Complex Monge-Ampère Equation and its Applications in Complex Geometry

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Requirements Statement

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ABSTRACT

The main threads of this thesis are related by the theme of the complex Monge-Ampère type equations. It consists of some analysis results from the partial differential equation aspect and several geometric consequences as applications.

In the first part, we study the *a priori* estimates for complex Hessian type equations on Hermitian manifolds. These estimates are the key ingredients for the solvability of the corresponding equations by virtue of the continuity method. In particular, we establish the first and second order derivative estimates for complex Monge-Ampère equations which are analogous to Yau's estimates on Kähler manifolds.

In Chapter 3, we investigate the interior Schauder estimates of the solutions to complex Monge-Ampère equations. Moreover, aiming to extend such regularity results to more general geometric setting, we also establish the classical Bedford-Taylor's interior second order estimate and a local version of Calabi's third order estimate on Hermitian manifolds.

The last two chapters of this thesis are devoted to the geometric problems related to complex Monge-Ampère type equations. In particular, we give some results on the nonnegative representation for the boundary class of Kähler cone and the existence of generalized Kähler-Einstein metrics.

ABRÉGÉ

Dans cette thèse, il est question de l'étude des équations de type Monge-Ampère complexes. On y présente une analyse basée sur les différentes techniques utilisées dans la théorie des équations aux dérivées partielles ainsi que certaines applications géométriques.

En premier lieu, nous présentons l'estimation à priori des équations de type Hessienne complexes sur des variétés hermitiennes. Ces estimations sont indispensables à la résolution de ces équations par le biais des méthodes de continuité. Au fait, nous établirons des estimations sur la première et la seconde dérivée des équations Monge-Ampère complexes de la même manière faite par Yau sur les variétés kählériennes.

Au troisième chapitre, nous étudions la régularité de Hölder intérieure des dérivées secondes de la solution pour les équations de type Monge-Ampère complexes. De plus, en visant la généralisation de ce type de résultats de régularité à des géométries plus généralee, on a obtenu une estimation de deuxième ordre de type Bedford-Taylor classique et une version locale des estimations de Calabi de troisième ordre sur des variétés hermitiennes.

Les deux derniers chapitres de cette thèse sont consacrés aux problèmes géométriques reliés aux équations de type Monge-Ampère complexes. Nous donnons quelques résultats sur la représentation non négative pour la classe de frontière du cône de Kähler et l'existence des métriques généralisée Kähler-Einstein.

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CHAPTER 1

Introduction

We begin by summarizing the main results of the thesis. In section 1.2 we collect some basic facts in Kähler geometry and Hermitian geometry, and in section 1.3 we give the outline of Yau's estimates for the complex Monge-Ampère equations.

1.1 Summary of the results

This thesis consists of four main parts which have come out my study in the field of geometric analysis. And they are all related by the theme of the complex Monge-Ampère equation type equations and its geometric applications. We begin with the results on the *a priori* estimates for Monge-Ampère equation on Hermitian manifolds which form the content of Chapter 2.

In 1976, Yau [81] gave an affirmative answer to the Calabi's conjecture by showing the existence of the solution to the complex Monge-Ampère equation:

$$\det(g_{i\bar{j}} + u_{i\bar{j}}) = f(z), \tag{1.1}$$

on a compact Kähler manifold (M, g). This work by Yau opened a vast field for the study of complex Monge-Ampère type equations (1.1). And it has proven to be a very powerful tool in understanding geometry and topology in Kähler setting. Given these successes, as a natural extension of Yau's result, one wants to consider the existence and uniqueness properties for the above equation on Hermitian manifolds and try to deduce geometric results from it. Our main result on the complex Monge-Ampère equation in Hermitian setting is as follows. We consider the Hermitian manifold (M, ω) of dimension $n \geq 2$ with smooth boundary ∂M and seek solutions to (1.1) in the *space of Hermitian metrics* defined as

$$PSH(\omega, M) = \{u \in C^2(M) \mid \omega_u = \omega + \sqrt{-1}\partial\bar{\partial}u > 0\}.$$

Theorem 1.1.1 ([87] Theorem 1 and 2)**.** *Let $u \in PSH(\omega, M) \bigcap C^4(M)$ be a solution of equation (1.1). Then there exist positive constants C_1, C_2 depending on $f, |u|_{C^0(M)}$ and geometric quantities of M (torsion and curvature) such that*

$$\max_{\bar{M}} |\nabla u| \leq C_1(1 + \max_{\partial M} |\nabla u|) \tag{1.2}$$

and

$$\max_{\bar{M}} |\Delta u| \leq C_2(1 + \max_{\partial M} |\Delta u|) \tag{1.3}$$

In particular, if the Hermitian manifold M is compact, i.e. $\partial M = \emptyset$, then one can get the estimates for gradient and Δu from (1.2) and (1.3).

Indeed, the above estimates are proved by the Bernstein type technique. A substantial difficulty in proving (1.3) is to control the extra terms involving third order derivatives which appear due to the nontrivial torsion. Similar estimates were also obtained independently by Guan-Li [44] and Tosatti-Weinkove [76].

We also study another important type of fully nonlinear geometric equations, complex Hessian equation, which includes (1.1) as a special case. We consider

$$\omega_u^k \wedge \omega^{n-k} = (\omega + \sqrt{-1}\partial\bar{\partial}u)^k \wedge \omega^{n-k} = f\omega^n \tag{1.4}$$

where $k = 1, 2, \cdots, n$, and f is a positive function on a Hermitian manifold (M, ω). Notice that if $k = n$, (1.4) is just the complex Monge-Ampère equation (1.1), while if $k = 1$, equation (1.4) becomes the Laplacian equation. So the complex Hessian type equation is a generalization of both complex Monge-Ampère equation and Laplacian equation over a compact Hermitian manifold.

Let $H(n)$ be the set of $n \times n$ Hermitian matrices and $\lambda(A)$ be the eigenvalues of A. For $k = 1, 2, \cdots, n$, we define

$$\sigma_k(A) = \sigma_k(\lambda(A)) \quad \text{ for } A \in H(n),$$

where $\sigma_k(\lambda)$ is the $k-th$ elementary symmetric function defined on $\mathbb{R}^n$ and let

$$\Gamma_k = \{A \in H(n) \mid \sigma_j(A) > 0, j = 1, \cdots, k\}.$$

It is well known that the $k-$positive cone Γ_k is an open convex cone for the admissible solutions of equation (1.4), i.e., the condition $(\omega + \sqrt{-1}\partial\bar{\partial}u) \in \Gamma_k$ is natural to guarantee equation (1.4) to be elliptic. Note that if $k = n$, Γ_n is just the *space of Hermitian metrics* $PSH(\omega, M)$.

Our main result gives the *a priori* gradient estimate for the complex Hessian equation (1.4) under a technique condition.

Theorem 1.1.2 ([87] Theorem 3)**.** *Let (M,ω) be a Hermitian manifold and $u \in C^3(M)$ be a solution of equation (1.4) with $(\omega + \sqrt{-1}\partial\bar{\partial}u) \in \Gamma_{k+1}$. Then there exist positive constant C_3 depending on $f, |u|_{C^0(M)}$ and geometric quantities of M (torsion and curvature) such that*

$$\max_{\bar{M}} |\nabla u| \leq C_3(1 + \max_{\partial M} |\nabla u|) \tag{1.5}$$

In particular, if M is compact, one can get the global gradient estimate from (1.5).

Next, we will briefly discuss the results of Chapter 3. The main objects of study are the Schauder type estimates to the complex Monge-Ampère equations on Kähler and Hermitian manifolds.

We consider the *a priori* $C^{2,\alpha}$ estimate for the complex Monge-Ampère equation

$$\det(u_{i\bar{j}}) = f \in C^{\alpha}. \tag{1.6}$$

Generally, if the right hand side data $f(z) \in C^2(M)$ (or even better), the uniform $C^{2,\alpha}$ estimate follows from the standard Evans-Krylov theory. One key point in the proof is to linearize equation (1.6) and use the Harnack inequality. However, if we only assume $f \in C^{\alpha}$, this argument does not work since one can not linearize the equation. By using a perturbation argument, we can prove

Theorem 1.1.3 ([85] Theorem 1)**.** *Let Ω be a domain in $\mathbb{C}^n$ and $u \in C^3(\Omega)$ is a pluri-subharmonic solution to the Monge-Ampère equation (1.6). Assume there exist positive constants K_0 and K_1 such that*

$$|u| + |Du| + |D^2u| \leq K_1, \quad K_0 \leq f(z) \in C^{\alpha}(\Omega),$$

for some constant $0 < \alpha < 1$. *Then, for any open domain* $\Omega' \subset\subset \Omega$, *there exists constant* C *depending only on* K_0, K_1, n, f, α *and a positive constant* C, *such that*

$$|D^2 u|_{C^\alpha(\Omega')} \leq C\Big(K_0, K_1, n, \|f\|_{C^\alpha}, \alpha, dist(\Omega', \partial\Omega)\Big) \tag{1.7}$$

In the proof we exploit a perturbation method and the crucial fact used is Bedford-Taylor interior $C^{1,1}$ estimate [6]. Another key ingredient in the proof is that we use a local version of Calabi's C^3 estimate in [64] to get the sharp $\alpha-$Hölder regularity of the second derivative.

The rest part of Chapter 3 is devoted to establish the corresponding $C^{2,\alpha}$ estimates for the complex Monge-Ampère equation on Hermitian manifolds. Let (M, ω) be a Hermitian manifold and we consider equation

$$(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = f(z)\omega^n, \tag{1.8}$$

where $0 < f(z) \in C^\infty(M)$. When the manifold (M, ω) is Kähler, that is $d\omega = 0$, by using the local potential for ω, one can deduce equation (1.8) to be (1.6) locally and the key tools also applicable. However, if ω is just a smooth positive $(1, 1)$-form (not necessarily closed), there is no local potential for ω anymore and thus Bedford-Taylor's result and the local Calabi's estimate can not be applied directly. In [83], we extend Bedford-Taylor's interior C^2 estimate to Hermitian setting by some modification of their original method.

Theorem 1.1.4 ([83] Theorem 1)**.** *Let* B *be the unit ball on* $\mathbb{C}^n$ *and* ω *be a smooth positive* $(1, 1)$*-form (not necessary closed) on* $\bar{B}$. *Let* $u \in C(\bar{B}) \cap PSH(\omega, B) \cap C^2(B)$

solve the Dirichlet problem

$$\begin{cases} (\omega + \sqrt{-1}\partial\bar{\partial}u)^n = f\omega^n & \text{in } B \\ u = \phi & \text{on } \partial B, \end{cases} \tag{1.9}$$

with $\phi \in C^{1,1}(\partial B)$ *and* $0 \le f(z) \in C^\infty(B)$. *Then, for arbitrary compact subset* $B' \subset\subset B$, *there exists a constant* C_4 *dependent only on* ω *and* $dist\{B', \partial B\}$ *such that*

$$||u||_{C^2(B')} \le C_4(||\phi||_{C^{1,1}(\partial B)} + ||f^{\frac{1}{n}}||_{C^{1,1}(B)}).$$

We also generalize the local Calabi's C^3 estimate in [64] to Hermitian manifolds.

Theorem 1.1.5 ([83] Theorem 2)**.** *Let* $u \in PSH(\omega, M) \cap C^4(M)$ *be a solution of the Monge-Ampère equation (1.8), satisfying*

$$\|\partial\bar{\partial}u\|_\omega \le K.$$

Let $\Omega' \subset\subset \Omega \subset M$. *Then the third derivatives of* $u(z)$ *of mixed type can be estimated in the form*

$$|\nabla_\omega \partial\bar{\partial}u|_\omega \le C_5 \quad \text{for } z \in \Omega',$$

where C_5 *is a constant depending on* K, $\|d\omega\|_\omega$, $\|R\|_\omega$, $\|\nabla R\|_\omega$, $\|T\|_\omega$, $\|\nabla T\|_\omega$, $dist(\Omega', \partial\Omega)$ *and* $\|\nabla^s f\|_\omega$, $s = 0, 1, 2, 3$. *Here* ∇ *is the Chern connection with respect to the Hermitian metric* ω, T *and* R *are the torsion tensor and curvature form of* ∇.

The local Calabi's C^3 estimate in the above theorem should be useful for studying the geometric problems on Hermitian manifolds, such as the Liouvelle type property. As a simple application, following the lines in [35], we prove the sharp interior $C^{2,\alpha}$ estimate for (1.8) on Hermitian manifolds.

Corollary 1.1.1 ([83] Corollary 1). *Let Ω be a domain in $\mathbb{C}^n$ and ω be a Hermitian form defined on Ω. Let $u(z) \in PSH(\omega, \Omega) \cap C^3(\Omega)$ be a solution of the Monge-Ampère equation (1.8). Suppose that $0 < f \in C^\alpha(\Omega)$ for some $0 < \alpha < 1$ and $|u| + |Du| + |D^2u| \leq L$. Then*

$$|D^2u|_{C^\alpha(\Omega')} \leq C$$

for some constant depending on $n, L, \|f\|_{C^\alpha}, \alpha, dist(\Omega', \partial\Omega)$ and the geometric quantities with respect to ω.

In the Chapter 4, we study some geometric properties of the boundary class of Kähler cones under the following *non-negative quadratic bisectional curvature* condition: for any orthogonal tangent frame $\{e_1, \cdots, e_n\}$ at any $x \in M$, and for any real numbers $a_1, \cdots, a_n$:

$$\sum_{i,j=1}^{n} R_{i\bar{i}j\bar{j}}(a_i - a_j)^2 \geq 0. \quad (*) \tag{1.10}$$

In [80], Wu-Yau-Zheng posted an interesting question to ask when the boundary class of Kähler cone can be represented by a closed, smooth (1,1) form that is everywhere nonnegative. And they also concluded that the curvature condition $(*)$ is sufficient, by proving the existence of a smooth solution u to the following homogeneous complex Monge-Ampère equation:

$$(\omega + \Phi + \sqrt{-1}\partial\bar{\partial}u)^n = 0, \tag{1.11}$$

with $(\omega+\Phi+\sqrt{-1}\partial\bar{\partial}u)\geq 0$ and the compatibility condition $\int_M(\omega+\Phi)^n=0$, where Φ is a $d-$closed $(1,1)$ form on M such that the cohomology class represented by $\omega+t\Phi$ is positive for each $0\leq t<1$.

In general, there is no smooth solutions for the degenerate complex Monge-Ampère equations like (1.11). By observing the special feature of equation (1.11) in this setting and some old geometric results related with curvature condition $(*)$, we obtain

Theorem 1.1.6 ([86] Theorem 1)**.** *Let (M^n,ω) be a compact Kähler manifold satisfying the curvature condition (1.10). Then, for any closed $(1,1)$ form Ψ on (M^n,g), we can find $\tilde{\Psi}\in[\Psi]$, such that $\tilde{\Psi}$ is parallel. In particular, for any closed $(1,1)$ form α, we have*

$$[\alpha]=[\beta+\lambda_s\omega_0]$$

where β is a nonnegative C^∞ closed $(1,1)$ form on the boundary of Kähler cone, λ_s is a constant depending on β and ω.

The main theorem of Wu-Yau-Zhang[80] can be recovered.

Corollary 1.1.2 ([86] Corollary 1)**.** *Let (M^n,ω) be a compact manifold satisfying the curvature condition (1.10). Then any boundary class of the Kähler cone of M^n can be represented by a C^∞, closed $(1,1)$ form that is parallel and everywhere nonnegative.*

If (M,ω) satisfies a $quasi-(*)$ curvature condition, namely, for any orthogonal tangent frame $e_1,\cdots,e_n$ at any $x\in M$, and for any real numbers $a_1,\cdots,a_n$, $\sum_{i,j=1}^n R_{i\bar{i}j\bar{j}}(a_i-a_j)^2\geq 0$ holds everywhere and strictly positive at least at one point unless $a_1=\cdots=a_n$, we get

Theorem 1.1.7 ([86] Theorem 2). *Let (M_n, ω) be a compact Kähler manifold satisfying the quasi $-$ $(*)$ curvature condition. Then,* $\dim h^{1,1}(M, \mathbb{R}) = 1$.

This theorem weakly generalizes a result of Bishop and Goldberg [10] that any compact Kähler manifold M^n with positive bisectional curvature must have its second Betti number equal to 1.

Finally, in Chapter 5, we consider the existence of generalized Kähler-Einstein metrics and properness of energy functionals. This is an analog of Tian's result [71] on the Kähler-Einstein metrics which asserts that existence of Kähler-Einstein metrics is equivalent to the properness of corresponding energy functional.

Let (M, J) be a $2n$-dimensional complex manifold, $[\omega_0] \in H^{1,1}(M, \mathbb{C}) \cap H^2(M, \mathbb{R})$ be a Kähler class on (M, J) and $[\alpha] = 2\pi c_1(M) - k[\omega_o]$ for some constant k. Fixing a closed $(1,1)$-form $\theta \in [\alpha]$, we consider the following generalized Kähler-Einstein equation

$$\rho(\omega) - \theta = k\omega, \tag{1.12}$$

where $\rho(\omega)$ is the Ricci form of the Kähler metric $\omega \in [\omega_0]$. If $\theta \equiv 0$, equation (1.12) is just the Kähler-Einstein equation. A Kähler metrics ω satisfying (1.12) will be called by a generalized Kähler-Einstein metric. Denote $\mathcal{K}_{\omega_0}$ to be the set of all Kähler forms on M cohomologous to ω_0.

It is easy to see that solving the generalized Kähler-Einstein equation (1.12) is equivalent to solving the following complex Monge-Ampère equation,

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^m = \exp(h_{\omega_0} - k\varphi)\omega_0^m, \tag{1.13}$$

where $\varphi \in PSH(\omega_0, M)$ and $h_{\omega_0} \in C^\infty(M)$ satisfying

$$\rho(\omega_0) - \theta = k\omega_0 + \sqrt{-1}\partial\bar{\partial}h_{\omega_0} \text{ and } \int_M \exp(h_{\omega_0})(\omega_0)^m = \int_M (\omega_0)^m = V.$$

If $k \leq 0$, the complex Monge-Ampère equation (1.13) can be solved by the work of Aubin [2] and Yau [81]. In [84], we consider the case $k > 0$, there should be obstructions to admit generalized Kähler-Einstein metrics. In fact, we show that the existence of generalized Kähler-Einstein metric with semi-positive twisting $(1,1)$-form θ is closely related to the properness of the twisted $\mathcal{K}$-energy functional $\mathcal{V}_{\theta,\omega_0}$ defined by Song-Tian [66].

Theorem 1.1.8 ([84] Theorem 2)**.** *Let (M, ω_0) be a Kähler manifold, and $\theta \in [\alpha] = 2\pi c_1(M) - k[\omega_0]$ is a real closed semipositive $(1,1)$-form for $k > 0$. If $\mathcal{V}_{\theta,\omega_0}$ is proper then there must exists a generalized Kähler-Einstein metric $\omega_{GKE} \in \mathcal{K}_{\omega_0}$. Assuming that the twisting form θ is strictly positive at a point or M admits no nontrivial Hamiltonian holomorphic vector field, if there exists a generalized Kähler-Einstein metric in $\omega_{GKE} \in \mathcal{K}_{\omega_0}$, then $\mathcal{V}_{\theta,\omega_0}$ must be proper. In fact, there exist uniform positive constants C_2, C_3 depending only on k and the geometry of (M, ω_0), such that*

$$\mathcal{V}_{\theta,\omega_0}(\varphi) \geq C_2 J_{\omega_0}(\varphi) - C_3, \tag{1.14}$$

for all $\varphi \in PSH(\omega_0, M)$.

In a special case, if $[\alpha] = (1-k)[\omega_0]$, where $0 < k < 1$, we set $\theta = (1-k)\omega_0$. Then the generalized Kähler-Einstein equation (1.12) is just the Aubin's equation

$$\rho(\omega) = (1-k)\omega_0 + k\omega. \tag{1.15}$$

As a corollary of previous theorem, we have

Corollary 1.1.3 ([83] Corollary 1.4)**.** *Let (M, ω_0) be a Kähler manifold with $2\pi c_1(M) = [\omega_0]$, and $0 < k < 1$. The following are equivalent:*

- *We can uniquely solve equation (1.15).*
- *There exists a Kähler metric $\omega \in [\omega_0]$ such that $\rho(\omega) > k\omega$.*
- *For any Kähler metric $\omega \in [\omega_0]$, $\mathcal{V}_\omega(\varphi) + (1-k)(I_\omega - J_\omega)(\varphi)$ is proper.*
- *For any Kähler metric $\omega \in [\omega_0]$, there exist uniform positive constants C_5 and C_6 such that $\mathcal{V}_\omega(\varphi) + (1-k)(I_\omega - J_\omega)(\varphi) \geq C_5 J_\omega(\varphi) - C_6$ for all $\varphi \in \mathcal{H}_\omega$.*

where I_ω and J_ω are the Aubin's energy functionals.

1.2 Basic Hermitian geometry

In this section, some basic definitions and facts about Hermitian geometry and Kähler geometry are stated. More information regarding this can be found in [43], [72].

• General notions of Hermitian geometry

Let (M, J) be a compact complex manifold of complex dimension n. A Riemannian metric g is called *Hermitian* if it satisfies

$$g(JX, JY) = g(X, Y), \qquad \text{for all } X, Y \in TM.$$

In this case, we then define a real $2-$ form ω by the formula

$$\omega(X, Y) = g(JX, Y).$$

If ω is closed, that is $d\omega = 0$, we call g a *Kähler* metric. Since ω and g are equivalent data, we will often refer to ω as the Kähler metric, or *Kähler form*. It is of type $(1,1)$, and if locally we write

$$g = g_{i\bar{j}} dz^i \otimes d\bar{z}^j,$$

then

$$\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

where here and henceforth we are using the Einstein summation convention. The Riemannian volume form of g is equal to $\frac{\omega^n}{n!}$, and we will denote by V the volume of M

$$V = \int_M \frac{\omega^n}{n!}.$$

Let ∇ be the *Chern connection* of g. It satisfies

$$\nabla_Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y), \quad \text{for } \forall X, Y, Z \in TM. \tag{1.16}$$

The *torsion tensor* and *curvature tensor* of ∇ are defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]; \tag{1.17}$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{1.18}$$

respectively. Since $\nabla J = 0$ we have

$$g(R(X,Y)JZ, JW) = g(R(X,Y)Z, W) \equiv R(X,Y,Z,W). \tag{1.19}$$

Therefore $R(X,Y,Z,W) = 0$ unless Z, W are of different type.

In local coordinates, define the Christoffel symbols Γ^l_{jk} by

$$\nabla_{\frac{\partial}{\partial z^j}} \frac{\partial}{\partial z^k} = \Gamma^l_{jk} \frac{\partial}{\partial z^l}.$$

Then,

$$\Gamma^l_{jk} = g^{l\bar{m}} \frac{\partial g_{k\bar{m}}}{\partial z^j} \tag{1.20}$$

and the torsion (1.17) is given by

$$T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji} = g^{k\bar{l}} \Big(\frac{\partial g_{j\bar{l}}}{\partial z^i} - \frac{\partial g_{i\bar{l}}}{\partial z^j} \Big), \tag{1.21}$$

while the curvature (1.18) is

$$\begin{aligned} R_{i\bar{j}k\bar{l}} &\equiv R\Big(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{z}^l} \Big) = -g_{m\bar{l}} \frac{\partial \Gamma^m_{ik}}{\partial \bar{z}^j} \\ &= -\frac{\partial^2 g_{k\bar{l}}}{\partial z^i \partial \bar{z}^j} + g^{p\bar{q}} \frac{\partial g_{k\bar{q}}}{\partial z^i} \frac{\partial g_{p\bar{l}}}{\partial \bar{z}^j}. \end{aligned} \tag{1.22}$$

Note that from (1.18), (1.19) and (1.20) that

$$R_{i\bar{j}kl} = R_{ijkl} = R_{ijk\bar{l}} = 0.$$

By (1.21) and (1.22) we have

$$R_{i\bar{j}k\bar{l}} - R_{k\bar{j}i\bar{l}} = g_{m\bar{l}} \frac{\partial T^m_{ki}}{\partial \bar{z}^j} = g_{m\bar{l}} \nabla_{\bar{j}} T^m_{ki}, \tag{1.23}$$

which also follows from the general Bianchi identity.

The trace of the curvature tensor

$$R_{i\bar{j}} = g^{k\bar{l}} R_{i\bar{j}k\bar{l}} = -\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log\det g_{k\bar{l}}. \tag{1.24}$$

is called the *Ricci curvature* of ω. And we associate to it the *Ricci form*

$$Ric(\omega) = \sqrt{-1} R_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

It is a closed real $(1,1)-$form that represents the cohomology class $c_1(M) \in H^2(M, 2\pi\mathbb{Z})$. The *scalar curvature* of ω is denote by

$$R = g^{i\bar{j}} R_{i\bar{j}}.$$

We use $\nabla^2 u$ to denote the *Hessian* of a function $u \in C^2(M)$:

$$\nabla^2 u(X,Y) \equiv \nabla_Y \nabla_X u = Y(Xu) - (\nabla_Y X)u, \quad X, Y \in TM. \tag{1.25}$$

In local coordinate, we see that

$$\nabla_{\frac{\partial}{\partial z^{j}}} \nabla_{\frac{\partial}{\partial \bar{z}^{j}}} u = \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}.$$

Consequently, the Laplacian of $u \in C^2(M)$ with respect to the Chern connection ∇ is

$$\Delta u = g^{i\bar{j}} \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j},$$

or equivalently,

$$\Delta u \frac{\omega^n}{n} = \sqrt{-1} \partial\bar{\partial} u \wedge \omega^{n-1}.$$

• **Kähler Geometry**

As defined above, a Hermitian manifold (M, ω) is Kähler if ω is closed, i.e., $d\omega = 0$. Below we give a basic example of Kähler manifolds.

Example 1.2.1 (The projective space $\mathbb{P}^n$)**.** *Consider the set $\mathbb{P}^n$ of all complex lines passing through 0 in $\mathbb{C}^{n+1}$. This set can be endowed with the (complex) manifold structure by using the natural projection from $\mathbb{C}^{n+1} \setminus \{0\}$ onto it. In order to define a Kähler form on $\mathbb{P}^n$ we consider the form*

$$dd^c \log\Big(|Z_0|^2 + \cdots + |Z_n|^2\Big),$$

where Z_i are the coordinates in $\mathbb{C}^{n+1} \setminus \{0\}$. Note that, when restricting to a complex line through 0, the form is invariant (because the function $\lambda \to \log|\lambda|$ is harmonic on $\mathbb{C} \setminus \{0\}$). Thus it descends onto a closed positive $(1,1)-$form on $\mathbb{P}^n$. The constructed form is called the Fubini-Study (Kähler) form and is often denoted by ω_{FS}.

From the definition of Kähler manifold, one can see Kähler geometry is a class of Hermitian geometry with one extra condition $d\omega = 0$. In order to emphasize this, the following is a well known example of non-Kähler Hermitian manifold.

Example 1.2.2 (Hopf Surface)**.** *Let $\phi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}^2 \setminus \{0\}$ defined by $\phi(z) = 2z$. Denote $< \phi >$ to be the group generated by the automorphism ϕ of $\mathbb{C}^2 \setminus \{0\}$. One can verify that the quotient $\mathbb{C}^2 \setminus \{0\}/ < \phi >$ has the complex manifold structure. This manifold is called Hopf surface. It can be proved that Hopf surface does not admit any Kähler structure.*

We continue to list some important notions on compact Kähler manifolds.

Lemma 1.2.1 ($\partial\bar{\partial}-$Lemma)**.** *Let (M, ω) be a Kähler manifold and let $\omega_1, \omega_2 \in H^{1,1}(M, \mathbb{R})$ and suppose that ω_1 is cohomology to ω_2. Then there exists a function $f \in C^\infty(M, \mathbb{R})$ such that $\omega_1 - \omega_2 = \sqrt{-1}\partial\bar{\partial}f$.*

As a direct corollary of the $\partial\bar{\partial}-$Lemma, we have

Corollary 1.2.1. *Given $\Omega \in H^{1,1}(M,\mathbb{C}) \cap H^2(M,\mathbb{R})$, define*

$$\mathcal{K}_\Omega = \{ \text{ all Kähler metrics } \omega \text{ with } [\omega] = \Omega\},$$

then,

$$\begin{aligned}\mathcal{K}_\Omega &= \{\omega + \sqrt{-1}\partial\bar{\partial}\phi : \phi \in C^\infty(M,\mathbb{R}), \int_M \phi\omega^n = 0\} \\ &\simeq \{\phi \in C^\infty(M,\mathbb{R}) : \int_M \phi\omega^n = 0, \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0\}.\end{aligned}$$

The set

$$\tilde{\mathcal{K}}_\Omega = \{\phi \in C^\infty(M,\mathbb{R}) : \int_M \phi\omega^n = 0, \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0\}$$

is called *space of Kähler potentials*. Then we can define a real-valued functional F^0_ω on the space of Kähler potentials by the formula

$$F^0_\omega(\phi) = -\frac{1}{V}\int_0^1\int_M \frac{\partial\phi_t}{\partial t}\frac{\omega^n_{\phi_t}}{n!},$$

where ϕ_t is any smooth path of Kähler potentials with $\phi_0 = 0$ and $\phi_1 = \phi$ (for example one can take $\phi_t = t\phi$). It can be written also as

$$F^0_\omega(\phi) = J_\omega(\phi) - \frac{1}{V}\int_M \phi\frac{\omega^n}{n!},$$

where the functional J_ω is defined by

$$J_\omega(\phi) = \frac{1}{V}\int_0^1\int_M \frac{\partial\phi_t}{\partial t}\Big(\frac{\omega^n}{n!} - \frac{\omega^n_{\phi_t}}{n!}\Big), \tag{1.26}$$

and the integration by parts shows that $J_\omega(\phi) \geq 0$. Moreover F^0_ω satisfies the following cocycle condition

$$F^0_\omega(\phi) = F^0_\omega(\psi) + F^0_{\omega_\psi}(\phi - \psi), \tag{1.27}$$

for all Kähler potentials ϕ, ψ. Another useful functional is the Mabuchi energy functional $\mathcal{M}_\omega(\phi)$, which is defined by

$$\mathcal{M}_\omega(\phi) = -\frac{1}{V}\int_0^1 \int_M \frac{\partial \phi_t}{\partial t}(R(\omega_{\phi_t}) - \underline{R})\frac{\omega^n_{\phi_t}}{n!}, \tag{1.28}$$

where ϕ_t is any smooth path of Kähler potentials with $\phi_0 = 0, \phi_1 = \phi$ and $\underline{R}$ denotes the average of scalar curvature R. It satisfies the same cocycle condition as F^0_ω, namely

$$\mathcal{M}_\omega(\phi) = \mathcal{M}_\omega(\psi) + \mathcal{M}_{\omega_\psi}(\phi - \psi). \tag{1.29}$$

1.3 Complex Monge-Ampère equation

In this section, we will recall the work by Yau [81] on Calabi's conjecture and briefly discuss his *a priori* estimates for the complex Monge-Ampère equation on Kähler manifold (M, ω).

Let (M, ω) be a compact Kähler manifold with complex dimension n. The Calabi's conjecture states that there is a unique Kähler metric in the same class whose Ricci form is any given $2-$form Ω representing the first Chern class. Indeed, this geometric problem can be translated to a Monge-Ampère equation. First, notice that both the Ricci curvature $Ric(\omega)$ and Ω represent the first Chern class and therefore the $\partial\bar{\partial}-$lemma 1.2.1 tells us that we can find F, only depending on ω and

Ω, such that

$$\Omega - Ric(\omega) = \sqrt{-1}\partial\bar{\partial}F,$$

where F is unique after normalizing to

$$\int_M (e^F - 1)\omega^n = 0.$$

It follows from the $\partial\bar{\partial}-$lemma, any Kähler metric $\tilde{\omega}$ cohomologous to ω has the form $\omega + \sqrt{-1}\partial\bar{\partial}\phi$. Suppose function ϕ satisfies

$$Ric(\omega + \sqrt{-1}\partial\bar{\partial}\phi) = \Omega = Ric(\omega) - \sqrt{-1}\partial\bar{\partial}F.$$

Now, by making use the local expression of Ricci curvature in local coordinate (1.24), this reads

$$-\sqrt{-1}\partial\bar{\partial}\log\det\left(g_{i\bar{j}} + \frac{\partial^2\phi}{\partial z_i\bar{\partial}z_j}\right) = -\sqrt{-1}\partial\bar{\partial}\log\det(g_{i\bar{j}}) - \sqrt{-1}\partial\bar{\partial}F.$$

Although this is only locally defined, the following is globally defined

$$\sqrt{-1}\partial\bar{\partial}\log\left(\frac{\det(g_{i\bar{j}} + \phi_{i\bar{j}})}{\det(g_{i\bar{j}})}\right) = \sqrt{-1}\partial\bar{\partial}F.$$

Therefore,

$$\frac{\det(g_{i\bar{j}} + \phi_{i\bar{j}})}{\det(g_{i\bar{j}})} = e^F, \tag{1.30}$$

where F is a smooth function on M with $\int_M (e^F - 1)\omega^n = 0$. Equation (1.30) is just a *complex Monge-Ampère equation* which is also equivalent to

$$(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^F\omega^n. \tag{1.31}$$

The Calabi's conjecture is equivalent to show equation (1.31) has a unique solution. The uniqueness part was proved in the 50's by Calabi himself [25] using maximum principle. To prove the existence of a solution, Yau derived *a priori* C^k estimates for ϕ and then applied the continuity method.

Define $F_t = tF + C_t$ for C_t constants and $0 \leq t \leq 1$. Requiring that $\int_M (e^{F_t} - 1)\omega^n = 0$ determines the constants uniquely. Observe that $F_0 = 0$ and $F_1 = F$. Consider the following family of equations

$$(\omega + \sqrt{-1}\partial\bar{\partial}\phi_t)^n = e^{F_t}\omega^n. \tag{1.32}$$

The solution of (1.32) is unique up to constants by Calabi's proof. Define

$$T = \{t \in [0,1] \mid (1.32) \text{ is solvable for } s \leq t\}.$$

To prove equation (1.31) is solvable, it suffices to prove that T is non-empty, open and closed. Clearly, $0 \in T$ (set ϕ=constant).

The openness follows from the invertibility of the linearized equation of (1.30) and the Implicit Function Theorem. The hard part for the solvability is the closeness. To prove this, Yau established the *a priori* estimates. The precise statement is:

Theorem 1.3.1 (Yau[81])**.** *Let (M, ω) by a closed $n-$dimensional Kähler manifold, and let F be a smooth real function on M that satisfies (1.30). Then there are constants $A_k, k = 0, 1, \cdots,$ that depend only on $k, F,$ and ω such that*

$$||\phi||_{C^k(\omega)} \leq A_k. \tag{1.33}$$

The desired estimates (1.33) were proved in four steps. In what follows we will always use C to denote a uniform constant, but this capital may mean many different constants.

Step 1. To prove an upper bound for $\Delta\phi$ which depends on the C_0-norm of ϕ, i.e., the inequality

$$tr_g\tilde{g} \le Ce^{A(\phi-\inf_M \phi)}, \tag{1.34}$$

holds for some uniform constants A, C. This inequality follows by applying maximum principle to the crucial estimate

$$\tilde{\Delta}(\log tr_g\tilde{g} - A\phi) \ge tr_{\tilde{g}}g - C$$

which is obtained by delicate computations.

Step 2. The second step is to show the C_0 estimate

$$\sup_M |\phi| \le C.$$

A Moser iteration argument is used to prove this estimate.

Step 3. From first two steps, one has $||\sqrt{-1}\partial\bar{\partial}\phi||_{C^0(\omega)} \le C$. The next is to deduce that

$$||\nabla_\omega\partial\bar{\partial}\phi||_{C^0(\omega)} \le C.$$

To establish step 3, one first considers the quantity $S = |\nabla\tilde{g}|^2_{\tilde{g}}$, where ∇ is the covariant derivative with respect to g. In terms of ϕ, it can be written as

$$S = \tilde{g}^{i\bar{p}}\tilde{g}^{q\bar{j}}\tilde{g}^{k\bar{r}}\phi_{i\bar{j}k}\phi_{\bar{p}q\bar{r}}, \tag{1.35}$$

where the lower indices are covariant derivatives with respect to g. After complicated computation, one gets

$$\tilde{\Delta} S \geq -CS - C. \tag{1.36}$$

On the other hand, as g and $\tilde{g}$ are equivalent which follows from the the C^2 estimate of ϕ,

$$\tilde{\Delta} tr_g \tilde{g} \geq \frac{1}{C} S - C.$$

One can then apply the maximum principle to $S + A tr_g \tilde{g}$, where A is large, to get the desired estimate $S \leq C$.

Step 4. Finally, one can get the higher order estimates following the standard elliptic PDE theory: $\forall\ k = 2, 3, \cdots$,

$$||\phi||_{C^k(\omega)} \leq A_k, \text{ for uniform } A_k.$$

CHAPTER 2
A priori estimates of Monge-Ampère equation on Hermitian manifolds

In this chapter, we study the *a priori* estimate for Monge-Ampère type equations on Hermitian manifolds. In section 2.1, we review some background of this problems and the notions related with complex Hessian equations. In section 2.2, we give the gradient estimate for the solution of Hessian type equation. As a corollary, C^1 bound for the complex Monge-Ampère equation in Hermitian setting follows. Finally, in section 2.3, we prove the C^2 *a priori* estimate by using the Pogorelov technique.

The results in this section can be found in [87].

2.1 Introduction

As discussed in Section 1.3, Yau proved the fundamental existence theorems of classical solutions of the complex Monge-Ampére equations on compact Kähler manifolds.

$$\det(g_{i\bar{j}} + \phi_{i\bar{j}}) = \det(g_{i\bar{j}})f(z) \tag{2.1}$$

where f is a smooth positive function on (M, g). Moreover, he also studied the generalized form of this equation when the right hand side function $f(z)$ may degenerate or have poles [81]. Later after that, Cheng and Yau [32], Tian and Yau [73, 74] solved equation (2.1) on complete non-compact Kähler manifolds which have natural applications to algebraic geometry. All these works essentially depend on deriving the *a priori* estimates up to C^2 by virtue of the continuity method.

From the steps recalled in Section 1.3 for Yau's *a priori* estimates (Theorem 1.3.1), we see that the crucial step is to derive the C^2 estimate which depends only on the C^0 norm. Then, the standard interpolation yields the gradient estimates from C^2 and C^0. However, it is of interest to have a C^1 bound directly from the C^0 estimate, without using a C^2 estimate. Such direct gradient estimate was obtained by Blocki [14] and Guan [46, 47] when the background manifold is compact Kähler. Using the same technique as [46], we give the direct C^1 estimate for complex Monge-Ampère equation in the Hermitian case. we consider the Hermitian manifold (M, ω) of dimension $n \geq 2$ with smooth boundary ∂M and seek solutions to (2.1) in the *space of Hermitian metrics* defined as

$$PSH(\omega, M) = \{u \in C^2(M) | \quad \omega_u = \omega + \sqrt{-1}\partial\bar{\partial}u > 0\}$$

Theorem 2.1.1 ([87] Theorem 1)**.** *Let $u \in PSH(\omega, M) \bigcap C^4(M)$ be a solution of equation (2.1). Then there exist positive constants C_1 depending on $f, |u|_{C^0(M)}$ and geometric quantities of M (torsion and curvature) such that*

$$\max_{\bar{M}} |\nabla u| \quad \leq \quad C_1(1 + \max_{\partial M} |\nabla u|) \tag{2.2}$$

In particular, if the Hermitian manifold M is compact, i.e. $\partial M = \emptyset$, then one can get the estimates for gradient from (2.2).

The complex Monge-Ampère equation on Hermitian manifolds has been studied extensively. In the eighties and nineties some results regarding equation (2.1) in the

Hermitian setting were obtained by Cherrier [28, 29] and Hanani [50]. For next few years there seems to be no activity on the subject until very recently, when the results were rediscovered and generalized by Guan-Li [44] and Zhang [87] independently. Later, Tosatti and Weinkove [76, 77] gave a more delicate *a priori* C^2 estimate and remove the conditions for the C^0 estimate in [44]. Moreover, Dinew-Kolodziej [33] also studied the equation in the weak sense and obtained the L^∞ estimate via suitably constructed pluripotential theory.

We also study another important type of fully nonlinear geometric equations, complex Hessian equation, which includes (2.1) as a special case. We consider

$$\omega_u^k \wedge \omega^{n-k} = (\omega + \sqrt{-1}\partial\bar{\partial}u)^k \wedge \omega^{n-k} = f\omega^n \tag{2.3}$$

where $k = 1, 2, \cdots, n$, and f is a positive function on a Hermitian manifold (M, ω). Notice that if $k = n$, (1.4) is just the complex Monge-Ampère equation (2.1), while if $k = 1$, equation (2.3) becomes the Laplacian equation. So the complex Hessian type equation is a generalization of both complex Monge-Ampère equation and Laplacian equation over a compact Hermitian manifold. Similar nonlinear equations have been studied extensively by many authors [13, 21, 23, 58, 52, 53, 46] and the references therein. Let us mention that the complex Hessian equation (2.3) is also closely related to the quaternionic version of the Calabi problem on a compact hypercomplex manifold with an HKT-metric proposed by Alesker [1]. To answer this analogous of Calabi problem, it is crucial to establish the estimates for the quaternionic Monge-Ampère type equation which can be reformulated as a special case of complex Hessian equation.

Let $H(n)$ be the set of $n \times n$ Hermitian matrices and $\lambda(A)$ be the eigenvalues of A. For $k = 1, 2, \cdots, n$, we define

$$\sigma_k(A) = \sigma_k(\lambda(A)) \quad \text{for } A \in H(n),$$

where $\sigma_k(\lambda)$ is the $k-th$ elementary symmetric function, that is, for $1 \le k \le n$ and $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$,

$$\sigma_k(\lambda) = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}.$$

We also define

$$\Gamma_k = \{A \in H(n) \mid \sigma_j(A) > 0, j = 1, \cdots, k\}.$$

It is well known that the $k-$positive cone Γ_k is open convex cone for the admissible solutions of equation (2.3), i.e., the condition $(\omega + \sqrt{-1}\partial\bar{\partial}u) \in \Gamma_k$ is natural to guarantee equation (2.3) to be elliptic by [23]. Note that if $k = n$, Γ_n is just the *space of Hermitian metrics* $PSH(\omega, M)$.

