Endoscopy, Topological Mirror Symmetry and Hitchin Systems

by

Yuzhen Cao

Department of Mathematics and Statistics
Faculty of Science
McGill University, Montréal, Québec, Canada

August 2024

A thesis submitted to McGill University in partial fulfilment of the requirements of the degree of Doctor of Philosophy

Abstract

The central topic of this thesis is the topological mirror symmetry conjecture proposed by Hausel-Thaddeus, which has been a theorem since 2017, due to the first proof using the method of p-adic integration provided by Groechenig-Wyss-Ziegler.

The conjecture states that the stringy E-polynomials of moduli spaces of stable SL_n-Higgs bundles and PGL_n-Higgs bundles should be the same. By a theorem from p-adic Hodge theory, one can reduce the matching of stringy E-polynomials to an equality of point countings of the moduli spaces of Higgs bundles. The point countings can then be expressed as orbital integrals, when restricted to fibres over the anisotropic locus of the Hitchin system. We then can show that the statement of topological mirror symmetry when we restrict ourselves in the anisotropic locus of Hitchin base, is equivalent to the statement of stabilization of regular elliptic terms of the trace formula. Fortunately, this stabilization has been done in the celebrated work of Ngô.

Then, by using support theorem of SL_n-Hitchin systems, one can extend the equality obtained in the anisotropic locus of the Hitchin base to the entire Hitchin base, thereby providing a different proof of the topological mirror symmetry when the degree of the chosen effective divisor is large enough and even.

Finally by the techniques developed of Maulik and Shen, one can reduce the general case to the case mentioned above.

Abrégé

Le sujet central de cette thèse est la conjecture de symétrie miroir topologique proposée par Hausel-Thaddeus, qui est devenue un théorème depuis 2017, grâce à la première preuve utilisant la méthode de l'intégration p-adique donnée par Groechenig-Wyss-Ziegler.

La conjecture stipule que les E-polynômes "stringy" des espaces de modules des fibrés de Higgs stables de SL_n et de PGL_n devraient être identiques. Par un théorème de la théorie de Hodge p-adique, on peut réduire la correspondance des E-polynômes "stringy" à une égalité du comptage des points des espaces de modules des fibrés de Higgs. Le comptage des points peut être exprimé en termes d'intégrales orbitales, lorsqu'il est restreint aux fibres sur le lieu anisotrope du système de Hitchin. On peut alors montrer que l'énoncé de la symétrie miroir topologique, lorsqu'on se restreint au lieu anisotrope de la base de Hitchin, est équivalent à l'énoncé de la stabilisation des termes elliptiques réguliers de la formule des traces. Heureusement, cette stabilisation a été réalisée dans le travail célèbre de Ngô.

Ensuite, en utilisant le théorème de support des systèmes de Hitchin de SL_n , on peut étendre l'égalité obtenue dans le lieu anisotrope de la base de Hitchin à l'ensemble de la base de Hitchin, fournissant ainsi une preuve différente de la symétrie miroir topologique lorsque le degré du diviseur effectif choisi est suffisamment grand.

Enfin, par les techniques développées par Maulik et Shen, on peut réduire le cas général au cas mentionné ci-dessus.

Acknowledgements

I would like to give my heartfelt thanks to my advisors, Professor Jacques Hurtubise and Professor Dmitry Jakobson, for their invaluable guidance. Their generous financial support has also been crucial, and I have learned so much from working with them.

I would also like to thank Joel Kamnitzer for his generous help and Brent Pym for organizing the geometric working group seminar in the department.

I would like to thank Michael Groechenig for his generous help; completing this thesis on time would not have been possible without conversation with him.

Additionally, I owe heartfelt thanks to my family—my parents and my sister, Xiang Cao—for their constant support. I am also grateful to my friends Jiuzhou Huang, Yujun Zhong, Houjie Li, Zi Wang, and Eleven Yin, whom I have met and re-met over the past few years.

I would also like to thank the beauty of the sunsets at Mount Royal and the vibrant autumn colours brought by the falling maple leaves.

Finally, my sincere thanks to the Schulich Scholarship and the ISM graduate fellowships for their support, which has been pivotal in my academic journey.

Contents

Introduction

Hitchin introduced the notions of Hitchin fibrations and Higgs bundles in [32] and [33], respectively. The moduli space of Higgs bundles, along with the associated Hitchin fibration – as already observed by Hitchin – possesses intricate geometric structures. Over the past three decades, these have emerged as compelling geometric subjects for study, intertwining with various areas of mathematics due to their rich structural complexity.

Now let us fix some notations to continue the story of the conjecture of topological mirror symmetry.

Let d, e be two integers which are coprime to n. Let X be a projective, smooth and geometrically connected curve. We fix an effective divisor D with $\deg(D) \geq 2g-2$, and $L \in Pic(X)$ a line bundle on X of degree d, and set $\mathcal{M}^L_{SL_n}$ to be the moduli space of stable SL_n-Higgs bundles (E, θ), where E is a vector bundle of rank n with an isomorphism of determinant line bundles $\det(E) \simeq L$ with $\deg(L) = d$, and θ is a trace-free Higgs field. Let $\mathcal{M}^e_{PGL_n}$ denote the moduli space of PGL_n-Higgs bundles, which come from stable GL_n-Higgs bundles of degree $e \pmod n$ over geometric points. It is well-known that SL_n and PGL_n are Langlands dual groups, and the Hitchin base $\mathcal{A}$ of them are the same. It was Hausel-Thaddeus [31] inspired by the formalism of [51], who observed that over a nice open dense subset of the Hitchin base $\mathcal{A}$, the SL_n-fibres and PGL_n-fibres of the following commutative diagram are dual abelian varieties,

$$\begin{tikzcd} \mathcal{M}^L_{SL_n} \arrow[dr, "f_{SL_n}"'] & & \mathcal{M}^e_{PGL_n} \arrow[dl, "f_{PGL_n}"] \\ & \mathcal{A} & \end{tikzcd}$$

where f_{SL_n} and f_{PGL_n} are Hitchin morphisms. This was generalized to other groups by Donagi and Pantev in [20], and for the parabolic case, by Biswas and Dey [9]. Hausel-Thaddeus then conjectured that $\mathcal{M}^L_{SL_n}$ and $\mathcal{M}^e_{PGL_n}$ should be mirror partners in the sense of [51], and should have the same E-stringy polynomial that was introduced in [6]. These stringy parts, in some sense, were introduced to reflect the presence of the orbifold singularities. Hausel-Thaddeus managed to verify their conjecture in the case of rank 2 and 3. As pointed out in [31] and [20], one also needs the input of gerbes, an extra structure, to formulate the duality properly and extend to the entire moduli space of stable vector bundles. For the SL_n and PGL_n cases, there is a natural lifting gerbe α_L on the moduli space $\mathcal{M}^e_{PGL_n}$, measuring the failure of lifting the universal projective bundle, as pointed out in section 3 of [31].

So finally, we can state the topological mirror symmetry theorem:

Theorem 0.0.1. *[**Topological Mirror Symmetry**] These two moduli spaces $\mathcal{M}^L_{SL_n}$ and $\mathcal{M}^e_{PGL_n}$ share the same stringy Hodge numbers*

$$h^{p,q}(\mathcal{M}^L_{SL_n}) = h^{p,q}_{str}(\mathcal{M}^e_{PGL_n}, \alpha_L).$$

Let us mention the existing proofs. The Theorem 0.0.1 was proved by Groechenig, Wyss and Ziegler in [27] by p-adic integration, Loeser and Wyss gave a proof by motivic integration in [40], and Maulik and Shen [41] gave another sheaf theoretic proof.

For the parabolic analogue of topological mirror symmetry, as we mentioned before, Biswas and Dey proved that the fibres are dual abelian varieties generically. Then Gothen and Oliveira gave the proof of the case of 2 and 3 in [25]. More recently, Shen gave a proof using p-adic integration in [49].

This thesis is devoted to providing a different proof by expressing the number of points on a Hitchin fibre as orbital integrals, and seeking to compare these orbital integrals over SL_n and PGL_n. The method we use here needs to stabilize these orbital integrals. Each chapter can be summarized as follows, thereby also providing an outline of this thesis.

1. In chapter 1, we introduce the notion of Higgs bundles and Hitchin systems. We define the anisotropic locus, which is pivotal to us in the sequel, and we introduce the celebrated product formula relating the number of Higgs bundles with those of local affine Springer fibres. Lastly, we introduce the adelic description of Higgs bundles: this is the foundation of the whole story. It turns out there is a nice

dictionary between adelic terms and notions in the theory of principal bundles, which allows one to relate the theory of orbital integral and adelic integrals to the geometry of vector bundles and Higgs bundles, see [28] for a classical example. After the adelic description, the counting formula in terms of adelic integral is introduced, which allows us to relate the point counting formula to the geometric side of Arthur's trace formula.

2. In chapter 2, we first follow section 2 of [27] to introduce the concepts of stringy E-polynomials, stringy counting formula and inertia stacks. Next, we define gerbes, and the line bundle induced by a μ_n-banded gerbe, where μ_n is the group of n-th roots of unity. A slight variant of stringy E-polynomials twisted by gerbes and counting formulas are introduced. Those variants are the main objects of this thesis. In the last section of chapter we explain why we only need to consider the case where the line bundle induced by the lifting gerbe is trivial, and the Γ-equivariant structure of the line bundle can be arbitrary taken from a μ_n-representation $\mathrm{Hom}(\Gamma, \mu_n)$, where $\Gamma = \mathrm{Pic}^0(X)[n]$.

   Also we mention the following theorem 2.19 in [27], which we present here for the reader's convenience. The theorem is of central importance to us, as that is the starting point of the whole story. It is by this theorem, one can reduce Theorem 0.0.1 to the equality of point counting.

   Theorem 0.0.2 (also see Theorem 2.2.5)**.** *Let $R \subset \mathbb{C}$ be a subalgebra of finite type over $\mathbb{Z}$. We fix an abstract isomorphism of $\mathbb{C}$ and $\overline{\mathbb{Q}}_\ell$. Let X_i be smooth Γ_i-varieties for two abstract finite abelian groups Γ_1 and Γ_2. Let $\mathcal{X}_i = [X_i/\Gamma_i]$ be the resulting quotient R-stacks and α_i be a μ_r-gerbe on $\mathcal{X}_i$ for $i = 1, 2$. We suppose that for every ring homomorphism $R \to \mathbb{F}_q$ to a finite field $\mathbb{F}_q$ we have*

$$\#^{\alpha_1}_{str}(\mathcal{X}_1 \times_R \mathbb{F}_q) = \#^{\alpha_2}_{str}(\mathcal{X}_2 \times_R \mathbb{F}_q).$$

   Then

$$E_{st}(\mathcal{X}_1 \times_R \mathbb{C}, \alpha_1; x, y) = E_{st}(\mathcal{X}_2 \times_R \mathbb{C}, \alpha_1; x, y).$$

   We will not present the proof of the above theorem, we refer readers to the paper [27] and the appendix written by Katz of the paper [30] for the details of proof of the above theorem.

So the rest of this thesis will be mainly devoted to proving the equality of point counting on the moduli spaces $\mathcal{M}^L_{SL_n}$ and $\mathcal{M}^e_{PGL_n}$, i.e., the following theorem, see Theorem 5.2.7 for more details.

Theorem 0.0.3. *Let the notations be as before, then*

$$\#\mathcal{M}^L_{SL_n} = \#^{\alpha_L}_{str}\mathcal{M}^e_{PGL_n}.$$

3. To prove the Theorem 0.0.3, we first noticed that in [26], there are equivalence theorems of coendoscopic decompositions and inertia stacks, more precisely, Theorem 5.14 and Theorem 5.23 in [26], or 3.2.14 in this thesis.

   The result of Groechenig-Wyss-Ziegler states that over the anisotropic locus $\mathcal{A}^{ani}$ of G-Higgs bundles, where G can be a general semisimple group, the coendoscopic decomposition is equivalent to the inertia stack of the moduli stack of Higgs bundles over the anisotropic locus. This enables us to write the contributions from the stringy parts as contributions from coendoscopic data. This builds a bridge between our problem and the fundamental lemma. These contributions from coendoscopic data are stable orbital integrals over the anisotropic locus, by the (non-standard) fundamental lemma [[46], theorem 1.12.7], those stable orbital integrals are the same for groups with dual root datum. In short, the idea is the slogan: to compare orbital integrals, one first has to write it as a sum of stable orbital integrals. The case of SL_n and PGL_n is more pleasant, since orbital integrals on PGL_n are naturally stable. Here by stable orbital integral we mean these orbital integrals which are invariant under the stable conjugation, which is conjugation by elements from the algebraic closure of the base field.

   This process is carried out in chapter 3.

4. We review the theory of weak abelian fibrations and support theorems in the first 2 sections of chapter 4. Next, we do a review of Cataldo's proof of the support theorem in SL_n-case. One of the key part of proving the support theorem is to do dimension estimating. Cataldo used the same stratification as Chaudouard-Laumon in [15]. Roughly, the idea is to argue that the dimension constraints for a point to be a generic point of certain supports force the irreducibility of the characteristic polynomials of Higgs fields.

Also we will review the basics of Grothendieck's function-sheaf dictionary and perverse continuation method in section 4.3, after an introduction to the support theorem.

In the last part of this chapter, we will review of Maulik-Shen's method and apply their constructions to our case to show that it is enough to consider the case where $\deg(D)$ is a large enough even integer.

5. In the last chapter, we will end the proof by showing that the main theorem is true in the case where $\deg(D)$ is a large enough even integer. We first argue that the group Γ is in a bijection with the set of equivalent classes of elliptic endoscopic data. Hence we can relate the decomposition of the complex $Rf_{SL_n,*}\overline{\mathbb{Q}}_\ell$ according to the action of Γ with the endoscopic decomposition given in the celebrated geometric stabilization theorem of Ngô. Then by restricting ourselves in the anisotropic locus, we can then get the desired comparison of point counts in the anisotropic locus, moreover, by using the non-standard fundamental lemma and taking the coendoscopic groups into account, we can reduce the point counting of the Hitchin fibres from the endoscopic groups to a counting on the SL_m-Hitchin fibres where m is a divisor of n. Then we can apply the support theorem of groups SL_n and SL_m to extend the endoscopic decomposition to the entire Hitchin base from the anisotropic locus. On the other side, the reduction of point counting on endoscopic groups of SL_n to SL_m allows us to assert that the matching of point counting on endoscopic groups and coendoscopic groups can also be extended to the entire Hitchin base. Overall, one then can get the desired equality of point countings, see Theorem 5.2.6.

Chapter 1

Higgs bundles and Hitchin systems

In this chapter, we review necessary backgrounds on Higgs bundles and Hitchin fibrations. The reader is referred to [33], [32] and [50] for original sources. The paper of Ngô [45] and the chapter 4 of [46] are also standard sources for the relation between Hitchin systems and orbital integrals.

Throughout this thesis, we will fix the following notations:

- $k = \mathbb{F}_q$ will be a finite field with characteristic $p > 0$ where p is large enough. More precisely, we will assume that p is larger than the order of the Weyl groups of the reductive groups we will consider in this thesis. We then will choose a fixed algebraic closure of k, and denote it by $\bar{k}$.

- X will be a smooth projective and geometrically connected curve over k of genus $g \geq 2$, and $|X|$ will denote the set of closed points of X.

- F will denote the function field of X. We let $\overline{F}$ be a fixed algebraic closure of F, and $\eta := \mathrm{Spec}(F)$ will denote the generic point of X.

- For all $v \in |X|$, we let F_v be the completion of F with respect to the v-adic topology, thus F_v will be a non-archimedean local field.

- An effective divisor $D = \sum_{v \in |X|} d_v[v]$ will be fixed and the degree $\deg(D)$ will be set to be an even integer which is strictly larger than $2g$.

- We will use $\mathbb{G}, \mathbb{T}, \mathbb{B}$ to denote a split reductive group, a maximal torus and a Borel subgroup over the field k and $\mathbb{g}, \mathbb{t}$ and $\mathbb{b}$ to denote the corresponding Lie algebras, respectively.
- Normally written letters such as G, T and B will be reserved for group schemes over a test scheme S and the curve X and $\mathfrak{g}, \mathfrak{t}$ and $\mathfrak{b}$ will be used to denote the Lie algebras of G, T and B, respectively.

1.1 Quasi-split reductive group schemes

We first give a rapid review on the notions of quasi-split reductive group schemes.

Let S be a k-scheme and G be a reductive group scheme over S, we first recall the following definitions.

Definition 1.1.1. A connected reductive group G over S, is called **split** if it contains a maximal torus that is split over S.

Definition 1.1.2. A **pinning** of G over S is a triple

$$(B, T, s)$$

where T is a maximal torus over S and $B \supset T$ is a Borel subgroup over S, s is a section of $\mathrm{Lie}(B)$ over S such that there is an étale covering S' of S with the following holds: over S', the groups T, B and G become constant group schemes and s admits a decompositions into simple roots, $s = \sum_{\alpha \in \Phi^+} s_\alpha$ for nowhere vanishing sections s_α over S', where α's are simple roots of G with respect to Borel subgroup B, and $0 \neq s_\alpha$ are sections of $\mathrm{Lie}(U)_\alpha$, where $\mathrm{Lie}(U)_\alpha$ are the corresponding eigenspace of the root α. Here U is the unipotent radical of B

A pinning (T, B, s) is called **split** if the torus T is split.

Definition 1.1.3. A reductive group scheme G over a scheme S is called **quasi-split** if it admits a pinning over S.

Now let $\mathbb{G}$ be a split reductive group scheme over k with a fixed pinning, and G be a group scheme over the curve X which is étale locally isomorphic to $X \times \mathbb{G}$.

According to section 7.3 of [24], we have the following fact

Fact 1.1.4. *The exact sequence*

$$1 \to \mathbb{G}^{ad} \to Aut(\mathbb{G}) \to Out(\mathbb{G}) \to 1$$

is split, where $\mathbb{G}^{ad}$ is the group of inner automorphisms of $\mathbb{G}$, and $Out(\mathbb{G})$ is the group of automorphisms which fix the chosen pinning of $\mathbb{G}$. The choice of a pinning gives a splitting of the short exact sequence above.

Let $x \in X$ be a geometrical point. The group scheme G over X is given by an $Aut(\mathbb{G})$-torsor τ_G, over X, whose isomorphism class is an element

$$[\tau_G] \in H^1(\pi_1(X,x), Aut(\mathbb{G})(\bar{k}))$$

which gives us an element $[\tau_G^{out}] \in H^1(\pi_1(X,x), Out(\mathbb{G}))$ by the morphism

$$\rho_G : \pi_1(X,x) \to \mathrm{Aut}(\mathbb{G}) \to Out(\mathbb{G}).$$

Hence we naturally have a covering X_Θ of X, given by the surjective homomorphism

$$\rho_G : \pi_1(X,x) \to \Theta \subset \mathrm{Out}(\mathbb{G}),$$

where we use Θ to denote the image of ρ_G.

Let us recall a lemma which is recorded in [26].

Lemma 1.1.5 (Lemma 4.8, [26])**.** *Let X be a smooth projective curve over k, and ρ_G be an $Out(\mathbb{G})$-torsor on X. There exists a finite étale covering X' of X over which the torsor ρ_G becomes trivial.*

Then by the lemma, we can have an finite étale covering (X_Θ, x_Θ) over (X,x) such that the torsor ρ_G splits.

On the other hand, notice that the splitting obtained by choosing a pinning, $\mathrm{Out}(\mathbb{G}) \to \mathrm{Aut}(\mathbb{G})$ gives a $\mathrm{Aut}(\mathbb{G})$-torsor over X,

$$\tau_G^{ext} : \pi_1(X,x) \xrightarrow{\rho_G} \mathrm{Out}(\mathbb{G}) \to \mathrm{Aut}(\mathbb{G}). \tag{1.1}$$

This gives us a quasi-split form G^* over X, with G being its inner form; clearly G is quasi-split over X iff τ_G and τ_G^{ext} are isomorphic as $\mathrm{Aut}(\mathbb{G})$-torsors over X.

Finally, we can now can introduce the following definition of Langlands dual groups taken from Definition 4.9 of [26]:

Definition 1.1.6. Let $\widehat{\mathbb{G}}/k$ be the reductive group given by the root datum dual to the root datum of $\mathbb{G}/k$. There is then a natural isomorphism $\mathrm{Out}(\widehat{\mathbb{G}}) = \mathrm{Out}(\mathbb{G})$.

For the curve X over k, and an outer form G of $\mathbb{G}$ over X given by a torsor $\rho_G : \pi_1(X, x) \to \mathrm{Out}(\mathbb{G})$ over X, we call the quasi-split outer form $\widehat{G}$ of $\widehat{\mathbb{G}}$ over X given by $\rho_G : \pi_1(X, x) \to \mathrm{Out}(\mathbb{G}) \to \mathrm{Out}(\widehat{\mathbb{G}})$ the **Langlands dual of** G over X.

1.2 Higgs bundles and Hitchin Systems

Now let G be a quasi-split group scheme over the curve X.

By a Higgs bundle on X, we mean a pair (E, φ) over X, where E is a G-torsor over X and $\varphi \in H^0(X, ad(E) \otimes O_X(D))$, where D is a divisor of even degree such that $\deg(D) > 2g$ and $D - K$ is an effective divisor.

We use $\mathcal{M}_G$ to denote the moduli stack classifying Higgs bundles over X, more precisely, $\mathcal{M}_G$ is the functor sending each test scheme S over k, to the groupoid $\mathcal{M}(S)$ consisting of Higgs bundles (E, φ) over $X \times S$, and we will reserve the notation M_G for corresponding coarse moduli spaces, or just simply M if there is no possible confusion.

Now consider $(E, \varphi) \in \mathcal{M}_G(S)$ where E is a G-torsor over $X \times S$ and φ is global section in $H^0(X \times S, ad(E) \otimes p_X^* O_X(D))$, where $p_X : X \times S \to X$ is the projection morphism. Following section 4.2.2 of [46], one has that the datum $(E, \varphi) \in \mathcal{M}(S)$ is equivalent to the morphism

$$h_{E,\varphi} : X \times S \to [\mathfrak{g}_D / G],$$

where $\mathfrak{g}_D := \mathfrak{g} \times_X^{\mathbb{G}_m} O_X(D)$, and G acts on $\mathfrak{g}_D$ by adjoint action.

Remark 1.2.1. By the definition of quotient stack, such a morphism is given by a G-torsor E over $X \times S$ and a G-equivariant map $\varphi' : E \to \mathfrak{g}_D$, which descends to $\varphi : X \times S \to E \times^G \mathfrak{g}_D$.

We now give a construction of Hitchin base.

If we use $k[\mathrm{g}]$ to denote the algebra of polynomial functions on g and $k[\mathrm{g}]^G$ for the subalgebra of $\mathbb{G}$-invariant functions, then we have the following Chevalley isomorphism

$$k[\mathrm{g}]^{\mathbb{G}} \xrightarrow{\simeq} k[\mathfrak{t}]^{\mathbb{W}},$$

where $k[\mathbb{t}]^{\mathbb{W}}$ is the algebra of $\mathbb{W}$-invariant functions on $\mathbb{t}$. Here $\mathbb{W}$ is the Weyl group of $(\mathbb{G}, \mathbb{T})$, see section 3.1 of [17] for more details.

Therefore, one has the following Chevalley morphism or characteristic morphism

$$\chi : \mathbb{g} \to \mathbb{c} := \mathrm{Spec}(k[\mathbb{t}]^{W}) = \mathrm{Spec}(k[\mathbb{g}]^{G})$$

which is induced from the natural morphism

$$k[\mathbb{g}]^{\mathbb{G}} \to k[\mathbb{g}].$$

Note that the action of $\mathbb{G}_m$ on $\mathbb{g}$ commutes with the action of the adjoint action of $\mathbb{G}$; here $\mathbb{G}_m$ means the multiplicative group. Hence we have the following morphism on the level of algebraic stacks,

$$[\chi] : [\mathbb{g}/\mathbb{G}] \to \mathbb{c},$$
$$[\chi/\mathbb{G}_m] : [\mathbb{g}/\mathbb{G} \times \mathbb{G}_m] \to [\mathbb{c}/\mathbb{G}_m].$$

Remark 1.2.2. For $G = GL(n)$ over k, the characteristic morphism is nothing but the association of a matrix in $\mathfrak{gl}(n)$ to its characteristic polynomial, which is clearly monic.

