Birational Semistability and the Isotriviality of Smooth Families of Canonically-Polarized Manifolds

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A thesis submitted to the faculty of Graduate Studies and Research
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To my parents.

Abstract

In the first part of the current thesis we prove that the fundamental group of a smooth complex projective fourfold with zero Kodaira dimension and non-vanishing holomorphic Euler characteristic is finite. This is a partial resolution of the so-called almost Abelianity conjecture in dimension 4 which predicts that the fundamental group of varieties with zero Kodaira dimension admits an Abelian subgroup of finite index. Our result is a consequence of another theorem where we prove that the Kodiara dimension of coherent subsheaves of the cotangent sheaf of fourfolds with non-negative Kodaira dimension is at most equal to the Kodaira dimension of the variety itself, that is the cotangent sheaf of these varieties is *birationally semistable*. In the second half of this thesis we prove that any smooth family of manifolds with ample canonical bundle over a *special* quasi-projective variety is isotrivial, i.e. there is no variation in the isomorphism classes of the fibers of the family. The special varieties were introduced by Campana as higher dimensional analogues of $\mathbb{C}$ and $\mathbb{C}^*$. From this perspective the above result is a resolution of the generalization of the classical conjecture of Shafarevich, settled by Parshin, which anticipated the isotriviality of smooth families of curves of genus $g \geq 2$ over $\mathbb{C}$ and $\mathbb{C}^*$.

Résumé

Dans la première partie de cette thèse, nous établissons la finitude du groupe fondamental d'une variété lisse projective complexe de dimension 4 dont la dimension de Kodaira est nulle et dont la caractéristique d'Euler holomorphe est non-nulle. Ceci est une résolution partielle de la soi-disant "almost Albelianity conjecture" en dimension 4, qui prédit que le groupe fondamental d'une variété dont la dimension de Kodaira est nulle admet un sous-groupe Abélien d'indice fini. Notre résultat est une conséquence d'un autre théorème, où nous prouvons que la dimension de Kodaira des sous-faisceaux cohérents du faisceau cotangent d'une variété de dimension 4 de dimension de Kodaira non-négative est au plus la dimension de Kodaira de la variété elle-même, c'est dire que le faisceau cotangent d'une telle variété est birationellement semistable. Dans la seconde partie de cette thèse, nous prouvons que n'importe quelle famille lisse de variétés ayant des faisceaux canoniques amples au-dessus d'une variété quasi-projective spèciale est isotriviale, i.e. il n'y a aucune variation dans les classes d'isomorphie des fibres de la famille. Les variétés spèciales ont été introduites par Campana comme des analogues de dimension supérieures de $\mathbb{C}$ et $\mathbb{C}^*$. D'après cette perspective, le résultat ci-haut est une généralisation des conjectures classiques de Shafarevich, ètablies par Parshin, qui anticipaient l'isotrivialité des familles lisses de courbes de genre $g \geq 2$ au-desuss de $\mathbb{C}$ et $\mathbb{C}^*$.

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Contents

Chapter 1

Introduction

The aim of the current thesis is to investigate two central themes in complex algebraic geometry: the classification of algebraic varieties and the global geometry of moduli spaces. In the classification problems we often like to know how global birational (or numerical) invariants such as the Kodaira dimension control topological properties like the fundamental group. These questions are well understood for a special class of surfaces, thanks to the classification of complex algebraic surfaces. In dimension 3 one often resorts to deep results in the minimal model program for similar results (See below). In the absence of such tools in dimension 4, we investigate the topology of fourfolds with zero Kodaira dimension and non-vanishing holomorphic Euler characteristic in chapter 2 by exploiting semistability of the cotangent sheaf Ω_X in a birational sense (See 1.0.6 for the definition). The results of this chapter have partially appeared in [Ta13a]. In the third chapter we investigate the deformation of canonically-polarized manifolds (manifolds with ample canonical bundle). The moduli theory of such manifolds have been heavily studied in the past two decades with striking success. We build upon the spectacular results of many people including Viehweg, Campana, Paun, Kebekus and Jabbusch to prove a conjecture of Campana 1.0.13 which is itself a far-reaching generalization of a conjecture due to Viehweg 1.0.15 and Shafarevich. Although the two aforementioned questions belong to somewhat different areas of algebraic geometry, the general techniques that are deployed largely (and perhaps surprisingly) overlap. These broadly consist of results in the minimal model theory and positivity results of Miyaoka-type concerning the (log-) cotangent sheaf (See 2.1 and 3.2.1).

The minimal model program (or MMP, for short) is one of the most important modern tools for classifying algebraic varieties by exploiting the numerical behaviour of the canonical sheaf under certain birational operations. Broadly speaking the program predicts that by contracting the non-nef locus of the canonical sheaf of a given variety (with "mild" singularities) through some well-known birational transformations, that leave the birational geometry of the initial variety intact, one can reach a final well-understood fibered variety: the *Iitaka fibered variety* (when the canonical sheaf is nef, i.e. when we have reached a minimal model) or the *Mori fiber space* (the fibered variety whose general fiber is log-Fano). For an in-depth discussion of the key definitions and background in the minimal model theory we refer to [KM98]. Here we briefly recall the main conjectures and results in this theory. Although the following statements have been formulated for a much larger class of varieties (or pairs), we restrict ourselves to those that shall be used in the rest of this thesis.

Conjecture 1.0.1 (The minimal model conjecture for smooth pairs)**.** *Let (X, D) be a smooth pair (See 3.1.1 for the definition) over a variety Z. If $K_X + D$ is pseudo-effective $/Z$, then X/Z has a minimal model. Otherwise it has a Mori fiber space $/Z$.*

We recall that by $K_X + D$ pseudo-effective over Z, we mean that the numerical class of $K_X + D$ can be realized as limit of effective classes in the relative Neron-Severi space $N^1(X/Z)$.

Conjecture 1.0.2 (The abundance conjecture for minimal models)**.** *Let (X, D) be a log-canonical (or lc, for short) pair. If $K_X + D$ is nef over Z, then it is semi-ample over Z, i.e. it is pull-back of a divisor that is ample $/Z$.*

These two conjectures put together is sometimes referred to as the good minimal model conjecture for smooth pairs. The good minimal model conjecture (or MMC, for short) is known up to dimension 3 through the works of many people including Mori, Miyaoka, Shokurov and Kollar (See [Ko92]). In dimension 4 the MMC 1.0.1 has been established by Birkar [Bir09] when $\lfloor D \rfloor = 0$, i.e. the boundary divisor D does not have any reduced components. The abundance conjecture in dimension 4 has not been completely settled, where the most intractable case is the case of pairs with zero Kodaira dimension. Finally when (X, D) is of log-general type and $\lfloor D \rfloor = 0$, the MMC is now known by the grace of the groundbreaking work of Birkar, Cascini, Hacon and McKernan [BCHM10].

1.0.A The Almost Abelianity Conjecture.

By the classical results of Iitaka, it is well-known that varieties with zero Kodaira dimension form an important class of algebraic varieties and therefore their classification lie at the heart of the classification theory. One of the most fundamental results in this direction is the so-called Bogomolov decomposition [Bea83] which shows that every nonsingular projective variety (or compact Kähler manifold) X with zero first Chern class $c_1(X) = 0$, admits a finite étale covering $\sigma : X' \to X$ by a smooth variety X' that decomposes into a product $X' \cong Y \times Z$ of an Abelian variety Y and a simply-connected variety Z with trivial canonical divisor $K_Z = 0$. Conjecturally a similar structural theorem should hold for all varieties with vanishing Kodaira dimension.

Conjecture 1.0.3 (Bogomolov decomposition for varieties with zero Kodaira dimension, cf. [Ko95, Conj. 4.16])**.** *Suppose X is a projective variety with zero Kodaira dimension $\kappa(X)$=0. Then X admits a finite covering $\sigma : X' \to X$, that is étale in codimension-one, such that X' is birational to a product of an Abelian variety and a simply-connected one. In particular the fundamental group $\pi_1(X)$ of X is almost (or virtually) Abelian, i.e. $\pi_1(X)$ has an Abelian subgroup of finite index.*

The second assertion in 1.0.3 is sometimes referred to as the **Almost Abelianity conjecture for varieties with Kodaira dimension zero** and is known in dimension 2 by the grace of the classification of algebraic surfaces. In the case of threefolds, the almost Abelianity of the fundamental group is a deep result of Namikawa and Steenbrink.

Theorem 1.0.4 (Almost Abelianity in dimension 3, cf. [NS95])**.** *The Almost Abelianity conjecture holds in dimension* 3*.*

The main ingredient of the proof of 1.0.4 is the existence of good minimal models in dimension 3 together with a certain smoothing technique for ($\mathbb{Q}$-factorial) Calabi-Yau threefolds. In the absence of these results in higher dimensions, and when $\chi(X, \mathscr{O}_X) \neq 0$, Campana has proposed the study of another birational invariant for smooth projective varieties defined by

$$\kappa^+(X) := \max\left\{\kappa(\det \mathscr{F}) \,\middle|\, \mathscr{F} \text{ is a coherent subsheaf of } \Omega^p_X, \text{ for } 1 \leq p \leq \dim X\right\},$$

where $\det \mathscr{F} := (\wedge^r \mathscr{F})^{**}$, $r = \operatorname{rank}(\mathscr{F})$, and proves how closely κ^+ controls the topology of X.

Theorem 1.0.5 (Finiteness of the fundamental groups, cf. [Cam95, Cor. 5.3])**.** *Let X be a nonsingular projective variety. If $\kappa^+(X) = 0$ and $\chi(X, \mathscr{O}_X) \neq 0$, then $\pi_1(X)$ is finite.*

More significantly Campana conjectures that the cotangent sheaf Ω_X of a non-uniruled variety is **birationally semistable**.

Definition 1.0.6 (Birational semistability of Ω_X). *Let X be a smooth projective variety. We say Ω_X is birationally semistable if the inequality*

$$\kappa(\mathscr{F}) \leq \kappa(X) \tag{1.1}$$

holds for every coherent $\mathscr{O}_X$-module subsheaf $\mathscr{F} \subseteq \Omega_X$.

Conjecture 1.0.7. *The cotangent sheaf Ω_X of a non-uniruled variety X is birationally semistable and the two birational invariants $\kappa(X)$ and $\kappa^+(X)$ coincide.*

Remark 1.0.8 (Generalization, cf. [Ta13a, Appendix]). *In fact the conjecture is slightly more general in the sense that the inequality 1.1 should hold for all subsheaves $\mathscr{F} \subseteq \Omega_X^{\otimes m}$, for any positive integer m.*

Furthermore Campana proves that the conjecture 1.0.7 holds assuming the validity of the good minimal model conjecture.

Theorem 1.0.9 (MMP and the equality of κ and κ^+ when $\kappa \geq 0$, cf. [Cam95, Prop. 3.10]). *Let X be a nonsingular projective variety in dimension n with non-negative Kodaira dimension. If the good minimal model conjecture holds for nonsingular projective varieties of dimension up to n and with vanishing Kodaira dimension, then Ω_X is birationally semistable and that $\kappa(X) = \kappa^+(X)$.*

This in particular refines the so-called Bogomolov-Sommese vanishing (the celebrated inequality asserting that $\kappa(\mathscr{L}) \leq p$, for every invertible subsehaf $\mathscr{L} \subseteq \Omega_X^p$ and every integer $1 \leq p \leq \dim X$), when the Kodaira dimension is relatively small.

We remark that when $c_1 = 0$, then we have $\kappa = \kappa^+$ by Bochner's vanishing coupled with Yau's solution [Yau77] to the Calabi's conjecture.