Our results in [87] give the *a priori* gradient estimate for the complex Hessian equation (2.3) under a technique condition.

Theorem 2.1.2 ([87] Theorem 3)**.** *Let (M, ω) be a Hermitian manifold and $u \in C^3(M)$ be a solution of equation (2.3) with $(\omega + \sqrt{-1}\partial\bar{\partial}u) \in \Gamma_{k+1}$. Then there exist positive constant C_3 depending on $f, |u|_{C^0(M)}$ and geometric quantities of M (torsion and curvature) such that*

$$\max_{\bar{M}} |\nabla u| \le C_3(1 + \max_{\partial M} |\nabla u|) \tag{2.4}$$

In particular, if M is compact, one can get the global gradient estimate from (2.4).

It is worthwhile to mention that the estimate (2.4) does not depend on the lower bound of f, we may use it to deal with degenerate case.

Remark 2.1.1. *In the above theorem, if $k = n$, one can see that equation (2.3) is just (2.1) and the condition $(\omega + \sqrt{-1}\partial\bar{\partial}u) \in \Gamma_{k+1}$ is the same as $u \in PSH(\omega)$. Thus, Theorem 2.1.1 is just a corollary of Theorem 2.1.2.*

Remark 2.1.2. *One would like to know whether the condition $(\omega+\sqrt{-1}\partial\bar{\partial}u) \in \Gamma_{k+1}$ can be weaken to $(\omega + \sqrt{-1}\partial\bar{\partial}u) \in \Gamma_k$. Indeed, this is the crucial part left for the solvability of complex Hessian equation (2.3).*

Our method of proving the gradient estimate is applicable for more general complex Hessian type equations. We consider the following complex Hessian equation with gradient term on the Hermitian manifolds.

$$\sigma_k(g_{i\bar{j}} + \phi_{i\bar{j}} + \mu(z)\phi_i\phi_{\bar{j}}) = f(z), \qquad z \in M \tag{2.5}$$

where$\mu(z)$ and $f(z)$ are smooth functions on (M, g) and f is positive.

Theorem 2.1.3 ([87] Theorem 4)**.** *Suppose $\phi \in C^3$ is a solution of equation (2.5) with $(g_{i\bar{j}} + \phi_{i\bar{j}} + \mu(z)\phi_i\phi_{\bar{j}}) \geq 0$ for some positive function f. Then there exist positive constant C_4 depending on $f, |u|_{C^0(M)}$ and geometric quantities of M (torsion and curvature) such that*

$$\max_{\bar{M}} |\nabla u| \leq C_4(1 + \max_{\partial M} |\nabla u|) \tag{2.6}$$

In particular, if M is compact, one can get the global gradient estimate from (2.6).

Remark 2.1.3. *Using the same method and test functions in the proof of Theorem 4, one can also consider the gradient estimate for the complex Hessian equation of the following general form:*

$$\sigma_k(g_{i\bar{j}} + \phi_{i\bar{j}} + a_i\phi_{\bar{j}} + b_{\bar{j}}\phi_i) = f$$

where a and b are some smooth functions on the Hermitian manifolds (M, g).

Using the gradient estimate established in Theorem 2.1.1, we have the following C^2-estimate for the complex Monge-Ampère equation (2.1) on Hermitian manifolds.

Theorem 2.1.4 ([87] Theorem 2)**.** *Let $u \in PSH(\omega, M) \bigcap C^4(M)$ be a solution of equation (2.1). Then there exist positive constants C_2 depending on $f, |u|_{C^0(M)}$ and geometric quantities of M (torsion and curvature) such that*

$$\max_{\bar{M}} |\Delta u| \quad \leq \quad C_2(1 + \max_{\partial M} |\Delta u|) \tag{2.7}$$

In particular, if the Hermitian manifold M is compact, i.e. $\partial M = \emptyset$, then one can get the estimates for Δu from (2.7).

Note that, in the *Step 1* (section 1.3) of Yau's *a priori* estimates, the C^2 estimate only depends on the C^0 norm. In Hermitian case, this type C^2 estimate can also be obtained, see [44, 76] which depends on a careful control of the third order terms. However, since we already established the C^1 estimate in Theorem 2.1.1, we just use the standard Pogorelov type test function $G = \log(m + \Delta\phi) + B|\nabla\phi|^2 - A\phi$ here.

2.2 Gradient estimate for complex Hessian type equation

In this section, we give the proof for Theorem 2.1.2 and Theorem 2.1.3. We write

$$F(\lambda) = \sigma_k(\lambda)$$

where $\lambda = (\lambda_1, \cdots, \lambda_n) \in \Gamma_k$, $F^{i\bar{i}} = \frac{\partial F}{\partial \lambda_i}$.

We recall a lemma from [46].

Lemma 2.2.1. *For each integer $k \geq 1$, there is constant $C_{n,k} > 0$ depending only on k,n such that for any $B \geq 0, \lambda \in \Gamma_k, 0 \leq s_i \in \mathbb{R}$ with $\sum_{i=1}^n s_i = 1$, we have*

$$\sum_{i=1}^{n} F^{i\bar{i}}(1 + Bs_i) \geq C_{n,k}\sigma_k^{\frac{k-1}{k}}(\lambda)(1+B)^{\frac{1}{k}}. \tag{2.8}$$

Proof. Notice that the lemma is trivial for the case $k = 1$. We will only consider $k \geq 2$. We may arrange $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. This yields

$$F^{1\bar{1}} \geq F^{2\bar{2}} \geq \cdots \geq F^{n\bar{n}}.$$

In turn,

$$\begin{aligned} \sum_i F^{i\bar{i}}(1 + Bs_i) &\geq \sum_i F^{i\bar{i}} + \sum_i F^{n\bar{n}} Bs_i = \sum_i F^{i\bar{i}} + F^{n\bar{n}} B \\ &= \sum_i \frac{\partial \sigma_k}{\partial \lambda_i}(\lambda)\tilde{\lambda}_i, \end{aligned}$$

where $\tilde{\lambda} = (1, \cdots, 1, 1 + B) \in \mathbb{R}^n$.

Now, we apply the Garding's inequality for polarized σ_k (see Appendix of [51]),

$$\begin{aligned}\sum_i F^{i\bar{i}}(1+Bs_i) &\geq \sum_i \frac{\partial \sigma_k}{\partial \lambda_i}(\lambda)\tilde{\lambda}_i \geq C_{n,k}\sigma_k^{\frac{k-1}{k}}(\lambda)\sigma_k^{\frac{1}{k}}(\tilde{\lambda}) \\ &\geq C_{n,k}\sigma_k^{\frac{k-1}{k}}(\lambda)(1+B)^{\frac{1}{k}}.\end{aligned}$$

□

Proof of Theorem 2.1.2:

Let's denote $\tilde{g}_{i\bar{j}} = g_{i\bar{j}} + \phi_{i\bar{j}}, W = |\nabla\phi|^2$ and $L = \sup_M |\phi|$. Suppose the maximum of

$$H = \log W + Ae^{L-\phi}$$

is attained at some interior point p. We pick a holomorphic orthonormal coordinate system at that point such that $(\tilde{g}_{i\bar{j}}) = (g_{i\bar{j}} + \phi_{i\bar{j}})$, is diagonal at that point. We may assume that $W(p) \geq 1$.

As $(\phi_{i\bar{j}})$ is diagonal at the point p, we differentiate H,

$$\frac{W_i}{W} - Ae^{L-\phi}\phi_i = 0, \qquad \frac{W_{\bar{i}}}{W} - Ae^{L-\phi}\phi_{\bar{i}} = 0 \tag{2.9}$$

Also, differentiating $W = |\nabla\phi|^2$, we have

$$W_i = \sum g^{\alpha\bar{\beta}}_{,i}\phi_\alpha\phi_{\bar{\beta}} + g^{\alpha\bar{\beta}}\phi_{\alpha i}\phi_{\bar{\beta}} + g^{\alpha\bar{\beta}}\phi_\alpha\phi_{\bar{\beta}i} = g^{\alpha\bar{\beta}}_{,i}\phi_\alpha\phi_{\bar{\beta}} + \phi_{\alpha i}\phi_{\bar{\alpha}} + \phi_i\phi_{i\bar{i}}, \tag{2.10}$$

$$W_{\bar{j}} = \sum g^{\alpha\bar{\beta}}_{,\bar{j}}\phi_\alpha\phi_{\bar{\beta}} + g^{\alpha\bar{\beta}}\phi_{\alpha\bar{j}}\phi_{\bar{\beta}} + g^{\alpha\bar{\beta}}\phi_\alpha\phi_{\bar{j}\bar{\beta}} = g^{\alpha\bar{\beta}}_{,\bar{j}}\phi_\alpha\phi_{\bar{\beta}} + \phi_\alpha\phi_{\bar{j}\bar{\alpha}} + \phi_{\bar{j}}\phi_{j\bar{j}} \tag{2.11}$$

and

$$
\begin{aligned}
W_{i\bar{j}} &= g^{\alpha\bar{\beta}}_{,i\bar{j}}\phi_\alpha\phi_{\bar{\beta}} + g^{\alpha\bar{\beta}}_{,i}\phi_{\alpha\bar{j}}\phi_{\bar{\beta}} + g^{\alpha\bar{\beta}}_{,i}\phi_\alpha\phi_{\bar{j}\bar{\beta}} + g^{\alpha\bar{\beta}}_{,\bar{j}}\phi_{\alpha i}\phi_{\bar{\beta}} + g^{\alpha\bar{\beta}}\phi_{\alpha i\bar{j}}\phi_{\bar{\beta}} \qquad (2.12)\\
&\quad + g^{\alpha\bar{\beta}}\phi_{\alpha i}\phi_{\bar{\beta}\bar{j}} + g^{\alpha\bar{\beta}}_{,\bar{j}}\phi_\alpha\phi_{\bar{\beta}i} + g^{\alpha\bar{\beta}}\phi_{\alpha\bar{j}}\phi_{\bar{\beta}i} + g^{\alpha\bar{\beta}}\phi_\alpha\phi_{\bar{\beta}i\bar{j}}\\
&= g^{\alpha\bar{\beta}}_{,i\bar{j}}\phi_\alpha\phi_{\bar{\beta}} + g^{j\bar{\beta}}_{,i}\phi_{j\bar{j}}\phi_{\bar{\beta}} + g^{\alpha\bar{\beta}}_{,i}\phi_\alpha\phi_{\bar{j}\bar{\beta}} + g^{\alpha\bar{\beta}}_{,\bar{j}}\phi_{\alpha i}\phi_{\bar{\beta}} + g^{\alpha\bar{i}}_{,\bar{j}}\phi_\alpha\phi_{i\bar{i}}\\
&\quad + \phi_{\alpha i}\phi_{\bar{j}\bar{\alpha}} + \phi^2_{i\bar{j}} + \phi_\alpha\phi_{\bar{\alpha}i\bar{j}} + \phi_{\alpha i\bar{j}}\phi_{\bar{\alpha}}
\end{aligned}
$$

By (2.10) and (2.11), one get

$$
\begin{aligned}
W_iW_{\bar{j}} &= g^{\alpha\bar{\beta}}_{,i}g^{\alpha\bar{\beta}}_{,\bar{j}}\phi^2_\alpha\phi^2_{\bar{\beta}} + g^{\alpha\bar{\beta}}_{,\bar{j}}|\phi_\alpha|^2\phi_{\bar{\beta}}\phi_{i\alpha} + g^{\alpha\bar{\beta}}_{,\bar{j}}\phi_\alpha\phi_{\bar{\beta}}\phi_i\phi_{i\bar{i}} \qquad (2.13)\\
&\quad + g^{\alpha\bar{\beta}}_{,i}|\phi_\alpha|^2\phi_\beta\phi_{\bar{j}\bar{\alpha}} + g^{\alpha\bar{\beta}}_{,i}\phi_\alpha\phi_{\bar{\beta}}\phi_{\bar{j}}\phi_{j\bar{j}} + |\phi_\alpha|^2\phi_{\alpha i}\phi_{\bar{j}\bar{\alpha}}\\
&\quad + \phi_\alpha\phi_i\phi_{\bar{j}\bar{\alpha}}\phi_{i\bar{i}} + \phi_{\bar{\alpha}}\phi_{\bar{j}}\phi_{\alpha i}\phi_{j\bar{j}} + \phi_i\phi_{\bar{j}}\phi_{i\bar{i}}\phi_{j\bar{j}}
\end{aligned}
$$

Again from (2.10) and (2.11) and equations (2.9),

$$
\begin{aligned}
\phi_{\bar{\alpha}}\phi_{i\alpha} &= AWe^{L-\phi}\phi_i - \phi_i\phi_{i\bar{i}} - g^{\alpha\bar{\beta}}_{,i}\phi_\alpha\phi_{\bar{\beta}} \qquad (2.14)\\
\phi_\alpha\phi_{\bar{i}\bar{\alpha}} &= AWe^{L-\phi}\phi_{\bar{i}} - \phi_{\bar{i}}\phi_{i\bar{i}} - g^{\alpha\bar{\beta}}_{,\bar{i}}\phi_\alpha\phi_{\bar{\beta}} \qquad (2.15)
\end{aligned}
$$

Combining this with (2.12), we may write

$$
\begin{aligned}
|W_i|^2 &= g^{\alpha\bar{\beta}}_{,i}g^{\alpha\bar{\beta}}_{,\bar{i}}\phi^2_\alpha\phi^2_{\bar{\beta}} + g^{\alpha\bar{\beta}}_{,i}\phi_\beta|\phi_\alpha|^2\phi_{\bar{i}\bar{\alpha}} + g^{\alpha\bar{\beta}}_{,\bar{i}}|\phi_\alpha|^2\phi_{\bar{\beta}}\phi_{\alpha i}\\
&\quad + |\phi_{\bar{\alpha}}\phi_{i\alpha}|^2 - |\phi_i\phi_{i\bar{i}}|^2 + 2AWe^{L-\phi}|\phi_i|^2\phi_{i\bar{i}}
\end{aligned}
$$

We pick $A \geq 1$ sufficient large, such that

$$
\Big(\frac{\sum g^{\alpha\bar{\beta}}_{,i\bar{i}}\phi_\alpha\phi_{\bar{\beta}}}{W} - \frac{\sum g^{\alpha\bar{\beta}}_{,i}g^{\alpha\bar{\beta}}_{,\bar{i}}\phi^2_\alpha\phi^2_{\bar{\beta}}}{W^2} - 100|g^{\alpha\bar{\beta}}_{,k}|g_{i\bar{j}} + \frac{A}{2}g_{i\bar{j}}\Big) \geq 0.
$$

Thus, at the maximal point p,

$$
\begin{aligned}
0 \;\geq\; & \sum_i F^{i\bar{i}} H_{i\bar{i}} = \sum_i F^{i\bar{i}} \Big(\frac{W_{i\bar{i}}}{W} - \frac{|W_i|^2}{W^2} - Ae^{L-\phi}\phi_{i\bar{i}} + Ae^{L-\phi}|\phi_i|^2 \Big) \qquad (2.16) \\
= \; & \sum_i F^{i\bar{i}} \Bigg[\Big(\frac{\sum g^{\alpha\bar{\beta}}_{,i\bar{i}} \phi_\alpha \phi_{\bar{\beta}}}{W} - \frac{\sum g^{\alpha\bar{\beta}}_{,i} g^{\alpha\bar{\beta}}_{,\bar{i}} \phi^2_\alpha \phi^2_{\bar{\beta}}}{W^2} + \frac{\sum (g^{i\bar{\beta}}_{,i} \phi_{\bar{\beta}} \phi_{i\bar{i}} + g^{\alpha\bar{i}}_{,\bar{i}} \phi_\alpha \phi_{i\bar{i}})}{W} \\
& + \frac{\sum (g^{\alpha\bar{\beta}}_{,i} \phi_\alpha \phi_{\bar{i}\bar{\beta}} + g^{\alpha\bar{\beta}}_{,\bar{i}} \phi_{\bar{\beta}} \phi_{i\alpha})}{W} - \frac{\sum (g^{\alpha\bar{\beta}}_{,i} |\phi_\alpha|^2 \phi_\beta \phi_{\bar{i}\bar{\alpha}} + g^{\alpha\bar{\beta}}_{,\bar{i}} |\phi_\alpha|^2 \phi_{\bar{\beta}} \phi_{i\alpha})}{W^2} \\
& + \frac{|\phi_{i\alpha}|^2}{W} - \frac{|\phi_{\bar{\alpha}} \phi_{i\alpha}|^2}{W^2} - Ae^{L-\phi} \Big(\phi_{i\bar{i}} - |\phi_i|^2 + \phi_{i\bar{i}} \frac{2|\phi_i|^2}{W}\Big) \\
& + \frac{\phi^2_{i\bar{i}}}{W} + \frac{|\phi_i \phi_{i\bar{i}}|^2}{W^2} + \frac{\phi_{\bar{\alpha}} \phi_{\alpha i\bar{i}} + \phi_\alpha \phi_{\bar{\alpha} i\bar{i}}}{W} \Bigg]
\end{aligned}
$$

We look for some cancelations and simplify the above terms.

$$
\begin{aligned}
& \sum_i F^{i\bar{i}} \frac{\sum (g^{i\bar{\beta}}_{,i} \phi_{\bar{\beta}} \phi_{i\bar{i}} + g^{\alpha\bar{i}}_{,\bar{i}} \phi_\alpha \phi_{i\bar{i}})}{W} = \sum_i F^{i\bar{i}} (\tilde{g}_{i\bar{i}} - g_{i\bar{i}}) \frac{g^{i\bar{\beta}}_{,i} \phi_{\bar{\beta}} + g^{\alpha\bar{i}}_{,\bar{i}} \phi_\alpha}{W} \\
\geq \; & \sum_i F^{i\bar{i}} \tilde{g}_{i\bar{i}} \frac{g^{i\bar{\beta}}_{,i} \phi_{\bar{\beta}} + g^{\alpha\bar{i}}_{,\bar{i}} \phi_\alpha}{W} - \frac{C}{W^{\frac{1}{2}}} \sum_i F^{i\bar{i}} \geq -\frac{C}{W^{\frac{1}{2}}} \sum_i F^{i\bar{i}} \tilde{g}_{i\bar{i}} - \frac{C}{W^{\frac{1}{2}}} \sum_i F^{i\bar{i}} \\
= \; & -\frac{C}{W^{\frac{1}{2}}} kf - \frac{C}{W^{\frac{1}{2}}} \sum_i F^{i\bar{i}}
\end{aligned}
$$

where C is some positive constant depending on $\sup_M |g^{\alpha\bar{\beta}}_{,i}|$.

Now, we deal with the main trouble term,

$$
\begin{aligned}
&\sum_i F^{i\bar i}\Big[\frac{\sum(g^{\alpha\bar\beta}_{,i}\phi_\alpha\phi_{\bar i\bar\beta}+g^{\alpha\bar\beta}_{,\bar i}\phi_{\bar\beta}\phi_{i\alpha})}{W}-\frac{\sum(g^{\alpha\bar\beta}_{,i}|\phi_\alpha|^2\phi_\beta\phi_{\bar i\bar\alpha}+g^{\alpha\bar\beta}_{,\bar i}|\phi_\alpha|^2\phi_{\bar\beta}\phi_{i\alpha})}{W^2} \qquad (2.17)\\
&+\frac{\sum|\phi_{i\alpha}|^2}{W}-\frac{\sum|\phi_{\bar\alpha}\phi_{i\alpha}|^2}{W^2}\Big]\\
= &\sum_{i,\alpha,\beta} F^{i\bar i}\Big[\frac{g^{\alpha\bar\beta}_{,i}\phi_\alpha\phi_{\bar i\bar\beta}}{W}(1-\frac{|\phi_\beta|^2}{W})+\frac{\overline{g^{\alpha\bar\beta}_{,i}\phi_\alpha\phi_{\bar i\bar\beta}}}{W}(1-\frac{|\phi_\beta|^2}{W})+\frac{|\phi_{i\beta}|^2}{W}(1-\frac{|\phi_\beta|^2}{W})\Big]\\
= &\sum_{i,\alpha,\beta} F^{i\bar i}\Big[(\frac{\phi_{\bar i\bar\beta}}{W^{\frac12}}+g^{\alpha\bar\beta}_{,i}\frac{\phi_\alpha}{W^{\frac12}})(\overline{\frac{\phi_{\bar i\bar\beta}}{W^{\frac12}}}+\overline{g^{\alpha\bar\beta}_{,i}\frac{\phi_\alpha}{W^{\frac12}}})-\frac{|g^{\alpha\bar\beta}_{,i}\phi_\alpha|^2}{W}\Big](1-\frac{|\phi_\beta|^2}{W})\\
\geq &-\sum_{i,\alpha,\beta} F^{i\bar i}\frac{|g^{\alpha\bar\beta}_{,i}\phi_\alpha|^2}{W}(1-\frac{|\phi_\beta|^2}{W})\\
\geq &-C\sum_i F^{i\bar i}
\end{aligned}
$$

where C is also a positive constant depending on $\sup_M |g^{\alpha\bar\beta}_{,i}|$.

By equation (2.3), we have,

$$
F^{i\bar j}(g_{i\bar j\alpha}+\phi_{i\bar j\alpha})=f_\alpha, \qquad F^{i\bar j}(g_{i\bar j\bar\alpha}+\phi_{i\bar j\bar\alpha})=f_{\bar\alpha} \qquad (2.18)
$$

Thus,

$$
\sum_i F^{i\bar i}\phi_{i\bar i\alpha}=f_\alpha-\sum_i F^{i\bar i}g_{i\bar i\alpha}, \qquad \sum_i F^{i\bar i}\phi_{i\bar i\bar\alpha}=f_{\bar\alpha}-\sum_i F^{i\bar i}g_{i\bar i\bar\alpha} \qquad (2.19)
$$

So,

$$
\begin{aligned}
\sum_i F^{i\bar i}\frac{\phi_{\bar\alpha}\phi_{\alpha i\bar i}+\phi_\alpha\phi_{\bar\alpha i\bar i}}{W} &= \frac{1}{W}\sum_i(\phi_\alpha f_{\bar\alpha}+\phi_{\bar\alpha}f_\alpha)-\frac{1}{W}\sum_i F^{i\bar i}(g_{i\bar i\alpha}\phi_{\bar\alpha}+g_{i\bar i\bar\alpha}\phi_\alpha)\\
&\geq -2\frac{|\nabla f|}{W^{\frac12}}-\frac{C}{W^{\frac12}}\sum_i F^{i\bar i} \qquad (2.20)
\end{aligned}
$$

where C is a positive constant depending on $\sup_M |g_{i\bar{j}\alpha}|$.

By combining all above estimates together, we get

$$\begin{aligned} 0 \;\geq\; & \sum_i F^{i\bar{i}}\Big[\frac{\sum g^{\alpha\bar{\beta}}_{,i\bar{i}}\phi_\alpha\phi_{\bar{\beta}}}{W} - \frac{\sum g^{\alpha\bar{\beta}}_{,i} g^{\alpha\bar{\beta}}_{,\bar{i}}\phi^2_\alpha\phi^2_{\bar{\beta}}}{W^2} - \frac{3C}{W^{\frac{1}{2}}} - Ae^{L-\phi}(\phi_{i\bar{i}} - |\phi_i|^2 \\ & +\phi_{i\bar{i}}\frac{2|\phi_i|^2}{W})\Big] - \frac{C}{W^{\frac{1}{2}}}kf - 2\frac{|\nabla f|}{W^{\frac{1}{2}}} \\ \geq\; & \sum_i F^{i\bar{i}}\Big[\frac{\sum g^{\alpha\bar{\beta}}_{,i\bar{i}}\phi_\alpha\phi_{\bar{\beta}}}{W} - \frac{\sum g^{\alpha\bar{\beta}}_{,i} g^{\alpha\bar{\beta}}_{,\bar{i}}\phi^2_\alpha\phi^2_{\bar{\beta}}}{W^2} - 3C - Ae^{L-\phi}((\tilde{g}_{i\bar{i}} - g_{i\bar{i}})(1 + \frac{2|\phi_i|^2}{W}) \\ & -|\phi_i|^2)\Big] - \frac{C}{W^{\frac{1}{2}}}kf - 2\frac{|\nabla f|}{W^{\frac{1}{2}}} \end{aligned} \tag{2.21}$$

Notice that $\omega + \sqrt{-1}\partial\bar{\partial}u \in \Gamma_{k+1}$ is equivalent to, in local coordinates, $\tilde{g}_{i\bar{j}} \in \Gamma_{k+1}$. It follows that

$$\frac{\partial\sigma_{k+1}(\tilde{g}_{m\bar{l}})}{\partial\tilde{g}_{i\bar{i}}} > 0 \;\text{ for } \forall i = 1, \cdots, n, \tag{2.22}$$

by the basic property of convex Γ_{k+1} cone:

$$\text{if } (\lambda) \in \Gamma_{k+1} \implies (\lambda \mid i) \in \Gamma_k \;\text{ for } \forall i = 1, \cdots, n.$$

where $(\lambda \mid i)$ means removing the $i - th$ element of $(\lambda) = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$.

Now, we are in the place to estimate the last trouble term in (2.21).

$$\begin{aligned}
&-\sum_i F^{i\bar{i}} A e^{L-\phi}\Big((\tilde{g}_{i\bar{i}} - g_{i\bar{i}})(1+\frac{2|\phi_i|^2}{W}) - |\phi_i|^2\Big) \\
= \;& A e^{L-\phi}\sum_i F^{i\bar{i}}(1+\frac{2|\phi_i|^2}{W} + |\phi_i|^2) - A e^{L-\phi}\sum_i F^{i\bar{i}}\tilde{g}_{i\bar{i}}(1+\frac{2|\phi_i|^2}{W}) \\
\geq \;& A e^{L-\phi}\sum_i F^{i\bar{i}}(1+|\phi_i|^2) - A e^{L-\phi}\sum_i F^{i\bar{i}}\tilde{g}_{i\bar{i}} - A e^{L-\phi}\sum_i F^{i\bar{i}}\tilde{g}_{i\bar{i}}\frac{2|\phi_i|^2}{W} \\
= \;& A e^{L-\phi}\sum_i F^{i\bar{i}}(1+|\phi_i|^2) - A e^{L-\phi} k f - 2A e^{L-\phi}\Big(f - \frac{\partial \sigma_{k+1}(\tilde{g}_{m\bar{l}})}{\partial \tilde{g}_{i\bar{i}}}\Big)\frac{|\phi_i|^2}{W} \\
\geq \;& A e^{L-\phi}\sum_i F^{i\bar{i}}(1+|\phi_i|^2) - (k+2) A e^{L-\phi} k f
\end{aligned}$$

where we have made use of (2.22) in the last inequality. Indeed, this is the only place where we need the Γ_{k+1} cone condition in the whole proof. Next, putting the above estimate back to inequality (2.21), we get

$$\begin{aligned}
0 \;\geq\; & \sum_i F^{i\bar{i}}\Big[\frac{\sum g^{\alpha\bar{\beta}}_{,i\bar{i}}\phi_\alpha\phi_{\bar{\beta}}}{W} - \frac{\sum g^{\alpha\bar{\beta}}_{,i} g^{\alpha\bar{\beta}}_{,\bar{i}}\phi^2_\alpha\phi^2_{\bar{\beta}}}{W^2} - 3C + A e^{L-\phi}(1+|\phi_i|^2)\Big] \\
& - A e^{L-\phi}(k+2) f - \frac{C}{W^{\frac{1}{2}}} k f - 2\frac{|\nabla f|}{W^{\frac{1}{2}}} \\
\geq\; & A e^{L-\phi}\Big[\frac{1}{2}\sum_i F^{i\bar{i}}(1+|\phi_i|^2) - (k+2) f\Big] - \frac{2|\nabla f| + C k f}{W^{\frac{1}{2}}}
\end{aligned}$$

Now by Lemma 2.2.1 (taking $B = W, \lambda_i = 1 + \phi_{i\bar{i}}, s_i = \frac{|\phi_i|^2}{W}$),

$$\begin{aligned}
0 \;\geq\; & A e^{L-\phi}\Big[\frac{D f^{1-\frac{1}{k}} W^{\frac{1}{k}}}{2} - (k+2) f\Big] - \frac{2|\nabla f| + C k f}{W^{\frac{1}{2}}} \\
=\; & f^{1-\frac{1}{k}}\Big[A e^{L-\phi}(\frac{D W^{\frac{1}{k}}}{2} - (k+2) f^{\frac{1}{k}}) - \frac{2|\nabla f^{\frac{1}{k}}| + C k f^{\frac{1}{k}}}{W^{\frac{1}{2}}}\Big] \\
\geq\; & f^{1-\frac{1}{k}}\Big[A e^{L-\phi}(\frac{D W^{\frac{1}{k}}}{4} - (k+2) f^{\frac{1}{k}}) + (A e^{L-\phi}\frac{D W^{\frac{1}{k}}}{4} - \frac{2|\nabla f^{\frac{1}{k}}| + C k f^{\frac{1}{k}}}{W^{\frac{1}{2}}})\Big] \qquad (2.23)
\end{aligned}$$

where D is a constant from Lemma 2.2.1.

So, either

$$\frac{DW^{\frac{1}{k}}}{4} - (k+2)f^{\frac{1}{k}} \leq 0,$$

or

$$\Big(Ae^{L-\phi}\frac{DW^{\frac{1}{k}}}{4} - \frac{2|\nabla f^{\frac{1}{k}}| + Ckf^{\frac{1}{k}}}{W^{\frac{1}{2}}}\Big) \leq 0.$$

In each case, we can get an upper bound for W at p, which depends on $\inf_M R_{i\bar{i}j\bar{j}}$, $\sup_M |g^{\alpha\bar{\beta}}_{,i}|, \sup_M f, \sup_M |\nabla f^{\frac{1}{k}}|$. $\square$

In the rest this section, we give the proof of the gradient estimate for the complex Hessian equation in a more general form:

$$\sigma_k(g_{i\bar{j}} + \phi_{i\bar{j}} + \mu\phi_i\phi_{\bar{j}}) = f.$$

The method is similar to the case above, but there are some extra terms which are not easy to handle. We need to modify the test function. On the other hand, because of the gradient term in $\tilde{g}_{i\bar{j}} = g_{i\bar{j}} + \phi_{i\bar{j}} + \mu(z)\phi_i\phi_{\bar{j}}$, we can not choose local coordinates to make $\tilde{g}_{i\bar{j}}$ diagonal, in turn $F^{i\bar{j}} := \frac{\partial\sigma_k(W)}{\partial W_{i\bar{j}}}$ can not be diagonalized anymore.

Proof of Theorem 2.1.3:

Let's also denote $\tilde{g}_{i\bar{j}} = g_{i\bar{j}} + \phi_{i\bar{j}} + \mu(z)\phi_i\phi_{\bar{j}}, W = |\nabla\phi|^2$ and $L = \sup_M |\phi|$. Suppose the maximum of

$$\tilde{H} = \log W + e^{A(L-\phi)}$$

is attained at some interior point p. We pick a holomorphic orthonormal coordinate system at that point such that $g_{i\bar{j}}$ and $\phi_{i\bar{j}}$ are diagonal at that point. We may assume that $W(p) \geq 1$ and pick A sufficient large.

As $(\phi_{i\bar{j}})$ is diagonal at the point p, we differentiate $\tilde{H}$,

$$\frac{W_i}{W} - Ae^{A(L-\phi)}\phi_i = 0, \qquad \frac{W_{\bar{j}}}{W} - Ae^{A(L-\phi)}\phi_{\bar{j}} = 0 \tag{2.24}$$

By the same way as (2.14) and (2.15), we have

$$\begin{aligned} \phi_{\bar{\alpha}}\phi_{i\alpha} &= AWe^{A(L-\phi)}\phi_i - \phi_i\phi_{i\bar{i}} - g^{\alpha\bar{\beta}}_{,i}\phi_\alpha\phi_{\bar{\beta}} && (2.25)\\ \phi_\alpha\phi_{\bar{j}\bar{\alpha}} &= AWe^{A(L-\phi)}\phi_{\bar{j}} - \phi_{\bar{j}}\phi_{j\bar{j}} - g^{\alpha\bar{\beta}}_{,\bar{j}}\phi_\alpha\phi_{\bar{\beta}} && (2.26) \end{aligned}$$

From this, we may write

$$\begin{aligned} W_iW_{\bar{j}} &= g^{\alpha\bar{\beta}}_{,i}g^{\alpha\bar{\beta}}_{,\bar{j}}\phi^2_\alpha\phi^2_{\bar{\beta}} + g^{\alpha\bar{\beta}}_{,\bar{j}}|\phi_\alpha|^2\phi_{\bar{\beta}}\phi_{i\alpha} + g^{\alpha\bar{\beta}}_{,i}|\phi_\alpha|^2\phi_\beta\phi_{\bar{j}\bar{\alpha}} + |\phi_\alpha|^2\phi_{\alpha i}\phi_{\bar{j}\bar{\alpha}} \quad (2.27)\\ &\quad + AWe^{A(L-\phi)}\left(\phi_{\bar{j}}\phi_i\phi_{i\bar{i}} + \phi_i\phi_{\bar{j}}\phi_{j\bar{j}}\right) + \phi_i\phi_{\bar{j}}\phi_{i\bar{i}}\phi_{j\bar{j}} \end{aligned}$$

We pick $A \geq 1$ sufficient large, such that

$$\left(\frac{\sum g^{\alpha\bar{\beta}}_{,i\bar{i}}\phi_\alpha\phi_{\bar{\beta}}}{W} - \frac{\sum g^{\alpha\bar{\beta}}_{,i}g^{\alpha\bar{\beta}}_{,\bar{i}}\phi^2_\alpha\phi^2_{\bar{\beta}}}{W^2} - 100|g^{\alpha\bar{\beta}}_{,k}|g_{i\bar{j}} + \frac{A}{2}g_{i\bar{j}}\right) \geq 0.$$

and

$$\frac{A}{4} \geq \sup_M |\mu|$$

Thus, at the maximal point p,

$$
\begin{aligned}
0 \;\geq\;& \sum_{i,j} F^{i\bar{j}} \left(\frac{W_{i\bar{j}}}{W} - \frac{W_i W_{\bar{j}}}{W^2} - Ae^{A(L-\phi)}\phi_{i\bar{j}} + A^2 e^{A(L-\phi)}\phi_i\phi_{\bar{j}} \right) \\
=\;& \sum_{i,j} F^{i\bar{j}} \Bigg[\Big(\frac{\sum g^{\alpha\bar{\beta}}_{,i\bar{j}}\phi_\alpha\phi_{\bar{\beta}}}{W} + \frac{\sum g^{\alpha\bar{\beta}}_{,i} g^{\alpha\bar{\beta}}_{,\bar{j}} \phi_\alpha^2 \phi_{\bar{\beta}}^2}{W^2} + \frac{\sum (g^{j\bar{\beta}}_{,i}\phi_{\bar{\beta}}\phi_{j\bar{j}} + g^{\alpha\bar{i}}_{,\bar{j}}\phi_\alpha\phi_{i\bar{i}})}{W} \\
&+ \frac{\sum (g^{\alpha\bar{\beta}}_{,i}\phi_\alpha\phi_{\bar{j}\bar{\beta}} + g^{\alpha\bar{\beta}}_{,\bar{j}}\phi_{\bar{\beta}}\phi_{i\alpha})}{W} - \frac{\sum (g^{\alpha\bar{\beta}}_{,i}|\phi_\alpha|^2\phi_\beta\phi_{\bar{j}\bar{\alpha}} + g^{\alpha\bar{\beta}}_{,\bar{j}}|\phi_\alpha|^2\phi_{\bar{\beta}}\phi_{i\alpha})}{W^2} \\
&+ \frac{\sum \phi_{i\alpha}\phi_{\bar{j}\bar{\alpha}}}{W} - \frac{\sum |\phi_\alpha|^2\phi_{i\alpha}\phi_{\bar{j}\bar{\alpha}}}{W^2} - Ae^{A(L-\phi)} \left(\phi_{i\bar{j}} - A\phi_i\phi_{\bar{j}} + \frac{\phi_i\phi_{\bar{j}}\phi_{i\bar{i}} + \phi_i\phi_{\bar{j}}\phi_{j\bar{j}}}{W} \right) \\
&+ \frac{\phi_{i\bar{j}}^2}{W} + \frac{\phi_i\phi_{\bar{j}}\phi_{i\bar{i}}\phi_{j\bar{j}}}{W^2} + \frac{\sum \phi_{\bar{\alpha}}\phi_{\alpha i\bar{j}} + \phi_\alpha\phi_{\bar{\alpha}i\bar{j}}}{W} \Bigg]
\end{aligned}
$$

Again, we look for cancelations and simplify the above terms, first

$$
\begin{aligned}
& \sum_{i,j} F^{i\bar{j}} \frac{\sum (g^{j\bar{\beta}}_{,i}\phi_{\bar{\beta}}\phi_{j\bar{j}} + g^{\alpha\bar{i}}_{,\bar{j}}\phi_\alpha\phi_{i\bar{i}})}{W} \\
=\;& \sum_{i,j} F^{i\bar{j}} (\tilde{g}_{i\bar{j}} - g_{i\bar{j}} - \mu\phi_i\phi_{\bar{j}}) \frac{\sum (g^{j\bar{\beta}}_{,i}\phi_{\bar{\beta}} + g^{\alpha\bar{i}}_{,\bar{j}}\phi_\alpha)}{W} \\
\geq\;& -\frac{C_0}{W^{\frac{1}{2}}} \sum_{i,j} F^{i\bar{j}}\tilde{g}_{i\bar{j}} - \frac{C_1}{W^{\frac{1}{2}}} \sum_{i,j} F^{i\bar{j}} - \frac{C_2}{W^{\frac{1}{2}}} \sum_{i,j} F^{i\bar{j}}\phi_i\phi_{\bar{j}} \\
=\;& -\frac{C_0}{W^{\frac{1}{2}}} kf - \frac{C_1}{W^{\frac{1}{2}}} \sum_i F^{i\bar{i}} - \frac{C_2}{W^{\frac{1}{2}}} \sum_{i,j} F^{i\bar{j}}\phi_i\phi_{\bar{j}}
\end{aligned}
$$

where C_0, C_1 and C_2 are some positive constants depending on $\sup_M |g_{\alpha\bar\beta,i}|$ and the function $\mu(z)$. In turn, we have

$$\begin{aligned}
& \sum_{i,j} F^{i\bar j}\left[\frac{\sum(g^{\alpha\bar\beta}_{,i}\phi_\alpha\phi_{\bar j\bar\beta} + g^{\alpha\bar\beta}_{,\bar j}\phi_{\bar\beta}\phi_{i\alpha})}{W} - \frac{\sum(g^{\alpha\bar\beta}_{,i}|\phi_\alpha|^2\phi_\beta\phi_{\bar j\bar\alpha} + g^{\alpha\bar\beta}_{,\bar j}|\phi_\alpha|^2\phi_{\bar\beta}\phi_{i\alpha})}{W^2}\right.\\
& \left. + \frac{\sum\phi_{i\alpha}\phi_{\bar j\bar\alpha}}{W} - \frac{\sum|\phi_\alpha|^2\phi_{i\alpha}\phi_{\bar j\bar\alpha}}{W^2}\right]\\
= & \sum_{i,j} F^{i\bar j}\left[\frac{g^{\alpha\bar\beta}_{,i}\phi_\beta\phi_{\bar j\bar\alpha}}{W}(1-\frac{|\phi_\alpha|^2}{W}) + \frac{g^{\alpha\bar\beta}_{,\bar j}\phi_{\bar\beta}\phi_{i\alpha}}{W}(1-\frac{|\phi_\alpha|^2}{W}) + \frac{\phi_{i\alpha}\phi_{\bar j\bar\alpha}}{W}(1-\frac{|\phi_\alpha|^2}{W})\right]\\
= & \sum_{i,j} F^{i\bar j}\left[(\frac{\phi_{\bar i\bar\alpha}}{W^{\frac12}} + g^{\alpha\bar\beta}_{,i}\frac{\phi_\beta}{W^{\frac12}})\overline{(\frac{\phi_{j\alpha}}{W^{\frac12}} + g^{\alpha\bar\beta}_{,j}\frac{\phi_\beta}{W^{\frac12}})} - \frac{|g^{\alpha\bar\beta}_{,i}\phi_\beta|^2}{W}\right](1-\frac{|\phi_\alpha|^2}{W})\\
\geq & -\sum_{i,j} F^{i\bar j}\frac{|g^{\alpha\bar\beta}_{,i}\phi_\beta|^2}{W}(1-\frac{|\phi_\alpha|^2}{W})\\
\geq & -C\sum_{i,j} F^{i\bar j}
\end{aligned}$$

where C is also a positive constant depending on $\sup_M |g^{\alpha\bar\beta}_{,i}|$.

