Normed Double Algebras, Local and Global Optimization, and Slow H^{∞} Adaptation

Le Yi Wang

M. Eng., (Shanghai Institute of Mechanical Engineering), 1982

Department of Electrical Engineering

McGill University

A thesis submitted to the Faculty of Graduate Studies and Research

in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

October 15, 1989

The larger the island of knowledge,

the longer the shore line of wonder.

RALPH W. SOCKMAN

Abstract

A normed space of input-output mappings equipped with two products, one global and the other local, called a normed double algebra (NDA), is introduced for the frozen-time analysis of stabilization and optimization of a class of slowly time-varying systems. Local-global relations within a normed double algebra are established, in the time and frequency domains, in systems which vary slowly. The local-global relations, applied to system properties such as stability, coprime factorization and optimization, enable global properties to be deduced from the local ones, especially in the frequency domain, by methods which are computationally tractable, at least in principle. Classical frozen-time stability is reinterpreted in terms of a relation between local and global resolvents in the NDA. Relations between local and global coprime factorizations and their implications to local and global robust stability are obtained.

An explicit double algebraic expression for adaptive BIBO sensitivity reduction is established. Notions of adaptive and robust (non-adaptive) sensitivity minimization are applied to an example involving rejection of narrowband disturbances of uncertain bandwidth and center frequency. The double algebra symbolism is employed to show that adaptive minimization can give better sensitivity than H^∞ optimal robust minimization.

To implement a design strategy of global sensitivity optimization using local H^∞ interpolation, Lipschitz continuity of optimal H^∞ interpolants on data is investigated. While optimal H^∞ interpolants in general do not depend Lipschitz continuously on data, δ-suboptimal interpolants based on AAK's maximal entropy solutions satisfy an appropriate Lipschitz continuity condition. These, applied to slowly time-varying

systems, achieve approximations to the globally optimal interpolants, which become accurate as the rates of variation approach zero.

Résumé

Un espace normé de relation entrée-sortie muni de deux produits, un global et un local, appelé à un temps donné double algèbre normé (ADN), est présenté pour une analyse et l'optimisation d'une classe de systèmes variant lentement dans le temps. Les relations local-global, appliqués aux propriétés du système tel que stabilité, factorisation première et optimisation, rendent possible la déduction des propriétés globales à partir des propriétés locales, en particulier dans le domaine fréquentielle, par des méthodes en principe calculables par ordinateur. La classique stabilité à un temps donné est réinterprétée en termes de relation entre les solutions globale et locale dans l'ADN. Les relations entre les factorisations premières globale et locale et leur implication avec la stabilité robuste, locale et globale, en résultent.

Une double expression algébrique pour la réduction de sensibilité adaptative (entrée et sortie bornées) est explicitement établie. Les notions de minimisation de sensibilité adaptative et robuste (non adaptative) sont appliquées à un exemple comprenant la rejection de perturbations à bande étroite d'une largeur de bande incertaine et d'une fréquence-centre. La symbolique d'algébre double est employé pour montrer que la minimisation adaptative peut donner une meilleure sensibilité que la minimisation robuste et optimale dans H^∞.

Pour mettre en place une stratégie pour la conception d'une optimisation en sensibilité globale utilisant une interpolation locale dans H^∞, la continuité Lipschitz d'interpolateurs optimaux dans H^∞ est étudiée. Ceci, appliqué aux systèmes variant lentement dans le temps, résulte dans l'approximation d'interpolateurs globalement optimaux, qui deviennent précis lorsque les taux de variation tendent vers zéro.

Acknowledgements

I would like to express my gratitude and appreciation to Prof. George Zames, my thesis supervisor, for his expert guidance, persistent encouragement and friendship during the course of this research. With his deep insight into systems theory, high standards in research, and kind understanding and patience, he not only contributed immensely to the direction of this work, but also made great efforts to develop my ability to think independently and logically, to be rigorous and disciplined in research and, not unimportantly, to write passable English.

I am very grateful to Prof. P.E. Caines for his friendship, continuous support and invaluable help in my course study and thesis research; to Prof. A. Tannenbaum for teaching me the subject of robust stability and operator theory; to Prof. M.C. Smith for many valuable discussions. I would like also to thank Profs. L. Gerenzer, M. Kaplan, P.J. Koosis, P. Makila, R. Mazumdar, K. Poolla, M.S. Verma for introducing me to many interesting topics in their areas.

I am particularly indebted to my friends and colleagues in the control group for providing me a warm, friendly, enthusiastic atmosphere. I have learnt a lot from many interesting discussions with H.M.J. Cantalloube, S. Meyn and J. Owen. P. Lie Chin Choeng and M. Readman rescued me from many computer puzzles. E. Desblancs made an expert French translation of the abstract. Sincere thanks to them and also to: M. Gaber, C. Houre, L. Lin, H. Ozbay, S.N. Wang, H. Wehn, F. Wredenhagen.

My special thanks go to my parents and sister. Although far from Montreal, they have extended generously their support and confidence all these years.

I wish to extend my deepest thanks to my wife, Hong, for her encouragement, patience and good spirits during this undertaking. She truly shared in each step of this thesis' gestation and always supported me with her love and understanding.

The financial support from McGill Graduate Faculty Fellowships is also greatly appreciated.

Contents

Notation

$\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$ denote the reals, complex numbers and integers. The complex conjugate of any $x \in \mathbb{C}$ is $\overline{x}$.

$\mathbb{K}^n$ and $\mathbb{K}^{n\times n}$ denote n-vectors and $n \times n$ matrices over a ring $\mathbb{K}$. $\mathbb{C}^n$ is viewed as a Euclidean space; for $x \in \mathbb{C}^n$ the conjugate transpose is x^* and norm $|x| = (x^*x)^{1/2}$. For a matrix $K \in \mathbb{C}^{n\times n}$, $|K|$ is its largest singular value.

$l^p_\sigma[a,b]$, $1 \le p \le \infty$, $\sigma \ge 0$, denotes the space of sequences $u(t)$, $t = a, a+1, \ldots, b$, $t \in \mathbb{Z}$, either of vectors in $\mathbb{C}^n$ or $n \times n$ matrices in $\mathbb{C}^{n\times n}$ for which

$$\|u\|_{l^p_\sigma} := \begin{cases} \left[\sum_{t=a}^{b}(|u(t)|\sigma^t)^p\right]^{1/p} < \infty, & \text{for } 1 \le p < \infty; \\ \sup_{t\in[a,b]} |u(t)|\sigma^t < \infty, & \text{for } p = \infty. \end{cases} \tag{0.1}$$

The dimension n will be fixed and omitted in notation except where it is to be emphasized, where the notation $\left(l^p_\sigma[a,b]\right)^n$ or $\left(l^p_\sigma[a,b]\right)^{n\times n}$ will be used.

H^p_σ, $1 \le p \le \infty$, $\sigma > 0$, denotes the H^p space of of $\mathbb{C}^n$-vector or $\mathbb{C}^{n\times n}$-matrix functions $K(z)$ on the disk $|z| < \sigma$ for which

$$\|K(\cdot)\|_{H^p_\sigma} := \|K(\sigma(\cdot))\|_{H^p} = \left(\frac{1}{2\pi}\int_0^{2\pi}\left|K(\sigma e^{i\theta})\right|^p d\theta\right)^{1/p}. \tag{0.2}$$

H^p_σ is viewed as a subspace of L^p_σ, the space of L^p (Lebesque-p spaces) functions of the circle of radius σ.

Note that for $p = 2$, $\sigma = 1$, $K \in \left(H^2\right)^{n\times n}$, the Banach norm employed in this thesis is

$$\|K\|_{H^2} = \left(\frac{1}{2\pi}\int_0^{2\pi}\left|K(e^{i\theta})\right|^2 d\theta\right)^{1/2}, \tag{0.3}$$

which is different from the usual definition of H^2-norm (Hilbert norm)

$$\|K\|_2 = \left(\frac{1}{2\pi} \int_0^{2\pi} Trace\left(K^*(e^{i\theta}) K(e^{i\theta}) \right) d\theta \right)^{1/2}, \tag{0.4}$$

where $Trace(A) = \sum_{i=1}^n a_{ii}$ for $A = [a_{ij}] \in \mathbb{C}^{n \times n}$.

Nevertheless, due to the matrix inequalities

$$|A|^2 \le Trace(A^* A) \le n|A|^2 \tag{0.5}$$

for any $A \in \mathbb{C}^{n \times n}$, $\|\cdot\|_{H^2}$ and $\|\cdot\|_2$ are equivalent norms.

$\mathcal{L}(u) \in L_\sigma^p$ denotes the z-transform of any $u \in l_\sigma^2(-\infty, \infty)$,

$$\mathcal{L}(u)(z) = \sum_{t=-\infty}^{\infty} u(t) z^t, \qquad |z| = \sigma. \tag{0.6}$$

$\mathcal{L}(u)$ will also be represented by $\hat{u}$. When $u \in l_\sigma^2[0, \infty)$, $\mathcal{L}(u)$ has analytic continuation into the disk of radius σ, i.e., $\mathcal{L}(u) \in H_\sigma^2$. $\mathcal{L}^{-1}(K) \in l_\sigma^2$ denotes the inverse transform of any $K \in L_\sigma^2$ defined for $t \in (-\infty, \infty)$, $t \in \mathbb{Z}$ by

$$\mathcal{L}^{-1}(K)(t) = \sigma^{-t} \left(\frac{1}{2\pi} \int_0^{2\pi} K(\sigma e^{i\theta}) e^{-i\theta t} d\theta \right). \tag{0.7}$$

If $K \in H_\sigma^2$, then $\mathcal{L}^{-1}(K)(t) = 0$ for $t < 0$. Functions in l_σ^p will be denoted by lower case letters, in H_σ^p by capitals, and operators in either space by boldface capitals.

Π_t, $t \in \mathbb{Z}$ denotes the truncation operator which maps any $f \in l_\sigma^p(-\infty, \infty)$ into f_t, where $f_t(\tau) = f(\tau)$ for $\tau \le t$ and 0 elsewhere.

The following constants (as a function of $\sigma > 1$) are fixed in the thesis:

$$\begin{aligned} \kappa_\sigma &= \left(\sum_{i=0}^{\infty} \sigma^{-2i} \right)^{1/2} = \frac{\sigma}{\sqrt{\sigma^2 - 1}}; \\ \kappa'_\sigma &= \left(\sum_{i=0}^{\infty} i^2 \sigma^{-2i} \right)^{1/2}; \\ \kappa_\sigma^{(p)} &= \kappa'_\sigma \quad \text{for } 2 \le p < \infty; \quad = \frac{1}{\sigma - 1} \quad \text{for } p = \infty. \end{aligned} \tag{0.8}$$

Chapter 1 Introduction

Our objective in this research is to develop a common systems framework for the frozen-time analysis and optimization of slowly time-varying MIMO systems. The main idea is to achieve stability and optimality (or near optimality) by means of notions of "local" stability and "local" optimality, especially in the frequency domain.

1.1 Problem and Approach

In order to get a nontrivial theory of adaptive stabilization and optimization for time-varying systems, the effect of persistent disturbances, say in l^∞, and causality constraints, i.e., causal dependence of control on time-varying data, have to be considered. Apart from some existence results for a related l^2 disturbance rejection problem without causality constraints ([Fei1][Fei2]), there is at present no such complete theory. Since it would appear that the ability to adapt or learn from experience is limited to those aspects of data which persist or, at most, vary slowly with time, it seems worthwhile to single out features of optimization which are peculiar to slowly time-varying systems.

There are conditions for the BIBO stability of slowly time-varying systems based on the ideas of frozen-time analysis and exponentially weighted l^2 spaces, going back to the 60's [Fre] [Des1]. It became apparent in the course of this research that certain features in these results could be abstracted and generalized to derive a common algebraic framework for frozen-time analysis of stability and optimization.

The framework introduced here involves the notion of a normed double algebra (NDA), i.e., a normed space of input-output mappings on which two products are defined, one local and the other global.

Local-global relations within a normed double algebra are established, in the time and frequency domains, in systems which vary sufficiently slowly. The local-global relations applied to system properties such as stability, coprime factorization and optimization, enable global versions of these properties to be deduced from the local ones, especially in the frequency domain, in a way which is computationally tractable at least in principle. Classical frozen-time stability is reinterpreted in terms of a relation between local and global resolvents in the NDA. Relations between local and global coprime factorizations and their implication to local and global robust stability are obtained.

One approach to solving a persistent disturbance rejection problem is to use direct l^1-kernel optimization. However, it might be desirable in systems analysis and synthesis to employ qualitative information provided by spectral data in the frequency domain, which would be lost in the l^1-kernel approach. Our alternative is to establish an "approximate isometry" between certain frequency and time domain norms to approximately evaluate l^1-kernel behavior from related H^∞ properties in the frequency domain.

An explicit double algebraic expression for adaptive BIBO sensitivity reduction is established. To implement a design strategy of global sensitivity optimization using local H^∞ interpolation, Lipschitz continuity of optimal H^∞ interpolants on data is investigated. While optimal H^∞ interpolants in general do not depend Lipschitz continuously on data, δ-suboptimal interpolants which satisfy a suitable Lipschitz condition can be obtained using AAK's maximal entropy solutions [Ada2]. These achieve an acceptable approximation to the globally optimal interpolants in systems whose variation rates are small enough.

Notions of adaptive and robust (non-adaptive) sensitivity minimization of [Zam4,6] are applied to an example involving rejection of narrowband disturbances of uncertain bandwidth and center frequency. The double algebra symbolism is employed to show that adaptive minimization can give better sensitivity than H^∞ optimal robust minimization.

1.2 A Brief Literature Review

Frozen-time stability analysis of slowly time-varying systems has been developed since the 60's, in both the frequency and the time domains. The "approximate isometry" based on exponential weighting was introduced by Zames [Zam1] to obtain an L^∞ version of the circle criterion. Freedman and Zames [Fre] introduced the notion of "frozen-time" analysis in an input-output setting, using a method of averaging for systems with exponentially decaying memories and slowly time-varying gains. Closely related results in a state space setting were obtained by Desoer [Des1] and Narendra[Nar], extending an early result of Rosenbrock [Ros]. Their results were later extended by students of Desoer, e.g., Barman [Bar] to nonlinear systems. The NDA scheme introduced in this thesis provides a unified framework for frozen-time stability analysis of slowly varying systems, which is capable of incorporating the previous work.

Algebraic approaches to input-ouput feedback go back to the 60's-70's, culminating in operator-norm sensitivity minimization of Zames [Zam2] and generalized coprime-factorization of Desoer [Des3]. [Zam2] and its further development (see [Fra1]), now collectively known as "H^∞ sensitivity optimization", forms a basis for "local" synthesis in the NDA. In fact, one motivation for the current work is to extend H^∞ optimization ideas to slowly time-varying systems.

Although the NDA framework is suitable for stability analysis, the main interest here is in performance analysis and system synthesis, especially sensitivity optimization in adaptive systems. Feintuch and Francis [Fei1][Fei2], employing the Arveson distance formula [Arv], proved the existence of an optimal controller in a l^2 disturbance rejection problem for linear time-varying systems. Their result does not include a causality assumption on the dependence of control on data. Ball, Foias, Helton, and Tannenbaum [Bal1][Bal2][Bal3], using local Volterra operator expansions, investigated the nonlinear sensitivity optimization problem. Major differences between these works and the present thesis is that they make no causality assumption, and are not concerned with persistent disturbances.

A simple example (chapter 5) shows that optimal H^∞ interpolants in general do not depend Lipschitz continuously on data, and hence local H^∞ *optimal* interpolation may yield a fast-varying feedback controller even though the plant and weighting are slowly time-varying. This problem is resolved here by using AAK's δ-suboptimal maximal entropy interpolants, which are shown to be Lipschitz continuously dependent on data. Smith [Smi1] discussed the norm sensitivity of H^∞ interpolants with respect to perturbations in data, and provided conditions for the well-posedness of H^∞ optimization. In Kreisselmeier [Kre], Dahleh and Dahleh [Dah], Cantalloube [Can1] [Can2], the continuity constraint is imposed as a hypothesis in their adaptive algorithms for slowly time-varying systems.

The relations between stability and coprime factorization are obtained in general setting developed by Francis, Schneider and Vidyasagar [Fra3] [Vid], and Desoer et al [Des3]. Although these approaches are well developed for time invariant systems, their counterparts for time-varying systems are not well understood. In the NDA framework, relations between local and global versions of robust stability and coprime factorization are obtained for systems which vary sufficiently slowly. Some related results were obtained by my colleagues Cantalloube, and Nahum and Caines [Can1] [Can2]. Recently, Verma [Ver1] established relations between robustness and coprime factorization for nonlinear systems.

Some preliminary results of this thesis were presented in [Wan1] [Wan2] [Wan3] [Wan4].

1.3 Outline of the Thesis

The thesis is organized as follows. Chapter 2 introduces the concept of a normed double algebra and its basic properties. The local-global coupling operator ∇ is introduced. Local-global relations within a NDA are established. An application of the NDA symbolism in Section 2.6 to state space models provides a unified framework for some previous frozen-time time-domain stability results. Then, in Chapter 3, an auxiliary frequency-domain norm $\mu_\sigma(\cdot)$ and a time-domain norm $\|\cdot\|_{\alpha(\sigma)}$ are introduced. Local frequency-domain bounds on the time-domain behavior are provided in Props. 3.6-3.9. An immediate application of the NDA framework is a unification of several classical frozen-time frequency-domain stability results for slowly time-varying systems, by means of a local-global resolvent relation in Corollary 3.1. Lower bounds on the time-domain norms $\|\cdot\|_{a(\sigma)}$ in terms of the frequency-domain norms $\mu_\sigma(\cdot)$ are established in Prop. 3.10 and 3.11. An "asymptotic isometry" between $\|\cdot\|_{a(\sigma)}$ and $\mu_\sigma(\cdot)$ is provided in Prop. 3.12.

Chapter 4 consists of a preliminary investigation of sensitivity optimization for feedback systems with slowly time-varying stable plants, using the local-global approximations established in Chapter 2 and 3. An explicit double algebra expression for adaptive BIBO sensitivity reduction is obtained. Notions of adaptive and robust (non-adaptive) sensitivity minimization are applied to an example, and the NDA symbolism is employed to show that adaptive minimization can give better sensitivity than H^∞ optimal robust minimization. The local sensitivity minimization problem is studied in Chapter 5. An example is first introduced to show that H^∞ optimal interpolants need not depend Lipschitz continuously on data. A controller constructed from slowly varying plants and weightings using optimal local interpolation may be quickly-varying and therefore not be amenable to frozen-time analysis. To avoid this difficulty, the issue of Lipschitz continuity is investigated and a δ-suboptimal interpolant which satisfies a suitable Lipschitz condition is achieved using the central (maximum entropy) solution in AAK's parametrization.

Coprime factorizations of unstable plants under assumptions of robustness are studied in both Chapter 6 and Chapter 7. General results are first presented, within the general framework of Francis, Schneider and Vidyasagar [Fra3] [Vid], and Desoer, Liu, Murray and Saeks [Des3], in Chapter 6 where relations among robust stability, separate coprimeness and joint coprimeness are explored in a general Banach algebra (Theorem 6.1). Robustly stabilizable plants in a small neighborhood of a nominal plant are parametrized by a fractional transformation (Theorem 6.2), which displays an equivalence between open and closed loop topologies. Then in a NDA framework, relations between local and global versions of robust stability and coprimeness are demonstrated in Chapter 7. In particular, we show, for systems which vary sufficiently slowly, that under certain assumptions there is an equivalence between local and global versions of robust stability and existence of a coprime factorization.

Finally, Chapter 8 summarizes briefly the main results in this thesis and points out some further research directions.

1.4 Main Contributions of the Thesis

Several mathematical concepts are introduced. It is shown that they can be used to produce a unified theory of frozen-time analysis. The main new concept is that of a normed double algebra of input-output mappings, incorporating local and global products, for the analysis of slowly varying systems, i.e., systems whose commutants with the shift are small. Based on that concept, notions of local stability, local optimization, local spectral and coprime factorization are introduced. It is shown that classical frozen-time stability results can be unified in the normed double algebra as relations between local and global spectra.

The thesis employs definitions of robust and adaptive control in an H^∞ context to show that under certain conditions adaptive control can achieve better sensitivity than an optimally robust control. The actual definitions of adaptive and robust control used here, as well as the idea of a double algebra were provided by Zames [Zam4,6]. However, these concepts are worked out here in detail for the first time.

New Lipschitz continuity conditions for H^∞ interpolants are derived.

New explicit double algebraic expressions for certain adaptive sensitivity optimization problems are obtained.

Relations between local and global coprime factorization and robustness of stability are derived.

Chapter 2 Preliminary

Let A denote the Banach Space of $\mathbb{C}^n$-valued functions in $l^\infty(-\infty,\infty)$. (Later in Chapter 3, A will be equipped with certain equivalent auxiliary norms.) Stable systems will belong to the Banach space $<\mathbb{B}>$ of bounded causal linear operators $\mathbf{K}: A \to A$ which have *convolution sum* representations,

$$(\mathbf{K}u)(t) = \sum_{\tau=-\infty}^{t} k(t,\tau)u(\tau), \quad t \in \mathbb{Z} \tag{2.1}$$

where the kernel $k : \mathbb{Z}^2 \to \mathbb{C}^{n\times n}$ is assumed, for each $t \in \mathbb{Z}$, to satisfy $k(t,\cdot) \in l^1(-\infty,\infty)$,

$$\sup_{t\in\mathbb{Z}} \|k(t,\cdot)\|_{l^1} =: \|\mathbf{K}\|_{\mathbf{B}} < \infty$$

and $k(t,\tau) = 0$ whenever $\tau > t$.

Unstable systems belong to a linear extension $<\mathbb{B}_e>$ of $<\mathbb{B}>$, defined as follows.

Let A^0 be the subspace of A,

$$A^0 := \left\{u \in A : u(t) = 0 \ \text{ for } t < t_u \ \text{ or } \ t > t'_u\right\}$$

where $t_u, t'_u \in \mathbb{Z}$ depend on u. $(A^0)_e$ is the linear space of functions whose truncations $\Pi_t(u)$ lie in A^0 for each $t \in \mathbb{Z}$. Then $<\mathbb{B}_e>$ is the space of causal linear operators in

$(A^0)_e$ which have convolution sum representations of the form (2.1). An operator $\mathbf{G}$ in $< \mathbb{B}_e >$ is said to be *bounded* if it satisfies that for all $u \in (A^0)_e$, $\mathbf{G}\Pi_t u \in A \quad \forall t \in \mathbb{Z}$ and

$$\sup_t \sup_{u \in (A^0)_e} \frac{\|\mathbf{G}\Pi_t u\|_{l^\infty}}{\|\Pi_t u\|_{l^\infty}} < \infty.$$

$< \mathbb{B} >$ can be viewed as a subspace of $< \mathbb{B}_e >$ modulo the following equivalence.

To each $\mathbf{K} \in < \mathbb{B} >$, assign the unique bounded operator $\mathbf{K}_e \in < \mathbb{B}_e >$ obtained by first restricting $\mathbf{K}$ from A down to A^0, and then extending to $(A^0)_e$; the map $\mathbf{K} \to \mathbf{K}_e$ is an equivalence between bounded operators in $< \mathbb{B} >$ and $< \mathbb{B}_e >$.

2.1 Local Systems

The local behavior of an operator $\mathbf{K} \in < \mathbb{B}_e >$ can be described in terms of a time-invariant "*frozen-time*" operator $\mathbf{K}_t$ with the property that $\mathbf{K}$ and $\mathbf{K}_t$, acting on any input in $(A^0)_e$, produce outputs which coincide at t.

If $\mathbf{K}$ is any linear operator in $< \mathbb{B}_e >$ defined, for $u \in (A^0)_e$, by a convolution sum

$$(\mathbf{K}u)(t) = \sum_{\theta=-\infty}^{t} k(t,\theta)u(\theta), \quad t \in \mathbb{Z},$$

then the *local system* of $\mathbf{K}$ at $\tau \in \mathbb{Z}$ is the (time-invariant) operator $\mathbf{K}_\tau$ with the same domain as $\mathbf{K}$ satisfying

$$(\mathbf{K}_\tau u)(t) = \sum_{\theta=-\infty}^{t} k\left(\tau, \tau - (t-\theta)\right) u(\theta), \quad t \in \mathbb{Z}. \tag{2.2}$$

The terms local and frozen-time will be used interchangeably.

For any $\mathbf{K} \in <\mathbf{B}>$ and $\tau \in \mathbb{Z}$, the convolution kernel $k(\tau, \tau - (\cdot))$ of $\mathbf{K}_\tau$ has a well-defined transform

$$\hat{\mathbf{K}}_\tau(z) = \sum_{\theta=0}^{\infty} k(\tau, \tau - \theta) z^\theta, \quad |z| < 1 \tag{2.3}$$

in H^∞ called the *transfer function* of $\mathbf{K}_\tau$. $\hat{\mathbf{K}}_\tau$ will be called the *local transfer function* (resp. *local transform*) of $\mathbf{K}$ (resp. of $k(\tau, \tau - (\cdot))$) at τ. The notation $k(\tau, \tau - (\cdot)) = k_\tau(\cdot)$ is used in the sequel.

2.2 Banach Double Algebra

We define two products on the space $<\mathbb{B}_e>$: (1) The usual operator composition product, which will be called the *global product*, and denoted explicitly by $*$, although that symbol, as usual, will mostly be suppressed in notation, i.e., $\mathbf{F} * \mathbf{G} = \mathbf{FG}$; and (2) a *local product*, denoted by $\otimes$ and defined as follows: For any $\mathbf{F}, \mathbf{K} \in <\mathbb{B}_e>$, $\mathbf{F} \otimes \mathbf{K}$ is the unique operator in $<\mathbb{B}_e>$ whose local operators satisfy

$$(\mathbf{F} \otimes \mathbf{K})_t = \mathbf{F}_t \mathbf{K}_t \qquad \forall t \in \mathbb{Z}. \tag{2.4}$$

We will naturally define the global summation $+$ by the usual operator summation and the local summation $\oplus$ by means of local operators: $(\mathbf{F} \oplus \mathbf{K})_t = \mathbf{F}_t + \mathbf{K}_t$. However, due to our choice of frozen-time systems, $\mathbf{F} \oplus \mathbf{K} = \mathbf{F} + \mathbf{K}$ for any $\mathbf{F}, \mathbf{K} \in \mathbb{B}_e$, and so we will make no distinction between local and global summations.

A *double algebra* is any subspace of $<\mathbb{B}_e>$ which is equipped with both products and is an algebra with respect to either one. In particular, the space $<\mathbb{B}_e>$ equipped with both products is clearly a double algebra which will be denoted by $\mathbb{B}_e$.

A double algebra is *normed*, called then a *normed double algebra* (NDA), if local and global norms, $\|\cdot\|_l$ and $\|\cdot\|_g$, are defined on it and satisfy

$$\|\mathbf{GK}\|_g \leq \|\mathbf{G}\|_g\|\mathbf{K}\|_g;$$

$$\|\mathbf{G}\otimes\mathbf{K}\|_l \leq \|\mathbf{G}\|_l\|\mathbf{K}\|_l. \tag{2.5}$$

In particular, the space $<\mathbb{B}>$ equipped with both products, and with local and global norms taken to be equal to $\|\cdot\|_{\mathbf{B}}$ is a normed double algebra, which will be denoted by $\mathbb{B}$.

If $\mathbb{A}$ is any (normed) double algebra, its restriction to one of its products (and norms) will be denoted by the prefixes $\mathbb{L}$ for local and $\mathbb{G}$ for global, as in $\mathbb{LB}_e$ and $\mathbb{GB}_e$, $\mathbb{LB}$ and $\mathbb{GB}$. $\mathbb{LA}$ and $\mathbb{GA}$ will be called the *local (normed) algebra* and *global (normed) algebra* respectively. If $\mathbb{LA}$ and $\mathbb{GA}$ are both Banach algebras then $\mathbb{A}$ is a *Banach double algebra* (BDA). $\mathbb{B}$ is an example.

$\mathbf{K}\in\mathbb{A}$ has a *local inverse* in $\mathbb{A}$, denoted by $\mathbf{K}^{\ominus}$, if $\mathbf{K}^{\ominus}$ is an inverse in $\mathbb{LA}$, i.e.

$$\mathbf{K}\otimes\mathbf{K}^{\ominus} = \mathbf{K}^{\ominus}\otimes\mathbf{K} = \mathbf{I}, \tag{2.6}$$

and a *global inverse*, denoted by $\mathbf{K}^{-1}$, if $\mathbf{K}^{-1}$ is an inverse in $\mathbb{GA}$, i.e.

$$\mathbf{K}\mathbf{K}^{-1} = \mathbf{K}^{-1}\mathbf{K} = \mathbf{I}. \tag{2.7}$$

Similarly any object defined in $\mathbb{LA}$ (in $\mathbb{GA}$) will be termed the *local (global) object* in $\mathbb{A}$.

2.3 Banach Double Algebra $\mathbb{E}_\sigma$

We introduce next a class of operators with exponentially decaying memories, called $\mathbb{E}_\sigma$, which forms a Banach double algebra.

For any $\sigma \geq 1$, introduce the function $\|\mathbf{K}\|_{(\sigma)}$ of $\mathbf{K} \in \mathbb{B}_e$ defined in terms of the kernel k of $\mathbf{K}$.

$$\|\mathbf{K}\|_{(\sigma)} := \sup_{t\in\mathbb{Z}} \sum_{\tau=-\infty}^{t} |k(t,\tau)|\sigma^{(t-\tau)} \tag{2.8}$$

(which equals $\sup_{t\in\mathbb{Z}} \|k(t,t-(\cdot))\|_{l^1_\sigma}$ and may be ∞). For any $\sigma > 1$, let $\underline{\mathbb{E}}_\sigma$ be the subspace of $\mathbb{B}_e$ consisting of operators $\mathbf{K}$ satisfying $\|\mathbf{K}\|_{(\sigma_k)} < \infty$ for some $\sigma_k > \sigma$, where σ_k depends on $\mathbf{K}$. Clearly $\|\cdot\|_{(\sigma)}$ is a norm for $\underline{\mathbb{E}}_\sigma$. Let $\mathbb{E}_\sigma$ be the closure of $\underline{\mathbb{E}}_\sigma$ with respect to $\|\cdot\|_{(\sigma)}$; i.e. $\mathbf{K} \in \mathbb{E}_\sigma$ iff $\|\mathbf{K}\|_{(\sigma)} < \infty$.

$\mathbb{E}_\sigma$ is a Banach space under that norm . We will show in Prop. 2.1 that $\underline{\mathbb{E}}_\sigma$ is a normed algebra under either product $*$ or $\otimes$, and therefore a normed double algebra. Similarly, $\mathbb{E}_\sigma$ is a Banach double algebra.

<u>**Proposition 2.1**</u>

The space $\underline{\mathbb{E}}_\sigma$ is a normed double algebra, and $\mathbb{E}_\sigma$ is a Banach double algebra, under the norm $\|\cdot\|_{(\sigma)}$ and either one of the products $*$ and $\otimes$.

□

Proof: In Appendix A.

2.4 The Local-Global Coupling

An essential issue in a normed double algebra is relations between its local and global properties. As we should proceed to see, a main purpose in introducing the normed double algebra symbolism is to address global properties, such as global stability and global performance, through local analysis and local synthesis, especially in the frequency domain. This strategy is valid only after the local-global coupling in the normed double algebra is established.

The local-global coupling consists of couplings between local and global summations, products and inversions. While local and global summations are always identical in our choice of local systems, the local-global product coupling is the main concern, which is expressed by the operator $\bigtriangledown$.

The *product-difference* binary operator $\bigtriangledown : \mathbb{B}_e \times \mathbb{B}_e \rightarrow \mathbb{B}_e$ is defined by

$$\mathbf{F} \bigtriangledown \mathbf{K} = \mathbf{F}\mathbf{K} - \mathbf{F} \otimes \mathbf{K}. \tag{2.9}$$

The $\bigtriangledown$ operator is also a pivotal element in the local-global inversion coupling, as shown in the following Inversion Lemmas I and II.

Let $\mathbb{A}$ be any Banach double subalgebra of $\mathbb{B}_e$. In particular $\mathbb{B}$ and $\mathbb{E}_\sigma$ are such Banach double subalgebras in which both global and local norms are taken to be $\|\cdot\|_{\mathbf{B}}$ and $\|\cdot\|_{(\sigma)}$.

We seek a relation between local and global invertibility in a Banach double algebra $\mathbb{A}$, as this determines stability. Observe first that $\mathbf{K} \in \mathbb{B}_e$ has a global inverse in $\mathbb{B}_e$ if and only if $k(t,t)$ is invertible in $\mathbb{C}^{n\times n}$ for each $t \in \mathbb{Z}$ (for then $\mathbf{K}$ decomposes into the sum of a memoryless invertible operator and a strictly causal one). Conditions for local invertibility in $\mathbb{B}_e$ are identical to global ones. However, in a general Banach double algebra $\mathbb{A}$ this is no longer true, and we get the following development.

Proposition 2.2 (Inversion Lemma I)

(a) If $\mathbf{K} \in \mathbb{A}$ has a local inverse, $\mathbf{K}^{\ominus} \in \mathbb{A}$, and $\|\mathbf{K}^{\ominus} \bigtriangledown \mathbf{K}\|_g < 1$ (or $\|\mathbf{K} \bigtriangledown \mathbf{K}^{\ominus}\|_g < 1$), then $\mathbf{K}$ has a global inverse in $\mathbb{A}$,

$$\mathbf{K}^{-1} = \left(\mathbf{K}^{\ominus}\mathbf{K}\right)^{-1}\mathbf{K}^{\ominus} = (\mathbf{I} + \mathbf{K}^{\ominus} \bigtriangledown \mathbf{K})^{-1}\mathbf{K}^{\ominus} \quad \text{if } \|\mathbf{K}^{\ominus} \bigtriangledown \mathbf{K}\|_g < 1, \tag{2.10}$$

$$\left(\text{or} \quad \mathbf{K}^{-1} = \mathbf{K}^{\ominus}\left(\mathbf{K}\mathbf{K}^{\ominus}\right)^{-1} = \mathbf{K}^{\ominus}(\mathbf{I} + \mathbf{K} \bigtriangledown \mathbf{K}^{\ominus})^{-1} \quad \text{if } \|\mathbf{K} \bigtriangledown \mathbf{K}^{\ominus}\|_g < 1\right).$$

Moreover $\|\mathbf{K}^{-1}\|_g$ is bounded by

$$\|\mathbf{K}^{-1}\|_g \le \|\mathbf{K}^{\ominus}\|_g(1 - \|\mathbf{K}^{\ominus} \bigtriangledown \mathbf{K}\|_g)^{-1}, \tag{2.11}$$

$$\left(\text{or} \quad \|\mathbf{K}^{-1}\|_g \le \|\mathbf{K}^{\ominus}\|_g(1 - \|\mathbf{K} \bigtriangledown \mathbf{K}^{\ominus}\|_g)^{-1}\right).$$

(b) Part (a) remains valid if global norms, products and inverses are interchanged with their local counterparts.

□

Proof:

(a) If $\|\mathbf{K}^{\ominus} \bigtriangledown \mathbf{K}\|_g < 1$, as

$$\mathbf{K}^{\ominus}\mathbf{K} = \mathbf{I} + \mathbf{K}^{\ominus}\mathbf{K} - \mathbf{K}^{\ominus} \otimes \mathbf{K} = \mathbf{I} + \mathbf{K}^{\ominus} \bigtriangledown \mathbf{K}, \tag{2.12}$$

$(\mathbf{K}^{\ominus}\mathbf{K})^{-1}$ exists in the Banach algebra $\mathbb{A}$ by the contraction principle. Therefore, $\left(\mathbf{K}^{\ominus}\mathbf{K}\right)^{-1}\mathbf{K}^{\ominus}\mathbf{K} = \mathbf{I}$, which proves that $\mathbf{K}$ has a global left inverse in $\mathbb{A}$. But as $\mathbf{K}^{\ominus} \in \mathbb{A}$, which is a subalgebra of $\mathbb{B}_e$, $\mathbf{K}$ has a global inverse in $\mathbb{B}_e$. Therefore the global left inverse $(\mathbf{K}^{\ominus}\mathbf{K})^{-1}\mathbf{K}^{\ominus}$ in $\mathbb{A}$ is in fact a global inverse in $\mathbb{A}$. The inequalities (2.11) are again the results of the contraction principle.

(b) The local counterpart is proved by interchanging $*$ and $\otimes$, $(\cdot)^{-1}$ and $(\cdot)^{\ominus}$, as well as $\|\cdot\|_g$ and $\|\cdot\|_l$.

Q.E.D.

The second Inversion Lemma addresses local-global inversion coupling in a normed subalgebra of $\mathbb{B}_e$. Let $\mathbb{A}$ be any normed subalgebra of $\mathbb{B}_e$ with norm $\|\cdot\|$ subject to the *Norm Characterization Property*

(NCP) $\quad \mathbf{K} \in \mathbb{A} \Longleftrightarrow \Pi_t\mathbf{K} \in \mathbb{A}\ \forall t \in \mathbb{Z}$ and $\sup_t \|\Pi_t\mathbf{K}\| < \infty$

where $\{\Pi_t, \quad t \in \mathbb{Z}\}$ is the family of truncation operators.

<u>Proposition 2.2′</u> (Inversion Lemma II)

Suppose $\mathbb{A}$ is an NDA satisfying (NCP) with respect to either $\|\cdot\|_g$ or $\|\cdot\|_l$.

(a) If $\mathbf{K} \in \mathbb{A}$ has a local inverse $\mathbf{K}^{\ominus} \in \mathbb{A}$, then it has a global inverse $\mathbf{K}^{-1} \in \mathbb{A}$ whenever either (1) $\|\mathbf{K}^{\ominus} \bigtriangledown \mathbf{K}\|_g < 1$, in which case

$$\mathbf{K}^{-1} = (\mathbf{K}^{\ominus}\mathbf{K})^{-1}\mathbf{K}^{\ominus} = (\mathbf{I} + \mathbf{K}^{\ominus} \bigtriangledown \mathbf{K})^{-1}\mathbf{K}^{\ominus} \tag{2.13a}$$

and

$$\|\mathbf{K}^{-1}\|_g \leq \|\mathbf{K}^{\ominus}\|_g \left\{1 - \|\mathbf{K}^{\ominus} \bigtriangledown \mathbf{K}\|_g\right\}^{-1}, \tag{2.13b}$$

or (2) $\|\mathbf{K} \bigtriangledown \mathbf{K}^{\ominus}\|_g < 1$, in which case

$$\mathbf{K}^{-1} = \mathbf{K}^{\ominus}(\mathbf{K}\mathbf{K}^{\ominus})^{-1} = \mathbf{K}^{\ominus}(\mathbf{I} + \mathbf{K} \bigtriangledown \mathbf{K}^{\ominus})^{-1} \tag{2.14a}$$

and

$$\|\mathbf{K}^{-1}\|_g \leq \|\mathbf{K}^{\ominus}\|_g \left\{1 - \|\mathbf{K} \bigtriangledown \mathbf{K}^{\ominus}\|_g\right\}^{-1}. \tag{2.14b}$$

(b) Part (a) remains valid under an interchange of global norms, products, and inverses with their local counterparts.

□

Proof:

(a) If $\mathbf{K}^{\ominus}$ is in $\mathbb{A}$, each matrix $k(t,t)$, $t \in \mathbb{Z}$, has an inverse in $\mathbb{C}^{n\times n}$, where k is the kernel of $\mathbf{K}$. Therefore, $\mathbf{K}^{-1}$ exists in $\mathbb{B}_e$. Furthermore, from the identities

$$\mathbf{K}^{\ominus}\mathbf{K} = \mathbf{I} + \mathbf{K}^{\ominus}\mathbf{K} - \mathbf{K}^{\ominus} \otimes \mathbf{K} = \mathbf{I} + \mathbf{K}^{\ominus} \bigtriangledown \mathbf{K}, \tag{2.15}$$

we get after multiplication by $\mathbf{K}^{-1}$ on the right,

$$\mathbf{K}^{-1} = \mathbf{K}^{\ominus} - (\mathbf{K}^{\ominus} \bigtriangledown \mathbf{K})\,\mathbf{K}^{-1}. \tag{2.16}$$

Subject to the norm characterization property (NCP) and causality of $\mathbf{K}$, the usual "small gain" argument applied in the global algebra $\mathbb{G}\mathbb{A}$ gives for all $t \in \mathbb{Z}$

$$\begin{aligned} \|\Pi_t\mathbf{K}^{-1}\|_g &\leq \|\Pi_t\mathbf{K}^{\ominus}\|_g \left\{1 - \|\Pi_t\mathbf{K}^{\ominus} \bigtriangledown \mathbf{K}\|_g\right\}^{-1} \\ &\leq \|\mathbf{K}^{\ominus}\|_g \left\{1 - \|\mathbf{K}^{\ominus} \bigtriangledown \mathbf{K}\|_g\right\}^{-1}, \end{aligned} \tag{2.17}$$

provided $\|\mathbf{K}^{\ominus} \bigtriangledown \mathbf{K}\|_g < 1$ in which case, since the bound (2.17) holds for all $t \in \mathbb{Z}$, $\mathbf{K}^{-1}$ is in $\mathbb{A}$. (2.13a,b) now follows from (2.15) and (2.17). The proof of (2.14a,b) is obtained similarly by multiplying $\mathbf{K}\mathbf{K}^{\ominus}$ by $\mathbf{K}^{-1}$ on the left.

