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ABSTRACT

A POTENTIAL FOR π-N SCATTERING

A Born approximation potential is derived from the one-boson-exchange model with the ρ-meson and the f°-meson exchanges, together with the direct and exchange pole terms of the π-N scattering. A 'cut off' mass is introduced to regularize the singular part of the potential. The potential is inserted into the Klein-Gordon equation and the equation is then solved for the phase shifts. The coupling constants $G^2/4\pi$, $\frac{g_{\rho\pi\pi}g_{\rho NN}}{4\pi}$, $\frac{g_{f\pi\pi}g_{fNN}}{4\pi}$ and the 'cut off' mass m_c are treated as adjustable parameters to fit the CERN[5] phase shifts. The best solution obtained in the 0-700 MEV energy range includes three resonances; P_{33}(194 MEV), P_{11}(600 MEV) and D_{13}(616 MEV). The solutions are consistent with the CERN phase-shift analysis especially for large positive phase shifts such as P_{11}, P_{33}, D_{15} and D_{13}. The P_{31}, P_{13} and D_{33} cannot be accurately predicted in our model.

A POTENTIAL FOR π-N SCATTERING

A Thesis

by

Lydia Sing-Chai Chan

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ABSTRACT

A Born approximation potential is derived from the one-boson-exchange model with the ρ-meson and the f°-meson exchanges, together with the direct and exchange pole terms of the π-N scattering. A 'cut off' mass is introduced to regularize the singular part of the potential. The potential is inserted into the Klein-Gordon equation and the equation is then solved for the phase shifts. The coupling constants $G^2/4\pi$, $\frac{g_{\rho\pi\pi}g_{\rho NN}}{4\pi}$, $\frac{g_{f\pi\pi}g_{fNN}}{4\pi}$ and the 'cut off' mass m_c are treated as adjustable parameters to fit the CERN[5] phase shifts. The best solution obtained in the 0-700 MEV energy range includes three resonances; P_{33}(194 MEV), P_{11} (600 MEV) and D_{13}(616 MEV). The solutions are consistent with the CERN phase-shift analysis especially for large positive phase shifts such as P_{11}, P_{33}, D_{15} and D_{13}. The P_{31}, P_{13} and D_{33} cannot be accurately predicted in our model.

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CHAPTER I

Review

1.1 Introduction

The study of the low-energy pion-nucleon scattering theory is one of the most basic subjects in the history of strong interaction physics. A diversity of approaches has been exploited towards a solution of this profound problem, in order to have a complete and consistent picture of the π-N interactions.

The first contribution to the low-energy π-N scattering theory was by Chew and Low[1], using the low-energy effective range approximation. On the basis of the cut off Yukawa Theory without nuclear recoil, it is found that the Chew-Low model provides a qualitative picture of the P-wave π-N scattering, particularly a (3,3) resonance consistent with the experimental data. However, Chew-Low's model fails to predict the correct information for the S-wave. By means of current commutator algebra and the hypothesis of PCAC, the S and P wave scattering lengths are also studied[2]. In the field theoretic approach, a Lagrangian model is used to understand the low-energy π-N interaction. In the framework of chiral symmetry[3], the scattering lengths can be calculated. Recently, a Padé approximation approach has been presented by Remiddi et al.[4], to parametrize the low-energy elastic π-N scattering. The expansion of the S-matrix in terms of the π-N coupling constants up to the fourth order is used to fit the CERN[5] phase shifts. However, the fit to the $T = \frac{1}{2}$ state is poor, with opposite signs in the S, P and D-wave phase shifts when compared with the corresponding ones from CERN.

The one-particle-exchange model has been considered as one of the most successful models in the past decade in the study of elementary particle physics. By taking the well-known existing elementary particles as exchange particles in a certain strong interaction process, it is possible, using simple mathematical tools, to calculate the physical observables to obtain good agreement with the experimental data. A more elaborate review of this model will be given in Chapter II, Section 2.1.

In connection with the study of the π-N scattering problem, we consider in the present work the so-called one-boson-exchange (OBE) model, to identify and estimate the contributions to the π-N forces. An attempt is made to derive an OBE Born potential due to the exchanges of nucleon, ρ , σ and f°. A 'cut off' method is introduced in order to regularize the singular potential. The details are given in Chapter II. The Klein-Gordon equation is used to calculate the π-N phase shifts in each channel. A discussion on the Klein-Gordon equation, including the justification for applying it to our problem, together with the basis of the assumptions inherent in our approach, is given in Chapter III. If the one-boson-exchange is the basic mechanism, the masses of the exchange particles are not adjustable parameters. However, the coupling constants of the exchange particles with the nucleon or pion, can be treated to a limited extent, as adjustable parameters. Proper adjustments in the parameters are made until a reasonable fit to the CERN phase shifts is secured. One of the successes achieved in our approach is the ability to reproduce all the characteristic features of the π-N resonances up to the energy of 700 MEV. We have thus far considered only the elastic scattering; an extension, which includes the effects of the absorption is more realistic and most desirable.

1.2 π-N Phase-Shift Analysis

Phase shift analysis has played a considerably important role in the π-N scattering problem in the past half decade, beginning from the evidence of Roper's[6] P_{11} resonance. The analysis of the phase shifts provides a meeting ground between the explanation of the experimental data and the various theoretical approaches. Moreover, it is also a powerful tool to detect resonant phenomena. Most resonances are not evident from the total cross-section, or from a Dalitz[7] plot.

Conventionally, there are two ways to deal with the problem of phase shift analysis. One is an energy-dependent approach and the other an energy-independent approach. In the former, one parametrizes the partial wave amplitudes as functions of energy in order to fit the experimental data. This approach is practical provided only a few partial waves are taken into account and at the same time the energy range is sufficiently restricted. This approach has been applied by three groups -- Yale[8], Livermore[9] and Chilton[10]. The disadvantage is that good statistical fits are not practically attainable; because of the inconsistencies among experimental data at different energies. Nevertheless, energy-dependent analysis gives a smooth behaviour of the solution at various energies.

For an energy-independent analysis, an extensive search at each energy for different solutions is performed. The disadvantage of this approach is that it is unable to produce smooth and continuous solutions at various energies. To impose continuity at different energies is quite a difficult task. However, sophisticated techniques have been attempted by the groups in Hawaii[11], Saclay[12] and CERN[13] to deal particularly with this problem in the energy-independent analyses.

1.3 Nucleon Resonances

About two years ago the pion-nucleon phase shift analysis of the groups at Berkeley[14], CERN and Saclay, was extended to the GEV region, with 19 or more resonances found below 2.2 GEV (C.M. total energy) in the π-N system. Recently, serious arguments have been raised to question the existence of a number of these previously unsuspected resonances. The traditional procedure to detect a resonance state is by means of the Argand diagram of the function $2qf_\ell$, where

$$f_\ell = \frac{1}{2iq} [\eta_\ell e^{2i\delta_\ell} - 1]$$

is the partial wave amplitude, q the C.M. momentum, η_ℓ the absorption parameters and δ_ℓ the phase shifts. The existence of a resonance will always give rise to a counter-clockwise circle in the Argand diagram. However, the inverse is not necessarily true. (A counter-clockwise circle in the Argand diagram does not necessarily imply that there is a resonance.) Much attention has been drawn to this criterion after Schmidt[15] proclaimed that the partial-wave projections of Regge-pole amplitudes freely exhibit resonance-type circles. A diversity of opinion has been expressed on this topic[16]. However, no concrete conclusion has been drawn so far at this point. Accordingly, Donnachie[17] is convinced that all the structure observed in the π-N scattering should be associated with resonances. To a greater or lesser extent, the resonances below 1.6 GEV (C.M. total energy) are quite reliable and have been confirmed by the five most recent phase-shift analysis groups[12], [13], [14], [18], [19]. (see Table 1).

Table 1

Conjectured pion-nucleon resonance assignments below 2.2 BEV* with the status of the corresponding structure observed in the five most recent phase-shift analyses.

Possible Resonances	Berkeley[14]	CERN I†[5]	Saclay[12]	Glasgow[18]	CERN II[19]
$P_{33}(1236)$		No argument about this one			
$S_{31}(1640)$	Definite	Definite	Definite	Definite	Definite
$D_{33}(1690)$	Possible	Possible	Ambiguous	Definite	Definite
$P_{33}(1690)$	Probable	Probable	Ambiguous	Possible	Definite
$F_{35}(1910)$	Probable	Probable	Ambiguous	Definite	Definite
$P_{31}(1930)$	Probable	Probable	Ambiguous	Definite	Definite
$F_{37}(1950)$	Definite	Definite	Definite	Definite	Definite
$D_{35}(1950)$	Doubtful	Doubtful	Ambiguous	No	Possible
$P_{11}(1470)$	Definite	Definite	Definite	Definite	-
$D_{13}(1520)$	Definite	Definite	Definite	Definite	-
$S_{11}(1550)$	Definite	Definite	Definite	Definite	-
$D_{15}(1680)$	Definite	Definite	Definite	Definite	-
$F_{15}(1690)$	Definite	Definite	Definite	Definite	-
$S_{11}(1710)$	Definite	Definite	Definite	Definite	-
$D_{13}(\sim 1730)$	No	Use imagination	No	No	-
$P_{11}(1750)$	No	Possible	No	Definite	-
$P_{13}(1860)$	No	Possible	No	Definite	-
$F_{17}(1980)$	No	Doubtful	No	Transferred to G_{17}	-
$D_{13}(\sim 2030)$	No	Probable	No	No	-
$G_{17}(2190)$	Ambiguous	Definite	-	-	-

* The energy used in Table 1 refers to the total C.M. energy.

† In order to avoid confusion, CERN I refers to ref. (5), and CERN II to ref. (19).

1.4 Generalized Interference Model

Very recent work intimately related to the resonance phenomena in π-N scattering is presented through a newly developed 'generalized interference model', by Donnachie and Kirsopp[20].

A pure Breit Wigner type of expression is proposed to fit the difference $D^{(T)}_{\ell\pm}$, between the partial wave amplitudes and the respective Regge amplitudes. Here, the difference $D^{(T)}_{\ell\pm}$ is defined by

$$D^{(T)}_{\ell\pm} = 2q\ (f^{(T)}_{\ell\pm} - \beta^{(T)}_{\ell\pm}) \tag{1-1}$$

and is parametrized by

$$D^{(T)}_{\ell\pm} = \sum_n \frac{q\ e^{i\phi^{(T)}_{n,\ell\pm}}\ \Gamma^{(T)E\ \ell}_{n,\ell\pm}}{(S^{(T)\ Res.}_{n,\ell\pm} - S) - \frac{i}{2}\, q\ \Gamma^{(T)Tot}_{n,\ell\pm}} \tag{1-2}$$

where $f^{(T)}_{\ell\pm}$ are the usual partial wave amplitudes, and $\beta^{(T)}_{\ell\pm}$ the projected Regge amplitudes; S and q are the total energy squared and momentum in the C.M. system. The summation in Equ. (1-2) indicates the number of resonances in a given partial wave. Here $S^{(T)\ Res.}_{n,\ell\pm}$ is the square of the mass of the n-th resonance, $\Gamma^{(T)E\ell}_{n,\ell\pm}$ and $\Gamma^{(T)Tot}_{n,\ell\pm}$ are the elastic and total widths of the n-th resonance respectively and $\phi^{(T)}_{n,\ell\pm}$ is the arbitrary phase.

Remarkable success has been shown by this model in describing the partial wave amplitudes and there is a consistency with the

Table II*

	Conventional Interpretation			Interference Model Interpretation		
	Mass	Width	Elasticity	Mass	Width	Elasticity
S_{31}	1620	140	0.25	1605	230	0.30
P_{31}	1905	300	0.25	Interpretation ambiguous		
P_{33}	1237	122.5	1.00	1241	120	1.00
	1690	280	0.10	1940	250	0.35
				(3 resonances improve fit)		
D_{33}	1670	225	0.13	1850	350	0.22
D_{35}	Interpretation ambiguous			1715	375	0.20
F_{35}	1880	250	0.18	Interpretation ambiguous		
F_{37}	1940	210	0.42	1870	250	0.32
S_{11}	1525	80	0.34	1440	240	0.76
	1715	280	0.66	1685	220	0.54
P_{11}	1460	260	0.57	1420	140	0.32
	1783	405	0.34	1815	175	0.16
P_{13}	1855	335	0.27	1665	370	0.38
D_{13}	1515	115	0.52	1520	105	0.47
	1730 ?	?	?	1980	200	0.32
	2030 ?	?	?			
D_{15}	1675	145	0.43	1650	135	0.29
F_{15}	1690	125	0.61	1705	120	0.67
F_{17}	-	-	-	-	-	-

* The masses in Table II refer to total energy in the C.M. system.

existence of all the conjectured pion-nucleon resonances up to 1.5 GEV* (C.M. total energy of 2.0 GEV), with the exception that the P_{31} resonance is absent and an extra D_{35} resonance is present. (See Table II).

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* From now onwards, energies in MEV mean pion lab. kinetic energy except when otherwise specified.

CHAPTER II

One-Particle-Exchange Model

2.1 The Development of the One-Particle-Exchange Model

The one-particle-exchange (OPE) model was first proposed by Hoshizaki et al.[21], in an attempt to explain nuclear forces in nucleon-nucleon (N-N) scattering. Since then similar models have been presented by a large number of authors on somewhat different grounds. There are two conventional approaches one can take with the OPE model. One is the dispersion theoretical approach and the other is using the one-particle-exchange potential (OPEP) to solve the non-relativistic Schrödinger equation. In the former, the great advantage is that the whole treatment can be made fully relativistic.

1. Partial Wave Dispersion Treatment. The dispersion treatment of the OPE model has been dominated by the analysis of the partial-wave dispersion relations. Extensive work on this approach in N-N scattering has been done by Scotti and Wong[22], Kantor[23], MacGregor[24] and Moravcsik[25]. Much of the early work on the partial wave dispersion relation in π-N scattering has been done by Hamilton et al.[26]

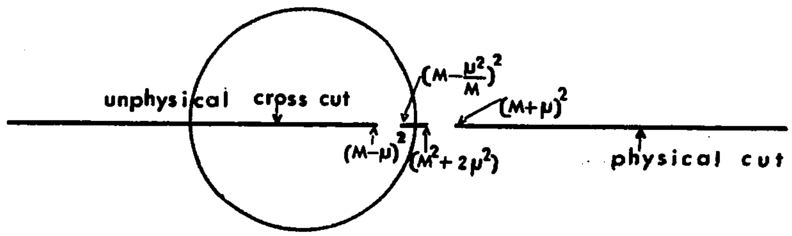

Fig. 1

We proceed to give a brief account of this matter. By definition, the π-N partial wave amplitudes with orbital angular momentum ℓ have the form

$$f_{\ell\pm}(S) = [\exp(2i\delta_{\ell\pm}) - 1]/2i\,q \quad . \tag{2-1}$$

The singularities of $f_{\ell\pm}(S)$ are shown in Fig. 1. The dispersion relation for $f_{\ell\pm}(S)$ is written as

$$f_{\ell\pm}(S) = \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} dS' \frac{I_m f_{\ell\pm}(S')}{S' - S} + \frac{1}{2\pi i}\int_{(\text{unphysical cuts})} dS' \frac{\Delta f_{\ell\pm}(S')}{S' - S} \tag{2-2}$$

where $\Delta f_{\ell\pm}(S)$ is the discontinuity in $f_{\ell\pm}(S)$ across the cut at S'. On the R.H.S. of Equ. (2-2), the first integral along the physical cut gives the rescattering, while the second integral with various unphysical cuts, can be regarded as the forces producing the π-N scattering. The short Born cut, $(M - \frac{\mu^2}{M})^2 \leqslant S \leqslant M^2 + 2\mu^2$, due to the cross Born term of the N-exchange, corresponds to the long range force. The contribution from the cut $0 \leqslant S \leqslant (M - \mu)^2$ arises from the N*-exchange and is a comparatively short range force. The circle

$|S| = M^2 - \mu^2$ arises from the channel $\pi + \pi \rightarrow N + \bar{N}$. The left half of the circle gives the short range part of the π-N interaction. However, the right half of the circle, and in particular the region nearest the physical threshold $S = (M + \mu)^2$, gives a comparatively long range interaction due to low-energy S-wave π-π contribution. The unphysical cut $-\infty \leqslant S \leqslant 0$ is due to very short range forces. Since little is known about this region, Donnachie and Hamilton[27] introduce a peripheral method, in which the very short range part of the interaction is almost suppressed. They define

$$F_{\ell\pm}(S) = \frac{f_{\ell\pm}(S)}{q^{2\ell}} \quad . \tag{2-3}$$

Instead of Equ. (2-2) the dispersion relation of $F_{\ell\pm}(S)$ is now written as

$$F_{\ell\pm}(S) = \frac{1}{\pi} \int_{(M+\mu)^2}^{\infty} dS' \frac{I_m F_{\ell\pm}(S')}{S' - S} + \frac{1}{2\pi i} \int_{(\text{unphysical cuts})} dS' \frac{\Delta F_{\ell\pm}(S')}{S' - S} \tag{2-4}$$

where

$$\Delta F_{\ell\pm}(S) = \frac{\Delta f_{\ell\pm}(S)}{q^{2\ell}(S)} \quad . \tag{2-5}$$

The factor $q^{-2\ell}$ suppresses the contribution from the unphysical cuts due to the unknown very short range force (i.e. from the left hand cut $-\infty \leqslant S \leqslant 0$, and the left half circle), at the same time it also ensures the proper threshold behaviour of $f_{\ell\pm}(s)$. However, at very high energy the dispersion relation of Equ.(2-4) breaks down. This is due to the presence of inelasticity at high energy and in addition the very short range interaction becomes important and can no longer be ignored. Further work to extend the peripheral method in the high energy region has also been done by Donnachie and Hamilton[28]. By using the unitary sum rule to estimate the short range part of the π-N interaction, an improved peripheral method can be achieved.

2. The One-Particle-Exchange Potential. We review the work based on the OPEP to solve the non-relativistic Schrödinger equation, which is similar in form to the Klein-Gordon equation used by us. The similarity is illustrated in Chapter III, Section 3.4 . In general, there are two methods of utilizing the OPEP. A direct method is to calculate the phase shifts by solving the Schrödinger equation with the OPEP. The other is an indirect method in which phenomenological potentials are analyzed in terms of OPEP.

Very extensive work in N-N scattering based on the direct OPEP has been performed by Bryan, Dismukes and Ramsay[29], Bryan and Scott[30] and Arndt, Bryan and MacGregor[31].

The first systematic analysis on the indirect OPEP in N-N scattering has been done by Hoshizaki et al.[21]. The difference between the phenomenological nuclear potentials and the pion theoretical potential in the range greater than μ^{-1} is examined in terms of OPEP. The hard core interaction in the inner region is discarded as it is too complicated to handle using this model.

Babikov[32] analyzed the phenomenological Hamada-Johnson[33] potential in terms of OPEP in N-N scattering due to ρ , ω and an $I = 0$ scalar meson with mass 2.5μ. An attempt to explain the hard core by a repulsive force due to the ω-meson was made.

Although the OPEP approach has been long used in N-N scattering,it has not been very popular in the π-N scattering problem. On the basis of the OPE model, Kikugawa[34] gives an analysis of the low energy π-N scattering, due to the exchange of N, N*, N**, a scalar meson and a vector meson. The isospin non-flip amplitude $\alpha^{(+)}_{\ell\pm}$ and the isospin flip amplitude $\alpha^{(-)}_{\ell\pm}$ are expressed in terms of the tangents of the phase shift $\delta^{2T}_{\ell}(T = \frac{1}{2}, \frac{3}{2})$, with the following forms

$$\alpha^{(+)}_{\ell\pm} = \frac{1}{3}[\tan\delta^{(1)}_{\ell\pm} + 2\tan\delta^{(3)}_{\ell\pm}]$$

$$\alpha^{(-)}_{\ell\pm} = \frac{1}{3}[\tan\delta^{(1)}_{\ell\pm} - \tan\delta^{(3)}_{\ell\pm}] ,$$

in order to analyze the experimental data. On the other hand, these amplitudes, due to the OPE model, are determined by matrix elements, corresponding to the lowest order Feynman diagrams. They depend on the C.M. momentum and the coupling constants. The resonances require that $\delta^{2T}_{\ell\pm} = \frac{\pi}{2}$ and hence that the $\alpha^{(\pm)}_{\ell\pm}$ be infinitely large. In the absence of N* and N** exchanges, this implies the coupling constants are infinite. Thus it was necessary to take into account the exchanges of N* and N** and identify the contributions of the first and second resonances with the resonance regions of N* and N** respectively. The Feynman amplitude for the resonance type (i.e. the exchanges of N* and N**) can be expressed in terms of the pole type contribution and the contact type contribution. The pole type contribution will be nearly equivalent to the pole approximation in the dispersion relations. The coupling constants G_{33}^2 and G_{13}^2 of the NN*π and NN** π vertices respectively are determined by fitting the experimental $P(\frac{3}{2})$ and $D(\frac{3}{2})$ phase shifts. For the isospin non-flip $S(\frac{1}{2})$ and $P(\frac{1}{2})$ amplitudes, in addition to the exchanges of N, N* and N** a scalar meson is considered. The scalar meson coupling constant is determined by minimizing the least square fit of the experimental isospin non-flip amplitudes. However, in the case of the isospin flip amplitudes for $S(\frac{1}{2})$ and $P(\frac{1}{2})$, instead of a scalar meson, a vector meson is included. By fitting the isospin flip $S(\frac{1}{2})$ and $P(\frac{1}{2})$ experimental data, Kikugawa obtains the two vector coupling constants.

$$G_V = -0.455$$

$$F_V = -8.23$$

Thus the ratio

$$\frac{F_V}{G_V} = \left(\frac{f}{g}\right)_V = 18$$

has a value ten times larger than the accepted value obtained from the electro-magnetic form factor of the nucleon. Except for the isospin non-flip $S(\frac{1}{2})$ amplitude, the fits for both the isospin flip and non-flip $P(\frac{1}{2})$ amplitudes and the isospin flip $S(\frac{1}{2})$, in the energy region greater than 1.5 μ, are rather poor. Moreover, Hiroshige et al[35], extend the work of Kikugawa, by taking into account the effects of the f°-meson exchange, in addition to the exchanges of N, N*, N**, a scalar meson and a vector meson. They conclude that if f°-meson be taken into account, another scalar meson will be necessary to improve both isospin non-flip $S(\frac{1}{2})$ and $P(\frac{1}{2})$ states. Our approach is quite different from the. We do not require the N* and N** exchanges, since we can reproduce the resonances by considering the exchanges of N, σ, ρ and f° mesons in our OPE potential approach. Further, instead of determining each coupling constant under a particular condition, as Kikugawa did, we try to determine the coupling constants once and for all by an overall fit to the S-, P- and D- wave phase shifts. It is hoped to obtain a set of coupling constants with values as close to the accepted values as possible. Recently, Dutta-Roy et al[36] have considered the N, N*, ρ and ε exchanges in low energy π-N scattering. An interaction Lagrangian model is used to calculate the scattering lengths for the S- and P- waves. The results are in good agreement with the experimental data.

2.2 Assumptions for the OBE Model

Before we embark on an elaborate analysis of the OBE model employed in our approach for the π-N scattering, we would like to outline the basis for the OBE model.

Assumptions for the OBE model:

(i) The dynamic behaviour of the interaction is determined from the matrix element corresponding to the lowest order Feynman diagrams with no closed loop in them.

(ii) The boson lines in the Feynman diagrams designate the existing mesons with known quantum numbers.

(iii) The higher order contributions are omitted.

(iv) The exchange of only even G parity mesons are considered in the OBE model.

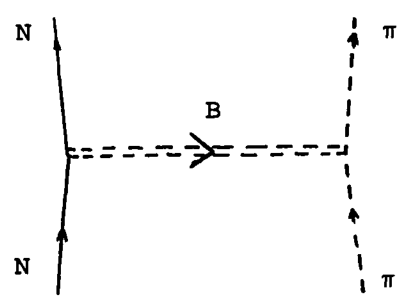

Fig. 2 Representation of π-N scattering assuming only OBE contributions

The relevant diagram is shown in Fig. 2, where B denotes the exchange boson, which can have spin 0, 1 and 2. For B the isobars with ($I = 0$, $J^P = \text{even}^+$) or ($I = 1$, $J^P = \text{odd}^-$) are allowed. We have assumed several invariant principles usually taken for strong interactions, such as time reversal, space inversion and charge conjugation, and the coupling constants can take only the real values. For convenience, we use natural units $\hbar = c = \mu = 1$.

Using the standard notation for elastic π-N scattering[37], the S-matrix is defined as

$$S_{fi} = \delta_{fi} + i(2\pi)^4 \delta^4 (P_1 + q_1 - P_2 - q_2) \frac{M}{\sqrt{4E_1 E_2 \omega_1 \omega_2}} T_{fi} \qquad (2\text{-}6)$$

where q_1, q_2 and P_1, P_2 are the initial and final four-momenta of the pions and the nucleons. Thus

$$E_i = (M^2 + \vec{P}_i^2)^{\frac{1}{2}}, \qquad \omega_i = (\mu^2 + \vec{q}_i^2)^{\frac{1}{2}} \qquad (i = 1, 2)$$

where M and μ are the nucleon and pion masses.