By equation (2.3),we have,

$$F^{i\bar j}\left(g_{i\bar j\alpha} + \phi_{i\bar j\alpha} + (\mu\phi_i\phi_{\bar j})_\alpha\right) = f_\alpha, \qquad F^{i\bar j}\left(g_{i\bar j\bar\alpha} + \phi_{i\bar j\bar\alpha} + (\mu\phi_i\phi_{\bar j})_{\bar\alpha}\right) = f_{\bar\alpha} \tag{2.28}$$

Thus,

$$\begin{aligned}
\sum_{i,j} F^{i\bar j}\phi_{\bar\alpha}\phi_{i\bar j\alpha} = & \ \phi_{\bar\alpha}f_\alpha - \sum_{i,j} F^{i\bar j}\phi_{\bar\alpha}\left(g_{i\bar j\alpha} + \mu_\alpha\phi_i\phi_{\bar j} + \mu\phi_{i\alpha}\phi_{\bar j} + \mu\phi_i\phi_{j\bar j}\right) \qquad (2.29)\\
\geq & \ \phi_{\bar\alpha}f_\alpha - C_3W^{\frac12}\sum_{i,j}F^{i\bar j} - C_4W^{\frac12}\sum_{i,j}F^{i\bar j}\phi_i\phi_{\bar j}\\
& -\mu\sum_{i,j}F^{i\bar j}\phi_{\bar\alpha}(\phi_{i\alpha}\phi_{\bar j} + \phi_i\phi_{j\bar j})
\end{aligned}$$

we need to control the last term in (2.29). By (2.25) and (2.26), we have

$$\begin{aligned}-\mu\sum_{i,j}F^{i\bar{j}}\phi_{\bar{\alpha}}\phi_{i\alpha}\phi_{\bar{j}} &= -\mu\sum_{i,j}F^{i\bar{j}}\phi_{\bar{j}}(AWe^{A(L-\phi)}\phi_i-\phi_i\phi_{i\bar{i}}-g^{\alpha\bar{\beta}}_{,i}\phi_\alpha\phi_{\bar{\beta}}) \\ &\geq -\mu AWe^{A(L-\phi)}\sum_{i,j}F^{i\bar{j}}\phi_i\phi_{\bar{j}}-C_5W\sum_{i,j}F^{i\bar{j}}|\phi_j| \\ &\quad -\mu\sum_{i,j}F^{i\bar{j}}\phi_{\bar{j}}\phi_i\phi_{i\bar{i}}\end{aligned} \tag{2.30}$$

$$\begin{aligned}-\mu\sum_{i,j}F^{i\bar{j}}\phi_{\bar{j}}\phi_i\phi_{i\bar{i}} &= -\mu\sum_{i,j}F^{i\bar{j}}(\tilde{g}_{i\bar{j}}-g_{i\bar{j}}-\mu\phi_i\phi_{\bar{j}})\phi_{\bar{i}}\phi_j \\ &\geq -C_6Wkf-C_7\sum_{i}F^{i\bar{i}}|\phi_i|^2+\mu^2\sum_{i,j}F^{i\bar{j}}|\phi_i|^2|\phi_{\bar{j}}|^2\end{aligned} \tag{2.31}$$

It follows that,

$$\begin{aligned}-\mu\sum_{i,j}F^{i\bar{j}}\phi_{\bar{\alpha}}(\phi_{i\alpha}\phi_{\bar{j}}+\phi_i\phi_{j\bar{j}}) &\geq -C_5W\sum_{i,j}F^{i\bar{j}}|\phi_j|-C_8W\sum_{i,j}F^{i\bar{j}}\phi_i\phi_{\bar{j}} \\ &\quad -\mu AWe^{A(L-\phi)}\sum_{i,j}F^{i\bar{j}}\phi_i\phi_{\bar{j}}-C_9Wkf\end{aligned} \tag{2.32}$$

Combining (2.29) and (2.32) together, we can get

$$\begin{aligned}\sum_{i,j}F^{i\bar{j}}\phi_{\bar{\alpha}}\phi_{i\bar{j}\alpha} &\geq -C_{10}W\sum_{i,j}F^{i\bar{j}}-C_{11}AWe^{A(L-\phi)}\sum_{i,j}F^{i\bar{j}}\phi_i\phi_{\bar{j}} \\ &\quad -C_9Wkf+\phi_{\bar{\alpha}}f_\alpha\end{aligned} \tag{2.33}$$

Moreover, similar argument as above also yields,

$$\begin{aligned}\sum_{i,j}F^{i\bar{j}}\phi_{\alpha}\phi_{i\bar{j}\bar{\alpha}} &\geq -C_{10}W\sum_{i,j}F^{i\bar{j}}-C_{11}AWe^{A(L-\phi)}\sum_{i,j}F^{i\bar{j}}\phi_i\phi_{\bar{j}} \\ &\quad -C_9Wkf+\phi_{\alpha}f_{\bar{\alpha}}\end{aligned} \tag{2.34}$$

where $C_3, \cdots, C_{11}$ are some constants depending only on $\sup_M |g^{\alpha\bar{\beta}}_{,i}|$ and the function μ.

Thus,

$$\begin{aligned}
&\sum_{i,j} F^{i\bar{j}} \frac{\sum \phi_{\bar{\alpha}}\phi_{\alpha i\bar{j}} + \phi_{\alpha}\phi_{\bar{\alpha}i\bar{j}}}{W} \qquad (2.35)\\
\geq\ & \frac{1}{W}\sum_{\alpha}(\phi_\alpha f_{\bar{\alpha}} + \phi_{\bar{\alpha}} f_\alpha) - C_{12}\sum_{i,j} F^{i\bar{j}} - C_{13}Ae^{A(L-\phi)}\sum_{i,j} F^{i\bar{j}}\phi_i\phi_{\bar{j}} - C_{14}kf\\
\geq\ & -2\frac{|\nabla f|}{W^{\frac{1}{2}}} - C_{12}\sum_{i,j} F^{i\bar{j}} - C_{13}Ae^{A(L-\phi)}\sum_{i,j} F^{i\bar{j}}\phi_i\phi_{\bar{j}} - C_{14}kf
\end{aligned}$$

where C_{12}, C_{13}, C_{14} are some constants depending only on $\sup_M |g^{\alpha\bar{\beta}}_{,i}|$ and the function μ.

By combining all the above estimates together and using the fact that $W > 1$ is large, we get

$$
\begin{aligned}
0 \geq & \sum_{i,j} F^{i\bar{j}} \Bigg[\Big(\frac{\sum g^{\alpha\bar{\beta}}_{,i\bar{j}} \phi_\alpha \phi_{\bar{\beta}}}{W} + \frac{\sum g^{\alpha\bar{\beta}}_{,i} g^{\alpha\bar{\beta}}_{,\bar{j}} \phi^2_\alpha \phi^2_{\bar{\beta}}}{W^2} - C_0' A e^{A(L-\phi)} \phi_i \phi_{\bar{j}} - C_1' \qquad (2.36) \\
& - A e^{A(L-\phi)} \left(\phi_{i\bar{j}} - A \phi_i \phi_{\bar{j}} + \frac{\phi_i \phi_{\bar{j}} \phi_{i\bar{i}} + \phi_j \phi_{\bar{i}} \phi_{i\bar{i}}}{W} \right) \Bigg] \\
& - c_2' kf - \frac{C}{W^{\frac{1}{2}}} kf - 2 \frac{|\nabla f|}{W^{\frac{1}{2}}} \\
\geq & \sum_{i,j} F^{i\bar{j}} \Bigg[\Big(\frac{\sum g^{\alpha\bar{\beta}}_{,i\bar{j}} \phi_\alpha \phi_{\bar{\beta}}}{W} + \frac{\sum g^{\alpha\bar{\beta}}_{,i} g^{\alpha\bar{\beta}}_{,\bar{j}} \phi^2_\alpha \phi^2_{\bar{\beta}}}{W^2} - C_0' A e^{A(L-\phi)} \phi_i \phi_{\bar{j}} - C_1' \\
& - A e^{A(L-\phi)} (\tilde{g}_{i\bar{j}} - g_{i\bar{j}} - \mu \phi_i \phi_{\bar{j}}) \left(1 + \frac{\phi_i \phi_{\bar{j}} + \phi_j \phi_{\bar{i}}}{W} \right) + A^2 e^{A(L-\phi)} \phi_i \phi_{\bar{j}} \Bigg] \\
& - C_2' kf - \frac{C}{W^{\frac{1}{2}}} kf - 2 \frac{|\nabla f|}{W^{\frac{1}{2}}} \\
\geq & \sum_{i,j} F^{i\bar{j}} \Bigg[\Big(\frac{\sum g^{\alpha\bar{\beta}}_{,i\bar{j}} \phi_\alpha \phi_{\bar{\beta}}}{W} + \frac{\sum g^{\alpha\bar{\beta}}_{,i} g^{\alpha\bar{\beta}}_{,\bar{j}} \phi^2_\alpha \phi^2_{\bar{\beta}}}{W^2} + \frac{A^2}{2} e^{A(L-\phi)} \phi_i \phi_{\bar{j}} - C_1' \\
& + A e^{A(L-\phi)} g_{i\bar{j}} + \mu A e^{A(L-\phi)} \phi_i \phi_{\bar{j}} \left(1 + \frac{\phi_i \phi_{\bar{j}} + \phi_j \phi_{\bar{i}}}{W} \right) \Bigg] \\
& - 4 A e^{A(L-\phi)} kf - \frac{C}{W^{\frac{1}{2}}} kf - 2 \frac{|\nabla f|}{W^{\frac{1}{2}}} \\
\geq & \sum_{i,j} F^{i\bar{j}} \Bigg[\Big(\frac{\sum g^{\alpha\bar{\beta}}_{,i\bar{j}} \phi_\alpha \phi_{\bar{\beta}}}{W} + \frac{\sum g^{\alpha\bar{\beta}}_{,i} g^{\alpha\bar{\beta}}_{,\bar{j}} \phi^2_\alpha \phi^2_{\bar{\beta}}}{W^2} + \frac{A^2}{4} e^{A(L-\phi)} \phi_i \phi_{\bar{j}} \\
& + \frac{A}{2} e^{A(L-\phi)} g_{i\bar{j}} \Bigg] - 4 A e^{A(L-\phi)} kf - \frac{C}{W^{\frac{1}{2}}} kf - 2 \frac{|\nabla f|}{W^{\frac{1}{2}}} \\
\geq & A e^{A(L-\phi)} \left[\frac{1}{2} \sum_{i,j} F^{i\bar{j}} (g_{i\bar{j}} + \phi_i \phi_{\bar{j}}) - 4kf \right] - \frac{2|\nabla f| + Ckf}{W^{\frac{1}{2}}}
\end{aligned}
$$

Let's denote $g_{i\bar{j}} + \phi_i\phi_{\bar{j}}$ by $Q_{i\bar{j}}$, then

$$\sigma_k(Q_{i\bar{j}}) = \sigma_k(\lambda\{Q_{i\bar{j}}\}) = 1 + W \tag{2.37}$$

Now, by Garding's inequality for polarized σ_k and (2.37), we have

$$\begin{aligned} \sum_{i,j} F^{i\bar{j}}(g_{i\bar{j}} + \phi_i\phi_{\bar{j}}) &= \sum_{i,j} \frac{\partial F}{\partial \tilde{g}_{i\bar{j}}} Q_{i\bar{j}} \geq C_{n,k}\sigma_k^{\frac{k-1}{k}}(\tilde{g}_{i\bar{j}})\sigma_k^{\frac{1}{k}}(Q_{i\bar{j}}) \\ &= C_{n,k} f^{1-\frac{1}{k}}(1+W)^{\frac{1}{k}} \\ &\geq C f^{1-\frac{1}{k}} W^{\frac{1}{k}} \end{aligned} \tag{2.38}$$

Finally, the inequalities (2.36) and (2.38) give that

$$\begin{aligned} 0 &\geq Ae^{L-\phi}\left[\frac{Cf^{1-\frac{1}{k}}W^{\frac{1}{k}}}{2} - 4kf\right] - \frac{2|\nabla f| + Ckf}{W^{\frac{1}{2}}} \\ &= f^{1-\frac{1}{k}}\left[Ae^{L-\phi}\left(\frac{CW^{\frac{1}{k}}}{2} - 4kf^{\frac{1}{k}}\right) - \frac{2|\nabla f^{\frac{1}{k}}| + Ckf^{\frac{1}{k}}}{W^{\frac{1}{2}}}\right] \\ &\geq f^{1-\frac{1}{k}}\left[Ae^{L-\phi}\left(\frac{CW^{\frac{1}{k}}}{4} - 4kf^{\frac{1}{k}}\right) + \left(Ae^{L-\phi}\frac{CW^{\frac{1}{k}}}{4} - \frac{2|\nabla f^{\frac{1}{k}}| + Ckf^{\frac{1}{k}}}{W^{\frac{1}{2}}}\right)\right]. \end{aligned}$$

So, either

$$\frac{CW^{\frac{1}{k}}}{4} - 4kf^{\frac{1}{k}} \leq 0,$$

or

$$\left(Ae^{L-\phi}\frac{CW^{\frac{1}{k}}}{4} - \frac{2|\nabla f^{\frac{1}{k}}| + Ckf^{\frac{1}{k}}}{W^{\frac{1}{2}}}\right) \leq 0.$$

In each case, we can get an upper bound for W at p, which depends on $\inf_M R_{i\bar{i}j\bar{j}}$, $\sup_M |g^{\alpha\bar{\beta}}_{,i}|, \sup_M f, \sup_M |\nabla f^{\frac{1}{k}}|$.

□

2.3 C^2 estimate for Monge-Ampère equation on Hermitian manifolds

In this section, we estimate the second derivatives $\phi_{i\bar{j}}$ assuming ϕ solves equation (2.1) and f is $C^3(M)$.

First, we want to fix one notation: in the following proof, we write $f = O(|\nabla\phi|)$, if there exist two nonnegative constants C_1, and C_2 such that

$$-C_1|\nabla\phi| \leq f \leq C_2|\nabla\phi|.$$

Proof of Theorem 2.1.4:

Let's denote $\tilde{g}_{i\bar{j}} = g_{i\bar{j}} + \phi_{i\bar{j}}$. Suppose the maximum of the test function

$$G = \log(m + \Delta\phi) + B|\nabla\phi|^2 - A\phi$$

is attained at some interior point p. We pick a holomorphic orthonormal coordinate system at that point such that $(\tilde{g}_{i\bar{j}}) = (g_{i\bar{j}} + \phi_{i\bar{j}})$, is diagonal at that point. We may assume $(m + \Delta\phi)$ is large.

As $\phi_{i\bar{j}}$ is diagonal at the point p, we have the following inequality at p,

$$\begin{aligned} 0 &\geq \sum_i F^{i\bar{i}} G_{i\bar{i}} \qquad (2.39) \\ &= \sum_i F^{i\bar{i}} \left[\frac{(m+\Delta\phi)_{i\bar{i}}}{m+\Delta\phi} - \frac{(m+\Delta\phi)_i (m+\Delta\phi)_{\bar{i}}}{(m+\Delta\phi)^2} + B|\nabla\phi|^2_{i\bar{i}} - A\phi_{i\bar{i}} \right] \end{aligned}$$

Now, we differentiate G at the maximal point p,

$$\begin{aligned}\frac{(m+\Delta\phi)_i}{m+\Delta\phi} &= -B|\nabla\phi|^2_i + A\phi_i \\ &= -Bg^{k\bar{l}}_{,i}\phi_k\phi_{\bar{l}} - B(\phi_{ki}\phi_{\bar{k}} + \phi_k\phi_{\bar{k}i}) + A\phi_i \\ &= -B\phi_i\phi_{i\bar{i}} - B\phi_{B_k}\phi_{ki} + A\phi_i - Bg^{k\bar{l}}_{,i}\phi_k\phi_{\bar{l}} \\ &= -B\phi_i\phi_{i\bar{i}} - B\phi_{B_k}\phi_{ki} + O(|\nabla\phi|^2)\end{aligned}$$

Similarly, by $\nabla_{\bar{i}}H = 0$, we have

$$\frac{(m+\Delta\phi)_{\bar{i}}}{m+\Delta\phi} = -B\phi_{\bar{j}}\phi_{j\bar{j}} - B\phi_l\phi_{\bar{l}\bar{j}} + O(|\nabla\phi|^2)$$

So,

$$\begin{aligned}\frac{(m+\Delta\phi)_i(m+\Delta\phi)_{\bar{i}}}{(m+\Delta\phi)^2} &= [B(\phi_i\phi_{i\bar{i}} + \phi_{B_k}\phi_{ki} + O(|\nabla\phi|^2)][B(\phi_i\phi_{i\bar{i}} + \phi_l\phi_{\bar{l}\bar{i}}) + O(|\nabla\phi|^2)] \\ &= B^2(\phi_i\phi_{i\bar{i}} + \phi_{B_k}\phi_{ki})(\phi_{\bar{i}}\phi_{i\bar{i}} + \phi_l\phi_{\bar{l}\bar{i}}) + B(\phi_i\phi_{i\bar{i}} + \phi_{\bar{i}}\phi_{i\bar{i}} \\ &\quad +\phi_{B_k}\phi_{ki} + \phi_l\phi_{\bar{l}\bar{i}})\cdot O(|\nabla\phi|^2) + O(|\nabla\phi|^4) \\ &= B^2(|\nabla\phi|^2\phi^2_{i\bar{i}} + \phi_i\phi_l\phi_{i\bar{i}}\phi_{\bar{l}\bar{i}} + \phi_{\bar{i}}\phi_{i\bar{i}}\phi_{B_k}\phi_{ki} + \phi_{B_k}\phi_l\phi_{ki}\phi_{\bar{l}\bar{i}}) \\ &\quad +B\phi_{i\bar{i}}\cdot O(|\nabla\phi|^3) + B\phi_{ki}\phi_{\bar{l}\bar{j}}\cdot O(|\nabla\phi|^3) + O(|\nabla\phi|^4) \\ &= B^2|\nabla\phi|^2\phi^2_{i\bar{i}} + B^2(\phi_{\bar{l}\bar{i}}\phi_{i\bar{i}} + \phi_{ki}\phi_{i\bar{i}})\cdot O(|\nabla\phi|^2) + B^2\phi_{ki}\phi_{\bar{l}\bar{i}}\cdot O(|\nabla\phi|^2) \\ &\quad +B\phi_{i\bar{i}}\cdot O(|\nabla\phi|^3) + B(\phi_{ki} + \phi_{\bar{l}\bar{i}})\cdot O(|\nabla\phi|^3) + O(|\nabla\phi|^4)\end{aligned}$$

$$\begin{aligned}|\nabla\phi|^2_{i\bar{i}} &= g^{k\bar{l}}_{,i\bar{i}}\phi_k\phi_{\bar{l}} + g^{k\bar{l}}_{,i}(\phi_{k\bar{i}}\phi_{\bar{l}} + \phi_k\phi_{\bar{i}\bar{l}}) + g^{k\bar{l}}_{,\bar{i}}(\phi_{ki}\phi_{\bar{l}} + \phi_k\phi_{i\bar{l}}) + (\phi_k\phi_{\bar{l}})_{i\bar{i}} \\ &= O(|\nabla\phi|^2) + (g^{k\bar{l}}_{,i}\phi_{\bar{l}}\phi_{i\bar{i}} + g^{k\bar{l}}_{,\bar{i}}\phi_k\phi_{i\bar{i}}) + (g^{k\bar{l}}_{,i}\phi_k\phi_{\bar{l}\bar{i}} + g^{k\bar{l}}_{,\bar{i}}\phi_{\bar{l}}\phi_{ki}) \\ &\quad +\phi_{ki}\phi_{\bar{l}\bar{i}} + \phi^2_{i\bar{i}} + \phi_{ki\bar{i}}\phi_{\bar{l}} + \phi_k\phi_{\bar{l}i\bar{i}}\end{aligned}$$

By equation (2.1),

$$F^{i\bar{i}} = \frac{\partial \sigma_n(\lambda)}{\partial \lambda_i} = \sigma_{n-1}(\lambda|i)$$

$$\sum_{i,j} F^{i\bar{j}}(g_{i\bar{j}\alpha} + \phi_{i\bar{j}\alpha}) = f_\alpha, \qquad \sum_{i,j} F^{i\bar{j}}(g_{i\bar{j}\bar{\alpha}} + \phi_{i\bar{j}\bar{\alpha}}) = f_{\bar{\alpha}}$$

So,

$$\sum_i F^{i\bar{i}}\phi_{i\bar{i}} = \sum_i \sigma_{n-1}(\lambda|i)\lambda_i = n\sigma_n(\lambda) = nf \tag{2.40}$$

$$\sum_i F^{i\bar{i}}(\phi_{ki\bar{i}}\phi_{\bar{l}} + \phi_k\phi_{\bar{l}i\bar{i}}) = \phi_{\bar{l}}f_k + \phi_k f_{\bar{l}} - \sum_i F^{i\bar{i}}\,(g_{i\bar{i}k} + g_{i\bar{i}\bar{l}}) \tag{2.41}$$

Combining the above expressions, we get the estimates for the third and fourth terms of (2.39),

$$\begin{aligned}
& \sum_i F^{i\bar{i}}\left(B|\nabla\phi|^2_{i\bar{i}} - \frac{(m+\Delta\phi)_i(m+\Delta\phi)_{\bar{i}}}{(m+\Delta\phi)^2}\right) \qquad (2.42)\\
= \; & \sum_i F^{i\bar{i}}\{[(B\phi^2_{i\bar{i}} + B^2|\nabla\phi|^2\phi^2_{i\bar{i}} + B(g^{k\bar{l}}_{,i}\phi_{\bar{l}} + g^{k\bar{l}}_{,\bar{i}}\phi_k)\phi_{i\bar{i}} + B\phi_{i\bar{i}}\cdot O(|\nabla\phi|^3))] \\
& +[B(\phi_{ki}\phi_{\bar{l}\bar{i}} + g^{k\bar{l}}_{,i}\phi_k\phi_{\bar{l}\bar{i}} + g^{k\bar{l}}_{,\bar{i}}\phi_{\bar{l}}\phi_{ki}) - B^2(\phi_{\bar{l}\bar{i}} + \phi_{ki})\phi_{i\bar{i}}\cdot O(|\nabla\phi|^2) \\
& -B^2\phi_{ki}\phi_{\bar{l}\bar{i}}\cdot O(|\nabla\phi|^2) - B(\phi_{ki} + \phi_{\bar{l}\bar{i}})\cdot O(|\nabla\phi|^3)] \\
& +[O|\nabla\phi|^2) + O|\nabla\phi|^4)] - (g_{i\bar{i}k} + g_{i\bar{i}\bar{l}})\} + (\phi_{\bar{l}}f_k + \phi_k f_{\bar{l}})
\end{aligned}$$

where we used (2.41) to get the last term.

Note that, all the terms in the second brackets involves the factors ϕ_{ki}, or $\phi_{\bar{l}\bar{i}}$. We can estimate these terms in the following way, if B is small enough.

$$
\begin{aligned}
& B(\phi_{ki}\phi_{\bar{l}\bar{i}} + g^{k\bar{l}}_{,i}\phi_k\phi_{\bar{l}\bar{i}} + g^{k\bar{l}}_{,\bar{i}}\phi_{\bar{l}}\phi_{ki}) - B^2(\phi_{\bar{l}\bar{i}} + \phi_{ki})\phi_{i\bar{i}} \cdot O(|\nabla\phi|^2) - B^2\phi_{ki}\phi_{\bar{l}\bar{i}} \cdot O(|\nabla\phi|^2) \\
& -B(\phi_{ki} + \phi_{\bar{l}\bar{i}}) \cdot O(|\nabla\phi|^3) \\
\geq\ & B\left[\frac{1}{4} - B \cdot O(|\nabla\phi|^2)\right]\phi_{ki}\phi_{\bar{l}\bar{i}} + B\left[\frac{1}{4}\phi_{ki}\phi_{\bar{l}\bar{i}} + g^{k\bar{l}}_{,i}\phi_k\phi_{\bar{l}\bar{i}} + g^{k\bar{l}}_{,\bar{i}}\phi_{\bar{l}}\phi_{ki}\right] \\
& +B\left[\frac{1}{4}\phi_{ki}\phi_{\bar{l}\bar{i}} - B\phi_{i\bar{i}}(\phi_{\bar{l}\bar{i}} + \phi_{ki}) \cdot O(|\nabla\phi|^2)\right] + B\left[\frac{1}{4}\phi_{ki}\phi_{\bar{l}\bar{i}} - O(|\nabla\phi|^3)(\phi_{ki} + \phi_{\bar{l}\bar{i}})\right] \\
\geq\ & -BC_1 \cdot O(|\nabla\phi|^2) - B^3\phi_{i\bar{i}}^2 \cdot O(|\nabla\phi|^4) - B \cdot O(|\nabla\phi|^6) \\
\geq\ & -C_2B^3\phi_{i\bar{i}}^2 - C_3
\end{aligned}
$$

where C_1, C_2 and C_3 depends on $|\nabla\phi|, |g^{k\bar{l}}_{,i}|$.

Thus, we have

$$
\begin{aligned}
& \sum_i F^{i\bar{i}}\left(B|\nabla\phi|^2_{i\bar{i}} - \frac{(m+\Delta\phi)_i(m+\Delta\phi)_{\bar{i}}}{(m+\Delta\phi)^2}\right) \qquad (2.43) \\
\geq\ & \sum_i F^{i\bar{i}}\left[(B\phi_{i\bar{i}}^2 + B^2|\nabla\phi|^2\phi_{i\bar{i}}^2 + B(g^{k\bar{l}}_{,i}\phi_{\bar{l}} + g^{k\bar{l}}_{,\bar{i}}\phi_k)\phi_{i\bar{i}} + B\phi_{i\bar{i}} \cdot O(|\nabla\phi|^3))\right] \\
& -\sum_i F^{i\bar{i}}C_2B^3\phi_{i\bar{i}}^2 - C_4 - 2|\nabla\phi||\nabla f| \\
\geq\ & \sum_i F^{i\bar{i}}\left[(B + B^2|\nabla\phi|^2 - C_2B^3)\phi_{i\bar{i}}^2 + B(g^{k\bar{l}}_{,i}\phi_{\bar{l}} + g^{k\bar{l}}_{,\bar{i}}\phi_k + O(|\nabla\phi|^3))\phi_{i\bar{i}}\right] \\
& -C_4 - 2|\nabla\phi||\nabla f| \\
\geq\ & -C_5nf - C_4 - 2|\nabla\phi||\nabla f|
\end{aligned}
$$

For the last inequality, we used equation (2.40) and the fact that B is small enough.

Now, we will consider the remaining two terms in (3.1).

$$\sum_i F^{i\bar{i}} \left[\frac{(m+\Delta\phi)_{i\bar{i}}}{m+\Delta\phi} - A\phi_{i\bar{i}} \right] = \frac{\tilde{\Delta}(m+\Delta\phi)}{m+\Delta\phi} - A\tilde{\Delta}\phi \tag{2.44}$$

where $\tilde{\Delta}$ is the Laplace operator with respect to the new metric $\tilde{g}_{i\bar{j}} = g_{i\bar{j}} + \phi_{i\bar{j}}$.

By directly computation, we get

$$\begin{aligned} \tilde{\Delta}(m+\Delta\phi) &= -\sum_i R_{i\bar{i}} + \Delta f + \tilde{g}^{k\bar{l}} R_{k\bar{l}} + \tilde{g}^{k\bar{l}} g^{i\bar{i}}_{,k\bar{l}} \phi_{i\bar{i}} - \tilde{g}^{k\bar{l}} |\partial_i g_{k\bar{l}}|^2 \\ &\quad + \tilde{g}^{k\bar{l}} \tilde{g}^{p\bar{q}} \partial_i \tilde{g}_{k\bar{q}} \partial_{\bar{i}} \tilde{g}_{p\bar{l}} + \tilde{g}^{k\bar{l}} \left(g^{i\bar{j}}_{,k} \phi_{i\bar{j}\bar{l}} + g^{i\bar{j}}_{,\bar{l}} \phi_{i\bar{j}k} \right) \end{aligned} \tag{2.45}$$

Everything is in order for the application of the maximum principle to get an upper bound of the test function except the last two terms.

Since $\tilde{g}_{i\bar{j}} = (g_{i\bar{j}} + \phi_{i\bar{j}})$ is diagonal at the maximal point, $\tilde{g}^{k\bar{l}} = \frac{1}{1+\phi_{k\bar{k}}} \delta_{kl}$.

$$\begin{aligned} & \tilde{g}^{k\bar{l}} \tilde{g}^{p\bar{q}} \partial_i \tilde{g}_{k\bar{q}} \partial_{\bar{i}} \tilde{g}_{p\bar{l}} + \tilde{g}^{k\bar{l}} \left(g^{i\bar{j}}_{,k} \phi_{i\bar{j}\bar{l}} + g^{i\bar{j}}_{,\bar{l}} \phi_{i\bar{j}k} \right) \\ = & \sum_{i,p,k} \frac{1}{1+\phi_{k\bar{k}}} \frac{1}{1+\phi_{p\bar{p}}} |\tilde{g}_{k\bar{p}i}|^2 + \frac{1}{1+\phi_{k\bar{k}}} \left(g^{i\bar{p}}_{,k} \tilde{g}_{i\bar{p}\bar{k}} + g^{i\bar{p}}_{,\bar{k}} \tilde{g}_{i\bar{p}k} \right) \\ & - \frac{1}{1+\phi_{k\bar{k}}} \left(g^{i\bar{j}}_{,k} g_{i\bar{j}\bar{k}} + g^{i\bar{j}}_{,\bar{k}} g_{i\bar{j}k} \right) \\ \geq & \sum_{i,p,k} \frac{1}{1+\phi_{k\bar{k}}} \frac{1}{1+\phi_{p\bar{p}}} |\tilde{g}_{k\bar{p}i}|^2 - \frac{1}{1+\phi_{k\bar{k}}} \left(|g^{i\bar{p}}_{,k} \tilde{g}_{i\bar{p}\bar{k}}| + |g^{i\bar{p}}_{,\bar{k}} \tilde{g}_{i\bar{p}k}| \right) \\ & + 2 \frac{|g_{i\bar{p}k}|^2}{1+\phi_{k\bar{k}}} \\ = & \sum_{i,p,k} \frac{1}{1+\phi_{k\bar{k}}} \frac{1}{1+\phi_{p\bar{p}}} |\tilde{g}_{k\bar{p}i}|^2 - 2 \frac{1}{1+\phi_{k\bar{k}}} |g^{i\bar{p}}_{,\bar{k}} \tilde{g}_{i\bar{p}k}| + 2 \frac{|g_{i\bar{p}k}|^2}{1+\phi_{k\bar{k}}} \end{aligned} \tag{2.46}$$

Let $T_{i\bar{p}k} = g_{i\bar{p}k} - g_{k\bar{p}i}$, then

$$\tilde{g}_{k\bar{p}i} = \tilde{g}_{i\bar{p}k} - (\tilde{g}_{i\bar{p}k} - \tilde{g}_{k\bar{p}i}) = \tilde{g}_{i\bar{p}k} - (g_{i\bar{p}k} - g_{k\bar{p}i}) = \tilde{g}_{i\bar{p}k} - T_{i\bar{p}k}$$

So,

$$\begin{aligned} & \tilde{g}^{k\bar{l}}\tilde{g}^{p\bar{q}}\partial_i\tilde{g}_{k\bar{q}}\partial_{\bar{i}}\tilde{g}_{p\bar{l}} + \tilde{g}^{k\bar{l}}\left(g^{i\bar{j}}_{,k}\phi_{i\bar{j}\bar{l}} + g^{i\bar{j}}_{,\bar{l}}\phi_{i\bar{j}k}\right) \\ \geq \quad & \sum_{i,p,k}\frac{1}{1+\phi_{k\bar{k}}}\frac{1}{1+\phi_{p\bar{p}}}|\tilde{g}_{i\bar{p}k} - T_{i\bar{p}k}|^2 - 2\frac{1}{1+\phi_{k\bar{k}}}|g_{i\bar{p}\bar{k}}\tilde{g}_{i\bar{p}k}| \end{aligned} \tag{2.47}$$

We will estimate the right hand side of (2.47) by divide it into two cases. For any fixed index i, j, k,

If $|\tilde{g}_{i\bar{p}k}| \leq \tilde{C}(m+\Delta\phi)\max_M\{|T_{i\bar{p}k}|, |g_{i\bar{p}\bar{k}}|\}$ for some constant $\tilde{C}$, then it follows from (2.46) and (2.47),

$$\begin{aligned} & \tilde{g}^{k\bar{l}}\tilde{g}^{p\bar{q}}\partial_i\tilde{g}_{k\bar{q}}\partial_{\bar{i}}\tilde{g}_{p\bar{l}} + \tilde{g}^{k\bar{l}}\left(g^{i\bar{j}}_{,k}\phi_{i\bar{j}\bar{l}} + g^{i\bar{j}}_{,\bar{l}}\phi_{i\bar{j}k}\right) \\ \geq \quad & -\frac{2}{1+\phi_{k\bar{k}}}\tilde{C}(m+\Delta\phi)\max_M\{|T_{i\bar{p}k}|^2, |g_{i\bar{p}\bar{k}}|^2\} \\ \geq \quad & -\frac{1}{1+\phi_{k\bar{k}}}C_6(m+\Delta\phi) \end{aligned} \tag{2.48}$$

where C_6 is a constant depending on $\tilde{C}, |g_{i\bar{p}\bar{k}}|$.

If $|\tilde{g}_{i\bar{p}k}| \geq \tilde{C}'(m+\Delta\phi)\max_M\{|T_{i\bar{p}k}|, |g_{i\bar{p}\bar{k}}|\}$ for some constant $\tilde{C}' \geq 4$, then it also follows from (2.46) and (2.47) that,

$$\begin{aligned}
& \tilde{g}^{k\bar{l}}\tilde{g}^{p\bar{q}}\partial_i\tilde{g}_{k\bar{q}}\partial_{\bar{i}}\tilde{g}_{p\bar{l}} + \tilde{g}^{k\bar{l}}\left(g^{i\bar{j}}_{,k}\phi_{i\bar{j}\bar{l}} + g^{i\bar{j}}_{,\bar{l}}\phi_{i\bar{j}k}\right) \qquad (2.49)\\
\geq\ & \frac{1}{1+\phi_{k\bar{k}}}\frac{1}{1+\phi_{p\bar{p}}}|\tilde{g}_{i\bar{p}k} - T_{i\bar{p}k}|^2 - 2\frac{1}{1+\phi_{k\bar{k}}}|g_{i\bar{p}\bar{k}}\tilde{g}_{i\bar{p}k}|\\
\geq\ & \frac{1}{1+\phi_{k\bar{k}}}\frac{1}{1+\phi_{p\bar{p}}}\left(|\tilde{g}_{i\bar{p}k}| - |T_{i\bar{p}k}|\right)^2 - 2\frac{1}{1+\phi_{k\bar{k}}}|g_{i\bar{p}\bar{k}}\tilde{g}_{i\bar{p}k}|\\
\geq\ & \frac{1}{1+\phi_{k\bar{k}}}\frac{1}{1+\phi_{p\bar{p}}}\left(\frac{1}{2}|\tilde{g}_{i\bar{p}k}|\right)^2 - 2\frac{1}{1+\phi_{k\bar{k}}}|g_{i\bar{p}\bar{k}}\tilde{g}_{i\bar{p}k}|\\
\geq\ & \frac{1}{1+\phi_{k\bar{k}}}\left(\frac{|\tilde{g}_{i\bar{p}k}|}{2(1+\phi_{p\bar{p}})} - 2|g_{i\bar{p}\bar{k}}|\right)|\tilde{g}_{i\bar{p}k}|\\
\geq\ & \frac{1}{1+\phi_{k\bar{k}}}\left(\frac{\tilde{C}'(m+\Delta\phi)|g_{i\bar{p}\bar{k}}|}{2(1+\phi_{p\bar{p}})} - 2|g_{i\bar{p}\bar{k}}|\right)|\tilde{g}_{i\bar{p}k}|\\
\geq\ & \frac{1}{1+\phi_{k\bar{k}}}\left(\frac{\tilde{C}'|g_{i\bar{p}\bar{k}}|}{2} - 2|g_{i\bar{p}\bar{k}}|\right)|\tilde{g}_{i\bar{p}k}| \geq 0
\end{aligned}$$

By combining the estimate (2.45), (2.48) and (2.49), we have

$$\begin{aligned}
\frac{\tilde{\Delta}(m+\Delta\phi)}{m+\Delta\phi} &\geq \frac{1}{m+\Delta\phi}\left(-C_7 - C_8\sum_k\frac{1}{1+\phi_{k\bar{k}}} - C_6(m+\Delta\phi)\right) \qquad (2.50)\\
&\geq -C_8\sum_k\frac{1}{1+\phi_{k\bar{k}}} - (C_6 + C_7)
\end{aligned}$$

where C_7 is a constant depending on $\inf_M R_{i\bar{i}j\bar{j}}$ and Δf, C_8 is a constant depending on $\inf_M R_{i\bar{i}j\bar{j}}$ and $|g_{i\bar{p}\bar{k}}|$.

On the other hand,

$$\tilde{\Delta}\phi = \sum_i\frac{\phi_{i\bar{i}}}{1+\phi_{i\bar{i}}} = m - \sum_i\frac{1}{1+\phi_{i\bar{i}}} \qquad (2.51)$$

Then it follows from (2.43), (2.50) and (2.51) that

$$
\begin{aligned}
0 &\geq \sum_i F^{i\bar{i}} \left[\frac{(m+\Delta\phi)_{i\bar{i}}}{m+\Delta\phi} - \frac{(m+\Delta\phi)_i (m+\Delta\phi)_{\bar{i}}}{(m+\Delta\phi)^2} + B|\nabla\phi|^2_{i\bar{i}} - A\phi_{i\bar{i}} \right] \quad (2.52)\\
&\geq \sum_i -C_8 \frac{1}{1+\phi_{i\bar{i}}} - (C_6 + C_7) - Am + A\frac{1}{1+\phi_{i\bar{i}}} - C_5 m f - C_4 \\
&\quad -2|\nabla\phi||\nabla f| \\
&\geq C_9 \sum_i \frac{1}{1+\phi_{i\bar{i}}} - C_{10}
\end{aligned}
$$

if A is large enough. Where C_9 is a constant depending on C_8, A and C_{10} is a constant depending on $C_5, C_4, C_6, C_7, m, f, |\nabla f|$.

Now, let us notice the following inequality:

$$\sum_i \frac{1}{1+\phi_{i\bar{i}}} \geq \left(\frac{\sum_i (1+\phi_{i\bar{i}})}{\Pi_i (1+\phi_{i\bar{i}})} \right)^{1/(m-1)} \tag{2.53}$$

Therefore, by equation (2.1) and (2.53),

$$\sum_i \frac{1}{1+\phi_{i\bar{i}}} \geq (m+\Delta\phi)^{1/(m-1)} f^{\frac{-1}{m-1}} \tag{2.54}$$

Thus it follows (2.52) and (2.54) that $(m+\Delta\phi)(p)$ has an upper bound C depending only on $\sup_M |\Delta f|, \sup_M |\inf_M R_{i\bar{i}j\bar{j}}|, \sup_M f, \sup_M |g_{i\bar{p}k}|, A, B, m.$

$\square$

CHAPTER 3
A priori $C^{2,\alpha}$ estimate of Complex Monge-Ampère equation

The *a priori* $C^{2,\alpha}$ estimate for the classical solutions of complex Monge-Ampère equation is a crucial step in the *continuity method*. In section 3.2, we use a perturbation argument to give the Schauder estimate when the right hand side function is only $C^{\alpha}(\Omega)$ for a domain $\Omega \subset \mathbb{C}^n$.

To establish this type estimate on Hermitian manifolds, we will generalize the crucial tools: Bedford-Taylor's interior C^2 estimate and a local Calabi's C^3 estimate in Hermitian setting in section 3.3 and section 3.4, respectively.

The results in this chapter are contained in joint works with Xi Zhang [85, 83].

3.1 Introduction

We consider *a priori* $C^{2,\alpha}$ estimate for the complex Monge-Ampère equation

$$\det(u_{i\bar{j}}) = f \in C^{\alpha}. \tag{3.1}$$

Let's recall that, to prove the closeness for continuity method, we always assume in *a priori* that the solution u is smooth enough and establish the uniform $C^{2,\alpha}$ estimate for it. Then, all the higher order estimates follows by the bootstrap argument.

Generally, if the right hand side function $f(z) \in C^2(\Omega)$ (or even better), the uniform $C^{2,\alpha}$ estimate of u follows from the standard Evans-Krylov theory (see [48]). One key point in the proof is to linearize equation (3.1) and use the Harnack inequalities for the non-divergent linear equations. Actually, following the main lines in the proof of the Evans-Krylov theorem, the requirement of $f(z)$ only need to be Lipschtz (see [12]) or even $f \in W^{1,p}$ for $p > n$ by using the Harnack inequalities for divergent linear equations. But this argument does not work for $f \in C^\alpha(\Omega)$, one can not linearize the equation to follow Evans-Krylov's proof.

On the other hand, for real Monge-Ampère equation, Caffarelli [19] proved the following interior regularity:

Theorem (Caffarelli [19])**.** *Let Ω be a convex domain in $\mathbb{R}^n$ and u is a convex solution (understood in the viscosity sense) of the problem*

$$\det(u_{ij}) = f, \tag{3.2}$$

where f is positive and α-Hölder continuous for some $\alpha \in (0,1)$. Assume moreover that u is equal to 0 on $\partial\Omega$. Then $u \in C^{2,\alpha}(\Omega)$.

However, Caffarelli's proof for this regularity result relies essentially on tools in convex analysis, like the geometric interpretation of the gradient image mappings, and *good shape* results for sublevel sets which are not available in the complex setting. In a joint work with Xi Zhang, by adopting some idea from [31], we can establish the *a priori* estimate under the weak regularity of f via a perturbation method. Our result is

Theorem 3.1.1 ([85] Theorem 1)**.** *Let Ω be a domain in $\mathbb{C}^n$ and $u \in C^3(\Omega)$ is a pluri-subharmonic solution to the Monge-Ampère equation (3.1). Assume there exist positive constants K_0 and K_1 such that*

$$|u| + |Du| + |D^2 u| \le K_1, \quad K_0 \le f(z) \in C^\alpha(\Omega),$$

for some constant $0 < \alpha < 1$. Then, for any open domain $\Omega' \subset\subset \Omega$, there exists constant C depending only on K_0, K_1, n, f, α and a positive constant C, such that

$$|D^2 u|_{C^\alpha(\Omega')} \le C\Big(K_0, K_1, n, \|f\|_{C^\alpha}, \alpha, dist(\Omega', \partial\Omega)\Big) \tag{3.3}$$

Remark 3.1.1. *Note that, in the above estimate, we consider $u \in C^3(\Omega)$ which is a classical solution to the complex Monge-Ampère equation (3.1). In the later joint work with S. Dinew and Xi Zhang [35], we also proved that any $C^{1,1}$ solution (in the weak sense of current) of (3.1) is indeed $C^{2,\alpha}$ by a similar perturbation argument as the proof of the above theorem.*

The key tools used in the proof are the Bedford-Taylor's interior $C^{1,1}$ estimate (Theorem 3.1.3) and the local Calabi's C^3 estimate (Theorem 3.1.4). In the rest of this section, we recall these important results that will be used.