(b) The proof remains valid under the specified interchange.

Q.E.D.

The condition that $\mathbf{K} \bigtriangledown \mathbf{G}$ is small will be related to the smallness of the commutant of $\mathbf{G}$ with the shift, i.e., to slow time variation, and later to slow variation in local transfer functions in the frequency domain. First, however, we summarize some elementary algebraic identities involving shift-invariant and memoryless operators.

2.5 Algebraic Preliminaries of $\bigtriangledown$ Operator

Let $\mathbf{T} \in \mathbb{B}_e$ denote the shift, $(\mathbf{T}u)(t) = u(t-1)$, $t \in \mathbb{Z}$. An operator $\mathbf{K} \in \mathbb{B}_e$ is *shift-invariant* iff its *commutant* $\mathbf{TK} - \mathbf{KT}$ vanishes.

Let $(\Delta\Pi)_\tau \in \mathbb{B}_e$, $\tau \in \mathbb{Z}$, denote the projection operator $(\Delta\Pi)_\tau = \Pi_\tau - \Pi_{\tau-1}$. An operator $\mathbf{F} \in \mathbb{B}_e$ has *no memory* if

$$(\Delta\Pi)_\tau \mathbf{F} = (\Delta\Pi)_\tau \mathbf{F} (\Delta\Pi)_\tau, \qquad \tau \in \mathbb{Z}. \tag{2.18}$$

The following properties are easy to prove.

Proposition 2.3

(1) For shift invariant $\mathbf{K}$ and arbitrary $\mathbf{G}$ in $\mathbb{B}_e$,

$$\mathbf{GK} = \mathbf{G} \otimes \mathbf{K}, \quad \text{i.e.} \quad \mathbf{G} \bigtriangledown \mathbf{K} = 0. \tag{2.19}$$

(2) All operators in $\mathbb{B}_e$ are locally shift-invariant, i.e.,

$$\mathbf{K} \otimes \mathbf{T} - \mathbf{T} \otimes \mathbf{K} = 0.$$

(3) For any $\mathbf{G}, \mathbf{H}$, and shift-invariant $\mathbf{K}$ in $\mathbb{B}_e$,

$$(\mathbf{G} \bigtriangledown \mathbf{H})\mathbf{K} = \mathbf{G} \bigtriangledown (\mathbf{HK}). \tag{2.20}$$

(4) If $\mathbf{F} \in \mathbb{B}_e$ has no memory, then for arbitrary $\mathbf{G}$, $\mathbf{H}$ in $\mathbb{B}_e$,

$$\mathbf{F} \otimes \mathbf{G} = \mathbf{F}\mathbf{G}, \quad \mathbf{F} \bigtriangledown \mathbf{G} = 0, \quad (\mathbf{F}\mathbf{G}) \bigtriangledown \mathbf{H} = \mathbf{F}(\mathbf{G} \bigtriangledown \mathbf{H}). \tag{2.21}$$

□

Proof:

(1) For shift invariant $\mathbf{K}$, $\mathbf{K}_t = \mathbf{K}$ for all t. Thus

$$(\mathbf{G}\mathbf{K})_t = (\mathbf{G}_t\mathbf{K})_t = (\mathbf{G}_t\mathbf{K}_t)_t = (\mathbf{G} \otimes \mathbf{K})_t$$

(2) It follows from the fact that $\mathbf{T}\mathbf{K}_t = \mathbf{K}_t\mathbf{T}$ for all $t \in \mathbb{Z}$.

(3) Since $\mathbf{K}$ is shift invariant, by (1) we obtain

$$\begin{aligned}
(\mathbf{G} \bigtriangledown \mathbf{H})\mathbf{K} &= (\mathbf{G}\mathbf{H} - \mathbf{G} \otimes \mathbf{H}) \otimes \mathbf{K} \\
&= (\mathbf{G}\mathbf{H}) \otimes \mathbf{K} - \mathbf{G} \otimes \mathbf{H} \otimes \mathbf{K} \\
&= \mathbf{G}\mathbf{H}\mathbf{K} - \mathbf{G} \otimes (\mathbf{H} \otimes \mathbf{K}) \\
&= \mathbf{G}(\mathbf{H}\mathbf{K}) - \mathbf{G} \otimes (\mathbf{H}\mathbf{K}) \\
&= \mathbf{G} \bigtriangledown (\mathbf{H}\mathbf{K})
\end{aligned} \tag{2.22}$$

(4) Trivial.

Q.E.D.

From Prop. 2.3 part (2), the term shift-invariant in $\mathbb{B}_e$ will be reserved for the global property.

Any $\mathbf{F} \in \mathbb{B}$ can be expressed as a linear combination of global powers of the shift,

$$\mathbf{F} \sim \sum_{r=0}^{\infty} \mathbf{F}^{(r)}\mathbf{T}^{r} \tag{2.23}$$

where $\mathbf{F}^{(r)} \in \mathbb{B}$, $r = 1, 2, \ldots$, are operators with no memory whose kernels satisfy: $\mathbf{F}^{(r)}(t,\tau) = f(t, t-r)$ when $\tau = t$, and 0 elsewhere; and the series converges in weak-l^1, defined as follows.

Definition

A sequence of operators $\mathbf{K}_m \in \mathbb{B}$ weakly-l^1 converges to $\mathbf{K} \in \mathbb{B}$ (as $m \to \infty$) iff given any $u \in l^\infty(-\infty,\infty)$ and any functional $\mathcal{F}$ (with kernel) in $l^1(-\infty,\infty)$, $\mathcal{F} : l^\infty(-\infty,\infty) \to \mathbb{C}$, the sequence $\mathcal{F}(\mathbf{K}_m u) \to \mathcal{F}(\mathbf{K}u)$ (as $m \to \infty$).

□

The weak-l^1 convergence of

$$\mathbf{F} \sim \sum_{r=0}^{\infty} \mathbf{F}^{(r)} \mathbf{T}^r$$

is proved by considering, for $u \in l^\infty$

$$\mathcal{G}\left(\mathbf{F} - \sum_{r=0}^{m-1} \mathbf{F}^{(r)} \mathbf{T}^r u\right) =: \mathcal{G}(\Delta_m),$$

where $\mathcal{G}$ is a functional with kernel g in l^1. By Lebesque's dominated convergence theorem,

$$\mathcal{G}(\Delta_m) = \sum_{t=-\infty}^{\infty} g(t) \sum_{\tau=m}^{\infty} f(t, t-\tau) u(\tau) \to 0$$

as $m \to \infty$, noting that

$$\left|\sum_{\tau=m}^{\infty} f(t, t-\tau) u(\tau)\right| \leq \|f\|_{l^1} \|u\|_{l^\infty}$$

and for every $t \in \mathbb{Z}$, $\sum_{\tau=m}^{\infty} f(t, t-\tau) u(\tau) \to 0$ as $m \to \infty$.

Remarks:

(a) The expression (2.23) means that $\mathbb{B}$ is a module spanned by powers of $\mathbf{T}$.

(b) The weak-l^1 convergence coincides with the weak operator convergence provided the domain of operators in $\mathbb{B}$ is taken to be A_0, the subspace of A consisting of signals x with finite starting time and $x(t) \to 0$ as $t \to \infty$, the dual space of A_0 being $l^1(-\infty, \infty)$.

We can now express the $\bigtriangledown$ operator in terms of commutants, after first observing that the commutant of an operator in $\mathbb{B}_e$ is precisely the difference between local and global products with the shift.

Proposition 2.4

$$\mathbf{T} \bigtriangledown \mathbf{K} = \mathbf{TK} - \mathbf{KT}, \quad \text{for } \mathbf{K} \in \mathbb{B}_e. \tag{2.24}$$

$$\mathbf{F} \bigtriangledown \mathbf{K} = \sum_{r=0}^{\infty} \mathbf{F}^{(r)}(\mathbf{T}^r \bigtriangledown \mathbf{K}), \quad \text{for } \mathbf{F},\ \mathbf{K} \in \mathbb{B}. \tag{2.25}$$

where $\mathbf{F}^{(r)}$ has no memory, and the series converges weakly-l^1, in $\mathbb{B}$.

□

Proof of Prop. 2.4:

By Prop. 2.3 parts (1) and (2),

$$\mathbf{T} \bigtriangledown \mathbf{K} = \mathbf{TK} - \mathbf{T} \otimes \mathbf{K} = \mathbf{TK} - \mathbf{K} \otimes \mathbf{T} = \mathbf{TK} - \mathbf{KT},$$

which proves (2.24).

From (2.23) we get

$$\begin{aligned} \mathbf{F} \bigtriangledown \mathbf{K} &= \left(\sum_{r=0}^{\infty} \mathbf{F}^{(r)} \mathbf{T}^r\right) \mathbf{K} - \left(\sum_{r=0}^{\infty} \mathbf{F}^{(r)} \mathbf{T}^r\right) \otimes \mathbf{K} \\ &= \sum_{r=0}^{\infty} \mathbf{F}^{(r)} \left(\mathbf{T}^r \mathbf{K} - \mathbf{T}^r \otimes \mathbf{K}\right) \\ &= \sum_{r=0}^{\infty} \mathbf{F}^{(r)} (\mathbf{T}^r \bigtriangledown \mathbf{K}) \end{aligned} \tag{2.26}$$

as $\mathbf{F}^{(r)}$ has no memory, and (2.25) follows. The weak-l^1 convergence of (2.25) follows from that of $\mathbf{FK}$ and $\mathbf{F} \otimes \mathbf{K}$, which can be easily proved.

Q.E.D.

(2.25) suggests that $\mathbf{F} \bigtriangledown \mathbf{K}$ will be small whenever $\mathbf{K}$ has a small commutant and the memory of $\mathbf{F}$ decays sufficiently fast, a motivation to work with $\mathbb{E}_\sigma$, i.e. systems with exponentially decaying memory.

2.6 Slowly Time-Varying Systems

Let $\mathbf{T} \in \mathbb{B}_e$ denote the shift, $(\mathbf{T}u)(t) = u(t-1)$, and $\mathbb{E}_\sigma$ the Banach double algebra defined in section 2.3. $\mathbf{K}$ *commutes with the shift approximately* in $\mathbb{E}_\sigma$, with rate $d_\sigma(\mathbf{K}) \geq 0$ if

$$d_\sigma(\mathbf{K}) := \|\mathbf{TK} - \mathbf{KT}\|_{(\sigma)} \leq \|\mathbf{K}\|_{(\sigma)}. \tag{2.27}$$

Although the interest here is primarily in H^∞-frequency domain conditions for slowly time-variation in the sense of (2.27), we note some time-domain results.

If $\mathbf{K}$ commutes with the shift approximately in $\mathbb{E}_\sigma$, and $\mathbf{F} \in \mathbb{E}_\sigma$ has a kernel f, we have the estimates:

<u>**Proposition 2.5**</u>

(a) If $\sup_t \sum_{\tau=0}^{\infty} |f(t, t-\tau)| \tau \sigma^\tau := \gamma < \infty$ then

$$\|\mathbf{F} \bigtriangledown \mathbf{K}\|_{(\sigma)} \leq \gamma d_\sigma(\mathbf{K}) \sigma^{-1}. \tag{2.28}$$

(b) If $\mathbf{F} \in \mathbb{E}_{\sigma_1}, \sigma_1 > \sigma$, then

$$\|\mathbf{F} \nabla \mathbf{K}\|_{(\sigma)} \leq \|\mathbf{F}\|_{(\sigma_1)} d_\sigma(\mathbf{K}) \ \ (e \ \ln(\sigma_1/\sigma))^{-1} \sigma^{-1}. \tag{2.29}$$

(c) For $\mathbf{F}$ and $\mathbf{K}$ in $\mathbb{E}_\sigma$

$$d_\sigma(\mathbf{FK}) \leq \|\mathbf{F}\|_{(\sigma)} d_\sigma(\mathbf{K}) + \|\mathbf{K}\|_{(\sigma)} d_\sigma(\mathbf{F}).$$

(d)

$$d_\sigma(\mathbf{K}) = \sigma \sup_t \|k_t - k_{t-1}\|_{l^1_\sigma}.$$

□

Proof: In Appendix A.

Most classical frozen-time stability conditions for slowly time-varying systems can be encompassed in a statement relating the existence of local and global inverses. Time domain conditions are contained in the following.

Corollary 2.1

If $\mathbf{G}$ and $\mathbf{K}$ are in $\mathbb{E}_{\sigma_0}$ ($\sigma_0 > 1$), and either $\mathbf{G}$ has no memory or $\mathbf{K}$ is shift-invariant, then existence of the local inverse $(\mathbf{I} + \mathbf{G} \otimes \mathbf{K})^{\ominus}$ in $\mathbb{E}_{\sigma_0}$ implies that of the global inverse $(\mathbf{I} + \mathbf{GK})^{-1}$ in $\mathbb{B}$, provided that

$$d_1(\mathbf{G} \otimes \mathbf{K}) \leq (e \ ln(\sigma_0)) \left\|(\mathbf{I} + \mathbf{G} \otimes \mathbf{K})^{\ominus} \otimes [(1-\alpha)\mathbf{I} - \alpha \mathbf{G} \otimes \mathbf{K}]\right\|^{-1}_{(\sigma_0)} \tag{2.30}$$

for some $\alpha \in \mathbb{R}$.

□

Proof:

The assumption that $\mathbf{G}$ is memoryless or $\mathbf{K}$ is shift-invariant implies that $\mathbf{G} \otimes \mathbf{K} = \mathbf{GK}$ and, by the Inversion Lemma 2.2, the Corollary is true provided that

$$\| (\mathbf{I} + \mathbf{G} \otimes \mathbf{K})^{\ominus} \bigtriangledown (\mathbf{I} + \mathbf{G} \otimes \mathbf{K}) \|_{\mathbf{B}} < 1. \tag{2.31}$$

As $\mathbf{I}$ has no memory and is shift invariant, $\mathbf{A} \bigtriangledown \mathbf{B} = (\mathbf{A} - \alpha\mathbf{I}) \bigtriangledown (\mathbf{B} - \mathbf{I})$ for any $\alpha \in \mathbb{R}$, and (2.31) is equivalent to

$$\left\| \left[(\mathbf{I} + \mathbf{G} \otimes \mathbf{K})^{\ominus} \otimes ((1-\alpha)\mathbf{I} - \alpha\mathbf{G} \otimes \mathbf{K}) \right] \bigtriangledown (\mathbf{G} \otimes \mathbf{K}) \right\|_{\mathbf{B}} < 1. \tag{2.32}$$

By (2.29) (with $\sigma_1 = \sigma_0$, $\sigma = 1$), (2.30) is sufficient for (2.31).

Q.E.D.

Remark: Some of classical frozen-time stability conditions are stated with $\alpha = 1$. Unfortunately, Corollary 2.1 involves the estimation of the l_1-kernel norm of an inverse, which is seldom an analytically tractable object, and we therefore move on to consider alternative methods in the frequency domain. First, however, an example of Desoer [Des1], which is nicely tractable, is included to illustrate the symbolism.

Example 2.1:

Stability of the difference equation

$$x(t) = G_t x(t-1) + F_t u(t), \qquad t \in \mathbb{Z} \tag{2.33}$$

$x(t), u(t) \in \mathbb{R}^n$; $G_t, F_t \in \mathbb{R}^{n \times n}$, is to be deduced from its local properties. If $u \in l^{\infty}(-\infty, \infty)$ and G_t, F_t are bounded functions of $t \in \mathbb{Z}$, (2.33) can be expressed in an operator form,

$$x = \mathbf{GT}x + \mathbf{F}u \tag{2.34}$$

with $\mathbf{G}, \mathbf{F} \in \mathbb{B}$, which has the solution $x = (\mathbf{I} - \mathbf{GT})^{-1}\mathbf{F}u$ where $(\mathbf{I} - \mathbf{GT})^{-1} \in \mathbb{B}_e$. The a priori assumption that the frozen-time system is exponentially stable means that $(\mathbf{I} + \mathbf{G} \otimes \mathbf{T})^{\ominus}$ is in $\mathbb{E}_{\sigma_0}$ for some σ_0, $1 < \sigma_0 < \lambda^{-1}$, where λ is the supremum of the spectral radii of the matrices $G_t, t \in \mathbb{Z}$, $\lambda < 1$ being necessary and sufficient for G_t to be stable.

The actual system is l^{∞}-stable if the global inverse $(\mathbf{I} + \mathbf{GT})^{-1}$ is in $\mathbb{B}$ which, by Corollary 2.1 ($\alpha = 1$), is ensured whenever the variation rate of $\mathbf{G}$ satisfies

$$d_{(1)}(\mathbf{G}) < (e \ln \sigma_0) \left\| (\mathbf{I} + \mathbf{G} \otimes \mathbf{T})^{\ominus} \otimes \mathbf{G} \right\|_{(\sigma_0)}^{-1} \tag{2.35}$$

for some $\sigma_0 \in (1, \lambda^{-1})$. The norm in (2.35) can be estimated as in [Des1] where it it shown that for any ν, $\lambda < \nu < 1$, (as $\mathbf{G}$ has no memory and G_t is finite dimensional), $\sup_{r \in \mathbb{Z}} \|(\mathbf{G}/\nu)^r\|_{(1)} =: \beta$ is a finite constant depending on ν. Therefore

$$\begin{aligned} \| (\mathbf{I} + \mathbf{G} \otimes \mathbf{T})^{\ominus} \otimes \mathbf{G}\|_{(\sigma_0)} &\leq \sup_{t \in \mathbb{Z}} \sum_{i=0}^{\infty} |G_t^{i+1}| \sigma_0^i \\ &\leq \beta \sum_{i=0}^{\infty} \nu^{i+1} \sigma_0^i \\ &\leq \beta \nu (1 - \sigma_0 \nu)^{-1} \end{aligned} \tag{2.36}$$

where σ_0 is chosen such that $\sigma_0 \nu < 1$.

The choice $\nu = \frac{1}{2}(1+\lambda)$, $\sigma_0 = \frac{1}{2}(1+\nu^{-1})$ and observation that $\ln \sigma_0 > 1 - \sigma_0^{-1}$ give a sufficient condition for stability, $d_{(1)}(\mathbf{G}) < \frac{e}{2\beta\nu} \frac{(1-\nu)^2}{(1+\nu)}$. If $\lambda = 1 - 2\varepsilon$ and $\nu = 1 - \varepsilon$, the rate bound is better than $\frac{e\varepsilon^2}{4\beta}$.

2.7 Nests of Normed Double Algebras

In preparation for the frequency-domain results, let's axiomatically introduce a concept which is common to the rest of the theory, and is exemplified by the parametrized family $\{\mathbb{E}_\sigma\}$ of double algebras.

Definition

A *nest* of NDAs is a one-parameter family $\{\mathbb{A}_\sigma,\ \sigma \geq 1\}$ of normed double subalgebras of $\mathbb{B}$ with these properties:

(1) $\{\mathbb{A}_\sigma\}$ is monotone by inclusion,

$$\mathbb{A}_{\sigma_0} \subset \mathbb{A}_{\sigma_1} \subset \mathbb{B} \tag{2.37}$$

whenever $1 \leq \sigma_1 \leq \sigma_0$, inclusion being strict if $1 \neq \sigma_1 \neq \sigma_0$.

(2) For $\mathbf{K} \in \mathbb{A}_{\sigma_1}$, $\sigma_1 > \sigma$, the local and global norms, $\|\mathbf{K}\|^l_\sigma$ and $\|\mathbf{K}\|^g_\sigma$, depend continuously on σ, and are monotone in σ, i.e.,

$$\|\mathbf{K}\|^g_{\mathbf{B}} \leq Const.\|\mathbf{K}\|^g_\sigma \leq Const.\|\mathbf{K}\|^g_{\sigma_1}, \tag{2.38a}$$

$$\|\mathbf{K}\|^l_{\mathbf{B}} \leq Const.\|\mathbf{K}\|^l_\sigma \leq Const.\|\mathbf{K}\|^l_{\sigma_1}, \tag{2.38b}$$

the constants being independent of $\mathbf{K}$.

(3) Each NDA $\mathbb{A}_\sigma$ is either a Banach double algebra or characterized by the global norm $\|\cdot\|^g_\sigma$ according to Property (NCP) (Section 2.4).

□

Let us show that $\{\mathbb{E}_\sigma,\ 1 \le \sigma < \sigma_0\}$ is a nest of NDAs. It obviously satisfies the conditions (1), (3) and (2.38a,b). As for the continuity of $\|\cdot\|_{(\sigma)}$ with respect to σ, observe that for $\mathbf{K} \in \mathbb{E}_{\sigma_1}$ and $1 \le \sigma < \sigma' < \sigma_1$,

$$
\begin{aligned}
&\|k_t\|_{l^1_{\sigma'}} - \|k_t\|_{l^1_\sigma} \\
&= \sum_{\tau=0}^{\infty} |k_t(\tau)|(\sigma'^\tau - \sigma^\tau) \\
&= \sum_{\tau=0}^{\infty} |k_t(\tau)|\sigma_1^\tau \left(\frac{\sigma'}{\sigma_1}\right)^\tau \left(1 - \left(\frac{\sigma}{\sigma'}\right)^\tau\right) \\
&= \sum_{\tau=0}^{\infty} |k_t(\tau)|\sigma_1^\tau \left(\frac{\sigma'}{\sigma_1}\right)^\tau \left(1 - \frac{\sigma}{\sigma'}\right)\left(1 + \left(\frac{\sigma}{\sigma'}\right) + \cdots + \left(\frac{\sigma}{\sigma'}\right)^{\tau-1}\right) \qquad (2.39) \\
&\le \left(1 - \frac{\sigma}{\sigma'}\right) \sum_{\tau=0}^{\infty} |k_t(\tau)|\sigma_1^\tau \left(\frac{\sigma'}{\sigma_1}\right)^\tau \tau \\
&\le \left(1 - \frac{\sigma}{\sigma'}\right)\left(e\ \ln\left(\frac{\sigma_1}{\sigma'}\right)\right)^{-1} \|k_t\|_{l^1_{\sigma_1}} \\
&\le \left(1 - \frac{\sigma}{\sigma'}\right)\left(e\ \ln\left(\frac{\sigma_1}{\sigma'}\right)\right)^{-1} \|\mathbf{K}\|_{(\sigma_1)}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\|\mathbf{K}\|_{(\sigma')} - \|\mathbf{K}\|_{(\sigma)} \\
&\le \left(1 - \frac{\sigma}{\sigma'}\right)\left(e\ \ln\left(\frac{\sigma_1}{\sigma'}\right)\right)^{-1} \|\mathbf{K}\|_{(\sigma_1)} \to 0 \quad \text{as } \sigma' \to \sigma
\end{aligned}
$$

which proves the required continuity.

The NDAs in this thesis all satisfy an additional inequality, linking rates of change in local norm to global behavior, which however is not part of the nest definition:

$$
\|\mathbf{F} \nabla \mathbf{K}\|_\sigma^g \le Const. \|\mathbf{F}\|_{\sigma_0}^l d_{\|\cdot\|_{\sigma_0}^l}(\mathbf{K}) \qquad \sigma_0 > \sigma \qquad (2.40)
$$

the constant being independent of $\mathbf{F},\mathbf{K}$.

An extension of the Inversion Lemmas to certain nest will be required.

Proposition 2.6 (Extended Inversion Lemma)

If $\{\mathbb{A}_\sigma\}$ is a nest of NDAs, and $\underline{\mathbb{A}}_\sigma$ is the normed double subalgebra of $\{\mathbb{A}_\sigma\}$,

$$\underline{\mathbb{A}}_\sigma = \left\{ \mathbf{K} \in \mathbb{A}_\sigma : \quad \mathbf{K} \in \mathbb{A}_{\sigma_0} \quad \text{for} \quad \text{some} \quad \sigma_0 > \sigma \right\}$$

where σ_0 may depend on $\mathbf{K}$, then the Inversion Lemmas hold with $\mathbb{A} = \underline{\mathbb{A}}_\sigma$.

□

Proof:

If $\mathbf{K}$ and $\mathbf{K}^\ominus$ are in $\underline{\mathbb{A}}_\sigma$, they are certainly in $\mathbb{A}_\sigma$. By Inversion Lemma I or II, $\mathbf{K}^{-1}$ is in $\mathbb{A}_\sigma$ and satisfies the inequalities (2.11) or (2.13b). All that remains to be shown is that $\mathbf{K}^{-1}$ is actually in $\underline{\mathbb{A}}_\sigma$. Under our hypothesis, $\mathbf{K}$ and $\mathbf{K}^\ominus$ are in some $\mathbb{A}_{\sigma_0}$, $\sigma_0 > \sigma$. There exists some σ_1, $\sigma < \sigma_1 < \sigma_0$, such that either $\|\mathbf{K} \bigtriangledown \mathbf{K}^\ominus\|_\sigma^g < 1$ implies $\|\mathbf{K} \bigtriangledown \mathbf{K}^\ominus\|_{\sigma_1}^g < 1$ by continuity of $\|\cdot\|_\sigma^g$ or, alternatively, $\|\mathbf{K}^\ominus \bigtriangledown \mathbf{K}\|_{\sigma_1}^g < 1$. In either case, the Inversion Lemmas imply that $\mathbf{K}^{-1} \in \mathbb{A}_{\sigma_1}$, and therefore $\mathbf{K}^{-1} \in \underline{\mathbb{A}}_\sigma$.

Q.E.D.

Chapter 3 Frequency Domain Auxiliary Norms

We would like to employ local frequency domain methods to obtain bounds on global time domain behavior. For this purpose, two kinds of auxiliary norms will now be introduced; one on $\mathbb{E}_\sigma$, evaluated in the frequency domain and computable in local operations; the second on $\mathbb{B} \supset \mathbb{E}_\sigma$, evaluated in the time-domain and computable in global operations. For slowly time-varying systems the two will be related.

3.1 The Local Algebra $\mathbb{LE}_\sigma$

As described in Section 2.1, for any $\mathbf{K} \in \mathbb{B}$ (or $\mathbb{E}_\sigma$), its local transfer functions satisfy $\widehat{\mathbf{K}}_\tau \in H^\infty$ (or H^∞_σ) for all τ and

$$\sup_\tau \|\widehat{\mathbf{K}}_\tau\|_{H^\infty} < \infty,$$

$$\left(\text{or } \sup_\tau \|\widehat{\mathbf{K}}_\tau\|_{H^\infty_\sigma} < \infty\right).$$

Although operators in $\mathbb{B}$ (or in $\mathbb{E}_\sigma$) have local transfer functions in H^∞ (or in H^∞_σ) the reverse is not true. $\mathbb{B}$ and $\mathbb{E}_\sigma$ have no precise characterizations in terms of transfer functions. To deal with operators initially specified in the frequency domain, we turn instead to the normed double subalgebras $\underline{\mathbb{E}}_\sigma$ of $\mathbb{B}$, defined in Section 2.3, which have such a specification.

For $2 \leq p \leq \infty$, $\sigma \geq 1$, define the functions of operators $\mathbf{K} \in \mathbb{E}_\sigma$

$$\mu_\sigma^{(p)}(\mathbf{K}) := \sup_{t \in \mathbb{Z}} \|\widehat{\mathbf{K}}_t\|_{H_\sigma^p}, \tag{3.1}$$

and in the case $p = \infty$ omit the superscript, i.e. $\mu_\sigma(\mathbf{K}) := \mu_\sigma^{(\infty)}(\mathbf{K})$.

Proposition 3.1

(a) The space $\underline{\mathbb{E}}_\sigma$ consists precisely of those operators $\mathbf{K} \in \mathbb{B}_e$ with the property that for some $\sigma_0 > \sigma$ each local transform $\widehat{\mathbf{K}}_t, t \in \mathbb{Z}$, is in $H_{\sigma_0}^\infty$ and $\mu_{\sigma_0}(\mathbf{K}) < \infty$.

(b) $< \mathbf{L}\underline{\mathbb{E}}_\sigma, \mu_\sigma(\cdot) >$ is a normed algebra.

□

Proof of Prop. 3.1:

(a) The hypothesis that $\mathbf{K} \in \underline{\mathbb{E}}_\sigma$ implies that for some $\sigma_0 > \sigma$, the kernel k_t of $\mathbf{K}_t$ is in $l_{\sigma_0}^1, \forall t \in \mathbb{Z}$, and $\|\mathbf{K}\|_{(\sigma_0)} < \infty$. Therefore the conclusion that $\widehat{\mathbf{K}}_t \in H_{\sigma_0}^\infty, \mu_{\sigma_0}(\mathbf{K}) < \infty$ is true.

Conversely, $\mu_{\sigma_0}(\mathbf{K}) < \infty$ implies that $\mu_{\sigma_0}^{(2)}(\mathbf{K}) \leq \mu_{\sigma_0}(\mathbf{K}) < \infty$. For each $t \in \mathbb{Z}$, $\widehat{\mathbf{K}}_t \in L_{\sigma_0}^2$ implies that $k_t \in l_{\sigma_0}^2$ and, by Parseval's Theorem, $\|k_t\|_{l_{\sigma_0}^2} = \|\widehat{\mathbf{K}}_t\|_{H_{\sigma_0}^2}$. For any σ_1 in (σ, σ_0) we have

$$\begin{aligned} \|k_t\|_{l_{\sigma_1}^1} &:= \sum_{i=0}^\infty \left| k_t(i)\sigma_1^i \right| \\ &\leq \left\{ \sum_{i=0}^\infty \left| k_t(i)\sigma_0^i \right|^2 \sum_{i=0}^\infty (\sigma_1/\sigma_0)^{2i} \right\}^{1/2} \\ &= \|k_t\|_{l_{\sigma_0}^2} \kappa(\sigma_1/\sigma_0) \\ &\leq \kappa(\sigma_1/\sigma_0) \mu_{\sigma_0}(\mathbf{K}) \end{aligned} \tag{3.2}$$

and as this is true $\forall t \in \mathbb{Z}$, $\mathbf{K} \in \underline{\mathbb{E}}_\sigma$.

(b) It follows from the inequality

$$\|(\widehat{\mathbf{G} \otimes \mathbf{K}})_t\|_{H_\sigma^\infty} \leq \|\widehat{\mathbf{G}}_t\|_{H_\sigma^\infty} \|\widehat{\mathbf{K}}_t\|_{H_\sigma^\infty}. \tag{3.3}$$

Q.E.D.

Although the space $\underline{\mathbb{E}}_\sigma$ has equivalent descriptions in terms of the kernel norm $\|\cdot\|_{(\sigma)}$ and transform norm $\mu_\sigma(\cdot)$, $\mu_\sigma(\cdot)$ yields a closure of $\underline{\mathbb{E}}_\sigma$ different from $\mathbb{E}_\sigma$, and is well behaved with respect to the local product only.

Henceforth assume $\underline{\mathbb{E}}_\sigma$ to be equipped with the global norm $\|\cdot\|_{(\sigma)}$ and local norm $\mu_\sigma(\cdot)$. Let $\mathbb{L}\underline{\mathbb{E}}_\sigma$ denote the restriction of $\underline{\mathbb{E}}_\sigma$ to its local product $\otimes$, and $\overline{\mathbb{L}\mathbb{E}_\sigma}$ the subalgebra of $\mathbf{B}_e$ consisting of operators $\mathbf{K} \in \mathbf{B}_e$ with the property that $\widehat{\mathbf{K}}_t \in H_\sigma^\infty$ for all $t \in \mathbb{Z}$ and $\mu_\sigma(\mathbf{K}) < \infty$. $\overline{\mathbb{L}\mathbb{E}_\sigma}$ will be abbreviated as $\overline{\mathbb{E}}_\sigma$ when the local product is not emphasized. We have the following obvious relations.

<u>**Proposition 3.2**</u>

For $\sigma > 1$,

$$\underline{\mathbb{E}}_\sigma \subset \mathbb{E}_\sigma \subset \overline{\mathbb{E}}_\sigma \subset \mathbb{B} \subset \overline{\mathbb{E}}_1 \quad \text{and} \quad \overline{\mathbb{E}}_{\sigma_0} \subset \underline{\mathbb{E}}_\sigma \quad \text{for} \quad \sigma < \sigma_0. \tag{3.4}$$

□

Remark : There is a precise time-domain condition for an operator $\mathbf{K}$ to be in $\mathbb{E}_\sigma$, namely that its kernel k satisfies $k(t, t - \cdot) \in l^1_\sigma$ uniformly in $t \in \mathbb{Z}$. However in the frequency domain there is only the sufficient condition that $\widehat{\mathbf{K}}_t \in H^\infty_{\sigma_0}$ for some $\sigma_0 > \sigma$, uniformly in t, which amounts to assuming that $\mathbf{K} \in \underline{\mathbb{E}}_\sigma$.

Of the three spaces $\underline{\mathbb{E}}_\sigma, \mathbb{E}_\sigma$ and $\overline{\mathbb{E}}_\sigma$, only $\underline{\mathbb{E}}_\sigma$ is a normed double algebra with both time and frequency domain characterization. Therefore, for problems requiring mixed local and global operations, $\underline{\mathbb{E}}_\sigma$ will be the algebra of choice. As we should proceed to see, local frequency norms $\mu_\sigma(\cdot)$ in $\underline{\mathbb{E}}_\sigma$ provide approximants and bounds to global norm behaviors.

3.2 The Global Algebra $\mathbb{GB}$ With An Auxiliary Time Domain Norm

Ultimately the interest here is in the time domain behavior of operators in $\mathbb{GE}_\sigma$ viewed as mappings from inputs in $l^\infty(-\infty, \infty) =: A$ to outputs in A, i.e., viewed as elements of the larger algebra $\mathbb{GB}$ of such mappings, $\mathbb{GB} \supset \mathbb{GE}_\sigma$.

The normed double algebra $\underline{\mathbb{E}}_\sigma$ has equivalent descriptions in the time domain via the kernel norm $\|\cdot\|_{(\sigma)}$ and in the frequency domain via the transfer-function norm $\mu_\sigma(\cdot)$. However, these norms are incommensurate, and inconvenient for the estimation of $l^\infty(-\infty, \infty)$ time domain behavior from local frequency domain properties, unlike, e.g., the time-invariant situation in $l^2(-\infty, \infty)$, where Parseval's theorem provides an isometry between kernel and transform representations. Instead, we introduce an auxiliary time-domain norm on $\mathbb{B}$, denoted by $\|\cdot\|_{a(\sigma)}$, which is equivalent to $l^\infty(-\infty, \infty)$ induced operator norm on $\mathbb{B}$.

This topology was introduced by Zames in [Zam1] and applied to slowly time-varying systems by Freedman and Zames in [Fre]. This weighs down the remote past and is accessible from the frequency domain, as follows.

Equip the space A (i.e., $l^\infty(-\infty,\infty)$) with the family of auxiliary norms

$$\|u\|_{a(\sigma)} = \kappa_\sigma^{-1} \sup_{t\in\mathbb{Z}} \left(\sum_{\tau=-\infty}^{t} \left| u(\tau)\sigma^{-(t-\tau)} \right|^2 \right)^{1/2}, \tag{3.5}$$

where $\kappa_\sigma := \left(\sum_{n=0}^{\infty} \sigma^{-2n}\right)^{1/2} = \left(1-\sigma^{-2}\right)^{1/2}$ depending on the parameter σ, $1 < \sigma \leq \infty$. Here $\|u\|_{a(\infty)}$ is interpreted as equal to $\|u\|_{l^\infty}$.

The $\|u\|_{a(\sigma)}$ norm is the l^∞ norm of the convolution of u with an exponential, smoothing kernel, the kernel normalized to have unit $l^2(-\infty,\infty)$ norm. The norms in this family obtained for various σ are equivalent to each other, i.e., for any $\sigma_2 > \sigma_1 > 1$,

$$\|u\|_{a(\sigma_1)} \leq Const. \|u\|_{a(\sigma_2)} \leq Const. \|u\|_{a(\sigma_1)} \tag{3.6}$$

and to the l^∞ norm; indeed,

$$\|u\|_{a(\sigma)} \leq \|u\|_{l^\infty} \leq \kappa_\sigma \|u\|_{a(\sigma)}. \tag{3.7}$$

Each $\|\cdot\|_{a(\sigma)}$ norm on A induces an auxiliary operator norm on the linear space $\mathbb{B}$ of operators; for $\mathbf{K} \in \mathbb{B}$, $\|\mathbf{K}\|_{a(\sigma)} := \sup\{\|\mathbf{K}u\|_{a(\sigma)} : u \in A, \|u\|_{a(\sigma)} \leq 1\}$. Assume the space $\mathbb{B}$ as well as the global algebra $\mathbb{GB}$ to be equipped with this family of auxiliary operator norms which, again, are equivalent to each other and to the principal norm $\|\cdot\|_{\mathbf{B}}$ on $\mathbb{B}$. The latter is the l^∞-induced norm, on operators $\mathbf{K} \in \mathbb{B}$, which equals the supreme of the l^1 norms of their kernels k_t, i.e.,

$$\|\mathbf{K}\|_{\mathbf{B}} = \|\mathbf{K}\|_{a(\infty)} = \sup_t \|k_t\|_{l^1}. \tag{3.8}$$

The following is obvious.

Proposition 3.3

The global algebra $\mathbb{GB}$ is a Banach algebra under the $\|\cdot\|_{a(\sigma)}$ norm, $1 < \sigma \leq \infty$. (However, as the constant in the inequality $\|\mathbf{K} \otimes \mathbf{G}\|_{a(\sigma)} \leq Const.\|\mathbf{K}\|_{a(\sigma)}\|\mathbf{G}\|_{a(\sigma)}$ differs from unity, the local algebra $\mathbb{LB}$ is not a normed algebra under $\|\cdot\|_{a(\sigma)}$.) □

The auxiliary norms are bounds on an operator which are uniform in time. Occasionally, we shall relate these to certain finer bounds emphasizing particular times. The (exponentially weighted) *recent past seminorms* $\|\mathbf{K}\|_{a(\sigma;t)}$ of $\mathbf{K} \in \mathbb{B}$ are defined by

$$\|\mathbf{K}\|_{a(\sigma;t)} := \kappa_\sigma^{-1}\sigma^{-t} \sup\{\|\Pi_t \mathbf{K} u\|_{l^2_\sigma} : u \in A, \|\mathbf{u}\|_{a(\sigma)} \leq 1\}, t \in \mathbb{Z}.$$

Then $\|\mathbf{K}\|_{a(\sigma)} = \sup_t \|\mathbf{K}\|_{a(\sigma;t)}$.

3.3 Slowly Time-Varying Transfer Functions

Our point in introducing the auxiliary $\|\cdot\|_{a(\sigma)}$ and $\mu_\sigma(\cdot)$ norms, is that the former is tractable for systems with persistent time-domain perturbations, the latter is computable in the frequency domain and, as we shall proceed to show, the latter gives an approximation on the former, i.e., $\mu_\sigma(\mathbf{K}) - \alpha \leq \|\mathbf{K}\|_{a(\sigma)} \leq \mu_\sigma(\mathbf{K}) + \beta$ where $\beta \to 0$ as the variation rate ρ of the local transfer functions of $\mathbf{K}$ approaches zero (in the sense of Section 3.6), and $\alpha \to 0$ as $\rho \to 0$ and $\sigma \to 1$.

Definition:

An operator $\mathbf{K} \in \mathbf{E}_\sigma$ has a *slowly time-varying transfer function with rate* $\partial_\sigma^{(p)}(\mathbf{K})$ if

$$\partial_\sigma^{(p)}(\mathbf{K}) := \sup_{t \in \mathbb{Z}} \|\widehat{\mathbf{K}}_t - \widehat{\mathbf{K}}_{t-1}\|_{H_\sigma^p} \leq \mu_\sigma^{(p)}(\mathbf{K}) \qquad 2 \leq p \leq \infty. \tag{3.9}$$

Denote $\partial_\sigma^{(\infty)}(\mathbf{K})$ by $\partial_\sigma(\mathbf{K})$. ($\partial_\sigma^{(p)}(\mathbf{K})$ will later be assumed small in relation to certain additional constants.)

□

For small enough $\partial_\sigma^{(p)}(\mathbf{K})$, the variation rate of the local transfer function of $\mathbf{K}$ provides a tractable sufficient condition for $\mathbf{K}$ to commute approximately with the shift, as well as a computable bound on the time-domain rate $d_\sigma(\mathbf{K})$.