The invariant transition scattering amplitude T is of the form

$$T = \overline{U}(P_2)[A(s, t, u) + \gamma^\mu Q_\mu B(s, t, u)] U(P_1) \qquad (2\text{-}7)$$

where $Q_\mu = \frac{1}{2}(q_1 + q_2)_\mu$ and γ^μ are the Dirac matrices.* A and B are invariant functions of two of the three Mandelstan variables

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* We adopt the notation developed in Ref.(38) except where specified otherwise.

$s = (P_1 + P_2)^2$, $t = (q_1 - q_2)^2$, $u = (P_1 - P_2)^2$. $U(P_1)$ and $U(P_2)$ are the Dirac spinors for the initial and final nucleon states.

The invariant T matrix in Equ. (2-6) is computed by drawing the Feynman diagram for the process in question. The transition amplitude T is related to the 2 x 2 Pauli scattering amplitude F by

$$F = \frac{M}{4\pi W} T \qquad (2\text{-}8)$$

where W is the total energy in the C.M. system.

In the C.M. frame we have

$$\vec{P}_i = -\vec{q}_i; \quad |\vec{q}_i| = |\vec{P}_i| = q \quad \text{C.M. Momentum}$$

and also

$$E_i = E, \quad \omega_i = \omega \quad .$$

To evaluate the matrix elements of T in Equ. (2-7) we must use the explicit representation of the two-component spinors for the nucleon given by[38]

$$U^{(r)}(P) = \sqrt{\frac{E + M}{2M}} \begin{bmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{P}}{E+M} \end{bmatrix} U^{(r)}(o) \qquad (r = 1, 2), \qquad (2\text{-}9)$$

in order to calculate

$$\overline{U}(P_2)\; U(P_1) = \frac{1}{2M}\left[(E + M)-(E - M)(\hat{q}_2\cdot\hat{q}_1 + i\, \vec{\sigma}\cdot\hat{q}_2 x\hat{q}_1)\right] \qquad (2\text{-}10)$$

and

$$\overline{U}(P_2)\gamma^{\mu}Q_{\mu}U(P_1) = \frac{1}{2M}\left[(E+M)(W-M)+(E-M)(W+M)(\hat{q}_2\cdot\hat{q}_1+i\vec{\sigma}\cdot\hat{q}_2 x\hat{q}_1)\right] \qquad (2\text{-}11)$$

where

$$\hat{q}_i = \frac{\vec{q}_i}{|\vec{q}_i\;|} \qquad (i = 1,\ 2).$$

In the C.M. frame we also get

$$\overline{U}(P_2)\ \not{q}_2\ U(P_1) = \overline{U}(P_2)\ \not{q}_1\ U(P_1).$$

2.3 Elastic Force in Pion-Nucleon Scattering

The purpose of this section is to study and investigate the significance of the elastic force in π-N scattering. We consider the direct pole term and the nucleon exchange pole term of π-N scattering through the Feynman diagrams in Fig. 3(a) and Fig. 3 (b) respectively.

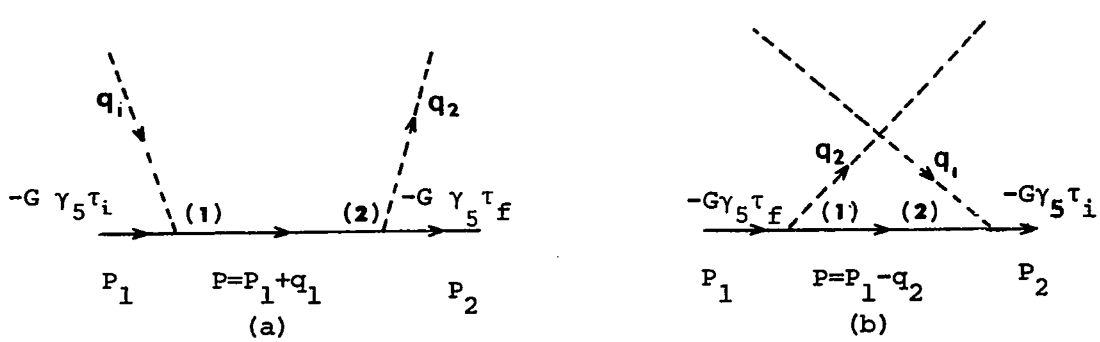

Fig. 3

Following the Feynman rules (see Appendix A) we can write down the T-matrix given by

$$T_{fi} = -G^2\bar{U}(P_2)[\gamma_5\tau_f \frac{1}{\not{P}_1+\not{q}_1-M} \gamma_5\tau_i + \gamma_5\tau_i \frac{1}{\not{P}_1-\not{q}_2-M} \gamma_5\tau_f]U(P_1) \quad (2\text{-}12)$$

where G is the π-N coupling constant. The first term in Equ. (2-12) refers to the direct pole term, while the second refers to the exchange pole term.

In terms of the isospin projection operators[39], we can rewrite Equ. (2-12) in the following form:

$$T = -G^2\bar{U}(P_2)\gamma^\mu Q_\mu \; U(P_1)[\frac{(1-\vec{\tau}.\vec{t})}{(P_1+q_1)^2-M^2} - \frac{(1+\vec{\tau}.\vec{t})}{(P_1-q_2)^2-M^2}], \quad (2\text{-}12a)$$

with

$$\langle\vec{\tau}.\vec{t}\rangle \equiv \{\begin{matrix}1\\-2\end{matrix} \quad \text{for } T = \{\begin{matrix}\frac{3}{2}\\ \frac{1}{2}\end{matrix} \quad \text{state,}$$

where $\frac{\vec{\tau}}{2}$ and $\vec{t}$ are the isospin operators of the nucleon and the pion respectively.

According to Equ. (2 − 7), it turns out that

$$A = 0$$

and

$$B = -G^2[\frac{(1-\vec{\tau}.\vec{t})}{(P_1+q_1)^2-M^2} - \frac{(1+\vec{\tau}.\vec{t})}{(P_1-q_2)^2-M^2}]. \quad (2\text{-}13)$$

If we use the scattering amplitude defined in Equ. (2-8), together with Equs. (2-12a) and (2-11), we get

$$F = -\frac{G^2}{8\pi W}\{\frac{(1-\vec{\tau}.\vec{t})}{(W+M)(W-M)}[(E+M)(W-M)+(E-M)(W+M)(\hat{q}_2.\hat{q}_1+i\vec{\sigma}.\hat{q}_2 x\hat{q}_1)]$$

$$-\frac{(1+\vec{\tau}.\vec{t})}{2(Z-\varkappa)}[\frac{(W-M)}{(E-M)}+\frac{(W+M)}{(E+M)}(\hat{q}_2.\hat{q}_1+i\vec{\sigma}.\hat{q}_2 \times \hat{q}_1)]\}$$

$$= \chi_f^+ [f_1 + f_2 (\hat{q}_2.\hat{q}_1 + i\vec{\sigma}.\hat{q}_2 x\hat{q}_1)] \chi_i. \quad (2\text{-}14)$$

where χ represents a two-component Pauli spinor, and

$$Z = \frac{\mu^2 - 2E\omega}{2q^2} < 0.$$

Thus we have

$$f_1 = -\frac{G^2}{8\pi W}[\frac{(1-\vec{\tau}.\vec{t})(E+M)}{(W+M)} - \frac{(1+\vec{\tau}.\vec{t})(W-M)}{2(Z-\varkappa)(E-M)}]$$

$$f_2 = -\frac{G^2}{8\pi W}[\frac{(1-\vec{\tau}.\vec{t})(E-M)}{(W-M)} - \frac{(1+\vec{\tau}.\vec{t})(W+M)}{2(Z-\varkappa)(E+M)}]. \quad (2\text{-}14a)$$

The partial wave scattering amplitudes can be projected out by means of the following operation[37].

$$f_{\ell\pm} = \frac{1}{2}\int_{-1}^{1} [f_1 P_\ell(\varkappa) + f_2 P_{\ell\pm1}(\varkappa)]\, d\varkappa \quad (2\text{-}15)$$

where $\varkappa = \cos\theta = \hat{q}_2.\hat{q}_1$ and θ is the scattering angle.

In a straightforward fashion, we substitute Equ. (2-14a) into Equ. (2-15) to obtain the partial wave amplitudes for Fig. 3.

$$f_{\ell\pm} = \frac{-G^2}{8\pi W}\left\{\frac{(1-\vec{\tau}.\vec{t})}{(W+M)(W-M)}\left[(E+M)(W-M)\frac{1}{2\ell+1}\delta_{\ell,o}+(E-M)(W+M)\frac{1}{2(\ell\pm1)+1}\delta_{\ell\pm1,o}\right]\right.$$

$$\left.-\frac{(1+\vec{\tau}.\vec{t})}{2(E+M)(E-M)}\left[(E+M)(W-M)Q_\ell(Z)+(E-M)(W+M)Q_{\ell\pm1}(Z)\right]\right\}. \qquad (2\text{-}16)$$

But since $Q_\ell(-Z)=(-1)^{\ell+1}Q_\ell(Z)$, therefore Equ. (2-16) can be rewritten as

$$f_{\ell\pm} = -\frac{G^2}{8\pi W}\left\{(1-\vec{\tau}.\vec{t})\left[\frac{(E+M)}{(W+M)}\frac{1}{(2\ell+1)}\delta_{\ell,o}+\frac{(E-M)}{(W-M)}\frac{1}{2(\ell\pm1)+1}\delta_{\ell\pm1,o}\right]\right.$$

$$\left.+(-1)^\ell\frac{(1+\vec{\tau}.\vec{t})}{2}\left[\frac{(W-M)}{(E-M)}Q_\ell(|Z|)-\frac{(W+M)}{(E+M)}Q_{\ell\pm1}(|Z|)\right]\right\}. \qquad (2\text{-}16a)$$

One remark concerning Equ. (2-16a) is worth mentioning. The first and second terms in the first parenthesis contribute respectively to the S_{11} and the P_{11} states only. The term with the factor $(-1)^\ell$ indicates the exchange character of the π-N interaction.

2.4 Translation of Born Amplitudes into Potentials

The conventional method of translating a Born scattering amplitude for a particular isospin state into a potential may be described as follows: The Born amplitude is rewritten as a function of variables W, E, $\vec{Q}$, and $\vec{P}$; where W and E are the total energy and and energy of nucleon in the C.M. system respectively; $\vec{Q}$ and $\vec{P}$ are respectively one half of the sum and the difference of the final and initial three-momenta of the pion in the C.M. system. All Dirac spinor contractions are then re-expressed in terms of the Pauli spin contraction. The amplitude in the $\vec{P}$ momentum space is Fourier transformed in the $\vec{r}$ configuration space to obtain the energy dependent potential, which in Born approximation gives back the original Born amplitude.

It has been customary to define the Born potential* corresponding to the scattering amplitude F as

$$U(\vec{r},E,W) = -4\pi \frac{1}{(2\pi)^3} \int F(E,W,\vec{Q},\vec{P}) e^{i\vec{P}\cdot\vec{r}} d^3P \tag{2-17}$$

where U has dimensions of energy squared, $\vec{Q} = (\vec{q}_1+\vec{q}_2)/2$ and $\vec{P} = (\vec{q}_2-\vec{q}_1)$. The function $U(\vec{r},E,W)$ is a sum of delta functions, Yukawa functions and their derivatives multiplied by simple rational functions of E, W, and spin and isospin factors.

In order to evaluate the Born approximation potential from the scattering amplitude, one considers the most general form of the scattering amplitude in the present case with the following expression:

$$F(E,W,\vec{Q},\vec{P}) = h_1(E,W) + h_2(E,W)(\vec{q}_2\cdot\vec{q}_1 + i\vec{\sigma}\cdot\vec{q}_2 x\vec{q}_1)$$
$$+ h_3(E,W) \frac{(\vec{q}_2\cdot\vec{q}_1 + i\vec{\sigma}\cdot\vec{q}_2 x\vec{q}_1)}{(\vec{q}_2-\vec{q}_1)^2 + m_x^2} \tag{2-18}$$

where h's are the energy dependent functions. In order to express the R.H.S. of Equ. (2-18) as a function of variables $W,E,\vec{Q}$ and $\vec{P}$, we make use of the relations

$$\vec{q}_2\cdot\vec{q}_1 = 2\vec{Q}^2 - (E^2-M^2)$$

and (2-19)

$$\vec{q}_2 x\vec{q}_1 = -\vec{Q} \times \vec{P}$$

---

* By Born potential we actually mean an "effective" Born potential, which gives the expression of the Born amplitude as Equ. (2-26).

The third term on the R.H.S. of Equ. (2-18) can now be rewritten, after some simplification, as

$$F_3(E,W,\vec{Q},\vec{P}) = h_3(E,W)\left[-\frac{1}{2} + \frac{(E^2-M^2) + m_x^2/2}{\vec{P}^2 + m_x^2} - i\,\frac{\vec{\sigma}\cdot\vec{Q}\times\vec{P}}{\vec{P}^2 + m_x^2}\right]. \quad (2\text{-}20)$$

We now take the Fourier transform of Equ. (2-20) according to Equ. (2-17). The first and second terms are straight-forward. They give rise to the delta function and the Yukawa function respectively. We now manipulate the third term. Consider the expression

$$\frac{1}{(2\pi)^3}\, i\vec{\sigma}\cdot\vec{Q}\times\int\frac{\vec{P}\,e^{i\vec{P}\cdot\vec{r}}}{\vec{P}^2 + m_x^2}\,d^3P$$

$$= \vec{\sigma}\cdot\vec{Q}\times\vec{\nabla}\frac{1}{(2\pi)^3}\int\frac{e^{i\vec{P}\cdot\vec{r}}}{\vec{P}^2 + m_x^2}\,d^3P$$

$$= \frac{1}{(4\pi)}\,\vec{\sigma}\cdot\vec{Q}\times\vec{\nabla}\left(\frac{e^{-m_x r}}{r}\right)$$

$$= \frac{1}{(4\pi)}\,\vec{\sigma}\cdot\vec{Q}\times\vec{r}\,\frac{1}{r}\,\frac{d}{dr}\left(\frac{e^{-m_x r}}{r}\right)$$

$$= -\frac{1}{(4\pi)}\,\vec{\sigma}\cdot\vec{L}\,\frac{1}{r}\,\frac{d}{dr}\left(\frac{e^{-m_x r}}{r}\right). \quad (2\text{-}21)$$

We have replaced $\vec{L} = \vec{r}\times\vec{Q}$ (for the proof of this see Appendix B I). Thus

$$U_3(\vec{r},E,W) = -\,4\pi\,\frac{1}{(2\pi)^3}\int F_3(E,W,\vec{Q}.\vec{P})\,e^{i\vec{P}\cdot\vec{r}}\,d^3P$$

$$= 2\pi\,h_3(E,W)\left[\delta^3(\vec{r}) - (2E^2-2M^2+m_x^2)\,\frac{1}{4\pi}\,\frac{e^{-m_x r}}{r} - \frac{1}{2\pi}\,\vec{\sigma}\cdot\vec{L}\,\frac{1}{r}\,\frac{d}{dr}\left(\frac{e^{-m_x r}}{r}\right)\right]. \quad (2\text{-}22)$$

The second term on the R.H.S. of Equ. (2-18) can be written in terms of $\vec{Q}$ and $\vec{P}$ as

$$F_2(E,W,\vec{Q}.\vec{P}) = h_2(E,W)[2\vec{Q}^2 - (E^2 - M^2) - i\vec{\sigma}.\vec{Q} \times \vec{P}] . \tag{2-23}$$

After Fourier transformation the first term of Equ. (2-23) will again give a delta function. We evaluate the second term here by considering

$$\frac{1}{(2\pi)^3} i \, \vec{\sigma}.\vec{Q} \times \int \vec{P} \, e^{i\vec{P}.\vec{r}} d^3P$$

$$= \vec{\sigma} \, .\vec{Q} \times \vec{\nabla} \frac{1}{(2\pi)^3} \int e^{i\vec{P}.\vec{r}} d^3P$$

$$= \vec{\sigma} \, .\vec{Q} \times \vec{\nabla} \, \delta^3(\vec{r})$$

$$= i\vec{\sigma}.\vec{Q} \times (\vec{P})_{oP} \, \delta^3(\vec{r}) \; . \tag{2-24}$$

In the last step of Equ. (2-24) we have replaced the operator $\vec{\nabla}$ by $i(\vec{P})_{oP}$. The operator $[-i\vec{\sigma}.\vec{Q}\times(\vec{P})_{oP}]$ will give the operator $(i\vec{\sigma}.\vec{q}_2 \times \vec{q}_1)$ again. (See Appendix B II)

Using Equs. (2-19) and (2-24), one can obtain the Born approximation potential from the scattering amplitude in Equ. (2-18), according to Equ. (2-17). This gives

$$U(\vec{r},E,W) = 4\pi \frac{1}{(2\pi)^3} \int F(E,W,\vec{Q},\vec{P})\ e^{i\vec{P}.\vec{r}}\ d^3P$$

$$= -4\pi \{[h_1(E,W) + h_2(E,W)(\vec{q}_2.\vec{q}_1 + i\vec{\sigma}.\vec{q}_2 x\vec{q}_1) - \frac{1}{2} h_3(E,W)]\ \delta^3(r)$$

$$- \frac{h_3(E,W)}{4\pi}[(\frac{2E^2 - 2M^2 + m_x^2}{2}) + \vec{\sigma}.\vec{L}\frac{1}{r}\frac{d}{dr}]\ \frac{e^{-m_x r}}{r}\}. \quad (2\text{-}25)$$

The Born amplitude is defined by[40]

$$F^B = - \frac{1}{4\pi} \int U(\vec{r},E,W)\ e^{-i\vec{P}.\vec{r}}\ d^3r \quad (2\text{-}26)$$

One can check the consistency of this calculation by carrying out the inverse transformation of Equation (2-26), and we indeed obtain the amplitude given by Equation (2-18).

In order to find the Born approximation potential corresponding to the scattering amplitude in Equ. (2-14), one cannot use the straightforward method by just substituting to Equ. (2-17). In view of the expression for the partial wave amplitudes in Equ. (2-16a), because of the angular momentum factors such as $(-1)^\ell$, $\delta_{\ell,o}$ etc., we have to find the Born potential for each particular angular momentum state. This 'partial wave' Born potential under the Fourier transformation and also by the aid of Equs. (2-14) and (2-15), will give back the original partial wave-scattering amplitude as Equ. (2-16a).

As shown in Appendix D, we find the expression for the 'partial wave' Born potential corresponding to Equ. (2-16a), in the following expression

$$V^\ell_{\pi N}(r) = \frac{G^2}{2W}\{\frac{(1-\vec{\tau}.\vec{t})}{(W+M)(W-M)}[(E+M)(W-M)\delta_{\ell,o} + (E-M)(W+M)(\hat{q}_2.\hat{q}_1 + i\vec{\sigma}.\hat{q}_2 x\hat{q}_1)\delta_{\ell-1,o}(\frac{3}{2} - J)]$$

$$+ (-1)^\ell \frac{(1+\vec{\tau}.\vec{t})(W+M)}{2(E+M)}\}\ \delta^3(\vec{r})$$

$$+(-1)^\ell \frac{G^2}{4\pi} \frac{(1+\vec{\tau}.\vec{t})}{2W}\{[(E+M)(W-M) - (E-M)(W+M)|Z|]\frac{e^{-m_t r}}{r} - \frac{(W+M)}{(E+M)}\frac{\vec{\sigma}.\vec{L}}{r}\frac{d}{dr}(\frac{e^{-m_t r}}{r})\}$$

(2-27)

We have replaced $(\frac{2E\omega - \mu^2}{2q^2})$ by $(1 + \frac{m_t^2}{2q^2})$ and obtain an energy dependent expression for the range $\frac{1}{m_t}$ with $m_t = \sqrt{2E\omega - \mu^2 - 2q^2}$ which is real.

Here we insert the factor $\delta_{\ell,o}$, $\delta_{\ell-1,o}$ $(\frac{3}{2} - J)$ into the first and second terms in the first parenthesis of Equ. (2-27) to ensure that the respective contributions are only for the S_{11} and P_{11} channels as dictated by Equ. (2-16a). It must be emphasized that in Equ. (2-27), the terms with the factor $(-1)^{\ell}$ refer to the exchange force in the π-N scattering.

2.5 The 'Cut Off' Method and Singular potentials

If we substitute the Born potential in Equ. (2-27) into the Klein-Gordon equation or the Schrödinger equation, immediately we are confronted with difficulties. The term $\frac{1}{r}\frac{d}{dr}(\frac{e^{-m_t r}}{r})$ appearing in the spin-orbit potential is singular because it varies as r^{-3} near the origin. In order to eliminate the r^{-3} divergence in the origin, it has been customary to introduce a 'cut off'. This has been done by Scotti and Wong[22] for the vector meson case with an exponential 'cut off' as suggested by the Regge-pole description for composite particles. Bryan et al.[29] have employed a zero 'cut off' uniformly in all states for their OBE potential with

$$V(r) = \begin{cases} 0 & 0<r<c \\ V_\pi(r) + V_S(r) + V_{\rho\omega}(r) & c<r \end{cases} \qquad (2\text{-}28)$$

where c = 0.54 fm.

However, in a somewhat different manner we introduce the 'cut off' by replacing

$$\frac{e^{-m_t r}}{r} \longrightarrow \frac{e^{-m_t r}}{r} - \frac{e^{-m_c r}}{r} \qquad (2\text{-}29)$$

not only for the spin-orbit potential term but also for the regular Yukawa potential term.

The 'cut off' will appear as a 'screening effect', the larger the 'cut off' mass m_c, the smaller the effect on the potential. Later on, m_c has been regarded as one of the adjustable parameters to fit the phase shifts. We confine ourselves to the condition $m_c > m_{max}$, where m_{max} is the largest mass among the exchange particles in our OBE potential. The purpose of doing so is to prevent the potential from changing sign when the 'cut off' is introduced.

Moreover, apart from the singularity appearing in the spin-orbit potential, the next question we would like to pose concerns the singularity from the $\delta^3(\vec{r})$ function. The $\delta^3(\vec{r})$ gives rise to a contact interaction, which is a very short range force. As the consequence of the $\delta^3(\vec{r})$ term which appears in the potential, the Klein-Gordon equation or the Schrödinger equation cannot be solved numerically. It is therefore plausible to replace it by a very short range Yukawa potential, i.e.,

$$\delta^3(\vec{r}) \rightarrow \frac{m_c^2}{4\pi} \frac{e^{-m_c r}}{r} \qquad (2\text{-}30)$$

here we use the same 'cut off' mass. The larger the m_c, the faster the term dies down. The factor $\frac{m_c^2}{4\pi}$ is the normalization constant; since

$$\frac{m_c^2}{4\pi} \int \frac{e^{-m_c r}}{r} d^3r = 1. \qquad (2\text{-}31)$$

As may be seen From Equ. (2-27) the factor $(x+i\vec{\sigma}\cdot\hat{q}_2 x\hat{q}_1)$ inside the parenthesis in the $\delta^3(\vec{r})$ function term also causes trouble, since its exact value is not known. In order to escape from this difficulty we parametrize it in order to fit the phase shifts.

Let

$$(x + i\vec{\sigma}\cdot\hat{q}_2 x\hat{q}_1) = G_p \lesssim 1. \qquad (2\text{-}32)$$

It is perhaps worth mentioning that the coefficient (E-M)(W+M) is a lot smaller than (E+M)(W-M) for moderate energy. In view of Equ. (2-27) the contribution from the term with coefficient $(E-M)(W+M)(x + i\vec{\sigma}\cdot\hat{q}_2 x\hat{q}_1)$ is negligible as compared with the term with coefficient (E+M)(W-M). The justification for making this approximation is that it does not affect our result at all in all other channels except the P_{11}. Even in the P_{11} channel, this approximation will have an insignificant effect on the potential.

After much effort, eventually we obtain a regular modified 'partial wave' Born approximation potential, which allows us to solve the Klein-Gordon equation numerically. Thus

$$V^{\ell}_{\pi N} = G_{\pi N}\frac{m_c^2}{2W}\Big\{(1-\vec{\tau}.\vec{t})\Big[\frac{(E+M)}{(W+M)}\delta_{\ell,o} + \frac{(E-M)}{(W-M)}G_p\delta_{\ell-1,o}(\tfrac{3}{2} - J)\Big]$$
$$+(-1)^{\ell}\frac{(1+\vec{\tau}.\vec{t})(W+M)}{2\ (E+M)}\Big\}\ \frac{e^{-m_c r}}{r}$$
$$+\ (-1)^{\ell}G_{\pi N}\frac{(1+\vec{\tau}.\vec{t})}{2W}\big[(E+M)(W-M)-(E-M)(W+M)\,|Z|\big]\Big(\frac{e^{-m_t r}}{r} - \frac{e^{-m_c r}}{r}\Big)$$
$$-(-1)^{\ell}\ G_{\pi N}\frac{(1+\vec{\tau}.\vec{t})}{2W}\frac{(W+M)}{(E+M)}\ \frac{1}{r}\ \frac{d}{dr}\Big(\frac{e^{-m_t r}}{r} - \frac{e^{-m_c r}}{r}\Big)\vec{\sigma}\ .\vec{L}, \qquad (2\text{-}27a)$$

where $G_{\pi N} = \frac{G^2}{4\pi}$.

