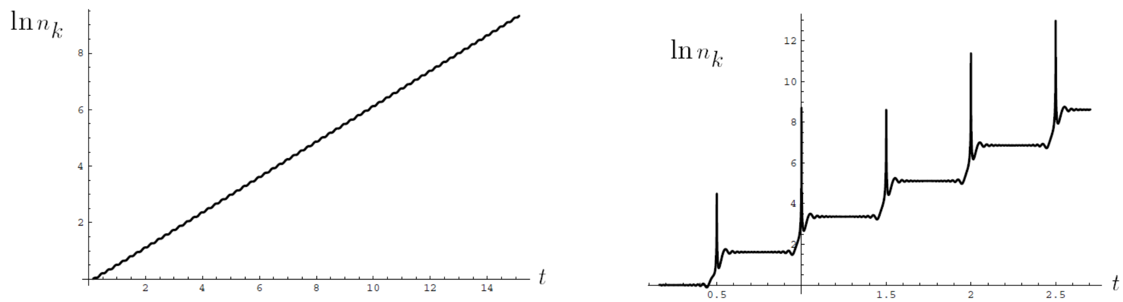

Figure 3–2: Number density of the produced particles in the case of narrow (left panel) and broad (right panel) resonance in a Minkowski background. Note the qualitative differences in the evolution of the resonance in the two cases[30].

In the expanding background, the situation is even more complicated, and resonance cannot begin in the narrow resonance regime. Next we are going to introduce a formalism to study the theory of broad resonance which in turn is applicable to both Minkowski and expanding backgrounds.

3.3.2 The Theory of Broad Resonance

In the more realistic case of an expanding background, resonance begins in the broad band regime and then, as the amplitude of the oscillations of the inflaton field decreases, it will be followed by a narrow resonance regime which in turn may eventually be followed by perturbative decay. In the spirit of Ref. [30] we are going to introduce a formalism to study the efficient particle production in the broad resonance regime. The formalism has roots in the conventional quantum field theory description of particle production in a time-dependent background [32, 33, 34], namely making use of adiabatic eigenfunctions and *Bogoliubov coefficients*[4] . Noting our discussion in previous subsection, it is clear why the adiabatic approximation is reasonable in the case of broad resonance and particle production happens only in a short period in the vicinity of $\phi = 0$. We will see that the process is similar to the case of scattering from a parabolic potential.

We begin with the more general case of an expanding background in which Eq. (3.9) has the form

$$\ddot{\chi}_k + 3H\dot{\chi}_k + (\frac{k^2}{a^2} + m_x^2 + g^2\Phi^2(t)\sin^2(m_\phi t))\chi_k = 0. \tag{3.13}$$

If we translate the violation of adiabaticity condition in terms of momenta, we get

$$\frac{k^2}{a^2} \leqslant \frac{2}{3\sqrt{3}} gm\Phi(t) - m_\chi^2. \tag{3.14}$$

By the field redefinition of $X_k = a^{\frac{3}{2}}\chi_k$ we can absorb the friction term in Eq. (3.13) and rewrite the equation as

$$\ddot{X}_k + \omega_k^2 X_k = 0, \tag{3.15}$$

where

$$\omega_k^2 = \frac{k^2}{a^2} + m_\chi^2 + g^2\Phi^2(t)\sin^2(m_\phi t) - \frac{3}{4}(\frac{\dot{a}}{a})^2 - \frac{3}{2}\frac{\ddot{a}}{a}. \tag{3.16}$$

[4] Bogoliubov coefficients transform creation and annihilation operators in a way that the Hamiltonian is diagonalized at each time t [33].

One can show that the last two terms cancel out after inflation. Also from now on we will consider $m_\chi = 0$ for simplicity. Therefore we can rewrite Eq. (3.15) as

$$\ddot{X}_k + (\frac{k^2}{a^2} + g^2\Phi^2(t)\sin^2(m_\phi t))X_k = 0. \tag{3.17}$$

The evolution of the preheat field is adiabatic for all times in each oscillation of the inflaton field except for a very short period of time in the vicinity of $\phi = 0$. Therefore one can write the solution of Eq. (3.17) in the adiabatic approximation

$$X_k(t) = \frac{\alpha_k(t)}{\sqrt{2\omega_k}} e^{-i\int \omega_k dt} + \frac{\beta_k(t)}{\sqrt{2\omega_k}} e^{+i\int \omega_k dt}, \tag{3.18}$$

where the coefficients α_k and β_k are the Bogoliubov transformation coefficients with the normalization

$$|\alpha_k|^2 - |\beta_k^2| = 1. \tag{3.19}$$

Note that vacuum initial condition at t_i implies $\alpha_k = 1$ and $\beta_k = 0$. Then the particle density in mode k at any time t is given by $n_k = |\beta_k|^2$. One can find the total number density per comoving volume

$$n_\chi = \frac{1}{(2\pi a)^3} \int_0^\infty d^3k |\beta_k|^2. \tag{3.20}$$

Also the energy density will be

$$\rho_\chi = \frac{1}{(2\pi a)^3} \int_0^\infty d^3k \frac{\omega_k}{a} |\beta_k|^2. \tag{3.21}$$

Therefore as we can see, the problem of particle production is reduced to finding the Bogoliubov coefficients. Next we will give an introduction to Bogoliubov transformations to make more clear the relation between particle production and Bogoliubov coefficients [33].

Bogoliubov transformations

One can write X_k in terms of the mode functions

$$X_k(t) = a_k v_k^*(t) + a_{-k}^\dagger v_k(t), \tag{3.22}$$

where v^*, v are a basis for the solution space of Eq.(3.15) and normalized in a way to satisfy the condition

$$v_k' v_k^* - v_k v_k'^* = 1. \tag{3.23}$$

Inserting Eq.(3.21) into the Fourier expansion of the rescaled field X_k gives

$$X(x,t) = \int \frac{d^3k}{(2\pi)^{3/2} V^{1/2}} (a_k v_k^* e^{ikx} + a_k^\dagger v_k e^{-ikx}). \tag{3.24}$$

Now we follow the canonical quantization procedure and promote X field to a quantum field operator $\hat{X}$ and introduce the canonical conjugated momentum $\hat{\pi}$ by imposing the commutation relations

$$[\hat{X}(x,t), \hat{\pi}(y,t)] = i\delta^3(x-y), \tag{3.25}$$

$$[\hat{X}(x,t), \hat{X}(y,t)] = [\hat{\pi}(x,t), \hat{\pi}(y,t)] = 0. \tag{3.26}$$

It is now straightforward to promote a_k and $a_k^\dagger$ to operators and find the commutation relations

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta^3(k-k'), \quad [\hat{a}_k, \hat{a}_{k'}] = [\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger] = 0. \tag{3.27}$$

Now it is obvious that we can interpret a_k and $a_k^\dagger$ as creation and annihilation operators. Given a_k and $a_k^\dagger$, we can construct a basis of the Hilbert space of quantum states. But the basis is not unique. If v_k and v_k^* solve the equation

$$\ddot{v}_k + \omega_k^2 v_k = 0, \tag{3.28}$$

then a linear combination of the two i.e.

$$u_k = \alpha_k v_k + \beta_k v_k^*, \tag{3.29}$$

will have the property that u_k and u_k^* also satisfy similar condition of Eq. (3.22) as long as the coefficients obey (3.19). Therefore the choice of creation and annihilation operators b_k and $b_k^\dagger$ corresponding to u_k mode functions is equally preferable to the set of a_k and $a_k^\dagger$, although they single out different vacua. Using Eq. (3.21)

$$a_k v_k^* + a_{-k}^\dagger v_k = b_k u_k^* + b_{-k}^\dagger u_k. \tag{3.30}$$

Then from Eq(3.28) we find the expression for the transformation between a_k and $a_k^\dagger$ and the set of b_k and $b_k^\dagger$

$$a_k = \alpha_k^* b_k + \beta_k b_k^\dagger, \quad a_k^\dagger = \alpha_k b_k^\dagger + \beta_k^* b_k, \tag{3.31}$$

which is the so-called Bogoliubov transformation, and α_k,β_k are the Bogoliubov coefficients. To compute the Bogoliubov coefficients we need to know u_k and v_k and their first derivatives at some time t_1

$$u_k(t_1) = \alpha_k v_k(t_1) + \beta_k v_k^*(t_1), \tag{3.32}$$

$$u'_k(t_1) = \alpha_k v'_k(t_1) + \beta_k v'^*_k(t_1). \tag{3.33}$$

Therefore we find the Bogoliubov coefficients

$$\alpha_k = \frac{W(u_k, v^*_k)}{2i}, \tag{3.34}$$

$$\beta_k = \frac{W(v_k, u_k)}{2i}, \tag{3.35}$$

where

$$W(v_k, u_k) = v_k u'_k - u_k v'_k,$$

is the Wronskian function.

Particle production in terms of wave propagation in parabolic potentials

Let us label the times when ϕ passes zero by t_j where j runs from 1 to N and counts the number of zero-crossings of the inflaton field. Therefore the adiabaticity condition is violated around t_j, and then before and after this time the adiabatic approximation is valid. So before scattering at t_j we have

$$X^j_k(t) = \frac{\alpha^j_k}{\sqrt{2\omega_k}} e^{-i\int \omega_k dt} + \frac{\beta^j_k}{\sqrt{2\omega_k}} e^{+i\int \omega_k dt}. \tag{3.36}$$

Then after scattering (zero-crossing) and in the time interval $t_j < t < t_{j+1}$ the mode function has the form

$$X^{j+1}_k(t) = \frac{\alpha^{j+1}_k}{\sqrt{2\omega_k}} e^{-i\int \omega_k dt} + \frac{\beta^{j+1}_k}{\sqrt{2\omega_k}} e^{+i\int \omega_k dt}. \tag{3.37}$$

The Bogoliubov coefficients only vary in the vicinity of t_j, t_{j+1} and so on, and during the rest of the time they are time-independent. Considering X^j as the incoming wave and X^{j+1} as the scattered outgoing wave, one can find the Bogoliubov coefficients with the label $j+1$ in terms of those with the label j and the reflection and transmission amplitudes R_k and D_k

$$\begin{pmatrix} \alpha_k^{j+1} e^{-i\int_0^{t_j}\omega_k dt} \\ \beta_k^{j+1} e^{+i\int_0^{t_j}\omega_k dt} \end{pmatrix} = \begin{pmatrix} \frac{1}{D_k} & \frac{R_k^*}{D_k^*} \\ \frac{R_k}{D_k} & \frac{1}{D_k^*} \end{pmatrix} \begin{pmatrix} \alpha_k^{j} e^{-i\int_0^{t_j}\omega_k dt} \\ \beta_k^{j} e^{+i\int_0^{t_j}\omega_k dt}. \end{pmatrix} \tag{3.38}$$

The integral $\int_0^{t_j}\omega_k dt$ measures the accumulated phase by the time t_j. Then to continue we note that in the vicinity of t_j one can Taylor expand and write $\sin^2(m_\phi t) \sim m_\phi^2(t-t_j)^2$ to quadratic order. Therefore Eq.(3.17) can be written as

$$, \ddot{X}_k^j + (\frac{k^2}{a(t_j)^2} + g^2\Phi_j^2 m^2(t-t_j^2))X_k^j = 0. \tag{3.39}$$

In the vicinity of t_j we can consider $\Phi(t) = \Phi_j$ to be time-independent. Then by introducing the time variable

$$\tau = (g\Phi_j m)^{\frac{1}{2}}(t-t_j),$$

and working with the rescaled momentum $\kappa_j = k(a^2(t_j)g\Phi_j m)^{-\frac{1}{2}}$, we get

$$\frac{d^2X_k^j}{d\tau^2} + (\kappa_j^2 + \tau^2)X_k^j = 0. \tag{3.40}$$

In the language of quantum mechanics, this equation describes scattering from a negative parabolic potential in analogy with the time-independent Schrödinger equation [3]

$$\nabla^2\psi + (E-V)\psi = 0, \tag{3.41}$$

where ψ is a wave function with energy E in a parabolic potential $V = -\tau^2$. For the ϕ values in the vicinity of t_j quantum tunneling occurs and one can solve the problem with standard methods of quantum mechanics. Therefore the reflection and transmission amplitudes are given [30] by

$$R_k = -\frac{ie^{i\varphi_k}}{\sqrt{1+e^{\pi\kappa^2}}}, \tag{3.42}$$

$$D_k = \frac{ie^{-i\varphi_k}}{\sqrt{1+e^{-\pi\kappa^2}}}, \tag{3.43}$$

where

$$|R_k|^2 + |D_k|^2 = 1, \tag{3.44}$$

and φ_k is given by

$$\varphi_k = \arg\Gamma(\frac{1+i\kappa^2}{2}) + \frac{\kappa^2}{2}(1+\ln\frac{2}{\kappa^2}). \tag{3.45}$$

Now we can rewrite Eq.(3.37)

$$\begin{pmatrix}\alpha_k^{j+1}\\ \beta_k^{j+1}\end{pmatrix} = \begin{pmatrix} e^{i\varphi_k}\sqrt{1+e^{-\pi\kappa^2}} & ie^{-(\pi/2)\kappa^2+2i\int_0^{t_j}\omega_k dt} \\ -ie^{-(\pi/2)\kappa^2-2i\int_0^{t_j}\omega_k dt} & e^{-i\varphi_k}\sqrt{1+e^{-\pi\kappa^2}} \end{pmatrix} \begin{pmatrix}\alpha_k^{j}\\ \beta_k^{j}\end{pmatrix}. \tag{3.46}$$

From this equation one can find the number of created particles after a zero-crossing of the inflaton field. Since $n_k^{j+1} = |\beta_k^{j+1}|^2$ we find

$$n_k^{j+1} = e^{-\pi\kappa^2} + (1+2e^{-\pi\kappa^2})n_k^j - 2e^{(-\pi/2)\kappa^2}\sqrt{1+e^{-\pi\kappa^2}}\sqrt{n_k^j(1+n_k^j)}\sin\theta_k^j, \tag{3.47}$$

where $\theta_k^j = 2\int_0^{t_j}\omega_k dt - \varphi_k + \arg\beta_k^j - \arg\alpha_k^j$ includes all the information about phase differences for the waves. We note that the exponential factor $e^{-\pi\kappa^2}$ is very important. If $\pi\kappa^2 > 1$ it is obvious that particle production is suppressed. So we need $\pi\kappa^2 \le 1$

for significant particle production. We can translate this condition in terms of the momentum k

$$\frac{k^2}{a^2} \leq \frac{g m_\phi \Phi}{\pi}, \tag{3.48}$$

which is consistent with our earlier result Eq.(3.14) and gives an estimate of the resonance band. Another point in Eq.(3.46) is that $\kappa^2 \propto g^{-1}$ therefore the exponential term $e^{-\pi\kappa^2}$ is not analytical at the limit of small coupling constant $g \to 0$. Therefore particle production in the broad resonance regime in such a nonconformal theory cannot be derived via a perturbative analysis, and hence the nonperturbative nature of the resonance is manifest.

We can define the growth index μ_k^j via

$$n_k^{j+1} = n_k^j e^{2\pi\mu_k^j}. \tag{3.49}$$

In the limit of $n_k^j \gg 1$ we can rewrite Eq.(3.46)

$$n_k^{j+1} \approx n_k^j (1 + 2e^{-\pi\kappa^2} - 2\sin(\theta_k^j) e^{-(\pi/2)\kappa^2} \sqrt{1 + e^{-\pi\kappa^2}}). \tag{3.50}$$

Comparing the last two equation gives

$$\mu_k^j = \frac{1}{2\pi} \ln(1 + 2e^{-\pi\kappa^2} - 2\sin(\theta_k^j) e^{-(\pi/2)\kappa^2} \sqrt{1 + e^{-\pi\kappa^2}}). \tag{3.51}$$

Let us have a careful look of this equation. If the argument of the logarithmic function is greater than 1 then the growth index is positive and particle production is efficient. This happens only if

$$e^{-(\pi/2)\kappa^2} > \sin(\theta_k^j) \sqrt{1 + e^{-\pi\kappa^2}},$$

$$\frac{1}{\sqrt{1+e^{\pi\kappa^2}}} > \sin\theta_k^j. \qquad (3.52)$$

Therefore particle production depends significantly on the phase θ_k^j and nature of the interference of the wave functions. The number of particles after a scattering can increase or even decrease, as is shown in Figure 3-3. The effect of particle production is maximal when $\sin\theta_k^j = -1$ ($\mu \approx 0.28$). If the interference is destructive and Eq.(3.51) does not hold, the number of particle decreases. This happens when $\sin\theta_k^j = 1$. One can find a typical value for the growth index $\mu_k^j \approx 0.175$ by considering $\sin\theta_k^j = 0$. In the case of Minkowski space where both $\Phi(t)$ and $a(t)$ are constant, the phase only depends on the momentum and one can expect separate instability/stability bands of momenta. However, in the case of an expanding background, the phase is time-dependent as well as being momentum-dependent, and hence the separation of the instability/stability bands is washed out as the bands become time-dependent.