By Theorem 1.0.9, the two invariants $\kappa(X)$ and $\kappa^+(X)$ coincide for non-uniruled threefolds as a consequence of the minimal model program. In the first chapter of the current thesis we prove Campana's conjecture in some higher dimensional cases.

Theorem 1.0.10 (Birational semistability in dimension 4 and 5, cf. [Ta13a, Thm. 1.4]). *Let X be a nonsingular projective variety. The cotangent sheaf Ω_X is birationally semistable and the equality $\kappa = \kappa^+$ holds in the following two cases.*

(1.0.10.1) When $\dim X = 4$ and $\kappa(X) \geq 0$.

(1.0.10.2) When $\dim X = 5$ *and* $\kappa(X) \geq 1$.

Theorem 1.0.10 is a consequence of a much more general result that we obtain in this thesis.

Theorem 1.0.11. *Let X be a nonsingular projective variety of dimension n. Assume that the good minimal model conjecture holds for terminal projective varieties with zero Kodaira dimension up to dimension $n - m$, where $m > 0$. If $\kappa(X) \geqslant m - 1$ then $\kappa = \kappa^+$.*

An important corollary of 1.0.10 and 1.0.5 is the resolution of the almost Abelianity conjecture in dimension 4 subject to the condition $\chi(X, \mathscr{O}_X) \neq 0$.

Theorem 1.0.12 (Almost Abelianity in dimension 4, cf. [Ta13a, Thm. 1.7])**.** *Let X be a nonsingular projective variety of dimension at most 4. Assume $\kappa(X) = 0$ and $\chi(X, \mathscr{O}_X) \neq 0$, then $\pi_1(X)$ is finite.*

1.0.B Smooth families of canonically-polarized manifolds and conjectures of Shafarevich and Campana

In 1962 Shafarevich conjectured that a smooth family of curves of genus $g \geq 2$ over non-hyperbolic algebraic curves, namely $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{P}^1$ and elliptic curve E, is isotrivial. More generally it was conjectured that any smooth family of canonically-polarized manifolds over these curves does not admit any variation. In moduli language this is equivalent to the prediction that given a quasi-projective variety Z°, parametrizing a smooth family of canonically-polarized manifolds, the induced muduli map $\mu : Z^\circ \to \mathfrak{M}$ contracts all the algebraic curves $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{P}^1$ and E in Z°, where $\mathfrak{M}$ is the quasi-projective scheme ([Vie95]) equipped with transformations

$$\Psi : \mathcal{M} \to \mathrm{Hom}(.,\mathfrak{M}),$$

in the sense that $\mathfrak{M}$ is the coarse moduli scheme of the moduli functor $\mathcal{M}$ of smooth family of canonically-polarized manifolds (with fixed Hilbert polynomial).

The question then naturally arises as to what other subvarieties of the base Z° behave the same under the moduli map μ. Campana conjectures that special manifolds are the natural candidates for such subvarieties.

Conjecture 1.0.13 (The isotriviality conjecture of Campana)**.** *Let Y° be a smooth quasi-projective variety parametrizing a smooth family of canonically polarized manifolds. If Y° is special (see the definition below), then the family is isotrivial.*

Definition 1.0.14 (Special logarithmic pairs)**.** *Let (Y, D) be a pair consisting of a smooth projective variety Y and a simple normal-crossing reduced boundary divisor D. We call (Y, D) special, if for every saturated coherent subsheaf of rank one $\mathscr{L} \subseteq \Omega^p_Y \log(D)$ and $p > 0$, we have $\kappa(\mathscr{L}) < p$. Moreover we shall call a smooth quasi-projective variety Y° special, if (Y,D) is special as a logarithmic pair, where Y is a smooth compactification with a simple normal-crossing (snc, for short) boundary divisor D.*

So by definition $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{P}^1$ and E is the list of all special quasi-projective curves (in fact conjecturally special varieties are also analytic higher dimensional analogues of these curves in the sense that through every point of a special variety must pass a subvariety along which the Kobayashi pseudo-metric identically vanishes). Rationally-connected varieties are obvious examples of special varieties. Less obvious, but equally important, are varieties with zero Kodaira dimension (See [Cam04]) and those with nef anti-canonical divisor [Lu02].

Conjecture 1.0.13 is generalization of the following celebrated conjecture of Viehweg.

Conjecture 1.0.15 (Viehweg's hyperbolicity conjecture)**.** *Let $f^\circ : X^\circ \to Y^\circ$ be a smooth family of canonically-polarized varieties over a quasi-projective variety Y°. Assume that Y is a smooth compactification of Y° with snc boundary divisor $D \cong Y \backslash Y^\circ$. If the moduli map $\mu : Y^\circ \to \mathfrak{M}$ is generically finite, then (Y, D) is of log-general type.*

This latter conjecture 1.0.15 has been recently established in [CP13] by using, among many other things, an important generalization of Miyaoka's generic semi-positivity (See 2.1) and the following remarkable result of Viehweg and Zuo.

Theorem 1.0.16 (Existence of pluri-logarithmic forms in the base, cf. [VZ02, Thm. 1.4])**.** *The notations are the same as in 1.0.15. If f° is not isotrivial, then for a positive integer $N \in \mathbb{N}^+$, there exists an invertible subsheaf $\mathscr{L} \subseteq Sym^N \Omega^1_Y \log D$ such that $\kappa(\mathscr{L}) \geq Var(f^\circ)$, where $Var(f^\circ)$ is defined by the dimension of the image of Y° in the coarse moduli scheme $\mathfrak{M}$.*

Clearly Viehweg's (and Shafarevich) hyperbolicity conjecture 1.0.15 when $\dim Y^\circ = 1$ is an immediate corollary of 1.0.16. In particular when the base Y° is 1-dimensional and the moduli map μ is not constant, then Y° is Brody-hyperbolic, that is there are no non-constant holomorphic maps $g : \mathbb{C} \to Y^\circ$ (one can also see this fact by noticing that in this case the invertible sheaf $\Omega_Y \log D$ is big and therefore Y° is Brody-hyperbolic by [Lu91]). In [VZ03] Viehweg and

Zuo generalize this observation and prove the Brody-hyperbolicity of the moduli stack $\mathcal{M}$: the quasi-projective variety Y° serving as the base of a family $(f^\circ : X^\circ \to Y^\circ) \in \mathcal{M}(Y^\circ)$ for which the induced moduli map $\mu : Y^\circ \to \mathfrak{M}$ is generically finite, is Brody-hyperbolic.

Viehweg's conjecture 1.0.15 and the isotriviality conjecture 1.0.13 were already known in $\dim(Y^\circ) \leq 3$ by [KK08, Thm. 1.1]. The stronger conjecture of Campana 1.0.13 is also known when $\dim(Y^\circ) \leq 3$, thanks to [JK11b, Thm. 1.5]. In the final section (section 3.4), and after following Campana and Paun's proof of Viehweg's conjecture very closely, we give a proof to the isotriviality conjecture 1.0.13. The proof heavily depends on a recent generic semi-positivity result of Campana and Paun (See 3.2.1), existence of log-minimal models for klt pairs with big boundary divisors established by [BCHM10, Thm. 1.1], and an important refinement of 1.0.16 given by [JK11a, Thm. 1.4].

Theorem 1.0.17 (cf. [Ta13b, Thm. 1.4])**.** *The isotriviality conjecture 1.0.13 holds in all dimensions.*

According to Campana's reduction theory for every projective variety there exists an almost holomorphic map $C_Y : Y \dashrightarrow Z$, called the core map, whose general fiber is special and contracts almost all special subvarieties of Y. By 1.0.17 it follows that the moduli stack factors through the core in the sense of the following corollary.

Corollary 1.0.18 (Factorization of the moduli stack through the core)**.** *Let Y° be a smooth quasi-projective variety admitting a morphism $\mu : Y^\circ \to \mathfrak{M}$ that factors through the moduli stack of smooth families of canonically-polarized manifolds, i.e. $\mu = \Psi\big(\mathcal{M}(Y^\circ)\big)$. Let $\widetilde{\mu}$ be the induced morphism between smooth compactifications Y, $\overline{\mathfrak{M}}$ of Y and $\mathfrak{M}$, respectively. Then $\widetilde{\mu}$ factors through the core $C_Y : Y \dashrightarrow Z$.*

Chapter 2

Birational semistability and the almost Abelianity conjecture in dimension 4

2.1 Generic semi-positivity and pseudo-effectivity of quotients of Ω_X

Let X be a non-uniruled nonsingular projective variety. It is a well known result of Miyaoka, cf. [Miy87a, Miy87b] that Ω_X is generically semi-positive. This means that the determinant line bundle of any torsion free quotient of Ω_X has non-negative degree on curves cut out by sufficiently ample divisors. Equivalently we can characterize this important positivity result by saying that Ω_X restricted to these general curves is nef unless X is uniruled. This property is sometimes called *generic nefness*. Since nefness is invariant under taking symmetric powers this result automatically generalizes to Ω_X^p. Using the same characteristic p arguments as Miyaoka and some deep results in differential geometry, Campana and Peternell have shown that in fact such a determinant line bundle is dual to the cone of moving curves, i.e. its restriction to these curves has non-negative degree. By [BDPP04] this is the same as saying that it is pseudo-effective.

Theorem 2.1.1 (Pseudo-effectivity of quotients of Ω_X^p, cf. [CPT07, Thm. 1.7])**.** *Let X be a non-uniruled nonsingular projective variety and let $\mathscr{F}$ be an $\mathscr{O}_X$- module torsion free quotient of Ω_X^p. Then* $\det \mathscr{F}$ *is a pseudo-effective line bundle.*

2.2 A refined Kodaira dimension

In this section we will use more or less the same ideas as Cascini [Cas06] to show that κ and κ^+ coincide for nonsingular projective varieties of dimension four with non-negative Kodaira dimension and also for varieties of dimension five with positive Kodaira dimension. The following proposition is a result of Campana, cf. [Cam95]. We include a proof for completeness.

Proposition 2.2.1. *Let X be a nonsingular projective variety with $\kappa(X) = 0$. If X has a good minimal model then $\kappa = \kappa^+$.*

Proof. Let Y be a $\mathbb{Q}$-factorial normal variety with at worst terminal singularities serving as a good minimal model for X. Note that K_Y is numerically trivial. Let $\pi : \widetilde{Y} \to Y$ be a resolution. Since $\kappa(\widetilde{Y}) = 0$, $\widetilde{Y}$ is not uniruled. Let $\mathscr{F} \subseteq \Omega^p_{\widetilde{Y}}$ be a coherent subsheaf with maximum Kodaira dimension, i.e. $\kappa(\det \mathscr{F}) = \kappa^+(\widetilde{Y})$.

Let C be an irreducible curve on Y cut out by sufficiently general hyperplanes and let $\widetilde{C}$ to be the corresponding curve in $\widetilde{Y}$. Now using the standard isomorphism: $\Omega^p_{\widetilde{Y}}|_{\widetilde{C}} \cong K_{\widetilde{Y}}|_{\widetilde{C}} \otimes \wedge^{n-p} \mathscr{T}_{\widetilde{Y}}|_{\widetilde{C}}$, we get $\mathscr{F}^*|_{\widetilde{C}}$ as a quotient of $K^*_{\widetilde{Y}}|_{\widetilde{C}} \otimes \Omega^{n-p}_{\widetilde{Y}}|_{\widetilde{C}}$. But $K^*_{\widetilde{Y}}$ is numerically trivial on $\widetilde{C}$ and $\Omega^{n-p}_{\widetilde{Y}}|_{\widetilde{C}}$ is nef by Miyaoka, so $\mathscr{F}^*|_{\widetilde{C}}$ must also be nef and we have

$$\deg(\det \mathscr{F}|_{\widetilde{C}}) \leq 0.$$

But this inequality holds for a covering family of curves and thus $\kappa(\mathscr{F}) \leq 0$.