For convenience, we write

$$V^{\ell}_{\pi N}(r) = A_1 \frac{e^{-m_c r}}{r} + A_2\left(\frac{e^{-m_t r}}{r} - \frac{e^{-m_c r}}{r}\right) + A_3 \frac{1}{r}\frac{d}{dr}\left(\frac{e^{-m_t r}}{r} - \frac{e^{-m_c r}}{r}\right) \quad (2\text{-}27b)$$

with

$$A_1 = G_{\pi N} \frac{m_c^2}{2W} \{(1-\vec{\tau}.\vec{t})[\frac{(E+M)}{(W+M)}\delta_{\ell,o}+\frac{(E-M)}{(W-M)}G_P\delta_{\ell-1,o}(\frac{3}{2}-J)]+(-1)^{\ell} \frac{(1+\vec{\tau}.\vec{t})(W+M)}{2(E+M)}]\} \quad (2\text{-}33a)$$

$$A_2 = (-1)^{\ell} G_{\pi N} \frac{(1+\vec{\tau}.\vec{t})}{2W}[(E+M)(W-M) - (E-M)(W+M)|Z|], \quad (2\text{-}33b)$$

$$A_3 = (-1)^{\ell+1} G_{\pi N} \frac{(1+\vec{\tau}.\vec{t})}{2W} \frac{(W+M)}{(E+M)} \vec{\sigma}.\vec{L} \quad (2\text{-}33c)$$

where the A's are the energy, spin and isospin dependent functions.

Now we introduce the 'partial wave' potential projection operator as Λ_ℓ , so that the total Born approximation potential corresponding to Fig. 3 can be written as

$$V_{\pi N}(r) = \sum_{\ell} V^{\ell}_{\pi N}(r)\Lambda_\ell \qquad \ell = 0, 1, 2, \ldots \quad (2\text{-}27c)$$

where Λ_ℓ has the property

$$\Lambda_{\ell'}|\ell\rangle = \delta_{\ell\ell'}|\ell\rangle$$

for a particular angular momentum state $|\ell\rangle$.

2.6 The Vector Coupling of ρ-Exchange in the π-N Scattering

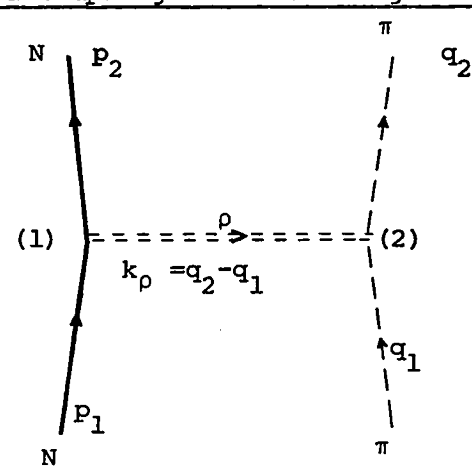

Fig. 4

In this section we discuss the ρ-exchange contribution in π-N scattering according to Fig. 4. Lowest order perturbation theory has been employed. The coupling of the vector meson to the nucleon consists of two parts, vector and tensor. First of all we consider the vector coupling. For the ρNN coupling, corresponding to vertex (1) in Fig. 4, we have the interaction Lagrangian density as follows:

$$L_{\rho NN} = g_{\rho NN} \bar{\psi} \gamma^{\mu} \frac{\tau^{\kappa}}{2} \psi B^{\kappa}{}_{\mu} \quad . \qquad (2\text{-}34)$$

For the ρππ coupling, corresponding to vertex (2) in Fig. 4, we have

$$L_{\rho\pi\pi} = g_{\rho\pi\pi} \, \varepsilon^{\kappa \ell m} \, \partial^{\mu}\phi^{\ell}\phi^{m}B^{\kappa}_{\mu} \qquad (2\text{-}35)$$

where B^{κ}_{μ} is the ρ-meson wave function with isospin component k(k = 1, 2, 3), and $\varepsilon^{\kappa \ell m}$ is the conventional totally antisymmetric tensor.

From the Lagrangian densities in Equs. (2-34) and (2-35), we can write down the vertices (1) and (2) according to the rules defined in Appendix A, i.e.,

$$\text{vertex} \quad (1) = -g_{\rho NN}\gamma^{\mu} \frac{\tau^{\kappa}}{2} \qquad (2\text{-}36)$$

and

$$\text{vertex (2)} = -ig_{\rho\pi\pi}\varepsilon^{\kappa\ell m}(q_1+q_2)^\nu . \qquad (2\text{-}37)$$

The propagator for ρ-exchange with 4-momentum k_ρ is given by[41]

$$-\frac{g_{\mu\nu} - (k_\rho)_\mu (k_\rho)_\nu /m_\rho^2}{k_\rho^2 - m_\rho^2} \qquad (2\text{-}38)$$

where m_ρ is the mass of the ρ-meson. By the conservation of 4-momentum

$$k_\rho = P_1 - P_2 = q_2 - q_1 .$$

The invariant T-matrix can be written out in straightforward fashion according to Appendix A, thus

$$T = g_{\rho NN} g_{\rho\pi\pi} \frac{\vec{\tau}.\vec{t}}{2} \; \overline{U}(P_2) \frac{\gamma^\mu[g_{\mu\nu} - (k_\rho)_\mu (k_\rho)_\nu/m_\rho^2]}{k_\rho^2 - m_\rho^2} (q_1+q_2)^\nu U(P_1) \qquad (2\text{-}39)$$

where we have made use of the relation[42]

$$<\ell|t_\kappa|m> = -it_{\kappa\ell m} \qquad (2\text{-}40)$$

for the matrix elements of the isospin of the pion between the states of $|\ell>$ and $|m>$.

Since

$$(k_\rho)_\nu(q_1 + q_2)^\nu = (q_2 - q_1)_\nu (q_1 + q_2)^\nu = 0 \qquad (2\text{-}41)$$

Equ. (2-39) reduces to

$$T = g_{\rho NN} g_{\rho\pi\pi} \frac{\vec{\tau}.\vec{t}}{2} \bar{U}(P_2) \frac{\gamma_\nu (q_1 + q_2)^\nu}{k_\rho^2 - m_\rho^2} U(P_1) \quad . \tag{2-39a}$$

Substituting Equ. (2-39a) together with Equ. (2-11) into Equ. (2-8), we obtain the expression for the scattering amplitude

$$F = -\frac{g_{\rho NN} g_{\rho\pi\pi}}{4\pi q^2 W} \frac{\vec{\tau}.\vec{t}}{4} \{-(E-M)(W+M) + \frac{(E+M)(W-M)+(E-M)(W+M)Z_\rho}{Z_\rho - x} + \frac{(E-M)(W+M).(i\vec{\sigma}.\hat{q}_2 x \hat{q}_1)}{Z_\rho - x}\} \tag{2-42}$$

where

$$Z_\rho = 1 + \frac{m_\rho^2}{2q^2} \cdot$$

Following the method adopted in Section 2.4, the Born approximation potential for the vector coupling of ρ-exchange is thus

$$V_\rho^v = -\frac{g_{\rho NN} g_{\rho\pi\pi}}{4\pi} \frac{\vec{\tau}.\vec{t}}{2W} \{ \frac{2\pi}{q^2} (E-M)(W+M)\delta^3(\vec{r}) - [(E+M)(W-M)+(E-M)(W+M)Z_\rho] \frac{e^{-m_\rho r}}{r}$$
$$- \frac{(E-M)(W+M)}{q^2} \frac{1}{r} \frac{d}{dr} (\frac{e^{-m_\rho r}}{r}) \vec{\sigma}.\vec{L} \} . \tag{2-43}$$

After introducing the 'cut off' to regularize the potential as before and also replacing the delta function

$$\delta^3(\vec{r}) \rightarrow \frac{m_c^2}{4\pi} \frac{e^{-m_c r}}{r} ,$$

we obtain the modified Born approximation potential for the vector coupling of ρ-exchange as follows:

$$V_\rho^V(r) = - \frac{g_{\rho NN} g_{\rho\pi\pi}}{4\pi} \frac{\vec{\tau}.\vec{t}}{2W} \{ \frac{m_c^2 (W+M)}{2(E+M)} \frac{e^{-m_c r}}{r} - [(E+M)(W-M) + (E-M)(W+M) Z_\rho] (\frac{e^{-m_\rho r}}{r}$$

$$- \frac{e^{-m_c r}}{r}) - \frac{(W+M)}{(E+M)} \frac{1}{r} \frac{d}{dr} (\frac{e^{-m_\rho r}}{r} - \frac{e^{-m_c r}}{r}) \vec{\sigma} . \vec{L} \} . \quad (2\text{-}44)$$

2.7 The tensor coupling of ρ-Exchange in π-N Scattering

In addition to the vector coupling of the ρ-meson to the nucleon, there is a tensor coupling part, which is due to the anomalous magnetic moment of the nucleon.

The corresponding interaction Lagrangian density is defined by

$$L_{\rho NN}^T = \frac{f_{\rho NN}}{2M} \bar{\psi} \sigma^{\mu\nu} \frac{\tau^\kappa}{2} \psi B_{\mu,\nu}^\kappa \quad (2\text{-}45)$$

where

$$B_{\mu,\nu}^\kappa = \partial_\mu B_\nu^\kappa - \partial_\nu B_\mu^\kappa . \quad (2\text{-}46)$$

The coupling constants $f_{\rho NN}$ and $g_{\rho NN}$ can be related to the corresponding residues γ_1 ,γ_2 of the annihilation amplitudes for

$N\bar{N} \to 2\pi$ used by Frazer and Fulco[43].

$$\gamma_\rho^V \equiv \frac{g_{\rho\pi\pi}\; g_{\rho NN}}{4\pi} = -3\gamma_1 \quad . \tag{2-47}$$

$$\gamma_\rho^T \equiv \frac{g_{\rho\pi\pi} f_{\rho NN}}{4\pi} = 3M\gamma_2 \quad .$$

In their study of the nucleon electromagnetic form factors, Ball and Wong[44] estimate that $\gamma_1 \simeq -1.0$ and that $M\gamma_2/\gamma_1 \simeq 1.83$.

Bowcock et al.[45] and Hamilton et al.[26] have estimated that

$$2 \leqslant \frac{g_{\rho\pi\pi} g_{\rho NN}}{4\pi} \leqslant 2.5 \quad . \tag{2-48}$$

From the ρ-width $\Gamma_\rho \sim 100 - 125$ MEV we obtain

$$2 \leqslant \frac{g^2_{\rho\pi\pi}}{4\pi} \leqslant 2.5 \quad . \tag{2-49}$$

Therefore a universal constant can be adopted

$$g_{\rho\pi\pi} \approx g_{\rho NN}$$

as postulated by Sakurai[46].

The T-matrix for the tensor coupling is thus

$$T=i\ \frac{f_{\rho NN}g_{\rho\pi\pi}}{2M}\ \vec{\tau}.\vec{t}\ \bar{U}(P_2)\sigma^{\mu\alpha}(q_2-q_1)_\mu\ \frac{[g_{\alpha\beta}-(k_\rho)_\alpha(k_\rho)_\beta/m_\rho^2\ ]}{k_\rho^2-m_\rho^2}\ (q_1+q_2)^\beta U(P_1)\ . \tag{2-50}$$

For the Gordon Reduction of the current we have[47]

$$\bar{U}(P_2)\gamma^\mu U(P_1)\ =\ \frac{1}{2M}\ \bar{U}(P_2)\,[(P_1+P_2)^\mu\ +\ i\sigma^{\mu\nu}\ (P_2-P_1)_\nu\ ]\ U(P_1)\ . \tag{2-51}$$

Substituting Equ. (2-51) into Equ. (2-50) gives

$$T=-\ \frac{f_{\rho NN}g_{\rho\pi\pi}}{2M}\ \vec{\tau}.\vec{t}\ \bar{U}(P_2)\frac{[2M\gamma^\mu(q_1+q_2)_\mu\ -\ (q_1+q_2)_\mu(P_1+P_2)^\mu]}{k_\rho^2-m_\rho^2}U(P_1)\ . \tag{2-52}$$

Let

$$T\ =\ T_1\ +\ T_2$$

where

$$T_1\ =\ -f_{\rho NN}g_{\rho\pi\pi}\ \vec{\tau}.\vec{t}\ \bar{U}(P_2)\frac{\gamma^\mu\ (q_1+q_2)_\mu}{k_\rho^2-m_\rho^2}U(\ P_1)\ . \tag{2-53}$$

This term has the same form as Equ. (2-39a) if we replace

$$-2f_{\rho NN}\ \rightarrow\ g_{\rho NN}\ .$$

$$T_2\ =\frac{f_{\rho NN}g_{\rho\pi\pi}}{2M}\ \vec{\tau}.\vec{t}\ \frac{\bar{U}(P_2)U(P_1)}{k_\rho^2-m_\rho^2}\ (q_1+q_2).(P_1+P_2)\ , \tag{2-54}$$

and also we have the corresponding scattering amplitude

$$F_2 = -\frac{2f_{\rho NN}g_{\rho\pi\pi}}{4\pi}\frac{\vec{\tau}.\vec{t}}{4W}\left[\frac{Y_\rho}{Z_\rho-x}-1\right]\bar{U}(p_2)U(p_1) \tag{2-55}$$

where $Y_\rho = 2 + \frac{4E\omega + m_\rho^2}{2q^2}$.

Substitution of Equ.(2-10) into Equ. (2-55) gives

$$F_2 = -\frac{2f_{\rho NN}g_{\rho\pi\pi}}{4\pi}\frac{\vec{\tau}.\vec{t}}{8MW}\left[\frac{Y_\rho}{Z_\rho-x}-1\right]\left[(E+M)-(E-M)(\hat{q}_2.\hat{q}_1+i\vec{\sigma}.\hat{q}_2x\hat{q}_1)\right] \tag{2-55a}$$

According to Equ. (2-14) we get

$$f_1 = -\frac{2fg}{4\pi}\frac{\vec{\tau}.\vec{t}}{8MW}(E+M)\left[\frac{Y_\rho}{Z_\rho-x}-1\right]$$

$$f_2 = \frac{2fg}{4\pi}\frac{\vec{\tau}.\vec{t}}{8MW}(E-M)\left[\frac{Y_\rho}{Z_\rho-x}-1\right].$$

The partial wave scattering amplitude corresponding to F_2 can be found by the aid of Equ.(2-15), thus

$$(f_{\ell\pm})_2 = \frac{2fg}{4\pi}\frac{\vec{\tau}.\vec{t}}{8MW}\left\{\left[\frac{(E+M)}{2\ell+1}\delta_{\ell,o}-\frac{(E-M)}{2(\ell\pm1)+1}\delta_{\ell\pm1,o}\right]-Y_\rho\left[(E+M)Q_\ell(Z_\rho)-(E-M)Q_{\ell\pm1}(Z_\rho)\right]\right\} . \tag{2-56}$$

Repeating the procedure as described in Appendix D, the 'partial wave' Born potential corresponding to Equ. (2-56) can be written as

$$V_2^\ell = -\frac{2fg\pi\vec{\tau}.\vec{t}}{4\pi\ 2MW}\left\{\left[(E+M)\delta_{\ell,o}-(E-M)(\hat{q}_2.\hat{q}_1+i\vec{\sigma}.\hat{q}_2x\hat{q}_1)\delta_{\ell-1,o}\left(\frac{3}{2}-J\right)-(E-M)Y_\rho\right]\delta^3(\vec{r})\right.$$
$$\left.-\frac{Y_\rho}{2\pi}\left[q^2((E+M)-(E-M)Z_\rho)\frac{e^{-m_\rho r}}{r}-(E-M)\frac{\vec{\sigma}.\vec{L}}{r}\frac{d}{dr}\left(\frac{e^{-m_\rho r}}{r}\right)\right]\right\}$$

As before, we write the Born potential corresponding to F_2 as

$$V_2(r) = \sum_\ell V_2^\ell(r)\,\Lambda_\ell$$

where Λ_ℓ is the partial wave projection operator.

After regularizing the potential as before, the modified Born approximation potential for both the vector and tensor coupling for ρ-exchange in the π-N interaction has the following expression

$$V_\rho = V_\rho^V + V_\rho^T$$

$$= B_1 \frac{e^{-m_c r}}{r} + B_2 \left(\frac{e^{-m_\rho r}}{r} - \frac{e^{-m_c r}}{r}\right) + B_3 \frac{1}{r}\frac{d}{dr}\left(\frac{e^{-m_\rho r}}{r} - \frac{e^{-m_c r}}{r}\right) \tag{2-57}$$

with

$$B_1 = - G_\rho \frac{\vec{\tau}\cdot\vec{t}}{4W} m_c^2 \Big\{\left(1-\frac{2f}{g}\right)\frac{(W+M)}{(E+M)} + \left(\frac{2f}{g}\right)\frac{1}{2M}\Big[\sum_\ell\big((E+M)\delta_{\ell,o} - (E-M)G_p\delta_{\ell\pm1,o}\left(\tfrac{3}{2}-J\right)\big)\Lambda_\ell - (E-M)Y_\rho\Big]\Big\} \tag{2-58a}$$

$$B_2 = G_\rho \frac{\vec{\tau}\cdot\vec{t}}{2W}\Big\{\left(1-\frac{2f}{g}\right)\left[(E+M)(W-M)+(E-M)(W+M)Z_\rho\right] + \left(\frac{2f}{g}\right)\frac{q^2 Y_\rho}{2M}\left[(E+M)-(E-M)Z_\rho\right]\Big\}, \tag{2-58b}$$

$$B_3 = G_\rho \frac{\vec{\tau}\cdot\vec{t}}{2W}\Big\{\left(1-\frac{2f}{g}\right)\frac{(W+M)}{(E+M)} - \left(\frac{2f}{g}\right)\frac{Y_\rho}{2M}(E-M)\Big\}\,\vec{\sigma}\cdot\vec{L}, \tag{2-58c}$$

where $G_\rho = \dfrac{g_{\rho NN}\,g_{\rho\pi\pi}}{4\pi}$, and $\dfrac{f}{g} = \dfrac{f_{\rho NN}}{g_{\rho NN}}$. Hence $\dfrac{f}{g}$ has a negative value with a magnitude of about 2 from the information obtained from the nucleon's electromagnetic structure.

2.8. The Scalar Meson σ-Exchange in π-N Scattering

The existence of a σ-meson is not well established, but there is evidence for a π-pair in the S-state (T = 0, J = 0). Thus in order to supply the necessary attractive part to the OBE potential, we take into account the exchange of the σ-meson. The mass and coupling constant of the σ are treated as two additional adjustable parameters in our potential.

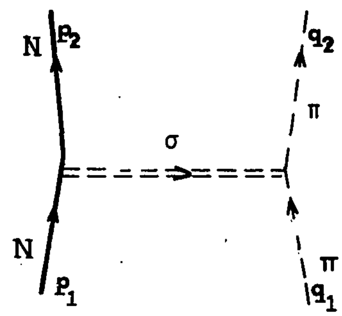

Fig. 5

The isoscalar σ-meson with the π-π and N-N couplings are as follows:

$$L_{\sigma\pi\pi} = \mu g_{\sigma\pi\pi} \phi\phi\chi \tag{2-59}$$

$$L_{\sigma NN} = g_{\sigma NN} \bar{\psi}\psi\chi \tag{2-60}$$

where g and χ are the coupling constants and wave function of the σ-meson.

The invariant T matrix for σ-exchange according to the Feynman diagram in Fig. 5 has the following expression

$$T = -\mu g_{\sigma\pi\pi} g_{\sigma NN} \frac{\bar{U}(P_2)U(P_1)}{k_\sigma^2 - m_\sigma^2} \tag{2-61}$$

where k_ρ and m_σ are the respective 4-momentum and mass of σ-meson.

In view of Equ. (2-8), we have the corresponding scattering amplitude

$$F = G_\sigma \frac{M\mu}{W} \frac{1}{2q^2} \frac{\bar{U}(P_2)U(P_1)}{Z_\sigma - x} \tag{2-62}$$

where

$$Z_\sigma = 1 + \frac{m_\sigma^2}{2q^2} \quad \text{and} \quad G_\sigma = \frac{g_{\sigma\pi\pi} g_{\sigma NN}}{4\pi} .$$

Regularizing the singular part, the modified Born potential we obtain

$$V_\sigma = C_1 \frac{e^{-m_c r}}{r} + C_2 \left(\frac{e^{-m_\sigma r}}{r} - \frac{e^{-m_c r}}{r}\right) + C_3 \frac{1}{r}\frac{d}{dr}\left(\frac{e^{-m_\sigma r}}{r} - \frac{e^{-m_c r}}{r}\right) \tag{2-63}$$

with

$$C_1 = -G_\sigma \frac{\mu}{4W} \cdot \frac{m_c^2}{(E+M)} \tag{2-64a}$$

$$C_2 = -G_\sigma \frac{\mu}{2W}\left[(E+M) - (E-M) Z_\sigma \right] \tag{2-64b}$$

$$C_3 = G_\sigma \frac{\mu}{2W} \frac{1}{(E+M)} . \tag{2-64c}$$

2.9 f^o-Meson Exchange in π-N Scattering

f^o-meson (I=0, $\pi^+\pi^-$ resonance at 1260 MEV) is first observed[48] in the reaction

$$\pi^- + P \to \eta + \pi^+ + \pi^- .$$

Later, Sodickson et al.[49] showed in the concerning experiment that the spin and parity of f^o are 2^+.

Hiroshige et al.(50) and Ino et al.(51) have taken into account the f^o-meson contribution in the P-P and N-N scattering respectively. They stressed that the effect of the tensor meson is to cause a repulsive $\vec{L}\vec{S}$ force for triplet odd states and to improve the fit of the ^3P_o phase shift. The f^o-exchange contribution to π-N scattering has also been investigated by Hiroshige et al.(35)

In this section we investigate to what extent the f^o-meson exchange will contribute to our OBE model. First of all let us discuss the basic theory for a tensor field. The tensor field is expressed by $T^{\mu\nu}(x)$ $(\mu,\nu = 0, 1, 2, 3)$ which satisfies the subsidary conditions

$$T^{\mu\nu}(x) = T^{\nu\mu}(x) \text{ symmetric in } \mu \text{ and } \nu \qquad (2\text{-}65a)$$

$$\partial_\mu T^{\mu\nu}(x) = 0 \text{ gauge invariance,} \qquad (2\text{-}65b)$$

$$g_{\mu\nu} T^{\mu\nu}(x) = 0 \text{ traceless condition.} \qquad (2\text{-}65c)$$

We take the interaction Lagrangian density between the tensor field and the nucleon as

$$L_{fNN} = i\frac{g_{fNN}}{2M} (\bar{\psi}\gamma_\mu\partial_\nu\psi - \partial_\nu\bar{\psi}\gamma_\mu\psi)\, T^{\mu\nu} \qquad (2\text{-}66)$$

For the $f^o_{\pi\pi}$ coupling we assume the interaction Lagrangian density to be

$$L_{f\pi\pi}=\frac{g_{f\pi\pi}}{\mu}[\partial_\mu\phi(q_1)\partial_\nu\phi(q_2)+\partial_\mu\phi(q_2)\partial_\nu\phi(q_1)-(\partial_\mu\partial_\nu\phi(q_1))\phi(q_2)$$

$$-\phi(q_1)(\partial_\mu\partial_\nu\phi(q_2))]T^{\mu\nu} \qquad (2\text{-}67)$$

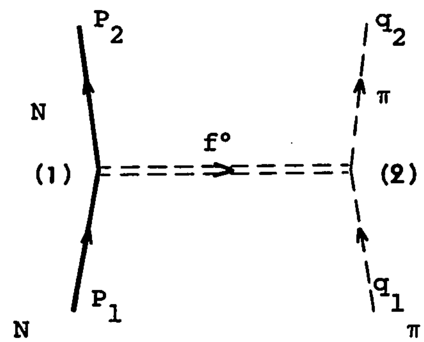

Fig. 6

The vertices (1) and (2) in Fig. 6 corresponding to the respective interactions in Equ. (2-66) and Equ. (2-67) are

$$\text{vertex (1)} = \frac{g_{fNN}}{2M}\gamma^\mu (P_1 + P_2)^\nu \qquad (2\text{-}68)$$

$$\text{vertex (2)} = -\frac{g_{f\pi\pi}}{\mu}(q_1+q_2)^\alpha (q_1+q_2)^\beta . \qquad (2\text{-}69)$$

The invariant T matrix is thus

$$T=\frac{g_{fNN}g_{f\pi\pi}}{2M\mu}\bar{U}(P_2)\gamma^\mu(P_1+P_2)^\nu\frac{N_{\mu\nu;\alpha\beta}}{P^2-m_f^2}(q_1+q_2)^\alpha(q_1+q_2)^\beta U(P_1) \qquad (2\text{-}70)$$

where $N_{\mu\nu;\alpha\beta}$ is the numerator of the propagator of spin 2 particle.