Proposition 3.4

For any $\sigma_0 > \sigma \geq 1$ and $p \geq 2$, if $\mathbf{K} \in \mathbf{E}_{\sigma_0}$ has a slowly time-varying local transfer function with rate $\partial_{\sigma_0}^{(p)}(\mathbf{K})$, then for $\partial_{\sigma_0}^{(p)}(\mathbf{K})$ small enough, $\mathbf{K}$ commutes approximately with the shift in $\mathbf{E}_\sigma$ (i.e. (2.27) holds), and

$$d_\sigma(\mathbf{K}) \leq \sigma \kappa_{(\sigma_0/\sigma)} \partial_{\sigma_0}^{(p)}(\mathbf{K}). \tag{3.10}$$

□

Proof of Prop. 3.4:

By Prop. 2.5 part (d),

$$\|\mathbf{TK} - \mathbf{KT}\|_{(\sigma)} = \sigma \sup_t \|k_t - k_{t-1}\|_{l_\sigma^1}. \tag{3.11}$$

$$
\begin{aligned}
\|k_t - k_{t-1}\|_{l^1_\sigma} &= \sum_{\tau=0}^{\infty} |k_t(\tau) - k_{t-1}(\tau)|\, \sigma^\tau \\
&= \sum_{\tau=0}^{\infty} |k_t(\tau) - k_{t-1}(\tau)|\, \sigma_0^\tau \,(\sigma/\sigma_0)^\tau \\
&\leq \|k_t - k_{t-1}\|_{l^2_{\sigma_0}} \kappa_{(\sigma_0/\sigma)} \qquad \text{by Schwartz Inequality} \\
&= \|\widehat{\mathbf{K}}_t - \widehat{\mathbf{K}}_{t-1}\|_{H^2_{\sigma_0}} \kappa_{(\sigma_0/\sigma)} \qquad \text{by Parseval's Theorem} \\
&\leq \|\widehat{\mathbf{K}}_t - \widehat{\mathbf{K}}_{t-1}\|_{H^p_{\sigma_0}} \kappa_{(\sigma_0/\sigma)} \qquad p \geq 2.
\end{aligned} \tag{3.12}
$$

The proof is completed after taking $\sup_t$ of both sides.

Q.E.D.

The time domain norm $\|\cdot\|_{a(\sigma)}$ is bounded by frequency domain auxiliary norms $\mu_\sigma(\cdot)$ through inequalities listed in the following propositions. The first one gives inequalities not dependent on slow variation:

<u>**Proposition 3.5**</u> **(rate-independent bounds on K)**

For any $\mathbf{K} \in \mathbb{E}_\sigma$ and $\tau \in \mathbb{Z}$, the following inequalities hold, $(1 < \sigma \leq \infty, 2 \leq p \leq \infty)$.

$$\|\mathbf{K}_\tau\|_{a(\sigma;t)} \leq \|\widehat{\mathbf{K}}_\tau\|_{H^\infty_\sigma}. \tag{3.13}$$

$$\|\mathbf{K}_\tau\|_{a(\infty)} \leq \kappa_\sigma \|\widehat{\mathbf{K}}_\tau\|_{H^p_\sigma}. \tag{3.14}$$

$$\|\mathbf{K}\|_{a(\sigma;t)} \leq \kappa_\sigma \mu_\sigma^{(p)}(\mathbf{K}) \leq \kappa_\sigma \mu_\sigma(\mathbf{K}). \tag{3.15}$$

□

Proof: In Appendix B.

The following inequalities depend on rates of time variation. All operators $\mathbf{G}, \mathbf{K}, \mathbf{F}$ in Props. 3.6-3.9 are assumed to be in $\mathbb{E}_\sigma$. Let $2 \le p \le \infty$, $1 < \sigma \le \infty$, $\kappa'_\sigma := \left(\sum_{i=1}^{\infty} i^2 \sigma^{-2i}\right)^{1/2}$.

Proposition 3.6 (rate-depending bounds on $\mathbf{K}$)

(a) For any $p \ge 2$,

$$\|\mathbf{K}\|_{a(\sigma;t)} \le \mu_\sigma(\mathbf{K}_t) + \kappa'_\sigma \partial_\sigma^{(p)}(\mathbf{K}). \tag{3.16}$$

(b) For $p = \infty$,

$$\|\mathbf{K}\|_{a(\sigma;t)} \le \mu_\sigma(\mathbf{K}_t) + \frac{1}{(\sigma - 1)} \partial_\sigma(\mathbf{K}). \tag{3.17}$$

□

Proof: In Appendix B.

Proposition 3.7 (bounds on $\mathbf{K} \bigtriangledown \mathbf{F}$)

(a) For any $p \ge 2$,

$$\|\mathbf{K} \bigtriangledown \mathbf{F}\|_{a(\sigma)} \le \kappa_\sigma \kappa'_\sigma \mu_\sigma^{(p)}(\mathbf{K}) \partial_\sigma^{(p)}(\mathbf{F}). \tag{3.18}$$

(b) For $p = \infty$,

$$\|\mathbf{K} \bigtriangledown \mathbf{F}\|_{a(\sigma)} \le \kappa_\sigma \frac{1}{(\sigma - 1)} \mu_\sigma(\mathbf{K}) \partial_\sigma(\mathbf{F}). \tag{3.19}$$

(c) If $\mathbf{K}$ is slowly time-varying,

$$\|\mathbf{K} \bigtriangledown \mathbf{F}\|_{a(\sigma)} \le \frac{1}{(\sigma - 1)} \left[\mu_\sigma(\mathbf{K}) \partial_\sigma(\mathbf{F}) + \partial_\sigma(\mathbf{K}) \|\mathbf{F}\|_{a(\sigma)} + \partial_\sigma(\mathbf{K} \otimes \mathbf{F})\right]. \tag{3.20}$$

□

Proof: In Appendix B.

Proposition 3.8 (local bounds on $\mathbf{GK}+\mathbf{F}$)

For slowly time-varying $\mathbf{K},\mathbf{F}$, let $\mathbf{S}=\mathbf{GK}+\mathbf{F}$ and $\mathbf{S}^{\mathrm{l}}=\mathbf{G}\otimes\mathbf{K}+\mathbf{F}$.

(a) For any $p\geq 2$,

$$\|\mathbf{S}\|_{a(\sigma;t)}-\mu_\sigma(\mathbf{S}^{\mathrm{l}}_t)\leq \kappa'_\sigma\left(\kappa_\sigma\mu^{(p)}_\sigma(\mathbf{G})\partial^{(p)}_\sigma(\mathbf{K})+\partial^{(p)}_\sigma(\mathbf{S}^{\mathrm{l}})\right). \tag{3.21}$$

(b) For $p=\infty$,

$$\|\mathbf{S}\|_{a(\sigma;t)}-\mu_\sigma(\mathbf{S}^{\mathrm{l}}_t)\leq \frac{1}{(\sigma-1)}\left(\kappa_\sigma\mu_\sigma(\mathbf{G})\partial_\sigma(\mathbf{K})+\partial_\sigma(\mathbf{S}^{\mathrm{l}})\right). \tag{3.22}$$

(c) If $\mathbf{G}$ is slowly time-varying,

$$\|\mathbf{S}\|_{a(\sigma;t)}-\mu_\sigma(\mathbf{S}^{\mathrm{l}}_t)\leq \frac{1}{(\sigma-1)}\left\{\mu_\sigma(\mathbf{G})\partial_\sigma(\mathbf{K})+\partial_\sigma(\mathbf{G})\left[\mu_\sigma(\mathbf{K})+\frac{1}{(\sigma-1)}\partial_\sigma(\mathbf{K})\right]+\partial_\sigma(\mathbf{F})\right\}. \tag{3.23}$$

□

Proof: In Appendix B.

Proposition 3.9 (bounds on $\partial^{(p)}_\sigma(\cdot)$)[3.1]

For any $p\geq 2$,

$$\partial^{(p)}_\sigma(\mathbf{K}\otimes\mathbf{F})\leq \mu_\sigma(\mathbf{K})\partial^{(p)}_\sigma(\mathbf{F})+\mu_\sigma(\mathbf{F})\partial^{(p)}_\sigma(\mathbf{K}), \tag{3.24}$$

$$\partial^{(p)}_\sigma(\mathbf{K}^{\ominus})\leq\left[\mu_\sigma(\mathbf{K}^{\ominus})\right]^2\partial^{(p)}_\sigma(\mathbf{K}). \tag{3.25}$$

□

[3.1] The bounds (3.25) remain valid for certain noncausal operators $\mathbf{K}^{\ominus}$; see the definition 4.1.

Proof: In Appendix B.

In summary , the global algebra $\mathbb{GE}_\sigma$ equipped with the $\|\cdot\|_{a(\sigma)}$ norm can be used to describe the global time domain behavior; the local algebra $\mathbb{LE}_\sigma, \sigma > 1$ equipped with the $\mu_\sigma(\cdot)$ norm can be used to generate local frequency domain approximants to that behavior.

3.4 Lower Bounds on $\|\cdot\|_{a(\sigma;t)}$

While Prop. 3.8 gives upper bounds of the global norm $\|\cdot\|_{a(\sigma)}$ by the local frequency auxiliary norm $\mu_\sigma(\cdot)$, lower bounds on $\|\cdot\|_{a(\sigma)}$ remains to be established, which, applied to local sub-optimal interpolations of sensitivity operators in Chapters 4 and 5, guarantee that the *sub-optimal* solution is actually *near-optimal.*

The first lower bound of $\|\cdot\|_{a(\sigma)}$ in Prop. 3.10 is valid for any shift-invariant and slowly time-varying $\mathbf{S}^l$. The second one in Prop. 3.11 depends on the radial growth property of $\mathbf{S}^l$.

Proposition 3.10

If the operators $\mathbf{S}$, $\mathbf{S}^l$ defined in Prop. 3.8 are in $\mathbb{E}_\sigma$, then

$$\mu_1(\mathbf{S}_t^l) - \kappa_\sigma^{(p)} \partial_\sigma^{(p)}(\mathbf{S}^l) - \|\mathbf{G} \bigtriangledown \mathbf{K}\|_{a(\sigma)} \leq \|\mathbf{S}\|_{a(\sigma;t)} \leq \mu_\sigma(\mathbf{S}_t^l) + \kappa_\sigma^{(p)} \partial_\sigma^{(p)}(\mathbf{S}^l) + \|\mathbf{G} \bigtriangledown \mathbf{K}\|_{a(\sigma)} \tag{3.26}$$

where $\kappa_\sigma^{(p)} = \kappa'_\sigma$ if $2 \leq p < \infty$ and $\frac{1}{\sigma - 1}$ if $p = \infty$.

□

Proof:

By the triangle inequality,

$$\|\mathbf{S}^1\|_{a(\sigma;t)} - \|\mathbf{S} - \mathbf{S}^1\|_{a(\sigma;t)} \leq \|\mathbf{S}\|_{a(\sigma;t)} \leq \|\mathbf{S}^1\|_{a(\sigma;t)} + \|\mathbf{S} - \mathbf{S}^1\|_{a(\sigma;t)}. \tag{3.27}$$

By Prop. 3.6,

$$\|\mathbf{S}^1\|_{a(\sigma;t)} \leq \mu_\sigma(\mathbf{S}^1_t) + \kappa_\sigma^{(p)} \partial_\sigma^{(p)}(\mathbf{S}^1). \tag{3.28}$$

Thus the upper bound is established as $\mathbf{S} - \mathbf{S}^l = \mathbf{G} \bigtriangledown \mathbf{K}$.

It remains only to prove

$$\mu_1(\mathbf{S}^l_t) \leq \|\mathbf{S}^1\|_{a(\sigma;t)} + \kappa_\sigma^{(p)} \partial_\sigma^{(p)}(\mathbf{S}^1). \tag{3.29}$$

Assume $\mu_1(\mathbf{S}^l_t)$ is achieved[3.2] by $\mu_1(\mathbf{S}^l_t) = \|\widehat{\mathbf{S}}^l_t\|_{H^\infty} = \left|\widehat{\mathbf{S}}^l_t(e^{i\theta})\right|$ for some $\theta \in [-\pi, \pi)$ with the largest (unit norm) singular vector $u \in \mathbb{C}^n$. Then the inequality (3.29) is obtained by noting that for exponential inputs $u_\theta \in l^\infty(-\infty, \infty)$, $u_\theta(\tau) = u \exp(i\theta\tau)$, $\tau \in \mathbb{Z}$, the output is

$$y_\theta(\tau) = (\mathbf{S}^l u_\theta)(\tau) = \widehat{\mathbf{S}^l_\tau}(e^{i\theta}) u_\theta(\tau).$$

By (B.5) in Appendix B, it follows that

$$\begin{aligned} \kappa_\sigma^{-1}\sigma^{-t} \left\|\Pi_t \left(y_\theta - \widehat{\mathbf{S}}^l_t(e^{i\theta}) u_\theta\right)\right\|_{l^2_\sigma} &= \kappa_\sigma^{-1}\sigma^{-t} \left\|\Pi_t \left(\widehat{\mathbf{S}}^l(e^{i\theta}) - \widehat{\mathbf{S}}^l_t(e^{i\theta})\right) u_\theta\right\|_{l^2_\sigma} \\ &\leq \kappa_\sigma^{(p)} \partial_\sigma^{(p)}(\mathbf{S}^1) \|u\|_{a(\sigma)} \\ &\leq \kappa_\sigma^{(p)} \partial_\sigma^{(p)}(\mathbf{S}^1) \end{aligned}$$

(3.2) A similar proof applies if $\mu_1(\mathbf{S}^l)$ is only approached but not achieved.

as $\|u\|_{a(\sigma)} = 1$, which implies that

$$\begin{aligned}\|y_\theta\|_{a(\sigma;t)} &\geq \kappa_\sigma^{-1}\sigma^{-t}\,\|\Pi_t y_\theta\|_{l^2_\sigma} \\ &\geq \kappa_\sigma^{-1}\sigma^{-t}\left\|\Pi_t \widehat{\mathbf{S}}_t^l(e^{i\theta})u_\theta\right\|_{l^2_\sigma} - \kappa_\sigma^{(p)}\partial_\sigma^{(p)}(\mathbf{S}^l) \\ &= \mu_1(\mathbf{S}_t^l) - \kappa_\sigma^{(p)}\partial_\sigma^{(p)}(\mathbf{S}^l)\end{aligned}$$

by the manipulation

$$\left\|\Pi_t \widehat{\mathbf{S}}_t^l(e^{i\theta})u_\theta\right\|_{l^2_\sigma} = \left\|\widehat{\mathbf{S}}_t^l(e^{i\theta})\Pi_t u_\theta\right\|_{l^2_\sigma} = \mu_1(\mathbf{S}_t^l)\,\|\Pi_t u_\theta\|_{l^2_\sigma}$$

and $\kappa_\sigma^{-1}\sigma^{-t}\,\|\Pi_t u_\theta\|_{l^2_\sigma} = 1$.

Thus

$$\|\mathbf{S}^l\|_{a(\sigma;t)} \geq \mu_1(\mathbf{S}_t^l) - \kappa_\sigma^{(p)}\partial_\sigma^{(p)}(\mathbf{S}^l).$$

Q.E.D.

We introduce next the concept of uniform radial growth, which will relate $\mu_\sigma(\cdot)$ to $\mu_1(\cdot)$. As a result, the lower bound in Prop. 3.10 can be expressed with respect to $\mu_\sigma(\cdot)$ instead of $\mu_1(\cdot)$.

If K is in $H^\infty_{\sigma_0}$, $\sigma_0 > 1$, $K \neq 0$, Hardy's Convexity Theorem (Duren [Dur]) implies the radial growth condition,

$$\|K\|_{H^\infty_\sigma}/\|K\|_{H^\infty} \leq \nu^{\left(\frac{\ln\sigma}{\ln\sigma_0}\right)} \tag{3.30}$$

where $\nu = \|K\|_{H^\infty_{\sigma_0}}/\|K\|_{H^\infty}$ and $1 \leq \sigma < \sigma_0$.

$\mathbf{K} \in \mathbf{E}_{\sigma_0}$ has *uniform (in t) radial growth* with constant $\nu_{\sigma_0}(\mathbf{K})$ iff

$$\nu_{\sigma_0}(\mathbf{K}) := \sup_{t\in\mathbb{Z}}\left\{\|\widehat{\mathbf{K}}_t\|_{H^\infty_{\sigma_0}}/\|\widehat{\mathbf{K}}_t\|_{H^\infty} : \widehat{\mathbf{K}}_t \neq 0\right\} < \infty, \tag{3.31}$$

in which case

$$\mu_\sigma(\mathbf{K}) \leq \mu_{(1)}(\mathbf{K}) \nu_{\sigma_0}^{\left(\frac{\ln \sigma}{\ln \sigma_0}\right)}. \tag{3.32}$$

Proposition 3.11

If the operator $\mathbf{S}$, $\mathbf{S}^l$, defined in Prop. 3.8, are in $\mathbb{E}_{\sigma_0}$, $\sigma_0 > 1$, and $\mathbf{S}^l$ has uniform radial growth, then

$$\|\mathbf{S}\|_{a(\sigma;t)} \geq \mu_\sigma(\mathbf{S}_t^1) - \mu_1(\mathbf{S}_t^l)\left\{\nu_{\sigma_0}(\mathbf{S}^l)^{\left(\frac{\ln \sigma}{\ln \sigma_0}\right)} - 1\right\} - \kappa_\sigma^{(p)}\partial_\sigma^{(p)}(\mathbf{S}^1) - \|\mathbf{G} \bigtriangledown \mathbf{K}\|_{a(\sigma)}. \tag{3.33}$$

□

Proof:

(3.33) follows immediately from Prop. 3.10 and (3.32), noting that

$$\begin{aligned}
\mu_1(\mathbf{S}_t^l) &= \mu_\sigma(\mathbf{S}_t^1) + \mu_1(\mathbf{S}_t^l) - \mu_\sigma(\mathbf{S}_t^1) \\
&\geq \mu_\sigma(\mathbf{S}_t^1) + \mu_1(\mathbf{S}_t^{\cdot}) - \mu_1(\mathbf{S}_t^l)\nu_{\sigma_0}(\mathbf{S}^l)^{\left(\frac{\ln \sigma}{\ln \sigma_0}\right)}.
\end{aligned}$$

Q.E.D.

Remarks: We can show that $\mu_\sigma(\cdot)$ norm is a continuous function of σ. Indeed, for $\mathbf{K} \in \mathbb{E}_{\sigma_1}$, $1 \leq \sigma < \sigma' < \sigma_1$,

$$\left|\mu_{\sigma'}(\mathbf{K}) - \mu_\sigma(\mathbf{K})\right| \leq \sup_t \left| \|\widehat{\mathbf{K}}_t\|_{H^\infty_{\sigma'}} - \|\widehat{\mathbf{K}}_t\|_{H^\infty_\sigma} \right|.$$

However

$$\begin{aligned}
&\left| \|\widehat{\mathbf{K}}_t\|_{H^\infty_{\sigma'}} - \|\widehat{\mathbf{K}}_t\|_{H^\infty_\sigma} \right| \\
&= \left| \sup_\theta \left| \sum_{\tau=0}^\infty k_t(\tau)(\sigma')^\tau e^{i\theta\tau} \right| - \sup_\theta \left| \sum_{\tau=0}^\infty k_t(\tau)\sigma^\tau e^{i\theta\tau} \right| \right| \\
&\leq \sup_\theta \left| \sum_{\tau=0}^\infty k_t(\tau)\left((\sigma')^\tau - \sigma^\tau\right) e^{i\theta\tau} \right| \\
&\leq \sum_{\tau=0}^\infty |k_t(\tau)| \left((\sigma')^\tau - \sigma^\tau\right) \\
&\leq \left(1 - \frac{\sigma}{\sigma'}\right)\left(e \ln\left(\frac{\sigma_1}{\sigma'}\right)\right)^{-1} \|\mathbf{K}\|_{(\sigma_1)}
\end{aligned}$$

by (2.39). After taking $\sup_t$ of the inequality, we obtain

$$\begin{aligned} &\left|\mu_{\sigma'}(\mathbf{K}) - \mu_\sigma(\mathbf{K})\right| \\ &\le \left(1 - \frac{\sigma}{\sigma'}\right)\left(e \ln\left(\frac{\sigma_1}{\sigma'}\right)\right)^{-1} \|\mathbf{K}\|_{(\sigma_1)} \to 0 \qquad \text{as} \quad \sigma' \to \sigma, \end{aligned}$$

which proves the continuity of $\mu_\sigma(\cdot)$ with respect to σ. Therefore as the rates of $\mathbf{G}$, $\mathbf{K}$, $\mathbf{F}$, and $\mathbf{S}^l$ approach zero (in the sense of Section 3.6) and $\sigma \to 1$, (3.26) (which is independent of radial growth condition) implies that the auxiliary time-domain norm $\|\mathbf{S}\|_{a(\sigma;t)}$ approaches the transfer-function norm $\mu_\sigma(\mathbf{S}^l_t)$; in this sense, the former norm is asymptotically isometric to the latter.

3.5 Applications to Frozen-Time Analysis

Although the main interest here is in adaptive optimization, we note in passing that many classical frequency domain stability conditions of the frozen-time variety (mainly linear systems with a time-varying gain matrix) can be summed up in a statement linking local and global resolvents, as follows.

The *resolvent set* $Res_{\mathbb{A}}(\mathbf{K})$ of an operator $\mathbf{K}$ in a normed algebra $\mathbb{A}$ is the set $\left\{\lambda \in \mathbb{C} : (\lambda\mathbf{I} + \mathbf{K})^{-1} \in \mathbb{A}\right\}$, and the *$\gamma$-sublevel set* of that resolvent ($\gamma > 0$) is

$$Res_{\mathbb{A};\gamma}(\mathbf{K}) = \left\{\lambda \in Res_{\mathbb{A}}(\mathbf{K}) : \|(\lambda\mathbf{I} + \mathbf{K})^{-1}\|_{\mathbb{A}} \le \gamma\right\}. \tag{3.34}$$

Let $\mathbf{G}, \mathbf{K}$ be operators in $\underline{\mathbb{E}}_\sigma$, where $\mathbf{G}$ has no memory and $\mathbf{K}$ is shift invariant. Take the local and global norms to be $\mu_\sigma(\cdot)$ and $\|\cdot\|_{a(\sigma)}$ respectively.

<u>Corollary 3.1</u>

$$Res_{\mathbf{L}\underline{\mathbf{E}}_\sigma;\gamma}(\mathbf{G} \otimes \mathbf{K}) \subset Res_{\mathbb{G}\mathbf{E}}(\mathbf{GK}), \tag{3.35}$$

provided $\partial_\sigma(\mathbf{G}) \le (\sigma - 1)/\left[\gamma\mu_\sigma(\mathbf{K})\left(1 + \gamma\mu_\sigma(\mathbf{G})\mu_\sigma(\mathbf{K})\right)\right]$.

□

Proof of Corollary 3.1:

Write $\mathbf{F}_\lambda := (\lambda \mathbf{I} + \mathbf{G} \otimes \mathbf{K})$. If $\lambda \in Res_{\mathbf{L}\underline{\mathbf{E}}_\sigma;\gamma}(\mathbf{G} \otimes \mathbf{K})$ then $\mu_\sigma(\mathbf{F}_\lambda{}^\ominus) \le \gamma$ by definition of $Res_{\mathbf{LE}_\sigma;\gamma}$. Now as $\mathbf{K}$ is shift-invariant $\mathbf{G} \otimes \mathbf{K} = \mathbf{GK}$; by the Inversion Lemma 2.2, $(\lambda \mathbf{I} + \mathbf{GK})^{-1}$ exists in $\mathbb{B}$, proving the Corollary, provided $\|\mathbf{F}_\lambda{}^\ominus \nabla \mathbf{F}_\lambda\|_{a(\sigma)} < 1$. Let us evaluate this.

As $\mathbf{K}$ is shift-invariant and $\lambda \mathbf{I}$ has no memory, by (3.20)

$$\begin{aligned} \|\mathbf{F}_\lambda{}^\ominus \nabla \mathbf{F}_\lambda\|_{a(\sigma)} &= \|\mathbf{F}_\lambda{}^\ominus \nabla (\mathbf{G} \otimes \mathbf{K})\|_{a(\sigma)} \\ &\le (\sigma - 1)^{-1}[\mu_\sigma(\mathbf{F}_\lambda{}^\ominus)\partial_\sigma(\mathbf{G} \otimes \mathbf{K}) \\ &\quad + \partial_\sigma(\mathbf{F}_\lambda{}^\ominus)\|\mathbf{G} \otimes \mathbf{K}\|_{a(\sigma)} + \partial_\sigma(\mathbf{F}_\lambda{}^\ominus \otimes \mathbf{F}_\lambda)]. \end{aligned} \tag{3.36}$$

The last term in (3.36) is null. The other terms are bounded, as $\mathbf{K}$ is shift-invariant,

$$\mu_\sigma(\mathbf{F}_\lambda{}^\ominus) \le \gamma; \tag{3.37}$$

$$\partial_\sigma(\mathbf{G} \otimes \mathbf{K}) \le \partial_\sigma(\mathbf{G})\mu_\sigma(\mathbf{K}); \tag{3.38}$$

$$\partial_\sigma(\mathbf{F}_\lambda{}^\ominus) \le \gamma^2\partial_\sigma(\mathbf{F}_\lambda) \le \gamma^2\partial_\sigma(\mathbf{G})\mu_\sigma(\mathbf{K}) \tag{3.39}$$

by (3.25). As $\mathbf{K}$ is shift-invariant and $\mathbf{G}$ has no memory.

$$\|\mathbf{G} \otimes \mathbf{K}\|_{a(\sigma)} \le \mu_\sigma(\mathbf{G})\mu_\sigma(\mathbf{K}) \tag{3.40}$$

by (3.13).

Therefore $\|\mathbf{F}_\lambda{}^\ominus \nabla \mathbf{F}_\lambda\|_{a(\sigma)} < 1$ if

$$(\sigma - 1)^{-1}\{\gamma\mu_\sigma(\mathbf{K})\,(1 + \gamma\mu_\sigma(\mathbf{G})\mu_\sigma(\mathbf{K}))\}\,\partial_\sigma(\mathbf{G}) < 1, \tag{3.41}$$

which implies the Corollary.

Q.E.D.

3.6 Variable Rates Approaching Zero

In adaptive problems, the variation rate of a system is often adjustable. We wish to describe the behavior of such a class of systems with variable rates as rates approach zero.

Definition

A slowly time-varying system $\mathbf{G} \in \mathbf{E}_\sigma$ has *variable rates* $\partial_\sigma^{(p)}(\cdot)$ (or $d_\sigma(\cdot)$) *approaching zero* if there exists an operator-valued function $\tilde{\mathbf{G}}(\rho)$ of the parameter ρ (variable rates) such that

(1) ρ takes values in $\mathbb{R}_+$ $(:= [0, \infty))$ with zero as a limit point; $\tilde{\mathbf{G}}(\rho) \in \mathbf{E}_\sigma$ for all ρ and for some ρ_1,

$$\tilde{\mathbf{G}}(\rho_1) = \mathbf{G}. \tag{3.42}$$

(2) $\mu_\sigma(\tilde{\mathbf{G}}(\rho))$ and $\mu_1(\tilde{\mathbf{G}}(\rho))$ (or $\|\tilde{\mathbf{G}}(\rho)\|_{(\sigma)}$) are invariant with ρ.

(3)

$$\partial_\sigma^{(p)}(\tilde{\mathbf{G}}(\rho)) \leq \rho \tag{3.43}$$

$$\text{or} \qquad d_\sigma(\tilde{\mathbf{G}}(\rho)) \leq \rho. \tag{3.44}$$

□

It is not specified in the previous definition the way to achieve $\widetilde{G}(\rho)$, which may come a priori from problem settings and can be complicated. An example of embedding G in some $\widetilde{G}(\rho)$ is via convex interpolation.

For $G \in \mathbf{E}_\sigma$ with normalized $\partial_\sigma^{(p)}(G) = 1$ (or $d_\sigma(G) = 1$), take $m \in \mathbb{Z}$, $m \geq 1$, define $\rho_m = \frac{1}{m}$ and $\widetilde{G}(\rho_m)$ as follows:

$$\widetilde{G}_t(\rho_m) = (a + 1 - \frac{t}{m})G_a + (\frac{t}{m} - a)G_{a+1}, \qquad t \in \mathbb{Z}, \tag{3.45}$$

where $a = \lfloor \frac{t}{m} \rfloor$, the largest integer below $\frac{t}{m}$.

For this choice of $\widetilde{G}(\rho_m)$, it is obvious that

$$\widetilde{G}(\rho_1) = G \tag{3.46}$$

and $\widetilde{G}(\rho_m) \in \mathbf{E}_\sigma$ for all m, $\rho_m \to 0$ as $m \to \infty$. So the axiom (1) is satisfied. For $m \in \mathbb{Z}$, $m \geq 1$, by definition (3.45) ($G_t = \widetilde{G}_{mt}(\rho_m), t \in \mathbb{Z}$)

$$\mu_\sigma(\widetilde{G}(\rho_m)) \geq \mu_\sigma(G). \tag{3.47}$$

On the other hand,

$$\|\widehat{\widetilde{G}}_t(\rho_m)\|_{H_\sigma^\infty} \leq (a + 1 - \frac{t}{m})\|\widehat{G}_a\|_{H_\sigma^\infty} + (\frac{t}{m} - a)\|\widehat{G}_{a+1}\|_{H_\sigma^\infty} \leq \mu_\sigma(G), \tag{3.48}$$

which, together with (3.47), implies

$$\mu_\sigma(\widetilde{G}(\rho_m)) = \mu_\sigma(G). \tag{3.49}$$

Similarly

$$\mu_1(\widetilde{G}(\rho_m)) = \mu_1(G), \qquad \|\widetilde{G}(\rho_m)\|_{(\sigma)} = \|G\|_{(\sigma)}.$$

Thus the axiom (2) is valid. It is easy to show from (3.45) that

$$\partial_\sigma^{(p)}(\widetilde{G}(\rho_m)) = \frac{1}{m}\partial_\sigma^{(p)}(G) = \rho_m, \qquad \text{or} \quad d_\sigma(\widetilde{G}(\rho_m)) = \frac{1}{m}d_\sigma(G) = \rho_m, \tag{3.50}$$

which verifies the axiom (3).

The main purpose in introducing the concept of "variable rates approaching zero" is to discuss the local-global coupling in the limit as rates of time variation approach zero and system memories approach infinity.

Let $\mathbf{S}, \mathbf{S}^l \in \mathbb{E}_{\sigma_0}$ be defined as in Prop. 3.8, (replacing $\mathbb{E}_\sigma$ by $\mathbb{E}_{\sigma_0}$), and $\widetilde{\mathbf{G}}(\rho), \widetilde{\mathbf{K}}(\rho), \widetilde{\mathbf{F}}(\rho)$ the operator-valued functions embedding $\mathbf{G}, \mathbf{K}, \mathbf{F}$ respectively, as in the previous definition. Define

$$\widetilde{\mathbf{S}}(\rho) = \widetilde{\mathbf{G}}(\rho)\widetilde{\mathbf{K}}(\rho) + \widetilde{\mathbf{F}}(\rho), \qquad \widetilde{\mathbf{S}}^l(\rho) = \widetilde{\mathbf{G}}(\rho) \otimes \widetilde{\mathbf{K}}(\rho) + \widetilde{\mathbf{F}}(\rho).$$

Proposition 3.12

If the operators $\mathbf{G}, \mathbf{K}, \mathbf{F}$ in Prop. 3.10 have variable rates approaching zero, then (3.26) has a limit version as

$$\lim_{\sigma \to 1} \lim_{\rho \to 0} \left| \|\widetilde{\mathbf{S}}(\rho)\|_{a(\sigma;t)} - \mu_\sigma(\widetilde{\mathbf{S}}_t^l(\rho)) \right| = 0 \quad \text{uniform in } t. \tag{3.51}$$

□

Proof:

Applying (3.26) to $\widetilde{\mathbf{S}}(\rho)$,

$$\begin{aligned} &\mu_1\left(\widetilde{\mathbf{S}}_t^l(\rho)\right) - \kappa_\sigma^{(p)} \partial_\sigma^{(p)}(\widetilde{\mathbf{S}}^l(\rho)) - \|\widetilde{\mathbf{S}}(\rho) - \widetilde{\mathbf{S}}^l(\rho)\|_{a(\sigma)} \\ \leq\ &\|\widetilde{\mathbf{S}}(\rho)\|_{a(\sigma;t)} \leq \mu_\sigma(\widetilde{\mathbf{S}}_t^l(\rho)) + \kappa_\sigma^{(p)} \partial_\sigma^{(p)}(\widetilde{\mathbf{S}}^l(\rho)) + \|\widetilde{\mathbf{S}}(\rho) - \widetilde{\mathbf{S}}^l(\rho)\|_{a(\sigma)}. \end{aligned} \tag{3.52}$$

As $\partial_\sigma^{(p)}(\widetilde{\mathbf{K}}(\rho)) \to 0$ as $\rho \to 0$, and so, by Prop. 3.7,

$$\|\widetilde{\mathbf{S}}(\rho) - \widetilde{\mathbf{S}}^l(\rho)\|_{a(\sigma)} = \|\widetilde{\mathbf{G}}(\rho) \bigtriangledown \widetilde{\mathbf{K}}(\rho)\|_{a(\sigma)} \to 0 \qquad \text{as} \quad \rho \to 0.$$

Therefore after taking $\lim_{\rho\to 0}$, we obtain

$$\lim_{\rho\to 0}\mu_1(\widetilde{S}^l_t(\rho)) \le \lim_{\rho\to 0}\|\widetilde{\mathbf{S}}(\rho)\|_{a(\sigma;t)} \le \lim_{\rho\to 0}\mu_\sigma(\widetilde{\mathbf{S}}^l_t(\rho)). \tag{3.53}$$

Now (3.51) follows as

$$\mu_\sigma(\widetilde{\mathbf{S}}^l_t(\rho)) \to \mu_1(\widetilde{S}^l_t(\rho)) \qquad \text{as} \quad \sigma \to 1 \qquad \text{uniform in } t,$$

since $\mu_\sigma(\cdot)$ norm is a continuous function of σ (recall $\mathbf{S}^l \in \mathbb{E}_{\sigma_0}$) (see the remark after Prop. 3.11).

Q.E.D.

Prop. 3.12 may be interpreted as asymptotic isometry between $\|\cdot\|_{a(\sigma)}$ and $\mu_\sigma(\cdot)$.

A possible application of the concept is adaptive design problems, where the variation rate of a system is often adjustable (e.g., to achieve slow adaptation). The simplest example is systems with a time-varying gain (matrix), its variation rate being reduced without changing its maximum value. Another application, as examplified by convex interpolation, is time-scaling technique, which has been used in adaptive systems and sampling data systems.

Since every $\mathbf{G} \in \mathbb{E}_\sigma$ can be embedded in a class of operators with variable rates approaching zero, e.g. via convex interpolation technique, Prop. 3.12 is a general coupling property between the global norm $\|\cdot\|_{a(\sigma)}$ and the local norm $\mu_\sigma(\cdot)$ rather than a property of an individual operator.

Chapter 4

Adaptive Design by Local Interpolation: Results for Stable Plants

The double algebra provides a symbolism in terms of which global stability and performance can be evaluated explicitly through local approximations. A simple example of this involves the stability analysis and the norm evaluation of a global sensitivity operator $\mathbf{S} := (\mathbf{I}+\mathbf{G})^{-1}$, $\mathbf{G} \in \mathbb{E}_\sigma$ (hence an open-loop stable system), from frequency-domain properties of its local approximant, $\mathbf{S}^l := (\mathbf{I}+\mathbf{G})^{\ominus}$. We have the following stability result, which is an immediate corollary to Props. 2.2 and 3.7.

Let $2 \leq p \leq \infty$.

Proposition 4.1

If $\mathbf{G}$ and $\mathbf{S}^l$ are in $\mathbb{E}_\sigma$ and the variation rate of $\mathbf{G}$ satisfies $\partial_\sigma^{(p)}(\mathbf{G}) < \left[\kappa_\sigma^{(p)}\kappa_\sigma\mu_\sigma^{(p)}(\mathbf{S}^l)\right]^{-1}$, then $\mathbf{S}$ is in $\mathbb{B}$,

$$\mathbf{S} = \left(\mathbf{I} + \mathbf{S}^l \triangledown \mathbf{G}\right)^{-1}\mathbf{S}^l \tag{4.1}$$

and

$$\|\mathbf{S}\|_{a(\sigma)} \leq (1-\alpha)^{-1}\mu_\sigma(\mathbf{S}^l) \tag{4.2}$$

where $\alpha := \kappa_\sigma\kappa_\sigma^{(p)}\mu_\sigma^{(p)}(\mathbf{S}^l)\partial_\sigma^{(p)}(\mathbf{G})$.

□

Although Prop. 4.1 provides an illustrative application of the double algebra symbolism, it is still little use in feedback system analysis, as a feedback system is often open-loop unstable. Instead, we seek global adaptive design via approximations by local interpolation. Again the double algebra symbolism is employed to describe the approximations. We start, in this chapter, with the case of stable plants.

4.1 Global Design by Local Interpolation

The main concern here is with the synthesis of a global sensitivity from a prescribed local (possibly locally optimal or suboptimal) behavior, as follows.

Suppose that $\mathbf{W}_1, \mathbf{W}_2 \in \underline{\mathbb{E}}_\sigma$ (and $\mathbf{W}_1^{-1} \in \underline{\mathbb{E}}_\sigma$) represent two weightings, and $\mathbf{G} \in \underline{\mathbb{E}}_\sigma$ represents a strictly causal plant. It is standard that the feedback controllers $\mathbf{F} \in \mathbf{B}_e$ stabilizing in $\underline{\mathbb{E}}_\sigma$, i.e., maintaining all closed-loop operators in $\underline{\mathbb{E}}_\sigma$, can be parametrized by a compensator $\mathbf{Q} \in \underline{\mathbb{E}}_\sigma$ which gives a sensitivity $(\mathbf{I}+\mathbf{GF})^{-1} = (\mathbf{I}-\mathbf{GQ})$, and a weighted sensitivity $\mathbf{S} \in \underline{\mathbb{E}}_\sigma$,

$$\mathbf{S} = \mathbf{W}_2(\mathbf{I}-\mathbf{GQ})\mathbf{W}_1 = \mathbf{W}_2\mathbf{W}_1 - \mathbf{W}_2\mathbf{GQW}_1, \tag{4.3}$$

($\mathbf{Q} = \mathbf{F}\,(\mathbf{I}+\mathbf{GF})^{-1}$ is itself a closed loop operator).

Denote $\mathbf{W}_2\mathbf{G}$ by $\mathbf{G}_\mathbf{W}$ and suppose that it has a local factorization

$$\mathbf{G}_\mathbf{W} = \mathbf{U} \otimes \mathbf{G}^{\mathrm{out}} \tag{4.4}$$

where $\mathbf{U}$ and $\mathbf{G}^{\mathrm{out}}$ are locally inner and locally outer in $\underline{\mathbb{E}}_\sigma$, i.e., for each $t \in \mathbb{Z}$, $\hat{\mathbf{U}}_t(\sigma(\cdot)) \in H^\infty$ is inner and $(\hat{\mathbf{G}}^{\mathrm{out}})_t(\sigma(\cdot)) \in H^\infty$ is outer. We are given a sensitivity $\mathbf{S}^l \in \underline{\mathbb{E}}_\sigma$ which locally interpolates $\mathbf{W} := \mathbf{W}_2 \otimes \mathbf{W}_1$ at $\mathbf{U}$ in $\underline{\mathbb{E}}_\sigma$, i.e., for which there exists $\mathbf{Q}_1 \in \underline{\mathbb{E}}_\sigma$ such that

$$\mathbf{S}^l = \mathbf{W} - \mathbf{U} \otimes \mathbf{Q}_1 \tag{4.5}$$

where $\mathbf{S}^l$ is assumed to be smaller than $\mathbf{W}$ in $\mu_\sigma(\cdot)$ norm to avoid the trivial case $\mathbf{Q}_1 = 0$, $\mathbf{Q}$ is now chosen to locally realize $\mathbf{S}^l$, i.e., to satisfy

$$\mathbf{S}^l = \mathbf{W} - \mathbf{U} \otimes \mathbf{G}^{\text{out}} \otimes (\mathbf{Q}\mathbf{W}_1) \tag{4.6}$$

(i.e., $\mathbf{Q}_1 = \mathbf{G}^{\text{out}} \otimes (\mathbf{Q}\mathbf{W}_1)$ or $\mathbf{Q} = ((\mathbf{G}^{\text{out}})^{\ominus} \otimes \mathbf{Q}_1)\mathbf{W}_1^{-1}$).

To describe the operator $\mathbf{Q}$ explicitly in the local algebra $\mathbb{L}\underline{\mathbb{E}}_\sigma$, we need to extend $\mathbb{L}\underline{\mathbb{E}}_\sigma$ to include (bounded) noncausal operators with kernels $k \in l^1_\sigma(-\infty, \infty)$.

<u>**Definition 4.1**</u>

$\widetilde{\mathbb{L}\underline{\mathbb{E}}}_\sigma$ consists of (time-varying convolution sum) operators $\mathbf{K}$ (possibly non-causal) with uniformly (in t) bounded frozen-time kernels $k_t \in l^1_{\sigma'}(-\infty, \infty)$ for all σ' in an open interval (depending on k) containing σ. The local product, local inverse etc. are extended to $\widetilde{\mathbb{L}\underline{\mathbb{E}}}_\sigma$ in an obvious way. For $\mathbf{K} \in \widetilde{\mathbb{L}\underline{\mathbb{E}}}_\sigma$

$$\tilde{\mu}_\sigma^{(p)}(\mathbf{K}) := \sup_t \|\widehat{\mathbf{K}}_t\|_{L^p_\sigma},$$

$$\tilde{\partial}_\sigma^{(p)}(\mathbf{K}) := \sup_t \|\widehat{\mathbf{K}}_t - \widehat{\mathbf{K}}_{t-1}\|_{L^p_\sigma}.$$

□

Then $\mathbf{K} \in \mathbb{L}\underline{\mathbb{E}}_\sigma$ iff $\mathbf{K} \in \widetilde{\mathbb{L}\underline{\mathbb{E}}}_\sigma$ and $\mathbf{K}$ is causal, or equivalently $\mathbf{K} \in \widetilde{\mathbb{L}\underline{\mathbb{E}}}_\sigma$ and $\widehat{\mathbf{K}}_t \in H^\infty_\sigma$ for all t.

