Partial Decode-Forward in Relay Networks

Yao Tang

Department of Electrical & Computer Engineering
McGill University
Montreal, Canada

August 2013

© 2013 Yao Tang

2013/09/12

Abstract

Cooperative transmission acts as a distributed solution for providing robust wireless communications. It pools available resources, such as power and bandwidth, across the network. The relay network is the simplest information theoretic model for a cooperative wireless network and a full understanding of communication limits over such a network can potentially shed light on the design of more efficient wireless networks. However, the capacity of the relay network is still unknown. As a step towards the goal of calculating the capacity, we derive the capacity bounds for a single-source single-destination relay network based on partial decode-forward.

In the first part of the thesis, we review existing bounds on the capacity of the discrete memoryless relay channel. We also review decode-forward and partial decode-forward in the relay network.

In the second part of the thesis, we first introduce a discrete memoryless relay network model consisting of one source, one destination and N relays. We then design a scheme based on partial decode-forward relaying. The source splits its message into one common part and $N+1$ private parts which are to be decoded at different relays. The source encodes split message parts using length-N block Markov coding, in which each private message part is independently superimposed on the common parts of the current and N previous blocks. Using joint sliding window decoding, each relay fully recovers the common message part and its intended private message part with the same block index, then forwards them together to the following nodes in the next block. We derive the achievable rate of this scheme in a compact form. The result is a generalization of and can be particularized to a known decode-forward lower bound for an N-relay network and partial decode-forward lower bound for a two-relay network. We then apply our proposed scheme to a Gaussian relay network and obtain its capacity lower bound considering power constraints at transmitting nodes.

In the third part of the thesis, we introduce the concept of exhaustive message splitting for partial decode-forward in a single-source single-destination relay network with N relays, in which the relays are divided into subsets, and each different relay subset has a distinct private message part to decode. We study this scheme in more depth in a three-relay network based on block Markov encoding. We derive its achievable rate. Finally, we apply this scheme to a Gaussian three-relay network and show that our scheme generalizes network decode-forward and the private message splitting scheme as shown in the second

part.

Abrégé

La transmission coopérative agit comme étant une solution diffusée, permettant de fournir une communication sans fil très robuste. Elle regroupe les ressources disponibles, comme la puissance et la bande passante, à travers le réseau. Le réseau à relais est le modèle théorique le plus simple pour un réseau sans fil coopératif et la compréhension des limites de communication à travers de ce réseau permet potentiellement de saisir la conception d'un réseau sans fil plus efficace. Par contre, la capacité du réseau à relais est encore inconnue. Dans cette thèse, nous adressons la conception du schéma du decode-forward partiel dans un réseau à relais ayant une source et une destination unique.

Dans la première partie de cette thèse, nous révisons le réseau à relais. Nous révisons aussi le decode-forward ainsi que le decode-forward partiel dans les réseaux à relais.

Dans la deuxième partie de cette thèse, nous introduisons un réseau à relais sans mémoire composé d'une source, d'une destination et de relais N. Par la suite, nous concevons un schéma à partir du réseau decode-forward partiel. La source divise son message en une partie commune et en parties privées $N+1$ qui sont destinées à des relais diffée length-N black Markov, dans lequel chaque partie du message privé est superposé indépendamment sur les parties communes du bloc actuel et des blocs précédents N. En utilisant le décodage conjoint fenêtre coulissante, chaque relais récupère la partie du message commune et sa partie du message privé ayant le même index bloc, et les envoi par la suite ensemble aux nœuds du bloc suivant. Nous dérivons son taux réalisable dans un format compact. Le résultat permet de réduire la borne inférieure connue du decode-forward d'un réseau à N-relais et la borne inférieure du decode-forward partiel pour un rseau à deux relais. Nous appliquons par la suite notre schéma propositionnel à un réseau à relais gaussien et d'obtenir sa capacité de borne inférieure en prenant en compte les contraints de puissance aux nœuds émetteurs.

Dans la troisième partie de cette thèse, nous introduisons le fractionnement exhaustif des messages pour le decode-forward partiel ayant une source et une destination unique dans un réseau à relais ayant N relais, dans lequel chaque sous-ensemble diffrent du relais contient un message priv distinct à décoder. Nous étudions profondément ce schéma dans un réseau à trois relais basé sur le codage un bloc Markov. Nous dérivons son taux réalisable. Nous fournissons un graphique orienté qui détaille le superpositionnement de la structure du codebook afin d'aider le lecteur à comprendre la hiérarchie de la génération du codeword.

Finalement, nous appliquons ce schéma à un réseau à trois relais gaussien et nous montrons que notre schéma généralise le relais decode-forward et le schéma du fractionnement du message privé comme démontré dans la seconde partie.

Acknowledgments

It has been an reward journey towards the end of my master studies. I would like to take this opportunity to acknowledge many individuals and teachers who supported me and inspired me to pursue my studies firmly.

First and foremost, I would like to thank and express my gratitude to my supervisor, Professor Fabrice Labeau. He gave me the opportunity to join his group and work with him towards the closing of my master studies. I am grateful to his generous support and humble attitude. I highly value his instruction on my thesis organization and his patient revision on my thesis writing.

I would like to express my gratitude to my former supervisor Professor Mai Vu who gave me the opportunity to start my journey in McGill and led me to understanding the topic of information theory. I am thankful for her funding throughout my master studies. I also admire Mai's outstanding research attitude and sharp problem-solving perspective. Her comments and feedback on my research inspired me a lot. I wish her great career success in Tufts.

I would like to express my appreciations to all my previous colleagues in the lab, including Ahmad, Peng, Zhuohua and Jun. They shared a lot with me in all the aspects of life. They bought fresh ideas and minds into my life so that I can adjust myself quickly. I would not be able to make such remarkable progress without their generosity and help. I enjoyed working with them.

Finally, I would like to express sincere gratitude to my families both in China and Canada. First, I am grateful to my family in Canada, including Jianqing, Ying Ying and Nancy. They treat me as their own son and provid me with great dinners, comfortable accommodations and meticulous cares. I believe that the precious time in Brossard would become a beautiful memory in my life. Next, I would like to thank my parents. They spare no effort to support me studying abroad. Without their love and encouragement, I would never have been able to finish my study at McGill.

Contents

List of Figures

List of Acronyms

RC	Relay Channel
RN	Relay Network
DM-RC	Discrete memoryless relay channel
PMF	Probability mass function
DF	Decode-forward
PDF	Partial decode-forward
AWGN	Additive White Gaussian Noise
FME	Fourier-Motzkin Elimination
CF	Compress-forward
NNC	Noisy network coding

Chapter 1

Introduction

1.1 Background

With the rapid development of wireless networks, cooperative communication is becoming more and more popular. In a wireless network with several active nodes including a source and a destination, due to the broadcast nature of wireless communication, several nodes in the network might overhear the signal transmitted from the source. If the direct transmission between the source and destination fails, those nodes which have copies of the transmitted signal can help to re-establish or enhance the coomunication between the intended source-destination pair. Nodes that take part in the transmission are called **relays**[1]. When relays cooperate, source messages are conveyed via multiple paths.

The relay channel was first introduced by van der Meulen [1]. It consists of a source aiming to communicate with a destination with the help of a relay. In [2], Cover and El Gamal innovatively introduce the cutset bound and two coding strategies, namely decode-forward and compress-forward, for the classical three-node relay channel. In decode-forward, the relay fully decodes the message, which requires high communication channel quality between the source and the relay. By allowing the relay to decode only a part of the transmitted message, **partial decode-forward** has the potential of yielding achievable rate gains through a better exploitation of the relay channel. Partial decode-forward can be considered as the

[1]In this thesis, we regard a node that has its own message to send as the source. We also regard the node that is interested in the message transmitted from the source as the destination. We assume that no message originates from the relay, and that the relay is not the intended destination of the transmitted message.

generalization of decode-forward as well [2], [3].

In this thesis, we are dedicated to extending the classical one-relay channel to a general relay network consisting of N communicating parties. An immediate question which emerges from the setup in the relay network is how to generalize decode-forward relaying to a general N-node relay network, for example, how to distribute split message parts among multiple relays. The main theme of this thesis is to investigate this question. In this thesis, we explore coding schemes based on partial decode-forward for a single-source single-destination relay network with N relays.

1.2 Thesis Contributions

In this thesis, we focus on how to extend partial decode-forward to a single source single destination relay network with N relays. We design two schemes based on the way the source splits its original message among different relays and the destination. Each split message part is conveyed through certain pre-assigned supposed relays, while other relays do not help the transmission of this message part.

We categorize our schemes into two schemes, namely, private message splitting scheme and exhaustive message splitting scheme.

In private message splitting scheme, the source splits its original message into $N + 1$ private parts and one common part. The word **private** means that only one relay is supposed to decode and re-transmit this specific message part while other relays do not decode it. On the other hand, the destination intends to decode all split message parts. We use this terminology throughout this thesis. The whole transmission process is over multiple transmission blocks[2]. The transmitter sends an n-symbol codeword in each block. Blocks are indexed consecutively. In this scheme, each relay fully recovers the common message part and its intended private message part of the current block, then forwards them to the following nodes when the last common message part of the same block index arrives. In exhaustive message splitting scheme, the source splits its original message considering all possible split message parts decoding situations that can occur between the destination and all relays. We use the term **exhaustive** to denote that each possible subset of relays has a distinct message part to decode, while any other relay subsets do not decode it. In

[2]The concept of block is illustrated in [4]. If we divide the whole transmission into nb channel uses and let each block consists of n channel uses, we then will have b transmission blocks.

this scheme, all relays are pre-separated to different subsets and each relay subset helps to decode a distinct split message part.

1.3 Thesis Organizations

This section outlines the thesis and summarizes main contributions.

Chapter 2

In Chapter 2, we review the capacity results for the discrete memoryless relay channel and the N-relay network. We first formally define the discrete memoryless relay channel and derive the cutset bound on capacity of the discrete memoryless relay channel. Next, we introduce four lower bounds on capacity of the relay channel. In the second half of Chapter 2, we review the capacity results for the single-source single-destination discrete memoryless relay network. These capacity results are based on decode-forward and partial decode-forward. Finally, we state the motivation of this thesis.

Chapter 3

In Chapter 3, we first introduce a discrete memoryless N-relay network model. We then propose a partial decode-forward scheme based on private message splitting in a single-source single-destination relay network with N relays. The source splits its message into one common part and $N + 1$ private parts. Each relay decodes the common part and its supposed private part. This scheme includes Aref's scheme [5] and Xie's scheme [6] as special cases. In the second half of this chapter, we first provide a Gaussian relay network model. We then derive our capacity results in the Gaussian relay network model and compare our scheme with full decode-forward.

Chapter 4

In Chapter 4, we first propose a partial decode-forward scheme based on exhaustive message splitting scheme for a discrete memoryless three-relay network. The source splits its message into eight parts. Each possible relay subset has its own private message part to decode. We then derive the capacity results of this scheme and show that it includes decode-forward as a special case. Next, we discuss the block Markov encoding structure of this scheme when it is extended in an N-relay single-source single-destination relay network. Finally, we derive the capacity result in the Gaussian three-relay network model and compare it with scheme shown in Chapter 3.

Chapter 5

Chapter 5 concludes this thesis.

1.4 Notations

In this thesis, we use the following notation conventions.

$\mathcal{M}$	A set of messages.
m	A message to be transmitted.
$\hat{m}$	A detected message when m is transmitted.
$p(x)$	probability mass function (pmf) of the random variable X.
$p(x,y)$	joint pmf of the random variables X and Y.
$p(x\|y)$	conditional pmf of the random variable X given $Y=y$.
$X \sim p(x)$	The random variable X is distributed according to $p(x)$.
$(\mathcal{X}, p(y\|x), \mathcal{Y})$	The channel with input alphabet $\mathcal{X}$, output alphabet $\mathcal{Y}$, and pmf $p(y\|x)$ where $x \in \mathcal{X}$ and $y \in \mathcal{Y}$.
$H(X)$	Entropy of the discrete random variable X.
$H(X,Y)$	Joint entropy of the discrete random variables X and Y.
$H(X\|Y)$	Conditional entropy of the discrete random variable X given Y.
$I(X;Y)$	Mutual information between the random variables X and Y.
$I(X;Y\|Z)$	Conditional mutual information between the random variables X and Y given Z.
$\mathcal{N}(\mu, \alpha^2)$	Gaussian pdf with the mean μ and the variance α^2.
x^n	$(x_1, x_2, \ldots, x_n)$.
$[1:n]$	The set $\{1, 2, \ldots, n\}$.
$\mathcal{T}_\epsilon^{(n)}$	ϵ-typical sequences of length n (see Appendix A).
$\mathcal{A} \subseteq \mathcal{B}$	$\mathcal{B}$ includes A.
$\|\mathcal{A}\|$	Cardinality of the set $\mathcal{A}$.
$\mathcal{X} \times \mathcal{Y}$	Cartesian product of $\mathcal{X}$ and $\mathcal{Y}$.
$\|\|x\|\|$	The Euclidean norm of the vector x.

$\mathbf{x}$	A vertical vector (bold-face lower case letter) .
$\mathbf{A}$	A matrix (bold-face capital letter).
min	Minimum.
max	Maximum.
sup	Supremum.

Chapter 2

Literature Review

Recently, cooperative communication through relay networks has received considerable research interests due to its potential for increasing wireless coverage and transmission reliability. The classical relay channel (RC) consists of three communicating parties: one source, one destination and one relay aiming to help the communication between the source and the destination [1]. This concept can be extended to large-scale network configuration. Due to the broadcasting nature of wireless communications, more than one relay in the network might overhear the transmitted signals from the source. They can cooperate with each other to strengthen the source-destination pair communication. The relay network (RN) is a general network consisting of multiple communicating parties, which can act as either sources, relays or destinations.

In this thesis, we study partial decode-forward schemes in single-source single-destination relay networks with N relays through different message splitting schemes. **Message splitting** was first introduced in [7], where the source splits its original message into two independent messages and uses superposition coding to superimpose one message onto another corresponding message.

Up to now, partial decode-forward in relay networks has not yet been studied much in the literature. But it has tight relationships with a number of coding schemes in classical relay channels and single-source single-destination relay networks, which will be reviewed in this chapter.

The arrangement in this chapter is as follows:

- In Section 2.1, we define the classical three-node relay channel and review several important capacity bounds in classical relay channel. The key coding strategies are: block Markov superposition encoding, backward decoding, sliding window decoding, binning and message splitting (for partial decode-forward). We will use these techniques as the fundamental tools in our relay networks analysis.

- In Section 2.3, we review works on extending decode-forward and partial decode-forward to single-source single-destination relay networks with more than one relay. The key coding strategy is joint sliding window decoding. This coding strategy sheds light on the coding scheme design that we will propose in Chapter 3 and Chapter 4.
- In Section 2.4, we review our contributions in this thesis and the improvements of our schemes over existing results.

2.1 Classical Relay Channel

In this section, we review the classical three-node relay channel and discuss several capacity bounds in classical relay channel. These bounds constitute the building blocks of capacity results in relay networks.

In our notations, a discrete random variable U is assumed to take values u in a finite set $\mathcal{U}$. We use $|\mathcal{U}|$ to denote the cardinality of $\mathcal{U}$, and $p(u)$[1] to denote the probability mass function (PMF) of U on $\mathcal{U}$. Vectors with length-n are denoted with lower-case letters, e.g. x^n, where the ith element of a vector x^n is denoted by x_i.

The discrete memoryless relay channel (DM-RC) was first introduced by Van der Meulen in 1971 [1]. The communication situation consists of three nodes: one source-destination pair and one relay node. We assume that the relay has no message to send and its role is just to assist the communication between the source and the destination. Figure 2.1 illustrates a relay channel consisting of four finite sets: two input finite sets $\mathcal{X}$ and $\mathcal{X}_r$, two output finite sets $\mathcal{Y}$ and $\mathcal{Y}_r$ and a probability mass function (PMF) $p(y, y_r|x, x_r)$, which represents the stochastic input-output property over the channel. The n-symbol input sequences of the source and the relay are denoted as x^n and x_r^n respectively. After n uses of the channel, the output sequences at the destination and the relay are denoted as y^n and y_r^n. The channel is assumed to be discrete memoryless and thus the channel transition

[1]For brevity, we denote pmf $p_U(u)$ as $p(u)$ when there is no risk of confusion

function is:

$$p(y^n, y_r^n | x^n, x_r^n) = \prod_{i=1}^{n} p(y_i, y_{ri} | x_i, x_{ri}),$$

which means that the received signals at the relay and the destination at time index i only depend on the transmitted signals from the source and the relay at time index i.

A $(2^{nR}, n)$ code for a DM-RC consists of the following: a message set $\mathcal{M} = [1 : 2^{nR}]$, from which the message m is uniformly drawn; an encoding function that assigns a length-n codeword $x^n(m)$ to each message $m \in [1 : 2^{nR}]$; a set of relay functions such that $x_{ri} = f_i(y_{r1}, y_{r2}, \ldots, y_{r,i-1}), \forall i \in [1 : n]$; a decoder function that maps the received signal y^n to an estimate of the message $\hat{m}$ or reports an error message e. The average probability of error is $P_e^{(n)} = P\{\hat{m} \neq m\}$.

The rate R is said to be achievable if there exists a sequence of $(2^{nR}, n)$ codes with $P_e^{(n)} \to 0$ as $n \to \infty$. The capacity C of the relay channel is the supremum of the set of achievable rates. The exact capacity of the relay channel is not known in general.

The rest of this section is organized as follows. In Section 2.1.1, we introduce cutset upper bound, which is up to now the best known upper bound on the capacity of the relay channel. We then present several best known lower bounds on the capacity in the relay channel: In Section 2.1.2, we introduce the direct transmission lower bound. In Section 2.1.3, we introduce the two hop lower bound. In Section 2.1.4, we introduce the decode-forward lower bound and corresponding coding techniques such as block Markov encoding, backward decoding and sliding window decoding. In Section 2.1.5, we introduce the partial decode-forward lower bound and message splitting scheme.

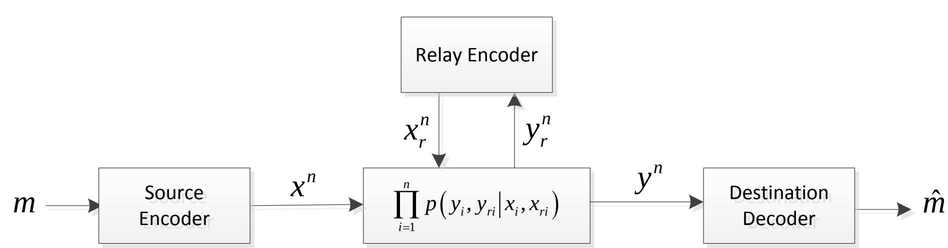

Fig. 2.1 Block diagram of Relay Channel.

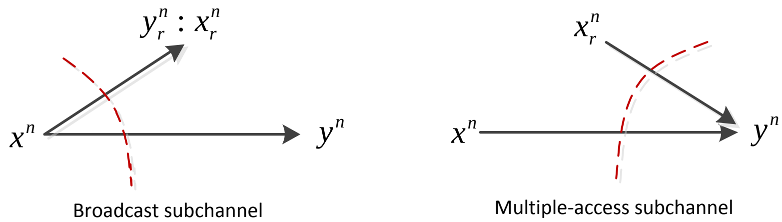

Fig. 2.2 Illustration of cutset upper bound.

2.1.1 Cutset Upper Bound

In this part, we present an upper bound on the capacity of the DM-RC. This upper bound of DM-RC is known as *max-flow min-cut* or *cutset* bound and it consists of two terms [2]:

- Broadcast subchannel bound: $C \leq I(X;Y,Y_r|X_r)$.
- Multiple-access subchannel bound: $C \leq I(X,X_r;Y)$.

Figure 2.2 illustrates the two bounds above. In the broadcast subchannel, the relay performs as another receiver. While in the multiple-access subchannel, the relay plays the role of another transmitter as it helps the message transmission from the source to the destination. Combining these two bounds, we would have:

$$C \leq \max_{p(x,x_r)} \min\{I(X,X_r;Y), I(X;Y,Y_r|X_r)\} \tag{2.1}$$

The cutset bound itself is the best known upper bound on the capacity of the relay channel.

2.1.2 Direct Transmision Lower Bound

If we do not use the relay in the communication protocol and just let it transmit a fixed symbol over all the blocks, which is $X_{ri} = x_r$ for all $i \in [1:n]$, then, the channel becomes the standard point-to-point DMC as in [4]. By using random coding argument [4], we can get the lower bound as:

$$C \geq \max_{p(x),x_r} I(X;Y|X_r = x_r). \tag{2.2}$$

This simple strategy achieves the capacity if the channel can be decomposed as *reversely degraded* relay channel, which means that the received signal at the relay is a degraded version of the received signal at the destination. This, loosely speaking, means that the quality of the source-destination link is fairly good and thus the destination receives a better copy of the transmitted message than the relay. The channel pmf can be decomposed as:

$$p(y, y_r|x, x_r) = p(y|x, x_r)p(y_r|y, x_r),$$

which means $X \to (Y, X_r) \to Y_r$ forms a Markov chain or equivalently $X \to Y \to Y_r$ forms a Markov chain conditioned on X_r. Note that by this definition bound in (2.1) simplify to $I(X;Y|X_r)$ and the rate obtained by direct transmission hence coincides with the cutset bound.

2.1.3 Two-hop Lower Bound

The direct transmission scheme in Section 2.1.2 can be improved by placing a relay in the middle of the source and the destination. Assume that the relay decodes the received signals in each block and then retransmits the decoded message in the next block. Assume also that there is no direct link between the source and the destination. This yields a lower bound on the capacity of DM-RC [8]:

$$C \geq \max_{p(x)p(x_r)} \min\{I(X_r;Y), I(X;Y_r|X_r)\}. \tag{2.3}$$

This bound is tight if the channel consists of two cascaded DMCs, i.e., $p(y, y_r|x, x_r) = p(y_r|x)p(y|y_r)$. We can improve this two-hop coding scheme by letting the source and the relay coherently cooperate with each other in transmitting their codewords, which gives another lower bound that has the same formula as (2.3), but its maximization is over the joint distribution $p(x, x_r)$ over the input sets $\mathcal{X} \times \mathcal{X}_r$ instead of the individual input PMFs $p(x)$ and $p(x_r)$.

2.1.4 Decode-forward Lower Bound

The Decode-Forward (DF) scheme combines the information received through the direct link (Section 2.1.2) with the information received from the relay (Section 2.1.3), which leads to a tighter lower bound in DM-RC. The DF lower bound does not coincide with the cutset upper bound in Section 2.1.1 for a general DM-RC.

The DF scheme allows the relay to transmit a fresh codeword by decoding signals received in the previous blocks. This scheme relies on the successful decoding at the relay and hence its performance is limited by the quality of the source-relay link. In [9], by using block Markov encoding at the source and backward decoding at the destination, the following rate is shown to be achievable with DF:

Theorem 1. *[9] For any relay channel ($\mathcal{X} \times \mathcal{X}_r$, $p(y, y_r|x, x_r)$, $\mathcal{Y} \times \mathcal{Y}_r$),by using decode-forward, the capacity C is lower bounded by:*

$$C \geq \max_{p(x,x_r)} \min\{I(X, X_r; Y), I(X; Y_r|X_r)\}, \tag{2.4}$$

where $I(X, X_r; Y)$ is the MAC bound on the capacity and $I(X; Y_r|X_r)$ represents the capacity of the link between source and relay.

The key coding techniques in this scheme are:

- **Block Markov encoding**: the codeword transmitted in a block depends on the codewords transmitted in the previous blocks;
- **Backward decoding**: decoding at the receiver is done backwards after all blocks are received.

We next briefly outline the achievability proof to illustrate how to apply block Markov encoding and backward decoding to a DM-RC. This method will be useful in future chapters.

Proof of achievability: In Figure 2.1, consider b transmission blocks, each consisting of n channel uses. In this scheme, a sequence of $b-1$ messages m_j[2], uniformly distributed over $[1 : 2^{nR}]$, for all $j \in [1 : b-1]$ is transmitted to the destination. Thus, the transmission rate is $\frac{n(b-1)R}{nb}$, which goes to R as $b \to \infty$.

[2] m_j denotes the message to be transmitted in block j.

Before the communication process, the codebook is generated according to block Markov encoding and then revealed to all communicating parties. The block Markovity between the source signal and the relay signal guarantees that the source and the relay coherently cooperate in transmitting their codewords. In block j, the source broadcasts $x^n(m_j|m_{j-1})$[3] to the relay and the destination. At the end of block j, the relay fully decodes m_j according to the joint typicality lemma (Appendix A) and retransmits it to the destination in the following block. After receiving signals from all blocks, the destination backward decodes messages block by block from the end of the last block.

Proof. (1) *Codebook generation*: Fix $p(x, x_r)$ achieving the lower bound in (2.4).

- For $j \in [1 : b]$, randomly and independently generate 2^{nR} sequences $x_r^n(m_{j-1})$ for all $m_{j-1} \in [1 : 2^{nR}]$, each according to $\prod_{i=1}^{n} p_{X_r}(x_{ri})$.
- For each $m_{j-1} \in [1 : 2^{nR}]$, randomly and conditionally independently generate 2^{nR} sequences $x^n(m_j|m_{j-1})$ for all $m_j \in [1 : 2^{nR}]$, each according to $\prod_{i=1}^{n} p_{X|X_r}(x_i|x_{ri}(m_{j-1}))$.

The codebook is then revealed to all the parties.

(2) *Encoding*: To send m_j in block j, the source encoder transmits $x^n(m_j|m_{j-1})$. At the end of block j, the relay has an estimate $\tilde{m}_j$ of message m_j and transmits $x_r^n(\tilde{m}_j)$ in block $j + 1$.

(3) *Decoding*:

- Decoding message m_j at the relay. At the end of block j, the relay receives $y_r^n(j)$ and knows m_{j-1}. It then looks for a unique message $\tilde{m}_j \in \mathcal{M}$ such that

$$(x^n(\tilde{m}_j|m_{j-1}), x_r^n(m_{j-1}), y_r^n(j)) \in \mathcal{T}_\epsilon^{(n)},$$

otherwise it declares an error. By the packing lemma (Lemma 2 in Appendix A), $P(\tilde{m}_j \neq m_j) \to 0$ as $n \to \infty$ if

$$R < I(X; Y_r|X_r) - \delta(\epsilon). \tag{2.5}$$

[3] $x^n(m_j|m_{j-1})$ denotes a length-n codeword that encodes m_j. The generation of this codeword is dependent on the codeword that encodes message m in block $j - 1$.

- The destination waits until it receives all blocks and backward decodes successively from the end of block b. Assume that destination at the block j knows m_{j+1} and has received signal $y^n(j)$. It then looks for a unique message $\hat{m}_j$ such that

$$(x^n(m_{j+1}|\hat{m}_j), x_r^n(\hat{m}_j), y^n(j)) \in \mathcal{T}_\epsilon^{(n)},$$

otherwise, it declares an error. By the packing lemma (Lemma 2 in Appendix A), $P(\hat{m}_j \neq m_j) \to 0$ as $n \to \infty$ if

$$R < I(X, X_r; Y) - \delta(\epsilon). \tag{2.6}$$

Combining (2.5) and (2.6), we can get Theorem 1. □

Up to now, there have been three common approaches to realize decode-forward strategies, namely: (a) irregular encoding/sequential decoding; (b) regular encoding/sliding window decoding; (c) regular encoding/backward decoding [10]. Here regular encoding refers to block Markov encoding. The irregular encoding is the strategy used in [2], where the encoding is done using codebooks of different size, hence the name. For many classes of relay networks [11], the second and third approaches can achieve the same rate, which are greater than that of the first approach. Furthermore, the second approach creates much smaller delay than the third one. In short, by far, sliding window decoding is the best decoding scheme that works in DF.

Sliding window decoding means that the decoder uses multiple consecutive blocks of channel outputs to decode one single message.

Next, we show how DF realizes **sliding window decoding** in a DM-RC, where the destination uses two consecutive blocks of channel outputs to decode the source message transmitted in the previous block.

Consider the channel configuration as shown in Figure 2.1. The codebook generation, the encoding operation and the relay decoding follow the same procedures shown in the earlier proof in this section. The only difference is that the destination uses sliding window decoding instead of backward decoding:

At the end of block $j+1$, upon having received signal $y^n(j)$ and $y^n(j+1)$ and having decoded m_{j-1}, the destination looks for a unique message $\hat{m}_j$ such that

$(x_r^n(\hat{m}_j), y^n(j+1)) \in \mathcal{T}_\epsilon^{(n)}$ and
$(x^n(\hat{m}_j|m_{j-1}), x_r^n(m_{j-1}), y_r^n(j)) \in \mathcal{T}_\epsilon^{(n)}$ simultaneously.