According to Appendix C, we have

$$N_{\mu\nu;\alpha\beta} = \frac{-1}{3} P_{\mu\nu}P_{\alpha\beta} + \frac{1}{2} P_{\mu\alpha}P_{\nu\beta} + \frac{1}{2} P_{\mu\beta}P_{\nu\alpha} \qquad (2\text{-}71)$$

where we define

$$P_{\mu\nu} = g_{\mu\nu} - \frac{P_\mu P_\nu}{m_f^2} \quad , \qquad (2\text{-}72)$$

here P is the 4-momentum of f^o-meson.

Since

$$P_\alpha (q_1 + q_2)^\alpha = 0$$

and also

$$P_\nu (p_1 + p_2)^\nu = 0 \ ,$$

we can now rewrite Equ. (2-70) as

$$T = \frac{g_{fNN} g_{f\pi\pi}}{2M\mu} \frac{1}{(P^2 - m_f^2)} \bar{U}(P_2)\left[\frac{1}{3}\gamma^\mu (P_1+P_2)_\mu (q_1+q_2)^2 - \gamma_\alpha (q_1+q_2)^\alpha (P_1+P_2)(q_1+q_2)\right] U(P_1) \ . \qquad (2\text{-}70a)$$

From Equ. (2-9) we obtain

$$\bar{U}(P_2)\gamma^\mu (P_1+P_2)_\mu U(P_1) = [(E+M) - (E-M)(x + i\vec{\sigma}.\hat{q}_2 x \hat{q}_1)] . \qquad (2\text{-}73)$$

Substituting Equs. (2-73) and (2-11) into Equ. (2-70a) together with (2-8) we obtain the following expression for the scattering amplitude

$$F=-G_f \frac{1}{2\mu W} \{[(E+M)(\frac{W}{M} - \frac{2}{3})+(E-M)(\frac{W}{M} + \frac{2}{3})(x + i\vec{\sigma}.\hat{q}_2 x\hat{q}_1)]$$

$$+ \frac{1}{(Z_f - x)}[(E+M)(Z_1+Z_2-\frac{W}{M}Z_2)-(E-M)(Z_1+Z_2+\frac{W}{M}Z_2)(x+i\vec{\sigma}.\hat{q}_2 x\hat{q}_1)]\}$$

(2-74)

with

$$G_f = \frac{g_{fNN}g_{f\pi\pi}}{4\pi} \quad ,$$

$$Z_f = 1 + \frac{m_f^2}{2q^2} \quad , \qquad (2\text{-}75)$$

$$Z_1 = \frac{4\mu^2 - m_f^2}{6q^2} \quad ,$$

and

$$Z_2 = \frac{4q^2 + 4E\omega + m_f^2}{2q^2} \quad .$$

According to Equ. (2-14) and Equ. (2-15) we can write the partial wave scattering amplitude corresponding to Equ. (2-74) as

$$f_{\ell\pm} = -\frac{G_f}{2\mu W}\{[(E+M)(\frac{W}{M}-\frac{2}{3})\frac{1}{2\ell+1}\delta_{\ell,o} + (E-M)(\frac{W}{M}+\frac{2}{3})\frac{1}{2(\ell\pm1)+1}\delta_{\ell\pm1,o}]$$

$$+[(E+M)(Z_1+Z_2-\frac{W}{M}Z_2)Q_\ell(Z_f)-(E-M)(Z_1+Z_2+\frac{W}{M}Z_2)Q_{\ell\pm1}(Z_f)]\} \qquad (2\text{-}76)$$

The 'partial wave' Born potential (see method described in Appendix D) corresponding to Equ.(2-76), is thus

$$V^\ell_{fo} = \frac{2G_f\pi}{\mu W}\{[(E+M)(\frac{W}{M}-\frac{2}{3})\delta_{\ell,o}+(E-M)(\frac{W}{M}+\frac{2}{3})\delta_{\ell-1,o}G_p(\frac{3}{2}-J)+(E-M)(Z_1+Z_2+\frac{W}{M}Z_2)]\delta^3(\vec{r})$$

$$+[(E+M)(Z_1+Z_2-\frac{W}{M}Z_2)-(E-M)(Z_1+Z_2+\frac{W}{M}Z_2)Z_f]\frac{q^2}{2\pi}\frac{e^{-m_f r}}{r}$$

$$-(E-M)(Z_1+Z_2+\frac{W}{M}Z_2)\frac{1}{2\pi}\frac{\vec{\sigma}.\vec{L}}{r}\frac{d}{dr}(\frac{e^{-m_f r}}{r})\} \qquad (2\text{-}77)$$

We employ the potential regularized method as described in Section 2.5 to modify the Born'partial wave' potential in Equ. (2-76). Eventually we obtain the following expression

$$V^\ell_{fo}(r) = D_1\frac{e^{-m_c r}}{r} + D_2(\frac{e^{-m_f r}}{r}-\frac{e^{-m_c r}}{r}) + D_3\frac{1}{r}\frac{d}{dr}(\frac{e^{-m_f r}}{r}-\frac{e^{-m_c r}}{r}),$$

where

$$D_1 = G_f\frac{m_c^2}{2\mu W}[(E+M)(\frac{W}{M}-\frac{2}{3})\delta_{\ell,o}+(E-M)(\frac{W}{M}+\frac{2}{3})G_p\delta_{\ell-1,o}(\frac{3}{2}-J) +$$

$$+(E-M)(Z_1+Z_2+\frac{W}{M}Z_2)] \qquad (2\text{-}78a)$$

$$D_2 = \frac{G_f q^2}{\mu W}[(E+M)(Z_1+Z_2 - \frac{W}{M}Z_2) - (E-M)(Z_1+Z_2 \frac{W}{M}Z_2) Z_f] \qquad (2\text{-}78b)$$

$$D_3 = - \frac{G_f(E-M)}{\mu W}(Z_1+Z_2 + \frac{W}{M}Z_2) \; \vec{\sigma}.\vec{L} \qquad (2\text{-}78c)$$

We thus have the Born potential for f°-exchange as

$$V_{f°} = \sum_{\ell} V^{\ell}_{f°} \; \Lambda_{\ell}. \qquad (2\text{-}79)$$

CHAPTER III

Discussion on Klein-Gordon Equation

The ideal solution in the study of π-N scattering problem is to find a complete and consistent theory, which is able to explain all the physical phenomena involved in this problem, and in addition does not violate any of the accepted laws. Unfortunately, up to now, no such theory exists. To find a way out, various approaches have been attempted, either by using some assumptions, or by using some approximation methods or both. One of the difficulties with these approaches is that some invariance principles or laws may have to be violated.

Our approach to this problem is to insert the sum of the OBE Born potentials derived in the previous chapter into the Klein-Gordon (K-G) equation and to solve for the phase shifts of the S-, P- and D-waves.

Immediately, a series of arguments can be raised to question the justification for using the K-G equation. The method requires some discussion.

3.1. Lorentz Covariance Property.

First of all, we would like to discuss the Lorentz covariant condition which the K-G equation should obey. The free particle K-G equation certainly does so, but if we want to put in an interaction through the use of a potential, then we must find a potential which can be absorbed into this equation in a Lorentz covariant manner. Thus we may search for a four-vector V_μ, and introduce the interaction into the K-G equation by modifying the operator from

$$q_\mu q^\mu - \mu^2$$

to

$$(q_\mu - V_\mu)(q^\mu - V^\mu) = \mu^2 \quad . \qquad (3\text{-}1)$$

The Lorentz covariance property is preserved, and we know that in the case of electrodynamics, this can be done by utilizing the four vector potential A_μ and the minimal coupling.

Furthermore, if we consider the case of a Lorentz scalar interaction U (for example, the interaction of a pion with a scalar (i.e. no spin) particle), the process can be regarded as a scattering problem, in which the pion is scattered due to an external field U. Then the K-G equation

$$(\Box + \mu^2)\phi = - U \phi \qquad (3\text{-}2)$$

is still manifestly covariant.

Now we pick the C.M. frame. For elastic scattering, we have $\omega_1 = \omega_2$, where ω_1 and ω_2 are the energies of the pion before and after scattering respectively. In the static approximation, the interaction U can be treated as time independent. The pion wave function ϕ in Equ.(3-2) is then separable with respect to $\vec{X}$ and t, and Equ.(3-2) reduces to the following form

$$(\nabla^2 + q^2)\phi = U\phi \qquad (3\text{-}3)$$

where $q = \sqrt{\omega^2 - \mu^2}$ is the relativistic C.M. momentum and U is now time independent.

At this point it should be noted that the Bethe-Salpeter equation(52) would give a Lorentz covariant description. By the assumption of a static interaction one loses the propagative character of the interaction.

The scattering amplitude $f(\theta)$ can be obtained from the asymptotic expression of $\phi(x)$, i.e.,

$$\phi(\infty) \rightarrow e^{i\vec{k}\,\vec{r}} + \frac{e^{ik\,r}}{r} f(\theta) \; .$$

On the other hand, one can obtain the invariant amplitude from field theory. Möller(53) showed that the invariant amplitude for zero spin is related to $f(\theta)$ by

$$A = \frac{W}{2} f \qquad (3\text{-}4)$$

where W is the total energy in the C.M. system. However, even though Equ.(3-3) is not an invariant form, through Equ.(3-4), one can still obtain the invariant amplitude, which also gives the correct unitarity condition.

In the case of π-N scattering, the problem becomes complicated because of the presence of the nucleon spin. There appears to be no simple procedure which would yield a manifestly covariant expression for the interaction. We consider the problem as if the pion were scattered by an external field which is due to the OBE potential. An examination of the Born potential U(r) that was derived in the previous chapter, shows that even apart from the assumption of the static approximation it contains terms such as $\vec{\sigma}\cdot\vec{L}$ for which covariance is not manifest. The energy dependence of U(r) may cause nonlocality in the coordinate space; in addition, the $\vec{\sigma}\cdot\vec{L}$ term is the non-relativistic form of a relativistic interaction. Nevertheless, a term such as $\vec{\sigma}\cdot\vec{L}$ is well defined in the C.M. System, where one may expect the form of the interaction to be simplified.

Further, if the π-N interaction is identified with the OBE potential, in the light of Feynman diagrams from field theory, we would be able to obtain the kinematical picture of the interaction. The object is to see, if to a certain extent, this "semi-phenomenological" potnetial will be able to reproduce some of the characteristic features of the π-N scattering problem. Since in the C.M. frame our Born potential does not depend on time explicitly, for elastic scattering the energies of the pion before and after scattering are the same. In the static approximation, the K-G equation can be reduced to the form of Equ.(3-3). A proper covariant form of the K-G equation is not achieved, nevertheless, the relativistic connection between the energy and momentum of the scattered particle, i.e. the pion, is correctly described. In addition, the $\vec{\sigma}\,\vec{L}$ term is uniquely defined, if we project out the Born potential into partial wave channels, We can match, in Born approximation, the partial wave scattering amplitude $f_{\ell\pm}$ with the $f_{\ell\pm}$ from the invariant matrix element given by field theory. The K-G equation can be used as a device to impose unitarity on a Born approximation.

3.2. One Particle Theory

We next proceed to the discussion of the one-particle theory in the K-G equation. One may argue that "the K-G equation has no place in a one-particle theory[54]". The major underlying difficulties are two-fold. First, the probability density is not positive definite. Second, there is the possibility of negative energy solutions. Obviously, the availability of negative states without lower bound would lead to collapse if the energy is allowed to transfer away from the particle in question. This difficulty will not arise if we stick to free particles, or particles in stationary states of static potentials. Suppose a particle is originally in a positive energy state. In the absence of any interaction, there will be no transfer of energy, and it will always remain in a positive energy state. Furthermore, from the expression of the probability density

$$\rho = \frac{i}{2\mu} \, \phi^* \overleftrightarrow{\partial_0} \, \phi = \frac{E}{\mu} \, \phi^* \phi \quad ,$$

for a free particle in a positive energy state, ρ remains positive definite for all times by virtue of the equation of motion. This is still true for a stationary state of positive energy in a static potential. The one-particle interpretation of the K-G equation in the presence of a non-static external field is no longer as simple as in the above special cases. However, even though it is not possible to give a complete satisfactory physical interpretation for the K-G equation in the presence of an external field, Pauli and Weisskopf[55] showed that there is no difficulty in the one particle interpretation if the K-G equation is regarded in the same sense as Maxwell's equation for electromagnetic field and quantized in the usual fashion. Furthermore, Feshbach and Villars[56] have presented a unified picture of the one-particle treatment of the K-G equation by employing a two-component wave function in a two dimensional charge space with an indefinite metric. The norm of the state vector is +1 for a positively charged particle and -1 for negatively charged particle. A treatment for the neutral particle is also described. In the light of the field theoretical reinterpretation the one-particle K-G equation with interaction continues to be of physical relevance. We shall proceed with the discussion as follows[57].

Classically, the probability current $\vec{j}(x)$ of a field is related to the probability density $\rho(x)$ through $\vec{j}(x) = \rho(x)\,\vec{v}(x)$, where $\vec{v}(x)$ is the velocity field. Together, they satisfy the continuity equation

$$\frac{\partial}{\partial t}\rho(x) + \vec{\nabla}\cdot\vec{j}(x) = 0 \quad .$$

In non-relativistic quantum mechanics, $\rho(x)$ is given by $|\phi(x)|^2$ and the velocity $\vec{v} = \frac{\vec{P}}{\mu} \rightarrow \frac{-i\vec{\nabla}}{\mu}$. Indeed, we can verify from Schrödinger equation that the obvious generalization $\vec{j}(x) = \frac{1}{2\mu} \phi^*(-i\overleftrightarrow{\nabla})\phi(x)$ is indeed the probability current, in the sense that the equation of continuity is satisfied.

If we take relativity into account, then $\vec{v} = \frac{\vec{P}}{P_o} = \frac{-i\vec{\nabla}}{\sqrt{-\nabla^2+\mu^2}}$, which is a non-local operator. Nevertheless, the energy-momentum operator $-i\partial_\mu$ is local. If we calculate, for example, the speed of propagation of a wave function $\phi(\vec{x},t)$ as a function of time (assuming say $\phi(\vec{x},0) = \delta(\vec{x})$), we will get the anomaly that although the energy density is propagating with a speed not greater than the speed of light, the same is not true for the probability density. We recall that similar anomalies occur in classical electromagnetic theory, in that group velocity (the velocity of energy propagation) never exceeds the speed of light, but the phase velocity (which carries no physical information) may. In a similar way, we insist that the speed of energy propagation is more physical than the speed of probability propagation, and we do not run into conflict with causality. If that is all we say, however, we will have to rule out probability $|\phi|^2$ as having any direct physical meaning at all, contrary to the usual assumption of quantum mechanics. This may be salvaged by noticing the following. The apparent violation of causality for the propagation of probability density is certainly related to the non-locality of the velocity operator

$-i\vec{\nabla}/\sqrt{-\nabla^2 + \mu^2}$. But the size of the non-local region of this operator is only of order of μ^{-1}, the Compton wavelength of the particle. Quite obviously then we may salvage the probability interpretation by assuming that a relativistic particle can never be localized to anything smaller than its Compton wavelength.

We can also verify from the K-G equation that the energy-momentum density

$$P^{\mu} = \frac{1}{2} \phi^{*}(x) (-i \overleftrightarrow{\partial}^{\mu}) \phi(x),$$

is divergenceless.

$$\partial_{\mu} P^{\mu}(x) = 0$$

i.e., that energy is conserved.

As a consequence of the foregoing discussion, one can conclude that the one-particle theory interpretation of the K-G equation is still adequate to a certain extent.

Now we would like to discuss, to what extent, the one-particle theory interpretation remains adequate in our approach. The OBE potential can be looked upon in two equivalent and complementary ways. One is as the description of a peripheral interaction, through the one-particle-exchange between the pion and the nucleon. The other is as the description of the behaviour of a field. Based on the field theoretic approach, the π-N scattering process, can be illustrated by the aid of Feynman diagrams.

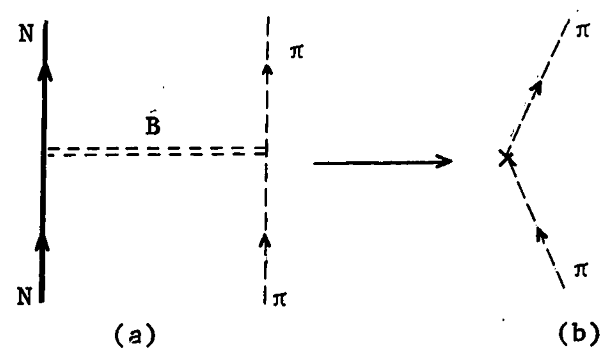

Fig. 7

Fig. 7 gives the lowest order Feynman diagram. When we consider only the case of the pion field, the interaction comes from the boson-pion vertex. This in effect, makes it a one-particle theory problem with an external field (see Fig. 7b), in which the pion is scattered.

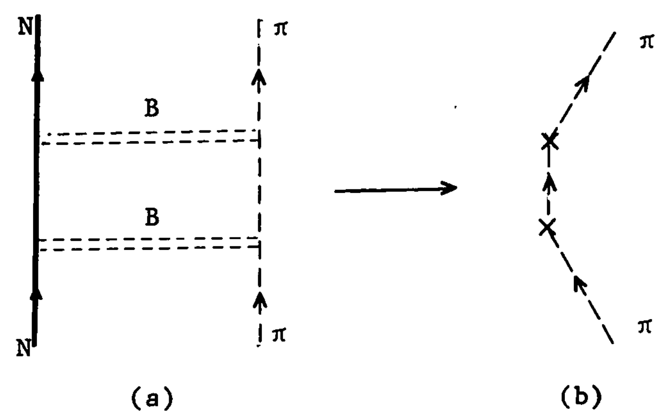

Fig. 8

Fig. 8 refers to a two-boson-exchange Feyman diagram for second order Born approximation. The interacting pion field is described by two external sources acting at different times. The picture (Fig.8b) shows that the problem can be regarded as the one-particle theory problem, in which the pion moving forward in time with positive energy is scattered at a earlier time by U_1 and at a later time by U_2.

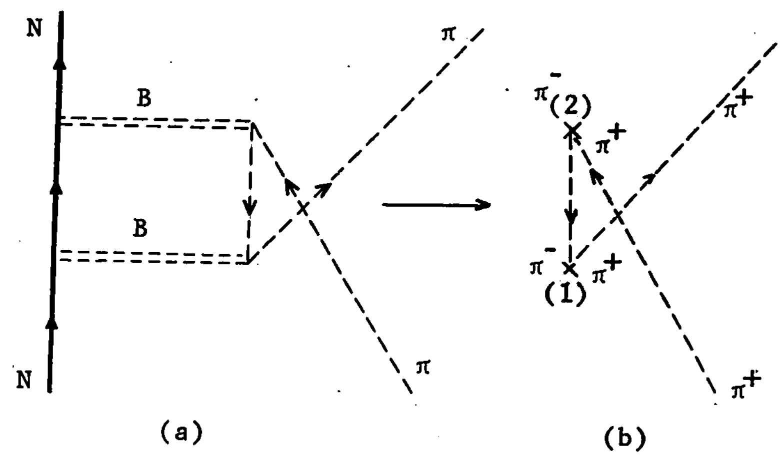

Fig. 9

In Fig. 9, the Feynman diagram is still from the second order Born approximation. However, the interaction is not an ordinary scattering process. At vertex (1) there is a virtual pair creation process; on the other hand, at vertex (2) there is a pair annihilation process. The particle and antiparticle pair creation or annihilation in the intermediate state of the interaction field, gives information beyond a one-particle theory interpretation. In the same manner, if we consider more complicated Feynman diagrams corresponding to higher Born approximations, the interacting field will involve the many-particle theory problem. We follow the method developed by Feynman[58] for the treatment of the K-G equation with an interacting field. In the present case, the Born potential is the interacting field. As mentioned before, the higher Born terms involve particles as well as antiparticles in the intermediate states. This treatment avoids the Klein paradox for the K-G equation in asymptotic states. A completely satisfactory expression for the interaction including higher Born terms is not feasible. For processes of increasingly higher orders, the complexity and difficulty increase rapidly, and the method becomes impractical in the present form. It does, however, seem satisfactory to a good approximation, to define the matrix elements of all real processes in the lowest order Born approximation (see Fig. 7). This has the advantage of reducing the many-particle problem to the one-particle problem, so that there is only a positive energy particle moving forward with time, and so that the causality condition of the Green's function is satisfied. It appears that we now have available a method for applying the K-G equation in the π-N scattering, where the interaction is analyzed in terms of invariant amplitudes in Born approximation, although a complete covariant form is not achieved.

3.3. Assumptions for the K-G Equation Approach

Based on the above discussion, we would like to summarize the assumptions leading to our potential approach for the π-N scattering while using the K-G equation.

1) We fix our frame of reference in the C.M. System, where the Born potential derived gives the correct kinematics of the π-N interaction.

2) The dynamic behaviour of the interaction is determined from matrix elements corresponding to the lowest order Born approximation.

3) To the extent that the interaction is static, we sacrifice the Lorentz covariance property*.

4) The whole treatment is not relativistic, but rather "semi-relativistic", in the sense that the energy and momentum of the scattered particle are treated relativistically.

In our approach, we use the Born potential for U(r) in Equ.(3-3), where

$$U(r) = V_{\pi N}(r) + V_{\sigma}(r) + V_{\rho}(r) + V_{f^{\circ}}(r). \qquad (3\text{-}5)$$

---

* As mentioned previously, only the Bethe-Salpeter equation could give an adequate representation of the Lorentz covariance property.

3.4. Why Not the Schrödinger Equation ?

Equ.(3-3) has a form similar to the time independent Schrödinger Equation . As a matter of fact, in the low energy limit, it gives the expression of the Schrödinger equation.

It may be argued that, if the Lorentz covariance condition has been violated in the K-G equation, it would be simpler to use the non-relativistic Schrödinger equation. As we know, the Schrödinger equation is adequate only for low energy problems. In the case of high energy problems, it breaks down. Many people[59] do use the Schrodinger equation for the high energy problem, with a relativistic correction. Instead of using the actual momentum $q = \sqrt{2 M_r E}$, they replace the q with the relativistic momentum $q = \sqrt{\omega^2 - \mu^2}$. Then the Schrödinger equation will have a form equivalent to Equ.(3-3), which is used in our approach, with $U = 2M_r V$, where V is the actual potential of the problem with dimensions of energy, and M_r is the reduced mass of the system. As Goldberger and Watson[60] observe, "Actually the difference between the K-G equation and the Schrödinger equation (except for a profound difference in principle) is not so great under any circumstances". It is curious to note that, the two equations begin with two quite different mathematical formulations. Their apparently dissimilar approaches, under certain circumstances, are shown to be mathematically equivalent.

CHAPTER IV

Phase Shift Solutions and Numerical Computations

4.1. Phase Shift Calculations

Since $U(r)$ is a symmetrical potential, the wave function Φ is separable.

This leads to the radial equation

$$\frac{d^2u_\ell}{dr^2} + \left[q^2 - \frac{\ell(\ell+1)}{r^2} - U(r)\right] u_\ell(r) = 0 \qquad (4\text{-}1)$$

Note that $\Phi(o)$ must be finite, Equ. (4-2) implies the boundary condition

$$u_\ell(o) = 0$$

for the radial wave function $u_\ell(r)$.

The phase shifts are found by matching the radialwave functions, the spherical Bessel functions $j_\ell(qr)$ and spherical Neumann functions $\eta_\ell(qr)$ in the asymptotic region. Thus the phase shifts are given by

$$\delta_\ell = \tan^{-1} \lim_{R \to \infty} \frac{\left[1-R\,\frac{u'_\ell(R)}{u_\ell(R)}\right] j_\ell(qR) + qR\, j'_\ell(qR)}{\left[1-R\,\frac{u'_\ell(R)}{u_\ell(R)}\right] \eta_\ell(qR) + qR\, \eta'_\ell(qR)} \qquad (4\text{-}2)$$

For elastic scattering δ_ℓ is real. A repulsive potential gives rise to negative phase shifts. On the other hand, an attractive potential yields positive phase shifts.

4.2 Numerical Computation

Since Equ. (4-1) cannot be solved in closed form with our potential, it has been customary to employ numerical calculation techniques. By using the conventional series expansion method[61], we will be able to obtain a good approximation for the initial value of $u_\ell(r_o)$, when r_o is sufficiently small, e.g. $r_o = 0.01$. Similarly we can also obtain the first and second derivatives of u_ℓ, i.e., $u_\ell'(r_o)$ and $u_\ell''(r_o)$ respectively.