Moreover, there is a section called **Kostant section** of the characteristic morphism

Proposition 1.2.3. *There is a section of $\chi : \mathbb{g} \to \mathbb{c}$, which we denote by $\epsilon : \mathbb{c} \to \mathbb{g}$, with image $im(\epsilon) \subset \mathbb{g}^{reg}$, where $\mathbb{g}^{reg} \subseteq \mathbb{g}$ is the dense subset consisting of regular elements.*

Remark 1.2.4. In the case of $G = GL(n)$ over k, the section ϵ is nothing but the morphism which sends every characteristic polynomial to its *companion matrix*.

Now let us move everything we mentioned above over X, and recall that G is a quasi-split group scheme over X.

By taking the $\mathrm{Aut}(\mathbb{G})$-torsor τ_G given by the group scheme G, it is natural to have

$$\mathfrak{g} := Lie(G) = \mathbb{g} \times^{\mathrm{Aut}(\mathbb{G})} \tau_G.$$

We set

$$\mathfrak{g}_D := \mathfrak{g} \times_X^{\mathbb{G}_m} O_X(D), \qquad \mathfrak{t}_D := \mathfrak{t} \times_X^{\mathbb{G}_m} O_X(D).$$

We can also twist $\mathbb{c}$ by the torsor τ_G, and set

$$\mathfrak{c} := \mathbb{c} \times^{\mathrm{Aut}(\mathbb{G})} \tau_G$$

and

$$\mathfrak{c}_D := \mathfrak{c} \times_X^{\mathbb{G}_m} O_X(D).$$

Definition 1.2.5. Let $\mathcal{A}_G$ be the functor sending a test k-scheme S to the groupoid of morphisms $a : X \times S \to \mathfrak{c}_D$.

Remark 1.2.6. Because of the existence of Kostant section, see proposition 1.2.8, it is equivalent to regard $\mathcal{A}_G$ as the affine space of global sections of $\mathfrak{c}_D$ over X.

Definition 1.2.7. Let S be a test scheme over k, then the so called **Hitchin morphism** $f : \mathcal{M}_G \to \mathcal{A}_G$ sends an Higgs bundle (E, φ) over $X \times S$ to $\mathcal{A}_G$ defined by the following commutative diagram:

$$\begin{tikzcd} X \times S \arrow[r] \arrow[d] & \mathfrak{c}_D \\ {[\mathfrak{g}_D/G].} \arrow[ur, "{[\chi]}"'] & \end{tikzcd}$$

We now end this section with the following proposition from [45], Proposition 2.5.

Proposition 1.2.8. *We suppose that G is quasi-split over X, which is saying that τ_G given by the data of G is isomorphic to the torsor τ_G^{ext} given in 1.1 as $Aut(\mathbb{G})$-torsors over X. Suppose that there is a square root L_D of $O_X(D)$,i.e., $L_D^2 = O_X(D)$. Then, there is a section of $f : \mathcal{M}_G \to \mathcal{A}_G$.*

Remark 1.2.9. This proposition gives the existence of Kostant section, which is pivotal in the following analysis of symmetries of Hitchin fibres and of importance in the process of stabilization of the counting formula of a Hitchin fibre. It also explains why Ngô need keep the degree of the fixed divisor D to be even in his proof [46].

1.3 Symmetries and product formula

1.3.1 Symmetries on Hitchin fibres

Let $\mathcal{M}_{a,G} := f^{-1}(a)$, sending a scheme S to the groupoid of pairs $(E,\varphi) \in \mathcal{M}(S)$ with $a \in \mathcal{A}(S)$, be the Hitchin fibre over the point a.

Consider the group scheme of centralizers I over $\mathbb{g}$ defined by

$$I_{\mathbb{G}} := \{(x,g) \in \mathfrak{g} \times \mathbb{G} \mid ad(g)x = x\}.$$

By twisting by the torsor τ_G and the line bundle $O_X(D)$ we obtain a group scheme $[I]$ over $[\mathfrak{g}_D/G]$.

Take $(E,\varphi) \in \mathcal{M}_{a,G}$, one has a group scheme $I_{E,\varphi}$ representing the functor sending S to the automorphism group sheaf $\underline{\mathrm{Aut}}(E,\varphi)$ over $X \times S$.

It is easy to see that

$$I_{E,\varphi} = h_{E,\varphi}^*[I],$$

where $h_{E,\varphi} : X \times S \to [\mathfrak{g}_D/G]$ is given by the datum of (E,φ).

Recall that for $(E,\varphi) \in \mathcal{M}_{a,G}(S)$ we have the following commutative diagram:

$$\begin{tikzcd} X \times S \arrow[r, "h_a"] \arrow[d, "h_{E,\varphi}"'] & \mathfrak{c}_D \\ {[\mathfrak{g}_D/G].} \arrow[ur, "{[\chi]}"'] & \end{tikzcd}$$

There is a Kostant section $\epsilon : \mathfrak{c}_D \to \mathfrak{g}_D$, and therefore we can define $J := \epsilon^* I$ and $J_a := h_a^* J$.

Take an $a \in \mathcal{A}(S)$, we shall consider the Picard groupoid $\mathcal{P}_a(S)$ of J_a-torsors over $X \times S$. This is a smooth group scheme over $X \times S$ since J is a group scheme over $\mathfrak{c}_D$. Now by lemma 2.1.1 in [46], we have a natural G-equivariant morphism

$$J_a \to \mathrm{Aut}_{X\times S}(E,\varphi) = h_{E,\varphi}^* I.$$

This defines a action of $\mathcal{P}_a(S)$ on $\mathcal{M}_a(S)$.

By the homotopy lemma 3.2.3 in [38], the induced action of $\mathcal{P}_a$ on the perverse cohomologies of $\mathcal{M}_a$ factors through the group of connected components of $\mathcal{P}_a$. For reader's convenience, we record the lemma in the following,

Lemma 1.3.1 (Lemme 3.2.3, [38]). *Let $f : X \to S$ be a morphism of k-schemes. Let $\pi : G \to S$ be a smooth S-group scheme with geometric connected fibres acting on X. Then the group of global sections $G(S)$ acts trivially on each perverse cohomology sheaf ${}^p\mathcal{H}^n(f_*\mathbb{Q}_\ell)$.*

Hence for the purpose of studying the action of $\mathcal{P}_a$ on the cohomology sheaves of the fibre $\mathcal{M}_a$, one needs to consider $\pi_0(\mathcal{P}_a)$. One of the most important and agreeable cases is where $\pi_0(\mathcal{P}_a)$ is finite. As we will see in the following, the condition of $\pi_0(\mathcal{P}_a)$ being finite defines an important locus of the Hitchin base $\mathcal{A}(\bar{k})$.

Example 1.3.2. In the linear group GL_n case, just as Hitchin showed in the paper [32], the Hitchin base $\mathcal{A}$ is given by

$$\mathcal{A} = \bigoplus_{i=1}^{r} H^0(X, O_X(D)^i).$$

The points $a = (a_1, ..., a_r)$ defines a polynomial equation

$$t^r - a_1 t^{r-1} + \ldots + a_r = 0$$

in the total space of the line bundle $O_X(D)$, hence cut out a so called spectral curve Y_a over X. There is a open dense subset $\mathcal{A}^\diamond$ such that the spectral curves Y_a are smooth, and over this locus, Hitchin showed the fibres over this locus is given by $\mathrm{Pic}(Y_a)$, i.e., the Picard group of the smooth curve Y_a. In the BNR paper [7], this result is generalised to locus of integral spectral curves.

Following Donagi [19] section 2, we now can define the cameral covers

Definition 1.3.3. Take $(x, a) \in \mathcal{A}_G \times \mathfrak{c}_D$, then one can consider the following cartesian diagram:

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & \mathfrak{t}_D \arrow[d, "\pi"] \\ X \times \mathcal{A} \arrow[r] & \mathfrak{c}_D. \end{tikzcd}$$

Here the right vertical arrow $\mathfrak{t}_D \to \mathfrak{c}_D$ is given by the quotient by the Weyl group W.

In particular, one can take the fibre over $a \in \mathcal{A}_G(S)$, where S is a k-scheme, and obtain the so called **cameral cover** $\pi_a : \widetilde{X}_a \to X \times_k S$.

Now we can introduce various open loci in $\mathcal{A}(\bar{k})$. We will keep the notations as same as in [46].

- $\mathcal{A}^{\heartsuit}(\bar{k})$ will be the subset of $\mathcal{A}(\bar{k})$ such that the cameral cover $\pi_a : \widetilde{X}_a \to \overline{X}$ is generically étale.

  If we view $a \in \mathcal{A}(\bar{k})$ as a morphism $h_a : \overline{X} \to \mathfrak{c}_D(\bar{k})$, then

  $$\mathcal{A}^{\heartsuit}(\bar{k}) = \{a \in \mathcal{A}(\bar{k}) \mid \ h_a(\overline{X}) \not\subset \mathfrak{D}_{G,D}\}$$

  where $\mathfrak{D}_{G,D}$ is the divisor obtained by twisting the divisor given by the discriminant function $\prod_{\alpha\in\Phi} d\alpha$ of the cover $\pi : \mathbb{t} \to \mathbb{c}$ with the torsor ρ_G and D.

- $\mathcal{A}^{\diamond}(\bar{k})$ is the subset of $\mathcal{A}^{\heartsuit}(\bar{k})$ such that the cameral curve $\widetilde{X}_a$ is smooth and in fact, $\mathcal{A}^{\diamond}(\bar{k})$ is the subset of $\mathcal{A}(\bar{k})$ where the action of $\mathcal{P}_a$ on the fibre $\mathcal{M}_{a,G}(\bar{k})$ is simply transitive and by corollary 4.10.4 in [46], for all $a \in \mathcal{A}^{\diamond}(\bar{k})$, one has $\pi_0(\mathcal{P}_a) = Z\widehat{\mathbb{G}}^{\Theta}$.

  Proposition 1.3.4 (proposition 4.7.1, [46])**.** *If* $\deg(D) > 2g$*, then the open subset* $\mathcal{A}^{\diamond}$ *is nonempty.*

- $\mathcal{A}^{ani}(\bar{k})$ will be defined to be the subset of $\mathcal{A}^{\heartsuit}(\bar{k})$ such that the group of connected components $\pi_0(\mathcal{P}_a)$ is finite.

Remark 1.3.5. By [46] 4.10.5 and 5.4.7, one knows that $\mathcal{A}^{ani}(\bar{k})$ is the set of $\bar{k}$-points of $\mathcal{A}^{ani}$ inside $\mathcal{A}^{\heartsuit}$.

By proposition 4.10.3 in [46] or corollary 6.7 in [45], the condition of $\pi_0(\mathcal{P}_a)$ being finite is equivalent to $|\widehat{\mathbb{T}}^{W_a}| < \infty$, where $W_a \subset \mathbb{W} \rtimes \mathrm{Out}(\mathbb{G}) = \mathbb{W} \rtimes \mathrm{Out}(\mathbb{G})$ is the image of $\pi_1(U_a, x) \to \mathbb{W} \rtimes \mathrm{Out}(\mathbb{G})$. Here U_a is the open locus where $\pi_a : \widetilde{X}_a \to \overline{X}$ is étale, and $\widehat{\mathbb{T}}$ is the $\bar{\mathbb{Q}}_\ell$-dual of the torus $\mathbb{T}$.

Remark 1.3.6. The morphism $\pi_1(U_a, x) \to \mathbb{W} \rtimes \mathrm{Out}(\mathbb{G})$ is coming from the map

$$\pi_1(U_a, x) \to \mathbb{W} \rtimes \pi_1(U_a, x)$$

and

$$\rho_G : \pi_1(X, x) \to \mathrm{Out}(\mathbb{G})$$

where the morphism

$$\pi_1(U_a, x) \to \mathbb{W} \rtimes \pi_1(U_a, x)$$

is coming from the cameral curve $\widetilde{U}_a \subset \widetilde{X}_a$, the inverse image of U_a. We know that $\widetilde{U}_a$ is a $\mathbb{W}$-torsor over U_a.

Overall, one can have the following commutative diagram:

$$\begin{tikzcd}
\pi_1(U_a, x) \arrow[r] \arrow[d] & \mathbb{W} \rtimes \mathrm{Out}(\mathbb{G}) \arrow[d] \\
\rho_G : \pi_1(X, x) \arrow[r] & \mathrm{Out}(\mathbb{G}).
\end{tikzcd}$$

Remark 1.3.7. In fact, the finiteness of $\widehat{\mathbb{T}}^{W_a}$ is equivalent to the finiteness of $\mathbb{T}^{W_a}$, hence equivalent to the finiteness of $\pi_0(\mathcal{P}_a)$. Therefore by the definition of $\mathcal{A}_G$ and $\mathcal{A}_{\widehat{G}}$, and the above remark, we immediately conclude that

$$\mathcal{A}_G^{ani} = \mathcal{A}_{\widehat{G}}^{ani}.$$

1.3.2 Automorphisms

Let $a \in \mathcal{A}^{\heartsuit}(\bar{k})$. As we mentioned at the beginning of this section, take $(E, \varphi) \in \mathcal{M}_{a,G}$, the sheaf of automorphisms $\underline{Aut}(E, \varphi)$ is represented by a group scheme $I_{E,\varphi} := h_{E,\varphi}^* I$, here $h_{E,\varphi} : \overline{X} \to [\mathfrak{g}_D/G](\bar{k})$.

The restriction of $I_{E,\varphi}$ to U_a is a torus, recall that $U_a \subset \overline{X}$ is the locus such that the cameral curve $\widetilde{X}_a \to \overline{X}$ is étale. But the group scheme $I_{E,\varphi}$ over $\overline{X}$ is not flat nor smooth in general, but [10] shows that there is a unique group scheme $I_{E,\varphi}^{lis}$ smooth over $\overline{X}$, such that for any $\overline{X}$-scheme S smooth over $\overline{X}$,

$$\mathrm{Hom}_{\overline{X}}(S, I_{E,\varphi}) = \mathrm{Hom}_{\overline{X}}(S, I_{E,\varphi}^{lis}).$$

Moreover, $I_{E,\varphi}^{lis}$ is isomorphic to $I_{E,\varphi}$ if restricted to the open locus U_a.

Now notice that by the universal property of $I_{E,\varphi}^{lis}$, we obtain a homomorphism

$$J_a \to I_{E,\varphi}^{lis}$$

which restricts to an isomorphism over U_a. Then by the universal property of the Néron model $J_a^{\flat}$ of J_a, one has

$$I_{E,\varphi}^{lis} \to J_a^{\flat}$$

which again restricts to an isomorphism above U_a.

Fix a point $\infty \in \mathcal{A}(\bar{k})$, we consider the open subset $\mathcal{A}^{\infty}$ of $\mathcal{A} \otimes_k \bar{k}$ which consisting of the points $a \in \mathcal{A}^{\bar{k}}$ such that $a(\infty) \in \mathfrak{c}_D^{rs}(\bar{k})$, where $\mathfrak{c}_D^{rs}$ is image of the covering $\mathfrak{t}^{rs} \to \mathfrak{c}^{rs}$, where $\mathfrak{t}^{rs}$ is the subset of regular semisimple elements of $\mathfrak{t}$. For $\widetilde{a} = (a, \infty) \in \mathcal{A}(\bar{k})$, we denote $W_{\widetilde{a}}$ to be the image of $\pi_1(U_a, \infty) \to \mathbb{W} \rtimes \mathrm{Out}(\mathbb{G})$.

Then one can have the following characterization of the automorphism groups:

Proposition 1.3.8 (section 4.11, [46])**.** *For all $(E, \varphi) \in \mathcal{M}_{a,G}(\bar{k})$ with $a \in \mathcal{A}^{\heartsuit}(\bar{k})$, then*

$$H^0(\overline{X}, J_a) \subset Aut(E, \varphi) \subset H^0(\overline{X}, J_a^{\flat}) = \mathbb{T}^{W_a},$$

where $Aut(E, \varphi)$ is the set of global sections of the sheaf of automorphisms $Aut(E, \varphi)$.

Corollary 1.3.9. *For all $(E, \varphi) \in \mathcal{M}_{a,G}(\bar{k})$ with $a \in \mathcal{A}^{\heartsuit}(\bar{k})$, then $Aut(E, \varphi)$ can be identified with a subgroup of $\mathbb{T}^{W_{\widetilde{a}}}$.*

Corollary 1.3.10. *For $(E, \varphi) \in \mathcal{M}_{a,G}(\bar{k})$ with $a \in \mathcal{A}^{ani}(\bar{k})$, the automorphism group $Aut(E, \varphi)$ is finite.*

1.3.3 Product formula

We will talk about the product formula of Hitchin fibres due to Ngô, the reference for this formula is theorem 4.6 of [45].

In this subsection, we will restrict ourselves in the locus $\mathcal{A}^{\heartsuit}(\bar{k})$. As before, $U_a \subset \overline{X}$ will be the open locus where the cameral cover $\pi_a : \widetilde{X}_a \to \overline{X}$ is étale.

Moreover, for places $v \in \overline{X} - U_a$, we will use $\overline{X}_v$ the formal disk at v, and $\overline{X}_v^{\bullet}$ the corresponding pointed formal disk.

As in the global case, we can consider the category $\mathcal{M}_{a,G,v}$ of pairs (E_v, φ_v) where E_v is a G-torsor above $\overline{X}_v$ and $\varphi_v \in H^0(\overline{X}_v, ad(E_v) \otimes O_X(D))$ which is mapped into a under the Hitchin morphism.

It is easy to see all the above constructions can be moved into the local case, one still has the notion of $J_{a,v}$ and $\mathcal{P}_{a,v}$ acting on $\mathcal{M}_{a,G,v}$.

Moreover, one can consider a slight variant of the above restriction. Let $(E^*, \varphi^*) \in \mathcal{M}_{a,G}(\bar{k})$ given by the Kostant section, where the G-torsor E^* is trivial.

Then we can consider the triples $(E_v, \varphi_v, \iota_v)$ with (E_v, φ_v) defined above $\overline{X}_v$ as before. The map $\iota_v : (E_v, \varphi_v) \to (E_v^*, \varphi_v^*)$ is an isomorphism over the pointed disk $\overline{X}_v^\bullet$. We use $\mathcal{M}_{a,G,v}^\bullet$ to denote the category of the triples.

Consider the group $\mathcal{P}_{a,v}^\bullet$ of $J_{a,v}$ torsors over $\overline{X}_v$ with a trivialization over $\overline{X}_v^\bullet$, then we still have the action of $\mathcal{P}_{a,v}$ on $\mathcal{M}_{a,G,v}^\bullet$.

We now are finally ready to state the product formula

Theorem 1.3.11 (theorem 4.6, [45])**.** *Suppose that the category $\mathcal{M}_a(\bar{k})$ is not empty, then we have the following equivalence of categories*

$$[\mathcal{M}_{a,G}(\bar{k})/\mathcal{P}_a(\bar{k})] = \prod_{v\in\overline{X}-U_a}[\mathcal{M}_{a,G,v}/\mathcal{P}_{a,v}] = \prod_{v\in\overline{X}-U_a}[\mathcal{M}_{a,G,v}^\bullet/\mathcal{P}_{a,v}^\bullet].$$

Sketch of the proof. One consider the following diagram first

$$\begin{tikzcd}
\prod_{v\in\overline{X}-U_a}[\mathcal{M}_{a,G,v}^\bullet/\mathcal{P}_{a,v}^\bullet] \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] & & {[\mathcal{M}_{a,G}(\bar{k})/\mathcal{P}_a(\bar{k})]} \arrow[dl, "\gamma"] \\
& \prod_{v\in\overline{X}-U_a}[\mathcal{M}_{a,G,v}/\mathcal{P}_{a,v}] &
\end{tikzcd}.$$

The arrow β is clearly the forgetful morphism which forgets the trivialization at generic points of each $\overline{X}_v$.

The map γ is obtained by restriction.

The map α is obtained by glueing torsors on each $\overline{X}_v$ with the torsor (E, φ) given by the trivial G-torsor on $\overline{X} - \overline{U}_a$ and the Kostant section φ on $\overline{X} - U_a$. It is clear that $\beta = \gamma \circ \alpha$.

We will mainly consider fully faithfulness the arrow α in the below, for other parts we refer readers to the original proof.

Let $m = (m_v)$ and $n = (n_v)$ be two points on $\mathcal{M}_{a,v}^\bullet$ with $v \in \overline{X} - U_a$, and suppose that

$$\alpha(m) = \alpha(n) \in [\mathcal{M}_a/P_a].$$

This isomorphism defines an object $p_a \in \mathcal{P}_a$ which sends $\alpha(m)$ to $\alpha(n)$ as objects on $\mathcal{M}_a$. Note that the restriction of $\alpha(m)$ and $\alpha(n)$ to $\overline{X} - U_a$ are already isomorphic to the chosen fixed (E^*, φ^*). Therefore the object p_a admits a trivialisation on $\overline{X} - U_a$ and defines an object in $\prod_v P_{a,v}$. This proves that α is fully faithful.

□

1.4 Adelic description of Higgs bundles

The references for this section are [45], [46], [12], [28] and appendix E of [37]. In [12], there is even a dictionary between adelic terms and bundle terms.

We will mainly deal with the case of groups SL_n and PGL_n in this thesis, for which we can use the definition of degrees and stability of vector bundles. As for the definition of the stability on general principal G-Higgs bundles, we refer readers to [21], [12], [14], [12], [53] and [39].

We begin with the following lemma,

Lemma 1.4.1 (Lemme 1.1, [45])**.** *Let G be a quasi-split reductive group scheme over the curve X, and let η be the generic point of X. Let E be a G-torsor over X. The element $cl_\eta(E) \in H^1(F, G)$ defined by the isomorphism class of G-torsor E_η will have trivial image in $H^1(F_v, G)$ for all $v \in |X|$. Conversely, let c be an element of the set*

$$Ker^1(F, G) = Ker\bigg(H^1(F, G) \to \prod_{v \in |X|} H^1(F_v, G)\bigg).$$

Then there is a G-torsor E over X, whose isomorphism class of the generic fibre $cl_\eta(E)$ is given by the element c.

Remark 1.4.2. According to a theorem by Kneser, Harder, and Chernousov, $\mathrm{Ker}^1(F, G) = 0$ for G simply connected or semisimple adjoint. See [48], Theorem 6.4 and 6.22. If $\mathrm{Ker}^1(F, G) = 0$, G is said to satisfy the **Hasse principle**.

In particular, one has $\mathrm{Ker}^1(F, SL_n) = 0$.

Take $c \in \mathrm{Ker}^1(F, G)$, one can choose a G-torsor model $\mathbb{E}_c$ over X with trivializations at all $\mathrm{Spec}(O_v)$ for all $v \in |X|$. Let G_c be the automorphism group of $\mathbb{E}_c$ and $\mathfrak{g}_c$ be the associated Lie algebra of G_c. Then we consider the category $\mathcal{M}_{c,G}(k)$ of Higgs bundles $(E, \varphi)$with an extra fixed trivialization at $\eta \in X$, $\iota : E_\eta \xrightarrow{\sim} \mathbb{E}_{c\eta}$.

Proposition 1.4.3 ([45])**.** *One has an equivalence of categories between the category $\mathcal{M}_{c,G}(k)$ and the category of the tuples $(\gamma, (g_v)_{v \in |X|})$, such that*

1. *$\gamma \in \mathfrak{g}_c(F)$, where F is the function field of curve X;*

2. *$g_v \in G_c(F_v)/G_c(O_v)$ such that for almost all $v \in |X|$, g_v is trivial.*

3. *moreover, $ad(g_v^{-1})(\gamma) \in \varpi_v^{-d_v}\mathfrak{g}(O_v)$, where $D = \sum_v d_v[v]$ and ϖ is the uniformiser at the place v.*

Remark 1.4.4. The idea is that g_v will serves as the gluing function to glue the generic trivialisation at the generic point with the torsor defined on the formal disk at place v.

We need to require the pole condition which Higgs fields should satisfy, therefore we need the condition

$$\gamma \in \varpi_v^{-d_v} ad(g_v)\mathfrak{g}(O_v).$$

Let us mention that there is also an adelic description of the degree

$$\begin{aligned} \deg : \mathbb{A}_F^\times &\to \mathbb{Z}, \\ a = (a_v)_{v\in|X|} &\mapsto -\sum_v \deg(v)\mathrm{val}(a_v), \end{aligned}$$

where we use $\mathbb{A}_F$ to denote the adele of the function field F of the curve X.

Then our desired degree notion for our bundles can be described as

$$\deg(\gamma, (g_v)_{v\in|X|}) := \deg(\det(g_v)_{v\in|X|})$$

Note that from the spirit of the above proposition 1.4.3, we know that a principal G-bundle E with a fixed trivialisation at $\eta \in X$ is equivalent to the datum $g \in G(\mathbb{A})/G(O)$, and we have

$$\deg(E) = \deg(\det(g)).$$

Moreover, we have that

$$\mathrm{Aut}(E) = \{\delta \in G(F) \mid g^{-1}\delta g \in G(O)\}.$$

So one knows that the stack $\mathrm{Bun}(G)$ of G-principal bundles with fixed trivialisation is equivalent to the set of double cosets $G(F)\backslash G(\mathbb{A})/G(O)$.