According to packing lemma (Lemma 2 in Appendix A), the decoding error probability as $n \to \infty$ if

$$R < I(X;Y|X_r) + I(X_r;Y) = I(X, X_r; Y) - \delta(\epsilon), \tag{2.7}$$

which is the same as (2.6). As can be understood for the above analysis, DF with sliding window decoding achieves the same lower bound as with backward decoding, but reduces decoding delay to only two blocks.

2.1.5 Partial Decode-forward Lower Bound

When the quality of the link between source and relay is worse than that of the link between source and destination, the channel to the relay can be a bottleneck and DF may in fact perform worse than direct transmission. In [2], Cover and El Gamal make some improvements by allowing the relay to decode only a part of the transmitted message, which is called *partial decode-forward* (PDF). This provides a more general lower bound on the capacity:

Theorem 2. *[2] For any relay channel ($\mathcal{X} \times \mathcal{X}_r$, $p(y, y_r|x, x_r)$, $\mathcal{Y} \times \mathcal{Y}_r$), by using partial decode-forward, the capacity C is lower bounded by:*

$$C \geq \max_{p(u,x,x_r)} \min\{I(X, X_r; Y), I(U; Y_r|X_r) + I(X; Y|X_r, U)\}, \tag{2.8}$$

where $U \to (X, X_r) \to (Y_r, Y)$ forms a Markov chains. The random variable U encodes the part of the transmitted message that the relay decodes. Note that, by choosing $U = X$, this scheme reduces to decode-forward in Theorem 1, and if we choose $U = \emptyset$, it simplifies to direct transmission lower bound in (2.2).

The key coding techniques in this scheme are:

- **Message Splitting**: The source splits the message m into two independent messages m' and m'' with rates R_1 and R_2 respectively, thus $R = R_1 + R_2$. Note that m'' is decoded only at the destination.

- **Superposition coding**: The source superposes m'' onto m' and broadcasts them together in the same block.

- **Binning**: The relay partitions the set of messages m' into 2^{nR_0} equal size bins. In block j, the relay decodes m'_j and then sends its binning index in the following block.

- **Successive cancellation decoding**: The destination decodes the stronger signal first, subtracts it from the combined signal, and extracts the weaker one from the residue.

We sketch the achievability proof to illustrate how to realize partial decode-forward in a DM-RC.

Proof of Achievability: In Figure 2.1, consider b blocks of transmission.

Before the communication process, the codebook is generated and revealed to all parties. The source splits the message m into two independent message parts m' and m'', which are encoded as $u^n(m'_j|l_{j-1})$[4] and $x^n(m''_j|m'_j, l_{j-1})$ respectively. In block j, the source sends $x^n(m''_j|m'_j, l_{j-1})$. At the end of block j, the relay decodes the bin index $\hat{l}_j$ of message m'_j and then sends decoded bin index as $x_r^n(\hat{l}_j)$ in the following block $j+1$. At the end of block $j+1$, the destination uses successive cancellation decoding on signals received in the previous consecutive two blocks to decode bin index $\hat{l}_j$, message part $\hat{m}'_j$ and message part $\hat{m}''_j$ successively.

Proof. (1) *Codebook generation*: Fix $p(u, x, x_r) = p(x_r)p(u|x_r)p(x|u, x_r)$ that achieves the lower bound in (2.8).

- For each $j \in [1 : b]$, generate 2^{nR_0} sequences $x_r^n(l_{j-1})$ each i.i.d $\sim \prod_{i=1}^n p_{X_r}(x_{ri})$, where l_{j-1} denotes bin index of message part m'_{j-1}.

- For each l_{j-1}, conditionally independently generate 2^{nR_1} sequences $u^n(m'_j|l_{j-1})$ i.i.d. $\sim \prod_{i=1}^n p_{U|X_r}(u_i|x_{ri})$.

[4] l_{j-1} denotes the binning index of message part m'_{j-1} in block $j-1$.

- For each set $m'_j \times l_{j-1}$, conditionally independently generate 2^{nR_2} sequences $x^n(m''_j|m'_j, l_{j-1})$ i.i.d. $\sim \prod_{i=1}^{n} p_{X|X_r,U}(x_i|x_{ri}, u_i)$.

Partition the set of messages m' into 2^{nR_0} equal size bins $B(l) = [(l-1)2^{n(R_1-R_0)} + 1 : l2^{n(R_1-R_0)}]$, $l \in [1 : 2^{nR_0}]$. Reveal the codebook and bin assignments to all parties.

(2) *Encoding*: U carries the message part m' (with rate R_1) to be decoded by the relay at the end of the current block. X carries the message part m'' (with rate R_2), which is decoded only by the destination at the end of the following block. R_0 denotes the binning rate at the relay. Let $m_j = (m'_j, m''_j) \in [1 : 2^{nR_1}] \times [1 : 2^{nR_2}]$ be the message to be sent in block j and assume that $m'_j \in B(l_j)$. Knowing l_{j-1}, the source sends $x^n(m''_j|m'_j, l_{j-1})$. At the end of block j, the relay has an estimate of $\hat{m}'_j$ of message m'_j. Assume that $\hat{m}'_j \in B(\tilde{l}_j)$, the relay sends $x_r^n(\hat{l}_j)$ in block $j+1$.

(3) *Decoding*: The decoding procedure for message m_j is as follows:

- Upon receiving $y_r^n(j)$, the relay declare that $\hat{m}'_j$ is sent if it is unique message such that $(u^n(\hat{m}'_j|\hat{l}_{j-1}), x_r^n(\hat{l}_{j-1}), y_r^n(j)) \in \mathcal{T}_\epsilon^n$, otherwise it declares an error. $P(\hat{m}'_j \neq m'_j) \to 0$ as $n \to 0$, and correspondingly $P(\hat{l}_j \neq l_j) \to 0$ as $n \to 0$ if

$$R_1 < I(U; Y_r|X_r) - \delta(\epsilon). \tag{2.9}$$

- The destination uses successive cancellation decoding. Upon receiving $y^n(j)$, the destination declares that $\hat{l}_{j-1}$ is sent if there is a unique message such that $(x_r^n(\hat{l}_{j-1}), y^n(j)) \in \mathcal{T}_\epsilon^n$, otherwise it declares an error. $P(\hat{l}_{j-1} \neq l_{j-1}) \to 0$ as $n \to 0$ if

$$R_0 < I(X_r; Y) - \delta(\epsilon). \tag{2.10}$$

- After knowing $\hat{l}_{j-1}$, the destination then declares that $\hat{m}'_{j-1}$ is sent in block $j-1$ if there is a unique message such that $(u^n(\hat{m}'_{j-1}), x_r^n(\hat{l}_{j-1}), y^n(j)) \in \mathcal{T}_\epsilon^n$ and $\hat{m}_j \in B(\hat{l}_j)$, otherwise it declares an error. $P(\hat{m}'_{j-1} \neq m'_{j-1}) \to 0$ as $n \to 0$ if

$$R_1 - R_0 < I(U; Y|X_r) - \delta(\epsilon). \tag{2.11}$$

- After knowing $\hat{l}_{j-2}$ and $\hat{m}'_{j-1}$, the receiver looks for a unique message $\hat{m}''_{j-1}$ such that

$$(x^n(\hat{m}''_{j-1}|\hat{m}'_{j-1}, \hat{l}_{j-2}), u^n(\hat{m}'_{j-1}), x_r^n(\hat{l}_{j-2}), y^n(j-1)) \in \mathcal{T}_\epsilon^n,$$

otherwise it declares an error. $P(\hat{m}''_{j-1} \neq m''_{j-1}) \to 0$ as $n \to 0$ if

$$R_2 < I(X;Y|U, X_r) - \delta(\epsilon). \tag{2.12}$$

Combining the above four equations, we can get the PDF lower bound in Theorem 2. □

2.2 Compress-forward Lower Bound

Decode-forward requires that the relay decodes the received signal correctly and therefore it results in a poor performance if the source-relay link is in poor quality. In the situation when successful decoding is not possible, the relay can help communication by sending a description of its received signal to the destination, which means the relay can transmit an estimate of the received signal $\hat{y}_r$ and the destination can first recover the estimate $\hat{y}_r$ using the side information and then decode the transmitted message. The codebook generation of compress-forward is much more complicated than that of partial decode-forward.

Theorem 3. *[12] For any relay channel ($\mathcal{X} \times \mathcal{X}_r$,$p(y, y_r|x, x_r)$,$\mathcal{Y} \times \mathcal{Y}_r$),the capacity C is lower bounded by:*

$$C \geq \max_{p(x)p(x_r)p(\hat{y}_r|x_r,y_r)} \min\{I(X, X_r; Y) - I(Y_r; \hat{Y}_r|X, X_r, Y), I(X; Y, \hat{Y}_r|X_r)\}. \tag{2.13}$$

Proof of Achievability. Assume that R_1 is binning rate and $\hat{R}_1$ is compression rate. We use block Markov superposition encoding and binning. In each block, the source sends a new message and the relay compresses its received signal and sends the bin index of the compression index to the receiver.

(1) *Codebook generation*: Fix $p(x)p(x_r)p(\hat{y}_r|y_r, x_r)$ that achieves lower bound.

- For each block $j \in [1:b]$, generate 2^{nR} sequences $x^n(m_j)$ each i.i.d $\sim \prod_{i=1}^n p_X(x_i)$.

- For each block $j \in [1:b]$, generate 2^{nR_1} sequences $x_r^n(l_{j-1})$ each i.i.d $\sim \prod_{i=1}^n p_{X_r}(x_{ri})$.

- For each $l_{j-1} \in [1 : 2^{nR_1}]$, generate $2^{n\hat{R}_1}$ sequences $\hat{y}_r^n(k_j|l_{j-1})$, each i.i.d. $\sim \prod_{i=1}^n p_{\hat{Y}_r|X_r}(\hat{y}_{ri}|x_{ri})$, where $p(\hat{y}_r|x_r) = \sum_{y_r \in \mathcal{Y}_r} p(y_r|x_r)p(\hat{y}_r|y_r, x_r)$.
- Partition the set $[1 : 2^{n\hat{R}_1}]$ into 2^{nR_1} equal size bins $B(l_j)$, $l_j \in [1 : 2^{nR_1}]$. Reveal the codebook and bin assignment to all the parties.

(2) *Encoding*: Let m_j be the message to be sent in block j, the sender sends codeword $x^n(m_j)$. Upon receiving $y_r^n(j)$, the relay finds an index k_j such that $(\hat{y}_r^n(k_j|l_{j-1}), y_r^n(j), x_r^n(l_{j-1})) \in \mathcal{T}_\epsilon^n$. Assume that such k_j is found and $k_j \in B(l_j)$, the relay sends $x_r^n(l_j)$ in block $j+1$. The probability that there there is no such k_j tends to 0 as $n \to \infty$, if

$$\hat{R}_r > I(\hat{Y}_r; Y_r|X_r) + \delta(\epsilon). \tag{2.14}$$

(3) *Decoding*: The receiver uses successive decoding.

- Upon receiving y_{j+1}^n, the receiver finds a unique $\hat{l}_j$ such that $(x_r^n(\hat{l}_j), y_{j+1}^n) \in \mathcal{T}_\epsilon^n$. Let l_j be the bin index chosen by the relay. $P(\hat{l}_j \neq l_j) \to 0$ as $n \to 0$, if

$$R_r < I(X_r; Y) - \delta(\epsilon). \tag{2.15}$$

- Knowing $\hat{l}_{j-1}$ and $y^n(j)$, the receiver finds a unique $\hat{m}_j$ such that

$$(x^n(\hat{m}_j), x_r^n(\hat{l}_{j-1}), \hat{y}_r^n(\hat{k}_j|\hat{l}_{j-1}), y^n(j)) \in \mathcal{T}_\epsilon^n \text{ for some } \hat{k}_j \in B(\hat{l}_j).$$

  Assume the previous step is correct, that is $(\hat{l}_j, \hat{l}_{j-1}) = (l_j, l_{j-1})$ and the relay chooses the estimate K_j. Define error events: $\mathcal{E}_1(j) = \{x^n(\hat{m}_j), x_r^n(l_{j-1}), \hat{y}_r^n(K_j|l_{j-1}), y^n(j) \in \mathcal{T}_\epsilon^n \text{ for some } \hat{m}_j \neq m_j\}$. $\mathcal{E}_2(j) = \{x^n(\hat{m}_j), x_r^n(l_{j-1}), \hat{y}_r^n(\hat{k}_j|l_{j-1}), y^n(j) \in \mathcal{T}_\epsilon^n \text{ for some } \hat{k}_j \in B(l_j), \hat{k}_j \neq K_j, \hat{m}_j \neq m_j\}$. $P(\mathcal{E}(j)) \leq P(\mathcal{E}_1(j)) + P(\mathcal{E}_1(j))$. Thus, $P(\mathcal{E}_1(j)) \to 0$ as $n \to 0$, if

$$R < I(X; Y, \hat{Y}_r, X_r) + \delta(\epsilon) = I(X; Y, \hat{Y}_r|X_r) + \delta(\epsilon). \tag{2.16}$$

  $P(\mathcal{E}_2(j)) \to 0$ as $n \to 0$, if

$$R + \hat{R}_r - R_r < I(X; Y|X_r) + I(\hat{Y}_r; X, Y|X_r) - \delta(\epsilon). \tag{2.17}$$

Combining (2.14), (2.15), (2.17), $P(\hat{m}_j \neq m_j) \to 0$ as $m \to 0$ for all j if

$$\begin{aligned}R &< I(X, X_r; Y) + I(\hat{Y}_r; X, Y|X_r) - I(\hat{Y}_r; Y_r|X_r) \\ &= I(X, X_r; Y) + I(\hat{Y}_r; X, Y|X_r) - I(\hat{Y}_r; Y_r, X, Y|X_r) \\ &= I(X, X_r; Y) - I(\hat{Y}_r; Y_r|X, X_r, Y).\end{aligned}$$

This completes the achievability proof.

2.3 Single-source Single-destination Relay Networks

In wireless networks, there can be more than one relay. So far, there have been several important works on extending decode-forward and partial decode-forward to relay networks which have more than one relay. Motivated by these works, in this thesis, we are focusing on finding a coding scheme which generalizes partial decode-forward to a single-source single-destination relay network with N relays, which provides a more general and tighter partial decode-forward lower bound that includes previous partial decode-forward lower bounds as special cases.

2.3.1 Decode-forward in N-relay networks

In [6], Xie and Kumar analyze an N-relay serial network with one source and one destination and give a new achievable rate for the discrete-memoryless case by using full decode-forward. Assuming that the relay network has N relays, which are labeled successively as $\{1, 2, \ldots, N\}$, the source is denoted as 0 and the destination is denoted as $N+1$. In the serial relay network, all relays are arranged in a feed-forward structure. Thus, messages transmitted by the ith relay cannot be decoded by the jth relay if $j < i$. In the schemes proposed in this thesis, we will adopt such serial relay network structure.

2.3.2 Partial Decode-forward in N-relay networks

In [5], Leila and Aref propose a partial decode-forward scheme for a two-relay serial network based on regular encoding/joint sliding window decoding and propose a new achievable rate that is tighter than the achievable rate proposed by the full decode-forward scheme. The authors split message considering all possible partial decoding states that can occur between the different parts of the messages among the source and two relays. Each relay in the network can partially decode the message transmitted by the sender and the previous relay. Each relay employs block Markov encoding and joint sliding window decoding to help the transmission of intended message parts.

The key coding technique in this scheme is **joint sliding window decoding**, which is a combination of joint decoding and sliding window decoding. Each decoder uses channel outputs of multiple consecutive blocks to decode independent splitted message parts jointly.

We use this technique at each relay decoders as well as the destination decoder in this thesis. This technique facilitates decoding different splitted message parts simultaneously with the shortest decoding delay.

In [13], partial decode-forward is studied in the Gaussian two-way relay channel, where each transmitter divides its message into two parts and the relay decodes only one part of each. The relay then generates a codeword as a function of the two decoded parts and then broadcasts it. The investigated Gaussian two-way relay channel gives us intuitions on validating our partial decode-forward schemes in the Gaussian environment.

In [14], the authors study partial decode-forward in a multiple-relay network, where each relay is parallel to each other. However, this scheme doesn't consider a non-line-of-sight or multihop context. In [15], partial decode-forward is tentatively extended to relay networks, in which all relays successively decode only part of the messages of the previous relay before they arrive at the destination. However, this scheme doesn't consider all possible message splitting conditions that can happen among the source and all relays. In Chapter 4, we discuss exhaustive message splitting in an N-relay network and show that our scheme provide a tighter lower bound compared with the result provided in [15] when $N = 3$.

2.4 Thesis Motivation and Contribution

In this section, we review our main contributions presented in this thesis: partial decode-forward in an N-relay network with single source and single destination. According to the way in which the source splits its original message among different relays and the destination, we categorize our proposed schemes into two types: private message splitting scheme and exhaustive message splitting scheme.

2.4.1 Private Message Splitting Scheme

To the best of our knowledge, up to now, except the work done in [15], no other work has been done to apply the partial decode-forward to the relay networks with arbitrary number of relays in which more than one relay partially decodes the message transmitted by the source.

In our schemes, instead of each decoder using backward decoding as shown in [6], each transmitter uses block superposition encoding and joint sliding window decoding, which reduces destination decoding delay to N blocks.

According to Section 2.1.4, backward decoding needs excessive delays to decode messages, which is difficult to implement in a relay network with more than one relay, thus we attempt to employ sliding window decoding in our scheme. However, as the number of relays increases, more independent message parts need to be decoded simultaneously at each relay, which makes sliding window decoding slow and inefficient in large-scale relay networks. Based on Section 2.3.2, we use joint decoding at each relay to implement decoding of different message parts simultaneously.

In our scheme, the source splits its message into one common part and $N+1$ private parts. Each relay helps forwarding the common message and the private message intended for itself. Each relay forwards its message parts to the following nodes when the last common message of the same block index arrives. We derive the achievable rate in a compact form to make it possible to plot rate regions and examine how each relay functions. We also show that this scheme includes network decode-forward of [2] and partial decode-forward in two-relay network of [5] as special cases. We also analyze an N-relay network as well as a two-relay network in AWGN environments and provide their achievable rates respectively.

2.4.2 Exhaustive Message Splitting Scheme

In this scheme, which has been presented in [16], for an N-relay network, we split the source message in such a way that every relay has its private message and a common message to decode.

When applying partial decode-forward to a large relay network, a key question is to identify the number of parts in which to split the message and the method to superimpose these message parts. Our exhaustive message splitting scheme deals with this problem in detail. To answer the first part of this question, we split the original source message in a way that each possible subset of relays has an individual message part to decode. Denote the set of all relays as $\mathcal{T} = \{1, \dots, N\}$, whose each element represents the corresponding relay index. We assign each relay subset $\mathcal{S} \subseteq \mathcal{T}$ with a different private message, which other relay subsets do not decode. Under such arrangement, the source message is split exhaustively for partial decode-forward. Enumeration of all possible relay subsets is a simple combinatorial problem. For an N-relay network with single source and single destination, the total number of relay subsets is $C_N^0 + C_N^1 + \cdots + C_N^N = 2^N$, where $C_N^r = \frac{N!}{r!(N-r)!}$. In relay networks with one or two relays [2] [17] [5], the private message scheme in [16] is already exhaustive, thus our focus is on a three-relay network scenario with respect to exhaustive partial decode-forward.

To answer the second part of the question, we design a block Markov superposition structure where message parts passing through fewer relays are superimposed on those passing through more relays and on message parts of all previous blocks which are decoded by the same relay subset. For ease of understanding, we assign all message parts that are decoded by the same number of relays to a so called layer. Specifically, a message part $m_{\mathcal{S}}$ located on layer l will be superimposed on all lower layer message parts $m_{\mathcal{S}'}$ of the current and previous blocks where $\mathcal{S} \subset \mathcal{S}'$.

In order to obtain some insights into exhaustive message splitting for partial decode-forward in a general N-relay network, we study in detail a network with three relays and consider all possible partial decoding cases that can occur between messages parts at the source and the relays. We introduce a directed graph to illustrate the superposition coding structure. The superposition coding structure for an N-relay network can be similarly obtained by inserting more layers under the bottom layer where messages are decoded by all relays and by expanding the graph horizontally to accommodate more relays. We also provide the corresponding achievable rate for the three-relay network and show that the proposed scheme includes network decode-forward in [6] and the private message splitting scheme in [16] as special cases.

In chapter 3, we will study the proposed private message splitting scheme in detail. In chapter 4, we will study the proposed exhaustive message splitting scheme in detail.

Chapter 3

Partial Decode-forward Scheme for N-relay Networks

In this chapter, we study partial decode-forward for an N-relay network with single source and single destination. One of our main contributions is to propose a private message splitting scheme based on block Markov encoding and joint sliding window decoding. We derive the achievable rate for this scheme in a compact form and show that this scheme includes network decode-forward [6] and partial decode-forward for two-relay networks [5] as special cases.

The remainder of this chapter is organized as follows. In Section 3.1, we present the discrete-memoryless N-relay network model used in this chapter. In Section 3.2, we illustrate the notations specified in this chapter. In Section 3.3, we present the private message splitting scheme in detail. In Section 3.4, we analyze the capacity results in the AWGN environments and provide the achievable rates.

3.1 Discrete Memoryless N-relay Network Models

In this section, we introduce the network model that we explore in this chapter. Consider an N-relay discrete memoryless relay network (DM-RN)[1] $(\mathcal{X}_0 \times \mathcal{X}_1 \times \cdots \times \mathcal{X}_N,$

[1] Note that if the source encoder 0 wants to send a message m to a set of destination nodes $\mathcal{D} \subseteq [1 : N{+}1]$, the DM-RN becomes a discrete memoryless multicast network (DM-MN), where each decoder $k \in \mathcal{D}$ assigns an estimate $\hat{m}_k$ to each received sequence $y_k^n \in \mathcal{Y}_k^n$, or declares an error message e. The DM-RN thus is a special case of the DM-MN as defined in [8].

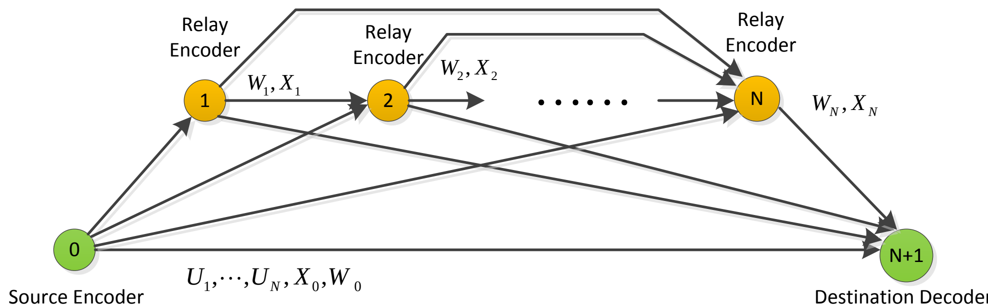

Fig. 3.1 A single-source single-destination network with N relays.

$p(y_1, y_2, \ldots, y_{N+1}|x_0, x_1, \ldots, x_N)$, $\mathcal{Y}_1 \times \mathcal{Y}_2 \times \cdots \times \mathcal{Y}_{N+1})$, where source node 0 wants to send a message m to the destination node $N+1$ with the help of relay nodes $1, \ldots, N$, as shown in Figure 3.1. A $(2^{nR}, n)$ code for this DM-RN consists of:

- A message set $\mathcal{M} = [1 : 2^{nR}]$.
- A source encoder that assigns a codeword $x_0^n(m)$ to each message $m \in \mathcal{M}$.
- A set of relay encoders $j \in [1 : N]$. At time index i, the relay encoder j transmits a single symbol based on receiving signals received during time interval 1 to $i-1$.
- A destination decoder, which assigns an estimate $\hat{m}_{N+1}$ to each received sequence $y_{N+1}^n \in \mathcal{Y}_{N+1}^n$, or declares an error message e.

The average probability of error $P_e^{(n)}$ is defined as

$$P_e^{(n)} = P\{\hat{m}_{N+1} \neq m\}.$$

The rate of a $(2^{nR}, n)$ code for this DM-RN is said to be achievable if there exists a sequence of $(2^{nR}, n)$ code such that $P_e^{(n)} \to 0$ as $n \to \infty$. The capacity C of the DM-RN is the supremum of all achievable rates.

Note that if $N = 1$, then the DM-RN reduces to the classical DM-RC introduced in Chapter 2.

3.2 Notations and Definitions

In this section, in order to make our following analysis more concise and readable, we first clarify mathematical notations and then introduce definitions, which are used throughout this chapter.

- Given a random variable M, we introduce the notation $M_a^b = \{M_a, M_{a+1}, \ldots, M_b\}$, where $b \geq a$. Note that the n-length codeword x_r^n (appearing in Chapter 2) is an exception and thus we won't use n as the upper subscript again in the following argument.

- Given a nonempty set $\mathcal{L}$ of integers and a random variable M, let $M_{\mathcal{L}} = \{M_a\}_{a\in\mathcal{L}}$. In addition, $|\mathcal{L}|$ signifies the cardinality of $\mathcal{L}$, which is the total number of elements in set $\mathcal{L}$.

- Define $\mathcal{T} = \{1, \ldots, N\}$ to be the complete set of all relays.

- Define $\mathcal{S}$ to be a subset of $\mathcal{T}$, that is $\mathcal{S} \subseteq \mathcal{T}$ and $\mathcal{S}^c = \mathcal{T} - \mathcal{S}$. Either $\mathcal{S}$ or $\mathcal{S}^c$ can be empty and the largest $\mathcal{S}$ is $\mathcal{T}$.

3.3 Private Message Splitting Scheme

In this section, we present our proposed private message splitting scheme for N-relay networks in detail.

Figure 3.1 shows a network consisting of one source, one destination and N relays. All transmitting nodes are ordered serially. We assume that each relay k, $k \in \mathcal{T}$, decodes information from all nodes below it, (i.e. $\{1, \ldots, k-1\}$) and forwards information to nodes above it (i.e. $\{k+1, \ldots, N\}$). In this network, the source (indexed as 0) has direct links to all relays and the destination (indexed as $N+1$). We design a novel transmission scheme for this relay network based on partial decode-forward.

The new idea of the scheme is in the way it performs rate splitting. The source pre-splits its message into multiple message parts. Each relay is responsible for the transmission of a certain distinct subset of split message parts. The destination decodes all the message parts and recombines them into an intact message. At each block transmission, the source splits its message into $N+2$ parts: one common message part m_0 and $N+1$ private message parts $(m_1, m_2 \dots, m_{N+1})$, where $(m_1, m_2 \dots, m_N)$ is decoded and re-transmitted only through the relay whose index is identical to corresponding message subscript (e.g., m_1 is decoded and re-transmitted only through relay 1, m_2 is decoded and re-transmitted only through relay 2, ...). m_{N+1} is decoded only at the destination. Each relay fully recovers the common message part and its own private message part with same block index as the common message part, then forwards them together in the next block.

Next, we introduce block index into split message parts. Specifically, in block j, let the source message be split as[2] $m_j = (m_{0,j}, m_{1,j}, \dots, m_{N+1,j})$, where $m_{0,j}$ denotes the common message part that is forwarded among all relays, $m_{k,j}$ denotes the message part supposed to be decoded at some relay k, $k \in \mathcal{T}$, while other relays do not decode it. $m_{N+1,j}$ denotes the message part supposed to be decoded only at the destination.

Consider that the whole communication process has b transmission blocks, each consisting of n channel uses. According to the property of block Markov decoding as shown in Chapter 2, each additional relay incurs one block decoding delay to the destination. Since there are N relays between the source and the destination, the whole network's delay is N blocks. Therefore, in this scheme, a sequence of $b - N$ messages $m_j = (m_{0,j}, m_{1,j}, \dots, m_{N+1,j})$, uniformly distributed over $[1 : 2^{nR_0}] \times [1 : 2^{nR_1}] \times \cdots \times [1 : 2^{nR_{N+1}}]$, for all $j \in [1 : b - N]$, is transmitted to the destination.

The average rate over b blocks is $R(b - N)/b$, which goes to R as $b \to \infty$. After being split, the rate becomes $R = \sum_{i=0}^{N+1} R_i$.

Next, we present the achievable rate for this scheme and prove its achievability.

3.3.1 Coding Scheme and Achievable Rate

The coding scheme for this relay network is illustrated in Figure 3.1.