With $u_\ell(r_o)$, $u_\ell'(r_o)$ and $u_\ell''(r_o)$ as initial values, we can solve Equ. (4-1) numerically by employing the Runge-Kutta-Nystrom method[62]. The logarithmic derivative is calculated up to the asymptotic region (i.e. $R \geqslant 4.5f$). The initial step size $h = 0.01$ is used. For sufficiently large r we change the step size to nh where n = 2, 3, 4 ... . At the beginning of each new step size we first calculate the phase shifts, and then compare them with the phase shifts calculated immediately before. When the two agree to within an acceptable amount we proceed with the calculations, otherwise we repeat the old step size for a greater range until we can change the step size. In this manner, we carry on our calculation up to r = 5f. In this region, the potential has no further effect on our phase shifts.

Very intensive computer work is involved, both time consuming and laborious. The actual calculation was performed by the McGill IBM 360/75 computer. We tried to fit the CERN phase shifts up to 700 MEV, for S-, P- and D- waves. Masses of nucleon and bosons are taken in terms of pion mass (140 MEV) as follows:

nucleon mass $M = 6.7$

vector meson mass $m_\rho = 5.5$

tensor meson mass $m_f = 9$

scalar meson mass $m_\sigma = 3.$

The search for a fit is performed and is subjected to the following restrictions:

i) The pseudoscalar π-N coupling constant is given a value between $14 \sim 15$.

ii) The vector meson ρ coupling constant must be bounded within the range between $2 \sim 3$, and we fix the value of $(\frac{f}{g})_\rho = -1.83$ in accord with Ball and Wong[44].

iii) The P_{33} phase shift has to pass through 90° at about 194 MEV.

iv) The contribution from $0 \lesssim G_P \lesssim 1$ does not affect the $J = \frac{1}{2}$ state P-wave significantly. We set $G_P = 0$ in order to eliminate one parameter.

v) The 'cut off' mass is subjected to a lower bound limit $m_c > m_f$ according to Chapter II, Section 2.5.

Under these conditions the phase shifts are expressed as a function of only three freely adjustable parameters (i.e. m_σ, G_σ and G_f) and one partially adjustable parameter (i.e. $m_c > m_f$).

It is found that if the P_{33} resonance is fitted, the potential is usually too strongly attractive to produce negative P_{31} phase shifts. The P_{13} phase shifts also have a tendency to be more positive as the energy increases. (A more elaborate discussion regarding this point will be given in the next chapter.) We abandon hope of a reasonable fit for the P_{31} phase shift if we insist on having a (3,3) resonance at the correct energy, but try to improve the P_{13} phase shift especially in the high energy region. The inclusion of a scalar meson in our OBE potential undoubtedly enhances the attractive potential in all channels. To improve the negative phase shifts of S_{31}, P_{31}, P_{13} and D_{35} we discard the scalar meson in our model. Furthermore, we eliminate two freely adjustable parameters m_σ and G_σ .

4.3 Computational Techniques

Much effort has been devoted in computational work to incorporate the requirement of a (3,3) resonance with the prediction of the overall qualitative and quantitative features of the S-, P- and D-wave phase shifts.

In this paragraph we would like to illustrate the procedure used to obtain the sets of parameters with a P_{33} resonance at the correct energy. We adopt a systematic way of searching.

Although we expect finally to obtain parameter values in agreement with those given in conditions (i) and (ii) of Section 4.2 , we start calculations with small values of parameters other than m_c (fixing $m_c > m_f$) and increase them one at a time to see the effects on the S-, P- and D-wave phase shifts and to ensure that no bound states of the system have been reached. After these checks have been completed, we pass on to the consideration of the P_{33} resonance. Now we choose a pair of values for $G_{\pi N}$ and G_ρ which are within the critical ranges (given in conditions (i) and (ii) of Section 4.2) and find values of m_c ($> m_f$) which will give the P_{33} resonance at the correct energy for given values of G_f. It should be remarked that for some values of G_f it is not possible to find any m_c to satisfy the constraints . We then repeat the procedure by choosing another pair of values for $G_{\pi N}$ and G_ρ .

Several sets of parameters which satisfy conditions (i), (ii), (iii) and (v) in Section 4.2 are given in Table III.

TABLE III

Values of m_c with fixed values of $G_{\pi N}$ and G_ρ

$G_{\pi N} = 14.0$					
G_f \ G_ρ	2.0	2.2	2.5	2.8	3.0
12				10.862	10.845
13	10.935	10.920	10.895	10.872	10.858
14	10.930	10.918	10.895	10.875	10.861
15	10.927	10.914	10.898	10.876	10.862
16	10.918	10.906	10.890		
17	10.908	10.898	10.882		

	$G_{\pi N} = 14.4$			$G_{\pi N} = 15.0$		
G_f \ G_ρ	2.0	2.2	2.5	2.0	2.2	2.5
11	10.875					
13	10.893					
14	10.893	10.880	10.860	10.838	10.825	10.805
15	10.890	10.878	10.858	10.838	10.828	10.808
16	10.885	10.873	10.855	10.834	10.825	10.808
17	10.878	10.867	10.850	10.830	10.820	10.805

	$G_\rho = 2.0$					
G_f \ $G_{\pi N}$	14.0	14.2	14.4	14.6	14.8	15.0
15	10.927	10.908	10.890	10.872	10.858	10.838
17	10.908	10.893	10.878	10.860	10.846	10.830

CHAPTER V

Results, Discussion and Conclusion

5.1. Results

As seen from Table I, within an energy range of 0 to 700 Mev, there are resonances conjectured from the phase shift analyses - a P_{33} at 194 Mev (1236 Mev*), a P_{11} at 532 Mev(1470 Mev) and a D_{13} at 611 Mev (1520 Mev). In order to study the effects on the P_{33}, P_{11} and D_{13} resonances due to variations of G_f and the corresponding m_c, we choose a random $G_{\pi N}$ - G_ρ pair, say $G_{\pi N}$ = 14.4, G_ρ = 2.0 from Table III. With these values fixed, we look at the behaviour of the phase shifts for different G_f and m_c. In Table IV, three values of G_f (with positions of corresponding resonances) are given.

Table IV

G_f	m_c	P_{33}	P_{11}	D_{13}
13	10.893	194	600	616
15	10.890	194	500	512
17	10.878	194	432	362

* The energies in the brackets refer to the total energies in the C.M. system.

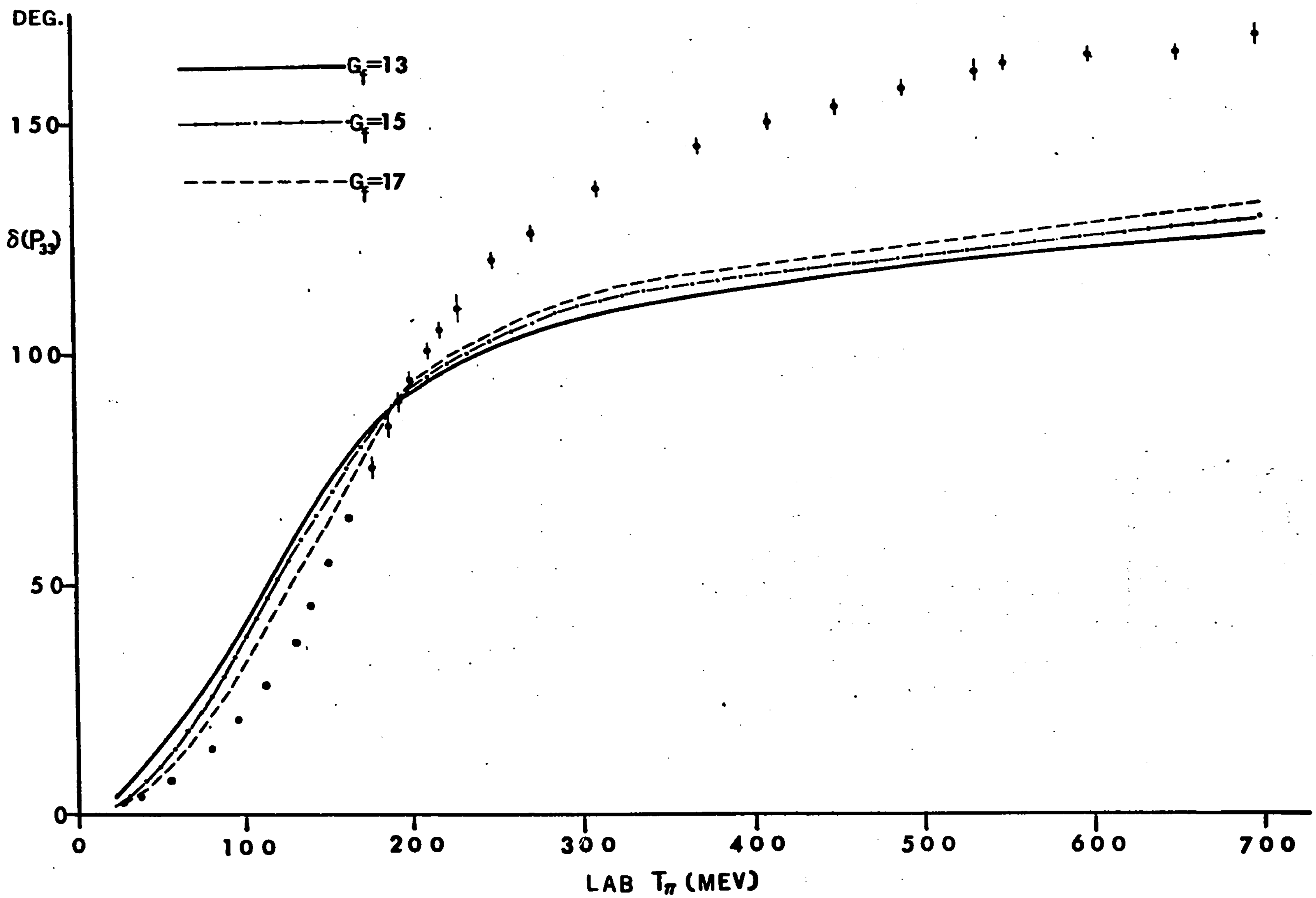
DEG.
150
100
50
0
δ(P33)
G_f=13
G_f=15
G_f=17
0
100
200
300
400
500
600
700
LAB Tπ (MEV)

FIG. 10

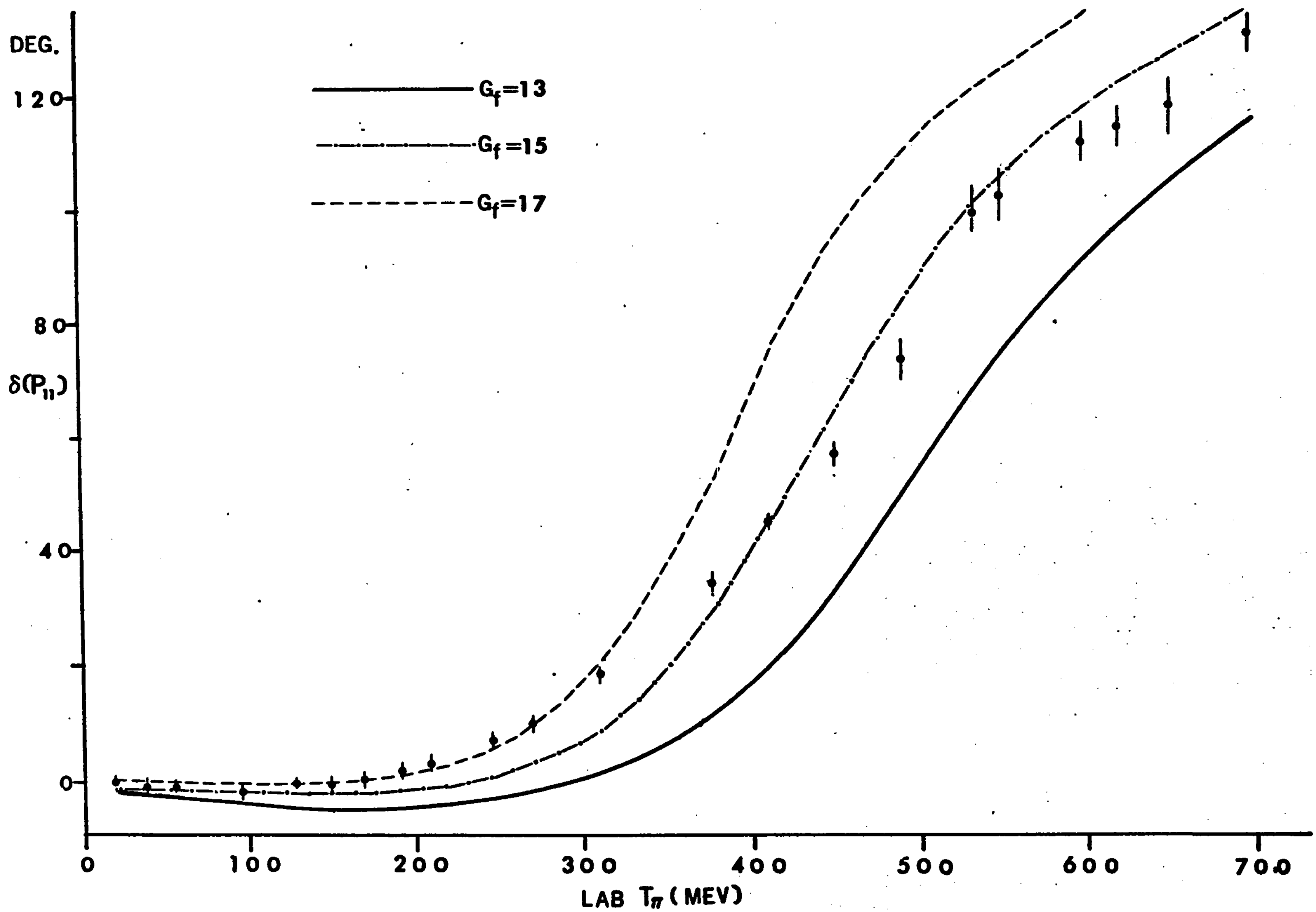
DEG.
120
80
40
0
δ(P11)
Gf=13
Gf=15
Gf=17
0
100
200
300
400
500
600
700
LAB Tπ (MEV)

FIG. 11

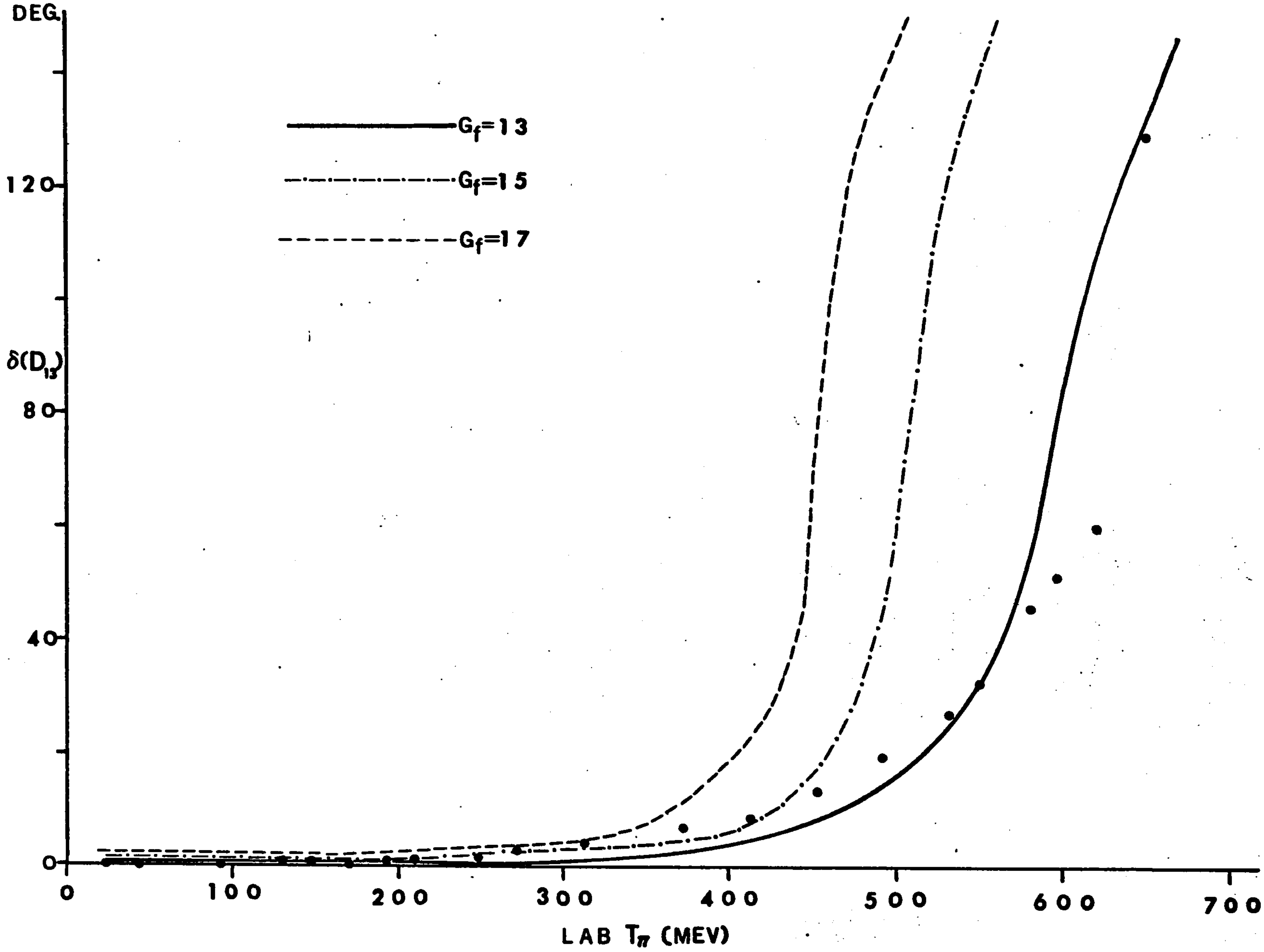
DEG.
120
δ(D13)
80
40
0
100
200
300
400
500
600
700
Gf=13
Gf=15
Gf=17
LAB Tπ (MEV)

FIG. 12

A plot for P_{33}, P_{11} and D_{13} phase shifts against the energy using the parameters given in Table IV are presented in Figs. 10,11 and 12 respectively.

As shown in Fig. 10, the smaller G_f value will give larger positive P_{33} phase shifts in the low energy region. The situation is reversed after passing the resonance. A larger G_f value will give a better fit for the P_{33} phase shifts. However, the improvement is not marked in the high energy region.

In the case of the P_{11} channel, as seen from Fig. 11, the change of the G_f value affects the P_{11} phase shifts considerably. The larger the G_f value, the lower the energy at which the P_{11} resonance occurs (see Table IV). At the same time, the P_{11} phase shifts in the low energy region tend to have smaller negative values initially and the phase shifts become positive at a lower energy. The best fit for the P_{11} state is when $G_f = 15$ and $m_c = 10.890$.

The effect of G_f on the D_{13} channel is the same as that in the P_{11} case. As shown in Fig. 12 the larger the G_f the more the deviation from the CERN phases. This is due to the very strong attractive contribution on the D_{13} state from the f^o-exchange. The D_{13} resonances for different values of G_f are given in Table IV.

Among the sets of parameters appearing in Table IV, we select the 'best' sets and list them in Table V together with the energies of the P_{33}, P_{11} and D_{13} resonances. These 'best' sets of parameters give a reasonable fit to the phase shifts in the maximum number of channels and reproduce the principle features. The most reasonable fit in Table V is found for the parameters of set I. The results for this set are presented in Figs. 13 to 22. There are three resonances $P_{33}(194)$, $P_{11}(600)$ and $D_{13}(616)$ which agree very well with the CERN and other phase shift analyses (see Table I).

Table V

Set	$G_{\pi N}$	G_ρ	G_f	m_c	P_{33}	P_{11}	D_{13}
I	14.0	3.0	12	10.845	194	600	616
II	14.0	2.8	12	10.862	194	606	614
III	14.0	2.5	13	10.895	194	532	560
IV	14.0	2.2	13	10.920	194	534	572
V	14.0	2.0	13	10.935	194	536	573
VI	14.4	2.5	15	10.858	194	488	502
VII	14.4	2.2	15	10.878	194	493	510
VIII	14.4	2.0	15	10.890	194	500	512

The eight sets of parameters, as shown in Table V, give more or less the same results. A few remarks concerning the small deviation among these eight sets of parameters are worth mentioning. The most significant feature of increasing $G_{\pi N}$ is to improve the fit of P_{13}, D_{35} and D_{15}, but destroy the fit of S_{11} and S_{31}. The increase of G_ρ will cause the most effective attractive potential in the $J = \ell - \frac{1}{2}$, $T = \frac{1}{2}$ states; a small repulsion in $J = \ell + \frac{1}{2}$, $T = \frac{1}{2}$ states; a greater repulsion in the $J = \ell - \frac{1}{2}$, $T = \frac{3}{2}$

states; and a slightly attractive potential in the $J = \ell + \frac{1}{2}$, $T = \frac{3}{2}$ states. In the case of f^o-exchange, since it has no isospin (I=0), both $T = \frac{1}{2}$ and $T = \frac{3}{2}$ isospin states receive repulsive contributions from the contact interaction term and attractive contribution from the Yukawa term. The attractive Yukawa contribution is stronger than the repulsive contact interaction contribution, therefore the S-wave receives attractive contribution from the f^o-meson weaker in low energy region, stronger as energy increases. For the P- and D-waves, the $\vec{L}\cdot\vec{S}$ term also contributes in addition to the contact interaction and the Yukawa terms. The $\vec{L}\cdot\vec{S}$ term gives rise to a strong repulsion for the $J = \ell + \frac{1}{2}$ states and a stronger attraction to the $J = \ell - \frac{1}{2}$ states. The total contribution to the $J = \ell + \frac{1}{2}$ states is moderately attractive in the low energy region and becomes more strongly attractive as energy increases. However, for the $J = \ell - \frac{1}{2}$ states the very strong attraction dominates, especially in the high energy region. One can retrieve the experimental features such as the resonances of P_{33} P_{11} and D_{13} by involving a large value of G_f. Nevertheless, in order to maintain a reasonable fit for the negative phase shifts P_{31}, P_{13} and D_{35} and the smaller positive phase shifts for D_{33} and D_{15}, the value of G_f is not allowed to be too large.

As shown in Talbe V, one can obtain better resonances for P_{11} and D_{13} from Sets IV and V. However, in Sets IV and V the phase shifts of P_{13} and D_{35} tend to be more positive as the energy increases. The parameters in Set I, on the whole provide a 'best' fit compared to the rest of Table V. (In view of demands on computer time a least square fit is not attempted.)

5.2. Discussion

The model developed in Chapters II, III and IV has shown a satisfactory fit to the CERN phase shifts except for the discrepancies appearing in a few channels. This section is devoted to a detailed examination of the results appearing in Figs. 13 to 22 pertaining to the parameters of Set I in Table V. Some of the results that we obtain have a much deeper significance than first meets the eye. The various aspects of these phase shifts that deserve special emphasis will now be discussed.

1. Results for S-wave

S_{31}:

Examination of Fig. 13 reveals that our fit for the S_{31} phase shifts has a larger negative value on the whole. As the energy increases, the S_{31} phase shift tends to approach the CERN value. The reason for this discrepancy can be explained as follows: the strong repulsion resulting from the 'nucleon-exchange'* gives very large negative phase shifts especially in the low energy region. As the energy increases, the attractive contribution from the f^{o}-exchange becomes evident and depresses the repulsion due to the 'nucleon-exchange' in the high energy region. This is why the S_{31} phase shift

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* The term 'nucleon-exchange' here refers to the direct and exchange pole terms for π-N scattering developed in Section 2.3.

decreases in the negative sense, at high energy region. The inclusion of a scalar meson, which gives strong attraction in low energy region, will be able to decrease the strongly repuslive effect at that region, and result in a better fit for the S_{31} phase shifts.

S_{11}:

In the case of the S_{11} phase shift as shown in Fig. 14, the fit is not satisfactory. Instead of having a plateau in the low energy region (< 300 MEV), the S_{11} phase shift increases monotonically and almost linearly with the energy. On the whole, the attractive contribution is not strong enough to give larger S_{11} phase shifts. The defect comes mainly from the strongly repuslive 'contact interaction'+ term of the 'nucleon-exchange'. If the value of $G_{\pi N}$ is allowed to decrease below the accepted value, and at the same time a scalar meson exchange is taken into account to provide a strongly attractive contribution especially in the low energy region, a better fit for the S_{11} phase shift can be achieved.

2. Results for the P-wave

P_{33}:

In the case of the P_{33} state, both 'nucleon-exchange' and ρ-exchange give attractive contribution. However, without the inclusion of the f^o-exchange, the attraction is not strong enough to produce a (3,3) resonance at the energy of 194 MEV. The P_{33} is dominated by the strong attractive force due to f^o-exchange. A look at Fig. 15,

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+ The 'contact interaction' term refers to the modified $\delta^3(\vec{r})$ term, i.e., $\frac{m_c^2}{4\pi} \frac{e^{-m_c r}}{r}$.

reveals that on one hand, in the low energy region (< 194 MEV), the attractive force is too strong, and on the other hand, in the high energy region (> 300 MEV), the attraction is not strong enough to give the required large positive phase shifts. By increasing the coupling constant G_f, we are able to improve the fit of the P_{33} phase shifts in the low energy region (see Fig. 10). However, in the high energy region the improvement is not marked.