So far we have considered particle production after only one zero-crossing of the inflaton field. Now we are going to consider the number density of produced particles after multiple oscillations of the inflaton field. We consider the initial condition $\alpha_k^0 = 1$, $\beta_k^0 = 0$, $n_k^0 = 0$ and random initial phases θ_k^0. After some oscillations of the inflaton field

$$n_k(t) = \frac{1}{2}e^{2\pi\sum_j \mu_k^j}. \qquad (3.53)$$

As an estimate one can rewrite $\sum_j \mu_k^j = \mu_k^{eff} N$, where N is the number of oscillations of the inflaton. If the period of oscillations is given by T, then at the time t after inflation

$$N = \frac{t}{T/2}, \qquad (3.54)$$

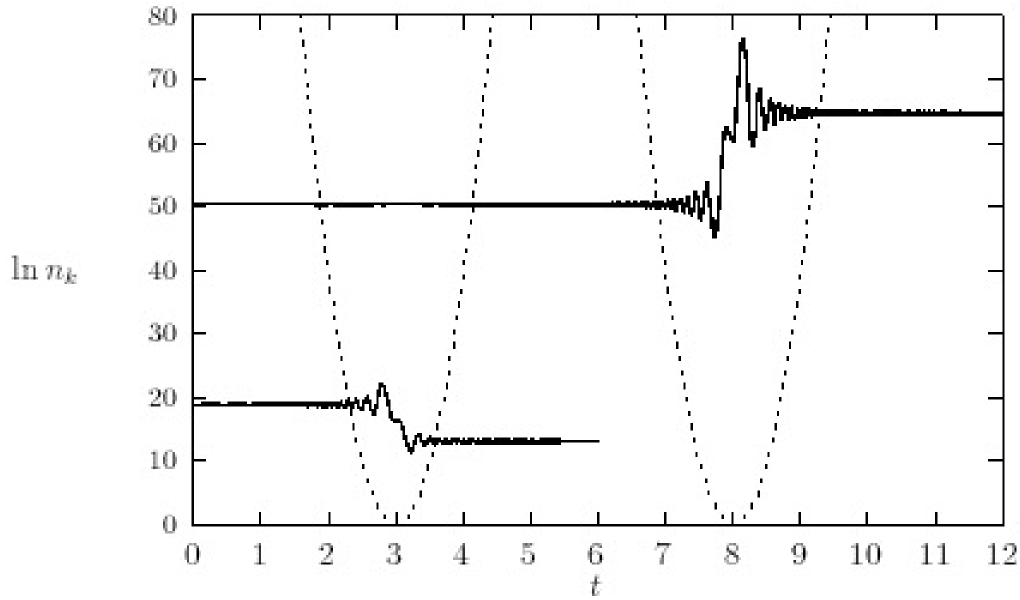

Figure 3–3: Number density after scattering for two different phases. Constructive interference corresponds to the case when the number density increases after scattering, while it is destructive when the number density decreases. The time is in the units of $2\pi/\kappa$ and the dotted lines show the time of the scattering event[30].

where $T = \frac{2\pi}{m_\phi}$. Therefore

$$\sum_j \mu_k^j = \mu_k \frac{m_\phi t}{\pi}, \tag{3.55}$$

where we dropped the index " *eff* " for simplicity. Then Eq.(3.52) gives

$$n_k(t) = \frac{1}{2} e^{2\mu_k m_\phi t}. \tag{3.56}$$

Therefore the total number density is given by integration of n_k over all momenta which experience the resonance:

$$n_X(t) = \frac{1}{2(2\pi a)^3} \int_0^{k_{max}} d^3k e^{2\mu_k m_\phi t}. \tag{3.57}$$

Since the integrand is angle-independent, the total number of produced particles during broad resonance regime is given by

$$n_X(t) = \frac{1}{4\pi^2 a^3}\int_0^{k_{max}} dk k^2 e^{2\mu_k m_\phi t}. \tag{3.58}$$

Then the energy density of the produced particles is given by integrating $n_k(t)\frac{k}{a}$ over all momenta which are in the resonance band

$$\rho_X(t) = \frac{1}{4\pi^2 a^4}\int_0^{k_{max}} dk k^3 e^{2\mu_k m_\phi t}. \tag{3.59}$$

In order to estimate the amount of the particle production, let us calculate the integral of Eq.(3.57) and find the number density of the produced particles as a function of time. To solve the integral approximately, we can rewrite the integrand as follows

$$k^2 e^{2\mu_k m_\phi t} = e^{2\ln k + 2\mu_k m_\phi t}.$$

We can Taylor expand the exponent around the maximal value of μ_k at k_* where $\mu_{k_*} \equiv \mu_*$. Then the integral has the form

$$n_X(t) \simeq \frac{k_*^2 e^{2m_\phi \mu t}}{4\pi^2 a^3}\int_0^{k_{max}} dk e^{-(m_\phi t|\mu_k''(k_*)| + \frac{1}{k_*^2})(k-k_*)^2}, \tag{3.60}$$

$$n_X(t) \simeq \frac{k_*^3}{(2\pi^{1/2}a)^3} e^{2m_\phi \mu t}(m_\phi t\mu_* + 1)^{-\frac{1}{2}} \operatorname{erf}(\sqrt{1 + m_\phi t\mu_*}). \tag{3.61}$$

Knowing that $\operatorname{erf}(\sqrt{1 + m_\phi t\mu_*}) \simeq 1$, our expression for the number density of produced particle during resonance is

$$n_X(t) \simeq \frac{k_*^3 e^{2m_\phi t\mu_*}}{(2\pi^{1/2}a)^3\sqrt{1 + m_\phi t\mu_*}}. \tag{3.62}$$

Then, from Eq. (3.58), the energy density will be

$$\rho_X(t) \simeq \frac{k_*^4 e^{2m_\phi t\mu_*}}{8\pi^{3/2}a^4\sqrt{\frac{3}{2}+m_\phi t\mu_*}}. \tag{3.63}$$

Therefore it is enough to just find k_* at which μ_k has its maximum and the value of the growth index at the maximum. Next will discuss the effects which terminate the resonance and hence particle production.

3.4 Backreaction and the End of Resonance

We will now consider some backreaction effects which could terminate or even suppress preheating and particle production. First of all, the fluctuations of χ which are generated in the preheating instability will back-react on the background and lead to correction terms both in the equation of motion for ϕ and even in that of χ_k fluctuations. We need to determine the length of time these effects can be neglected. Secondly, fluctuations in the inflaton field themselves could be amplified and then back-react both on the equation of motion for ϕ, shutting off the oscillations which drive preheating, and also on the equation of motion for χ, providing corrections to the mass term which will prevent the violation of adiabaticity which is required to obtain the resonance. When these backreaction effects become important then our linear analysis won't be valid and nonlinear dynamics is needed to study the problem.

3.4.1 Corrections Due to Amplification of χ Field Fluctuations

Amplified χ_k fluctuations will lead to corrections in the effective mass of the inflaton field which in turn will terminate coherent oscillations of the inflaton field and hence shut off the resonance. From Eq.(3.6) the equation of motion for the background inflaton field during reheating while the χ_k fluctuations get amplified is

$$\ddot{\phi} + 3H\dot{\phi} + (m_\phi^2 + g^2 \left\langle \chi_k^2 \right\rangle)\phi = 0, \tag{3.64}$$

where $\langle \chi_k^2 \rangle$ is the averaged squared of the preheat field fluctuations and is given by [2]

$$\left\langle \chi_k^2 \right\rangle = \int_{k_{min}}^{k_{max}} d^3k |\chi_k|^2, \tag{3.65}$$

and k_{min},k_{max} determine the resonance band. In our case $k_{min} = 0$ and k_{max} is given by Eq.(3.47). This correction may affect the coherent nature of the inflaton oscillations. Therefore we can derive a criterion for the time when the backreaction effect becomes important:

$$\left\langle \chi_k^2 \right\rangle \sim \frac{m_\phi^2}{g^2}. \tag{3.66}$$

The same effect (amplification of χ_k fluctuations) will induce a correction term to the evolution of preheat field fluctuations. Due to nonlinearities which will be present due to renormalization considerations there is a term of the from $\frac{1}{4}\lambda_\chi \chi^4$ in the potential. Then the equation of motion for the χ field fluctuations (in the Hartree approximation) becomes

$$\ddot{\chi}_k + (k^2 + m_\chi^2 + g^2\phi^2(t) + \lambda_\chi \left\langle \chi_k^2 \right\rangle)\chi_k = 0, \tag{3.67}$$

where for simplicity we considered the case of a non-expanding background. The criterion from this effect will be

$$g^2\phi^2(t)\lambda_\chi^{-1} \sim \left\langle \chi_k^2 \right\rangle. \tag{3.68}$$

As a rough estimate we can consider the value of the inflaton field at the end of inflation to be $\phi_{end} \sim 0.1 M_P$. One can rewrite the criterion as

$$\langle \chi_k^2 \rangle \sim 10^{-2} \frac{g^2}{\lambda_\chi} M_P^2. \tag{3.69}$$

From naturalness at the one loop level we know that $\lambda_\chi \sim g^4$. Also, considering $m_\phi \sim 10^{-6} M_P$, one can easily see that the former criterion (3.65) will happen much sooner than the latter (3.68). Thus means that in the model we discuss, amplification of preheat field fluctuations has a larger contribution to the equation of motion for the inflaton field.

3.4.2 Corrections Due to Amplification of the Inflaton Field Fluctuations

Depending on the potential, parametric resonance may happen for inflaton field fluctuations as well. Then the amplified fluctuations will back-react on the background evolution. In our toy model, the equation of motion for inflaton field fluctuations is

$$\ddot{\delta}\phi_k + (k^2 + m_\phi^2)\delta\phi_k = 0. \tag{3.70}$$

It is obvious that there is no instability in this equation, and $\delta\phi_k$ fluctuations just oscillate. We will see in the next chapters that self-resonance will happen in some models.

3.5 Conclusion

We have studied the reheating mechanism after inflation. We observed that reheating is an integral part of any inflationary theory without which inflation leaves the universe cold and empty of matter. We saw that to connect the inflationary phase to the hot Big Bang Cosmology, during reheating energy which is initially stored in the inflaton field should be transferred to other degrees of freedom and particle production should happen. We presented the perturbative analysis of reheating and discussed the problems with this approach. Then we introduced parametric resonance and preheating and discussed the scenario in detail. We saw that coherent oscillation of the inflaton field play an essential role in the scenario, and periodic time dependency of the frequency is a key feature in the analysis. The linear stage of the resonance will terminate when backreaction of the produced particles on the background dynamics becomes important. At this time resonance might be terminated by backreaction or resonance may transit to a non-linear stage when the linear analysis introduced here would not work.

So far in what we discussed the frequency has a periodic time-dependency, in which the Floquet theory applies. In the next chapter we will introduce another formalism to do the analysis of reheating and particle production even in the cases with non-periodic time-dependent frequency using a *Born Approximation Method* which involves the Green's function method for solving inhomogeneous linear differential equations.

Chapter 4

REHEATING IN G-INFLATION

4.1 Introduction

In all the models we have discussed in previous chapters, inflation is so-called potential-driven. Now we are going to introduce a model in which inflation is driven kinetically and study particle production in the model. Although there are different classes of models in which inflation is not potential driven, it was shown [35], that by introducing Galileon type terms (in particular kinetic terms) in the action of a scalar field ϕ, it is possible to obtain an inflationary model in which matter violates the *Null Energy Condition* (NEC)[1] and hence a blue tensor tilt is possible [2]. This model is called *G-inflation*. In this model, inflation is driven by the kinetic term in the action which at early times has the "wrong" sign and hence can lead to the violation of the NEC. Nevertheless thanks to the Galileon-type terms, the stability of fluctuations is maintained even in the presence of NEC violation contrary to the case of k-inflation [36]. Inflation terminates at a scalar field value above which the sign of the kinetic term reverts back to its canonical form. This leads to a transition from an inflationary phase to a standard kinetic-driven phase with equation of state $w = 1$, where w is the ratio of pressure to energy density. The energy density in ϕ then decreases as $a(t)^{-6}$, where $a(t)$ is the cosmological scale factor.

[1] $T_{\mu\nu}n^{\mu}n^{\nu} > 0$ where n^{μ} is any null (light-like) vector.

[2] In the context of inflationary cosmology with vacuum initial conditions and with matter obeying the (NEC), one always obtains a red tilt for tensor spectrum. This is different than the scalar spectrum for which either a red or a blue tilt can be obtained, although the simplest slow-roll models of inflation also predict a red tilt of the scalar spectrum.

Since there is no phase during which $\phi(t)$ oscillates there is no possibility of preheating. As discussed in [35], the production of regular matter particles after Galileon inflation is still possible by the gravitational Parker particle production mechanism [32, 33, 34]. The question one can ask is whether in the presence of a coupling between matter and the inflaton there is nevertheless some non-gravitational channel which transfers energy to matter faster than what can be achieved by gravitational effects.

In the following we will see that there is indeed a channel for direct particle production, and we derive conditions on the coupling constant for which this direct channel is more efficient than Parker particle production [3] . The analysis is based on the general framework set out in [29].

The analysis of this chapter is based on our previous work [1]. We begin with a brief review of G-inflation, move on to a discussion of the particle production mechanism we use, before presenting the calculations applied to the model. The notation is in natural units in which the speed of light, Planck's constant and Boltzmann's constant are set to 1.

[3] A similar channel is operative in the "emergent Galileon" scenario of [37] - see [38]. Particle-induced particle production has also recently been studied in a bouncing cosmology in [39].

4.2 G-inflation

The original G-inflation [35] is based on a scalar field ϕ minimally coupled to gravity with an action

$$\mathcal{L} = K(\phi, X) - G(\phi, X)\Box\phi\,, \tag{4.1}$$

where X is the standard kinetic operator

$$X = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi\,, \tag{4.2}$$

and K and G are general functions of ϕ and X. See [40] for its generalized version. The special property of this class of Lagrangians is that the resulting equations of motion contain no higher derivative terms than second order [41]. In the case $K = K(X)$ and $G(\phi, X) \propto X$ the action has an extra shift symmetry ("Galilean symmetry") and these Lagrangians were introduced and studied in [42].

The model of kinetically driven G-inflation [35] is based on choosing

$$K(\phi, X) = -A(\phi)X + \delta K\,, \tag{4.3}$$

with

$$A(\phi) = \tanh\big[\lambda(\phi_e - \phi)\big]\,, \tag{4.4}$$

and

$$G(\phi, X) = \tilde{g}(\phi)X = \tilde{g}X\,. \tag{4.5}$$

Here λ and $\tilde{g}$ are coupling constants and δK includes higher order terms in X which are important during inflation. After the sign of the linear kinetic term in the action is flipped at $\phi = \phi_e$, the higher order corrections soon become negligible and do not affect our analysis of reheating.

We consider homogeneous and isotropic cosmological solutions resulting from this action. As shown in [35], for $\phi < \phi_e$ there are inflationary trajectories for which the quasi-exponential expansion of space is driven by the wrong-sign kinetic term. Inflation ends at $\phi = \phi_e$, and for $\phi > \phi_e$ the background becomes that of a kinetic-driven phase with $w = 1$, $a(t) \sim t^{1/3}$ and

$$\dot{\phi}(t) \sim \frac{1}{t} \,. \tag{4.6}$$

We call this stage the *kination regime* of the model. Since the energy density in ϕ decays so rapidly, eventually the kination regime will end and regular radiation and matter will begin to dominate. The energy density at which this transition happens determines the *reheating temperature* of the Universe.

Knowledge of the reheating temperature is important for various post-inflationary processes such as baryogenesis or the possibility of production of topological defects. It may also be possible to directly probe the physics of the phase between the initial thermal stage and the hot Big Bang phase with precision observations (see e.g. [43]).

Regular matter and radiation are produced by gravitational particle production. However, if this is the only mechanism, then the reheating temperature will be low as it is suppressed by

$$\frac{H^4}{\rho_I} \sim \frac{H^2}{M_P^2} \,, \tag{4.7}$$

where ρ_I the energy density during inflation and M_P is the Planck mass. This ratio is bounded to be smaller than 10^{-8} based on the upper bound on the strength of gravitational radiation produced during inflation [94]

In the following we will assume that there is a direct coupling between matter (described by a free massless scalar field χ) and the inflaton field ϕ. We consider two possible couplings. The first is of the form

$$\mathcal{L}_I = \frac{1}{2} g^2 \dot{\phi} \chi^2 \, , \tag{4.8}$$

where g is a dimensionless coupling constant. Note that we have chosen a derivative coupling of ϕ with χ to preserve the invariance of the interaction Lagrangian under shifting of the value of ϕ (which is part of the Galilean symmetry. The disadvantage of this coupling is that it violates the symmetry $\phi \to -\phi$. The second coupling obeys this symmetry but involves non-renormalizable interactions. It is

$$\mathcal{L}_I = -\frac{1}{2} M^{-2} \dot{\phi}^2 \chi^2 \, , \tag{4.9}$$

where M is a new mass scale which is expected to be smaller than the Planck mass. These couplings open up non-gravitational channels for the production of χ particles [4] .In the following we will study the conditions under which these direct production

[4] Since the shift symmetry is already broken (in order to have inflation) there is no a priori reason to exclude couplings which do not respect the symmetry. However we take a more conservative choice and work with couplings which respect the shift symmetry.

channels are more efficient than the gravitational particle production channel.

4.3 Inflaton-Driven Particle Production

Assuming that the Lagrangian for the matter field χ has canonical kinetic term, then the Lagrangian for χ is that of a free scalar field with a time dependent mass, the time dependence being given by the interaction Lagrangians (4.8) or (4.9). Each Fourier mode χ_k of χ evolves independently, the equation of motion is

$$\ddot{\chi}_k + 3H\dot{\chi}_k + (\frac{k^2}{a^2} - g^2\dot{\phi})\chi_k \, = \, 0 \, . \tag{4.10}$$

or

$$\ddot{\chi}_k + 3H\dot{\chi}_k + (\frac{k^2}{a^2} + M^{-2}\dot{\phi}^2)\chi_k \, = \, 0 \, , \tag{4.11}$$

depending on the form of the interaction Lagrangian. The effects of the expansion of space can be pulled out by rescaling the field

$$X_k \, \equiv \, a^{-1}\chi_k \, . \tag{4.12}$$

Then, in terms of conformal time τ (which is related to physical time t by $dt = a(t)d\tau$), the equation of motion becomes

$$X_k'' + (k^2 - g^2\dot{\phi}a^2 - \frac{a''}{a})X_k \, = \, 0 \, , \tag{4.13}$$

or

$$X_k'' + (k^2 + M^{-2}\dot{\phi}^2a^2 - \frac{a''}{a})X_k \, = \, 0 \, , \tag{4.14}$$

where a prime denotes a derivative with respect to τ.

The qualitative features of the equations of motion (4.13) or (4.14) are well known from the theory of cosmological perturbations (see e.g. [10] for an in-depth review and [44] for a brief overview). In the absence of the interaction term, X_k will oscillate on sub-Hubble scales, i.e. scales for which

$$k^2 > \frac{a''}{a} \sim \mathcal{H}^2 \,, \tag{4.15}$$

whereas the mode function X_k is squeezed on super-Hubble scales, i.e.

$$X_k \sim a \,. \tag{4.16}$$

Following [29], we will treat the effects of the interaction term in leading order Born approximation, i.e. we write

$$X \equiv X_0 + X_1 \,, \tag{4.17}$$

(here and in the following we will drop the subscript k) where X_0 is the solution of the equation in the absence of interactions, i.e. a solution of

$$X_0'' + (k^2 - \frac{a''}{a})X_0 = 0 \,, \tag{4.18}$$

solving the initial conditions of the problem, and X_1 is the solution of the inhomogeneous equation

$$X_1'' + (k^2 - \frac{a''}{a})X_1 = g^2\dot{\phi}a^2X_0, \tag{4.19}$$

or

$$X_1'' + (k^2 - \frac{a''}{a})X_1 = -M^{-2}\dot{\phi}^2a^2X_0, \tag{4.20}$$

(with vanishing initial conditions) obtained by taking the interaction term in (4.13) or (4.14) to the right hand side of the equation and replacing X by the "unperturbed" solution X_0.