□

As was mentioned in the introduction (Theorem 1.0.9), assuming the good Minimal Model conjecture for varieties up to dimension n and with zero Kodaira dimension, we have $\kappa = \kappa^+$ in the case of n-dimensional varieties of positive Kodaira dimension as well. See [Cam95, Prop. 3.10] for a proof. The main result of this paper is concerned with replacing this assumption with the abundance conjecture in lower dimensions.

Remark 2.2.2. *Following the recent developments in the minimal model program, we now know that we have a good minimal model when numerical Kodaira dimension is zero. The proposition 3.1 shows that $\kappa^+(X)$ also vanishes in this case. By [Cam95] this implies in particular that nonsingular varieties with vanishing numerical dimension have finite fundamental groups as long as they have non-trivial holomorphic Euler characteristic (See 1.0.5).*

We will need the following lemmas in the course of the proof of our main result.

Lemma 2.2.3. *Let $f : X \to Z$ be a surjective morphism with connected fibers between normal projective varieties X and Z. Let D be an effective $\mathbb{Q}$-Cartier divisor in X that is numerically trivial on the general fiber of f. If D is f-nef, then there exist biratioanl morphism $\pi : \widetilde{Z} \to Z$ verifying the following properties:*

2.1. Let $\widetilde{X}$ denote the normalization of the fiber product $X \times_Z \widetilde{Z}$ with resulting commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & \widetilde{X} \arrow[l, "\mu"'] \arrow[d, "\widetilde{f}"] \\ Z & \widetilde{Z}, \arrow[l, "\pi"'] \end{tikzcd}$$

$\mu : \widetilde{X} \to X$ being the naturally induced biraitonal morphism. Then the induced fibration $\widetilde{f} : \widetilde{X} \to \widetilde{Z}$ is equi-dimensional.

2.2. There exists a $\mathbb{Q}$-Cartier divisor G in $\widetilde{Z}$ such that $\mu^*(D) = \widetilde{f}^*(G)$.

Proof. The fact that we can modify the base of our fibration to get a morphism whose fibers are of constant dimension is guaranteed by [Ray72]. This is called *flattening* of f. Let $\widetilde{X}$ be a normal birational model of X and $\widetilde{Z}$ a smooth birational model for Z such that $\widetilde{f} : \widetilde{X} \to \widetilde{Z}$ is flat.

If general fibers are curves, by assumption the degree of $\mu^*(D)$ on general fibers of $\widetilde{f}$ is zero. On the other hand $\mu^*(D)$ is effective and relatively nef, so it must be trivial on all fibers. This implies the existence of the required $\mathbb{Q}$-Cartier divisor G in $\widetilde{Z}$.

In the case of higher dimensional fibers, $\mu^*(D)$ must still be numerically trivial on all fibers of $\widetilde{f}$. To see this, let C be an irreducible curve contained in a d-dimensional non-general fiber $\widetilde{F}_0$ of $\widetilde{f}$. Then for a sufficiently general members D_i of the linear system of an ample divisor H containing C, we have

$$D_1 \cap \ldots \cap D_{d-1} \cap \widetilde{F}_0 = mC + C',$$

where C' is an effective curve and m accounts for the multiplicity of the irreducible component of F_0 containing C. Now since $\mu^*(D)$ is numerically trivial on the general fiber of $\widetilde{f}$, we have

$$\mu^* D \cdot (mC + C') = 0.$$

But $\mu^*(D)$ is $\widetilde{f}$-nef, so that $\mu^*(D) \cdot C = 0$.

We know that $\mu^*(D)$ is effective, so $\mu^*(D)$ must be trivial on all fibers. Again this ensures the existence of a $\mathbb{Q}$-Cartier divisor G in $\widetilde{Z}$ such that $\mu^*(D) = \widetilde{f}^*(G)$. $\square$

In the course of the proof of Lemma 2.2.3 we repeatedly used the standard fact that given a surjective morphism with connected fibers $f : X \to Z$ between normal varieties X and Z, where Z is $\mathbb{Q}$-factorial, and an effective $\mathbb{Q}$-Cartier divisor D that is trivial on all fibers, we can always find a $\mathbb{Q}$-Cartier divisor G in Z such that $D = f^*(G)$. One can verify this by reducing it to the case where X is a surface and Z is a curve. Here the negative semi-definiteness of the intersection matrix of the irreducible components of singular fibers establishes the claim.

For applications, a natural setting for lemma 2.2.3 is the relative minimal model program. The following is a reformulation of this lemma in this context.

Lemma 2.2.4. *Let $f : X \to Z$ be a surjective morphism with connected fibers between nonsingular projective varieties X and Z with dimension n and m respectively. Assume $\kappa(X) \geq 0$ and that X/Z has a minimal model model Y/Z. Denote the morphism between Y and Z by ψ. Also assume that the abundance conjecture for varieties of vanishing Kodaira dimension holds in dimension $n - m$. If the Kodaira dimension of the general fiber of f is zero, then there exist birational morphisms $\pi : \widetilde{Z} \to Z$, $\mu : \widetilde{Y} \to Y$, a $\mathbb{Q}$-Cartier divisor G in $\widetilde{Z}$, and an equidimensional morphism $\widetilde{\psi} : \widetilde{Y} \to \widetilde{Z}$ such that $\mu^*(K_Y) = \widetilde{\psi}^*(G)$.*

$$\begin{tikzcd} X \arrow[r, dotted] \arrow[dr] & Y \arrow[d, "\psi"] & \widetilde{Y} \arrow[l, "\mu"'] \arrow[d, "\widetilde{\psi}"] \\ & Z & \widetilde{Z} \arrow[l, "\pi"] \end{tikzcd}$$

Proof. Since K_Y is ψ-nef and that the dimension of the general fibers is $n - m$, we find that the canonical of the general fiber is torsion by the abundance assumption. Now apply Lemma 2.2.3 to $\psi : Y \to Z$ and take D to be K_Y. $\square$

We now turn to another crucial ingredient that we shall use in the proof of 2.2.7.

Lemma 2.2.5. *Let $f : X \to Z$ be a surjective morphism with connected fibers between normal projective varieties X and Z of dimension n and k respectively. Let D be a $\mathbb{Q}$-Cartier divisor in Z. If $(f^*D) \cdot C_X \geq 0$, for all $C_X \in \overline{\mathrm{Mov}}(X)$, then $D \cdot C_Z \geq 0$, for any $C_Z \in \overline{\mathrm{Mov}}(Z)$, $\overline{\mathrm{Mov}}(X) \subseteq \mathrm{N}_1(X)_{\mathbb{R}}$ and $\overline{\mathrm{Mov}}(Z) \subseteq \mathrm{N}_1(Z)_{\mathbb{R}}$ being the* movable cones *of X and Z.*

Proof. First assume that f is birational. Let C be a moving curve in Z and let $\mu : \widetilde{Z} \to Z$, be a birational morphism such that $\mu_*(\widetilde{C}) = C$, where $\widetilde{C}$ is a complete intersection curve cut out by hyperplanes. Let $\pi : \widetilde{X} \to X$ be a suitable modification such that $\widetilde{f} : \widetilde{X} \to \widetilde{Z}$ is a morphism and we have the following commutative diagram.

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Z} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\mu} \\ X & \xrightarrow{f} & Z \end{array}$$

Now let $\widetilde{C} = H_1 \cdot \ldots \cdot H_{k-1}$, where $H_1, \ldots, H_{k-1}$ are ample divisors in $\widetilde{Z}$. We have

$$\begin{aligned} \mu^*D \cdot \widetilde{C} &= \mu^*D \cdot H_1 \cdot \ldots \cdot H_{k-1} \\ &= \widetilde{f}^*(\mu^*D) \cdot \widetilde{f}^*H_1 \cdot \ldots \cdot \widetilde{f}^*H_{k-1} \\ &= \pi^*(f^*D) \cdot \widetilde{f}^*H_1 \cdot \ldots \cdot \widetilde{f}^*H_{k-1} \quad \text{by commutativity of the diagram.} \end{aligned}$$

Clearly $\pi^*(f^*D)$ is pseudo-effective. Now since nef divisors are numerically realized as limit of ample ones we have

$$\pi^*(f^*D) \cdot \widetilde{f}^*H_1 \cdot \ldots \cdot \widetilde{f}^*H_{k-1} \geq 0,$$

which implies $\mu^*(D) \cdot \widetilde{C} \geq 0$. So that $D \cdot C \geq 0$ as required.

Now assume that f is not birational and let $C = H_1 \cap \ldots \cap H_{k-1}$ be an irreducible curve cut out by general members of basepoint-free linear systems defined by very ample divisors in Z. In particular C is of constant dimension along the image of fibers. After cutting down by general hyperplanes $H'_1, \ldots, H'_{n-k}$, we can find an irreducible curve

$$C' = H'_1 \cdot \ldots \cdot H'_{n-k} \cdot f^*(H_1) \cdot \ldots \cdot f^*(H_{k-1})$$

that maps surjectively onto C. Thus we have $(\deg f|_{C'})D \cdot C = f^*D \cdot C' \geq 0$.

For a moving curve that is not given by intersections of hyperplanes, we repeat the same argument as above after going to a suitable modification. □

Remark 2.2.6. *We know by [BDPP04] that for nonsingular projective varieties, pseudo-effective divisors are dual to the cone of moving curves. Using the lemma above, we can easily extend this to normal varieties by going to a resolution. This fact is of course already well known. For convenience we rephrase Lemma 2.2.5 as follows:*

Let $f : X \to Z$ be a surjective morphism with connected fibers between normal projective varieties. Let D be a $\mathbb{Q}$-Cartier divisor in Z. If f^*D is *pseudo-effective* then so is D.

We shall prove Theorem 1.0.11 as a consequence of the following proposition:

Proposition 2.2.7. *Let X be a nonsingular projective variety of dimension n with $\kappa(X) \geqslant 0$. Assume that the good minimal model conjecture holds for terminal projective varieties with zero Kodaira dimension up to dimension $n - m$, where $m > 0$. Let $\mathscr{F} \subseteq \Omega_X^p$ be a coherent subsheaf and define the line bundle $L = \det \mathscr{F}$. If $\kappa(K_X + L) \geqslant m$, then $\kappa(L) \leqslant \kappa(X)$.*

Proof. First a few observations. The isomorphism $K_X^* \otimes \Omega_X^p \cong \wedge^{n-p} \mathscr{T}_X$ implies that $K_X^* \otimes \mathscr{F}$ is a subsheaf of $\wedge^{n-p} \mathscr{T}_X$. But X is not uniruled and so by 2.1.1 $rK_X - L$ is pseudo-effective as a Cartier divisor, where r is the rank of $\mathscr{F}$.

We can of course assume that X is not general type. Now if we assume that $K_X + L$ is big then by using the equality $(r+1)K_X = (rK_X - L) + (K_X + L)$ and pseudo-effectivity of $rK_X - L$, we conclude that K_X must be big as well. So we may also assume that $K_X + L$ is not big and that $\kappa(L) > 0$.

Without loss of generality we can also assume that the rational map $X \dashrightarrow Z$ corresponding to $K_X + L$ is a morphism, since we can always go to a suitable modification, pull back L and prove the theorem at this level. Denote this map by i_{K_X+L} and note that by definition we have $\kappa((K_X + L)|_F) = 0$, where F is the general fiber of i_{K_X+L}. Finally, we observe that $\kappa(F) \leq \kappa((K_X + L)|_F) = 0$ and as we are assuming that X has non-negative Kodaira dimension, we have $\kappa(F) = 0$.