First, recall below the comparison principle due to Bedford and Taylor:

Theorem 3.1.2 ([6] Comparison principle). *Given a domain $\Omega \subset \mathbb{C}^n$, let u and v be $C^{1,1}(\Omega) \cap C(\bar{\Omega})$ plurisubharmonic functions.*[1] *Suppose that*

$$\begin{cases} \det(u_{i\bar{j}}) \geq \det(u_{i\bar{j}}) & \text{in } \Omega \\ u \leq v & \text{on } \partial\Omega. \end{cases}$$

Then $u \leq v$ in the whole Ω.

Building on this result and using the transitivity of the automorphism group of the unit ball in $\mathbb{C}^n$ Bedford and Taylor were able to prove the following interior estimate:

Theorem 3.1.3 ([6] Interior C^2 estimate). *Let B be the unit ball in $\mathbb{C}^n$ and let $B' \subset\subset B$ be arbitrary compact subset of B. Let $u \in PSH(B) \cap C(\bar{B})$ solve the Dirichlet problem*

$$\begin{cases} \det(u_{i\bar{j}}) = f & \text{in } B \\ u = \phi \text{ on} & \text{on } \partial B, \end{cases}$$

where $\phi \in C^{1,1}(\partial B)$ and $0 \leq f^{1/n} \in C^{1,1}(B)$. Then $u \in C^{1,1}(B)$ and moreover there exist a constant C dependent only on dist$\{B', \partial B\}$ such that

$$||u||_{C^{1,1}(B')} \leq C(||\phi||_{C^{1,1}(\partial B)} + ||f^{\frac{1}{n}}||_{C^{1,1}(B)}).$$

Remark 3.1.2. *Note that no strict positivity of f is needed. Observe also that this estimate is scaling and translation invariant, i.e. the same constant will work if we*

[1] Actually the theorem holds for merely locally bounded u and v, see [6]. Here we state it in this form for the sake of simplicity.

consider the Dirichlet problem in any ball with arbitrary radius (and suitably rescaled set B').

Finally let us mention an interior C^3 estimate which (in the real case) is due to Calabi [24] (the complex version due to Yau ([81]) for the global case and to Riebesehl and Schulz ([64]) for a local estimate).

Here we state the complex version which will be the one we shall use:

Theorem 3.1.4 ([64] A local Calabi's estimate)**.** *Let Ω be a domain in $\mathbb{C}^n$ and suppose that $u \in \mathrm{PSH}(\Omega) \cap C^4(\Omega)$ satisfy the Monge-Ampère equation*

$$\det(u_{i\bar{j}}) = f(z).$$

Then one has the interior third order estimate

$$||\nabla \Delta u||_{\Omega'} \leq C$$

where C is a constant depending only on n, $||\Delta u||_{\Omega}$, $\inf_{\Omega} f$, $||\nabla^1 f||_{\Omega}$, $||\nabla^2 f||_{\Omega}$ and $dist\{\Omega', \partial\Omega\}$.

3.2 *A priori* $C^{2,\alpha}$ estimate

In this section, we will prove the *a priori* Schauder estimate (Theorem 3.1.1) via a perturbation method by using the Bedford-Taylor's interior C^2 estimate (Theorem 3.1.3) and the local Calabi's C^3 estimate (Theorem 3.1.4). Besides these two key tools, we still need the following elementary lemmas. (The proof of these lemmas can be found in [31].)

Lemma 3.2.1. *If $u \in C^{\alpha}_{loc}(\mathbb{C}^n)$ for some $0 < \alpha \le 1$, then there exists a constant $C = C(n, \alpha, k, \rho)$, such that*

$$|D^k \tilde{u}(z,\tau)| \le C\tau^{\alpha-k}|u|_{C^{\alpha}(B_\tau(z))}$$

where $\rho \in C_0^{\infty}(\mathbb{C}^n)$ is a mollifier and

$$\tilde{u}(z,\tau) = \tau^{-2n}\int_{\mathbb{C}^n} \rho\Big(\frac{z-w}{\tau}\Big)u(w)dw$$

is the mollified function of $u(z)$. In the case of $\alpha = 0$, the same conclusion is true if $u \in C^{\alpha}_{loc}(\mathbb{C}^n)$ is replaced by $u \in L^{\infty}_{loc}(\mathbb{C}^n)$.

Lemma 3.2.2. *Suppose $u \in C(\mathbb{C}^n)$ and $R > 0$. If for any $0 < \alpha \le 1$,*

$$\sup_{w \in B_R(z), 0<\tau<R} \tau^{1-\alpha}|D\tilde{u}(w,\tau)| < \infty$$

then u is Hölder continuous at z in $B_R(z)$ and

$$|u|_{C^{\alpha}(B_R(z))} \le C \sup_{w \in B_R(z), 0<\tau<R} \tau^{1-\alpha}|D\tilde{u}(w,\tau)|,$$

for some constant C only depends on n, α and ρ, where ρ is the mollifier and $\tilde{u}(z,\tau)$ is the mollified function of $u(z)$ defined as in Lemma 3.2.1.

Lemma 3.2.3. *Suppose $\phi(t)$ is a bounded and nonnegative function on $[T_0, T_1]$ with $T_1 > T_0 \ge 0$. If for any s, t with $T_0 \le t < s \le T_1$, ϕ satisfies*

$$\phi(t) \le \theta\phi(s) + \frac{A}{(s-t)^{\alpha}} + B,$$

where θ, A, B and α are nonnegative constants, and $\theta < 1$. Then, there exits a constant C depends only on α and θ, such that

$$\phi(\rho) \leq C\Big[\frac{A}{(R-\rho)^{\alpha}} + B\Big], \quad \forall T_0 \leq \rho < R \leq T_1.$$

Proof of Theorem 3.1.1:

For any fixed point z_0, we may assume $z_0 = 0$ and $u(0) = \nabla u(0) = 0$ (if necessary, replace u by $u(z) - u(0) - D_i u(0) z_1 - D_{\bar{i}} u(0) z_{\bar{i}}$). For any ball $B_{2R}(0) \subset\subset \Omega$, consider the following Dirichlet problem:

$$\begin{cases} \det(v_{i\bar{j}}) = f(0), & \text{in } B_{2R}(0) \\ v|_{\partial B_{2R}} = u & \text{on } \partial B_{2R}(0) \end{cases} \tag{3.4}$$

Without lost of generality, we may assume $f(0) = 1$. Moreover, let

$$v^R(z) = \frac{1}{(2R)^2} v(2Rz),$$

then we just need to consider the following Dirichlet problem instead of (3.4):

$$\begin{cases} \det(v^R_{i\bar{j}}) = 1, & \text{in } B_1(0) \\ v^R|_{\partial B_1} = w^R(z) & \text{on } \partial B_1(0) \end{cases} \tag{3.5}$$

where $w^R(z) = \frac{1}{(2R)^2} u(2Rz)$.

Under the original assumption on u,

$$w^R(z) \in C^{1,1} \text{ and } |D^2 w^R(z)| \leq K, \ \forall z \in \overline{B_1(0)}.$$

Here, K is a constant depending only on K_1, n, not on R.

By Bedford-Taylor interior estimate (Theorem 3.1.3), it follows that the solution v^R of (3.63) satisfies

$$|v^R|_{C^{1,1}(B_{3/4})} < C_1, \tag{3.6}$$

where $C_1 = C(n, K_0, K_1)$ is a constant independent on R. Note that we may assume v^R is smooth (by approximating w^R in (3.63) with smooth functions and make use of regularity of complex Monge-Ampère equation in [21]). By the Calabi's interior C^3 estimate (Theorem 3.1.4),

$$|D^2 v^R(z)|_{C^\gamma(B_{\frac{1}{2}})} < C_2, \ \forall\ 0 < \gamma < 1.$$

The standard Schauder estimate implies that there exists a constant C_3 such that

$$|D^3 v^R(z)| \leq C_3(n, K_0, K_1) \quad \text{ for any } z \in B_{\frac{1}{2}}(0).$$

Rescaling back to $B_R(0)$, we get the following interior estimate for the solution of the Dirichlet problem (3.4):

$$|D^2 v(z)| \leq C, \quad |D^3 v(z)|_{B_R(0)} < \frac{C}{R} \tag{3.7}$$

for some constant C depending only on n, K_0 and K_1.

Let

$$q(z) = \frac{1}{2} u_{i\bar{j}}(0) z_i z_{\bar{j}}$$

and also denote

$$\hat{v}(z) = v(z) - q(z), \quad \hat{u}(z) = u(z) - q(z)$$

From the first inequality in (3.7), v is a smooth function in $B_R(0)$ satisfies the *uniform elliptic* complex Monge-Ampère equation

$$\det(v_{i\bar{j}}(z)) = 1, \quad \text{in } B_R(0).$$

Thus, by standard interior estimate for uniform elliptic concave equations (e.g., (6.10) in chapter 7 in [31]), for any $0 < \rho < r \leq R$,

$$|D^3 v(z)|_{B_\rho(0)} \leq \frac{C}{(r-\rho)} osc_{B_r(0)} D^2 v \leq \frac{2C}{(r-\rho)} |D^2 \hat{v}|_{B_r(0)} \tag{3.8}$$

And the interpolation inequality yields,

$$|D^3 v(z)|_{B_\rho(0)} \leq \frac{C}{r-\rho} \left[\epsilon |D^3 \hat{v}|_{B_r(0)} + \frac{C}{\epsilon^2} |\hat{v}|_{B_r(0)} \right] \tag{3.9}$$

Choosing ϵ small enough that $\frac{\epsilon C}{r-\rho} = \frac{1}{2}$. By Lemma 3.2.3,

$$|D^3 v|_{B_\rho(0)} \leq \frac{C}{(R-\rho)^3} |\hat{v}|_{B_R(0)} \leq \frac{C}{R^3} |\hat{v}|_{B_{2R}(0)} \tag{3.10}$$

From equation (3.4) and the definition of $\hat{v}$, the function $\hat{v}$ satisfies the Dirichlet problem:

$$\begin{cases} \det(\hat{v}_{i\bar{j}}(z) + u_{i\bar{j}}(0)) = f(0), & \text{in } B_{2R}(0) \\ \hat{v} = \hat{u}, & \text{on } \partial B_{2R}(0) \end{cases} \tag{3.11}$$

Also, notice that $\det(u_{i\bar{j}}(0)) = f(0)$. Thus,

$$\det(\hat{v}_{i\bar{j}}(z) + u_{i\bar{j}}(0)) - \det(u_{i\bar{j}}(0)) = 0 \implies F^{i\bar{j}} \hat{v}_{i\bar{j}}(z) = 0$$

where

$$
\begin{aligned}
F^{i\bar{j}} &= \int_0^1 \frac{\partial F}{\partial r_{i\bar{j}}}\Big(t(\hat{v}_{i\bar{j}}(z) + u_{i\bar{j}}(0)) + (1-t)u_{i\bar{j}}(0)\Big)dt \qquad (3.12)\\
&= \int_0^1 \frac{\partial F}{\partial r_{i\bar{j}}}\Big(t\hat{v}_{i\bar{j}}(z) + u_{i\bar{j}}(0)\Big)dt\\
&= \int_0^1 \frac{\partial F}{\partial r_{i\bar{j}}}\Big(tv_{i\bar{j}}(z) + (1-t)u_{i\bar{j}}(0)\Big)dt
\end{aligned}
$$

By the assumption $|D^2u(z)| < K_1$ and $u(z)$ is the solution of (3.1) with $f(z) > 0$, there exists $\Lambda > \lambda > 0$, such that

$$\lambda I \leq u_{i\bar{j}}(0) \leq \Lambda I$$

Hence, we have

$$
\begin{aligned}
&\frac{\partial F}{\partial r_{i\bar{j}}}\Big(tv_{i\bar{j}}(z) + (1-t)u_{i\bar{j}}(0)\Big) \geq (1-t)^{n-1}\lambda^{n-1}I \qquad (3.13)\\
\Longrightarrow\quad & F^{i\bar{j}}\xi_i\xi_{\bar{j}} \geq \lambda^{n-1}|\xi|^2\int_0^1 (1-t)^{n-1}dt \geq \delta_0 > 0
\end{aligned}
$$

for any unit vector $\xi = (\xi_i) \in \mathbb{C}^n$. It follows that

$$F^{i\bar{j}}\hat{v}_{i\bar{j}}(z) = 0$$

is an uniform elliptic equation. By the maximal principle,

$$|\hat{v}|_{B_{2R}(0)} \leq |\hat{u}|_{B_{2R}(0)}$$

Putting this estimate back into (3.10), we get, for any $\gamma < 1$,

$$|D^3v|_{B_\rho(0)} \leq \frac{C}{R^3}|\hat{v}|_{B_{2R}(0)} \leq \frac{C}{R^3}|\hat{u}|_{B_{2R}(0)} \leq \frac{C}{R^{1-\gamma}}|D^2u|_{C^\gamma(B_{2R}(0))} \qquad (3.14)$$

where the last inequality follows from an interpolation.

Let $w = u - v$, then w satisfies the equation

$$a^{i\bar{j}} w_{i\bar{j}} = f(z) - f(0)$$

where

$$a^{i\bar{j}} = \int_0^1 \frac{\partial F}{\partial r_{i\bar{j}}} \Big((1-t) v_{i\bar{j}}(z) + t u_{i\bar{j}}(0) \Big) dt.$$

By the same reason and estimates as (3.12, 3.13), for any unit vector $\xi = (\xi_i) \in \mathbb{C}^n$,

$$a^{i\bar{j}} \xi_i \xi_{\bar{j}} \geq \lambda^{n-1} |\xi|^2 \int_0^1 t^{n-1} dt \geq \delta_1 > 0$$

Now, by the Alexandrov-Bakelman-Pucci estimate and the condition $f(z) \in C^\alpha(\Omega)$,

$$\sup_{B_{2R}(0)} w \leq CR \Big|\Big| \frac{f(z) - f(0)}{\delta_1} \Big|\Big|_{L^{2n}(B_{2R}(0))} \leq CR^{2+\alpha} \tag{3.15}$$

where $C = C(n, \delta_1)$ is a constant independent of R.

$$|D^3 \tilde{w}|_{B_\tau(0)} \leq C\tau^{-3} \sup_{B_\tau(0)} w \leq C \Big(\frac{R}{\tau} \Big)^2 R^\alpha \tau^{-1}$$

Thus,

$$\begin{aligned} \tau^{1-\alpha} |D^3 \tilde{u}|_{B_\tau(0)} &\leq \tau^{1-\alpha} |D^3 v|_{B_\tau(0)} + \tau^{1-\alpha} |D^3 \tilde{w}|_{B_\tau(0)} \\ &\leq C \Big[\Big(\frac{\tau}{R} \Big)^{1-\alpha} |D^2 u|_{C^\alpha(B_{2R}(0))} + \Big(\frac{R}{\tau} \Big)^{2+\alpha} \Big] \end{aligned} \tag{3.16}$$

Let $2R = N\tau$, $N > 0$ is a constant to be determined. Then

$$\tau^{1-\alpha} |D^3 \tilde{u}|_{B_\tau(0)} \leq C \Big[N^{\alpha-1} |D^2 u|_{C^\alpha(B_{2R}(0))} + N^{2+\alpha} \Big] \tag{3.17}$$

Now, for any $0 < s < t < d_0 = \text{dist}\{0, \partial\Omega\}$,

- If $t - s \geq 2R,$ i.e. $\tau = \frac{2R}{N} \leq \frac{1}{N}(t-s)$,

$$\tau^{1-\beta}|D^3\tilde{u}|_{B_\tau(0)} \leq C\Big[N^{\beta-1}|D^2u|_{C^\alpha(B_{t-s}(0))} + N^{2+\alpha}\Big]$$

- If $t - s \leq 2R,$ i.e. $\tau = \frac{2R}{N} \geq \frac{1}{N}(t-s)$(we can extend $u(z)$ to outside of Ω by defining $u(z) = 0$), by Lemma 3.2.1,

$$\tau^{1-\alpha}|D^3\tilde{u}|_{B_\tau(0)} \leq \tau^{1-\alpha}\cdot C\tau^{-3}|u|_{B_{2R}(0)} \leq \frac{CN^{2+\alpha}}{(t-s)^{2+\alpha}}|u|_{B_{2R}(0)}$$

Combining above two cases together, it follows from Lemma 3.2.2 that

$$\begin{aligned}
\sup_{\tau>0}\tau^{1-\alpha}|D^3\tilde{u}|_{B_\tau(0)} &\leq C\Big[N^{\alpha-1}|D^2u|_{C^\alpha(B_{t-s}(0))} + N^{2+\alpha}\Big(1+\frac{|u|_{B_{2R}(0)}}{(t-s)^{2+\alpha}}\Big)\Big] \\
&\leq C\Big[N^{\alpha-1}\sup_{\tau>0,y\in B_{t-s}(0)}\tau^{1-\alpha}|D^3\tilde{u}|_{B_\tau(y)} + N^{2+\alpha}\Big(1+\frac{|u|_{B_{2R}(0)}}{(t-s)^{2+\alpha}}\Big)\Big].
\end{aligned}$$

In turn,

$$\sup_{\tau>0,y\in B_s(0)}\tau^{1-\alpha}|D^3\tilde{u}|_{B_\tau(y)} \leq C\Big\{N^{\alpha-1}\sup_{\tau>0,y\in B_t(0)}\tau^{1-\alpha}|D^3\tilde{u}|_{B_\tau(y)}+N^{2+\alpha}\Big(1+\frac{|u|_{B_{2R}(0)}}{(t-s)^{2+\alpha}}\Big)\Big\}$$

Set $CN^{\alpha-1} = \frac{1}{2}$, by Lemma 3.2.3,

$$\sup_{\tau>0,y\in B_\rho(0)}\tau^{1-\alpha}|D^3\tilde{u}|_{B_\tau(y)} \leq C\Big(1+\frac{|u|_{B_{2R}(0)}}{(R-\rho)^{2+\alpha}}\Big), \quad \forall\ 0<\rho<R\leq d_0.$$

Again, by Lemma 3.2.2,

$$|D^2u|_{C^\alpha(B_\rho(0))} \leq C\Big(1+\frac{|u|_{B_{2R}(0)}}{(R-\rho)^{2+\alpha}}\Big), \quad \forall\ 0<\rho<R\leq d_0. \tag{3.18}$$

where C is a constant depending only on $n, K_0, K_1, \alpha, f,$ and Ω. $\square$

3.3 *A priori* $C^{2,\alpha}$ estimate on Hermitian manifolds

The regularity estimates of the complex Monge-Ampère equation are closed related to the study of existence and uniqueness of the Kähler-Einstein metric and constant scalar curvature metric in the given Kähler class (see [30]). This motivates us to extend the results established in section 3.1 (Theorem 3.1.1 and Remark 3.1.1) from a domain in $\mathbb{C}^n$ to general Hermitian manifolds.

Let (M, ω) be a smooth Hermitian manifold and we consider the equation

$$(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = f(z)\omega^n, \tag{3.19}$$

where $0 < f(z) \in C^\infty(M)$. When the manifold (M, ω) is Kähler, that is $d\omega = 0$, one can always find a local potential function $\rho \in C^\infty(M)$ such that

$$\omega = \sqrt{-1}\partial\bar{\partial}\rho$$

Let $v = \rho + u$, we can deduce equation (3.19) to be (3.1) locally. Moreover, the key tools (Theorem 3.1.3 and Theorem 3.1.4) are also applicable. Thus, as a corollary of Theorem 3.1.1, we get the interior $C^{2,\alpha}$ estimate of v. And the estimate of u also follows since ρ only depends on ω which is smooth.

Corollary 3.3.1. *Let Ω be a domain on a Kähler manifold (M, ω). Let $u \in PSH(\omega, \Omega) \cap C^3(\Omega)$ be a solution of the Monge-Ampère equation (3.19). Suppose that $0 < f \in C^\alpha(\Omega)$ for some $0 < \alpha < 1$ and $|u| + |Du| + |D^2u| \le L$. Then, for any domain $\Omega' \subset\subset \Omega$, we have*

$$|D^2u|_{C^\alpha(\Omega')} \le C$$

for some constant depending on $n, L, \|f\|_{C^\alpha}, \alpha, dist(\Omega', \partial\Omega)$ *and the geometric quantities (curvature and torsion) with respect to* ω*.*

However, if ω is just a smooth positive $(1,1)$-form (not necessarily closed), no local potentials for ω anymore which means one can not deal with this case as on Kähler manifolds. On the other hand, Bedford-Taylor's interior estimate and the local Calabi's estimate can not be applied directly, neither. This force us to extend these two important estimates to Hermitian manifolds. Once the crucial tools established, following the lines of the proof for Theorem 3.1.1, we can prove the following corollary:

Corollary 3.3.2. *Let* Ω *be a domain in* $\mathbb{C}^n$ *and* ω *be a Hermitian form defined on* Ω*. Let* $\phi(z) \in PSH(\omega, \Omega) \cap C^3(\Omega)$ *be a solution of the Monge-Ampère equation*

$$(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n = f(z)\omega^n.$$

Suppose that $0 < f \in C^\alpha(\Omega)$ *for some* $0 < \alpha < 1$ *and* $|u| + |Du| + |D^2u| \leq L$*. Then, for any domain* $\Omega' \subset\subset \Omega$*, we have*

$$|D^2u|_{C^\alpha(\Omega')} \leq C$$

for some constant depending on $n, L, \|f\|_{C^\alpha}, \alpha, dist(\Omega', \partial\Omega)$ *and the curvature with respect to* ω*.*

This corollary gives the *a priori* $C^{2,\alpha}$ estimate of the complex Monge-Ampère equation with C^α right hand side on Hermitian manifolds. Moreover, the interior $C^{2,\alpha}$ regularity for the weak solutions mentioned in Remark 3.1.1 could also be extended to the Hermitian setting via the same method.

3.3.1 Bedford-Taylor's interior C^2 estimate on Hermitian manifolds

The interior estimate for second order derivatives is an important and difficult topic in the study of complex Monge-Ampère equation. It has many fundamental applications in complex geometric problems. In the cornerstone work of Bedford and Taylor [6], by using the transitivity of the automorphism group of the unit ball $B \subset \mathbb{C}^n$, they obtained the interior C^2-estimate (Theorem 3.1.3) for the following Dirichlet problem:

$$\begin{cases} \det(u_{i\bar{j}}) = f & \text{in } B \\ u = \phi & \text{on } \partial B, \end{cases}$$

where $\phi \in C^{1,1}(\partial B)$ and $0 \leq f^{\frac{1}{n}} \in C^{1,1}(B)$.

Unfortunately for generic domains $\Omega \subset \mathbb{C}^n$, due to the non-transitivity of the automorphism group of Ω, Bedford and Taylor's method is not applicable and the analogous estimate is still open. Here, we exploit the method of Bedford-Taylor to study the interior estimate for the Dirichlet problem of the complex Monge-Ampère equation in the unit ball in the Hermitian setting (notice that for local arguments the shape of the domain is immaterial and hence it suffices to consider balls). We consider the following Dirichlet problem:

$$\begin{cases} (\omega + \sqrt{-1}\partial\bar{\partial}u)^n = f\omega^n & \text{in } B, \\ u = \phi \quad \text{on } \partial B, \end{cases} \tag{3.20}$$

where $0 \leq f^{\frac{1}{n}} \in C^{1,1}(B)$ and ω is a smooth positive $(1,1)$-form (not necessarily closed) defined on $\bar{B}$. We denote $PSH(\omega, \Omega)$ be the set of all integrable, upper semicontinuous functions satisfying $(\omega + \sqrt{-1}\partial\bar{\partial}u) \geq 0$ in the current sense on the domain Ω. Our result is as following:

Theorem 3.3.1 ([83] Theorem 1)**.** *Let B be the unit ball on $\mathbb{C}^n$ and ω be a smooth positive $(1,1)$-form (not necessary closed) on $\bar{B}$. Let $u \in C^{1,1}(B) \cap C(\bar{B}) \cap PSH(\omega, B)$ solve the Dirichlet problem (3.20) with $\phi \in C^{1,1}(\partial B)$. Then, for arbitrary compact subset $B' \subset\subset B$, there exists a constant C dependent only on ω and* $\operatorname{dist}\{B', \partial B\}$ *such that*

$$||u||_{C^{1,1}(B')} \leq C(||\phi||_{C^{1,1}(\partial B)} + ||f^{\frac{1}{n}}||_{C^{1,1}(B)}).$$

Remark 3.3.1. *Observe that this estimate is scale and translation invariant, i.e. the same constant will work if we consider the Dirichlet problem in any ball with arbitrary radius (and suitably rescaled set B').*

In the proof of interior C^2-estimates, the comparison theorem will play the key role. Following the same idea as in [21], it's easy to see that the comparison theorem is still true for the complex Monge-Ampère equation on Hermitian manifold (M, ω).

Lemma 3.3.1. *Let $\Omega \subset M$ be a bounded set and $u, v \in C^2(\bar{\Omega})$, with $\omega + \sqrt{-1}\partial\bar{\partial}u \geq 0$, $\omega + \sqrt{-1}\partial\bar{\partial}v > 0$ be such that*

$$(\omega + \sqrt{-1}\partial\bar{\partial}v)^n \geq (\omega + \sqrt{-1}\partial\bar{\partial}u)^n$$

and

$$v \leq u \quad \text{on } \partial\Omega,$$

then $v \leq u$ in $\bar{\Omega}$.

Proof of Theorem 3.3.1:

As mentioned above, we will follow the idea of Bedford and Taylor from [6]. For $a \in B^n$, let $T_a \in$ Aut (B^n) be defined by

$$T_a(z) = \Gamma(a)\frac{z-a}{1-\bar{a}^t z},$$

where $\Gamma(a) = \frac{a^t \bar{a}}{1-v(a)} - v(a)I$ and $v(a) = \sqrt{1-|a|^2}$.

Note that $T_a(a) = 0, T_{-a} = T_a^{-1}$, and $T_a(z)$ is holomorphic in z, and a smooth function in $a \in B^n$. For any $a \in B(0, 1-\eta) = \{a : |a| < 1-\eta\}$ set

$$L(a, h, z) = T_{a+h}^{-1} T_a(z)$$

and

$$U(a, h, z) = L_1^* u(z), \quad U(a, -h, z) = L_2^* u(z),$$
$$\Phi(a, h, z) = L_1^* \phi(z), \quad \Phi(a, -h, z) = L_2^* \phi(z), \qquad \text{for } z \in \partial B^n.$$

where L_i^* means the pull-back of L_i for $i = 1, 2$ and $L_1 = L(a, h, z), L_2 = L(a, -h, z)$. Since $U(a, h, z) = \Phi(a, h, z)$ for $z \in \partial B^n$, it follows that

$$U \in C^{1,1}(B(0, 1-\eta) \times B(0, \eta) \times \partial B^n).$$

Consequently, for a suitable constant K_1, depending on $\eta > 0$, we have

$$\frac{1}{2}(U(a, h, z) + U(a, -h, z)) - K_1|h|^2 \leq U(a, 0, z) = \phi(z) \tag{3.21}$$

for all $|a| \leq 1-\eta, |h| \leq \frac{1}{2}\eta,$ and $z \in \partial B^n$. If it can be shown that $v(a, h, z)$ satisfies

$$(\omega + \sqrt{-1}\partial\bar{\partial} v)^n \geq f(z)\omega^n, \tag{3.22}$$

where

$$v(a,h,z)=\frac{1}{2}\Big[U(a,h,z)+U(a,-h,z)\Big]-K_1|h|^2+K_2(|z|^2-1)|h|^2, \tag{3.23}$$

then it follows from the comparison theorem in the Hermitian case that $v(a,h,z)\le u(z)$. Thus, if we set $a=z$, we conclude that

$$\frac{1}{2}[u(z+h)+u(z-h)]\le u(z)+(K_1+K_2)|h|^2$$

which would prove the theorem.

Let now

$$\begin{aligned} F(\omega+\sqrt{-1}\partial\bar\partial v) &= \Big(\frac{(\omega+\sqrt{-1}\partial\bar\partial v)^n}{(\sqrt{-1})^n dz^1\wedge d\bar z^1\wedge\cdots\wedge dz^n\wedge d\bar z^n}\Big)^{\frac{1}{n}} \\ &= \Big(\det(g_{i\bar j}+v_{i\bar j})\Big)^{\frac{1}{n}}, \end{aligned} \tag{3.24}$$

where $g_{i\bar j}$ is the local expression of ω under the standard coordinate $\{z_i\}_{i=1}^n$ in $\mathbb{C}^n$.

By the concavity of F, we have

$$\begin{aligned} F(\omega+\sqrt{-1}\partial\bar\partial v) &= F\Big(\omega+\frac{\sqrt{-1}}{2}(\partial\bar\partial L_1^*u+\partial\bar\partial L_2^*u+2K_2|h|^2\partial\bar\partial|z|^2)\Big) \\ &= F\Big(\frac{1}{2}(\omega-L_1^*\omega)+\frac{1}{2}(\omega-L_2^*\omega)+K_2|h|^2\sqrt{-1}\partial\bar\partial|z|^2 \\ &\quad +\frac{1}{2}(L_1^*\omega+\sqrt{-1}\partial\bar\partial L_1^*u)+\frac{1}{2}(L_2^*\omega+\sqrt{-1}\partial\bar\partial L_2^*u)\Big) \\ &\ge \frac{1}{2}F(L_1^*\omega+\sqrt{-1}\partial\bar\partial L_1^*u)+\frac{1}{2}F(L_2^*\omega+\sqrt{-1}\partial\bar\partial L_2^*u) \\ &\quad +\frac{1}{2}F\Big((\omega-L_1^*\omega)+(\omega-L_2^*\omega)+2K_2|h|^2\sqrt{-1}\partial\bar\partial|z|^2\Big). \end{aligned} \tag{3.25}$$

Since the Hermitian metric ω is smooth, one can find K_2 large enough, such that

$$(\omega-L_1^*\omega)+(\omega-L_2^*\omega)+K_2|h|^2\sqrt{-1}\partial\bar\partial|z|^2\ge 0. \tag{3.26}$$

On the other hand, since $L(a,h,z)$ is holomorphic in z, it follows from equation (3.20) that

$$\begin{aligned} F(L_1^*\omega + \sqrt{-1}\partial\bar\partial L_1^* u) &= F(L_1^*(\omega + \sqrt{-1}\partial\bar\partial u)) \\ &= \Big(\frac{L_1^*(\omega+\sqrt{-1}\partial\bar\partial u)^n}{(\sqrt{-1})^n dz^1\wedge d\bar z^1\wedge\cdots\wedge dz^n\wedge d\bar z^n}\Big)^{\frac{1}{n}} \\ &= \Big(\frac{L_1^*(f(z)\omega^n)}{(\sqrt{-1})^n dz^1\wedge d\bar z^1\wedge\cdots\wedge dz^n\wedge d\bar z^n}\Big)^{\frac{1}{n}} \\ &= F(L_1^*(f^{\frac{1}{n}}\omega)) = L_1^*(f^{\frac{1}{n}})F(L_1^*(\omega)). \end{aligned} \tag{3.27}$$

Similarly, we can get

$$F(L_2^*\omega + \sqrt{-1}\partial\bar\partial L_2^* u) = F(L_2^*(f^{\frac{1}{n}}\omega)) = L_2^*(f^{\frac{1}{n}})F(L_2^*(\omega)).$$

Thus,

$$\begin{aligned} F(\omega+\sqrt{-1}\partial\bar\partial v) &\geq \frac{1}{2}\Big(F(L_1^*(f^{\frac{1}{n}}\omega)) + F(L_2^*(f^{\frac{1}{n}}\omega))\Big) + \frac{1}{2}F(K_2|h|^2\sqrt{-1}\partial\bar\partial|z|^2) \\ &= F(f^{\frac{1}{n}}\omega) + \frac{1}{2}\Big(F(L_1^*(f^{\frac{1}{n}}\omega)) + F(L_2^*(f^{\frac{1}{n}}\omega)) - 2F(f^{\frac{1}{n}}\omega)\Big) \\ &\quad + \frac{1}{2}F(K_2|h|^2\sqrt{-1}\partial\bar\partial|z|^2). \end{aligned} \tag{3.28}$$

Again, since ω is smooth and $f^{1/n} \in C^{1,1}$, choosing K_2 large enough, we have

$$F(L_1^*(f^{\frac{1}{n}}\omega)) + F(L_2^*(f^{\frac{1}{n}}\omega)) - 2F(f^{\frac{1}{n}}\omega) \leq F(K_2|h|^2\sqrt{-1}\partial\bar\partial|z|^2). \tag{3.29}$$

Finally, we obtain

$$F(\omega+\sqrt{-1}\partial\bar\partial v) \geq F(f^{\frac{1}{n}}\omega), \tag{3.30}$$

and thus, inequality (3.22) follows. ∎

3.3.2 A local Calabi's C^3 estimate on Hermitian manifolds

Calabi's C^3-estimate for the real Monge-Ampère equation was first proved by Calabi himself in [24]. After that many mathematicians paid a lot of attention to this estimate. In Yau's work [81], he gave a detailed proof of the C^3-estimate for the complex Monge-Ampère equation on Kähler manifolds, which was generalized to the Hermitian case by Cherrier [28].

Most of these C^3-estimates are global. However, in some situations, a local C^3-estimate is needed. For example Riebesehl and Schulz [64] gave a local version of Calabi's estimate in order to study the Liouville property of Monge-Ampère equations on $\mathbb{C}^n$. And also, for the result in section 3.2 (Theorem 3.1.1, Remark 3.1.1), the local result in [64] played an important role to get the optimal value of α in the $\mathcal{C}^{2,\alpha}$ estimate of solutions to the complex Monge-Ampère equations. Thus, it is also natural to generalize this local estimate to Hermitian manifolds and hope to find some interesting geometric applications.

Let (M,ω) be a Hermitian manifold. We consider the following complex Monge-Ampère equation

$$(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^f \omega^n, \tag{3.31}$$

where $f(z) \in C^\infty(M)$.

Theorem 3.3.2 ([83] Theorem 2)**.** *Let $\phi(z) \in PSH(\omega, M) \cap \mathcal{C}^4(M)$ be a solution of the Monge-Ampère equation (3.31), satisfying*

$$|d\phi|_\omega + |\partial\bar{\partial}\phi|_\omega \leq K. \tag{3.32}$$

Let $\Omega' \subset\subset \Omega \subset M$. Then the third derivatives of $\phi(z)$ of mixed type can be estimated in the form

$$|\nabla_\omega \partial\bar{\partial}\phi|_\omega \leq C \quad \text{for } z \in \Omega',$$

where C is a constant depending on $K, |d\omega|_\omega, |R|_\omega, |\nabla R|_\omega, |T|_\omega, |\nabla T|_\omega, dist(\Omega', \partial\Omega)$ and $|\nabla^s f|_\omega$, $s = 0, 1, 2, 3$. Here ∇ is the Chern connection with respect to the Hermitian metric ω, T and R are the torsion tensor and curvature form of ∇.

From the detailed proof in Yau's paper [81] (see also [62]), in the Kähler case, we know that the quantity considered by Calabi

$$S = \tilde{g}^{j\bar{r}}\tilde{g}^{s\bar{k}}\tilde{g}^{m\bar{l}}\phi_{j\bar{k}m}\phi_{\bar{r}s\bar{l}}$$

satisfies the following elliptic inequality:

$$\tilde{\triangle} S \geq -C_1 S - C_2. \tag{3.33}$$

Here ϕ is a smooth solution of equation (3.31), $\tilde{g}$ denotes the Hermitian metric with respect to the form $\omega_\phi = \omega + \sqrt{-1}\partial\bar{\partial}\phi$, $\phi_{i\bar{j}k}$ denotes the covariant derivative with respect to the Chern connection ∇.

Riebesehl and Schulz [64] used the above elliptic inequality to get the L^p estimate for S. Then, a standard theorem for linear elliptic equations gave the L^∞ estimate. For the Hermitian case, due to the non-vanishing torsion term, the estimates are more complicated.

Thus, aim to get the local Calabi's estimate, one should establish the similar inequality as (3.33) on Hermitian manifolds. Indeed, Cherrier [28] gave such an inequality:

$$\tilde{\triangle} S \geq -C_1 S^{\frac{3}{2}} - C_2, \tag{3.34}$$

where $\tilde{\triangle}$ is the canonical Laplacian with respect to the Hermitian metric $\tilde{g}$ (i.e. $\tilde{\triangle} f = 2\tilde{g}^{i\bar{j}} f_{i\bar{j}}$), positive constants C_1 and C_2 depend on $K, |R|_\omega, |\nabla R|_\omega, |T|_\omega, |\nabla T|_\omega$, and $|\nabla^s f|_\omega$, $s = 0, 1, 2, 3$. Cherrier's proof for (3.34) follows closely to Yau's [81] computation in the Kähler case. Here, by a geometric understanding of the Calabi quantity S, similar to [62], we give a simpler proof for the elliptic inequality (3.34).

Proof of the elliptic inequality (3.34):

Let (M, J, ω) be a Hermitian manifold and ∇ denote the Chern connection with respect to the metric ω. Let locally $\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge dz^{\bar{j}}$, then the local formula for the connection 1-form reads $\theta = \partial g \cdot g^{-1}$. We also denote

$$\theta_\alpha = \partial_\alpha g \cdot g^{-1}, \quad \theta^\gamma_{\alpha\beta} = \frac{\partial g_{\beta\bar{\delta}}}{\partial z^\alpha} g^{\gamma\bar{\delta}}.$$

The torsion tensor of ∇ is defined by

$$\begin{aligned} T\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta}\right) &= \nabla_{\frac{\partial}{\partial z^\alpha}} \frac{\partial}{\partial z^\beta} - \nabla_{\frac{\partial}{\partial z^\beta}} \frac{\partial}{\partial z^\alpha} - \left[\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta}\right] \\ &= \left(\frac{\partial g_{\beta\bar{\delta}}}{\partial z^\alpha} - \frac{\partial g_{\alpha\bar{\delta}}}{\partial z^\beta}\right) g^{\gamma\bar{\delta}}. \end{aligned}$$

Notice that $T = 0 \iff \omega$ is Kähler (and ∇ is the Levi-Civita connection on M).

The curvature form of ∇ is defined by $R = \bar{\partial}\theta = d\theta - \theta \wedge \theta = \bar{\partial}(\partial g \cdot g^{-1})$. In local coordinates, we have

$$
\begin{aligned}
R^j_{i\alpha\bar{\beta}} &= -\bar{\partial}_\beta(\partial_\alpha g \cdot g^{-1})^j_i = -g^{j\bar{k}} \frac{\partial^2 g_{i\bar{k}}}{\partial z^\alpha \partial \bar{z}^\beta} + \frac{\partial g_{i\bar{k}}}{\partial z^\alpha} g^{j\bar{s}} \frac{\partial g_{t\bar{s}}}{\partial \bar{z}^\beta} g^{t\bar{k}}, \\
R_{i\bar{j}\alpha\bar{\beta}} &= g_{k\bar{j}} R^k_{i\alpha\bar{\beta}}.
\end{aligned}
$$

Note that $R^{(2,0)} = R^{(0,2)} = 0$ and $T^{(1,1)} = T^{(0,2)} = 0$, since the almost complex structure J is integrable and ∇ is the Chern connection.

Let ∇ and $\tilde{\nabla}$ denote the Chern connections corresponding to the Hermitian metrics ω and $\omega + \sqrt{-1}\partial\bar{\partial}\phi$ respectively. Define

$$h = \tilde{g} \cdot g^{-1} \tag{3.35}$$

and

$$h^j_i = \tilde{g}_{i\bar{k}} g^{j\bar{k}}, \quad (h^{-1})^j_i = g_{i\bar{k}} \tilde{g}^{j\bar{k}}.$$

In fact, h can be thought to be an endomorphism $h : T^{1,0}(M) \longrightarrow T^{1,0}(M)$, such that $\tilde{g}(X, Y) = g(h(X), Y)$.

Set

$$S = \tilde{g}^{j\bar{r}} \tilde{g}^{s\bar{k}} \tilde{g}^{m\bar{l}} \phi_{j\bar{k}m} \phi_{\bar{r}s\bar{l}}, \tag{3.36}$$

where $\phi_{j\bar{k}m} = \nabla_m \nabla_{\bar{k}} \nabla_j \phi$.

By (3.35), we have

$$\begin{aligned}\tilde{\theta} &= \partial\tilde{g}\cdot\tilde{g}^{-1} = \partial(h\cdot g)\cdot g^{-1}h^{-1} \\ &= \partial h\cdot g\cdot g^{-1}\cdot h^{-1} + h\cdot\partial g\cdot g^{-1}\cdot h^{-1} \\ &= \partial h\cdot h^{-1} + h\cdot\theta\cdot h^{-1} \\ &= \partial h\cdot h^{-1} + h\cdot\theta\cdot h^{-1} - \theta\cdot h\cdot h^{-1} + \theta \\ &= \theta + (\nabla^{1,0}h)\cdot h^{-1}.\end{aligned} \tag{3.37}$$

$$\begin{aligned}\tilde{R} &= \bar{\partial}\tilde{\theta} = \bar{\partial}(\theta + (\nabla^{1,0}h)\cdot h^{-1}) \\ &= R + \bar{\partial}((\nabla^{1,0}h)\cdot h^{-1}).\end{aligned} \tag{3.38}$$

By similar computation, we can get

$$\theta = \partial g\cdot g^{-1} = \tilde{\theta} - h^{-1}(\tilde{\nabla}^{1,0}h), \tag{3.39}$$

$$R = \tilde{R} - \bar{\partial}(h^{-1}\cdot(\tilde{\nabla}^{1,0}h)). \tag{3.40}$$

Now, using the definitions, one can see that

$$\phi_{j\bar{k}m} = (\nabla_m\tilde{g})(\partial_j, \bar{\partial}_k) = \tilde{g}_{j\bar{k},m}.$$

Thus,

$$S = \tilde{g}^{j\bar{r}}\tilde{g}^{s\bar{k}}\tilde{g}^{m\bar{l}}\phi_{j\bar{k}m}\phi_{\bar{r}s\bar{l}} = |\nabla^{1,0}\tilde{g}|^2_{\tilde{g}}. \tag{3.41}$$

On the other hand,

$$\nabla_m\tilde{g} = \nabla_m(h\cdot g) = \nabla_m h\cdot g = \Big(\frac{\partial}{\partial z^m}h + h\cdot\theta_m - \theta_m\cdot h\Big)\cdot g,$$

so

$$\begin{aligned}\tilde{\nabla}_m h &= \frac{\partial}{\partial z^m} h + h \cdot \tilde{\theta}_m - \tilde{\theta}_m \cdot h \\ &= \frac{\partial}{\partial z^m} h + h \cdot \theta_m - \theta_m \cdot h + h \cdot (\nabla_m h) \cdot h^{-1} - \nabla_m h \\ &= h \cdot (\nabla_m h) \cdot h^{-1}.\end{aligned}$$

Thus,

$$\nabla_m \tilde{g} = \nabla_m h \cdot g = h^{-1} \cdot (\tilde{\nabla}_m h) \cdot h \cdot g = h^{-1} \cdot (\tilde{\nabla}_m h) \cdot \tilde{g}.$$

Finally we end up with the formula

$$S = |\nabla^{1,0} \tilde{g}|^2_{\tilde{g}} = |h^{-1} \cdot (\tilde{\nabla}^{1,0} h)|^2_{\tilde{g}} = |\tilde{\theta} - \theta|^2_{\tilde{g}} \tag{3.42}$$

i.e. S can be thought as the $\tilde{g}$-norm of the difference between the two connection 1-forms.