The inhomogeneous equation (4.19) (or (4.20)) can be solved by the Green's function method

$$X_1(\tau) \,=\, \int_{\tau_i}^{\tau} d\tau' G(\tau,\tau') g^2 a^2(\tau') \dot{\phi}(\tau') X_0(\tau')\,, \tag{4.21}$$

or

$$X_1(\tau) \,=\, -\int_{\tau_i}^{\tau} d\tau' G(\tau,\tau') M^{-2} a^2(\tau') \dot{\phi}^2(\tau') X_0(\tau')\,, \tag{4.22}$$

where the Green's function $G(\tau,\tau')$ is determined in terms of the two fundamental solutions u_1 and u_2 of the homogeneous equation via

$$G(\tau,\tau') \,=\, W^{-1}\big(u_1(\tau)u_2(\tau') - u_2(\tau)u_1(\tau')\big)\,, \tag{4.23}$$

where W is the Wronskian

$$W \,=\, u_1(\tau)u_2'(\tau) - u_2(\tau)u_1'(\tau)\,. \tag{4.24}$$

In the above, the time τ_i is the time when the initial conditions are imposed. In our case it is the end of the period of inflation.

The condition that direct particle production is more efficient than gravitational particle production is

$$X_1(\tau) \,>\, X_0(\tau), \tag{4.25}$$

at some time $\tau > \tau_i$ before the time when the kinetic phase would be terminated by gravitational particle production alone.

4.4 Analysis

We now apply the formalism of the previous section to our specific Galileon inflation model. We are interested in super-Hubble modes for which the k^2 term in the equation of motion (4.10) can be neglected. The fundamental solutions are then

$$\begin{aligned} u_1(\tau) &= (\frac{\tau}{\tau_i})^{1/2}, \\ u_2(\tau) &= (\frac{\tau}{\tau_i})^{1/2}\ln(\frac{\tau}{\tau_i}), \end{aligned} \tag{4.26}$$

and hence the Wronskian becomes

$$W = \frac{1}{\tau_i}, \tag{4.27}$$

and the Green's function is

$$G(\tau, \tau') = \left(\tau\tau'\right)^{1/2}\ln(\frac{\tau'}{\tau}). \tag{4.28}$$

The contribution $X_1(\tau)$ induced by the direct coupling between ϕ and χ thus becomes

$$X_1(\tau) = g^2 \int_{\tau_i}^{\tau} d\tau' \left(\tau\tau'\right)^{1/2}\ln(\frac{\tau'}{\tau})\dot{\phi}(\tau')a^2(\tau')X_0(\tau'), \tag{4.29}$$

or

$$X_1(\tau) = \frac{-1}{M^2} \int_{\tau_i}^{\tau} d\tau' (\tau\tau')^{1/2}\ln(\frac{\tau'}{\tau})\dot{\phi}(\tau')^2 a^2(\tau')X_0(\tau'), \tag{4.30}$$

For $X_0(\tau)$ we can take the dominant solution of the homogeneous equation

$$X_0(\tau') \,=\, X_0(\tau_i)(\frac{\tau'}{\tau_i})^{1/2}\ln(\frac{\tau'}{\tau_i})\,. \tag{4.31}$$

Making use of the scaling (4.6) of $\dot{\phi}$ and after a couple of lines of algebra we obtain the approximate result (keeping only the contribution from the upper integration limit)

$$X_1(\tau) \,\simeq\, g^2\dot{\phi}_i(\tau_i)\tau^2 X_0(\tau_i)\,. \tag{4.32}$$

or

$$X_1(\tau) \,\simeq\, -M^{-2}\dot{\phi}_i^{\,2}(\tau_i)\tau_i^2(\frac{\tau}{\tau_i})^{1/2}X_0(\tau_i)\ln(\frac{\tau}{\tau_i})\,. \tag{4.33}$$

If we take the initial time τ_i to correspond to the end of inflation, we have

$$\dot{\phi}(\tau_i) \,\simeq\, H(\tau_i)M_P\,, \tag{4.34}$$

where $H(\tau_i)$ is the value of H at the end of inflation. In this case

$$X_1(\tau) \,\sim\, g^2 M_P\tau_i(\frac{\tau}{\tau_i})^{3/2}\,, \tag{4.35}$$

or

$$X_1(\tau) \,\simeq\, -(\frac{M_P}{M})^2 X_0(\tau)\,. \tag{4.36}$$

The criterion (4.25) for direct particle production to dominate over gravitational particle production then becomes (up to logarithmic factors)

$$g^2 \,>\, \frac{H(\tau_i)}{M_P}(\frac{t_i}{t})\,. \tag{4.37}$$

or

$$(\frac{M_P}{M})^2 \, > \, 1 \, . \tag{4.38}$$

Note that for the second interaction term, particle production via direct interactions dominates within one Hubble expansion time (the time interval after which the contribution from the lower integration end can be neglected), provided that $M < M_P$, a condition which has to be satisfied if we are to trust the effective field justification of the interaction term.

Once $X_1(\tau)$ starts to dominate over $X_0(\tau)$, the Born approximation ceases to be valid. At that point, the coupling term in the equation of motion for X will become the dominant one, and an approximation to (4.13) (we will first focus on the case of the first interaction term) which is self-consistent for long wavelength modes (for which the k^2 term in the equation is negligible) is

$$X'' - g^2 \dot{\phi} a^2 X \, = \, 0 \, . \tag{4.39}$$

An approximate solution of this equation is

$$X(\tau) \, = \, \mathcal{A}(\tau) e^{\tilde{f}(\frac{\tau}{\tau_i})^{3/4}\tau_i} \tag{4.40}$$

with

$$\tilde{f} \, \equiv \, \frac{4}{3}\big(g^2 \dot{\phi}(\tau_i)\big)^{1/2} \, . \tag{4.41}$$

Inserting this ansatz (4.40 and 4.41) into (4.39) we find an equation for the amplitude $\mathcal{A}(\tau)$

$$\mathcal{A}'' + \frac{3}{2}\tilde{f}\tau^{-1/4}\tau_i^{-3/4}\mathcal{A}' - \frac{3}{16}\tilde{f}\tau^{-5/4}\tau_i^{-3/4}\mathcal{A} \, = \, 0 \, , \tag{4.42}$$

which both for $\tilde{f}\tau_i \ll 1$ and $\tilde{f}\tau_i \gg 1$ has a dominant solution which is constant in time.

From (4.40) and (4.41) we see that there is quasi-exponential growth of X which becomes important once

$$\tilde{f}\tau_i(\frac{\tau}{\tau_i})^{3/4} \; > \; 1\,, \tag{4.43}$$

which in terms of physical time is

$$\frac{t}{t_i} \; > \; \tilde{f}^{-2}\tau_i^{-2}\,. \tag{4.44}$$

In the above we are implicitly assuming that $\tilde{f}\tau_i < 1$. If $\tilde{f}\tau_i > 1$ then reheating via direct particle production is instantaneous on a Hubble time scale and the reheating temperature is given by the energy density at the end of inflation.

Returning to the case $\tilde{f}\tau_i < 1$, the we see that once the time t is larger than the one given by (4.44), the energy transfer from the inflaton to matter is exponentially fast and will immediately drain all of the energy from the inflaton. Hence, the "reheating time" t_{RH} is

$$t_{RH} \; \sim \; t_i(\tilde{f}\tau_i)^{-2}\,, \tag{4.45}$$

and since the energy density between t_i and t_{RH} decreases as $a(t)^{-6} \sim t^{-2}$ we have

$$\rho(t_{RH}) \; \sim \; \rho(t_i)(\tilde{f}\tau_i)^4\,. \tag{4.46}$$

Making use of $\rho(t_i) = H^2(t_i)M_P^2$ (up to a numerical factor) and $\rho(t_{RH}) \sim T_{RH}^4$ we finally obtain the reheating temperature T_{RH} to be

$$T_{RH} \; \sim \; \tilde{f}\tau_i(H(t_i)M_P)^{1/2}\,, \tag{4.47}$$

which is larger than the reheating temperature $H(t_i)$ which would be obtained if only gravitational particle production were effective, provided that

$$\tilde{f}\tau_i > (\frac{H(t_i)}{M_P})^{1/2} . \tag{4.48}$$

In the case of the second coupling, the conclusions are similar. Once the coupling term in the equation of motion dominates over the expansion term, the equation can be approximated as (changing the sign of the coupling term)

$$X'' - M^{-2}\dot{\phi}^2 a^2 X = 0 . \tag{4.49}$$

Since

$$\dot{\phi}^2(\tau)a^2(\tau) = \dot{\phi}^2(\tau_i)(\frac{\tau_i}{\tau})^2 , \tag{4.50}$$

the equation has power law solutions with an exponent Δ given by

$$\Delta = \frac{1}{2}\big[1 \pm \sqrt{1+4R^2}\big] , \tag{4.51}$$

where

$$R \equiv \frac{M_P}{M} . \tag{4.52}$$

We see that if $M \ll M_P$, then the power of the dominant solution is $\Delta \gg 1$ and this means that there is complete energy transfer from the inflaton to χ within one Hubble expansion time. Hence, the reheating temperature is given by the energy density at the end of inflation, i.e.

$$T_{RH} \sim (H(t_i)M_P)^{1/2} . \tag{4.53}$$

4.5 Conclusion

In this chapter we have introduced G-inflation model and derived the condition under which direct particle production in the model dominates over gravitational particle production. The discussion also applies to k-inflation [36]. We consider two possible interaction Lagrangians, namely (4.8) and (4.9). We first study the onset of matter particle production from the direct coupling using the Born approximation. We find that for both interaction terms we consider, the direct particle production channel eventually dominates. This happens within one Hubble expansion time for the coupling (4.9), whereas in the case of (4.8) the time when direct particle production starts to dominate depends on the coupling constant g.

Once direct particle production begins to dominate over gravitational particle production we must use a different approximation scheme to solve the equation of motion. We can now neglect the squeezing term in the equation of motion. We provide solutions of the resulting approximate equations of motion and show that once direct particle production begins to dominate, the energy transfer from the inflaton to the matter fields will be almost instantaneous. This allows us to estimate the value of the reheating temperature, the temperature of matter once the inflaton field has lost most of its energy density to particle production. In the case of the second interaction term (4.9), the reheating temperature is given by the energy density at the end of inflation, in the case of the first interaction term (4.8), it is reduced by a factor which involves the interaction coupling constant g.

Chapter 5

PREHEATING OF COSMOLOGICAL PERTURBATIONS

5.1 Introduction

As we promised before, in this chapter we are going to include gravitational effects in our analysis of reheating in three different inflationary models. Note that we are interested in long wavelength modes where linear analysis is valid. Non-linear evolution of the modes is beyond the scope of our work. In the first model, we discuss the parametric instability in the amplitude of cosmological perturbations during massless preheating. We show how instabilities in the matter sector are connected to *entropy perturbations* which in turn will induce curvature perturbations, as it was pointed out initially in [45, 46, 47, 48]. Here we will apply the covariant formalism (introduced in chapter 2) to study the evolution of entropy perturbations in a two-field inflationary model the so-called *massless preheating*. As we see in the next section, the result is consistent with earlier works and confirms the significant enhancement of cosmological perturbations even on large scales.

In the second model, we will study *axion monodromy* inflation [49] with a derivative coupling between the axion field and the $U(1)$ gauge field. We will show that although the resonance is efficient in producing gauge field fluctuations, the induced curvature perturbation is consistent with the current observation data.

Then we will show preheating in an *Asymptotically Safe Quantum Field Theory* [50]. We will discuss that parametric resonance of the spectator scalar fields is efficient in this model. This observation has crucial consequences for the evolution of cosmological perturbations in the theory.

5.2 Preheating of Entropy Perturbations in Massless Preheating

The key aspect of preheating is that the oscillations of the inflaton field ϕ after the end of inflation lead to a periodically varying contribution to the mass term of the χ field. The equation of motion for χ thus falls into the category of those described by Floquet theory [31], which states that there are bands of Fourier modes of χ which experience exponential growth. Since the inflaton field ϕ also couples to gravity, the oscillations of ϕ lead to a periodically varying contribution to the mass term in the equation of motion for cosmological perturbations, as was first pointed out in [51] and [52]. Hence, there is the possibility that preheating can lead to a parametric instability in the amplitude of cosmological perturbations, even on scales which are super-Hubble at the end of inflation. If this were true, it would completely change the usual predictions of inflationary models. In fact, the presence of a resonant instability of cosmological fluctuation modes could lead to an amplitude of fluctuations which is much larger than the observed value, thus placing constraints on inflationary models.

In [53] it was shown that in models with only adiabatic fluctuations there is no instability of curvature fluctuations on super-Hubble scales. This is related to the conservation of the comoving curvature fluctuation variable ζ on super-Hubble scales [54, 55]. However, in certain two field models of inflation it was argued in [45, 46, 47, 48] that in the case of "massless preheating" (the inflaton having vanishing mass) there will be a preheating instability for the entropy fluctuation mode, even on super-Hubble scales [1] . In fact, it was shown in [57, 58] that back reaction effects will not be strong enough to shut off the instability before the entropy mode has become dominant. The presence of a preheating instability for the entropy mode was confirmed in the analyses of [59] and [60], extended to the case of multi-field generalized Einstein models in [61], and applied to certain examples in [62].

However, there remain concerns about the conclusions of [53]. An analysis using the "separate universe" method [63] argues that the preheating of entropy fluctuations is less effective. An analysis using the δN formalism, a method which is closely related to the separate universe approach, finds that there is parametric resonance of the entropy mode [64], although in a subsequent paper [65] the same authors do not find substantial effects on the curvature fluctuations . In addition, little amplification of the curvature fluctuations is observed in the numerical work of [66] which was based on a numerical implementation of the δN formalism.

Since the entropy mode seeds a growing curvature fluctuation on super-Hubble scales, any parametric resonance instability of an entropy mode can lead to an exponential growth of the curvature fluctuation during the preheating stage. This is a potentially disastrous effect since the fluctuations could well grow to become larger than the observed values. Hence, from the point of view of inflation model building it is very important to determine whether the parametric resonance instability of the entropy mode is robust.

[1] In the case of "massive preheating" (the potential of the inflaton being dominated by the mass term) there is no preheating of the entropy mode at linear order in cosmological perturbation theory, as shown in [56].

In this section we are going to study evolution of entropy perturbations in massless preheating. The goal is to reconsider this question using different methods than have been used before. Specifically we will make use of the “covariant formalism", a formalism developed in [19] (see also [20] for earlier related work), a formalism which can be applied even non-perturbatively (we, however, will use a perturbative truncation of the formalism). Our goal is to demonstrate an instability of the model on cosmological scales. Our methods obviously break down once the nonlinear regime is reached. At this point, numerical methods used to study nonlinear preheating effects (see e.g. [67] for the first numerical code for studying preheating which includes the metric fluctuations, and [66, 68, 69, 70, 71, 72, 73]) would have to be applied. These nonlinear effects, however, cannot reduce the amplitude of curvature perturbations on cosmological scales, and hence we do not consider them.

Our study shows that the preheating of entropy modes is indeed effective in the massless preheating toy model which we consider, and that this leads to an exponentially growing contribution to the curvature fluctuation. Since we have shown that our equations are the perturbative limit of a consistent non-perturbative formalism, we now have a better reason for arguing that the instability we find will extend beyond the perturbative treatment.

5.2.1 Setup for The Model

In [53] necessary conditions for the effectiveness of preheating of the entropy mode of metric fluctuations have been discussed. One of the conditions is that there is efficient parametric resonance in the matter sector in the absence of gravitational fluctuations. A model of massless inflation satisfies this condition. Hence, we consider a toy model containing a massless inflaton field ϕ coupled to a massive matter field χ with a potential

$$V(\phi, \chi) = \frac{\lambda}{4}\phi^4 + \frac{1}{2}g^2\phi^2\chi^2 \,. \tag{5.1}$$

The interaction term $\phi^2\chi^2$ allows for the decay of the coherent inflaton configuration ϕ into massive χ excitations.

Up to the mass term for χ (which we will neglect in this section) our model is conformally invariant. Thus, via a conformal transformation we can map our model into one living in Minkowski space-time, and then study preheating in Minkowski space-time. This is technically much simpler than performing the calculation in the original variables in an expanding universe. As pointed out first by [74], the structure of resonance in the matter sector depends in a crucial way on the relation between the coupling constants λ and g^2. Indeed the only parameter responsible for the structure of resonance is the ratio $\frac{g^2}{\lambda}$.

We are interested in studying the evolution of the entropy perturbation in this two field model. To do this we first need to study the evolution of the quantum field χ in the background of the classical field ϕ. This will be done in the rest of this section.

We consider the case in which the zero mode of the χ field is zero and therefore χ has no effect on the inflationary dynamics in the absence of quantum fluctuations. However, quantum fluctuation of this field are continuously excited. Since the equation of motion for χ is linear in χ each Fourier mode evolves separately. We are interested in modes whose wavelength today corresponds to cosmological scales.

The wavelength is smaller than the Hubble radius early during the inflationary phase, grows relative to the Hubble radius as a consequence of the accelerated expansion of space during the period of inflation, and exits the Hubble radius a number of e-folding times before the end of inflation. At that point, the oscillations of the quantum fluctuations freeze out and the modes can be squeezed.

Later on we will have to take into account the fact that the ensemble of large scale fluctuations of the χ field will generate an effective χ background in which smaller scale χ modes live. We will find this background by averaging over the large-scale fluctuations.

5.2.2 Background Evolution

First we review the dynamics of ϕ after the end of the period of inflation. During inflation the effective potential for the inflaton is $\frac{\lambda}{4}\phi^4$. Therefore the Klein-Gordon equation for the classical inflaton field ϕ is

$$\ddot{\phi} + 3H\dot{\phi} + \lambda\phi^3 = 0\,, \tag{5.2}$$

where H is the Hubble parameter and " $\dot{}$ " is the derivative with respect to physical time. For further simplification we work with conformal time η defined via

$$a d\eta = dt\,.$$

Then as we mentioned before we do a conformal transformation

$$a\phi = \varphi\,.$$

If we rewrite Eq.(5.2), we find

$$\varphi'' + \lambda\varphi^3 - \frac{a''}{a}\varphi = 0\,, \tag{5.3}$$

where " ′ " denotes the derivative with respect to conformal time.