Theorem 4. *For a single-source single-destination network with N relays ($\mathcal{X}_0 \times \mathcal{X}_1 \times \cdots \times \mathcal{X}_N$, $p(y_1, y_2, \dots, y_{N+1} | x_0, x_1, \dots, x_N)$, $\mathcal{Y}_1 \times \mathcal{Y}_2 \times \cdots \times \mathcal{Y}_{N+1}$), by using the partial decode-forward scheme with private message splitting defined in Section 3.3, the capacity C is lower bounded by (3.1):*

[2] Among two subscripts of message part $m_{k,j}$, the front subscript k denotes the message part index and the back subscript j denotes the block index. This notation is used throughout the thesis.

$$C \geq \sup_{p} \min_{\mathcal{S}\subseteq\mathcal{T}} \begin{cases} I\left(X_0, X_{\mathcal{S}}, U_{\mathcal{S}}; Y_{N+1}|X_{\mathcal{S}^c}, U_{\mathcal{S}^c}, W_0^N\right) \\ +\min_{j\in\mathcal{S}^c} I\left(W_0^{j-1}; Y_j|X_j, W_j^N\right) \\ +\sum_{j\in\mathcal{S}^c} I\left(U_j; Y_j|W_0^N, X_j\right) \end{cases} \tag{3.1}$$

where

$$p = p(U_1^N, W_0^N, X_0^N) = p(X_0|U_1^N, W_0^N, X_1^N)\prod_{k=1}^{N} p(U_k|W_0^N, X_k)p(W_0|W_1^N) \prod_{k=1}^{N} p(X_k|W_k^N)p(W_k|W_{k+1}^N). \tag{3.2}$$

(3.1) *gives a lower bound on the capacity of the single-source single-destination network with* N *relays as depicted in Figure 3.1. The supremum is taken oven all possible input distributions defined in* (3.2). *We relate the message parts with the meaning of random variables appearing in* (3.1) *and* (3.2) *in the following:*

- W_k, $k \in \{0\}\cup\mathcal{T}$, *carries common message part* $m_{0,j-k}$ *from the transmitting node* k *in each block* j. *The codeword* w_k *is generated according to the probability function* $p(W_k|W_{k+1}^N)$.

- U_k, $k \in \mathcal{T}$, *carries private message part* $m_{k,j}$ *to be decoded at relay* k *and not decoded at other relays in the block* j. *All* u_k*s are transmitted by the source. The codeword* u_k *is generated according to the probability function* $p(U_k|W_0^N, X_k)$.

- X_k, $k \in \mathcal{T}$, *is sent by the relay* k. X_k *carries the forwarding of the private message part contained in* U_k *and the common message parts contained in all* W_l $(l \leq k)$. *The codeword* x_k *is generated according to the probability function* $p(X_k|W_0^N)$.

- X_0 *is sent by the source.* X_0 *carries the remaining message part* $m_{N+1,j}$ *in the block* j, *which is decoded only at the destination. The codeword* x_0 *is generated according to the probability function* $p(X_0|U_1^N, W_0^N, X_1^N)$.

Proof. The source sends $b - N$ messages over b blocks of n symbols. Each relay and the destination use block Markov superposition coding to generate independent codewords in each block and employ simultaneous sliding window decoding to decode its supposed message parts.

Before the communication process, the codebook is generated and revealed to all parties. For each block j, the source sends[3] $x_0(m_{N+1,j}|m_{1,j}^{N,j}, m_{0,j-N}^{0,j}, \{m_{k,j-k}\}_{k\in\mathcal{T}})$ (here message parts $m_{1,j}^{N,j} = \{m_{1,j}, m_{2,j}, \dots, m_{N,j}\}$ and message parts $m_{0,j-N}^{0,j} = \{m_{0,j-N}, m_{0,j-N+1}, \dots, m_{0,j}\}$). x_0 is generated conditioned on codewords containing all private message parts in the current block and all common message parts in the previous N blocks. At the end of block j, each relay k decodes $\hat{m}_{k,j-k+1}$ and $\hat{m}_{0,j-k+1}$. Then in block $j+1$, the relay k broadcasts $x_k(\hat{m}_{k,j-k+1}|\hat{m}_{0,j-N+1}^{0,j-k+1})$ (here message parts $\hat{m}_{0,j-N+1}^{0,j-k+1} = \{\hat{m}_{0,j-N+1}, \hat{m}_{0,j-N+2}, \dots, \hat{m}_{0,j-k+1}\}$), which contains its private message part $\hat{m}_{k,j-k+1}$ and previous $N-k$ blocks' common message parts. The destination uses joint decoding simultaneously over signals received in all previous N blocks. Specifically, it waits until the end of the last block arrives and then decode all message parts simultaneously using signals received in the last N blocks.

Codebook generation

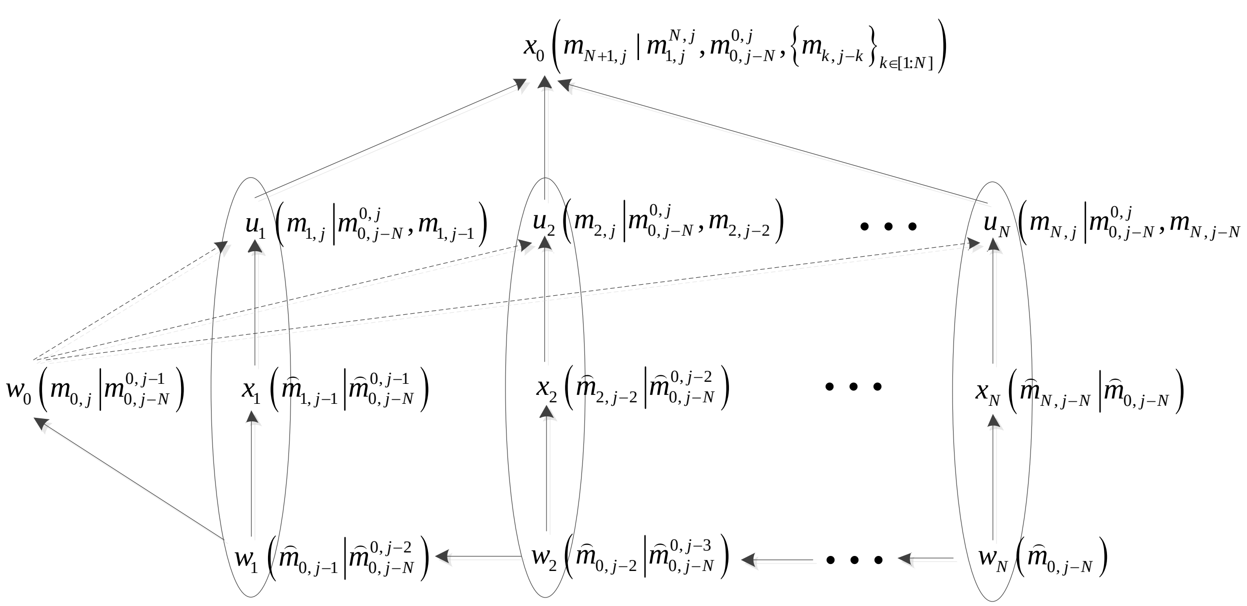

Fig. 3.2 Encoding diagram of a single-source single-destination network with N relays in block j.

Figure 3.2 illustrates the superposition encoding of each split message part. In the

[3] We write n-length codeword x_0^n as x_0 to avoid confusion with $\{x_0, x_1, \dots, x_n\}$.

figure, codewords are connected by an arrow where the codeword at the end of the arrow is generated conditioned on the one at the beginning of the arrow. Codewords which are to be decoded by the same relay are aligned on the same vertical line and kept in the same circle. Following the arrow direction, we can find that: at relay k, the codeword w_k is conditioned upon codewords $w_{k+1}, w_{k+2}, \ldots, w_N$; the codeword x_k is conditioned upon codewords $w_k, w_{k+1}, \ldots, w_N$. At the source, the codeword w_0 is conditioned upon codewords $w_1, w_2, \ldots, w_N$; each codeword u_k ($k \in [1 : N]$) is conditioned upon codewords $x_k, w_k, \ldots, w_N$; finally, the codeword x_0 is conditioned upon codewords $u_1, u_2, \ldots, u_N, x_1, x_2, \ldots, x_n, w_0, w_1 \ldots, w_N$.

Next, we standardize our codebook generation using mathematical notation. We generate independent codebook for each block according to block Markov encoding. In block j, after the source splits its message as $m_j = (m_{0,j}, m_{1,j}, \ldots, m_{N,j}, m_{N+1,j})$, the generation of codebook $\mathcal{C}_j$ of block j is as follows:

For every relay $k = N, \ldots, 1$ and corresponding message parts $m_{0,j-k}$ and $m_{k,j-k}$:

- Randomly and independently generate 2^{nR_0} sequences $w_k(m_{0,j-k}|m_{0,j-N}^{0,j-k-1})$ for all $m_{0,j-k} \in [1 : 2^{nR_0}]$, each i.i.d $\sim p(w_k|w_{k+1}^N)$.
- Randomly and independently generate 2^{nR_k} sequences $x_k(m_{k,j-k}|m_{0,j-N}^{0,j-k})$ for all $m_{k,j-k} \in [1 : 2^{nR_k}]$, each i.i.d. $\sim p(x_k|w_k^N)$.

For source node $k = 0$ and corresponding message part $m_{0,j}^{N+1,j}$:

- For all sequences $w_k(m_{0,j-k}|m_{0,j-N}^{0,j-k-1})$ with $k \in \mathcal{T}$, randomly and independently generate 2^{nR_0} sequences $w_0(m_{0,j}|m_{0,j-N}^{0,j-1})$ for all $m_{0,j} \in [1 : 2^{nR_0}]$ each i.i.d $\sim p(w_0|w_1^N)$.
- For all relays $k \in \mathcal{T}$, randomly and independently generate 2^{nR_k} sequences $u_k(m_{k,j}|m_{0,j-N}^{0,j}, m_{k,j-k})$ for all $m_{k,j} \in [1 : 2^{nR_k}]$, each i.i.d $\sim p(u_k|w_0^N, x_k)$,
- Randomly and independently generate $2^{nR_{N+1}}$ sequences

  $x_0(m_{N+1,j}|m_{1,j}^{N,j}, m_{0,j-N}^{0,j}, \{m_{k,j-k}\}_{k\in\mathcal{T}})$ for all $m_{N,j} \in [1 : 2^{nR_N}]$, each according to $\sim p(x_0|u_1^N, w_0^N, x_1^N)$.

The above constitutes the codebook $\mathcal{C}_j$ of block j. The codebook $\mathcal{C}_j$ is then revealed to all the parties. Likewise, we generate codebooks $\mathcal{C}_{j+1}$, $\mathcal{C}_{j+2}$, $\ldots$ in the same way.

Encoding

To send $m_{N+1,j},\dots,m_{0j}$ in block j, the source encoder transmits $x_0(m_{N+1,j}|m_{1,j}^{N,j},m_{0,j-N}^{0,j},\{m_{k,j-k}\}_{k\in\mathcal{T}})$ from codebook $\mathcal{C}_j$. At the end of block j, each relay $k\in\mathcal{T}$ has an estimate $\hat{m}_{k,j-k+1}$ of message $m_{k,j-k+1}$ and $\hat{m}_{0,j-k+1}$ of message $m_{0,j-k+1}$. In the block $j+1$, each relay $k\in\mathcal{T}$ transmits $x_k(\hat{m}_{k,j-k+1}|\hat{m}_{0,j-N+1}^{0,j-k+1})$ from codebook $\mathcal{C}_{j+1}$.

Decoding

Simultaneous decoding at the first relay

At the end of block j, by knowing $\hat{m}_{0,j-N}^{0,j-1}$ and $\hat{m}_{1,j-1}$, the first relay $k=1$ will decode $m_{1,j}$ and $m_{0,j}$ such that:

$$\begin{aligned}\big(u_1(m_{1,j}|\hat{m}_{0,j-N+1}^{0,j},\hat{m}_{1,j-1}),w_0(m_{0,j}|\hat{m}_{0,j-N+1}^{0,j-1}),w_1(\hat{m}_{0,j-1}|\hat{m}_{0,j-N}^{0,j-2}),\\ x_1(\hat{m}_{1,j-1}|\hat{m}_{0,j-N}^{0,j-1}),\,w_2,w_3,\dots,w_N,y_1(j)\big)\in\mathcal{T}_\epsilon^{(n)}.\end{aligned}\tag{3.3}$$

In (3.3), the first relay decodes $\hat{m}_{1,j}$ and $\hat{m}_{0,j}$ from output $y_1(j)$ when $y_1(j)$ is jointly typical with $u_1(m_{1,j}|\hat{m}_{0,j-N+1}^{0,j})$ and $w_0(m_{0,j}|\hat{m}_{0,j-N+1}^{0,j-1})$ given the knowledge of $x_1,w_1,w_2,\dots,w_N$. According to joint typicality lemma (Appendix A Lemma 1), the decoding error probability goes to 0 as $n\to\infty$, if following rate constraints are satisfied:

$$R_1 < I(U_1;Y_1|W_0^N,X_1),\tag{3.4}$$

$$R_1+R_0 < I(U_1,W_0;Y_1|X_1,W_1^N).\tag{3.5}$$

Simultaneous sliding window decoding at other relays $k\in[2:N]$

At the end of block j, by knowing $\hat{m}_{0,j-N-k+1}^{0,j-k}$ and $\hat{m}_{k,j-2k+1}$, the relay node k will decode $m_{k,j-k+1}$ and $m_{0,j-k+1}$ such that the following conditions hold simultaneously:

$$\begin{aligned}\Big(w_{k-1}(m_{0,j-k+1}|\hat{m}_{0,j-N}^{0,j-k}),w_k(\hat{m}_{0,j-k}|\hat{m}_{0,j-N}^{0,j-k-1}),\\ x_k(\hat{m}_{k,j-k}|\hat{m}_{0,j-N}^{0,j-k}),w_{k+1},w_{k+2},\dots,w_N,y_k^n(j)\Big)\in\mathcal{T}_\epsilon^{(n)},\end{aligned}$$

and

$$\Big(w_{k-2}(m_{0,j-k+1}|\hat{m}_{0,j-N-1}^{0,j-k}), w_{k-1}(\hat{m}_{0,j-k}|\hat{m}_{0,j-N-1}^{0,j-k-1}),$$
$$w_k(\hat{m}_{0,j-k-1}|\hat{m}_{0,j-N-1}^{0,j-k-2}), x_k(\hat{m}_{k,j-k-1}|\hat{m}_{0,j-N-1}^{0,j-k-1}),$$
$$w_{k+1}, w_{k+2}, \ldots, w_N, y_k(j-1)) \in \mathcal{T}_\epsilon^{(n)},$$

$$\vdots$$

$$\Big(w_1(m_{0,j-k+1}|\hat{m}_{0,j-k-N+2}^{0,j-k}), w_2, \ldots, w_{k-1}, w_k, x_k,$$
$$w_{k+1}, \ldots, w_N, y_k(j-k+2)) \in \mathcal{T}_\epsilon^{(n)},$$

and

$$\Big(u_k(m_{k,j-k+1}|\hat{m}_{0,j-N-k+1}^{0,j-k+1}, \hat{m}_{k,j-2k+1}),$$
$$w_0(m_{0,j-k+1}|\hat{m}_{0,j-N-k+1}^{0,j-k}), w_1, w_2, \ldots, w_{k-1},$$
$$x_k, w_k, w_{k+1}, \ldots, w_N, y_k(j-k+1)) \in \mathcal{T}_\epsilon^{(n)}. \tag{3.6}$$

In (3.6), k decoding rules should be satisfied simultaneously at the relay k. In the lth decoding rule ($l < k$), the relay k decodes $\hat{m}_{k,j-k+1}$ from output $y_k(j-l+1)$ when $y_k(j-l+1)$ is jointly typical with $w_{k-l}(m_{0,j-k+1}|\hat{m}_{0,j-N-l+1}^{0,j-k})$ given the knowledge of $w_{k-l+1}, \ldots, w_N, x_k$. In the last decoding rule, the relay k decodes $\hat{m}_{k,j-k+1}$ and $\hat{m}_{0,j-k+1}$ from $y_k(j-k+1)$ when $y_k(j-k+1)$ is jointly typical with $u_k(m_{k,j-k+1}|\hat{m}_{0,j-N-k+1}^{0,j-k+1}, \hat{m}_{k,j-2k+1})$ and $w_0(m_{0,j-k+1}|\hat{m}_{0,j-N-k+1}^{0,j-k})$ given the knowledge of $w_1, \ldots, w_N$ and x_k. The decoding error probability goes to 0, as $n \to \infty$, if

$$R_k < I(U_k; Y_k|W_0^N, X_k), \tag{3.7}$$
$$R_k + R_0 < I(U_k, W_0^{k-1}; Y_k|X_k, W_k^N). \tag{3.8}$$

Detailed error analysis of the relay k is in Appendix B.

Simultaneous sliding window decoding at destination node $k = N + 1$

At the end of block j, the destination node $k = N + 1$ will decode $m_{k,j-N}$ for all $k \in \mathcal{T} \cup \{N + 1\}$ and $m_{0,j-N}$ such that the following conditions hold simultaneously:

$$\big(x_N(m_{N,j-N}|m_{0,j-N}), w_N(m_{0,j-N}), y^n_{N+1}(j)\big) \in \mathcal{T}^{(n)}_\epsilon.$$
$$\vdots$$
$$\Big(x^n_1(m_{1,j-N}|\hat{m}^{0,j-N}_{0,j-2N+1}), w^n_1(m_{0,j-N}|\hat{m}^{0,j-N-1}_{0,j-2N+1}),$$
$$x^n_2, w^n_2, \ldots, x^n_N, w^n_N, y^n_{N+1}(j - N + 1)\big) \in \mathcal{T}^{(n)}_\epsilon.$$

There are N expressions in the above decoding rules. The lth expression can be written as:

$$\Big(x^n_{N-l+1}(m_{N-l+1,j-N}|\hat{m}^{0,j-N}_{0,j-N+1-l}),\ w^n_{N-l+1}(m_{0,j-N}|\hat{m}^{0,j-N}_{0,j-N+1-l}),$$
$$x_{N-l+1}, w_{N-l+2}, \ldots, x_N, w_N, y_{N+1}(j - l + 1)) \in \mathcal{T}^{(n)}_\epsilon.$$

And, the last expression is:

$$\Big(\{u_k(m_{k,j-N}|\hat{m}^{0,j-N}_{0,j-2N}, \hat{m}_{k,j-N-k})\}_{k\in\mathcal{T}},\ x_0(m_{N+1,j-N}|m^{N,j-N}_{1,j-N}, , \hat{m}^{0,j-N}_{0,j-2N},$$
$$\{\hat{m}_{k,j-N-k}\}_{k\in\mathcal{T}}), w_0(m_{0,j-N}|\hat{m}^{0,j-N-1}_{0,j-2N}),$$
$$x_1, w_1, x_2, w_2, \ldots, x_N, w_N, y_{N+1}(j - N)) \in \mathcal{T}^{(n)}_\epsilon. \tag{3.9}$$

In (3.9), $N + 1$ decoding rules should be satisfied simultaneously. In the lth decoding rule ($l < N + 1$), the destination $N + 1$ decodes $\hat{m}_{N-l+1,j-N}$ and $\hat{m}_{0,j-N}$ from output $y_{N+1}(j-l+1)$ when $y_{N+1}(j-l+1)$ is jointly typical with $x_{N-l+1}(m_{N-l+1,j-N}|\hat{m}^{0,j-N}_{0,j-N+1-l})$ and $w_{N-l+1}(m_{0,j-N}|\hat{m}^{0,j-N}_{0,j-N+1-l})$, given the knowledge of $w_{N-l+2}, \ldots, w_N, x_{N-l+2}, \ldots, x_N$. In the last decoding rule, the destination $N + 1$ decodes $\hat{m}_{k,j-N}$ for all $k \in \mathcal{T}$, $\hat{m}_{N+1,j-N}$ and $m_{0,j-N}$ from $y_{N+1}(j - N)$ when $y_{N+1}(j - N)$ is jointly typical with u_k, x_0 and w_0 given the knowledge of $w_1, \ldots, w_N$ and $x_1, \ldots, x_N$. The decoding error probability goes to 0, as

$n \to \infty$, if

$$\sum_{i=0}^{N+1} R_i < I(U_1^N, X_0^N, W_0^N; Y_{N+1}), \tag{3.10}$$

$$\sum_{i\in\mathcal{S}} R_i + R_{N+1} < I(X_0, X_{\mathcal{S}}, U_{\mathcal{S}}; Y_N | X_{\mathcal{S}^c}, U_{\mathcal{S}^c}, W_0^N), \tag{3.11}$$

$$R_{N+1} < I(X_0; Y_{N+1} | U_1^N, X_1^N, W_0^N), \tag{3.12}$$

$$\sum_{i=1}^{N+1} R_i < I(U_1^N, X_0^N; Y_{N+1} | W_0^N), \tag{3.13}$$

where $\mathcal{S}$ is a subset of $\mathcal{T}$. Detailed error analysis at the destination is in Appendix C.

Combination Process

In this section, we illustrate the combination process of all the inequalities that we have derived.

We first restate all the rate constraints derived throughout the error analysis:

$$\begin{cases} R_k & < I(U_k; Y_k | W_0^N, X_k), \forall k \in \mathcal{T}, \\ R_k + R_0 & < I(U_k, W_0^{k-1}; Y_k | X_k, W_k^N), \forall k \in \mathcal{T}, \\ \sum_{i=0}^{N+1} R_i & < I(U_1^N, X_0^N, W_0^N; Y_{N+1}), \\ \sum_{i\in\mathcal{S}} R_i + R_{N+1} & < I(X_0, X_{\mathcal{S}}, U_{\mathcal{S}}; Y_N | X_{\mathcal{S}^c}, U_{\mathcal{S}^c}, W_0^N), \forall \mathcal{S} \\ R_{N+1} & < I(X_0; Y_{N+1} | U_1^N, X_1^N, W_0^N), \\ \sum_{i=1}^{N+1} R_i & < I(U_1^N, X_0^N; Y_{N+1} | W_0^N). \end{cases}$$

The combination process is as follows:

From (3.10), we can directly get that:

$$R < I(U_1^N, X_0^N, W_0^N; Y_{N+1}). \tag{3.14}$$

From (3.11), (3.7) and (3.8), we get

$$\begin{aligned} R &= (R_{N+1} + \sum_{i\in\mathcal{S}} R_i) + \min_{j\in\mathcal{S}^c}[(R_j + R_0) + (\sum_{i\in\mathcal{S}^c, i\neq j} R_i)] \\ &< I(X_0, X_{\mathcal{S}}, U_{\mathcal{S}}; Y_{N+1}|X_{\mathcal{S}^c}, U_{\mathcal{S}^c}, W_0^N) \\ &\quad + \min_{j\in\mathcal{S}^c} I(W_0^{j-1}; Y_j|X_j, W_j^N) + \sum_{i\in\mathcal{S}^c} I(U_i; Y_i|W_0^N, X_i), \end{aligned} \tag{3.15}$$

for all $\mathcal{S} \subset \mathcal{T}$ and $\mathcal{S} \neq \mathcal{T}$.

From (3.7), (3.8), (3.12)and (3.13), we get

$$\begin{aligned} 2R &< \sum_{i=1}^{N+1} R_i + R_{N+1} + \sum_{l\in\mathcal{T}} R_l \\ &\quad + \min_{i,j\in\mathcal{T}}[(R_i + R_0) + (R_j + R_0) + \sum_{l\in\mathcal{T}, l\neq i,j} R_l] \\ &< I(U_1^N, X_0^N; Y_{N+1}|W_0^N) + I(X_0; Y_{N+1}|U_1^N, X_1^N, W_0^N) \\ &\quad + 2\min_{j\in\mathcal{T}} I(W_0^{j-1}; Y_j|X_j, W_j^N) + 2\sum_{i\in\mathcal{T}} I(U_i; Y_i|W_0^N, X_i). \end{aligned} \tag{3.16}$$

However, if we let $\mathcal{S} = \emptyset$ in (3.15) and double its right-hand-side (RHS) expression, then we can get a smaller expression than the RHS of (3.16). Thus, (3.16) is redundant.

After this combination process, we can get the rate in (3.1). □

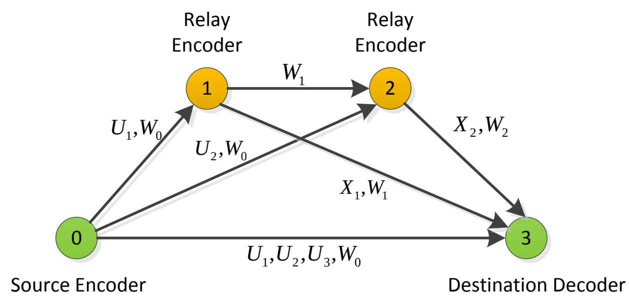

Fig. 3.3 Two-level relay network with partial-decode-forward.

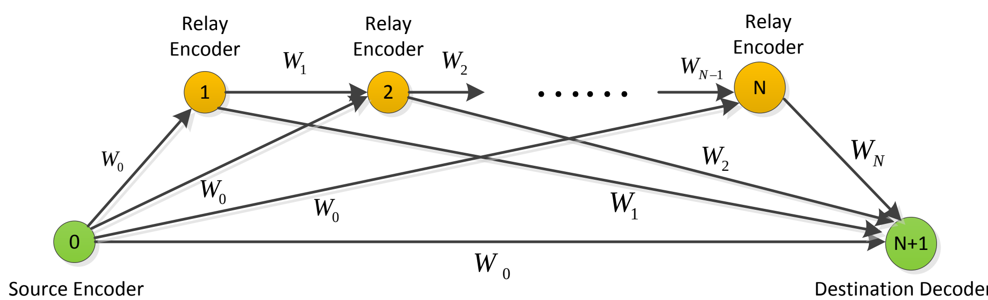

Fig. 3.4 Decode-forward relay network.

3.3.2 Discussions

If $N = 2$, we have the partial decode-forward lower bound for a two-relay network as shown in Figure 3.3, which coincides with the result in [5]. In Appendix D, we show the Fourier-Motzkin elimination of the rate constraints for a two-relay network in detail.

Corollary 1. *For any relay channel* $(\mathcal{X}_0 \times \mathcal{X}_1 \times \mathcal{X}_2, p(y_1, y_2, y_3|x_0, x_1, x_2), \mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{Y}_3)$*, the* *capacity C is lower bounded by:*

$$\begin{aligned}C \geq \sup_p \min\{ & I(U_1; Y_1|W_0, W_1, W_2, X_1) + I(U_2, W_0, W_1; Y_2|X_2, W_2) \\ & + I(X_0; Y_3|U_1, X_1, U_2, X_2, W_0, W_1, W_2), \\ & I(U_1, W_0; Y_1|X_1, W_1, W_2) + I(U_2; Y_2|W_0, W_1, W_2, X_2) \\ & + I(X_0; Y_3|U_1, X_1, U_2, X_2, W_0, W_1, W_2), \\ & I(U_2, W_0, W_1; Y_2|X_2, W_2) + I(X_0, U_1, X_1; Y_3|U_2, W_0, W_1, W_2, X_2) \\ & I(U_1, W_0; Y_1|X_1, W_1, W_2) + I(X_0, U_2, X_2; Y_3|U_1, W_0, X_1, W_1, W_2), \\ & I(X_0, X_1, X_2, U_1, U_2, W_0, W_1, W_2; Y_4), \}\end{aligned} \quad (3.17)$$

where the supremum is over all joint pmf $p(x_0, x_1, x_2, u_1, u_2, w_0, w_1, w_2)$ and $(U_1, U_2, W_0, W_1, W_2) \to (X_0, X_1, X_2) \to (Y_1, Y_2, Y_3)$ forms a Markov chain.

In (3.1), if we set private parts $X_1 = \emptyset$ and $U_1^N = \emptyset$, we can get Xie and Kumar's [6] network decode-forward lower bound as shown in Figure 3.4.

Corollary 2. *The capacity of the discrete memoryless relay network based on decode-forward is lower bounded as:*

$$C \geq \max_{p(x_0, x_1, \ldots, x_N)} \min_{k \in [1:N+1]} I(W_0^{k-1}; Y_k|W_k^N) \quad (3.18)$$

Furthermore, in (3.18), if $N = 1$, it reduces to the decode-forward lower bound [2] for the discrete-memoryless relay channel.

3.3.3 Illustration of Encoding and Decoding

In this section, we present encoding and decoding tables detailing relay operations in a few blocks to illustrate the message part transmission procedure clearly.

Assume that we have a discrete memoryless relay network consisting of one source (labeled as 0), one destination (labeled as 200) and 199 relays (labeled consecutively from 1 to 199).