P_{31}:

The most noticeable violation occurs in the fit of the P_{31} phase shift. We are obviously faced with a number of glaring disagreements. The phase-shift analysis by CERN shows that in the P_{31} channel the potential must be moderately repulsive, which makes it possible to produce negative phase shifts. The enormous positive P_{31} phase shift appearing in Fig. 16 reflects the presence of a strong attractive potential. Careful study has been given to this channel, in connection with the contribution of each exchange particle to the potential, in order to analyze the discrepancy. In the 'nucleon-exchange' contribution; the 'contact interaction' term gives a strong attractive force; the Yukawa interaction provides a moderate attraction; while the $\vec{L}\cdot\vec{S}$ force produces a strong repulsion and exhibits an even stronger repulsive force as the energy increases. Consequently, the 'nucleon-exchange' term contributes an appropriate repulsive force necessary for this channel. For the ρ-exchange, the potential receives repulsive contributions from both vector and tensor terms, but it is not as strong as the 'nucleon-exchange' contribution. However, in the case of the f^o-exchange, the repulsive contribution from the 'contact interaction' term is not strong enough to outweigh the very strong attractive force due to both the Yukawa and $\vec{L}\cdot\vec{S}$ terms. The effect of the tensor-meson is to produce such a strong attractive force, that it suppresses all the repulsive effects from other contributions. Needless to say, this results in the large positive P_{31} phase shift. The P_{31} phase shift can be improved considerably provided the f^o-exchange is not taken into account in this channel. Otherwise a very strong repulsive core is recommended to be added to the P_{31} channel to overcome the strongly attractive effect due to the f^o-exchange.

P_{13}:

As shown in Fig. 17 the fit of the P_{13} phase shift is reasonably good up to 200 MEV. As the energy increases the phase shift deviates markedly from that of the CERN results and tends to be more positive. The disparity is mainly due to the strong attractive force contributed from the f^o-exchange. The repulsive contributions from both the 'nucleon-exchange' and ρ-exchange are incapable of cancelling this attractive effect in the high energy region (> 300 MEV).

P_{11}:

The most successful fit in the model is the P_{11} phase shift. Most theoretical models of π-N scattering fail to produce the correct resonance feature in this channel. As shown in Fig. 18 the P_{11} fit is not only good qualitatively but alse quantitatively. The P_{11} phase shift remains small and negative in low energy region, and has a shallow dip at about 150 MEV at a slightly higher energy than that of CERN (i.e. 100 MEV). It becomes positive beyond 250 MEV. The small negative phase shift region is slightly larger than the one from CERN. As the energy increases the phase shift increases swiftly. Eventually, a resonance at approximately 600 MEV is achieved. The CERN P_{11} resonance is 530 MEV which is slightly less than the Roper[9] P_{11} resonance (580 MEV). The P_{11} resonance in our fit agrees very well with both of them, but is near to the Roper one. The P_{11} phase shift can be improved considerably by increasing the value of G_f from 12 to 13 or 14 (e.g. see Fig. 11). By so doing, however, we shall cause more deviation for the P_{13} and D_{35} phase shifts from those of the CERN. The main contribution to the very strong attractive force especially in the high energy region comes from the f^o-exchange. The attractive effects from both 'nucleon-exchange' and ρ-exchange are insufficient to produce a P_{11} resonance at the correct energy without the inclusion of the f^o-meson. To this end, the f^o-exchange is quite necessary in the P_{11} channel. A point to be stressed is that, according to the CERN and other phase shift analyses, the P_{11} state is highly absorptive above 300 MEV. In our mode, no absorption is taken into account. Nevertheless, a P_{11} resonance at 600 MEV is reproduced.

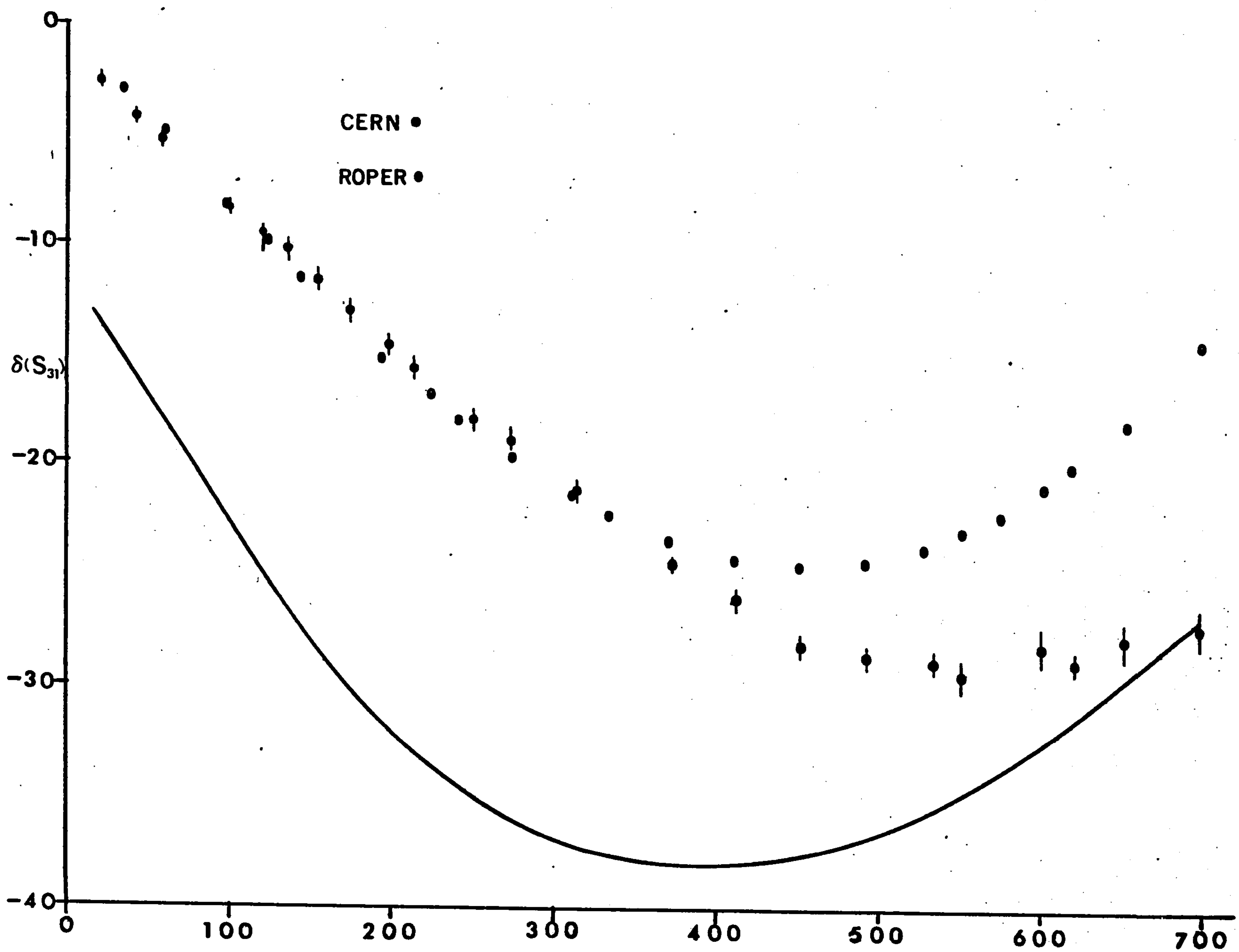
CERN
ROPER
δ(S31)
0
-10
-20
-30
-40
0
100
200
300
400
500
600
700
LAB Tπ (MEV)

FIG. 13

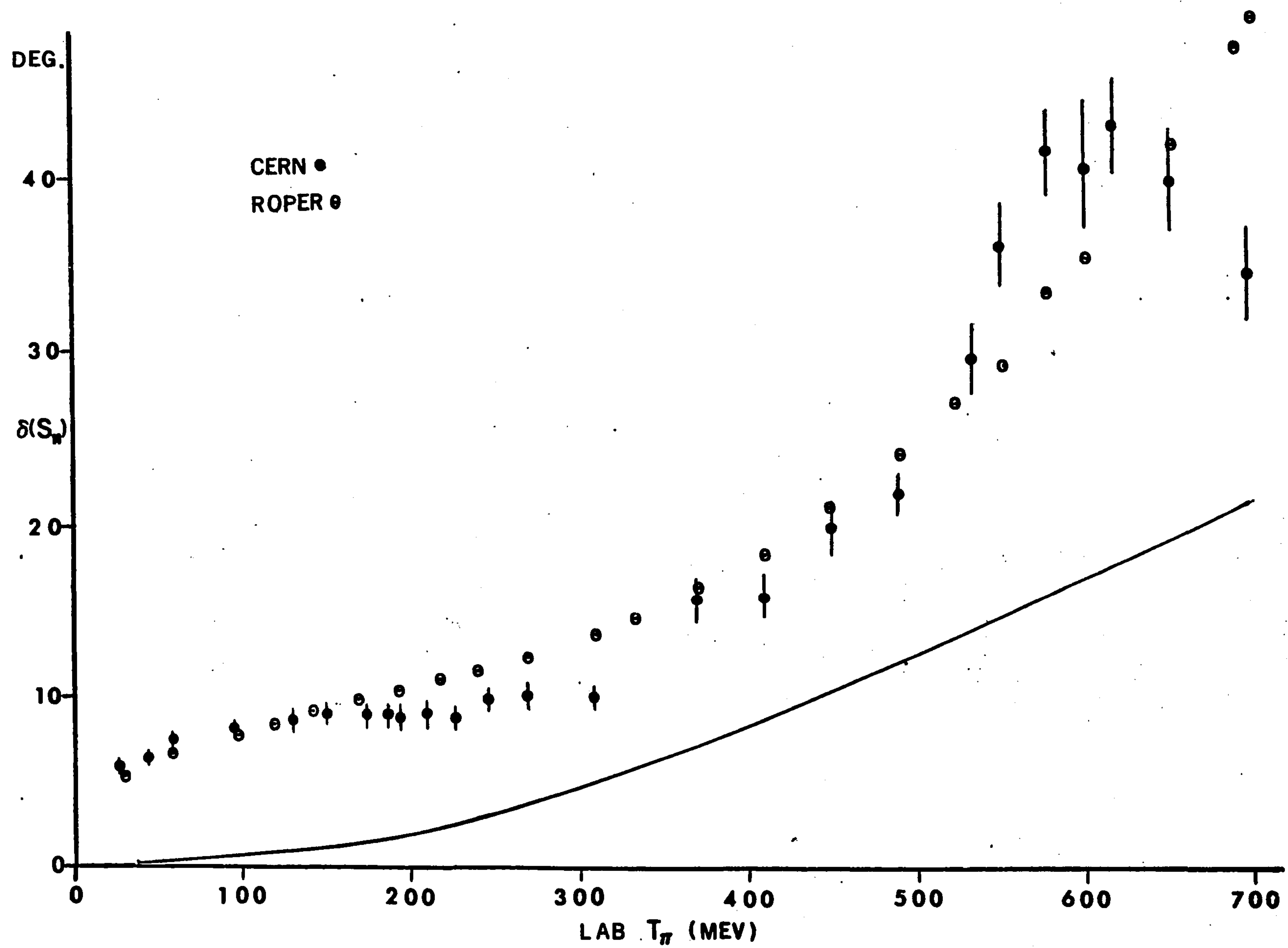
DEG.
40
30
20
10
0
δ(S_π)
CERN ●
ROPER ⊖
0
100
200
300
400
500
600
700
LAB T_π (MEV)

FIG. 14

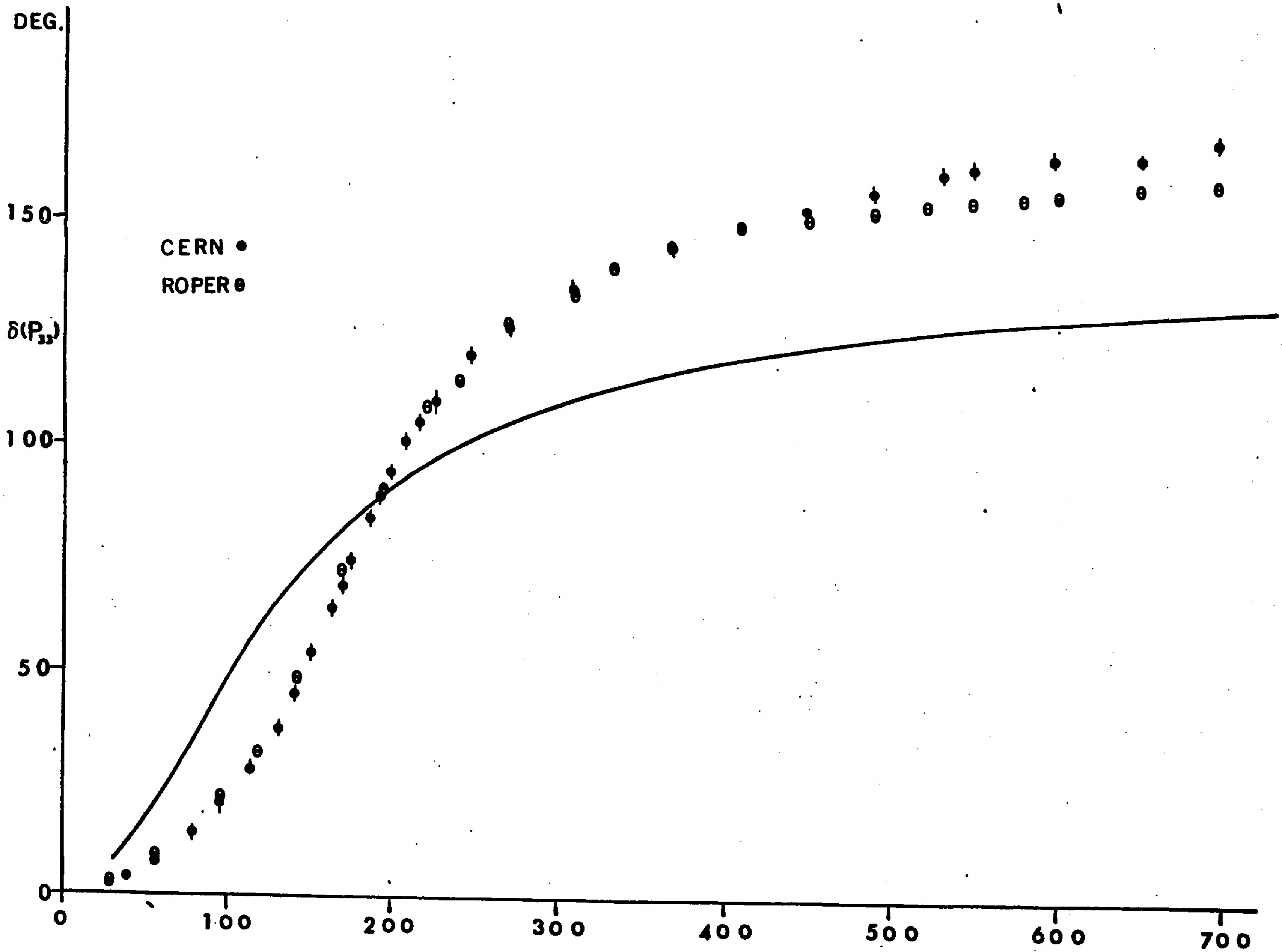
DEG.
150
100
50
0
δ(P33)
CERN ●
ROPER θ
0
100
200
300
400
500
600
700
LAB Tπ (MEV)

FIG. 15

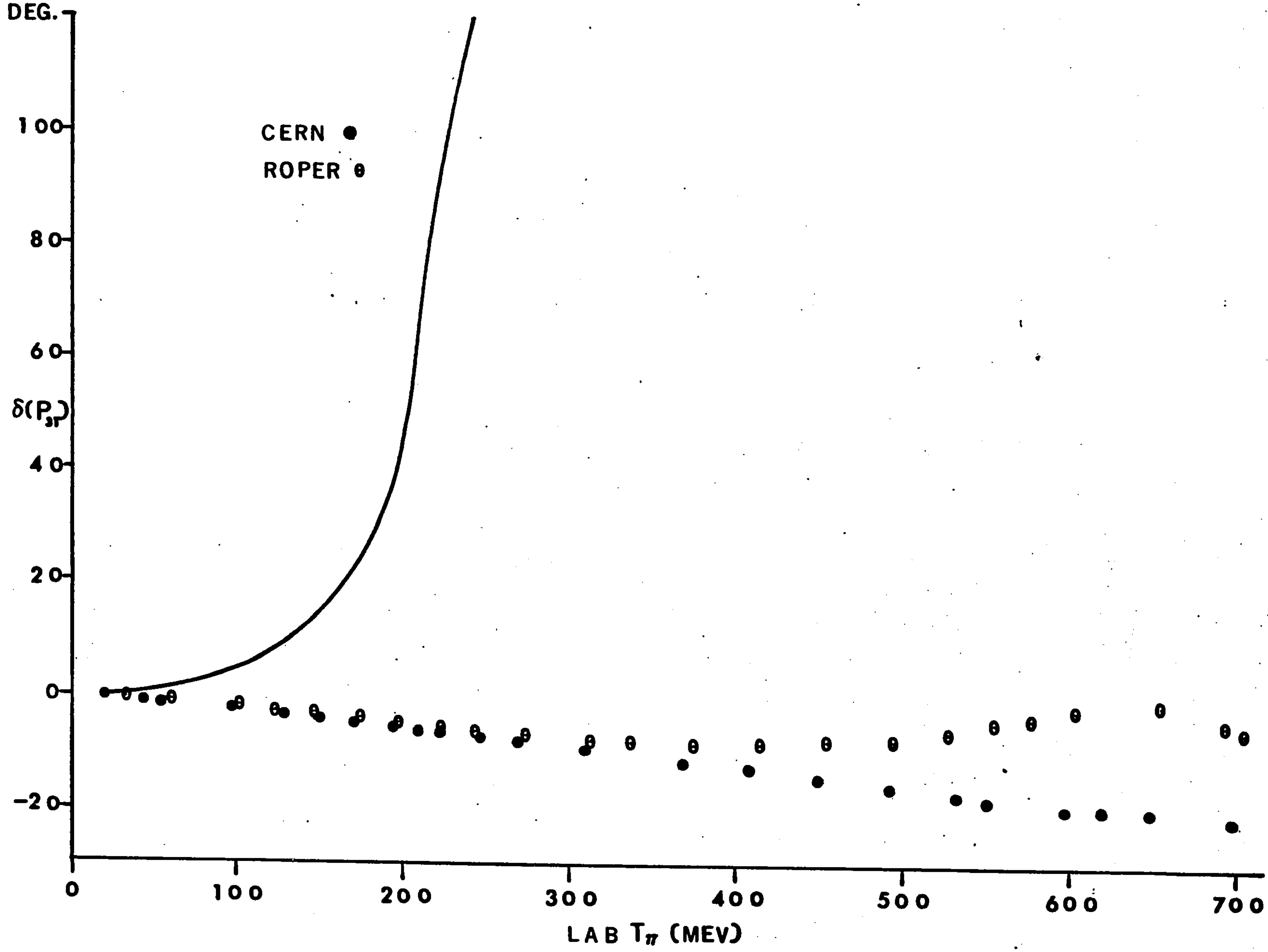
DEG.
100
80
60
δ(P31)
40
20
0
-20
CERN
ROPER
0
100
200
300
400
500
600
700
LAB Tπ (MEV)

FIG. 16

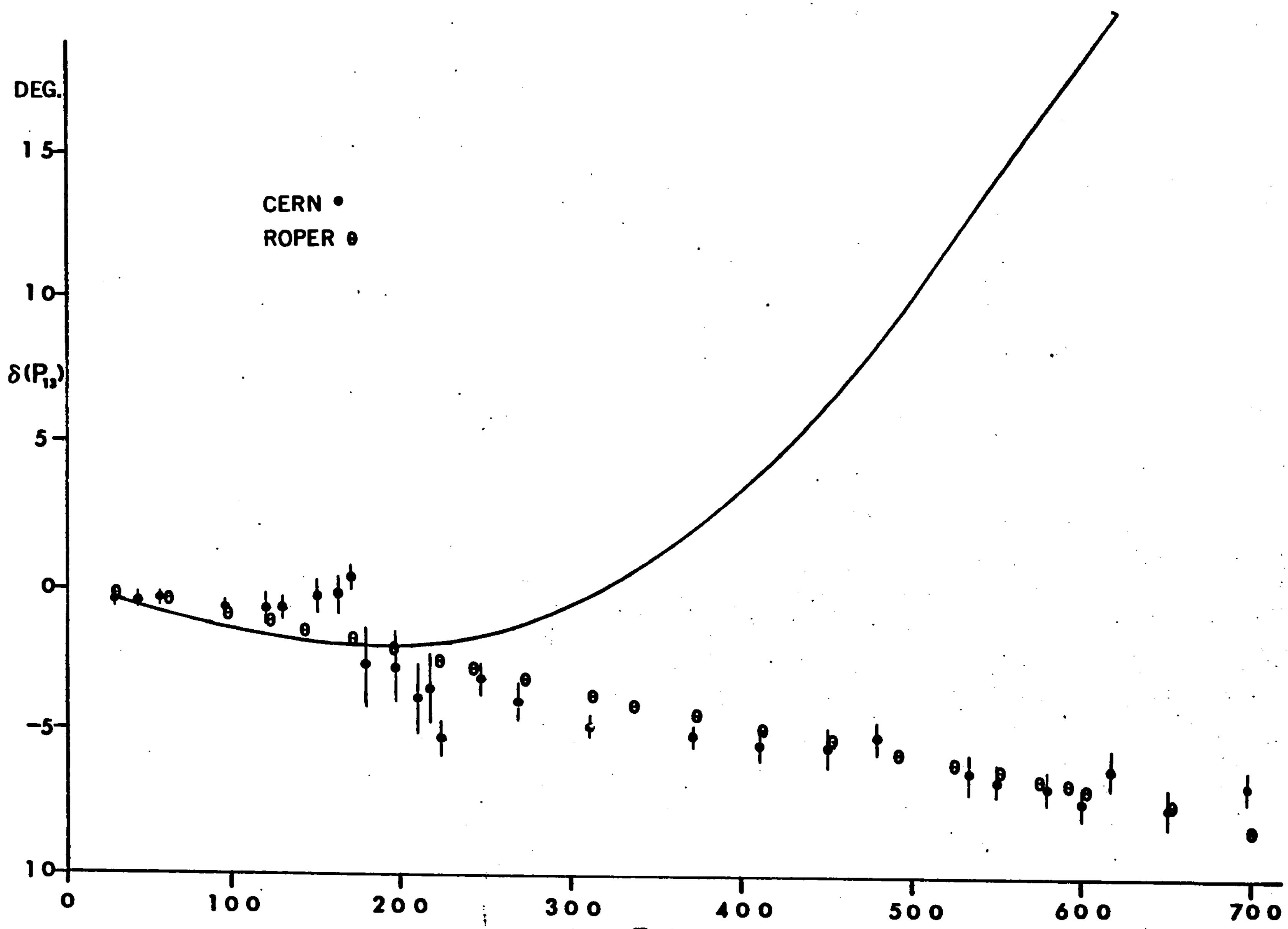
DEG.
15
10
5
0
-5
10
δ(P13)
CERN •
ROPER θ
0
100
200
300
400
500
600
700
LAB Tπ (MEV)