With the designated extension, the choice of $\mathbf{Q}$ is explicitly given in $\widetilde{\mathbb{L}\underline{\mathbb{E}}}_\sigma$ by

$$\mathbf{Q} := \left[\mathbf{G_W}^{\ominus} \otimes (\mathbf{W} - \mathbf{S}^l)\right] \mathbf{W}_1^{-1} \tag{4.7}$$

where $\mathbf{G_W}^{\ominus} \in \widetilde{\mathbb{L}\underline{\mathbb{E}}}_\sigma$.

The problem is to determine whether (4.7) is stabilizing and makes $\mathbf{S}^l$ a good approximant to the (true global) sensitivity $\mathbf{S}$ for slowly time-varying $\mathbf{G}, \mathbf{W}_i$ $(i = 1,2)$, and $\mathbf{S}^l$.

Assumptions for Theorem 4.1:

(a) $\mathbf{S}^l$ locally interpolates $\mathbf{W}$ at $\mathbf{U}$ in $\underline{\mathbb{E}}_\sigma$.

(b) $\mathbf{W}_1^{-1}$ and $\mathbf{W}_2^{-1}$ are in $\underline{\mathbb{E}}_\sigma$.

(c) $\left(\mathbf{G}^{\text{out}}\right)^\ominus \in \underline{\mathbb{E}}_\sigma$ and $\widehat{\mathbf{U}}_t^{-1}$ is uniformly bounded in an annulus $\sigma \le |z| \le \sigma_0$ for some $\sigma_0 > \sigma$, i.e.

$$\sup\{|\widehat{\mathbf{U}}_t^{-1}(z)| : \sigma \le |z| < \sigma_0, t \in \mathbb{Z}\} < \infty. \tag{4.8}$$

<u>**Theorem 4.1**</u>

(a) $\mathbf{Q}$ defined by (4.7) stabilizes $\mathbf{G}$ in $\mathbb{B}$.

(b) If $\mathbf{G}_W$, and $\mathbf{G}_W{}^\ominus$, and $\mathbf{S}^l$ are slowly time-varying, then the weighted sensitivity $\mathbf{S} \in \underline{\mathbb{E}}_\sigma$ is explicitly given by

$$\mathbf{S} = \mathbf{S}^l + \mathbf{G}_W \bigtriangledown (\mathbf{G}_W{}^\ominus \otimes \mathbf{M}) + \mathbf{W}_2 \bigtriangledown \mathbf{W}_1, \tag{4.9}$$

where $\mathbf{M} := \mathbf{W} - \mathbf{S}^l$ and $\mathbf{G}_W = \mathbf{U} \otimes \mathbf{G}^{\text{out}}$

Moreover, $\mathbf{S}$ satisfies, for $2 \le p \le \infty$,

$$\|\mathbf{S} - \mathbf{S}^l\|_{a(\sigma)} \le \beta,$$

where

$$\beta = \kappa_\sigma \kappa_\sigma^{(p)} \left[\mu_\sigma(\mathbf{G}_W{}^\ominus)\partial_\sigma^{(p)}(\mathbf{G}_W{}^\ominus \otimes \mathbf{M}) + \mu_\sigma^{(p)}(\mathbf{W}_2)\partial_\sigma^{(p)}(\mathbf{W}_1)\right]; \tag{4.10}$$

and

$$\mu_1(\mathbf{S}_t^l) - \kappa_\sigma^{(p)}\partial_\sigma^{(p)}(\mathbf{S}^l) - \beta \le \|\mathbf{S}\|_{a(\sigma;t)} \le \mu_\sigma(\mathbf{S}_t^l) + \kappa_\sigma^{(p)}\partial_\sigma^{(p)}(\mathbf{S}^l) + \beta, \tag{4.11}$$

where

$$\partial_\sigma^{(p)}(\mathbf{M}) \le \partial_\sigma^{(p)}(\mathbf{W}) + \partial_\sigma^{(p)}(\mathbf{S}^l) \le \mu_\sigma(\mathbf{W}_2)\partial_\sigma^{(p)}(\mathbf{W}_1) + \mu_\sigma(\mathbf{W}_1)\partial_\sigma^{(p)}(\mathbf{W}_2) + \partial_\sigma^{(p)}(\mathbf{S}^l)$$

and $\mu_\sigma(\mathbf{M}) \le \mu_\sigma(\mathbf{W}) + \mu_\sigma(\mathbf{S}^l) \le 2\mu_\sigma(\mathbf{W}_2)\mu_\sigma(\mathbf{W}_1)$.

(c) If in addition $\mathbf{S}^l$ has uniform radial growth $\nu_{\sigma_0}(\mathbf{S}^l)$, $\sigma_0 > \sigma$, then

$$\left|\|\mathbf{S}\|_{a(\sigma;t)} - \mu_\sigma(\mathbf{S}_t^l)\right| \le \mu_1(\mathbf{S}_t^l)\left\{\nu_{\sigma_0}(\mathbf{S}^l)^{\left(\frac{\ln\sigma}{\ln\sigma_0}\right)} - 1\right\} + \beta + \kappa_\sigma^{(p)}\partial_\sigma^{(p)}(\mathbf{S}^l). \tag{4.12}$$

(d) If $\mathbf{W}_1, \mathbf{W}_2, \mathbf{U}, \mathbf{G}^{out}$ have variable rates approaching zero, then

$$\lim_{\sigma\to 1}\lim_{\rho\to 0}\left|\|\widetilde{S}(\rho)\|_{a(\sigma;t)} - \mu_\sigma(\widetilde{\mathbf{S}}_t^l(\rho))\right| = 0 \qquad \text{uniform in } t$$

where $\widetilde{\mathbf{S}}(\rho) = \widetilde{\mathbf{W}}_1(\rho)\widetilde{\mathbf{W}}_2(\rho) - \widetilde{\mathbf{G}}_W(\rho)\widetilde{\mathbf{Q}}(\rho)\widetilde{\mathbf{W}}_1(\rho)$.

□

Proof:

(a) If $\mathbf{Q}$ satisfies (4.7) then

$$\mathbf{Q} = \left[\left(\mathbf{G}^{\text{out}}\right)^{\ominus} \otimes \mathbf{U}^{\ominus} \otimes (\mathbf{W} - \mathbf{S}^l)\right]\mathbf{W}_1^{-1}. \tag{4.13}$$

Assumption (c) ensures that $\left(\mathbf{G}^{\text{out}}\right)^{\ominus} \in \underline{\mathbb{E}}_\sigma$. Also $\mathbf{W}_1^{-1} \in \underline{\mathbb{E}}_\sigma$ by hypothesis. It is enough therefore to establish that

$$\mathbf{K} := \mathbf{U}^{\ominus} \otimes (\mathbf{W} - \mathbf{S}^l) \tag{4.14}$$

is in $\underline{\mathbb{E}}_\sigma$ to prove that $\mathbf{Q} \in \underline{\mathbb{E}}_\sigma$, which would mean that $\mathbf{Q}$ stabilizes $\mathbf{G}$.

Now as $\mathbf{S}^l$ locally interpolates $\mathbf{W}$ at $\mathbf{U}$ in $\underline{\mathbf{E}}_\sigma$ by Assumption (a), $\widehat{\mathbf{U}}_t$ divides $(\mathbf{W}-\mathbf{S}^l)_t$ in $H^\infty_{\sigma_1}$ for some $\sigma_1 > \sigma$, for each $t \in \mathbb{Z}$, and therefore $\widehat{\mathbf{K}}_t \in H^\infty_{\sigma_1}$. To conclude that $\mathbf{K} \in \underline{\mathbf{E}}_\sigma$ it is enough to show that $\mu_{\sigma_1}(\mathbf{K}) < \infty$ for some $\sigma_1 > \sigma$. This follows from the existence of some $\sigma_1 > \sigma$ in which $\widetilde{\mu}_{\sigma_1}(\mathbf{U}^\ominus) < \infty$ by Assumption (c), $\mu_{\sigma_1}(\mathbf{W}-\mathbf{S}^l) < \infty$ as $\mathbf{W} = \mathbf{W}_1 \otimes \mathbf{W}_2$ and $\mathbf{S}^l \in \underline{\mathbf{E}}_\sigma$, and the inequality holds

$$\mu_{\sigma_1}(\mathbf{K}) \le \widetilde{\mu}_{\sigma_1}(\mathbf{U}^\ominus)\mu_{\sigma_1}(\mathbf{W}-\mathbf{S}^l) \tag{4.15}$$

where $\widetilde{\mu}_{\sigma_1}(\mathbf{U}^\ominus) := \sup_{t\in\mathbb{Z}} \|\widehat{\mathbf{U}}^\ominus_t\|_{L^\infty_{\sigma_1}}$. Therefore $\mathbf{Q} \in \underline{\mathbf{E}}_\sigma$, as claimed.

(b) From (4.3) and (4.6) the identities

$$\begin{aligned}(\mathbf{S}-\mathbf{S}^l) &= \mathbf{W}_2\mathbf{W}_1 - \mathbf{W}_2 \otimes \mathbf{W}_1 + \mathbf{G}_W\mathbf{Q}\mathbf{W}_1 - \mathbf{G}_W \otimes (\mathbf{Q}\mathbf{W}_1)\\ &= \mathbf{W}_2 \bigtriangledown \mathbf{W}_1 + \mathbf{G}_W \bigtriangledown (\mathbf{Q}\mathbf{W}_1)\end{aligned} \tag{4.16}$$

holds. Therefore, (4.9) follows as $\mathbf{Q}\mathbf{W}_1 = \mathbf{G_W}^\ominus \otimes \mathbf{M}$, and

$$\|\mathbf{S}-\mathbf{S}^l\|_{a(\sigma)} \le \|\mathbf{G}_W \bigtriangledown (\mathbf{G_W}^\ominus \otimes \mathbf{M})\|_{a(\sigma)} + \|\mathbf{W}_2 \bigtriangledown \mathbf{W}_1\|_{a(\sigma)}$$

$$\le \kappa_\sigma\kappa_\sigma^{(p)} \left[\mu_\sigma^{(p)}(\mathbf{G}_W)\partial_\sigma^{(p)}(\mathbf{G_W}^\ominus \otimes \mathbf{M}) + \mu_\sigma^{(p)}(\mathbf{W}_2)\partial_\sigma^{(p)}(\mathbf{W}_1)\right] \qquad \text{(by Prop. 3.7).} \tag{4.17}$$

From which (4.10) follows, and (4.11) follows by Prop. 3.10.

(c) (4.12) follows from Prop. 3.11.

(d) It follows from Prop. 3.12.

Q.E.D.

By Theorem 4.1, global synthesis of the sensitivity operator $\mathbf{S}$ can be approximately realized by slowly time-varying local interpolants $\mathbf{S}^l$.

4.2 Local H^∞ Adaptive Optimization

A natural idea for adaptive compensation is to make the sensitivity $\mathbf{S}_t$ at time $t \in \mathbb{Z}$ depend on the local behavior $\mathbf{G}_t$ of the plant and $(\mathbf{W}_i)_t$ of the weightings, which are either found by identification schemes, or given a priori. In frozen-time adaptive design a local approximation $\mathbf{S}_t^l$ to $\mathbf{S}_t$ is generated by local interpolation for which the adaptive relationship can be represented by a map $S^l : \mathbb{Z} \times H^\infty_{\sigma_0} \times H^\infty_{\sigma_0} \times H^\infty_{\sigma_0} \to H^\infty_{\sigma_0}$, $\sigma_0 > \sigma$, $\widehat{\mathbf{S}}_t^l = S^l(t, \widehat{\mathbf{U}}_t, \mathbf{W}_{1t}, \mathbf{W}_{2t})$ as in (4.5).

Theorem 4.1 provides a basis for frozen-time designs to be valid, provided that $\mathbf{S}^l$ varies slowly. A sufficient condition for slow variation of $\mathbf{S}^l$, when $\mathbf{U}, \mathbf{W}_1$ and $\mathbf{W}_2$ are slowly time-varying in $\mu_\sigma(\cdot)$ norm, is that at each $t \in \mathbb{Z}$, $S^l(t,\cdot,\cdot,\cdot)$ be Lipschitz continuous in its variables, i.e., there are constants $\gamma_{W_1}^{(p)}$, $\gamma_{W_2}^{(p)}$ and $\gamma_U^{(p)}$ such that for all $t \in \mathbb{Z}$

$$\begin{aligned}\|\widehat{\mathbf{S}}_t^l - \widehat{\mathbf{S}}_{t-1}^l\|_{H^p_{\sigma_0}} \le \gamma_U^{(p)} \|\widehat{\mathbf{U}}_t - \widehat{\mathbf{U}}_{t-1}\|_{H^\infty_{\sigma_0}} \\ + \gamma_{W_1}^{(p)} \|\widehat{\mathbf{W}}_{1t} - \widehat{\mathbf{W}}_{1(t-1)}\|_{H^\infty_{\sigma_0}} + \gamma_{W_2}^{(p)} \|\widehat{\mathbf{W}}_{2t} - \widehat{\mathbf{W}}_{2(t-1)}\|_{H^\infty_{\sigma_0}},\end{aligned} \tag{4.18}$$

where $2 \le p \le \infty$.

In particular, we may try to design $\mathbf{S}^l$ by local $H^\infty_{\sigma_0}$ optimization, which gives a local optimal weighted sensitivity $\mathbf{S}^l_{opt}$ satisfying

$$\|(\widehat{\mathbf{S}}^l_{opt})_t\|_{H^\infty_{\sigma_0}} = \inf_{\widehat{\mathbf{Q}}_t \in H^\infty_{\sigma_0}} \|\widehat{\mathbf{W}}_t - \widehat{\mathbf{U}}_t \widehat{\mathbf{Q}}_t\|_{H^\infty_{\sigma_0}} \tag{4.19}$$

for each $t \in \mathbb{Z}$, or

$$\mu_{\sigma_0}(\mathbf{S}^l_{\mathrm{opt}}) = \inf_{\mathbf{Q} \in \overline{\mathbf{E}}_{\sigma_0}} \mu_{\sigma_0}(\mathbf{W} - \mathbf{U} \otimes \mathbf{Q}). \tag{4.20}$$

However it will be shown in Chapter 5 that S^l obtained in this way is not always Lipschitz continuous in the sense of (4.18), and therefore not a suitable candidate for frozen-time design. Nevertheless it will be shown that for any $\delta > 0$, the central (maximum entropy) interpolant in AAK's parametrization provides an adaptive scheme $\mathbf{S}^l(\delta)$ in $\overline{\mathbf{E}}_{\sigma_0}$ which is δ-suboptimal, i.e.,

$$\mu_{\sigma_0}(\mathbf{S}^l(\delta)) \leq \mu_{\sigma_0}(\mathbf{S}^l_{\text{opt}}) + \delta \tag{4.21}$$

and is Lipschitz continuous, with constants $\gamma^{(2)}_{W_1}(\delta)$, $\gamma^{(2)}_{W_2}(\delta)$, $\gamma^{(2)}_{U}(\delta)$, whose dependence on δ will be evaluated.

For such a δ-suboptimal adaptation scheme we get the following.

Corollary 4.1

Given any $\delta' > \delta$, the global sensitivity $\mathbf{S} \in \underline{\mathbf{E}}_\sigma$ (realized using such a δ-suboptimal Lipschitz continuous local interpolation by (4.7) as well as (4.3)) satisfies

$$\|\mathbf{S}\|_{a(\sigma)} \leq \mu_\sigma(\mathbf{S}^1_{\text{opt}}) + \delta', \tag{4.22}$$

provided that

$$\kappa_\sigma \kappa'_\sigma \mu^{(2)}_\sigma(\mathbf{G}_W)\left[2\mu_\sigma(\mathbf{W}_1)\mu_\sigma(\mathbf{W}_2)\tilde{\partial}^{(2)}_\sigma(\mathbf{G}_{\mathbf{W}}{}^{\ominus}) + \beta\right] + \kappa_\sigma \kappa'_\sigma \mu^{(2)}_\sigma(\mathbf{W}) \leq \delta' - \delta, \tag{4.23}$$

where $\mathbf{W} = \mathbf{W}_2 \otimes \mathbf{W}_1$ and

$$\beta = \tilde{\mu}_\sigma(\mathbf{G}_{\mathbf{W}}{}^{\ominus})\left[\mu_\sigma(\mathbf{W}_1)\partial_\sigma(\mathbf{W}_2)\left(1 + \gamma^{(2)}_{W_2}\right) + \mu_\sigma(\mathbf{W}_2)\partial_\sigma(\mathbf{W}_1)\left(1 + \gamma^{(1)}_{W_1}\right) + \partial_\sigma(\mathbf{U})\gamma^{(2)}_U\right].$$

□

Proof:

(4.22) follows immediately from (4.23) and Theorem 4.1, inequality (4.11), noting that

$$\tilde{\partial}_\sigma^{(2)}(\mathbf{G_W}^\ominus \otimes \mathbf{M}) \leq \tilde{\mu}_o(\mathbf{G_W}^\ominus)\tilde{\partial}_\sigma^{(2)}(\mathbf{M}) + \tilde{\mu}_\sigma(\mathbf{M})\tilde{\partial}_\sigma^{(2)}(\mathbf{G_W}^\ominus),$$

$$\begin{aligned}\partial_\sigma^{(2)}(\mathbf{M}) &\leq \partial_\sigma^{(2)}(\mathbf{W}) + \partial_\sigma^{(2)}(\mathbf{S}^1)\\ &\leq \partial_\sigma^{(2)}(\mathbf{W}) + \mu_\sigma(\mathbf{W}_2)\gamma_{W_1}^{(2)}\partial_\sigma(\mathbf{W}_1) + \mu_\sigma(\mathbf{W}_1)\gamma_{W_2}^{(2)}\partial_\sigma(\mathbf{W}_2) + \gamma_U^{(2)}(\delta)\partial_\sigma(\mathbf{U}),\end{aligned}$$

$$\partial_\sigma^{(2)}(\mathbf{W}) \leq \mu_\sigma(\mathbf{W}_1)\partial_\sigma^{(2)}(\mathbf{W}_2) + \mu_\sigma(\mathbf{W}_2)\partial_\sigma^{(2)}(\mathbf{W}_1),$$

$$\mu_\sigma(\mathbf{M}) \leq 2\mu_\sigma(\mathbf{W}) \leq 2\mu_\sigma(\mathbf{W}_1)\mu_\sigma(\mathbf{W}_2).$$

Q.E.D.

(4.23) is satisfied for small enough rates $\partial_\sigma(\mathbf{W}_1)$, $\partial_\sigma(\mathbf{W}_2)$, and $\partial_\sigma(\mathbf{U})$. In other words, for slow-enough systems, the upper bound (4.22) on the global sensitivity approximates the supreme of the local H_σ^∞ minima.

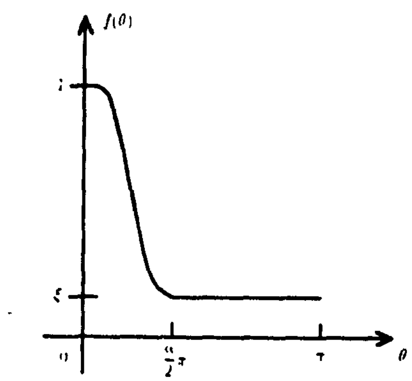
4.3 Robust vs Adaptive Sensitivity Minimization

Information about uncertain perturbations or disturbances is represented by a weighting operator $\mathbf{W} \in \mathbf{E}_\sigma$, $\sigma > 1$. At time t, disturbance pasts are assumed to lie in the image under $\mathbf{W}_t$ of the unit ball of $\ell_\sigma^2(-\infty, t)$ in the case of noise, or of H_σ^∞ in the case of transfer function uncertainty. We distinguish *apriori* information at some starting time t_o, and *aposteriori* information at time $\tau \geq t_o$ represented by operators $\mathbf{W}^o$ and $\mathbf{W}^\tau$. The difference between $\mathbf{W}^o$ and $\mathbf{W}^\tau$ represents a reduction of uncertainty or acquisition of information in the interval $[t_o, \tau]$, and this reduction is reflected in a shrinkage of weighting, $|(\mathbf{W}^\tau)_t(z)| \leq |(\mathbf{W}^o)_t(z)|$ for at least some $t \geq \tau$ and z in some subset of the circle $|z| = \sigma$ of non-zero length. A sensitivity reduction scheme will be called *robust* or *adaptive* if based on *apriori or aposteriori* information respectively. A controller which achieves a sensitivity which is better than an optimal robust one is necessarily adaptive, and the question arises how much advantage adaptation provides. For slowly time-varying systems, this can be answered independently of how the information was obtained.

Example 4.1:

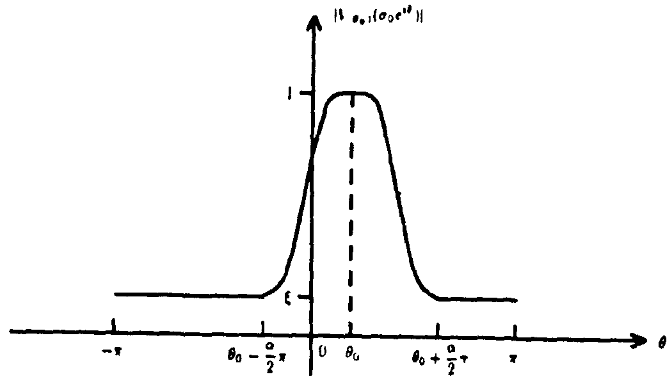
We will introduce a family of "narrow band" disturbance weighting functions whose center frequencies become known with increasing accuracy, and whose envelope is easy to compute.

Let $f(\cdot) : [0,\pi] \to \mathbb{R}$ be a differentiable monotone decreasing function satisfying $f(0) = 1, f(\theta) = \xi$ for $\theta \geq \frac{\alpha}{2}\pi$, where $0 < \xi \ll 1,\ 0 < \alpha \ll 1$ are constants, $f(\cdot)$ will be fixed.



Let $\sigma_0 > 1$ be fixed. A narrowband weighting $V_{(\theta_0)} \in H^\infty_{\sigma_0}$, $\sigma_0 > 1$ with center $\theta_0, \frac{\alpha}{2}\pi \leq \theta_0 \leq (1 - \frac{\alpha}{2})\pi$ is a function such that $V_{(\theta_0)}(\sigma_0(\cdot))$ is outer in H^∞, defined in terms of its boundary magnitude by

$$|V_{(\theta_0)}(\sigma_0 e^{i\theta})| = \begin{cases} f(|\theta - \theta_0|) & \text{for } 0 \le \theta \le \pi, \\ \left|V_{(\theta_0)}(\sigma_0 e^{-i\theta})\right| & \text{for } -\pi \le \theta \le 0. \end{cases}$$



Narrowband disturbances with uncertain center frequencies will be represented as elements of a family of such narrowband weightings,

$$\mathcal{F}(\beta, c) = \left\{ V_{(\theta_0)} \in H^\infty_{\sigma_0} : |\theta_0 - c| \le \beta \le 1 - \alpha \right\}.$$

The center frequencies lie in an interval with midpoint c and width β; β is a measure of uncertainty about center frequencies.

Let $\widetilde{V}_{(\beta,c)} \in H^\infty_{\sigma_0}$ denote the *envelope* weighting of the family, $\widetilde{V}_{(\beta,c)}(\sigma_0(\cdot))$ outer in H^∞, and satisfying

$$\left|\widetilde{V}_{(\beta,c)}(\sigma_0 e^{i\theta})\right| = \sup_{V \in \mathcal{F}(\beta,c)} \left|V(\sigma_0 e^{i\theta})\right|, \quad -\pi \le \theta \le \pi.$$

Apriori information about the disturbances is that they belong to the family $\mathcal{F}(\beta_0, c_0)$. (The apriori weighting is assumed to be time invariant.)

Sensitivity is to be minimized in a SISO time-invariant plant $\mathbf{G} \in \mathbb{E}_{\sigma_0}$, whose inner part consists of one zero at the origin. For $1 < \sigma < \sigma_0$, the inner factor in $\mathbb{E}_\sigma$ is $U(z) = \sigma^{-1}z$. A robust control based on the apriori envelope, $\widehat{\mathbf{W}}_0 = \widetilde{V}_{(\beta_0,c_0)}$ achieves

$$\mu_0(\mathbf{S}_{rbst}) = \inf_{\mathbf{Q}\in\mathbf{E}_\sigma} \mu_\sigma(\mathbf{W}_0 - \mathbf{UQ}) = \widehat{\mathbf{W}}_0(0) \tag{4.24}$$

In an interval $[0,t]$, additional information is received about the disturbances, and results in a shrinkage in aposteriori uncertainty about the center frequency parameter, i.e., β_t is monotone decreasing as $t \to \infty$. An adaptive local optimization of the worst case sensitivity, based on the aposteriori envelope

$$\widehat{\mathbf{W}}_t := \begin{cases} \widetilde{V}_{(\beta_t,c_t)} & \text{for } t \geq 0, \\ \widetilde{V}_{(\beta_0,c_0)} & \text{for } t \leq 0, \end{cases}$$

based on Theorem 4.1, achieves

$$\mu_\sigma\left[\left(\mathbf{S}^{\ell}_{adpt}\right)_t\right] = \inf_{\mathbf{Q}\in\mathbf{E}_\sigma} \|\widehat{\mathbf{W}}_t - \widehat{\mathbf{G}}_t\widehat{\mathbf{Q}}_t\|_{H^\infty_\sigma} = \widehat{\mathbf{W}}_t(0) \tag{4.26}$$

and the resulting adaptive sensitivity achieved is

$$\mathbf{S}_{adpt} = \mathbf{S}^{\ell}_{adpt} + \mathbf{U}\nabla\left(\mathbf{U}^{\ominus} \otimes (\mathbf{W} - \mathbf{S}^{\ell})\right).$$

The constants in (4.24-4.25) can be expressed in terms of the logarithmic bandwidth $\phi(t)$ of the envelope at time t, defined by

$$\log\phi(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|\widetilde{V}_{(\beta_t,c_t)}(\sigma_0 e^{i\theta})\right| d\theta.$$

From the assumption that $f(\theta) = \xi$ for $\theta \geq \frac{\alpha}{2}\pi$ and the fact that $|\widetilde{V}_{(\beta_0,c_0)}(\cdot)|$ is a widening of $|\widetilde{V}_{(\beta_t,c_t)}(\cdot)|$ by $(\beta_0 - \beta_t)$, we deduce that

$$\log\phi(t) = \log\phi(0) + (\beta_0 - \beta_t)\log\xi.$$

By Jensen's Theorem,

$$\widetilde{V}_{(\beta_t,c_t)}(0) = e^{\log\phi(t)} = \phi(0)\xi^{(\beta_0-\beta_t)}. \tag{4.26}$$

Let us evaluate the recent past norms $\|\cdot\|_{a(\sigma;t)}$ of the sensitivity for the robust and adaptive controllers. In the robust case

$$\mu_1(\mathbf{S}_{rbst}) \leq \|\mathbf{S}_{rbst}\|_{a(\sigma)} \leq \mu_\sigma(\mathbf{S}_{rbst}) = \phi(0).$$

In this simple example, the $\mu_\sigma(\mathbf{S}_{rbst})$ norm is independent of σ and we get, from (4.24) and (4.26)

$$\|\mathbf{S}_{rbst}\|_{a(\sigma)} = \phi(0). \tag{4.27}$$

In the adaptive case, (4.25) and (4.26) give

$$\mu_\sigma\left[\left(\mathbf{S}^{\ell}_{adpt}\right)_t\right] = \phi(0)\xi^{(\beta_0-\beta_t)}. \tag{4.28}$$

Suppose now that β_t and c_t change slowly, $|\beta_t - \beta_{t-1}| \leq \rho_\beta, |c_t - c_{t-1}| \leq \rho_c$, and $|\frac{df(\theta)}{d\theta}| \leq \rho_f$. The rates of $\mathbf{W}$ and $\mathbf{S}^{\ell}_{adpt}$ are

$$\partial_\sigma(\mathbf{W}) \leq \rho := \rho_f(\rho_\beta + \rho_c)$$

$$\partial_\sigma(\mathbf{S}^{\ell}_{adpt}) = \partial_\sigma(\widetilde{V}_{(\beta_t,c_t)}(0)) \leq \partial_\sigma(\mathbf{W}).$$

As $\mathbf{S}^{\ell}_{adpt}$ depends Lipschitz continuously ($L^\infty \rightarrow L^\infty$) on $\mathbf{W}$, the rate of the local optimal sensitivity becomes small as $\rho \rightarrow 0$, and we can base our solution on it rather than on the δ-suboptimal one. To evaluate the upper bound in Theorem 4.1, we note that $\widehat{\mathbf{G}}_W(z) = \widehat{\mathbf{U}}(z) = \sigma^{-1}z$, $\mu_\sigma(\widehat{\mathbf{G}}_W) = 1$,

$$\begin{aligned}\theta(\infty,\sigma) = \partial_\sigma(\widehat{\mathbf{F}}) &\leq \partial_\sigma(\mathbf{W}) + \partial_\sigma(\mathbf{S}^{\ell}_{adpt}) \\ &\leq 2\rho\end{aligned}$$

which gives

$$\|\mathbf{S}_{adpt}\|_{a(\sigma;t)} \leq \mu_\sigma(\mathbf{S}^{\ell}_{adpt}) + \rho\kappa_\sigma^{(\infty)}[1 + 2\kappa_\sigma]. \tag{4.29}$$

A lower bound is computed using (3.26), giving

$$\|\mathbf{S}_{adpt}\|_{a(\sigma;t)} \geq \mu_1(\mathbf{S}^{\ell}_{adpt}) - \rho\kappa^{(\infty)}_{\sigma}[1 + 2\kappa_{\sigma}]. \tag{4.30}$$

As μ_σ is independent of σ in (4.29-4.30), and using (4.28),

$$\left| \|\mathbf{S}_{adpt}\|_{a(\sigma;t)} - \phi(0)\xi^{(\beta_0 - \beta_t)} \right| \leq \rho\kappa^{(\infty)}_{\sigma}[1 + 2\kappa_{\sigma}]$$

and by (4.27),

$$\frac{\|\mathbf{S}_{adpt}\|_{a(\sigma;t)}}{\|\mathbf{S}_{rbst}\|_{a(\sigma;t)}} \leq \xi^{(\beta_0 - \beta_t)} + \rho\kappa^{(\infty)}_{\sigma}[1 + 2\kappa_{\sigma}]\phi^{-1}(0). \tag{4.31}$$

In the limit of slow time variation, as $\rho \to 0$, (4.31) shows that adaptive sensitivity is better than robust sensitivity by a factor $\xi^{(\beta_0 - \beta_t)}$, where $(\beta_0 - \beta_t)$ is the reduction in log-bandwidth of the disturbance weighting resulting from extra information about disturbances acquired in the intervening interval $[0, t]$.

Chapter 5 δ-suboptimal Lipschitz Continuous Designs

Theorem 4.1 and its corollary suggest that the global synthesis of the sensitivity operator can be approximately realized by slowly time-varying local interpolants. A sufficient condition for slow variation is that $\mathbf{S}^l$ defined in (4.5) be Lipschitz continuous with respect to $\mathbf{W}$ $(= \mathbf{W}_2 \otimes \mathbf{W}_1)$ and $\mathbf{U}$, i.e.,

$$\|\widehat{\mathbf{S}}_t^l - \widehat{\mathbf{S}}_{t-1}^l\|_{H^p_{\sigma_0}} \leq \gamma_W^{(p)} \|\widehat{\mathbf{W}}_t^l - \widehat{\mathbf{W}}_{t-1}^l\|_{H^\infty_{\sigma_0}} + \gamma_U^{(p)} \|\widehat{\mathbf{U}}_t^l - \widehat{\mathbf{U}}_{t-1}^l\|_{H^\infty_{\sigma_0}} \quad \forall t \in \mathbb{Z}. \tag{5.1}$$

Since a change of variable from z to $\sigma_0 z$ will transfer the results on H^∞ to $H^\infty_{\sigma_0}$, we will concentrate here on H^∞. Thus assume $\mathbf{W}$, $\mathbf{U} \in \mathbf{E}_1$ and $\mathbf{U}$ is locally inner in $\mathbf{E}_1$ (i.e., $\widehat{\mathbf{U}}_t^*(e^{i\theta})\widehat{\mathbf{U}}_t(e^{i\theta}) = \mathbf{I}$ for $\theta \in [-\pi, \pi)$). The H^∞-norm $\|\cdot\|_{H^\infty}$ will be abbreviated as $\|\cdot\|_\infty$ in this chapter.

Suppose now $\mathbf{S}^l$ is a *local interpolant* of $\mathbf{W}$ at $\mathbf{U}$ in $\overline{\mathbf{E}}_1$, i.e., there exists $\mathbf{Q} \in \overline{\mathbf{E}}_1$ such that

$$\mathbf{S}^l = \mathbf{W} - \mathbf{U} \otimes \mathbf{Q}. \tag{5.2}$$

$\mathbf{S}^l$ is said to be a *local optimal interpolant* of $\mathbf{W}$ at $\mathbf{U}$ if $\mathbf{S}_t^l$ is in fact the optimal solution to the local interpolation problem,

$$\mu_t = \inf_{\widehat{\mathbf{Q}}_t \in H^\infty} \|\widehat{\mathbf{W}}_t - \widehat{\mathbf{U}}_t \widehat{\mathbf{Q}}_t\|_\infty \qquad t \in \mathbb{Z}. \tag{5.3}$$

We are interested here in the Lipschitz continuity of the local optimal or sub-optimal solution $\mathbf{S}^l = \mathbf{W} - \mathbf{U} \otimes \mathbf{Q}$ with respect to $\mathbf{W}$ and $\mathbf{U}$.

5.1 Lipschitz Continuity and Lipschitz Continuity In Norm

Given $\mathbf{W}$, $\mathbf{U} \in \overline{\mathbf{E}}_1$ with $\mathbf{U}$ locally inner in $\overline{\mathbf{E}}_1$. Suppose $\mathbf{S}^l$ is a local interpolant of $\mathbf{W}$ at $\mathbf{U}$ in $\overline{\mathbf{E}}_1$.

$\mathbf{S}^l$ is said to be *Lipschitz continuous* ($L^\infty \to L^2$) with constants γ_W and γ_U if constants γ_W and γ_U can be found for which

$$\|\widehat{\mathbf{S}}_t - \widehat{\mathbf{S}}_{t-1}\|_{H^2} \le \gamma_W \|\widehat{\mathbf{W}}_t - \widehat{\mathbf{W}}_{t-1}\|_\infty + \gamma_U \|\widehat{\mathbf{U}}_t - \widehat{\mathbf{U}}_{t-1}\|_\infty \qquad \forall t \in \mathbb{Z}. \tag{5.8}$$

If we have only

$$\left| \|\widehat{\mathbf{S}}_t\|_\infty - \|\widehat{\mathbf{S}}_{t-1}\|_\infty \right| \le \gamma_W \|\widehat{\mathbf{W}}_t - \widehat{\mathbf{W}}_{t-1}\|_\infty + \gamma_U \|\widehat{\mathbf{U}}_t - \widehat{\mathbf{U}}_{t-1}\|_\infty, \tag{5.9}$$

then $\mathbf{S}^l$ is said to be *Lipschitz continuous in norm* with constants γ_W and γ_U.

The problem (5.3) can be transferred into an equivalent Nehari distance problem in L^∞:

$$\begin{aligned} \mu_t &= \inf_{\widehat{\mathbf{Q}}_t \in H^\infty} \|\widehat{\mathbf{W}}_t - \widehat{\mathbf{U}}_t \widehat{\mathbf{Q}}_t\|_\infty \\ &= \inf_{\widehat{\mathbf{Q}}_t \in H^\infty} \|\widehat{\mathbf{U}}_t \left(\widehat{\mathbf{U}}_t^* \widehat{\mathbf{W}}_t - \widehat{\mathbf{Q}}_t \right) \|_{L^\infty} \\ &= \inf_{\widehat{\mathbf{Q}}_t \in H^\infty} \| \left(\widehat{\mathbf{U}}_t^* \widehat{\mathbf{W}}_t - \widehat{\mathbf{Q}}_t \right) \|_{L^\infty} \qquad t \in \mathbb{Z} \end{aligned} \tag{5.10}$$

where $\widehat{\mathbf{U}}_t^*$ denotes the complex conjugate and transpose of $\widehat{\mathbf{U}}_t$.

We will show later in section 5.3 that the local optimal interpolant $\mathbf{S}^l$ is Lipschitz continuous in norm. But for the moment, we will demonstrate by an example that the local optimal interpolant $\mathbf{S}^l$ is in general not Lipschitz continuous.

Example 5.1:

Consider the problem of optimally interpolating (W_ω, U) in H^∞, where $U \in H^\infty$ is fixed, $U(z) = \frac{\beta_1 - z}{\beta_1 z - 1}\frac{\beta_2 - z}{\beta_2 z - 1}$, $0 < \beta_i < 1$, $(i = 1,2)$, and $W_\omega \in H^\infty$ is variable depending on a parameter $\omega > 0$. By the Nevanlinna-Pick theory, the optimal interpolant of (W_ω, U) has the form

$$S_\omega = \mu_\omega \frac{(\alpha - z)}{(\alpha z - 1)}, \; |\alpha| < 1, \; \mu_\omega \in \mathbb{R}, \tag{5.11}$$

where S_ω satisfies the interpolation constraints $S_\omega(\beta_i) = W_\omega(\beta_i)$, $i = 1,2$. Consider any W_ω for which the ratio $W_\omega(\beta_2)/W_\omega(\beta_1) =: \rho_\omega$ approaches 1 as $\omega \to 0$, and which satisfies the inequality

$$\left|\frac{d\rho_\omega}{d\omega}\right| \geq \theta \frac{\|dW\|}{|d\omega|}, \qquad \theta > 0.$$

For example, $W_\omega := 1 + \omega W'$, where $W' \in H^\infty$, $\|W'\|_\infty < 1$, $W'(\beta_1) = 0$, $W'(\beta_2) > 0$ will have these properties. We will show that as $\omega \to 0$, $\|dS_\omega\|_{H^2}/\|dW_\omega\|_\infty \to \infty$, implying that the optimal interpolant is not Lipschitz.

As $\omega \to 0$ we get

$$\frac{dS}{\|dW\|_\infty} = \mu \frac{(z^2 - 1)}{(\alpha z - 1)^2}\frac{d\alpha}{\|dW\|_\infty} + \frac{(\alpha - z)}{(\alpha z - 1)}\frac{d\mu}{\|dW\|_\infty} \tag{5.12}$$

where S, W, α, μ, all depend on ω. The term proportional to $d\mu$ is ≤ 1 for $|z| \leq 1$, so it is enough to establish the unboundedness of the term proportional to $d\alpha$. Now $\omega \to 0$ implies that $\rho_\omega \to 1$ which, it is not hard to show, implies that $\alpha \to -1$ from the right, and $|d\alpha/d\omega| \to |\beta_2 - \beta_1|^{-1}\sqrt{(1-\beta_2^2)(1-\beta_1^2)}W'(\beta_2) > 0$. Therefore, for ω small enough

$$\begin{aligned}\frac{|dS(z)|}{\|dW\|_\infty} &\geq \left|\mu \frac{(z^2-1)}{(\alpha z - 1)^2}\frac{d\alpha}{d\omega}\frac{d\omega}{\|dW\|_\infty}\right| - 1 \\ &\geq Const. \frac{|z^2 - 1|}{|\alpha z - 1|^2} - 1, \qquad |z| \leq 1,\end{aligned} \tag{5.13}$$

as $d\omega/\|dW\|_\infty = (\|W'\|_\infty)^{-1}$. Contour integration now gives

$$\|dS\|_{H^2}/\|dW\|_\infty \geq Const.(1-\alpha^2)^{-\frac{1}{2}} - 1, \tag{5.14}$$

which grows without bound as $\alpha \to -1$, and therefore as $\omega \to 0$.

□

In this example, the optimal sensitivity S becomes very sensitive to perturbations in U and W when W takes values close to each other at the zeros of the plant inside the unit disk. How general this phenomenon is will be a task of future research. This example shows that the local optimal interpolant is not a suitable candidate for the local interpolation outlined in Chapter 4.

Although the optimal solution is not Lipschitz continuous in general, we will show that a δ-suboptimal Lipschitz continuous solution can be constructed. This suboptimal solution is based on the AAK's parametrization, which will be presented in the next section.

5.2 Lipschitz Continuity of AAK's Suboptimal Central Interpolants

Before the description of AAK's parametrization of optimal and suboptimal interpolants to the Nehari distance problem (5.10), an important operator will first be introduced.