As shown in Table 3.1, in the first block, the source 0 broadcasts codeword $x_0^n(m_{200,1}|m_{1,1}^{199,1}, m_{0,1})$ to all following relays and all relays of this network remain silent. At the end of the first block, the first relay decodes message parts $\hat{m}_{1,1}$ and $\hat{m}_{0,1}$ from its output while other relays

	Block 1	Block 2
X_0	$x_0^n(m_{200,1}\|m_{1,1}^{199,1}, m_{0,1})$	$x_0^n(m_{200,2}\|m_{1,2}^{199,2}, m_{0,2}, m_{0,1}, m_{1,1})$
X_1	$\emptyset$	$x_1^n(\hat{m}_{1,1}\|\hat{m}_{0,1})$
Y_1	$\hat{m}_{1,1},\ \hat{m}_{0,1}$	$\hat{m}_{1,2},\ \hat{m}_{0,2}$
Y_2	$\emptyset$	$\hat{m}_{2,1},\ \hat{m}_{0,1}$

Table 3.1 Message parts transmission in Block 1 and Block 2.

keep their output received in the first block in their memories. In the second block, the source 0 broadcasts codeword $x_0^n(m_{200,2}|m_{1,2}^{199,2}, m_{0,2}, m_{0,1}, m_{1,1})$ to all following relays while the first relay broadcasts codeword $x_1^n(\hat{m}_{1,1}|\hat{m}_{0,1})$ to its following relays. At the end of the second block, the first relay will decode message parts $\hat{m}_{1,2}$ and $\hat{m}_{0,2}$ from its output of current block while the second relay decodes message parts $\hat{m}_{2,1}$ and $\hat{m}_{0,1}$ from its outputs of current and previous blocks.

	Block 200
X_0	$x_0^n(m_{200,200}\|m_{1,200}^{199,200}, m_{0,1}^{0,200}, \{m_{l,200-l}\}_{l\in\mathcal{T}})$
X_1	$x_1^n(\hat{m}_{1,199}\|\hat{m}_{0,1}^{0,199})$
X_2	$x_2^n(\hat{m}_{2,198}\|\hat{m}_{0,1}^{0,198})$
...	...
X_{100}	$x_{100}^n(\hat{m}_{100,100}\|\hat{m}_{0,1}^{0,100})$
...	...
X_{199}	$x_{199}^n(\hat{m}_{199,1}\|\hat{m}_{0,1})$
Y_1	$\hat{m}_{1,200},\ \hat{m}_{0,200}$
Y_2	$\hat{m}_{2,199},\ \hat{m}_{0,199}$
...	...
Y_{100}	$\hat{m}_{100,101},\ \hat{m}_{0,101}$
...	...
Y_{200}	$\{\hat{m}_{200,1}\}_{k\in\mathcal{T}\cup\{200\}},\ \hat{m}_{0,1}$

Table 3.2 Message parts transmission in Block 200.

As shown in Table 3.2, in the 200th block, the source 0 broadcasts codeword $x_0^n(m_{200,200}|m_{1,200}^{199,200}, m_{0,1}^{0,200}, \{m_{l,200-l}\}_{l\in\mathcal{T}})$ to all following relays while the relay in this network (for example, the relay indexed with k) broadcasts codeword $x_k^n(\hat{m}_{k,200-k}|\hat{m}_{0,1}^{0,200-k})$ to its following relays. At the end of the 200th block, the relay k decodes message parts $\hat{m}_{k,201-k}$ and $\hat{m}_{0,201-k}$ from its outputs of current block and previous $k-1$ blocks. And, the destination 200 starts implementing decoding. It decodes message parts $\{\hat{m}_{200,1}\}_{k\in\mathcal{T}\cup\{200\}}$

and $\hat{m}_{0,1}$ from its outputs of current and previous 199 blocks.

As shown in Figure 3.5, we give an example of message parts transmission in the two-relay network to illustrate the private message splitting scheme in a smaller network infrastructure. Assume that $N = 2$, so the relay network consists of source 0, relay 1, relay 2 and destination 3. In block j, the source wants to send message parts $m_{0,j}$, $m_{1,j}$, $m_{2,j}$ and $m_{3,j}$ to the destination via the help from two relays. The transmission process of message parts from block index j spans over three blocks: block j, block $j+1$, block $j+2$. At the beginning of block j, the source sends $x_0(m_{3,j})$ to the destination 0, $u_2(m_{2,j})$ to the relay 2 and $u_1(m_{1,j}), w_0(m_{0,j})$ to the relay 1 respectively. By the end of block j, the relay 1 decodes $m_{0,j}$ and $m_{1,j}$ from the received signal $y_1(j)$. At the beginning of block $j+1$, the relay 1 sends $w_1(m_{0,j})$ to the relay 2 and $x_1(m_{1,j})$ to the destination respectively. By the end of block $j+1$, the relay 2 decodes $m_{0,j}, m_{2,j}$ from received signals $y_2(j), y_2(j+1)$. Finally, at the beginning of block $j+2$, the relay 2 sends $w_2(m_0, j)$ and $x_2(m_2, j)$ to the destination. At the end block $j+2$, the destination decodes message parts $m_{0,j}$, $m_{1,j}$, $m_{2,j}$ and $m_{3,j}$ from received signals $y_3(j), y_3(j+1), y_3(j+2)$.

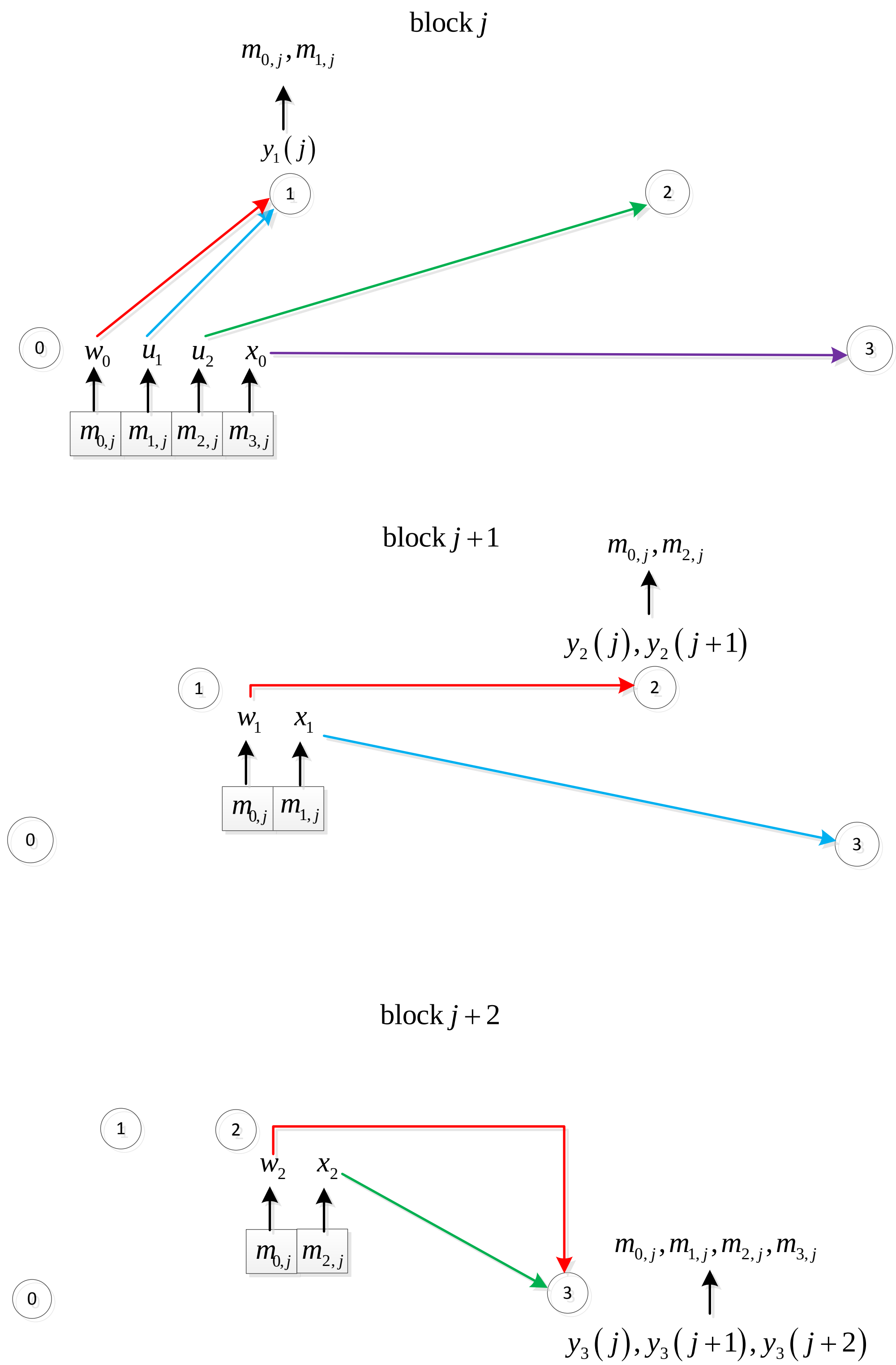

Fig. 3.5 Message parts transmission in the two-relay network.

3.4 Gaussian Relay Networks

In this section, we investigate our achievable rates in Gaussian relay networks. We also provide the achievable rate for Gaussian two-relay networks.

3.4.1 Signaling and Rates for Gaussian Relay Networks

The most important continuous alphabet relay network is the Gaussian relay network. The network infrastructure is shown in Figure 3.1. It is an additive time-discrete relay network, that is, for any receiver in this network, at time i, the output Y_i is the sum of the input X_i and the noise Z_i. The noise Z_i is assumed to be independent of the input X_i. Z_is in each time i are drawn identically and independently from a Gaussian distribution with variance N. This network is a model for some common communication networks, such as wired base station networks and satellite networks. If the noise variance is zero or there is no power constraint on the input, each receiver receives the transmitted symbol perfectly.

In our scheme, the Gaussian relay network can be modeled as:

$$\mathbf{y} = \mathbf{G}\mathbf{x} + \mathbf{z}, \tag{3.19}$$

where[4] $\mathbf{y}$, $\mathbf{x}$ and $\mathbf{z} \in \Re^{(N+1)\times 1}$. $\mathbf{y}$ is the received signal vector. $\mathbf{x}$ is the transmitted signal vector. $\mathbf{z}$ is the noise vector. $\mathbf{G}$ is the channel coefficient matrix, which is a $(N+1)\times(N+1)$ real upper diagonal matrix with all diagonal elements being 0. Each element in $\mathbf{z}$ is a noise signal at the decoder satisfying the Gaussian distribution[5] $\mathcal{N}(0, 1)$. There is a power constraint at each transmitting node.

From the codebook generation, we get the following Gaussian signaling:

For each relay $k \in \mathcal{T}$,

$$X_k = \alpha_k^{\mathbf{T}}\mathbf{w} + \beta_{kk}V_k, \tag{3.20}$$

where $\alpha_k = [0, \ldots, 0, \alpha_{kk}, \ldots, \alpha_{k,N}]^{\mathbf{T}}$ and $\mathbf{w} = [W_0, \ldots, W_N]^{\mathbf{T}} \in \Re^{(N+1)\times 1}$. $[\cdot]^{\mathbf{T}}$ denotes the vector transpose. In vector $\mathbf{w}$, elements W_l $(l \in \mathcal{T} \cup \{0\})$ carry the message at the lth transmitting node which supports the transmission of the common message part $m_{0,j-i}$.

[4] Here, we denote vertical vectors using boldface lower case letters to avoid confusion. We denote matrices using boldface upper case letters.

[5] $\mathcal{N}(0, 1)$ is the Gaussian distribution with mean 0 and variance 1.

W_i is successively superimposed on $W_{i+1}, W_{i+1}, \ldots, W_N$. V_k is the codeword sent by the relay k, which supports the forwarding of the message in U_k (sent by the source) and all W_l ($l \leq k$) respectively. Elements $\{\alpha_{kl}\}_{l\in[k:N]}$ and β_{kk} are power allocations satisfying the following constraint:

$$||\alpha_k||_2^2 + \beta_{kk}^2 = P_k, \tag{3.21}$$

where P_k is power constraint at the relay k.

For source node $k = 0$,

$$x_0 = \alpha_0^{\mathbf{T}}\mathbf{w} + \beta_0^{\mathbf{T}}\mathbf{v} + \phi_0^{\mathbf{T}}\mathbf{u}, \tag{3.22}$$

where $\beta_0 = [0, \beta_{01}, \ldots, \beta_{0,N}]^{\mathbf{T}}$ and $\mathbf{v} = [V_0, \ldots, V_N]^{\mathbf{T}} \in \Re^{(N+1)\times 1}$. $\phi_0 = [0, \phi_{01}, \ldots, \phi_{0,N+1}]^{\mathbf{T}}$ and $\mathbf{u} = [U_0, \ldots, U_{N+1}]^{\mathbf{T}} \in \Re^{(N+2)\times 1}$. $\{W_l\}_{l\in\mathcal{T}\cup\{0\}}$, $\{V_l\}_{l\in\mathcal{T}}$ and $\{U_l\}_{l\in\mathcal{T}\cup\{N+1\}}$ are independent, normalized Gaussian random variables satisfying $\mathcal{N}(0,1)$ respectively. In vector $\mathbf{u}$, elements U_l ($l \in \mathcal{T}$) carry private message parts m_l, which is decoded by relay l and not decoded by other relays. Each U_l is superimposed on all W_l. Element U_{N+1} carries private message part m_{N+1}, which is decoded at the destination $N+1$ and not decoded by all the relays. U_{N+1} is superimposed on all W_l and $\{U_k\}_{k\in\mathcal{T}}$. $\{\alpha_{0l}\}_{l\in[k:N]}$, $\{\beta_{0l}\}_{l\in\mathcal{T}}$ and $\{\phi_{0l}\}_{l\in\mathcal{T}\cup\{N+1\}}$ are power allocations satisfying the following constraints:

$$||\alpha_0||^2 + ||\beta_0||_2^2 + ||\phi_0||_2^2 = P_0, \tag{3.23}$$

where P_0 is the power constraint at the source 0.

By synthesizing Gaussian signaling from (3.20) and (3.22) with Gaussian relay network model from (3.19), (3.1) will become the lower bound for the capacity of Gaussian relay networks in the following:

$$C \geq \min_{\mathcal{S}\subset\mathcal{T}} \left\{ I_1 + \min_{j\in\mathcal{S}^c} I_2 + \sum_{j\in\mathcal{S}^c} I_3 \right\}, \tag{3.24}$$

where

$$I_1 = \frac{1}{2}\log\left(\sum_{i\in\mathcal{S}}(g_{N+1,0}\beta_{0i}+g_{N+1,i}\beta_{ii})^2+g_{N+1,0}^2\left(\sum_{i\in\mathcal{S}}\phi_{0i}^2+\phi_{0,N+1}^2\right)+1\right)$$
$$= \frac{1}{2}\log\left(||\mathbf{l}_{N+1}||_2^2+g_{N+1,0}^2||\phi_0||^2_{\mathcal{S}\cup\{N+1\}}+1\right),$$
$$I_2 = \frac{1}{2}\log\left(\frac{||g_{j0}\alpha_0^{\mathbf{T}}+\sum_{i=1}^{j-1}g_{ji}{\alpha'}_i^{\mathbf{T}}||_2^2}{\sum_{i=1}^{j-1}(g_{j1}\beta_{1i}+g_{ji}\beta_{ii})^2+g_{j0}^2\left(\sum_{i=j+1}^{N}\beta_{0i}^2+\sum_{i=1}^{N+1}\phi_{0i}^2\right)+1}+1\right)$$
$$= \frac{1}{2}\log\left(\frac{||g_{j0}\alpha_0^{\mathbf{T}}+\mathbf{e}_j^{\mathbf{T}}\mathbf{G}\mathbf{A}'||_2^2}{||\mathbf{l}_j||_2^2+g_{j0}^2(||\beta_0||^2_{\mathcal{P}}+||\phi_0||_2^2)+1}+1\right),$$
$$I_3 = \frac{1}{2}\log\left(\frac{\phi_{0j}^2g_{j0}^2}{\sum_{i=1}^{j-1}(g_{j1}\beta_{1i}+g_{ji}\beta_{ii})^2+g_{j0}^2\left(\sum_{i=j+1}^{N}\beta_{0i}^2+\sum_{i=1,i\neq j}^{N+1}\phi_{0i}^2\right)+1}+1\right),$$
$$= \frac{1}{2}\log\left(\frac{\phi_{0j}^2g_{j0}^2}{||\mathbf{l}_j||_2^2+g_{j0}^2(||\beta_0||^2_{\mathcal{P}}+||\phi_0||_2^2-\phi_{0j}^2)+1}+1\right).$$

$\mathbf{I}_n = \mathbf{I} = [\mathbf{e}_1, \ldots, \mathbf{e}_n]$ is the identity matrix;
$\mathbf{A} = [\alpha_0, \ldots, \alpha_N, \mathbf{0}]^{\mathbf{T}}$;
$\mathbf{A}' = [\mathbf{0}, {\alpha'}_1, \ldots, {\alpha'}_{j-1}, \mathbf{0}, \ldots, \mathbf{0}]^{\mathbf{T}}$ and ${\alpha'}_i = [\mathbf{0}, \ldots, \mathbf{0}, \alpha_{ii}, \ldots, \alpha_{i,j-1}, \mathbf{0}, \ldots, \mathbf{0}]^{\mathbf{T}}$;
$\mathbf{l}_j = [g_{j1}\beta_{11} + g_{j0}\beta_{01}, \ldots, g_{j,j-1}\beta_{j-1,j-1} + g_{j0}\beta_{0,j-1}]$.
We also define $||\mathbf{x}||^2_{\mathcal{S}} = \sum_{i\in\mathcal{S}} x_i^2$ and $\mathcal{P} = \{j+1, \ldots, N\}$.

3.4.2 Signaling and Rates for Gaussian Two-relay Networks

As shown in Figure 3.3, the Gaussian two-relay network can be modeled as:

$$\begin{aligned} Y_3 &= g_{03}X_0 + g_{13}X_1 + g_{23}X_2 + Z_3, \\ Y_2 &= g_{02}X_0 + g_{12}X_1 + Z_2, \\ Y_1 &= g_{01}X_0 + Z_1, \end{aligned} \tag{3.25}$$

where Z_3, Z_2 and Z_1 are independent AWGN noise according to the normal distribution $\mathcal{N}(0,1)$. The signaling at each node can be written as:

$$\begin{aligned}
x_2 &= \alpha_{22}W_2(w_{0,j-2}) + \beta_{22}V_2(w_{2,j-2}),\\
x_1 &= \alpha_{12}W_2(w_{0,j-2}) + \alpha_{11}W_1(w_{0,j-1}) + \beta_{11}V_1(w_{1,j-1}),\\
x_0 &= \alpha_{00}W_0(w_{0,j}) + \alpha_{01}W_1(w_{0,j-1}) + \alpha_{02}W_2(w_{0,j-2})\\
&\quad + \beta_{01}V_1(w_{1,j-1}) + \beta_{02}V_2(w_{2,j-2}) + \phi_{01}U_1(w_{1,j}) + \phi_{02}U_2(w_{2,j}) + \phi_{03}U_3(w_{3,j}),
\end{aligned} \tag{3.26}$$

where W_2, V_2, W_1, V_1, W_0, U_1, U_2, U_3 are independent, normalized Gaussian random variables satisfying $\mathcal{N}(0,1)$ respectively; $\{\alpha_\cdot, \beta_\cdot, \phi_\cdot\}$ are power allocations satisfying the following constraints:

$$\begin{aligned}
\alpha_{22}^2 + \beta_{22}^2 &= P_2,\\
\alpha_{11}^2 + \alpha_{12}^2 + \beta_{11}^2 &= P_1,\\
\alpha_{00}^2 + \alpha_{01}^2 + \alpha_{02}^2 + \beta_{01}^2 + \beta_{02}^2 + \phi_{01}^2 + \phi_{02}^2 + \phi_{03}^2 &= P_0,
\end{aligned} \tag{3.27}$$

where P_0, P_1 and P_2 are power constraints at the corresponding node and they can be equal to each other without loss of generality.

Corollary 3. *The capacity for a Gaussian two-relay network in (3.25) is lower bounded by:*

$$C \geq \min\{I_1 + I_4 + I_5, I_2 + I_3 + I_5, I_2 + I_7, I_4 + I_6, I_8\}, \tag{3.28}$$

where

$$I_1 = \frac{1}{2}\log\left(1 + \frac{g_{01}^2\phi_{01}^2}{g_{01}^2(\beta_{02}^2 + \phi_{02}^2 + \phi_{03}^2) + 1}\right),$$
$$I_4 = \frac{1}{2}\log\left(1 + \frac{g_{02}^2(\alpha_{00} + \phi_{02})^2 + (g_{02}\alpha_{01} + g_{12}\alpha_{11})^2}{(g_{02}\beta_{01} + g_{12}\beta_{11})^2 + g_{02}^2(\phi_{01}^2 + \phi_{03}^2) + 1}\right),$$
$$I_3 = \frac{1}{2}\log\left(1 + \frac{g_{02}^2\phi_{02}^2}{(g_{02}\beta_{01} + g_{12}\beta_{11})^2 + g_{02}^2(\phi_{01}^2 + \phi_{03}^2) + 1}\right),$$
$$I_2 = \frac{1}{2}\log\left(1 + \frac{g_{01}^2(\alpha_{00}^2 + \phi_{01}^2)}{g_{01}^2(\beta_{02}^2 + \phi_{02}^2 + \phi_{03}^2) + 1}\right),$$
$$I_5 = \frac{1}{2}\log\left(1 + g_{03}^2\phi_{03}^2\right),$$
$$I_6 = \frac{1}{2}\log\left(1 + (g_{03}\beta_{01} + g_{13}\beta_{11})^2 + g_{03}^2(\phi_{01}^2 + \phi_{03}^2)\right),$$
$$I_7 = \frac{1}{2}\log\left(1 + (g_{03}\beta_{02} + g_{23}\beta_{22})^2 + g_{03}^2(\phi_{02}^2 + \phi_{03}^2)\right),$$
$$\begin{aligned}I_8 = \frac{1}{2}\log\big(&1 + g_{03}^2P_0 + g_{13}^2P_1 + g_{23}^2P_2\\ &+ 2g_{03}g_{13}(\alpha_{01}\alpha_{11} + \alpha_{02}\alpha_{12} + \beta_{01}\beta_{11})\\ &+2g_{03}g_{23}(\alpha_{02}\alpha_{22} + \beta_{02}\beta_{22}) + 2g_{13}g_{23}\alpha_{12}\alpha_{22}\big),\end{aligned}$$

and $\alpha_{ij}, \beta_{ij}, \phi_{ij}$ *(*$i \in \{0,1,2\}$*,*$j \in \{0,1,2,3\}$*) are power allocations satisfying (3.27).*

Remark 1. *By setting* $N = 3$ *in* (3.24)*, we can get the capacity lower bound for Gaussian two-relay networks in* (3.28)*.*

3.4.3 Numerical Comparison

We numerically compare the achievable rates of private message splitting based partial decode-forward scheme and pure decode-forward scheme in the two-relay network. For both schemes, we assume that transmitting powers at source 0, relay 1 and relay 2 are the same. As shown in Figure 3.6, we consider the comparison of two schemes in four possible channel conditions: all links are in the same condition ($g_{01} = g_{02} = g_{03} = g_{12} = g_{13} = g_{23} = 1$, shown in left upper plot), the link between two relays is worse than other links ($g_{01} = g_{02} = g_{03} = g_{13} = g_{23} = 1$, $g_{12} = 0.3$, shown in right upper plot), the source-to-relay links are better than in-relay links as well as relay-to-destination links ($g_{01} = g_{02} = g_{03} = g_{12} = 1$, $g_{13} = g_{23} = 4$, shown in left lower plot), the relay-to-destination links are better than in-relay links as well as source-to-relay links ($g_{01} = g_{02} = 4$, $g_{03} = g_{12} = g_{13} = g_{23} = 1$, shown in right lower plot). Such channel condition setups are common communication scenarios. We also simulate two random channel conditions ($g_{01} = 1$, $g_{02} = 2$, $g_{03} = 3$, $g_{12} = 4$, $g_{13} = 5$, $g_{23} = 6$ and $g_{01} = 6$, $g_{02} = 5$, $g_{03} = 4$, $g_{12} = 3$, $g_{13} = 2$, $g_{23} = 1$) in Figure 3.7. Note that the private message splitting scheme is depicted with red solid line and the decode-forward scheme is depicted with blue dashed line. We can see that the private message splitting scheme outperforms the decode-forward scheme in all channel conditions. Especially, when the link between two relays suffers from poor channel condition, the private message splitting scheme performs much better than decode-forward. The reason is that when common message part transmission is error prone, source will take advantage of the channel condition by allocating more power transmitting private message parts. This result agrees with the analysis in Theorem 4 that the private message splitting scheme can result in a tighter bound than decode-forward.

Message splitting scheme provides us with a feasible way to facilitate partial decode-forward in the relay network with N relays, while previous literatures are only able to extend partial decode-forward to the relay network with one or two relays. With private message splitting scheme, each relay in the network is responsible for the transmission of a distinct message part. The source pre-splits its message according to the source-relay channel condition. Thus, if the link between the source and some relay is in poor channel condition (e.g., deep fading or high noises), the source will split less message parts to that relay and will re-allocate more message parts to some other relay with better channel condition to the source. This is the reason that private message splitting scheme provides a tighter lower bound on capacity than the schemes shown in [5] and [15].

3.5 Summary of The Chapter

In this chapter, we propose partial decode-forward scheme based on private message splitting for an N-relay network with single source and single destination. We use block Markov encoding and joint sliding window decoding at each relay. We derive the achievable rate for this scheme as shown in Theorem 4. We also show that this result includes network full decode-forward and two-relay networks as special cases. We next analyze this result in AWGN environments. We derive capacity results for Gaussian relay networks as shown in (3.24) and capacity results for Gaussian two-relay networks as shown in (3.28). We then compare the private message splitting scheme with pure decode-forward in the Gaussian two-relay network.

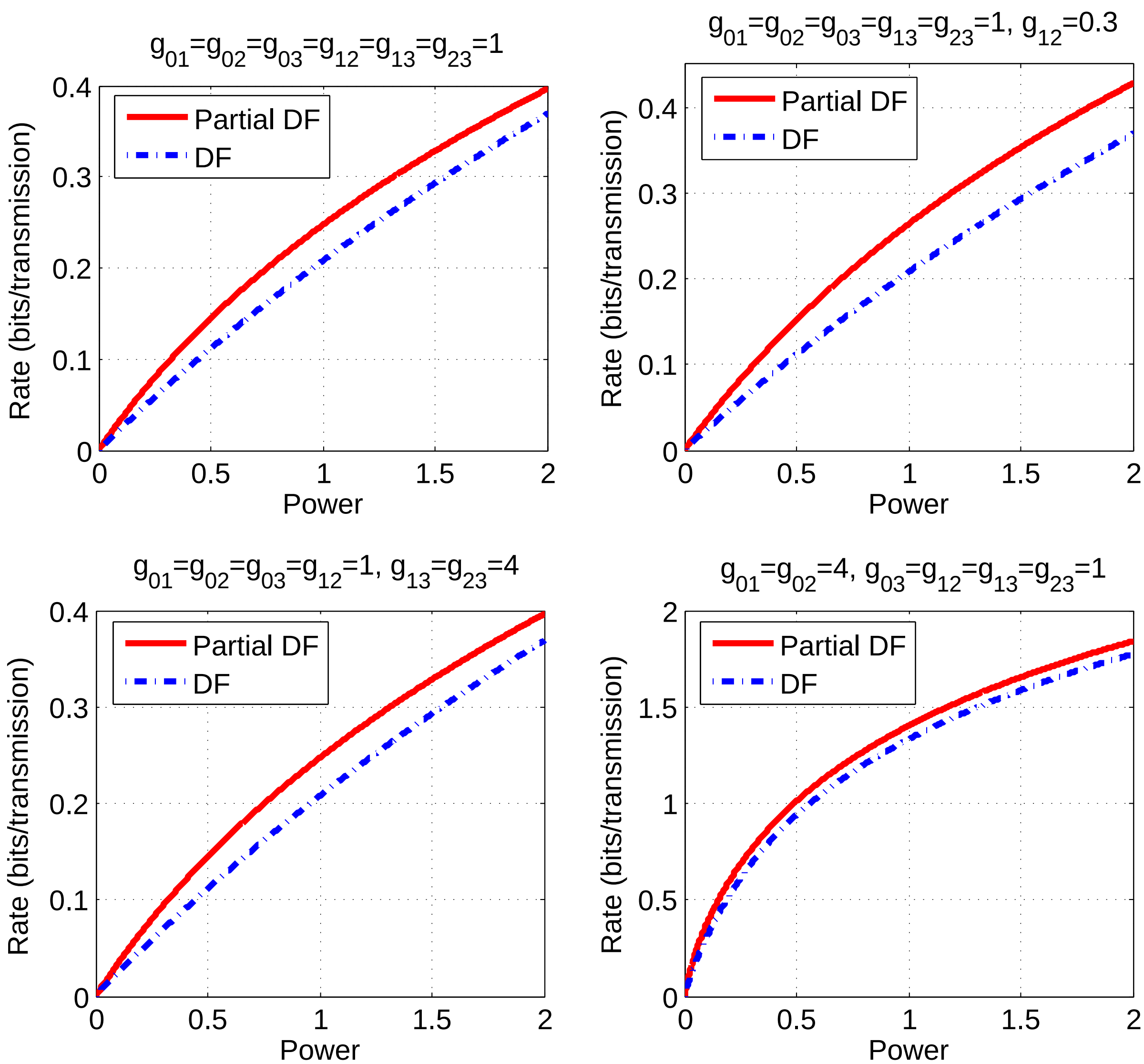

Fig. 3.6 Rate comparison between partial decode-forward and decode-forward in the two-relay network.