Remark 1.4.5. As we will see in the following chapters, for SL_n-Higgs bundles we would like to consider the twisted version, i.e., we would like to consider SL_n-Higgs bundles of degree d, and we shall require that $g.c.d(d, n) =$

1. The adelic description above can only give us the moduli stack of SL_n-Higgs bundles of degree 0. But one can remedy this situation, following [28], by replacing the maximal compact subgroup $G(O)$ with $\mathfrak{K}$, which we will define in the following.

Let E_0 be a vector bundle of rank n over our fixed curve X. We assume that $L = \det(E_0)$ is of degree d. Then by considering the space V of all meromorphic sections of E_0 over X, we see that $\dim V = n$.

If $E_{0,x}$ denotes the completion of $\lim_{U \ni x} \Gamma(U, E_0)$, we see that $E_{0,x}$ is an O_x lattice in $V \otimes_F F_x$, we define locally $\mathfrak{K}_x := SL(E_{0,x}) \subset SL(V \otimes K_x)$. hence we have a maximal compact subgroup $\mathfrak{K}$ of $SL_V(\mathbb{A})$. We will see that the proposition 1.4.3 will not be affected if we replace the group $G(O)$ by $\mathfrak{K}$, hence so is the dictionary between adelic terms and vector bundles we mentioned above. As we will see that the groups $G(O)$ and $\mathfrak{K}$ will be implicit in the counting formula as well as orbital integrals, we will not stress the difference again, if there is no possible of confusion. However, it does not mean that the counting of Higgs bundles of degree d will be the same as the counting of Higgs bundles of degree 0. Indeed, in the process of writing out the counting integral, we will normalize the measure on $G(\mathbb{A})$ to make $\mathrm{vol}(\mathfrak{K}) = 1$ or $\mathrm{vol}(G(O)) = 1$, and the constants of normalization will be different.

To study the moduli of stable (semistable) G-principal Higgs bundles, we still need to have a description of stability and subbundles in terms adelic language, and that is what we will do in the remainder of this section, by following [12], [14] and [39]. And we refer readers to section IV.1 of [54] for an explicit computation in the case of GL_n of all the following notations and morphisms.

For simplicity, we will restrict ourselves to the case where G is SL_n, PGL_n or GL_n splitting over the curve X, from now on in this section. The principal G-Higgs bundles we will consider in the following can be viewed as Higgs bundles with possible extra structures on the underlying vector bundles and Higgs fields. For instance, SL_n-Higgs bundles are Higgs bundles whose determinant line bundle is fixed and whose Higgs fields have trace 0.

Most of these constructions will remain valid in the case of more general groups by considering the corresponding moduli of principal Higgs bundles.

We now fix a minimal parabolic subgroup $B = P_0$ of G which should be a Borel subgroup of G if G is quasi-split, and a Levi subgroup M_0. We denote by $\mathcal{P}$ the set of standard parabolic subgroups of G, which by definition contain P_0. For $P \in \mathcal{P}$, we use N_P and M_P to denote the unipotent radical of P and the unique levi component of P containing M_0.

We denote by A_P the maximal split torus in the centre Z_P of M_P, i.e.,

$$A_P := \mathrm{Hom}_k(\mathbb{G}_{m,k}, Z_P) \otimes \mathbb{G}_{m,k},$$

Here we use $\mathbb{G}_{m,k}$ to denote the multiplicative group with values in k.

For each $P \in \mathcal{P}$, Let $X_*(A_P) := \mathrm{Hom}_k(\mathbb{G}_{m,k}, Z_P)$, and we set

$$\mathfrak{a}_P = \mathbb{R} \otimes X_*(A_P).$$

One has a surjective morphism

$$H_P : P(\mathbb{A}) \to \mathfrak{a}_P$$

such that for all $\chi \in X^*(P)$ and $p \in P(\mathbb{A})$,

$$\langle \chi, H_P(p) \rangle = \deg(\chi(p)).$$

If $P \subset Q$ are two standard parabolic subgroups of G, then we have a canonical retraction from $A_Q \hookrightarrow A_P$

$$\mathfrak{a}_P \twoheadrightarrow \mathfrak{a}_Q.$$

as well as the canonical splitting, since they are both vector spaces,

$$\mathfrak{a}_P = \mathfrak{a}_Q \oplus \mathfrak{a}_P^Q,$$

where $\mathfrak{a}_P^Q$ is the kernel of the canonical retraction. Taking the dual, we get a splitting

$$\mathfrak{a}_P^* = \mathfrak{a}_P^{Q*} \oplus \mathfrak{a}_Q^*.$$

Let Φ_P be the set of nontrivial characters of A_P which occur in the Lie algebra $\mathfrak{g}$ of G, and let $\Phi_P^+ \subset \Phi_P$ be the set of the nontrivial characters of A_P which occur in the Lie algebra $\mathfrak{n}_P$ of the unipotent radical $N_P \subset P$. It is well-known that $\Phi_0 = \Phi_{P_0}$ is a root system, and we set $\Delta_P \subset \Phi_P^+$ to be the set of nontrivial restriction to A_P of the simple roots in Δ_0, where Δ_0 is defined to the set of simple roots in Φ_0. Then Δ_P is a basis of the real vector space $\mathfrak{a}_P^{G*}$. For each $\alpha \in \Delta_P$, there is a corresponding coroot $\alpha^\vee \in \mathfrak{a}_P^G$ such that $(\alpha^\vee)_{\alpha\in\Delta_P}$ is a basis of the real vector space $\mathfrak{a}_P^G$.

If $P \subset Q$ are two standard parabolic subgroups of G , let $\Phi_P^Q := \Phi_{P\cap M_Q}$ be the set of α in Φ_P which occur in the Lie algebra $\mathfrak{m}_Q$ of M_Q.

On the one hand, Δ_P^Q, which is defined to be $\Delta_{P\cap M_Q}$, is contained in $\mathfrak{a}_P^{Q*} \subset \mathfrak{a}_P^{G*}$ and is a basis of the real vector space $\mathfrak{a}_P^{Q*}$. On the other hand, the projection of $(\alpha^\vee)_{\alpha\in\Delta_P^Q}$ onto $\mathfrak{a}_P^Q$ is a basis of the real vector space $\mathfrak{a}_P^Q$, by taking the dual, we obtain the basis $(\varpi_\alpha^Q)_{\alpha\in\Delta_P^Q}$ of $\mathfrak{a}_P^{Q*}$.

We now follow Arthur [4] to define the following characteristic functions τ_P^Q of the following so called acute Weyl chamber,

$$\mathfrak{a}_P^{Q+} := \{H \in \mathfrak{a}_P^Q \mid \langle \alpha, H\rangle > 0, \forall \alpha \in \Delta_P^Q\}$$

and the characteristic function $\widehat{\tau}_P^Q$ of the following so called obtuse Weyl chamber

$${}^+\mathfrak{a}_P^Q := \{H \in \mathfrak{a}_P^Q \mid \langle \varpi_\alpha^Q, H\rangle > 0, \forall \alpha \in \Delta_P^Q\}.$$

Remark 1.4.6. We will simply write $\tau_P^Q, \widehat{\tau}_P^Q$ as τ_P and $\widehat{\tau}_P$, respectively, if $Q = G$.

We now recall the Langlands lemma, following lemme 2.4.2 in [12]

Lemma 1.4.7. *For all vector bundle $E \neq 0$ over the curve X, one has*

$$\sum_{F^\bullet} (-1)^{length(F^\bullet)-1} = \begin{cases} 1 & \text{if } E \text{ is semistable} \\ 0 & \text{otherwise,} \end{cases}$$

where $F^\bullet$ runs through the set of the so called destabilizing flag of subbundles of E

$$0 = F_0 \subsetneq F_1 \subsetneq \ldots \subsetneq F_r = E$$

where by destabilizing, we mean $\mu(F_i) > \mu(E)$ for $1 \le i \le r-1$, and we define $length(F^\bullet) = r$.

Now, we know that a principal G-bundle E with a fixed trivialization at the generic point $\eta \in X$, can be represented by $g \in G(\mathbb{A})/G(O)$, where $\mathbb{A}$ is the adele ring of the function field F of X. Then a flag $F^\bullet$ of subbundles of E is equivalent to a pair (P, δ), where P is a standard parabolic subgroup of G, and $\delta \in P(F)\backslash G(F)$. By the spirit of the section 2.6 of [14], we have the following geometric interpretation of $\widehat{\tau}_P$,

Lemma 1.4.8. *$\widehat{\tau}_P(H_P(\delta g)) = 1$ if and only if the flag of subbundles $F^\bullet$ given by the pair (P, δ) of the vector bundle E is destabilizing, where we assume that E can be represented by $g \in G(\mathbb{A})/G(O)$.*

Hence E is semistable if and only if

$$F^G(g) := \sum_P (-1)^{\dim(\mathfrak{a}_P^G)} \sum_{\delta \in P(F)\backslash G(F)} \widehat{\tau}_P(H_P(\delta g)) = 1.$$

Hence one has the above characteristic function supported on the locus of semistable bundles inside the stack of G-bundles.

Remark 1.4.9. Moreover, if we denote $\mathrm{Bun}_n^{e,ss}$ to be the stack of semistable vector bundles of degree e, n, (so that here we are taking our G to be GL_n), we fix a Haar measure on $G(\mathbb{A})$ such that $G(O)$ has measure 1($\mathrm{vol}(\mathfrak{K}) = 1$ for other degrees in the case of $G = SL_n$), and we consider the mass of the stack is given by

$$|\mathrm{Bun}_n^{e,ss}| := \sum \frac{1}{|\mathrm{Aut}(E)|}$$

where E runs through isomorphisms of semistable bundles. We see that, E can be represented by the double coset $G(F)\backslash G(\mathbb{A})^e/G(O)$, here by $G(\mathbb{A})^e$. Here by $g \in G(\mathbb{A}^e)$, we mean the elements $g \in G(\mathbb{A})$ with $\deg(\det(g)) = e$

$$\mathrm{Aut}(E) = \{\delta \in G(F) \mid g^{-1}\delta g \in G(O)\}.$$

Hence

$$|\mathrm{Aut}(E)| = |G(F) \cap g^{-1}G(O)g|.$$

Hence

$$\begin{aligned}
|\mathrm{Bun}_n^{e,ss}| &:= \sum_{E \text{ semistable}} \frac{1}{|\mathrm{Aut}(E)|} \\
&= \sum_{\substack{g \in G(F)\backslash G(\mathbb{A})^e/G(O) \\ g \text{ represents a semistable vector bundle}}} |g^{-1}G(F)g \cap G(O)\backslash G(O)| \\
&= \int_{G(F)\backslash G(\mathbb{A})^e} \sum_P (-1)^{\dim(\mathfrak{a}_P^G)} \sum_{\delta \in P(F)\backslash G(F)} \widehat{\tau}_P(H_P(\delta g)) dg
\end{aligned}$$

by our choice of the Haar measure dg on $G(\mathbb{A})$.

Now we need to do some minor modifications of the above constructions to count the moduli of Higgs bundles.

Following section 3.5 of [12], one can have that

$$|\mathcal{M}_G^d(k)| := \sum_{(E,\varphi)} \frac{1}{|\mathrm{Aut}(E,\varphi)|} = \sum_E \frac{|\mathrm{Hom}(E,E(D))|}{|\mathrm{Aut}(E)|}.$$

Note that $Aut(E,\varphi)$ is the stabilizer of φ inside $Aut(E)$, this makes senses of the second equality above.

Recall here we use $\mathcal{M}_G^d$ to denote the moduli stack of G-Higgs bundles of fixed degree d. Recall that the divisor D is what we used to define the Hitchin system, and $E(D)$ is an abbreviation for $E \otimes O_X(D)$.

We now recall Lemme 3.10.1 of [12],

Lemma 1.4.10. *For all vector bundle E, one has*

$$|Hom^{ss}(E,E(D))| = \sum_{F^\bullet} (-1)^{length(F^\bullet)-1} |Hom(F^\bullet, F^\bullet(D))|,$$

where the sum is taking over the set of destabilizing flags $F^\bullet$ of E.

Hence overall, we can have a practical counting formula of stable (we are assuming that $g.c.d(d,n) = 1$) Higgs bundles

$$|\mathcal{M}_G^{d,s}(k)| = \sum_E \sum_{F^\bullet} (-1)^{\mathrm{length}(F^\bullet)-1} \frac{|\mathrm{Hom}(F^\bullet, F^\bullet(D))|}{|\mathrm{Aut}(E)|},$$

where $|\mathrm{Hom}(F^\bullet, F^\bullet(D))|$ will be given by

$$K_{P,D}(\delta g) = \sum_{\gamma \in \mathfrak{p}(F)} 1_D((\delta g)^{-1} \gamma (\delta g)).$$

Here the flat $F^\bullet$ is given by (P,δ) and 1_D is the characteristic function supported on the set $\varpi_v^{-d_v} ad(g_v)\mathfrak{g}(O_v)$.

Overall, combining what we have obtained in the case of vector bundles, we have the following counting formula for Higgs bundles

$$|\mathcal{M}_G^{d,s}(k)| = \int_{G(F)\backslash G(\mathbb{A})^d} \sum_P (-1)^{\dim(\mathfrak{a}_P^G)} \sum_{\delta \in P(F)\backslash G(F)} \widehat{\tau}_P(H_P(\delta g)) K_{P,D}(\delta g) dg. \tag{1.2}$$

Remark 1.4.11. In fact, one cannot obtain the above counting formula naively as what we did above. Unlike the vector bundle case in which the moduli stack of (semi)stable vector bundles is of finite type, the stack of Higgs bundles is not of finite type, so one must take care of the convergence problem. To remedy this, Chaudouard used the so called T-stability to do a truncation on the moduli stack of Higgs bundles, like what Arthur did in [4]. It turns out that the counting of $\mathcal{M}_G^{d,s,T}(k)$ behaves like a polynomial in T, as the parameter T varies, where $\mathcal{M}_G^{d,s,T}$ is the moduli space of T-stable Higgs bundles, see section 4 of [12] and also [4]. Hence one can safely take $T = 0$, and get the above counting formula for $|\mathcal{M}_G^{d,s}(k)|$.

Chapter 2

Stringy invariants and gerbes

2.1 Stringy E-polynomials and stringy counting

This section is following section 2 of [27].

Let $\mathcal{X}$ be a Deligne-Mumford stack.

Definition 2.1.1.
1. We say that $\mathcal{X}$ is a finite quotient stack if there is an algebraic space Y with a generically free action of a finite group Γ such that $\mathcal{X} \simeq [Y/\Gamma]$.
2. If moreover Γ is abelian, we say that $\mathcal{X}$ is a finite abelian quotient stack.

We now introduce the inertia stacks,

Definition 2.1.2. Recall that for a stack $\mathcal{X}$, the inertia stack is defined to be

$$\mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X}.$$

More explicitly, $I\mathcal{X}$ sends a test scheme S to the groupoids $I\mathcal{X}(S)$ whose objects are the pairs (x, α) with $x \in \mathcal{X}(S)$ and $\alpha \in Aut_{\mathcal{X}}(x)$.

In particular if $\mathcal{X} = [Y/\Gamma]$ is a finite quotient stack, one has the following equivalence

$$I\mathcal{X} \simeq \coprod_{[\gamma]\in\Gamma/\mathrm{conj}} [Y^{\gamma}/C(\gamma)],$$

where $C(\gamma)$ is the centralizer of γ in Γ and $Y^{\gamma} \subset Y$ is the locus consisting of fixed points of γ.

Definition 2.1.3. Let $x \in Y^\gamma$, then the tangent space T_xY admits a representation of the group generated $I := \langle \gamma \rangle$, over $\overline{k}$, we can choose a basis of eigenvectors, hence get a sequence of eigenvalues $(\lambda_1, \ldots, \lambda_k)$ with respect to the chosen basis of eigenvectors. Note that γ is of finite order, hence we can choose a basis of eigenvectors here.

We choose a primitive root of unity ζ of order $|I|$ in $\overline{k}$, for each λ_i, there is a $0 \leq c_i < r$ such that $\zeta^{c_i} = \lambda_i$. Then we define

$$F(\gamma, x) = \sum_{i=1}^{k} \frac{c_i}{r}.$$

This number is locally constant on Y^γ, and therefore defines a function on $\pi_0(Y^\gamma)$. Moreover, $F(\gamma)$ is constant on the $C(\gamma)$-orbit in $\pi_0(Y^\gamma)$. Hence one actually have a function

$$F(\gamma, \cdot) : \pi_0[Y^\gamma / C(\gamma)] \to \mathbb{Q}.$$

Now we are ready to introduce the following stringy invariants.

Definition 2.1.4. Let $\mathcal{X} = [Y/\Gamma]$ be a smooth finite quotient stack over a field k, we can choose a representative of Y to be smooth.

1. If $k = \mathbb{C}$, then the so called stringy E-polynomial is defined as

$$E_{str}(\mathcal{X}; u, v) = \sum_{\gamma \in \Gamma/\mathrm{conj}} \left(\sum_{\mathcal{Z} \in \pi_0(Y^\gamma / C(\gamma))} E(\mathcal{Z}; u, v)(uv)^{F(\gamma, \mathcal{Z})} \right),$$

   where for $\mathcal{Z} = [W/C(\gamma)]$, we define

$$E(\mathcal{Z}; u, v) = \sum_{p,q,k} (-1)^k \dim \left(H_c^{p,q,k}(W)^{C(\gamma)} \right) u^p v^q.$$

   Note that $H_c^{p,q,k}$ are the space $Gr^W_{p+q}(H^k_c(W))^{p,q}$ given by the mixed Hodge structure on the compactly supported cohomology of W.

2. If $k = \mathbb{F}_q$ is a finite field, one can define the stringy counting

$$\#_{str}(\mathcal{X}) = \sum_{\gamma \in \Gamma/\mathrm{conj}} \left(\sum_{\mathcal{Z} \in \pi_0(Y_\gamma / C(\gamma))} q^{F(\gamma, \mathcal{Z})} \# \mathcal{Z}(k) \right),$$

where

$$\#\mathcal{Z}(k) := \sum_{x \in \mathcal{Z}(k)_{iso}} \frac{1}{|\mathrm{Aut}(x)|}$$

is the groupoid mass of $\mathcal{Z}(k)$.

Recall that for $\mathcal{X} = [Y/\Gamma]$, one has

$$I\mathcal{X} = \coprod_{[\gamma] \in \Gamma/\mathrm{conj}} [Y^{\gamma}/C(\gamma)].$$

So in fact, one can write the stringy counting of $\mathcal{X}$ more compactly as

$$\#_{str}(\mathcal{X}) = \sum_{x \in I\mathcal{X}(k)_{iso}} \frac{q^{F(x)}}{|\mathrm{Aut}_{I\mathcal{X}(k)}(x)|}.$$

2.2 Gerbes and twisted stringy invariants

As one can see in [31], [20], to fully formulate the topological mirror symmetry, one has to take the notion of gerbes into account.

Definition 2.2.1. Let S be a Deligne-Mumford stack and A be a commutative group scheme over S. Then a gerbe is a morphism of algebraic stacks $\alpha : \mathcal{W} \to \mathcal{S}$ such that

1. For any scheme S' over S, $\forall x, y \in Ob(\mathcal{W}(S'))$ there is an étale covering $\mathcal{S}''$ of S' such that x, y become isomorphic in $\mathcal{W}(\mathcal{S}'')$.

2. There is an étale covering $\mathcal{S}'$ of $\mathcal{S}$ such that $\mathcal{W}(\mathcal{S}')$ is not empty.

A banding of a gerbe $\mathcal{W}$ over $\mathcal{S}$ by A consists of isomorphisms $A_{\mathcal{S}'} \simeq \underline{\mathrm{Aut}}_{\mathcal{S}'}(x)$ of étale group sheaves for every $\mathcal{S}$-stack $\mathcal{S}'$, and every object $x \in \mathcal{W}(\mathcal{S}')$. A gerbe banded by a group A is called an A-gerbe.

Remark 2.2.2. Descent data for A-gerbes are given by 2-cocyles with values in A, one knows that the set of isomorphism classes of A-gerbes is $H^2_{et}(\mathcal{S}, A)$.

We now recall the construction of the induced A-torsor on $I\mathcal{X}$ following [31] and [27].

Note

$$I\mathcal{X} = \coprod_{\gamma \in \Gamma/\mathrm{conj}} [Y^\gamma / C(\gamma)].$$

Let $\alpha \in H^2_{et}(\mathcal{X}, A)$, which corresponds to a Γ-equivariant A-gerbe α on Y.

The Γ-equivariant structure of α is given by isomorphisms

$$\eta_\gamma : \gamma^* \alpha \xrightarrow{\simeq} \alpha$$

for every $\gamma \in \Gamma$. Now consider those isomorphisms to corresponding γ-fixed locus Y^γ, one obtains an automorphism

$$\eta_\gamma|_{Y^\gamma} : \alpha|_{Y^\gamma} = (\gamma|_{Y^\gamma})^* \alpha|_{Y^\gamma} \xrightarrow{\simeq} \alpha.$$

This automorphism of α gives rise to an A-torsor L_γ on Y^γ, and we know that those torsors L_γ should be $C(\gamma)$-equivariant since α is Γ-equivariant. Finally, we obtain an A-torsor L_γ on $[Y^\gamma / C(\gamma)]$.

Now we are ready to introduce the following gerbe twisted stringy invariants.

Definition 2.2.3. Let $\mathcal{X} = [Y/\Gamma]$ be a quotient stack of a smooth complex variety Y by a finite group Γ, then for a positive integer n and a μ_n-gerbe $\alpha \in H^2_{et}(\mathcal{X}, \mu_r)$ we define the α-twisted stringy E-polynomial of $\mathcal{X}$ as

$$E_{str}(\mathcal{X}, \alpha; u, v) = \sum_{\gamma \in \Gamma/\mathrm{conj}} \left(\sum_{\mathcal{Z} \in \pi_0([Y^\gamma / C * \gamma])} E(\mathcal{Z}, L_\gamma; u, v)(uv)^{F(\gamma, \mathcal{Z})} \right)$$

where L_γ denotes the μ_n-torsor induced from the trivialisation of α on the fixed locus Y^γ and

$$E(\mathcal{Z}, L_\gamma : u, v) = E^\chi(L_\gamma; u, v)$$

where $\chi : \mu_r(\mathbb{C}) \to \mathbb{C}^\times$ is the standard character, and E^χ denotes the part of E-polynomial corresponding to the χ-isotopic component of the total space $H^*_c(L_\gamma)$.

Definition 2.2.4. For a positive integer n which is prime to the characteristic of the field k, and $\alpha \in H^2_{et}(\mathcal{X}, \mu_n)$, we define

$$\#^\alpha_{str}(\mathcal{X}) = \sum_{\gamma \in \Gamma/\mathrm{conj}} \left(\sum_{\mathcal{Z} \in \pi_0([Y^\gamma / C(\gamma)])} q^{F(\gamma, \mathcal{Z})} \#^{L_\gamma} \mathcal{Z}(k) \right),$$

where L_γ is the induced ℓ-adic local system on Y^γ obtained from the μ_n-torsor $P_\alpha|_{[Y^\gamma / C(\gamma)]}$. To make sense of this construction and to calculate the trace of the Frobenius, we always fix a isomorphism $\mu_n(k) \subset \mathbb{C} \simeq \overline{\mathbb{Q}_\ell}$.

$$\#^{L_\gamma} \mathcal{Z}(k) = \sum_{z \in \mathcal{Z}(k)} \frac{\mathrm{Tr}(Fr_z, L_{\gamma, z})}{|\mathrm{Aut}(z)|},$$

where Fr_z denotes the Frobenius at z.

Finally, we record Theorem 2.19 in [27], which allows us reduce the topological mirror symmetry into an equality of point counts.