FIG. 17

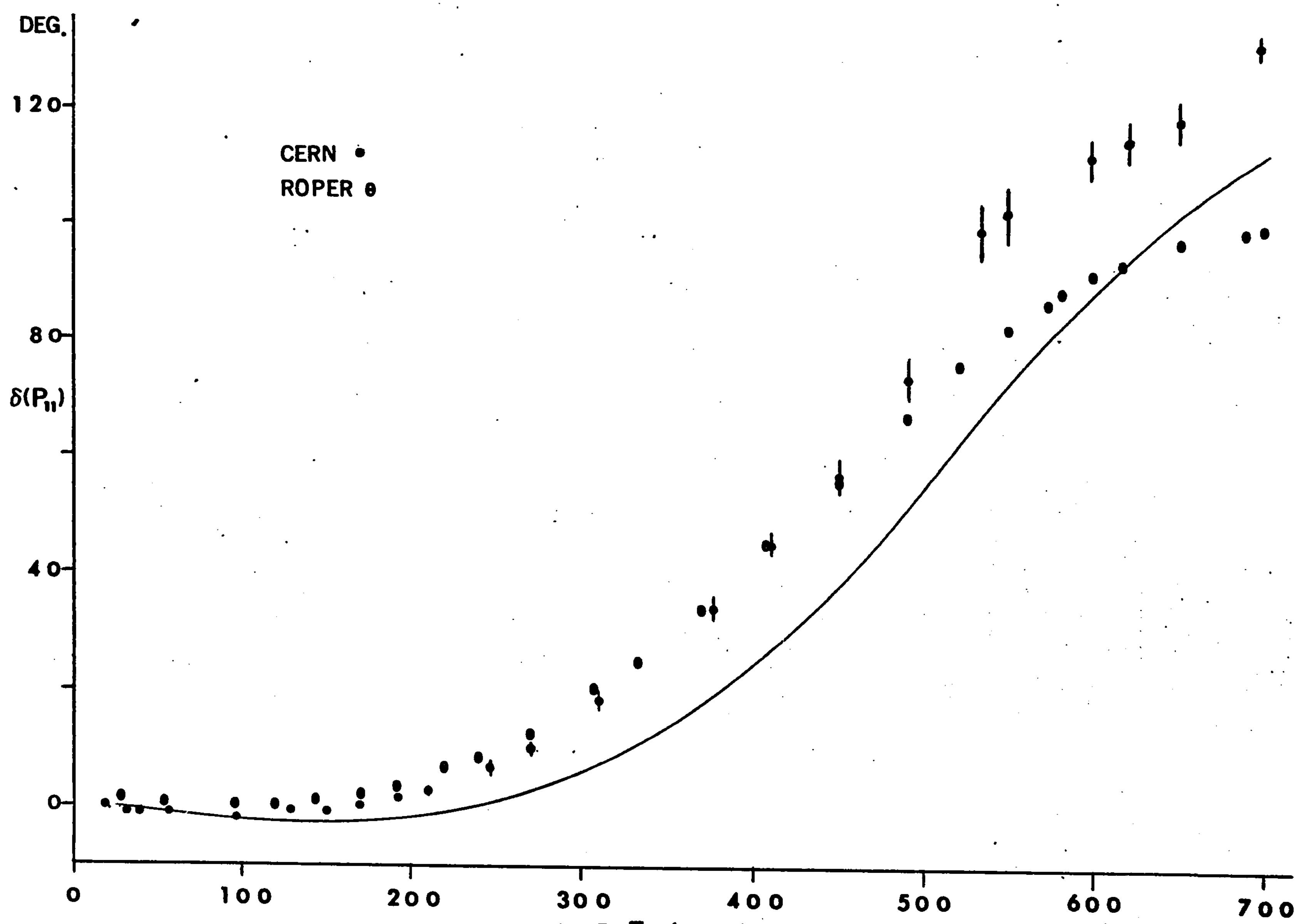
DEG.
120
80
$\delta(P_{11})$
40
0
CERN
ROPER
0
100
200
300
400
500
600
700
LAB T_π (MEV)

FIG. 18

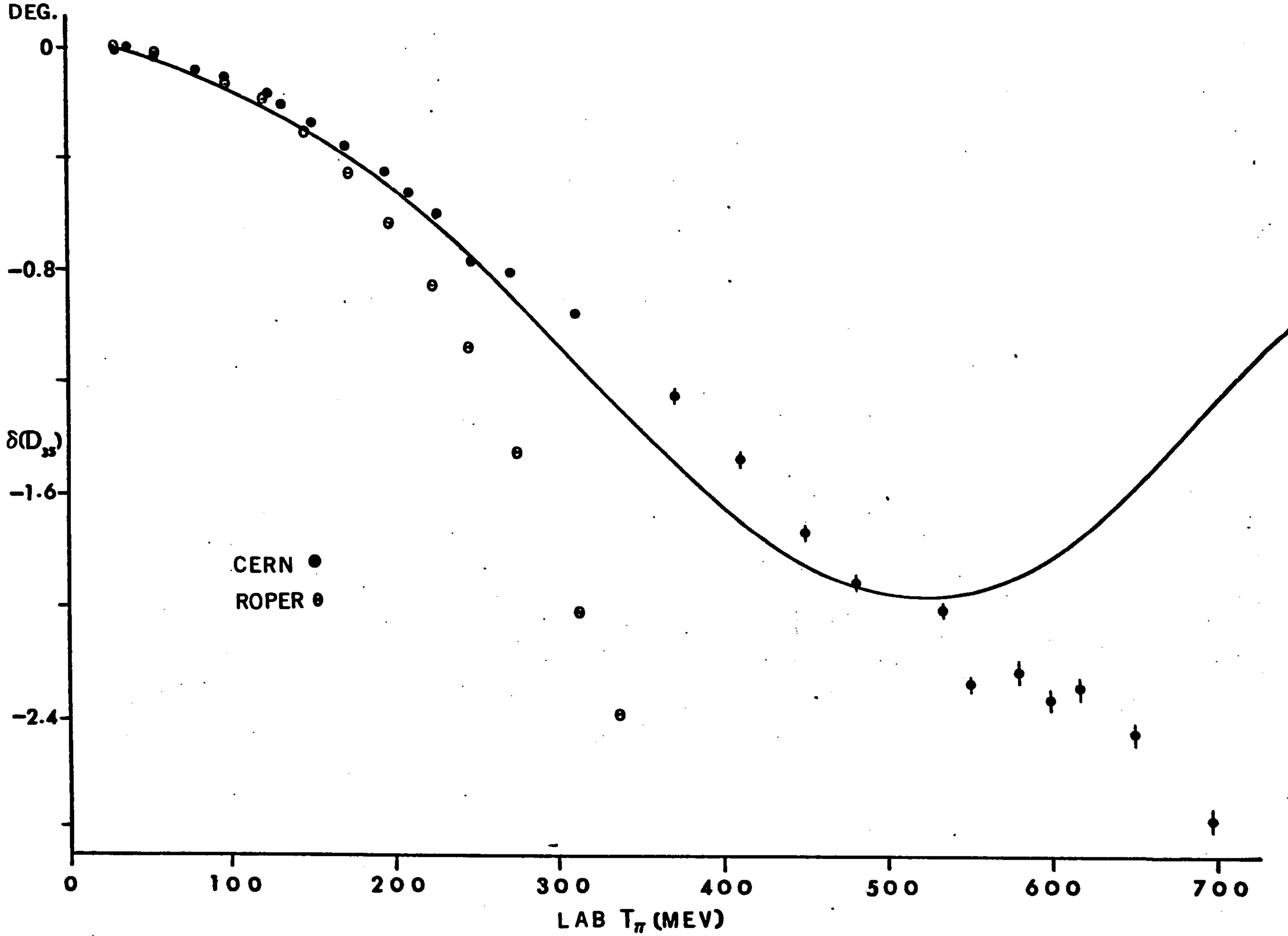
DEG.
0
-0.8
$\delta(D_{35})$
-1.6
-2.4
CERN ●
ROPER θ
0
100
200
300
400
500
600
700
LAB T_π (MEV)

FIG. 19

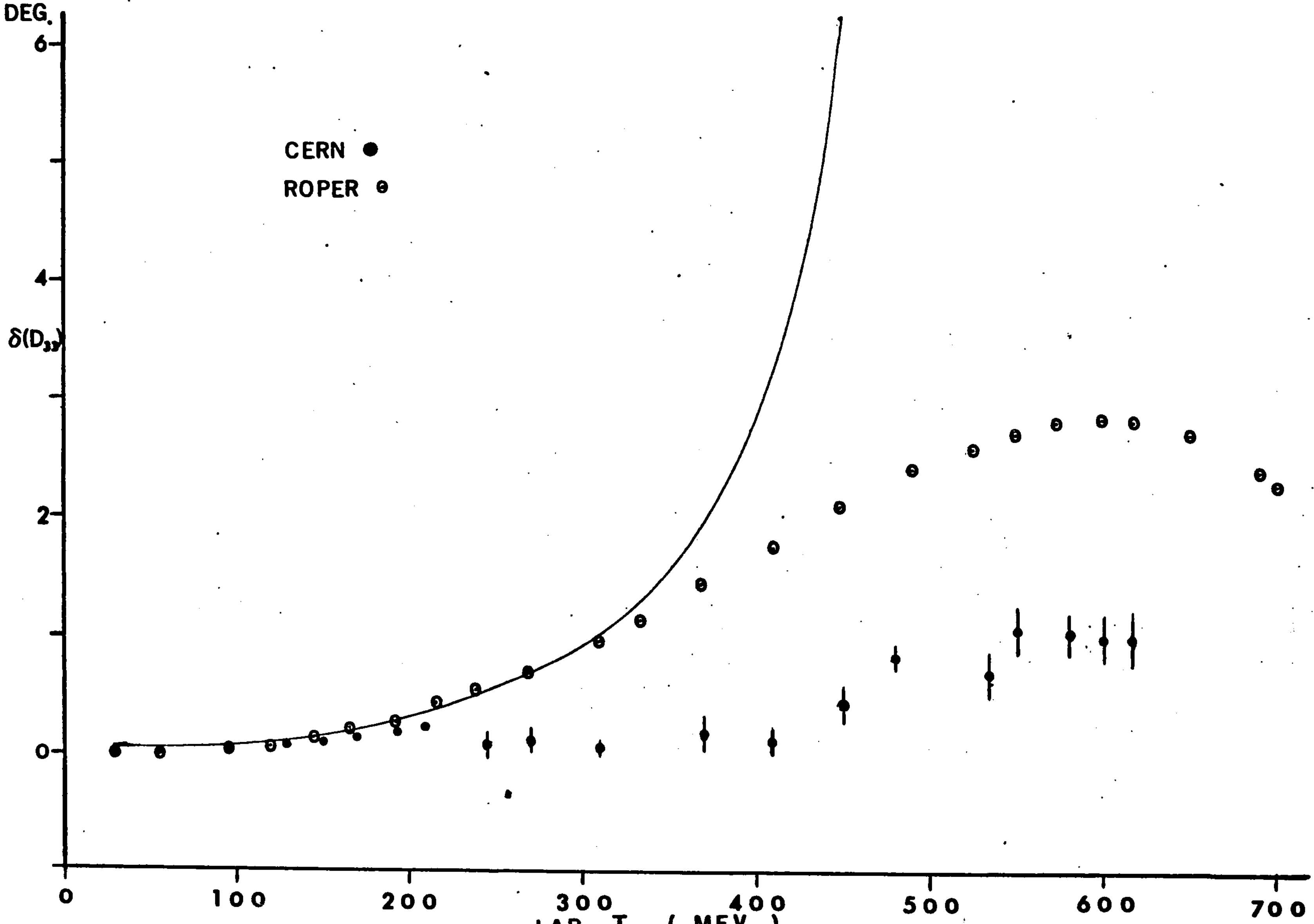
DEG.
6
4
2
0
δ(D33)
CERN
ROPER
0
100
200
300
400
500
600
700
LAB Tπ (MEV)

FIG. 20

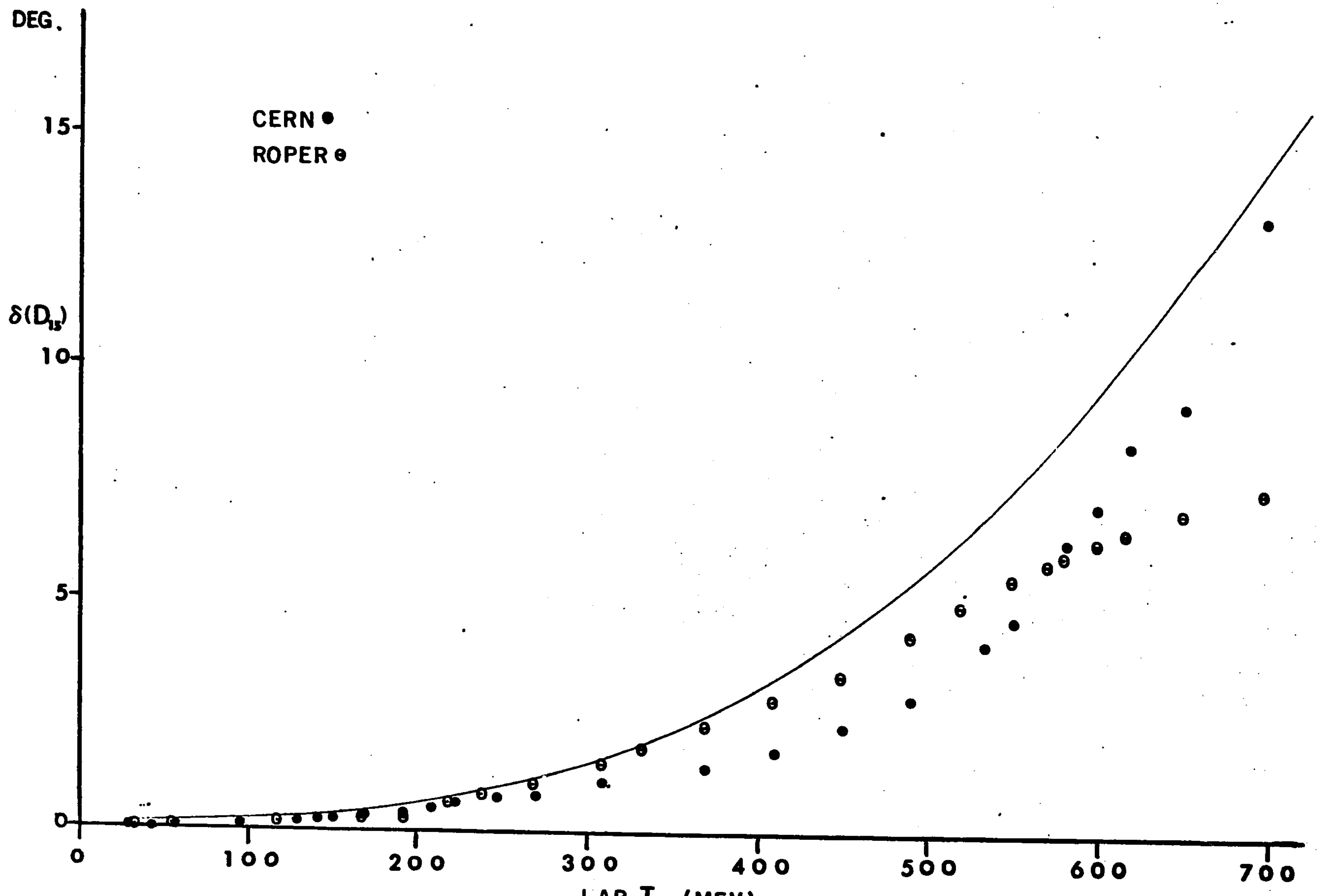
DEG.
$\delta(D_{15})$
15
10
5
0
CERN ●
ROPER ⊖
0
100
200
300
400
500
600
700
LAB T_π (MEV)

FIG. 21

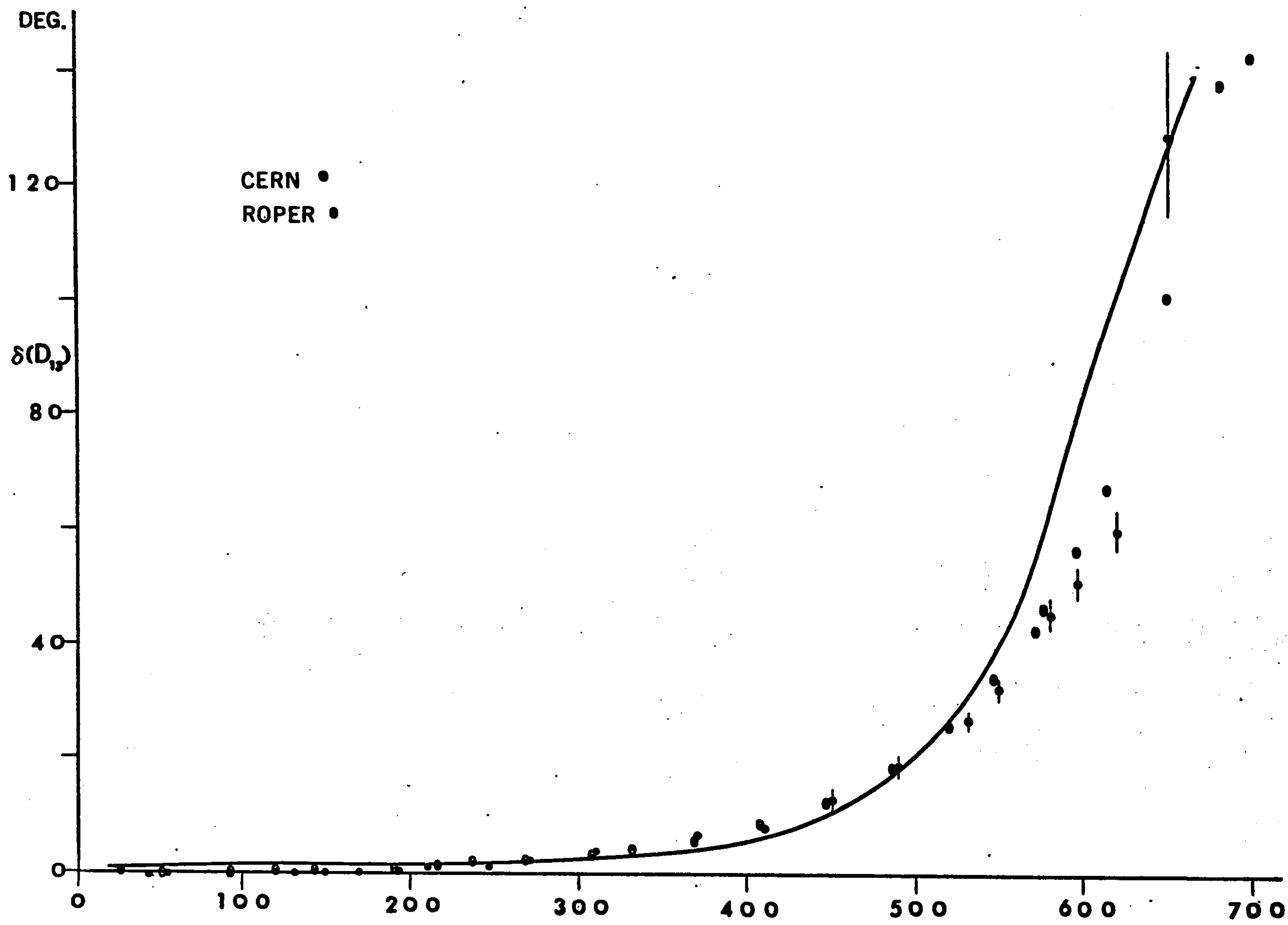
DEG.
120
$\delta(D_{13})$
80
40
0
CERN
ROPER
0
100
200
300
400
500
600
700
LAB T_π (MEV)

FIG. 22

3. Results for D-wave

D_{35}:

As seen from Fig. 19, the fit of D_{35} is reasonably good up to 450 MEV. However, we are unable to obtain a larger negative phase shift beyond this region, where the phase shift tends to diminish in a negative sense as the energy increases. The disagreement in the high energy region (>450 MEV) can be explained by the strongly attractive contribution due to the f^o-exchange. The D_{35} phase shift can be improved considerably by reducing the value of G_f. However, a smaller value of G_f is incapable of reproducing the resonance phenomena for P_{33}, P_{11} and D_{13} at the correct energies.

D_{33}:

As shown in Fig. 20, the CERN D_{33} phase shifts have rather small positive values up to 700 MEV, with small fluctuations throughout this range. On the other hand, the D_{33} phase shifts analyzed by Roper et al.[(9)] have a rather smooth behaviour in this region and are slightly larger than the CERN phases. The D_{33} phase shifts fit the Roper results better than the CERN results. The D_{33} phases up to 360 MEV are close to the Roper analysis. Beyond this region they increase rapidly as the energy increases. The discrepancy can be explained as follows. In the low energy region the strong repulsion due to the ρ-exchange cancels part of the attraction due to the f^o-exchange.

As the energy increases the attractive contribution from f^o-exchange becomes very strong. On the other hand, the repulsion from the ρ-exchange is weakened. This is why the D_{33} has very large positive phase shifts in the high energy region. To improve the fit of D_{33} phases, one can either reduce the value of G_f or take into account the exchange of a high ρ-meson (1650 MEV). The treatment for the high ρ-exchange is the same as the ρ-exchange that the mass of the high ρ is much larger. It is hoped that, due to this large mass value, the high ρ-exchange will be able to produce an effective repulsion in the high energy region to weaken the strong attraction due to the f^o-exchange.

D_{15}:

As seen from Fig. 21 a very good fit to the CERN D_{15} phase shifts is attained. The D_{15} phase shifts remain small positive values throughout the energy range from 0 to 700 MEV. As the energy increases the D_{15} increases slowly. The presence of the f^o-exchange gives the attractive contribution which is able to cancel the repulsive effect produced by the ρ-exchange. In addition, the attractive contribution produced by the 'nucleon-exchange' is not strong enough to attain the correct D_{15} phases in the high energy region without consideration of the f^o-exchange. Oneconcludes that, the f^o-exchange indeed improves the fit for the D_{15} phase shifts.

D_{13}:

The fit of the D_{13} phase shifts, as shown in Fig. 22 is in good agreement with those of the CERN. One of the characteristic features is a D_{13} resonance appearing approximately at 616 MEV, which is consistent with the results from the phase shift analyses as shown in Table I. The strongly attractive contributions from both ρ-exchange and f^o-exchange, especially the latter, improve the D_{13} phase shift considerably. This is one of the successes which result from taking into account the f^o-exchange in our model. One remark should be stressed is that, according CERN and other phase shift analyses the D_{13} phase shift is highly absorptive above 400 MEV. In our model no absorption is taken into account. The inclusion of the absorption effect in our model will be more realistic.

5.3. Claim of Originality

An OBE potential model approach has never been very fashionable in the high energy π-N scattering problem, altough such an approach has long been used in N-N scattering problem. We have derived an OBE Born potential by taking into account the f^o-exchange, in addition to the usual ρ and N exchanges. The effect of the f^o-exchange is the main source contributing to the strong attractive force required to reproduce resonance phenomena in the appropriate channels of π-N scattering in the energy range 0-700 MEV.

It has been customary to use the Schrödinger equation as a device to calculate the phase shifts with the OBE potential. However, as mentioned before, the validity of utilizing the Schrödinger equation breaks down as soon as the problem enters into moderate or high energy regions. Our work is based on the OBE potential approach, to calculate π-N scattering phase shifts by means of the Klein-Gordon equation. The K-G equation forms a viable means of generating unitarity. The relativistic connection between the energy and momentum is correctly described, instead of imposing the relativistic correction for the high energy problem when using the Schrödinger equation.

Even though a fully covariant form is not achieved, the K-G equation has its essence in the static approximation. A fully relativistically covariant form of the π-N scattering problem can be described only through the use of the Bethe Salpeter[52] equation. A manifestly covariant expression for the corresponding matrix element of the π-N scattering amplitude can be obtained by means of the Feynman diagram of the OBE model. If one can evaluate the 4-dimensional Fourier transformation of the invariant amplitude instead of the three-dimensional one described by us in Equ.(2-17), a covariant form of the Born potential can be attained, provided the Dirac wave function and the γ matrices can be transformed in a covariant manner by making use of the Foldy-Wouthuysen transformation[64]. This approach is complicated and difficult, perhaps is not feasible in practice.

The original contribution of this work to general knowledge is that, such a simple model, based on the idea of the ρ, f^o and 'nucleon' exchanges is able to successfully reproduce the observed resonance phenomena. However, an extension of the general idea to include inelastic channels would be more realistic.

5.4. Conclusion

The model gives a good fit for the S-, P- and D-wave phase shifts with the exception of P_{31}. In addition, it is capable of reproducing all the existing resonances conjectured in the π-N scattering within the energy range from 0 to 700 MEV. The pseudoscalar pion-nucleon coupling constant of $G^2/4\pi = 14.0$ and the resultant value of $(\frac{f}{g})_{\rho NN} = -1.83$ from ρ-exchange, are in good agreement with the experiments. The ρ-exchange coupling constant $g_{\rho\pi\pi}\, g_{\rho NN}/4\pi = 3.0$ is slightly larger than the accepted value $2 \sim 2.5$. The f^o coupling constant is not well determined. One may be able to obtain the value of G_f from the life time of 2π decay of the f^o-meson.

In N-N scattering, Kantor[23] gives $G_I^2/4\pi = 5.71$ and $G_{II}^2/4\pi = 4.45$; and Ino et al.[51] obtain $G_I^2/4\pi = 10 \sim 20$. However, we obtain the f^o-coupling constant of $g_{f\pi\pi}\, g_{fNN}/4\pi = 12$ which is in agreement with the value obtained by Ino et al.. The 'cut off' mass of $m_c = 10.845\mu$ is reasonable. With this large 'cut off' mass, the effects on the long range components of the potential are insignificant. It is able to produce very short range force to describe the hard core type behaviour.

The resulls indicate that the ρ-exchange, f^{o}-exchange and 'nucleon-exchange' play essential roles for π-N scattering in the energy range 0-700 MEV. Of even greater importance, the model provides us with a theory that appears to be in accord with our empirical knowledge of all the qualitative phenomena of π-N scattering within the energy range 0-700 MEV except for absorption.

If we go one step further, a consideration of inelastic processes with an optical potential would give a more realistic model especially in the high energy region (> 300 MEV), and would perhaps improve the fit in some channels.

Figure Captions

Fig. 1 The singularities of the partial-wave amplitudes $f_{\ell\pm}(s)$ in the complex plane.

Fig. 2 The one-boson-exchange diagram of the π-N scattering.