After the end of the period of slow-roll inflation, the background φ field will start anharmonic oscillations about $\varphi = 0$ since it lives in a confining potential. It is well known that for a quartic potential $\lambda\phi^4$ the time-averaged energy momentum tensor is traceless and hence the equation of state (averaged over an oscillation period) is the same as for radiation. Hence $a(\eta) \sim \eta$, and therefore the last term in the equation (5.3) vanishes. Thus we obtain

$$\varphi'' + \lambda\varphi^3 = 0 \tag{5.4}$$

which has periodic solutions. To find these solutions we introduce the dimensionless conformal time

$$x \equiv \sqrt{\lambda}\tilde{\varphi}\eta\,,$$

where $\tilde{\varphi}$ is the constant amplitude of the oscillations of $\varphi = \tilde{\varphi}f(x)$. The solution of equation (5.4) can be written in terms of Jacobi elliptic functions:

$$\varphi = \tilde{\varphi}cn(x - x_0, \frac{1}{\sqrt{2}})\,. \tag{5.5}$$

We can approximate the elliptic cosine function by the leading term in its series expansion, $\cos(x)$, which is very good approximation as discussed in [74]. Therefore the solution of the Klein-Gordon equation for the φ field is

$$\varphi \simeq \tilde{\varphi} \cos(x) \,. \tag{5.6}$$

Before moving on to next section, for further reference it is useful to find the form of scale factor in this theory [74]. At the end of inflation and beginning of preheating the effective potential is $\lambda\phi^4/4$ and the homogeneous value of χ field is zero. Thus the Friedman equation is

$$H^2 \, = \, \frac{8\pi}{3M_p^2}(\frac{1}{2}\dot{\phi}^2 + \frac{\lambda\phi^4}{4}) \,, \tag{5.7}$$

where H is the Hubble parameter. When averaging over several oscillations of the inflaton field while $\phi \ll M_p$ we then find

$$a(x) \, \sim \, \sqrt{\frac{2\pi}{3}} \frac{\tilde{\varphi}}{M_p} x \,. \tag{5.8}$$

5.2.3 Evolution of the Preheat Field χ Fluctuations

In this subsection we study the evolution of the linear mode functions of the χ field, and use the results to determine an effective background χ field which a fixed Fourier mode of the fluctuations will feel. As discussed e.g. in [2, 5, 58], this background is obtained by integrating over fluctuations of wavelength larger than the one we are considering.

At the classical level, the homogeneous value of the χ field is zero. The effective background which a mode with wavenumber k will feel is generated by the quantum fluctuations of larger wavelengths which have exited the inflationary Hubble radius earlier, have been squeezed and decohered and hence become classical (see e.g. [75, 76]). To find this effective background, we must first solve the equation for the quantum fluctuations of the χ field. For simplicity we take the spatial sections to be flat.

As it is standard in the field, we use the formalism of quantum field theory in curved space-time. Since we are considering a free quantum field $\hat{\chi}$, we can expand the field in Fourier modes, and each Fourier mode in creation and annihilation operators $\hat{a}_k^+$ and $\hat{a}_k$, respectively

$$\hat{\chi}(t,\vec{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3k \quad \Big[\quad \hat{a}_k \chi_k(t) \exp(-ik.x) + \quad \hat{a}_k^+ \chi_k^*(t) \exp(ik.x) \Big] , \tag{5.9}$$

where the mode functions χ_k satisfy the following Fourier space Klein-Gordon equation

$$\ddot{\chi}_k + 3H\dot{\chi}_k + (\frac{k^2}{a^2} + g^2\phi^2)\chi_k = 0 . \tag{5.10}$$

In conformal time and considering conformal transformation

$$a\chi \equiv X \tag{5.11}$$

as well as a conformal transformation of the ϕ field

$$a\phi \equiv \varphi \tag{5.12}$$

we find

$$X_k'' + [\frac{k^2}{\lambda\tilde{\varphi}^2} + \frac{g^2}{\lambda} f(x) - \frac{a''}{a}]X_k = 0 , \tag{5.13}$$

where " ′ " denotes the derivative with respect to dimensionless conformal time, and $f(x)$ is the periodic with amplitude 1. As mentioned before, the last term in this equation vanishes during massless preheating and the equation becomes:

$$X_k'' + [\frac{k^2}{\lambda\tilde{\varphi}^2} + \frac{g^2}{\lambda}\cos^2(x)]X_k \,=\, 0\,, \tag{5.14}$$

where we have also inserted the approximate form of $f(x)$. This equation has the structure of a Mathieu equation. To make it clear we rewrite equation (5.14) as follows:

$$X_k'' + [(\frac{k^2}{\lambda\tilde{\varphi}^2} + \frac{g^2}{2\lambda}) + \frac{g^2}{2\lambda}\cos(2x)]X_k \,=\, 0\,. \tag{5.15}$$

As we mentioned before, we will consider values of the coupling constant for which it is known that there is preheating in the matter sector. Hence, we consider the case $\frac{g^2}{\lambda} \simeq 2$ since in this case all long wavelength modes of the χ field are located in the instability region of the Mathieu equation [74]. Therefore the solution for X_k will be of the form

$$X_k(x) \,=\, A_1\exp(\mu_k x)P_1(x,k) + A_2\exp(-\mu_k x)P_2(x,k)\,, \tag{5.16}$$

where μ_k is the so-called Floquet exponent [31] which in this case has a positive real component (the Lyapunov exponent), and P_1 and P_2 are periodic functions of x with amplitude 1 which appear in the solution of the Mathieu equation [31]. Note that the period is determined by the period of the inflaton field, and is independent of k.

Considering only the growing mode we need to determine the constant A_1 by fixing the initial conditions. Since preheating is preceded by a phase of inflationary expansion, the initial conditions for preheating are determined by the evolution of the field during inflation. This slow-roll inflation is given by a quasi exponential expansion of the universe, where the Hubble parameter is almost constant. During inflation, quantum χ field perturbations (as well as ϕ perturbations) are created from vacuum initial conditions on sub-Hubble scales. As the wavelengths of these fluctuations are amplified in this phase relative to the Hubble radius, they eventually exit the Hubble radius where they "freeze out" and may undergo squeezing.

To see whether squeezing occurs, we have to compare the magnitude of the induced mass term in (5.13), the term $\frac{g^2}{\lambda} f(x) X_k$, with the squeezing term $\frac{a''}{a} X_k$. Thus, the condition for squeezing is

$$\frac{g^2}{\lambda} < \frac{a''}{a} \,. \tag{5.17}$$

Re-expressing the derivative with respect to the rescaled time in terms of the regular time derivates, the condition (5.17) becomes (making use of $|\dot{H}| \ll H^2$)

$$H^2 > g^2 \Phi^2 \,. \tag{5.18}$$

Since during slow-roll it follows from (5.7) that $H^2 \sim \Phi^4$, we see that (5.18) will be more easily satisfied for large values of the inflaton field. Thus, to see if we get squeezing we need to determine the range of values of Φ during slow-roll inflation. Specifically, we need to determine the value of Φ at the end of the slow-roll period.

During slow-roll, the second derivative term in (5.2) is neglected and the kinetic term is negligible in the Friedmann equation (5.7). Solving for the evolution of Φ in the slow-roll approximation and using the result to check when the kinetic contribution to H ceases to be sub-dominant yields the result

$$\varphi_{end}^2 \sim 2(6\pi G)^{-1} \tag{5.19}$$

for the value of φ at the end of the slow-roll period. We are interested whether there is squeezing for modes which exit the Hubble radius a number N Hubble times before the end of inflation (for scales of cosmological interest we have $N \sim 50$). Making a Taylor expansion in the evolution of φ about the endpoint of the slow-roll phase yields the lower bound

$$\varphi(N) \, > \, \varphi_{end}\left(1 + N\frac{3}{2}\right) \tag{5.20}$$

for the value of φ N Hubble expansion times before the end of inflation. Inserting this result into (5.18), we can see that the condition (5.18) is satisfied provided $N > 2$. This means that the quantum fluctuations of the entropy modes will be squeezed between when they exit the Hubble radius and when $N = 2$, i.e. essentially to the end of the slow-roll phase.

Since during the squeezing period $X_k \sim a$, the power spectrum of the entropy modes from inflation becomes scale invariant and the initial conditions for the X modes at the beginning of the preheating phase are given by:

$$A_1(k) = \frac{1}{\sqrt{2k}}\frac{a(t_R)}{a(t_H(k))} = H_I k^{-3/2}\,, \tag{5.21}$$

where t_R is the time when inflation ends. We will normalize the scale factor such that $a(t_R) = 1$. Also, $t_H(k)$ is the time of horizon crossing, with $a(t_{H(k)}) = k/H_I$

and H_I the Hubble rate during inflation. With that, the solution for X_k is:

$$X_k(x) \simeq H_I k^{-3/2} \exp(\mu_k x) P_1(x, k) . \tag{5.22}$$

Thus we can observe clearly that the X_k mode function is exponentially growing, which in turn leads to an exponentially growing number density of χ particles. The value of the Floquet exponent as a function of k is shown in Fig. 5-1 [2]. In this figure, the vertical axis denotes the value of the Floquet exponent, the horizontal axis labels k. Note that for the infrared modes which we are interested in, the value of the Floquet exponent is about 0.2. Even though $\mu_k < 1$, the time scale of the exponential instability is (while long compared to the oscillation time) short compared to the Hubble expansion time. Note that the analysis has so far been in the absence of gravitational fluctuations.

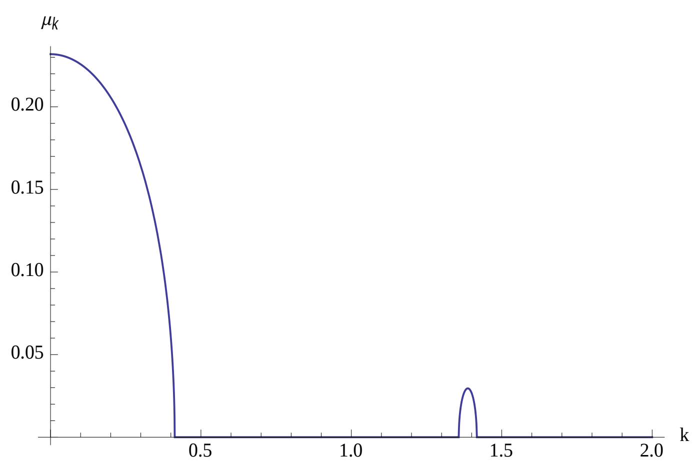

Figure 5–1: Value of the Floquet exponent (vertical axis) as a function of k in units of $k^2/(\lambda\tilde{\phi}^2)$.

The quantum fluctuations on large wavelength which were squeezed and classicalized after exiting the Hubble radius will form a background in which the metric fluctuations evolve. To find the effective background value which a mode with wavenumber k feels is given by [2]

$$\chi_{\rm eff}(k) = \left(\int_0^k d^3k' |X_{k'}|^2 \right)^{1/2} . \tag{5.23}$$

To see this, we begin from the expression for the contribution of long wavelength (i.e. longer than k^{-1}) modes to the X field at a fixed point x in space

$$X(x) \,=\, V^{1/2} \int_0^k d^3k' X_{k'} e^{ik'x} \,, \tag{5.24}$$

in terms of the Fourier modes. Here, V is the cutoff spatial volume which we introduced such that the Fourier modes have the mass dimension of a harmonic oscillator. Without loss of generality we can take the point x to be $x = 0$. We now consider the expectation value of the square of the absolute value of (5.24)

$$< |X(0)|^2 >= V \int_0^k d^3k' \int_0^k d^3k'' < X_{k'} X^*_{k''} > . \tag{5.25}$$

Since the Fourier modes are uncorrelated we have

$$< X_{k'} X^*_{k''} > \ \delta^3(k' - k'') V^{-1} |X_{k'}|^2 \,. \tag{5.26}$$

Inserting (5.26) into (5.25) then yields (5.23).

[2] To further justify this, imagine that χ contains fluctuations with two Fourier modes, a mode k we are interested in, and a longer wavelength mode k'. The mode k can locally be viewed as a mode which fluctuates not about 0, but about the local value of the k' mode.

We are interested in infrared modes k which lie in the instability band of the Mathieu equation. From Figure 5-1 we see that for these values of k the infrared modes k' which appear inside the integral also lie in the instability band, and that we can approximate the Floquet exponent by a constant $\mu_{k'} = \mu$. Inserting (5.22) into (5.23) we see that there is a potential logarithmic infrared divergence of the integral. There is, however, an infrared cutoff: the form (5.22) does not apply for modes which are outside the Hubble radius at the beginning of inflation. Therefore the integral can be estimated by

$$\chi_{\text{eff}}(k) \sim \sqrt{\pi} H_I \exp(\mu x) \left(\ln \left(\frac{k}{k_{min}} \right) \right)^{1/2} P_1(x,k)\,, \tag{5.27}$$

where k_{min} is the value of k which corresponds to Hubble radius crossing at the beginning of the period of inflation[3] . The logarithm is given by N_I, the number of e-foldings of inflation which in the rest of analysis of this section we consider $\sqrt{N_I} \sim O(1)$. The periodic function P_1 is [31] a series of sine and cosine functions with different coefficients and the leading term is $\sin(x)$ with unit coefficient Therefore, the leading term for P_1 gives us

$$\chi_{\text{eff}}(k) \sim \sqrt{\pi} H_I \exp(\mu x) \sin(x), \tag{5.28}$$

[3] This is a key point in our analysis. At strictly linear order in perturbation theory the background value of χ would vanish. However, in a particular patch of length k^{-1} the average value of χ will not vanish because of the presence of fluctuations on scales which are in the infrared with respect to k. They will produce a non-vanishing effective background. Our procedure corresponds to integrating out infrared modes which are unobservable from the point of view of the patch of interest.

for the effective background value of the preheat field.

5.2.4 Application of The Covariant Formalism to Massless Preheating

Our goal in this section is to show that in massless preheating entropy fluctuations are indeed parametrically amplified, and that this in turn leads to an exponentially growing contribution to the curvature fluctuations. As we shall see, the effect on the curvature fluctuations is quadratic in the amplitude of the quantum fluctuations. The entropy fluctuations themselves have an exponentially growing term which is linear in the fluctuation amplitude. However, the coupling between the entropy and the adiabatic mode is suppressed by an additional power which comes from the fact that the background of the entropy field vanishes at zero order.

For the case of massless preheating the equation for long wave linearized entropy fluctuations (2.79) becomes

$$\begin{aligned} \ddot{s}_a &+ 3H\dot{s}_a + \Big[(3\lambda\phi^2 + g^2\chi^2)\sin^2\theta - 2g^2\phi\chi\sin 2\theta \\ &+ g^2\phi^2\cos^2\theta + 3\dot{\theta}^2\Big] s_a \simeq 0\,. \end{aligned} \tag{5.29}$$

To analyze this equation of motion we make a couple of approximations and use the following setup:

- As mentioned before, at the beginning of preheating the overall homogeneous value of the χ field is zero (the effective χ field on a scale k will be non-vanishing but of linear order in the amplitude of the fluctuations).

– We use the relation for θ in the small angle approximation introduced in Eq.(2.68) in chapter 2:

$$\theta \equiv \frac{\dot{\chi}}{\dot{\phi}} \tag{5.30}$$

for the instantaneous angle between the background trajectory and the ϕ field direction in field space. Therefore at the beginning of preheating the angle is of linear order in the fluctuations.

– We will use the result from [17] for $\dot{\theta}$ in the large scale limit

$$\dot{\theta} = -\frac{V_{,s}}{\dot{\sigma}}, \tag{5.31}$$

where V, s is the derivative of the potential with respect to the entropy component. We use another result which is shown in [17]

$$V_{,s} = -V_{,\phi} \sin\theta + V_{,\chi} \cos\theta \,. \tag{5.32}$$

At the beginning of preheating we have $V_{,\chi} \simeq \chi$ which like χ is of first order. Thus, $\dot{\theta}$ is of first order and we can drop the last term in equation (5.29) as it is of second order.

– The perturbative expression for the expansion parameter is

$$\Theta = 3H + \epsilon F + \epsilon^2 G + \text{higher order terms}, \tag{5.33}$$

where we use ϵ to track the order of perturbations. We will only need to consider the zero order term since effects of the other terms would be of higher order in perturbation theory.

Making use of the above points, the equation (2.79) at leading order becomes

$$\ddot{s}_a + 3H\dot{s}_a + g^2\phi^2 s_a \;=\; 0\,. \tag{5.34}$$

If we do a conformal transformation

$$\begin{aligned} as_a &\equiv S_a \\ a\phi &\equiv \varphi \end{aligned} \tag{5.35}$$

and work with conformal time instead of physical time we get

$$S''_a + g^2\varphi^2 S_a \;=\; 0\,, \tag{5.36}$$

which has the same structure as the equations (5.14, 5.15) for the χ background field. From the discussion of the solutions of this equation in the early section of this article it thus follows that

$$S_a \;=\; B_1 u_{1a} + B_2 u_{2a}\,, \tag{5.37}$$

where u_{1a} is exponentially growing and u_{2a} is exponentially damped. Considering only the growing mode and remembering the same form for u_{1a} as we used in equation (5.22), we find

$$S_a \;=\; Hk^{-3/2}\exp(\mu x)P_{1a}(x,k)\,, \tag{5.38}$$

where P_{1a} indicate periodic functions of x with unit amplitude, and we hence conclude that the entropy component is exponentially growing due to parametric resonance at the beginning of preheating. This is one of the main results of this section.

We are interested in the process of conversion of the entropy fluctuation into a curvature fluctuation. This process happens continuously throughout the preheating phase. In the rest of this section we will study the evolution of curvature covector due to this conversion process.