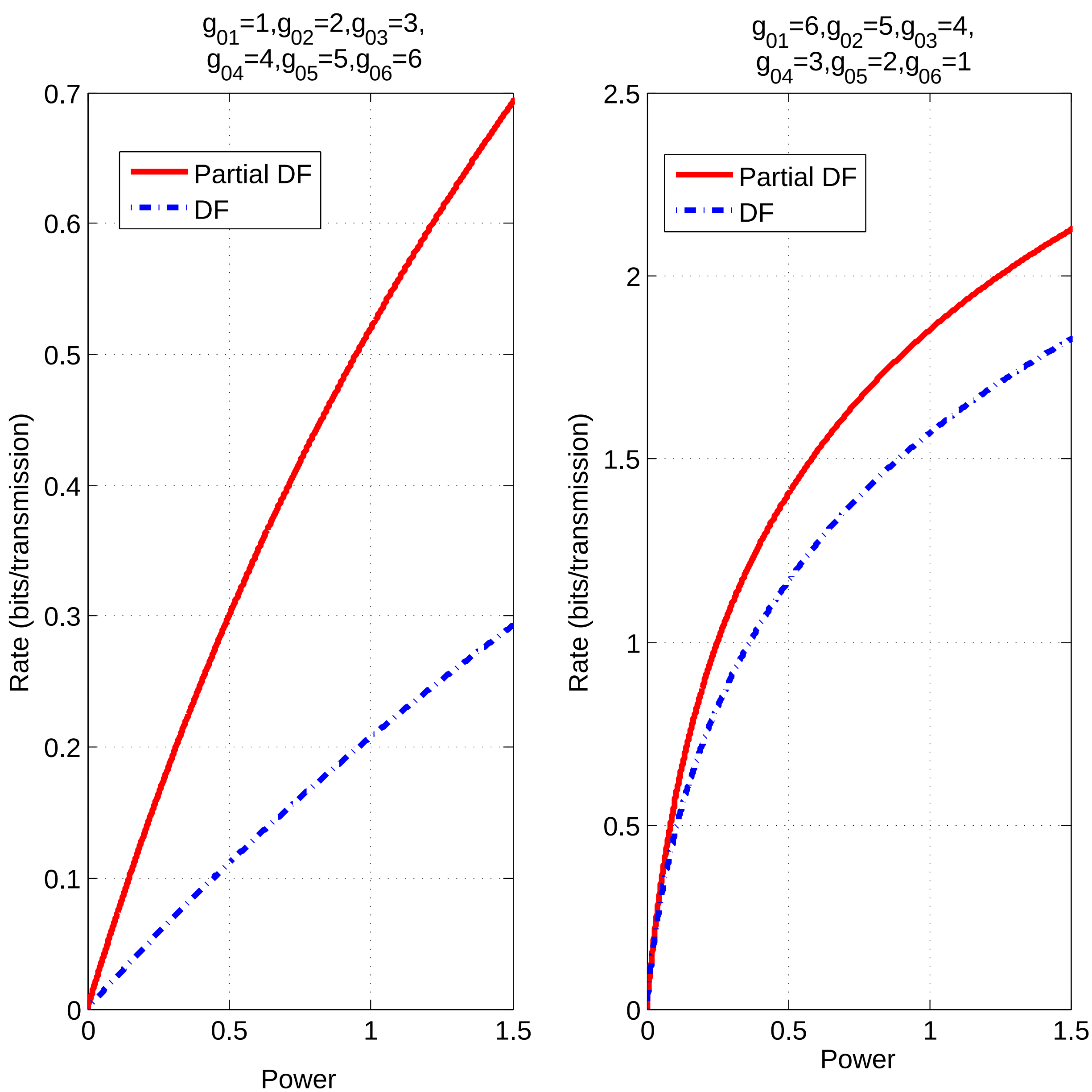

Fig. 3.7 Rate comparison between partial decode-forward and decode-forward in the two-relay network.

Chapter 4

Partial Decode-Forward Scheme for Three-relay Networks

In this chapter, we mainly study partial decode-forward based on exhaustive message splitting scheme for three-relay networks. In this relay network, we pre-assign each relay to different relay subsets. Each relay subset is responsible for the transmission of a distinct message part. This scheme is based on block Markov encoding and joint sliding window decoding. We then extend this scheme to a single-source single-destination relay network with multiple relays. We investigate the codebook generation for the multi-relay network. In the last part of this chapter, we analyze the capacity results of three-relay networks in the AWGN environment and make comparison with the capacity results from the scheme in Chapter 3.

4.1 Exhaustive Message Splitting Scheme

In this section, we propose a partial decode-forward scheme for a three-relay network as shown in Figure 4.1. Different from the private message splitting scheme discussed in Chapter 3, we consider all possible partial decoding cases that can occur among message parts at the source and the relays, which is called **exhaustive** message splitting as shown in Chapter 1 and Chapter 2.

In each transmission block, the source splits its original message m into eight parts as shown in Table 4.1. In this table, the subscript of $m_{\cdot}$ denotes the relay indices which help the transmission of that message part (except that message part m_0 is decoded and re-transmitted among all relays $1, 2, 3$). $\{W., Q., S., V., X.\}$ represent the corresponding codewords for different message parts and their subscripts denote the transmitting node indices. $U.$ is transmitted directly by the source and its subscript denotes the supposed helping relay. $U.$ is re-transmitted by $X.$ at each relay.

Then let's look at Table 4.1 in detail. In the whole communication process, the source sends X_0, V_0, Q_0, S_0, W_0 to carry the message parts m_4, m_{12}, m_{23}, m_{13} and m_0 respectively. The source also sends U_1, U_2, U_3 to carry the message parts m_1, m_2 and m_3 respectively. The relay 1 sends X_1, V_1, S_1, W_1 to carry the message parts m_1, m_{12}, m_{13} and m_0 respectively. The relay 2 sends X_2, V_2, Q_2, W_2 to carry the message parts m_2, m_{12}, m_{23} and m_0 respectively. Finally, The relay 3 sends X_3, Q_3, S_3, W_3 to carry the message parts m_3, m_{23}, m_{13} and m_0 respectively.

Block Markov superposition coding is used to generate the independent codewords in each block. To better understand the superposition coding structure, we use a directed graph (Figure 4.2) to represent the superposition structure among the generated codewords. Codewords are connected by an arrow where the codeword at the end of the arrow is superimposed on the one at the beginning of the arrow.

From Figure 4.2, we can find the meaning of all variables in the following:

- w_k, $k \in \{0, 1, 2, 3\}$, carries common message part $m_{0,j-k}$ of different blocks in relay k. The codeword w_k is superimposed upon codewords $w_{k+1}, \ldots, w_3$.

- v_k, $k \in \{0, 1, 2\}$, carries private message part m_{12} to be decoded only at relays $\{1, 2\}$. q_k, $k \in \{0, 2, 3\}$, carries private message part m_{23} to be decoded only at relays $\{2, 3\}$. s_k, $k \in \{0, 1, 3\}$, carries private message part m_{13} to be decoded only at relays $\{1, 3\}$. Each codeword $\{v_k, q_k, s_k\}$ is superimposed on all codewords w_l $(l \geq k)$ and corresponding codewords $\{v_l\}$ $(l \geq k)$, $\{q_l\}$ $(l \geq k)$ and $\{s_l\}$ $(l \geq k)$ respectively.

- u_k, $k \in \{1, 2, 3\}$, carries private message part m_k, which is to be decoded at relay k and not decoded at other relays. The codeword u_1 is superimposed on all codewords $\{s_l, v_l, w_l\}$ $(l \geq 1)$. The codeword u_2 is superimposed on all codewords $\{q_l, v_l, w_l\}$ $(l \geq 2)$. u_3 is superimposed on all codewords $\{s_3, q_3, w_3\}$.

- x_k, $k \in \{1, 2, 3\}$, is the codeword sent by relay k which supports the forwarding of the message in u_k.

- x_0 is the codeword sent by the source which carries the remaining message part m_4, which is to be decoded only at the destination. The codeword x_0 is conditioned upon all codewords $\{w_k, u_k, s_k, v_k, x_k, q_k\}$ $(k \in \{1, 2, 3\})$.

Next, we present the achievable rate and prove its achievablity.

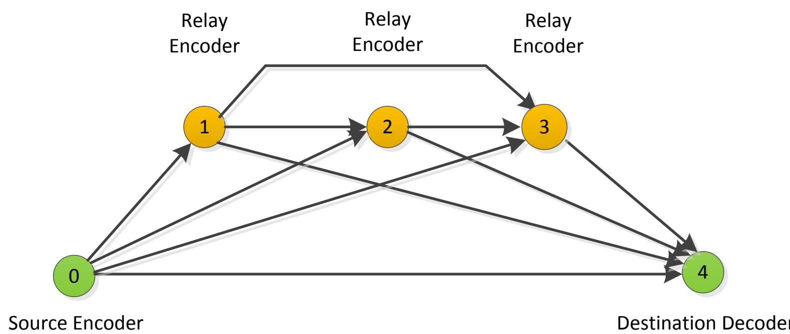

Fig. 4.1 Partial-decode-forward for three-relay network.

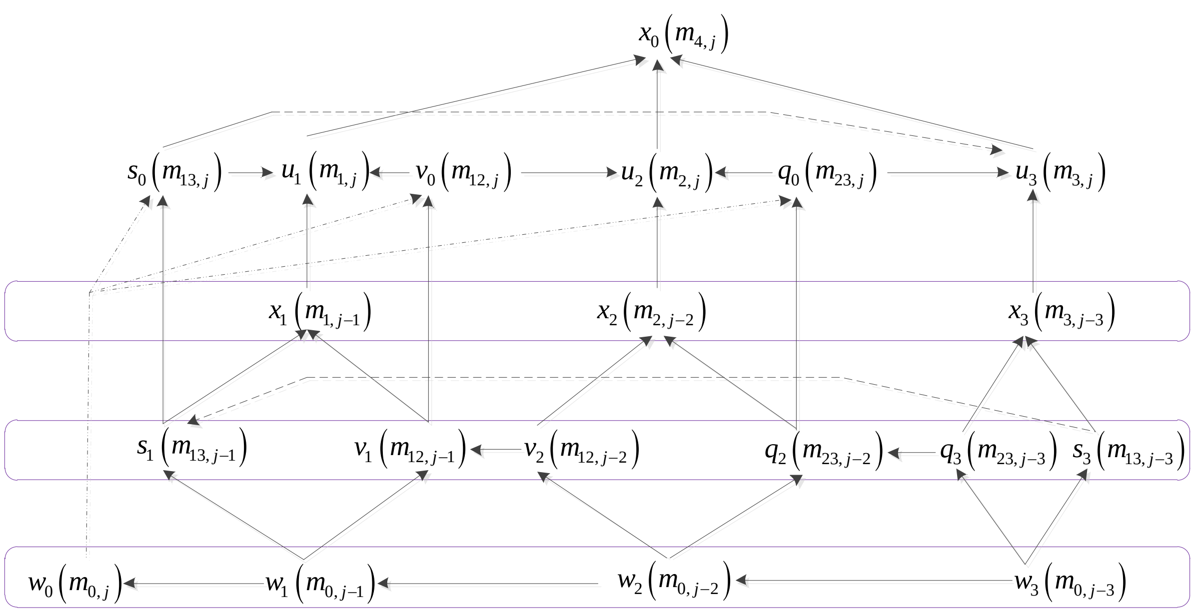

Fig. 4.2 Coding structure at the source and the relay.

Table 4.1 Message parts and transmitting relays

Message parts	Relays	Variables
m_0	1,2,3	W_0, W_1, W_2, W_3
m_3	3	U_3, X_3
m_{12}	1, 2	V_0, V_1, V_2
m_2	2	U_2, X_2
m_{23}	2, 3	Q_0, Q_2, Q_3
m_1	1	U_1, X_1
m_{13}	1, 3	S_0, S_1, S_3
m_4	$\emptyset$	X_0

4.1.1 Coding Scheme and Achievable Rate

Theorem 5. *For a single-source single-destination network with three relays ($\mathcal{X}_0 \times \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3$, $p(y_1, y_2, y_3, y_4|x_0, x_1, x_2, x_3)$, $\mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{Y}_3 \times \mathcal{Y}_4$), by using partial decode-forward, the capacity C is lower bounded as follows:*

$$\begin{aligned} C \geq \sup_{P^*} \min\{ & I_{33}, I_1 + I_{23}, I_6 + I_{25}, I_{11} + I_{27}, I_6 + I_{13} + I_{18}, I_7 + I_{11} + I_{18}, \\ & I_1 + I_{12} + I_{17}, I_2 + I_{11} + I_{17}, I_1 + I_8 + I_{16}, I_3 + I_6 + I_{16}, \\ & I_5 + I_6 + I_{13} + I_{34}, I_6 + I_3 + I_{15} + I_{34}, I_1 + I_8 + I_{15} + I_{34}, \\ & I_1 + I_{10} + I_{12} + I_{34}, I_5 + I_7 + I_{11} + I_{34}, I_2 + I_{10} + I_{11} + I_{34}, \\ & I_3 + I_7 + I_{12} + I_{34}, I_2 + I_8 + I_{13} + I_{34}\}, \end{aligned} \tag{4.1}$$

for some joint distribution that factors as

$$
\begin{aligned}
P^* \triangleq & p(w_0^3, u_1^3, x_0^3, s_0, s_1, s_3, v_0, v_1, v_2, q_0, q_2, q_3) \\
& p(w_3)p(q_3|w_3)p(s_3|w_3)p(x_3|s_3, q_3, w_3) \\
& p(w_2|w_3)p(v_2|w_2, w_3)p(q_2|q_3, w_2, w_3)p(x_2|v_2, q_2, q_3, w_2, w_3) \\
& p(w_1|w_2, w_3)p(v_1|v_2, w_1, w_2, w_3)p(s_1|s_3, w_1, w_2, w_3)p(x_1|v_1, v_2, s_1, s_3, w_1, w_2, w_3) \\
& p(w_0|w_1, w_2, w_3)p(q_0|q_2, q_3, w_0, w_1, w_2, w_3)p(v_0|v_1, v_2, w_0, w_1, w_2, w_3) \\
& p(s_0|s_1, s_3, w_0, w_1, w_2, w_3) \\
& p(u_3|x_3, s_0, s_1, s_3, q_0, q_2, q_3, w_0, w_1, w_2, w_3) \\
& p(u_2|x_2, v_0, v_1, v_2, q_0, q_2, q_3, w_0, w_1, w_2, w_3) \\
& p(u_1|x_1, v_0, v_1, v_2, s_0, s_1, s_3, w_0, w_1, w_2, w_3) \\
& p(x_0|u_1, x_1, u_2, x_2, u_3, x_3, v_0, v_1, v_2, q_0, q_2, q_3, s_0, s_1, s_3, w_0, w_1, w_2, w_3),
\end{aligned} \tag{4.2}
$$

where I_j *are defined in* (4.4), (4.6), (4.8) *and* (4.10), *which can be found in the following Decoding section.* (4.1) *shows a lower bound on the capacity of the single-source single-destination three-relay network based on partial decode-forward with exhaustive message splitting. The supremum is taken oven all possible input distributions defined in* (4.2).

Proof. We use a block Markov coding scheme in which the source sends $b-3$ messages over b blocks of n symbols each. Each relay and the destination employ simultaneous sliding window decoding.

Before the communication process, the codebook is generated and revealed to all parties. For each block j, the source sends x_0, which contains corresponding message parts in the blocks j, $j-1$, $j-2$ and $j-3$. At the end of block j, the first relay $k=1$ has estimates $\hat{m}_{0,j}, \hat{m}_{1,j}, \hat{m}_{12,j}, \hat{m}_{13,j}$ of the message parts $m_{0,j}, m_{1,j}, m_{12,j}, m_{13,j}$, the second relay $k=2$ has estimates $\hat{m}_{0,j-1}, \hat{m}_{2,j-1}, \hat{m}_{12,j-1}, \hat{m}_{23,j-1}$ of the message parts $m_{0,j-1}, m_{2,j-1}, m_{12,j-1}, m_{23,j-1}$ and the third relay $k=3$ has estimates $\hat{m}_{0,j-2}, \hat{m}_{3,j-2}, \hat{m}_{13,j-2}$, $\hat{m}_{23,j-2}$ of the message parts $m_{0,j-2}, m_{3,j-2}, m_{13,j-2}, m_{23,j-2}$. In the block $j+1$, the relays node $j = 1,2,3$ broadcasts $x_1(\hat{m}_{1,j}|\hat{m}_{12,j}, \hat{m}_{0,j}, \hat{m}_{13,j})$, $x_2(\hat{m}_{2,j-1}|\hat{m}_{23,j-1}, \hat{m}_{12,j-1}, \hat{m}_{0,j-1})$ and $x_3(\hat{m}_{3,j-2}|\hat{m}_{13,j-2}, \hat{m}_{23,j-2}, \hat{m}_{0,j-2})$ to their following nodes. The destination uses joint decoding simultaneously over signals received in all current and previous three blocks.

Codebook Generation

For relay node $k = 3$

- For each $j \in [1 : b-3]$, independently generate 2^{nR_0} sequences $w_3(m_{0,j-3})$ each i.i.d $\sim p(w_3)$.
- For each message $m_{0,j-3}$, randomly and conditionally independently generate $2^{nR_{23}}$ sequences $q_3(m_{23,j-3}|m_{0,j-3})$ each i.i.d $\sim p(q_3|w_3)$.
- For each message $m_{0,j-3}$, randomly and conditionally independently generate $2^{nR_{13}}$ sequences $s_3(m_{13,j-3}|m_{0,j-3})$ each i.i.d $\sim p(s_3|w_3)$.
- For each message set $m_{0,j-3}, m_{13,j-3}, m_{23,j-3}$, randomly and conditionally independently generate 2^{nR_3} sequences
  $x_3(m_{3,j-3}|m_{13,j-3}, m_{23,j-3}, m_{0,j-3})$ each i.i.d $\sim p(x_3|s_3, q_3, w_3)$.

For relay node $k = 2$

- For each message $m_{0,j-3}$, randomly and conditionally independently generate 2^{nR_0} sequences $w_2(m_{0,j-2}|m_{0,j-3})$ each i.i.d $\sim p(w_2|w_3)$.
- For each message set $m_{0,j-3}, m_{0,j-2}$, randomly and conditionally independently generate $2^{nR_{12}}$ sequences $v_2(m_{12,j-2}|m_{0,j-2}, m_{0,j-3})$ each i.i.d $\sim p(v_2|w_2, w_3)$.
- For each message set $m_{0,j-3}, m_{0,j-2}, m_{23,j-3}$, randomly and conditionally independently generate $2^{nR_{23}}$ sequences $q_2(m_{23,j-2}|m_{23,j-3}, m_{0,j-2}, m_{0,j-3})$ each i.i.d $\sim p(q_2|q_3, w_2, w_3)$.
- For each message set $m_{0,j-3}$, $m_{0,j-2}$, $m_{12,j-2}$, $m_{23,j-3}$, $m_{23,j-2}$, randomly and conditionally independently generate 2^{nR_2} sequences
  $x_2(m_{2,j-2}|m_{23,j-2}, m_{12,j-2}, m_{0,j-2}, m_{0,j-3}, m_{23,j-3})$ each i.i.d $\sim p(x_2|v_2, w_2, w_3, q_2, q_3)$.

For relay node $k = 1$

- For each message set $m_{0,j-3}$, $m_{0,j-2}$, randomly and conditionally independently generate 2^{nR_0} sequences $w_1(m_{0,j-1}|m_{0,j-2}, m_{0,j-3})$ each i.i.d $\sim p(w_1|w_2, w_3)$.
- For each message set $m_{0,j-3}$, $m_{0,j-2}$, $m_{0,j-1}$, $m_{12,j-2}$, randomly and conditionally independently generate $2^{nR_{12}}$ sequences

  $v_1(m_{12,j-1}|m_{0,j-1}, m_{12,j-2}, m_{0,j-2}, m_{0,j-3})$ each i.i.d $\sim p(v_1|w_1, v_2, w_2, w_3)$.

- For each message set $m_{0,j-3}$, $m_{0,j-2}$, $m_{0,j-1}$, $m_{13,j-3}$, randomly and conditionally independently generate $2^{nR_{13}}$ sequences

  $s_1(m_{13,j-1}|m_{13,j-3}, m_{0,j-1}, m_{0,j-2}, m_{0,j-3})$ each i.i.d $\sim p(s_1|w_1, s_3, w_2, w_3)$.

- For each message set $m_{0,j-3}$, $m_{0,j-2}$, $m_{0,j-1}$, $m_{12,j-2}$, $m_{12,j-1}$, $m_{13,j-3}$, $m_{13,j-1}$, randomly and conditionally independently generate 2^{nR_2} sequences

  $x_1(m_{1,j-1}|m_{12,j-1}, m_{12,j-2}, m_{0,j-1}, m_{0,j-2}, m_{0,j-3}, m_{13,j-3}, m_{13,j-1})$
  each i.i.d $\sim p(x_1|v_1, v_2, w_1, w_2, w_3, s_1, s_3)$.

For source node $k = 0$

- For each message set $m_{0,j-3}$, $m_{0,j-2}$, $m_{0,j-1}$, randomly and conditionally independently generate 2^{nR_0} sequences $w_0(m_{0,j}|m_{0,j-1}, m_{0,j-2}, m_{0,j-3})$ each i.i.d $\sim p(w_0|w_1, w_2, w_3)$.

- For each message set $m_{0,j-3}$, $m_{0,j-2}$, $m_{0,j-1}$, $m_{0,j}$, $m_{23,j-3}$, $m_{23,j-2}$, randomly and conditionally independently generate $2^{nR_{23}}$ sequences
  $q_0(m_{23,j}|m_{23,j-2}, m_{23,j-3}, m_{0,j}, m_{0,j-1}, m_{0,j-2}, m_{0,j-3})$
  each i.i.d $\sim p(q_0|q_2, q_3, w_0, w_1, w_2, w_3)$.

- For each message set $m_{0,j-3}$, $m_{0,j-2}$, $m_{0,j-1}$, $m_{0,j}$, $m_{12,j-2}$, $m_{12,j-1}$, randomly and conditionally independently generate $2^{nR_{12}}$ sequences

  $v_0(m_{12,j}|m_{12,j-1}, m_{0,j}, m_{0,j-1}, m_{12,j-2}, m_{0,j-2}, m_{0,j-3})$
  each i.i.d $\sim p(v_0|v_1, w_0, w_1, v_2, w_2, w_3)$.

- For each message set $m_{0,j-3}$, $m_{0,j-2}$, $m_{0,j-1}$, $m_{0,j}$, $m_{13,j-3}$, $m_{13,j-1}$, randomly and conditionally independently generate $2^{nR_{13}}$ sequences

  $s_0(m_{13,j}|m_{13,j-1}, m_{0,j}, m_{13,j-3}, m_{0,j-1}, m_{0,j-2}, m_{0,j-3})$
  each i.i.d $\sim p(s_0|s_1, w_0, w_1, s_3, w_2, w_3)$.

- For each message set $m_{0,j-3}$, $m_{0,j-2}$, $m_{0,j-1}$, $m_{0,j}$, $m_{13,j-3}$, $m_{13,j-1}$, $m_{13,j}$, $m_{23,j-3}$, $m_{23,j-2}$, $m_{23,j}$, randomly and conditionally independently generate 2^{nR_3} sequences
  $u_3(m_{3,j}|m_{3,j-3}, m_{13,j-3}, m_{13,j-1}, m_{13,j}, m_{23,j-3}, m_{23,j-2}, m_{23,j}, m_{0,j-3}, m_{0,j-2}, m_{0,j-1}, m_{0,j})$
  each i.i.d $\sim p(u_3|x_3, s_0, s_1, s_3, q_0, q_2, q_3, w_0, w_1, w_2, w_3)$.

- For each message set $m_{0,j-3}$, $m_{0,j-2}$, $m_{0,j-1}$, $m_{0,j}$, $m_{12,j-2}$, $m_{12,j-1}$, $m_{12,j}$, $m_{23,j-3}$, $m_{23,j-2}$, $m_{23,j}$, randomly and conditionally independently generate 2^{nR_2} sequences

$u_2(m_{2,j}|m_{2,j-2}, m_{12,j-1}, m_{12,j-2}, m_{12,j}, m_{23,j-3}, m_{23,j-2}, m_{23,j}, m_{0,j-3}, m_{0,j-2}, m_{0,j-1}, m_{0,j})$ each i.i.d $\sim p(u_2|x_2, v_0, v_1, v_2, q_0, q_2, q_3, w_0, w_1, w_2, w_3)$.

- For each message set $m_{0,j-3}$, $m_{0,j-2}$, $m_{0,j-1}$, $m_{0,j}$, $m_{12,j-2}$, $m_{12,j-1}$, $m_{12,j}$, $m_{13,j-3}$, $m_{13,j-1}$, $m_{13,j}$, randomly and conditionally independently generate 2^{nR_1} sequences $u_1(m_{1,j}|m_{1,j-1}, m_{12,j-1}, m_{12,j-2}, m_{12,j}, m_{13,j-3}, m_{13,j-1}, m_{13,j}, m_{0,j-3}, m_{0,j-2}, m_{0,j-1}, m_{0,j})$ each i.i.d $\sim p(u_1|x_1, v_0, v_1, v_2, s_0, s_1, s_3, w_0, w_1, w_2, w_3)$.

- For each message set $m_{2,j}$, $m_{23,j}$, $m_{13,j}$, $m_{3,j}$, $m_{1,j}$, $m_{1,j-1}$, $m_{0,j-3}$, $m_{0,j-2}$, $m_{0,j-1}$, $m_{0,j}$, $m_{12,j-2}$, $m_{12,j-1}$, $m_{12,j}$, $m_{13,j-1}$, $m_{2,j-2}$, $m_{23,j-2}$, $m_{3,j-3}$, $m_{13,j-3}$, $m_{23,j-3}$, randomly and conditionally independently generate 2^{nR_4} sequences $x_0(m_{4,j}|m_{2,j}, m_{23,j}, m_{13,j}, m_{3,j}, m_{0,j}, m_{12,j}, m_{1,j}, m_{1,j-1}, m_{12,j-1}, m_{0,j-1}, m_{12,j-2}, m_{0,j-2}, m_{0,j-3}, m_{13,j-1}, m_{2,j-2}, m_{23,j-2}, m_{3,j-3}, m_{13,j-3}, m_{23,j-3})$ each i.i.d $\sim p(x_0|u_1, u_2, u_3, x_1, x_2, x_3, w_0, w_1, w_2, w_3, v_0, v_1, v_2, s_0, s_1, s_3, q_0, q_2, q_3)$.

The above constitutes the codebook $\mathcal{C}_j$ of block j. The codebook $\mathcal{C}_j$ is then revealed to all the parties. The codebook is then revealed to all parties.

Encoding

To send $m_{0,j}$, $m_{3,j}$, $m_{12,j}$, $m_{2,j}$, $m_{23,j}$, $m_{1,j}$, $m_{13,j}$ and $m_{4,j}$ in block j, the source sends $x_0(m_{4,j}|m_{2,j}, m_{23,j}, m_{13,j}, m_{3,j}, m_{0,j}, m_{12,j}, m_{1,j})$ from codebook $\mathcal{C}_j$. At the end of block j, the first relay $k = 1$ has an estimate $\hat{m}_{0,j}, \hat{m}_{1,j}, \hat{m}_{12,j}, \hat{m}_{13,j}$ of the message $m_{0,j}, m_{1,j}, m_{12,j}, m_{13,j}$, the second relay $k = 2$ has an estimate $\hat{m}_{0,j-1}, \hat{m}_{2,j-1}, \hat{m}_{12,j-1}, \hat{m}_{23,j-1}$ of the message $m_{0,j-1}, m_{2,j-1}, m_{12,j-1}, m_{23,j-1}$ and the third relay $k = 3$ has an estimate $\hat{m}_{0,j-2}, \hat{m}_{3,j-2}, \hat{m}_{13,j-2}$, $\hat{m}_{23,j-2}$ of the message $m_{0,j-2}, m_{3,j-2}, m_{13,j-2}, m_{23,j-2}$. In the block $j+1$, the relays node $j = 1, 2, 3$ transmits $x_1(\hat{m}_{1,j}|\hat{m}_{12,j}, \hat{m}_{0,j}, \hat{m}_{13,j})$, $x_2(\hat{m}_{2,j-1}|\hat{m}_{23,j-1}, \hat{m}_{12,j-1}, \hat{m}_{0,j-1})$ and $x_3(\hat{m}_{3,j-2}|\hat{m}_{13,j-2}, \hat{m}_{23,j-2}, \hat{m}_{0,j} - 2)$ respectvely from codebook $\mathcal{C}_{j+1}$.