Theorem 2.2.5. *Let $R \subset \mathbb{C}$ be a subalgebra of finite type over $\mathbb{Z}$. Let Y_1 and Y_2 be two smooth R-varieties acted on by two finite abelian groups Γ_1, Γ_2 respectively. For $i = 1, 2$, let $\mathcal{X}_i = [Y_i/\Gamma_i]$ be the corresponding quotient stack, and let α_i be a μ_n-gerbe on $\mathcal{X}_i$. If for any ring homomorphism $R \to \mathbb{F}_q$, the following is true*

$$\#^{\alpha_1}_{str}(\mathcal{X}_1 \times_R \mathbb{F}_q) = \#^{\alpha_2}_{str}(\mathcal{X}_2 \times_R \mathbb{F}_q).$$

We also have the following equality of twisted stringy E-polynomials:

$$E_{str}(\mathcal{X}_1 \times_R \mathbb{C}, \alpha_1; u, v) = E_{str}(\mathcal{X}_2 \times_R \mathbb{C}, \alpha_2; u, v).$$

2.3 Lifting gerbe on moduli space of SL_n-Higgs bundles

In this section, we will follow [40] and section 4 of [31] to reduce the gerbe twisted stringy counting to the counting given by fundamental lemma on the Hitchin fibres.

Let us now specialize to the setting of SL_n and PGL_n-Higgs moduli stacks. It is well know that

$$\mathcal{M}^d_{PGL_n} = [\mathcal{M}^L_{SL_n}/\mathrm{Pic}^0(X)[n]],$$

where $\mathcal{M}^d_{PGL_n}$ is moduli stack of PGL_n-Higgs bundle which admit a presentation as a stable Higgs bundle over each geometric point, note that the degree is well defined in $\mathbb{Z}/n\mathbb{Z}$. Let $\mathcal{M}^L_{SL_n}$ denote the stack of stable SL_n-Higgs bundle with fixed determinant line bundle L whose moduli space will be denoted by $M^L_{SL_n}$, and let us say that degree of L is d. We shall assume that d is coprime to n. In this case $M^L_{SL_n}$ is a smooth projective variety.

As we mentioned in the introduction, there is a lifting gerbe we now describe in the following. Let $(\mathbf{PE}, \mathbf{\Phi}) \to M^L_{SL_n} \times X$ be the universal projective bundle and a universal endomorphism bundle. Then the restriction $\mathbf{PE}|_{M^L_{SL_n}\times\{x\}}$ to the base-point in X is a projective bundle Ψ on $M^L_{SL_n}$. Then the gerbe α_L can be described as the gerbe of liftings of Ψ, in fact, it is easy to write a 2-cocycle on $M^L_{SL_n}$ representing it. It takes an étale neighbourhood to the category of liftings on the neighbourhood of Ψ to an SL_n-bundle. It is clear that α_L is a μ_n-torsor, where μ_n is the group of n-th root of unity. And by the construction we mentioned in section 2.2, we know that α_L induces a μ_n-torsor on the fixed locus $M^{L,\gamma}_{SL_n}$. By the same idea, one can define the lifting gerbe α_L on the moduli stack $\mathcal{M}^L_{SL_n}$.

Now let $\Gamma = \mathrm{Pic}^0(X)[n]$, and $\rho := \langle -, - \rangle : \Gamma \times \Gamma \to \mu_n$ be the Weil pairing and $\rho_\gamma := \langle \gamma, - \rangle : \Gamma \to \mu_n$ the character defined by γ with the aid of the pairing. Then ρ defines a isomorphism class $[\rho] \in H^2(\Gamma, \mu_n)$.

As for the lifting gerbe α_L one has the following proposition,

Proposition 2.3.1 (Proposition A.1 [40]). *The μ_n-torsor L_γ induced by the natural lifting gerbe α_L on $\mathcal{M}^L_{SL_n}$ is trivial, and the Γ-equivariant structure is given by the character ρ_γ^{-q} where q is the multiplicative inverse of d modulo n.*

Proof. To see that the line bundle is trivial. Consider a stable Higgs bundle (E, θ); then the automorphism given by

$$E \otimes L_\gamma = \lambda E.$$

For some scalar λ, this implies that γ acts trivially on the universal projective Higgs bundle restricted on $M_n^{L,\gamma}$, hence preserves any lifting of **PE** over $M_n^{L,\gamma}$.

As for the equivariant structure part, we refer readers to the original proof due to Loeser-Wyss [40] for arbitrary rank case and Hausel [31] section 8 for prime rank case.

□

- First let us consider a Γ-equivariant μ_n-gerbe α which is a pullback of a Γ-equivariant gerbe over a point, i.e. given by a element $[\rho'] \in H^2(\Gamma, \mu_n)$, then by the argument in the proof of the Proposition 2.3.1, one can write the twisted stringy E-polynomial of moduli of PGL_n-Higgs bundles in this case as the following

  $$E_{str}^{\rho'} = \sum_{\gamma \in \Gamma} (uv)^{F(\gamma)} E^{\rho'_\gamma}(M_n^{L,\gamma}/\Gamma; u, v).$$

  Here $E^{\rho'_\gamma}(M_n^{L,\gamma}; u, v)$ is the ρ'_γ-isotropic part of the E-polynomial .

- Now let us write the twisted string E-polynomial with respect to the lifting gerbe α_L, following Proposition 2.3.1. We have

  $$E_{str}^{\alpha_L} = \sum_{\gamma \in \Gamma} (uv)^{F(\gamma)} E^{\rho_\gamma^{-q}}(M_n^{L,\gamma}/\Gamma; u, v).$$

  where q is the multiplicative inverse of d modulo n.

  As we will show by the support theorems that will be mentioned in chapter 4, the counting result of SL_n-Higgs bundles will be independent of the degree d of the fixed determinant L, as long as $g.c.d(d, n) = 1$, see Remark 4.2.13.

  Therefore by varying the degree d, one can set $q = -1$. The E-polynomial of moduli space of SL_n-Higgs bundles will not change, since the counting does not rely on the degree d.

Therefore

$$E_{str}^{\alpha_L} = \sum_{\gamma\in\Gamma} (uv)^{F(\gamma)} E^{\rho_\gamma^{-q}} (M_n^{L,\gamma}/\Gamma; u, v) = \sum_{\gamma\in\Gamma} (uv)^{F(\gamma)} E^{\rho_\gamma} (M_n^{L,\gamma}/\Gamma; u, v).$$

Remark 2.3.2. The above argument shows that it would be equivalent to consider only gerbes which are pullbacks of an element $[\rho'] \in H^2(\Gamma, \mu_n)$. We know that for any two pairing of the above form

$$\rho_1 : \Gamma \times \Gamma \to \mu_n,$$
$$\rho_2 : \Gamma \times \Gamma \to \mu_n,$$

there is an integer s such that $\rho_1^s = \rho_2$.

Therefore by the same argument of varying degrees, we can always find a suitable degree d of the fixed determinant line bundle L such that

$$\rho_1^{-q} = \rho_2.$$

Finally, just as the statement in section 1.2 of [40], we can restrict ourselves to prove the following identity of stringy E-polynomials twisted by the Weil pairing ρ on $\Gamma \times \Gamma$,

$$E(\mathcal{M}_{SL_n}^L; u, v) = E_{str}^{\rho}(\mathcal{M}_{PGL_n}^d; u, v)$$

where

$$E_{str}^{\rho}(\mathcal{M}_{PGL_n}^d; u, v) = \sum_{\gamma\in\Gamma} (uv)^{F(\gamma)} E^{\rho_\gamma} (\mathcal{M}_{SL_n}^{L,\gamma}/\Gamma; u, v)$$

provided that the stringy E-polynomials do not depend on the degree of Higgs bundles under the assumption that the coprime condition, i.e., $g.c.d(d, n) = 1$ is satisfied.

Eventually, we can show that

Corollary 2.3.3. *$E_{str}^{\rho}(\mathcal{M}_{PGL_n}^d; u, v) = E_{str}^{\rho}(\mathcal{M}_{PGL_n}^e; u, v)$ as long as $g.c.d(d, n) = 1 = g.c.d(e, n)$.*

This is provided that the counting of $\mathcal{M}_{SL_n}^L$ is independent of the degree d of the fixed determinant L.

Chapter 3

Endoscopy and Coendoscopy Decomposition of Inertia Stacks

In this chapter, we study the relation between the (co)endoscopy decomposition of the moduli stack of stable G-Higgs bundles (the meaning of the decomposition will be clarified in Proposition 3.2.10) and the inertia stack of the moduli stack. For simplicity, groups considered in this chapter will mainly be SL_n and PGL_n, even though most of the results should be applicable to other groups after minor modifications.

The story begins with an observation in the paper of Narasimhan and Ramanan [44].

Theorem 3.0.1 (Proposition 3.3,[44])**.** *Let $Y \to X$ be a cyclic covering over a complete nonsingular algebraic curve of Galois group $G \simeq \mathbb{Z}/m\mathbb{Z}$, i.e. the cyclic group of order m. Let $M^Y(r,d)$ be the moduli space of stable vector bundles over Y, of rank r and degree d, and $U \subset M^Y(r,d)$ be the open subset of points in $M^Y(r,d)$ not fixed by any nontrivial element of G. Then the quotient variety U/G is canonically isomorphic to the nonsingular subvariety of $M^X(mr,d)$ consisting precisely of the fixed points of the action of $\widehat{G}$ on $M^X(mr,d)$.*

Remark 3.0.2. There are already some generalizations of this classical theorem above to other groups; we refer readers to [23] and [5].

Hausel-Thaddeus then noticed in [31] that their argument can be applied to the moduli spaces of stable Higgs bundles without change, which means that the set of fixed points of the action of $\Gamma = \mathrm{Pic}^0(X)[n]$ consists of the direct image of stable Higgs bundles over a cyclic unramified covering of the curve X. For details we refer readers to section 7 of [31], and this leads to the generalization of endoscopic decomposition of the Hitchin complex given by Maulik and Shen, and then a proof of topological mirror symmetry, see [41].

More recently, Groechenig-Wyss-Ziegler proved that there is an equivalence of groupoids between inertia stacks and the (co)endoscopic decomposition of the stacks of Higgs bundles in [26], Theorem 5.14, see 3.2.14.

At first glance, the theorem of Groechenig-Wyss-Ziegler appears quite different from the theorem of Narasimhan-Ramanan and Hausel-Thaddeus. In this chapter, we will recast the classic theorem of Narasimhan-Ramanan. More precisely, we are going to extend this equivalence of groupoids of Groechenig-Wyss-Ziegler over the entire moduli of stable Higgs bundles. The key observation is that for stable Higgs bundles, at least in the case of PGL_n and SL_n, the automorphism group is of finite order, see section 7 of [47] and [21], hence by the spirit of Frenkel-Witten [22], these stable Higgs bundles should come from elliptic endoscopic data.

3.1 Stable conjugacy classes and endoscopy datum

We now give a quick review of the notion of endoscopy datum and, of the notion of stable conjugacy class with emphasis on the case of SL_n and PGL_n in this section. We will give an explicit computation of endoscopic groups of SL_n at the end of this section, which shall be used to understand the phenomenon we are describing in this chapter.

For a introduction to endoscopy datum and stable conjugacy the readers are referred to [34] and [36].

To motivate the introduction of endoscopy datum for our purposes in this thesis, let us recall in section 1.4, one can have an adelic description of Higgs bundles, i.e., $(E, \varphi) \in \mathcal{M}_{a,G}$ can be represented by a pair $((g_v)_{v\in|X|}, \gamma)$ where $\gamma \in \mathfrak{g}(F)$ such that $\chi(\gamma) = a$ and $g_v \in G(F_v)/G(O_v)$, where F is the function field of X, such that for almost all v, g_v is in fact in $G(O_v)$. Moreover, we shall require that the pair $(\gamma, (g_v)_{v\in|X|})$ satisfies

$$g_v^{-1}(\gamma)g_v \in \varpi_v^{-d_v}\mathfrak{g}(O_v).$$

If one further assume that $a \in \mathcal{A}_G^{ani}$, then the essential part of the counting formula over a single fibre of the Hitchin morphism as we will see in chapter 5 is given by the orbital integral of the following form

$$O_\gamma(1_D) = \int_{G_\gamma(F)\backslash G(\mathbb{A}_F)} 1_D(g^{-1}\gamma g)dg$$

where

$$1_D = \bigotimes_{x\in|X|} 1_{\varpi^{-d_v}\mathfrak{g}(O_v)}$$

with $1_{\varpi^{-d_v}\mathfrak{g}(O_v)}$ being the characteristic function of the set $\varpi^{-d_v}\mathfrak{g}(O_v)$.

One can see from above that the counting problem on a Hitchin fibre is roughly an integral over the conjugacy class of the Higgs field $\gamma \in \mathfrak{g}(F)$. Thus if one seek to compare the counting on fibres of moduli space of SL_n-Higgs bundles and PGL_n-Higgs bundles, a natural thing to do is to compare the conjugacy classes in these two different groups.

More specifically, what may happen is that $\gamma_1 \in \mathfrak{g}(F)$ is conjugate to $\gamma_2 \in \mathfrak{g}(F)$ in the algebraic closure of F (here F is the function field of the curve X), but not conjugate to each other in F. In the special case of SL_n and PGL_n, this means that γ_1 is conjugate to γ_2 in $PGL_n(F)$, but not in SL_n. Roughly, this may leads to the phenomenon that two Higgs bundles (over a same fibre $a \in \mathcal{A}^{ani}$) which may be counted as a single isomorphism class in the moduli of PGL_n-Higgs bundles, but as two different isomorphism classes in the moduli of SL_n-Higgs bundles.

The above discussion, hence leads us naturally to the consideration of the notion of stable conjugacy classes.

Now let G be a group over a field F.

Definition 3.1.1. Two elements $\gamma_1, \gamma_2 \in G(F)$ are called **stably conjugate** if there is $g \in G(\overline{F})$ such that

1. $g\gamma_1 g^{-1} = \gamma_2$
2. for every $\sigma \in Gal(\overline{F}|F)$, the element $g^{-1}\sigma(g) \in G^0_{\gamma_{1s}}$ where $G^0_{\gamma_{1s}}$ is the identity component of the centralizer of the semisimple part of γ_1, i.e., γ_{1s}.

Remark 3.1.2. In the sequel, we will call conjugacy classes over F "rational conjugacy classes" in opposition to “stable conjugacy classes” which means the conjugacy is taken over $\overline{F}$.

Remark 3.1.3. In the case of SL_n, centralizer $G_{\gamma_{1s}}$ is connected. For general groups, we use the notion of being strongly regular to ensure the connectedness of the centralizer.

Here is a well-known toy example showing the differences of rational conjugacy and stable conjugacy.

Example 3.1.4 (Example 2.14, [34])**.** The elements

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

of $SL_2(\mathbb{R})$ are conjugate in $SL_2(\mathbb{C})$ but not in $SL_2(\mathbb{R})$.

Now assume that γ_1, γ_2 are (strongly) regular elements, which means that the centralizers of γ_1 and γ_2 are tori. Then from the definition of being stably conjugate, we have a 1-cocyle

$$\begin{aligned} \mathrm{inv}(\gamma_1, \gamma_2) : Gal(\overline{F}|F) &\to T := G_{\gamma_{1s}} \\ \sigma &\mapsto g^{-1}\sigma(g). \end{aligned}$$

and one can check that the cohomology class of $\mathrm{inv}(\gamma_1, \gamma_2) \in H^1(Gal(\overline{F}|F), T)$ is well-defined.

Moreover, we have

Proposition 3.1.5. *The map $\gamma_2 \mapsto inv(\gamma_1, \gamma_2)$ gives a bijection between the set of F-conjugacy classes (referred to as rational conjugacy class from now on) inside the stable conjugacy class of γ_2 and the set $ker(H^1(Gal(\overline{F}|F), T) \to H^1(Gal(\overline{F}|F), G))$.*

Remark 3.1.6. Recall in chapter 1 we say that a connected reductive group G defined over F satisfies the **Hasse principle** if the set

$$\ker^1(F, G) := \ker(H^1(Gal(\overline{F}|F), T) \to H^1(Gal(\overline{F}|F), G)) = 0.$$

It is known that all simply connected groups satisfy the Hasse principle, hence our group SL_n satisfies Hasse principle. For more details, we refer readers to the paragraph under the Definition 5.2.3 in [34], and references therein.

In particular, one knows that for a regular element $\gamma \in \mathfrak{sl}_n$, the rational conjugacy classes in the stable conjugacy class of γ is parametrized by $H^1(\mathrm{Gal}(\overline{F}|F), T)$, with T being the centralizer of γ in SL_n.

In the following, let us pick $a \in \mathcal{A}^{ani}$, and consider (E, γ) as points on the fibre $\mathcal{M}_a := f^{-1}(a)$, where $f : \mathcal{M} \to \mathcal{A}$ is the Hitchin morphism of G-Higgs bundles; here we are taking $G = SL_n$. We believe the special case of $G = SL_n$ and $a \in \mathcal{A}^{ani}$ should be enough to convince the readers that it is natural to introduce endoscopic groups in the sequel.

By the definition of the set $\mathcal{A}^{ani}$, one knows that the centralizer of $\gamma \in \mathfrak{g}(F)$ will be an elliptic maximal torus, see Remark 1.3.7. From the global Tate-Nakayama duality,

$$H^1(\Gamma, T(\mathbb{A}_F)/T(F)) \simeq \pi_0([\widehat{T}/Z(\widehat{G})]^\Gamma)^* = \pi_0([\widehat{T}]^\Gamma)^*$$

where $\Gamma = \mathrm{Gal}(\overline{F}|F)$. Note that by the choice of $a \in \mathcal{A}^{ani}$, we know that $[\widehat{T}]^\Gamma$ is already finite, hence $\pi_0([\widehat{T}]^\Gamma) = [\widehat{T}]^\Gamma$.

Hence one knows that the rational conjugacy classes in a single stable conjugacy class can be parametrized by a finite order element in $\widehat{T}(F)$, and this can be generalized to general reductive groups with finer techniques.

This leads one to introduce the notion of endoscopic datum following [46] section 1.8.

Definition 3.1.7. Let $\widehat{\mathbb{G}}$ be with pinning $(\widehat{\mathbb{T}}, \widehat{\mathbb{B}}, \widehat{s}_+)$.

Let $\kappa \in \widehat{\mathbb{T}}$ be an element of finite order. The identity component of the centralizer of κ, in $\widehat{\mathbb{G}}$ is a reductive subgroup $\widehat{\mathbb{H}}_\kappa$. The pinning of $\widehat{\mathbb{G}}$ can give $\widehat{\mathbb{H}}_\kappa$ a natural choice of pinning with maximal torus $\widehat{\mathbb{T}}$.

By taking the dual root datum of that of $\widehat{\mathbb{H}}_\kappa$, we obtain a split group scheme $\mathbb{H}_\kappa$ over k.

Then one has the following short exact sequence

$$1 \to \widehat{\mathbb{H}}_\kappa \to (\widehat{\mathbb{G}} \rtimes \mathrm{Out}(\mathbb{G}))_\kappa \to \pi_0(\kappa) \to 1,$$

where $\pi_0(\kappa)$ is the group of connected components of $(\widehat{\mathbb{G}} \rtimes \mathrm{Out}(\mathbb{G}))_\kappa$, with $(\widehat{\mathbb{G}} \rtimes \mathrm{Out}(\mathbb{G}))_\kappa$ the centralizer of κ in $\widehat{\mathbb{G}} \rtimes \mathrm{Out}(\mathbb{G})$. Note that the group $\pi_0(\kappa)$ has natural homomorphisms

$$o_{\mathbb{G}} : \pi_0(\kappa) \to \mathrm{Out}(\mathbb{G})$$

and

$$o_{\mathbb{H}} : \pi_0(\kappa) \to \mathrm{Out}(\mathbb{H}_\kappa).$$

Now we consider a quasi-split form G over X given by a torsor $\rho_G : \pi_1(X, x) \to \mathrm{Out}(\mathbb{G})$. Then we say that an endoscopic datum of G over

X is a couple (κ, ρ_κ) with $\kappa \in \widehat{\mathbb{T}}$ and $\rho_k : \pi_1(X, x) \to \pi_0(\kappa)$ which has a $o_{\mathbb{G}}$-equivariant homomorphism $\rho_k \to \rho_G$. Let $\rho_H : \pi_1(X, x) \to \mathrm{Out}(\mathbb{H}_\kappa)$ induced by ρ_k.

We can have the following commutative diagram of all these torsors introduced above.

$$\begin{tikzcd}
& \pi_1(X,x) \arrow[d, "\rho_\kappa"] \arrow[ddl, "\rho_H"'] \arrow[ddr, "\rho_G"] & \\
& \pi_0(\kappa) \arrow[dl, "o_{\mathbb{H}}"] \arrow[dr, "o_{\mathbb{G}}"'] & \\
\mathrm{Out}(\mathbb{H}) & & \mathrm{Out}(\mathbb{G}).
\end{tikzcd}$$

Now we finally define the endoscopic group associated to the endoscopic datum (κ, ρ_κ) to be H over X obtained by twisting $\mathbb{H}$ with the torsor ρ_H.

We now end this section, with example of a calculation of endoscopic groups of the form of SL_n over the curve X, given by the trivial torsor $\rho : \pi_1(X, x) \to \mathrm{Out}(SL_n)$. The following calculation can be found in [36], chapter 2.

Example 3.1.8. Let $\mathbb{G} = SL_n(k)$, where $k = \mathbb{F}_q$ is the finite field with q elements. Let G be the form of $\mathbb{G}$ on the curve X given by the trivial torsor $\rho_G : \pi_1(X, x) \to \mathrm{Out}(\mathbb{G})$. Here we use F to denote the function field of X.

Hence $\widehat{\mathbb{G}} = PGL_n(k)$, so if we pick up a finite order element $\kappa \in \widehat{\mathbb{T}}$, and consider its centralizer, then $\widehat{\mathbb{H}}_\kappa$ is the image of $g \in GL_n(k)$ such that $g\kappa g^{-1} = \kappa$, with the centralizer itself being the image of $g \in GL_n(k)$ such that $g\kappa g^{-1} = \lambda\kappa$ for some $\lambda \in k^*$, where λ is of finite order. Let us denote the order of λ by m, it is easy to see that $m \mid n$.

By the definition of $\pi_0(\kappa)$

$$1 \to \widehat{\mathbb{H}}_\kappa \to (\widehat{\mathbb{G}} \times \mathrm{Out}(\mathbb{G}))_\kappa \to \pi_0(\kappa) \to 1,$$

we get that $\pi_0(\kappa) = (\mathbb{Z}/m\mathbb{Z}) \times \mathrm{Out}(\mathbb{G})$.

Now ρ_κ is equivalent to a morphism $\pi_1(X, x) \to (\mathbb{Z}/m\mathbb{Z})$, and this map should factor through a finite covering X' of X such that ρ_κ splits over X', and the degree l of covering of X' over X should be a factor of m, i.e.,

$l \mid m$. We denote the function field of the covering X' by E. Hence the degree of the field extension $[E : F]$ is l by our convention.

We consider the following situations under the assumption that

$$\kappa = \mathrm{diag}(\underbrace{1, ..., 1}_{d \text{ copies}}, \underbrace{s, ..., s}_{d \text{ copies}}, \dots, \underbrace{s^{m-1}, ..., s^{m-1}}_{d \text{ copies}}).$$

Remark 3.1.9. Recall we imposed a condition on κ, i.e., there is a $\lambda \in k^*$ of finite order, such that $g\kappa g^{-1} = \lambda\kappa$. Note that the adjoint action on κ will permute the eigenvalues of κ, therefore the condition will force κ to have the form above in some sense, when $\lambda \neq 1$.

- We first consider the case where $l = m$ and we denote $d = \frac{n}{m}$. Then one knows that in this case $\widehat{\mathbb{H}}_\kappa$ is the image of matrices $g \in GL_n$ such that $g^{-1}\kappa g = \kappa$ in PGL_n, hence $\widehat{\mathbb{H}}_\kappa$ is the image of the set of the matrices of the following form in PGL_n,

$$\underbrace{GL_d \times GL_d \times \ldots \times GL_d}_{m \text{ copies}}.$$

  So one can take the endoscopic group H on X to be the form on X defined by the kernel of the norm map $\mathrm{Res}_{E/F}(GL_d) \to GL_1$, where Res is the Weil restriction of scalars functor, along with the trivial torsor $\rho : \pi_1(X, x) \to \{1\}$.

- The case where $l < m$, here we assume that $l \cdot \frac{m}{l} \cdot d = n$, we will see that in this case, $\widehat{\mathbb{H}}_\kappa$ will be the image of the set of the matrices of the following form in PGL_n,

$$\underbrace{\overbrace{GL_d \times \ldots \times GL_d}^{l \text{ copies}} \times \ldots \times \overbrace{GL_d \times \ldots \times GL_d}^{l \text{ copies}}}_{\frac{m}{l} \text{ copies}}.$$

  Hence one can take the endoscopic group H on X to be the form on X defined by the kernel of the norm map on the following group

$$\underbrace{GL_d \times \ldots \times GL_d}_{\frac{m}{l} \text{ copies}},$$

  along with the trivial torsor $\rho : \pi_1(X, x) \to \{1\}$.