Claim 2.2.8. *Without loss of generality, we can assume L is the pull back of a $\mathbb{Q}$-Cartier divisor L_1 in Z.*

Assuming this claim for the moment, our aim is now to show that after a modification $\pi : \widetilde{Z} \to Z$, we can find a big divisor in $\widetilde{Z}$ whose Kodaira dimension matches that of X. This will imply that $\kappa(L) = \kappa(L_1) \leq \kappa(X)$, as required.

To this end take Y to be a relative minimal model for X over Z and denote the birational map between X and Y by ϕ and the induced morphism $Y \to Z$ by ψ (See the diagram below). Observe that we can assume that ϕ is a morphism without losing generality. Denote $\psi^*(L_1)$ by L_Y. Fix K_Y to be the cycle theoretic push forward of K_X.

Now by lemma 2.2.4 and the abundance assumption after modifying the base by $\pi : \widetilde{Z} \to Z$, we can find a morphism $\widetilde{\psi} : \widetilde{Y} \to \widetilde{Z}$ such that the dimension of the fibers of this new fibration are all the same and $\mu^*(K_Y) = \widetilde{\psi}^*(G)$ for some $\mathbb{Q}$-Cartier divisor G in $\widetilde{Z}$.

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[dr, "i_{K_X+L}"'] & Y \arrow[d, "\psi"] & \widetilde{Y} \arrow[l, "\mu"'] \arrow[d, "\widetilde{\psi}"] \\ & Z & \widetilde{Z} \arrow[l, "\pi"] \end{tikzcd}$$

Noting that Y is at worst terminal, i.e. $K_X + L = \phi^*(K_Y + L_Y) + E$ for an effective exceptional divisor E, we have $\kappa(K_X + L) = \kappa(K_Y + L_Y)$. We also observe that $rK_Y - L_Y$ must be pseudo-effective.

Define $\widetilde{L}_1 := \pi^*(L_1)$, so that $\mu^*(L_Y) = \widetilde{\psi}^*(\widetilde{L}_1)$ and $\widetilde{\psi}^*(G + \widetilde{L}_1) = \mu^*(K_Y + L_Y)$. This implies that $G + \widetilde{L}_1$ is big in $\widetilde{Z}$. We also know that $\mu^*(rK_Y - L_Y)$ is pseudo-effective and $\mu^*(rK_Y - L_Y) = \widetilde{\psi}^*(rG - \widetilde{L}_1)$. Thus by lemma 2.2.5 (See also Remark 2.2.6) $rG - \widetilde{L}_1$ is pseudo-effective too. Additionally we have

$$(r+1)G = (rG - \widetilde{L}_1) + (G + \widetilde{L}_1),$$

where the right hand side is a sum of pseudo-effective and big divisors. This implies that G is big and we have

$$\kappa(L) = \kappa(\widetilde{L}_1) \leq \kappa(G) = \kappa(\mu^*(K_Y)) = \kappa(K_X).$$

□

Now it remains to prove 3.29.

Proof of 3.29. Let $X \dashrightarrow Z'$ be the map given by the global sections of large enough multiple of L, and let $i_L : X' \to Z'$ be the Iitaka fibration corresponding to L, where $\mu : X' \to X$ is a suitable modification of X. As $\kappa(K_X) \geq 0$, we have $\kappa(L) \leq \kappa(K_X + L)$, where the right hand side of this inequality is zero on the general fiber of i_{K_X+L}. On the other hand since we have assumed $\kappa(L)$ to be positive, we find that $\kappa(L|_F) = 0$. Hence i_{K_X+L} factors through i_L via a rational map g and we have the following commutative diagram:

$$\begin{tikzcd} X' \arrow[r, "i_L"] \arrow[d, "\mu"'] & Z' \\ X \arrow[r, "i_{K_X+L}"'] \arrow[ur, dotted] & Z \arrow[u, dotted, "g"'] \end{tikzcd}$$

Now by considering suitable modifications of X, Z and X', we can assume that g is a morphism. Define the line bundle $L' := \mu^*(L) - A = i_L^*(H)$, where A is an effective divisor and H is an ample $\mathbb{Q}$-Cartier divisor in Z'. Let L'' be the pull back of H in X via g and i_{K_X+L}, so that $\mu^*(L'') = L'$ and that $\mu^*(L'') + A = \mu^*(L)$. We claim that we don't lose generality if we replace L by L''. To see this we need to check the following two properties: (i) $rK_X - L''$ is pseudo-effective and (ii) $\kappa(L'') = \kappa(L)$.

To see that (i) holds, note that we have $\mu^*(rK_X - L'') = \mu^*(rK_X) - (\mu^*(L) - A) = \mu^*(rK_X - L) + A$. Now since $rK_X - L$ is pseudo-effective and A is effective, $rK_X - L''$ must also be pseudo-effective.

For (ii) it suffices to show $\kappa(L) = \kappa(L')$ which is a consequence of the following inequality:

$$\kappa(L) = \kappa(\mu^* L) \leq dim Z' = \kappa(L').$$

This finishes off the proof of Claim 3.29 after a possible base change corresponding to $K_X + L''$.

□

Now our main result immediately follows:

Proof of Theorem 1.0.11. Let $\mathscr{F} \subseteq \Omega_X^p$ be a coherent subsheaf with maximum Kodaira dimension, i.e. $\kappa(L) = \kappa^+(X)$, where $L = \det(\mathscr{F})$. Assume that $\kappa(L) > \kappa(X)$. Then $\kappa(L) \geq m$ and in particular we have $\kappa(K_X + L) \geq m$. Now the proposition above implies that $\kappa(L) \leq \kappa(X)$, which is a contradiction.

□

As we discussed in the introduction, this greatly improves the Bogomolov's inequality for projective varieties of dimension at most five and with relatively small Kodaira dimension.

Remark 2.2.9. *Theorem 1.0.10 can be further strengthened by replacing κ^+ by a stronger birational invariant $\omega(X)$ (See the appendix of [Ta13a] for the definition) which measures the maximal positivity of coherent rank one subsheaves of ${\Omega_X^1}^{\otimes m}$, for any $m > 0$, i.e. $\kappa(X)$ and $\omega(X)$ coincide for fourfolds with non-negative Kodaira dimension. The proof is identical to that of Theorem 1.0.10 by observing that pseudo-effectivity of $rK_X - L$ in 2.2.7 can be be replaced by that of $mK_X - L$, where m denotes the tensorial power of cotangent bundle containing the line bundle L .*

Remark 2.2.10. *We would like to point out that when $\kappa(X) \geq \dim X - 3$, we have $\kappa = \kappa^+$ by [Cam95, Prop. 10.9] , where 3 in this inequality comes from the abundance result for varieties of dimension at most 3. So the real improvement provided by 1.0.10 is when $\kappa = 0$ in dimension 4 and $\kappa = 1$ in dimension 5.*

Chapter 3

Smooth families of canonically-polarized manifolds over a special base

3.1 Preliminaries

To approach the isotriviality conjecture 1.0.13, it is essential to work with pairs (or the orbifold pairs in the sense of Campana) instead of just logarithmic ones. We refer the reader to [Cam08] and [JK11b] for an in-depth discussion of the definitions and background. In the present section we give a brief overview of the key ingredients of this theory to the extent that is necessary for our arguments in the rest of the paper.

Definition 3.1.1 (Smooth Pairs)**.** *Let X be an n-dimensional normal (quasi-) projective variety and $D = \sum d_i D_i$, where $d_i \in \mathbb{Q} \cap [0,1]$, a $\mathbb{Q}$-Weil divisor in X. We shall call the pair (X, D) a* smooth pair, *if X is smooth and $supp(D)$ is simple normal-crossing.*

Definition 3.1.2 ($\mathcal{C}$-Multiplicity)**.** *Let (X, D) be a smooth pair as in Definition 3.1.1. When $d_i \neq 1$, let a_i and b_i be the positive integers for which the equality $1 - \frac{b_i}{a_i} = d_i$ holds. For every i, we define the $\mathcal{C}$-multiplicity of the irreducible component D_i of D by*

$$m_D(D_i) := \begin{cases} \frac{1}{1-d_i} = \frac{a_i}{b_i} & \text{if } d_i \neq 1 \\ \infty & \text{if } d_i = 1. \end{cases}$$

A classical result of Kawamata (See [Laz04, Prop. 1.12]) proves that given a collection of smooth prime divisors $\{D_1, \dots, D_l\}$ and positive integers $\{c_1, \dots, c_l\}$, one can always construct a *smooth* variety Y together with a finite, flat morphism $\gamma : Y \to X$ such that

$$\gamma^*(D_i) = c_i \sum D_{ij},$$

where $(\sum D_{ij})$ is a simple normal-crossing divisor in Y. In particular, given a smooth pair (X, D), we may take the coefficients c_i to be equal to a_i (a_i being the numerator of $m_D(D_i)$, as in Definition 3.1.2), so that the resulting Kawamata cover $\gamma : Y \to X$ is, in a sense, *adapted* to the structure of the pair (X, D).

Definition 3.1.3 (Adapted Covers)**.** *Let (X, D) be a smooth pair, Y a smooth variety, and $\gamma : Y \to X$ a finite, flat, cyclic cover with Galois group G such that if $m_D(D_i) = \frac{a_i}{b_i} < \infty$, then every prime divisor in Y that appears in $\gamma^*(D_i)$ has multiplicity exactly equal to a_i. We call γ an adapted cover for the pair (X, D), if it additionally satisfies the following properties:*

(3.1.3.1) The branch locus is given by

$$supp(H + \bigcup_{m_D(D_i) \neq \infty} D_i),$$

where H is a general member of a linear system $|L|$ of a very ample divisor L in X.

(3.1.3.2) γ is totally branched over H.

(3.1.3.3) γ is not branched at the general point of $supp(\lfloor D \rfloor)$.

Notation 3.1.4. *Let $\gamma : Y \to X$ be an adapted cover of a smooth pair (X, D), where $D = \sum d_i D_i$, $d_i = 1 - \frac{b_i}{a_i}$ as in Definition 3.1.2. For every prime component D_i of D with $m_D(D_i) \neq \infty$, let $\{D_{ij}\}_{j(i)}$ be the collection of prime divisors that appear in $\gamma^{-1}(D_i)$. We define new divisors in Y by*

$$D_Y^{i,j} := b_i D_{ij}\ , \quad m_D(D_i) \neq \infty, \tag{3.1}$$

$$D_\gamma := \gamma^*(\lfloor D \rfloor). \tag{3.2}$$

Definition 3.1.5 ($\mathcal{C}$-Cotangent Sheaf)**.** *Given a smooth pair (X, D) with an adapted cover $\gamma : Y \to X$, define the $\mathcal{C}$-cotangent sheaf Ω_{Y^∂} to be the unique maximal locally-free subsheaf of $\Omega_Y \log(D_\gamma)$ for which the sequence*

$$0 \longrightarrow \Omega_{Y^\partial}|_{(Y\setminus D_\gamma)} \longrightarrow \gamma^*\big(\Omega_X \log(\ulcorner D \urcorner)\big)|_{(Y\setminus D_\gamma)} \xrightarrow{\ \rho\ } \bigoplus_{i,j(i)} \mathscr{O}_{D_Y^{i,j}} \longrightarrow 0,$$

induced by the natural residue map, is exact.