Fig. 3 Lowest order π-N scattering a) corresponding to the direct pole term and b) the nucleon exchange pole term.

Fig. 4 The lowest order ρ-exchange diagram of π-N scattering.

Fig. 5 The σ-exchange diagram of π-N scattering.

Fig. 6 The f^{o}-exchange diagram of π-N scattering.

Fig. 7 The one-boson-exchange Feynman diagram corresponding to the lowest Born approximation in π-N scattering.

Fig. 8 Two-boson-exchange Feynman diagram in π-N scattering.

Fig. 9 Two-boson-exchange Feynman diagram with virtual pair creation and pair annihilation processes in π-N scattering.

Fig.10 P_{33} : the solid circles are the CERN phases

———————— $G_f=13$

—·—·—·—·—·— $G_f=15$

-------------------- $G_f=17$

Fig.11 P_{11} : Notations as for Fig.10.

Fig.12 D_{13} : Notations as for Fig.10.

Fig.13 S_{31}: the solid circles are the CERN phases, the open circles ⊖ are the Roper phases. The solid line our OBE model fit to them.

Fig.14 S_{11}: notations as for Fig.13.

Fig.15 P_{33}: notations as for Fig.13 , showing a resonance at 194 MEV.

Fig.16 P_{31}: notations as for Fig.13.

Fig.17 P_{13}: notatkons as for Fig.13.

Fig.18 P_{11}: showing a resonance at approximately 600 MEV. Notations as for Fig.i3.

Fig.19 D_{35}: Notations as for Fig.13.

Fig.20 D_{33}: Notations as for Fig.13.

Fig.21 D_{15}: Notations as for Fig.13.

Fig.22 D_{13}: Notations as for Fig.13, showing a resonance at approximately 616 MEV.

APPENDIX A

Feynman Rules for T Matrix

The S-matrix for elastic π-N scattering is defined in Equ. (2-6) as

$$S = 1 + i(2\pi)^4\ \delta(P_1 + q_1 - P_2 - q_2)\frac{M}{\sqrt{4E_1E_2\omega_1\omega_2}}\, T \qquad \text{(A-1)}$$

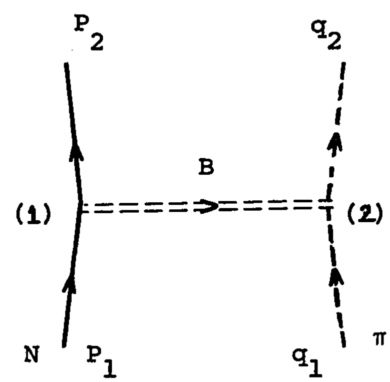

Fig. A1

Before we write down the rules for T-matrix, it is worthwhile to mention the procedures for obtaining the vertex from the interaction Lagrangian density in question. Here we outline the main points as follows:

1) If the Lagrangian density contains no derivatives for the wave functions, extract all the factors, except the wave functions from the interaction Lagrangian density.

2) If the Lagrangian density involves the derivatives*
of the wave functions, we simply replace

$$\partial_\mu \psi(P) \rightarrow iP_\mu \psi(P) ,$$

where ψ (P) is the incoming wave function towards the vertex with four-momentum P.
and

$$\partial_\mu \overline{\psi}(P') \rightarrow -iP'_\mu \overline{\psi}(P')$$

where $\overline{\psi}(P')$ is the out-going wave function leaving the vertex, with four-momentum P'.

3) Multiply by a factor

$$(-i)^{n}i$$

where n is the number of particles involved in the interaction at the vertex.

Once the vertices are specified, we can write down the T matrix at once, provided the propagator of the exchange particle is known†, thus

$$T = \overline{U}(P_2) \text{ vertex (1). propagator. vertex (2) } U(P_1)$$

---

* We do not consider derivatives higher than the first order.

† Here we take a "-" sign in front of the propagator.

APPENDIX B

Born Approximation Potential - Fourier Transformation

From Equ. (2-19) we have

$$\vec{q}_2 \times \vec{q}_1 = - \vec{Q} \times \vec{P} \tag{B-1}$$

One can also express

$$\vec{q}_2 \times \vec{q}_1 = - \vec{q}_1 \times \vec{P} \tag{B-2}$$

I) Since Equs. (B-1) and (B-2) are equal, therefore

$$\frac{1}{(2\pi)^3} i\vec{\sigma}.\vec{q}_1 \times \int \frac{\vec{P}e^{i\vec{P}.\vec{r}}}{\vec{P}^2+m^2} d^3P = \frac{1}{(2\pi)^3} i\vec{\sigma}.\vec{Q} \times \int \frac{\vec{P}\, e^{i\vec{P}.\vec{r}}}{\vec{P}^2+m^2} d^3P \tag{B-3}$$

Consider

$$I = \frac{1}{(2\pi)^3} i\vec{\sigma}.\vec{q}_1 \times \int \frac{\vec{P}e^{i\vec{P}.\vec{r}}}{\vec{P}^2+m^2} d^3P$$

$$= \frac{1}{(2\pi)^3} \vec{\sigma}.\vec{q}_1 \times \vec{\nabla} \int \frac{e^{i\vec{P}.\vec{r}}}{\vec{P}^2+m^2} d^3P$$

$$= \frac{1}{4\pi} \vec{\sigma}.\vec{q}_1 \times \vec{\nabla} \left(\frac{e^{-mr}}{r}\right)$$

$$= \frac{1}{4\pi} \vec{\sigma}.\vec{q}_1 \times \vec{r} \frac{1}{r} \frac{d}{dr} \left(\frac{e^{-mr}}{r}\right) \tag{B-4}$$

The orbital angular momentum of the pion-nucleon system is defined by

$$\vec{L} = \vec{r}_{\pi} \times \vec{q}_1 + \vec{r}_N \times \vec{P}_1$$

In the C.M. system,

$$\vec{q}_1 = - \vec{P}_1 ,$$

$$\therefore \vec{L} = (\vec{r}_{\pi} - \vec{r}_N) \times \vec{q}_1$$

$$= \vec{r} \times \vec{q}_1 \qquad \text{(B-5)}$$

where $\vec{r} = \vec{r}_{\pi} - \vec{r}_N$ is the relative coordinate.

By substituting Equ. (B-5) into Equ. (B-4), we attain

$$\frac{1}{(2\pi)^3} i\vec{\sigma}.\vec{q}_1 \times \int \frac{\vec{P}e^{i\vec{P}.\vec{r}}}{\vec{P}^2 + m^2} d^3p$$

$$= - \frac{1}{4\pi} \vec{\sigma}.\vec{L} \frac{1}{r} \frac{d}{dr} \left(\frac{e^{-mr}}{r}\right) \qquad \text{(B-6)}$$

From the relation of Equs. (B-6) and (B-3), gives

$$\frac{1}{4\pi} \vec{\sigma}.\vec{Q} \times \vec{r} \frac{1}{r} \frac{d}{dr}\left(\frac{e^{-mr}}{r}\right) = \frac{-1}{4\pi} \vec{\sigma}.\vec{L} \frac{1}{r} \frac{d}{dr} \left(\frac{e^{-mr}}{r}\right) \qquad \text{(B-7)}$$

Equ. (B-7) implies that

$$\vec{L} = \vec{r} \times \vec{Q} \qquad \text{(B-8)}$$

II) From Equ. (2-24) we have

$$\frac{1}{(2\pi)^3} \; i\vec{\sigma} \cdot \vec{Q} \times \int \vec{p} \; e^{i\vec{p}\cdot\vec{r}} \; d^3p$$

$$= \vec{\sigma} \cdot \vec{Q} \times \vec{\nabla} \, \delta^3(\vec{r}) \qquad \text{(B-10)}$$

Thus the Born potential due to the second term of F_2 in Equ.(2-23) will have the form

$$\vec{\sigma} \cdot \vec{Q} \times \vec{\nabla} \, \delta^3(\vec{r}) \; ,$$

multiplied by a scalar function.

If we now work backwards to obtain the corresponding Born scattering amplitude, using Equ.(2-26), we have

$$F^B \propto \vec{\sigma} \cdot \vec{Q} \times \int e^{-i\vec{p} \cdot \vec{r}} \vec{\nabla} \, \delta^3(\vec{r}) d^3r \qquad \text{(B-11)}$$

After integration by parts, Equ.(B-11) can be written as

$$\vec{\sigma} \cdot \vec{Q} \times \int e^{-i\vec{p} \cdot \vec{r}} \vec{\nabla} \, \delta^3(\vec{r}) d^3r$$

$$= - \vec{\sigma} \cdot \vec{Q} \times \int \delta^3(\vec{r}) \; \vec{\nabla} \; e^{-i\vec{p} \cdot \vec{r}} \; d^3r$$

$$= i \, \vec{\sigma} \cdot \vec{Q} \times \vec{p} \int e^{-i\vec{p} \cdot \vec{r}} \; \delta^3(\vec{r}) \; d^3r \qquad \text{(B-12)}$$

Therefore in this particular case, we can replace

$$\vec{\nabla}\delta^3(\vec{r}) \rightarrow i\vec{p}\ \delta^3(\vec{r})$$

or

$$\vec{\sigma} \cdot \vec{Q} \times \vec{\nabla}\ \delta^3(\vec{r}) \rightarrow i\vec{\sigma} \cdot \vec{Q} \times \vec{p}\,\delta^3(\vec{r}) \quad .$$

Making use of Equ.(B-1), we can write

$$i\vec{\sigma} \cdot \vec{Q} \times \vec{p}\ \delta^3(\vec{r}) = -i\vec{\sigma} \cdot \vec{q}_2 \times \vec{q}_1\ \delta^3(\vec{r}) \ .$$

APPENDIX C

Propagator of Spin 2 Particle

For spin 2 particle the field transforms as a Lorentz group tensor of rank 2, with the subsidiary conditions as follows[63]:

i) $(P^2 - m^2)\, T_{\mu\nu} = 0$ (C-1a)

ii) $T_{\mu\nu} = T_{\nu\mu}$ (C-1b) symmetric in μ and ν (C-1)

iii) $\partial^\mu\, T_{\mu\nu} = 0$ (C-1c) gauge invariance

iv) $g^{\mu\nu}\, T_{\mu\nu} = 0$ (C-1d) traceless condition

where $T_{\mu\nu}$ is the tensor field for spin 2 particle.

In order to calculate the spin 2 propagator, we define a most general form for the numerator of the propagator as

$$N_{\mu\nu;\alpha\beta}=a_1 g_{\mu\nu}g_{\alpha\beta}+a_2(g_{\mu\beta}g_{\alpha\nu}+g_{\mu\alpha}g_{\nu\beta})+a_3(g_{\mu\nu}P_\alpha P_\beta+g_{\alpha\beta}P_\mu P_\nu)$$

$$+ a_4(g_{\mu\alpha}P_\nu P_\beta+ g_{\mu\beta}P_\nu P_\alpha+ g_{\alpha\nu}P_\mu P_\beta+g_{\beta\nu}P_\mu P_\alpha) + a_5 P_\mu P_\nu P_\alpha P_\beta \tag{C-2}$$

which also satisfies the subsidary conditions

i) $N_{\mu\nu;\alpha\beta} = N_{\alpha\beta,\mu\nu} = N_{\nu\mu;\alpha\beta} = N_{\mu\nu;\beta\alpha}$ (C-3a)

ii) $P^{\mu}N_{\mu\nu;\alpha\beta} = 0$ (C-3b)

iii) $g^{\mu\nu}N_{\mu\nu;\alpha\beta} = 0$ (C-3c)

The constants a_i's can be evaluated by employing these subsidary conditions (C-3).

By equating zero coefficients of $g_{\alpha\beta}$ and $P_\alpha P_\beta$ from (C-3a) we get

$$4a_1 + 2a_2 + P^2a_3 = 0 \quad \text{(C-4a)}$$

$$4a_3 + 4a_4 + P^2a_5 = 0 \quad \text{(C-4b)}$$

From (C-3b), by equating the coefficients of $P_\nu g_{\alpha\beta}$ (where ν,α,β in all permutations) to zero gives

$$a_1 + P^2a_4 = 0 \quad \text{(C-4c)}$$

$$a_2 + P^2a_4 = 0 \quad \text{(C-4d)}$$

$$a_3 + 2a_4 + P^2a_5 = 0 \quad \text{(C-4e)}$$

There are five equations (C-4a-e) with five unknowns, a_1, a_2, a_3, a_4 and a_5. The solutions are trivial. On the mass shell we have

$$P^2 = m_f^2$$

If we let $a_1 = A$, where A is any arbitrary constant, we are able to solve the above five equations simultaneously.

As the result we have

$$a_1 = A; \quad a_2 = -\frac{3}{2}A; \quad a_3 = -\frac{A}{m_f^2}; \quad a_4 = \frac{3}{2}\frac{A}{m_f^2}; \quad a_5 = -\frac{2A}{m_f^4} \tag{C-5}$$

Thus (C-2) becomes

$$N_{\mu\nu;\alpha\beta} = A[g_{\mu\nu}g_{\alpha\beta} - \frac{3}{2}(g_{\mu\beta}g_{\alpha\nu} + g_{\mu\alpha}g_{\nu\beta}) - \frac{1}{m_f^2}(g_{\mu\nu}P_\alpha P_\beta + g_{\alpha\beta}P_\mu P_\nu)$$
$$+ \frac{3}{2}\frac{1}{m_f^2}(g_{\mu\alpha}P_\nu P_\beta + g_{\mu\beta}P_\nu P_\alpha + g_{\alpha\nu}P_\mu P_\beta + g_{\beta\nu}P_\mu P_\alpha) - \frac{2}{m_f^4}P_\mu P_\nu P_\alpha P_\beta] \tag{C-2a}$$

The next step is to evaluate A, subjecting to the normalized condition

$$\sum_{spin} N_{\mu\nu;\alpha\beta} N^{\mu\nu;\alpha\beta} = 1 \tag{C-7}$$

where $\sum_{spin} \to \frac{1}{2S+1}$. Here we have S = 2.

From Equ. (C-7) we have

$$A^2 = \frac{1}{9}\,; \quad \text{or } A = \pm\frac{1}{3} \qquad \text{(C-8)}$$

According to our convention to derive the T matrix we have to take the negative value i.e., $A = -\frac{1}{3}$.

If we define

$$P_{\mu\nu} = g_{\mu\nu} - \frac{P_\mu P_\nu}{mf^2}\,, \qquad \text{(C-9)}$$

(C-2a) can now be written as

$$N_{\mu\nu;\alpha\beta} = -\frac{1}{3}P_{\mu\nu}P_{\alpha\beta} + \frac{1}{2}P_{\mu\alpha}P_{\nu\beta} + \frac{1}{2}P_{\mu\beta}P_{\nu\alpha}\,. \qquad \text{(C-10)}$$

APPENDIX D

'Partial Wave' Born Potential

The partial wave scattering amplitudes for the direct pole term and the nucleon exchange pole term of π-N scattering, according to Equ. (2-16a), have the following form

$$f_{\ell\pm} = - \frac{G^2}{8\pi W} (1-\vec{\tau}\cdot\vec{t})\left[\frac{(E+M)}{(W+M)}\frac{1}{2\ell+1}\delta_{\ell,0} + \frac{(E-M)}{(W-M)}\frac{1}{2(\ell\underline{+}1)+1}\delta_{\ell\underline{+}1,0}\right]$$

$$+ (-1)^{\ell}\frac{(1+\vec{\tau}\cdot\vec{t})}{2}\left[\frac{(W-M)}{(E-M)}Q_{\ell}(|z|) - \frac{(W+M)}{(E+M)}Q_{\ell\underline{+}1}(|z|)\right] \quad \text{(D-1)}$$

with $|z| = \dfrac{2EW-\mu^2}{2q^2} > 0$.

For simplicity, we rewrite Equ. (D-1) as

$$f_{\ell\underline{+}} = - \left\{\frac{a}{2\ell+1}\delta_{\ell,0} + \frac{b}{2(\ell\underline{+}1)+1}\delta_{\ell\underline{+}1,0} + (-1)^{\ell}\left[c\,Q_{\ell}(|z|) - d\,Q_{\ell\underline{+}1}(|z|)\right]\right\}$$

(D-1a)

where

$$a = \frac{G^2}{8\pi W}\frac{(1-\vec{\tau}\cdot\vec{t})(E+M)}{(W+M)}$$

$$b = \frac{G^2}{8\pi W}\frac{(1-\vec{\tau}\cdot\vec{t})(E-M)}{(W-M)}$$

$$c = \frac{G^2}{8\pi W}\frac{(1+\vec{\tau}\cdot\vec{t})(W-M)}{2(E-M)} \quad \text{(D-2)}$$

and

$$d = \frac{G^2}{8\pi W} \; \frac{(1+\vec{\tau}\cdot\vec{t})(W+M)}{2(E+M)}$$

(A) For S-wave, $\ell = 0$, we have $J = \frac{1}{2}$ State only, thus

$$f_{0+} = - [\, a + c\, Q_o(|z|) - d\, Q_1(|z|)\,] \tag{D-3}$$

From Equ. (2-16), the partial wave scattering amplitude is given by

$$f_{0+} = \frac{1}{2} \int_{-1}^{1} [f_1\, P_o(x) + f_2 P_1(x)]\, dx \tag{D-4}$$

Equ. (D-3) can be rewritten as

$$f_{0+} = \frac{1}{2} \int_{-1}^{1} \left\{ - [a + \frac{c}{|z|-x}]\, P_o(x) + \frac{d}{|z|-x}\, P_1(x) \right\} dx \tag{D-5}$$

where we have used the relations

$$\int_{-1}^{1} P_\ell(x)\, P_{\ell'}(x) dx = \frac{2}{2\ell+1}\, \delta_{\ell,\ell'} \;;\quad \frac{1}{2} \int_{-1}^{1} \frac{P_\ell(x) dx}{z - x} = Q_\ell(z)$$

Comparison of Equ.(D-5) with Equ.(D-4), yields

$$f_1 = - (a + \frac{c}{|z| - x})$$

$$f_2 = \frac{d}{|z| - x} \tag{D-6}$$

One remark concerning Equ.(D-6) should be made when we substitute Equ. (D-6) into the expression of the total scattering amplitute in Equ.(2-14), it gives the correct S-wave scattering amplitude only, and it might not be adequate for other partial waves. This is because the term with 'a' in Equ.(D-1a), contributes only to the S-wave.

Substitution of Equ.(D-6) into Equ (2-14), gives

$$F = - \left[(a+d) + (c-d|z|) \frac{1}{|z|-x} - \frac{d}{|z|-x} i\vec{\sigma}\, \hat{q}_2 \times \hat{q}_1 \right] \qquad \text{(D-7)}$$

Consequently, Equ.(D-7) indeed reproduces the correct form of f_{0+}. Following the procedure outlined in Section 2.4, we thus obtain the Born approximation potential for the S-wave, by the Fourier transformation of Equ.(D-7). Thus

$$V_o(r) = 4\pi \left\{ (a+d)\, \delta^3(\vec{r}) + \frac{q^2}{2\pi} (c-d|z|) \frac{e^{-m_t r}}{r} \right.$$

$$\left. - \frac{d}{2\pi} \frac{\vec{\sigma}\,\vec{L}}{r} \frac{d}{dr} \left(\frac{e^{-m_t r}}{r} \right) \right\} , \qquad \text{(D-8)}$$

where $m_t = \sqrt{(2E\omega) - \mu^2 - 2q^2} > 0$.

(B) For p-wave, we have $\ell = 1$.

i) $J = \frac{3}{2}$ state, Equ.(D-1a) gives

$$f_{1+} = c\,Q_1(\,z\,) - d\,Q_2\,(|z|)$$

$$= \frac{1}{2} \int_{-1}^{1} [c\,P_1(x) - d\,P_2(x)] \frac{1}{|z| - x}\, dx \qquad \text{(D-10)}$$

Comparing Equ.(D-10) with Equ.(2-16), we obtain

$$f_1 = \frac{c}{|z| - x}$$

$$f_2 = \frac{-d}{|z| - x} \qquad \text{(D-11)}$$

As before, Equ.(D-11) indeed gives the correct f_{1+} but not necessarily the correct results for other partial waves.

Substituting Equ.(D-11) into Equ.(2-14) we find an expression for the total scattering amplitude, which gives the correct f_{1+}. The Born potential for the ℓ=1, $J = \frac{3}{2}$ state can be obtained, by the Fourier transformation of this total scattering amplitude. Thus

$$V_{1+} = -\,4\pi \left\{ d\,\delta^3(\vec{r}) + (c-d|z|)\,\frac{q^2}{2\pi}\,\frac{e^{-m_t r}}{r} - \frac{d}{2\pi}\frac{\vec{\sigma}\cdot\vec{L}}{r}\frac{d}{dr}\left(\frac{e^{-m_t r}}{r}\right) \right\} \qquad \text{(D-12)}$$

ii) For $\ell = 1$, $J = \frac{1}{2}$ state, according to Equ.(D-1a), we have

$$f_{1-} = -\left\{ b - [\,c\,Q_1(|z|) + d\,Q_o(|z|)\,] \right\}$$

$$= \frac{1}{2} \int_{-1}^{1} \left\{ \frac{c}{|z|-x} P_1(x) - \left[\, b + \frac{d}{|z|-x} \,\right] P_o(x) \right\} dx \qquad \text{(D-13)}$$

Thus

$$f_1 = \frac{c}{|z|-x}$$

$$f_2 = -\left(b + \frac{d}{|z|-x}\right) \qquad \text{(D-14)}$$

Comparing Equ.(D-14) with Equ.(D-11), we notice that they are almost the same, except for an extra term "-b" in the expression for f_2 in Equ.(D-14). This extra term, on substitution into the total scattering amplitude and then upon Fourier transformation, will give a delta function with coefficient $-d\,(\hat{q}_2 \cdot \hat{q}_1 + i\,\vec{\sigma}\cdot\hat{q}_2 \times \hat{q}_1)$ in the Born potential. Thus the complete Born potential for $\ell = 1$, $J = \frac{1}{2}$ state can be written as

$$V_{1^-} = -4\pi \left\{ [d - b(x + i\,\vec{\sigma}\cdot\hat{q}_2 \times \hat{q}_1)]\,\delta^3(\vec{r}) + \frac{q^2}{2\pi}(c - d|z|)\frac{e^{-m_t r}}{r} \right.$$

$$\left. - d\,\frac{1}{2\pi}\,\frac{\vec{\sigma}\,\vec{L}}{r}\,\frac{d}{dr}\left(\frac{e^{-m_t r}}{r}\right) \right\} \qquad \text{(D-15)}$$

Combining Equ.(D-12) and (D-15), we obtain an expression for the Born potential for the P-wave as

$$V_{1\pm} = -4\pi \left\{ [d - b(x + i\,\vec{\sigma}\cdot\hat{q}_2 \times \hat{q}_1)(\tfrac{3}{2} - J)]\,\delta^3(\vec{r}) + \frac{q^2}{2\pi}(c - d|z|)\frac{e^{-m_t r}}{r} \right.$$

$$\left. - \frac{d}{2\pi}\,\frac{\vec{\sigma}\cdot\vec{L}}{r}\,\frac{d}{dr}\left(\frac{e^{-m_t r}}{r}\right) \right\} . \qquad \text{(D-16)}$$

Here we insert the factor $(\frac{3}{2} - J)$ into the second term of the first parenthesis in Equ.(D-16) to ensure that this term contributes only for the $J = \frac{1}{2}$ state. c) For $\ell \geq 2$, repeating the same procedure as before, we obtain a general form for the partial wave Born potential with orbital angular momentum $\ell \geq 2$, thus

$$V_{\ell\pm} = (-1)^{\ell} 4\pi \left\{ d\, \delta^3(\vec{r}) + \frac{q^2}{2\pi} (c-d|z|) \frac{e^{-m_t r}}{r} - \frac{d}{2\pi} \frac{\vec{\sigma}\cdot\vec{L}}{r} \frac{d}{dr} \left(\frac{e^{-m_t r}}{r} \right) \right\} \tag{D-17}$$

We can obtain a general expression for the 'partial wave' Born potential for any angular momentum state $|\ell\rangle$, by combining Equs. (D-8), (D-16) and (D-17),

$$V_{\ell\pm} = 4\pi \left\{ \left[a\, \delta_{\ell,o} + b(x + i\, \vec{\sigma}\cdot\hat{q}_2 \times \hat{q}_1) \delta_{\ell-1,o} (\tfrac{3}{2} - J) + (-1)^{\ell} d \right] \delta^3(\vec{r}) \right.$$

$$\left. + (-1)^{\ell} \left[(c-d|z|) \frac{q^2}{2\pi} \frac{e^{-m_t r}}{r} - \frac{d}{2\pi} \frac{\vec{\sigma}\cdot\vec{L}}{r} \frac{d}{dr} \left(\frac{e^{-m_t r}}{r} \right) \right] \right\}. \tag{D-18}$$

The factors $\delta_{\ell,o}$ and $\delta_{\ell-1,o}(\frac{3}{2} - J)$ appearing in the first and second terms of the first parenthesis of Equ.(D-18), are to ensure that the respective contributions are for the S_{11} and P_{11} states only.

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