Suppose $M \in L^\infty$. Then the following (negative Fourier) coefficients $m_k \in \mathbb{C}^{n\times n}$ $(k = 1, 2, \cdots)$ are well defined,

$$m_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\theta k} M(e^{i\theta})d\theta \qquad k = 1, 2, \cdots.$$

An operator $\Gamma_M : l_+^2 \to l_+^2$ ($l_+^2 := l^2[0,\infty)$) is called an Hankel operator with symbol M if it is defined by an (infinite) Hankel matrix,

$$\Gamma_M = [m_{j+k-1}] \qquad j,k \in [1,\infty). \tag{5.15}$$

Hankel operators play an important role in harmonic analysis of the functions in L^∞. By Nehari theorem, the distance between an function $M \in L^\infty$ and the space H^∞ is precisely the norm of the Hankel operator with symbol M, i.e.,

$$\mu = \inf_{Q \in H^\infty} \|M - Q\|_{L^\infty} = \|\Gamma_M\|. \tag{5.16}$$

For $\delta > 0$, a function $S \in L^\infty$ is said to be a δ-suboptimal interpolant of $M \in L^\infty$ in (5.16) if

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\theta k} S(e^{i\theta})d\theta = m_{-k} \qquad k = 1,2,\cdots$$

and

$$\|S\|_{L^\infty} \leq \mu + \delta. \tag{5.17}$$

Adamjan, Arov and Krein [Ada2] give a complete parametrization of all δ-suboptimal interpolants of $M \in L^\infty$. To describe the parametrization, we will first define, following AAK's notation, the following operators. Let $M \in L^\infty$, $\delta > 0$, $\rho =$

$\mu + \delta$, where μ is defined in (5.16).

$$
\begin{aligned}
\Gamma &:= \Gamma_M: \quad l^2_+ \to l^2_+; \\
\mathbf{R}_{\rho^2} &= (\rho^2 \mathbf{I} - \Gamma^* \Gamma)^{-1}: \quad l^2_+ \to l^2_+; \\
\widetilde{\mathbf{R}}_{\rho^2} &= (\rho^2 \mathbf{I} - \Gamma \Gamma^*)^{-1}: \quad l^2_+ \to l^2_+; \\
\mathbf{G}(\Gamma, \rho) &= [\Pi_{\mathbf{C}^n} \mathbf{R}_{\rho^2}|_{\mathbf{C}^n}]^{-\frac{1}{2}}: \quad \mathbf{C}^n \to \mathbf{C}^n; \\
\widetilde{\mathbf{G}}(\Gamma, \rho) &= [\Pi_{\mathbf{C}^n} \widetilde{\mathbf{R}}_{\rho^2}|_{\mathbf{C}^n}]^{-\frac{1}{2}}: \quad \mathbf{C}^n \to \mathbf{C}^n; \\
\mathbf{P} &= \rho \mathbf{R}_{\rho^2} \mathbf{G}(\Gamma, \rho): \quad \mathbf{C}^n \to l^2_+; \\
\mathbf{Q} &= \mathbf{T} \Gamma \mathbf{R}_{\rho^2} \mathbf{G}(\Gamma, \rho): \quad \mathbf{C}^n \to l^2_+; \\
\widetilde{\mathbf{P}} &= \rho \widetilde{\mathbf{R}}_{\rho^2} \widetilde{\mathbf{G}}(\Gamma, \rho): \quad \mathbf{C}^n \to l^2_+; \\
\widetilde{\mathbf{Q}} &= \mathbf{T} \Gamma^* \widetilde{\mathbf{R}}_{\rho^2} \widetilde{\mathbf{G}}(\Gamma, \rho): \quad \mathbf{C}^n \to l^2_+,
\end{aligned}
\tag{5.18}
$$

where $\Pi_{\mathbf{C}^n}$ is the projection operator from l^2 onto $\mathbf{C}^n$, $\mathbf{T}$ is the right shift operator in l^2_+: for $u \in l^2_+$, $(\mathbf{T}u)(t) = u(t-1)$ for $t \geq 1$ and 0 for $t = 0$, and $(\rho^2 \mathbf{I} - \Gamma^* \Gamma)^{-1}$ (and $(\rho^2 \mathbf{I} - \Gamma \Gamma^*)^{-1}$) exists since $\rho > \|\Gamma\|$. For the same reason $\mathbf{G}(\Gamma, \rho)$ (and $\widetilde{\mathbf{G}}(\Gamma, \rho)$) exists. For simplicity of notation, write $\mathbf{R} = \mathbf{R}_{\rho^2}$; $\widetilde{\mathbf{R}} = \widetilde{\mathbf{R}}_{\rho^2}$; $\mathbf{G} = \mathbf{G}(\Gamma, \rho)$; $\widetilde{\mathbf{G}} = \widetilde{\mathbf{G}}(\Gamma, \rho)$, in the rest of Chapter 5.

Let $\mathcal{L}[x]$, for $x \in l^2_+$, denote the usual z-transform of x, i.e., if $x = \{x_m,\ m = 0, 1, \cdots\}$ then $\mathcal{L}[x](z) = \sum_{m=0}^{\infty} x_m z^m$, $|z| = 1$. Then for $\mathbf{P}$ (similarly for $\mathbf{Q}, \widetilde{\mathbf{P}}, \widetilde{\mathbf{Q}}$) defined in (5.18), $\mathcal{L}[\mathbf{P}h]$ defines an operator from $\mathbf{C}^n$ to L^2. If $\{\xi_1, \xi_2, \cdots, \xi_n\}$ is the axis for $\mathbf{C}^n$ (i.e., $\xi_1 = (1,0,0,\cdots,0)^T$, $\xi_2 = (0,1,0,\cdots,0)^T$, etc.), then

$$
P_+(z) := [\mathcal{L}[\mathbf{P}\xi_1](z), \cdots, \mathcal{L}[\mathbf{P}\xi_n](z)] \qquad |z| = 1 \tag{5.19}
$$

defines uniquely (in L^2-sense) an function P_+ in L^2 which satisfies

$$
P_+(z)h = \mathcal{L}[\mathbf{P}h](z) \qquad h \in \mathbf{C}^n,\ |z| = 1. \tag{5.20}
$$

Similarly, we define functions Q_+, P_-, $Q_- \in L^2$ via, for $|z| = 1$, $h \in \mathbb{C}^n$

$$
\begin{aligned}
Q_+(z)h &= \mathcal{L}[\widetilde{\mathbf{Q}}h](z), \\
P_-(z)h &= \mathcal{L}[\widetilde{\mathbf{P}}h](\bar{z}), \qquad (5.21) \\
Q_-(z)h &= \mathcal{L}[\mathbf{Q}h](\bar{z}).
\end{aligned}
$$

P_+, Q_+ have analytic continuation into the unit disk, and hence $P_+, Q_+ \in H^2$. Similarly $P_-, Q_- \in L^2 \ominus H^2$, the orthogonal complement of H^2 in L^2.

By AAK [Ada2,p150], the following identity holds: for $|z| = 1$

$$P_+^*(z)P_+(z) - Q_+^*(z)Q_+(z) = I \qquad a.e. \text{ (in Lebesque measure)}, \qquad (5.22)$$

which implies that for any $E \in H^\infty$, $\|E\|_{H^\infty} \leq 1$, $P_+(z)$ and $P_+(z) + Q_+(z)E(z)$ are invertible (a.e. $|z| = 1$), and also

$$\left|P_+^{-1}(z)\right| \leq 1 \qquad (a.e). \qquad (5.23)$$

From now on, the specification "a.e." (almost everywhere in Lebesque measure) will be dropped from notation.

By Adamjan, Arov and Krein [Ada2, Theorem 6.1], the formulae

$$S_E(z) = \rho\,(Q_-(z) + P_-(z)E(z))\,(P_+(z) + Q_+(z)E(z))^{-1} \qquad (5.24)$$

where $E \in H^\infty$ and $\|E\|_\infty \leq 1$ gives a complete parametrization of all δ-suboptimal ($\rho = \mu + \delta$) interpolants $S_E \in L^\infty$ of $M \in L^\infty$ in (5.16).

Take especially $E = 0$, called the centre solution (or the maximum entropy solution) by AAK, we have

$$S_0(z) = \rho Q_-(z)P_+^{-1}(z) \qquad |z| = 1. \qquad (5.25)$$

Next, let M be a variable in a subset $\mathbf{M} \subset L^\infty$. Define a mapping Φ : $\mathbb{R}_+ \times \mathbf{M} \rightarrow L^\infty$ by

$$\Phi(\delta, M) = S(M) \tag{5.26}$$

where $S(M)$ is a δ-suboptimal interpolant of M.

$\Phi(\cdot,\cdot)$ is said to be Lipschitz continuous in $\mathbf{M} \subset L^\infty$ if a constant γ_δ can be found for which

$$\|\Phi(\delta, M_1) - \Phi(\delta, M_2)\|_{L^2} \leq \gamma_\delta \|\Gamma_{M_1} - \Gamma_{M_2}\| \quad \forall M_1, M_2 \in \mathbf{M}. \tag{5.27}$$

The mapping $\Phi(\cdot,\cdot)$ is called central, denoted by $\Phi_0(\cdot,\cdot)$, if

$$\Phi_0(\delta, M) = S_0(M)$$

where $S_0(M)$ is the δ-suboptimal central interpolant of M defined in (5.25).

Let $\delta > 0$ and

$$\mu_M = \inf_{Q \in H^\infty} \|M - Q\|_{L^\infty},$$

$$\rho_M = \mu_M + \delta,$$

$$\mu = \sup_{M \in \mathbf{M}} \mu_M,$$

$$\rho = \mu + \delta.$$

Theorem 5.1

The central mapping $\Phi_0(\cdot,\cdot)$ is Lipschitz continuous with constant

$$\gamma_\delta = 2n^{1/2} \frac{\sqrt{\rho^2 + \mu^2}}{\delta} \left[1 + \frac{2\mu}{\delta}\right]. \tag{5.28}$$

□

To prove the theorem we need some intermediate results.

First of all, the expression of $S_0(z)$ in (5.25) can be simplified. Define

$$\begin{aligned} \mathbf{P}^o &= \rho \mathbf{R}|_{\mathbb{C}^n} \qquad \mathbb{C}^n \to l_+^2, \\ \mathbf{Q}^o &= \mathbf{T}\Gamma\mathbf{R}|_{\mathbb{C}^n} \qquad \mathbb{C}^n \to l_+^2. \end{aligned} \tag{5.29}$$

Then

$$\mathbf{P} = \mathbf{P}^o\mathbf{G}, \qquad \mathbf{Q} = \mathbf{Q}^o\mathbf{G} \tag{5.30}$$

where $\mathbf{G} : \mathbb{C}^n \to \mathbb{C}^n$ is memoryless and hence admits a matrix representation $G \in \mathbb{C}^{n\times n}$.

Similar to (5.19), (5.21), define

$$\begin{aligned} P^o(z) &:= [\mathcal{L}[\mathbf{P}^o\xi_1](z), \cdots, \mathcal{L}[\mathbf{P}^o\xi_n](z)] \qquad |z| = 1, \\ Q^o(z) &:= [\mathcal{L}[\mathbf{Q}^o\xi_1](\overline{z}), \cdots, \mathcal{L}[\mathbf{Q}^o\xi_n](\overline{z})] \qquad |z| = 1. \end{aligned} \tag{5.31}$$

Then for $|z| = 1$,

$$\begin{aligned} P_+(z) &= P^o(z)G, \\ Q_-(z) &= Q^o(z)G. \end{aligned} \tag{5.32}$$

Now

$$\begin{aligned} S_o(z) &= \rho Q_-(z)P_+^{-1}(z) \\ &= \rho Q^o(z)(P^o(z))^{-1}. \end{aligned} \tag{5.33}$$

The subscript o in $S_o(z)$ will be omitted in the rest of the chapter since the central solution is the only solution involved.

To simplify notation, we will always use $\|\cdot\|$ to denote operator norms although its precise meaning will depend on individual input-output spaces.

<u>**Lemma 5.1**</u>

Suppose $\mathbf{K} : \mathbb{C}^n \to l_+^2$ and $K(z)h = \mathcal{L}[\mathbf{K}h](z)$, or $K(z)h = \mathcal{L}[\mathbf{K}h](\overline{z})$, $|z| = 1$, $h \in \mathbb{C}^n$. Then the following norm inequality holds:

$$\|\mathbf{K}\|_{L^2} \leq n^{1/2}\|\mathbf{K}\|. \tag{5.34}$$

□

Proof:

By definition, for any $h \in \mathbb{C}^n$

$$K(z)h = \mathcal{L}[\mathbf{K}h](z) \quad \in L^2.$$

By Parseval's theorem

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} h^* K^*(e^{i\theta})K(e^{i\theta})h d\theta &= h^*\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} K^*(e^{i\theta})K(e^{i\theta})d\theta\right)h \\ &= \|\mathbf{K}h\|^2_{l^2_+}.\end{aligned} \tag{5.35}$$

Thus

$$\|\mathbf{K}\|^2 = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} K^*(e^{i\theta})K(e^{i\theta})d\theta\right|. \tag{5.36}$$

We must prove that

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\pi}^{\pi} |K^*(e^{i\theta})K(e^{i\theta})|d\theta \\ &\leq n\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} K^*(e^{i\theta})K(e^{i\theta})d\theta\right|.\end{aligned}$$

From the matrix inequalities: for any $A \in \mathbb{C}^{n\times n}$

$$|A|^2 \leq Trace(A^*A) \leq n|A|^2,$$

we obtain

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\pi}^{\pi} |K^*(e^{i\theta})K(e^{i\theta})|d\theta \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} Trace\left(K^*(e^{i\theta})K(e^{i\theta})\right)d\theta \\ &= \frac{1}{2\pi}Trace\left(\int_{-\pi}^{\pi} K^*(e^{i\theta})K(e^{i\theta})d\theta\right) \\ &\leq n\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} K^*(e^{i\theta})K(e^{i\theta})d\theta\right|,\end{aligned}$$

which proves the lemma.

Q.E.D.

In the rest of this section, the subscript i ($i = 1, 2$) in all operators defined in (5.18) and (5.29) indicates the operators corresponding to the symble $M_i \in L^\infty$, and $\Delta\mathbf{K} := \mathbf{K}_2 - \mathbf{K}_1$.

Without loss of generality, assume $\rho_2 \leq \rho_1$ in this section.

Lemma 5.2

For the operators $\mathbf{P}^o, \mathbf{P}_1^o, \mathbf{P}_2^o$, defined in (5.29),

$$\|\mathbf{P}^o\| \leq \frac{1}{\delta}, \tag{5.37}$$

$$\|\mathbf{P}_1^o - \mathbf{P}_2^o\| \leq \frac{1}{\rho_2 + \mu_2}\frac{1}{\delta}\left(1 + \frac{\mu_1 + 3\mu_2}{\delta}\right)\|\Gamma_2 - \Gamma_1\|. \tag{5.38}$$

□

Proof:

By definition,

$$\mathbf{P}^o = \rho\mathbf{R}|_{\mathbb{C}^n}. \tag{5.39}$$

Since for $x \in l_+^2$,

$$\begin{aligned}\|(\rho^2\mathbf{I} - \Gamma^*\Gamma)x\|_{l_+^2} &\geq \rho^2\|x\| - \mu^2\|x\| \\ &= (\rho^2 - \mu^2)\|x\|,\end{aligned} \tag{5.40}$$

by Banach algebra inverse mapping theorem,

$$\|\mathbf{R}\| = \|(\rho^2\mathbf{I} - \Gamma^*\Gamma)^{-1}\| \leq \frac{1}{\rho^2 - \mu^2} \tag{5.41}$$

Now, by (5.39) and (5.41), as well as $\rho - \mu = \delta$,

$$\begin{aligned} \|\mathbf{P}^o\| &= \|\rho \mathbf{R}|_{\mathbf{C}^n}\| \\ &\leq \rho \|\mathbf{R}\| \\ &\leq \frac{\rho}{\rho^2 - \mu^2} \\ &= \frac{\rho}{\delta(\rho+\mu)} \\ &\leq \frac{1}{\delta}, \end{aligned} \tag{5.42}$$

which proves (5.37).

To prove (5.38), define

$$\widetilde{\mathbf{K}}_i = \frac{1}{\rho_i^2}\Gamma_i^*\Gamma_i \qquad i = 1, 2. \tag{5.43}$$

Then

$$\begin{aligned} \rho_2 \mathbf{R}_2 - \rho_1 \mathbf{R}_1 &= \frac{1}{\rho_2}(\mathbf{I} - \widetilde{\mathbf{K}}_2)^{-1} - \frac{1}{\rho_1}(\mathbf{I} - \widetilde{\mathbf{K}}_1)^{-1} \\ &= (\frac{1}{\rho_2} - \frac{1}{\rho_1})(\mathbf{I} - \widetilde{\mathbf{K}}_2)^{-1} + \frac{1}{\rho_1}\left[(\mathbf{I} - \widetilde{\mathbf{K}}_2)^{-1} - (\mathbf{I} - \widetilde{\mathbf{K}}_1)^{-1}\right]. \end{aligned} \tag{5.44}$$

The RHS of (5.44) is bounded via,

$$\begin{aligned} \|(\frac{1}{\rho_2} - \frac{1}{\rho_1})(\mathbf{I} - \widetilde{\mathbf{K}}_2)^{-1}\| &\leq \frac{|\rho_2 - \rho_1|}{\rho_2\rho_1}\frac{1}{1 - \frac{\mu_2^2}{\rho_2^2}} \\ &= \frac{|\rho_2 - \rho_1|}{\rho_1}\frac{\rho_2}{\rho_2^2 - \mu_2^2} \\ &= \frac{|\rho_2 - \rho_1|}{\rho_1}\frac{\rho_2}{\delta(\rho_2 + \mu_2)} \\ &\leq \frac{1}{\delta(\rho_2 + \mu_2)}\|\Gamma_2 - \Gamma_1\|, \end{aligned} \tag{5.45}$$

by the inequalities $\rho_2 \leq \rho_1$ and

$$\begin{aligned} |\rho_2 - \rho_1| &= |\mu_2 - \mu_1| \\ &= |\|\Gamma_2\| - \|\Gamma_1\|| \\ &\leq \|\Gamma_2 - \Gamma_1\|. \end{aligned} \tag{5.46}$$

Also

$$\begin{aligned}
&\|\frac{1}{\rho_1}\left[(\mathbf{I}-\widetilde{\mathbf{K}}_2)^{-1}-(\mathbf{I}-\widetilde{\mathbf{K}}_1)^{-1}\right]\| \\
&\leq \frac{1}{\rho_1}\|(\mathbf{I}-\widetilde{\mathbf{K}}_2)^{-1}\|\|(\mathbf{I}-\widetilde{\mathbf{K}}_1)^{-1}\|\|\widetilde{\mathbf{K}}_2-\widetilde{\mathbf{K}}_1\| \\
&\leq \frac{1}{\rho_1}\frac{\rho_1^2}{\rho_1^2-\mu_1^2}\frac{\rho_2^2}{\rho_2^2-\mu_2^2}\|\widetilde{\mathbf{K}}_2-\widetilde{\mathbf{K}}_1\| \\
&\leq \frac{\rho_1}{\delta(\rho_1+\mu_1)}\frac{\rho_2^2}{\delta(\rho_2+\mu_2)}\|\widetilde{\mathbf{K}}_2-\widetilde{\mathbf{K}}_1\| \\
&\leq \frac{1}{\delta^2}\frac{\rho_1\rho_2}{\rho_2+\mu_2}\|\widetilde{\mathbf{K}}_2-\widetilde{\mathbf{K}}_1\|,
\end{aligned} \tag{5.47}$$

as $\rho_2 \leq \rho_1 \leq \rho_1+\mu_1$.

However

$$\begin{aligned}
&\rho_1\rho_2\|\widetilde{\mathbf{K}}_2-\widetilde{\mathbf{K}}_1\| \\
&= \rho_1\rho_2\|\frac{1}{\rho_2^2}\Gamma_2^*\Gamma_2-\frac{1}{\rho_1^2}\Gamma_1^*\Gamma_1\| \\
&\leq \rho_1\rho_2\left[\left|\frac{1}{\rho_2^2}-\frac{1}{\rho_1^2}\right|\mu_2^2+\frac{1}{\rho_1^2}\|\Gamma_2^*\Gamma_2-\Gamma_1^*\Gamma_1\|\right] \\
&\leq \rho_1\rho_2\left[\frac{|\rho_2-\rho_1|(\rho_2+\rho_1)}{\rho_2^2\rho_1^2}\mu_2^2+\frac{\mu_1+\mu_2}{\rho_1^2}\|\Gamma_2-\Gamma_1\|\right] \\
&\leq \left[|\rho_2-\rho_1|\frac{\rho_2+\rho_1}{\rho_2\rho_1}\mu_2^2+(\mu_1+\mu_2)\frac{\rho_2}{\rho_1}\|\Gamma_2-\Gamma_1\|\right] \\
&\leq \left[|\rho_2-\rho_1|\frac{\mu_2}{\rho_2}(1+\frac{\rho_2}{\rho_1})\mu_2+(\mu_1+\mu_2)\|\Gamma_2-\Gamma_1\|\right] \\
&\leq [|\rho_2-\rho_1|2\mu_2+(\mu_1+\mu_2)\|\Gamma_2-\Gamma_1\|] \\
&\leq (\mu_1+3\mu_2)\|\Gamma_2-\Gamma_1\|
\end{aligned} \tag{5.48}$$

by (5.43) and the inequalities $\mu_2 \leq \rho_2$, $\rho_2 \leq \rho_1$ and $\|\Gamma_i\| = \mu_i$.

Therefore, from (5.44), (5.45) and (5.48),

$$\begin{aligned}
&\|\rho_2\mathbf{R}_2-\rho_1\mathbf{R}_1\| \\
&\leq \frac{1}{\delta(\rho_2+\mu_2)}\|\Gamma_2-\Gamma_1\|+\frac{\mu_1+3\mu_2}{\delta^2(\rho_2+\mu_2)}\|\Gamma_2-\Gamma_1\| \\
&= \frac{1}{\delta(\rho_2+\mu_2)}\left(1+\frac{\mu_1+3\mu_2}{\delta}\right)\|\Gamma_2-\Gamma_1\|.
\end{aligned} \tag{5.49}$$

Now, (5.38) follows from (5.49) as

$$\|\Delta \mathbf{P}^o\| = \|\Delta(\rho \mathbf{R})|_{\mathbb{C}^n}\| \leq \|\Delta(\rho \mathbf{R})\|. \tag{5.50}$$

Q.E.D.

Lemma 5.3

$$|(P^o)^{-1}(z)| \leq \sqrt{\rho^2 + \mu^2} \qquad |z| = 1. \tag{5.51}$$

□

Proof:

By (5.32), $P_+(z) = P^o(z)G$, then

$$(P^o(z))^{-1} = GP_+^{-1}(z) \qquad |z| = 1. \tag{5.52}$$

From (5.25)

$$|P_+^{-1}(z)| \leq 1 \qquad |z| = 1. \tag{5.53}$$

Since $\mathbf{G}$ is memoryless and self-adjoint,

$$\begin{aligned} |G| &= \|\mathbf{G}\| \\ &= \|[\Pi_{\mathbb{C}^n}\mathbf{R}|_{\mathbb{C}^n}]^{-\frac{1}{2}}\| \\ &= \|[\Pi_{\mathbb{C}^n}\mathbf{R}|_{\mathbb{C}^n}]^{-1}\|^{\frac{1}{2}} \\ &= \rho\|[\Pi_{\mathbb{C}^n}(\mathbf{I} - \frac{1}{\rho^2}\Gamma^*\Gamma_J{}^{-1}|_{\mathbb{C}^n}]^{-\frac{1}{2}}\|. \end{aligned} \tag{5.54}$$

For $\alpha := \frac{\mu^2}{\rho^2} < 1$, $\widetilde{\mathbf{K}} := \frac{1}{\rho^2}\Gamma^*\Gamma$, $(\mathbf{I} - \widetilde{\mathbf{K}})^{-1}$ can be written as

$$(\mathbf{I} - \widetilde{\mathbf{K}})^{-1} = \frac{1}{1-\alpha^2}(\mathbf{I} - \mathbf{K}) \tag{5,55}$$

where $\mathbf{K}$ is self-adjoint and $\|\mathbf{K}\| \leq \alpha$.

Let $\mathbf{K}_1 = \Pi_{\mathbb{C}^n}\mathbf{K}|_{\mathbb{C}^n}$. Then

$$\begin{aligned}
\|[\Pi_{\mathbb{C}^n}(\mathbf{I}-\widetilde{\mathbf{K}})^{-1}|_{\mathbb{C}^n}]^{-1}\| &= (1-\alpha^2)\|(\mathbf{I}-\mathbf{K}_1)^{-1}\| \\
&\leq \frac{1-\alpha^2}{1-\|\mathbf{K}_1\|} \\
&\leq \frac{1-\alpha^2}{1-\|\mathbf{K}\|} \\
&\leq \frac{1-\alpha^2}{1-\alpha} \\
&= 1+\alpha.
\end{aligned} \tag{5.56}$$

Therefore

$$\begin{aligned}
|G| &\leq \rho\sqrt{1+\alpha} \\
&= \rho\sqrt{1+\frac{\mu^2}{\rho^2}} \\
&= \sqrt{\rho^2+\mu^2}.
\end{aligned} \tag{5.57}$$

Now, (5.51) follows from (5.52) and (5.57).

Q.E.D.

Proof of Theorem 5.1:

Consider the cent: al solution (5.33),

$$S_i(z) = \rho_i Q_i^o(z)(P_i^o(z))^{-1} \qquad |z|=1, \quad i=1,2. \tag{5.58}$$

$$\begin{aligned}
S_2(z)-S_1(z) &= (\rho_1 Q_1^o(z) - \rho_2 Q_2^o(z))(P_1^o(z))^{-1} + \rho_2 Q_2^o(z)\left((P_1^o(z))^{-1} - (P_2^o(z))^{-1}\right) \\
&= \left[(\rho_1 Q_1^o(z) - \rho_2 Q_2^o(z)) + \rho_2 Q_2^o(z)(P_2^o(z))^{-1}(P_2^o(z) - P_1^o(z))\right](P_1^o(z))^{-1}.
\end{aligned} \tag{5.59}$$

Recall that for $|z|=1$,

$$|\rho_2 Q_2^o(z)(P_2^o(z))^{-1}| = |S_2(z)| \leq \rho_2. \tag{5.60}$$

By Lemma 5.3 and (5.59), (5.60),

$$\|S_2 - S_1\|_{L^2} \leq \left[\|\rho_1 Q_1^o - \rho_2 Q_2^o\|_{L^2} + \rho_2 \|P_2^o - P_1^o\|_{L^2}\right] \sqrt{\rho_1^2 + \mu_1^2}. \tag{5.61}$$

By Lemma 5.1,

$$\begin{aligned} &\|S_2 - S_1\|_{L^2} \\ &\leq n^{1/2}\sqrt{\rho_1^2 + \mu_1^2}\,[\|\rho_1 \mathbf{Q}_1^o - \rho_2 \mathbf{Q}_2^o\| + \rho_2 \|\mathbf{P}_2^o - \mathbf{P}_1^o\|]\,. \end{aligned} \tag{5.62}$$

But

$$\rho_i \mathbf{Q}_i^o = \mathbf{T}\Gamma_i \mathbf{P}_i^o \qquad i = 1,2. \tag{5.63}$$

Then

$$\begin{aligned} &\|\rho_1 \mathbf{Q}_1^o - \rho_2 \mathbf{Q}_2^o\| \\ &= \|\mathbf{T}\Gamma_1 \mathbf{P}_1^o - \mathbf{T}\Gamma_2 \mathbf{P}_2^o\| \\ &\leq \|\mathbf{P}_1^o\|\|\Gamma_1 - \Gamma_2\| + \mu_2 \|\mathbf{P}_1^o - \mathbf{P}_2^o\|. \end{aligned} \tag{5.64}$$

So

$$\begin{aligned} &\|S_2 - S_1\|_{L^2} \\ &\leq n^{1/2}\sqrt{\rho_1^2 + \mu_1^2}\,[\|\mathbf{P}_1^o\|\|\Gamma_1 - \Gamma_2\| + (\rho_2 + \mu_2)\|\mathbf{P}_2^o - \mathbf{P}_1^o\|]\,. \end{aligned} \tag{5.65}$$

Finally, by Lemma 5.2,

$$\begin{aligned} &\|S_2 - S_1\|_{L^2} \\ &\leq n^{1/2}\sqrt{\rho_1^2 + \mu_1^2}\left[\frac{1}{\delta} + \frac{1}{\delta}(1 + \frac{\mu_1 + 3\mu_2}{\delta})\right]\|\Gamma_1 - \Gamma_2\| \\ &\leq n^{1/2}\sqrt{\rho_1^2 + \mu_1^2}\frac{1}{\delta}\left[2 + \frac{\mu_1 + 3\mu_2}{\delta}\right]\|\Gamma_1 - \Gamma_2\|, \end{aligned}$$

which proves the theorem after bounding ρ_i and μ_i by their maximum values ρ and μ, respectively.

Q.E.D.

5.3 Application to Adaptive Design

Back to the design problem (5.10).

Proposition 5.1

The local optimal interpolant $\mathbf{S}^l$ of $\mathbf{W}$ at $\mathbf{U}$ in $\mathbb{E}_1$ is Lipschitz continuous in norm with $\gamma_W = 1$ and $\gamma_U = k_W := \sup_t \|\widehat{\mathbf{W}}_t\|_\infty$.

□

Proof:

Let $M_t = \widehat{\mathbf{U}}_t^* \widehat{\mathbf{W}}_t \in L^\infty$. It is easy to show that for $M_1, M_2 \in L^\infty$, $\|\Gamma_{M_1}\| \leq \|M_1\|_{L^\infty}$ and $\Gamma_{M_1} - \Gamma_{M_2} = \Gamma_{M_1 - M_2}$. Therefore

$$\begin{aligned}
\|\Gamma_t - \Gamma_{t-1}\| &\leq \|\widehat{\mathbf{U}}_t^* \widehat{\mathbf{W}}_t - \widehat{\mathbf{U}}_{t-1}^* \widehat{\mathbf{W}}_{t-1}\|_{L^\infty} \\
&= \|(\widehat{\mathbf{U}}_t^* - \widehat{\mathbf{U}}_{t-1}^*)\widehat{\mathbf{W}}_t + \widehat{\mathbf{U}}_{t-1}^*(\widehat{\mathbf{W}}_t - \widehat{\mathbf{W}}_{t-1})\|_{L^\infty} \\
&\leq k_W \|\widehat{\mathbf{U}}_t^* - \widehat{\mathbf{U}}_{t-1}^*\|_{L^\infty} + \|\widehat{\mathbf{W}}_t - \widehat{\mathbf{W}}_{t-1}\|_\infty \\
&= k_W \|\widehat{\mathbf{U}}_t - \widehat{\mathbf{U}}_{t-1}\|_\infty + \|\widehat{\mathbf{W}}_t - \widehat{\mathbf{W}}_{t-1}\|_\infty .
\end{aligned} \tag{5.66}$$

We have the inequality

$$\begin{aligned}
\big|\|\widehat{\mathbf{S}}_t^l\|_\infty - \|\widehat{\mathbf{S}}_{t-1}^l\|_\infty\big| &= \big|\|\Gamma_t\| - \|\Gamma_{t-1}\|\big| \\
&\leq \|\Gamma_t - \Gamma_{t-1}\| \\
&\leq k_W \|\widehat{\mathbf{U}}_t - \widehat{\mathbf{U}}_{t-1}\|_\infty + \|\widehat{\mathbf{W}}_t - \widehat{\mathbf{W}}_{t-1}\|_\infty
\end{aligned} \tag{5.67}$$

as required.

Q.E.D.

Consider now the local H^∞ adaptive optimization problem in Chapter 4. The main concern here is to synthesize a sensitivity $\mathbf{S}^l \in \underline{\mathbb{E}}_\sigma$ which locally interpolates $\mathbf{W}$ $(:= \mathbf{W}_2\mathbf{W}_1)$ at $\mathbf{U}$ in $\underline{\mathbb{E}}_\sigma$, i.e., for which there exists $\mathbf{Q}_1 \in \underline{\mathbb{E}}_\sigma$ such that

$$\mathbf{S}^l = \mathbf{W} - \mathbf{U} \otimes \mathbf{Q}_1 \tag{5.68}$$

and also $\mathbf{S}^l$ is Lipschitz continuous in the sense of (4.18).

Since $\mathbf{W},\ \mathbf{U} \in \underline{\mathbb{E}}_\sigma$, there exists some $\sigma_0 > \sigma$ for which $\mathbf{W},\ \mathbf{U} \in \mathbb{E}_{\sigma_0}$. Hence for each $t \in \mathbb{Z}$, $\widehat{\mathbf{W}}_t,\ \widehat{\mathbf{U}}_t \in H^\infty_{\sigma_0}$. Define $M_t = \widehat{\mathbf{U}}_t^* \widehat{\mathbf{W}}_t \in L^\infty_{\sigma_0}$, and

$$\mu_t = \inf_{\widetilde{Q} \in H^\infty} \|\widehat{\mathbf{U}}_t^*(\sigma_0(\cdot))\widehat{\mathbf{W}}_t(\sigma_0(\cdot)) - \widetilde{Q}\|_\infty,$$

$$\mu = \sup_t \mu_t,$$

$$\rho = \mu + \delta,$$

for some $\delta > 0$.

For each $t \in \mathbb{Z}$, let $(S_0)_t(\sigma_0(\text{cot}))$ be the central δ-suboptimal interpolant of M_t in L^∞ as defined in (5.25). Thus, there are $\widetilde{\mathbf{Q}}_t \in H^\infty$ such that

$$(S_0)_t(\sigma_0(\cdot)) = \widehat{\mathbf{U}}_t^*(\sigma_0(\cdot))\widehat{\mathbf{W}}_t(\sigma_0(\cdot)) - \widetilde{Q}_t \ \in L^\infty, \tag{5.69}$$

which means that

$$\widehat{\mathbf{U}}_t(\sigma_0(\cdot))(S_0)_t(\sigma_0(\cdot)) = \widehat{\mathbf{W}}_t(\sigma_0(\cdot)) - \widehat{\mathbf{U}}_t(\sigma_0(\cdot))\widetilde{Q}_t \tag{5.70}$$

has analytic continuation into the unit disk, i.e., $\widehat{\mathbf{U}}_t(\sigma_0(\cdot))(S_0)_t(\sigma_0(\cdot)) \in H^\infty$, $\forall t \in \mathbb{Z}$. Since

$$\begin{aligned}\|\widetilde{Q}_t\|_{H^\infty} &\le \|\widehat{\mathbf{W}}_t\|_{H^\infty_{\sigma_0}} + \|(S_0)_t\|_{H^\infty_{\sigma_0}} \\ &\le \mu_{\sigma_0}(\mathbf{W}) + \rho \ < \infty \qquad \forall t \in \mathbb{Z},\end{aligned}$$

the operator $\mathbf{Q}_1 \in \mathbb{B}_e$ defined by the frozen-time formula

$$\widehat{\mathbf{Q}}_{1t}(\cdot) = \widetilde{Q}_t(\sigma_0^{-1}(\cdot)) \tag{5.71}$$

is in $\overline{\mathbb{E}}_{\sigma_0} \subset \underline{\mathbb{E}}_\sigma$. Then a sensitivity $\mathbf{S}^l \in \overline{\mathbb{E}}_{\sigma_0} \subset \underline{\mathbb{E}}_\sigma$ can be constructed via

$$\mathbf{S}^l = \mathbf{W} - \mathbf{U} \otimes \mathbf{Q}_1, \tag{5.72}$$

which, by definition, locally interpolates $\mathbf{W}$ at $\mathbf{U}$ in $\underline{\mathbb{E}}_\sigma$, and also, by construction, locally δ-suboptimal in $\overline{\mathbb{E}}_{\sigma_0}$.

Let γ_δ be the Lipschitz constant in (5.28).

Theorem 5.2

The local central δ-suboptimal interpolant $\mathbf{S}^l$ of $\mathbf{W}$ at $\mathbf{U}$ in $\underline{\mathbb{E}}_\sigma$, constructed in (5.72), is Lipschitz continuous in the sense of (4.18) with constants

$$\begin{aligned}
\gamma_U^{(2)} &= \mu_{\sigma_0}(\mathbf{W})\gamma_\delta + \rho \le \mu_{\sigma_0}(\mathbf{W}_1)\mu_{\sigma_0}(\mathbf{W}_2)\gamma_\delta + \rho, \\
\gamma_{W_1}^{(2)} &= \mu_{\sigma_0}(\mathbf{W}_2)\gamma_\delta, \\
\gamma_{W_2}^{(2)} &= \mu_{\sigma_0}(\mathbf{W}_1)\gamma_\delta.
\end{aligned}$$

□

Proof:

$$\begin{aligned}
\|\widehat{\mathbf{S}}_t^l - \widehat{\mathbf{S}}_{t-1}^l\|_{H^2_{\sigma_0}} &= \|\widehat{\mathbf{U}}_t(S_0)_t - \widehat{\mathbf{U}}_{t-1}(S_0)_{t-1}\|_{L^2_{\sigma_0}} \\
&\le \|(S_0)_t\|_{L^\infty_{\sigma_0}}\|\widehat{\mathbf{U}}_t - \widehat{\mathbf{U}}_{t-1}\|_{L^2_{\sigma_0}} + \|(S_0)_t - (S_0)_{t-1}\|_{L^2_{\sigma_0}} \\
&\le \rho\|\widehat{\mathbf{U}}_t - \widehat{\mathbf{U}}_{t-1}\|_{H^2_{\sigma_0}} + \gamma_\delta\|\widehat{\mathbf{U}}_t^*\widehat{\mathbf{W}}_t - \widehat{\mathbf{U}}_{t-1}^*\widehat{\mathbf{W}}_{t-1}\|_{L^\infty_{\sigma_0}} \\
&\qquad \text{by Theorem 5.1 and (5.66)} \\
&\le \rho\|\widehat{\mathbf{U}}_t - \widehat{\mathbf{U}}_{t-1}\|_{H^\infty_{\sigma_0}} + \gamma_\delta\mu_{\sigma_0}(\mathbf{W})\|\widehat{\mathbf{U}}_t - \widehat{\mathbf{U}}_{t-1}\|_{H^\infty_{\sigma_0}} \\
&\qquad + \gamma_\delta\|\widehat{\mathbf{W}}_t - \widehat{\mathbf{W}}_{t-1}\|_{H^\infty_{\sigma_0}} \\
&= \gamma_U^{(2)}\|\widehat{\mathbf{U}}_t - \widehat{\mathbf{U}}_{t-1}\|_{H^\infty_{\sigma_0}} + \gamma_\delta\|\widehat{\mathbf{W}}_t - \widehat{\mathbf{W}}_{t-1}\|_{H^\infty_{\sigma_0}}.
\end{aligned}$$

Since $\mathbf{W} = \mathbf{W}_2 \otimes \mathbf{W}_1$,

$$\begin{aligned}
\|\widehat{\mathbf{W}}_t - \widehat{\mathbf{W}}_{t-1}\|_{H^\infty_{\sigma_0}} &= \|\widehat{\mathbf{W}}_{2t}\widehat{\mathbf{W}}_{1t} - \widehat{\mathbf{W}}_{2,t-1}\widehat{\mathbf{W}}_{1,t-1}\|_{H^\infty_{\sigma_0}} \\
&\le \mu_{\sigma_0}(\mathbf{W}_2)\|\widehat{\mathbf{W}}_{1t} - \widehat{\mathbf{W}}_{1,t-1}\|_{H^\infty_{\sigma_0}} + \mu_{\sigma_0}(\mathbf{W}_1)\|\widehat{\mathbf{W}}_{2t} - \widehat{\mathbf{W}}_{2,t-1}\|_{H^\infty_{\sigma_0}},
\end{aligned}$$

which proves the theorem.

Q.E.D.

Chapter 6 Coprimeness Vs. Robustness: Generalities

In this chapter some preliminary algebraic and topological results on unstable plants will be developed. The development proceeds within the general topological and algebraic framework of Francis, Schneider and Vidyasagar [Fra3] [Vid], and Desoer et al [Des3]. For related work on the relation between robustness and coprimeness, see Verma [Ver1].

First, the properties of $\mathbb{B}_e$ on which these results depend will be abstracted. Let $\mathbb{A}$ be a normed algebra with identity $\mathbf{I}$, contained in some larger algebra $\mathbb{A}_e$. The elements of $\mathbb{A}$ and $\mathbb{A}_e$ represent stable and possibly unstable systems. $\mathbb{A}_e$ is the direct sum of two subalgebras of elements called *memoryless* and *strictly causal*, and denoted by $(\mathbb{A}_e)_{nm}$ and $(\mathbb{A}_e)_{sc}$ respectively, with the following properties: If $\mathbf{K} \in (\mathbb{A}_e)_{sc}$ then $(\mathbf{I}+\mathbf{K})^{-1}$ exists in $\mathbb{A}_e$, and for any $\mathbf{G} \in \mathbb{A}_e$, $\mathbf{KG}$ as well as $\mathbf{GK}$ are in $(\mathbb{A}_e)_{sc}$, i.e., the strictly causal elements form an ideal in $\mathbb{A}_e$; the memoryless subalgebra $(\mathbb{A}_e)_{nm}$ is a proper subalgebra of $\mathbb{A}_e$ containing the identity $\mathbf{I}$ and $(\mathbf{KG})_{nm} = (\mathbf{K})_{nm}(\mathbf{G})_{nm}$ for any $\mathbf{K}, \mathbf{G} \in \mathbb{A}_e$; if $\mathbf{K} \in \mathbb{A}$ then the memoryless and strictly causal components of $\mathbf{K}$ are in $\mathbb{A}$.

It follows from these assumptions that $\mathbf{K} \in \mathbb{A}_e$ has an inverse in $\mathbb{A}_e$ iff the memoryless component of $\mathbf{K}$ has such an inverse, whereupon $((\mathbf{K})_{nm})^{-1} = (\mathbf{K}^{-1})_{nm}$.