Remark 3.1.6. *The $\mathcal{C}$-cotangent sheaf defined in 3.1.5 coincides with Campana and Paun's notion [CP13, Sec. 1.1] of the coherent sheaf on Y which they denote by $\gamma^*\Omega^1(X, D)$. It is also identical with the sheaf defined in [Lu02, Lem. 4.2]. See also [JK11b, Def. 2.13] for an equivalent definition in the classical setting, i.e. when the $\mathcal{C}$-multiplicities are all integral.*

Notation 3.1.7. *We shall denote the dual of the $\mathcal{C}$-cotangent sheaf by T_{Y^∂}, i.e.*

$$T_{Y^\partial} := (\Omega_{Y^\partial})^*$$

.

Remark 3.1.8 (Determinant of $\mathcal{C}$-Cotangent Sheaf)**.** *Given a smooth pair (X, D), let $\gamma : Y \to X$ be an adapted cover of degree d. There exists a natural isomorphism between the two invertible sheaves $\det(\Omega_{Y^\partial})$ and $\mathscr{O}_Y\big(\gamma^*(K_X + D)\big)$*

$$\det(\Omega_{Y^\partial}) \cong \mathscr{O}_Y\big(\gamma^*(K_X + D)\big). \tag{3.3}$$

This follows from the ramification formula for the adapted cover γ:

$$\begin{aligned}
K_Y + D_\gamma &= \gamma^*(K_X + \lfloor D \rfloor) + \sum_{\substack{i \\ m_D(D_i)\neq\infty}} \sum_{j(i)} (a_i - 1) D_{ij} + (d-1)\widetilde{H} \\
&= \gamma^*(K_X + D) - \gamma^*(D - \lfloor D \rfloor) + \sum_{\substack{i \\ m_D(D_i)\neq\infty}} \sum_{j(i)} (a_i - 1) D_{ij} + (d-1)\widetilde{H} \\
&= \gamma^*(K_X + D) - \sum_{\substack{i \\ m_D(D_i)\neq\infty}} \sum_{j(i)} (a_i - b_i) D_{ij} + \sum_{i, m_D(D_i)\neq\infty} \sum_{j(i)} (a_i - 1) D_{ij} \\
&\qquad + (d-1)\widetilde{H} \\
&= \gamma^*(K_X + D) + \sum_{\substack{i \\ m_D(D_i)\neq\infty}} \sum_{j(i)} (b_i - 1) D_{ij} + (d-1)\widetilde{H},
\end{aligned}$$

for $\widetilde{H} := \gamma^* H$*, where* H *is the very ample divisor given in Definition 3.1.3. As a consequence, we find that the isomorphism (3.3) holds by construction:*

$$\begin{aligned} \det \Omega_{X^\partial} &\cong \mathscr{O}_Y\big((K_Y + D_\gamma) - \sum_{\substack{i \\ m_D(D_i) \neq \infty}} \sum_{j(i)} (b_i - 1) D_{ij} - (d-1)\widetilde{H}\big) \quad \textit{by definition} \\ &\cong \mathscr{O}_Y\big(\gamma^*(K_X + D)\big), \end{aligned}$$

where the last isomorphism follows form the ramification formula. Clearly, the isomorphism (3.3) implies that the $\mathcal{C}$*-cotangent sheaf* Ω_{Y^∂} *can be seen as the unique locally-free subsheaf of* $\Omega_Y \log(D_\gamma)$ *whose determinant is isomorphic to the pull-back bundle* $\mathscr{O}_Y\big(\gamma^*(K_X + D)\big)$*.*

Definition 3.1.9 (Symmetric $\mathcal{C}$-Differential Forms, cf. [Cam08, Sect. 2.6-7])**.** *Let* (X, D) *be a smooth pair,* $D = \sum d_i D_i$*, and* V_x *an open neighbourhood of a given point* $x \in X$ *equipped with a coordinate system* $z_1, \ldots, z_n$ *such that* $supp(D) \cap V_x = \{z_1 \cdot \ldots \cdot z_l = 0\}$*, for a positive integer* $1 \leq l \leq n$*. For every* $N \in \mathbb{N}^+$*, define the sheaf of symmetric* $\mathcal{C}$*-differential forms* $Sym^N_{\mathcal{C}}\big(\Omega_X \log(D)\big)$ *by the locally-free subsheaf of* $Sym^N\big(\Omega_X \log(\ulcorner D \urcorner)\big)$ *that is locally-generated, as an* $\mathscr{O}_{V_x}$*-module, by the elements*

$$\frac{dz_1^{k_1}}{z_1^{\lfloor d_1 \cdot k_1 \rfloor}} \cdot \ldots \cdot \frac{dz_l^{k_l}}{z_l^{\lfloor d_l \cdot k_l \rfloor}} \cdot dz_{l+1}^{k_{l+1}} \cdot \ldots \cdot dz_n^{k_n},$$

where $\sum k_i = N$*.*

Remark 3.1.10 (An Equivalent Definition)**.** *There is an alternative definition for the sheaf of* $\mathcal{C}$*-differential forms: Let* V_x *be an open neighbourhood of* $x \in X$ *as in Definition 3.1.9 and take* $\gamma : W \to V_x$ *to be an adapted cover for* $(V_x, D|_{V_x})$*. Let* $\sigma \in \Gamma\big(V_x, Sym^N\big(\Omega_X(*\ulcorner D \urcorner)\big)\big)$*, that is* σ *is a local rational section of* $Sym^N(\Omega_X)$ *with poles along* $\ulcorner D \urcorner$*. Then,*

$$\sigma \in \Gamma\big(V_x, Sym^N_{\mathcal{C}}\big(\Omega_X \log(D)\big)\big) \iff \gamma^*(\sigma) \in \Gamma\big(W, Sym^N(\Omega_{W^\partial})\big), \tag{3.4}$$

So that, in particular, $\gamma^*(\sigma)$ *has at worst logarithmic poles* only *along those prime divisors in* W *that dominate* $(\lfloor D \rfloor \cap V_x)$*, and is regular otherwise.*

Explanation 3.1.11. *Assume that* $\sigma \in \Gamma\big(V_x, Sym^N_{\mathcal{C}}(\Omega_X \log(D))\big)$ *is a local* $\mathcal{C}$*-differential form in the sense of (3.4). By the classical result of Iitaka [Iit82, Chap. 11], it follows that* $\sigma \in \Gamma\big(V_x, Sym^N(\Omega_X \log(\ulcorner D \urcorner))\big)$. *In particular we find that along the reduced component of* D *the equivalence between the two definitions trivially holds. So assume, without loss of generality, that* $m_D(D_i) \neq \infty$, *for all irreducible components* D_i *of* D. *Furthermore let us assume, for simplicity, that*

$$\sigma = f \cdot \frac{dz_1^{k_1}}{z_1^{e_1}} \cdot \ldots \cdot \frac{dz_l^{k_l}}{z_l^{e_l}} \cdot dz_{l+1}^{k_l+1} \cdot \ldots \cdot dz_n^{k_n} \quad \in \Gamma\big((V_x, Sym^N(\Omega_X \log(\ulcorner D \urcorner))\big),$$

where $f \in \mathscr{O}_{V(x)}$ *with no zeros along* D_i*'s, is the local explicit description of* σ. *Since* $\gamma^*(\sigma) \in Sym^N(\Omega_{W^\partial})$, *the inequality*

$$k_i \cdot (a_i - 1) - a_i \cdot e_i \geq k_i(b_i - 1)$$

holds for $1 \leq i \leq l$, *where* $d_i = 1 - (b_i/a_i)$, *i.e.*

$$e_i \leq k_i.d_i, \quad \textit{for all } 1 \leq i \leq l.$$

In particular σ *is a symmetric* $\mathcal{C}$*-differential form on* V_x *in the sense of Definition 3.1.9.*

Remark 3.1.12 (Tensorial $\mathcal{C}$-Differential Forms)**.** *Similar to the Definitions 3.1.9 and (3.4), we can define the sheaf of tensorial* $\mathcal{C}$*-differential forms* $\big(\Omega_X \log(D)\big)^{\otimes_{\mathcal{C}} N}$, *that is, roughly-speaking* $\big(\Omega_X \log(D)\big)^{\otimes_{\mathcal{C}} N}$ *is the maximal subsheaf of* $\big(\Omega_X \log(\ulcorner D \urcorner)\big)^{\otimes N}$ *such that*

$$\gamma^*\Big(\big(\Omega_X \log(D)\big)^{\otimes_{\mathcal{C}} N}\Big) \subseteq (\Omega_{Y^\partial})^{\otimes N}.$$

As we shall see in section 3.3, the Viehweg-Zuo subsheaves generically come from the coarse moduli space, as long as we extend the sheaf of symmetric differential forms to that of $\mathcal{C}$-differential forms associated to the naturally imposed $\mathcal{C}$-structures or orbifold structures (See Definition 3.1.15 below or [Cam08, Sect. 3]) that appear over the moduli variety. But, as the usual Kodaira dimension of subsheaves of symmetric $\mathcal{C}$-differential forms is not sensitive to the fractional positivity of the non-reduced components of the bounder divisor (See Remark 3.1.14 below), a new birational notion is needed to measure the positivity of the Viehweg-Zuo subsheaves in the moduli.

Definition 3.1.13 ($\mathcal{C}$-Kodaira Dimension, cf. [Cam08, Sect. 2.7])**.** *Let* (X, D) *be a smooth pair and* $\mathscr{L} \subseteq Sym^r_{\mathcal{C}}(\Omega_X \log(D))$ *a saturated coherent subsheaf of rank one.*

Define the $\mathcal{C}$-product $Sym^m_{\mathcal{C}}(\mathscr{L})$ of $\mathscr{L}$, to the order of m, to be the saturation of the image of $Sym^m(\mathscr{L})$ inside $Sym^{(m\cdot r)}_{\mathcal{C}}\big(\Omega_X \log(D)\big)$ and define the $\mathcal{C}$-Kodaira dimension of $\mathscr{L}$ by

$$\kappa_{\mathcal{C}}(X, \mathscr{L}) := \max\{k \mid \limsup_{m\to\infty} \frac{h^0\big(X, Sym^m_{\mathcal{C}}(\mathscr{L})\big)}{m^k} \neq 0\},$$

and when $h^0\big(X, Sym^m_{\mathcal{C}}(\mathscr{L})\big) = 0$ for all $m \in \mathbb{N}^+$, then, by convention, we define $\kappa_{\mathcal{C}}(X, \mathscr{L}) = -\infty$.

Remark 3.1.14 (Comparing Kodaira Dimensions)**.** *When $D = 0$ or when D is reduced the sheaf of symmetric $\mathcal{C}$-differential forms $Sym^r_{\mathcal{C}}\big(\Omega_X \log(D)\big)$ is equal to $Sym^r(\Omega_X)$ and $Sym^r\big(\Omega_X \log(D)\big)$, respectively, so that the $\mathcal{C}$-Kodaira dimension $\kappa_{\mathcal{C}}(X, \mathscr{L})$ of a rank one coherent subsheaf $\mathscr{L}$ of $Sym^r_{\mathcal{C}}\big(\Omega_X \log(D)\big)$ coincides with the usual Kodaira dimension $\kappa(X, \mathscr{L})$ of $\mathscr{L}$.*

Let (Y, D) be a smooth pair, Z a smooth variety, and $f : Y \to Z$ a fibration with connected fibres. Assume that every f-exceptional prime divisor F, that is, $\mathrm{codim}_Z\big(f(F)\big) \geq 2$, is a reduced component of D. Then, simple local calculations show that there exists a maximal—in the sense of multiplicities of the irreducible components—divisorial structure Δ on Z, whose support coincides with the codimension-1 closed subset of the log-discriminant locus B of $f : (Y, D) \to Z$ (recall that B is the smallest closed subset of Z such that f is smooth over its complement, and that for every point $z \in Z\backslash\Delta$, the set-theoretic fibre $f^{-1}(z)$ is not contained in D, and that the scheme-theoretic intersection of the fibre Y_z with D is a simple normal-crossing divisor in Y_z) and that the natural pull-back map

$$(df)^m : f^*\big(\mathrm{Sym}^m_{\mathcal{C}}\big(\Omega_Z \log(\Delta)\big)\big) \to \mathrm{Sym}^m_{\mathcal{C}}\big(\Omega_Y \log(D)\big)$$

is well-defined. We call Δ the *$\mathcal{C}$-base* (or the orbifold-base) of the fibration $f : (Y, D) \to Z$.