As is well known, entropy fluctuations can seed a growing curvature fluctuation mode on super-Hubble scales. In the linear approximation which we use (and working under the small angle θ assumption) the induced curvature fluctuation ζ_k^{ent} is given by

$$\zeta_k^{ent} \simeq \frac{H}{\dot{\varphi}^2}\dot{\chi}S_k\,, \tag{5.39}$$

where S_k is the Fourier space entropy fluctuation determined above in (5.38). Making use of (5.38), inserting the result for χ given in (5.28), and taking care of the change in the temporal variable from x to η we find

$$\zeta_k^{ent} \simeq \sqrt{\pi\lambda}\frac{H^3\varphi}{\dot{\varphi}^2}e^{2\mu x}k^{-3/2}P(x,k) \tag{5.40}$$

where P is a periodic function of unit amplitude. This clearly shows the exponential growth of the induced curvature fluctuations. A derivation of this result from first principles making use of the covariant formalism is given in the Appendix. Note that even though $\mu_k < 1$, the time scale of the exponential instability is (while long compared to the oscillation time) short compared to the Hubble expansion time, the time scale relevant to the conversion of entropy fluctuations to adiabatic ones.

We can evaluate the power spectrum for the curvature perturbation given by the entropy mode from at horizon crossing:

$$P_k^{ent} = \frac{k^3}{2\pi^2}|\zeta_k^{ent}|^2 \sim \lambda\frac{H^6}{\dot{\varphi}^4}\varphi^2 e^{4\mu x}\,. \tag{5.41}$$

Hence, we conclude that the entropy perturbations give a scale invariant contribution to the power spectrum of the curvature perturbation from inflation, but with an amplitude which is exponentially increasing.

We have derived our result in a simple two field inflation model, the conclusions will carry over to other multi-field models. This analysis suggests that in any inflationary model in which the inflaton satisfies the massless preheating condition, then if low mass entropy fields are present which couple to the inflaton, then parametric resonance of the entropy perturbation indeed happens. Due to the conversion process of entropy perturbation into adiabatic perturbation (as studied in the context of coupled scalar fields in [17]), parametric resonance of entropy perturbation may lead to a rapidly growing adiabatic mode which could have a large impact on the spectrum of curvature perturbation we observe today. The spectrum will remain approximately scale-invariant, but there is the danger that the exponential growth will cause the fluctuations to become non-linear (which would rule out the model). To see whether this is a serious concern, we must however first consider backreaction issues. Backreaction might cut off the instability before the induced curvature fluctuations become too large. However, below we find that at least the backreaction effects which we consider are not strong enough to shut off the resonance in time for the induced curvature fluctuations to remain small enough.

5.2.5 Backreaction Effects

In the previous sections we considered the parametric resonance of entropy perturbations during preheating neglecting any backreaction effects. However, the exponential instability of the entropy field leads to an exponential creation of χ particles that are expected to back react in the background. The study of backreaction is important, since the cumulative effect of the creation of particle eventually becomes important affecting the resonance and even terminating preheating, as already noticed in [57, 74, 102] (see also [79] for earlier numerical work).

We will consider two backreaction effects that can affect preheating in the $g^2/\lambda = 2$ case. Other backreaction effects and parameters choices were studied in [57, 74]. The first effect is the backreaction of the parametrically amplified χ on the evolution of the inflaton background. If the force induced by χ is larger than the force present in the absence of χ, then the condition for massless preheating will no longer be satisfied and the broad parametric resonance will terminate. This will happen when

$$g^2\langle\chi^2\rangle \sim \lambda\phi^2 \,. \tag{5.42}$$

Using $\langle\chi^2\rangle = \chi^2_{eff}$ from (5.28) and setting $\phi = \phi_{end}$, this condition implies that

$$e^{2\mu\Delta x} \simeq \lambda^{-1} \,. \tag{5.43}$$

This gives us the time interval before this backreaction effect becomes important. We can use this result to evaluate the power spectrum of the curvature perturbations induced by entropy modes at the time that the resonance shuts off by this backreaction effect. Using (5.41) and (5.43), the power spectrum at this time is given by:

$$P_k^{ent} \sim \lambda^{-1} \frac{H^6}{\dot{\varphi}^4} \varphi^2 \,. \tag{5.44}$$

Since at the end of inflation $\dot{\varphi}^2 = V$ and $\varphi = \varphi_{end}$, we can estimate the power spectrum for the curvature perturbations from the super-Hubble amplified entropy perturbations as:

$$P_k^{ent} \sim \frac{1}{5} \,. \tag{5.45}$$

This is considerably larger than the observed values, exceeding by many orders of magnitude the COBE normalization measurement [80]. We thus conclude that the parametric amplification of entropy perturbations can lead to a serious problem for models like the one we consider, unless other effects are found which shut off the resonance earlier.

We can also consider the backreaction of the produced χ particles on the Friedmann equation. We find that demanding that the induced χ terms remain subdominant leads to precisely the condition (5.42).

The second effect considered in this paper is the influence of the produced $\delta\phi$ particles on the $\delta\chi$ resonance. If the creation of ϕ particles is large enough, increasing significantly the effective mass of the $\delta\chi$ field, this could damp or even stop the resonance of the χ field. Thus we need to know if

$$V_{,\chi} < \; g^2 \chi \langle \delta\phi^2 \rangle \;\; \Rightarrow \;\; \phi^2 < \langle \delta\phi^2 \rangle \,, \tag{5.46}$$

at some point during preheating, altering the effective time dependent mass and consequently the χ resonance. However, we can see from the equation for the eigenmodes

$\phi_k(t)$ during preheating:

$$\delta\ddot{\phi}_k + 3H\delta\dot{\phi}_k + \left(\frac{k^2}{a^2} + 3\lambda\delta\phi^2\right)\delta\phi_k \,=\, 0\,, \tag{5.47}$$

that for $\delta\phi_k$ the resonance is always narrow, since it is equivalent to the case of parametric resonance with $g^2/\lambda = 3$. This leads to a very small characteristic exponent μ [74] and a very inefficient creation of ϕ particles. This effect will always be less important than the parametric resonance for the χ field, that is very broad and has a large characteristic exponent. Thus, the second backreaction effect studied in this paragraph does not have the potential of shutting off the resonant amplification of entropy fluctuations early, and does not change the conclusion from (5.45) that curvature perturbations are too large when the resonance is finally shut off.

We have to stress, though, that the full theory of backreaction and rescattering during preheating is not fully developed. However, this result represents an advance with respect to previous investigations since the covariant formalism allows for a full non-linear analysis including metric fluctuations (although we here considered only the linear limit).

To conclude this section we note that we have considered the preheating of entropy fluctuations in a two field model in which an inflaton field with vanishing bare mass is coupled to a massless entropy field. In the absence of metric fluctuations, it is known that in this model there is efficient preheating ("massless preheating"). We find, using a covariant formulation of the theory of cosmological fluctuations which in principle can be extended to a full nonlinear analysis, that the entropy fluctuations experience a period of broad parametric resonance. At quadratic order in the amplitude of fluctuations, the entropy modes seed a growing curvature fluctuation. Hence, we find a curvature fluctuation mode which is growing exponentially during the preheating phase. In agreement with previous studies [57] we find that backreaction effects are too weak to shut off the resonance before the power spectrum of the induced curvature fluctuations has reached an amplitude close to 1, i.e. many orders of magnitude larger than the observational value. Hence we see that models of the type we consider here are phenomenologically ruled out, unless there are backreaction effects not considered here which manage to truncate the resonance earlier than the ones we have studied.

5.3 Entropy Perturbations in Axion Monodromy with a Derivative Coupling

There has been a lot of recent interest in large field models of inflation in which the potential is given by a fractional power of the field. An example which is currently attracting much attention is axion monodromy inflation (see [81] for the initial paper and [82] for a recent review). Axion monodromy models are attractive since they may provide a natural realization of large field inflation in the context of superstring theory. Large field models of inflation [83, 84] are advantageous since in such a context the slow-roll trajectory is a local attractor in initial condition space [85], even including metric fluctuations [86]. In contrast, for small field models the initial velocity of the inflaton field needs to be fine tuned, thus creating a potential initial condition problem [87]. As is well known [88], from the point of view of observations, large field models of inflation are interesting since they may lead to a significant tensor to scalar ratio.

There are challenges to obtain large field inflation models. For field values $|\phi| > M_P$ corresponding to large field inflation (M_P is the Planck mass), there is the danger that gravitational corrections will lift the potential and prevent slow rolling of the field, unless there is a symmetry such as shift symmetry [89] which protects the small mass required for large field inflation. In string theory, there is a further challenge of obtaining large field inflation: we expect the field range of the candidate inflatons, e.g. the moduli fields or the fields associated with brane separations, to be small, and hence incompatible with large field inflation. Monodromy inflation [90] provides a possible resolution of this problem, and axion monodromy is currently regarded as the most promising implementation of the idea of monodromy inflation in the context of string theory [82]. For this reason, there has been a lot of recent activity on this topic. The axion of axion monodromy inflation is a bulk field, and thus couples to and can lose energy to each sector. There may be phenomenological issues which arise when the energy loss of the axion into other sectors is considered, but we will not address this issue here.

In this section we will focus on a minimal setup of axion monodromy inflation in which we only consider the axion field and its associated $U(1)$ gauge field. The Lagrangian we are considering is (in (-,+,+,+) signature)

$$\mathcal{L} \, = \, -(1/2)(\partial\phi)^2 + V(\phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{\Lambda} \, \phi \;\, F_{\mu\nu} \widetilde{F}^{\mu\nu} , \tag{5.48}$$

where Λ is a UV scale, and is different from the axion decay constant. The potential $V(\phi)$ is the monodromy potential:

$$V(\phi) = \mu^3 \sqrt{\phi_c^2 + \phi^2}\,, \tag{5.49}$$

where μ is an energy scale whose value can be determined from the observed magnitude of the cosmic microwave background anisotropies, and where $\phi_c < M_P$ is a constant, M_P denoting the Planck mass. The field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{5.50}$$

is that of the abelian gauge field. From the point of view of effective field theory, we would expect Λ to be given either by the string scale or the Planck scale. More stringent, though indirect, constraints come from models of early universe cosmology based upon this coupling. The gaussianity of the CMB constrains the parameter ξ, which we will define shortly, to be $\xi_* \lesssim 2.2$ at the moment when the pivot scale k_* exits the horizon [94], see also [95, 96], which can be translated to a bound $\Lambda^{-1} \leq 12\, M_P$. Recent results on the validity of perturbation theory during inflation [97] constrain ξ to be $\xi \leq 3.5$, which correspond to an even tighter constraint on ξ_* (if the whole inflationary trajectory is to be treated perturbatively). Given these considerations we will take a conservative approach, and work with an upper bound $\Lambda^{-1} \leq \mathcal{O}(1) {M_P}^{-1}$.

5.3.1 Background Evolution

We assume that the axion starts out in the large field region $\phi \gg M_P$ where the slow-roll approximation

$$3H\dot{\phi} \;=\; -V'(\phi) \;\simeq\; -\mu^3 \tag{5.51}$$

of the equation of motion is self-consistent. The end of inflation occurs at the field value when the slow-roll approximation breaks down, at which point $(1/2)\dot{\phi}^2 = V$. This takes place when

$$|\phi| \;\equiv\; \phi_e \;=\; \frac{1}{\sqrt{6}} M_P \,, \tag{5.52}$$

and the kinetic energy at this point is

$$\frac{1}{2}\dot{\phi}^2_{\phi=\phi_e} \;=\; \frac{1}{\sqrt{6}}\mu^3 M_P \,. \tag{5.53}$$

The value of the Hubble constant at the end of inflation is $H = H_e$ with

$$H_e \;=\; 2^{-1/4} 3^{-3/4} M_P^{-1/2} \mu^{3/2} \,. \tag{5.54}$$

After inflation ends ϕ begins anharmonic motion about the ground state $\phi = 0$. As long as we can neglect the expansion of space and the loss of energy by particle production, the motion is periodic but anharmonic.

The value of μ is set by the observed amplitude of the cosmic microwave background (CMB) anisotropies. A simple application of the usual theory of cosmological perturbations (see e.g. [10] for a detailed review, and [98] for an overview) shows that the power spectrum $\mathcal{P}_\zeta$ of the primordial[4] curvature fluctuation ζ has the amplitude

$$\mathcal{P}_\zeta \sim \big(\frac{\mu}{M_P}\big)^3, \tag{5.55}$$

from which it follows that

$$\mu \sim 6 \times 10^{-4} M_P\,. \tag{5.56}$$

5.3.2 Preheating of Gauge Field Fluctuations

As first pointed out in [29] and [99], a periodic axion background can lead to explosive particle production for all fields coupled to the axion. This effect is called "preheating" [100, 101, 102] (see also [27, 103] for reviews). Here we will consider the resonance of the gauge field fluctuations [5]

The equation of motion for the linear fluctuations of A_μ is (see e.g. [95, 105, 106])

$$\frac{d^2 A_{k\pm}}{d\tau^2} + \left(k^2 \pm 2k\frac{\xi}{\tau}\right) A_{k\pm} \,=\, 0\,, \tag{5.57}$$

[4] We add the word "primordial" to make a distinction between the original fluctuations and the induced ones which will be the focus of this paper.

[5] There is the also a possibility that there is an efficient self-resonance of the inflaton, leading to oscillons [104]. Oscillon formation occurs once the amplitude of ϕ oscillations falls below ϕ_c, as defined in equation (5.49). Provided that ϕ_c is small compared to the initial amplitude of oscillations, which is indeed the case in realistic string embeddings, oscillon formation will not occur until preheating in to gauge fields has ceased to be efficient, and will not occur at all if preheating into gauge fields is efficient enough to halt the oscillatory motion of ϕ. Given this, we will not consider oscillon formation in this work, although this does deserve further attention.

where $\pm$ denote the two polarizations of the gauge field, τ is conformal time, k indicates a comoving mode, and ξ is given by[6]

$$\xi = \frac{2\dot{\phi}}{\Lambda H}, \tag{5.58}$$

where H is the Hubble expansion rate and ϕ is the background field, and an overdot denotes the derivative with respect to physical time. As long as the slow-roll approximation is valid, ξ can be taken to be constant. This is the equation relevant during the inflationary period.

As Eq. (5.57) shows, for one of the polarization states there is a tachyonic instability (see e.g. [107] for an initial discussion of tachyonic instabilities in reheating) already during inflation for long wavelength modes, i.e. modes which obey

$$k - \frac{2\xi}{|\tau|} = k - \frac{4|\dot{\phi}_I|}{\Lambda H|\tau|} \; < \; 0\,, \tag{5.59}$$

where the subscript I indicates that the time derivative is evaluated during slow-roll inflation. The critical wavelength beyond which there is a tachyonic instability has a fixed value in physical coordinates if we take H and $\dot{\phi}$ to be constant in time. The critical wavelength can be called a "gauge horizon" and it plays a similar role as the Hubble radius (Hubble horizon) for cosmological perturbations. The gauge horizon is proportional to the Hubble radius, its physical wavenumber k_p being given by

$$k_p = 2\xi H\,. \tag{5.60}$$

[6] Our definition of ξ is equivalent to the definition used in [95, 105, 106] with the identification $\alpha/f = 4/\Lambda$.

For modes which start in their vacuum state deep inside the horizon, the tachyonic resonance [107] leads to squeezing of the mode function. The Floquet exponent is proportional to k, and hence, among all the modes which become super-horizon (meaning super-gauge horizon) by the end of inflation, the ones which undergo the most squeezing are the ones which exit shortly before the end of inflation, i.e. whose comoving wavenumbers is given by

$$k = k_* \equiv 2\xi H , \tag{5.61}$$

if we normalize the cosmological scale factor to be $a(t) = 1$ at the end of inflation. The value of k_* is determined by the Hubble rate and the axion field velocity at the end of the period of inflation.

It can be shown [105] that the mode function prepared by inflation is

$$\begin{aligned} A^{(0)}_{k+} &= \frac{2^{-1/4}}{\sqrt{2k}} \left(\frac{k}{\xi a H} \right)^{1/4} e^{\pi\xi - 4\xi\sqrt{k/2\xi a H}} \\ A^{(0)}_{k-} &= 0, \end{aligned} \tag{5.62}$$

where $+/-$ denote the positive/negative chirality mode (the $-$ mode is not amplified during inflation). This corresponds to a highly blue spectrum of gauge field fluctuations with an ultraviolet cutoff which is set by the gauge horizon; the cutoff comes from the second term in the exponential on the right hand side of (5.62). The major amplification factor F_I of the amplitude is

$$F_I = e^{\pi\xi} . \tag{5.63}$$

For the specific potential (5.49) of axion monodromy inflation the values of k_* and ξ are (making use of (5.53) and (5.54))

$$k_* = 4\left(\frac{2}{3}\right)^{1/4} M_P^{1/2} \mu^{3/2} \Lambda^{-1} \quad (5.64)$$

$$\xi = 2\sqrt{6}\frac{M_P}{\Lambda}\,. \quad (5.65)$$

This shows that if $\Lambda \ll M_P$ there is a large enhancement of the amplitude of A_k during inflation. On the other hand, if $\Lambda \gg M_P$, then the growth is negligible. For small values of Λ (i.e. large values of ξ), the "gauge horizon" is smaller than the Hubble horizon, whereas for large values of Λ the opposite is true.

As mentioned above, the power spectrum $\mathcal{P}_A$ of gauge field fluctuations is blue. On length scales larger that the gauge horizon we have

$$\mathcal{P}_A(k) \equiv k^3 |A_k|^2 \sim k^{5/2}\,. \quad (5.66)$$

During reheating the expansion of space can be neglected [29] and the equation (5.57) becomes

$$\ddot{A}_{k\pm} + \left(k^2 \pm 4\frac{k}{\Lambda}\dot{\phi}\right) A_{k\pm} = 0. \quad (5.67)$$

We immediately see that the tachyonic resonance which was present during the period of inflation persists during the preheating period when ϕ undergoes damped anharmonic oscillations about $\phi = 0$. While $\dot{\phi}$ is negative, then the same polarization mode gets amplified as during inflation. During the second half cycle, when $\dot{\phi} > 0$, it is the other mode which is amplified while the original mode oscillates.