Decoding

Simultaneous decoding at the first relay

At the end of block j, the first relay $k=1$, will decode $\hat{m}_{0,j}, \hat{m}_{1,j}, \hat{m}_{12,j}, \hat{m}_{13,j}$ such that:

$$\begin{aligned}(w_0(\hat{m}_{0,j}), v_0(\hat{m}_{12,j}|\hat{m}_{0,j}), s_0(\hat{m}_{13,j}|\hat{m}_{0,j}),\ u_1(\hat{m}_{1,j}|\hat{m}_{12,j}, \hat{m}_{13,j}, \hat{m}_{0,j}),\\ w_1(\hat{m}_{0,j-1}), v_1(\hat{m}_{12,j-1}|\hat{m}_{0,j-1}), s_1(\hat{m}_{13,j-1}|\hat{m}_{0,j-1}), x_1(\hat{m}_{1,j-1}|\hat{m}_{12,j-1}, \hat{m}_{0,j-1}, \hat{m}_{13,j-1}),\\ w_2(\hat{m}_{0,j-2}), v_2(\hat{m}_{12,j-2}|\hat{m}_{0,j-2}), w_3(\hat{m}_{0,j-3}), s_3(\hat{m}_{13,j-3}|\hat{m}_{0,j-3}), y_1(j)) \in \mathcal{T}_\epsilon^{(n)}.\end{aligned} \tag{4.3}$$

In (4.3), the first relay decodes $\hat{m}_{1,j}$, $\hat{m}_{12,j}$, $\hat{m}_{13,j}$ and $\hat{m}_{0,j}$ from output $y_1(j)$ when $y_1(j)$ is jointly typical with $w_0(m_{0,j})$, $v_0(m_{12,j}|m_{0,j})$, $s_0(m_{13,j}|m_{0,j})$ and $u_1(m_{1,j}|m_{12,j}, m_{13,j}, m_{0,j})$ given the knowledge of w_1, w_2, w_3, v_1, v_3, s_1 and s_3. According to joint typicality lemma (Appendix A Lemma 1), the decoding error probability goes to 0 as $n \to \infty$, if following rate constraints are satisfied:

$$\begin{aligned}R_0 + R_{12} + R_{13} + R_1 &< I(W_0, V_0, S_0, U_1; Y_1|W_1, W_2, W_3, S_1, S_3, V_1, V_2, X_1) \triangleq I_1,\\ R_{12} + R_1 &< I(V_0, U_1; Y_1|W_0, W_1, W_2, W_3, S_0, S_1, S_3, V_1, V_2, X_1) \triangleq I_2,\\ R_{13} + R_1 &< I(S_0, U_1; Y_1|W_0, W_1, W_2, W_3, S_1, S_3, V_0, V_1, V_2, X_1) \triangleq I_3,\\ R_{12} + R_{13} + R_1 &< I(V_0, S_0, U_1; Y_1|W_0, W_1, W_2, W_3, S_1, S_3, V_1, V_2, X_1) \triangleq I_4,\\ R_1 &< I(U_1; Y_1|W_0, W_1, W_2, W_3, S_0, S_1, S_3, V_0, V_1, V_2, X_1) \triangleq I_5.\end{aligned} \tag{4.4}$$

Simultaneous sliding window decoding at the second relay

At the end of block $j+1$, the second relay $k=2$, will decode $\hat{m}_{0,j}, \hat{m}_{2,j}, \hat{m}_{12,j}, \hat{m}_{23,j}$ such that:

$$\begin{aligned}(v_1(\hat{m}_{12,j}|\hat{m}_{0,j}), w_1(\hat{m}_{0,j}),\ w_2(\hat{m}_{0,j-1}), v_2(\hat{m}_{12,j-1}|\hat{m}_{0,j-1}),\\ q_2(\hat{m}_{23,j-1}|\hat{m}_{0,j-1}), x_2(\hat{m}_{2,j-1}|\hat{m}_{23,j-1}, \hat{m}_{12,j-1}),\\ w_3(\hat{m}_{0,j-2}), q_3(\hat{m}_{23,j-2}|\hat{m}_{0,j-2}), y_2(j+1)) \in \mathcal{T}_\epsilon^{(n)},\end{aligned}$$

and,

$$\begin{aligned}(w_0(\hat{m}_{0,j}),\ v_0(\hat{m}_{12,j}|\hat{m}_{0,j}), q_0(\hat{m}_{23,j}|\hat{m}_{0,j}), u_2(\hat{m}_{2,j}|\hat{m}_{12,j}, \hat{m}_{23,j}, \hat{m}_{0,j}),\\ w_1(\hat{m}_{0,j-1}), v_1(\hat{m}_{12,j-1}|\hat{m}_{0,j-1}), w_2(\hat{m}_{0,j-2}), v_2(\hat{m}_{12,j-2}|\hat{m}_{0,j-2}),\\ q_2(\hat{m}_{23,j-2}|\hat{m}_{0,j-2}), x_2(\hat{m}_{2,j-2}|\hat{m}_{23,j-2}, \hat{m}_{12,j-2}, \hat{m}_{0,j-2}),\\ w_3(\hat{m}_{0,j-3}), q_3(\hat{m}_{23,j-3}|\hat{m}_{0,j-3}),\ y_2(j)) \in \mathcal{T}_\epsilon^{(n)}.\end{aligned} \tag{4.5}$$

In (4.5), two decoding rules should be satisfied simultaneously at the second relay. In the first decoding rule, the second relay $k = 2$ decodes $\hat{m}_{12,j}$ and $\hat{m}_{0,j}$ from output $y_2(j+1)$ when $y_2(j+1)$ is jointly typical with v_1 and w_1 given the knowledge of w_2, w_3, v_2, q_2, x_2 and q_3,. In the second decoding rule, the second relay $k = 2$ decodes $\hat{m}_{12,j}$, $\hat{m}_{23,j}$, $\hat{m}_{2,j}$ and $\hat{m}_{0,j}$ from $y_2(j)$ when $y_2(j)$ is jointly typical with w_0, v_0, q_0 and u_2 given the knowledge of w_1, v_1, w_2, v_2, q_2, x_2, w_3 and q_3. The decoding error probability goes to 0, as $n \to \infty$, if

$$\begin{aligned}R_0 + R_{12} + R_{23} + R_2 &< I(W_0, W_1, V_0, V_1, Q_0, U_2; Y_2|W_2, W_3, V_2, Q_2, Q_3, X_2) \triangleq I_6,\\ R_{12} + R_2 &< I(V_0, V_1, U_2; Y_2|W_0, W_1, W_2, W_3, V_2, Q_0, Q_2, Q_3, X_2) \triangleq I_7,\\ R_{23} + R_2 &< I(Q_0, U_2; Y_2|W_0, W_1, W_2, W_3, V_0, V_1, V_2, Q_2, Q_3, X_2) \triangleq I_8,\\ R_{12} + R_{23} + R_2 &< I(V_0, V_1, Q_0, U_2; Y_2|W_0, W_1, W_2, W_3, V_2, Q_2, Q_3, X_2) \triangleq I_9,\\ R_2 &< I(U_2; Y_2|W_0, W_1, W_2, W_3, V_0, V_1, V_2, Q_0, Q_2, Q_3, X_2) \triangleq I_{10}.\end{aligned} \tag{4.6}$$

Simultaneous sliding window decoding at the third relay

At the end of block $j+2$, the third relay $k = 3$, will decode $\hat{m}_{0,j}, \hat{m}_{3,j}, \hat{m}_{13,j}, \hat{m}_{23,j}$ such that:

$$\begin{aligned}(w_2(\hat{m}_{0,j}),\ q_2(\hat{m}_{23,j}|\hat{m}_{0,j}), w_3(\hat{m}_{0,j-1}), q_3(\hat{m}_{23,j-1}|\hat{m}_{0,j-1}), s_3(\hat{m}_{13,j-1}|\hat{m}_{0,j-1}),\\ x_3(\hat{m}_{3,j-1}|\hat{m}_{13,j-1}, \hat{m}_{23,j-1}, \hat{m}_{0,j-1}),\ y_3(j+2)) \in \mathcal{T}_\epsilon^{(n)},\end{aligned}$$

and,

$$\begin{aligned}(s_1(\hat{m}_{13,j}|\hat{m}_{0,j}),\ w_1(\hat{m}_{0,j}), w_2(\hat{m}_{0,j-1}), q_2(\hat{m}_{23,j-1}|\hat{m}_{0,j-1}),\\ w_3(\hat{m}_{0,j-2}), q_3(\hat{m}_{23,j-2}|\hat{m}_{0,j-2}), s_3(\hat{m}_{13,j-2}|\hat{m}_{0,j-2}),\\ x_3(\hat{m}_{3,j-2}|\hat{m}_{13,j-2}, \hat{m}_{23,j-2}, \hat{m}_{0,j-2}),\ y_3(j+1)) \in \mathcal{T}_\epsilon^{(n)},\end{aligned}$$

and,

$$\begin{aligned}w_0(\hat{m}_{0,j}),)\ s_0(\hat{m}_{13,j}|\hat{m}_{0,j}), q_0(\hat{m}_{23,j}|\hat{m}_{0,j}), u_3(\hat{m}_{3,j}|\hat{m}_{13,j}, \hat{m}_{23,j}, \hat{m}_{0,j}),\\ w_1(\hat{m}_{0,j-1}), s_1(\hat{m}_{13,j-1}|\hat{m}_{0,j-1}), w_2(\hat{m}_{0,j-2}), q_2(\hat{m}_{23,j-2}|\hat{m}_{0,j-2}),\\ w_3(\hat{m}_{0,j-3}), q_3(\hat{m}_{23,j-3}|\hat{m}_{0,j-3}), s_3(\hat{m}_{13,j-3}|\hat{m}_{0,j-3}),\\ x_3(\hat{m}_{3,j-3}|\hat{m}_{13,j-3}, \hat{m}_{23,j-3}, \hat{m}_{0,j-3}),\ y_3(j)) \in \mathcal{T}_\epsilon^{(n)}.\end{aligned} \tag{4.7}$$

In (4.7), three decoding rules should be satisfied simultaneously at the third relay. In the first decoding rule, the third relay $k = 3$ decodes $\hat{m}_{23,j}$ and $\hat{m}_{0,j}$ from output $y_3(j+2)$ when $y_3(j+2)$ is jointly typical with w_1 and q_2 given the knowledge of w_3, q_3, s_3 and x_3. In the second decoding rule, the third relay $k = 3$ decodes $\hat{m}_{13,j}$ and $\hat{m}_{0,j}$ from $y_3(j+1)$ when $y_3(j+1)$ is jointly typical with w_1 and s_1 given the knowledge of w_2, q_2, w_3, q_3, s_3 and x_3. In the third decoding rule, the third relay $k = 3$ decodes $\hat{m}_{13,j}$, $\hat{m}_{23,j}$, $\hat{m}_{3,j}$ and $\hat{m}_{0,j}$ from $y_3(j)$ when $y_3(j)$ is jointly typical with w_0, q_0, u_3 and s_0 given the knowledge of w_1, s_1, w_2, q_2, w_3, q_3, s_3 and x_3. The decoding error probability goes to 0 as $n \to \infty$, if (4.8) is satisfied.

$$\begin{aligned}R_0 + R_{13} + R_{23} + R_3 &< I(W_0, W_1, W_2, S_0, S_1, Q_0, Q_2, U_3; Y_3|W_3, S_3, Q_3, X_3) \triangleq I_{11},\\ R_{23} + R_3 &< I(Q_0, Q_2, U_3; Y_3|W_0, W_1, W_2, W_3, S_0, S_1, S_3, Q_3, X_3) \triangleq I_{12},\\ R_{13} + R_3 &< I(S_0, S_1, U_3; Y_3|W_0, W_1, W_2, W_3, S_3, Q_0, Q_2, Q_3, X_3) \triangleq I_{13},\\ R_{13} + R_{23} + R_3 &< I(S_0, S_1, Q_0, Q_2, U_3; Y_3|W_0, W_1, W_2, W_3, S_3, Q_3, X_3) \triangleq I_{14},\\ R_3 &< I(U_3; Y_3|W_0, W_1, W_2, W_3, S_0, S_1, S_3, Q_0, Q_2, Q_3, X_3) \triangleq I_{15},\end{aligned} \tag{4.8}$$

Simultaneous sliding window decoding at the destination

At the end of block $j+3$, the destination $k=4$, will decode $\hat{m}_{0,j}, \hat{m}_{1,j}, \hat{m}_{2,j}, \hat{m}_{3,j}, \hat{m}_{4,j}, \hat{m}_{12,j}, \hat{m}_{13,j}, \hat{m}_{23,j}$ such that:

$$(w_3(\hat{m}_{0,j}),\ q_3(\hat{m}_{23,j}|\hat{m}_{0,j}), s_3(\hat{m}_{13,j}|\hat{m}_{0,j}), x_3(\hat{m}_{3,j}|\hat{m}_{13,j}, \hat{m}_{23,j}, \hat{m}_{0,j}),\ y_4(j+3)) \in \mathcal{T}_\epsilon^{(n)},$$

and,

$$\begin{aligned} w_2(\hat{m}_{0,j}), v_2(\hat{m}_{12,j}|\hat{m}_{0,j}), q_2(\hat{m}_{23,j}|\hat{m}_{0,j}), x_2(\hat{m}_{2,j}|\hat{m}_{23,j}, \hat{m}_{12,j}, \hat{m}_{0,j}), \\ w_3(\hat{m}_{0,j-1}), q_3(\hat{m}_{23,j-1}|\hat{m}_{0,j-1}), s_3(\hat{m}_{13,j-1}|\hat{m}_{0,j-1}), \\ x_3(\hat{m}_{3,j-1}|\hat{m}_{13,j-1}, \hat{m}_{23,j-1}, \hat{m}_{0,j-1}),\ y_4(j+2)) \in \mathcal{T}_\epsilon^{(n)}, \end{aligned}$$

and,

$$\begin{aligned} (s_1(\hat{m}_{13,j}|\hat{m}_{0,j})\ w_1(\hat{m}_{0,j}), v_1(\hat{m}_{12,j}|\hat{m}_{0,j}), x_1(\hat{m}_{1,j}|\hat{m}_{12,j}, \hat{m}_{13,j}, \hat{m}_{0,j}), \\ w_2(\hat{m}_{0,j-1}), v_2(\hat{m}_{12,j-1}|\hat{m}_{0,j-1}), q_2(\hat{m}_{23,j-1}|\hat{m}_{0,j-1}), x_2(\hat{m}_{2,j-1}|\hat{m}_{23,j-1}, \hat{m}_{12,j-1}, \hat{m}_{0,j-1}), \\ w_3(\hat{m}_{0,j-2}), q_3(\hat{m}_{23,j-2}|\hat{m}_{0,j-2}), s_3(\hat{m}_{13,j-2}|\hat{m}_{0,j-2}), \\ x_3(\hat{m}_{3,j-2}|\hat{m}_{13,j-2}, \hat{m}_{23,j-2}, \hat{m}_{0,j-2}),\ y_4(j+1)) \in \mathcal{T}_\epsilon^{(n)}, \end{aligned}$$

and,

$$\begin{aligned} (w_0(\hat{m}_{0,j})\ s_0(\hat{m}_{13,j}|\hat{m}_{0,j}), q_0(\hat{m}_{23,j}|\hat{m}_{0,j}), v_0(\hat{m}_{12,j}|\hat{m}_{0,j}), u_3(\hat{m}_{3,j}|\hat{m}_{13,j}, \hat{m}_{23,j}, \hat{m}_{0,j}), \\ u_2(\hat{m}_{2,j}|\hat{m}_{12,j}, \hat{m}_{23,j}, \hat{m}_{0,j}), u_1(\hat{m}_{1,j}|\hat{m}_{12,j}, \hat{m}_{13,j}, \hat{m}_{0,j}), \\ x_0(\hat{m}_{4,j}|\hat{m}_{2,j}, \hat{m}_{23,j}, \hat{m}_{13,j}, \hat{m}_{3,j}, \hat{m}_{0,j}, \hat{m}_{12,j}, \hat{m}_{1,j}), \\ w_1(\hat{m}_{0,j-1}), s_1(\hat{m}_{13,j-1}|\hat{m}_{0,j-1}), v_1(\hat{m}_{12,j-1}|\hat{m}_{0,j-1}), x_1(\hat{m}_{1,j-1}|\hat{m}_{12,j-1}, \hat{m}_{0,j-1}, \hat{m}_{13,j-1}), \\ w_2(\hat{m}_{0,j-2}), q_2(\hat{m}_{23,j-2}|\hat{m}_{0,j-2}), v_2(\hat{m}_{12,j-2}|\hat{m}_{0,j-2}), x_2(\hat{m}_{2,j-2}|\hat{m}_{23,j-2}, \hat{m}_{12,j-2}, \hat{m}_{0,j-2}), \\ w_3(\hat{m}_{0,j-3}), q_3(\hat{m}_{23,j-3}|\hat{m}_{0,j-3}), s_3(\hat{m}_{13,j-3}|\hat{m}_{0,j-3}), \\ x_3(\hat{m}_{3,j-3}|\hat{m}_{13,j-3}, \hat{m}_{23,j-3}, \hat{m}_{0,j-3}),\ y_4(j)) \in \mathcal{T}_\epsilon^{(n)}. \end{aligned} \tag{4.9}$$

In (4.9), four decoding rules should be satisfied simultaneously at the third relay. In the first decoding rule, the destination decodes $\hat{m}_{23,j}$, $\hat{m}_{13,j}$, $\hat{m}_{3,j}$ and $\hat{m}_{0,j}$ from output

$y_4(j+3)$ when $y_4(j+3)$ is jointly typical with w_3, q_3, s_3 and x_3. In the second decoding rule, the destination decodes $\hat{m}_{23,j}$, $\hat{m}_{12,j}$, $\hat{m}_{2,j}$ and $\hat{m}_{0,j}$ from $y_4(j+2)$ when $y_4(j+2)$ is jointly typical with w_2, q_2, v_2 and x_2 given the knowledge of w_3, q_3, s_3 and x_3. In the third decoding rule, the destination decodes $\hat{m}_{13,j}$, $\hat{m}_{12,j}$, $\hat{m}_{1,j}$ and $\hat{m}_{0,j}$ from $y_4(j+1)$ when $y_4(j+1)$ is jointly typical with w_1, v_1, x_1 and s_1 given the knowledge of w_2, v_2, x_2, q_2, w_3, q_3, s_3 and x_3. In the last decoding rule, the destination decodes $\hat{m}_{23,j}$, $\hat{m}_{13,j}$, $\hat{m}_{12,j}$, $\hat{m}_{1,j}$, $\hat{m}_{2,j}$, $\hat{m}_{3,j}$, $\hat{m}_{4,j}$ and $\hat{m}_{0,j}$ from $y_4(j)$ when $y_4(j)$ is jointly typical with w_0, s_0, q_0, v_0, u_3, u_2, u_1 and x_0 given the knowledge of w_1, s_1, v_1, x_1, w_2, v_2, x_2, q_2, w_3, q_3, s_3 and x_3. The decoding error probability goes to 0 as $n \to \infty$, if (4.10) is satisfied:

$$
\begin{aligned}
R_3 + R_4 &< I(X_3, U_3, X_0; Y_4 | W_0^3, U_1^2, X_1^2, S_0^1, S_3, V_0^2, Q_0, Q_2^3) \triangleq I_{16},\\
R_2 + R_4 &< I(X_2, U_2, X_0; Y_4 | W_0^3, U_1, U_3, X_1, X_3, S_0^1, S_3, V_0^2, Q_0, Q_2^3) \triangleq I_{17},\\
R_1 + R_4 &< I(X_1, U_1, X_0; Y_4 | W_0^3, U_2^3, X_2^3, S_0^1, S_3, V_0^2, Q_0, Q_2^3) \triangleq I_{18},\\
R_2 + R_3 + R_4 &< I(X_2^3, U_2^3, X_0; Y_4 | W_0^3, U_1, X_1, S_0^1, S_3, V_0^2, Q_0, Q_2^3) \triangleq I_{19},\\
R_1 + R_3 + R_4 &< I(X_1, X_3, U_1, U_3, X_0; Y_4 | W_0^3, U_2, X_2, S_0^1, S_3, V_0^2, Q_0, Q_2^3) \triangleq I_{20},\\
R_1 + R_2 + R_4 &< I(X_1^2, U_1^2, X_0; Y_4 | W_0^3, U_3, X_3, S_0^1, S_3, V_0^2, Q_0, Q_2^3) \triangleq I_{21},\\
R_1 + R_2 + R_3 + R_4 &< I(U_1^3, X_0^3; Y_4 | W_0^3, S_0^1, S_3, V_0^2, Q_0, Q_2^3) \triangleq I_{22},\\
R_{23} + R_2 + R_3 + R_4 &< I(Q_0, Q_2^3, U_2^3, X_2^3, X_0; Y_4 | W_0^3, U_1, X_1, S_0^1, S_3, V_0^2) \triangleq I_{23},\\
R - R_{12} - R_{13} - R_0 &< I(Q_0, Q_2^3, U_1^3, X_0^3; Y_4 | W_0^3, S_0^1, S_3, V_0^2) \triangleq I_{24},\\
R_{13} + R_1 + R_3 + R_4 &< I(S_0^1, S_3, U_1, U_3, X_0^1, X_3; Y_4 | W_0^3, U_2, X_2, Q_0, Q_2^3, V_0^2) \triangleq I_{25},\\
R - R_{12} - R_{23} - R_0 &< I(S_0^1, S_3, U_1^3, X_0^3; Y_4 | W_0^3, Q_0, Q_2^3, V_0^2) \triangleq I_{26},\\
R_{12} + R_1 + R_2 + R_4 &< I(V_0^2, U_1^2, X_1^2, X_0; Y_4 | W_0^3, U_1, X_1, S_0^1, S_3, Q_0, Q_2^3) \triangleq I_{27},\\
R - R_{23} - R_{13} - R_0 &< I(V_0^2, U_1^3, X_0^3; Y_4 | W_0^3, S_0^1, S_3, Q_0, Q_2^3) \triangleq I_{28},\\
R - R_{13} - R_0 &< I(V_0^2, Q_0, Q_2^3, U_1^3, X_0^3; Y_4 | W_0^3, S_0^1, S_3) \triangleq I_{29},\\
R - R_{23} - R_0 &< I(S_0^1, S_3, V_0^2, U_1^3, X_0^3; Y_4 | W_0^3, Q_0, Q_2^3) \triangleq I_{30},\\
R - R_{12} - R_0 &< I(S_0^1, S_3, Q_0, Q_2^3, U_1^3, X_0^3; Y_4 | W_0^3, V_0^2) \triangleq I_{31},\\
R - R_0 &< I(S_0^1, S_3, Q_0, Q_2^3, V_0^2, U_1^3, X_0^3; Y_4 | W_0^3) \triangleq I_{32},\\
R &< I(W_0^3, U_1^3, X_0^3, S_0^1, S_3, V_0^2, Q_0, Q_2^3; Y_4) \triangleq I_{33},\\
R_4 &< I(X_0; Y_4 | W_0^3, U_1^3, X_1^3, S_0^1, S_3, V_0^2, Q_0, Q_2^3) \triangleq I_{34}.
\end{aligned} \tag{4.10}
$$

In combination process, by applying Fourier-Motzkin elimination to (4.4), (4.6), (4.8) and (4.10), the rate in Theorem 5 is achievable. □

Remark 2. *In order to simultaneously decode different message parts, each relay waits until all its intended message parts with the same block index arrive.*

Remark 3. *We can interpret this scheme from another perspective, called repetitive message splitting scheme.*

We maintain relay network infrastucture as shown in Figure 4.2.

In this scheme, m_0 is the common message part that is decoded by all the nodes. Message part m_{ij} is decoded and re-transmitted only at relay nodes i and j.

In the first stage, for the purpose of transmiting message m to the destination, the source splits messages into five parts, each message part is directed to its following nodes independently:

$$m = m_{01} + m_{02} + m_{03} + m_{04} + m_0.$$

Thus, the source broadcasts m_{01}, m_{02}, m_{03}, m_{04} simultaneously.

In the second stage, the first relay $k = 1$ decodes m_{01} and m_0. It then splits m_{01} further into three parts:

$$m_{01} = m_{12} + m_{13} + m_{14}.$$

Relay node 1 then broadcasts m_0, m_{12}, m_{13} and m_{14} simultaneously. After this stage, the rate $R_{01} = R_{12} + R_{13} + R_{14}$.

In the third stage, the second relay $k = 2$ decodes m_0, m_{02}, m_{12} and splits m_{02} further into two parts:

$$m_{02} = m_{23} + m_{34}.$$

Relay 2 then sends m_{12}, m_{23}, m_{24} and m_0 simultaneously. After this stage, the rate $R_{02} = R_{23} + R_{24}$.

In the fourth stage, the third relay $k = 3$ decodes $m_{03}, m_{13}, m_{23}, m_0$ and retransmits m_{03}

as:

$$m_{03} = m_{34}.$$

Relay 3 sends m_0, m_{13}, m_{23} and m_{34} simultaneously. The rate $R_{03} = R_{34}$.

In the last stage, the destination decodes m_{04}, m_{14}, m_{24}, m_{34}, m_{12}, m_{13}, m_{23}, m_0 and recombines them together to get the source-intended originl message:

$$m = m_{04} + m_{14} + m_{24} + m_{34} + m_{12} + m_{13} + m_{23} + m_0.$$

We can see that we still have eight message parts in total and this repetitive scheme achieves the same rate as the exhaustive scheme.

Remark 4. *By setting $\{S.\} = \{V.\} = \{Q.\} = \emptyset$, the rate in Theorem 5 reduces to the rate in Chapter 3, where no message parts are decoded by more than one relay. In addition, by setting $\{U.\} = \{X.\} = \emptyset$, it reduces to the decode-forward lower bound in [6].*

4.2 Implications on Relay Networks

In the previous analysis of three-relay networks, we introduce a directed graph to explain how block Markov superimposition is implemented among exhaustively split message parts. In this section, by expanding this graph, we show block Markov superimposition structure of exhaustive split message parts in an N-relay single-source single-destination network.

Define:

- Let $\mathcal{T} = \{1, 2, \ldots, N\}$ be the complete relay set.
- For any relay index $k \in \mathcal{T}$, let $\mathcal{O}_k = \mathcal{T} - \{k\}$ be the set of other relays.
- Let $\mathcal{S}$ to be a subset of $\mathcal{T}$, that is $\mathcal{S} \subseteq \mathcal{T}$.
- In block j, let message part $m_{k,\mathcal{S}}(j-k)$ be the message re-transmitted by relay k. Note that the union of the message indices, $\{k\} \cup \mathcal{S}$, denotes a specific relays group that will help the transmission of this message part. For example, in block $j+1$, the relay $k+1$ sends $m_{k+1,\mathcal{S}\cup\{k\}-\{k+1\}}(j-k)$ to help the re-transmission of message part $m_{k,\mathcal{S}}(j-k)$.

Next, we introduce the construction of directed graph in block j:

- Graph *layer*: Assume that the codebook graph of the N-relay networks has N layers. We index each of them as $l = 1, \ldots, N$ successively, beginning from the top. We insert each message $m_{k,\mathcal{S}}(j-k)$ into layer l when the cardinality of the union $\{k\} \cup \mathcal{S}$ equals to layer index l, that is $l = |\{k\} \cup \mathcal{S}|$. Note that $k \in \mathcal{T}, \mathcal{S} \subseteq \mathcal{O}_k$. Thus, in general, layer index l means that for each message part in the layer l, l relays will help to transmit such certain message part.

- Among adjacent layers, assume that message part $m_{k,\mathcal{S}}(j-k)$ is located at layer l and message part $m_{k,\mathcal{S}'}(j-k)$ is located at level $l+1$, there exists an directed edge from $m_{k,\mathcal{S}'}(j-k)$ to $m_{k,\mathcal{S}}(j-k)$ if $\mathcal{S} \subset \mathcal{S}'$.