Definition 3.1.15 ($\mathcal{C}$-Base of a Fibration)**.** *Given a smooth pair (Y, D), let $f : Y \to Z$ be a fibration with connected fibres onto a smooth variety Z. Let $\{\Delta_i\}_i$ be the set of the irreducible components of the divisorial part of the log-discriminant locus of f. For every i, define $\{\Delta_{ij}\}_j$ to be the collection of prime divisors in $f^{-1}(\Delta_i)$ that are* not *f-exceptional. To each divisor Δ_i, assign a positive rational number $m_\Delta(\Delta_i)$ defined by*

$$m_\Delta(\Delta_i) := \min_j\{d_j \cdot m_\Delta(\Delta_{ij})\},$$

d_j being the positive integer verifying the equality

$$f^*(\Delta_i) = \sum_j d_j \Delta_{ij} + E.$$

We define the $\mathcal{C}$-base of the fibration $f : (Y, D) \to Z$ by the divisor

$$\Delta := \sum_i \Big(1 - \frac{1}{m_\Delta(\Delta_i)}\Big)\Delta_i.$$

We finish this section by collecting the various notations that we have introduced in the following table.

Table 3.1: Notations

Ω_{Y^∂}	$\mathcal{C}$-cotangent sheaf 3.1.5
$\mathrm{Sym}^N_{\mathcal{C}}(\Omega_X \log(D))$	Symmetric $\mathcal{C}$-differential forms 3.1.9
$(\Omega_X \log(D))^{\otimes_{\mathcal{C}} N}$	Tensorial $\mathcal{C}$-differential forms 3.1.12
$\mathrm{Sym}^N_{\mathcal{C}}(\mathscr{L})$	$\mathcal{C}$-product 3.1.13
$\kappa_{\mathcal{C}}(X, \mathscr{L})$	$\mathcal{C}$-Kodaira dimension 3.1.13

3.2 The orbifold generic semi-positivity

According to [BDPP04] Miyaoka's generic semi-positivity result (see [Miy87a] and [Miy87b]) can be interpreted as a characterization of positivity of the canonical bundle by the generic nefness of the cotangent sheaf. This positivity result was achieved by certain characteristic p-arguments which cannot be adapted to the context of pairs. In [CP13] Campana and Paun have overcome this obstacle by deploying an important refinement of Miyaoka's theorem, due to Bogomolov and McQuillen, concerning the algebraicity of leaves of foliations induced by positive subsheaves of the tangent sheaf. The generic semi-postivity result of [CP13] has been formulated for G-linearized quotients of the $\mathcal{C}$-cotangent sheaf (See [CP13, Def. 1.2]). We recall that given a finite surjective morphism $\gamma : Y \to X$ with $G := \mathrm{Gal}(Y/X)$, we say that an $\mathscr{O}_Y$-module coherent sheaf $\mathscr{F}$ on Y is G-linearized, if there exists a sheaf isomorphism

$$\psi : \mathrm{pr}_1^*(\mathscr{F}) \to \rho^*(\mathscr{F}),$$

where $\mathrm{pr}_1 : Y \times G \to Y$ is the natural projection onto the first factor and $\rho : Y \times G \to Y$ is the gruop action G, that verifies the natural cocycle conditions.

Theorem 3.2.1 (Generic Semi-Positivity of $\mathcal{C}$-Cotangent Sheaf [CP13, Thm. 2.1])**.** *Let (X, D) be a smooth pair with an adapted cover $\gamma : Y \to X$, whose Galois group we denote by G. If $(K_X + D)$ is pseudo-effective, then every G-linearized, torsion-free, coherent, $\mathscr{O}_Y$-module quotient $\mathscr{F}$ of $(\Omega_{Y\partial})^{\otimes N}$ verifies the inequality*

$$\mathrm{c}_1(\mathscr{F}) \cdot \gamma^*(H_1) \cdot \ldots \cdot \gamma^*(H_{n-1}) \geq 0, \tag{3.5}$$

for all $(n-1)$-tuples of ample divisors $(H_1, \ldots, H_{n-1})$ in X.

Corollary 3.2.2. *Let (X, D) be a smooth pair. Let $\mathscr{L} \subseteq \left(\Omega^1_X \log(D)\right)^{\otimes_{\mathcal{C}} N}$ be an invertible subseheaf and L a divisor in X such that $\mathscr{O}_X(L) \cong \mathscr{L}$. If $(K_X + D)$ is pseudo-effective, then for every collection of $(n-1)$ $\mathbb{Q}$-Cartier nef divisors $P_1, \ldots, P_{n-1}$ the following inequality holds:*

$$(N(K_X + D) - L) \cdot P_1 \cdot \ldots \cdot P_{n-1} \geq 0.$$

Proof. Since for any fixed ample divisor H in X the equality

$$(N(K_X+D)-L) \cdot P_1 \cdot \ldots \cdot P_{n-1} = (N(K_X+D)-L) \cdot (P_1 + \frac{1}{t}H) \cdot \ldots \cdot (P_{n-1} + \frac{1}{t}H)$$

holds as $t \to \infty$, it suffices to prove that

$$(N(K_X + D) - L) \cdot H_1 \cdot \ldots \cdot H_{n-1} \geq 0 \tag{3.6}$$

for any collection of $(n-1)$ $\mathbb{Q}$-Cartier ample divisors $H_1, \ldots, H_{n-1}$.

Take $\gamma : Y \to X$ to be an adapted cover with the corresponding Galois group G. We observe that according to the Remark 3.1.8, the sheaf isomorphism

$$(\Omega_{Y\partial})^{\otimes N} \cong \left(\det(\Omega_{Y\partial}) \otimes \wedge^{n-1} T_{Y\partial}\right)^{\otimes N}$$

reads as

$$(\Omega_{Y\partial})^{\otimes N} \cong \left(\mathscr{O}_Y(\gamma^*(K_X + D)) \otimes \wedge^{n-1} T_{Y\partial}\right)^{\otimes N}. \tag{3.7}$$

Furthermore, we know, by definition, that $\gamma^*(\mathscr{L})$ is a subsheaf of $(\Omega_{Y\partial})^{\otimes N}$. Let us for the moment assume that this is a saturated inclusion. As a result $(\gamma^*(\mathscr{L}))^*$ is a quotient of the dual of the locally-free sheaf given in the right-hand side of the isomorphism (3.7). Thus we arrive at the following exact sequence of locally free sheaves:

$$(\Omega^{n-1}_{Y^\partial})^{\otimes N} \longrightarrow \gamma^*(\mathscr{L}^* \otimes \mathscr{O}_X(N(K_X+D))) \longrightarrow 0.$$

Here the generic semi-positivity result (Theorem 3.2.1) applies and we find that

$$\gamma^*\big(N(K_X+D)-L\big)\cdot\gamma^*(P_1)\cdot\ldots\cdot\gamma^*(P_{n-1})\geq 0 \tag{3.8}$$

holds. The required inequality (3.6) then follows from the projection formula.

Now, if $\gamma^*\mathscr{L}$ is not saturated inside $(\Omega_{Y^\partial})^{\otimes N}$, define $\overline{\mathscr{L}}$ to be its saturation. Since $\overline{\mathscr{L}}$ is not necessarily G-invariant anymore, consider $(\overline{\mathscr{L}})^{\otimes r}$, for any positive multiple of the degree of the cyclic cover γ. Local calculation shows that $(\overline{\mathscr{L}})^{\otimes r}$ is G-invariant. Moreover, $(\overline{\mathscr{L}})^{\otimes(r\cdot N)}$ is saturated inside $(\Omega_{Y^\partial})^{\otimes(r\cdot N)}$. This is because tensorial powers of saturated subsheaves of locally-free sheaves remain saturated. At this point we can argue, as we did in the case of $\gamma^*\mathscr{L}$, to find that the inequality

$$\mathrm{c}_1\Big((\overline{\mathscr{L}})^*\otimes\mathscr{O}_Y\big(\gamma^*(rN(K_X+D))\big)\Big)\cdot\gamma^*(H_1)\cdot\ldots\cdot\gamma^*(H_{n-1})\geq 0, \tag{3.9}$$

holds. From the inequality (3.9) we can readily establish the inequality (3.8). Again, the required inequality (3.6) will follow from the projection formula.

□

3.3 Viehweg-Zuo subsheaves in the parametrizing space

The fundamental result of Jabbusch and Kebekus [JK11b] shows that the symmetric $\mathcal{C}$-differential forms is the correct framework to study the positivity of subsheaves of forms in the coarse moduli space of canonically-polarized manifolds. In this section we give a brief explanation of how one can then reduce the isotriviality conjecture (Conjecture 1.0.13) to the problem of showing that existence of rank one subsheaves of the sheaf of symmetric $\mathcal{C}$-differential forms, attached to a smooth pair, with maximal $\mathcal{C}$-Kodaira dimension implies that the given pair is of log-general type (see Theroem 3.3.3 below). To prepare the correct setting for this reduction, we introduce a notion that, as far as the author is aware, is originally due to Campana.

Definition 3.3.1 (Neat Model of a Pair)**.** *Let (Y, D) be a normal logarithmic pair (Y is normal and the Weil divisor D is reduced) and $h : Y \to Z$ a fibration with connected fibers onto an algebraic base Z. We call a smooth pair (Y_h, D_h) a neat model for (Y, D) and h, if there exists a fibration $\widetilde{h} : Y_h \to Z_h$ that is birationally equivalent to h, that is, there are birational morphisms $\mu : Y_h \to Y$ and $\alpha : Z_h \to Z$ such that the diagram*

$$\begin{tikzcd} Y \arrow[d, "h"'] & Y_h \arrow[l, "\mu"'] \arrow[d, "\widetilde{h}"] \\ Z & Z_h \arrow[l, "\alpha"'] \end{tikzcd}$$

commutes, for which the following conditions are satisfied:

3.10. D_h is the extension of the μ-birational transform $\widetilde{D}$ of D by some reduced μ-exceptional divisor, i.e. $D_h = \widetilde{D} + E'$, where E' is μ-exceptional.
3.11. (Z_h, Δ_h) is a smooth pair, Δ_h being the $\mathcal{C}$-base (See Definition 3.1.15) of the fibration $\widetilde{h} : (Y_h, D_h) \to Z_h$.
3.12. Every $\widetilde{h}$-exceptional prime divisor P in Y_h (P verifies the inequality $codim_{Z_h}(\widetilde{h}(P)) \geq 2$) is contained in $supp(D_h)$.