The first case in the above example in fact reveals an important type of endoscopic groups, which we now define.

Definition 3.1.10. An endoscopic group H will be called **elliptic**, if $Z(\widehat{H})^{\Gamma,0} = Z(\widehat{G})^{\Gamma,0}$. Here by $Z(\widehat{H})^{\Gamma,o}$ we mean the identity component of Galois invariants of the centre of $\widehat{H}$, and similarly for $Z(\widehat{G})^{\Gamma,0}$, here Γ is the absolute Galois group $\mathrm{Gal}(\overline{F}|F)$.

Remark 3.1.11. It is easy to see that the endoscopic groups that come from the second case of Example 3.1.8 will not be elliptic. The endoscopic groups that come from the first case of Example 3.1.8 will be elliptic.

Remark 3.1.12. One can always take a suitable finite extension of k to make the extensions E over F, where F is the function field of the curve X, appeared in Example 3.1.8 to be unramified. In particular, if we were working over algebraic closure of k in Example 3.1.8, the covering X' will be an finite étale covering of the origin curve X.

3.2 Coendoscopic decomposition of inertia stacks

In this section, we will review the theorem of Groechenig-Wyss-Ziegler, and prove a slight generalization of it. Basically, their proof of the theorem 5.14 of [26] can be carried over. To do this, we will follow Groechenig-Wyss-Ziegler to introduce the notion of coendoscopic datum. As Groechenig-Wyss-Ziegler pointed out, it seems more natural to decompose inertia stacks according to coendoscopic datum, instead of endoscopic datum.

We begin with some notations,

Notations 3.2.1. Here we follow section 2.4 of [26], and make the following conventions,

- By $\widehat{\mu}$ we mean that profinite étale group $\varprojlim_r \mu_r$, with $r \in \mathbb{N}$, which should be a affine group scheme of infinity type over $\mathrm{Spec}(k)$, here μ_r denotes the group of r-th roots of unity.
- If $\mathcal{X}$ is a stack, by $I_{\widehat{\mu}}\mathcal{X}(k)$ we mean the following groupoid

$$\{(x,\alpha) \mid x \in \mathcal{X}(k), \alpha \in \mathrm{Hom}_{\mathrm{cts}}(\widehat{\mu}(k), \mathrm{Aut}_{\mathcal{X}(\bar{k})}(x)), a \circ \varphi = \varphi \circ \alpha\},$$

where $\varphi : \mathrm{Aut}_{\mathcal{X}(\bar{k})}(x) \to \mathrm{Aut}_{\mathcal{X}(\bar{k})}(x)$ induced by the Frobenius element $\varphi \in \mathrm{Gal}(\bar{k}|k)$.

We introduce the notion of coendoscopic datum by following [26] section 5.3.

Definition 3.2.2. A coendoscopic datum for G over k is a triple $\mathcal{E} := (\kappa, \rho_k, \rho_k \to \rho)$ consisting of

1. a homomorphism $\kappa : \widehat{\mu} \to \mathbb{T}$, and
2. a $\pi_0(\kappa)$-torsor ρ_k over X
3. with a $o_{\mathbb{G}}$-equivariant morphism $\rho_k \to \rho$.

Following the same procedure as in Definition 3.1.7, we can define an coendoscopic group $H_{\mathcal{E}}$ over X which is obtained by twisting the centralizer $\mathbb{H}_\kappa$ with the torsor ρ_H in definition 3.1.7. Clearly, the Langlands dual (see Definition 1.1.6) $\widehat{H}_{\mathcal{E}}$ is the endoscopic group of $\widehat{G}$.

Similarly, one can then define the notion of elliptic coendoscopic groups by imposing the condition defined in Definition 3.1.10.

Example 3.2.3. Let the notations be as in Example 3.1.8, we immediately know that the elliptic coendoscopic groups of PGL_n will be the Langlands dual groups of elliptic endoscopic groups of SL_n we gave in Example 3.1.8, in other words, it should be given by the restriction of scalars of GL_d over X' to X, and then take images in PGL_n, and we recall that X' is a m-th covering of X, and $n = md$.

Then for coendoscopic datum (essentially the same for endoscopic datum), one has the following natural constructions

$$[\mathfrak{h}_{\mathcal{E},D}/H_{\mathcal{E}}] = [\mathbb{h}_{\mathcal{E}} \times \rho_k \times D/\mathbb{H}_\kappa \rtimes (\pi_0(\kappa) \times \mathbb{G}_m)],$$

and

$$[\mathfrak{g}_D/G] = [(\mathbb{g} \times \rho_k \times D)/\mathbb{G} \rtimes (\pi_0(\kappa) \times \mathbb{G}_m)],$$

where we use $\mathfrak{h}$ and $\mathbb{h}$ for the Lie algebras of H and $\mathbb{H}$.

We refer readers to construction 5.1 in [26], for a more detailed explanation of the equivalences of the above various stack quotients.

Now we can have the following natural morphism

$$\nu_{\mathcal{E}} : \mathfrak{c}_{\mathcal{E}} = \mathfrak{h}_{\mathcal{E},D}/H_{\mathcal{E}} \to \mathfrak{c} = \mathfrak{g}_D/G,$$

which induces a natural morphism

$$\nu_{\mathcal{E}} : \mathcal{A}_{H_{\mathcal{E}}} \to \mathcal{A}_G.$$

Then one has the natural morphisms

$$\begin{array}{ccc} \mathcal{M}_{H_{\mathcal{E}}} & \longrightarrow & \mathcal{M}_G \\ \downarrow & & \downarrow \\ \mathcal{A}_{H_{\mathcal{E}}} & \longrightarrow & \mathcal{A}_G. \end{array}$$

Construction 3.2.4 ([26])**.** Consider a Higgs bundle (E, φ) which is a image of a $H_{\mathcal{E}}$-Higgs bundle, say (F, ϕ) through the morphism $\mathcal{M}_{H_{\mathcal{E}}} \to \mathcal{M}_G$, then the homomorphism $\kappa : \widehat{\mu} \to \mathbb{T}$ induces a homomorphism $\gamma_{\mathcal{E}} : \widehat{\mu} \to Z(H_{\mathcal{E}}) \subset \mathrm{Aut}(F, \phi)$, by functoriality, we obtain

$$\gamma : \widehat{\mu} \to Aut(E, \varphi)$$

This gives rise an element $((E, \varphi), \gamma) \in I_{\widehat{\mu}}\mathcal{M}_G$.

Overall, we obtain a morphism

$$\mu_{\mathcal{E}} : \mathcal{M}_{H_{\mathcal{E}}} \to I_{\widehat{\mu}}\mathcal{M}_G.$$

Remark 3.2.5. In the special case of SL_n and PGL_n, $I_{\widehat{\mu}}\mathcal{M}_{PGL_n}$ will be the usual inertia stack $I\mathcal{M}_{PGL_n}$.

Remark 3.2.6. If we further assume that the $H_{\mathcal{E}}$-Higgs bundle (F, ϕ) is stable, then we will see that $Z(H_{\mathcal{E}}) \subset \mathrm{Aut}(F, \phi)$ has to be finite modulo the centre of $Z(G)$. As Faltings proved in the paper [21], stable G-Higgs bundles have only finite automorphisms modulo centre of G.

Hence $H_{\mathcal{E}}$ has to be an elliptic coendoscopic group provided that (F, ϕ) is a stable $H_{\mathcal{E}}$-Higgs bundle.

Notation 3.2.7. Before we continue, we need make the following choice. We fix a point $\infty \in X(k)$, let ∞_{ρ_κ} be a point in $\rho_\kappa(k)$ above ∞. Then the point

∞_{ρ_κ} induces identifications $G_\infty \simeq \mathbb{G}$ and $H_{\mathcal{E},\infty} \simeq \mathbb{H}_\kappa$, compatible with the pinnings on the group , as well as the following identification on $\mathfrak{t}$,

$$\begin{array}{ccc} \mathfrak{t}_{\mathcal{E},D,\infty} & \xrightarrow{\simeq} & \mathfrak{t}_{D,\infty} \\ \downarrow & & \downarrow \\ \mathfrak{c}_{\mathcal{E},\infty} & \xrightarrow{\nu_{\mathcal{E}}} & \mathfrak{c}_\infty. \end{array}$$

From now on we consider the stack of stable Higgs bundles only, which should contain the subset of Higgs bundles over the anisotropic locus, since Higgs bundles over the anisotropic locus have no parabolic reduction which implies that those Higgs bundles are stable. For simplicity, we would like to take $\mathbb{G} = PGL_n(k)$ in the remaining of this section, unless otherwise mentioned.

Construction 3.2.8 ([26])**.** We let $\mathcal{M}_G^{s,e}$ be the stack of stable G-Higgs bundles with fixed degree e and similarly $\mathcal{M}_{H_{\mathcal{E}}}^{s,e}$ for stable $H_{\mathcal{E}}$-Higgs bundles with fixed degree e, and we assume that $g.c.d(e,n) = 1$. Note that in the case of $\mathbb{G} = PGL_n(k)$, the degree e is a well-defined element in $\mathbb{Z}/n\mathbb{Z}$.

Now let $((E,\varphi),\gamma) \in I_{\widehat{\mu}}\mathcal{M}^{s,e}$. We know that γ should be of finite order since stable Higgs bundles admits only finitely many automorphism (modulo centre, but we will not need to consider the problem of centres since we are restricting ourselves to the case of PGL_n here).

Suppose that the homomorphism $\gamma : \widehat{\mu} \to \mathrm{Aut}(E,\varphi)$ is the same as $\kappa \in \mathbb{T}$ after passing to $\mathbb{T}$ through $\mathrm{Aut}(E,\varphi) \to \mathbb{T}$, where the morphism $\mathrm{Aut}(E,\varphi) \to \mathbb{T}$ can be obtained by evaluating at a k-rational point $\infty \in X(k)$. In other words, the following diagram commutes,

$$\begin{tikzcd} \widehat{\mu} \arrow[r, "\gamma"] \arrow[dr, "\kappa"'] & \mathrm{Aut}(E,\varphi) \arrow[d, "\infty"] \\ & \mathbb{T}. \end{tikzcd}$$

Note that (E,φ) corresponds to a morphism from X to $[\mathfrak{g}_D/G] = [\mathrm{g} \times \rho_k \times O_X(D)/\mathbb{G} \rtimes (\pi_0(\kappa) \times \mathbb{G}_m))]$, (E,φ) can then be viewed as a $\mathbb{G} \rtimes (\pi_0(\kappa) \times \mathbb{G}_m)$-torsor $\widetilde{E}$ over $X \times S$ together with an equivariant morphism $\widetilde{\theta} : \widetilde{E} \to \mathrm{g} \times \rho_k \times O_X(D)$.

In this way, the automorphism γ can be viewed as an automorphism of the torsor $\widetilde{E}$ which stabilizes $\widetilde{\theta}$.

Fix an étale covering X' of $X \times S$ over which $\widetilde{E}$ has a section. By also fixing such a section we can identify γ with a homomorphism

$$\widehat{\mu}_{X'} \to (\mathbb{G} \rtimes (\pi_0(\kappa) \times \mathbb{G}_m))_{X'}$$

up to a conjugation.

Note that the image of γ fixes $\widetilde{\theta}$, while the group $\pi_0(\kappa) \times \mathbb{G}_m$ acts without fixed points on $\mathfrak{g} \times \rho_k \times O_X(D)$, hence the homomorphism

$$\widehat{\mu}_{X'} \to (\mathbb{G} \rtimes (\pi_0(\kappa) \times \mathbb{G}_m))_{X'}$$

will factor through $\mathbb{G} \subset \mathbb{G} \rtimes (\pi_0(\kappa) \times \mathbb{G}_m)$.

Lemma 3.2.9 (Lemma 5.11. [26])**.** *The homomorphism $\gamma, \kappa : \widehat{\mu}_{X'} \to \mathbb{G}_{X'}$ are conjugate étale locally over X'.*

Proof. The following proof is taken from [26].

By replacing with an étale cover of X' if necessary, the homomorphism γ factors through maximal torus of $\mathbb{G}$. Indeed, let us consider the centralizer $Z := C_{\mathbb{G}}(\gamma)^o$ of γ in $\mathbb{G}_{X'}$. Then Z is a reductive subgroup scheme of $\mathbb{G}_{X'}$, through which γ factors, and this holds in every geometric fibre. Then any maximal torus of Z which exists after replacing X' by a suitable covering is a maximal torus of $\mathbb{G}_{X'}$ through which γ factors.

Hence by the étale local conjugacy of maximal tori of $\mathbb{G}$, we may assume that γ factors through $\mathbb{T}_{X'}$.

Let $r \geq 1$, and assume both γ and κ factors through a homomorphism $\mu_r \to \mathbb{T}_{X'}[r]$. After replacing X' with suitable étale covering, we may assume that $\mathbb{T}_{X'}[r]$ is a constant group scheme.

By construction, the homomorphisms γ and κ are conjugate above the chosen point $\infty \in X(k)$. Let X'_1 be connected, with image in $X \times S$ containing ∞. Note that $\mathbb{T}[r]$ is a constant group scheme, then that γ is conjugate to κ at some point in X'_1 means that they are conjugate over all of X'_1.

If X'_2 is any other connected component, still, by going up to another étale covering $X'_{12} = X'_1 \times_{X \times S} X_2$, which has open dense image in both X'_1 and X'_2, by the construction of γ, $\gamma|_{X'_1}$ is conjugate to $\gamma|_{X'_2}$ in $\mathbb{T}[r](X'_{12})$, hence the conjugacy of γ and κ on X_1 implies conjugacy over X'_2 by the same argument given in the above paragraph.

□

Now let $\widetilde{F} \subset \widetilde{E}$ be the subsheaf consisting of those sections of $\widetilde{E}$ on which the action of γ and κ coincide. By the lemma just proved, this is a $(\mathbb{G} \rtimes \mathrm{Out}(\mathbb{G}))_\kappa \times \mathbb{G}_m$-subtorsor of $\widetilde{E}$. Let ρ_k be the $\pi_0(\kappa)$-torsor $\pi_0(F)$ over $X \times S$. The composition

$$\widetilde{F} \to \widetilde{E} \to \mathfrak{g} \times \rho \times O_X(D) \to \rho$$

induces an $o_{\mathbb{G}}$-equivariant morphism $\rho_k \to \rho$ over $X \times S$.

Next we consider the fibre product

$$\begin{array}{ccc} \widehat{E} & \longrightarrow & \widetilde{E} \\ \downarrow & & \downarrow \\ \mathfrak{g} \times \rho_k \times O_X(D) & \longrightarrow & \mathfrak{g} \times \rho \times O_X(D) \end{array}$$

This diagram is naturally equivariant under the fibre product

$$\begin{array}{ccc} \mathbb{G} \rtimes (\pi_0(\kappa) \times \mathbb{G}_m) & \longrightarrow & \mathbb{G} \rtimes (\mathrm{Out}(\mathbb{G}) \times \mathbb{G}_m) \\ \downarrow & & \downarrow \\ \mathbb{G} \rtimes (\pi_0(\kappa) \times \mathbb{G}_m) & \longrightarrow & \mathbb{G} \rtimes (\mathrm{Out}(\mathbb{G}) \times \mathbb{G}_m) \end{array}$$

which makes $\widehat{E}$ into a $\mathbb{G} \rtimes (\pi_0(\kappa) \times \mathbb{G}_m)$ torsor. The quotient morphism $\widetilde{F} \to \rho_k$ induces a closed embedding $\widetilde{F} \hookrightarrow \widehat{E}$. Hence one can see that $\widetilde{F}$ is an $\mathbb{H}_k \rtimes (\pi_0(\kappa) \times \mathbb{G}_m)$-subtorsor of $\widehat{E}$.

By the definition of $\widetilde{F}$, the action of γ and κ on sections coincide. Since the image of γ will fix $\widetilde{\theta}$, it follows that the composition

$$\widetilde{F} \to \widetilde{E} \to \mathfrak{g} \times \rho \times O_X(D) \to \mathfrak{g}$$

factors through $\mathbb{h}_\kappa$. Finally we obtain an $\mathbb{H}_k \rtimes (\pi_0(\kappa) \times \mathbb{G}_m)$-equivariant morphism

$$\widetilde{F} \to \mathbb{h}_\kappa \times \rho_\kappa \times O_X(D).$$

So eventually, we claim the following equivalence of groupoids

Proposition 3.2.10. *Let us still restrict ourselves in the case of $G = PGL_n$,*

$$\coprod_{\mathcal{E}} \mu_{\mathcal{E}} : \coprod_{\mathcal{E}} \mathcal{M}^{s,e}_{H_{\mathcal{E}}}(k) \to I_{\widehat{\mu}} \mathcal{M}^{s,e}(k),$$

where $\mathcal{E}$ runs through the set of elliptic coendoscopic datum which occurs in the inertia stack.

Proof. From the above discussion, the following morphism has been constructed

$$\mu_{\mathcal{E}} : \mathcal{M}^s_{H_{\mathcal{E}_\kappa}} \to I_{\widehat{\mu}}\mathcal{M}^s(k),$$

as well as the inverse map construction 3.2.8, i.e., starting from an element $((E,\theta),\gamma)$ we get a $H_{\mathcal{E}_\kappa}$-Higgs bundle.

Clearly by the construction, one has $\mathrm{rank}(E) = \mathrm{rank}(F)$, as $H_{\mathcal{E}_\kappa}$ admits the same maximal torus with G. Note that the reduction of structure group is achieved by identifying $\widetilde{F}$ as eigen-sections under the action of κ over some étale covering of X, and κ is conjugate to γ over same étale covering over X, which means that γ and κ are stably conjugate. Overall, one gets that F is obtained from by conjugating (E,θ) over a suitable étale covering of X, this leads to the conclusion that $\deg(F) = \deg(E)$.

In the discussion of the construction 3.2.8 a choice of $\infty \in X(k)$ hence a series of identifications has been made, our moduli stacks appeared in the decomposition should respect these identifications in notation 3.2.7.

One needs to show that the construction of F will respect the choice of the identification in notation 3.2.7. This has been done in lemma 5.16 in [26].

To finish the proof of the proposition, we need to prove that (E,φ) is stable as a PGL_n-Higgs bundle if and only if (F,ϕ) is stable and the associated endoscopic datum is elliptic. This will be done in the following proposition.

□

Proposition 3.2.11. *Let the notations be as above, then (E,φ) is stable if and only if (F,ϕ) is stable and the corresponding coendoscopic datum of (F,ϕ) is elliptic.*

Proof. In fact, one knows that (F,ϕ) is obtained from (E,φ) by reduction of structure groups. Therefore we know that (E,φ) is stable iff (F,ϕ) is stable.

It remains to show that (E,φ) being stable implies that H is elliptic.

In fact, if H is not elliptic, then $H \subset P$ for some parabolic subgroup $P \subset G$ over X. Note that P will give rise to a subbundle say (F_1,ϕ_1) of (F,ϕ).

If we then pull back (F_1,ϕ_1) and (F,ϕ) along the finite étale covering $\pi_\kappa : X' \to X$ given by the torsor ρ_κ in the coendoscopic datum. We will see that the fact that $H \subset P$ will give rise to a decomposition $(F,\phi) = (F_1,\phi_1) \oplus (F_2,\phi_2)$ over X'. Hence we will get that this decomposition of

(F, ϕ) is still valid over X, since P is defined on X, which contradicts with the fact that (F, ϕ) is stable.

Hence H has to be elliptic. $\square$

Remark 3.2.12. The proof of the proposition 3.2.11 in essence follows the proof of the Proposition 3.1 of [44]. Indeed, (F, ϕ) can be viewed as a Higgs bundle (F_1, ϕ_1) of rank n/m on the finite étale covering $\pi_\kappa : X' \to X$ given by the torsor ρ_k in the coendoscopic datum. But (F, φ) is isomorphic to $\pi_{\kappa *}(F_1, \phi_1)$, and what Narasimhan-Ramanan proved is that (E, φ) is stable iff (F_1, ϕ_1) is stable and H is elliptic.

Remark 3.2.13. In fact, the above proposition is the classical theorem of Narasimhan-Ramanan, see Theorem 3.0.1. It seems more natural to view their theorem in the setting of PGL_n instead of GL_n or SL_n, since the action of $\gamma \in \mathrm{Pic}^0(X)[n]$ on PGL_n-Higgs bundles can be viewed as finite order automorphisms, more precisely, take a PGL_n-Higgs bundle coming from a projectivization of a GL_n-Higgs bundle (E, φ), $\gamma \in \mathrm{Pic}^0(X)[n]$ acts on E by tensorisation, i.e., $E \mapsto \gamma \otimes E$, in turn $\gamma \otimes E$ gives us the same PGL_n-Higgs bundle that E gives.

The above extensions of constructions in [26] to the moduli stack of stable Higgs bundles, as one can see, can be generalized to other semisimple groups without much modifications. Therefore, one can view Theorem 5.14 of [26] as a natural generalization of the classical theorem 3.0.1 of Narasimhan-Ramanan.

If we restrict ourselves in the anisotropic locus, then we have the following the groupoid equivalence of Groechenig-Wyss-Ziegler.

Corollary 3.2.14 (Theorem 5.14 [26])**.** *One has the following equivalence of groupoids*

$$\coprod_{\mathcal{E}} \mu_{\mathcal{E}} : \coprod_{\mathcal{E}} \mathcal{M}^{ani}_{H_{\mathcal{E}}}(k) \to I_{\widehat{\mu}} \mathcal{M}^{ani}_G(k).$$

Remark 3.2.15. Again, here we are abusing notations, all the moduli stack above should respect the choice of the rational point $\infty \in X(k)$ and hence all the identifications we made in 3.2.7.

Remark 3.2.16. The proof actually relies on the following observation made by Ngô in section 4.17.2, [46]. A necessary change has been made in lemma 5.7 [26] to adapt to the case of coendoscopic datum.

Lemma 3.2.17. *Let $\mathcal{E}$ be an coendoscopic datum for G over k, the morphism $\nu_{\mathcal{E}}$: $\mathcal{A}_{\mathcal{E}} \to \mathcal{A}$ satisfies*

$$\nu_{\mathcal{E}}^{-1}(\mathcal{A}^{ani}) = \mathcal{A}_{\mathcal{E}}^{ani}.$$

Proof. Note that the anisotropic locus $\mathcal{A}_{\mathcal{E}}^{ani}$ is the same for both $H_{\mathcal{E}}$ and $\widehat{H}_{\mathcal{E}}$, the latter is the endoscopic group of $\widehat{G}$.

For the latter one, it suffices to show that $\pi_0(\mathcal{P}_{\widehat{H}_{\mathcal{E}},a_{\mathcal{E}}})$ is finite iff $\pi_0(\mathcal{P}_a)$ is finite, where $a_{\mathcal{E}} \in \mathcal{A}_{\mathcal{E}}^{ani}(\bar{k})$ with image $a \in \mathcal{A}(\bar{k})$.

Ngô showed that there is a canonical surjective homomorphism

$$\mathcal{P}_a \to \mathcal{P}_{\widehat{H}_{\mathcal{E}},a_{\mathcal{E}}}$$

whose kernel is an affine group scheme of finite type over Spec(k). This induces a surjective homomorphism $\pi_0(\mathcal{P}_a) \to \pi_0(\mathcal{P}_{\widehat{H}_{\mathcal{E}},a_{\mathcal{E}}})$ with finite kernel. □

Remark 3.2.18. As mentioned in the last paragraph of section 5.3 in [26], for a non-algebraically closed field k, sometimes the splitting $(\rho_\kappa)_\infty$ may not well defined over k, however, it will be always defined over some finite extension of k. So it would be convenient to have the following notation

$$(I_{\widehat{\mu}}\mathcal{M}_G^{ani}(k))_\kappa := I_{\widehat{\mu}}\mathcal{M}_G^{ani}(k) \cap \coprod_{\mathcal{E} \text{ defined by } \kappa} \mathcal{M}_{H_{\mathcal{E}}}^{ani}(k).$$

It was shown by Ngô, see proposition 6.3.3, [46], that for a given κ, and a point $a \in \mathcal{A}_G^{ani}(k)$, there is at most one coendoscopic datum $\mathcal{E}_a$ of type κ occurs in the Hitchin fibre $(I_{\widehat{\mu}}\mathcal{M}_G)_a$.

