Survey of Some Developments in the Gross-Neveu Model

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November 1983

A Thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements of the degree of Master.

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I. SOMMAIRE

La théorie classique du modèle de Gross-Neveu est revue en détail. De plus, pour la première fois, croyons nous, on montre la solution à un pôle associée à la solution à deux pôles conjugées présentée par Zakharov et Mikhaïlov. De même, la solution comprenant un nombre arbitraire de solitons et de doublets est montrée explicitement (matrice X). Pour le cas où un fermion est présent (N = 1), on calcule analytiquement les solutions comprenant un soliton, deux solitons et un doublet. On trouve un soliton singulier. Les résultats sont généralisés pour un N arbitraire. Il est démontré que le champ scalaire σ est indépendant de N (résultat conjecturé par Neveu et Papanicolaou). En conclusion, on discute brièvement de la nécessité de pouvoir prédire les propriétés topologiques d'une solution à partir de la matrice X.

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II. ABSTRACT

A detailed review of the Gross-Neveu model at the classical level is presented. Moreover, the one-pole solution associated with the conjugate-pole solution presented by Zakharov and Mikhailov is shown for what is believed to be the first time. The solution with an arbitrary number of solitons and doublets present is also displayed explicitly (X Matrix). For the one-fermion case (N = 1), we analytically calculate soliton, two-soliton and doublet solutions. A singular soliton is found. Results are extended to the arbitrary N case. The scalar field σ is demonstrated to be N-independent (results conjectured by Neveu and Papanicolaou). In the conclusion, the necessity of being able to predict topological properties of a solution from the X-matrix is briefly discussed.

III. ACKNOWLEDGEMENTS

I am grateful to Professor T.F. Morris, my thesis director, for suggesting a stimulating field of research. My mother, Hélène Trudeau, a translator, helped me to improve the text. I am also greatly indebted to Stratos Mahmourides who valiantly struggled to type the manuscript and did a great job. I also thank Lise Desrochers who helped finishing the typing of the manuscript. I wish also to thank my family and friends for providing support in the course of my work.

Needless to say, the responsibility for any remaining error lies with me.

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1. Introduction

The Gross-Neveu model was first introduced in quantum field theory in 1974 by D.J. Gross and A. Neveu [16][1]. It is a model in (1 + 1) dimensions of N fermions interacting through a scalar interaction. [see equation (2,1)] This model exhibited some features that were under close scrutiny at that time: asymptotic freedom, dynamical spontaneous symmetry breaking (degenerate vacuum), dimensional transmutation of the coupling constant, etc. Perhaps more important is the fact that this paper served to introduce the method of the 1/N expansion which would play a significant role in subsequent years in helping to provide information on some as yet unsolvable field theories. It is based on the fact that, in the Gross-Neveu model, aside from the overall mass scale there exists no adjustable parameters. Hence ratios of particle masses depend only on N. When the number of particles is large, 1/N can be used as a small parameter to obtain peturbative results. As $N \to \infty$, results are exact.

Almost at the same time, R.F. Dashen, B. Hasslacher and the same A. Neveu (DHN) were devising a powerful semiclassical functional method [10]. (For a good introduction to the subject see [13] and [14]). Their first important application was to the sine - Gordon equation. Surprisingly, they found the exact particle spectrum of the theory. The infinite set of conservation laws of the sine - Gordon equation were responsible for this peculiar behavior. In 1975, the WKB method of DHN was applied to the Gross-Neveu model [22]. Before starting they had to find classical solutions which are the necessary input to the method. The standard

[1] Reference numbers are put in brackets [ ]

inverse scattering method gave them the time - independent solution while the time - dependent solution was found by a clever trick. They started with a solution which mimicked the sine - Gordon time - dependent solution (or doublet) but contained a certain number of parameters to be fitted. Introducing this solution into the Gross-Neveu equations of motion, they were able to find the parameters. However, unable to find this solution's Floquet indices (see [10] to [14]), they had to be content with finding the particle spectrum to zeroth order of the 1/N expansion (exact as $N = \infty$). The fact that the Gross-Neveu model possessed time-dependent solutions in close analogy to the sine - Gordon equation led theoreticians to believe that the model had an infinite set of conservation laws and was exactly integrable at the classical level.