- In the same layer l, there exists an directed edge from $m_{k+1,\mathcal{S}\cup\{k\}-\{k+1\}}(j-k-1)$ to $m_{k,\mathcal{S}}(j-k)$ for any $k \in \mathcal{T}$, which means $m_{k,\mathcal{S}}(j-k)$ is successively superimposed on messages of previous blocks.

After we have above superposition coding strucutre, we can get the codeword generation for this relay network in the following.

The codeword generation at relay $k \in \mathcal{T}$ in block j is:

- For each message part $m_{k,\mathcal{S}}(j-k)$ at layer l, where $l \in [1:N]$, generate $2^{nR_{\{k\}\cup\mathcal{S}}}$ sequences $v_{k,\{k\}\cup\mathcal{S}}[m_{k,\mathcal{S}}(j-k)]$, which is superposed on all codewords whose message parts have a path to $m_{k,\mathcal{S}}(j-k)$ along with entire edge directions.

- Generate $x_k(j-k)$ as the function of all codewords $v_{k,\{k\}\cup\mathcal{S}}[m_{k,\mathcal{S}}(j-k)]$ for all $\mathcal{S} \subseteq \mathcal{O}_k$.

The codeword generation at source 0 in block j is:

- Generate $2^{nR_{\{k\}\cup\mathcal{S}}}$ sequences $u_{\{k\}\cup\mathcal{S}}[m_{k,\mathcal{S}}(j-k)]$, which is superimposed on all relay codewords $v_{k',\{k'\}\cup\mathcal{S}'}[m_{k',\mathcal{S}}(j-k')]$ that help the transmission of message part $m_{k,\mathcal{S}}$ where $(k \cup S) = (k' \cup S')$ and on source codewords $u_{\{k'\}\cup\mathcal{S}'}[m_{k',\mathcal{S}'}(j-k')]$ for all $(k \cup S) \subset (k' \cup S')$.

- Generate $x_0(j)$ as the function of codewords $u_{\{k\}\cup\mathcal{S}}[m_{k,\mathcal{S}}(j-k)]$ for all $\{k\} \cup \mathcal{S} = \mathcal{T}$.

Remark 5. *By setting $N = 2$, the codebook generation in [5] can be deduced from our superposition directed graph. By settting $N = 1$, this graph reduces to partial decode-forward structure for the classical one-relay channel.*

After we have the above codebooks, we then reveal them to all the communication parties in the N-relay network. Each nodes in the relay network use this universally generated codebooks to do encoding and decoding message parts. We will then make error analysis at each receiving node to deduce achievable rates inequalities. Finally, we synthesize all achievable rates inequalities to get the capacity results of exhaustive message splitting scheme in a single-source single destination relay network with N relays.

4.3 Gaussian Three-relay Networks

In this section, we will see capacity results in a continuous alphabet relay network, Gaussian relay network. Following the network infrastructure shown in Figure 4.2, we study exhaustive message splitting scheme for Gaussian three-relay network in detail here. This network is a model for simple source-destination communication networks with three relays. There is energy or power constraint on the input of each transmitting node.

4.3.1 Signaling and Rates for Gaussian Three-relay Networks

As shown in Figure 4.2, the Gaussian three-relay network can be modeled as:

$$\begin{aligned}
Y_4 &= g_{04}X_0 + g_{14}X_1 + g_{24}X_2 + g_{34}X_3 + Z_4,\\
Y_3 &= g_{03}X_0 + g_{13}X_1 + g_{23}X_2 + Z_3,\\
Y_2 &= g_{02}X_0 + g_{12}X_1 + Z_2,\\
Y_1 &= g_{01}X_0 + Z_1,
\end{aligned} \tag{4.11}$$

where Z_4, Z_3 and Z_2, Z_1 are independent AWGN noise according to the normal distribution $\mathcal{N}(0,1)$. An achievable rate for the Gaussian three-relay network can be obtained by applying Theorem 5 with the following signaling:

$$
\begin{aligned}
x_3 &= \alpha_{33}W_3(w_{0,j-3}) + \gamma_{33}Q_3(w_{23,j-3}) + \delta_{33}S_3(w_{13,j-3}) + \beta_{33}T_3(w_{3,j-3}),\\
x_2 &= \alpha_{23}W_3(w_{0,j-3}) + \alpha_{22}W_2(w_{0,j-2}) + \rho_{22}V_2(w_{12,j-2}) + \gamma_{23}Q_3(w_{23,j-2}) +\\
&\quad \gamma_{22}Q_2(w_{23,j-2}) + \beta_{22}T_2(w_{2,j-2}),\\
x_1 &= \alpha_{11}W_1(w_{0,j-1}) + \alpha_{12}W_2(w_{0,j-2}) + \alpha_{13}W_3(w_{0,j-3}) + \rho_{12}V_2(w_{12,j-2}) +\\
&\quad \rho_{11}V_1(w_{12,j-1}) + \delta_{13}S_3(w_{13,j-3}) + \delta_{11}S_1(w_{13,j-1}) + \beta_{11}T_1(w_{1,j-1}),\\
x_0 &= \alpha_{00}W_0(w_{0,j}) + \alpha_{01}W_1(w_{0,j-1}) + \alpha_{02}W_2(w_{0,j-2}) + \alpha_{03}W_3(w_{0,j-3}) +\\
&\quad \gamma_{03}Q_3(w_{23,j-3}) + \gamma_{02}Q_2(w_{23,j-2}) + \gamma_{00}Q_0(w_{23,j}) + \rho_{02}V_2(w_{12,j-2}) +\\
&\quad \rho_{01}V_1(w_{12,j-1}) + \rho_{00}V_0(w_{12,j}) + \delta_{03}S_3(w_{13,j-3}) + \delta_{01}S_1(w_{13,j-1}) +\\
&\quad \delta_{00}S_0(w_{13,j-0}) + \beta_{03}T_3(w_{3,j-3}) + \phi_{03}U_3(w_{3,j}) + \beta_{02}T_2(w_{2,j-2}) +\\
&\quad \phi_{02}U_2(w_{2,j}) + \beta_{01}T_1(w_{1,j-1}) + \phi_{01}U_1(w_{1,j}) + \phi_{04}V_4(w_{4,j}),
\end{aligned} \tag{4.12}
$$

where $\{W., Q., U., S., T., V.\}$ are independent, normalized Gaussian random variables satisfying $\mathcal{N}(0,1)$. $\{\alpha., \gamma., \delta., \rho., \beta., \phi.\}$ are power allocation coefficients, which satisfy the following power constraints at each relay:

$$
\begin{aligned}
\alpha_{33}^2 + \gamma_{33}^2 + \delta_{33}^2 + \beta_{33}^2 &= P_3,\\
\alpha_{23}^2 + \alpha_{22}^2 + \rho_{22}^2 + \gamma_{23}^2 + \gamma_{22}^2 + \beta_{22}^2 &= P_2,\\
\alpha_{11}^2 + \alpha_{12}^2 + \alpha_{13}^2 + \rho_{12}^2 + \rho_{11}^2 + \delta_{13}^2 + \delta_{11}^2 + \beta_{11}^2 &= P_1,\\
\alpha_{00}^2 + \alpha_{01}^2 + \alpha_{02}^2 + \alpha_{03}^2 + \gamma_{03}^2 + \gamma_{02}^2 + \gamma_{00}^2 + \rho_{02}^2 + \rho_{01}^2 &\\
+\rho_{00}^2 + \delta_{03}^2 + \delta_{01}^2 + \delta_{00}^2 + \beta_{03}^2 + \phi_{03}^2 + \beta_{02}^2 + \phi_{02}^2 + \beta_{01}^2 + \phi_{01}^2 + \phi_{04}^2 &= P_0,
\end{aligned} \tag{4.13}
$$

where P_0, P_1, P_2 and P_3 are power constraints at the corresponding nodes $k = 0$, $k = 1$, $k = 2$ and $k = 3$, which can be made equal to each other without loss of generality.

By considering (4.11) and (4.12) in (4.1), we can get the lower bound for the capacity of a Gaussian three-relay network in the following:

$$
\begin{aligned}
C \geq \min\{& I_{33}, I_1 + I_{23}, I_6 + I_{25}, I_{11} + I_{27}, I_6 + I_{13} + I_{18}, I_7 + I_{11} + I_{18}, I_1 + I_{12} + I_{17}, \\
& I_2 + I_{11} + I_{17}, I_1 + I_8 + I_{16}, I_3 + I_6 + I_{16}, I_5 + I_6 + I_{13} + I_{34}, I_6 + I_3 + I_{15} + I_{34}, \\
& I_1 + I_8 + I_{15} + I_{34}, I_1 + I_{10} + I_{12} + I_{34}, I_5 + I_7 + I_{11} + I_{34}, I_2 + I_{10} + I_{11} + I_{34}, \\
& I_3 + I_7 + I_{12} + I_{34}, I_2 + I_8 + I_{13} + I_{34}\}, \qquad (4.14)
\end{aligned}
$$

where

$$
\begin{aligned}
I_{33} \leq F\Big(& (g_{04}\alpha_{01} + g_{14}\alpha_{11})^2 + (g_{04}\alpha_{02} + g_{14}\alpha_{12} + g_{24}\alpha_{22})^2 \\
& + (g_{04}\alpha_{03} + g_{14}\alpha_{13} + g_{24}\alpha_{23} + g_{34}\alpha_{33})^2 + (g_{04}\gamma_{02} + g_{24}\gamma_{22})^2 \\
& + (g_{04}\gamma_{03} + g_{24}\gamma_{23} + g_{34}\gamma_{33})^2 + (g_{04}\rho_{01} + g_{14}\rho_{11})^2 \\
& + (g_{04}\rho_{02} + g_{14}\rho_{12} + g_{24}\rho_{22})^2 + (g_{04}\delta_{01} + g_{14}\delta_{11})^2 + (g_{04}\delta_{03} + g_{14}\delta_{13} + g_{34}\delta_{33})^2 \\
& + (g_{04}\beta_{03} + g_{34}\beta_{33})^2 + (g_{04}\beta_{02} + g_{24}\beta_{22})^2 + (g_{04}\beta_{01} + g_{14}\beta_{11})^2 \\
& + g_{04}^2(\alpha_{00}^2 + \gamma_{00}^2 + \rho_{00}^2 + \delta_{00}^2 + \phi_{03}^2 + \phi_{02}^2 + \phi_{01}^2 + \phi_{04}^2)\Big), \\
I_{16} \leq F\Big(& (g_{04}\beta_{03} + g_{34}\beta_{33})^2 + g_{04}^2\phi_{03}^2 + g_{04}^2\phi_{04}^2\Big), \\
I_{17} \leq F\Big(& (g_{04}\beta_{02} + g_{24}\beta_{22})^2 + g_{04}^2\phi_{02}^2 + g_{04}^2\phi_{04}^2\Big), \\
I_{18} \leq F\Big(& (g_{04}\beta_{01} + g_{14}\beta_{11})^2 + g_{04}^2\phi_{01}^2 + g_{04}^2\phi_{04}^2\Big), \\
I_{23} \leq F\Big(& (g_{04}\gamma_{02} + g_{24}\gamma_{22})^2 + (g_{04}\gamma_{03} + g_{24}\gamma_{23} + g_{34}\gamma_{33})^2 + \\
& (g_{04}\beta_{03} + g_{34}\beta_{33})^2 + (g_{04}\beta_{02} + g_{24}\beta_{22})^2 + g_{04}^2(\gamma_{00}^2 + \phi_{03}^2 + \phi_{02}^2 + \phi_{04}^2)\Big), \\
I_{25} \leq F\Big(& (g_{04}\delta_{01} + g_{14}\delta_{11})^2 + (g_{04}\delta_{03} + g_{14}\delta_{13} + g_{34}\delta_{33})^2 \\
& + (g_{04}\beta_{03} + g_{34}\beta_{33})^2 + (g_{04}\beta_{01} + g_{14}\beta_{11})^2 + g_{04}^2(\delta_{00}^2 + \phi_{03}^2 + \phi_{01}^2 + \phi_{04}^2)\Big), \\
I_{27} \leq F\Big(& (g_{04}\rho_{01} + g_{14}\rho_{11})^2 + (g_{04}\rho_{02} + g_{14}\rho_{12} + g_{24}\rho_{22})^2 \\
& + (g_{04}\beta_{02} + g_{24}\beta_{22})^2 + (g_{04}\beta_{01} + g_{14}\beta_{11})^2 + g_{04}^2(\phi_{02}^2 + \phi_{01}^2 + \phi_{04}^2 + \rho_{00}^2)\Big), \\
I_{34} \leq F\Big(& g_{04}^2\phi_{04}^2\Big), \\
I_1 \leq F\Bigg(& \frac{g_{01}^2(\alpha_{00}^2 + \rho_{00}^2 + \delta_{00}^2 + \phi_{01}^2)}{g_{01}^2(\gamma_{03}^2 + \gamma_{02}^2 + \gamma_{00}^2 + \beta_{03}^2 + \beta_{02}^2 + \phi_{03}^2 + \phi_{02}^2 + \phi_{04}^2) + 1}\Bigg),
\end{aligned}
$$

$$I_2 \le F\left(\frac{g_{01}^2(\rho_{00}^2+\phi_{01}^2)}{g_{01}^2(\gamma_{03}^2+\gamma_{02}^2+\gamma_{00}^2+\beta_{03}^2+\beta_{02}^2+\phi_{03}^2+\phi_{02}^2+\phi_{04}^2)+1}\right),$$
$$I_3 \le F\left(\frac{g_{01}^2(\delta_{00}^2+\phi_{01}^2)}{g_{01}^2(\gamma_{03}^2+\gamma_{02}^2+\gamma_{00}^2+\beta_{03}^2+\beta_{02}^2+\phi_{03}^2+\phi_{02}^2+\phi_{04}^2)+1}\right),$$
$$I_5 \le F\left(\frac{g_{01}^2\phi_{01}^2}{g_{01}^2(\gamma_{03}^2+\gamma_{02}^2+\gamma_{00}^2+\beta_{03}^2+\beta_{02}^2+\phi_{03}^2+\phi_{02}^2+\phi_{04}^2)+1}\right),$$
$$I_6 \le F\left(\frac{g_{02}^2(\alpha_{00}^2+\rho_{00}^2+\gamma_{00}^2+\phi_{02}^2)+(g_{02}\alpha_{01}+g_{12}\alpha_{11})^2+(g_{02}\rho_{01}+g_{12}\rho_{11})^2}{1+(g_{02}\delta_{01}+g_{12}\delta_{11})^2+(g_{02}\delta_{03}+g_{12}\delta_{13})^2+g_{02}^2(\delta_{00}^2+\beta_{03}^2+\beta_{01}^2+\phi_{03}^2+\phi_{01}^2+\phi_{04}^2)}\right),$$
$$I_7 \le F\left(\frac{g_{02}^2(\rho_{00}^2+\phi_{02}^2)+(g_{02}\rho_{01}+g_{12}\rho_{11})^2}{1+(g_{02}\delta_{01}+g_{12}\delta_{11})^2+(g_{02}\delta_{03}+g_{12}\delta_{13})^2+g_{02}^2(\delta_{00}^2+\beta_{03}^2+\beta_{01}^2+\phi_{03}^2+\phi_{01}^2+\phi_{04}^2)}\right),$$
$$I_8 \le F\left(\frac{g_{02}^2(\gamma_{00}^2+\phi_{02}^2)}{1+(g_{02}\delta_{01}+g_{12}\delta_{11})^2+(g_{02}\delta_{03}+g_{12}\delta_{13})^2+g_{02}^2(\delta_{00}^2+\beta_{03}^2+\beta_{01}^2+\phi_{03}^2+\phi_{01}^2+\phi_{04}^2)}\right),$$
$$I_{10} \le F\left(\frac{g_{02}^2\phi_{02}^2}{1+(g_{02}\delta_{01}+g_{12}\delta_{11})^2+(g_{02}\delta_{03}+g_{12}\delta_{13})^2+g_{02}^2(\delta_{00}^2+\beta_{03}^2+\beta_{01}^2+\phi_{03}^2+\phi_{01}^2+\phi_{04}^2)}\right),$$
$$I_{11} \le F\left(\frac{g_{03}^2(\alpha_{00}^2+\delta_{00}^2+\gamma_{02}^2+\phi_{03}^2)+(g_{03}\alpha_{01}+g_{13}\alpha_{11})^2+(g_{03}\alpha_{02}+g_{13}\alpha_{12}+g_{23}\alpha_{22})^2}{1+(g_{03}\rho_{01}+g_{13}\rho_{11})^2+(g_{03}\rho_{02}+g_{13}\rho_{12}+g_{23}\rho_{22})^2+g_{03}^2(\rho_{00}^2+\beta_{02}^2+\beta_{01}^2+\phi_{02}^2+\phi_{01}^2+\phi_{04}^2)}\right.$$
$$\left.+\frac{(g_{03}\delta_{01}+g_{13}\delta_{11})^2+(g_{03}\gamma_{02}+g_{23}\gamma_{22})^2}{1+(g_{03}\rho_{01}+g_{13}\rho_{11})^2+(g_{03}\rho_{02}+g_{13}\rho_{12}+g_{23}\rho_{22})^2+g_{03}^2(\rho_{00}^2+\beta_{02}^2+\beta_{01}^2+\phi_{02}^2+\phi_{01}^2+\phi_{04}^2)}\right),$$
$$I_{12} \le F\left(\frac{g_{03}^2(\gamma_{02}^2+\phi_{03}^2)+(g_{03}\gamma_{02}+g_{23}\gamma_{22})^2}{1+(g_{03}\rho_{01}+g_{13}\rho_{11})^2+(g_{03}\rho_{02}+g_{13}\rho_{12}+g_{23}\rho_{22})^2+g_{03}^2(\rho_{00}^2+\beta_{02}^2+\beta_{01}^2+\phi_{02}^2+\phi_{01}^2+\phi_{04}^2)}\right),$$
$$I_{13} \le F\left(\frac{g_{03}^2(\delta_{00}^2+\phi_{03}^2)+(g_{03}\delta_{01}+g_{13}\delta_{11})^2}{1+(g_{03}\rho_{01}+g_{13}\rho_{11})^2+(g_{03}\rho_{02}+g_{13}\rho_{12}+g_{23}\rho_{22})^2+g_{03}^2(\rho_{00}^2+\beta_{02}^2+\beta_{01}^2+\phi_{02}^2+\phi_{01}^2+\phi_{04}^2)}\right),$$
$$I_{15} \le F\left(\frac{g_{03}^2\phi_{03}^2}{1+(g_{03}\rho_{01}+g_{13}\rho_{11})^2+(g_{03}\rho_{02}+g_{13}\rho_{12}+g_{23}\rho_{22})^2+g_{03}^2(\rho_{00}^2+\beta_{02}^2+\beta_{01}^2+\phi_{02}^2+\phi_{01}^2+\phi_{04}^2)}\right),$$

and $F(x) = \frac{1}{2}\log(1+x)$.

4.4 Numerical Comparison

We numerically compare the achievable rate of the exhaustive message splitting scheme with the achievable rate of private message splitting scheme introduced in Chapter 3 in the three-relay network. For both schemes, we assume that transmitting powers at source 0, relay 1, relay 2 and relay 3 are the same. As shown in Figure 4.3, we consider two schemes in four possible channel conditions: all link conditions are equal ($g_{01} = g_{02} = g_{03} = g_{04} =$

$g_{12} = g_{13} = g_{14} = g_{23} = g_{24} = g_{34} = 1$, shown in the first plot); the source-to-relay and source-to-destination links are stronger than any other links ($g_{12} = g_{13} = g_{14} = g_{23} = g_{24} = g_{34} = 1$, $g_{01} = g_{02} = g_{03} = g_{04} = 3$, shown in the second plot); the relay-to-destination links are stronger than any other links ($g_{01} = g_{02} = g_{03} = g_{04} = g_{12} = g_{13} = g_{23} = 1$, $g_{14} = g_{24} = g_{34} = 3$, shown in the third plot); the in-relay links are stronger than any other links ($g_{01} = g_{02} = g_{03} = g_{04} = g_{14} = g_{24} = g_{34} = 1$, $g_{23} = g_{12} = g_{13} = 3$, shown in the last plot). Note that the exhaustive message splitting scheme is depicted with red solid line and the private message splitting scheme is depicted with blue dashed line. We can see that the exhaustive message splitting scheme results in a tighter bound than the private message splitting scheme does in all channel conditions.

As shown in the last plot of Figure 4.3, when in-relay links are stronger than any other links, exhaustive message splitting scheme produces the same lower bound as private message splitting scheme does with increasing transmitting power. In such case, we may prefer private message splitting scheme since it has less split message parts with decreased encoding/decoding complexity. As shown in the second plot of Figure 4.3, when the source has stronger links to relays and the destination as well as encoding/decoding complexity is tolerable, we may prefer the exhaustive message splitting scheme since the gap between two bounds is large under such channel conditions. In short, in different communication scenarios, we can adopt a combined scheme depending on different channel conditions.

4.5 Summary of the Chapter

In this chapter, we analyze partial decode-forward scheme based on exhaustive message splitting in a three-relay network. We derive its achievable rate as shown in Theorem 5 and provide corresponding achievability proof. We show that this scheme can include Chapter 3's results on network partial decode-forward as special cases. We next propose partial decode-forward scheme based on private message splitting for an N-relay network with single source and single destination. We discuss the superposition coding structure and codebook generation for the N-relay network. We next analyze the discrete-memoryless capacity results in AWGN environments. We derive capacity results for Gaussian three-relay networks as shown in (4.14). Finally, we compare the exhaustive message splitting scheme and private message splitting scheme in the three-relay network.

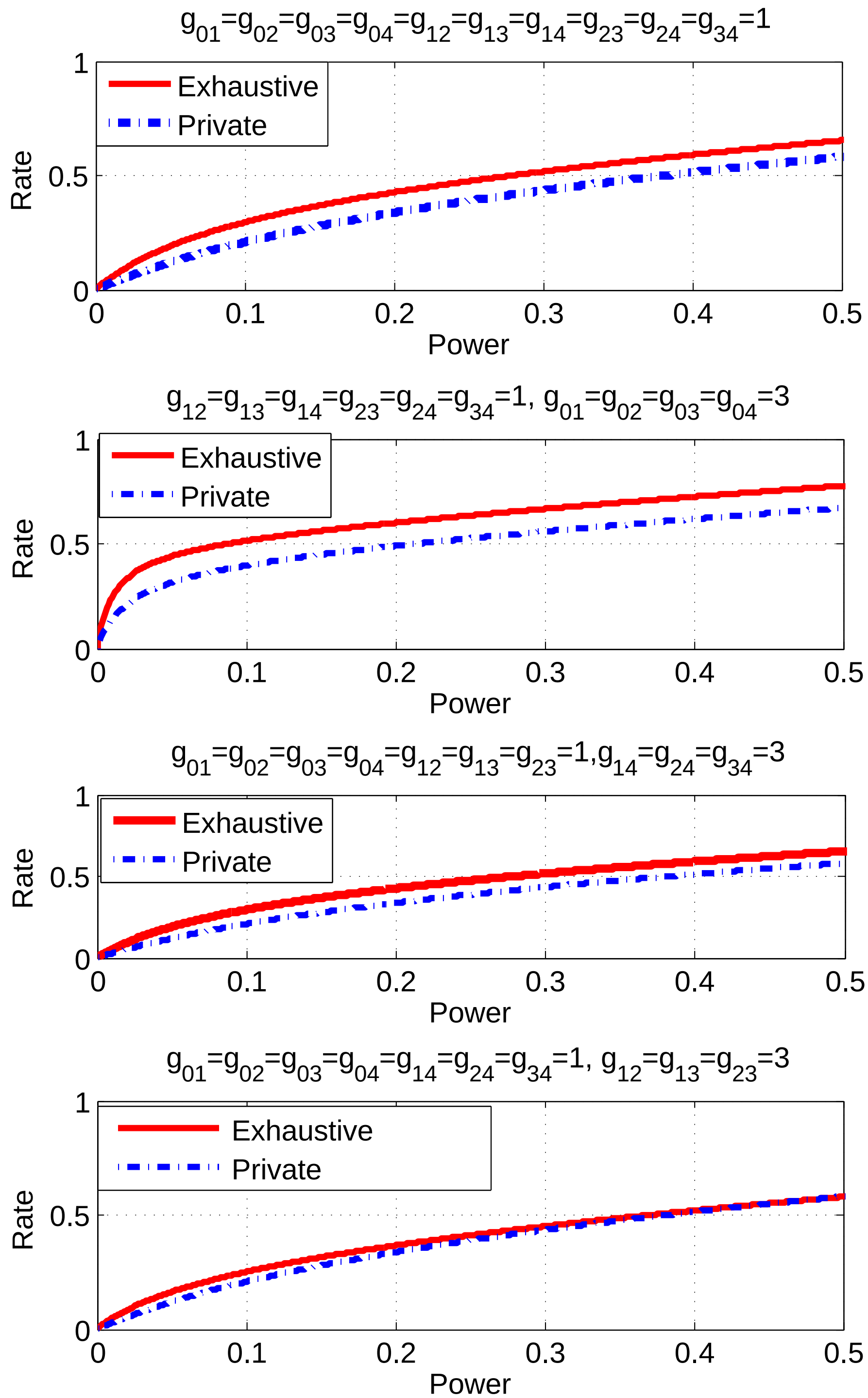

Fig. 4.3 Rate comparison between exhaustive message splitting scheme and private message splitting scheme in the three-relay network.

Chapter 5

Conclusions

5.1 Summary

In this thesis, we have proposed two new coding schemes for partial decode-forward relaying in a single-source single-destination network with N relays. These two schemes are based on message splitting, block Markov encoding and joint sliding window decoding.

First, we review capacity results in the classical relay channel and the discrete memoryless relay network. We also introduce several coding schemes used throughout the thesis, including block Markov encoding, backward decoding, sliding window decoding, message splitting, joint decoding and superposition coding.

We then design a scheme in which each relay forwards the common message part and a specific private part to the following nodes. The achievability proof is based on block Markov encoding and joint sliding window decoding. When extending to larger relay networks, we introduce the idea that each relay decodes and forwards its private part only when the last common part with the same block index arrives, which reduces the decoding delay in the destination to be linearly proportional with the number of relays. We then obtain the achievable rate for this scheme and novelly express the achievable rate in a compact form over all cutsets of relays. The capacity result is shown to contain existing results for an N-relay network with decode-forward and a two-relay network with partial decode-forward considering all message splitting cases.

We further study exhaustive message splitting scheme in a three-relay network and provide the corresponding achievable rate. We show that this scheme generalizes all existing decode-forward results. We also provide a graphical illustration to make the block Markov encoding structure easily understandable. Finally, we expand this graphical illustration to accommodate more relays in the N-node relay network, which makes exhaustive message splitting based partial decode-forward in the multi-relay network feasible.

5.2 Conclusions

Message splitting scheme provides us with a flexible way to facilitate partial decode-forward in the relay network with N relays, while previous literatures are only able to extend partial decode-forward to the relay network with one or two relays. With private message splitting scheme, each relay in the network is responsible for the transmission of a separate message part. The source pre-splits its message depending on the source-relay channel condition. Thus, if the link between the source and some relay is in poor channel condition (e.g., deep fading or high noises), the source will devote fewer message parts to that relay and will re-allocate more message parts to some other relay with better channel condition to the source. This is the reason why private message splitting scheme provides a tighter lower bound on capacity than the schemes shown in [5] and [15]. In the three-relay network, when the source has stronger links to relays and the destination , we can use the exhaustive message splitting scheme since it outperforms private message splitting scheme in terms of capacity bound. However, when in-relay links are stronger than any other links, exhaustive message splitting scheme produces the same lower bound as private message splitting scheme does. In such case, we may prefer private message splitting scheme since it has fewer split message parts with decreased encoding/decoding complexity. In short, in different communication scenarios, we can adopt a combined scheme depending on different channel conditions.

5.3 Future works

We have split source messages exhaustively in Chapter 4. Although the approach provides more general lower bound on the capacity, the question remains as to whether there exist other solutions with fewer split message parts but achieving the same lower bound. The increasing number of relays boosts the number of split message parts quickly, which adds to the decoding complexity at the destination. More message parts decoding at the higher-index relay may put more constraints on the final achievable rate as well. We will first derive a compact form of the achievable rate in the single-source single-destination N-

relay network with exhaustive message splitting scheme. We then try simplifying this rate by employing mutual information property or other mathematical approximation tools. By observing the simplified achievable rate, it should be possible to determine redundant variables and its corresponding message parts. We will also relate the final result with relay network infrastructure.