Now, we can deduce the elliptic inequality:

$$\begin{aligned}\tilde{\triangle} S &= \tilde{\triangle} |h^{-1} \cdot (\tilde{\nabla}^{1,0} h)|^2_{\tilde{g}} \qquad (3.43) \\ &= \tilde{g}^{i\bar{j}} \partial_i \partial_{\bar{j}} < h^{-1} \cdot (\tilde{\nabla}^{1,0} h), \overline{h^{-1} \cdot (\tilde{\nabla}^{1,0} h)} >_{\tilde{g}} \\ &= \tilde{g}^{i\bar{j}} \partial_i \Big(< \tilde{\nabla}_{\bar{j}} (h^{-1} \cdot (\tilde{\nabla}^{1,0} h)), \overline{h^{-1} \cdot (\tilde{\nabla}^{1,0} h)} >_{\tilde{g}} \\ &\quad + < (h^{-1} \cdot (\tilde{\nabla}^{1,0} h)), \overline{\tilde{\nabla}_j h^{-1} \cdot (\tilde{\nabla}^{1,0} h)} >_{\tilde{g}} \Big) \\ &= \tilde{g}^{i\bar{j}} < \tilde{\nabla}_i \tilde{\nabla}_{\bar{j}} (h^{-1} \cdot (\tilde{\nabla}^{1,0} h)), \overline{h^{-1} \cdot (\tilde{\nabla}^{1,0} h)} >_{\tilde{g}} \\ &\quad + \tilde{g}^{i\bar{j}} < h^{-1} \cdot (\tilde{\nabla}^{1,0} h), \overline{\tilde{\nabla}_{\bar{i}} \tilde{\nabla}_j (h^{-1} \cdot (\tilde{\nabla}^{1,0} h))} >_{\tilde{g}} \\ &\quad + |\tilde{\nabla}^{1,0} (h^{-1} \cdot (\tilde{\nabla}^{1,0} h))|^2_{\tilde{g}} + |\tilde{\nabla}^{0,1} (h^{-1} \cdot (\tilde{\nabla}^{1,0} h))|^2_{\tilde{g}}.\end{aligned}$$

Using the relation $R = \tilde{R} - \bar{\partial}(h^{-1} \cdot (\tilde{\nabla}^{1,0} h))$, we have

$$\tilde{g}^{i\bar{j}} \tilde{\nabla}_i \tilde{\nabla}_{\bar{j}} (h^{-1} \cdot (\tilde{\nabla}_t^{1,0} h))_m^l = \tilde{g}^{i\bar{j}} \tilde{\nabla}_i \Big(\tilde{R}^l_{mt\bar{j}} - R^l_{mt\bar{j}} \Big). \tag{3.44}$$

Recall the Bianchi identities of curvature forms which can be found in [57](p. 135):

$$\sum (R(X,Y)Z) = \sum T(T(X,Y),Z) + (\nabla_X T)(Y,Z); \tag{3.45}$$

$$\sum \{\nabla_X R(Y,Z) + R(T(X,Y),Z)\} = 0, \tag{3.46}$$

where $X, Y, Z \in TM$ and T is the torsion of the connection ∇ (recall that ∇ is not necessarily the Levi-Civita connection), while $\sum$ denotes the cyclic sum with respect to X, Y, Z.

By the first Bianchi identity (3.45), one obtains

$$\begin{aligned} & \tilde{R}(\partial_i, \partial_{\bar{j}})\partial_m + \tilde{R}(\partial_{\bar{j}}, \partial_m)\partial_i + \tilde{R}(\partial_m, \partial_i)\partial_{\bar{j}} \\ = \; & \tilde{T}\Big(\tilde{T}(\partial_i, \partial_{\bar{j}}), \partial_m\Big) + \tilde{T}\Big(\tilde{T}(\partial_{\bar{j}}, \partial_m), \partial_i\Big) + \tilde{T}\Big(\tilde{T}(\partial_m, \partial_i), \partial_{\bar{j}}\Big) \\ & + (\tilde{\nabla}_i \tilde{T})(\partial_{\bar{j}}, \partial_m) + (\tilde{\nabla}_{\bar{j}} \tilde{T})(\partial_m, \partial_i) + (\tilde{\nabla}_m \tilde{T})(\partial_i, \partial_{\bar{j}}). \end{aligned}$$

Recall the fact that $\tilde{R}^{2,0} = \tilde{R}^{0,2} = 0$, $\tilde{T}^{1,1} = 0$ (since $\tilde{\nabla}$ is the Chern connection) and $\tilde{T}(\partial_m, \partial_i) \in T^{1,0}(M)$. Also

$$\begin{aligned} & \tilde{T}(\partial_i, \partial_{\bar{j}}) = \tilde{T}(\partial_{\bar{j}}, \partial_m) = (\tilde{\nabla}_i \tilde{T})(\partial_{\bar{j}}, \partial_m) = (\tilde{\nabla}_m \tilde{T})(\partial_i, \partial_{\bar{j}}) = 0, \\ & \tilde{R}(\partial_m, \partial_i)\partial_{\bar{j}} = 0. \end{aligned}$$

Thus,

$$\tilde{R}(\partial_i, \partial_{\bar{j}})\partial_m + \tilde{R}(\partial_{\bar{j}}, \partial_m)\partial_i = (\tilde{\nabla}_{\bar{j}}\tilde{T})(\partial_m, \partial_i).$$

By definition $\tilde{R}(\partial_i, \partial_{\bar{j}})\partial_m = \tilde{R}^l_{mi\bar{j}}\partial_l$ and $\tilde{R}^l_{mi\bar{j}} = -\tilde{R}^l_{m\bar{j}i}$, so we get

$$\tilde{R}^l_{mi\bar{j}} = \tilde{R}^l_{im\bar{j}} + \tilde{T}^l_{mi,\bar{j}}. \tag{3.47}$$

Similarly, one can also obtain

$$\tilde{R}^{\bar{l}}_{\bar{k}i\bar{j}} = \tilde{R}^{\bar{l}}_{\bar{j}i\bar{k}} + \tilde{T}^{\bar{l}}_{\bar{j}\bar{k},i}. \tag{3.48}$$

Moreover, by the second Bianchi identity (3.46) and following the same step as above we have

$$\tilde{R}^l_{mt\bar{j},i} + \tilde{R}^l_{m\bar{j}i,t} + \tilde{R}^l_{mit,\bar{j}} = -\tilde{R}(\tilde{T}(\partial_i, \partial_t), \partial_{\bar{j}}) - \tilde{R}(\tilde{T}(\partial_t, \partial_{\bar{j}}), \partial_i) - \tilde{R}(\tilde{T}(\partial_{\bar{j}}, \partial_i), \partial_t)$$

and $\tilde{R}^l_{mit,\bar{j}} = 0, \tilde{T}(\partial_t, \partial_{\bar{j}}) = \tilde{T}(\partial_{\bar{j}}, \partial_i) = 0$. Thus,

$$\tilde{R}^l_{mi\bar{j},t} = \tilde{R}^l_{mt\bar{j},i} + \tilde{T}^s_{it}\tilde{R}^l_{ms\bar{j}}. \tag{3.49}$$

Now, using the identities (3.47), (3.48) and (3.49), we obtain

$$\begin{aligned}
\tilde{g}^{i\bar{j}}\tilde{\nabla}_i\tilde{R}^l_{mt\bar{j}} &= \tilde{g}^{i\bar{j}}\tilde{R}^l_{mt\bar{j},i} = \tilde{g}^{i\bar{j}}\tilde{R}^l_{mi\bar{j},t} - \tilde{g}^{i\bar{j}}\tilde{T}^s_{it}\tilde{R}^l_{ms\bar{j}} \\
&= \tilde{g}^{i\bar{j}}\tilde{R}_{m\bar{k}i\bar{j},t}\tilde{g}^{l\bar{k}} - \tilde{g}^{i\bar{j}}\tilde{T}^s_{it}\tilde{R}^l_{ms\bar{j}} \\
&= \tilde{g}^{i\bar{j}}(\tilde{R}_{i\bar{k}m\bar{j},t} + \tilde{T}^s_{mi,\bar{j}t}\tilde{g}_{s\bar{k}})\tilde{g}^{l\bar{k}} - \tilde{g}^{i\bar{j}}\tilde{T}^s_{it}\tilde{R}^l_{ms\bar{j}} \\
&= -\tilde{g}^{i\bar{j}}\tilde{R}_{\bar{k}im\bar{j},t}\tilde{g}^{l\bar{k}} + \tilde{g}^{i\bar{j}}\tilde{T}^l_{mi,\bar{j}t} - \tilde{g}^{i\bar{j}}\tilde{T}^s_{it}\tilde{R}^l_{ms\bar{j}} \\
&= -\tilde{g}^{i\bar{j}}\tilde{R}_{\bar{j}im\bar{k},t}\tilde{g}^{l\bar{k}} - \tilde{g}^{i\bar{j}}\tilde{T}^{\bar{l}}_{\bar{j}\bar{k},mt}\tilde{g}_{i\bar{l}}\tilde{g}^{l\bar{k}} + \tilde{g}^{i\bar{j}}\tilde{T}^l_{mi,\bar{j}t} - \tilde{g}^{i\bar{j}}\tilde{T}^s_{it}\tilde{R}^l_{ms\bar{j}} \\
&= \tilde{g}^{i\bar{j}}\tilde{R}_{i\bar{j}m\bar{k},t}\tilde{g}^{l\bar{k}} - \tilde{g}^{i\bar{j}}\tilde{T}^{\bar{l}}_{\bar{j}\bar{k},mt}\tilde{g}_{i\bar{l}}\tilde{g}^{l\bar{k}} + \tilde{g}^{i\bar{j}}\tilde{T}^l_{mi,\bar{j}t} - \tilde{g}^{i\bar{j}}\tilde{T}^s_{it}\tilde{R}^l_{ms\bar{j}} \\
&= \tilde{R}^i_{im\bar{k},t}\tilde{g}^{l\bar{k}} - \tilde{g}^{i\bar{j}}\tilde{T}^{\bar{l}}_{\bar{j}\bar{k},mt}\tilde{g}_{i\bar{l}}\tilde{g}^{l\bar{k}} + \tilde{g}^{i\bar{j}}\tilde{T}^l_{mi,\bar{j}t} - \tilde{g}^{i\bar{j}}\tilde{T}^s_{it}\tilde{R}^l_{ms\bar{j}}
\end{aligned} \tag{3.50}$$

From the Monge-Ampère equation (3.31), it follows that

$$\tilde{R}^i_{im\bar{k},t} = \tilde{\nabla}_t R^i_{im\bar{k}} - \tilde{\nabla}_t f_{m\bar{k}}. \tag{3.51}$$

In the following, we denote $\epsilon = O(S^\alpha)$ if there is a constant C depending only on $K, |d\omega|_\omega, |R|_\omega, |\nabla R|_\omega, |T|_\omega, |\nabla T|_\omega$ and $|\nabla^s f|_\omega$, $s = 0, 1, 2, 3$, such that $\epsilon \leq CS^\alpha$. Note that $\tilde{\nabla}$ is $O(S^{\frac{1}{2}})$, so

$$\tilde{R}^i_{im\bar{k},t}\tilde{g}^{l\bar{k}} = O(S^{\frac{1}{2}}) + O(1). \tag{3.52}$$

For the second term in (3.50)

$$\begin{aligned}
\tilde{T}^{\bar{s}}_{\bar{j}\bar{k},mt} &= \Big((\partial_{\bar{j}} g_{n\bar{k}} - \partial_{\bar{k}} g_{n\bar{j}})\tilde{g}^{n\bar{s}}\Big)_{mt} \\
&= (T_{\bar{j}\bar{k}n}\tilde{g}^{n\bar{s}})_{mt} = \tilde{\nabla}_t\tilde{\nabla}_m T_{\bar{j}\bar{k}n}\tilde{g}^{n\bar{s}} \\
&= \tilde{\nabla}_t(\nabla_m T_{\bar{j}\bar{k}n} - (\tilde{\theta}_m - \theta_m)^l_n T_{\bar{j}\bar{k}l})\tilde{g}^{n\bar{s}} \\
&= \Big(\nabla_t(\nabla_m T_{\bar{j}\bar{k}n}) - (\tilde{\theta}_t - \theta_t)^l_m \nabla_l T_{\bar{j}\bar{k}n} - \tilde{\nabla}_t((\tilde{\theta}_m - \theta_m)^l_n)T_{\bar{j}\bar{k}l} \\
&\quad -(\tilde{\theta}_t - \theta_t)^l_n \nabla_m T_{\bar{j}\bar{k}l} - (\tilde{\theta}_m - \theta_m)^l_n(\nabla_t T_{\bar{j}\bar{k}l} - (\tilde{\theta}_t - \theta_t)^s_l T_{\bar{j}\bar{k}s})\Big)\tilde{g}^{n\bar{s}}.
\end{aligned} \tag{3.53}$$

Again, by the fact that $\tilde{\nabla}$ is $O(S^{\frac{1}{2}})$ and $|h^{-1}\cdot(\tilde{\nabla}^{1,0}h)|_{\tilde{g}}$ is also $O(S^{\frac{1}{2}})$, we have

$$|\tilde{g}^{i\bar{j}}\tilde{T}^{\bar{l}}_{\bar{j}\bar{k},mt}\tilde{g}_{i\bar{l}}\tilde{g}^{l\bar{k}}| \leq O(S^{\frac{1}{2}}) + O(S) + C|\tilde{\nabla}^{1,0}(h^{-1}\cdot(\tilde{\nabla}^{1,0}h))| + O(1). \tag{3.54}$$

Similarly, we can get the estimate for the last two terms in (3.50)

$$|\tilde{g}^{i\bar{j}}\tilde{T}^l_{mi,\bar{j}t}| \leq O(S^{\frac{1}{2}}) + O(S) + C|\tilde{\nabla}^{0,1}(h^{-1}\cdot(\tilde{\nabla}^{1,0}h))| + O(1), \tag{3.55}$$

$$|\tilde{g}^{i\bar{j}}\tilde{T}^s_{it}\tilde{R}^l_{ms\bar{j}}| \leq C|\tilde{\nabla}^{0,1}(h^{-1}\cdot(\tilde{\nabla}^{1,0}h))| + O(1). \tag{3.56}$$

Put the above estimates (3.50)-(3.56) into (3.44), we can conclude that

$$\begin{aligned}
&|\tilde{g}^{i\bar{j}}\tilde{\nabla}_i\tilde{\nabla}_{\bar{j}}(h^{-1}\cdot(\tilde{\nabla}^{1,0}_t h))^l_m| \\
\leq\ & O(S^{\frac{1}{2}}) + O(S) + C|\tilde{\nabla}^{1,0}(h^{-1}\cdot(\tilde{\nabla}^{1,0}h))| + C|\tilde{\nabla}^{0,1}(h^{-1}\cdot(\tilde{\nabla}^{1,0}h))|.
\end{aligned} \tag{3.57}$$

One the other hand,

$$\tilde{g}^{i\bar{j}}\tilde{\nabla}_{\bar{i}}\tilde{\nabla}_j(h^{-1}\cdot(\tilde{\nabla}^{1,0}h)) = \tilde{g}^{i\bar{j}}\tilde{\nabla}_j\tilde{\nabla}_{\bar{i}}(h^{-1}\cdot(\tilde{\nabla}^{1,0}h)) - (\tilde{g}^{i\bar{j}}\tilde{R}^l_{mi\bar{j}})\#(h^{-1}\cdot(\tilde{\nabla}^{1,0}h))$$

where

$$
\begin{aligned}
& (\tilde{g}^{i\bar{j}}\tilde{R}^{l}_{mi\bar{j}})\#(h^{-1}\cdot(\tilde{\nabla}^{1,0}h)) \\
= \;& \tilde{g}^{i\bar{j}}\{h^{-1}\cdot(\tilde{\nabla}^{1,0}_t h)^s_m \tilde{R}^l_{si\bar{j}} - h^{-1}\cdot(\tilde{\nabla}^{1,0}_s h)^l_m \tilde{R}^s_{ti\bar{j}} - h^{-1}\cdot(\tilde{\nabla}^{1,0}_t h)^l_s \tilde{R}^s_{mi\bar{j}}\} \\
& dz^t\otimes dz^m\otimes \tfrac{\partial}{\partial z^l}
\end{aligned}
$$

and

$$
\tilde{g}^{i\bar{j}}\tilde{R}^l_{mi\bar{j}} \;=\; \tilde{g}^{i\bar{j}}\tilde{R}^l_{im\bar{j}} + \tilde{g}^{i\bar{j}}\tilde{T}^l_{mi,\bar{j}} = \tilde{g}^{i\bar{j}}\tilde{R}_{i\bar{j}m\bar{k}}\tilde{g}^{l\bar{k}} + \tilde{g}^{i\bar{j}}\tilde{T}^{\bar{s}}_{\bar{j}\bar{k},m}\tilde{g}_{i\bar{s}}\tilde{g}^{l\bar{k}} + \tilde{g}^{i\bar{j}}\tilde{T}^l_{mi,\bar{j}}.
$$

Thus

$$
|\tilde{g}^{i\bar{j}}\tilde{R}^l_{mi\bar{j}}| \;\le\; O(S^{\frac{1}{2}}) + O(1).
$$

Hence we conclude that

$$
\begin{aligned}
& |\tilde{g}^{i\bar{j}}\tilde{\nabla}_{\bar{i}}\tilde{\nabla}_j(h^{-1}\cdot(\tilde{\nabla}^{1,0}h))| \qquad (3.58)\\
\le\; & |\tilde{g}^{i\bar{j}}\tilde{\nabla}_j\tilde{\nabla}_{\bar{i}}(h^{-1}\cdot(\tilde{\nabla}^{1,0}h))| + |(\tilde{g}^{i\bar{j}}\tilde{R}^l_{mi\bar{j}})\#(h^{-1}\cdot(\tilde{\nabla}^{1,0}h))| \\
\le\; & O(S^{\frac{1}{2}}) + O(S) + C|\tilde{\nabla}^{1,0}(h^{-1}\cdot(\tilde{\nabla}^{1,0}h))| + C|\tilde{\nabla}^{0,1}(h^{-1}\cdot(\tilde{\nabla}^{1,0}h))|.
\end{aligned}
$$

Finally, by (3.43) and (3.57), (3.58), we obtain the elliptic inequality:

$$
\tilde{\triangle} S \ge -C_1 S^{\frac{3}{2}} - C_2 \qquad (3.59)
$$

where C_1, C_2 are positive constants depending only on $K, |d\omega|_\omega, |R|_\omega, |\nabla R|_\omega, |T|_\omega, |\nabla T|_\omega$ and $|\nabla^s f|_\omega$, $s = 0, 1, 2, 3$.

$\square$

Now, we are in the place to prove the local Calabi's estimate. We will use inequality (3.34) and delicate integration by parts to get a L^p estimate for u. Then, applying the Moser's iteration technique to complete the proof.

Proof of Theorem 3.3.2:

By the assumption (3.32) for the solution of equation (3.31), we know that

$$\frac{1}{\lambda} g \leq g_\phi \leq \lambda g \qquad \text{for some constant} \quad \lambda > 0,$$

where λ depends only on K and $\|f\|_{C^0}$, and g_ϕ denotes the Hermitian metric with respect to the form $\omega_\phi = \omega + \sqrt{-1}\partial\bar{\partial}\phi$. Thus,

$$S = (g_\phi)^{j\bar{r}}(g_\phi)^{s\bar{k}}(g_\phi)^{m\bar{l}}\phi_{j\bar{k}m}\phi_{\bar{r}s\bar{l}} \leq \lambda (g_\phi)^{j\bar{r}}(g_\phi)^{s\bar{k}} g^{m\bar{l}}\phi_{j\bar{k}m}\phi_{\bar{r}s\bar{l}}. \tag{3.60}$$

On the other hand, we have

$$\begin{aligned} g_\phi^{j\bar{k}} g^{m\bar{l}}\phi_{j\bar{k}m\bar{l}} &= \left(g_\phi^{j\bar{k}} g^{m\bar{l}}\phi_{j\bar{k}m}\right)_{\bar{l}} - (g_\phi^{j\bar{k}})_{\bar{l}} g^{m\bar{l}}\phi_{j\bar{k}m} \\ &= g^{m\bar{l}} f_{m\bar{l}} + g_\phi^{j\bar{s}}\phi_{t\bar{s}\bar{l}} g_\phi^{t\bar{k}} g^{m\bar{l}}\phi_{j\bar{k}m}, \end{aligned}$$

where we used equation (3.31) in the last equality above. Thus

$$S \leq \lambda \Big[g_\phi^{j\bar{k}} g^{m\bar{l}}\phi_{j\bar{k}m\bar{l}} - \triangle f \Big]. \tag{3.61}$$

Notice that $g_\phi^{j\bar{k}} g^{m\bar{l}}\phi_{j\bar{k}m\bar{l}} = \wedge_{g_\phi}(g^{m\bar{l}}\nabla_{\bar{l}}\nabla_m(\sqrt{-1}\partial\bar{\partial}\phi))$ is a globally defined quantity. Therefore we can estimate for every sufficiently large exponents ρ, σ, and every non-negative test function $\eta(z) \in C_0^1(\Omega)$:

$$\int_\Omega S^\sigma \eta^{p+1}\frac{\omega^n}{n!} \leq \lambda \int_\Omega S^{\sigma-1}\eta^{p+1}[g_\phi^{j\bar{k}} g^{m\bar{l}}\phi_{j\bar{k}m\bar{l}} - \triangle f]\frac{\omega^n}{n!}. \tag{3.62}$$

Now, using the following identity:

$$\begin{aligned}
\phi_{j\bar{k}m\bar{l}} &= \phi_{j\bar{k}\bar{l}m} + \phi_{s\bar{k}}R^{s}_{jm\bar{l}} - \phi_{j\bar{t}}R^{\bar{t}}_{\bar{k}m\bar{l}} \\
&= \phi_{j\bar{l}m\bar{k}} + \phi_{s\bar{l}}R^{s}_{jm\bar{k}} + \phi_{s\bar{k}}R^{s}_{jm\bar{l}} - \phi_{j\bar{t}}R^{\bar{t}}_{\bar{l}m\bar{k}} - \phi_{j\bar{k}}R^{\bar{t}}_{\bar{k}m\bar{l}} \\
&= \phi_{m\bar{l}j\bar{k}} + C_1,
\end{aligned}$$

where C_1 is a constant depending on K and $|R|_\omega$. Therefore, we have

$$\begin{aligned}
\int_\Omega S^\sigma \eta^{p+1}\frac{\omega^n}{n!} &\leq \lambda\Big(\int_\Omega S^{\sigma-1}\eta^{p+1} g_\phi^{j\bar{k}} g^{m\bar{l}}\phi_{m\bar{l}j\bar{k}}\frac{\omega^n}{n!} + \\
&\qquad \int_\Omega S^{\sigma-1}\eta^{p+1}(C_1 - \triangle f)\frac{\omega^n}{n!}\Big) \\
&\leq \lambda\int_\Omega S^{\sigma-1}\eta^{p+1} g_\phi^{j\bar{k}}(\triangle\phi)_{j\bar{k}}\frac{\omega^n}{n!} + C_2\int_\Omega S^{\sigma-1}\eta^{p+1}\frac{\omega^n}{n!},
\end{aligned} \tag{3.63}$$

where C_2 is a constant depending on C_1 and $\triangle f$.

Now, using integration by parts, it is easy to see that

$$\begin{aligned}
&\int_\Omega S^{\sigma-1}\eta^{p+1} g_\phi^{j\bar{k}}(\triangle\phi)_{j\bar{k}}\frac{\omega^n}{n!} \\
= &\int_\Omega e^{-f} S^{\sigma-1}\eta^{p+1} g_\phi^{j\bar{k}}(\triangle\phi)_{j\bar{k}}\frac{\omega_\phi^n}{n!} \\
= &\int_\Omega e^{-f} S^{\sigma-1}\eta^{p+1}\sqrt{-1}\partial\bar{\partial}(\triangle\phi)\wedge\frac{\omega_\phi^{n-1}}{(n-1)!} \\
= &\int_\Omega \sqrt{-1}d(e^{-f} S^{\sigma-1}\eta^{p+1}\bar{\partial}(\triangle\phi))\wedge\frac{\omega_\phi^{n-1}}{(n-1)!} \\
&-\int_\Omega \sqrt{-1}d(e^{-f} S^{\sigma-1}\eta^{p+1})\wedge\bar{\partial}(\triangle\phi)\wedge\frac{\omega_\phi^{n-1}}{(n-1)!} \\
=: &\ I - II.
\end{aligned}$$

Next, we will estimate I and II. First,

$$\begin{aligned} I &= \int_\Omega \sqrt{-1} d(e^{-f} S^{\sigma-1}\eta^{p+1}\bar\partial(\triangle\phi)) \wedge \frac{\omega_\phi^{n-1}}{(n-1)!} \\ &= -\int_\Omega \sqrt{-1} e^{-f} S^{\sigma-1}\eta^{p+1}\bar\partial(\triangle\phi) \wedge d\omega_\phi \wedge \frac{\omega_\phi^{n-2}}{(n-2)!}. \end{aligned} \tag{3.64}$$

By the equivalence of two forms ω and ω_ϕ (i.e., the assumption (1.2) on ϕ), we know

$$\begin{aligned} \bar\partial(\triangle\phi) \wedge d\omega_\phi \wedge \frac{\omega_\phi^{n-2}}{(n-2)!} &= \bar\partial(\triangle\phi) \wedge d\omega \wedge \frac{\omega_\phi^{n-2}}{(n-2)!} \\ &\le C_3 |\bar\partial(\triangle\phi)|_{g_\phi} |d\omega|_{g_\phi} \frac{\omega^n}{n!} \\ &\le C_4 S^{\frac{1}{2}} \frac{\omega^n}{n!}, \end{aligned} \tag{3.65}$$

where C_4 is a constant depending on $|d\omega|_g$, $\|f\|_{C^0}$ and K (for the justification of the last inequality we refer to the formula of S given in the appendix). This estimate yields

$$I \le C_5 \int_\Omega S^{\sigma-\frac{1}{2}} \eta^{p+1} \frac{\omega^n}{n!} \tag{3.66}$$

for some constant C_5 dependent on ω, $\|f\|_{C^0}$ and K.

Let us now estimate the second term:

$$\begin{aligned} II &= \int_\Omega \sqrt{-1} d(e^{-f}) S^{\sigma-1}\eta^{p+1} \wedge \bar\partial(\triangle\phi) \wedge \frac{\omega_\phi^{n-1}}{(n-1)!} \\ &\quad +(\sigma-1)\int_\Omega \sqrt{-1} e^{-f} S^{\sigma-2}\eta^{p+1} dS \wedge \bar\partial(\triangle\phi) \wedge \frac{\omega_\phi^{n-1}}{(n-1)!} \\ &\quad +(p+1)\int_\Omega \sqrt{-1} e^{-f} S^{\sigma-1}\eta^{p} d\eta \wedge \bar\partial(\triangle\phi) \wedge \frac{\omega_\phi^{n-1}}{(n-1)!} \\ &\le C_6 \Big(\int_\Omega S^{\sigma-\frac{1}{2}}\eta^{p+1}\frac{\omega^n}{n!} + (\sigma-1)\int_\Omega S^{\sigma-\frac{3}{2}} |\nabla S| \eta^{p+1} \frac{\omega^n}{n!} \\ &\quad +(p+1)\int_\Omega S^{\sigma-\frac{1}{2}}\eta^p |\nabla\eta| \frac{\omega^n}{n!} \Big), \end{aligned} \tag{3.67}$$

where C_6 is a constant depending on $\|f\|_{C^1(\omega)}$ and K.

By the estimates (3.66), (3.67) and using Cauchy's inequality

$$(\sigma-1)\eta^{p+1}S^{\sigma-\frac{3}{2}}|\nabla S| \leq \frac{(\sigma-1)^2}{4\epsilon}\eta^{p+1}S^{\sigma-3}|\nabla S|^2 + \epsilon\eta^{p+1}S^{\sigma}$$

we have, for $\epsilon > 0$ small enough,

$$\begin{aligned}\int_\Omega S^\sigma\eta^{p+1}\frac{\omega^n}{n!} \;\leq\; & C_7\Big((\sigma-1)^2\int_\Omega S^{\sigma-3}|\nabla S|^2\eta^{p+1}\frac{\omega^n}{n!} + \int_\Omega S^{\sigma-1}\eta^{p+1}\frac{\omega^n}{n!} \\ & +(p+1)\int_\Omega S^{\sigma-\frac{1}{2}}\eta^p|\nabla\eta|\frac{\omega^n}{n!} + \int_\Omega S^{\sigma-\frac{1}{2}}\eta^{p+1}\frac{\omega^n}{n!}\Big),\end{aligned} \tag{3.68}$$

where C_7 is a constant depending on $|d\omega|_\omega, |R|_\omega, K, \|f\|_{C^1(\omega)}$ and $\triangle f$.

Now we are in the place to use the elliptic inequality (3.34) in the introduction. Recall that

$$\triangle_\phi S \geq -CS^{\frac{3}{2}} - C_0. \tag{3.69}$$

Multiplying by $S^{\sigma-2}\eta^{p+1}$ on both sides of the above inequality and integrating over Ω, we have

$$-C\int_\Omega S^{\sigma-\frac{1}{2}}\eta^{p+1}\frac{\omega^n}{n!} - C_0\int_\Omega S^{\sigma-2}\eta^{p+1}\frac{\omega^n}{n!} \leq \int_\Omega S^{\sigma-2}\eta^{p+1}\triangle_\phi S\frac{\omega^n}{n!}. \tag{3.70}$$

The right hand side of above inequality can be estimated as follows

$$\begin{aligned}
& \int_\Omega S^{\sigma-2}\eta^{p+1}\triangle_\phi S\frac{\omega^n}{n!} \\
= & \int_\Omega e^{-f}S^{\sigma-2}\eta^{p+1}\sqrt{-1}\partial\bar\partial S\wedge\frac{\omega_\phi^{n-1}}{(n-1)!} \\
= & \int_\Omega \sqrt{-1}d(e^{-f}S^{\sigma-2}\eta^{p+1}\bar\partial S)\wedge\frac{\omega_\phi^{n-1}}{(n-1)!}-\int_\Omega\sqrt{-1}d(e^{-f}S^{\sigma-2}\eta^{p+1})\wedge\bar\partial S\wedge\frac{\omega_\phi^{n-1}}{(n-1)!} \\
= & -\int_\Omega\sqrt{-1}e^{-f}S^{\sigma-2}\eta^{p+1}\bar\partial S\wedge d\omega\wedge\frac{\omega_\phi^{n-2}}{(n-2)!}-\sqrt{-1}\int_\Omega d(e^{-f})S^{\sigma-2}\eta^{p+1}\wedge\bar\partial S\wedge\frac{\omega_\phi^{n-1}}{(n-1)!} \\
& -(\sigma-2)\int_\Omega\sqrt{-1}e^{-f}S^{\sigma-3}\eta^{p+1}\partial S\wedge\bar\partial S\wedge\frac{\omega_\phi^{n-1}}{(n-1)!} \\
& -(p+1)\int_\Omega\sqrt{-1}e^{-f}S^{\sigma-2}\eta^{p}\partial\eta\wedge\bar\partial S\wedge\frac{\omega_\phi^{n-1}}{(n-1)!} \\
\leq & -C_8(\sigma-2)\int_\Omega S^{\sigma-3}\eta^{p+1}|\nabla S|^2\frac{\omega^n}{n!}+C_9\int_\Omega S^{\sigma-2}\eta^{p+1}|\nabla S|\frac{\omega^n}{n!} \\
& +C_9(p+1)\int_\Omega S^{\sigma-2}\eta^p|\nabla\eta||\nabla S|\frac{\omega^n}{n!}.
\end{aligned}$$

From this, we obtain,

$$\begin{aligned}
& (\sigma-2)\int_\Omega S^{\sigma-3}\eta^{p+1}|\nabla S|^2\frac{\omega^n}{n!} \\
\leq & C_{10}\Big((p+1)\int_\Omega S^{\sigma-2}\eta^p|\nabla\eta||\nabla S|\frac{\omega^n}{n!}+\int_\Omega S^{\sigma-2}\eta^{p+1}|\nabla S|\frac{\omega^n}{n!} \\
& +\int_\Omega S^{\sigma-\frac{1}{2}}\eta^{p+1}\frac{\omega^n}{n!}+\int_\Omega S^{\sigma-2}\eta^{p+1}\frac{\omega^n}{n!}\Big).
\end{aligned} \tag{3.71}$$

Now, by Cauchy's inequality again,

$$\begin{aligned}
S^{\sigma-2}\eta^{p+1}|\nabla S| & \leq \epsilon|\nabla S|^2S^{\sigma-3}\eta^{p+1}+\frac{1}{4\epsilon}\eta^{p+1}S^{\sigma-1} \\
(p+1)S^{\sigma-2}\eta^p|\nabla\eta||\nabla S| & \leq \epsilon|\nabla S|^2S^{\sigma-3}\eta^{p+1}+\frac{(p+1)^2}{4\epsilon}\eta^{p-1}S^{\sigma-1}|\nabla\eta|^2.
\end{aligned}$$

These two inequalities, together with (3.71) and (3.68) yield

$$\begin{aligned}
&\int_\Omega S^\sigma \eta^{p+1}\frac{\omega^n}{n!} \\
\leq\ & C_{11}\sigma^2(p+1)^2\Big(\int_\Omega S^{\sigma-\frac{1}{2}}\eta^{p+1}\frac{\omega^n}{n!}+\int_\Omega S^{\sigma-\frac{1}{2}}\eta^{p+1}\frac{\omega^n}{n!} \\
&+\int_\Omega S^{\sigma-\frac{1}{2}}\eta^{p}|\nabla\eta|\frac{\omega^n}{n!}+\int_\Omega S^{\sigma-2}\eta^{p+1}\frac{\omega^n}{n!}\int_\Omega S^{\sigma-1}\eta^{p-1}|\nabla\eta|^2\frac{\omega^n}{n!}\Big)
\end{aligned} \tag{3.72}$$

for $p \geq 2, \sigma \geq 4$.

Now, let $B_{R_0}(z) \subset\subset \Omega$ be a ball, and let $0 < R \leq r < t \leq R_0, R_0 - R \leq 1$. By choosing an appropriate testing function $\eta(z)$, with $0 \leq \eta \leq 1, \eta|_{B_r} = 1, \eta|_{M/B_t} = 0, |\nabla\eta| \leq \frac{C}{t-r}$, and putting $p = \sigma - 1$, we conclude that

$$\begin{aligned}
\int_{B_t(z)}(S\eta)^\sigma\frac{\omega^n}{n!} \ \leq\ & C_{12}\sigma^4\int_{B_t(z)}\Big\{\frac{1}{(t-r)^2}(S\eta)^{\sigma-2}S \\
&+\frac{1}{t-r}(S\eta)^{\sigma-1}S^{\frac{1}{2}}+(S\eta)^{\sigma-\frac{1}{2}}\eta^{\frac{1}{2}}+(S\eta)^{\sigma-1}\eta+(S\eta)^{\sigma-2}\eta^2\Big\}\frac{\omega^n}{n!}
\end{aligned} \tag{3.73}$$

By Young's inequality

$$ab \leq \epsilon\frac{a^\alpha}{\alpha}+\frac{1}{\epsilon^{\beta/\alpha}}\frac{b^\beta}{\beta}, \qquad \text{for } \epsilon > 0, \frac{1}{\alpha}+\frac{1}{\beta} = 1.$$

It follows that,

$$
\begin{aligned}
\frac{1}{t-r}(S\eta)^{\sigma-1}S^{\frac{1}{2}} &\leq \frac{\epsilon}{\frac{\sigma}{\sigma-1}}\Big((S\eta)^{\sigma-1}\Big)^{\frac{\sigma}{\sigma-1}} + \frac{1}{\epsilon^{\sigma-1}\sigma}\Big(\frac{1}{t-r}S^{\frac{1}{2}}\Big)^{\sigma}; \alpha = \frac{\sigma}{\sigma-1}, \beta = \sigma \\
\frac{1}{(t-r)^2}(S\eta)^{\sigma-2}S &\leq \frac{\epsilon}{\frac{\sigma}{\sigma-2}}\Big((S\eta)^{\sigma-2}\Big)^{\frac{\sigma}{\sigma-2}} + \frac{1}{\epsilon^{\frac{\sigma-2}{2}}\frac{\sigma}{2}}\Big(\frac{1}{(t-r)^2}S\Big)^{\frac{\sigma}{2}}; \alpha = \frac{\sigma}{\sigma-2}, \beta = \frac{\sigma}{2} \\
(S\eta)^{\sigma-2} &\leq \frac{\epsilon}{\frac{\sigma}{\sigma-4}}\Big((S\eta)^{\sigma-4}\Big)^{\frac{\sigma}{\sigma-4}} + \frac{1}{\epsilon^{\frac{\sigma-4}{4}}\frac{\sigma}{4}}\Big((S\eta)^{2}\Big)^{\frac{\sigma}{4}}; \alpha = \frac{\sigma}{\sigma-4}, \beta = \frac{\sigma}{4} \\
(S\eta)^{\sigma-1} &\leq \frac{\epsilon}{\frac{\sigma}{\sigma-2}}\Big((S\eta)^{\sigma-2}\Big)^{\frac{\sigma}{\sigma-2}} + \frac{1}{\epsilon^{\frac{\sigma-2}{2}}\frac{\sigma}{2}}\Big(S\eta\Big)^{\frac{\sigma}{2}}; \alpha = \frac{\sigma}{\sigma-2}, \beta = \frac{\sigma}{2} \\
(S\eta)^{\sigma-\frac{1}{2}} &\leq \frac{\epsilon}{\frac{\sigma}{\sigma-1}}\Big((S\eta)^{\sigma-1}\Big)^{\frac{\sigma}{\sigma-1}} + \frac{1}{\epsilon^{\sigma-1}\sigma}\Big((S\eta)^{\frac{1}{2}}\Big)^{\sigma}; \alpha = \frac{\sigma}{\sigma-1}, \beta = \sigma.
\end{aligned}
$$

All the above inequalities combined with (3.73), lead to

$$
\begin{aligned}
\int_{B_r(z)} S^{\sigma}\frac{\omega^n}{n!} &\leq C_{13}B(\epsilon)^{\sigma}\Big(\frac{1}{(t-r)^{\sigma}} + \frac{1}{(t-r)^{\frac{\sigma}{2}}} + 1\Big)\int_{B_t(z)} S^{\frac{\sigma}{2}}\frac{\omega^n}{n!} \qquad (3.74) \\
&\leq C_{13}\frac{B(\epsilon)^{\sigma}t^n}{(t-r)^{\sigma}}\Big(\int_{B_t(z)} S^{\sigma}\frac{\omega^n}{n!}\Big)^{\frac{1}{2}},
\end{aligned}
$$

where $B(\epsilon)$ is a constant depending on ϵ which comes from the coefficients in the Young's inequalities above.

Now we can apply the Meyers' lemma:

Lemma 3.3.2 ([60])**.** *If $u = u(x)$ is a nonnegative, non-decreasing continuous function in the interval $[0, d)$, which satisfies the functional inequality:*

$$
u(s) \leq \frac{c}{r-s}\Big(u(r)\Big)^{1-\alpha}, \qquad \text{forany } 0 \leq s < r < d,
$$

with α and c being constants $(0 < \alpha < 1)$, then

$$
u(0) \leq \Big(\frac{2^{\alpha+1}c}{(2^{\alpha}-1)d}\Big)^{\frac{1}{\alpha}}.
$$

Using (3.74) and applying the Meyers' lemma with $d = R_0 - R, s = r - R$ and $\phi(s) = \Big(\int_{B_{R+s}(z)} S^\sigma \frac{\omega^n}{n!}\Big)^{\frac{1}{\sigma}}$, one can obtain

$$\phi(0) \leq \frac{C^{\frac{1}{\sigma}} B(\epsilon) R_0^{\frac{1}{\sigma}}}{(R_0 - R)^2},$$

and thus

$$\Big(\int_{B_R(z)} S^\sigma \frac{\omega^n}{n!}\Big)^{\frac{1}{\sigma}} \leq \frac{(CR_0)^{\frac{1}{\sigma}}}{(R_0 - R)^2} B(\epsilon). \tag{3.75}$$

From this, we obtain the L^p estimate of S for arbitrary p. However, by tracking the constant $B(\epsilon)$, one can find that $B(\epsilon) \sim \sigma^4$. Thus, we cannot get the estimate for $\sup_\Omega S$ by letting $\sigma \longrightarrow \infty$. We should instead use the standard Moser iteration to finish the L^∞ estimate for S.