In 1978, the research on the Gross-Neveu model expanded on many fronts. Results were published which fell in the fields of classical theory, group theory and quantum theory (S-matrix theory).

At the quantum level, Alexander B. Zamolodchikov and Alexey B. Zamolodchikov (ZZ) presented their important work on certain relativistic quantum field theory models (which included the Gross-Neveu model) [19], [20]. They presented the exact factorized S-matrix of the Gross-Neveu model. The factorization of the S-matrix means that multi-particle scattering can always be reduced to a succession of two-particle scatterings. They examined the relationship between factorization and the existence of an infinite set of conservation laws at the quantum level. Other work was also done at the quantum level (see references [15], [17] and [18]).

Group theory articles put the emphasis on the isotopic spin symmetry group of the model (the internal symmetry of the N fermions) at the quantum level. This division is somewhat arbitrary since the symmetry group of the Gross-Neveu model is always more or less ever present in the worker's mind. E. Witten published interesting properties of the model [23]. For $N = 3$, the model was shown to be equivalent to the supersymetric sine-Gordon equation while for $N = 4$, the model was shown equivalent to two decoupled sine-Gordon equations (Fermi-Bose symmetry). Some higher conservation laws responsible for the exact solubility of the model were also constructed. References [21] and [22] provided more insights on the topic of the long-time- puzzling Fermi-Bose relationship as applied to the Gross-Neveu model [a fermion system (e.g. the Gross-Neveu model) was shown to be equivalent to a boson system (e.g. the sine-Gordon equation)]. The simultaneity of the work on the three fronts previously described is shown by the fact that article [20] (quantum level) and article [23] (group theory) both referred to a preprint of an article to be published by A. Neveu and N. Papanicolaou (NP) [5]. In this article for the first time, the infinite set of conservation laws (classical level) were displayed. The model was shown to be completely integrable for the case $N = 1$ and $N = 2$. An interesting fact is that the scalar fields σ that they found are identical to those displayed by DHN in [12]. This led them to conjecture rightfully (as we shall see in this work) that the field σ is N-independent.

Shortly following (1980) is a paper by V.E. Zakharov and A.V. Mikhailov (ZM) which showed the integrability of the model for any N [7].

Now that the Gross-Neveu model is almost fully understood it is most often used as a research tool. That is, the author uses it to illustrate some point or help him in some task which is not the comprehension of the model itself. Reference [26] to [29] fall within this class. For example, in reference [27], T.E. Clark and S.T. Love applied West's proof of confinement by contradiction to the Gross-Neveu Model (which is non confining), and to the confining Schwinger model. In both cases they showed that the contradiction found by West for four-dimensional Q C D is averted.

Originally, this work was intended as a review of know important results at the classical (NP and ZM), semi-classical (DHN) and quantum (ZZ) level in the field of soliton theory. Working our way in the classical theory we found a wealth of new expected and unexpected results. The original project was then modified. In this work, we shall provide a thorough introduction to the vesture method of Zakharov, Shabat and Mikhailov [4], [6], [7], [8], [9]. Throughout the presentation we keep our approach as general as possible and even use methods much too powerful for the present problem (e.g. the reduction problem). The aim of this procedure is to constantly remind the reader of the possibility of applying this method to numerous other equations and to keep the discussion rigourous. After presenting the vesture method and the work of NP and ZM, we derive and calculate soliton solutions.