Theorem 3.2.19 (Corollary 5.17, [26])**.** *Assume that the torsor $\infty_{\rho_{\mathcal{E}_a}}(k)$ is nonempty. Then there is a equivalence of groupoids*

$$\mu_{\mathcal{E}_a} : \mathcal{M}_{H_{\mathcal{E}_a},a}^{ani}(k) \to (I_{\widehat{\mu}}\mathcal{M}_G^{ani}(k))_{a,\kappa}.$$

Chapter 4

Support theorems and reduction of the Main theorem

In this chapter, we will review several support theorems of SL_n and its endoscopic groups, as well as the perverse continuation method. Cataldo proved a satisfying support type theorem in the case of SL_n when the base field is of characteristic 0 in [18]. Additionally, by incorporating the Severi-type inequality demonstrated in Theorem 3.3 of [43], Cataldo's approach can be seamlessly extended to cases involving positive characteristics. The same method of stratifying the Hitchin base according to the decomposition types of the characteristic polynomials of Higgs fields can be also applied to prove that the supports of endoscopic groups of SL_n are also contained in the anisotropic locus. We will review the method developed by Maulik-Shen which allows us to reduce the desired theorem in which the degree of the divisor D is allowed to satisfy $\deg(D) > 2g-2$ or $D = K_X$, where K_X is the canonical divisor of the curve X to the case where $\deg(D) > 2g-2$ and $\deg(D)$ is even with D being effective. Recall the divisor D is part of the definition of the Hitchin systems we considered in this thesis.

The support theorems we will review in this chapter allow us to reduce the problem of comparing point counts over the entire Hitchin base to the problem of comparing point counting over the anisotropic locus, inside which the spectral curve is integral and the stabilization of orbital integrals that arises from the point counting has been well-developed. The reason why we can do this is explained in section 4.3. The rough idea is that once we know that supports of two Hitchin systems say $f_1 : \mathcal{M}_1 \to \mathcal{A}$ and

$f_2 : \mathcal{M}_2 \to \mathcal{A}$ are contained in the anisotropic locus, and we know that

$$Rf_{1,*}\overline{\mathbb{Q}_\ell}|_{\mathcal{A}^{ani}} \simeq Rf_{2,*}\overline{\mathbb{Q}_\ell}|_{\mathcal{A}^{ani}},$$

then the isomorphism can be extended to the entire Hitchin base $\mathcal{A}$, i.e.,

$$Rf_{1,*}\overline{\mathbb{Q}_\ell} \simeq Rf_{2,*}\overline{\mathbb{Q}_\ell}.$$

4.1 Weak abelian fibration

Let $f : M \to S$ be a proper map of varieties over a finite field k. Let $g : P \to S$ be a smooth commutative group scheme over S. Further more we assume that P acts on M relative to S and that the stabilizers of this action are affine.

Let $P^0 \subset P$ be the open group scheme such that for any geometric point $s \in S$, P^0_s is the identity component of P_s, then by the structure theorem of Chevalley

Theorem 4.1.1 (Chevalley)**.** *Let G be a connected algebra group. Then G has a largest connected affine normal subgroup G_{aff}. Further, the quotient group G/G_{aff} is an abelian variety.*

One has the following exact sequence

$$1 \to R_s \to P^0_s \to A_s \to 1$$

where R_s is the largest connected affine normal subgroup of P^0_s, and A_s is an abelian variety.

It induces a decomposition of Tate modules

$$0 \to T_{\overline{\mathbb{Q}}_\ell}(R_s) \to T_{\overline{\mathbb{Q}}_\ell}(P^0_s) \to T_{\overline{\mathbb{Q}}_\ell}(A_s) \to 0$$

where

$$T_{\overline{\mathbb{Q}}_\ell}(P^0) = H^{2d-1}(g^0_!\overline{\mathbb{Q}}_\ell)(d).$$

Definition 4.1.2. We say that $T_{\overline{\mathbb{Q}}_\ell}(P^0)$ is polarizable if étale locally on S, there is an alternating bilinear form

$$\psi : T_{\overline{\mathbb{Q}}_\ell}(P^0) \times T_{\overline{\mathbb{Q}}_\ell}(P^0) \to \overline{\mathbb{Q}}_\ell$$

whose fibre at every geometric point $s \in S$ has kernel $T_{\overline{\mathbb{Q}}_\ell}(R_s)$, and ψ induces a perfect pairing on $T_{\overline{\mathbb{Q}}_\ell}(A_s)$.

We also need consider the $\mathbb{N}$-valued function $\delta(s) := \dim R_s$ defined for the topological points of S, which is upper-semicontinuous by section 5.6.2 in [46].

Suppose that δ is constructible, then it induces a locally closed stratification

$$S = \coprod_{\delta \in \mathbb{N}} S^{\delta},$$

where $S^{\delta} = \{s \in S \mid \delta(s) = \delta\}$.

Definition 4.1.3. One says that (f, g) is a **weak abelian fibration** if the following are true:

1. f and g have the same relative dimension d;
2. for any geometric point $s \in S$ and any $m \in M$, the stabilizer in P_s is affine;
3. The Tate module $T_{\overline{\mathbb{Q}}_{\ell}}(P^o)$ is polarizable.

Definition 4.1.4. A weak abelian fibration (f, g) is called δ**-regular** if for every $\delta \in \mathbb{N}$, $\mathrm{codim}(S^{\delta}) \geq \delta$. This is equivalent to say that for any irreducible closed subset $Z \subset S$, then $\mathrm{codim}_S(Z) \geq \delta_Z$, where δ_Z is the minimum of δ over Z.

4.2 Support theorem

Definition 4.2.1. Let $f : M \to S$ be as before, i.e., a proper map over a scheme of finite type. Then the constant sheaf $\overline{\mathbb{Q}}_{\ell}$ on M is clearly self dual and pure, the complex $f_*\overline{\mathbb{Q}}_{\ell}$ is also pure. The BBD theorem [8] gives us a decomposition of the complex $f_*\overline{\mathbb{Q}}_{\ell}$ into a direct sum of semisimple perverse cohomology sheaves over $S \otimes_k \overline{k}$, i.e.,

$$f_*\overline{\mathbb{Q}}_{\ell} \simeq \bigoplus_n {}^pH^n(f_*\overline{\mathbb{Q}}_{\ell})[-n].$$

Each semisimple ${}^pH^n$ appeared in the above decomposition is isomorphism to a direct sum of simple perverse sheaves

$$\bigoplus_{\alpha} K_{\alpha}$$

For every geometric simple K_α, there exists a irreducible reduced closed subset $i : Z_\alpha \hookrightarrow S \otimes_k \bar{k}$ and an open dense subset $U_\alpha \subset Z_\alpha$ with a irreducible local system $\mathcal{K}_\alpha$ over U_α such that

$$K_\alpha = i_{\alpha *} j_{\alpha ! *} \mathcal{K}_\alpha [\dim(Z)].$$

The closed subscheme Z_α is completely determined by K_α and the set $\{Z_\alpha\}$ is called the support of f.

Definition 4.2.2. Let the notations be as above, then the set

$$\mathrm{Socle}(f_* \overline{\mathbb{Q}}_\ell) := \{\eta_S \mid \eta_S \text{ is the generic point of } S\}.$$

Ngô proved a support theorem for Hitchin systems in order to prove the fundamental lemma, it implies the following:

Theorem 4.2.3. *For a reductive group G, if we consider the Hitchin morphism $f : M_G \to \mathcal{A}_G$ with degree of the fixed divisor $\deg(D) > 2g$, and if we denote the restriction of f to $\mathcal{A}^{ani}$ by f^{ani},then the $\mathrm{Socle}(f_*^{ani} \overline{\mathbb{Q}}_\ell)$ is the generic point $\eta_{\mathcal{A}_G}$ of the Hitchin base $\mathcal{A}_G$.*

Then Chaudouard and Laumon extended Ngô's results to generic semisimple locus, see [13]:

Theorem 4.2.4. *Let $f^\heartsuit$ be the restriction of $f : M_G \to \mathcal{A}_G$ to $\mathcal{A}_G^\heartsuit$, then one has*

$$Socle(f_*^\heartsuit \overline{\mathbb{Q}}_\ell) \subset \mathcal{A}_G^{ani}.$$

Finally, they extend this result to the entire Hitchin base $\mathcal{A}$ for characteristic 0 fields or fields with large enough characteristics but only for $G = GL_n$, see [15] and [16]

Theorem 4.2.5. *For $G = GL_n$, if we use $M_{GL_n}^{st}$ to denote the space of stable Higgs vector bundles of degree d, and if we let $f : M_{GL_n}^{s} \to \mathcal{A}_n$ be the Hitchin morphism, then we have*

$$Socle(f_* \overline{\mathbb{Q}}_\ell) = \{\eta_{\mathcal{A}_n}\}.$$

Remark 4.2.6. This theorem is for the moduli space of **stable** Higgs moduli.

As a side product, they obtained in the following counting result.

Theorem 4.2.7 (Corollaire 2.1. [16]). *If X is a geometrically connected, smooth projective curve over a finite field with characteristic strictly larger than n, then the number of isomorphism classes of stable Higgs bundles over X, does not depend on e, if $g.c.d(n, e) = 1$.*

Remark 4.2.8. The idea is that we first count the number of stable Higgs bundles over the anisotropic locus. One can easily see that those number do not depend on e, provided that $g.c.d(e, n) = 1$, since they corresponds to compactified Jacobians. Then, from the support theorem above, we deduce that the point counting of stable GL_n-Higgs bundles is determined by the number of stable GL_n-Higgs bundles live in the anisotropic locus, which is independent of the chosen degree, hence the total number of isomorphism classes of stable Higgs bundles is independent with the degree e.

Remark 4.2.9. The obstruction to extend the result of [15] over positive characteristic fields is the Severi-type inequality. This has been proved in the positive characteristic case, see Theorem 3.3 in [43].

In the sequel we will use the following theorem, see section 7.2.2 in [46] as well as Theorem 2.6.4 in [18], which provides "remarkably useful" restrictions on the dimension of the supports.

If $s \in S$ then we define $d_s := \overline{\dim\{s\}}$ the dimension of the closed subvariety of S with generic point s.

Theorem 4.2.10. *Let (M, S, P) be a weak abelian fibration, where M and S are nonsingular with $f : M \to S$ projective of pure relative dimension d_f. We also assume that $g : P \to S$ has geometrically connected fibres. If $s \in Socle(f_*\overline{\mathbb{Q}}_\ell)$, then*

$$d_f - d_P + d_s \geq d_s^{ab}(P).$$

Recall that for P, we have the Chevalley exact sequence

$$0 \to R_s \to P_s \to A_s \to 0.$$

We set $d_s^{ab}(P) := d(P_s) - \delta(s)$.

Remark 4.2.11. Note that the above inequality can be rewritten as

$$\delta(s) \geq \operatorname{codim}(\overline{\{s\}}).$$

We now conclude this section by the support theorem for the moduli space of stable SL_n-Higgs bundles, proven by Cataldo in [18].

Theorem 4.2.12 (Theorem 1.0.2, [18]). *For $f^{st}: M^{st}_{SL_n} \to \mathcal{A}_{SL_n}$, the set $Socle(f^{st}_* \overline{\mathbb{Q}}_\ell)$ is contained in $\mathcal{A}^{ani}_{SL_n}$.*

Now one of the immediate corollary of the support theorem is that the $\#\mathcal{M}^{L}_{SL_n}$ will not depend on the degree L, indeed if $\mathcal{M}^{L_1}_{SL_n}$ and $\mathcal{M}^{L_2}_{SL_n}$ are two moduli spaces of Higgs bundles of degree $d_1 = \deg(L_1)$ and $d_2 = \deg(L_2)$, respectively. We assume that $g.c.d(d_1, n) = 1 = g.c.d(d_2, n)$.

Then by considering their restrictions to the anisotropic locus, we have the following identity

$$\#\mathcal{M}^{L_1}_{SL_n}|_{\mathcal{A}^{ani}_{SL_n}} = \#\mathcal{M}^{L_2}_{SL_n}|_{\mathcal{A}^{ani}_{SL_n}}. \tag{4.1}$$

since Higgs bundles living in $\mathcal{A}^{ani}_{SL_n}$ correspond to compactified Jacobians of spectral curves by the well-known result of [7]. By the support theorem the above identity 4.1 can be extend to the entire Hitchin base $\mathcal{A}_{SL_n}$, i.e.,

$$\#\mathcal{M}^{L_1}_{SL_n} = \#\mathcal{M}^{L_2}_{SL_n}.$$

In conclusion, we have the following corollary of the support theorem of SL_n,

Corollary 4.2.13. *Let $d_1, d_2 \in \mathbb{N}$ be coprime with n, and let L_1, L_2 be two line bundles on X with degree d_1 and d_2, respectively.*

$$\#\mathcal{M}^{L_1}_{SL_n} = \#\mathcal{M}^{L_2}_{SL_n}.$$

4.3 Function-sheaf dictionary and perverse continuation method

Before we continue on support theorems, we would like to first recall the theory of function-sheaf dictionary and perverse continuation method.

Let $k = \mathbb{F}_q$, and $\bar{k}$ be its algebraic closure, and $\Gamma = \mathrm{Gal}(\bar{k}/k)$ be its Galois group which is generated by the Frobenius element σ. We now consider a scheme X over k, and the derived category of bounded complexes of constructible sheaves $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ on X. For a point $x \in X(k)$, we use $\bar{x}$ to denote the geometric point of $X(\bar{k})$ which lies above x, we set

$$\mathrm{Tr}_{\mathcal{F}}(x) := \sum_i (-1)^i \mathrm{Tr}(\sigma, \mathcal{H}^i(\mathcal{F})_{\bar{x}}), \tag{4.2}$$

Here $\mathcal{H}^i$ are the cohomology sheaves of $\mathcal{F}$.

We then have the following theorem translating Grothendieck's six functors on sheaves into operations on the functions Tr.

Theorem 4.3.1 (Theorem 12.1, chapter 3 [35]). *Let X be an algebraic scheme over the finite field k, then the functions $Tr_{\mathcal{F}}$ for $\mathcal{F} \in D_c^b(X, \overline{\mathbb{Q}}_\ell)$ have the following properties*

1. $Tr_{\mathcal{F}} = Tr_{\mathcal{F}_1} + Tr_{\mathcal{F}_2}$ *for $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}$ if there exists a distinguished triangle $(\mathcal{F}_1, \mathcal{F}, \mathcal{F}_2)$.*
2. $Tr_{\mathcal{F}\otimes\mathcal{G}} = Tr_{\mathcal{F}} \cdot Tr_{\mathcal{G}}$ *for $\mathcal{F}, \mathcal{G} \in D_c^b(X, \overline{\mathbb{Q}}_\ell)$.*
3. *Let $\mathcal{F}, \mathcal{G}$ be semisimple perverse sheaves on X_0, then the equality*
$$Tr^m_{\mathcal{F}}(x) = Tr^m_{\mathcal{G}}(x), \forall x \in X(k_m)$$
for all finite field extension k_m of k holds iff $\mathcal{F} \simeq \mathcal{G}$.
4. *Let $g : X \to Y$ be a morphism defined over k, let $\mathcal{F} \in D_c^b(X, \overline{\mathbb{Q}}_\ell)$, then*
$$Tr_{Rg_!(\mathcal{F})}(y) = \sum_{\substack{x\in X(k)\\ g(x)=y}} Tr_{\mathcal{F}}(x).$$

We now follow Ngô's note [11], to give a review on perverse continuation method.

Now let $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ be proper morphisms of k-schemes of finite type. Assume that both X_1 and X_2 are smooth and Y is irreducible.

Suppose now that

1. there is an open subset of Y such that for every $y \in U(k')$ for some finite extension k' of k, there are the same number of k'-points on the fibres $f_1^{-1}(y)$ and $f_2^{-1}(y)$;
2. Both supports of f_1 and f_2 are $\{Y \otimes_k \bar{k}\}$

Claim 4.3.2.
$$\# f_1^{-1}(y)(k') = \# f_2^{-1}(y)(k')$$
for all $y \in Y(k')$ of Y with some values in a finite extension k' of k

Proof of claim 4.3.2. Let $\mathcal{F}_i = f_{i*}\overline{\mathbb{Q}}_\ell$, where $i = 1, 2$.

One may assume that there are local systems L_1^i, L_2^i on U such that

$$^pH^i(\mathcal{F}_n) = j_{!*}L_n^i, \quad \text{where } n = 1, 2$$

by restricting to a smaller open subset U if necessary, where $j : U \to X$ is the open embedding, since both of the supports of f_1 and f_2 are assumed to be $\{Y \otimes_k \bar{k}\}$.

The first assumption implies that for every $y \in U(k')$,

$$\sum_i (-1)^i \mathrm{Tr}(\sigma, L_1^n) = \sum_i (-1)^i \mathrm{Tr}(\sigma, L_2^n).$$

Since L_1^i and L_2^i are pure local systems of weight i, one can separate the above identity of alternating trace for each i,

$$\mathrm{Tr}(\sigma, L_1^i) = \mathrm{Tr}(\sigma, L_2^i).$$

By the Chebotarev density theorem, L_1^i are isomorphic to L_2^i up to semi-simplification.

For a short exact sequence of local systems on U, we have

$$0 \to j_{!*}A \xrightarrow{\alpha} j_{!*}B \xrightarrow{\beta} j_{!*}C \to 0$$

where α is injective and β is surjective, but the image of α may be strictly smaller than to the kernel of β. But if we assume that B is geometrically semisimple and the support of B is $\{X \otimes_k \bar{k}\}$, the subquotient

$$\ker(\beta)/\mathrm{im}(\alpha)$$

is supported on $X - U$, which should vanish, and hence the sequence

$$0 \to j_{!*}A \xrightarrow{\alpha} j_{!*}B \xrightarrow{\beta} j_{!*}C \to 0$$

is exact, even enough the $j_{!*}$ functor may not be exact in general.

By the above argument on the $j_{!*}$ functor we now have

$$^pH^i(\mathcal{F}_1) \simeq {}^pH^i(\mathcal{F}_2).$$

This gives us the identity

$$\# f_1^{-1}(y)(k') = \# f_2^{-1}(y)(k')$$

for all $y \in Y(k')$ of Y with some values in finite extension k' of k, by the Grothendieck-Lefschetz trace formula.

□

4.4 The support theorem for the endoscopic groups

In order to give an endoscopic decomposition of $\#\mathcal{M}^{st,L}_{SL_n}$, where we set $\deg(L) = d$, we need to investigate the supports for H_{ρ_κ}-Hitchin systems, where we use H_{ρ_κ} for the endoscopic group of SL_n over X associated to (κ, ρ_κ). Let us assume that $\kappa \in \widehat{\mathbb{T}}$ is of order m and $n = mr$. It turns out that the methods used in [41] and [18] are enough to tell us that the supports of H_{ρ_κ}-Hitchin systems are also contained in the anisotropic locus of $\mathcal{A}_{H_{\rho_\kappa}}$, where $\mathcal{A}_{H_{\rho_\kappa}}$ denotes the Hitchin base of H_{ρ_κ}-Hitchin system. More precisely, the main goal of this section is to show that the supports of the Hitchin system $f^{st}_{\rho_\kappa} : \mathcal{M}^{st,L}_{H_{\rho_\kappa}} \to \mathcal{A}_{H_{\rho_\kappa}}$ are contained in the anisotropic locus of $\mathcal{A}_{H_{\rho_\kappa}}$, where we use $\mathcal{M}^{st,L}_{H_{\rho_\kappa}}$ to denote the moduli space of stable H_{ρ_κ}-Higgs bundles with fixed determinant L.

In the sequel, all moduli spaces considered will be moduli spaces of *stable* Higgs bundles, hence we will omit the superscript "st" in the notations.

Recall that in example 3.1.8, we get that the torsor ρ_κ in the endoscopic datum (κ, ρ_κ) is given by a cyclic m-finite étale covering $\pi_{\rho_\kappa} : X_{\rho_\kappa} \to X$. Then H_κ can be obtained by first pushing-forward the constant group scheme GL_d on X_{ρ_κ} and then taking the kernel of the norm of $\det : \mathrm{Res}_{E/F}(GL_d) \to GL_1(F)$, where E and F are function fields of the curves X_{ρ_κ} and X, respectively.

First, we notice that the moduli space of H_{ρ_κ}-Higgs bundles is smooth for arbitrary π_κ, and when $m = 1$, we recover the moduli space of stable SL_n-Higgs bundles, of which we already know the smoothness.

Following [41] and [31], one can embed $\mathcal{M}^L_{H_{\rho_\kappa}}$ over X into the moduli space $\mathcal{M}^{d'}_{GL_r}$ over X_{ρ_κ}, i.e., the moduli space of stable GL_r-Higgs bundles with degree d' such that the pushforward of those Higgs bundles are of degree d, and one recover $\mathcal{M}^L_{H_{\rho_\kappa}}$ by taking the fibre of the following map over the point $(L, 0)$and then pushforward along the covering map $\pi_{\rho_\kappa} : X_{\rho_\kappa} \to X$

$$M^{d'}_{GL_r}(X_{\rho_\kappa}) \to \mathcal{M}^{d'}_{GL_1}(X_{\rho_\kappa}) \to \mathcal{M}^{d}_{GL_1}(X) = \mathrm{Pic}^d(X) \times H^0(X, O_X(D)),$$

where the first arrow defined by

$$(E, \varphi) \mapsto (\det(E), \mathrm{tr}(\varphi))$$

and the second arrow is defined by

$$(L, \theta) \mapsto (\det(\pi_{\rho_\kappa,*}L), \mathrm{tr}(\pi_{\rho_\kappa,*}\theta)).$$

Each of these morphism is smooth, hence the moduli space $\mathcal{M}^L_{H_{\rho_\kappa}}$ is smooth, since it is the fibre of the above smooth morphism over $(L, 0) \in \mathrm{Pic}^d(C) \times H^0(X, O_X(D))$.

Note that for the moduli space $\mathcal{M}^L_{H_{\rho_\kappa}}$, one has a smooth commutative group scheme $P_{H_{\rho_\kappa}}$ over $\mathcal{A}_{H_{\rho_\kappa}}$ acting on $\mathcal{M}^L_{H_{\rho_\kappa}}$. In fact, one can define

$$g_{H_{\rho_\kappa}} : P_{H_{\rho_\kappa}} \to \mathcal{A}_{H_{\rho_\kappa}}$$

as the sub-Picard scheme of degree 0 of the smooth Picard group scheme P_{GL_r} associated to the moduli space $\mathcal{M}^{d'}_{GL_r}$ on the curve X_{ρ_κ} by taking the kernel of the norm map

$$\mathrm{Pic}^0(X_{\rho_\kappa}) \to \mathrm{Pic}^0(X).$$

One has the following Proposition 2.6 in [41]

Proposition 4.4.1. *The triple*

$$\left(\mathcal{M}_{H_{\rho_\kappa}}, \mathcal{A}_{H_{\rho_\kappa}}, P_{H_{\rho_\kappa}}\right)$$

is a weak abelian fibration which is δ-regular when restricted to $\mathcal{A}^{ani}_{H_{\rho_\kappa}}$, see definition 4.1.3, with notations as above.

We then have the following theorem concerning about the supports of the Hitchin systems $f_{\rho_\kappa} : \mathcal{M}_{H_{\rho_\kappa}} \to \mathcal{A}_{H_{\rho_\kappa}}$, we refer readers to the Theorem 2.3 of [41].

Theorem 4.4.2. *Let D be an effective divisor on X of degree $\deg(D) > 2g - 2$, and let us consider the m-th cyclic étale covering $\pi_{\rho_\kappa} : X_{\rho_\kappa} \to X$, where κ is of order m, with $n = mr$. For each π_{ρ_κ}, one has the corresponding Hitchin system $f_{\rho_\kappa} : \mathcal{M}_{H_{\rho_\kappa}} \to \mathcal{A}_{H_{\rho_\kappa}}$. Then*

$$Socle(Rf_{\rho_\kappa,*}\overline{\mathbb{Q}}_\ell) \subset \mathcal{A}^{ani}_{H_{\rho_\kappa}}.$$

For the definition of $Socle(Rf_{\rho_\kappa,*}\overline{\mathbb{Q}}_\ell)$, we refer readers to Definition 4.2.2.

Remark 4.4.3. The main ingredient in the proof of the support theorems of SL_n and H_{ρ_κ}, and originally for GL_n, is the stratification on the Hitchin bases according to the decomposition type of characteristic polynomials of Higgs fields. More precisely one define $\mathcal{A}_{(\underline{m},\underline{n})}$ where $\underline{m} = (m_1, \dots, m_s)$ and $\underline{n} = (n_1, \dots, n_s)$ to be the set of characteristic polynomials of Higgs fields which admit the following form

$$P = \prod_{i=1}^{s} P_i, \quad \deg(P_i) = m_i.$$

One can then obtain a contradiction by considering the enhanced δ-regular inequality, see Theorem 4.2.10, if we assume that the point $a \in$ Socle is not elliptic.