Recall that by inequality (3.71) we have

$$\begin{aligned}
& (\sigma - 2) \int_\Omega S^{\sigma-3} \eta^{p+1} |\nabla S|^2 \frac{\omega^n}{n!} \\
\leq \;& C_{10} \Big((p+1) \int_\Omega S^{\sigma-2} \eta^p |\nabla \eta| |\nabla S| \frac{\omega^n}{n!} + \int_\Omega S^{\sigma-2} \eta^{p+1} |\nabla S| \frac{\omega^n}{n!} \\
& + \int_\Omega S^{\sigma - \frac{1}{2}} \eta^{p+1} \frac{\omega^n}{n!} + \int_\Omega S^{\sigma-2} \eta^{p+1} \frac{\omega^n}{n!} \Big).
\end{aligned}$$

Coupling this with Young inequalities

$$\begin{aligned}
S^{\sigma-2} \eta^{p+1} |\nabla S| &\leq \epsilon |\nabla S|^2 S^{\sigma-3} \eta^{p+1} + \frac{1}{4\epsilon} \eta^{p+1} S^{\sigma-1}, \\
(p+1) S^{\sigma-2} \eta^p |\nabla \eta| |\nabla S| &\leq \epsilon |\nabla S|^2 S^{\sigma-3} \eta^{p+1} + \frac{(p+1)^2}{4\epsilon} \eta^{p-1} S^{\sigma-1} |\nabla S|^2
\end{aligned}$$

we have

$$
\begin{aligned}
& (\sigma-2)\int_\Omega S^{\sigma-3}\eta^{p+1}|\nabla S|^2\frac{\omega^n}{n!} && (3.76)\\
\leq\ & C_{14}\int_\Omega \frac{(p+1)^2}{\sigma-2}\eta^{p-1}S^{\sigma-1}|\nabla\eta|^2+\frac{1}{\sigma-2}S^{\sigma-1}\eta^{p+1}+S^{\sigma-\frac{1}{2}}\eta^{p+1}+S^{\sigma-2}\eta^{p+1}\frac{\omega^n}{n!}.
\end{aligned}
$$

Let now $q=\sigma-1\geq 2,$ and $p=1$, then one obtains

$$
\begin{aligned}
& \int_\Omega S^{q-2}\eta^2|\nabla S|^2\frac{\omega^n}{n!} && (3.77)\\
\leq\ & C_{15}\int_\Omega \frac{1}{(q-1)^2}S^q|\nabla\eta|^2+\frac{1}{(q-1)^2}S^q\eta^2+\frac{1}{q-1}S^{q+\frac{1}{2}}\eta^2+\frac{1}{q-1}S^{q-1}\eta^2\frac{\omega^n}{n!}.
\end{aligned}
$$

By the Sobolev inequality

$$
\Big(\int_\Omega v^{\frac{2m}{m-1}}\frac{\omega^n}{n!}\Big)^{\frac{m-1}{2m}}\leq C\Big(\int_\Omega|\nabla v|^2\frac{\omega^n}{n!}\Big)^{\frac{1}{2}}+C\Big(\int_\Omega v^2\frac{\omega^n}{n!}\Big)^{\frac{1}{2}}
$$

applied to $v=\eta S^{\frac{q}{2}}$, we conclude that

$$
\begin{aligned}
& \Big(\int_\Omega(\eta S^{\frac{q}{2}})^{\frac{2m}{m-1}}\frac{\omega^n}{n!}\Big)^{\frac{m-1}{2m}} && (3.78)\\
\leq\ & C_{16}\Big[\Big(\int_\Omega|\nabla(\eta S^{\frac{q}{2}})|^2\frac{\omega^n}{n!}\Big)^{\frac{1}{2}}+\Big(\int_\Omega(\eta S^{\frac{q}{2}})^2\frac{\omega^n}{n!}\Big)^{\frac{1}{2}}\Big]\\
\leq\ & C_{17}\Big[\Big(\int_\Omega S^q|\nabla\eta|^2+(\frac{q}{2})^2S^{q-2}\eta^2|\nabla S|^2\frac{\omega^n}{n!}\Big)^{\frac{1}{2}}+\Big(\int_\Omega\eta^2S^q\frac{\omega^n}{n!}\Big)^{\frac{1}{2}}\Big].
\end{aligned}
$$

Using inequality (3.77), we have

$$
\begin{aligned}
& \Big(\int_\Omega(\eta^2S^q)^{\frac{m}{m-1}}\frac{\omega^n}{n!}\Big)^{\frac{m-1}{m}} && (3.79)\\
\leq\ & C_{18}\int_\Omega\Big(|\nabla\eta|^2S^q+\eta^2S^q+\frac{q^2}{(q-1)^2}S^q|\nabla\eta|^2+\frac{q^2}{(q-1)^2}S^q\eta^2\\
& +\frac{q^2}{q-1}S^{q+\frac{1}{2}}\eta^2+\frac{q^2}{q-1}S^{q-1}\eta^2\Big)\frac{\omega^n}{n!}
\end{aligned}
$$

for any $q > 4$.

Again, let $B_{R_0}(z) \subset\subset \Omega$ be a ball, and let $0 < R \leq r_1 < r_2 \leq R_0, R_0 - R \leq 1$. By choosing an appropriate testing function $\eta(z)$, with $0 \leq \eta \leq 1, \eta|_{B_{r_1}} = 1, \eta|_{M/B_{r_2}} = 0, |\nabla \eta| \leq \frac{C}{r_2 - r_1}$, we conclude that

$$\begin{aligned}
& \Big(\int_{B_{r_1}(z)} S^{q\frac{m}{m-1}} \frac{\omega^n}{n!}\Big)^{\frac{m-1}{m}} \qquad (3.80)\\
\leq\ & C_{19} \int_{B_{r_2}(z)} \Big((1 + \frac{q^2}{(q-1)^2})(\frac{1}{(r_2 - r_1)^2} + 1) S^q + \frac{q^2}{q-1} S^{q+\frac{1}{2}} + \frac{q^2}{q-1} S^{q-1}\Big) \frac{\omega^n}{n!} \\
\leq\ & qC_{20}(\frac{1}{(r_2 - r_1)^2} + 1) \int_{B_{r_2}(z)} (S^q + S^{q-1} + S^{q+\frac{1}{2}}) \frac{\omega^n}{n!} \\
\leq\ & qC_{21}(\frac{1}{(r_2 - r_1)^2} + 1) \int_{B_{r_2}(z)} S^{q+\frac{1}{2}} \frac{\omega^n}{n!}.
\end{aligned}$$

Thus,

$$||S||_{L^{\frac{qm}{m-1}}(B_{r_1}(z))} \leq \Big[Cq(\frac{1}{(r_2 - r_1)^2} + 1)\Big]^{\frac{1}{q}} ||S||^{\frac{q+\frac{1}{2}}{q}}_{L^{q+\frac{1}{2}}(B_{r_2}(z))} \qquad (3.81)$$

for any $0 < R \leq r_1 < r_2 \leq R_0$.

Let $\frac{q_k m}{m-1} = q_{k+1} + \frac{1}{2}$ and $r_k = R + (R_0 - R)2^{-k}$. Then,

$$q_k = \Big(\frac{m}{m-1}\Big)^k + \frac{m-1}{2}, \qquad \text{and } |r_k - r_{k-1}| = (R_0 - R)2^{-k}$$

By (3.81), we have

$$\begin{aligned}
||S||_{L^{q_{k+1}+\frac{1}{2}}(B_{r_{k+1}}(z))} &\leq \Big[Cq_k(1 + \frac{1}{(r_{k+1} - r_k)^2}\Big]^{\frac{1}{q_k}} ||S||^{a_k}_{L^{q_k+\frac{1}{2}}(B_{r_k}(z))} \qquad (3.82)\\
&\leq q_k^{\frac{1}{q_k}} \Big(C(1 + \frac{1}{(R_0 - R)^2})\Big)^{\frac{1}{q_k}} 2^{\frac{2k}{q_k}} ||S||^{a_k}_{L^{q_k+\frac{1}{2}}(B_{r_k}(z))}.
\end{aligned}$$

where $a_k := \frac{q_k+\frac{1}{2}}{q_k}$. By iteration, it follows from (3.82) that

$$
\begin{aligned}
& ||S||_{L^{q_{k+1}+\frac{1}{2}}(B_{r_{k+1}}(z))} \qquad (3.83)\\
\leq \; & \Big[\prod_{i=1}^{k} q_i^{\frac{1}{q_i}}\Big(C(1+\frac{1}{(R_0-R)^2})\Big)^{\frac{1}{q_i}} 2^{\frac{2i}{q_i}}\Big]^{\prod_{i=1}^{k} a_i} ||S||^{\prod_{i=1}^{k} a_i}_{L^{q_1+\frac{1}{2}}(B_{r_1}(z))}.
\end{aligned}
$$

Notice that $a_k = \frac{q_k+\frac{1}{2}}{q_k} = \frac{\frac{q_{k-1}m}{m-1}}{q_k} = \frac{m}{m-1}\frac{q_{k-1}}{q_k}$, so

$$\prod_{i=1}^{k} a_i = \Big(\frac{m}{m-1}\Big)^k \frac{q_0}{q_1}\cdots\frac{q_{k-1}}{q_k} = \Big(\frac{m}{m-1}\Big)^k \frac{q_0}{q_k}$$

and thus

$$\lim_{k\to\infty}\prod_{i=1}^{k} a_i = q_0 = \frac{m+1}{2}.$$

Moreover,

$$\prod_{i=1}^{k} q_i^{\frac{1}{q_i}}\Big(C(1+\frac{1}{(R_0-R)^2})\Big)^{\frac{1}{q_i}} 2^{\frac{2i}{q_i}} = \prod_{i=1}^{k} q_i^{\frac{1}{q_i}}\Big(C(1+\frac{1}{(R_0-R)^2})\Big)^{\sum_{i=1}^{k}\frac{1}{q_i}} 2^{\sum_{i=1}^{k}\frac{2i}{q_i}}.$$

When $k\to\infty$, it is easy to show that $\sum_{i=1}^{\infty}\frac{1}{q_i} < \infty$ and $\sum_{i=1}^{\infty}\frac{2i}{q_i} < \infty$. Notice also that $\log(\prod_{i=1}^{\infty} q_i^{\frac{1}{q_i}}) < \infty$. Thus,

$$\lim_{k\to\infty}\prod_{i=1}^{k} q_i^{\frac{1}{q_i}}\Big(C(1+\frac{1}{(R_0-R)^2})\Big)^{\frac{1}{q_i}} 2^{\frac{2i}{q_i}} < \infty.$$

It follows from (3.83), by letting $k\to\infty$,

$$||S||_{L^\infty} \leq C||S||^{\frac{m+1}{2}}_{L^{q_1+\frac{1}{2}}(B_{R_0}(z))}. \qquad (3.84)$$

Choosing now $\sigma = q_1+\frac{1}{2} = \frac{m}{m-1}+\frac{m}{2}$ in (3.75), we finally obtain

$$||S||_{L^\infty} \leq C, \qquad (3.85)$$

where C is a positive constant depending on $K, |d\omega|_\omega, |R|_\omega, |\nabla R|_\omega, |T|_\omega, |\nabla T|_\omega, dist(\Omega', \partial\Omega)$ and $|\nabla^s f|_\omega$, $s = 0, 1, 2, 3$. $\square$

CHAPTER 4
On the boundary of Kähler cone

Kähler cone is the convex cone formed by all the cohomology classes that can be represented by smooth closed $(1,1)$ forms which are everywhere positive. It is interesting to investigate the question that whether any boundary class of the Kähler cone can always be representable by a smooth closed $(1,1)$ form that is everywhere nonnegative. In section 4.1, we introduce the background and recall some works done by Wu-Yau-Zheng [80] on this geometric problem where they related it to a degenerate complex Monge-Ampère equation.

In section 4.2, we deduce some geometric results for the manifolds with the non-negative quadratic orthogonal bisectional curvature condition (see Definition 4.1.1). As a direct corollary, we recover the main result in [80] which asserts that any boundary class of the Kähler cone of (M, ω) can be represented by a C^∞ closed $(1,1)$ form that is everywhere nonnegative and parallel if M satisfies the non-negative quadratic orthogonal bisectional curvature condition. A result on the rigidity of $h^{1,1}(M, \mathbb{R})$ under this curvature condition is given in section 4.3.

The results in this chapter can be found in [86].

4.1 A degenerate complex Monge-Ampère equation

Let (M, ω) be a compact Kähler manifold. Denote by

$$H(M) = H^{1,1}_{\mathbb{R}}(M) = H^{1,1}(M) \bigcap H^2(M, \mathbb{R})$$

the vector space of real $(1,1)$ classes. Write $\mathcal{K}(M)$ for the Kähler cone in $H(M)$, namely, the convex cone formed by all the cohomology classes that can be represented by smooth closed $(1,1)$ forms that are everywhere positives. We are interested in the boundary set $\mathcal{B} = \overline{\mathcal{K}} \backslash \mathcal{K}$ of the Kähler cone. We will call a (non-trivial) cohomology class α in $\mathcal{B}$ a boundary Kähler class of M. We want to know when α can be represented by a closed, smooth (1,1) form that is everywhere nonnegative.

In general, such a result does not hold without any extra conditions. This follows from the well known fact that a numerically effective line bundle on a compact complex manifold M may not admit any smooth Hermitian metric whose curvature is everywhere nonnegative. The first such example were discovered by Demailly, Peternell and Schneider in 1994 [34]. They showed that for a non-splitting extension

$$0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O} \to 0$$

on an elliptic curve C, the line bundle L dual to the tautological line bundle of the projective bundle $M^2 = \mathbb{P}(\mathcal{E})$ does not admit any smooth Hermitian metric with nonnegative curvature. In fact, they prove that any singular Hermitian metric on L with nonnegative curvature must have logarithmic singularity, so the metric cannot even by continuous. Clearly, L is a numerically effective line bundle on the ruled surface M, since $\mathcal{E}$ is numerically effective on C.

Given this failure, one need to seek extra condition to guarantee that any numerically effective line bundles, or more generally any boundary classes of the Kähler cone, will always be representable by a smooth closed $(1,1)$ form that is everywhere nonnegative. In [80], the authors found a sufficient curvature condition for this assertion to hold.

Definition 4.1.1. *A Kähler manifold (M,ω) of complex dimension $n \geq 2$ is said to have nonnegative quadratic orthogonal bisectional curvature (NQOBC) condition at $p \in M$ if: for any unitary frame $\{e_1, \cdots, e_n\}$ of $T_p^{1,0}(M)$ and any real numbers $a_1, \cdots, a_n$ we have*

$$\sum_{i,j=1}^{n} R_{i\bar{i}j\bar{j}}(a_i - a_j)^2 \geq 0. \tag{4.1}$$

And we say that a manifold (M,ω) satisfies NQOBC if it does for any point $p \in M$.

In fact, the curvature condition NQOBC comes out naturally from the Bernstein type technique and it was studied in some previous works [10, 54]. It is weaker than requiring M to have nonnegative orthogonal bisectional curvature:

$$R(V, \bar{V}, W, \bar{W}) \geq 0$$

for any orthogonal unitary pair $V, W \in T^{1,0}(M)$, while the two conditions are equivalent on complex surfaces, i.e. $\dim M = 2$.

Also note that if a product of Kähler manifolds $M_1 \times M_2$ has NQOBC then so must each factor M_1 and M_2. However, the reverse implication may be false in general: M_1 and M_2 may both have NQOBC which $M_1 \times M_2$ may not.

Under the NQOBC condition, Wu-Yau-Zheng proved the following result:

Theorem ([80]). *Let (M, ω) be a compact Kähler manifold satisfying the NQOBC condition, then any boundary class of the Kähler cone of M^n can be represented by a C^∞ closed $(1,1)$ form that is everywhere nonnegative.*

The key point of the proof in [80] is that they deduced the original geometric problem to a special form of degenerate complex Monge-Ampère equation as following.

Let ω be the Kähler form. Consider a path in $H(M)$ from $[\omega]$ to the boundary class $\alpha \in \mathcal{B}$:

$$\alpha_t := (1-t)[\omega] + t\alpha, \quad \text{for } t \in [0,1].$$

Note that for $0 \le t < 1, \alpha_t$ lies in the Kähler cone. Then

$$a(t) := \frac{1}{V} \int_M \alpha_t^n$$

is positive for $0 \le t < 1$, and $a(1) \ge 0$. Here we denote by $V = \int_M \omega^n$. Let us fix a smooth $(1,1)$ form η in the class α (certainly if η happens to be nonnegative then we are done.). Since $a(t)\omega^n$ defines a smooth volume form on M, by the result of Yau on the solvability of complex Monge-Ampère equations, there exists a smooth function u_t on M, unique up to a constant, satisfying the following equations

$$\begin{cases} (\omega + t(\eta - \omega) + dd^c u_t)^n = a(t)\omega^n \\ \omega + t(\eta - \omega) + dd^c u_t > 0, \end{cases} \tag{4.2}$$

for all $0 \le t < 1$. If there is a smooth limit of u_t, say u_1, as $t \to 1$, then $\eta + dd^c u_1$ will be a desired nonnegative $(1,1)$ form representing the boundary class α.

In fact, this feature is equivalent to the following partial differential equation problem. Let Φ be a $d-$closed $(1,1)$ form on M, such that the cohomology class represented by $\omega + t\Phi$ is positive for each $0 \leq t < 1$. In other words, for each $0 \leq t < 1$, there is a smooth function f_t on M such that

$$\omega + t\Phi + dd^c f_t > 0, \quad \text{on } M.$$

We assume that

$$\int_M (\omega + \Phi)^n = 0. \tag{4.3}$$

The goal is to find a smooth solution v to the following equations

$$\begin{cases} (\omega + \Phi + dd^c v)^n = 0 \\ \omega + \Phi + dd^c v \geq 0. \end{cases} \tag{4.4}$$

Here, we remark that the condition (4.3) is exactly the compatibility condition for equation (4.4).

In [80], the authors proved:

Theorem ([80])**.** *Let ω and Φ be given as above. Suppose that the compact Kähler manifold (M, ω) satisfies the NQOBC condition. Then there exists a smooth solution v for the problem (4.4) and (4.3).*

The approach for the above theorem is by the perturbation method. Instead solving the degenerate equation like (4.4), one consider

$$\begin{cases} (\omega + t\Phi + dd^c v_t)^n = \gamma(t)\omega^n \\ \omega + t\Phi + dd^c v_t > 0 \\ \int_M v_t \omega^n = 0 \end{cases} \tag{4.5}$$

where

$$\gamma(t) = \frac{1}{V}\int_M (\omega + t\Phi)^n, \qquad \text{for all } t \in \mathbb{R}.$$

It is easy to see that $\gamma(t)$ is a smooth function which is positive on $[0, 1)$ and (4.3) is equivalent to $\gamma(1) = 0$. To solve (4.4), it suffice to show that there is a smooth limit for a subsequence of $\{v_t\}$ as $t \to 1$.

In general, one can not hope to solve the degenerate Monge-Ampère equations like

$$\det(u_{i\bar{j}}) = f(z) \quad \text{ with } f \geq 0$$

smoothly. The counter-example can be found in [7]. In general, one can only hope for $C^{1,1}$ regularity in the degenerate case (e.g., [45]). Thus, some special properties should be involved here to insure the existence of the smooth solution to equation (4.5).

The key observation in [80] is that, under the condition NQOBC, they can prove

$$\tilde{\Delta} \log(tr_{\omega_t}\omega) \geq 0 \tag{4.6}$$

where $\tilde{\Delta}$ is the Laplacian operator with respect to the Kähler metric $\omega_t = \omega + t\Phi + dd^c v_t$. This inequality follows from the Chern-Lu formula [20, 59]. Applying the maximum principle on inequality (4.6), we conclude that $\log(tr_{\omega_t}\omega)$ depends only on t.

On the other hand, linearizing equation (4.5) with respect to t, we have

$$\tilde{\Delta}(v_t - t\dot{v}_t) = C(t) - \log(tr_{\omega_t}\omega), \tag{4.7}$$

where $C(t)$ is a constant depending only on t.

Thus, the function $v_t - t\dot{v}_t$ depends only on t by the maximum principle, since the right hand side $C(t) - \log(tr_{\omega_t}\omega)$ does. By the normalization condition,

$$\int_M \dot{v}_t \omega^n = 0.$$

Thus, we obtain

$$v_t - t\dot{v}_t = 0 \quad \text{ on } M.$$

Solving this ordinary differential equation, we conclude that

$$v_t = th$$

for some smooth function h on M with $\omega + t\Phi + dd^c h > 0$ for all $0 \leq t < 1$. Finally, let $t \to 1$, it gives a nonnegative, smooth $(1,1)$ form

$$\omega_1 = \omega + \Phi + dd^c h$$

which satisfies $\omega_1^n = 0$.

4.2 On the Boundary class of Kähler cone

In the previous section, we reviewed Wu-Yau-Zheng [80]'s main result and their proof by solving the degenerate complex Monge-Ampère equation in which the curvature condition NQOBC plays the crucial role. In fact, this condition has been studied in many old works [10, 54]. In particular, a nice result related with this curvature condition was given in [54] that,

Lemma 4.2.1 ([54])**.** *If a compact Kähler manifold M satisfies NQOBC condition, then all harmonic forms of type $(1,1)$ are parallel.*

Observation this lemma, we prove a geometric result about the cohomology classes in the Kähler cone.

Theorem 4.2.1. *Let (M^n, g) be a compact Kähler manifold satisfying the curvature condition $(*)$. Then, for any closed $(1,1)$ form Φ on (M^n, g), we can find $\tilde{\Phi} \in [\Phi]$, such that $\tilde{\Phi}$ is parallel. In particular, for any closed $(1,1)$ form α, we have*

$$[\alpha] = [\beta + \lambda_s \omega_0]$$

where β is a nonnegative closed $(1,1)$ form on the boundary of Kähler cone, λ_s is a constant depending on β and ω_0 is the Kähler form on (M^n, g).

Proof of Theorem 4.2.1.

Consider the equation

$$\sigma_1(\omega + \Phi + \partial\bar{\partial} v) = C \tag{4.8}$$

where C is some constant to be determined. The above equation is equivalent to

$$\triangle v = C - (n + Tr_g \Phi)$$

By standard theory of partial differential equation, we know $\triangle u = f$ is solvable if and only if $\int_M f = 0$. So, if we choose

$$C = n + \frac{1}{Vol(M)} \int_M \omega^{n-1} \wedge \Phi,$$

there is a smooth solution to equation (4.8).

Let $\tilde{\Phi} = \Phi + \partial\bar{\partial} v$, then the equation is

$$\sigma_1(\omega + \tilde{\Phi}) = C \tag{4.9}$$

Recall a well-known fact that a closed $(1,1)$ form Φ on a compact Kähler manifold is harmonic if and only if its trace is constant (see [8] (2.33)). It follows that $\omega + \tilde{\Phi}$ is harmonic. And thus, $\tilde{\Phi}$ is parallel by Lemma 4.2.1.

Let λ_s be the smallest eigenvalue of the $(1,1)$ form $\tilde{\Phi}$ (under the fixed orthonormal frame) and define $\phi = \tilde{\Phi} - \lambda_s \omega_0$. Then, it's easy to see that ϕ is nonnegative everywhere on M^n and on the boundary of Kähler cone.

Thus, for any closed $(1,1)$ form α, we can find a nonnegative closed $(1,1)$ form β on the boundary of Kähler cone such that

$$[\alpha] = [\beta + \lambda_s \omega_0]$$

where ω_0 is the Kähler form. $\square$

As an application of above theorem, we give a new proof for the main theorem in [80].

Corollary 4.2.1. *Let (M, ω) be a compact manifold satisfying the curvature condition NQOBC. Then any boundary class of the Kähler cone of M^n can be represented by a C^∞ closed (1,1) form that is parallel and everywhere nonnegative.*

Proof of Corollary 4.2.1.

Suppose α is a closed (1,1) form on the boundary of Kähler cone. By the definition of boundary Kähler class, we know there exists a sequence of smooth closed (1,1) forms ω_m which are everywhere positive such that

$$\omega_m \longrightarrow \alpha.$$

Now, consider the integration to be a continuous functional on the form space $H(M)$, and by the convergence, we get

$$\int_M \omega_m^k \wedge \omega_0^{n-k} \longrightarrow \int_M \alpha^k \wedge \omega_0^{n-k}, \quad k = 0, 1, \cdots, n. \tag{4.10}$$

Consequently, we have

$$\int_M \alpha^k \wedge \omega_0^{n-k} \geq 0 \quad \text{ for } \forall k = 0, 1, \cdots, n.$$

By the result of Theorem 4.2.1, there is a parallel closed $(1, 1)$ form $\tilde{\alpha} \in [\alpha]$. Thus, the eigenvalues of $\tilde{\alpha}$ are all constant on M^n and

$$0 \leq \int_M \alpha^k \wedge \omega_0^{n-k} = \int_M \tilde{\alpha}^k \wedge \omega_0^{n-k} = \sigma_k(\tilde{\alpha}) \int_M \omega_0^n. \tag{4.11}$$

In turn, we obtain the fact that

$$\sigma_k(\tilde{\alpha}) \geq 0, \quad \text{for } \forall k = 0, 1, \cdots, n.$$

This means $\tilde{\alpha}$ is in the Γ_n convex cone. So, $\tilde{\alpha} \in [\alpha]$ is nonnegative everywhere.

□

Remark 4.2.1. *Notice that, in our result, the nonnegative* $(1,1)$ *form can also be parallel on* (M, ω)*.*

4.3 Rigidity result on Hodge number

The *Hodge number* is an important topological invariant in the study of algebraic geometry. We consider a real vector space W, a Hodge structure of integer k on W is a direction sum decomposition of $W^{\mathbb{C}} = W \otimes \mathbb{C}$, the *complexification* of W, into graded pieces $W^{p,q}$ where $k = p + q$.

For a compact Kähler manifold (M, ω), let W be the tangent space of M. We defined the dimension of the complex subspaces $W^{p,q}$ to be the *Hodge number* and denote it by $h^{p,q}(M)$. And also define

$$b_k = dim H^k(M) = \sum_{p+q=k} h^{p,q}(M)$$

to be the $k-$th *Betti number* of M.

In [10], Bishop and Goldberg showed that any compact Kähler manifold M with positive bisectional curvature must have its second Betti number equal to 1. Later, Goldberg and Kobayashi [42] introduced the conception of holomorphic bisectional curvature and proved that the second Betti number of a compact connected Kähler manifold M with positive holomorphic bisectional curvature is 1.

It would be interesting to know what kind restriction the second Betti number must obey under the NQOBC condition (see Definition 4.1.1). Unfortunately, we cannot expect the second Betti number to always be 1, even when $n = 2$ case. This can be see from a very quick example: the product of $\mathbb{CP}^1$ (equipped with a sufficiently positively curved metric) with another curve always satisfies the NQOBC curvature condition. Thus, to get some rigidity, we introduce a so-called *Quasi-NQOBC* condition.

Definition 4.3.1. *We say a compact manifold (M, ω) satisfying the Quasi-NQOBC curvature condition if it satisfies: for any orthogonal tangent frame $e_1, \cdots, e_n$ at any $x \in M$, and for any real numbers $a_1, \cdots, a_n$,*

$$\sum_{i,j=1}^{n} R_{i\bar{i}j\bar{j}}(a_i - a_j)^2 \geq 0$$

is nonnegative everywhere and strictly positive at least at one point unless the real numbers $a_1 = \cdots = a_n$.

Under this Quasi-NQOBC condition, we can get some restriction on the Hodge number $h^{1,1}(M)$. Indeed, we can prove

Theorem 4.3.1 ([86] Theorem 2)**.** *Let (M, ω) be a compact Kähler manifold satisfying the Quasi-NQOBC curvature condition. Then, $h^{1,1}(M, \mathbb{R}) = 1$.*

The key observation for Theorem 4.3.1 is that we notice that the constant rank theorem in [9] still holds under the curvature condition Quasi-NQOBC for some special equations.

Proof of Theorem 4.3.1.

Let ξ be a closed (1,1) form on a compact Kähler manifold M^n. For any fixed point on M^n, we can choose a local coordinate $\{z_1, \cdots, z_n\}$ such that

$$g_{\alpha\bar{\beta}} = \delta_{\alpha\beta}, \ \frac{\partial g_{\alpha\bar{\beta}}}{\partial z_i} = \frac{\partial g_{\alpha\bar{\beta}}}{\partial z_{\bar{i}}} = 0$$

and $\xi = \xi_{i\bar{j}} dz^i \wedge dz^{\bar{j}}$. Let $\phi = \sigma_2(g^{i\bar{l}}\xi_{l\bar{j}})$ and $W = (w_{i\bar{j}}) = (g^{i\bar{l}}\xi_{l\bar{j}})$, we have

$$\begin{aligned} \phi_\alpha &= \sigma_1(W|i)(g^{i\bar{l}}\xi_{l\bar{i}})_\alpha \qquad (4.12)\\ \phi_{\alpha\bar{\beta}} &= \sigma_1(W|i)(g^{i\bar{l}}\xi_{l\bar{i}})_{\alpha\bar{\beta}} + (g^{i\bar{l}}\xi_{l\bar{i}})_\alpha (g^{i\bar{k}}\xi_{k\bar{i}})_{\bar{\beta}} - (g^{i\bar{l}}\xi_{l\bar{i}})_\alpha (g^{i\bar{k}}\xi_{k\bar{i}})_{\bar{\beta}} \end{aligned}$$

By the proof of Theorem 4.2.1, we know that there is a closed $(1,1)$ form (which we still denote by ξ) in $[\xi]$ such that

$$F(\xi) = \sigma_1(g^{i\bar{l}}\xi_{l\bar{j}}) = C \qquad (4.13)$$

where C is some constant.

By (4.13), we have

$$\begin{aligned} F^{\alpha\bar{\beta}} = \frac{\partial \sigma_1(\xi)}{\partial \xi_{\alpha\beta}} &= g^{\alpha\bar{\beta}} = \delta_{\alpha\beta} \qquad (4.14)\\ \sigma_1(g^{\alpha\bar{\eta}}\xi_{\eta\bar{\beta}}) = C &\Longrightarrow \sum_i^n \xi_{i\bar{i},\alpha} = 0 \\ \sigma_1(g^{\alpha\bar{\eta}}\xi_{\eta\bar{\beta}}) = C &\Longrightarrow g^{\alpha\bar{\eta}}_{,i\bar{i}}\xi_{\eta\bar{\beta}} + g^{\alpha\bar{\eta}}\xi_{\eta\bar{\beta},i\bar{i}} = 0 \qquad (4.15)\\ &\Longrightarrow \xi_{\alpha\bar{\alpha},i\bar{i}} = -g^{\alpha\bar{\eta}}_{,i\bar{i}}\xi_{\eta\bar{\alpha}} = -g^{\alpha\bar{\alpha}}_{,i\bar{i}}\xi_{\alpha\bar{\alpha}} \end{aligned}$$

Thus, by directly computation, we have

$$
\begin{aligned}
F^{\alpha\bar{\beta}}\phi_{\alpha\bar{\beta}} &= \sigma_1(W|i)(g^{i\bar{l}}\xi_{l\bar{i}})_{\alpha\bar{\alpha}} + (g^{i\bar{l}}\xi_{l\bar{i}})_{\alpha}(g^{i\bar{k}}\xi_{k\bar{i}})_{\bar{\alpha}} - (g^{i\bar{l}}\xi_{l\bar{i}})_{\alpha}(g^{i\bar{k}}\xi_{k\bar{i}})_{\bar{\alpha}} && (4.16)\\
&= \sigma_1(W|i)(g^{i\bar{i}}_{\alpha\bar{\alpha}}\xi_{i\bar{i}} + \xi_{i\bar{i},\alpha\bar{\alpha}}) + \xi_{i\bar{i},\alpha}\xi_{j\bar{j},\bar{\alpha}} - \xi_{i\bar{j},a}\xi_{i\bar{j},\bar{\alpha}} \\
&= \sigma_1(W|i)(g^{i\bar{i}}_{\alpha\bar{\alpha}}\xi_{i\bar{i}} + \xi_{\alpha\bar{\alpha},i\bar{i}}) + \xi_{i\bar{i},\alpha}\xi_{j\bar{j},\bar{\alpha}} - \xi_{i\bar{j},a}\xi_{i\bar{j},\bar{\alpha}} && (4.17)\\
&= \sigma_1(W|i)(g^{i\bar{i}}_{\alpha\bar{\alpha}}\xi_{i\bar{i}} - g^{\alpha\bar{\alpha}}_{,i\bar{i}}\xi_{\alpha\bar{\alpha}}) - \sum_{i,j}|\nabla\xi_{i\bar{j}}|^2 + (\sum_{i=1}^{n}\xi_{i\bar{i},\alpha})^2 \\
&= \sigma_1(W|i)(-R_{i\bar{i}\alpha\bar{\alpha}}\xi_{i\bar{i}} + R_{i\bar{i}\alpha\bar{\alpha}}\xi_{\alpha\bar{\alpha}}) - \sum_{i,j}|\nabla\xi_{i\bar{j}}|^2 \\
&= -\frac{1}{2}\sum_{i\alpha}R_{i\bar{i}\alpha\bar{\alpha}}(\sigma_1(W|i) - \sigma_1(W|\alpha))(\xi_{i\bar{i}} - \xi_{\alpha\bar{\alpha}}) - \sum_{i,j}|\nabla\xi_{i\bar{j}}|^2 \\
&= -\frac{1}{2}\sum_{i\alpha}R_{i\bar{i}\alpha\bar{\alpha}}(\xi_{i\bar{i}} - \xi_{\alpha\bar{\alpha}})^2 - \sum_{i,j}|\nabla\xi_{i\bar{j}}|^2
\end{aligned}
$$

In equality (4.17) above, we have used the fact that ξ is a closed $(1,1)$ form which gives us

$$\xi_{\alpha\bar{\alpha},i} = \xi_{i\bar{\alpha},\alpha}, \quad \xi_{\alpha\bar{\alpha},\bar{i}} = \xi_{\alpha\bar{i},\bar{\alpha}} \Longrightarrow \xi_{i\bar{i},\alpha\bar{\alpha}} = \xi_{\alpha\bar{i},i\bar{\alpha}} = \xi_{\alpha\bar{i},\bar{\alpha}i} = \xi_{\alpha\bar{\alpha},i\bar{i}} \tag{4.18}$$

From equality (4.16), we can see if the compact Kähler manifold M^n satisfies the curvature condition Quasi-NQOBC, then

$$F^{\alpha\bar{\beta}}\phi_{\alpha\bar{\beta}} = -\frac{1}{2}\sum_{i,\alpha}R_{i\bar{i}\alpha\bar{\alpha}}(\xi_{i\bar{i}} - \xi_{\alpha\bar{\alpha}})^2 - \sum_{i,j}|\nabla\xi_{i\bar{j}}|^2 \leq 0 \tag{4.19}$$

By strong maximal principle, we have ϕ=constant. Thus, $\sigma_1(g^{i\bar{l}}\xi_{l\bar{j}})$=constant and $\sigma_2(g^{i\bar{l}}\xi_{l\bar{j}})$= constant, from which we can get $\xi_{i\bar{i}}$ are constants on M^n. Furthermore, if the manifold satisfies the Quasi-NQOBC condition, by (4.19), we can see that

$\xi_{i\bar{i}} = \xi_{\alpha\bar{\alpha}}$ for $i, \alpha = 1, \cdots, n$. Thus, $\xi = \lambda\omega_0$, where λ is a constant. In turn, we conclude that

$$h^{1,1}(M) = 1.$$

$\square$

Remark 4.3.1. *For the results in [10], [42], [54], the restriction of the bisectional curvature makes the Ricci tensor of M to be positive. So there is no nontrivial holomorphic 2-forms on M (cf. Bochner [15]), i.e. $H^{2,0}(M) = H^{0,2}(M) = 0$. Thus, $b_2(M) = dim H^2(M) = dim h^{1,1}(M)$.*

CHAPTER 5

Generalized Kähler-Einstein metrics and Energy functionals

The existence of canonical metrics in any given Kähler classes was conjectured by Calabi in 1950. By Aubin [2] and Yau [81], we know that $[\omega]$ admits a Kähler-Einstein metric when the first Chern class $c_1(M) < 0$ or $c_1(M) = 0$ and $[\omega] = -kc_1(M)$. For the case $c_1(M) > 0$, the existence question is still open. In a remarkable work [71], Tian introduced the $\mathcal{K}$ stability and showed that the existence of Kähler-Einstein metrics is equivalent to the properness of corresponding energy function. In this chapter, we consider the generalized Kähler-Einstein metrics which is the case that $[\omega]$ is not proportional to $c_1(M)$.

In Section 5.2, we give some preliminary results about energy functionals and prove the existence result for generalized Kähler-Einstein metric, i.e. the properness of twisted $\mathcal{K}$ energy implies the existence of generalized Kähler-Einstein. In section 5.3, we obtain a Moser-Trudinger type inequality on the generalized Kähler-Einstein manifolds. As an application of this inequality, we get a strictly slope stability result in section 5.4.

The results in this chapter can be found in [84], a joint work with a visiting professor Xi Zhang. The problem was suggested by him and came out from our discussion.

5.1 Introduction

An important problem in Kähler geometry is that of finding a canonical kähler metric in a given Kähler class. By Aubin and Yau's work ([2], [81]), we know that $[\omega]$ admits a Kähler-Einstein metric when $c_1(M) = 0$, or $c_1(M) < 0$ and $[\omega] = -kc_1(M)$. For the remained case, i.e. $c_1(M) > 0$, the existence question is still open. Important progress was made by Tian [69, 70, 71], Tian and Yau [75], Siu [65], Ding [36] and others. In [71], Tian introduce the notion of $\mathcal{K}$ stability and show that the existence of Kähler-Einstein metrics is equivalent to the properness of corresponding energy functional. For the case that the given Kähler class is not proportion to the first Chern class, we can consider constant scalar curvature Kähler metrics or more general extremal Kähler metrics which was first raised by Calabi [26]. It is well known that the existence of canonical Kähler metrics is related to stability in the sense of Hilbert schemes and geometric invariant theory by a conjecture of Yau [82], Tian [37] and Donaldson [38].

Let (M, J) be a m-dimensional complex manifold, $[\omega_0] \in H^{1,1}(M, C) \cap H^2(M, R)$ be a Kähler class on (M, J), and

$$[\alpha] = 2\pi c_1(M) - k[\omega_0],$$

where k is a constant. Fixing a closed $(1,1)$-form $\theta \in [\alpha]$, we consider the following genenralized Kähler-Einstein equation

$$\rho(\omega) - \theta = k\omega, \tag{5.1}$$

where $\rho(\omega)$ is the Ricci form of the Kähler metric $\omega \in [\omega_0]$.

If $\theta \equiv 0$, the above equation (5.1) is just the Kähler-Einstein equation. A Kähler metrics ω satisfying (5.1) will be called by a *generalized Kähler-Einstein metric*. Let's denote $\mathcal{H}_{\omega_0}$ to be the set of all smooth strictly ω_0-plurisubharmonic functions, i.e.

$$\mathcal{H}_{\omega_0} = \{\varphi \in C^\infty(M) : \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi \geq 0\}, \tag{5.2}$$

and $\mathcal{K}_{\omega_0}$ to be the set of all Kähler forms on M cohomologous to ω_0. It is easy to see that solving the above generalized Kähler-Einstein equation (5.1) is equivalent to solving the following complex Monge-Ampère equation,

$$\frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^m}{\omega_0^m} = \exp(h_{\omega_0} - k\varphi), \tag{5.3}$$

where $\varphi \in \mathcal{H}_{\omega_0}$ and h_{ω_0} is a smooth function which satisfies

$$\rho(\omega_0) - \theta = k\omega_0 + \sqrt{-1}\partial\bar{\partial}h_{\omega_0}$$

and

$$\int_M \exp(h_{\omega_0})(\omega_0)^m = \int_M (\omega_0)^m = V.$$

If $k \leq 0$, by Aubin and Yau's work ([2], [81]), the above complex Monge-Ampère equation (5.3) can be solved. In this chapter, we consider the case $k > 0$, there should be obstructions to admit generalized Kähler-Einstein metrics. Through the work of Bando and Mabuchi[5], Ding and Tian[37], Tian[71], Donaldson [38] and others, it is well known that the Mabuchi $\mathcal{K}$-energy is very useful in Kähler geometry. Let's recall the following twisted $\mathcal{K}$-energy which was first introduced by Song and Tian in [66].

Definition 5.1.1. *For every $(\varphi_0, \varphi_1) \in \mathcal{H}_{\omega_0} \times \mathcal{H}_{\omega_0}$, we define*

$$\mathcal{M}_\theta(\varphi_0, \varphi_1) = -\frac{1}{V}\int_0^1 \int_M \dot{\varphi}_t (S(\omega_t) - \Lambda_{\omega_{\varphi_t}}\theta - \bar{S}_\theta)\omega_{\varphi_t}^m dt, \tag{5.4}$$

where $\{\varphi_t | 0 \leq t \leq 1\}$ is an arbitrary piecewise smooth path in $\mathcal{H}_{\omega_0}$ such that $\varphi_t|_{t=0} = \varphi_0$ and $\varphi_t|_{t=1} = \varphi_1$, $S(\omega_{\varphi_t})$ is the scalar curvature of ω_{φ_t} , $\Lambda_{\omega_{\varphi_t}}$ is the contraction with ω_{φ_t} and $\bar{S}_\theta = \frac{1}{V}\int_M m(2\pi c_1(M) - [\theta]) \cup [\omega_0]^{m-1}$. For every $\varphi \in \mathcal{H}_{\omega_0}$, we define

$$\mathcal{V}_{\theta,\omega_0}(\varphi) = \mathcal{M}_\theta(0, \varphi). \tag{5.5}$$

Song and Tian ([66], proposition 6.1) have shown that the integral in (5.4) is independent of the choice of the path φ_t. Thus, $\mathcal{M}_\theta$ is well defined. By the definition, it is easy to check that $\mathcal{M}_\theta$ satisfies the 1-cocycle condition, i.e.

$$\mathcal{M}_\theta(\varphi_0, \varphi_1) + \mathcal{M}_\theta(\varphi_1, \varphi_0) = 0, \tag{5.6}$$

$$\mathcal{M}_\theta(\varphi_0, \varphi_1) + \mathcal{M}_\theta(\varphi_1, \varphi_2) + \mathcal{M}_\theta(\varphi_2, \varphi_0) = 0, \tag{5.7}$$

and

$$\mathcal{M}_\theta(\varphi_0 + C_0, \varphi_1 + C_1) = \mathcal{M}_\theta(\varphi_1, \varphi_0), \tag{5.8}$$

for all $\varphi_0, \varphi_1, \varphi_2 \in \mathcal{H}_{\omega_0}$ and all $C_0, C_1 \in R$. By the above properties, we know that $\mathcal{M}_\theta$ (or $\mathcal{V}_{\theta,\omega_0}$) can also be defined on the space $\mathcal{K}_{\omega_0} \times \mathcal{K}_{\omega_0}$ ($\mathcal{K}_{\omega_0}$).