To obtain an order of magnitude estimate of the amplification of A_k during preheating, we focus on the first oscillation period (when the Floquet exponent of the instability is largest). We focus on the first quarter of the oscillation period T when ϕ is decreasing from $\phi = \phi_e$ to $\phi = 0$. The velocity during most of this time interval is approximately $\dot{\phi}_e$ (see (5.53)). The amplitude of A_k grows exponentially at a rate (for $k/k_* < 1$),

$$\mu_k \,=\, 2\left(\frac{k}{\Lambda}\right)^{1/2}\sqrt{\dot{\phi}_e} \,=\, 2\left(\frac{2}{3}\right)^{1/8}\left(\frac{k}{\Lambda}\right)^{1/2} M_P^{1/4}\mu^{3/4}\,. \qquad (5.68)$$

The factor F_k by which the amplitude of A_k is amplified is

$$F_k \,=\, e^{X_k}\,, \qquad (5.69)$$

with

$$X_k \,=\, \frac{1}{4}T\mu_k\,, \qquad (5.70)$$

where T is the period. The quarter period is given by

$$\frac{1}{4}T \,=\, \frac{\phi_e}{\dot{\phi}_e}\,. \qquad (5.71)$$

Combining these equations yields

$$X_k \,=\, X_{k_*}(\frac{k}{k_*})^{1/2}, \qquad (5.72)$$

with

$$X_{k_*} \,=\, 2\left(\frac{2}{3}\right)^{1/2}\frac{M_P}{\Lambda}\,. \qquad (5.73)$$

Comparing the amplification factors F_I and F_k (see (5.63) and (5.73)) one sees that at the value $k = k_*$ they have similar magnitudes.

The mode function after one period of oscillation of ϕ is thus given by

$$A_{k+} = \frac{2^{-1/4}}{\sqrt{2k}} e^{X_k} \left(\frac{k}{\xi aH}\right)^{1/4} e^{\pi\xi - 4\xi\sqrt{k/2\xi aH}} . \tag{5.74}$$

As long as the expansion of the universe can be neglected, and before backreaction shuts off the resonance, the gauge field fluctuations grow by the same factor in each period. Hence, after N periods we obtain

$$A_{k+} = \frac{2^{-1/4}}{\sqrt{2k}} e^{NX_k} \left(\frac{k}{\xi aH}\right)^{1/4} e^{\pi\xi - 4\xi\sqrt{k/2\xi aH}} . \tag{5.75}$$

Comparing the expressions for the period T and the Hubble expansion rate H at the end of inflation we see that right at the end of inflation $T \sim H^{-1}$ and hence the expansion of space cannot be neglected. However, once reheating starts, ϕ decreases and hence T decreases and the expansion of space becomes negligible. The Floquet exponent can be taken to be approximately constant during half of each period, and vanishing for the other half. Hence, over a period $(0, t)$ of reheating, the increase in the amplitude is

$$F_k \sim e^{\frac{1}{2}\mu_k t} , \tag{5.76}$$

and the gauge field amplitude becomes

$$A_{k+} = \frac{2^{-1/4}}{\sqrt{2k}} e^{\frac{1}{2}\mu_k t} \left(\frac{k}{\xi aH}\right)^{1/4} e^{\pi\xi - 2\sqrt{2}\xi\sqrt{k/\xi aH}} . \tag{5.77}$$

There is also an amplification for the $(-)$ polarization, A_{k-}, but this mode is suppressed during inflation, and enters preheating with a different mode function.

5.3.3 Gauge Field Energy Density Fluctuations

We have thus far computed the gauge fields produced during preheating. This sources an energy density perturbation, $\delta\rho_A$, which we will now focus on. The gauge field energy density is defined as (in (-,+,+,+) signature)

$$\rho_A(x,t) \;=\; -T^0_0\,, \tag{5.78}$$

where $T_{\mu\nu}$ is given by (again in (-,+,+,+) signature, and assuming a Lagrangian $\mathcal{L} = (1/4)F^2$),

$$T_{\mu\nu} \;=\; -\frac{1}{4}g_{\mu\nu}F^2 + F_{\mu\lambda}F^{\lambda}_{\nu}\,. \tag{5.79}$$

In terms of the gauge field A_μ, and without any gauge fixing, this reduces to

$$\rho_A(x,t) \;=\; -\frac{1}{2}(\partial^0 A^i - \partial^i A^0)(\partial_0 A_i - \partial_i A_0) + \frac{1}{4}(\partial^i A^j - \partial^j A^i)(\partial_i A_j - \partial_j A_i)\,. \tag{5.80}$$

We can fix the gauge by setting $A_0 = 0$. The leading term on cosmological scales is given by

$$\rho_A(x,t) \;\simeq\; -\frac{1}{2}\partial^0 A^i \partial_0 A_i\,. \tag{5.81}$$

To find the Fourier modes of $\rho_A(x,t)$, we first expand A_μ in terms of *classical* oscillators

$$A_\mu(x,t) \;=\; \sum_{\lambda=+,-}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\left[\epsilon^\lambda_\mu A_\lambda(k,t)\alpha_k e^{ikx} + \epsilon^{\lambda\,*}_\mu A_\lambda(k,t)\alpha^\dagger_k e^{-ikx}\right]\,, \tag{5.82}$$

where α_k are classical oscillators drawn from a nearly Gaussian distribution, satisfying

$$\langle \alpha_k \alpha_{k'}\rangle \;=\; (2\pi)^3\delta^3(k+k')\,, \tag{5.83}$$

where the angular brackets stand for ensemble averaging. We can expand ρ in a similar fashion

$$\rho_A(x,t) \;=\; \int \frac{\mathrm{d}^3k}{(2\pi)^3}\rho_{Ak}\beta_k e^{ikx} + c.c.\,, \tag{5.84}$$

where β_k are a different set of classical oscillators, whose distribution function can be determined in terms of the α_k. The Fourier modes of $\rho(x,t)$ are simply a convolution of Fourier modes of the gauge field A_μ

$$\rho_{Ak}\beta_k \;=\; +\frac{1}{2}a^{-2}\int \frac{\mathrm{d}^3k'}{(2\pi^3)}\dot{A}_{k'+}\dot{A}_{(k-k')+}\alpha_{k'}\alpha_{k-k'}\,, \tag{5.85}$$

where the mode function A_k is given by equation (5.77). There is a gradient term $k^2A_k^2$ which is comparable in magnitude to the time-derivative term, and thus changes ρ_{Ak} by a factor of two.

We can use the above expression to straightforwardly calculate the background energy density in the gauge field and the spectrum of the gauge fluctuations. The homogenous background energy density is simply $\langle\rho_A(x,t)\rangle$, and we define the fluctuations $\delta\rho_A(x,t)$ about this background as $\delta\rho_A = \rho_A - \langle\rho_A\rangle$, such that $\langle\delta\rho_A\rangle = 0$, and the variance of fluctuations is simply $\langle\delta\rho_A^2\rangle = \langle\rho_A^2\rangle - \langle\rho_A\rangle^2$. A simple calculation shows that the background is given by

$$\langle\rho_A(x,t)\rangle \;=\; \frac{1}{2}a^{-2}\int \mathrm{d}^3k|\dot{A}_{k+}|^2\,. \tag{5.86}$$

The dominant contribution to the integral comes from the maximally amplified mode $k = k_*$, and we can hence approximate it as

$$\langle\rho_A(x,t)\rangle \;\sim\; \sqrt{2}\,a^{-2}\,e^{2\mu_* t}(\mu_* k_*)^2 e^{-2\sqrt{2}}\cdot e^{2\pi\xi}\,, \tag{5.87}$$

where $\mu_* \equiv \mu_{k_*}$ From this we see that the amplitude of $\langle\rho\rangle$ depends inversely on the UV scale Λ, since a smaller Λ means an increased k_*.

The mode function of fluctuations can be straightforwardly computed using the definition $\delta\rho_A = \rho_A - \langle\rho_A\rangle$ in conjunction with equation (5.85) and the approximation that the β_k are drawn from a nearly Gaussian distribution, i.e. $\langle\beta_k\beta_{k'}\rangle = (2\pi)^3\delta^3(k+k')$. The exact β_k are *not* drawn from a Gaussian distribution, but as we have the modest goal of computing power spectra (i.e. two-point statistics), this is not an important distinction. The dominant term in $\delta\rho_{Ak}$ is

$$|\delta\rho_{Ak}|^2 \simeq \frac{1}{4} a^{-4} \int \mathrm{d}^3q\, |\dot{A}_q|^2 |\dot{A}_{k-q}|^2 \,. \tag{5.88}$$

For modes in the IR, i.e. $k \ll k_*$, this integral is highly peaked at $q = k_*$ and we can find

$$|\delta\rho_{Ak}|^2 \simeq \frac{\langle\rho_A\rangle^2}{k_*^3} \,. \tag{5.89}$$

Note, in particular, that the resulting power spectrum of gauge field fluctuations is highly blue. The spectral index is $n_s = 4$.

5.3.4 Backreaction Considerations

The exponential increase in the gauge field value cannot continue forever. Eventually, the tachyonic resonance will be shut off by backreaction effects. backreaction in a two field toy model of parametric resonance was considered in [108], where it was concluded that backreaction does not prevent the exponential production of entropy fluctuations before these perturbations become important. In this subsection we estimate how long the tachyonic resonance in our model will last until backreaction becomes important.

We will consider the two most important backreaction effects involving gauge field production. The first is the effect of gauge field production on the axion field dynamics, the dynamics driving the instability. Recall that the axion equation of motion is given by

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} \; = \; \frac{1}{\Lambda}\langle F\tilde{F}\rangle, \tag{5.90}$$

where $\langle F\tilde{F}\rangle$ refers to enseble or spatial averaging as was done to determine $\langle \rho_A \rangle$ in the previous section. To obtain an order of magnitude estimate of when backreaction becomes important, we can compare the term on the right hand side of (5.90) with the force driving the oscillations. The first condition of 'small backreaction' comes from demanding that the force term dominates. This translates to

$$\langle V_{,\phi}\rangle_{rms} \gg \langle\frac{F\tilde{F}}{\Lambda}\rangle_{RMS} \,. \tag{5.91}$$

We can estimate the order of magnitude of the right-hand side of the above equation by $\Lambda^{-1}\rho_A$, and hence the condition (5.91) becomes

$$V' \; \gg \; \frac{1}{\Lambda}\rho_A \,. \tag{5.92}$$

The second backreaction condition comes from demanding that the energy density is dominated by the scalar field, i.e.

$$V \; \gg \; \rho_A \,. \tag{5.93}$$

In this equation, the value of ϕ appears. We will use the value at the end of inflation.

For the axion monodromy potential we are using, the two conditions differ by a factor Λ/M_P. Inserting the expression (5.87) into the first backreaction criterium (5.92) yields

$$2\mu_* t \;=\; -2\pi\xi + 3\ln\left(\frac{\Lambda}{\mu}\right) + 2\ln\left(\frac{\Lambda}{M_P}\right) \tag{5.94}$$

for the time interval t before backreaction becomes important, whereas the second condition (5.93) yields

$$2\mu_* t \;=\; -2\pi\xi + 3\ln\left(\frac{\Lambda}{\mu}\right) + 3\ln\left(\frac{\Lambda}{M_P}\right) \tag{5.95}$$

which is a stronger condition if $\Lambda < M_P$ and weaker otherwise.

The amplitude of the gauge field energy density fluctuations when backreaction becomes important then is bounded from above by

$$\begin{aligned} \delta\rho_{Ak} &\sim \frac{V}{k_*^{3/2}} \quad \text{for } \Lambda > M_P \\ \delta\rho_{Ak} &\sim \frac{V}{k_*^{3/2}}\frac{\Lambda}{M_P} \quad \text{for } \Lambda < M_P\,. \end{aligned} \tag{5.96}$$

Note that there can be backreaction effects from the production of other fields which may turn off the resonance much earlier. Since we are interested in obtaining upper bounds on the effects generated by gauge field production, we will work with the above upper bounds.

5.3.5 Induced Curvature Perturbations

During reheating purely adiabatic fluctuations on super-Hubble scales cannot be amplified since it can be shown that the curvature fluctuation variable ζ is conserved.

This can be shown in linear cosmological perturbation theory [88, 109, 110, 111], but the result holds more generally (see e.g. [54, 55]). On the other hand, entropy fluctuations can be parametrically amplified during reheating [46, 47] (see also [113]). Entropy fluctuations inevitably seed a growing curvature perturbation. Thus, in the presence of entropy modes it is possible to obtain an exponentially growing curvature fluctuation on super-Hubble scales (see e.g. [114] for some studies of this question in earlier string-motivated models of inflation).

Consider ζ, the curvature perturbation on uniform density hypersurfaces. This is the variable which determines the amplitude of the CMB anisotropies at late times (see [10] for a detailed overview of the theory of cosmological perturbations). In the absence of entropy fluctuations, this variable is conserved on super-Hubble scales [55, 88, 109, 110]. However, in the presence of entropy perturbations, a growing mode of ζ is induced on super-Hubble scales, as already discussed in the classic review articles on cosmological perturbations [10, 115] and as applied to axion inflation in [116]. For more modern discussions the reader is referred to [118, 138]. The equation of motion for ζ on large scales (of cosmological interest) is given by (2.43)

$$\dot{\zeta} = -\frac{H}{p+\rho}\delta P_{nad}\,. \qquad (5.97)$$

Note that ζ is dimensionless.

From our discussion in chapter 2, we know that the relative entropy perturbations is related to the non-adiabatic pressure perturbations. Then we can write non-adiabatic pressure perturbations as

$$\delta P_{nad} \,=\, \dot{p}\left(\frac{\delta p}{\dot{p}} - \frac{\delta\rho}{\dot{\rho}}\right)\,, \tag{5.98}$$

where in our case the total pressure is the sum of the contributions from the ϕ field and from the gauge field, i.e. $p = p_\phi + p_A$, and similarly for ρ. For a background that is dominated by ϕ, and with $\delta\rho_A > \delta\rho_\phi$, the above non-adiabatic pressure perturbation is simply

$$\delta P_{nad} \,\simeq\, \dot{p_\phi}\left(\frac{\delta p_A}{\dot{p}_\phi} - \frac{\delta\rho_A}{\dot{\rho}_\phi}\right)\,, \tag{5.99}$$

and the evolution equation of ζ is given by

$$\dot{\zeta} \,\simeq\, -\frac{H}{\rho_\phi + p_\phi}\left(\frac{1}{3} - c_{s\phi}^2\right)\delta\rho_A\,, \tag{5.100}$$

where $c_{s\phi}^2 = \frac{\dot{P}_\phi}{\dot{\rho}_\phi}$ is the speed of sound for the inflaton field.

In our case, the gauge field energy density fluctuations $\delta\rho_A$ is increasing exponentially with a Floquet exponent $2\mu_*$ during the preheating phase, as shown in earlier sections. Hence, integrating over time, we get

$$\Delta\zeta_k \,=\, -\mu_*^{-1}\frac{H}{\rho_\phi + p_\phi}\left(\frac{1}{3} - c_{s\phi}^2\right)\delta\rho_{Ak}\,, \tag{5.101}$$

where the wavenumber k and the density fluctuation $\delta\rho_A$ are Fourier space quantities. However, since $\delta\rho_A$ is independent of k, we find that the power spectrum of the induced fluctuations of ζ is

$$\mathcal{P}_{\Delta\zeta}(k) \,\sim\, k^3\,, \tag{5.102}$$

which corresponds to a highly blue tilted spectrum with index $n_s = 4$. Since the spectrum has such a large blue tilt, there are no constraints on our model coming from demanding that the induced curvature fluctuations do not exceed the observational upper bounds.

5.3.6 Primordial Black Hole Constraints

Since the power spectrum of induced curvature fluctuations is highly blue, we have to worry about the possible constraints on the model coming from overproduction of primordial black holes. Primordial black holes are constrained by a set of cosmological observations, beginning with the original constraints coming from the observational bounds on cosmic rays produced by radiating black holes [119]. Primordial black hole production during reheating has been considered in simple two field inflation models in [120], and in models with spectra with a distinguished scale in [121].

In the context of an inflationary cosmology, primordial black holes of mass M can form when the length scale associated with this mass (i.e. the length l for which the mass inside a sphere of radius l equals M) enters the Hubble radius. The number density of black holes of this mass will depend on the amplitude of the primordial power spectrum [7] .

Since in our case the power spectrum is highly blue, the tightest constraints will come from the smallest mass for which cosmological constraints exist. These correspond to black holes with a mass such that they evaporate during nucleosynthesis.

[7] The are numerous subtleties in computing the precise number density, which tend to suppress the number of primordial black holes formed, see e.g. [122] and references therein. These details will not be important for our analysis.

The extra radiation from these black holes would act as an extra species of radiation, and would destroy the agreement between the theory of nucleosynthesis and observations (see [123] for reviews). The smallest length scale (i.e. largest wavenumber k) for which constraints exist is [124]

$$k_{max} \sim 10^{19}\mathrm{Mpc}^{-1}, \tag{5.103}$$

and the approximate bound on the power spectrum is

$$\mathcal{P}_\zeta(k_{max}) < 10^{-1.5}\,. \tag{5.104}$$

In fact, the bound for smaller values of k has comparable amplitude.

The power spectrum including the induced curvature perturbations is given by

$$P_\zeta(k) = \frac{k^3}{(2\pi)^2}|\mathcal{A}_0 k^{-3/2} + \Delta\zeta_k|^2, \tag{5.105}$$

where $\mathcal{A}_0 \sim 10^{-10}$ is the amplitude of the power spectrum at the pivot scale $k = k_0 = 0.05\mathrm{Mpc}^{-1}$, and we have approximated the spectrum of curvature perturbations from inflation to be scale invariant. We already computed the value of the induced curvature fluctuations $\Delta\zeta$ in the previous subsection in Eq. (5.101). Inserting the values from (5.96), (5.64) and (5.65) we obtain the following expressions for the leading order correction to the power spectrum of curvature fluctuations

$$\begin{aligned} \Delta\mathcal{P}_\zeta(k) &= \mathcal{O}(10^{-3})\sqrt{\mathcal{A}_0}\;k^{3/2}\,\frac{\Lambda^{5/2}}{M_P^{7/4}\mu^{9/4}} \;\text{ for }\; \Lambda > M_P \\ \Delta\mathcal{P}_\zeta(k) &= \mathcal{O}(10^{-3})\sqrt{\mathcal{A}_0}\;k^{3/2}\,\frac{\Lambda^{7/2}}{M_P^{11/4}\mu^{9/4}} \;\text{ for }\; \Lambda < M_P\,, \end{aligned} \tag{5.106}$$

corresponding to a spectrum with index $n_s = 5/2$. These expressions hold if the exponential growth of the curvature fluctuations is only limited by the backreaction effects studied before. Other effects may terminate the growth earlier. Hence, the above equations provide upper bounds on the amplitude of the induced curvature perturbations.

For the largest value of k for which the primordial black hole constraints apply we have

$$\frac{k}{M_P} \sim 10^{-39} \,. \tag{5.107}$$

Inserting this value into (5.106) we find that the primordial black hole constraint (5.104) is trivially satisfied for the realistic range of values of Λ.