$\mathbb{A}_I$ will denote the set of operators in $\mathbb{A}$ which have inverse in $\mathbb{A}_e$. An operator $\mathbf{G} \in \mathbb{A}_e$ has a *right factorization* in $\mathbb{A}$ if $\mathbf{G} = \mathbf{N}\mathbf{D}^{-1}$, where $(\mathbf{N}, \mathbf{D}) \in \mathbb{A} \times \mathbb{A}_I$. The factorization as well as the pair $(\mathbf{N}, \mathbf{D})$ are *right coprime* in $\mathbb{A}$ if for some $(\widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}}) \in \mathbb{A} \times \mathbb{A}_I$

$$(\widetilde{\mathbf{X}}\mathbf{N} + \widetilde{\mathbf{Y}}\mathbf{D})^{-1} \in \mathbb{A}. \tag{6.1}$$

Similarly, $\mathbf{F} \in \mathbb{A}_e$ has a *left factorization* in $\mathbb{A}$ if $\mathbf{F} = \widetilde{\mathbf{Y}}^{-1}\widetilde{\mathbf{X}}$ for some $(\widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}}) \in \mathbb{A} \times \mathbb{A}_I$. The factorization and pair $(\widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}})$ are *left coprime* in $\mathbb{A}$ if (6.1) holds with some $(\mathbf{N}, \mathbf{D}) \in \mathbb{A} \times \mathbb{A}_I$. The pairs $(\mathbf{N}, \mathbf{D})$ and $(\widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}})$ as well as the corresponding factorizations are called *jointly right-left coprime* if (6.1) holds, or *jointly coprime* when there is no ambiguity. A coprime factorization is *normalized* if the inverse (6.1) equals the identity $\mathbf{I}$.

Note that if $\mathbf{G}$ (or $\mathbf{F}$) $\in \mathbb{A}_e$ is strictly causal, then the condition "$\widetilde{\mathbf{Y}} \in \mathbb{A}_I$" (or $\mathbf{D} \in \mathbb{A}_I$) can be replaced by "$\widetilde{\mathbf{Y}} \in \mathbb{A}$" (or $\mathbf{D} \in \mathbb{A}$) in the right (or left) coprimeness definition. Indeed, if $\mathbf{G} \in (\mathbb{A}_e)_{sc}$, then $\mathbf{N} \in (\mathbb{A}_e)_{sc}$ (as $(\mathbf{N})_{nm} = (\mathbf{G})_{nm}(\mathbf{D})_{nm} = 0$), which implies $\widetilde{\mathbf{X}}\mathbf{N} \in (\mathbb{A}_e)_{sc}$. Hence, $\mathbf{I} - \widetilde{\mathbf{X}}\mathbf{N} = \widetilde{\mathbf{Y}}\mathbf{D}$ is invertible in $\mathbb{A}_e$, and $\widetilde{\mathbf{Y}}^{-1} = \mathbf{D}\left(\mathbf{I} - \widetilde{\mathbf{X}}\mathbf{N}\right)^{-1} \in \mathbb{A}_e$, i.e., $\widetilde{\mathbf{Y}} \in \mathbb{A}_I$. The left coprime counterpart can be argued similarly.

The interconnection of a feedback $\mathbf{F}$ and plant $\mathbf{G}$ in $\mathbb{A}_e$ is *well posed* if all four operators in the matrix

$$\mathbf{K} = [\mathbf{K}_{ij}] := \begin{pmatrix} (\mathbf{I} + \mathbf{F}\mathbf{G})^{-1} & \mathbf{G}(\mathbf{I} + \mathbf{F}\mathbf{G})^{-1} \\ \mathbf{F}(\mathbf{I} + \mathbf{G}\mathbf{F})^{-1} & (\mathbf{I} + \mathbf{G}\mathbf{F})^{-1} \end{pmatrix} \tag{6.2}$$

are in $\mathbb{A}_e$, and $\mathbb{A}$-stable iff all four operators are in $\mathbb{A}$. If such an interconnection is stable then $\mathbf{G}$ can be expressed as a ratio of the closed-loop operators appearing in (6.2) in two ways, i.e., has right and left factorizations in $\mathbb{A}, \mathbf{G} = \mathbf{N}\mathbf{D}^{-1} = \widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{N}}$, and similarly $\mathbf{F} = \tilde{\mathbf{Y}}^{-1}\tilde{\mathbf{X}} = \mathbf{X}\mathbf{Y}^{-1}$. $\mathbf{G}$ and $\mathbf{F}$ can therefore be represented, albeit nonuniquely, by pairs of elements of $\mathbb{A}$, which will be denoted by $\mathbf{G} \sim (\mathbf{N},\mathbf{D}), \mathbf{F} \sim (\tilde{\mathbf{X}},\tilde{\mathbf{Y}})$ in the case of, e.g., the right factors of $\mathbf{G}$ and left factors of $\mathbf{F}$.

6.1 Robustness of Stability

Suppose the plant $\mathbf{G} \sim (\mathbf{N},\mathbf{D}) \in \mathbb{A} \times \mathbb{A}_I$ is stabilized by a feedback $\mathbf{F}$. $\mathbf{F}$ is held fixed while $\mathbf{G}_1 \sim (\mathbf{N}_1,\mathbf{D}_1)$ is allowed to vary in some neighbourhood of $(\mathbf{N},\mathbf{D})$. View the plant representations $(\mathbf{N}_1,\mathbf{D}_1)$ as elements of $\mathbb{A} \times \mathbb{A}_I$ under the open-loop norm

$$\|(\mathbf{N},\mathbf{D})\|_{ol(\mathbb{A})} = \max(\|\mathbf{N}\|_{\mathbb{A}}, \|\mathbf{D}\|_{\mathbb{A}}). \tag{6.3}$$

Define $\mathbf{K} \in \mathbb{A}^{2\times 2}$ (of closed loop operators specified in (6.2)) to be $\|\mathbf{K}\|_{cl(\mathbb{A})} := \max_{i,j=1,2} \|\mathbf{K}_{ij}\|_{\mathbb{A}}$. Denote the map from plant pairs to closed loop operators by $\mathcal{K} : \mathbb{A}^2 \to \mathbb{A}^4$, $\mathcal{K}(\mathbf{N},\mathbf{D}) = \mathbf{K}$.

Definition 6.1

Stabilization of $\mathbf{G}$ by $\mathbf{F}$ is *robust* in the $\|\cdot\|_{\mathbb{A}}$ norm if there is a neighborhood in $\mathbb{A} \times \mathbb{A}_I$ of $(\mathbf{N},\mathbf{D})$ in the open-loop $\|\cdot\|_{ol(\mathbb{A})}$ norm such that for any $(\mathbf{N}_1,\mathbf{D}_1)$ in it a constant β can be found such that $\mathcal{K}(\mathbf{N}_1,\mathbf{D}_1) \in \mathbb{A}^{2\times 2}$ and

$$\|\mathcal{K}(\mathbf{N}_1,\mathbf{D}_1) - \mathcal{K}(\mathbf{N},\mathbf{D})\|_{cl(\mathbb{A})} \le \beta\|(\mathbf{N}_1,\mathbf{D}_1) - (\mathbf{N},\mathbf{D})\|_{ol(\mathbb{A})}. \tag{6.4}$$

6.2 A Coprimeness vs. Robustness Result

Although not every operator in $\mathbb{A}_e$ admits a factorization, every stabilizable plant $\mathbf{G} \in \mathbb{A}_e$ (and feedback controller $\mathbf{F} \in \mathbb{A}_e$) has, as described in (6.2), right and left factorizations in $\mathbb{A}$. We will denote by $\mathbb{A}_e^f$ the subset of $\mathbb{A}_e$ consisting of all operators admitting both left and right factorizations in $\mathbb{A}$. For linear time-invariant systems, it is well understood that joint coprimeness is sufficient for robust stability. Recently, using the Corona Theorem, Smith [Smi2] showed that in H^∞ all stabilizable systems admit coprime factorizations. However, a linear time-varying operator may admit no coprime factorization even if it is robustly stabilizable. For the discussion of coprimeness vs. robustness in the following sections, we restrict our plants and feedback controllers to those admitting some (possibly unknown) coprime factorizations.

Let $\mathbb{A}_e^c$ be the subset of $\mathbb{A}_e^f$ consisting of all operators admitting some (possibly unknown) left and right coprime factorizations in $\mathbb{A}$. [(6.1)] Operators in $\mathbb{A}_e^c$ has the following division property.

<u>**Proposition 6.1**</u>[(6.2)]

(1) If $\mathbf{G} \in \mathbb{A}_e^c$ and has factorization representations (not assumed coprime) $\mathbf{G} = \mathbf{N}\mathbf{D}^{-1} = \tilde{\mathbf{D}}^{-1}\tilde{\mathbf{N}}$, then there exist $\mathbf{Q}$, $\tilde{\mathbf{Q}} \in \mathbb{A}_I$ such that

$$\mathbf{N} = \mathbf{N}_0\mathbf{Q}, \quad \mathbf{D} = \mathbf{D}_0\mathbf{Q}, \quad \tilde{\mathbf{N}} = \tilde{\mathbf{Q}}\tilde{\mathbf{N}}_0, \quad \tilde{\mathbf{D}} = \tilde{\mathbf{Q}}\tilde{\mathbf{D}}_0 \tag{6.5}$$

where $(\mathbf{N}_0, \mathbf{D}_0)$ is right coprime, and $(\tilde{\mathbf{N}}_0, \tilde{\mathbf{D}}_0)$ left coprime.

(6.1) See [Des3] and [Vid] for some examples of $\mathbb{A}_e^c$.

(6.2) For (1), see also [Des3] property 2.

(2) If in addition, $(\mathbf{D})_{nm}^{-1} \in \mathbb{A}$ (or $(\tilde{\mathbf{D}})_{nm}^{-1} \in \mathbb{A}$), then $(\mathbf{Q})_{nm}^{-1} \in \mathbb{A}$ (or $(\tilde{\mathbf{Q}})_{nm}^{-1} \in \mathbb{A}$).

□

Proof:

(1) Since $\mathbf{G} \in \mathbb{A}_e^c$, there exist some $(\mathbf{N}_0,\ \mathbf{D}_0)$, $(\tilde{\mathbf{N}}_0,\ \tilde{\mathbf{D}}_0) \in \mathbb{A} \times \mathbb{A}_I$ for which $\mathbf{G} = \mathbf{N}_0\mathbf{D}_0^{-1} = \tilde{\mathbf{D}}_0^{-1}\tilde{\mathbf{N}}_0$ and

$$\begin{aligned} \tilde{\mathbf{X}}_0\mathbf{N}_0 + \tilde{\mathbf{Y}}_0\mathbf{D}_0 &= \mathbf{I}, \\ \tilde{\mathbf{N}}_0\mathbf{X}_0 + \tilde{\mathbf{D}}_0\mathbf{Y}_0 &= \mathbf{I} \end{aligned} \tag{6.6}$$

with $(\mathbf{X}_0, \mathbf{Y}_0)$, $(\tilde{\mathbf{X}}_0, \tilde{\mathbf{Y}}_0) \in \mathbb{A} \times \mathbb{A}_I$.

Define $\mathbf{Q} = \mathbf{D}_0^{-1}\mathbf{D}$, $\tilde{\mathbf{Q}} = \tilde{\mathbf{D}}\tilde{\mathbf{D}}_0^{-1}$, both invertible in $\mathbb{A}_e$. We only need to verify that $\mathbf{Q}$, $\tilde{\mathbf{Q}} \in \mathbb{A}$. But

$$\begin{aligned} \mathbf{Q} &= (\tilde{\mathbf{X}}_0\mathbf{N}_0 + \tilde{\mathbf{Y}}_0\mathbf{D}_0)\mathbf{Q} \\ &= \tilde{\mathbf{X}}_0\mathbf{N}_0\mathbf{Q} + \tilde{\mathbf{Y}}_0\mathbf{D}_0\mathbf{Q} \\ &= \tilde{\mathbf{X}}_0\mathbf{N} + \tilde{\mathbf{Y}}_0\mathbf{D} \qquad \in \mathbb{A} \end{aligned} \tag{6.7}$$

where the identity $\mathbf{N}_0\mathbf{D}_0^{-1} = \mathbf{N}\mathbf{D}^{-1}$ has been used. Similarly $\tilde{\mathbf{Q}} \in \mathbb{A}$.

(2) It follows the hypothesis and the identities:

$$\begin{aligned} (\mathbf{Q})_{nm}^{-1} &= (\mathbf{Q}^{-1})_{nm} \\ &= (\mathbf{D}^{-1}\mathbf{D}_0)_{nm} \\ &= (\mathbf{D}^{-1})_{nm}(\mathbf{D}_0)_{nm} \end{aligned}$$

and $(\tilde{\mathbf{Q}})_{nm}^{-1} = (\tilde{\mathbf{D}}_0)_{nm}(\tilde{\mathbf{D}})_{nm}^{-1}$.

Q.E.D.

The following results are stated in the case of left factors of **G** and right factors of **F**, but the results hold after interchanging "left" and "right".

Proposition 6.2[6.3]

Suppose $\mathbf{G} = \tilde{\mathbf{D}}^{-1}\tilde{\mathbf{N}}$ and $\mathbf{F} = \mathbf{X}\mathbf{Y}^{-1}$ are separately coprime in $\mathbb{A}$, i.e., there exist $(\mathbf{X}_0, \mathbf{Y}_0)$, $(\tilde{\mathbf{N}}_0, \tilde{\mathbf{D}}_0) \in \mathbb{A} \times \mathbb{A}_I$ such that

$$\begin{aligned} \tilde{\mathbf{N}}\mathbf{X}_0 + \tilde{\mathbf{D}}\mathbf{Y}_0 &= \mathbf{I}, \\ \tilde{\mathbf{N}}_0\mathbf{X} + \tilde{\mathbf{D}}_0\mathbf{Y} &= \mathbf{I}. \end{aligned} \tag{6.8}$$

Then the following statements are equivalent.

(1) $\mathbf{G}$ and $\mathbf{F}$ are mutually stabilizing.

(2) $\mathbf{G}$ and $\mathbf{F}$ are mutually robustly stabilizing.

(3) $\mathbf{G}$ and $\mathbf{F}$ are jointly coprime.

□

Proof:

(3) $\Longrightarrow$ (2): If $\mathbf{G}$ and $\mathbf{F}$ are jointly coprime, then

$$\mathbf{R}^{-1} = (\tilde{\mathbf{N}}\mathbf{X} + \tilde{\mathbf{D}}\mathbf{Y})^{-1} \in \mathbb{A}. \tag{6.9}$$

Since $[\mathbf{K}_{ij}]$ can be expressed as

$$[\mathbf{K}_{ij}] = \begin{pmatrix} \mathbf{I} - \mathbf{X}\mathbf{R}^{-1}\tilde{\mathbf{N}} & \mathbf{Y}\mathbf{R}^{-1}\tilde{\mathbf{N}} \\ \mathbf{X}\mathbf{R}^{-1}\tilde{\mathbf{D}} & \mathbf{Y}\mathbf{R}^{-1}\tilde{\mathbf{D}} \end{pmatrix}, \tag{6.10}$$

(2) follows, noting that every component $\mathbf{K}_{ij}$ in (6.10) depends on its variables continuously.

(2) $\Longrightarrow$ (1): By definition.

[6.3] See also Lemma 3.1 in [Vid].

(1) $\Longrightarrow$ (3): If

$$[\mathbf{K}_{ij}] = \begin{pmatrix} \mathbf{I} - \mathbf{X}\mathbf{R}^{-1}\tilde{\mathbf{N}} & \mathbf{Y}\mathbf{R}^{-1}\tilde{\mathbf{N}} \\ \mathbf{X}\mathbf{R}^{-1}\tilde{\mathbf{D}} & \mathbf{Y}\mathbf{R}^{-1}\tilde{\mathbf{D}} \end{pmatrix} \in \mathbb{A}^{2\times 2},$$

then

$$\mathbf{R}^{-1} = \tilde{\mathbf{N}}_0(\mathbf{I} - \mathbf{K}_{11})\mathbf{X}_0 + \tilde{\mathbf{D}}_0\mathbf{K}_{12}\mathbf{X}_0 + \tilde{\mathbf{N}}_0\mathbf{K}_{21}\mathbf{Y}_0 + \tilde{\mathbf{D}}_0\mathbf{K}_{22}\mathbf{Y}_0 \in \mathbb{A}, \tag{6.11}$$

which means **G** and **F** are jointly coprime.

Q.E.D.

For $\mathbf{G}$, $\mathbf{F} \in \mathbb{A}_e^c$, a stronger result holds.

Theorem 6.1

Suppose $\mathbf{G}$, $\mathbf{F} \in \mathbb{A}_e^c$ and have factorization representations $\mathbf{G} = \mathbf{N}\mathbf{D}^{-1} = \tilde{\mathbf{D}}^{-1}\tilde{\mathbf{N}}$, $\mathbf{F} = \mathbf{X}\mathbf{Y}^{-1} = \tilde{\mathbf{Y}}^{-1}\tilde{\mathbf{X}}$ (not assumed coprime) with $(\mathbf{D})_{nm}^{-1}, (\tilde{\mathbf{D}})_{nm}^{-1}, (\mathbf{Y})_{nm}^{-1}, (\tilde{\mathbf{Y}})_{nm}^{-1} \in \mathbb{A}$. Then the following statements are equivalent.

(1) **G** and **F** are mutually robustly stabilizing.

(2) The factorizations of **G** and **F** are jointly coprime.

□

Proof:

(2) $\Longrightarrow$ (1): This is Prop. 6.2, part (3) $\Longrightarrow$ (2).

(1) $\Longrightarrow$ (2): Let $\mathbf{R}_1 = \tilde{\mathbf{N}}\mathbf{X} + \tilde{\mathbf{D}}\mathbf{Y}$ and $\mathbf{R}_2 = \tilde{\mathbf{X}}\mathbf{N} + \tilde{\mathbf{Y}}\mathbf{D}$. We will prove $\mathbf{R}_1^{-1} \in \mathbb{A}$. The proof for $\mathbf{R}_2^{-1} \in \mathbb{A}$ is similar.

Since $\tilde{\mathbf{D}}^{-1}$, $\mathbf{Y}^{-1} \in \mathbb{B}_e$, and $(\mathbf{I}+\mathbf{GF})^{-1} \in \mathbb{A}$,

$$\mathbf{R}_1^{-1} = \mathbf{Y}^{-1}(\mathbf{I}+\mathbf{GF})^{-1}\tilde{\mathbf{D}}^{-1} \in \mathbb{B}_e. \tag{6.12}$$

From this identity, we obtain

$$(\mathbf{I}+\mathbf{GF})^{-1} = \mathbf{Y}\mathbf{R}_1^{-1}\tilde{\mathbf{D}} = \mathbf{Y}(\tilde{\mathbf{N}}\mathbf{X}+\tilde{\mathbf{D}}\mathbf{Y})^{-1}\tilde{\mathbf{D}}. \tag{6.13}$$

Since $\mathbf{G}$, $\mathbf{F} \in \mathbb{A}_e^c$, by Prop. 6.1, there are $\tilde{\mathbf{Q}}$, $\mathbf{Q} \in \mathbb{A}_I$ such that

$$\mathbf{X} = \mathbf{X}_0\mathbf{Q}, \quad \mathbf{Y} = \mathbf{Y}_0\mathbf{Q}, \quad \tilde{\mathbf{N}} = \tilde{\mathbf{Q}}\tilde{\mathbf{N}}_0, \quad \tilde{\mathbf{D}} = \tilde{\mathbf{Q}}\tilde{\mathbf{D}}_0, \tag{6.14}$$

where $(\mathbf{X}_0, \mathbf{Y}_0)$, $(\tilde{\mathbf{N}}_0, \tilde{\mathbf{D}}_0) \in \mathbb{A} \times \mathbb{A}_I$ are coprime.

After substitution, noting $\mathbf{Q}^{-1}$, $\tilde{\mathbf{Q}}^{-1} \in \mathbb{A}_e$,

$$(\mathbf{I}+\mathbf{GF})^{-1} = \mathbf{Y}_0(\tilde{\mathbf{N}}_0\mathbf{X}_0+\tilde{\mathbf{D}}_0\mathbf{Y}_0)^{-1}\tilde{\mathbf{D}}_0. \tag{6.15}$$

As $\tilde{\mathbf{Q}}$, $\mathbf{Q} \in \mathbb{A}$, robustness of $(\mathbf{I}+\mathbf{GF})^{-1}$ with respect to perturbations in $(\mathbf{X}, \mathbf{Y}, \tilde{\mathbf{N}}, \tilde{\mathbf{D}})$ implies that with respect to $(\mathbf{X}_0, \mathbf{Y}_0, \tilde{\mathbf{N}}_0, \tilde{\mathbf{D}}_0)$. By Prop. 6.2, robustness plus separate coprimeness imply joint coprimeness, and henceforth $\mathbf{R}_0 = \tilde{\mathbf{N}}_0\mathbf{X}_0 + \tilde{\mathbf{D}}_0\mathbf{Y}_0$ has inverse in $\mathbb{A}$. Without loss of generality, assume $\mathbf{R}_0 = \mathbf{I}$. We will prove that $\tilde{\mathbf{Q}}^{-1} \in \mathbb{A}$ (similarly $\mathbf{Q}^{-1} \in \mathbb{A}$) which means that $(\tilde{\mathbf{N}}, \tilde{\mathbf{D}})$ is in fact coprime.

Now, robustness of

$$\mathbf{K} = \begin{pmatrix} \mathbf{I} - \mathbf{X}(\tilde{\mathbf{N}}\mathbf{X}+\tilde{\mathbf{D}}\mathbf{Y})^{-1}\tilde{\mathbf{N}} & \mathbf{Y}(\tilde{\mathbf{N}}\mathbf{X}+\tilde{\mathbf{D}}\mathbf{Y})^{-1}\tilde{\mathbf{N}} \\ \mathbf{X}(\tilde{\mathbf{N}}\mathbf{X}+\tilde{\mathbf{D}}\mathbf{Y})^{-1}\tilde{\mathbf{D}} & \mathbf{Y}(\tilde{\mathbf{N}}\mathbf{X}+\tilde{\mathbf{D}}\mathbf{Y})^{-1}\tilde{\mathbf{D}} \end{pmatrix} \tag{6.16}$$

with respect to $(\mathbf{X}, \mathbf{Y}, \tilde{\mathbf{N}}, \tilde{\mathbf{D}})$,

$\Longrightarrow$ robustness of

$$\mathbf{M} = \begin{pmatrix} \mathbf{X}_0(\tilde{\mathbf{N}}\mathbf{X}_0+\tilde{\mathbf{D}}\mathbf{Y}_0)^{-1}\tilde{\mathbf{N}} & \mathbf{Y}_0(\tilde{\mathbf{N}}\mathbf{X}_0+\tilde{\mathbf{D}}\mathbf{Y}_0)^{-1}\tilde{\mathbf{N}} \\ \mathbf{X}_0(\tilde{\mathbf{N}}\mathbf{X}_0+\tilde{\mathbf{D}}\mathbf{Y}_0)^{-1}\tilde{\mathbf{D}} & \mathbf{Y}_0(\tilde{\mathbf{N}}\mathbf{X}_0+\tilde{\mathbf{D}}\mathbf{Y}_0)^{-1}\tilde{\mathbf{D}} \end{pmatrix}$$

$$:= \begin{pmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{pmatrix} \tag{6.17}$$

with respect to $(\mathbf{X}_0, \mathbf{Y}_0, \tilde{\mathbf{N}}, \tilde{\mathbf{D}})$ (as $\mathbf{Q} \in \mathbb{A}$),

$$\begin{aligned} \Longrightarrow \qquad & \text{robustness of} \\ & \tilde{\mathbf{N}}_0(\mathbf{M}_{11})\mathbf{X}_0 + \tilde{\mathbf{D}}_0\mathbf{M}_{12}\mathbf{X}_0 + \tilde{\mathbf{N}}_0\mathbf{M}_{21}\mathbf{Y}_0 + \tilde{\mathbf{D}}_0\mathbf{M}_{22}\mathbf{Y}_0 \\ & = \mathbf{R}_0 \\ & = (\tilde{\mathbf{N}}\mathbf{X}_0 + \tilde{\mathbf{D}}\mathbf{Y}_0)^{-1}\tilde{\mathbf{Q}} \end{aligned} \tag{6.18}$$

(as $\mathbf{R}_0 = \mathbf{I}$) with respect to $(\mathbf{X}_0, \mathbf{Y}_0, \tilde{\mathbf{N}}, \tilde{\mathbf{D}})$.

Consider the special perturbation

$$\tilde{\mathbf{N}} + \delta\tilde{\mathbf{N}}_0, \qquad \tilde{\mathbf{D}} + \delta\tilde{\mathbf{D}}_0 \tag{6.19}$$

with small non-zero $\delta \in \mathbb{R}$. By Prop. 6.1 part (2), $(\tilde{\mathbf{Q}})^{-1}_{nm} \in \mathbb{A}$. By contraction principle, for small enough δ, $(\tilde{\mathbf{Q}} + \delta\mathbf{I})^{-1}_{nm} \in \mathbb{A}$, which implies $(\tilde{\mathbf{Q}} + \delta\mathbf{I}) \in \mathbb{A}_I$. Define

$$\begin{aligned} \mathbf{L}(\delta) &= \left((\tilde{\mathbf{N}} + \delta\tilde{\mathbf{N}}_0)\mathbf{X}_0 + (\tilde{\mathbf{D}} + \delta\tilde{\mathbf{D}}_0)\mathbf{Y}_0\right)^{-1}\tilde{\mathbf{Q}} \\ &= (\tilde{\mathbf{Q}} + \delta\mathbf{I})^{-1}\tilde{\mathbf{Q}} \end{aligned} \tag{6.20}$$

or $\tilde{\mathbf{Q}}\,(\mathbf{I} - \mathbf{L}(\delta)) = \delta\mathbf{L}(\delta)$.

By robustness, $\mathbf{L}(\delta)$ is a continuous function of δ and $\mathbf{L}(0) = \mathbf{I}$. Thus by contraction principle and continuity, for small $\delta \neq 0$, $\delta^{-1}\mathbf{L}^{-1}(\delta) \in \mathbb{A}$ and

$$\tilde{\mathbf{Q}}\,(\mathbf{I} - \mathbf{L}(\delta))\,\delta^{-1}\mathbf{L}^{-1}(\delta) = \mathbf{I}. \tag{6.21}$$

Since $(\mathbf{I} - \mathbf{L}(\delta))\,\delta^{-1}\mathbf{L}^{-1}(\delta) \in \mathbb{A}$, $\tilde{\mathbf{Q}}^{-1} \in \mathbb{A}$ and $(\tilde{\mathbf{N}}, \tilde{\mathbf{D}})$ is coprime.

By reciprocity, $(\mathbf{X}, \mathbf{Y})$ is also coprime. By Prop. 6.2, we conclude that the factorizations of $\mathbf{G}$ and $\mathbf{F}$ are mutually coprime.

Q.E.D.

A fundamental question in this development is: what kind of plants are in $\mathbb{A}_e^c$? While there is at present no complete answer to this question, we will restate a known result, which claims that any plant, which is stabilizable by a feedback with a coprime factorization representation, must be in $\mathbb{A}_e^c$.

<u>Lemma 6.1</u> [Des3][Vid]

If $\mathbf{F} = \mathbf{X}\mathbf{Y}^{-1} = \tilde{\mathbf{Y}}^{-1}\tilde{\mathbf{X}}$ are coprime in $\mathbb{A}$ and $\mathbf{G} \in \mathbb{A}_e$ is stabilized by $\mathbf{F}$, then $\mathbf{G} \in \mathbb{A}_e^c$.

□

Proof: See [Des3] or Lemma 3.2 of [Vid].

<u>Corollary 6.1</u>

If $\mathbf{F} = \mathbf{X}\mathbf{Y}^{-1} = \tilde{\mathbf{Y}}^{-1}\tilde{\mathbf{X}}$ are coprime in $\mathbb{A}$ and $\mathbf{G} \in \mathbb{A}_e$, then the following statements are equivalent:

(a) $\mathbf{G} = \mathbf{N}\mathbf{D}^{-1}$ is robustly stabilized in $\mathbb{A}$ by $\mathbf{F} = \tilde{\mathbf{Y}}^{-1}\tilde{\mathbf{X}}$.

(b) $\mathbf{N}\mathbf{D}^{-1}$ and $\tilde{\mathbf{Y}}^{-1}\tilde{\mathbf{X}}$ are jointly coprime in $\mathbb{A}$.

□

Proof:

(b) $\Longrightarrow$ (a): Theorem 6.1.

(a) $\Longrightarrow$ (b): Since $\mathbf{G}$ is stabilized by $\mathbf{F}$, by Lemma 6.1, $\mathbf{G} \in \mathbb{A}_e^c$. The implication follows from Theorem 6.1.

Q.E.D.

6.3 Characterization of Robustly Stabilizing Operators

Consider a plant $\mathbf{G} \in \mathbb{A}_e^f$ with factorizations $\mathbf{G} = \mathbf{N}\mathbf{D}^{-1} = \tilde{\mathbf{D}}^{-1}\tilde{\mathbf{N}}$ (not assumed coprime) and a feedback controller $\mathbf{F} \in \mathbb{A}_e^c$ with (separately but not assumed jointly with $\mathbf{G}$) coprime factorizations $\mathbf{F} = \mathbf{X}\mathbf{Y}^{-1} = \tilde{\mathbf{Y}}^{-1}\tilde{\mathbf{X}}$.

<u>Δ – Notation</u>

In the following theorem if $\mathbf{G}_1 \sim (\mathbf{N}_1, \mathbf{D}_1), \mathbf{G}_2 \sim (\mathbf{N}_2, \mathbf{D}_2)$ are any two given plants and $\mathbf{M} = \mathcal{F}(\mathbf{N}_i, \mathbf{D}_i)$ is any function of pairs $(\mathbf{N}_i, \mathbf{D}_i) \in \mathbb{A}^2$, the notation $\Delta\mathbf{M}$ represents

$$\mathcal{F}(\mathbf{N}_1, \mathbf{D}_1) - \mathcal{F}(\mathbf{N}_2, \mathbf{D}_2). \tag{6.22}$$

We want to characterize all plants in a small neighbourhood which can be robustly stabilized by $\mathbf{F}$.

<u>**Theorem 6.2**</u>

If $\mathbf{G}$ is robustly stabilized by $\mathbf{F}$, then there is a neighbourhood of $(\mathbf{N}, \mathbf{D}) \in \mathbb{A} \times \mathbb{A}_I$ of radius $\delta > 0$ in the open loop norm $\|\cdot\|_{ol(\mathbb{A})}$ in which any two operators $\mathbf{G}_i \sim (\mathbf{N}_i, \mathbf{D}_i)$, $i = 0, 1$, stabilized by $\mathbf{F}$, are related by

$$\mathbf{G}_1 = (\mathbf{N}_0 + \tilde{\mathbf{Y}}\mathbf{W})(\mathbf{D}_0 - \tilde{\mathbf{X}}\mathbf{W})^{-1} \tag{6.23}$$

where $\mathbf{W} \in \mathbb{A}$, and the inequality

$$\|\mathbf{W}\|_{\mathbb{A}} \leq \lambda \|\Delta\mathbf{K}\|_{cl(\mathbb{A})} \tag{6.24}$$

holds for some constant $\lambda \geq 0$, where $\Delta\mathbf{K}$ denotes the difference in the corresponding closed-loop operator matrices, which are defined in (6.2).

Moreover there are normalized representations

$$\mathbf{G}_i \sim (\mathbf{N}_i\mathbf{R}_i^{-1}, \mathbf{D}_i\mathbf{R}_i^{-1}) \in \mathbb{A} \times \mathbb{A}_I \qquad i = 1, 2 \tag{6.25}$$

for some $\mathbf{R}_i \in \mathbb{A}_I$, which are unique and satisfy

$$\|\Delta(\mathbf{N}\mathbf{R}^{-1}, \mathbf{D}\mathbf{R}^{-1})\|_{ol(\mathbb{A})} \leq Const.\|\Delta\mathbf{K}\|_{cl(\mathbb{A})} \leq Const.\|\Delta(\mathbf{N}\mathbf{R}^{-1}, \mathbf{D}\mathbf{R}^{-1})\|_{ol(\mathbb{A})}$$
$$\leq Const.\|\Delta(\mathbf{N}, \mathbf{D})\|_{ol(\mathbb{A})}. \tag{6.26}$$

□

Remarks: In Theorem 6.2 the factorizations of $\mathbf{G}$ and $\mathbf{G}_i$ are assumed to be right, and those of $\mathbf{F}$ left. However the theorem obviously holds with right and left interchanged.

In preparation for the proof, a lemma will first be introduced.

Lemma 6.2

For any $\mathbf{G}_1 \in \mathbb{A}_e$ with factorization $\mathbf{G}_1 = \mathbf{N}_1\mathbf{D}_1^{-1}$ satisfying the assumptions of Theorem 6.2, and for which $(\mathbf{I} + \mathbf{F}\mathbf{G}_1)^{-1}$ exists in $\mathbb{A}_e$, the relation (6.23) holds with $\mathbf{W} \in \mathbb{A}_e$. $\mathbf{W}$ is expressible as a linear form (see (6.40) below) in the closed-loop perturbations $\Delta\mathbf{K}_{ij} \in \mathbb{A}_e$, with coefficients in $\mathbb{A}$.

□

Proof of Lemma 6.2:

Denote $(\mathbf{N},\mathbf{D})$ by $(\mathbf{N}_0, \mathbf{D}_0)$ and let

$$\tilde{\mathbf{X}}\mathbf{N}_i + \tilde{\mathbf{Y}}\mathbf{D}_i =: \mathbf{R}_i, \qquad i = 0, 1. \tag{6.27}$$

$\mathbf{R}_i^{-1}$ exists in $\mathbb{A}_e$, as $\mathbf{R}_i = \tilde{\mathbf{Y}}_i(\mathbf{I}+\mathbf{F}\mathbf{G}_i)\mathbf{D}_i$, whence

$$\tilde{\mathbf{X}}\mathbf{N}_i\mathbf{R}_i^{-1} + \tilde{\mathbf{Y}}\mathbf{D}_i\mathbf{R}_i^{-1} = \mathbf{I} \tag{6.28}$$

$$\begin{aligned} \Longrightarrow \quad & \tilde{\mathbf{X}}(\mathbf{N}_1\mathbf{R}_1^{-1} - \mathbf{N}_0\mathbf{R}_0^{-1}) = -\tilde{\mathbf{Y}}(\mathbf{D}_1\mathbf{R}_1^{-1} - \mathbf{D}_0\mathbf{R}_0^{-1}) \\ \Longrightarrow \quad & \mathbf{X}\mathbf{Y}^{-1}(\mathbf{N}_1\mathbf{R}_1^{-1} - \mathbf{N}_0\mathbf{R}_0^{-1}) = -(\mathbf{D}_1\mathbf{R}_1^{-1} - \mathbf{D}_0\mathbf{R}_0^{-1}) \end{aligned} \tag{6.29}$$

as $\tilde{\mathbf{Y}}^{-1}\tilde{\mathbf{X}} = \mathbf{X}\mathbf{Y}^{-1}$. Let

$$\mathbf{W} := \mathbf{Y}^{-1}(\mathbf{N}_1\mathbf{R}_1^{-1} - \mathbf{N}_0\mathbf{R}_0^{-1})\mathbf{R}_0. \tag{6.30}$$

It follows from (6.29) that

$$-(\mathbf{D}_1\mathbf{R}_1^{-1} - \mathbf{D}_0\mathbf{R}_0^{-1})\mathbf{R}_0 = \mathbf{X}\mathbf{W} \tag{6.31}$$

and

$$\mathbf{N}_1\mathbf{R}_1^{-1} = \mathbf{N}_0\mathbf{R}_0^{-1} + \mathbf{Y}\mathbf{W}\mathbf{R}_0^{-1}, \tag{6.32}$$

$$\mathbf{D}_1\mathbf{R}_1^{-1} = \mathbf{D}_0\mathbf{R}_0^{-1} - \mathbf{X}\mathbf{W}\mathbf{R}_0^{-1}. \tag{6.33}$$

Obviously $\mathbf{D}_0 - \mathbf{X}\mathbf{W} = \mathbf{D}_1\mathbf{R}_1^{-1}\mathbf{R}_0$ has an inverse in $\mathbb{A}_e$. From which (6.23) follows with $\mathbf{W} \in \mathbb{A}_e$.

Next, the closed loop perturbations $[\Delta\mathbf{K}_{ij}]$ will be related to $\mathbf{W}$. We have from (6.2)

$$\begin{aligned} \mathbf{K}_{11} := (\mathbf{I}+\mathbf{F}\mathbf{G})^{-1} &= (\mathbf{I} + \tilde{\mathbf{Y}}^{-1}\tilde{\mathbf{X}}\mathbf{N}\mathbf{D}^{-1})^{-1} \\ &= \mathbf{D}(\tilde{\mathbf{X}}\mathbf{N} + \tilde{\mathbf{Y}}\mathbf{D})^{-1}\tilde{\mathbf{Y}} \\ &= \mathbf{D}\mathbf{R}^{-1}\tilde{\mathbf{Y}}. \end{aligned} \tag{6.34}$$

Similarly,

$$\begin{aligned} \mathbf{K}_{12} &:= \mathbf{G}(\mathbf{I}+\mathbf{F}\mathbf{G})^{-1} = \mathbf{N}\mathbf{R}^{-1}\tilde{\mathbf{Y}}, \\ \mathbf{K}_{21} &:= \mathbf{F}(\mathbf{I}+\mathbf{G}\mathbf{F})^{-1} = \mathbf{D}\mathbf{R}^{-1}\tilde{\mathbf{X}}, \\ \mathbf{K}_{22} &:= (\mathbf{I}+\mathbf{G}\mathbf{F})^{-1} = \mathbf{I} - \mathbf{N}\mathbf{R}^{-1}\tilde{\mathbf{X}}. \end{aligned} \tag{6.35}$$

Since $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$ are fixed,

$$\begin{aligned}
\Delta\mathbf{K}_{11} &:= \left(\Delta(\mathbf{D}\mathbf{R}^{-1})\right)\tilde{\mathbf{Y}},\\
\Delta\mathbf{K}_{12} &:= \left(\Delta(\mathbf{N}\mathbf{R}^{-1})\right)\tilde{\mathbf{Y}},\\
\Delta\mathbf{K}_{21} &:= \left(\Delta(\mathbf{D}\mathbf{R}^{-1})\right)\tilde{\mathbf{X}},\\
\Delta\mathbf{K}_{22} &:= \left(\Delta(\mathbf{N}\mathbf{R}^{-1})\right)\tilde{\mathbf{X}}.
\end{aligned} \tag{6.36}$$

From (6.32) and (6.33),

$$\mathbf{X}\mathbf{W} = -\Delta\mathbf{K}_{11}\mathbf{D}_0 - \Delta\mathbf{K}_{21}\mathbf{N}_0, \tag{6.37}$$

$$\mathbf{Y}\mathbf{W} = \Delta\mathbf{K}_{12}\mathbf{D}_0 + \Delta\mathbf{K}_{22}\mathbf{N}_0. \tag{6.38}$$

By separate coprimeness of $(\mathbf{X},\ \mathbf{Y})$, there are $(\tilde{\mathbf{N}}_F, \tilde{\mathbf{D}}_F) \in \mathbb{A} \times \mathbb{A}_I$ such that

$$\tilde{\mathbf{N}}_F\mathbf{X} + \tilde{\mathbf{D}}_F\mathbf{Y} = \mathbf{I}. \tag{6.39}$$

Therefore

$$\mathbf{W} = -\tilde{\mathbf{N}}_{\mathbf{F}}\Delta\mathbf{K}_{11}\mathbf{D}_0 - \tilde{\mathbf{N}}_{\mathbf{F}}\Delta\mathbf{K}_{21}\mathbf{N}_0 + \tilde{\mathbf{D}}_{\mathbf{F}}\Delta\mathbf{K}_{12}\mathbf{D}_0 + \tilde{\mathbf{D}}_{\mathbf{F}}\Delta\mathbf{K}_{22}\mathbf{N}_0, \tag{6.40}$$

which means $\mathbf{W} \in \mathbb{A}$ provided $[\Delta\mathbf{K}_{ij}] \in \mathbb{A}^{2\times 2}$.

Q.E.D.

Proof of Theorem 6.2:

The proof will be carried out as a series of implications. In the rest of the proof omit the $\mathbb{A}$ subscript from all norms, i.e. $\|\cdot\| = \|\cdot\|_{\mathbb{A}}$, $\|\cdot\|_{cl} = \|\cdot\|_{cl(\mathbb{A})}$, etc.

By Corollary 6.1, robust stability of $(\mathbf{G}, \mathbf{F})$ implies the joint coprimeness of $\mathbf{N}\mathbf{D}^{-1}$ and $\tilde{\mathbf{Y}}^{-1}\tilde{\mathbf{X}}$, i.e., $\mathbf{R}^{-1} \in \mathbb{A}$. For small enough $\delta = 0$ and $\|(\Delta\mathbf{N})\| \le \delta$, $\|(\Delta\mathbf{D})\| \le \delta$,

$\mathbf{R}_0^{-1}, \mathbf{R}_1^{-1}$ are also in $\mathbb{A}$, which implies that $\Delta\mathbf{K}_{ij} \in \mathbb{A}$ for $i,j = 1,2$. It follows from (6.40) that $\mathbf{W} \in \mathbb{A}$. As the parametrization (6.23) has been shown to hold with $\mathbf{W} \in \mathbb{A}_e$, it now holds with $\mathbf{W} \in \mathbb{A}$.