For details, we refer readers to [15], [16], [18] and section 2 of [41]

Moreover, we would like to consider the action of $\Gamma = \mathrm{Pic}^0(X)[n]$, i.e., the torsion points of Jacobians of X, on the corresponding moduli space of H_{ρ_κ}-Higgs bundles. More precisely, let us choose $s \in \Gamma$, which is of order m. Let $L \in \Gamma$ be a torsion line bundle, we have

$$\det(\pi_{\rho_\kappa,*}(\pi_{\rho_\kappa}^* L \otimes E)) = \det(L \otimes \pi_* E) = L^{mr} \otimes \det(\pi_{\rho_\kappa} E) = \det(\pi_* E)$$

for E being a vector bundle over X_{ρ_κ} of rank r. Therefore we have an action of Γ on moduli space of H_{ρ_κ}-Higgs bundles by the tensorisation of the pullback of $L \in \Gamma$ along π_{ρ_κ}.

Therefore one can have the following decomposition

$$Rf_{H_{\rho_\kappa},*}\overline{\mathbb{Q}}_\ell = \bigoplus_{\kappa \in \widehat{\Gamma}} \left(Rf_{H_{\rho_\kappa},*}\overline{\mathbb{Q}}_\ell \right)_\kappa, \tag{4.3}$$

where $\widehat{\Gamma}$ is isomorphic to Γ through the Weil pairing. When $\kappa = 1$, we will write

$$\left(Rf_{H_{\rho_\kappa},*}\overline{\mathbb{Q}}_\ell \right)_{\mathrm{stab}} = \left(Rf_{H_{\rho_\kappa},*}\overline{\mathbb{Q}}_\ell \right)_{\kappa=1}.$$

Then we have the following property of supports of the complexes $\left(Rf_{H_{\rho_\kappa},*}\overline{\mathbb{Q}}_\ell \right)_{\mathrm{stab}}$, see Theorem 2.3 (b), in [41].

Proposition 4.4.4. $\left(Rf_{H_{\rho_\kappa},*}\overline{\mathbb{Q}}_\ell \right)_{stab}$ *has full support* $\mathcal{A}_{H_{\rho_\kappa}}$.

Now let us summarize what we can obtain from the support theorems for SL_n and H_{ρ_κ} and the function sheaf dictionary in section 4.3. When we restrict to the anisotropic locus of $\mathcal{A}_{SL_n}$, we will have

$$\#\mathcal{M}^L_{SL_n}|_{\mathcal{A}^{ani}_{SL_n}} = \sum_{(\kappa,\rho_\kappa)} \left(\#^{\text{stab}}\mathcal{M}_{H_{\rho_\kappa}}|_{\mathcal{A}^{ani}_{H_{\rho_\kappa}}}\right) q^{2d_\gamma},$$

where $\#^{\text{stab}}\mathcal{M}_{H_{\rho_\kappa}}$ is the stable part of the point counting $\#\mathcal{M}_{H_{\rho_\kappa}}$ in the sense of stabilization given by the geometric stabilisation theorem of Ngô in section 6.4 of [46] and here $2d_\gamma = \operatorname{codim}_{\mathcal{M}^L_{SL_n}}(\mathcal{M}_{H_{\rho_\kappa}})$ which coincides with the fermion shift of the locus fixed by $\gamma \in \Gamma = \operatorname{Pic}^0(X)[n]$, with γ corresponds to κ through Weil pairing.

In fact, over the anisotropic locus, the decomposition of $\#\mathcal{M}_{H_{\rho_\kappa}}$ into $\widehat{\Gamma}$-isotropic loci according to the action described above coincides with the endoscopic decomposition of $\#\mathcal{M}_{H_{\rho_\kappa}}$ given by geometric stabilisation theorem of Ngô.

In the spirit of the stabilization of the trace formula, one would like to extend the above decomposition of point counting restricted to the anisotropic locus to the entire Hitchin base.

Now Proposition 4.4.4 shows that the $\#^{stab}\mathcal{M}_{H_{\rho_\kappa}}$ has full support on $\mathcal{A}_{H_{\rho_\kappa}}$, then one has

Corollary 4.4.5.

$$\#\mathcal{M}^L_{SL_n} = \sum_{(\kappa,\rho_\kappa)} \left(\#^{stab}\mathcal{M}_{H_{\rho_\kappa}}\right) q^{2d_\gamma}.$$

which is the counting version of the result of Maulik-Shen in [41], by taking the alternating sum of traces of Frobenius of their correspondences on the level of complexes.

4.5 Maulik-Shen's reduction

This section is taken from section 4 of [41]. We will review the method developed by Maulik and Shen in [41], which allows us to reduce the general case where the effective divisor satisfying $\deg(D) > 2g-2$ or $D = K_X$, where K_X is the canonical divisor of the curve X to the case when the degree of the effective divisor D satisfies $\deg(D) > 2g-2$ and $\deg(D)$ is even. The group schemes over X in this section are all assumed to be split.

Let p be a k-rational point on the curve X, with a closed embedding

$$i_p : \{p\} \to X.$$

Then one can consider the space given by the quotient

$$\mathcal{M}_{n,p} = [\mathfrak{sl}_n / SL_n]$$

which can be viewed as the moduli space of SL_n-Higgs bundle at a point p, as well as the Hitchin morphism

$$h_p : \mathcal{M}_{n,p} \to \mathcal{A}_p$$

where $\mathcal{A}_p = \mathfrak{sl}_n /\!\!/ SL_n$ is the affine GIT quotient parametrizing all the characteristic polynomials which come from a trace 0 matrix.

Let us denote by $\mathcal{M}_n^{D,L}$ the moduli space of SL_n-Higgs bundles then one has the following commutative diagram:

$$\begin{array}{ccc} \mathcal{M}^{D,L}_{SL_n} & \xrightarrow{\mathrm{ev}_p} & \mathcal{M}_{SL_n,p} \\ \Big\downarrow{\scriptstyle h^D} & & \Big\downarrow{\scriptstyle h_p} \\ \mathcal{A}^D & \longrightarrow & \mathcal{A}_p. \end{array}$$

Also, we have the same diagram for the moduli space $\mathcal{M}^{D,L}_{H_\kappa}$ of H_{ρ_κ}-Higgs bundles, where H_κ is coming from the endoscopic datum (κ, ρ_κ), with $\kappa \in \widehat{\mathbb{T}}$ of order m, and ρ_κ defines a m-th cyclic finite étale covering $\pi_{\rho_\kappa} : X_{\rho_\kappa} \to X$ with $n = rm$,

$$\begin{array}{ccc} \mathcal{M}^{D,L}_{H_{\rho_\kappa}} & \xrightarrow{\mathrm{ev}_p} & \mathcal{M}_{\rho_\kappa,p} \\ \Big\downarrow{\scriptstyle h^D} & & \Big\downarrow{\scriptstyle h_p} \\ \mathcal{A}^D & \longrightarrow & \mathcal{A}_p \end{array}$$

where $\mathcal{M}_{\rho_\kappa,p} = [\mathfrak{h}_\kappa / H_{\pi_\kappa}]$, here we let $\mathfrak{h}_{\rho_\kappa}$ be the Lie algebra of H_{ρ_κ}.

Maulik-Shen first prove the following proposition concerning about the smoothness of the evaluation map ev_p,

Proposition 4.5.1. *Assume that D is a divisor on X satisfying*

1. $D - p = K_X$ *or*

2. $D - p$ *is effective and* $\deg(D-p) < 2g - 2$.

The evaluation map $ev_p : \mathcal{M}^{D,L}_{H_{\rho_\kappa}} \to \mathcal{M}_{\rho_\kappa, p}$ *is smooth.*

Now if $\deg(D)$ satisfies these two conditions in proposition 4.5.1, then let us consider a stable $(D-p)$-Higgs bundle (E, θ) on X. Then one may view θ as

$$\theta' : E \xrightarrow{\theta} E \otimes O_C(D-p) \to E \otimes O_C(D).$$

Clearly the stableness is preserved, hence we obtain a closed embedding

$$\mathcal{M}^{D-p,L}_{H_{\rho_\kappa}} \to \mathcal{M}^{D,L}_{H_{\rho_\kappa}}, \qquad (E,\theta) \mapsto (E, \theta').$$

Hence, a Higgs bundle $(E, \theta) \in \mathcal{M}^{D,L}_{H_{\rho_\kappa}}$ from $\mathcal{M}^{D-p,L}_{H_{\rho_\kappa}}$ can be characterized as the vanishing of the Higgs field over the point p.

Recall that for $(a_2, \ldots, a_n) \in \mathfrak{sl}_n /\!\!/ SL_n$ determines a characteristic polynomial. The term a_i defines polynomial function of degree i, on the Lie algebra $\mathfrak{sl}_n$, which by definition is SL_n-invariant. We take the quadratic function,

$$\mu = a_2 : \mathfrak{sl}_n \to \mathbb{A}^1.$$

We have

$$\mu_{\rho_\kappa} := \mathfrak{h}_{\rho_\kappa} \hookrightarrow \mathfrak{sl}_n \xrightarrow{\mu} \mathbb{A}^1$$

Now we consider the critical locus of the function μ_2.

Lemma 4.5.2 (Lemma 4.3, [41])**.** *The critical locus of the function* μ_2 *is the isolated reduced point* $0 \in \mathfrak{sl}_n$*, i.e.,*

$$\{d\mu_{\rho_\kappa} = 0\} =: Crit(\mu_{\rho_\kappa}) = \{0\} \subset \mathfrak{sl}_n.$$

Consequently, the perverse sheaf of vanishing cycles $\varphi_{\mu_2}(\overline{\mathbb{Q}}_\ell[\dim(\mathfrak{sl}_n)])$ *is the skyscraper sheaf supported on the closed point* $0 \in \mathfrak{sl}_n$*.*

The $\mathfrak{sl}_n$-invariant function μ_2 induces the function

$$\mu_{\rho_\kappa} : [\mathfrak{h}_{\rho_\kappa} / H_{\rho_\kappa}] \to \mathbb{A}^1$$

which form the commutative diagram

$$\begin{tikzcd} \mathcal{M}_{\rho_\kappa,p} \arrow[d, "h_p"] \arrow[dr, "\mu_1"] & \\ \mathcal{A}_p \arrow[r, "\mu_2"] & \mathbb{A}^1. \end{tikzcd}$$

Once we pullback along the previous evaluation diagram,

$$\begin{tikzcd} \mathcal{M}^{D,L}_{H_{\rho_\kappa}} \arrow[r, "\mathrm{ev}_p"] \arrow[d, "h^D"] & \mathcal{M}_{\rho_\kappa,p} \arrow[d, "h_p"] \\ \mathcal{A}^D \arrow[r] & \mathbb{A}_p, \end{tikzcd}$$

we get the functions

$$\mu_{\rho_\kappa,\mathcal{M}} : \mathcal{M}^{D,L}_{H_{\rho_\kappa}} \to \mathbb{A}^1, \quad \mu_{\rho_\kappa,\mathcal{A}} : \mathcal{A}^D_{H_{\rho_\kappa}} \to \mathbb{A}^1$$

which satisfies the following commutative diagram

$$\begin{tikzcd} \mathcal{M}^{D,L}_{H_{\rho_\kappa}} \arrow[d, "h^D"] \arrow[dr, "\mu_{\rho_\kappa,\mathcal{M}}"] & \\ \mathcal{A}^D_{H_{\rho_\kappa}} \arrow[r, "\mu_{\rho_\kappa,\mathcal{A}}"] & \mathbb{A}^1. \end{tikzcd}$$

Now finally, we may introduce some useful properties that we shall use in the sequel.

Lemma 4.5.3. *Let $f : V \to \mathbb{A}^1$ be a regular function, where $\mathbb{A}^1$ is the affine line.*

1. *Assume that V admits an action of a finite group G which is fibrewise with respect to f, then the nearby and vanishing cycle functors Φ_f, φ_f are G-equivariant.*

2. *Assume that $\mathcal{F} \in D^b_c(V)$ and assume that $g = \lambda \cdot id \in End(\mathcal{F})$ with $\lambda \in \overline{\mathbb{Q}}_\ell^\times$, then the following hold:*

$$\Phi_f(g) = \lambda \cdot id : \Phi_f\mathcal{F} \xrightarrow{\sim} \Phi_f\mathcal{F}$$

and

$$\varphi_f(g) = \lambda \cdot id : \varphi_f \mathcal{F} \xrightarrow{\sim} \varphi_f \mathcal{F}.$$

3. *Assume that* $g : W \to V$ *is smooth with* $f' = f \circ g : W \to \mathbb{A}^1$*, then*

$$g^* \circ \varphi_f = \varphi_{f'} \circ g^* : D_c^b(V) \to D_c^b(f'^{-1}(0_{\mathbb{A}^1})).$$

Theorem 4.5.4 (theorem 4.5, [41])**.** *Assume that the divisor* D *satisfies these two conditions in 4.5.1,*

1. *The closed embedding can be realized as the critical locus of the function* $\mu_{\rho_\kappa,\mathcal{M}} : \mathcal{M}_{H_{\rho_\kappa}}^{D,L} \to \mathbb{A}^1$.

2. *Let* r_0 *be the codimension of the embedding* $\mathcal{M}_{H_{\rho_\kappa}}^{D-p,L} \hookrightarrow \mathcal{M}_{H_{\rho_\kappa}}^{D,L}$*. There is a natural isomorphism*

$$\varphi_{\mu_{\rho_\kappa,\mathcal{M}}} \overline{\mathbb{Q}}_\ell = \overline{\mathbb{Q}}_\ell[-r_0].$$

   Here $\overline{\mathbb{Q}}_\ell$ *denotes the constant sheaves on* $\mathcal{M}_{r,L}^D$ *and* $\mathcal{M}_{r,L}^{D-p}$*, respectively.*

3. *For any character* κ *of the group* Γ*, one has the following natural isomorphism*

$$\varphi_{\mu_{\rho_\kappa},\mathbb{A}}(Rf_{\rho_\kappa,*}^D \overline{\mathbb{Q}}_\ell)_\kappa = (Rf_{\rho_\kappa,*}^{D-p} \overline{\mathbb{Q}}_\ell)_\kappa[-r_0].$$

Let us now consider the following cases,

1. The effective divisor D satisfies that $\deg(D)$ is even and $\deg(D) > 2g-2$, then that is what we will prove in chapter 5. We need to reduce the general case of D being effective and $\deg(D) > 2g-2$ or $D = K_X$, where K_X is the canonical divisor of X.

2. Let us assume that the degree of the effective divisor D is odd and large than $2g-2$, then we set

$$D_p = D + p$$

   where $p \in X$ is a closed point.

   Then one can compute the codimension $r_1 := \operatorname{codim}_{\mathcal{M}_{SL_n}^{D_p,L}}(\mathcal{M}_{SL_n}^{D,L})$ and $r_2 := \operatorname{codim}_{\mathcal{M}_{H_{\rho_\kappa}}^{D_p}}(\mathcal{M}_{H_{\rho_\kappa}}^{D})$, according to section 4.4 of [41], we have

$$r_1 - r_2 = 2(d_\gamma^D - d_\gamma^D)$$

here $2d_\gamma^D = \mathrm{codim}_{\mathcal{M}_{SL_n}^{D,L}}(\mathcal{M}_{H_{\rho_\kappa}}^D)$ which coincides with the fermion shift of the locus fixed by $\gamma \in \Gamma = \mathrm{Pic}^0(X)[n]$, with γ corresponds to κ through Weil pairing.

Thus in this case if we already have

$$(\#\mathcal{M}_{SL_n}^{D_p,L})_\kappa = q^{2d_\gamma^{D_p}}(\#\mathcal{M}_{H_{\rho_\kappa}}^{D_p})_{st}$$

After applying that natural morphism in the third part of the above theorem we have

$$(\#\mathcal{M}_{SL_n}^{D_p,L})_\kappa = q^{r_1}(\#\mathcal{M}_{SL_n}^{D,L})_\kappa$$

and

$$(\#\mathcal{M}_{H_{\rho_\kappa}}^{D_p})_{st} = q^{r_2}(\#\mathcal{M}_{H_{\rho_\kappa}}^{D})_{st}.$$

Now suppose that for D_p, we have

$$(\#\mathcal{M}_{SL_n}^{D_p,L})_\kappa = q^{2d_\gamma^{D_p}}(\#\mathcal{M}_{H_{\rho_\kappa}}^{D_p})_{st}.$$

One can have

$$(\#\mathcal{M}_{SL_n}^{D,L})_\kappa = q^{2d_\gamma^{D}}(\#\mathcal{M}_{H_{\rho_\kappa}}^{D})_{st}.$$

3. Then one can consider the case of $D = K_X$, where K_X is the canonical divisor of the curve X.

   Then one can consider the divisor

$$K_{p,q} = K_X + p + q,$$

   where p, q are two closed points of X.

   Then by applying the natural morphism in third part of Theorem 4.5.4 twice, one gets the desired reduction to the first case.

Chapter 5

Counting points on Hitchin systems

In this chapter, we will work under the assumption that the effective divisor we use to definition Hitchin systems satisfies $\deg(D)$ is even and $\deg(D) > 2g-2$. We will use the fundamental lemma, i.e., the stabilization of the trace formula over the anisotropic locus and the supports theorems we mentioned in chapter 4 to show that

$$\#\mathcal{M}^{L}_{SL_n} = \#_{str}\mathcal{M}^{e}_{PGL_n}.$$

In this chapter, we will take $G = SL_n$ split over the curve X, and H_{ρ_κ} to be the endoscopic groups of SL_n on X associated to the endoscopic datum (κ, ρ_κ), since the split case would be enough for the original Hausel-Thaddeus conjecture. And we will use $\widehat{G}, \widehat{H}_{\rho_\kappa}$ to denote the corresponding Langlands dual groups over X. Recall that there is a finite étale cyclic covering $\pi_{\rho_\kappa} : X_{\rho_\kappa} \to X$, associated to ρ_κ. We will always assume that the cyclic covering π_{ρ_κ} is always of degree m if not otherwise mentioned, and we will always assume that $mr = n$.

Let us mention the quasi-split case briefly. We consider the base change to X_Θ, where X_Θ is the covering of X given by the torsor $\rho_G : \pi_1(X, x) \to \mathrm{Out}(\mathbb{G})$, see section 1 in chapter 1. Then ρ_G is trivial over X_Θ and by Lemma 1.1.5 one knows that X_Θ is finite over X. Then as discussed in the section 3.1 of Yun [55], by pulling everything back to X_Θ and taking Θ-invariants of the cohomological sheaves, it is reasonable for one to expect similar result like Theorem 3.2.8 in Yun [55] holds over the entire Hitchin base for the curve X_Θ.

5.1 Counting on fibres in the anisotropic locus

We start by recalling the action of $\Gamma = \mathrm{Pic}^0(X)[n]$ on the complex $Rf_{SL_n,*}\overline{\mathbb{Q}}_\ell$ induced by the action of Γ on the moduli space $\mathcal{M}^L_{SL_n}$ by tensorisation, hence Γ acts on the cohomology groups of $\mathcal{M}^L_{SL_n}$ through $s : \Gamma \to \overline{\mathbb{Q}}_\ell^\times$. One then has the following decomposition,

$$Rf_{SL_n,*}\overline{\mathbb{Q}}_\ell = \bigoplus_{s \in \widehat{\Gamma}} \left(Rf_{SL_n,*}\overline{\mathbb{Q}}_\ell \right)_s, \tag{5.1}$$

see Lemme 3.2.5 in [38]. We denote

$$\left(Rf_{SL_n,*}\overline{\mathbb{Q}}_\ell \right)_{stab} := \left(Rf_{SL_n,*}\overline{\mathbb{Q}}_\ell \right)_{s=1}.$$

Hence by taking the alternating sum of the Frobenius trace (see equation 4.2), we get the corresponding point count $\#^{stab}\mathcal{M}^L_{SL_n}$. From the support theorem 4.2.12 of SL_n, we know that the support of $f_{SL_n} : \mathcal{M}^L_{SL_n} \to \mathcal{A}_{SL_n}$ is contained in $\mathcal{A}^{ani}_{SL_n}$.

Also we have the similar decomposition of $\left(Rf_{H_{\rho_\kappa},*}\overline{\mathbb{Q}}_\ell \right)$, see (4.3), as well as the notion of stable point counting $\#^{stab}\mathcal{M}_{H_{\rho_\kappa}}$.

Let us mention that there is an one-to-one correspondence between the equivalence classes of elliptic coendoscopic datum (κ, ρ_κ) of SL_n over X and the group $\Gamma = \mathrm{Pic}^0(X)[n]$.

- On the one hand, if one is given a $\gamma \in \Gamma$ of order m, there is an associated degree m étale covering $X_\gamma \to X$ such that γ is trivial over X_γ. By taking the trivialization of γ over X_γ and the torsor $\rho_\gamma : X_\gamma \to X$ given by the m-th étale covering, we get an elliptic coendoscopic datum (κ, ρ_κ), by the spirit of Example 3.1.8, κ should be of the following form
$$\kappa = \mathrm{diag}(\underbrace{1, ..., 1}_{r \text{ copies}}, \underbrace{s, ..., s}_{r \text{ copies}}, \ldots, \underbrace{s^{m-1}, ..., s^{m-1}}_{r \text{ copies}}).$$
- On the other hand, an elliptic coendoscopic datum (κ, ρ_κ) gives us an m-th étale cyclic covering $\pi_{\rho_\kappa} : X_{\rho_\kappa} \to X$, and we can view κ as an element $s \in \mu_m$, an m-th root of unity, which is constant over X_{ρ_κ}, hence by pushforward we get a line bundle of order m.

It is easy to see that the two operations above are inverse to each other, hence we have the following diagram:

$$
\begin{array}{ccc}
\Gamma & \xrightarrow{\quad\quad\quad\rho\quad\quad\quad} & \widehat{\Gamma} \\
\updownarrow & & \updownarrow \\
\left\{ \begin{array}{r} \text{equivalent classes of} \\ \text{elliptic coendoscopic datum} \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{r} \text{equivalent classes of} \\ \text{elliptic endoscopic datum} \end{array} \right\}
\end{array}. \tag{5.2}
$$

Then by the celebrated geometric stabilisation theorem of Ngô, see Theorem 6.4.2 in [46], we have the following decomposition of $Rf^{ani}_{SL_n,*}\overline{\mathbb{Q}}_\ell$,

$$\bigoplus_n {}^pH^n(Rf^{ani}_{SL_n,*}\overline{\mathbb{Q}}_\ell)[2d_\gamma](d_\gamma) \simeq \bigoplus_n \bigoplus_{(\kappa,\rho_\kappa)} \nu^* \, {}^pH^n(Rf^{ani}_{H_{\rho_\kappa},*}\overline{\mathbb{Q}}_\ell)_{stab},$$

where $\nu : \mathcal{A}_{H_{\rho_\kappa}} \to \mathcal{A}_{SL_n}$ is the canonical map on Hitchin bases.

Therefore we have the corresponding decomposition of point counting

$$\#\mathcal{M}^L_{SL_n}|_{\mathcal{A}^{ani}} = \sum_{(\kappa,\rho_\kappa)} q^{2d_\gamma}\#^{stab}\mathcal{M}_{H_{\rho_\kappa}}|_{\mathcal{A}^{ani}_{H_{\rho_\kappa}}},$$

where the sum is taken over all equivalent classes of elliptic endoscopic datum of SL_n, note that $2d_\gamma$ is the fermionic shift, which is also the difference of dimensions, i.e., $\dim \mathcal{M}^L_{SL_n} - \dim \mathcal{M}_{H_{\rho_\kappa}}$, where γ corresponds to (κ, ρ_κ) via the commutative diagram 5.2.

5.2 Comparison of point counts

From the first section, we have the following decomposition of the point counting on $\mathcal{M}^L_{SL_n}$,

$$\#\mathcal{M}^L_{SL_n}|_{\mathcal{A}^{ani}_{SL_n}} = \sum_{(\kappa,\rho_\kappa)} \left(\#^{\text{stab}}\mathcal{M}^L_{H_{\rho_\kappa}}|_{\mathcal{A}^{ani}_{H_{\rho_\kappa}}}\right) q^{2d_\gamma}, \tag{5.3}$$

where the sum is taking over the set of equivalence classes of endoscopic datum (κ, ρ_κ) of SL_n over the curve X, or equivalently, the group $\Gamma \simeq \widehat{\Gamma}$, as we discussed at the beginning of the first section.

For simplicity, we will omit the superscript L in $\mathcal{M}^L_{H_{\rho_\kappa}}$ in this section.