We will need to compare the private message splitting scheme with exhaustive message splitting scheme in the Gaussian relay network scenario. We will analyze the channel conditions between the source and each relay to see under which conditions one scheme outperforms the other one in order to decide whether the system should adopt the combined scheme. This becomes an optimization problem. []

Appendix A

Joint Typicality Lemma and Packing Lemma

In this section, we define ϵ-typical and jointly ϵ-typical sequences. The definitions and lemmas in this section can be found in [4] and [8].

Definition 1. (ϵ-typical sequences) Let $X \sim p(x)$ be a random variable with finite alphabet $\mathcal{X}$. Let x^n be a sequence with elements drawn from $\mathcal{X}$. Define the empirical pmf of x^n as

$$\pi(x|x^n) = \frac{1}{n}|\{i : x_i = x\}|, \forall x \in \mathcal{X}. \tag{A.1}$$

Let $\epsilon > 0$. Then the set $\mathcal{T}_\epsilon^{(n)}(X)$ of ϵ-typical sequences of length n is defined as[1]

$$\mathcal{T}_\epsilon^{(n)}(X) = \{x^n : |\pi(x|x^n) - p(x)| \leq \epsilon \cdot p(x), \forall x \in \mathcal{X}\}. \tag{A.2}$$

Definition 2. (Jointly ϵ-typical sequences) Let $(X, Y) \sim p(x, y)$ be a pair of random variables with finite alphabets $\mathcal{X} \times \mathcal{Y}$. Let (x^n, y^n) be a pair of sequences with elements drawn from $\mathcal{X} \times \mathcal{Y}$. Define the joint empirical pmf as

$$\pi(x, y|x^n, y^n) = \frac{1}{n}|\{i : (x_i, y_i) = (x, y)\}|, \forall (x, y) \in \mathcal{X} \times \mathcal{Y}. \tag{A.3}$$

[1]For brevity, we denote $\mathcal{T}_\epsilon^{(n)}(X)$ as $\mathcal{T}_\epsilon^{(n)}$ in the analysis.

Let $\epsilon > 0$. Then the set $\mathcal{T}_\epsilon^{(n)}(X,Y)$ of jointly ϵ-typical sequences of length n is defined as[2]

$$\mathcal{T}_\epsilon^{(n)}(X,Y) = \{(x^n, y^n) : |\pi(x,y|x^n,y^n) - p(x,y)| \leq \epsilon \cdot p(x,y), \forall (x,y) \in \mathcal{X} \times \mathcal{Y}\}\,. \quad \text{(A.4)}$$

We then show some consequences of the above definitions.

Lemma 1. *(Joint typicality lemma) Let X and Y be random variables with joint pmf $p(x,y)$ and $\epsilon > 0$. If x^n and $\hat{y}^n$ are distributed according to the joint pmf $p(x^n)p(\hat{y}^n)$, where $p(x^n) = \prod_{i=1}^n p(x_i)$ and $p(\hat{y}^n) = \prod_{i=1}^n p(y_i)$, each with the same marginal pmf as that of $p(x,y)$, then for n sufficiently large, there exists a function $\delta(\epsilon) \to 0$ as $\epsilon \to 0$ such that*

$$Pr\{(x^n, \hat{y}^n) \in \mathcal{T}_\epsilon^{(n)}(X,Y)\} \leq 2^{-n(I(X;Y)-\delta(\epsilon))} \quad \text{(A.5)}$$

Lemma 2. *(Packing lemma) Let X and Y be random variables with the joint pmf $p(x,y)$ and $\epsilon > 0$. Let y^n be distributed according to an arbitrary pmf $p(y^n)$ over alphabet $\mathcal{Y}$. Let $x^n(m)$, $m \in \mathcal{M}$ where $|\mathcal{M}| \leq 2^{nR}$ be random sequences, each distributed according to $\prod_{i=1}^n p(x_i)$. Assume that $x^n(m)$ is independent of y^n. Then there exists a function $\delta(\epsilon) \to 0$ as $\epsilon \to 0$ such that*

$$Pr\{(x^n(m), y^n) \in \mathcal{T}_\epsilon^{(n)}(X,Y),\ \textit{for some}\ \ m \in \mathcal{M}\} \to 0, \quad \text{(A.6)}$$

as $n \to 0$, if $R < I(X;Y) - \delta(\epsilon)$.

[2] For brevity, we denote $\mathcal{T}_\epsilon^{(n)}(X,Y)$ as $\mathcal{T}_\epsilon^{(n)}$ in the analysis.

Appendix B

Error analysis at the relay k in private message splitting scheme

Following the standard proof in [4], assume without loss of generality that $(m_{k,j-k+1}, m_{0,j-k+1}) = (1,1)$ is sent in block j.

We first define the following events:

- $E_i(m_{k,j-k+1}, m_{0,j-k+1})$ for all $i \in [0 : k-1]$, where only the ith decoding rule is satisfied. We simplify the event notation as E_i in the following analysis.
- $E(m_{k,j-k+1}, m_{0,j-k+1})$ such that all the decoding rules are satisfied simultaneously.

Then, by the union of event bounds, the probability of error is bounded as:

$$\begin{aligned} P_e^{(n)} &\leq P(E^c(1,1)) \\ &\quad + \sum_{m_{k,j-k+1}\neq 1, m_{0,j-k+1}=1} P(E(m_{k,j-k+1}, 1)) \\ &\quad + \sum_{m_{k,j-k+1}\neq 1, m_{0,j-k+1}\neq 1} P(E(m_{k,j-k+1}, m_{0,j-k+1})), \end{aligned}$$

where P is the conditional probability given that $(1,1)$ was sent.

By the law of large numbers[1], $P(E^c(1,1)) \to 0$ as $n \to \infty$.

[1]In weak law of large numbers, for any nonzero margin ϵ, within a sufficiently large sample, the sample average $\bar{X}_n$ converges in probability to the expected value μ, that is, $\lim_{n\to\infty} P(|\bar{X}_n - \mu| > \epsilon) = 0$

By the joint typicality lemma, we have

$$\sum_{m_{k,j-k+1}\neq 1, m_{0,j-k+1}=1} P(E(m_{k,j-k+1},1)) \leq 2^{nR_k} \times 2^{-n(I(U_k;Y_k|W_0^N,X_k)-\delta(\epsilon))}.$$

Thus, decoding error goes to 0 as $n \to \infty$, if

$$R_k < I(U_k;Y_k|W_0^N,X_k) - \delta(\epsilon).$$

Since according to the random code construction, $m_{k,j-k+1}$ represented by X_k is superimposed onto $m_{0,j-k+1}$ represented by W_k, it is impossible to correctly decode $m_{k,j-k+1}$ when $m_{0,j-k+1}$ is not decoded correctly. And, the joint PMFs of the tuple (U_k, W_0^N, X_k, Y_k) in events $E(1, m_{0,j-k+1})$ and $E(m_{k,j-k+1}, m_{0,j-k+1})$ are the same.

According to independence of the codebooks and the joint typicality lemma,

$$\begin{aligned}
&\sum_{m_{k,j-k+1}\neq 1, m_{0,j-k+1}\neq 1} P(E(m_{k,j-k+1},m_{0,j-k+1})) \\
&= P(\cup_{m_{k,j-k+1}\neq 1} \cup_{m_{0,j-k+1}\neq 1} (E_0 \cap E_1 \cap \cdots \cap E_{k-1})) \\
&\leq \sum_{m_{k,j-k+1}\neq 1} \sum_{m_{0,j-k+1}\neq 1} P(E_0) \times P(E_1) \times \cdots \times P(E_{k-1}) \\
&\leq 2^{nR_k} \times 2^{nR_0} \times 2^{-n(I(W_{k-1};Y_k|X_k,W_k^N)-\delta(\epsilon))} \\
&\quad \times 2^{-n(I(W_{k-2};Y_k|X_k,W_{k-1}^N)-\delta(\epsilon))} \times \cdots \times \\
&\quad 2^{-n(I(W_1;Y_k|X_k,W_2^N)-\delta(\epsilon))} \times 2^{-n(I(U_k,W_0;Y_k|X_k,W_1^N)-\delta(\epsilon))},
\end{aligned}$$

which tends to 0 as $n \to \infty$ if

$$R_k + R_0 < I(U_k, W_0^{k-1}; Y_k|X_k, W_k^N) - k\delta(\epsilon).$$

Appendix C

Error analysis at destination $k = N + 1$ in private message scheme

Assume without loss of generality that $(\{m_{k,j-N}\}_{k\in\mathcal{T}}, m_{0,j-N}, m_{N+1,j-N}) = (1, 1, \ldots, 1)$ is sent in block j.

We first define the following events:

- $E_i(\{m_{k,j-N}\}_{k\in\mathcal{T}}, m_{0,j-N}, m_{N+1,j-N})$, $i \in [1 : N + 1]$, where only the ith decoding rule is satisfied. We simplify each event notation as E_i in the following analysis.
- $E(\{m_{k,j-N}\}_{k\in\mathcal{T}}, m_{0,j-N}, m_{N+1,j-N})$, such that all the $N+1$ decoding rules are satisfied simultaneously.

In this appendix, we define the set $\mathcal{S}$ to be the set of wrongly decoded relay private messages and the set $\mathcal{S}^c$ to be the set of correctly decoded relay private messages. Then, by the union of event bounds, the probability of error is bounded as in (C.1), where P is the conditional probability given that $(\{m_{k,j-N}\}_{k\in\mathcal{T}}, m_{0,j-N}, m_{N+1,j-N}) = (1, 1, \ldots, 1)$ was sent.

$$\begin{aligned} P_e^{(n)} \leq & P(E^c(\{1\}_{N+1}, 1)) + \sum_{\{m_{i,j-N}\}_{i\in\mathcal{T}}, m_{N+1,j-N}, m_{0,j-N}\neq 1} P(E(\{m_{i,j-N}\}_{i\in T}, m_{N+1,j-N}, m_{0,j-N})) \\ & + \sum_{\{m_{i,j-N}\}_{i\in\mathcal{S}}\neq 1, \{m_{i,j-N+2}\}_{i\in\mathcal{S}^c}=1, m_{N+1,j-N}\neq 1, m_{0,j-N}=1} P(E(\{m_{i,j-N}\}_{i\in\mathcal{S}}, \{1\}_{|\mathcal{S}^c|}, m_{N+1,j-N}, 1)) \end{aligned} \tag{C.1}$$

By the law of large numbers, $P(E^c(\{1\}_{N+1}, 1)) \to 0$ as $n \to \infty$.

According to the random code construction, $\{m_{k,j-N}\}_{k\in\mathcal{T}}$ represented by $\{U_k, X_k\}_{k\in\mathcal{T}}$ is superimposed onto $m_{0,j-N}$ represented by $\{W_k\}_{k\in\{1\}\cup\mathcal{T}}$, it is impossible to correctly decode any of $\{m_{k,j-N}\}_{k\in\mathcal{T}}$ when $m_{0,j-N}$ is not decoded correctly. The joint PMFs of the tuple $(U_1^N, W_0^N, X_1^N, Y_{N+1})$ in events $E(\{m_{i,j-N}\}_{i\in\mathcal{S}}, \{1\}_{|\mathcal{S}^c|}, 1, m_{0,j-N})$, $E(\{m_{i,j-N}\}_{i\in\mathcal{S}}, \{1\}_{|\mathcal{S}^c|}, m_{N+1,j-N}, m_{0,j-N})$, $E(\{1\}_{|\mathcal{S}|}, \{1\}_{|\mathcal{S}^c|}, m_{N+1,j-N}, m_{0,j-N})$, $E(\{m_{i,j-N}\}_{i\in\mathcal{S}}, \{m_{i,j-N}\}_{i\in\mathcal{S}^c}, m_{N+1,j-N}, m_{0,j-N})$, $E(\{m_{i,j-N}\}_{i\in\mathcal{S}}, \{m_{i,j-N}\}_{i\in\mathcal{S}^c}, 1, m_{0,j-N})$ are the same. According to independence of the codebooks and the joint typicality lemma, we can get (C.2), which tends to 0 as $n \to \infty$ if

$$\sum_{i=1}^{N+1} R_i + R_0 < I(U_1^N, X_0^N, W_0^N; Y_{N+1}) - (N+1)\delta(\epsilon).$$

$$\begin{aligned}
&\sum_{\{m_{i,j-N}\}_{i\in\mathcal{T}}, m_{N+1,j-N}, m_{0,j-N}\neq 1} P(E(\{m_{i,j-N}\}_{i\in\mathcal{S}}, \{m_{i,j-N}\}_{i\in\mathcal{S}^c}, m_{N+1,j-N}, m_{0,j-N}))\\
&= P(\cup_{m_{\{k,j-N\}_{k\in\mathcal{T}\cup\{N+1\}}}\neq 1} \cup_{m_{0,j-N}\neq 1} (E_1 \cap E_2 \cap \cdots \cap E_{N+1}))\\
&\leq \sum_{m_{\{k,j-N\}_{k\in\mathcal{T}\cup\{N+1\}}}\neq 1} \sum_{m_{0,j-N}\neq 1} P(E_1) \times P(E_2) \times \cdots \times P(E_{N+1})\\
&\leq 2^{nR_1} \times 2^{nR_2} \times \cdots \times 2^{nR_{N+1}} \times 2^{nR_0} \times 2^{-n(I(X_N, W_N; Y_{N+1}) - \delta(\epsilon))} \times\\
&\quad 2^{-n(I(X_{N-1}, W_{N-1}; Y_{N+1}|X_{N-2}, W_{N-2}) - \delta(\epsilon))}\\
&\quad \times \cdots \times 2^{-n(I(X_1, W_1; Y_{N+1}|X_2^N, W_2^N) - \delta(\epsilon))} \times 2^{-n(I(U_1^N, X_0, W_0; Y_{N+1}|X_1^N, W_1^N) - \delta(\epsilon))} \qquad \text{(C.2)}
\end{aligned}$$

$$\sum_{\{m_{i,j-N}\}_{i\in\mathcal{S}}\neq 1, \{m_{i,j-N}\}_{i\in\mathcal{S}^c}=1, m_{N+1,j-N}\neq 1, m_{0,j-N}=1} P(E(\{m_{i,j-N}\}_{i\in\mathcal{S}}, \{1\}_{|\mathcal{S}^c|}, m_{N+1,j-N}, 1))$$
$$= P(\cup_{\{m_{i,j-N}\}_{i\in\mathcal{S}}\neq 1} \cup_{m_{N+1,j-N}\neq 1} (\cap_{i\in\mathcal{S}} E_i \cap E_{N+1}))$$
$$\leq \sum_{\{m_{i,j-N}\}_{i\in\mathcal{S}}\neq 1} \sum_{m_{N+1,j-N}\neq 1} \prod_{i\in\mathcal{S}} P(E_i) \times P(E_{N+1})$$
$$\leq \prod_{i\in\mathcal{S}} \{2^{nR_i}\} \times 2^{nR_{N+1}} \times \prod_{i\in S} 2^{-n(I(X_i;Y_{N+1}|W_i^N, X_{i+1}^N)-\delta(\epsilon))} \times 2^{-n(I(X_0,U_{\mathcal{S}};Y_{N+1}|W_0^N,X_1^N,U_{\mathcal{S}^c})-\delta(\epsilon))}$$
$$\leq \prod_{i\in\mathcal{S}} \{2^{nR_i}\} \times 2^{nR_{N+1}} \times 2^{-n(I(X_0,X_{\mathcal{S}},U_{\mathcal{S}};Y_{N+1}|X_{\mathcal{S}^c},U_{\mathcal{S}^c},W_0^N)-(|\mathcal{S}|+1)\delta(\epsilon))}, \quad \text{(C.3)}$$

Similarly, it is impossible to correctly decode $m_{N+1,j-N}$ if any of $\{m_{k,j-N}\}_{k\in\mathcal{T}}$ isn't decoded correctly. According to independence of the codebooks and the joint typicality lemma, the third term in (C.1) becomes (C.3), which tends to 0 as $n \to \infty$ if

$$\sum_{i\in\mathcal{S}} R_i + R_{N+1} < I(X_0, \{X_i, U_i\}_{i\in\mathcal{S}}; Y_{N+1}|\{X_i, U_i\}_{i\in\mathcal{S}^c}, W_0^N) - (|\mathcal{S}| + 1)\delta(\epsilon).$$

Appendix D

Fourier-Motzkin Elimination for the two-relay network

The Fourier-Motzkin elimination gives a systematic procedure for finding the system of a tuple of linear inequalities (see [18] for details). In this thesis, we apply Fourier-Motzkin elimination to a group of linear rate inequalities. After such elimination process, we will find the capacity results of the relay network with N relays. In this Appendix, for example, we use Fourier-Motzkin elimination to find the achievable rate R from split message part rates R_0, R_1, R_2 and R_3 in a two-relay network. This example can extend to a larger tuple of linear rate inequalities associated with more relays.

From the error analysis in two-relay network, we have the following rate constraints:

$$
\begin{aligned}
R_1 &< I(U_1;Y_1|W_0,W_1,W_2,X1) \triangleq I_1,\\
R_1+R_0 &< I(U_1,W_0;Y_1|X_1,W_1,W_2) \triangleq I_2,\\
R_2 &< I(U_2;Y_2|W_0,W_1,W_2,X_2) \triangleq I_3,\\
R_2+R_0 &< I(U_2,W_0,W_1;Y_2|X_2,W_2) \triangleq I_4,\\
R_3 &< I(X_0;Y_3|U_1,X_1,U_2,X_2,W_0,W_1,W_2) \triangleq I_5,\\
R_3+R_1 &< I(X_0,U_1,X_1;Y_3|U_2,W_0,W_1,W_2,X_2) \triangleq I_6,\\
R_3+R_2 &< I(X_0,U_2,X_2;Y_3|U_1,W_0,W_1,W_2,X_1) \triangleq I_7,\\
R_3+R_1+R_2 &< I(X_0,X_1,X_2,U_1,U_2;Y_3|W_0,W_1,W_2) \triangleq I_8,\\
R_3+R_1+R_2+R_0 &< I(X_0,X_1,X_2,U_1,U_2,W_0,W_1,W_2;Y_3) \triangleq I_9.
\end{aligned}
$$

Next, we show the Fourier-Motzkin Elimination procedure for the above rate constraints step by step. The standard Fourier-Motzkin elimination is illustrated in [8].

Step 1 Eliminate R_0

Rearrange rate constraints, we have

$$R < I_9,$$

and,

$$\begin{aligned} R_1 + R_0 &< I_2, \\ R_2 + R_0 &< I_4, \\ R - R_2 - R_0 - R_3 &< I_1, \\ R - R_1 - R_0 - R_3 &< I_3, \\ R - R_1 - R_2 - R_0 &< I_5, \\ R - R_2 - R_0 &< I_6, \\ R - R_1 - R_0 &< I_7, \\ R - R_0 &< I_8, \end{aligned}$$

Then, we get

$$\begin{aligned} R + R_1 - R_2 - R_3 &< I_1 + I_2, \\ R - R_3 &< I_2 + I_3, \\ R - R_2 &< I_2 + I_5, \\ R + R_1 - R_2 &< I_2 + I_6, \\ R &< I_2 + I_7, \\ R + R_1 &< I_2 + I_8, \\ R - R_3 &< I_4 + I_1, \\ R + R_2 - R_1 - R_3 &< I_4 + I_3, \\ R - R_1 &< I_4 + I_5, \\ R &< I_4 + I_6, \\ R + R_2 - R_1 &< I_4 + I_7, \\ R + R_2 &< I_4 + I_8, \end{aligned}$$

Step 2 Eliminate R_1

Rearrange rate constraints, we have

$$\begin{aligned} R &< I_9, \\ R &< I_2 + I_7, \\ R &< I_4 + I_6, \end{aligned}$$

and,

$$\begin{aligned} R - R_3 &< I_2 + I_3, \\ R - R_3 &< I_4 + I_1, \\ R + R_1 - R_2 - R_3 &< I_1 + I_2, \\ R + R_1 - R_2 &< I_2 + I_6, \\ R + R_1 &< I_2 + I_8, \\ R + R_2 - R_1 - R_3 &< I_4 + I_3, \\ R - R_1 &< I_4 + I_5, \\ R + R_2 - R_1 &< I_4 + I_7, \\ R - R_2 &< I_2 + I_5, \\ R + R_2 &< I_4 + I_8, \end{aligned}$$

Then, we get

$$R - R_3 < I_2 + I_3,$$
$$R - R_3 < I_4 + I_1,$$
$$2R - 2R_3 < I_1 + I_2 + I_4 + I_3,$$
$$2R - R_2 - R_3 < I_1 + I_2 + I_4 + I_5,$$
$$2R - R_3 < I_1 + I_2 + I_4 + I_7,$$
$$2R - R_3 < I_2 + I_6 + I_4 + I_3,$$
$$2R - R_2 < I_2 + I_6 + I_4 + I_5,$$
$$2R < I_2 + I_6 + I_4 + I_7,$$
$$2R + R_2 - R_3 < I_2 + I_8 + I_4 + I_3,$$
$$2R < I_2 + I_8 + I_4 + I_5,$$
$$2R + R_2 < I_2 + I_8 + I_4 + I_7,$$
$$R - R_2 < I_2 + I_5,$$
$$R + R_2 < I_4 + I_8,$$

Step 3 Eliminate R_2

Rearrange rate constraints, we have

$$R < I_9,$$
$$R < I_2 + I_7,$$
$$R < I_4 + I_6,$$
$$2R < I_2 + I_8 + I_4 + I_5,$$

and,

$$R - R_3 < I_2 + I_3,$$
$$R - R_3 < I_4 + I_1,$$
$$2R - R_3 < I_1 + I_2 + I_4 + I_7,$$
$$2R - R_3 < I_2 + I_6 + I_4 + I_3,$$
$$2R - 2R_3 < I_1 + I_2 + I_4 + I_3,$$
$$2R - R_2 - R_3 < I_1 + I_2 + I_4 + I_5,$$
$$2R - R_2 < I_2 + I_6 + I_4 + I_5,$$
$$R - R_2 < I_2 + I_5,$$
$$2R + R_2 - R_3 < I_2 + I_8 + I_4 + I_3,$$
$$2R + R_2 < I_2 + I_8 + I_4 + I_7,$$
$$R + R_2 < I_4 + I_8,$$

Then, we get

$$R - R_3 < I_2 + I_3,$$
$$R - R_3 < I_4 + I_1,$$
$$2R - R_3 < I_1 + I_2 + I_4 + I_7,$$
$$2R - R_3 < I_2 + I_6 + I_4 + I_3,$$
$$2R - 2R_3 < I_1 + I_2 + I_4 + I_3,$$
$$4R - 2R_3 < I_1 + I_2 + I_4 + I_5 + I_2 + I_8 + I_4 + I_3,$$
$$4R - R_3 < I_1 + I_2 + I_4 + I_5 + I_2 + I_8 + I_4 + I_7,$$
$$3R - R_3 < I_1 + I_2 + I_4 + I_5 + I_4 + I_8,$$
$$4R - R_3 < I_2 + I_6 + I_4 + I_5 + I_2 + I_8 + I_4 + I_3,$$
$$4R < I_2 + I_6 + I_4 + I_5 + I_2 + I_8 + I_4 + I_7,$$
$$3R < I_2 + I_6 + I_4 + I_5 + I_4 + I_8,$$
$$3R - R_3 < I_2 + I_5 + I_2 + I_8 + I_4 + I_3,$$
$$3R < I_2 + I_5 + I_2 + I_8 + I_4 + I_7,$$
$$2R < I_2 + I_5 + I_4 + I_8,$$

Step 4 Delete redundant rate constraints

After deleting redundant rate constraints, we have

$$
\begin{aligned}
R <& I_9,\\
R <& I_2 + I_7,\\
R <& I_4 + I_6,\\
2R <& I_2 + I_8 + I_4 + I_5,\\
R - R_3 <& I_2 + I_3,\\
R - R_3 <& I_4 + I_1,\\
2R - R_3 <& I_1 + I_2 + I_4 + I_7,\\
2R - R_3 <& I_2 + I_6 + I_4 + I_3,\\
4R - R_3 <& I_1 + I_2 + I_4 + I_5\\
&+ I_2 + I_8 + I_4 + I_7,\\
3R - R_3 <& I_1 + I_2 + I_4 + I_5 + I_4 + I_8,\\
4R - R_3 <& I_2 + I_6 + I_4 + I_5 + I_2\\
&+ I_8 + I_4 + I_3,\\
3R - R_3 <& I_2 + I_5 + I_2 + I_8 + I_4 + I_3,
\end{aligned}
$$

Step 5 Add R_3 back

After adding $R_3 < I_5$ to each rate constraints, we have

$$
\begin{aligned}
R <& I_9,\\
R <& I_2 + I_7,\\
R <& I_4 + I_6,\\
2R <& I_2 + I_8 + I_4 + I_5,\\
R <& I_2 + I_3 + I_5,\\
R <& I_4 + I_1 + I_5,\\
2R <& I_1 + I_2 + I_4 + I_7 + I_5,\\
2R <& I_2 + I_6 + I_4 + I_3 + I_5,\\
4R <& I_1 + I_2 + I_4 + I_5 + I_2 + I_8 + I_4 + I_7 + I_5,\\
3R <& I_1 + I_2 + I_4 + I_5 + I_4 + I_8 + I_5,\\
4R <& I_2 + I_6 + I_4 + I_5 + I_2 + I_8 + I_4 + I_3 + I_5,\\
3R <& I_2 + I_5 + I_2 + I_8 + I_4 + I_3 + I_5,
\end{aligned}
$$

Step 6 Delete redundant rata constraints

After deleting redundant rate constraints, we have

$$
\begin{aligned}
R <& I_9,\\
R <& I_2 + I_7,\\
R <& I_4 + I_6,\\
R <& I_2 + I_3 + I_5,\\
R <& I_4 + I_1 + I_5.
\end{aligned}
$$

References

[1] E. van der Meulen, "Three-terminal communication channels," *Advances in Applied Probability*, vol. 3, no. 1, pp. 120–154, 1971.

[2] T. Cover and A. El Gamal, "Capacity theorems for the relay channel," *IEEE Trans. on Info. Theory*, vol. 25, no. 5, pp. 572–584, 1979.

[3] A. E. Gamal and M. R. Aref, "The capacity of the semideterministic relay channel," *IEEE Trans. on Info. Theory*, vol. 28, no. 3, p. 536, 1982.

[4] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. John Wiley, 2006.

[5] L. Ghabeli and M. R. Aref, "On achievable rate for relay networks based on partial decode-and-forward," in *IEEE Int'l Symp. on Info. Theory (ISIT)*, June 2010.

[6] L.-L. Xie and P. R. Kumar, "An achievable rate for the multiple level relay channel," *IEEE Trans. on Info. Theory*, vol. 51, pp. 1348–1358, April 2005.

[7] T. S. Han and K. Kobayashi, "A new achievable rate region for the interference channel," *IEEE Trans. on Info. Theory*, vol. 27, no. 1, pp. 49–60, 1981.

[8] A. E. Gamal and Y.-H. Kim, *Network information theory*. Cambridge University Press, 2011.

[9] F. Willems and E. van der Meulen, "The discrete memoryless multiple-access channel with cribbing encoders," *IEEE Trans. on Info. Theory*, vol. 31, pp. 313–327, May 1985.

[10] G. Kramer, M. Gastpar, and P. Gupta, "Cooperative strategies and capacity theorems for relay networks," *IEEE Trans. on Info. Theory*, vol. 51, pp. 3037–3063, Sept. 2005.

[11] L. Ghabeli and M. R. Aref, "Symmertric semideterministic relay networks with no interference at the relays," *IEEE Trans. on Info. Theory*, vol. 57, pp. 5673–5681, September 2011.

[12] A. El Gamal, M. Mohseni, and S. Zahedi, "Bounds on capacity and minimum energy-per-bit for AWGN relay channels," *IEEE Trans. on Info. Theory*, vol. 52, pp. 1545–1561, April 2006.

[13] P. Zhong and M. Vu, "Partial decode-forward coding schemes for the gaussian two-way relay channel," in *IEEE International Conference on Communication*, June 2012.

[14] G. R. O. C. Hucher and A. Saadani, "A new partial decode-and-forward protocol," in *Wireless Communications and Networking Conference*, 2008.

[15] L. Ghabeli and M. R. Aref, "A new achievable rate and the capacity of a class of semideterministic relay networks," in *IEEE Int'l Symp. on Info. Theory (ISIT)*, June 2007.

[16] Y. Tang and M. Vu, "A partial decode-forward scheme for a network with N relays," in *Conference on Information Sciences and Systems*, March 2013.

[17] L. Ghabeli and M. R. Aref, "Simultaneous partial and backward decoding approach for two-level relay networks," in *IEEE Int'l Symp. on Info. Theory (ISIT)*, June 2009.

[18] G. M. Ziegler, *Lecture on Polytopes*. New York: Springer-Verlag, 1995.