The interest in the neat models of pairs (that are equipped with fibrations), is two-fold. First, the conditions (3.11) and (3.12) ensure that $(\widetilde{h})^*$ defines a well-defined pull-back map from symmetric $\mathcal{C}$-differential forms $\mathrm{Sym}^N_{\mathcal{C}}(\Omega_{Z_h} \log(\Delta_h))$ attached to (Z_h, Δ_h) to the sheaf of symmetric logarithmic forms $\mathrm{Sym}^N(\Omega_{Y_h} \log(D_h))$ (see the discussion before the Definition 3.1.15). Secondly, according to the property (3.10), the neat model (Y_h, D_h) inherits the birational properties of the original pair (Y, D). For example if (Y, D) special, then so is (Y_h, D_h). These attributes will be crucial to the proof of the main result (Theorem. 3.3.3) of this section.

Proposition 3.3.2 (Construction of Neat Models, cf. [JK11b, Sect. 10])**.** *Every normal logarithmic pair (Y, D) and a surjective morphism with connected fibers $h : Y \to Z$, where Z is a projective variety, admits a neat model.*

Proof. Let $\alpha_1 : Z_1 \to Z$ be a suitable modification of the base of the fibration h such that the normalization of the induced fiber product $Y \times_Z Z_1$, which we denote by Y_1, givers rise to an equidimensional fibration $h_1 : Y_1 \to Z_1$, i.e. a *flattening* of h, and a birational map $\mu_1 : Y_1 \to Y$ (see the diagram below). Define D_1 to be the maximal reduced divisor contained in the $\mathrm{supp}(\mu_1^{-1} D)$ and let

$$D_1 = D_1^{\mathrm{ver}} + D_1^{\mathrm{hor}}$$

be the decomposition of D_1 into sum of its vertical D_1^{ver} and horizontal D_1^{hor} components. Introduce a closed subset in Z_1 by $D_{Z_1} := h_1(D_1^{\text{ver}})$. Let $\Delta_1 \subset Z_1$ denote the log-discriminant locus defined by the fibration h_1 and the divisor D_1. Now, let $\alpha_2 : Z_h \to Z_1$ be a desingularization of Z_1 such that the maximal reduced divisor in the $\text{supp}(\alpha_2^{-1}\Delta_1 \cup \alpha_2^{-1}D_{Z_1})$ is snc. Set Y_2 to be the normalization of the fiber product $Y_1 \times_{Z_1} Z_2$, and μ_2 the naturally induced birational morphism. Define D_2 in Y_2 by the maximal reduced divisor contained in the $\text{supp}(\mu_2^{-1}D_1)$. Finally let $\mu_3 : Y_h \to Y_2$ be a log-resolution of (Y_2, D_2) and take $\widetilde{h} : Y_h \to Z_h$ to be the induced fibration.

$$\begin{tikzcd}
Y \arrow[d, "h"] & Y_1 \arrow[l, "\mu_1"'] \arrow[d, "h_1"] & Y_2 \arrow[l, "\mu_2"'] \arrow[d, "h_2"] & Y_h \arrow[l, "\mu_3"'] \arrow[dl, "\widetilde{h}"] \\
Z & Z_1 \arrow[l, "\alpha_1"'] & Z_h \arrow[l, "\alpha_2"'] &
\end{tikzcd}$$

Now set $\widetilde{D}_2$ to be the maximal reduced divisor in $\text{supp}(\mu_3^{-1})$. Note that h_1 remains equidimensional under the base change of α_2, i.e. h_2 is also equidimensional. This implies that when we desingularize Y_2 by μ_3, every $\widetilde{h}$-exceptional divisor is μ_3-exceptional. Let E_3 be the sum of all $\widetilde{h}$-exceptional prime divisors in Y_h and define $D_h := \widetilde{D}_2 + E_3$ to be the extension of $\widetilde{D}_2$ by E_3. We finish by defining the birational morphisms μ and α in Definition 3.3.1 by $(\mu_3 \circ \mu_2 \circ \mu_1)$ and $(\alpha_2 \circ \alpha_1)$, respectively. Now by construction, the $\mathcal{C}$-structure Δ_h on Z_h induced by D_h and $\widetilde{h}$ defines a smooth pair (Z_h, Δ_h), as required.

$\square$

Theorem 3.3.3 (Reduction of the Isotriviality Conjecture)**.** *The isotriviality conjecture 1.0.13 holds, if the following assertion is true:*

> *3.13. Let (T, B) be a smooth pair. If $\text{Sym}^N_{\mathcal{C}}(\Omega_T \log(B))$ admits a saturated rank-one subsheaf $\mathscr{L}$ with $\kappa_{\mathcal{C}}(T, \mathscr{L}) = \dim T$, then (T, B) is of log-general type.*

Proof. Let $f^\circ : X^\circ \to Y^\circ$ be a smooth family of canonically-polarized manifolds, where Y° is a special quasi-projective variety, and let Y be a smooth compactification with boundary divisor D such that $D \cong Y \backslash Y^\circ$ and that the induced map $\widetilde{\mu} : Y \to \overline{\mathfrak{M}}$ to a compactification of $\mathfrak{M}$ is a morphism. Aiming for a contradiction, assume that the family $f^\circ : X^\circ \to Y^\circ$ is *not* isotrvial. Now, if $\widetilde{\mu}$ is generically finite, then thanks to Campana and Paun's solution to Viehweg's conjecture (Conjecture 1.0.15), we find that $(K_Y + D)$ is big, contradicting the assumption that (Y, D) is special. Therefore to prove the theorem, we only need to treat the case where

$\widetilde{\mu} : Y \to \overline{\mathfrak{M}}$ is *not* generically finite. In this case, by the Stein factorization, we can find a projective variety Z such that the morphism $\widetilde{\mu}$ factors through a fibration with connected fibres $h : Y \to Z$ and a finite morphism $Z \to \overline{\mathfrak{M}}$. According to Proposition 3.3.2, we can find a neat model (Y_h, D_h) of the pair (Y, D) and the fibration $h : Y \to Z$.

$$\begin{tikzcd} & Y \arrow[dl, "\widetilde{\mu}"'] \arrow[d, "h"'] & Y_h \arrow[l, "\mu"'] \arrow[d, "\widetilde{h}"] \\ \overline{\mathfrak{M}} & Z \arrow[l, "\text{finite}"'] & Z_h \arrow[l, "\alpha"'] \end{tikzcd}$$

We observe that since $Y_h \backslash D_h$ is isomorphic to an open subset of Y°, it also parametrizes a smooth family of canonically polarized manifolds. Thus by [VZ02, Thm. 1.4], for some positive integer N, we can find a line subbundle $\mathscr{L} \subseteq \mathrm{Sym}^N(\Omega_{Y_h} \log(D_h))$ such that $\kappa(Y_h, \mathscr{L}) \geq \dim Z_h$. Moreover by [JK11a, Thm. 1.4], we know that the Viehweg-Zuo subsheaf $\mathscr{L}$ generically comes form the coarse moduli space. More precisely, there exists an inclusion $\mathscr{L} \subseteq \mathrm{Sym}^N \mathscr{B}$, where $\mathscr{B}$ is the saturation of the image of

$$d\widetilde{h} : (\widetilde{h})^*(\Omega_{Z_h}) \to \Omega_{Y_h} \log(D_h).$$

Let us now collect the various properties of the pairs (Y_h, D_h) and (Z_h, Δ_h), and the fibration $\widetilde{h} : Y_h \to Z_h$ (recall that, by definition, the divisor Δ_h is the $\mathcal{C}$-base of the fibration $\widetilde{h} : (Y_h, D_h) \to Z_h$), that we have found so far:

3.14. (Y_h, D_h) and (Z_h, Δ_h) are both smooth pairs (property (3.11)).

3.15. D_h contains all $\widetilde{h}$-exceptional prime divisors (property (3.12)).

3.16. There exists a saturated rank-one subsheaf $\mathscr{L} \subseteq \mathrm{Sym}^N \mathscr{B}$, for some positive integer N, such that $\kappa(Y_h, \mathscr{L}) \geq \dim Z_h$.

With these conditions, we can apply [JK11a, Cor. 5.8] to find a saturated rank-one subsheaf $\mathscr{L}_{Z_h} \subseteq \mathrm{Sym}^N_{\mathcal{C}}(\Omega_{Z_h} \log(\Delta_h))$ such that

$$\kappa_{\mathcal{C}}(Z_h, \mathscr{L}_{Z_h}) = \kappa(Y_h, \mathscr{L}) \geq \dim(Z_h). \tag{3.17}$$

Finally, if the statement (3.13) holds, then (Z_h, Δ_h) is of log-general type. On the other hand by property (3.10), for every $1 \leq p \leq n$, we can push-forward invertible subsheaves of $\Omega^p_{Y_h} \log(D_h)$ to those of $\Omega^p_Y \log(D)$. In particular, since (Y, D) is special, then so is (Y_h, D_h). But, this is a contradiction to our previous finding that $(K_{Z_h} + \Delta_h)$ is big (recall that for a neat model $\widetilde{h} : Y_h \to Z_h$, and for sufficiently divisible positive integer m, we always have

$$h^0(Y_h, \mathscr{G}^{\otimes m}) = h^0\big(Z_h, \mathscr{O}_{Z_h}(K_{Z_h} + \Delta_h)^{\otimes m}\big), \tag{3.18}$$

where $\mathscr{G}$ denotes the saturation of the pull-back bundle $(\widetilde{h})^*\big(\mathscr{O}_{Z_h}(K_{Z_h})\big)$ inside $\Omega^{\dim(Z_h)}_{Y_h} \log(D_h)$).

□

3.4 The Isotriviality conjecture: The approach of Campana and Paun

In this section we prove the statement (3.13) in the previous section. The isotriviality conjecture will then follow from Theorem 3.3.3. The proof is completely based on the solution of [CP13, Sect. 4] to the Viehweg's hyperbolicity conjecture 1.0.15. In particular, Theorem 3.4.2 should be taken as the generalization of [CP13, Thm. 4.1] from the category of purely logarithmic smooth pairs (the boundary divisor is reduced) to that of smooth pairs in general.

For the ease of notation we have replaced the pair (T, B) in the reduction statement (3.13) by (X, D) with the warning that D should not be confused with the boundary divisor of the compactification of Y° that was introduced in the previous sections.

Proposition 3.4.1. *Let (X, D) be a smooth pair and $\mathscr{L} \subseteq Sym^N_{\mathcal{C}}\big(\Omega_X \log(D)\big)$ a saturated rank one subsheaf with $\kappa_{\mathcal{C}}(X, \mathscr{L}) = \dim X$. For every ample divisor A in X, there exists a rational number $c \in \mathbb{Q}^+(A, \mathscr{L})$, depending on A and $\mathscr{L}$, such that the inequality*

$$\mathrm{vol}(K_X + D + G) \geq c \cdot \mathrm{vol}(A), \tag{3.19}$$

holds for every $\mathbb{Q}$-Cartier divisor G satisfying the following properties:

3.20. $(D + G) \sim_{\mathbb{Q}} P$, for some big $\mathbb{Q}$-Cartier divisor P such that $\lfloor P \rfloor = 0$.

3.21. $(X, D + G)$ and (X, P) are both smooth pairs.