Now let us pick an arbitrary term in the right hand side of the above identity 5.3, i.e., $\#^{stab}\mathcal{M}_{H_{\rho_\kappa}}|_{\mathcal{A}^{ani}_{H_{\rho_\kappa}}}$. We first note that by the non-standard fundamental lemma that Ngô proved in [46], Theorem 1.12.7

Theorem 5.2.1. *Suppose that G_1 and G_2 are two groups with isogeny root datum, then for $a \in \mathcal{A}^{ani}_{G_1} \simeq \mathcal{A}^{ani}_{G_2}$, one has*

$$\#^{stab}\mathcal{M}_{G_1,a} = \#^{stab}\mathcal{M}_{G_2,a}.$$

Thus one has the following identity

$$\#^{stab}\mathcal{M}_{H_{\rho_\kappa}}|_{\mathcal{A}^{ani}_{\rho_\kappa}} = \#^{stab}\mathcal{M}_{\widehat{H}_\gamma}|_{\mathcal{A}^{ani}_{\rho_\kappa}},$$

where $\widehat{H}_\gamma \subset PGL_n$ is the coendoscopic group that given by the coendoscopic datum (η, ρ_κ) which is obtained from applying the Weil pairing $\rho : \Gamma \times \Gamma \to \mu_n$. Here we use $\gamma \in \mathrm{Pic}^0(X)[n]$ to denote the line bundle that one can obtain from the coendoscopic datum (η, ρ_κ), and $s \in \widehat{\Gamma}$ to denote the corresponding character obtained from (κ, ρ_κ), and we remark that s corresponds to γ through ρ.

Recall again in Example 3.1.8, we have an explicit description of endoscopic groups of SL_n and therefore a description of coendoscopic groups of PGL_n, in this case we know that

$$\widehat{H}_\gamma = \left(\mathrm{Res}_{E_{\rho_\kappa}/F} GL_r\right)/GL_1,$$

where F is the function field of X, and E_{ρ_κ} is the function field of the finite étale covering X_{ρ_κ} of X. One immediately sees that any orbital integral over the group $\widehat{H}_\gamma$ should be stable, since the notion of stable conjugacy and rational conjugacy coincide on the group $\widehat{H}_\gamma$.

Hence we have the following equality of point counts,

$$\#^{stab}\mathcal{M}_{H_{\rho_\kappa}}|_{\mathcal{A}^{ani}_{\rho_\kappa}} = \#\mathcal{M}_{\widehat{H}_\gamma}|_{\mathcal{A}^{ani}_{\rho_\kappa}}.$$

Now the observation that should come in is the following short exact sequence of groups

$$1 \to \left(\mathrm{Res}_{E_{\rho_\kappa}/F} GL_1\right)/GL_1 \to \widehat{H}_\gamma \to PGL_r(E_{\rho_\kappa}) \to 1.$$

Then one has the following identity of the point counts.

Proposition 5.2.2. *Let $f_1 : \mathcal{M}^e_{PGL_r} \to \mathcal{A}_{PGL_r}$ and $f_2 : \mathcal{M}^e_{\widehat{H}_\gamma} \to \mathcal{A}_{\rho_\kappa}$ be two Hitchin systems of PGL_r-Higgs bundles and $\widehat{H}_\gamma$-Higgs bundles of degree e, respectively. Then one has the following*

$$\#\mathcal{M}^e_{\widehat{H}_\gamma} = q^{\Delta_{\rho_\kappa}} \#\mathcal{M}^e_{PGL_r},$$

where Δ_{ρ_κ} is a constant which only depends on the covering $\rho_\kappa : X_{\rho_\kappa} \to X$.

Proof. For simplicity, we write $G_1 = \widehat{H}_\gamma$, $G_2 = PGL_r$ and $E = E_{\rho_\kappa}$. Also, we will write $T = (\mathrm{Res}_{E/F} GL_1)/GL_1$, we remark that T is in the centre of G_1.

Then we have the following short exact sequence

$$1 \to T \to G_1 \to G_2 \to 1.$$

We recall from the full counting formula 1.2 in section 1.4.

$$\#\mathcal{M}^e_{G_1} = \int_{G_1(E)\backslash G_1(\mathbb{A}_E)^e} \sum_{P_1} (-1)^{\dim(\mathfrak{a}^{G_1}_{P_1})} \sum_{\delta \in P_1(E)\backslash G_1(E)} \widehat{\tau}_{P_1}(H_{P_1}(\delta g)) \sum_{\gamma \in \mathfrak{p}_1(E)} 1_D((\delta g)^{-1}\gamma(\delta g)) dg.$$

By the spirit of [12], one knows that the above integral converges absolutely, we have

$$\begin{aligned}
\#\mathcal{M}^e_{G_1} &= \sum_{P_1} (-1)^{\dim(\mathfrak{a}^{G_1}_{P_1})} \int_{P_1(E)\backslash G_1(\mathbb{A}_E)^e} \widehat{\tau}_{P_1}(H_{P_1}(g)) \sum_{\gamma \in \mathfrak{p}_1(E)} 1_D(g^{-1}\gamma g) dg \\
&= \sum_{P_2} (-1)^{\dim(\mathfrak{a}^{G_2}_{P_2})} q^{d(T)} \mathrm{vol}(T(F)\backslash T(\mathbb{A}_E)^e) \\
&\left(\int_{P_2(E)\backslash G_2(\mathbb{A}_E)^e} \widehat{\tau}_{P_2}(H_{P_2}(g)) \sum_{\gamma \in \mathfrak{p}_2(E)} 1_D(g^{-1}\gamma g) dg \right) \\
&= q^{d(T)} \mathrm{vol}(T(F)\backslash T(\mathbb{A}_E)^e) \#\mathcal{M}^e_{G_2}.
\end{aligned}$$

Since for each $P_1 \subset G_1$, we have

$$P_2 = (T\backslash P_1) \subset G_2$$

and for each $\gamma_1 \in \mathfrak{p}_1$, we can write it as

$$\gamma_1 = \gamma_2 + U, \qquad \text{where } \gamma_1 \in \mathfrak{p}_1, \gamma_2 \in \mathfrak{p}_1, \text{ and } U \in \mathrm{Lie}(T),$$

note that U should be central.

Hence

$$1_D(g^{-1}\gamma_1 g) = 1_D(g^{-1}(\gamma_2 + U)g) = 1_D(g^{-1}\gamma_2 g)1_D(U).$$

Hence we can define $q^{d(T)} := \sum_{U\in \mathrm{Lie}(T)} 1_D(U)$, which is a finite number by Riemann-Roch theorem.

For simplicity, we will define Δ_{ρ_κ} such that

$$q^{\Delta_{\rho_\kappa}} := q^{d(T)}\mathrm{vol}(T(F)\backslash T(\mathbb{A}_E)^e).$$

$\square$

Remark 5.2.3. One has the following inclusion,

$$\mathcal{A}^{ani}_{PGL_r} \hookrightarrow \mathcal{A}^{ani}_{\widehat{H}_\gamma}.$$

We know that $\mathcal{A}^{ani}_{\widehat{H}_\gamma}$ is a trivial affine bundle over $\mathcal{A}^{ani}_{PGL_r}$. Let $a \in \mathcal{A}^{ani}_{PGL_r}$, we should note that every b above a should be also living in $\mathcal{A}^{ani}_{\widehat{H}_\gamma}$. Indeed, one has

$$\pi_0(\mathcal{P}_b) < \infty \iff \pi_0(\mathcal{P}_a) < \infty,$$

since $T = (\mathrm{Res}_{E/F}GL_1)/GL_1$ is finite. Then by the paragraph under remark 1.3.5, we have that $b \in \mathcal{A}^{ani}_{\widehat{H}_\gamma}$ if and only if $a \in \mathcal{A}^{ani}_{PGL_r}$.

If we exam the proof above more closely, we know that for each b above a, the point counting of each Hitchin fibre

$$\#\mathcal{M}^e_{G_1,b} = \mathrm{vol}(T(F)\backslash T(\mathbb{A}_E)^e)\#\mathcal{M}^e_{G_2,a}.$$

Therefore we have the following sequences of identities

Corollary 5.2.4.

$$\#^{stab}\mathcal{M}|_{\mathcal{A}^{ani}_{H_{\rho_\kappa}}} = q^{\Delta_{\rho_\kappa}}\#\mathcal{M}_{PGL_r}|_{\mathcal{A}^{ani}_{PGL_r}} = q^{\Delta_{\rho_\kappa}}\#^{stab}\mathcal{M}_{SL_r}|_{\mathcal{A}^{ani}_{SL_r}}, \tag{5.4}$$

where the 2nd equality comes from the definition of $\#^{stab}\mathcal{M}_{SL_r}$.

Hence we have the following consequence

$$\#\mathcal{M}^L_{SL_n}|_{\mathcal{A}^{ani}_{SL_n}} = \sum_{\widehat{\Gamma}} \left(q^{\Delta_{\rho_\kappa}} \#^{\text{stab}} \mathcal{M}_{SL_r(E_{\rho_\kappa})}|_{\mathcal{A}^{ani}_{SL_r(E_{\rho_\kappa})}} \right) q^{2d_\gamma}. \tag{5.5}$$

Now the observation is that the both sides of the decomposition 5.5 above are supported on the anisotropic loci of their Hitchin bases, and the image of $\mathcal{A}^{ani}_{SL_r(E_{\rho_\kappa})}$ is in $\mathcal{A}^{ani}_{SL_n}$. Hence, we can extend the above decomposition to full Hitchin bases by the spirit of perverse continuation method, and get the following sequence of identities,

$$\#\mathcal{M}^L_{SL_n} = \sum_{\widehat{\Gamma}} \left(q^{\Delta(\rho_\kappa)} \#^{\text{stab}} \mathcal{M}_{SL_r(E_{\rho_\kappa})} \right) q^{2d_\gamma} \tag{5.6}$$

$$= \sum_{\Gamma} \left(q^{\Delta(\rho_\kappa)} \#\mathcal{M}_{PGL_r(E_{\rho_\kappa})} \right) q^{2d_\gamma} \tag{5.7}$$

$$= \sum_{\Gamma} \left(\#\mathcal{M}_{\widehat{H}_\gamma} \right) q^{2d_\gamma} \tag{5.8}$$

$$= \#_{str}\mathcal{M}^d_{PGL_n}. \tag{5.9}$$

Here we use $\#_{str}$ to denote the stringy point counting formula we defined for inertial stacks in chapter 2, see Definition 2.1.4. Here for the last equality above we used the equivalence between the inertia stacks and the coendoscopic decomposition of $\mathcal{M}^d_{PGL_n}$.

Corollary 5.2.5. *From the above argument, if we take $\kappa \in \widehat{\Gamma}$ and the corresponding $\gamma \in \Gamma$, which is obtained from κ through the Weil pairing, we see the following*

$$(\#\mathcal{M}^L_{SL_n})_\kappa = (\#_{str}\mathcal{M}^d_{PGL_n})_\gamma.$$

To conclude, we have the following theorems first conjectured by Hausel-Thaddeus in [31],

Theorem 5.2.6. *Let d, e be two integers, both of them are supposed to be coprime with $n \in \mathbb{N}$. Let the effective divisor D on X satisfy $\deg(D) > 2g$ and $\deg(D)$ being even.*

Let $\mathcal{M}^L_{SL_n}$ and $\mathcal{M}^e_{PGL_n}$ be the moduli stack of stable SL_n-Higgs bundles and PGL_n-Higgs bundles, respectively. Let α_L be the lifting gerbe defined in section 2.3, then one has the following

$$\#\mathcal{M}^L_{SL_n} = \#^{\alpha_L}_{str}\mathcal{M}^e_{PGL_n}.$$

In section 4.5, we showed how the method of utilizing vanishing cycle functors developed by Maulik and Shen in section 4 of [41] can reduce the following general case from Theorem 5.2.6.

Theorem 5.2.7. *Let d, e be two integers coprime with $n \in \mathbb{N}$. Let the effective divisor D on X satisfy $\deg(D) \geq 2g-2$ or $K = D$.*

Let $\mathcal{M}^L_{SL_n}$ and $\mathcal{M}^e_{PGL_n}$ be the moduli stack of stable SL_n-Higgs bundles and PGL_n-Higgs bundles, respectively. Let α_L be the lifting gerbe defined in section 2.3, then one has the following

$$\#\mathcal{M}^L_{SL_n} = \#^{\alpha_L}_{str}\mathcal{M}^e_{PGL_n}.$$

We now conclude this section with the following corollary, which is Conjecture 3.27 in [29], also see Theorem 3.2 in [41].

Corollary 5.2.8. *For $\gamma = \rho(\kappa)$, we have*

$$E_\kappa(\mathcal{M}^d_{SL_n}; u, v) = (uv)^{F(\gamma)} E(\mathcal{M}^\gamma_{PGL^e_n}/\Gamma, L_{\alpha^d_L}; u, v),$$

where $L_{\alpha^d_L}$ is the induced Local system.

Chapter 6

Conclusion

This thesis reproves the topological mirror symmetry conjecture posed by Hausel-Thaddeus in [31]. Starting with the idea that point counts reveal topological information which was used in previous proof by Groechenig, Wyss and Ziegler, we use a different way of proving the desired agreement of point counts. More precisely, we use the fundamental lemma to decompose point counts on SL_n-Hitchin systems according to the set of endoscopic data and then used the same set of (co)endoscopic data to decompose the stringy point counts on PGL_n-Hitchin systems. Then we get the desired equality by proving that each part of these two decompositions match, on the anisotropic locus of the Hitchin base. We then reduce the point counts on endoscopic groups of SL_n to SL_r-Hitchin systems, where $r \mid n$. Finally, we use the support theorem of Hitchin systems of structure group SL_n to get the desired equality over the entire Hitchin base.

This thesis, inevitably, leaves some ends untied. From the viewpoint of the proof we present here, it would be interesting to compare the full stabilization of the trace formula already established by Arthur in his series of papers [2], [1] and [3] with this proof, to see whether the support theorem of SL_n can be avoid by using the techniques invented by Arthur, perhaps by some delicate induction methods on Levi subgroups.

It would be intriguing to investigate whether the methodologies used in this thesis, along with the weighted fundamental lemma as proved by Chaudouard and Laumon [13] could be applied to establish topological mirror symmetry in the parabolic case. There are already some related counting results established by Yu in [52], but he only dealt with the coarse expansion of the geometric side of the trace formulas only.

It would also be interesting to see whether the support theorem proved by Maulik and Shen in [42] without the coprimality condition can be used to extends the equality of point counting we concern in this thesis to the case where the degree and rank of the Higgs bundle are not necessarily coprime.

Also, since it has been observed that there should be rich geometric structures under the phenomenon of endoscopy in [22], it would be interesting to further investigate the geometry that hiding underlying endoscopy. Additionally, it has been pointed out to the author that the classical fixed point theorem of Narasimhan and Ramanan, see theorem 3.0.1, has been generalized to other groups in [23] and [5]. It would be interesting to see whether the endoscopy phenomenon we discussed in the context of the classical theorem of Narasimhan and Ramanan reoccurs in their setting.

Furthermore, it would be interesting to look at the behaviours of nilpotent terms in the context of mirror symmetry. As one can already see from the explicit computation in chapter 6 of [34], the (regular) nilpotent parts of SL_2 can be decomposed according to the set of equivalent classes endoscopic datum, hence transferred into the nilpotent part of those Hitchin systems given by the elliptic torus of SL_2. More generally in prime rank case, one would expect that only regular nilpotent Higgs bundles of SL_n contributes to the stringy part of the cohomology of the moduli space of PGL_n-Higgs bundles. This is a conjecture of Hausel-Thaddeus made in the last paragraph of their paper [31], and they verified that this conjecture is true in the case of rank being 2 and 3, but now it is a direct consequence of the topological mirror symmetry. In the most general case, it would be reasonable to expect that one can do a direct computation on integrals of point countings over the nilpotent fibres to prove that only nilpotent orbits that have equal size blocks contributes to the stringy part of the cohomology of moduli space PGL_n-Higgs bundles. In fact, the explicit computation for those orbits in the case of GL_n has been done by Chaudouard in [14].

Lastly, one would like to ask the possibility of utilizing the automorphic method on the point counting integral on nilpotent fibres to reduce the case of general divisor D to the special case we mentioned, i.e., D is effective and $\deg(D)$ is a large enough even integer.

Bibliography

[1] J. Arthur. A stable trace formula II. Global descent. *Inventiones mathematicae*, 143(1):157, 2001.

[2] J. Arthur. A stable trace formula. I. General expansions. *Journal of the Institute of Mathematics of Jussieu*, 1(2):175–277, 2002.

[3] J. Arthur. A stable trace formula III. Proof of the main theorems. *Annals of mathematics*, 158(3):769–873, 2003.

[4] J. G. Arthur. A trace formula for reductive groups I terms associated to classes in g (q). 1978.

[5] G. Barajas and O. García-Prada. A Prym-Narasimhan-Ramanan construction of principal bundle fixed points. *arXiv preprint arXiv:2211.12812*, 2022.

[6] V. V. Batyrev and D. I. Dais. Strong McKay correspondence, string-theoretic Hodge numbers and mirror symmetry. *Topology*, 35(4):901–929, 1996.

[7] A. Beauville, S. Ramanan, and M. S. Narasimhan. Spectral curves and the generalised theta divisor. 1989.

[8] A. Beilinson, J. Bernstein, P. Deligne, and O. Gabber. *Faisceaux pervers*, volume 4. Société mathématique de France Paris, 2018.

[9] I. Biswas and A. Dey. Syz duality for parabolic Higgs moduli spaces. *Nuclear Physics B*, 862(1):327–340, 2012.

[10] S. Bosch, W. Lütkebohmert, and M. Raynaud. *Néron models*, volume 21. Springer Science & Business Media, 2012.

[11] N. B. Châu. Perverse sheaves and fundamental lemmas.

[12] P.-H. Chaudouard. Sur le comptage des fibrés de Hitchin. *Astérisque*, 369:223–284, 2015.

[13] P.-H. Chaudouard and G. Laumon. Le lemme fondamental pondéré. ii. énoncés cohomologiques. *Annals of Mathematics*, pages 1647–1781, 2012.

[14] P.-H. Chaudouard and G. Laumon. Sur le comptage des fibrés de Hitchin nilpotents. *Journal of the Institute of Mathematics of Jussieu*, 15(1):91–164, 2016.

[15] P.-H. Chaudouard and G. Laumon. Un théorème du support pour la fibration de Hitchin. In *Annales de l'Institut Fourier*, volume 66, pages 711–727, 2016.

[16] P.-H. Chaudouard and G. Laumon. Addendum à «un théorème du support pour la fibration de Hitchin». In *Annales de l'Institut Fourier*, volume 67, pages 1005–1008, 2017.

[17] N. Chriss and V. Ginzburg. *Representation Theory and Complex Geometry*, volume 42. Springer, 1997.

[18] M. A. de Cataldo. A support theorem for the Hitchin fibration: the case of SLn. *Compositio Mathematica*, 153(6):1316–1347, Apr. 2017.

[19] R. Donagi. Decomposition of spectral covers. *Astérisque*, 218:145–175, 1993.

[20] R. Donagi and T. Pantev. Langlands duality for Hitchin systems. *Inventiones mathematicae*, 189(3):653–735, 2012.

[21] G. Faltings. Stable G-bundles and projective connections. *J. Algebraic Geom*, 2(3):507–568, 1993.

[22] E. Frenkel and E. Witten. Geometric endoscopy and mirror symmetry. *arXiv preprint arXiv:0710.5939*, 2007.

[23] O. García-Prada and S. Ramanan. Involutions and higher order automorphisms of Higgs bundle moduli spaces. *Proceedings of the London Mathematical Society*, 119(3):681–732, 2019.

[24] J. R. Getz and H. Hahn. An introduction to automorphic representations with a view towards trace formulae. *Graduate Studies in Mathematics*, 6, 2019.

[25] P. B. Gothen and A. G. Oliveira. Topological mirror symmetry for parabolic Higgs bundles. *Journal of Geometry and Physics*, 137:7–34, 2019.

[26] M. Groechenig, D. Wyss, and P. Ziegler. Geometric stabilisation via p-adic integration. *Journal of the American Mathematical Society*, 33(3):807–873, 2020.

[27] M. Groechenig, D. Wyss, and P. Ziegler. Mirror symmetry for moduli spaces of Higgs bundles via p-adic integration. *Inventiones mathematicae*, 221(2):505–596, 2020.

[28] G. Harder and M. S. Narasimhan. On the cohomology groups of moduli spaces of vector bundles on curves. *Mathematische Annalen*, 212(3):215–248, 1975.

[29] T. Hausel. Global topology of the Hitchin system. *arXiv preprint arXiv:1102.1717*, 2011.

[30] T. Hausel and F. Rodriguez-Villegas. Mixed Hodge polynomials of character varieties: With an appendix by Nicholas M. Katz. *Inventiones mathematicae*, 174(3):555–624, 2008.

[31] T. Hausel and M. Thaddeus. Mirror symmetry, Langlands duality, and the Hitchin system. *Inventiones mathematicae*, 153:197–229, 2003.

[32] N. Hitchin. Stable bundles and integrable systems. *Duke Mathematical Journal*, 54(1):91 – 114, 1987.

[33] N. J. Hitchin. The self-duality equations on a Riemann surface. *Proceedings of the London Mathematical Society*, 3(1):59–126, 1987.

[34] T. Kaletha. Lectures on the stable trace formula with emphasis on SL2.

[35] R. Kiehl and R. Weissauer. *Weil conjectures, perverse sheaves and l-adic Fourier transform*, volume 42. Springer Science & Business Media, 2013.

[36] R. P. Langlands. *Les débuts d'une formule des traces stable*, volume 13. UER de mathématiques & LA 212 du CNRS, 1982.

[37] G. Laumon. Cohomology of Drinfeld modular varieties. Part II. Automorphic forms, trace formulas and Langlands correspondence, with an appendix by Jean-Loup Waldspurger. *Cambridge Studies in Advanced Mathematics*, 56, 1997.

[38] G. Laumon and B. C. Ngô. Le lemme fondamental pour les groupes unitaires. *Annals of mathematics*, pages 477–573, 2008.

[39] G. Laumon and M. Rapoport. The Langlands lemma and the Betti numbers of stacks of G–bundles on a curve. *arXiv preprint alg-geom/9503006*, 1995.

[40] F. Loeser and D. Wyss. Motivic integration on the Hitchin fibration. *Algebraic Geometry*, page 196–230, Mar. 2021.

[41] D. Maulik and J. Shen. Endoscopic decompositions and the Hausel–Thaddeus conjecture. In *Forum of Mathematics, Pi*, volume 9, page e8. Cambridge University Press, 2021.

[42] D. Maulik and J. Shen. On the intersection cohomology of the moduli of sl n sl_n-higgs bundles on a curve. *Journal of Topology*, 15(3):1034–1057, 2022.

[43] M. Melo, A. Rapagnetta, and F. Viviani. Fourier–Mukai and autoduality for compactified Jacobians. I. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2019(755):1–65, 2019.

[44] M. Narasimhan and S. Ramanan. Generalised Prym varieties as fixed points. *J. Indian Math. Soc*, 39(1-4):1–19, 1975.

[45] B. C. Ngô. Fibration de Hitchin et endoscopie. *Inventiones mathematicae*, 164:399–453, 2006.

[46] B. C. Ngô. Le lemme fondamental pour les algebres de lie. *Publications Mathématiques de l'IHÉS*, 111(1):1–169, 2010.

[47] N. Nitsure. Moduli space of semistable pairs on a curve. *Proceedings of the London Mathematical Society*, 3(2):275–300, 1991.

[48] V. Platonov, A. Rapinchuk, and R. Rowen. *Algebraic groups and number theory*. Academic press, 1993.

[49] S. Shen. Mirror symmetry for parabolic Higgs bundles via p-adic integration. *arXiv preprint arXiv:2302.02817*, 2023.

[50] C. T. Simpson. Higgs bundles and local systems. *Publications Mathématiques de l'IHÉS*, 75:5–95, 1992.

[51] A. Strominger, S.-T. Yau, and E. Zaslow. Mirror symmetry is T-duality. *Nuclear Physics B*, 479(1–2):243–259, Nov. 1996.

[52] H. Yu. Number of cuspidal automorphic representations and Hitchin's moduli spaces. *arXiv preprint arXiv:2110.13858*, 2021.

[53] H. Yu. A coarse geometric expansion of a variant of Arthur's truncated traces and some applications. *Pacific Journal of Mathematics*, 321(1):193–237, 2023.

[54] H. Yu. Comptage des systémes locaux l-adiques sur une courbe. *Annals of Mathematics*, 197(2):423–531, 2023.

[55] Z. Yun. Towards a global Springer theory iii: Endoscopy and Langlands duality. *arXiv preprint arXiv:0904.3372*, 2009.