To conclude, we note that in this section we have considered a minimal axion monodromy model and have calculated the spectrum of curvature perturbations induced by the entropy modes associated with the gauge field to which the axion couples. We find that the leading correction to the curvature spectrum is blue with spectral index $n_s = 5/2$. Hence, there are no constraints from large scale cosmological observations. On the other hand, since the spectrum is blue, there is a danger of overproduction of primordial black holes. We find, however, that the amplitude of the spectrum is too low even on the smallest scales for which cosmological constraints exist.

5.4 Entropy Perturbation in an Asymptotic-safe Model

Recently, interesting results have come up in the area of asymptotically safe quantum field theories [128, 129], where it has been shown that a gauge-Yukawa theory can exhibit an ultraviolet (UV) safe (i.e. non-trivial) fixed point. Being UV safe these models could be interesting for early time cosmology. In particular, it has been realized that these models admit a period of cosmological inflation at very early times [130].

As we emphasized throughout the current thesis, in order to connect the inflationary phase with the late time cosmology, a period of “reheating" at the end of inflation is required. The early phase of reheating, namely the preheating process, typically leads to a non-thermal state in which modes in certain wavelength intervals are highly excited whereas the rest are not excited at all. However, it leads to a phase in which the equation of state of matter is approximately that of radiation. For early universe considerations such as baryogenesis or the production of topological defects it is important to know the energy density when the radiation phase of expansion begins. Hence, it is important to know whether the preheating process is operative or not [8] . The first motivation for the work presented in this section is to find out whether in the asymptotically safe quantum field models discussed in [130] preheating occurs.

[8] Preheating does not arise in all inflationary models.

If an inflationary universe model admits a preheating instability at the end of the period of inflation, there is the danger that the instability will also affect the cosmological perturbations [133]. The period of inflation produces fluctuations [16] which have the right spectral shape to explain the observed distribution of matter in the universe and the observed cosmic microwave background anisotropies. A parametric growth of these fluctuations at the end of inflation would destroy the agreement between theory and observation. As we saw earlier in this chapter, in the presence of a second scalar field a preheating instability of curvature fluctuations is possible [2]. It is important to point out why the growth of super-Hubble scale perturbations during reheating is compatible with causality. This is already discussed in [113].The key point is that in inflationary cosmology there is an exponentially large difference between the horizon (the forward light cone of a point on the initial condition surface, e.g. a point at the beginning of inflation) which grows exponentially in time, and the Hubble radius, the inverse expansion rate (which is constant during a period of exponential inflation). The reason why inflation can provide a solution of the horizon problem of Standard Big Bang cosmology is precisely the fact that the physical scale of a region of causal contact and homogeneity expands exponentially and becomes much larger than the Hubble radius. At the end of inflation, the inflaton field is in the coherent state and the correlation length of the field would set the maximum wavelength of fluctuations which can causally be amplified. Inflation will provide that the background inflaton field is coherent over a distance much larger than the Hubble radius during reheating and therefore guarantees possibility of causal amplification of super Hubble modes. This possibility is the case for both adiabatic and entropy fluctuations. Also we would like to mention the fact that field equations are relativistic, hence causality is mathematically build-in and the result from field equations will not violate causality. In this regard there is no distinction between supper-Hubble and sub-Hubble modes.

The asymptotically safe quantum field models studied in [130] contains many scalar fields. A second goal of our study is to see whether there is a parametric amplification of entropy modes in our models, and what the resulting amplitude of the induced curvature fluctuations is [9] .

5.4.1 Model

We will here give a short recap of the model at hand. The full Lagrangian density is composed of adjoint $SU(N_c)$ gauge fields, N_f Dirac fermions in the fundamental of $SU(N_c)$ and an $N_f \times N_f$ neutral complex scalar matrix, H. We will here only present the scalar part of the Lagrangian.

$$\mathcal{L}_{scalar} = \mathrm{Tr}\left(\partial_\mu H^\dagger \partial^\mu H\right) - u\,\mathrm{Tr}\left(H^\dagger H H^\dagger H\right) - v\left(\mathrm{Tr}\,H^\dagger H\right)^2 , \tag{5.108}$$

where u and v are dimensionless coupling constants.

[9] This question has recently been addressed in other two field models of inflation in [2, 4].

As we will be working in the UV regime of this model it is noteworthy that the scalar couplings at the UV fixed point [128] are given by

$$\alpha_u^* = \frac{u^* N_f}{(4\pi)^2} = \frac{\sqrt{23}-1}{19}\delta \tag{5.109}$$

$$\alpha_v^* = \frac{v^* N_f^2}{(4\pi)^2} = -\frac{1}{19}\left(2\sqrt{23} - \sqrt{20+6\sqrt{23}}\right)\delta \tag{5.110}$$

where the constant

$$\delta \equiv \frac{N_f}{N_c} - \frac{11}{2} \tag{5.111}$$

(which must be positive) can be made arbitrarily small by adjusting N_c and N_f.

We will in this work take a simplified version of (5.108) as we assume H to be symmetric and real. The parametrization of H is given by

$$H_{ij} = \begin{cases} \frac{1}{\sqrt{2N_f}}\phi & \text{if } i = j \\ \frac{1}{2}\chi_{(ij)} & \text{else}\,, \end{cases} \tag{5.112}$$

where (ij) indicates that this part is symmetric in i and j.

With this normalization the kinetic term in (5.108) is given by

$$\mathrm{Tr}\left(\partial_\mu H^\dagger \partial^\mu H\right) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\sum_l^{N_p} \partial_\mu\chi_l\partial^\mu\chi_l \tag{5.113}$$

where $N_p = N_f(N_f-1)/2$ is the number of different off-diagonal fields, χ_{ij}. It is clear from the kinetic term why we chose the normalization of H in (5.112). Similarly the double trace potential is given by

$$v\left(\mathrm{Tr}\, H^\dagger H\right)^2 = \frac{v}{4}\phi^4 + \frac{v}{4}\sum_l^{N_p}\chi_l^4 + \frac{v}{2}\phi^2\sum_l^{N_p}\chi_l^2 + \frac{v}{2}\sum_{l>k}^{N_p}\chi_l^2\chi_k^2. \tag{5.114}$$

For a general complex matrix H the potential can be fully written in terms of the structure constant of $U(N_f)$, see Appendix B of Ref. [135]. We will however here use the following grouping of terms for the single trace potential

$$\begin{aligned} u \,\mathrm{Tr}\, H^\dagger H H^\dagger H = & \frac{u}{4N_f}\phi^4 + \frac{3u}{2N_f}\phi^2 \sum_l^{N_p} \chi_l^2 \\ & + \frac{3u}{\sqrt{2N_f}}\phi \sum_{i<j<k}^{N_f} \chi_{ij}\chi_{jk}\chi_{ki} + \frac{u}{8}\sum_l^{N_p}\chi_l^4 \\ & + \frac{u}{8}\sum_{i\neq j\ k>i}^{N_f} \chi_{ij}^2\chi_{jk}^2 + \frac{u}{16}\sum_{i,j,k,l}^{N_f} \chi_{ij}\chi_{jk}\chi_{kl}\chi_{li}. \end{aligned} \tag{5.115}$$

It is worth noting that there is no cubic term for ϕ. This is a consequence of the fact that ϕ appears only on the diagonal of H.

At first glance this model seems overly complicated, however a short motivation why this model is relevant for discussion will be given here. A central parameter in the study for parametric resonance is the ratio of the quartic inflaton coupling (λ) to the portal coupling (g^2). A study similar to the present can be given in a toy model with an inflaton and a scalar field coupled to the inflaton. In these toy models the relevant parameter (g^2/λ) will need to be fixed arbitrarily, see e.g. [2, 3, 102, 132]. This is in contrast to the model presented in this section. With the model given by (5.108) this ratio is given by the model it self and is thereby not an arbitrarily chosen number. Furthermore, as was shown in [129] the running of the couplings follow that of the gauge coupling (in [129] called α_g) along the UV uni-dimensional stable trajectory. This implies that the ratio g^2/λ stays constant even including running away from the UV fixed point. This is a remarkable fact that solidifies our future choice for this ratio to be that at the UV fixed point. This is a priori not a feature of a toy model, hence the model in (5.108) is an interesting scenario.

5.4.2 Recap of Results on Inflation

We will in this section recap some of the results of Ref. [130] as this is the inflationary scenario we have in mind for our investigation. Inflation is driven by the diagonal element of H where all diagonal elements are taken to be the same [129]. The inflationary effective potential can be derived from (5.108) with the couplings given by (5.109) and (5.110),

$$V(\phi) = \frac{\lambda\phi^4}{4(1+W(\phi))}\left(\frac{W(\phi)}{W(\mu_0)}\right)^{\frac{18}{13\delta}} \tag{5.116}$$

where $\lambda = v^* + \frac{u^*}{N_f}$ and $W(\cdot)$ is related to the product logarithm, see Ref [130] for details. This is a ϕ^4-theory including renormalisation of the inflaton operator expanded near the UV fixed point.

The potential in equation (5.116) is valid for a large range in ϕ. However, for the study of parametric resonance small field values are considered as this minimum of the potential is located here. This means that for this analysis the potential will be approximated by $\lambda\phi^4$. This full ϕ dependence is included for completeness.

It was shown that this model can provide a viable scenario for large field inflation. The inflationary slow-roll condition ceases to be satisfied and thus quasi-exponential expansion stops at a field value

$$\phi_{end} = \sqrt{(4-\frac{16}{19}\delta)(3-\frac{16}{19}\delta)}M_P \simeq \sqrt{12}M_P. \tag{5.117}$$

The phenomenological predictions for this model, for very small δ, lie just outside the Planck'15 2σ contours for the tensor-to-scalar ratio and scalar spectral index. It was noted that in this perturbative regime a very large number of flavors was needed to produce the measured amplitude of curvature perturbations by the inflaton fluctuations alone. However this number drops rapidly as the perturbative parameter δ is pushed close to and beyond the radius of convergence of the underlying model.

The inflationary phase will quasi-exponentially redshift the wavelength of any fluctuations existing before the onset of reheating, and will produce a homogeneous inflaton condensate. Once the field value of this condensate decreases to below the value given by (5.117), accelerated expansion of space ends and the reheating period begins. We will be working in terms of the usual metric

$$ds^2 \;=\; dt^2 - a(t)^2 d\mathbf{x}^2 \qquad (5.118)$$

of space-time, where t is physical time and $\mathbf{x}$ are the Euclidean comoving coordinates of the expanding space. It will often be useful to work in terms of conformal time η defined via $dt = ad\eta$. Since we are interested in the period of reheating, we will normalize the scale factor to be $a(t_R) = 1$ at the time t_R corresponding to the beginning of reheating.

5.4.3 Parametric Resonance

We will here discuss parametric resonance of the inflaton and the off-diagonal scalar fields. One could also investigate parametric production of fermionic fields; however this will be left for a later discussion.

At the end of the inflationary epoch and before significant production of any other field has happend, the equation of motion (EoM) for the homogeneous inflaton field is given by

$$\ddot{\phi} + 3H\dot{\phi} + \lambda\phi^3 = 0\,, \tag{5.119}$$

where H is the Hubble expansion parameter. It is obvious that the solution for ϕ will correspond to damped oscillation. The expansion of space can be factored out by introducing a rescaled field $\tilde{\varphi} \equiv a\phi$ and working in terms of conformal time. In terms of this field the solution is, as noted in [74], oscillatory, however not sinusoidal but proportional to the elliptic cosine, $cn(x)$, where x is a rescaled dimensionless conformal time which will be defined below. The equation of state of these oscillations is (upon time averaging) that of radiation. Hence the amplitude, which is asymptotically given by

$$\tilde{\varphi} = a\frac{1}{\sqrt{t}}\left(\frac{3M_P^2}{8\pi\lambda}\right)^{\frac{1}{4}}\,, \tag{5.120}$$

is constant.

The EoM for the ij off diagonal component is given by

$$\begin{aligned} 0 = \quad & \ddot{\chi}_{ij} + 3H\dot{\chi}_{ij} + g^2\phi^2\chi_{ij} - a^{-2}\nabla^2\chi_{ij} \\ & + \frac{3u}{\sqrt{2N_f}}\phi\sum_k^{N_f}\chi_{ik}\chi_{kj} + \mathcal{O}(\chi^3)\,, \end{aligned} \tag{5.121}$$

where ∇ stands for the gradient operator with respect to the comoving spatial coordinates. Here

$$g^2 = v + \frac{3u}{N_f}\,. \tag{5.122}$$

Note that first line is leading order in $\chi \sim 0$, and anything beyond this is subleading. In our study of parametric resonance we consider the first three terms only, and the rest we regard as back reactions in the next section. The first line is a linear equation for χ_{ij} which we will solve for each Fourier mode independently.

Fourier transforming this linear EoM and rescaling the field by $a\chi = X$ yields [10]

$$\tilde{X}_k'' + \left(\kappa^2 + \frac{g^2}{\lambda} cn^2\left(x, \frac{1}{\sqrt{2}}\right) - \frac{a''}{a}\right)\tilde{X}_k = 0\,, \tag{5.123}$$

where we have dropped the ij indices since the equation is identical for each component as long as we do not include mixing terms from the higher orders in χ. We have defined

$$\kappa^2 = \frac{k^2}{\lambda\tilde{\varphi}^2} \tag{5.124}$$

and primes are derivatives w.r.t the dimensionless conformal time

$$x \equiv \sqrt{\lambda}\tilde{\varphi}\eta\,. \tag{5.125}$$

In (5.123) we have signified the Fourier component by a tilde, however they will be omitted from now on. The term $\frac{a''}{a}$ is zero during the phase of parametric resonance as can be checked from the Friedman equation.

[10] We are neglecting terms in the equation of motion for X_k containing the metric perturbations. As discussed in Chapter 19 of the review article [10] this is justified as long as the energy density in the entropy field is smaller than that of the inflaton field. We are not the first to use this approximation. It is the basis of the *curvaton scenario* [141] and related scenarios such as the *New Ekpyrotic model [142].*

We will now approximate the elliptic cosine by the first term of a cosine expansion [74] as follows

$$cn\left(x, \frac{1}{\sqrt{2}}\right) \simeq \frac{8\sqrt{2}\pi}{T}\frac{e^{-\frac{\pi}{2}}}{1+e^{-\frac{\pi}{2}}}\cos\left(\frac{2\pi}{T}x\right) \tag{5.126}$$

where T is giving by the complete elliptic integral

$$T = 4K\left(\frac{1}{\sqrt{2}}\right) = 4F(\frac{\pi}{2}|\frac{1}{2})\,.$$

Here

$$F(\theta|m^2) = \int_0^\theta \frac{d\phi}{\sqrt{1-m^2\sin^2(\phi)}}$$

is the elliptic integral. Numerically we have $\beta \equiv \frac{2\pi}{T} \simeq 0.8472$. As an approximation to the elliptic cosine we will use

$$cn\left(x, \frac{1}{\sqrt{2}}\right) \simeq \cos\left(\beta x\right). \tag{5.127}$$

With this we can rewrite (5.123) in terms of the rescaled dimensionless conformal time, $y \equiv \beta x$ (which we normalize to be $y = 0$ at the beginning of the reheating period, i.e. $t = t_R$) as

$$X_k'' + \beta^{-2}\left(\kappa^2 + \frac{g^2}{2\lambda} + 2\frac{g^2}{4\lambda}\cos\left(2y\right)\right)X_k = 0 \tag{5.128}$$

where the derivative is with respect to y.

Written on this form, the equation of motion for χ takes the form of a Mathieu equation with parameters $q = \frac{g^2}{4\beta^2\lambda}$ and $A = \beta^{-2}(\kappa^2 + 2q)$ (see e.g. [**?**]). The Matheiu equation has a solution given by

$$X(y) = A_1 \exp(\mu_k y) P_1(y,k) + A_2 \exp(-\mu_k y) P_2(y,k), \qquad (5.129)$$

that is, a growing and decaying exponential solution, with periodic behavior captured by P_1 and P_2. Here μ_k is the Mathieu characteristic exponent, and the two periodic functions have amplitude 1 and the frequency which is given by the frequency of the inflaton condensate, i.e. independent of k (see [31, 136]).

The characteristic exponent μ_k is in general a complex number. However, for some parameters it has a real part called the Floquet index. Given our model parameters

$$\frac{g^2}{\lambda} = \frac{\alpha_v^* + 3\alpha_u^*}{\alpha_v^* + \alpha_u^*} \simeq 7.4\,, \qquad (5.130)$$

we present the Floquet index in Figure 5-2.

Before discussing the parametric amplification of χ after inflation, we must determine the initial conditions for $\chi(y)$ at the end of the inflationary phase. Similar to our discussion in section 2 of the current chapter, squeezing condition satisfies here as well. Hence the spectrum of the X perturbations will be scale invariant at the beginning of the reheating period ($t = t_R$), i.e.

$$X_k(t_R) \simeq H_I k^{-3/2} P_1(y=0,k)\,, \qquad (5.131)$$

where H_I is the value of H during inflation (more precisely when the scales of interest exit the Hubble radius during inflation). In the following we will not make a difference between H_I and the Hubble expansion rate at the end of the inflationary phase, i.e. $H(t_R)$.