3.22. The $\mathbb{Q}$-Cartier divisor $(K_X + D + G)$ is pseudo-effective.

Proof. First, let us fix an ample divisor A. We notice that by an argument similar to that of Kodaira's lemma [Laz04, Prop. 2.2.6]), we can always find a (sufficiently large) positive integer m such that

$$H^0\big(X, \mathrm{Sym}^m_{\mathcal{C}}(\mathscr{L}) \otimes \mathscr{O}_X(-A)\big) \neq 0.$$

Let the invertible subsheaf $\mathscr{L}' \subseteq \mathrm{Sym}^{m\cdot N}_{\mathcal{C}}\big(\Omega_X \log(D)\big)$ denote the line-bundle $\mathrm{Sym}^m_{\mathcal{C}}(\mathscr{L})$, so that the inequality

$$A \leq L' \tag{3.23}$$

holds between Cartier divisors L' and A, L' being the divisor verifying the isomorphism $\mathscr{O}_X(L') \cong \mathscr{L}'$. We shall prove the proposition in two steps. First, we run the log-minimal model program (or LMMP, for short) for the smooth pair (X, P). We notice that since P is big and has no reduced components (assumption (3.20)), according to [BCHM10, Thm. 1.1], after a finite number of divisorial contractions and log-flips, the program terminates in a log-minimal model (X', P'), i.e. $(K_{X'} + P')$ is nef. Here, at the minimal level, we shall find a lower-bound for $\mathrm{vol}(K_{X'} + P')$ in terms of $\mathrm{vol}(A)$ and *independent of* G. The second step of the proof is standard; we will just use the negativity lemma in the minimal model theory and replace $\mathrm{vol}(K_{X'} + P')$ by $\mathrm{vol}(K_X + P)$ to establish the required inequality (3.19).

Step. 1: Log-minimal model of (X, P) and the volume of its log-canonical divisor. Let $\pi : (X, P) \dashrightarrow (X', P')$ be the birational map defined by the LMMP. Take $\mu : \widetilde{X} \to X$ to be a modification of X resolving the indeterminacy of π, with resulting morphism $\widetilde{\pi} : \widetilde{X} \to X'$, and such that $\mathrm{supp}(\mathrm{Exc}(\mu) \cup \widetilde{D} \cup \widetilde{G})$, where $\widetilde{D}$, $\widetilde{G}$ are the μ-birational transforms of D and G, respectively, is simple normal-crossing in $\widetilde{X}$:

$$\begin{tikzcd}
\widetilde{Y} \arrow[r, "\gamma", "\text{adapted cover}"'] & \widetilde{X} \arrow[d, "\mu"'] \arrow[dr, "{\widetilde{\pi}, \text{ birational}}"] & \\
 & (X, P) \arrow[r, dashed, "\pi", "LMMP"'] & (X', P')
\end{tikzcd}$$

Let $\gamma : \widetilde{Y} \to \widetilde{X}$ be an adapted cover for the pair $(\widetilde{X}, \widetilde{D} + \widetilde{G} + E)$, where E is the maximal reduced divisor contained in $\mathrm{Exc}(\mu)$. We notice that, as $\mathscr{L}'$ is a subsheaf of $\mathrm{Sym}^{m\cdot N}_{\mathcal{C}}\big(\Omega_X \log(D)\big)\big(\subseteq \mathrm{Sym}^{m\cdot N}\big(\Omega_X \log(\ulcorner D \urcorner)\big)\big)$, the inclusion

$$\mu^*(\mathscr{L}') \subseteq \mathrm{Sym}^{m\cdot N}_{\mathcal{C}}\big(\Omega_{\widetilde{X}} \log(\widetilde{D} + \widetilde{G} + E)\big).$$

follows from the definition. Now, in order for us to use the generic semi-positvity result (Corollary. 3.2.2), we need $(K_{\widetilde{X}} + \widetilde{D} + \widetilde{G} + E)$ to be pseudo-effective. This is indeed the case: from the ramification formula for μ we have $(K_{\widetilde{X}} + \widetilde{D} + \widetilde{G}) = \mu^*(K_X + D + G) + \widetilde{E}$, $\widetilde{E}$ being an effective exceptional divisor (the effectivity follows from our assumption that $(X, D + G)$ is a smooth pair (3.20)). So, from the pseudo-effectivity of $(K_X + D + G)$ (assumption (3.22)) it follows that $(K_{\widetilde{X}} + \widetilde{D} + \widetilde{G})$ is pseudo-effective, and thus so is $(K_{\widetilde{X}} + \widetilde{D} + \widetilde{G} + E)$, as required. Therefore Corollary 3.2.2 applies and the inequality

$$\mu^*(L') \cdot P^{n-1} \leq (m \cdot N)(K_{\widetilde{X}} + \widetilde{D} + \widetilde{G} + E) \cdot P^{n-1}$$

holds, for any nef divisor P in $\widetilde{X}$. In particular, for any fixed ample divisor H' in X' and positive integer r, we have

$$\mu^*(L') \cdot \widetilde{\pi}^*(K_{X'} + P' + \frac{1}{r}H')^{n-1} \leq (m \cdot N)(K_{\widetilde{X}} + \widetilde{D} + \widetilde{G} + E) \cdot \\ \cdot \widetilde{\pi}^*(K_{X'} + P' + \frac{1}{r}H')^{n-1}. \tag{3.24}$$

Now, let U be a Zariski open subset of X' of $\operatorname{codim}_{X'}(X' \backslash U) \geq 2$ where $\pi^{-1}|_U$ and $\widetilde{\pi}^{-1}|_U$ are both isomorphisms. For every $r \in \mathbb{N}^+$, define d_r to be a sufficiently large positive integer such that the linear system $|d_r(K_{X'} + P' + \frac{1}{r}H')|$ is basepoint-free and that the irreducible curve $C_r := B_r^1 \cap \ldots \cap B_r^{n-1}$, cut out by general members $B_r^i \in |d_r(K_{X'} + P' + \frac{1}{r}H')|$, is a subset of U. We notice that as $C_r \subset U$, and because of our assumption (3.20), the left-hand side of the inequality (3.24) is equal to $(\frac{1}{d_r})^{n-1}(m \cdot N)(K_{X'} + P') \cdot (K_{X'} + P' + \frac{1}{r})^{n-1}$. Therefore, we may write the inequality (3.24) as

$$(d_r)^{n-1}\mu^*(L') \cdot \widetilde{\pi}^*(K_{X'} + P' + \frac{1}{r}H')^{n-1} \leq (m \cdot N)(K_{X'} + P' + \frac{1}{r}H')^n,$$

so that

$$\mu^*(L') \cdot \widetilde{\pi}^*(K_{X'} + P' + \frac{1}{r}H')^{n-1} \leq (m \cdot N)\mathrm{vol}(K_{X'} + P' + \frac{1}{r}H'). \tag{3.25}$$

Next, we notice that, as $(L' - A) \geq 0$ (inequality (3.23)), the pull-back $\mu^*(L' - A)$ is also effective. Therefore, and again by using the fact that the nef cone in the Néron-Severi space $\mathrm{N}^1(\widetilde{X})_{\mathbb{R}}$ is equal to the closure of the ample one, we have $\mu^*(L'-A)\cdot\widetilde{\pi}^*(K_{X'}+P'+\frac{1}{r}H')^{n-1}\geq 0$. Hence we can rewrite the inequality (3.25) as

$$\mu^*(A)\cdot\widetilde{\pi}^*(K_{X'}+P'+\frac{1}{r}H')^{n-1}\leq (m\cdot N)\mathrm{vol}(K_{X'}+P'+\frac{1}{r}H') \tag{3.26}$$

Now, by applying the Teissier's inequality [Laz04, Thm. 1.6.1] (to the left-hand side of the inequality (3.26)), we have

$$\mathrm{vol}(A)^{\frac{1}{n}}\cdot\mathrm{vol}(K_{X'}+P'+\frac{1}{r}H')^{\frac{n-1}{n}}\leq (m\cdot N)\mathrm{vol}(K_{X'}+P'+\frac{1}{r}H'),$$

i.e.

$$\mathrm{vol}(A)^{\frac{1}{n}}\leq (m\cdot N)\mathrm{vol}(K_{X'}+P'+\frac{1}{r}H')^{\frac{1}{n}}. \tag{3.27}$$

Finally, thanks to the continuity of vol(.), by taking $r\to\infty$ in the inequality (3.27) we have

$$\frac{1}{(m\cdot N)^n}\cdot\mathrm{vol}(A)\leq\mathrm{vol}(K_{X'}+P'), \tag{3.28}$$

that is, the inequality (3.19) holds for the log minimal model (X',P'), if we take $c:=\frac{1}{(m\cdot N)^n}$.

Step. 2: Lower-bound for the volume of (K_X+P). By the negativity lemma in the minimal model theory, we know that $H^0(X,m(K_X+P))\cong H^0(X',m(K_{X'}+P'))$, for all $m\in\mathbb{N}^+$. In particular the equality $\mathrm{vol}(X,K_X+P)=\mathrm{vol}(X',K_{X'}+P')$ holds. The required inequality (3.19) now follows form the inequality (3.28) in the previous step and assumption (3.20).

□

Theorem 3.4.2. *Let (X,D) be a smooth pair and $\mathscr{L}\subseteq Sym^N_{\mathcal{C}}(\Omega_X\log(D))$ a saturated rank-one subsheaf. If $\kappa_{\mathcal{C}}(X,\mathscr{L})=\dim X$, then (K_X+D) is big.*

Proof. Let H be a very ample divisor such that $H-D$ is ample, and let r be a (fixed) sufficiently large positive integer for which the divisor $\big(r(H-D)\big)$ is very ample. Define the hyperplane section B_D to be a general member of the linear system $|r(H-D)|$. From construction it follows that, for every integer $M>r$, the

$\mathbb{Q}$-divisor $(D + \frac{1}{M}B_D)$ is $\mathbb{Q}$-linearly equivalent to an snc divisor, which we denote by P_M, with no reduced components:

$$\begin{aligned} D + \frac{1}{M}B_D &\sim_{\mathbb{Q}} D + \frac{1}{M}\big(r(H - D)\big) \\ &= (1 - \frac{r}{M})D + \frac{r}{M}H =: P_M. \end{aligned}$$

Claim 3.29. *The divisor $(K_X + P_M)$ is pseudo-effective, for all integers M verifying the inequality $M > r$.*

Let us for the moment assume that the claim holds. Define the $\mathbb{Q}$-Cartier divisor G in Proposition 3.4.1 by $G := \frac{1}{M}B_D$. As the conditions (3.20), (3.21) and (3.22) in Proposition 3.4.1 are all satisfied, it follows from the inequality (3.19) that for any fixed ample divisor A, there exists a constant c such that

$$\mathrm{vol}(K_X + D + \frac{1}{M}B_D) \geq c \cdot \mathrm{vol}(A), \quad \forall M \in \mathbb{N} \text{ such that } \; M > r. \tag{3.30}$$

Therefore, by taking $M \to \infty$, the continuity property of vol(.) and the fact that *the constant c in Proposition 3.4.1 is independent of M*, it follows that the divisor $(K_X + D)$ is big.

It now remains to prove the claim 3.29.

Proof of claim 3.29. Aiming to extract a contradiction, suppose that $(K_X + P_M)$ is not pseudo-effective for some positive integer $M > r$. Let H' be a suitably-chosen very ample divisor such that the effective log-threshold given by

$$\epsilon := min\{t \in \mathbb{R}^+ : K_X + P_M + tH' \; \text{ is pseudo-effective}\},$$

is smaller than 1. According to [BCHM10, Cor. 1.1.7] ϵ is rational. Now by applying Proposiotion 3.4.1 to the pair (X, D) with $G := \frac{1}{M}B_D + \epsilon H'$, we find that $K_X + P_M + \epsilon H'$ is big. But as the big cone forms the interior of the cone of pseudo-effective $\mathbb{Q}$-Cartier classes, for sufficiently small δ, $K_X + D_M + (\epsilon - \delta)H'$ is also pseudo-effective, contradicting the minimality assumption on ϵ.

□

The isotriviality conjecture (Conjecture 1.0.13) now follows from Theorem 3.4.2 together with Theorem 3.3.3 in the previous section.

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