From (6.40) and the fact that $\|\mathbf{K}_{ij}\| \leq \|\mathbf{K}\|_{cl}$, we get after some rearrangement

$$\|\mathbf{W}\| \leq (\|\mathbf{N}_0\| + \|\mathbf{D}_0\|)(\|\tilde{\mathbf{N}}_F\| + \|\tilde{\mathbf{D}}_F\|)\|\Delta\mathbf{K}\|_{cl}, \tag{6.41}$$

which implies (6.24).

By (6.30) and (6.31) it now follows that

$$\|\Delta\mathbf{N}\mathbf{R}^{-1}\| = \|\mathbf{Y}\mathbf{W}\mathbf{R}_0^{-1}\| \leq Const.\|\mathbf{W}\| \leq Const.\|\Delta\mathbf{K}\|_{cl}, \tag{6.42}$$

$$\|\Delta\mathbf{D}\mathbf{R}^{-1}\| \leq Const.\|\Delta\mathbf{K}\|_{cl}. \tag{6.43}$$

The reverse of inequalities (6.42-6.43) also hold by (6.36) and the bounds on $\|\mathbf{R}_i^{-1}\|_{cl}$, $i = 0,1$. For example

$$\|\Delta\mathbf{K}_{11}\| = \left\| \left(\Delta(\mathbf{D}\mathbf{R}^{-1})\right) \tilde{\mathbf{Y}}\right\| \leq \|\tilde{\mathbf{Y}}\|\|\Delta(\mathbf{N}\mathbf{R}^{-1}, \mathbf{D}\mathbf{R}^{-1})\|_{ol}. \tag{6.44}$$

Similar bounds on the remaining $\|\Delta\mathbf{K}_{ij}\|$ hold also. Moreover,

$$\begin{aligned}
\|\Delta(\mathbf{N}\mathbf{R}^{-1})\| &:= \|\mathbf{N}_1\mathbf{R}_1^{-1} - \mathbf{N}_0\mathbf{R}_0^{-1}\| \\
&\leq \|(\mathbf{N}_1 - \mathbf{N}_0)\mathbf{R}_1^{-1}\| + \|\mathbf{N}_1(\mathbf{R}_1^{-1} - \mathbf{R}_0^{-1})\| \\
&\leq Const.\|\Delta\mathbf{N}\| + Const.\|\Delta\mathbf{R}\| \qquad (6.45) \\
&\leq Const.\|\Delta\mathbf{N}\| + Const.\|\Delta\mathbf{D}\| \\
&\leq Const.\|\Delta(\mathbf{N},\mathbf{D})\|_{ol}.
\end{aligned}$$

A similar bound on $\|\Delta(\mathbf{D}\mathbf{R}^{-1})\|$ gives (6.26). Uniqueness of the representation follows from the fact that $\|\Delta\mathbf{K}\| = 0 \Longrightarrow \Delta(\mathbf{N}\mathbf{R}^{-1}, \mathbf{D}\mathbf{R}^{-1}) = 0$, by (6.26).

Q.E.D.

Chapter 7 Local vs. Global Coprimeness and Stability

We return now to the operator double algebra $\mathbb{B}_e$. The strictly causal (resp. invertible) elements of $\mathbb{B}_e$ are those $\mathbf{G} \in \mathbb{B}_e$ whose kernels satisfy $k(t,t) = 0$ (resp. $[k(t,t)]^{-1} \in \mathbb{C}^{n\times n}$) for all $t \in \mathbb{Z}$.

The normed double algebra in this chapter will be $\underline{\mathbb{E}}_\sigma$. Let $(\underline{\mathbb{E}}_\sigma)_J$ denote operators $\mathbf{D}$ in $\underline{\mathbb{E}}_\sigma$ with memoryless part invertible in $\underline{\mathbb{E}}_\sigma$, i.e., the kernel of $\mathbf{D}$ satisfies $\left|d(t,t)^{-1}\right| \leq Const.$ for all $t \in \mathbb{Z}$.[7.1]

An operator $\mathbf{K} \in \mathbb{B}_e$ has a *global right factorization* (resp. $\mathbf{K}^l \in \mathbb{B}_e$ has a *local right factorization*) in $\underline{\mathbb{E}}_\sigma$ iff $\mathbf{K}$ has the form $\mathbf{K} = \mathbf{N}\mathbf{D}^{-1}$ (resp. iff $\mathbf{K}^l$ has the form $\mathbf{K}^l = \mathbf{N} \otimes \mathbf{D}^{\ominus}$) where $(\mathbf{N}, \mathbf{D})$ is in $\underline{\mathbb{E}}_\sigma \times (\underline{\mathbb{E}}_\sigma)_J$.

More generally, any object defined on the global algebra $\mathbb{G}\underline{\mathbb{E}}_\sigma$ (resp. local algebra $\mathbb{L}\underline{\mathbb{E}}_\sigma$) will be designated as the global object (resp. local object) in the double algebra $\underline{\mathbb{E}}_\sigma$. For example the global factorization $\mathbf{K} = \mathbf{N}\mathbf{D}^{-1}$ (resp. local factorization

---

[7.1] Note that the stronger condition $(\mathbf{D}_{nm})^{-1} \in \underline{\mathbb{E}}_\sigma$ rather than $(\mathbf{D}_{nm})^{-1} \in \mathbb{B}_e$ (as in Chapter 6) is used here in definition.

$\mathbf{K}^l = \mathbf{N} \otimes \mathbf{D}^{\ominus}$) is globally (resp. locally) right coprime in $\underline{\mathbb{E}}_\sigma$ if there are operators $\tilde{\mathbf{X}}, \tilde{\mathbf{Y}} \in \underline{\mathbb{E}}_\sigma$ for which the following inverse exists

$$\text{globally} \qquad \left(\tilde{\mathbf{X}}\mathbf{N} + \tilde{\mathbf{Y}}\mathbf{D}\right)^{-1} \in \underline{\mathbb{E}}_\sigma, \tag{7.1}$$

$$\text{(resp. locally} \qquad \left(\tilde{\mathbf{X}} \otimes \mathbf{N} + \tilde{\mathbf{Y}} \otimes \mathbf{D}\right)^{\ominus} \in \underline{\mathbb{E}}_\sigma). \tag{7.2}$$

Let $\mathbf{G}$, $\mathbf{F} \in \mathbb{B}_e$ represent plant and feedback operators, respectively. Their feedback interconnection is well-posed in $\mathbb{B}_e$ if all four operators in either one of the matrices (7.3 - 7.4) are in $\mathbb{B}_e$

$$\text{Globally :} \qquad \mathbf{K} = [\mathbf{K}_{ij}] := \begin{pmatrix} (\mathbf{I} + \mathbf{F}\mathbf{G})^{-1} & \mathbf{G}\,(\mathbf{I} + \mathbf{F}\mathbf{G})^{-1} \\ \mathbf{F}\,(\mathbf{I} + \mathbf{G}\mathbf{F})^{-1} & (\mathbf{I} + \mathbf{G}\mathbf{F})^{-1} \end{pmatrix}, \tag{7.3}$$

$$\text{Locally :} \qquad \mathbf{K}^l = [\mathbf{K}^l_{ij}] := \begin{pmatrix} (\mathbf{I} + \mathbf{F} \otimes \mathbf{G})^{\ominus} & \mathbf{G} \otimes (\mathbf{I} + \mathbf{F} \otimes \mathbf{G})^{\ominus} \\ \mathbf{F} \otimes (\mathbf{I} + \mathbf{G} \otimes \mathbf{F})^{\ominus} & (\mathbf{I} + \mathbf{G} \otimes \mathbf{F})^{\ominus} \end{pmatrix}. \tag{7.4}$$

There is no distinction between local and global well-posedness, the two are equivalent.[7.2] Let $\mathbb{A}$ be a normed double subalgebra of $\mathbb{B}_e$. $\mathbf{F}$ and $\mathbf{G}$ globally (resp. locally) stabilize each other in $\mathbb{A}$ if all four operators in (7.3) (resp. in (7.4)) are in $\mathbb{A}$. In general, local and global stabilization may not be equivalent.

The matrices (7.3) and (7.4) will be termed the global and local matrices respectively.

[7.2] A sufficient condition for well-posedness is that the memoryless part of $\mathbf{I} + \mathbf{F}\mathbf{G}$ be invertible in $\mathbf{B}_e$.

Global robustness in $\underline{\mathbf{E}}_\sigma$ of stabilization is defined as in Chapter 6, as neighbourhood boundedness of the global open-to-closed-loop map $\mathcal{K}(\cdot,\cdot)$, with the algebra $\mathbb{A}$ identified as $\underline{\mathbf{E}}_\sigma$. Similarly, local robustness in $\underline{\mathbf{E}}_\sigma$ is defined as neighbourhood boundedness of the map $\mathcal{K}^l$ which takes open-loop plant pairs into closed-loop matrices, i.e., $\mathcal{K}^l : \underline{\mathbf{E}}_\sigma^2 \to \underline{\mathbf{E}}_\sigma^{2\times 2}, \mathcal{K}^l(\mathbf{N},\mathbf{D}) = \mathbf{K}^l$, the matrix $\mathbf{K}^l$ specified by (7.4).

It should be noted that the maps $\mathcal{K}$ and $\mathcal{K}^l$ coincide for shift invariant operators and therefore $\mathcal{K}^l(\mathbf{N}_t,\mathbf{D}_t) = \mathcal{K}(\mathbf{N}_t,\mathbf{D}_t)$ for each $t \in \mathbb{Z}$. Consequently local stabilizations and local robustness are in fact properties of the global map $\mathcal{K}(\cdot,\cdot)$ restricted to shift-invariant variables.

7.1 Relations between Local and Global Properties

Consider a strictly causal plant $\mathbf{G} \in \mathbf{B}_e$ which is to be stabilized in $\mathbb{A}$ by a feedback $\mathbf{F} \in \mathbf{B}_e$, and which therefore necessarily has a factorization in $\mathbb{A}$, say $\mathbf{G} = \mathbf{N}\mathbf{D}^{-1}$. The stabilization is to be designed on the basis of a local approximation to $\mathbf{G}$, $\mathbf{G}^l := \mathbf{N} \otimes \mathbf{D}^{\ominus}$, which is used to select a feedback $\mathbf{F}^l := \tilde{\mathbf{Y}}^{\ominus} \otimes \tilde{\mathbf{X}}$ which locally stabilizes $\mathbf{G}^l$. We would like to answer the following question for slowly time-varying $\mathbf{G}$ and $\mathbf{F}$: Do local properties of $(\mathbf{G}^l,\mathbf{F}^l)$, such as joint coprimeness and robust stability, extend to global properties of $(\mathbf{G},\mathbf{F})$?

As pointed out in Chapter 6, strict causality of $\mathbf{G} \in \mathbf{B}_e$ implies that of $\mathbf{N}$. It follows that for any $\tilde{\mathbf{X}},\tilde{\mathbf{Y}} \in \underline{\mathbf{E}}_\sigma$ satisfying the Bezout equation

$$\tilde{\mathbf{X}} \otimes \mathbf{N} + \tilde{\mathbf{Y}} \otimes \mathbf{D} = \mathbf{I}, \quad \text{or} \quad \tilde{\mathbf{X}}\mathbf{N} + \tilde{\mathbf{Y}}\mathbf{D} = \mathbf{I},$$

we have

$$(\tilde{\mathbf{Y}})_{nm}(\mathbf{D})_{nm} = (\mathbf{I} - \tilde{\mathbf{X}}\mathbf{N})_{nm} = \mathbf{I}.$$

Thus $(\widetilde{\mathbf{Y}})_{nm}^{-1} = (\mathbf{D})_{nm} \in \underline{\mathbf{E}}_\sigma$, i.e., $(\widetilde{\mathbf{Y}})_{nm} \in (\underline{\mathbf{E}}_\sigma)_J$. Henceforth, the verification of $(\widetilde{\mathbf{Y}})_{nm} \in (\underline{\mathbf{E}}_\sigma)_J$ will be omitted in the following discussion.

A pair $(\mathbf{N},\mathbf{D})$ in $\underline{\mathbf{E}}_\sigma \times \underline{\mathbf{E}}_\sigma$ will be said to satisfy the *uniform corona conditions* if there is $\sigma_0 > \sigma$ such that for all $t \in \mathbb{Z}$ and for each vector e_i, $i = 1, \cdots, n$ of an orthonormal basis in $\mathbb{C}^n$,

$$|\widehat{\mathbf{N}}_t(z)e_i| + |\widehat{\mathbf{D}}_t(z)e_i| \geq \alpha \qquad |z| < \sigma_0. \tag{7.5}$$

We may wish to consider coprimeness in a variable rate situation defined in Chapter 3, in which $\mathbf{N},\mathbf{D},\widetilde{\mathbf{X}},\widetilde{\mathbf{Y}}$ are embedded in sets of operators $\mathbf{N}(\gamma),\mathbf{D}(\gamma),\widetilde{\mathbf{X}}(\gamma),\widetilde{\mathbf{Y}}(\gamma)$, in some subalgebra $\mathbb{A}$ of $\mathbb{B}_e$, and depending on a parameter $\gamma > 0$. If $(\mathbf{N}(\gamma),\mathbf{D}(\gamma))$, $(\widetilde{\mathbf{X}}(\gamma),\widetilde{\mathbf{Y}}(\gamma)) \in \mathbb{A} \times \mathbb{A}_J$ are jointly coprime for all γ in some $(0,\gamma_0]$, joint coprimeness will be called *uniform in rate* on $(0,\gamma_0]$ if $\|(\widetilde{\mathbf{X}}(\gamma)\mathbf{N}(\gamma) + \widetilde{\mathbf{Y}}(\gamma)\mathbf{D}(\gamma))^{-1}\|_{\mathbb{A}}$ is bounded on $(0,\gamma_0]$; similarly, robust stabilization will be called *uniform in rate* if the constants appearing in the definition (6.4) is independent of γ, $\gamma \in (0,\gamma_0]$.

Assumptions for Theorem 7.1: (Assumptions for the case of variable rates are expressed as (and $\cdots$) in parentheses).

$1 \leq \sigma < \sigma_0$ are constants. For the operators $\mathbf{N},\mathbf{D},\widetilde{\mathbf{X}},\widetilde{\mathbf{Y}}$ in $\underline{\mathbf{E}}_\sigma$, the maximal variation-rate of either their transforms, $(2 \leq p \leq \infty)$

$$\gamma^{(p)} = \max\left\{\partial_{\sigma_0}^{(p)}(\mathbf{K}) : \mathbf{K} \in \{\mathbf{N},\mathbf{D},\widetilde{\mathbf{X}},\widetilde{\mathbf{Y}}\}\right\}, \tag{7.6}$$

or their kernels,

$$d = \max\left\{d_{\sigma_0}(\mathbf{K}) : \mathbf{K} \in \{\mathbf{N},\mathbf{D},\widetilde{\mathbf{X}},\widetilde{\mathbf{Y}}\}\right\} \tag{7.7}$$

is equal to γ_0.

Recall that $(\mathbb{G}\underline{\mathbb{E}}_\sigma)_e^c$ (or $(\mathbb{L}\underline{\mathbb{E}}_\sigma)_e^c$) is the subset of $\mathbb{B}_e$ consisting of operators admitting some (maybe unknown) coprime factorization representations in $\mathbb{G}\underline{\mathbb{E}}_\sigma$ (or in $\mathbb{L}\underline{\mathbb{E}}_\sigma$).

Theorem 7.1

(1) The following statements (a), (b) and (c) are equivalent in $\mathbb{L}\underline{\mathbb{E}}_\sigma$.

(a) $\mathbf{G}^l$ and $\mathbf{F}^l$ are in $(\mathbb{L}\underline{\mathbb{E}}_\sigma)_e^c$ and have factorizations (and uniform in rate)

$$\mathbf{G}^l = \mathbf{N} \otimes \mathbf{D}^{\ominus}, \qquad \mathbf{F}^l = \tilde{\mathbf{Y}}^{\ominus} \otimes \tilde{\mathbf{X}}, \tag{7.8}$$

which are mutually robustly stabilizing in the local algebra $\mathbb{L}\underline{\mathbb{E}}_\sigma$ (and uniform in rate).

(b) The factorizations (7.8) are jointly coprime in $\mathbb{L}\underline{\mathbb{E}}_\sigma$ (and uniform in rate).

(c) $(\mathbf{N},\mathbf{D})$ and $(\tilde{\mathbf{X}},\tilde{\mathbf{Y}})$ satisfy the uniform corona conditions and are mutually stabilizing (and with constants independent of γ).

(2) The following statements (d) and (e) are equivalent in $\mathbb{G}\underline{\mathbb{E}}_\sigma$.

(d) The factorizations

$$\mathbf{G} = \mathbf{N}\mathbf{D}^{-1} \quad \text{and} \quad \mathbf{F} = \tilde{\mathbf{Y}}^{-1}\tilde{\mathbf{X}} \tag{7.9}$$

are jointly coprime in the global algebra $\mathbb{G}\underline{\mathbb{E}}_\sigma$ (and uniform in rate).

(e) $\mathbf{G}$ and $\mathbf{F}$ are in $(\mathbb{G}\underline{\mathbb{E}}_\sigma)_e^c$ and have factorizations (7.9), and $\mathbf{G}$ and $\mathbf{F}$ are mutually globally stabilizing in $\mathbb{G}\underline{\mathbb{E}}_\sigma$ (and uniform in rate).

(3) There exists a variation-rate bound $\gamma_0 > 0$ for which (1) and (2) are equivalent (and uniform in rate).

□

Two lemmas will proceed the proof of Theorem 7.1. Since **G** is assumed strictly causal, **N** is strictly causal.

<u>Lemma 7.1</u>[7.3]

For $(\mathbf{N},\mathbf{D})$ in $\underline{\mathbb{E}}_\sigma \times (\underline{\mathbb{E}}_\sigma)_J$, if the uniform corona conditions (7.5) are satisfied then $(\mathbf{N},\mathbf{D})$ is right coprime in $\mathbb{L}\underline{\mathbb{E}}_\sigma$. Converse is also true.

□

Proof of Lemma 7.1:

By Fuhrmann-Vasyunin Theorem [Nik, p293], under the uniform corona conditions, there exist $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$ in $\overline{\mathbb{E}}_{\sigma_0}$ such that

$$\tilde{\mathbf{X}} \otimes \mathbf{N} + \tilde{\mathbf{Y}} \otimes \mathbf{D} = \mathbf{I}, \tag{7.10}$$

and for α defined in (7.5), $\beta := \alpha/(\mu^2_{\sigma_0}(\mathbf{N}) + (\mu^2_{\sigma_0}(\mathbf{D}))^{1/2} \le 1$, $\mu_\sigma(\tilde{\mathbf{X}})$ and $\mu_\sigma(\tilde{\mathbf{Y}})$ are bounded by

$$(\mu^2_{\sigma_0}(\tilde{\mathbf{X}}) + \mu^2_{\sigma_0}(\tilde{\mathbf{Y}}))^{1/2} \le \sqrt{n}\left(\frac{1}{\beta^n} + \frac{1}{\beta^{2n}}(7\sqrt{\log\frac{1}{\beta^n}} + 20\log\frac{1}{\beta^n})\right)$$

(see [Nik, pp. 292-293]) i.e., $(\mathbf{N},\mathbf{D})$ is right coprime in $\overline{\mathbb{L}\mathbb{E}}_{\sigma_0}$, and hence in $\mathbb{L}\underline{\mathbb{E}}_\sigma$.

Conversely, if $\tilde{\mathbf{X}} \otimes \mathbf{N} + \tilde{\mathbf{Y}} \otimes \mathbf{D} = \mathbf{I}$ for $(\tilde{\mathbf{X}},\tilde{\mathbf{Y}}),(\mathbf{N},\mathbf{D}) \in \mathbb{L}\underline{\mathbb{E}}_\sigma \times (\mathbb{L}\underline{\mathbb{E}}_\sigma)_J$, then there exists $\sigma_0 > \sigma$ such that $(\tilde{\mathbf{X}},\tilde{\mathbf{Y}}),(\mathbf{N},\mathbf{D}) \in \overline{\mathbb{L}\mathbb{E}}_{\sigma_0} \times (\overline{\mathbb{L}\mathbb{E}}_{\sigma_0})_J$. Thus for any unit vector $e \in \mathbb{C}^n$, and $|z| < \sigma_0$

$$e = \tilde{\mathbf{X}}_t(z)\mathbf{N}_t(z)e + \tilde{\mathbf{Y}}_t(z)\mathbf{D}_t(z)e.$$

[7.3] For related work on the relation between corona condition and robust stability, see [Can1,2]

Therefore,

$$1 \leq \mu_\sigma(\widetilde{\mathbf{X}})|\mathbf{N}_t(z)e| + \mu_\sigma(\widetilde{\mathbf{Y}})|\mathbf{D}_t(z)e|$$
$$\leq \max\left\{\mu_{\sigma_0}(\widetilde{\mathbf{X}}), \mu_{\sigma_0}(\widetilde{\mathbf{Y}})\right\}(|\mathbf{N}_t(z)e| + |\mathbf{D}_t(z)e|).$$

As $\max\left\{\mu_{\sigma_0}(\widetilde{\mathbf{X}}), \mu_{\sigma_0}(\widetilde{\mathbf{Y}})\right\} > 0$ (by the Bezout equation (7.10)), the conclusion follows.

Q.E.D.

In the following lemma, let $\sigma_0 > \sigma$ and $\mathbf{N}, \mathbf{D}, \widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}} \in \mathbb{E}_{\sigma_0}$ with

$$k = \max\{\|\widetilde{\mathbf{X}}\|_{(\sigma_0)}, \|\widetilde{\mathbf{Y}}\|_{(\sigma_0)}, \|\mathbf{N}\|_{(\sigma_0)}, \|\mathbf{D}\|_{(\sigma_0)}\},$$

$$\gamma = \max\{d_\sigma(\widetilde{\mathbf{X}}), d_\sigma(\widetilde{\mathbf{Y}}), d_\sigma(\mathbf{N}), d_\sigma(\mathbf{D})\},$$

$$\widetilde{\mathbf{X}} \otimes \mathbf{N} + \widetilde{\mathbf{Y}} \otimes \mathbf{D} = \mathbf{R}_l, \tag{7.11}$$

$$\widetilde{\mathbf{X}}\mathbf{N} + \widetilde{\mathbf{Y}}\mathbf{D} = \mathbf{R}_g. \tag{7.12}$$

<u>**Lemma 7.2**</u>

(a) If $\mathbf{R}_l^\ominus$ exists in $\mathbb{E}_\sigma$, then $\mathbf{R}_g^{-1} \in \mathbb{E}_\sigma$ provided the following inequalities hold

$$\|\mathbf{R}_l \bigtriangledown \mathbf{R}_l^\ominus\|_{(\sigma)} < 1, \tag{7.13}$$

$$\|(\widetilde{\mathbf{X}} \bigtriangledown \mathbf{N} + \widetilde{\mathbf{Y}} \bigtriangledown \mathbf{D})\mathbf{R}_l^{-1}\|_{(\sigma)} < 1. \tag{7.14}$$

Moreover (7.13) and (7.14) are valid provided

$$0 < \alpha\gamma(1 - \beta\gamma)^{-1} < 1 \tag{7.15}$$

where $\alpha = 2\left(e \ln\left(\frac{\sigma_0}{\sigma}\right)\right)^{-1} \sigma^{-1} k \|\mathbf{R}_l^\ominus\|_{(\sigma)}$, $\beta = 4k^2\alpha\|\mathbf{R}_l^\ominus\|_{(\sigma)}$, and certainly for small enough rate γ.

(b) If $\mathbf{R}_g^{-1}$ exists in $\mathbf{E}_\sigma$, then $\mathbf{R}_l^\ominus \in \mathbf{E}_\sigma$, provided the following inequalities hold:

$$\|\mathbf{R}_g \bigtriangledown \mathbf{R}_g^{-1}\|_{(\sigma)} < 1, \tag{7.16}$$

$$\|(\widetilde{\mathbf{X}} \bigtriangledown \mathbf{N} + \widetilde{\mathbf{Y}} \bigtriangledown \mathbf{D}) \otimes \mathbf{R}_g^\ominus\|_{(\sigma)} < 1. \tag{7.17}$$

Moreover (7.16) and (7.17) are valid provided (7.15) is satisfied with $\mathbf{R}_l^\ominus$ replaced by $\mathbf{R}_g^{-1}$.

□

Proof of Lemma 7.2:

(a): The existence of $\mathbf{R}_l{}^\ominus \in \mathbf{E}_\sigma$ and (7.13) implies the existence of $\mathbf{R}_l{}^{-1} \in \mathbf{E}_\sigma$ by Prop. 2.2 provided $\|\mathbf{R}_l \bigtriangledown \mathbf{R}_l^\ominus\|_{(\sigma)} < 1$. Therefore, as

$$\begin{aligned}\mathbf{R}_g &= \mathbf{R}_l + \widetilde{\mathbf{X}} \bigtriangledown \mathbf{N} + \widetilde{\mathbf{Y}} \bigtriangledown \mathbf{D}\\ &= \left[\mathbf{I} + (\widetilde{\mathbf{X}} \bigtriangledown \mathbf{N} + \widetilde{\mathbf{Y}} \bigtriangledown \mathbf{D})\mathbf{R}_l^{-1}\right]\mathbf{R}_l\end{aligned}$$

by the contraction principle, (7.14) ensures existence of $\mathbf{R}_g{}^{-1} \in \mathbf{E}_\sigma$, where

$$\mathbf{R}_g^{-1} = \mathbf{R}_l^{-1}\left[\mathbf{I} + (\widetilde{\mathbf{X}} \bigtriangledown \mathbf{N} + \widetilde{\mathbf{Y}} \bigtriangledown \mathbf{D})\mathbf{R}_l^{-1}\right]^{-1}. \tag{7.18}$$

Now by Prop. 2.5 (b),

$$\begin{aligned}\|\mathbf{R}_l \bigtriangledown \mathbf{R}_l^\ominus\|_{(\sigma)} &\le \left(e\ln\left(\frac{\sigma_0}{\sigma}\right)\right)^{-1}\sigma^{-1}\|\mathbf{R}_l\|_{(\sigma)}d_\sigma(\mathbf{R}_l^\ominus)\\ &\le 2k^2\left(e\ln\left(\frac{\sigma_0}{\sigma}\right)\right)^{-1}\sigma^{-1}d_\sigma(\mathbf{R}_l^\ominus).\end{aligned} \tag{7.19}$$

By Prop. 2.5 (d) and the fact that $(\mathbf{R}_l^\ominus)_t = (\widetilde{\mathbf{X}}_t\mathbf{N}_t + \widetilde{\mathbf{Y}}_t\mathbf{D}_t)^{-1}$,

$$\begin{aligned}\sigma^{-1}d_\sigma(\mathbf{R}_l^\ominus) &\le \|\mathbf{R}_l^\ominus\|_{(\sigma)}^2 \sup_t \|(r_l)_t - (r_l)_{t-1}\|_{l^1_\sigma}\\ &\le \|\mathbf{R}_l^\ominus\|_{(\sigma)}^2\sigma^{-1}4k\gamma,\end{aligned} \tag{7.20}$$

where r_l is the kernel of $\mathbf{R}_l$. Therefore

$$\begin{aligned}
&\|\mathbf{R}_l \nabla \mathbf{R}_l^{\ominus}\|_{(\sigma)} \\
&\leq 8k^3 \left(e \ln\left(\frac{\sigma_0}{\sigma}\right)\right)^{-1} \sigma^{-1} \|\mathbf{R}_l^{\ominus}\|^2_{(\sigma)} \gamma := \beta\gamma \qquad (7.21) \\
&< 1 \qquad \text{for small enough } \gamma,
\end{aligned}$$

where $\beta = 8k^3 \left(e \ln\left(\frac{\sigma_0}{\sigma}\right)\right)^{-1} \sigma^{-1} \|\mathbf{R}_l^{\ominus}\|^2_{(\sigma)}$. By Prop. 2.2,

$$\|\mathbf{R}_l^{-1}\|_{(\sigma)} \leq (1-\beta\gamma)^{-1} \|\mathbf{R}_l^{\ominus}\|_{(\sigma)}. \qquad (7.22)$$

(7.22) implies, by Prop. 2.5 (b) again,

$$\begin{aligned}
&\left\|\left(\widetilde{\mathbf{X}} \nabla \mathbf{N} + \widetilde{\mathbf{Y}} \nabla \mathbf{D}\right) \mathbf{R}_l^{-1}\right\|_{(\sigma)} \\
&\leq 2 \left(e \ln\left(\frac{\sigma_0}{\sigma}\right)\right)^{-1} \sigma^{-1} k\gamma (1-\beta\gamma)^{-1} \|\mathbf{R}_l^{\ominus}\|_{(\sigma)} \qquad (7.23) \\
&= \alpha\gamma (1-\beta\gamma)^{-1}
\end{aligned}$$

where $\alpha = 2 \left(e \ln\left(\frac{\sigma_0}{\sigma}\right)\right)^{-1} \sigma^{-1} k \|\mathbf{R}_l^{\ominus}\|_{(\sigma)}$. Therefore, (7.13) and (7.14) are satisfied provided (7.15) is valid.

(b) The proof is similar to that of part (a), with $\mathbf{R}_l^{\ominus}$ replaced by $\mathbf{R}_g^{-1}$, the global product $*$ by $\otimes$ etc., and also

$$d_\sigma(\mathbf{R}_g^{-1}) \leq \|\mathbf{R}_g^{-1}\|^2_{(\sigma)} 4k\gamma.$$

Q.E.D.

Proof of Theorem 7.1:

Suppose $\mathbf{N}, \mathbf{D}, \widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}}$ are fixed in $\underline{\mathbb{E}}_\sigma$ and $\gamma \in (0, \gamma_0]$ is a constant.

(1) (a) $\Longleftrightarrow$ (b): This is Theorem 6.1, part (a) $\Longleftrightarrow$ (b), and with the algebra $\mathbb{A}$ identified as $\mathbb{L}\underline{\mathbb{E}}_\sigma$.

(b) $\Longleftrightarrow$ (c): This follows from Lemma 7.1 and Prop. 6.2.

(2) (d) $\Longleftrightarrow$ (e): This is Theorem 6.1 with the identification of $\mathbb{A}$ as $\mathbb{G}\underline{\mathbb{E}}_\sigma$.

(3) For small γ_0,

(1) $\Longrightarrow$ (2): If (b) is true, then there exists $\sigma_0 > \sigma_1 > \sigma$ such that (7.6,7.7) are defined and $\mathbf{R}_l^\ominus \in \mathbb{E}_{\sigma_1}$. Lemma 7.2 part (a) (replacing σ by σ_1 in Lemma 7.2 and noting that $\gamma \leq \gamma_0$) implies that for small enough γ_0, $\mathbf{R}_g^{-1} \in \mathbb{E}_{\sigma_1} \subset \mathbb{G}\underline{\mathbb{E}}_\sigma$. So (d) is true.

(2) $\Longrightarrow$ (1): Similarly applying Lemma 7.2 part (b) to (d), we conclude that $\mathbf{R}_l^\ominus \in \mathbb{L}\underline{\mathbb{E}}_\sigma$.

Q.E.D.

Chapter 8 Conclusion

8.1 A Discussion of the Results

The problem of robust stabilization and sensitivity optimization for slowly time-varying systems has been investigated in this thesis. The problem formulation reflects three main features of adaptive systems: persistent external noises, time-varying plant model and disturbance data, and causal dependence of feedback controller design on that data.

The local-global double algebra symbolism, introduced for the first time in this research, provides a common framework for stability and performance analysis in slowly time-varying systems. Within this framework, the coupling between local and global properties is described via a ∇ operator, which is small for small rates of time variation.

Slowly time-varying systems are characterized as operators with small commutants (with the shift). The norm of the commutant is bounded by the variation rate of a local transfer function, which is tractable in the frequency domain. For systems with small variation rates in local transfer functions, the validity of the local-global coupling is established. Within certain prescribed limits, stability and performance analysis can be carried out locally in the frequency domain.

Although details are worked out for discrete-time systems and certain sensitivity optimization problems, the normed double algebra symbolism provides a general approximation framework for slowly time-varying systems. The symbolism can be applied to other system settings provided the axioms of the normed double algebra are satisfied and smallness of the ∇ operator is established. As for performance criteria, the sensitivity optimization imposed in this thesis serves as only one choice (though not an unimportant one) for analysis and synthesis. Other design criteria can certainly be employed, such as mixed sensitivity minimization, ex., Jonckheere and Verma [Jon], parameter optimization, etc. A critical issure in such synthesis problems is that of Lipschitz dependence on data, which is resolved in this thesis by using δ-suboptimal maximum entropy solutions.

8.2 Some Further Research Directions

Other design criteria, especially mixed sensitivity or general four block sensitivity optimization, may be worth considering in the NDA framework.

A task complementary to this research is to develop modeling and identification schemes compatible with the underlying design problem. Integration of these schemes with the local-global double algebra would be a major step towards comprehensive operator-system adaptive theory.

References

[Ada1] V.M. Adamian, D.Z. Arov, and M.G. Krein, Analytic properties of Schmidt pairs for a Hankel operator and the generalized Schur-Takagi problem, *Mat. Sb.*, 86(128), 1971, pp.34-75.

[Ada2] V.M. Adamian, D.Z. Arov, and M.G. Krein, Infinite Hankel block matrices and related extension problems, *Amer. Math. Soc. Transl.* 2, Vol. 111, 1978, pp.133-156.

[Akh] N.I. Akhiezer and I.M. Glazman, Theory of linear operators in Hibert space, Vol.1, New York: Ungar, 1961.

[Arv] W. Arveson, Interpolation problems in nest algebras, *J. Functional Analysis*, 20, 1975, pp.208-233.

[Bal1] J.A. Ball and J.W. Helton, A Beurling-Lax theory for the Lie group U(m,n) which contains most classical interpolation theory, preprint.

[Bal2] J.A. Ball, C. Foias, J.W. Helton, and A. Tannenbaum, On a local nonlinear commutant lifting theorem, preprint.

[Bal3] J.A. Ball, C. Foias, J.W. Helton, and A. Tannenbaum, A Poincare-Dulac approach to a nonlinear Beurling-Lax-Halmos theorem, preprint.

[Bar] J.F. Barman, Wellposedness of feedback systems and singular perturbations, Ph.D. thesis, Univ. California, 1973.

[Can1] H.M.J. Cantalloube, Robust adaptive control: a Bezout approach, Ph.D dissertation, McGill University, 1989.

[Can2] H.M.J. Cantalloube, C.E. Nahum, and P.E. Caines, Robust adaptive control: a direct factorization approach, preprint.

[Dah] M. Dahleh and M. Dahleh, A class of adaptive controllers with application to robust adaptive control, preprint.

[Des1] C.A. Desoer, Slowly varying discrete system $x_{i+1} = Ax_i$, *Electronics Lett.*, 6, 1970, pp.339-340.

[Des2] C.A. Desoer and M. Vidyasagar, Feedback Systems: Input-output Properties, Academic Press, New York, 1975.

[Des3] C.A. Desoer, R.W. Liu, J. Murray, and R. Saeks, Feedback system design: the fractional representation approach to analysis and synthesis, *IEEE Trans. on Automat. Control*, Vol. AC-25, 1980, pp. 399-412.

[Des4] C.A. Desoer, A.N. Gundes and M.G. Kabuli, Coprime factorizations in stable linear systems, *Proc. 1988 ACC conference*, Atlanta, Georgia, U.S.A., 1988, pp.1252-1253.

[Doy] J.C. Doyle and G. Stein, Multivariable feedback design: concepts for a classical modern synthesis, *IEEE Trans. on Automat. Control*, Vol. AC-26, 1981, pp. 4-16.

[Dur] P.L. Duren, Theory of H^p spaces, Academic Press, New York, 1970.

[Dym1] H. Dym and I. Gohberg, A maximum entropy principle for contractive interpolants, *J. of Functional Analysis*, 65, 1986, pp. 83-125

[Dym2] H. Dym and I. Gohberg, A new class of contractive interpolants and maximum entropy principles,

[Fei1] A. Feintuch and B.A. Francis, Uniformly optimal control of linear systems, *Automatica*, 21(5),1985,pp.563-574.

[Fei2] A. Feintuch and B.A. Francis, Uniformly optimal control of linear time-varying systems, *Systems and Control Letters*, 5, 1984, pp.67-71.

[Fra1] B.A. Francis, A course in H^∞ control theory, *Lecture Notes in Control and Information Sciences*, Vol.88, Springer-Verlag: New York, 1985.

[Fra2] B.A. Francis, J.W. Helton, and G. Zames, H^∞ optimal feedback controllers for linear multivariable systems, *IEEE Trans. Automat. Control*, Vol. AC-29, 1984, pp.888-900.

[Fra3] B.A. Francis and M. Vidyasagar, Algebraic and topological aspects of the servo problem for lumped systems, Dep. Eng. Appl. Sci., Yale Univ., New Haven, CI, S & SI Rep. 8003, 1980.

[Fre] M. Freedman and G. Zames, Logarithmic variation criteria for the stability of systems with time-varying gains, *SIAM J. Control* Vol.6, No.3, 1968, pp.487-507.

[Gar] J.B. Garnett, Bounded analytic functions, Academic Press, New York, 1981.

[Geo] T.T. Georgiou and M.C. Smith, Optimal robustness in the gap metric, preprint.

[Hel] H. Helson, Lectures on invariant subspaces, Academic Press, New York, 1964.

[Hof] K. Hoffman, Banach spaces of analytic functions, Englewood Cliffs, NJ: Pretice-Hall, 1962.

[Jon] E.A. Jonckheere and M.S. Verma, A spectral characterization of H^∞-optimal feedback performance and its efficient computation. *Systems and Control Letters* 8, 1986, pp. 13-22.

[Kam] E.W. Kamen, P.P. Khargonekar, K.R. Poolla, A transfer-function approach to linear time-varying discrete-time systems, *SIAM J. Control and Optimization*, Vol.23, No.4, July 1985, pp.550-565.

[Kha1] P. Khargonekar, T. Georgiou, and A. Pascoal, On the robust stabilizability of linear unstructured uncertainty, *IEEE Trans. on Automat. Control*, Vol. AC-32, 1987, pp. 201-207.

[Kha2] P.P. Khargonekar and M.A. Rotea, Coprime factorization for linear time-varying systems, *Proc. 1988 ACC conference*, Atlanta, Georgia, U.S.A., 1988, pp.848-851.

[Kha3] P.P. Khargonekar and K.R. Poolla, Uniformly optimal control of linear time-invariant plants: nonlinear time-varying controllers, *Syst. & Contr. Letters*, 6, 1986, pp.303-308.

[Kim] H. Kimura, Robust stabilization for a class of transfer functions, *IEEE Trans. on Automat. Control*, Vol. AC-29, 1984, pp. 788-793.

[Koo] P.J. Koosis, Introduction to H^p spaces, Cambridge University Press, 1980.

[Kre] G. Kreisselmeier, Adaptive control of a class of slowly time-varying plants, *Systems and Control Letters*, 8, 1986, pp.97-103.

[Nag] B. Sz.-Nagy and C. Foias, Harmonic analysis of operators on Hilbert space, North-Holland: Amsterdam, 1970.

[Nar] K.S. Narendra, Frequency domain criteria for absolute stability, New York: Academic Press, 1973.

[Nik] N.K. Nikol'skii, Treatise on the shift operator, Springer-Verlag: Berlin, 1986.

[Prz] K.M. Przyluski and S. Rolewicz, On stability of linear time-varying infinite-dimentional discrete-time systems, *Systems and Control Letters*, 4, 1984, pp.307-315.

[Ros] H.H. Rosenbrock, The stability of linear time-dependent control systems, *J. Electron. Contr.*, 15, 1963, pp.73-80.

[Rud1] W. Rudin, Real and complex analysis, McGraw-Hill, 1974.

[Rud2] W. Rudin, Functional analysis, McGraw-Hill, 1973.

[Sar] D. Sarason, Generalized interpolation in H^∞, *Amer. Math. Soc. Transl.* 127, 1967, pp.179-203.

[Smi1] M.C. Smith, Well-posedness of H^∞ optimal sensitivity problem, preprint.

[Smi2] M.C. Smith, On stabilization and the existence of coprime factorizations, *IEEE Trans. on Automat. Control*, Vol. AC-34, 1989, pp. 1005-1007.

[Ver1] M.S. Verma, Robust stability of linear feedback systems under time-varying and nonlinear perturbations in the plant, preprint.

[Ver2] M.S. Verma and E.A. Jonckheere, L^∞-compensation with mixed sensitivity as a broadband matching problem, *Systems and Control Letters* 4, 1984, pp. 125-129.

[Vid] M. Vidyasagar, H. Schneider and B. Francis, Algebraic and topological aspects of feedback stabilization, *IEEE Trans. on Automat. Control*, Vol. AC-27, 1982, pp. 880-894.

[Wan1] L.Y. Wang and G. Zames, H^∞ optimization and slowly time-varying systems, *Proc. of 26th IEEE CDC conference*, Los Angeles, U.S.A., Dec. 9-11, 1987, pp.81-83.