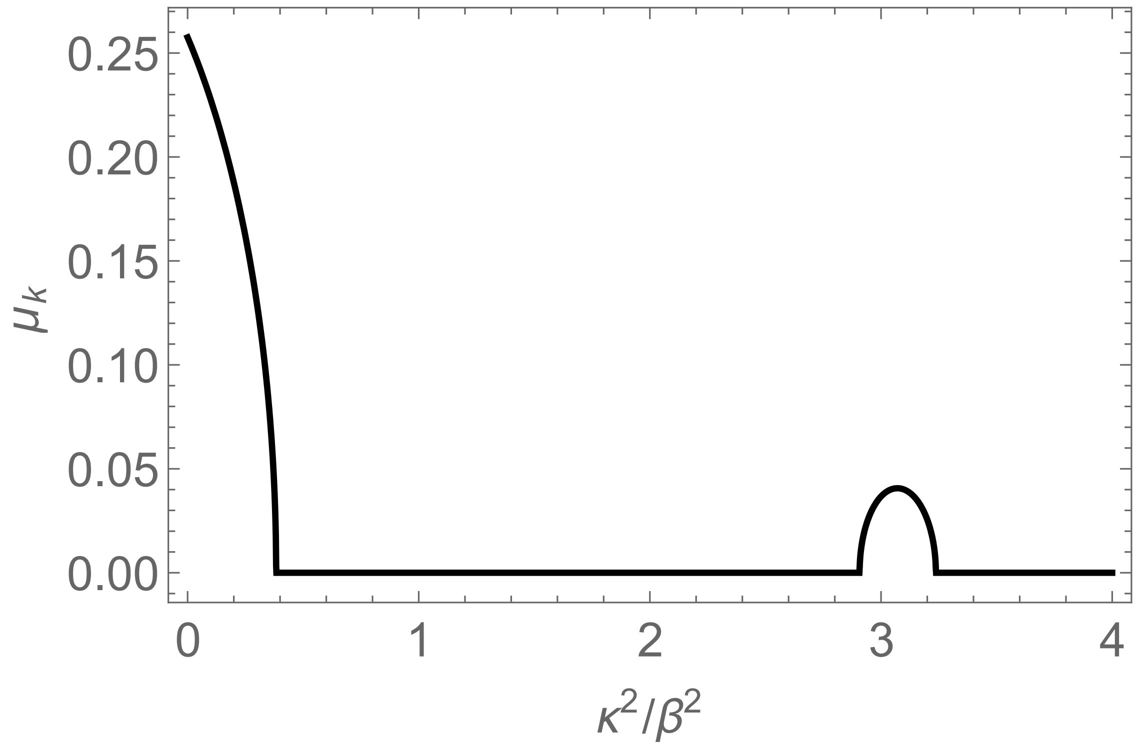

Figure 5–2: Positive real part of the Mathieu characteristic exponent for $q = \frac{g^2}{4\beta^2\lambda}$ and $A = \beta^{-2}(\kappa^2 + 2q)$.

During the preheating phase, the above value of X is exponentially amplified, yielding

$$X_k \simeq H_I k^{-3/2} \exp(\mu_k y) P_1(y, k)\,. \tag{5.132}$$

Since we have normalized the scale factor to be $a(t_R) = 1$, and since we can ignore the growth of $a(t)$ during the initial preheating period, we can identify X and χ.

The fluctuations of χ_k computed above yield the entropy fluctuations generated in the model. In order to compute the induced curvature fluctuations, we need the background value of the χ field. In the case of two field inflationary models with scalar fields having classical background, it is clear how to identify the background value of the entropy field. In the case of a model like the one we are considering, in which there is no classical zero mode of the entropy field, the situation is more complicated. Working strictly at first order in perturbation theory there is no background, and hence there will be no induced curvature fluctuations. However, this is clearly not the correct result, since if we were to argue in this way then cosmic string formation in an early universe phase transition would never lead to curvature fluctuations on super-Hubble scales, and it is well known that such curvature perturbations are formed (see e.g. [137] for reviews on cosmic strings and structure formation). A way to address this issue was recently suggested in [2, 4]: each k mode of the fluctuations lives in an *effective background* $\chi_{ij}^{eff}(k)$ which is generated by all perturbation modes with smaller wavelengths.

The effective background is given by

$$\chi_{ij}^{eff}(k) = \left(\int_0^k d^3k' |X_{k'}|^2 \right)^{1/2} . \tag{5.133}$$

Following our analysis in section 2, we can estimate the effective background for the χ mode fluctuations

$$\chi_{ij}^{eff}(k) \sim H_I N_I^{1/2} \exp(\mu y) P_1(y, k) \, , \tag{5.134}$$

where again k_{min} is the value of k which corresponds to Hubble crossing at the beginning of inflation. Note that N_I is the logarithm function as in (5.27).

5.4.4 Backreaction Effects

In our analysis of the preheating in the UV-safe theory, we have neglected any backreaction mechanism. Having done that we observed that exponential amplification sets in, and that the induced curvature fluctuations might have potential dangerous consequences for the theory. However it is crucial to consider backreaction effects since they will eventually terminate the resonance. Backreaction effects are nonlinear and are often studied numerically. However, for the questions of large-scale curvature fluctuations, an analytical analysis is preferable (see [57] for an initial study of backreaction effects during preheating in a two field inflation model).

Here we will consider two kinds of backreaction effects:

– The effect of produced χ particles on the evolution of the inflaton field .

– The contribution of amplified modes on the effective mass for χ field fluctuations.

Other backreaction effects are studied in [74, 3] but are not relevant in our case.

Effects on the evolution of the inflation field

The leading order term in χ, for small χ, in the inflaton action is

$$\frac{g^2}{2}\phi^2 \sum_{l}^{N_p} \chi_l^2 \,, \tag{5.135}$$

where the index l runs over all of the χ fields. This must remain sub-dominant to the main interaction term in the inflaton action which is

$$\frac{\lambda}{4}\phi^4 \,. \tag{5.136}$$

This leads to the condition

$$\sum_{l}^{N_p} \langle \chi_l \rangle_{eff}^2 < \frac{\lambda}{2g^2} \phi_{end}^2 \,, \tag{5.137}$$

which needs to be satisfied in order to justify neglecting backreaction effects. In the case when all preheat fields are excited equally, which we show is the case, we get

$$\langle \chi_l \rangle_{eff}^2 < \frac{\lambda}{2g^2 N_p} \phi_{end}^2 \,. \tag{5.138}$$

Naturally, the more fields we have, the less each must be excited to interfere. Note that all χ modes which are excited contribute to the left hand side of (5.138), and hence the expression is proportional to N_I.

Contribution to the effective mass of the preheat field fluctuation

Now we focus on the mass term in the equation of motion for one of the χ_k field modes. At linear order in χ, the mass term comes from the coupling of χ to the inflaton. This is the mass term which we have considered and which leads to the parametric instability, and its value is

$$g^2 \phi_{end}^2 \,. \tag{5.139}$$

However, beyond linear order there is a contribution to the mass which comes from the interactions between all χ fields. As is evident from (5.114) and (5.115) each χ field couples quadratically to a fixed χ. Assuming that all χ fields are excited equally we get a contribution to the mass which is

$$\lambda' N_p \langle \chi \rangle_{eff}^2 \tag{5.140}$$

where we have taken into account that each mode of χ which are excited contribute to the effective mass of the χ field, and hence the above expression is proportional to N_I. In the above, λ' is a coupling constant made up of the constants appearing in (5.114) and (5.115). The condition, that backreaction can be neglected, then becomes

$$\langle\chi\rangle_{eff}^2 < \frac{g^2}{2\lambda' N_p}\phi_{end}^2. \tag{5.141}$$

Since λ' is of the same order of magnitude as λ we find that (using the value of g^2/λ which our model predicts) the first backreaction condition (5.138) is slightly stronger than the second one (5.141).

Note that when the parametric resonance stops all preheating fields have been excited equally as the sole non-democratic couplings enter in the χ^4 terms relevant only for the second backreaction effect considered.

5.4.5 Induced Curvature Perturbations

Having shown that parametric resonance of the spectator scalar fields is efficient in this model, we move on to investigate the resulting amplification of the entropy fluctuations which in turn leads to a contribution to the curvature perturbation with an exponentially growing amplitude.

Fluctuations in a spectator scalar field will induce a contribution to the curvature whenever the equation of state of the spectator field mode is different from that of the adiabatic mode. The magnitude of the induced curvature fluctuation is proportional to the energy density in the spectator field. Thus, a background value of the spectator field is required in order to obtain a growing curvature mode. If we are considering curvature fluctuations on a fixed scale k, we will use the effective background χ field constructed earlier as this background. The conversion of entropy field fluctuations into curvature perturbations has been studied in many works (see e.g. [138, 139] for some classic papers). We may also use the covariant formalism of [140] as applied to study preheating in [2] and presented in this chapter. Following our analysis in section 2, the induced curvature fluctuation is given by

$$\zeta_k \simeq \frac{H}{\dot{\phi}^2}\dot{\chi}S_k\,, \tag{5.142}$$

where S_k is the entropy field perturbation which, according to our analysis, is given by

$$S_k = Hk^{-3/2}\exp(\mu y)P_1(y,k)\,, \tag{5.143}$$

where $P_1(y,k)$ is the same periodic function as in (5.132).

The expression for $\dot{\chi}$ can be found using (5.134), yielding

$$\dot{\chi} = \beta\sqrt{\lambda}\Phi H\sqrt{N_I}\exp(\mu y)\frac{\partial P_1(y,k)}{\partial y}\,, \tag{5.144}$$

where Φ is the amplitude of ϕ, and where we have neglected the derivative of the exponential factor, as it carries a factor of μ which is small compared to the order 1 frequency of P_1. Hence, we obtain

$$\zeta_k = \frac{H^3}{\dot{\phi}^2}\sqrt{\lambda}\beta\sqrt{N_I}\exp(2\mu y)k^{-3/2}\Phi P(y,k) \tag{5.145}$$

where $P(y,k) = P_1\frac{\partial P_1}{\partial y}$ is a periodic function. The most important feature of this result is exponential growth of curvature perturbation which is induced by the entropy perturbation (see also the Appendix of Ref [2] for a more detailed derivation of this result).

Using this result we can evaluate the power spectrum of induced curvature perturbations, yielding

$$P_k = \frac{k^3}{2\pi^2}|\zeta_k|^2 \simeq \frac{H^6\Phi^2}{\dot{\phi}^4}\exp(4\mu y)\frac{\beta^2\lambda}{4\pi^2}N_I\,, \tag{5.146}$$

where we used $|P|^2 \sim \frac{1}{2}$. To estimate this expression we will use the fact that at the end of inflation kinetic energy is of the same order of magnitude as the potential energy. We also use $\Phi = \phi_{end}$ and introduce the number σ via

$$\phi_{end} \equiv \sigma M_P. \tag{5.147}$$

We can use the result of the previous section on backreaction to yield an estimate for the value of $\exp(4\mu y)$ when the resonance stops. As discussed in the previous section one can show the first backreaction effect shuts off the resonance before the second. Therefore using (5.138) for the time when the backreaction becomes important, we get

$$\exp(4\mu y) \sim \left(\frac{2g^2}{\lambda}\right)^{-2}\frac{M_P^4}{H^4}N_I^{-2}N_p^{-2}\sigma^4 \tag{5.148}$$

Inserting (5.148) and (5.147) into (5.146) and taking into account that the potential energy at the end of inflation is

$$V = \frac{\lambda}{4}\sigma^4 M_P^4 \tag{5.149}$$

we obtain our final result

$$P_k \sim \frac{\beta^2\sigma^2}{N_p^2 N_I}\left(\frac{g^2}{\lambda}\right)^{-2} \tag{5.150}$$

for the power spectrum of the induced curvature fluctuations.

For our model with $\frac{g^2}{\lambda} \simeq 7.4$ and $\sigma^2 \simeq 12$ and $\beta \simeq 1$ we get

$$P_k \sim \frac{1}{N_p^2 N_I}. \tag{5.151}$$

For this not to exceed the observed value with amplitude of order 10^{-10} we need a large number of flavors and/or a large number of e-foldings of inflation.

To conclude this section, we note that our first result is that the parametric instability indeed arises in the UV-safe model we introduced, hence the energy transfer from the inflaton condensate to fluctuating fields is rapid. Our second result concerns the demand that the curvature fluctuations induced by the parametrically amplified entropy modes do not exceed the upper observational bounds. We have seen that this puts a lower bound on the product $N_p^2 N_I$, where N_p is the number of scalar fields which the model of [128, 129] contains, and N_I is the total number of e-foldings of the inflationary phase. The reason that the power spectrum of the induced curvature fluctuations decreases as N_p^2 is that backreaction effects turn off the parametric instability earlier as N_p increases. It is a linear effect in N_p on the fluctuation modes, and hence a quadratic effect in the power spectrum. The reason that the bound depends on N_I is that the energy density in the effective entropy field background (which determines the strength of the conversion of entropy to adiabatic mode) is proportional to $\sqrt{N_I}$, and that as N_I increases the backreaction is shut off earlier due to more modes being super-Hubble. The combination of these effects gives the net scaling of the power spectrum as N_I^{-1}.

5.5 Conclusion

In this chapter gravitational effects are considered during preheating. We observed that the enhancement of cosmological perturbations indeed happens during preheating. Depending on the model and the conditions during preheating, this enhancement can occur even on large scales which are of cosmological interest.

In the first model we considered here, we applied the covariant formalism to study the evolution of entropy perturbations in massless preheating. We observed that the amplifications of the induced curvature perturbations is a serious problem for the model as it exceeds the observational constraints by many orders of magnitude.

In the second model, we studied preheating into the gauge field fluctuations in the axion monodromy setup with a derivative coupling to the gauge field via a Pontryagin term. Although preheating is efficient in the model to produce gauge field fluctuations, we observed that the spectrum of induced curvature perturbation is very blue. This means that the enhancement on large scales is not significant and preheating does not have an observational impact in this regard in the model. Since the spectrum is blue, overproduction of primordial black holes might a problem for the model. However our analysis showed that the production of primordial black holes in the model satisfies the current observational constraints.

Then we studied preheating in a UV-safe field theory and observed that preheating happening in the model and enhancement of the cosmological perturbations on large scales may happen depending on the parameters of the model.

Chapter 6

CONCLUSION

Inflation leaves behind a non-thermal state in the universe. The average temperature of the universe is very low and space is almost empty of matter. The reheating mechanism is the connecting phase between inflation and Big Bang cosmology. It explains high temperatures at Big Bang Nucleosynthesis and is considered as the origin of the Standard Matter particles. Therefore, reheating is an integral part of any inflationary scenario.

In this thesis, we reviewed both inflation as the standard paradigm of the early universe and also the theory of cosmological perturbations. Then we gave a detailed review of the reheating mechanism. We saw that to connect the inflationary phase to the hot Big Bang Cosmology, during reheating the energy which is stored in the inflaton field should be transferred to other degrees of freedom and particle production happens. We studied the linear stage of reheating in which explosive particle production occurs due to parametric resonance. We saw that the linear stage of reheating will terminate when backreaction of the produced particles on the background dynamics becomes important.

In chapter 4, we studied reheating in a class of G-inflation models. Since there is no phase of oscillation of the inflaton field after inflation, there no possibility of parametric resonance. We studied the onset of matter particle production from direct couplings using a Born approximation approach. We derived the conditions under which direct particle production dominates over gravitational particle production.

In chapter 5, we included gravitational effects in our analysis of reheating. We studied particle production in a massless preheating model, in axion monodromy inflation and in an asymptotically safe quantum field theory. We observed that in all three models, parametric resonance is efficient and explosive particle production occurs. Then we calculated the curvature perturbations which are induced by growing modes of entropy perturbations. We confirmed that in the massless preheating model and in the asymptotic safe QFT model this enhancement happens even on large cosmological scales. Due to observations, this enhancement put severe constraints on the models. In the case of axion monodromy inflation with a derivative coupling to a $U(1)$ gauge field, we observed that the induced curvature perturbations on large scales have less power in comparison to small scales, and hence the model is safe regarding the amplification of entropy perturbations. We also studied formation of primordial black holes and showed that the observational constraints are satisfied in the model.

Appendix: Growth of Induced Curvature Fluctuations

In this appendix we derive the growth of the induced curvature fluctuations from first principles, making use of the covariant formalism.

In the long wavelength limit, the growth of ζ is given by [19, 21]

$$\dot{\zeta}_a \equiv \mathcal{L}_u \zeta_a = \frac{2}{3} \frac{\Theta}{\dot{\sigma}^2} V_{,s} S_a \,. \tag{6.1}$$

Note that the induced growing mode of ζ is quadratic in the magnitude of fluctuations.

The equation (6.1) can be understood in the following way: For the Lie derivative of curvature covector we have

$$\mathcal{L}_u \zeta_a = u^c \nabla_c \zeta_a + \zeta_c \nabla_a u^c \,. \tag{6.2}$$

Considering $u^c = \{1/a, 0, 0, 0\}$ we get

$$\mathcal{L}_u \zeta_a = \frac{1}{a} \partial_t \zeta_a \,. \tag{6.3}$$

Then one can find the relation between the curvature covector and the conventional coordinate based curvature perturbation as follows [23] (up to first order)

$$\zeta_i = \partial_i \zeta \,. \tag{6.4}$$

Note that the quantity ζ we introduced here is the same as the conventional curvature perturbation in the large scale limit which we are interested in (see the discussion in [23]). The same relation holds for S_a, and considering $\partial_i \to ik$ in Fourier space leads to

$$\frac{1}{a}\partial_t\zeta \;=\; \frac{\sqrt{2}H}{\dot{\sigma}^2}V_{,s}H\,k^{-3/2}\exp(\mu x)P_1(x)\,. \tag{6.5}$$

Up to first order in perturbation theory

$$\dot{\sigma}^2 \;\simeq\; \dot{\phi}^2 \tag{6.6}$$

and

$$V_{,s} \;=\; -\lambda\phi^3\theta + g^2\phi^2\chi\,. \tag{6.7}$$

Using dimensionless conformal time and again applying the conformal transformation $\varphi = a\phi$ and the same for the preheat field χ, we obtain for the angle θ

$$\theta \;=\; \frac{\frac{\chi'}{a} - \frac{a'}{a^2}\chi}{\frac{\varphi'}{a} - \frac{a'}{a^2}\varphi}\,. \tag{6.8}$$

Therefore using equation (6.1) gives

$$\zeta' = \frac{2\sqrt{\pi}}{x}k^{-3/2}cos^2(x)e^{2\mu x}F(x)\,, \tag{6.9}$$

where

$$F(x) \equiv \lambda\frac{\mu x\sin^2(x)\cos(x) + x\cos^2(x)\sin(x) - \sin^2(x)\cos(x)}{(x\sin(x)+\cos(x))^3} + g^2\frac{\sin^2(x)}{(x\sin(x)+\cos(x))^2}\,, \tag{6.10}$$

which clearly shows that

$$\zeta \;\propto\; \exp(2\mu x)\,. \tag{6.11}$$

Hence, we conclude that parametric resonance of entropy perturbations induces an exponentially growing curvature mode.

Thus, we see that taking into account the squeezing of the modes and the solution (5.22), curvature perturbation induced by the entropy modes after inflation, using (6.5), is given by:

$$\zeta_k^{ent} \sim \sqrt{2\pi\lambda}\,\frac{H^3}{\dot{\varphi}^2}\varphi\, k^{-3/2} e^{2\mu x}\,, \tag{6.12}$$

where we made the approximation of $V_{,s} \simeq \lambda\varphi^2\chi_{eff}$.

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