We say the $\mathcal{K}$-energy functional $\mathcal{V}_{\theta,\omega_0}$ is *proper* if

$$\limsup_{i\to+\infty} \mathcal{V}_{\theta,\omega_0}(\varphi_i) = +\infty \text{ whenever } \lim_{i\to+\infty} J_{\omega_0}(\varphi_i) = +\infty,$$

where $\varphi_i \in \mathcal{H}_{\omega_0}$ and J_{ω_0} is the Aubin's functional (see (1.26)).

By using Tian's method in [71], we can prove that the existence of generalized Kähler-Einstein metric is closely related to the properness of the twisted $\mathcal{K}$-energy functional. Moreover, we also follow the discussion in Phong-Song-Sturm-Weinkove's ([63]) to deduce a Moser-Trudinger type inequality. In fact, we obtain the following theorem.

Theorem 5.1.1 ([84]). *Let (M, ω_0) be a compact Kähler manifold, and $\theta \in [\alpha] = 2\pi c_1(M) - k[\omega_o]$ is a real closed semipositive $(1,1)$-form, where $k > 0$. If $\mathcal{V}_{\theta,\omega_0}$ is proper then there must exists a generalized Kähler-Einstein metric $\omega_{GKE} \in \mathcal{K}_{\omega_0}$.*

Furthermore, assuming that the twisting form θ is strictly positive at a point or M admits no nontrivial Hamiltonian holomorphic vector field, if there exists a generalized Kähler-Einstein metric in $\omega_{GKE} \in \mathcal{K}_{\omega_0}$, then $\mathcal{V}_{\theta,\omega_0}$ must be proper.

In fact, there exist uniform positive constants C_2, C_3 depending only on k and the geometry of (M, ω_0), such that

$$\mathcal{V}_{\theta,\omega_0}(\varphi) \geq C_2 J_{\omega_0}(\varphi) - C_3, \tag{5.9}$$

for all $\varphi \in \mathcal{H}_{\omega_0}$.

5.2 Generalized Kähler-Einstein metric

In this section, we prove some properties for the twisted $\mathcal{K}-$energy functional defined in [66] and show the relation between the properness of this functional and the existence of generalized Kähler-Einstein metric.

5.2.1 Twisted $\mathcal{K}$-energy functional

Let (M, ω_0) be a Kähler manifold, and $[\alpha] \in H^{1,1}(M, C) \cap H^2(M, R)$. Fixed a real closed $(1,1)$ form $\theta \in [\alpha]$, the twisted $\mathcal{K}$-energy functional can be expressed by

$$\begin{aligned} \mathcal{M}_\theta(\varphi_0, \varphi_1) &= -\frac{1}{V}\int_M \sum_{j=0}^{m-1} (\varphi_1 - \varphi_0)(\rho(\omega_{\varphi_0}) - \theta) \wedge \omega_{\varphi_0}^j \wedge \omega_{\varphi_1}^{m-j-1} \\ &+ \frac{\bar{S}_\theta}{(m+1)V} \sum_{j=0}^{m} \int_M (\varphi_1 - \varphi_0) \omega_{\varphi_0}^j \wedge \omega_{\varphi_1}^{m-j} \\ &+ \frac{1}{V}\int_M \log \frac{\omega_{\varphi_1}^m}{\omega_{\varphi_0}^m} \omega_{\varphi_1}^m, \end{aligned} \tag{5.10}$$

and

$$\begin{aligned} \mathcal{V}_{\theta,\omega_0}(\varphi) &= -\frac{1}{V}\int_M \sum_{j=0}^{m-1} \varphi(\rho(\omega_0) - \theta) \wedge \omega_0^j \wedge \omega_\varphi^{m-j-1} \\ &+ \frac{1}{V}\int_M \log \frac{\omega_\varphi^m}{\omega_0^m} \omega_\varphi^m + \frac{\bar{S}_\theta}{(m+1)V} \sum_{j=0}^{m} \int_M \varphi \omega_0^j \wedge \omega_\varphi^{m-j}, \end{aligned} \tag{5.11}$$

for all $\varphi, \varphi_0, \varphi_1 \in \mathcal{H}_{\omega_0}$. Let's recall the Aubin's functionals

$$\begin{aligned} I_{\omega_0}(\varphi) &= \frac{1}{V}\int_M \varphi\{(\omega_0)^m - (\omega_\varphi)^m\} \\ J_{\omega_0}(\varphi) &= \int_0^1 \frac{1}{s} I_{\omega_0}(s\varphi) ds. \end{aligned} \tag{5.12}$$

Let φ_s be a smooth curve in $\mathcal{H}_{\omega_0}$, by direct calculation, we have

$$\frac{d}{ds} I_{\omega_0}(\varphi_s) = \frac{1}{V}\int_M \dot{\varphi}_s\{(\omega_0)^m - (\omega_{\varphi_s})^m\} - \frac{1}{2V}\int_M \varphi_s \triangle_{\varphi_s} \dot{\varphi}_s (\omega_{\varphi_s})^m, \tag{5.13}$$

$$\frac{d}{ds} J_{\omega_0}(\varphi_s) = \frac{1}{V}\int_M \dot{\varphi}_s\{(\omega_0)^m - (\omega_{\varphi_s})^m\}, \tag{5.14}$$

and then

$$\frac{d}{ds}\{I_{\omega_0}(\varphi_s) - J_{\omega_0}(\varphi_s)\} = -\frac{1}{2V}\int_M \varphi_s(\triangle_s\dot{\varphi}_s)(\omega_{\varphi_s})^m. \tag{5.15}$$

Moreover, we also have the following properties for I and J, the proof can be found in [5]. Let C be a constant, then

$$I_{\omega_0}(\varphi + C) = I_{\omega_0}(\varphi), \quad J_{\omega_0}(\varphi + C) = J_{\omega_0}(\varphi), \tag{5.16}$$

and

$$0 \le I_{\omega_0}(\varphi) \le (m+1)\{I_{\omega_0}(\varphi) - J_{\omega_0}(\varphi)\} \le mI_{\omega_0}(\varphi), \tag{5.17}$$

for all $\varphi \in \mathcal{H}_{\omega_0}$. Let ω' be an another Kähler form in $[\omega_0]$, and assume that $\omega' = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi$ for some function ϕ. It is easy to check that

$$|I_{\omega'}(\varphi - \phi) - I_{\omega_0}(\varphi)| \le (m+1)Osc(\phi) \tag{5.18}$$

for all $\varphi \in \mathcal{H}_{\omega_0}$. If $\theta_1 - \theta_2 = \sqrt{-1}\partial\bar{\partial}f$, then we have

$$\begin{aligned} \mathcal{V}_{\theta_1,\omega_0}(\varphi) - \mathcal{V}_{\theta_2,\omega_0}(\varphi) &= \frac{1}{V}\int_M \sum_{j=0}^{m-1} \varphi(\theta_1 - \theta_2) \wedge \omega_0^j \wedge \omega_\varphi^{m-j-1} \\ &= \frac{1}{V}\int_M \sum_{j=0}^{m-1} \varphi\sqrt{-1}\partial\bar{\partial}f \wedge \omega_0^j \wedge \omega_\varphi^{m-j-1} \\ &= \frac{1}{V}\int_M \sum_{j=0}^{m-1} f(\omega_\varphi - \omega_0) \wedge \omega_0^j \wedge \omega_\varphi^{m-j-1} \\ &= \frac{1}{V}\int_M f(\omega_\varphi^m - \omega_0^m). \end{aligned} \tag{5.19}$$

Lemma 5.2.1. *Let* $\theta_1 - \theta_2 = \sqrt{-1}\partial\bar{\partial} f$*, then*

$$|\mathcal{V}_{\theta_1,\omega_0}(\varphi) - \mathcal{V}_{\theta_2,\omega_0}(\varphi)| \leq Osc(f) \tag{5.20}$$

for all $\varphi \in \mathcal{H}_{\omega_0}$.

Now, we suppose that $\theta \in [\alpha] = 2\pi c_1(M) - k[\omega_o]$. Let h_{ω_0} is the smooth function which satisfies

$$\rho(\omega_0) - \theta = k\omega_0 + \sqrt{-1}\partial\bar{\partial} h_{\omega_0} \text{ and } \int_M \exp(h_{\omega_0})(\omega_0)^m = \int_M (\omega_0)^m = V.$$

Let's recall the Ding-Tian's functional

$$\begin{aligned} F^0_{\omega_0}(\varphi) &= J_{\omega_0}(\varphi) - \frac{1}{V}\int_M \varphi(\omega_0)^m, \\ F_{\omega_0}(\varphi) &= F^0_{\omega_0}(\varphi) - k^{-1}\log\{\frac{1}{V}\int_M e^{h_{\omega_0} - k\varphi}(\omega_0)^m\}. \end{aligned} \tag{5.21}$$

Denote φ_s to be a smooth path in $\mathcal{H}_{\omega_0}$, then

$$\frac{d}{ds} F^0_{\omega_0}(\varphi_s) = -\frac{1}{V}\int_M \dot{\varphi}_s (\omega_{\varphi_s})^m, \tag{5.22}$$

and

$$\begin{aligned} \frac{d}{ds} F_{\omega_0}(\varphi_s) &= -\frac{1}{V}\int_M \dot{\varphi}_s(\omega_{\varphi_s})^m \\ &\quad + (\int_M e^{h_{\omega_0} - k\varphi}(\omega_0)^m)^{-1} \int_M \dot{\varphi}_s e^{h_{\omega_0} - k\varphi}(\omega_0)^m. \end{aligned} \tag{5.23}$$

From (5.23), it is easy to check that the critical points of F_{ω_0} are generalized Kähler-Einstein metrics. As that in [69], one can easily check that F_{ω_0} satisfies the following cocycle property, i.e.

$$F_{\omega_0}(\psi) + F_{\omega'}(\phi - \psi) = F_{\omega_0}(\phi), \tag{5.24}$$

and

$$F_{\omega_0}(\psi) = -F_{\omega'}(-\psi) \tag{5.25}$$

for all $\phi, \psi \in \mathcal{H}_{\omega_0}$ and $\omega' = \omega_0 + \sqrt{-1}\partial\bar{\partial}\psi$. Moreover, $F^0_{\omega_0}$ also has the same cocycle condition.

By the definitions and direct calculation, we have

$$V(I_{\omega_0} - J_{\omega_0})(\varphi) = -\frac{m}{m+1}\int_M \varphi\omega_\varphi^m + \frac{1}{m+1}\int_M \sum_{j=1}^{m} \varphi\omega_0^j \wedge \omega_\varphi^{m-j}. \tag{5.26}$$

and

$$\begin{aligned}\int_M h_{\omega_0}(\omega_0^m - \omega_\varphi^m) &= -\int_M h_{\omega_0}(\sqrt{-1}\partial\bar{\partial}\varphi) \wedge \sum_{j=0}^{m-1} \omega_0^j \wedge \omega_\varphi^{m-j-1} \\ &= -\int_M \varphi(\sqrt{-1}\partial\bar{\partial}h_{\omega_0}) \wedge \sum_{j=0}^{m-1} \omega_0^j \wedge \omega_\varphi^{m-j-1} \\ &= -\int_M \varphi(\rho(\omega_0) - \theta + k\omega_0) \wedge \sum_{j=0}^{m-1} \omega_0^j \wedge \omega_\varphi^{m-j-1}\end{aligned} \tag{5.27}$$

Noting that $\bar{S}_\theta = km$, by (5.11), it's easy to check that

$$\begin{aligned}\mathcal{V}_{\theta,\omega_0}(\varphi) &= -k(I_{\omega_0} - J_{\omega_0})(\varphi) \\ &\quad + \frac{1}{V}\int_M h_{\omega_0}(\omega_0^m - \omega_\varphi^m) + \frac{1}{V}\int_M \log(\frac{\omega_\varphi^m}{\omega_0})\omega_\varphi^m.\end{aligned} \tag{5.28}$$

We also have the following relation between the Ding-Tian's functional and the twisted Mabuchi $\mathcal{K}$-energy functional.

Lemma 5.2.2. *Let (M, ω_0) be a Kähler manifold, and $\theta \in [\alpha] = 2\pi c_1(M) - k[\omega_o]$, then*

$$\mathcal{V}_{\theta,\omega_0}(\varphi) - kF_{\omega_0}(\varphi) = \frac{1}{V}\int_M h_{\omega_0}(\omega_0)^m - \frac{1}{V}\int_M h_{\omega_\varphi}(\omega_\varphi)^m \tag{5.29}$$

for any $\varphi \in \mathcal{H}_{\omega_0}$, where h_ω is the smooth function which satisfies

$$\rho(\omega) - \theta = k\omega + \sqrt{-1}\partial\bar{\partial}h_\omega$$

and the normalized condition $\int_M \exp(h_\omega)(\omega)^m = V$. Further more, we have

$$\mathcal{V}_{\theta,\omega_0}(\varphi) \geq kF_{\omega_0}(\varphi) + \frac{1}{V}\int_M h_{\omega_0}(\omega_0)^m. \tag{5.30}$$

Proof of Lemma 5.2.2:

By the definition of h_ω, it is easy to check that

$$-\log\frac{(\omega_\varphi)^m}{(\omega_0)^m} - k\varphi + c_\varphi = h_{\omega_\varphi} - h_{\omega_0} \tag{5.31}$$

for all $\varphi \in \mathcal{H}_{\omega_0}$, where the constant $c_\varphi = -\log(\frac{1}{V}\int_M e^{h_{\omega_0} - k\varphi}\omega_0)$. Then, by (5.28) and (5.31), we have

$$\begin{aligned}
& \mathcal{V}_{\theta,\omega_0}(\varphi) && (5.32)\\
= \;& kJ_{\omega_0}(\varphi) - kI_{\omega_0}(\varphi) - \frac{k}{V}\int_M \varphi\omega_\varphi^m + c_\varphi + \frac{1}{V}\int_M h_{\omega_0}\omega_0^m - \frac{1}{V}\int_M h_{\omega_\varphi}\omega_\varphi^m \\
= \;& k(J_{\omega_0}(\varphi) - \frac{1}{V}\int_M \varphi\omega_0^m + k^{-1}c_\varphi) + \frac{1}{V}\int_M h_{\omega_0}\omega_0^m - \frac{1}{V}\int_M h_{\omega_\varphi}\omega_\varphi^m.
\end{aligned}$$

So, (5.32) implies (5.29). By the normalized condition of h_{ω_φ}, we known that

$$\int_M h_{\omega_\varphi}\omega_\varphi^m \leq 0,$$

then we have (5.30). $\square$

5.2.2 Existence result for the generalized Kähler-Einstein metrics

Let (M, ω_0) be a Kähler manifold, and $\theta \in [\alpha] = 2\pi c_1(M) - k[\omega_0]$. Finding generalized Kähler-Einstein metric can be reduced to solving the complex Monge-Ampère equation (5.3). As in Kähler-Einstein case, we consider a family of complex Monge-Ampère equation

$$\frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^m}{\omega_0^m} = \exp(h_{\omega_0} - tk\varphi), \tag{5.33}$$

and set

$$S = \{t \in [0,1] \mid (5.33) \ \text{ is solvable for } \ t\}. \tag{5.34}$$

By [81], we know that (5.33) is solvable for $t = 0$, and then S is not empty. If we can prove that S is open and closed, then we must have $S = [0,1]$, and so the complex Monge-Ampère equation (5.3) can be solved.

In the proof of the openness and closeness of S, we need the assumption that θ is semipositive. The key point is that the semipositivity of θ will lead a lower bound of the Ricci curvature by a positive constant, then we can use the implicity function theorem to prove the openness and obtain a lower bound of the Green's function which is very important to get C^0 estimate. We follow Aubin's discussion ([2]) in the proof of the openness and follow Tian's method ([71]) to prove the closeness. We first obtain the following proposition and the proof is similar as that in [5].

Proposition 5.2.1. *Let (M, ω_0) be a compact Kähler manifold and $\theta \in [\alpha] = 2\pi c_1(M) - k[\omega_0]$ is a real closed semipositive $(1,1)$-form, where $k > 0$.*

Let $0 < \tau \leq 1$, and suppose that (5.33) has a solution φ_τ at $t = \tau$. If $0 < \tau < 1$, then there exists some $\epsilon > 0$ such that φ_τ uniquely extends to a smooth family of solution $\{\varphi_t\}$ of (5.33) for $t \in (0,1) \cap (\tau - \epsilon, \tau + \epsilon)$. S is also open near $t = 0$, i.e. there exists a small positive number ϵ such that there is a smooth family solution of (5.33) for $t \in (0, \epsilon)$.

Furthermore, if M admits no nontrivial Hamiltonian holomorphic vector field or the twisting form θ is strictly positive at a point, φ_1 can also be extended uniquely to a smooth family of solution $\{\varphi_t\}$ of (5.33) for $t \in (1 - \epsilon, 1]$.

Proof of Proposition 5.2.1:

Let $H^{\gamma,\alpha}$ be the set of all function $\varphi \in C^{\gamma,\alpha}(M)$ such that $\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$ is positive definite, where $2 \leq \gamma \in \mathbb{Z}^+$ and $0 < \alpha < 1$. Consider the operator

$$\Xi : H^{\gamma,\alpha} \times R \to C^{\gamma-2,\alpha}(M)$$

defined by

$$\Xi(\varphi, t) := \log \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^m}{(\omega_0)^m} + tk\varphi - h_{\omega_0}. \tag{5.35}$$

The linearized operator is

$$D_\varphi \Xi(\psi) = \frac{1}{2}\triangle_\varphi \psi + tk\psi, \tag{5.36}$$

where $\psi \in C^{\gamma,\alpha}(M)$. By the implicit function theorem, it is sufficiently to prove that $D_\varphi \Xi$ is invertible. For further consideration, let's recall the Bochner-Kodaira formula,

$$2\int_M |\nabla^{1,0}(\nabla^{1,0}_\omega u)|^2_\omega \omega^m = \int_M (\triangle_\omega u)^2 - 2\rho(\omega)(\nabla_\omega u, J(\nabla_\omega u))\omega^m \tag{5.37}$$

for any $u \in C^2(M)$.

In the case of $\tau \in (0,1)$. Since φ_τ is a solution of (5.33), we have

$$\rho_{\varphi_\tau} = \theta + k\omega_0 + \tau k\sqrt{-1}\partial\bar{\partial}\varphi_\tau > \tau k \omega_{\varphi_\tau}. \tag{5.38}$$

If $\psi \in ker D_{\varphi_\tau}\Xi$, the Bochner-Kodaira formula (5.37) implies $\nabla_{\omega_{\varphi_\tau}}\psi \equiv 0$, and then $\psi \equiv 0$. This shows that $D_{\varphi_\tau}\Xi$ is invertible.

When $\tau = 0$, we consider the following operator

$$\tilde{\Xi}(\varphi, t) \;\; := \;\; \log \frac{(\omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi)^m}{(\omega_0)^m} + tk\varphi - h_{\omega_0} + \beta \int_M \varphi(\omega_0)^m, \tag{5.39}$$

where $\beta > 0$ is a constant. Its linearized operator is given by

$$D_\varphi \tilde{\Xi}(\psi) = \frac{1}{2}\triangle_\varphi \psi + tk\psi + \beta \int_M \psi(\omega_0)^m. \tag{5.40}$$

It's easy to check that $D_\varphi \tilde{\Xi}$ is invertible at $t = 0$. By the implicit function theorem, there is a smooth one parameter family

$$\{\tilde{\varphi}_t \mid t \in [0, \epsilon)\}$$

such that $\tilde{\Xi}(\tilde{\varphi}_t, t) = 0$. Then

$$\varphi_t = \tilde{\varphi}_t + \frac{\beta}{tk} \int_M \tilde{\varphi}_t (\omega_0)^m \tag{5.41}$$

is a family solution of (5.33) for $t \in (0, \epsilon)$. So, S is open near $t = 0$.

When $\tau = 1$. Let φ_1 be a solution of (5.33) for $t = 1$, and $\psi \in \ker D_{\varphi_1}\Xi$, i.e.

$$\triangle_{\omega_{\varphi_1}} \psi = -2k\psi.$$

Replacing ω and u in (5.37) by ω_{φ_1} and ψ, we have

$$\int_M |\nabla^{1,0}(\nabla^{1,0}_{\omega_{\varphi_1}} \psi)|^2_{\omega_{\varphi_1}} \omega^m_{\varphi_1} = -\int_M \theta(\nabla_{\omega_{\varphi_1}} \psi, J(\nabla_{\omega_{\varphi_1}} \psi)) \omega^m_{\varphi_1} \tag{5.42}$$

If θ is positive at some point, then $\nabla_{\varphi_1}\psi = 0$ on some open domains. Since the Laplace-Beltrami operator $\triangle_{\varphi_1}$ is real, Aronszajin's unique continuation theorem implies $\nabla_{\varphi_1}\psi \equiv 0$. If M admits no nontrivial Hamiltonian holomorphic vector field, since θ is semi positive, (5.42) implies that $\nabla^{1,0}_{\varphi_1}\psi \equiv 0$. So, $D_{\varphi_1}\Xi$ is invertible.

□

Using the generalized Aubin's equations and discussing as that in Bando-Mabuchi's paper [5], we can obtain the uniqueness of the solution of equation (5.3) (i.e. the uniqueness of generalized Kähler-Einstein metric). So, we omit the proof of the following lemma.

Lemma 5.2.3. *Let (M, ω_0) be a Kähler manifold, and $\theta \in [\alpha] = 2\pi c_1(M) - k[\omega_o]$ is a real closed semipositive $(1,1)$-form, where $k > 0$. If M admits no nontrivial Hamiltonian holomorphic vector field or the twisting form θ is strictly positive at a point, then there exists at most one solution of (5.3).*

Let $\{\varphi_t\}$ be a smooth family of solution of (5.33) for $t \in (0,1]$. Differentiating (5.33) with respect to t, we have

$$\frac{1}{2}\triangle_t \dot{\varphi}_t = -t(m+1)\dot{\varphi}_s - (m+1)\varphi_t. \tag{5.43}$$

Using (5.37) and (5.43), by the same discussion as in [5] we have the following lemma.

Lemma 5.2.4. *Let $\{\varphi_t\}$ be a smooth family of solution of (5.33) for $t \in (0,1]$, then*

$$\frac{d}{dt}(I_{\omega_0} - J_{\omega_0})(\varphi_t) \geq 0. \tag{5.44}$$

Now, we consider the existence problem of generalized Kähler-Einstein metrics. The following theorem gives the proof of the first part of our main result Theorem 5.1.1.

Theorem 5.2.1. *Let (M, ω_0) be a Kähler manifold, and $\theta \in [\alpha] = 2\pi c_1(M) - k[\omega_o]$ is a real closed semipositive $(1,1)$-form, where $k > 0$. If $\mathcal{V}_{\theta,\omega_0}$ (or F_{ω_0}) is proper then there exists a generalized Kähler-Einstein metric $\omega \in \mathrm{K}_{\omega_0}$.*

Proof of Theorem 5.2.1:

From inequality (5.30) in Lemma 5.2.2, we only need to prove the case when $\mathcal{K}$-energy is proper.

By Proposition 5.2.1, we can suppose that there exists a smooth family of solution $\{\varphi_t\}$ of (5.33) for $t \in (0, \tau)$ with some $\tau \in (0,1)$. From equation (5.33), we know that

$$\triangle_t \varphi_t \leq 2m \quad \text{and} \quad \rho(\omega_{\varphi_t}) \geq tk\omega_{\varphi_t}.$$

Using the Green's formula and the lower bound of the Green's function by Bando-Mabuchi [5], we have

$$\frac{1}{V}\int_M \varphi_t(\omega_{\varphi_t})^m \le \inf_M \varphi_t + \frac{\epsilon_1(m)}{tk}. \tag{5.45}$$

where positive constant $\epsilon_1(m)$ depends only on m. Using the fact $\triangle_{\omega_0}\varphi_t \ge -2m$ and the Green's formula, we have

$$\sup_M \varphi_t \le \frac{1}{V}\int_M \varphi_t(\omega_0)^m + \epsilon_2 \tag{5.46}$$

where ϵ_2 is a positive constant depends only on the geometry of (M, ω_0). By the normalization condition, it's easy to see that

$$\sup_M \varphi_t \ge 0 \quad \text{and} \quad \inf_M \varphi_t \le 0.$$

Then

$$\begin{aligned} \|\varphi_t\|_{C^0} &\le \sup_M \varphi_t - \inf_M \varphi_t \\ &\le I_{\omega_0}(\varphi_t) + \frac{\epsilon_1(m)}{tk} + \epsilon_2. \end{aligned} \tag{5.47}$$

By (5.17) and (5.44), it follows that

$$I_{\omega_0}(\varphi_{t_1}) \le (m+1)(I_{\omega_0} - J_{\omega_0})(\varphi_{t_2}) \tag{5.48}$$

for any $0 < t_1 \le t_2 < \tau$. Combining (5.47) and (5.48), we get

$$t\|\varphi_t\|_{C^0} \le t_0(m+1)(I_{\omega_0} - J_{\omega_0})(\varphi_{t_0}) + \epsilon_3 \tag{5.49}$$

for any $0 < t \leq t_0 < \tau$, where ϵ_3 is a positive constant depends only on k and the geometry of (M, ω_0). Thus, we obtain an uniform bound on

$$\Big|\frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_t)^m}{(\omega_0)^m}\Big|$$

for $0 < t \leq t_0 < \tau$. By Yau's C^0 estimate ([81]) for complex Monge-Ampère equations, there exists a uniform constant ϵ_4 such that

$$\|\varphi_t\|_{C^0} \leq \epsilon_4 \tag{5.50}$$

for $0 < t \leq t_0 < \tau$.

On the other hand, it is easy to see that along the solutions φ_t of (5.33), we have

$$S(\omega_{\varphi_t}) = k(m - \frac{(1-t)}{2}\triangle_{\omega_{\varphi_t}}\varphi_t) + \Lambda_{\omega_{\varphi_t}}\theta, \tag{5.51}$$

and

$$\mathcal{V}_{\theta,\omega_0}(\varphi_t) = -k(I_{\omega_0} - J_{\omega_0})(\varphi_t) + \frac{1}{V}\int_M h_{\omega_0}\omega_0^m - \frac{tk}{V}\int_M \varphi_t(\omega_{\varphi_t})^m. \tag{5.52}$$

Then, by (5.15) and (5.51), one obtain

$$\begin{aligned} \frac{d}{dt}\mathcal{V}_{\theta,\omega_0}(\varphi_t) &= -\frac{1}{V}\int_M \dot{\varphi}_t(S(\omega_{\varphi_t}) - \Lambda_{\omega_{\varphi_t}}\theta - km)(\omega_{\varphi_t})^m \\ &= \frac{k}{V}\int_M \dot{\varphi}_t\frac{(1-t)}{2}\triangle_{\omega_{\varphi_t}}\varphi_t(\omega_{\varphi_t})^m \\ &= k(t-1)\frac{d}{dt}((I_{\omega_0} - J_{\omega_0})(\varphi_t)). \end{aligned} \tag{5.53}$$

From (5.52) and (5.53), we have

$$\frac{d}{dt}\left(\frac{t}{V}\int_M \varphi_t(\omega_{\varphi_t})^m + t(I_{\omega_0} - J_{\omega_0})(\varphi_t)\right) = (I_{\omega_0} - J_{\omega_0})(\varphi_t). \tag{5.54}$$

By the uniform estimate (5.50) near $t = 0$, it is easy to check that

$$\frac{t}{V}\int_M \varphi_t(\omega_{\varphi_t})^m + t(I_{\omega_0} - J_{\omega_0})(\varphi_t) \to 0, \quad \text{as} \quad t \to 0.$$

In turn, the identity (5.54) implies that

$$\frac{1}{V}\int_M \varphi_t(\omega_{\varphi_t})^m + (I_{\omega_0} - J_{\omega_0})(\varphi_t) \geq 0, \tag{5.55}$$

and

$$\begin{aligned} \mathcal{V}_{\theta,\omega_0}(\varphi_t) &\leq -k(1-t)(I_{\omega_0} - J_{\omega_0})(\varphi_t) + \frac{1}{V}\int_M h_{\omega_0}\omega_0^m \\ &\leq \frac{1}{V}\int_M h_{\omega_0}\omega_0^m. \end{aligned} \tag{5.56}$$

Then the properness of $\mathcal{V}_{\theta,\omega_0}$ implies that $J_{\omega_0}(\varphi_t)$ and $I_{\omega_0}(\varphi_t)$ is uniformly bounded. Using (5.47), we obtain a uniform C^0 estimate on φ_t for $t \in [\epsilon, \tau)$.

Again, by Yau's estimates ([81]) for complex Monge-Ampère equations, the C^0-estimate implies the $C^{2,\alpha}$-estimate, and the elliptic Schauder estimates give higher order estimates. Therefore, equation (5.3) can be solved, i.e. there is a generalized Kähler-Einstein metric in $\mathcal{K}_{\omega_0}$. $\square$

5.3 A Moser-Trudinger type inequality

In this section, we will establish a Moser-Trudinger type inequality on the generalized Kähler-Einstein manifolds which will finish the proof for the rest part of our main result Theorem 5.1.1.

First, we consider the following generalized Kähler-Ricci flow

$$\frac{\partial \omega_s}{\partial s} = -(\rho(\omega_s) - \theta - k\omega_s) \tag{5.57}$$

with $\omega_s|_{s=0} = \tilde{\omega}_0 \in [\omega_0]$. Solving the above equation is equivalent to solve the following parabolic version of complex Mong-Ampere equation

$$\frac{\partial v}{\partial s} = \log \frac{(\tilde{\omega}_0 + \sqrt{-1}\partial\bar{\partial} v)^m}{\tilde{\omega}_0^m} + kv - h_{\tilde{\omega}_0}, \tag{5.58}$$

with $v|_{s=0} \equiv 0$. It is well known that the long-time existence of the above parabolic equation follows from Cao's result [27].

Let v_s be a smooth solution of (5.58), and $\tilde{\omega}_s = \tilde{\omega}_0 + \sqrt{-1}\partial\bar{\partial} v_s$. By direct calculation, we have

$$\frac{\partial}{\partial s}\dot{v}_s = \frac{1}{2}\triangle_{\tilde{\omega}_s}\dot{v}_s + k\dot{v}_s, \tag{5.59}$$

$$\begin{aligned} \frac{\partial}{\partial s}|d\dot{v}_s|^2_{\tilde{\omega}_s} &= \frac{1}{2}\triangle_{\tilde{\omega}_s}|d\dot{v}_s|^2_{\tilde{\omega}_s} + k|d\dot{v}_s|^2_{\tilde{\omega}_s} - |\nabla_{\tilde{\omega}_s} d\dot{v}_s|^2_{\tilde{\omega}_s} \\ &\quad -\theta(\nabla_{\tilde{\omega}_s}\dot{v}_s, J(\nabla_{\tilde{\omega}_s}\dot{v}_s)), \end{aligned} \tag{5.60}$$

$$
\begin{aligned}
& (\frac{\partial}{\partial s} - \frac{1}{2}\triangle_{\tilde{\omega}_s})(\dot{v}_s^2 + s|d\dot{v}_s|^2_{\tilde{\omega}_s}) \\
= \;& 2k\dot{v}_s^2 + sk|d\dot{v}_s|^2_{\tilde{\omega}_s} - s|\nabla_{\tilde{\omega}_s} d\dot{v}_s|^2_{\tilde{\omega}_s} - s\theta(\nabla_{\tilde{\omega}_s}\dot{v}_s, J(\nabla_{\tilde{\omega}_s}\dot{v}_s)) \\
\leq \;& 2k(\dot{v}_s^2 + s|d\dot{v}_s|^2_{\tilde{\omega}_s}),
\end{aligned} \tag{5.61}
$$

and

$$
(\frac{\partial}{\partial s} - \frac{1}{2}\triangle_{\tilde{\omega}_s})(\triangle_{\tilde{\omega}_s}\dot{v}_s) = k\triangle_{\tilde{\omega}_s}\dot{v}_s - |\partial\bar{\partial}\dot{v}_s|^2_{\tilde{\omega}_s} \tag{5.62}
$$

where $\dot{v}_s = \frac{\partial}{\partial s}v_s$. Note that we have used the semi-positivity of θ in (5.61). Applying the maximum principle to the above equalities and discussing as that in [4] (or Lemma 4 in [63]), we have the following lemmas.

Lemma 5.3.1. *The following inequalities*

$$
\|\frac{\partial v_s}{\partial s}\|_{C^0} \leq e^{ks}\|h_{\tilde{\omega}_0}\|_{C^0}, \tag{5.63}
$$

$$
\sup_M(|h_{\tilde{\omega}_s}|^2 + s|dh_{\tilde{\omega}_s}|^2_{\tilde{\omega}_s}) \leq 4e^{2ks}\|h_{\tilde{\omega}_0}\|^2_{C^0}, \tag{5.64}
$$

$$
e^{-ks}\triangle_{\tilde{\omega}_s}h_{\tilde{\omega}_s} \geq \triangle_{\tilde{\omega}_0}h_{\tilde{\omega}_0}, \tag{5.65}
$$

hold for all $s \geq 0$.

Lemma 5.3.2. *Suppose there exists a generalized Kähler-Einstein metric $\omega_{GKE} \in [\omega_0]$. Let $v_{t,s}$ be a solution of (5.58) with $\tilde{\omega}_0 = \omega_{\varphi_t}$. Let*

$$
\tilde{h} = h_{\tilde{\omega}_1} - \frac{1}{V}\int_M h_{\tilde{\omega}_1}(\tilde{\omega}_1)^m
$$

and assume that

$$\frac{1}{2}\omega_{GKE} \le \tilde{\omega}_1 \le \omega_{GKE}. \tag{5.66}$$

Then for any $p > 2m$*, there exist positive constant* $\bar{C}_1$ *depending only on* p*,* k *and* (M, ω_{GKE}) *such that*

$$\|\tilde{h}\|_{C^0} \le \bar{C}_1(1-t)^{\frac{1}{p-1}}\|h_{\omega_{\varphi_t}}\|_{C^0}^{\frac{p-2}{p-1}}. \tag{5.67}$$

Proof of Lemma 5.3.2:

By the condition $\tilde{\omega}_0 = \omega_{\varphi_t}$, we have

$$\rho(\tilde{\omega}_0) = \theta + k\omega_0 + tk\sqrt{-1}\partial\bar{\partial}\varphi_t \ge \theta + tk\tilde{\omega}_0 \tag{5.68}$$

and

$$\triangle_{\tilde{\omega}_0} h_{\tilde{\omega}_0} \ge 2mk(t-1).$$

Thus, it follows from (5.65) that

$$-\triangle_{\tilde{\omega}_1} h_{\tilde{\omega}_1} \le 2mke^k(1-t). \tag{5.69}$$

Integrating by parts, we have

$$\begin{aligned} \int_M |d\tilde{h}|^2_{\tilde{\omega}_1}(\tilde{\omega}_1)^m &= -\int_M \tilde{h}\triangle_{\tilde{\omega}_1}\tilde{h}(\tilde{\omega}_1)^m \\ &\le \int_M (\tilde{h} - \inf \tilde{h}) \sup_M(-\triangle_{\tilde{\omega}_1}\tilde{h})(\tilde{\omega}_1)^m \\ &\le \bar{C}_2(1-t)\|\tilde{h}\|_{C^0}, \end{aligned} \tag{5.70}$$

where $\bar{C}_2$ depends only on k, m and the volume V.

On the other hand, (5.64) implies that

$$\|\tilde{h}\|_{C^0} \leq 4e^k \|h_{\tilde{\omega}_0}\|_{C^0}.$$

Let $p \geq 2m+1$, by the Sobolev imbedding theorem (Lemma 2.22 of [3]), the Poincaré inequality, (5.64) and the condition (5.66), we have

$$\begin{aligned}
\|\tilde{h}\|^p_{C^0} &\leq \bar{C}_3 \int_M |\tilde{h}|^p + |d\tilde{h}|^p_{\omega_{GKE}} (\omega_{GKE})^m \\
&\leq \bar{C}_4 \|h_{\tilde{\omega}_0}\|^{p-2}_{C^0} \int_M |\tilde{h}|^2 + |d\tilde{h}|^2_{\omega_{GKE}} (\omega_{GKE})^m \\
&\leq \bar{C}_5 \|h_{\tilde{\omega}_0}\|^{p-2}_{C^0} \int_M |d\tilde{h}|^2_{\omega_{GKE}} (\omega_{GKE})^m \\
&\leq \bar{C}_6 \|h_{\tilde{\omega}_0}\|^{p-2}_{C^0} \int_M |d\tilde{h}|^2_{\tilde{\omega}_1} (\tilde{\omega}_1)^m,
\end{aligned} \tag{5.71}$$

where constants $\bar{C}_i$ depends only on p, m and the geometry of (M, ω_{GKE}). Then (5.70) and (5.71) imply (5.67). □

Lemma 5.3.3. *Let $v_{t,s}$ be a solution of (5.58) with initial data $\tilde{\omega}_0 = \omega_{\varphi_t}$, and $u_t = v_{t,1}$. We have the inequality*

$$\|u_t\|_{C^0} \leq \frac{1}{k} e^k \|h_{\omega_{\varphi_t}}\|_{C^0}, \quad \textit{for all} \;\; t \in [0, 1]. \tag{5.72}$$

Moreover, assume that

$$\frac{1}{2}\omega_{GKE} \leq \omega_{\varphi_t + u_t} \leq \omega_{GKE} \;\; \textit{for all} \;\; t \in [t_1, 1],$$

where $t_1 \in [0,1)$. *Then for any* $p > 2m$ *and* $0 \le \delta < 1$, *there exists a constant* $\bar{C}_7$ *depending only on* p, k *and* (M, ω_{GKE}) *such that*

$$\|h_{\omega_{\varphi_t+u_t}}\|_{C^{0,\delta}(\omega_{GKE})} \le \bar{C}_7(1-t)^{1-\beta}(1+\|h_{\omega_{\varphi_t}}\|_{C^0})^{\beta} \tag{5.73}$$

for all $t \in [t_1, 1]$, *where* $\beta = \frac{p+\delta-2}{p-1}$.

Proof of Lemma 5.3.3:

Inequality (5.72) can be easily deduced from (5.63).

By the condition $\frac{1}{2}\omega_{GKE} \le \omega_{\varphi_t+u_t} \le \omega_{GKE}$, we have

$$|dh_{\omega_{\varphi_t+u_t}}|_{\omega_{GKE}} \le \sqrt{2}|dh_{\omega_{\varphi_t+u_t}}|_{\omega_{\varphi_t+u_t}}.$$

In the following, let $d(x,y)$ be the distance between x and y with respect to the metric ω_{GKE}.

If $d(x,y) \le (1-t)^{\frac{1}{p-1}}(1+\|h_{\omega_{\varphi_t}}\|_{C^0})^{-\frac{1}{p-1}}$, by (5.64) in lemma 4.1, we have

$$\begin{aligned}
|h_{\omega_{\varphi_t+u_t}}(x) - h_{\omega_{\varphi_t+u_t}}(y)| &\le d(x,y)\sup_M |dh_{\omega_{\varphi_t+u_t}}|_{\omega_{GKE}} \qquad (5.74)\\
&\le \sqrt{2}d(x,y)\sup_M |dh_{\omega_{\varphi_t+u_t}}|_{\omega_{\varphi_t+u_t}}\\
&\le 4\sqrt{2}e^k d(x,y)(1+\|h_{\omega_{\varphi_t}}\|_{C^0})\\
&\le 4\sqrt{2}e^k(1-t)^{\frac{1-\delta}{p-1}}(1+\|h_{\omega_{\varphi_t}}\|_{C^0})^{\frac{p+\delta-2}{p-1}}d(x,y)^{\delta}.
\end{aligned}$$

If $d(x,y) \geq (1-t)^{\frac{1}{p-1}}(1+\|h_{\omega_{\varphi_t}}\|_{C^0})^{-\frac{1}{p-1}}$, then the estimate (5.67) in lemma 4.2 implies

$$\begin{aligned} & |h_{\omega_{\varphi_{t+u_t}}}(x) - h_{\omega_{\varphi_{t+u_t}}}(y)| \leq 2\|\tilde{h}\|_{C^0} \\ \leq\ & 2\bar{C}_1(1-t)^{\frac{1}{p-1}}(\|h_{\omega_{\varphi_t}}\|_{C^0})^{\frac{p-2}{p-1}} \\ \leq\ & 2\bar{C}_1(1-t)^{\frac{1-\delta}{p-1}}(1+\|h_{\omega_{\varphi_t}}\|_{C^0})^{\frac{p+\delta-2}{p-1}}d(x,y)^{\delta}. \end{aligned} \tag{5.75}$$

On the other hand, the integral normalization $\int_M e^{h_{\omega_{\varphi_{t+u_t}}}}(\omega_{\varphi_{t+u_t}})^m = V$ implies $h_{\omega_{\varphi_{t+u_t}}}$ change signs, so we have

$$\begin{aligned} \|h_{\omega_{\varphi_{t+u_t}}}\|_{C^0} &\leq Osc(h_{\omega_{\varphi_{t+u_t}}}) = Osc(\tilde{h}) \leq 2\|\tilde{h}\|_{C^0} \\ &\leq 2\bar{C}_1(1-t)^{\frac{1}{p-1}}(\|h_{\omega_{\varphi_t}}\|_{C^0})^{\frac{p-2}{p-1}}. \end{aligned} \tag{5.76}$$

It is easy to see that (5.74), (5.75) and (5.76) imply the estimate (5.73). ∎

Set $\zeta := 1 - \frac{1}{4m} > \frac{1}{2}$ and define the function f_{ω_0} by

$$f_{\omega_0}(t) := (1-t)^{1-\zeta}(k^{-1} + 2(1-t)\|\varphi_t\|_{C^0})^{\zeta}. \tag{5.77}$$

Discussing as that in [69] (or lemma 1 in [63]), we have the following proposition. We write out the proof just for reader's convenience.