[Wan2] L.Y. Wang and G. Zames, slowly time-varying systems and H^∞ optimization, *Proc. of 8th IFAC symp. on identif. and param. est.*, Beijing, China, 1988, (1), pp.492-495.

[Wan3] L.Y. Wang and G. Zames, Banach double algebra and H^∞ optimization, *MTNS conference*, Amsterdam, Nethelands, June 22-28, 1989.

[Wan4] L.Y. Wang and G. Zames, Local-global double algebras for slow H^∞ adaptation, *Proc. of IEEE 28th CDC conference*, Tampa, Florida, U.S.A., Dec. 13-15, 1989.

[Wan5] L.Y. Wang and G. Zames, Lipschitz continuity of H^∞ interpolation, Submitted for publication.

[Wil] J.C. Willems, The analysis of feedback systems, Cambridge, MA.: MIT Press, 1971.

[You1] D.C. Youla, J.J. Bongiorno,Jr., and H.A. Jabr, Modern Wiener-Hopf design of optimal controllers part I: the single-input-output case, *IEEE Trans. on Automat. Control*, Vol. AC-21, 1976, pp. 3-13.

[You2] D.C. Youla, J.J. Bongiorno,Jr., and H.A. Jabr, Modern Wiener-Hopf design of optimal controllers part II: the multivariable case, *IEEE Trans. on Automat. Control*, Vol. AC-21, 1976, pp. 319-338.

[Zam1] G. Zames, Nonlinear time-varying feedback systems: Conditions for L^∞-boundedness derived using conic operators on exponentially weighted spaces, *Proc. 3rd Allerton Conference on Circuit and System Theory*, University of Illinois, Urbana, 1965, pp.460-471.

[Zam2] G. Zames, Feedback and optimal sensitivity: model reference transformations, multiplicative seminorms, and approximate inverses, Proc. 1979 Allerton Conference, and *IEEE Trans. Automat. Control*, Vol. AC-26, 1981, pp.301-320.

[Zam3] G. Zames, On the input-output stability of time-varying nonlinear feedback systems, Part I and Part II, *IEEE Trans. Automat. Control*, Vol. AC-11, 1966, pp.228-238 and 465-476.

[Zam4] G. Zames, Feedback organization, learning and complexity in H^∞, Plenary speech, 1988 ACC conference, Atlanta, June 1988.

[Zam5] G. Zames, private communication.

[Zub] Zubov, Mathematical methods for the study of automatic control systems, Pergamon, 1962.

Appendix A. Proofs of Props. 2.1 and 2.5

(1) Proof of Prop. 2.1:

For any $\mathbf{F}, \mathbf{K} \in \underline{\mathbf{E}}_\sigma$, let $\mathbf{M} := \mathbf{FK}$, $\mathbf{M}^l := \mathbf{F} \otimes \mathbf{K}$ and denote the kernels of those operators by m, f, k and m^l. $\mathbf{F}$ and $\mathbf{K}$ have a common σ_0 ($\sigma_0 > \sigma$) for which $\|\mathbf{F}\|_{(\sigma_0)} < \infty$, $\|\mathbf{K}\|_{(\sigma_0)} < \infty$ by hypothesis. We will show that for any $\sigma_1 \leq \sigma_0$

$$\|\mathbf{M}\|_{(\sigma_1)} \leq \|\mathbf{F}\|_{(\sigma_1)} \|\mathbf{K}\|_{(\sigma_1)}, \tag{A.1}$$

which means that $\mathbf{M} \in \underline{\mathbf{E}}_\sigma$. Therefore $\underline{\mathbf{E}}_\sigma$ is a normed algebra under $*$.

To prove (A.1) observe that as $f, k \in l^1_{\sigma_1}$, the following changes of summation and bounds are valid.

$$\begin{aligned}(\mathbf{M}u)(t) = (\mathbf{FK}u)(t) &= \sum_{\eta=-\infty}^{\infty} f(t,\eta) \sum_{\theta=-\infty}^{\infty} k(\eta,\theta) u(\theta) \\ &= \sum_{\theta=-\infty}^{\infty} \left(\sum_{\eta=-\infty}^{\infty} f(t,\eta) k(\eta,\theta) \right) u(\theta).\end{aligned}$$

Thus, m, the kernel of $\mathbf{M}$, is

$$m(t,\theta) = \sum_{\eta=-\infty}^{\infty} f(t,\eta) k(\eta,\theta),$$

and

$$\begin{aligned}\sum_{\theta=-\infty}^{\infty} |m(t,\theta)| \sigma_1^{(t-\theta)} &= \sum_{\theta=-\infty}^{\infty} \left| \sum_{\eta=-\infty}^{\infty} f(t,\eta) k(\eta,\theta) \right| \sigma_1^{(t-\theta)} \\ &\leq \sum_{\eta=-\infty}^{\infty} |f(t,\eta)| \sigma_1^{(t-\eta)} \sum_{\theta=-\infty}^{\infty} |k(\eta,\theta)| \sigma_1^{(\eta-\theta)} \qquad (A.2) \\ &\leq \sum_{\eta=-\infty}^{\infty} |f(t,\eta)| \sigma_1^{(t-\eta)} \|\mathbf{K}\|_{(\sigma_1)}.\end{aligned}$$

After taking $\sup_{t \in \mathbb{Z}}$ of (A.2) we get (A.1).

As for the local operation $\otimes$, we will prove the inequality

$$\|\mathbf{F} \otimes \mathbf{K}\|_{(\sigma_1)} \leq \|\mathbf{F}\|_{(\sigma_1)} \|\mathbf{K}\|_{(\sigma_1)}. \tag{A.3}$$

In fact, for any $u \in (A^0)_e$,

$$\begin{aligned}
(\mathbf{F} \otimes \mathbf{K})u(t) &= (\mathbf{F} \otimes \mathbf{K})_t u(t) \\
&= (\mathbf{F}_t \otimes \mathbf{K}_t) u(t) \\
&= \sum_{\eta=-\infty}^{\infty} f(t,\eta) \sum_{\theta=-\infty}^{\infty} k\,(t, t-(\eta-\theta))\, u(\theta) \\
&= \sum_{\theta=-\infty}^{\infty} \sum_{\eta=-\infty}^{\infty} f(t,\eta) k\,(t, t-(\eta-\theta))\, u(\theta).
\end{aligned}$$

So the kernel $m^l(t,\theta)$ of $\mathbf{F} \otimes \mathbf{K}$ is

$$m^l(t,\theta) = \sum_{\eta=-\infty}^{\infty} f(t,\eta) k\,(t, t-(\eta-\theta)).$$

As a result,

$$\begin{aligned}
&\sum_{\theta=-\infty}^{\infty} |m(t,\theta)| \sigma_1^{(t-\theta)} \\
&= \sum_{\theta=-\infty}^{\infty} \left| \sum_{\eta=-\infty}^{\infty} f(t,\eta) k\,(t, t-(\eta-\theta)) \right| \sigma_1^{(t-\theta)} \\
&\leq \sum_{\eta=-\infty}^{\infty} |f(t,\eta)| \sigma_1^{(t-\eta)} \sum_{\theta=-\infty}^{\infty} |k\,(t, t-(\eta-\theta))|\, \sigma_1^{(\eta-\theta)} \\
&\leq \|\mathbf{F}\|_{(\sigma_1)} \|\mathbf{K}\|_{(\sigma_1)}.
\end{aligned} \tag{A.4}$$

(A.3) follows after taking $\sup_{t \in \mathbb{Z}}$ of (A.4). Therefore $\underline{\mathbb{E}}_\sigma$ is a normed double algebra. By replacing σ_1 with σ in (A.1) and (A.3), we conclude that $\mathbb{E}_\sigma$ is also a normed double algebra.

To prove $\mathbf{E}_\sigma$ is a Banach algebra, it remains only to show that $\mathbf{E}_\sigma$ is a Banach space under the $\|\cdot\|_{(\sigma)}$ norm (note here $\|\cdot\|_g = \|\cdot\|_l = \|\cdot\|_\sigma$), which is true since l^1_σ is a Banach space under $\|\cdot\|_{l^1_\sigma}$ norm. This completes the proof.

Q.E.D.

(2) Proof of Prop. 2.5:

(a)

$$((\mathbf{TK}-\mathbf{KT})u)(t) = \sum_{\theta=-\infty}^{t-1} (k(t-1,\theta) - k(t,\theta+1))\, u(\theta)$$
$$= \sum_{\theta=-\infty}^{t-1} m(t,\theta)u(\theta),$$

where

$$m(t,\theta) = \begin{cases} k_{t-1}(t-1-\theta) - k_t(t-1-\theta), & \theta \le t-1; \\ 0, & \textit{elsewhere.} \end{cases}$$

The hypothesis $\|\mathbf{TK}-\mathbf{KT}\|_{(\sigma)} \le \rho_\sigma(\mathbf{K})$ implies that

$$\begin{aligned} &\sum_{\theta=-\infty}^{t-1} |k_{t-1}(t-1-\theta) - k_t(t-1-\theta)|\, \sigma^{(t-1-\theta)} \\ &= \sum_{\theta=-\infty}^{t-1} |m(t,\theta)|\sigma^{(t-\theta)}\sigma^{-1} \\ &\le \sigma^{-1} \sum_{\theta=-\infty}^{t-1} |m(t,\theta)|\sigma^{(t-\theta)} \\ &\le \sigma^{-1}\rho_\sigma(\mathbf{K}). \end{aligned} \qquad (A.5)$$

Now

$$((\mathbf{F} \bigtriangledown \mathbf{K})u)(t) = \sum_{\tau=\infty}^{\infty} f(t,\tau) \sum_{\theta=\infty}^{\infty} (k_\tau(\tau-\theta) - k_t(\tau-\theta))\, u(\theta)$$
$$= \sum_{\theta=\infty}^{\infty} \left(\sum_{\tau=\infty}^{\infty} f(t,\tau)\,(k_\tau(\tau-\theta) - k_t(\tau-\theta)) \right) u(\theta)$$
$$= \sum_{\theta=\infty}^{\infty} n(t,\theta)u(\theta)$$

where

$$n(t,\theta) = \sum_{\tau=\infty}^{\infty} f(t,\tau)\left(k_\tau(\tau-\theta) - k_t(\tau-\theta)\right).$$

Then

$$\sum_{\theta=\infty}^{\infty} |n(t,\theta)|\sigma^{(t-\theta)}$$

$$\leq \sum_{\tau=\infty}^{\infty} |f(t,\tau)|\sigma^{(t-\tau)} \sum_{\theta=\infty}^{\infty} |k_\tau(\tau-\theta) - k_t(\tau-\theta)|\,\sigma^{(\tau-\theta)}.$$

By (A.5) and the hypothesis

$$\leq \sum_{\tau=\infty}^{\infty} |f(t,\tau)|\sigma^{(t-\tau)}|t-\tau|\sigma^{-1}\rho_\sigma(\mathbf{K})$$

$$\leq \gamma\sigma^{-1}\rho_\sigma(\mathbf{K}). \tag{A.6}$$

The inequality (2.28) follows after taking $\sup_t$ of (A.6).

(b) Note that under the conditions of (b)

$$\begin{aligned}
&\sum_{\tau=\infty}^{t} |f(t,\tau)|\sigma^{(t-\tau)}|t-\tau| \\
&= \sum_{\tau=\infty}^{t} |f(t,\tau)|\sigma_1^{(t-\tau)}(\sigma/\sigma_1)^{t-\tau}|t-\tau| \\
&\leq \sum_{\tau=\infty}^{t} |f(t,\tau)|\sigma_1^{(t-\tau)} \sup_{\tau\leq t}\left((\sigma/\sigma_1)^{t-\tau}|t-\tau|\right) \\
&\leq \|\mathbf{F}\|_{(\sigma_1)} \sup_{\tau\leq t}\left((\sigma/\sigma_1)^{t-\tau}|t-\tau|\right) \\
&\leq \|\mathbf{F}\|_{(\sigma_1)} \left(e\ln(\sigma_1/\sigma)\right)^{-1}.
\end{aligned} \tag{A.7}$$

By taking $\sup_t$ of (A.8), (2.29) follows from the inequality (2.28).

(c)

$$\begin{aligned} d_\sigma(\mathbf{FK}) &= \|\mathbf{TFK} - \mathbf{FKT}\|_{(\sigma)} \\ &= \|(\mathbf{TF} - \mathbf{FT})\mathbf{K} + \mathbf{F}(\mathbf{TK} - \mathbf{KT})\|_{(\sigma)} \\ &\le \|\mathbf{K}\|_{(\sigma)}\|\mathbf{TF} - \mathbf{FT}\|_{(\sigma)} + \|\mathbf{F}\|_{(\sigma)}\|\mathbf{TK} - \mathbf{KT}\|_{(\sigma)} \\ &= \|\mathbf{F}\|_{(\sigma)} d_\sigma(\mathbf{K}) + \|\mathbf{K}\|_{(\sigma)} d_\sigma(\mathbf{F}). \end{aligned}$$

(d) Since for $u \in l^\infty$,

$$((\mathbf{TK} - \mathbf{KT})u)(t) = -\sum_{\tau=-\infty}^{t-1} (k_t(t-1-\tau) - k_{t-1}(t-1-\tau))u(\tau)$$

the kernel q_t of $(\widehat{\mathbf{TK} - \mathbf{KT}})_t$ is

$$q_t(\tau) = \begin{cases} 0, & \tau \le 0; \\ k_t(\tau-1) - k_{t-1}(\tau-1), & \tau \ge 1. \end{cases}$$

Thus

$$\begin{aligned} \|\mathbf{TK} - \mathbf{KT}\|_{(\sigma)} &= \sup_t \|q_t\|_{l^1_\sigma} \\ &= \sup_t \sum_{\tau=1}^{\infty} |k_t(\tau-1) - k_{t-1}(\tau-1)|\sigma^\tau \\ &= \sigma \sup_t \sum_{\tau=1}^{\infty} |k_t(\tau-1) - k_{t-1}(\tau-1)|\sigma^{\tau-1} \\ &= \sigma \sup_t \|k_t - k_{t-1}\|_{l^1_\sigma}. \end{aligned}$$

Q.E.D.

Appendix B. Proofs of Props. 3.5 - 3.9

(1) Proof of Prop. 3.5:

Inequality 3.13:

For any $t, \tau \in \mathbb{Z}$ and $u \in A$ we get, for $\theta < t$,

$$\begin{aligned}(\Pi_t \mathbf{K}_\tau u)(\theta)\sigma^\theta &= (\Pi_t \mathbf{K}_\tau \Pi_t u)(\theta)\sigma^\theta \\ &= \sum_{\eta=-\infty}^{\theta} \left[k_\tau(\theta-\eta)\sigma^{(\theta-\eta)}\right]\left[(\Pi_t u)(\eta)\sigma^\eta\right] \qquad (B.1)\end{aligned}$$

where k_τ is the kernel of $\mathbf{K}_\tau$ and causality of $\mathbf{K}_\tau$ has been used. This is a convolution of $k_\tau(\eta)\sigma^{(\eta)}$ and $y(\eta) := (\Pi_t u)(\eta)\sigma^{(\eta)}$. By Parseval's Theorem we therefore have

$$\begin{aligned}\|\Pi_t \mathbf{K}_\tau u\|_{l^2_\sigma} &= \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\widehat{\mathbf{K}}_\tau(\sigma e^{j\theta})y(e^{j\theta})\right|^2 d\theta\right\}^{1/2} \\ &\leq \|\widehat{\mathbf{K}}_\tau\|_{H^\infty_\sigma}\|y\|_{L^2} \qquad (B.2) \\ &= \|\widehat{\mathbf{K}}_\tau\|_{H^\infty_\sigma}\|\Pi_t u\|_{l^2_\sigma},\end{aligned}$$

(3.13) is obtained.

Inequality (3.14):

We have, for $t,\ \tau \in \mathbb{Z}$, $u \in A$,

$$\begin{aligned}|(\mathbf{K}_\tau u)(t)| &= \left|\sum_{\eta=-\infty}^{t} k_\tau(t-\eta)u(\eta)\right| \\ &\leq \left(\sum_{\eta=-\infty}^{t}\left|k_\tau(t-\eta)\sigma^{(t-\eta)}\right|^2\right)^{1/2}\left(\sum_{\eta=-\infty}^{t}\left|u(\eta)\sigma^{-(t-\eta)}\right|^2\right)^{1/2} \qquad (B.3) \\ &\leq \|k_\tau\|_{l^2_\sigma}\kappa_\sigma\|u\|_{a(\sigma)}.\end{aligned}$$

Therefore, as $\|\cdot\|_{a(\infty)}$ coincides with $\|\cdot\|_{l^\infty}$,

$$\begin{aligned} \|\mathbf{K}_\tau\|_{a(\infty)} &\le \kappa_\sigma \|\widehat{\mathbf{K}}_\tau\|_{H^2_\sigma} && \text{by Parseval's Theorem} \\ &\le \kappa_\sigma \|\widehat{\mathbf{K}}_\tau\|_{H^p_\sigma} && \text{for } p \ge 2, \end{aligned} \tag{B.4}$$

which proves (3.14).

Inequality (3.15):

As $\|\mathbf{K}_\tau u\|_{a(\sigma)} \le \|\mathbf{K}_\tau u\|_{a(\infty)}$, (3.15) also follows from (B.3) and the inequality $\mu_\sigma^{(p)}(\mathbf{K}) \le \mu_\sigma(\mathbf{K})$.

Q.E.D.

(2) Proof of Prop. 3.6:

We prove first the following inequality:

$$\|\Pi_t(\mathbf{K} - \mathbf{K}_t)u\|_{l^2_\sigma} \sigma^{-t} \le \kappa_\sigma \kappa_\sigma^{(p)} \partial_\sigma^{(p)}(\mathbf{K}) \|u\|_{a(\sigma)} \tag{B.5}$$

where $\kappa_\sigma^{(p)} = \kappa'_\sigma = \left(\sum_{i=1}^{\infty} i^2 \sigma^{-2i}\right)^{1/2}$ for $2 \le p < \infty$ and $\frac{1}{\sigma - 1}$ for $p = \infty$. For $2 \le p < \infty, \tau \le t$

$$\begin{aligned} \left|\sigma^{-t}\left((\mathbf{K} - \mathbf{K}_t)u\right)(\tau)\right| &= \sigma^{-t} \left| \sum_{\xi=-\infty}^{\tau} \left(k_\tau(\tau - \xi) - k_t(\tau - \xi)\right) u(\xi) \right| \\ &= \sigma^{-t} \left| \sum_{\xi=-\infty}^{\tau} \left(k_\tau(\tau - \xi) - k_t(\tau - \xi)\right) \sigma^{\tau - \xi} \sigma^{-(\tau - \xi)} u(\xi) \right| \\ &\le \sigma^{-t} \|\mathbf{K}_\tau - \mathbf{K}_t\|_{H^2_\sigma} \kappa_\sigma \|u\|_{a(\sigma)}. \end{aligned} \tag{B.6}$$

It follows that

$$\begin{aligned}
\|\Pi_t(\mathbf{K}-\mathbf{K}_t)u\|_{l^2_\sigma}\sigma^{-t} &\le \kappa_\sigma\left(\sum_{\tau=-\infty}^{t}\left(\sigma^{-t}\sigma^{\tau}\|\mathbf{K}_\tau-\mathbf{K}_t\|_{H^2_\sigma}\right)^2\right)^{1/2}\|u\|_{a(\sigma)}\\
&\le \kappa_\sigma\left(\sum_{\tau=-\infty}^{t-1}\left(\sigma^{-(t-\tau)}|t-\tau|\right)^2\right)^{1/2}\partial^{(2)}_\sigma(\mathbf{K})\|u\|_{a(\sigma)} \qquad (B.7)\\
&\le \kappa_\sigma\kappa'_\sigma\partial^{(2)}_\sigma(\mathbf{K})\|u\|_{a(\sigma)}\\
&\le \kappa_\sigma\kappa'_\sigma\partial^{(p)}_\sigma(\mathbf{K})\|u\|_{a(\sigma)}.
\end{aligned}$$

For $p=\infty$, $\Pi_t(\mathbf{K}-\mathbf{K}_t)$ can be resolved into a sum and then summed by parts,

$$\begin{aligned}
\Pi_t(\mathbf{K}-\mathbf{K}_t) &= \sum_{\tau=-\infty}^{t}\Delta\Pi_\tau(\mathbf{K}-\mathbf{K}_t)\\
&= \lim_{\tau\to-\infty}\Pi_\tau(\mathbf{K}_\tau-\mathbf{K}_t)-\sum_{\tau=-\infty}^{t}\Pi_{\tau-1}(\mathbf{K}_\tau-\mathbf{K}_{\tau-1}) \qquad (B.8)
\end{aligned}$$

where $\Delta\Pi_t=\Pi_t-\Pi_{t-1}$ and lim denotes a weak-l^1 operator limit, which is null. Observe that the weak-l^1 convergence implies that for $u\in A$ $\lim_{\tau\to-\infty}\|\Pi_\tau(\mathbf{K}_\tau-\mathbf{K}_t)u\|_{l^2_\sigma}=0$, which will be implicitly used in the following proofs to reach required norm inequalities. Therefore, for $u\in A_\sigma$, by causality of $\mathbf{K}_\tau$ and the inequality $\|\Pi_t u\|_{l^2_\sigma}\le\kappa_\sigma\sigma^t\|u\|_{a(\sigma)}$,

$$\begin{aligned}
\|\Pi_t(\mathbf{K}-\mathbf{K}_t)u\|_{l^2_\sigma} &\le \sum_{\tau=-\infty}^{t}\|\Pi_{\tau-1}(\mathbf{K}_\tau-\mathbf{K}_{\tau-1})u\|_{l^2_\sigma}\\
&= \sum_{\tau=-\infty}^{t}\|\Pi_{\tau-1}(\mathbf{K}_\tau-\mathbf{K}_{\tau-1})\Pi_{\tau-1}u\|_{l^2_\sigma} \qquad (B.9)\\
&\le \kappa_\sigma\sum_{\tau=-\infty}^{t}\sigma^{\tau-1}\|\mathbf{K}_\tau-\mathbf{K}_{\tau-1}\|_{H^\infty_\sigma}\|u\|_{a(\sigma)},
\end{aligned}$$

which yields (B.5).

Inequality (3.16) and (3.17):

$$\begin{aligned} &\kappa_\sigma^{-1}\|(\Pi_t \mathbf{K} u)\|_{l^2_\sigma}\sigma^{-t} \\ &\le \kappa_\sigma^{-1}\|(\Pi_t \mathbf{K}_t u)\|_{l^2_\sigma}\sigma^{-t} + \kappa_\sigma^{-1}\|\Pi_t(\mathbf{K}-\mathbf{K}_t)u\|_{l^2_\sigma}\sigma^{-t} \\ &\le \kappa_\sigma^{-1}\|(\Pi_t \mathbf{K}_t \Pi_t u)\|_{l^2_\sigma}\sigma^{-t} + \kappa_\sigma^{-1}\|\Pi_t(\mathbf{K}-\mathbf{K}_t)u\|_{l^2_\sigma}\sigma^{-t} \\ &\le \mu_\sigma(\mathbf{K}_t)\|u\|_{a(\sigma)} + \kappa_\sigma^{(p)}\partial_\sigma^{(2)}(\mathbf{K})\|u\|_{a(\sigma)} \end{aligned} \tag{B.10}$$

by (B.5).

Thus for $p \geq 2$

$$\begin{aligned} \|\mathbf{K}\|_{a(\sigma;t)} &\le \mu_\sigma(\mathbf{K}_t) + \kappa_\sigma^{(p)}\partial_\sigma^{(2)}(\mathbf{K}) \\ &\le \mu_\sigma(\mathbf{K}_t) + \kappa_\sigma^{(p)}\partial_\sigma^{(p)}(\mathbf{K}), \end{aligned} \tag{B.11}$$

which proves (3.16) and (3.17).

Q.E.D.

(3) Proof of Prop. 3.7:

Inequality (3.18):

For any $u \in A$, let $y := (\mathbf{K} \bigtriangledown \mathbf{F})u = (\mathbf{KF} - \mathbf{K} \otimes \mathbf{F})u$

$$\begin{aligned} |y(t)| &= \left| \sum_{\tau=-<\infty}^{t} k(t,\tau) \sum_{\xi=-\infty}^{\tau} (f_\tau(\tau-\xi) - f_t(\tau-\xi))\, u(\xi) \right| \\ &\le \sum_{\tau=-\infty}^{t} |k(t,\tau)|\sigma^{(t-\tau)} \left\{ \sum_{\xi=-\infty}^{\tau} |(f_\tau(\tau-\xi) - f_t(\tau-\xi))|\, \sigma^{(\tau-\xi)} u(\xi)\sigma^{-(\tau-\xi)} \right\} \sigma^{-(t-\tau)}. \end{aligned} \tag{B.12}$$

The part in { }-brackets in (B.12) is bounded using Schwartz's inequality and Parseval's theorem:

$$\left\{ \sum_{\xi=-\infty}^{\tau} \left| (f_\tau(\tau-\xi) - f_t(\tau-\xi))\, \sigma^{(\tau-\xi)} \right|^2 \sigma^{(\tau-\xi)} \sum_{\xi=-\infty}^{\tau} \left(|u(\xi)|\sigma^{-(\tau-\xi)} \right)^2 \right\}^{1/2}$$

$$= \|\widehat{\mathbf{F}}_\tau - \widehat{\mathbf{F}}_t\|_{H^2_\sigma} \kappa_\sigma \|u\|_{a(\sigma)} \leq \kappa_\sigma \partial^2_\sigma(\mathbf{F})|t-\tau|\|u\|_{a(\sigma)}. \tag{B.13}$$

From (B.12) and (B.13) we get

$$\begin{aligned}
|y(t)| &\leq \kappa_\sigma \partial^{(2)}_\sigma(\mathbf{F}) \sum_{\tau=-\infty}^{t} |k(t,\tau)|\sigma^{(t-\tau)}|t-\tau|\sigma^{-(t-\tau)}\|u\|_{a(\sigma)} \\
&\leq \kappa_\sigma \partial^{(2)}_\sigma(\mathbf{F}) \left(\sum_{\tau=-\infty}^{t} \left|k(t,\tau)\sigma^{(t-\tau)}\right|^2 \sum_{\tau=-\infty}^{t-1} \left(|t-\tau|\sigma^{-(t-\tau)}\right)^2 \right)^{1/2} \|u\|_{a(\sigma)} \\
&= \kappa_\sigma \kappa'_\sigma \partial^{(2)}_\sigma(\mathbf{F}) \|\widehat{\mathbf{K}}_t\|_{H^2_\sigma} \|u\|_{a(\sigma)}
\end{aligned} \tag{B.14}$$

by Schwartz's inequality. We now get

$$\begin{aligned}
\|y\|_{a(\sigma)} \leq \|y\|_{a(\infty)} &\leq \kappa_\sigma \kappa'_\sigma \partial^{(2)}_\sigma(\mathbf{F}) \mu^{(2)}_\sigma(\mathbf{K}) \|u\|_{a(\sigma)} \\
&\leq \kappa_\sigma \kappa'_\sigma \partial^{(p)}_\sigma(\mathbf{F}) \mu^{(p)}_\sigma(\mathbf{K}) \|u\|_{a(\sigma)},
\end{aligned} \tag{B.15}$$

which implies (3.18).

Inequality (3.19):

Since $\|y\|_{a(\sigma)} \leq \|y\|_{l^\infty} = \sup_t \sigma^{-t} \|(\Delta\Pi_t)(y)\|_{l^2_\sigma}$, we obtain

$$\|(\mathbf{K} \nabla \mathbf{F})u\|_{a(\sigma)} = \sup_t \sigma^{-t} \|(\Delta\Pi_t)(\mathbf{K} \nabla \mathbf{F})u\|_{l^2_\sigma}. \tag{B.16}$$

However,

$$\begin{aligned}
\|(\Delta\Pi_t)(\mathbf{K} \nabla \mathbf{F})u\|_{l^2_\sigma} &= \|(\Delta\Pi_t)\mathbf{K}_t \sum_{\tau=-\infty}^{t} \Pi_{\tau-1}(\mathbf{F}_\tau - \mathbf{F}_{\tau-1})u\|_{l^2_\sigma} \\
&\leq \kappa_\sigma \sigma^t \frac{1}{\sigma-1} \mu_\sigma(\mathbf{K}) \partial_\sigma(\mathbf{F}) \|u\|_{a(\sigma)}
\end{aligned} \tag{B.17}$$

by (B.5) and the fact that $\|\mathbf{K}_t\|_{H^\infty_\sigma} \leq \mu_\sigma(\mathbf{K})$. (3.19) follows from (B.16) and (B.17).

Q.E.D.

Before going on to the rest of the proofs, we will first show the following series expansions for $\mathbf{G} \bigtriangledown \mathbf{K}$ and $\mathbf{GK} - (\mathbf{G} \otimes \mathbf{K})_t$.

$$\Pi_t \left[\mathbf{GK} - (\mathbf{G} \otimes \mathbf{K})_t\right] = -\Pi_t \mathbf{G}_t \sum_{\tau=-\infty}^{t} \Pi_{\tau-1}(\mathbf{K}_\tau - \mathbf{K}_{\tau-1}) - \sum_{\tau=-\infty}^{t} \Pi_{\tau-1}(\mathbf{G}_\tau - \mathbf{G}_{\tau-1})\mathbf{K}, \tag{B.18}$$

$$\Pi_t[\mathbf{G} \bigtriangledown \mathbf{K}] = \Pi_t \left[\mathbf{GK} - (\mathbf{G} \otimes \mathbf{K})_t\right] + \sum_{\tau=-\infty}^{t} \Pi_{\tau-1}(\mathbf{G}_\tau \mathbf{K}_\tau - \mathbf{G}_{\tau-1}\mathbf{K}_{\tau-1}), \tag{B.19}$$

$$\begin{aligned}(\Delta\Pi_t)\left[\mathbf{GK} - (\mathbf{G} \otimes \mathbf{K})_t\right] &= (\Delta\Pi_t)(\mathbf{G} \bigtriangledown \mathbf{K}) \\ &= (\Delta\Pi_t)\mathbf{G}_t \sum_{\tau=-\infty}^{t} \Pi_{\tau-1}(\mathbf{K}_\tau - \mathbf{K}_{\tau-1}),\end{aligned} \tag{B.20}$$

and the series of operators are weakly-l^1 convergent.

To prove the series expansions, observe the identities,

$$\begin{aligned}\Pi_t \left[\mathbf{GK} - (\mathbf{G} \otimes \mathbf{K})_t\right] &= \Pi_t \left[(\mathbf{G} - \mathbf{G}_t)\mathbf{K} + \mathbf{G}_t(\mathbf{K} - \mathbf{K}_t)\right] \\ &= \sum_{\tau=-\infty}^{t} (\Delta\Pi_\tau)(\mathbf{G} - \mathbf{G}_t)\mathbf{K} + \Pi_t \mathbf{G}_t \sum_{\tau=-\infty}^{t} (\Delta\Pi_\tau)(\mathbf{K} - \mathbf{K}_t)\end{aligned} \tag{B.21}$$

where Π_t has been resolved into $\sum \Delta\Pi_\tau$. Now for any $\mathbf{F} \in \mathbb{B}$, $(\Delta\Pi)_\tau \mathbf{F} = (\Delta\Pi)_\tau \mathbf{F}_\tau$, so $\mathbf{G}$ and $\mathbf{K}$ can be replaced by $\mathbf{G}_\tau$ and $\mathbf{K}_\tau$ in the sums, which can be summed by parts to give (B.18), after the observation that $\Pi_\tau(\mathbf{G}_\tau - \mathbf{G}_t)$ and $\Pi_\tau(\mathbf{K}_\tau - \mathbf{K}_t)$ both weakly-l^1 converge to $\mathbf{0}$ as $\tau \to -\infty$.

By definition of $\bigtriangledown$,

$$\Pi_t[\mathbf{G} \bigtriangledown \mathbf{K}] = \Pi_t \left[(\mathbf{GK} - (\mathbf{G} \otimes \mathbf{K})_t) - (\mathbf{G} \otimes \mathbf{K} - (\mathbf{G} \otimes \mathbf{K})_t)\right]. \tag{B.22}$$

Resolution followed by partial summation gives

$$\begin{aligned}-\Pi_t \left[(\mathbf{G} \otimes \mathbf{K} - (\mathbf{G} \otimes \mathbf{K})_t)\right] &= \sum_{\tau=-\infty}^{t} (\Delta\Pi_\tau)(\mathbf{G}_\tau \mathbf{K}_\tau - \mathbf{G}_t \mathbf{K}_t) \\ &= \sum_{\tau=-\infty}^{t} \Pi_{\tau-1}(\mathbf{G}_\tau \mathbf{K}_\tau - \mathbf{G}_{\tau-1}\mathbf{K}_{\tau-1}).\end{aligned} \tag{B.23}$$

(B.23) app'ied to (B.22) proves (B.19).

(B.20) is obtained from the identities

$$\begin{aligned}\Delta\Pi_t\left(\mathbf{GK}-(\mathbf{G}\otimes\mathbf{K})_t\right) &= \Delta\Pi_t\left((\mathbf{GK})_t-(\mathbf{G}\otimes\mathbf{K})_t\right)\\ &= \Delta\Pi_t\left(\mathbf{G}\bigtriangledown\mathbf{K}\right)\\ &= \Delta\Pi_t\mathbf{G}\Pi_t(\mathbf{K}-\mathbf{K}_t)\end{aligned}$$

followed by resolution and summation by parts.

Inequality (3.20):

By the definition of $\bigtriangledown$ and the triangle inequality,

$$\begin{aligned}\|\Pi_t(\mathbf{K}\bigtriangledown\mathbf{F})u\|_{l^2_\sigma} &= \|\Pi_t(\mathbf{KF}-\mathbf{K}\otimes\mathbf{F})u\|_{l^2_\sigma}\\ &\le \|\Pi_t\left(\mathbf{KF}-(\mathbf{K}\otimes\mathbf{F})_t\right)u\|_{l^2_\sigma}+\|\Pi_t\left(\mathbf{K}\otimes\mathbf{F}-(\mathbf{K}\otimes\mathbf{F})_t\right)u\|_{l^2_\sigma}.\end{aligned} \tag{B.24}$$

The first norm is bou~~n~~ded by

$$\begin{aligned}&\|\Pi_t\left(\mathbf{KF}-(\mathbf{K}\otimes\mathbf{F})_t\right)u\|_{l^2_\sigma}\\ &\le \kappa_\sigma\|\widehat{\mathbf{K}}_t\|_{H^\infty_\sigma}\sum_{\tau=-\infty}^{t}\sigma^{(\tau-1)}\|\mathbf{F}_\tau-\mathbf{F}_{\tau-1}u\|_{a(\sigma)}+\kappa_\sigma\sum_{\tau=-\infty}^{t}\sigma^{(\tau-1)}\|(\mathbf{K}_\tau-\mathbf{K}_{\tau-1})\mathbf{F}u\|_{a(\sigma)}.\end{aligned} \tag{B.25}$$

The series are summed, and the inequalities

$$\|\widehat{\mathbf{K}}_t\|_{H^\infty_\sigma}\le\mu_\sigma(\mathbf{K}),\quad \|(\mathbf{K}_\tau-\mathbf{K}_{\tau-1})\mathbf{F}u\|_{a(\sigma)}\le\partial_\sigma(\mathbf{K})\|\mathbf{F}u\|_{a(\sigma)}$$

used, to obtain the bound:

$$\le\kappa_\sigma\sigma^t\frac{1}{\sigma-1}\left\{\mu_\sigma(\mathbf{K})\partial_\sigma(\mathbf{F})+\partial_\sigma(\mathbf{K})\|\mathbf{F}\|_{a(\sigma)}\right\}\|u\|_{a(\sigma)}. \tag{B.26}$$

After bounding the second norm using (B.5), we get (3.20).

Q.E.D.

(4) Proof of Prop. 3.8:

Ineqalities (3.21) and (3.22):

By the triangle inequality,

$$
\begin{aligned}
&\|\Pi_t(\mathbf{GK} + \mathbf{F})u\|_{l^2_\sigma} - \|\Pi_t(\mathbf{G} \otimes \mathbf{K} + \mathbf{F})_t u\|_{l^2_\sigma} \\
&\le \|\Pi_t(\mathbf{GK} - \mathbf{G} \otimes \mathbf{K})u\|_{l^2_\sigma} + \|\Pi_t\left[(\mathbf{G} \otimes \mathbf{K} + \mathbf{F}) - (\mathbf{G} \otimes \mathbf{K} + \mathbf{F})_t\right] u\|_{l^2_\sigma} \\
&\le \|\Pi_t(\mathbf{G} \bigtriangledown \mathbf{K})u\|_{l^2_\sigma} + \kappa_\sigma \sigma^t \kappa_\sigma^{(p)} \partial_\sigma^{(p)}(\mathbf{G} \otimes \mathbf{K} + \mathbf{F}) \|u\|_{a(\sigma)}
\end{aligned}
\tag{B.27}
$$

(by the definition of $\bigtriangledown$ and (B.5)). By prop. 3.7,

$$
\|(\mathbf{G} \bigtriangledown \mathbf{K})\|_{a(\sigma)} \le \kappa_\sigma \kappa_\sigma^{(p)} \mu_\sigma^{(p)}(\mathbf{G}) \partial_\sigma^{(p)}(\mathbf{K}), \tag{B.28}
$$

and we obtain (3.21) and (3.22).

Inequality (3.23):

By the triangle inequality,

$$
\begin{aligned}
&\|\Pi_t(\mathbf{GK} + \mathbf{F})u\|_{l^2_\sigma} \\
&\le \|\Pi_t(\mathbf{G} \otimes \mathbf{K} + \mathbf{F})_t u\|_{l^2_\sigma} + \|\Pi_t\left(\mathbf{GK} - (\mathbf{G} \otimes \mathbf{K})_t\right) u\|_{l^2_\sigma} + \|\Pi_t(\mathbf{F} - \mathbf{F}_t)u\|_{l^2_\sigma}.
\end{aligned}
\tag{B.29}
$$

On the RHS, the first norm is bounded by $\kappa_\sigma \sigma^t \mu_\sigma(\mathbf{G} \otimes \mathbf{K} + \mathbf{F}) \|u\|_{a(\sigma)}$.

By (B.26) and (3.17), the second norm is bounded by

$$
\begin{aligned}
&\|\Pi_t\left(\mathbf{GK} - (\mathbf{G} \otimes \mathbf{K})_t\right) u\|_{l^2_\sigma} \\
&\le \kappa_\sigma \sigma^t \frac{1}{\sigma - 1} \left\{ \mu_\sigma(\mathbf{G}) \partial_\sigma(\mathbf{K}) + \partial_\sigma(\mathbf{G}) \left(\mu_\sigma(\mathbf{K}) + \frac{1}{\sigma - 1} \partial_\sigma(\mathbf{K}) \right) \right\} \|u\|_{a(\sigma)}.
\end{aligned}
\tag{B.30}
$$

The third is bounded by $\kappa_\sigma \sigma^t \frac{1}{\sigma - 1} \partial_\sigma(\mathbf{F})$ by (B.5), and (3.23) is obtained.

Q.E.D.

(5) Proof of Prop. 3.9:

Inequality (3.24):

For any $t,\ \tau \in \mathbb{Z}$, we have, for $p \geq 2$,

$$
\begin{aligned}
&\|(\widehat{\mathbf{K}\otimes\mathbf{F}})_t - (\widehat{\mathbf{K}\otimes\mathbf{F}})_{t-1}\|_{L^p_\sigma} \\
&= \|\widehat{\mathbf{K}}_t\widehat{\mathbf{F}}_t - \widehat{\mathbf{K}}_{t-1}\widehat{\mathbf{F}}_{t-1}\|_{L^p_\sigma} \\
&= \|\widehat{\mathbf{K}}_t(\widehat{\mathbf{F}}_t - \widehat{\mathbf{F}}_{t-1}) + (\widehat{\mathbf{K}}_t - \widehat{\mathbf{K}}_{t-1})\widehat{\mathbf{F}}_{t-1}\|_{L^p_\sigma} \\
&\leq \mu_\sigma(\mathbf{K})\partial^{(p)}_\sigma(\mathbf{F}) + \mu_\sigma(\mathbf{F})\partial^{(p)}_\sigma(\mathbf{K}),
\end{aligned}
\tag{B.31}
$$

which proves (3.24).

Inequality (3.25):

It is implied by the following inequality,

$$
\begin{aligned}
\|\widehat{\mathbf{G}}^\ominus_t - \widehat{\mathbf{G}}^\ominus_{t-1}\|_{L^p_\sigma} &= \|\widehat{\mathbf{G}}^{-1}_t(\widehat{\mathbf{G}}_t - \widehat{\mathbf{G}}_{t-1})\widehat{\mathbf{G}}^\ominus_{t-1}\|_{L^p_\sigma} \\
&\leq \|\widehat{\mathbf{G}}^{-1}_t\|_{L^p_\sigma}\|\widehat{\mathbf{G}}_t - \widehat{\mathbf{G}}_{t-1}\|_{L^\infty_\sigma}\|\widehat{\mathbf{G}}^{-1}_{t-1}\|_{L^\infty_\sigma}
\end{aligned}
\tag{B.32}
$$

which holds for all $t \in \mathbb{Z}$.

Q.E.D.