Proposition 5.3.1. *Let φ_t be a smooth family of solutions of equation (5.33) for $t \in (0,1]$, and $\omega_{GKE} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_1$. There exist a constant $D > 0$ depending only on k and (M, ω_{GKE}) such that*

$$\|\varphi_1 - \varphi_t\|_{C^0} \leq A(1-t)\|\varphi_t\|_{C^0} + 1 \tag{5.78}$$

for all $t \in [t_0, 1]$*, where* $t_0 \in [0,1)$ *satisfies* $f_{\omega_0}(t_0) = \max_{[t_0,1]} f_{\omega_0} = D$ *and* A *depending only on* m *and* k*.*

Proof of Proposition 5.3.1:

Let's rewrite (5.33) as the following complex Monge-Ampère equation with ω_{GKE} as reference metric

$$\frac{(\omega_{GKE} + \sqrt{-1}\partial\bar{\partial}(\varphi_t - \varphi_1))^m}{(\omega_{GKE})^m} = \exp(-k(\varphi_t - \varphi_1) + (1-t)k\varphi_t). \tag{5.79}$$

It is easy to see that $h_{\omega_{\varphi_t}} = (t-1)k\varphi_t + c_t$, for some constant c_t. The integrate normalization of the potential function $h_{\omega_{\varphi_t}}$ implies

$$|c_t| \leq k(1-t)\|\varphi_t\|_{C^0}, \tag{5.80}$$

and

$$\|h_{\omega_{\varphi_t}}\|_{C^0} \leq 2k(1-t)\|\varphi_t\|_{C^0}. \tag{5.81}$$

Then, Lemma 5.3.3 implies that

$$\|u_t\|_{C^0} \leq 2e^k(1-t)\|\varphi_t\|_{C^0}. \tag{5.82}$$

Note that

$$\omega_{\varphi_t + u_t} = \omega_0 + \sqrt{-1}\partial\bar{\partial}(\varphi_t + u_t) = \omega_{GKE} + \sqrt{-1}\partial\bar{\partial}(\varphi_t + u_t - \varphi_1),$$

and then

$$\frac{(\omega_{GKE} + \sqrt{-1}\partial\bar{\partial}(\varphi_t + u_t - \varphi_1))^m}{(\omega_{GKE})^m} = \exp(-k(\varphi_t + u_t - \varphi_1) - h_{\omega_{\varphi_t + u_t}} - \tilde{c}_t) \tag{5.83}$$

for some constant $\tilde{c}_t$. Let $\tilde{\varphi}_t = \varphi_t + u_t - \varphi_1 + \frac{\tilde{c}_t}{k}$, from (5.83), (5.79) and (5.82), we have

$$\begin{aligned}\int_M e^{h_{\omega_{\varphi_t+u_t}}}(\omega_{\varphi_t+u_t})^m &= \int_M e^{-k\tilde{\varphi}_t}(\omega_{GKE})^m \\ &= \int_M e^{-k\tilde{\varphi}_t+tk\varphi_t-k\varphi_1}(\omega_{\varphi_t})^m \\ &= \int_M e^{(t-1)k\varphi_t-ku_t-\tilde{c}_t}(\omega_{\varphi_t})^m,\end{aligned} \tag{5.84}$$

and then

$$\begin{aligned}|\tilde{c}_t| &\leq (1-t)k\|\varphi_t\|_{C^0} + k\|u_t\|_{C^0} \\ &\leq (1-t)k(1+2e^k)\|\varphi_t\|_{C^0}.\end{aligned} \tag{5.85}$$

Recall that $\varphi_t - \varphi_1 = \tilde{\varphi}_t - u_t - \frac{\tilde{c}_t}{k}$, it follows from (5.82) and (5.85) that

$$\|\varphi_t - \varphi_1\|_{C^0} = \|\tilde{\varphi}_t\|_{C^0} + (1-t)(4e^k+1)\|\varphi_t\|_{C^0}. \tag{5.86}$$

From above estimates, it will suffice to show that

$$\|\tilde{\varphi}_t\|_{C^0} \leq 1.$$

Let's consider the following complex Monge-Ampère equation

$$\log\{\frac{(\omega_{GKE} + \sqrt{-1}\partial\bar{\partial}\psi)^m}{(\omega_{GKE})^m}\} + k\psi = \tilde{\psi}. \tag{5.87}$$

The linearization of the left side of (5.87) at $\psi = 0$ is

$$\delta\psi \mapsto \frac{1}{2}\triangle_{\omega_{GKE}}\delta\psi + k\delta\psi. \tag{5.88}$$

If M doesn't have non-trivial Hamiltonian holomorphic vector fields or θ is strictly positive at a point, by (5.42), we know that

$$ker(\frac{1}{2}\triangle_{\omega_{GKE}} + k) = 0,$$

then the operator $(\frac{1}{2}\triangle_{\omega_{GKE}} + k) : C^{i+2,\epsilon}(M) \to C^{i+2,\epsilon}(M)$ is invertible. Applying the implicit function theorem, there exist positive constants $\epsilon(\omega_{GKE})$ and $C^*(\omega_{GKE})$ which depend only on δ and the geometry of (M, ω_{GKE}), so that

$$\text{if} \quad \|\tilde{\psi}\|_{C^{0,\delta}} \le \epsilon(\omega_{GKE}) \quad \text{then} \quad \|\psi\|_{C^{2,\delta}} \le C^*(\omega_{GKE})\|\tilde{\psi}\|_{C^{0,k}}. \tag{5.89}$$

Let

$$D = \frac{\epsilon k^{-\zeta}}{2(\bar{C}_7 + 1)(C^* + 1)(\epsilon + 1)},$$

where $\epsilon = \epsilon(\omega_{GKE})$, $C^* = C^*(\omega_{GKE})$ are chosen as in (5.89), $\zeta = 1 - \frac{1}{4m}$, $\bar{C}_7$ is defined as in Lemma 5.3.3 (by choosing $\delta = \frac{1}{2}$ and $p = 2m + 1$). Let $t_0 \in [0, 1)$ satisfies $f_{\omega_0}(t_0) = \max_{[t_0,1]} f_{\omega_0} = D$. Now, we only need to prove the following claim:

Claim For all $t \in [t_0, 1]$, we have

$$\|\tilde{\varphi}_t\|_{C^{2,\frac{1}{2}}} < \frac{1}{2}. \tag{5.90}$$

We assume the contrary. Since $\tilde{\varphi}_1 = 0$, there exists $t_1 \in [t_0, 1)$ such that

$$\|\tilde{\varphi}_{t_1}\|_{C^{2,\frac{1}{2}}(\omega_{GKE})} = \frac{1}{2}, \quad and \quad \|\tilde{\varphi}_t\|_{C^{2,\frac{1}{2}}(\omega_{GKE})} < \frac{1}{2} \quad if \quad t_1 < t < 1. \tag{5.91}$$

In particular $-\frac{1}{4}\omega_{GKE} \leq \sqrt{-1}\partial\bar{\partial}\tilde{\varphi}_t \leq \frac{1}{4}\omega_{GKE}$, and then

$$\frac{3}{4}\omega_{GKE} \leq \omega_{\varphi_t+u_t} \leq \frac{5}{4}\omega_{GKE} \tag{5.92}$$

for all $t \in [t_1, 1]$. By applying (5.73) in Lemma 5.3.3 (by choosing $p = 2m + 1$) and (5.81), we have

$$\begin{aligned} \|h_{\omega_{\varphi_t+u_t}}\|_{C^{0,\frac{1}{2}}(\omega_{GKE})} &\leq \bar{C}_7(1-t)^{1-\zeta}(1+\|h_{\omega_{\varphi_t}}\|_{C^0})^{\zeta} \\ &\leq \bar{C}_7(1-t)^{1-\zeta}(1+2(1-t)k\|\varphi_t\|_{C^0})^{\zeta} \\ &\leq \bar{C}_7k^{\zeta}(1-t)^{1-\zeta}(k^{-1}+2(1-t)\|\varphi_t\|_{C^0})^{\zeta} \\ &\leq \bar{C}_7k^{\zeta}D \\ &= \frac{\bar{C}_7\epsilon}{2(\bar{C}_7+1)(C^*+1)(\epsilon+1)} < \epsilon, \end{aligned} \tag{5.93}$$

for all $t \in [t_1, 1]$. Using (5.89) again, we get

$$\begin{aligned} \|\tilde{\varphi}_{t_1}\|_{C^{2,\frac{1}{2}}(d\eta_{SE})} &\leq C^*\|h_{d\eta_{\varphi_t+u_t}}\|_{C^{0,\frac{1}{2}}(\omega_{GKE})} \\ &\leq \frac{C^*\bar{C}_7\epsilon}{2(\bar{C}_7+1)(C^*+1)(\epsilon+1)} < \frac{1}{2}. \end{aligned} \tag{5.94}$$

This gives a contradiction and complete the proof of the claim. Thus, the proof of the proposition is complete. $\square$

Using Proposition 5.3.1 and discussing as that in [63] (Theorem 1), we establish a Moser-Trudinger type inequality for functional $F_{\omega_{GKE}}$. In fact, we obtain the following theorem. We write out the proof in details just for reader's convenience.

Theorem 5.3.1. *Let (M, ω_0) be a Kähler manifold, and $\theta \in [\alpha] = 2\pi c_1(M) - k[\omega_0]$ is a real closed semipositive $(1,1)$-form, where $k > 0$. Assuming that the twisting form*

θ is strictly positive at a point or M admits no nontrivial Hamiltonian holomorphic vector field. If there exists a generalized Kähler-Einstein metric $\omega_{GKE} \in \mathrm{K}_{\omega_0}$, then there exist uniform positive constants $\tilde{C}_1$, $\tilde{C}_2$ depending only on k and the geometry of (M, ω_{GKE}), such that

$$F_{\omega_{GKE}}(\varphi) \geq \tilde{C}_1 J_{\omega_{GKE}}(\varphi) - \tilde{C}_2, \tag{5.95}$$

for all $\varphi \in \mathcal{H}_{\omega_{GKE}}$.

Proof of Theorem 5.3.1:

Fix a function $\phi \in \mathcal{H}_{\omega_{GKE}}$, and set $\omega_0 = \omega_{GKE} + \sqrt{-1}\partial\bar{\partial}\phi$. We consider the complex Monge-Ampère equation (5.33). Since M admits no nontrivial Hamiltonian holomorphic vector fields or the twisting form θ is strictly positive at a point , by the uniqueness of generalized Kähler-Einstein structure (Lemma 5.2.3) and Proposition 5.3.1, a unique solution φ_t exists for all $t \in (0, 1]$ and $\omega_{\varphi_1} = \omega_{GKE}$. In particular φ_1 and $-\phi$ differ by a constant.

For further consideration, we give the following estimates for functionals F, I and J. From (5.12), (5.14) and (5.43), we have

$$\frac{d}{ds}(I_{\omega_0} - J_{\omega_0})(\varphi_s) = -\frac{d}{ds}\left(\frac{1}{V}\int_M \varphi_s(\omega_{\varphi_s})^m\right) - \frac{1}{V}\int_M \dot{\varphi}_s(\omega_{\varphi_s})^m. \tag{5.96}$$

The uniform C^0 estimate (5.50) of φ_t implies that

$$s\frac{1}{V}\int_M \varphi_s(\omega_{\varphi_s})^m \to 0, \quad \text{as} \quad s \to 0. \tag{5.97}$$

By integrating on $[0,t]$, we get

$$t(I_{\omega_0} - J_{\omega_0})(\varphi_t) - \int_0^t (I_{\omega_0} - J_{\omega_0})(\varphi_s)ds = -\frac{t}{V}\int_M \varphi_t(\omega_{\varphi_t})^m, \tag{5.98}$$

and then

$$\begin{aligned} F^0_{\omega_0}(\varphi_t) &= -(I_{\omega_0} - J_{\omega_0})(\varphi_t) - \frac{1}{V}\int_M \varphi_t(\omega_{\varphi_t})^m \\ &= \frac{-1}{t}\int_0^t (I_{\omega_0} - J_{\omega_0})(\varphi_s)ds. \end{aligned} \tag{5.99}$$

Taking $t = 1$ and considering $F_{\omega_0}(\varphi_1) = -F_{\omega_{GKE}}(\phi)$, so that

$$F_{\omega_{GKE}}(\phi) = \int_0^1 (I_{\omega_0} - J_{\omega_0})(\varphi_s)ds. \tag{5.100}$$

By the definitions (5.21) and the cocycle property of $F^0_{\omega_0}$, it is easy to check

$$|J_{\omega_0}(\varphi_1) - J_{\omega_0}(\varphi_t)| \leq Osc(\varphi_1 - \varphi_t) \tag{5.101}$$

and

$$|(I_{\omega_0} - J_{\omega_0})(\varphi_t) - (I_{\omega_0} - J_{\omega_0})(\varphi_1)| \leq m \cdot Osc(\varphi_1 - \varphi_t). \tag{5.102}$$

Using the relationship $F_{\omega_0}(\varphi_1) = -F_{\omega_{GKE}}(\phi)$, we have

$$\begin{aligned}
J_{\omega_0}(\varphi_1) &= F_{\omega_0}(\varphi_1) + \frac{1}{V}\int_M \varphi_1(\omega_0)^m \qquad (5.103)\\
&= -F_{\omega_{GKE}}(\phi) + \frac{1}{V}\int_M \varphi_1(\omega_0)^m\\
&= -J_{\omega_{GKE}}(\phi) + \frac{1}{V}\int_M \phi\{(\omega_{GKE})^m - (\omega_0)^m\}\\
&= (I_{\omega_{GKE}} - J_{\omega_{GKE}})(\phi)\\
&\geq \frac{1}{m} J_{\omega_{GKE}}(\phi),
\end{aligned}$$

where we have used inequality (5.17). On the other hand, since $(I_{\omega_0} - J_{\omega_0})(\varphi_t)$ is nondecreasing in t, (5.100) implies that

$$F_{\omega_{GKE}}(\phi) \geq (1-t)(I_{\omega_0} - J_{\omega_0})(\varphi_t) \geq \frac{1-t}{m} J_{\omega_0}(\varphi_t). \qquad (5.104)$$

Combining this inequality with (5.103) and (5.101), we have

$$F_{\omega_{GKE}}(\phi) \geq \frac{1-t}{m^2} J_{\omega_{GKE}}(\phi) - \frac{1-t}{m} Osc(\varphi_t - \varphi_1). \qquad (5.105)$$

In the following, we choose t_0 as that in Proposition 5.3.1.

If $2(1-t_0)\|\varphi_{t_0}\|_{C^0} \leq k^{-1}$, by the definition of t_0, we have

$$D \leq (1-t_0)^{1-\zeta} 2^{\zeta} k^{-\zeta},$$

which gives

$$(1-t_0) \geq 2^{-\frac{\zeta}{1-\zeta}} k^{\frac{\zeta}{1-\zeta}} D^{\frac{1}{1-\zeta}}. \qquad (5.106)$$

Similarly, if $2(1-t_0)\|\varphi_{t_0}\|_{C^0} \geq k^{-1}$, we have

$$D \leq 4^{\zeta}(1-t_0)\|\varphi_t\|_{C^0}^{\zeta},$$

then

$$(1-t_0) \geq \frac{D}{4^{\zeta}\|\varphi_{t_0}\|_{C^0}^{\zeta}}. \tag{5.107}$$

For the second case, we may assume that $1 - t_0 < \frac{A^{-1}}{2}$, the inequality implies that

$$\|\varphi_{t_0}\|_{C^0} \leq 2\|\varphi_1\|_{C^0} + 2, \tag{5.108}$$

then

$$(1-t_0) \geq \frac{D}{4^{\zeta}(2\|\varphi_1\|_{C^0} + 2)^{\zeta}}. \tag{5.109}$$

Again, since $\sup\varphi_1 \cdot \inf\varphi_1 \leq 0$, we always have the following inequality

$$(1-t_0) \geq \frac{C'}{(\|\varphi_1\|_{C^0} + 1)^{\zeta}} \geq \frac{C'}{(Osc(\phi) + 1)^{\zeta}}, \tag{5.110}$$

where C' is a positive constant depending only on k and (M, ω_{GKE}). On the other hand, using Proposition 5.3.1 again, we have

$$(1-t_0)\|\varphi_1 - \varphi_{t_0}\|_{C^0} \leq (1-t_0)^2 A\|\varphi_{t_0}\|_{C^0} + 1 \leq AD^{\frac{1}{\zeta}} + 1. \tag{5.111}$$

By inequalities (5.105), (5.110) and (5.111), we obtain

$$F_{\omega_{GKE}}(\phi) \geq \tilde{C}_3 \frac{J_{\omega_{GKE}}(\phi)}{(Osc(\phi) + 1)^{\zeta}} - \tilde{C}_4, \tag{5.112}$$

for all $\phi \in \mathcal{H}_{\omega_{GKE}}$, where $\tilde{C}_3$ and $\tilde{C}_4$ are positive constants depending only on k and the geometry of (M, ω_{GKE}).

Notice that $\varphi_t - \varphi_1 \in \mathcal{H}_{\omega_{GKE}}$ and $\rho(\omega_{\varphi_t}) \geq \theta + tk\omega_{\varphi_t}$, we can use (5.47) to obtain the following estimate

$$Osc(\varphi_t - \varphi_1) \leq I_{\omega_{GKE}}(\varphi_t - \varphi_1) + \tilde{C}_5, \tag{5.113}$$

for $t \in [\frac{1}{2}, 1]$, where $\tilde{C}_5$ is a constant depending only on on k and the geometry of (M, ω_{GKE}). By(5.17), (5.112) and (5.113), we have

$$F_{\omega_{GKE}}(\varphi_t - \varphi_1) \geq \tilde{C}_6 \frac{J_{\omega_{GKE}}(\varphi_t - \varphi_1)}{(J_{\omega_{GKE}}(\varphi_t - \varphi_1) + 1)^{\zeta}} - \tilde{C}_4, \tag{5.114}$$

for $t \in [\frac{1}{2}, 1]$, where $\tilde{C}_6$ is a positive constant depending only on k and the geometry of (M, ω_{GKE}).

Finally, by the cocycle property of the functional F, (5.98), (5.99), (5.47), non-decreasing of $(I_{\omega_0} - J_{\omega_0})(\varphi_t)$ and the concavity of the log function, we have

$$\begin{aligned} F_{\omega_{GKE}}(\varphi_t - \varphi_1) &= F_{\omega_0}(\varphi_t) - F_{\omega_0}(\varphi_1) \\ &\leq m(1-t)\{(m+1)J_{\omega_{GKE}}(\varphi_t - \varphi_1) + \frac{\tilde{C}_7}{tk} + \tilde{C}_8\} \end{aligned} \tag{5.115}$$

By a same discussion in [63] (Page 1083), we know that (5.105), (5.113), (5.114) and (5.115) imply the Moser-Trudinger inequality (5.95). ∎

In view of the cocycle identity of F_ω and properties of I_ω, J_ω (see (5.24), (5.18) and (5.17)), inequality (5.9) holds for every Kähler metric ω which is cohomology to ω_{GKE}. Moreover, the relation (5.30) implies that the Moser-Trudinger type inequality (5.95) also be valid for the $\mathcal{K}$-energy $\mathcal{V}_{\theta,\omega}$.

Corollary 5.3.1. *Let (M, ω_0) be a Kähler manifold, and $\theta \in [\alpha] = 2\pi c_1(M) - k[\omega_o]$ is a real closed semipositive $(1,1)$-form, where $k > 0$. Assuming that the twisting form θ is strictly positive at a point or M admits no nontrivial Hamiltonian holomorphic vector field. If there exists a generalized Kähler-Einstein metric in K_{ω_0}. Then, for any Kähler metric $\omega \in \mathrm{K}_{\omega_0}$ there exist uniform positive constants $\{\tilde{D}_i\}_{i=1}^4$ depending only on k and the geometry of (M, ω), such that*

$$F_\omega(\varphi) \geq \tilde{D}_1 J_\omega(\varphi) - \tilde{D}_2, \tag{5.116}$$

and

$$\mathcal{V}_{\theta,\omega}(\varphi) \geq \tilde{D}_3 J_\omega(\varphi) - \tilde{D}_4, \tag{5.117}$$

for all $\varphi \in \mathcal{H}_\omega$.

Remark 5.3.1. *Theorem 5.2.1 and Corollary 5.3.1 imply the main result Theorem 5.1.1.*

5.4 A result on Slope stability

In [67], Stoppa discussed the so called twisted cscK equation, i.e. finding a metric $\omega \in [\omega_0]$ such that

$$S(\omega) - \Lambda_\omega \theta = \bar{S}_\theta \tag{5.118}$$

where θ is a real closed semipositive $(1,1)$-form. In particularly, if $\theta \in 2\pi c_1(M) - k[\omega_o]$, then the above twisted cscK equation is equivalent to the generalized Kähler-Einstein equation (5.1). By the definition of the twisted $\mathcal{K}$-energy, it is easy to check that the second derivative along a path $\varphi_t \in \mathcal{H}_{\omega_0}$ is given by

$$V\frac{d^2}{dt^2}\mathcal{V}_{\theta,\omega_0}(\varphi_t) = \|\bar{\partial}\nabla^{1,0}_{\omega_{\varphi_t}}\dot{\varphi}_t\|^2_{\varphi_t} + (\partial\dot{\varphi}_t \wedge \bar{\partial}\dot{\varphi}_t, \theta)_{\varphi_t} - \int_M (\ddot{\varphi}_t - \frac{1}{2}|\nabla^{1,0}_{\omega_{\varphi_t}}\dot{\varphi}_t|^2_{\varphi_t})(S(\omega_t) - \Lambda_{\omega_{\varphi_t}}\theta - \bar{S}_\theta)\omega^m_{\varphi_t}. \tag{5.119}$$

If either the twisting form θ is strictly positive at a point or M admits no nontrivial Hamiltonian holomorphic vector field, $\mathcal{V}_\theta$ is strictly convex along geodesics in $\mathcal{H}_{\omega_0}$. Then, the results of Chen and Tian [30] on the regularity of weak geodesics imply uniqueness of solution of the twisted cscK equation (5.118) and that the twisted $\mathcal{K}$-energy $\mathcal{V}_{\theta,\omega_0}$ has a lower bound. The above facts were pointed out by Stoppa in [67], where he used the lower bound of $\mathcal{V}_{\theta,\omega_0}$ to get a slope stability condition.

Let $D \subset M$ be an effective divisor. The Seshadri constant of D with respect to the Kähler class $[\omega_0]$ is given by

$$\epsilon(D, [\omega_0]) = \sup\{x \mid [\omega_0] - x2\pi c_1(D) \in \mathcal{K}\}, \tag{5.120}$$

where $\mathcal{K}$ is the Kähler cone. Stoppa also defined the twisted Ross-Thomas polynomial of $(M, [\omega_0])$ with respect to D and θ by

$$\mathcal{F}_{\theta,D}(\lambda) = \int_0^\lambda (\lambda - x)\alpha_2(x)dx + \frac{\lambda}{2}\alpha_1(0) - \frac{\bar{S}_\theta}{2}\int_0^\lambda (\lambda - x)\alpha_1(x)dx, \tag{5.121}$$

where

$$\alpha_1(x) = \frac{1}{(m-1)!}\int_M 2\pi c_1(D) \cup ([\omega_0] - 2x\pi c_1(D))^{m-1}, \tag{5.122}$$

$$\alpha_2(x) \tag{5.123}$$
$$= \frac{\int_M 2\pi c_1(D) \cup (2\pi c_1(M) - [\theta] - 2\pi c_1(D)) \cup ([\omega_0] - x 2\pi c_1(D))^{m-2}}{2(m-2)!}.$$

In [67], it was proved that if (5.118) is solvable in $[\omega_0]$ then $\mathcal{F}_{\theta,D}(\lambda) \geq 0$ for all effective divisors $D \subset M$ and $0 \leq \lambda \leq \epsilon(D, [\omega_0])$. In fact, see Theorem 3.1 in [67], we can find a family of Kähler metrics $\omega_\epsilon \in [\omega_0]$ with $\omega_\epsilon|_{\epsilon=1} = \omega_0$ such that as $\epsilon \to 0$

$$\mathcal{V}_{\theta,\omega_0}(\omega_\epsilon) = -\pi \mathcal{F}_\theta(\lambda) \log(\epsilon) + l \cdot o \cdot t. \tag{5.124}$$

By the calculation in [67] (Lemma 3.12, Lemma 3.15), we also have the following asymptotic behavior of the Aubin's functional

$$J_{\omega_0}(\omega_\epsilon) = -\frac{\pi}{2} \int_0^\lambda (\lambda - x)\alpha_1(x) dx \log(\epsilon) + l \cdot o \cdot t. \tag{5.125}$$

By the above Moser-Trudinger inequality (5.9) in Theorem 5.1.1, we can obtain a strictly slope stability. In fact, we have the following corollary.

Corollary 5.4.1. *Let (M, ω_0) be a Kähler manifold, and $\theta \in [\alpha] = 2\pi c_1(M) - k[\omega_o]$ is a real closed semipositive $(1,1)$-form, where $k > 0$. Assuming that the twisting form θ is strictly positive at a point or M admits no nontrivial Hamiltonian holomorphic vector field. If there exists a generalized Kähler-Einstein metric in $\omega \in \mathbb{K}_{\omega_0}$, then there exists a uniform positive constant C_4 such that*

$$\mathcal{F}_{\theta,D}(\lambda) \geq C_4 \int_0^\lambda (\lambda - x)\alpha_1(x) dx > 0 \tag{5.126}$$

for all effective divisors $D \subset M$ and $0 < \lambda \leq \epsilon(D, [\omega_0])$.

In a special case of the generalized Kähler-Einstein equation (5.1), if

$$[\alpha] = (1-k)[\omega_0],$$

where $0 < k < 1$, we set $\theta = (1-k)\omega_0$. Then the generalized Kähler-Einstein equation (5.1) is just the Aubin's equation

$$\rho(\omega) = (1-k)\omega_0 + k\omega. \tag{5.127}$$

The twisted $\mathcal{K}$-energy $\mathcal{V}_{(1-k)\omega_0,\omega_0}$ can be expressed by

$$\mathcal{V}_{(1-k)\omega_0,\omega_0}(\varphi) = \mathcal{V}_{\omega_0}(\varphi) + (1-k)(I_{\omega_0} - J_{\omega_0})(\varphi), \tag{5.128}$$

for all $\varphi \in \mathcal{H}_{\omega_0}$, where $\mathcal{V}_{\omega_0}$ is the Mabuchi $\mathcal{K}$-energy, I_{ω_0} and J_{ω_0} are the Aubin's energy functionals. If there exists a Kähler metric $\omega \in [\omega_0]$ such that

$$\rho(\omega) - k\omega > 0. \tag{5.129}$$

Let $\theta = (1-k)\omega' = \rho(\omega) - k\omega > 0$, we know that the generalized Kähler-Einstein equation (5.1) can be solved in $[\omega_0]$. By Theorem 5.1.1, we know that

$$\mathcal{V}_{(1-k)\omega',\omega_0} \quad \text{is proper.}$$

In fact, it satisfies the Moser-Trudinger type inequality (5.9).

On other hand, by Lemma 5.2.1, the cocycle identity of $\mathcal{M}_\theta$ and properties of I_ω, J_ω (see (5.7), (5.18) and (5.17)), it is easy to see that the properness of the twisted $\mathcal{K}$-energy $\mathcal{V}_{\theta,\omega}$ is independent on the choice of the twisting form $\theta \in [\alpha]$ and Kähler metric $\omega \in [\omega_0]$. So, we have the following corollary which also was proved by G. Székelyhidi in [68].

Corollary 5.4.2. *Let (M, ω_0) be a Kähler manifold with $2\pi c_1(M) = [\omega_0]$, and $0 < k < 1$. The following are equivalent.*

(1) We can uniquely solve equation (5.127).

(2) There exists a Kähler metric $\omega \in [\omega_0]$ such that $\rho(\omega) > k\omega$.

(3) For any Kähler metric $\omega \in [\omega_0]$, $\mathcal{V}_\omega(\varphi) + (1-k)(I_\omega - J_\omega)(\varphi)$ is proper.

(4) For any Kähler metric $\omega \in [\omega_0]$, there exist uniform positive constants C_5 and C_6 such that

$$\mathcal{V}_\omega(\varphi) + (1-k)(I_\omega - J_\omega)(\varphi) \geq C_5 J_\omega(\varphi) - C_6, \tag{5.130}$$

for all $\varphi \in \mathcal{H}_\omega$.

REFERENCES

[1] S. Alesker, *Quaternionic Monge-Ampère equation and Calabi problem for HKT-manifolds*, Israel Journal of Math., 176 (2010), 109-138.

[2] T.Aubin, *Équation du type Monge-Ampère sur les variétés Kählerienes compactes*, C. R. Acad. Sci. Paris Sér. A-B **283**, (1976), A119-A121.

[3] T.Aubin, *Nonlinear analysis on manifolds, Monge-Ampère equation* Springer-Verlag, Berlin, New York, 1982.

[4] S.Bando, *The K-energy map, almost Einstein Kähler metrics and an inequality of the Miyaoka-Yau type*, Tohuku Math.J. **39**(1987), 231-235.

[5] S.Bando and T.Mabuchi, *Uniqueness of Einstein-Kähler metrics modulo connected group actions*, Algebraic Geometry, Adv.Studies in Pure math. **10**(1987).

[6] E. Bedford, B. A. Taylor, *The Dirichlet problem for a complex MongeAmpère equation*, Inventiones Mathematicae 37, no. 1 (1976), 1-44.

[7] E. Bedford and J. Fornaess, *Counterexamples to regularity for the complex Monge-Ampère equation*, Invent. Math. 50 (1978/79), no. 2, 129-134.

[8] A. Besse, *Einstein manifolds*, Ergeb. Math. Grenzgeb. Band 10. Springer-Verlag, Berlin and New York, (2009).

[9] B.J. Bian, P.F. Guan, *A Microscopic Convexity Principle for Nonlinear Partial Differential Equations*, Invent. Math., V. 177, (2009), 307-335.

[10] R.L.Bishop and S.Goldberg *On the second cohomology group of a Kähler manifold of positive curvature*. Proc. Amer. Math. Soc. **16**, (1965), 119–122.

[11] Z. Blocki, On the regularity of the complex Monge-Ampère operator, Contemporary math. **222**, Complex Geometric Analysis in Pohang, ed. K.-T.Kim, S.G.Krantz, (1999), 181-189.

[12] Z. Blocki, *Interior regularity of the complex Monge-Ampère equation in convex domains*, Duke Mathematical Journal, 105 (2000), 167-181.

[13] Z. Blocki, *Weak solutions to the complex Hessian equation*, Annales de L'Institut Fourier (Grenoble), 55, no. 5 (2005) 1735-56.

[14] Z. Blocki, *A gradient estimate in the Calabi-Yau theorem*, Math. Annalen, 344 (2009), 317-327.

[15] S.Bochner and K.Yano, *Curvature and Betti numbers*. Annals of Math. Studies. No.32, Princeton University Press, Princeton, 1953.

[16] X. Cabré, *Nondivergent elliptic equations on manifolds with nonnegative curvature*, Comm. Pure Appl. Math., 50, (1997), 623-665.

[17] X. Cabré, L. Caffarelli, *Fully nonlinear elliptic equations*, Colloquium Publications, volume 43, AMS, 1995.

[18] L. Caffarelli, *Interior a priori estimates for solutions of fully nonlinear equations*, Ann. of Math., (2) 130 (1989), no. 1, 189-213.

[19] L. Caffarelli, *Interior $W^{2,p}$ estimates for solutions of Monge-Ampère equations*,Ann. of Math., 131 (1990), 135-150.

[20] S. S. Chern, *On holomorphic mappings of Hermitian manifolds of the same dimension*, Proc. Symp. Pure Math. 11, Amer. Math. Soc. (1968), 157-170.

[21] Caffarelli, L., J. J. Kohn, L. Nirenberg, and J. Spruck, *The Dirichlet problem for nonlinear second-order elliptic equations. II*, Communications on Pure and Applied Mathematics 38, no. 2 (1985), 209-52.

[22] S. Campanato, *Proprieta di una famiglia di spazzi funzionali,* Ann. Sc. Norm. Super. Pisa Cl. Sci. **18** (1964) (3), 137-160.

[23] Caffarelli, L., L. Nirenberg, and J. Spruck *The Dirichlet problem for nonlinear second-order elliptic equations. III*, Acta Mathematica 155, no. 34 (1985): 261-301.

[24] E. Calabi, *Improper affine hyperspheres and a generalization of a theorem of K. Jörgens,* Mich. Math. J. **5** (1958), 105-126.

[25] E. Calabi, *On Kähler manifolds with vanishing canonical class, in Algebraic geometry and topology.* A symposium in honor of S. Lefschetz, pp. 78-89. Princeton University Press, Princeton, N. J., 1957.

[26] E. Calabi, *Extremal Kähler metrics*, Seminar on Differential Geometry, pp. 259–290, Ann. of Math. Stud., **102**, Princeton Univ. Press, Princeton, N.J., 1982.

[27] H.D.Cao, *Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds*, Invent. Math. **81** (1985), no. 2, 359–372.

[28] P. Cherrier, *Équations de Monge-Ampère sur les variétés hermitiennes compactes. (French)*, Bull. Sci. Math. (2) **111**(1987), no.4, 343-385.

[29] P. Cherrier, *Le probléme de Dirichlet pour des équations de Monge-Ampère complexes modifiées*, J. Funct. Anal. 156 (1998), 208-251.

[30] X.X. Chen, G. Tian, *Geometry of Kähler metrics and foliations by holomorphic discs*, Publ. Math. Inst. Hautes Études Sci. No. 107 (2008), 1-107.

[31] Y. Z. Chen, L. C. Wu, *Second order elliptic equations and Systems of elliptic equations,* Science press, Beijing, 1991.

[32] S.Y.Cheng, S.T. Yau, *On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman's equation*, Comm.Pure Appl. Math. **33**(1980), no.4, 507-544.

[33] S. Dinew, S. Kolodziej, *Pluri-potential estimates on compact Hermitian manifolds*, arXiv: 0910.3937.

[34] J.P. Demailly, T. Peternell and M. Schneider, *Compact complex manifolds with numerically effective tangent bundles*, J. Algebraic Geom. 3 (1994), no. 2, 295-345.

[35] S. Dinew, Xi Zhang and X.W. Zhang, *The* $\mathcal{C}^{2,\alpha}$ *estimate of complex Monge-Ampère equation*, to appear on Indiana Univ. Math. J.

[36] W.Y. Ding, *Remarks on the existence problem of positive Kähler-Einstein metrics*, Math. Ann. **282**, 463-471.

[37] W.Y. Ding and G. Tian, *Kähler-Einstein metrics and the generalized Futaki invariant*, Invent. Math. **110**(1992), no. 2, 315–335.

[38] S.K. Donaldson, *Symmetric spaces, Kähler geometry and Hamiltonian dynamics*, Northern California Symplectic Geometry Seminar, 13–33, Amer. Math. Soc. Transl. Ser. 2, **196**, Amer. Math. Soc., Providence, RI, 1999.

[39] S.K. Donaldson, *Scalar curvature and stability of toric varieties*, J. Diff. Geom. **62** (2002), no.2, 289-349.

[40] S.K. Donaldson, *Two-forms on four-manifolds and elliptic equations*, in Inspired by S.S. Chern, World Scientific, 2006.

[41] L.C. Evans, *Classical solutions of fully nonlinear, convex, second order elliptic equations*, Comm. Pure Appl. Math. 25 (1982), 333-363.

[42] S.Goldberg and S.Kobayashi *Holomorphic bisectional curvature*, J.Differential Geometry. **1**, (1967), 225–233.

[43] P. Griffiths, A. Harris, *Principles of algebraic geometry*, Hohn Wiley and Sons, New York, (1978).

[44] B. Guan, Q. Li , *Complex Monge-Ampère equations on Hermitian manifolds*, (225), Issue 3, (2010), 1185-1223.

[45] P. Guan, *Extremal function associated to intrinsic norms*, Annals of Mathematics, 156(2002), 197-211.

[46] P. Guan, *On the Gradient Estimate for Monge-Ampère Equation on Kähler Manifold*, Private Notes.

[47] P. Guan, *Remarks on the homogeneous complex Monge-Ampère equation*, Proceedings of Conference in Complex Analysis, 2008.

[48] D. Gilbarg, N. S. Trudinger, *Elliptic partial differential equations of second order,* Springer 1983.

[49] R.S. Hamilton, *The Harnack estimate for the Ricci flow*, J. Differential Geometry, 37 (1993), 225-243.

[50] A.Hanani, *Equations du type de Monge-Ampère sur les varieties hermitiennes compactes. (French)*, J. Funct. Anal. **137** (1996), no. 1, 49–75.

[51] L. Hörmander, *Notions of Convexity* (Progress in Mathematics), Birkhäuser Boston, 1994.

[52] Z.L. Hou, *Complex Hessian equation on Kähler manifold*, IMRN, Vol. 2009, no.16, 3098-3111.

[53] Z.L. Hou, X.N. Ma, D.M. Wu, *A second order estimate for complex Hessian equations on a compact Kähler manifold*, Math. Res. Lett. 17 (2010), no. 3, 547-561.

[54] A. Howard, B. Smyth and H. Wu, *On compact Kähler manifolds of nonnegative bisectional curvature, I*, Acta Math. **147**, (1981), 51-56.

[55] H.-Y. Jian, X.-J. Wang, *Continuity estimates for the Monge-Ampère equation,* SIAM J. Math. Anal. **39** (2007), 608-626.

[56] N. V. Krylov, *Boundedly inhomogeneous elliptic and parabolic equations*, Izv. Akad. Nauk. SSSR Ser. Mat. 46 (1982), 487-523; English translation in Math USSR Izv. 20 (1983), 459-492.

[57] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, Vol. I.

[58] S.Y. Li, *On the Dirichlet problems for symmetric function equations of the eigenvalues of the complex Hessian*, Asian Journal of Mathematics 8, no. 1 (2004), 87-106.

[59] Y. Lu, *Holomorphic mappings of complex manifolds*, J. Diff. Geom. (1968), 299-312.

[60] N.G. Meyers, *On a class of non-uniformly elliptic quasi-linear equations in the plane*, Arch. Rational Mech. Anal. **12**, (1963), 367-391.

[61] A. V. Pogorelov, *The multidimensional Minkovski problems,* Wiley and Sons , New York, 1978.

[62] D.H. Phong, N. Sesum, J. Sturm, *Multiplier ideal sheaves and the Kähler-Ricci flow*, Communications in Analysis and Geometry, **15**, (2007), 613-632.

[63] D.H.Phong, J.Song, J.Sturm and B.Weinkove, *The Moser-Trudinger inequality on Kähler-Einstein manifolds*, Amer.J.Math., 130 (2008), 1067-1085.

[64] D. Riebesehl, F. Schulz, *A Priori Estimates and a Liouville Theorem for Complex Monge-Ampère Equations,* Math. Z. **186** (1984), 57-66.

[65] S.T. Siu, *The existence of Kähler-Einstein metrics on manifolds with positive anticanonical line bundle and a suitable finite symmetry group*, Ann. of Math. (2) **127** (1988), no. 3, 585-627.

[66] J. Song and G. Tian, *Canonical measures and Kähler-Ricci flow*, to appear on J. Amer. Math. Soc. .

[67] J. Stoppa, *Twisted constant scalar curvature Kähler metrics and Kähler slope stability*, J. Diff. Geom. **83** (2009), 663-691.

[68] G. Szekelyhidi, *Greatest lower bounds on the Ricci curvature of Fano manifolds*, Compositio Math. 147 (2011), 319-331.

[69] G.Tian, *On Kähler-Einstein metrics on certain Kähler manifolds with* $C_1(M) > 0$, Invent.Math. **89**, 1987, 225-246.

[70] G.Tian, *On Calabi's conjecture for complex surfaces with positive first Chern class*, Invent. Math. **101** (1990), no. 1, 101–172.

[71] G.Tian, *Kähler-Einstein metrics with positive scalar curvature.* Invent.Math. **137**, 1997, 1-37.

[72] G.Tian, *Canonical metrics in Kähler geometry*, Lecture Notes in Mathematics, Birkhäuser (2000).

[73] G.Tian, S.T. Yau, *Complete Kähler manifolds with zero Ricci curvature. I*, J. Amer. Math. Soc. **3**(1990), no.3, 579-609.

[74] G.Tian, S.T. Yau, *Complete Kähler manifolds with zero Ricci curvature. II*, Invent. Math. **106**(1991), no.1, 27-60.

[75] G.Tian and S.T. Yau, *Kähler-Einstein metrics on complex surfaces with* $C_1 > 0$, Comm. Math. Phys. **112** (1987), no. 1, 175-203.

[76] V. Tosatti, B. Weinkove, *Estimates for the complex Monge-Ampère equation on Hermitian and balanced manifolds*, Asian J. Math. 14 (2010), no.1, 19-40.

[77] V. Tosatti, B. Weinkove, *The complex Monge-Ampère equation on compact Hermitian manifolds*, J. Amer. Math. Soc. 23 (2010), no.4, 1187-1195.

[78] V. Tosatti, B. Weinkove, S.T. Yau, *Taming symplectic forms and the Calabi-Yau equation*, Proc. London Math. Soc. (3) **97** (2008), 401-424.

[79] X.-J. Wang, *Schauder estimates for elliptic and parabolic equations,* Chin. Ann. Math., 27 (2006)(2), 637-642.

[80] D.Wu, S-T.Yau and F.Zheng, *A degenerate Monge-Ampère equation and the boundary classes of Kähler cones*. Math. Res. Lett. **16**, (2009), no.2, 365–374.

[81] S.T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation*, Comm.Pure Appl. Math. **31**(1978), 339-411.

[82] S.T. Yau, *Open problems in differential geometry*, Internat. Press, 1992.

[83] Xi Zhang, X.W. Zhang, *Some regularity estimates for the complex Monge-Ampère equations on Hermitian manifolds*, Journal of Functional Analysis, Vol. 260, Issue 7, 2004-2026, (2011).

[84] Xi Zhang, X.W. Zhang, *Generalized Kähler-Einstein metrics and Energy functionals*, submitted.

[85] Xi Zhang and X.W. Zhang, *Schauder estimate of complex Monge-Ampère equation*, Private Notes.

[86] X.W. Zhang, *On the boundary of Kähler cone*, to appear on Proc. Amer. Math. Soc..

[87] X.W. Zhang, *A priori estimates for complex Monge-Ampère equation on Hermitian manifolds*, International Mathematics Research Notices, Vol. 2010, Issue 19, 3814-3836, (2010).