We found the one-pole solution to the model which oorrresponds to the "static" solution. The conjugute-pole solution given by ZM correspond to the doublet. Having found the one-pole solution, we are able to give the solution containing an arbitrary number of solitons containing an arbitrary number of solitons and doublets. As an application, we give the soliton-doublet solution. We then calculate soliton, two-soliton and *doublet solutions analytically for the case N = 1 using the path prescribed by ZM. An unknown singular soliton of the Gross–Neveu model is found (and the corresponding two–soliton and doublet solution). The calculation is extended to the case of arbitrary N. We find that the vacuum fermions are degenerate (they have the same amplitude and velocity). As a consequence the fermionic fields for arbitrary N are simply related to the Fermion fields for N = 1. Moreover, the field σ [see equation (2,8)] is shown to be N-independent. This result is a consequence of the degeneracy of the vacuum fermions. Therefore whatever the field σ that we find within the framework of this theory is, it is N–independent.*

For the sake of clarity, here are the chapters that represent original work and those that do not. Chapters 2, 3, 4, 5, sections (6.2) and (6.3), sections (7.1), (7.2) and (7.5) are borrowed material from the litterature (references included in the text). Our personal contribution to this work is included in section (6.3), sections (7.3), (7.4), (7.6) and (7.7), chapter 8 and chapter 9. The topic of Appendix A is most probably treated in an original way.

2. The Gross–Neveu Model

2.1 The Lagrangian

The Gross–Neveu model is a model of N massless Dirac fermions interacting through a scalar interaction. It is defined by the Lagrangian

$$L = \sum_{k=1}^{N} i\bar{\Psi}^{(k)} \not{\partial} \Psi^{(k)} - \frac{g^2}{2} \left(\sum_{k=1}^{N} \bar{\Psi}^{(k)} \Psi^{(k)}\right)^2 \qquad (2,1)$$

where $\not{\partial} = \gamma^0 \partial_t + \gamma^1 \partial_x$ and γ^0, γ^1 are two-dimensional Dirac matrices.

We can suppress the particle-type index k using the notation

$$i\bar{\Psi} \not{\partial} \Psi = \sum_{k=1}^{N} i \bar{\Psi}^{(k)} \not{\partial} \Psi^{(k)}$$

$$\text{and} \qquad \bar{\Psi}\Psi = \sum_{k=1}^{N} \bar{\Psi}^{(k)} \Psi^{(k)} \qquad (2,2)$$

The Lagrangian:

$$L = \Psi\, i \not{\partial}\Psi - g\, \sigma\, \bar{\Psi}\Psi + 1/2\, \sigma^2 \qquad (2,3)$$

is equivalent to the Lagrangian (1). Using the equation of motion for σ

$$\sigma = g \bar{\Psi}\Psi$$

the equivalence is easily established. Ψ *is a two-component*

$$\text{spinor:}\quad \Psi^{\alpha} = \begin{bmatrix} \psi^{\alpha} \\ \phi^{\alpha} \end{bmatrix}, \quad \overline{\Psi}^{\alpha} = \Psi^{*}\gamma^{0} \qquad (2,4)$$

The coupling constant can be scaled out. Letting $\Psi \to 1/g\,\Psi$ *in equation (1) we can factorize g which becomes irrelevant. The Lagrangian (1), without g, written in full is*

$$L = \Big[\sum_{\alpha} \{ i\,\phi^{*\alpha} (\partial_t \phi^{\alpha} + \partial_x \phi^{\alpha}) + i(\psi^{*\alpha} (\partial_t \psi^{\alpha} - \partial_x \psi^{\alpha}))\} - \frac{1}{2} \{\sum_{\alpha} (\psi^{*\alpha} \phi^{\alpha} + \psi^{\alpha} \phi^{*\alpha})\}^{2} \Big] \qquad (2,5)$$

Introducing light-cone coordinates

$$\eta = 1/2(t + x) \qquad \xi = 1/2(t - x) \qquad (2,6)$$

$$\partial_{\eta} = \partial_t + \partial_x \qquad \partial_{\xi} = \partial_t - \partial_x$$

the Lagrangian (2,5) simply becomes

$$L = \Big[\sum_{\alpha} (i\,\phi^{*\alpha} \partial_{\eta} \phi^{\alpha} + i\,\psi^{*\alpha} \partial_{\xi} \psi^{\alpha}) - \frac{1}{2} \{\sum_{\alpha} (\psi^{*\alpha} \phi^{\alpha} + \psi^{\alpha} \phi^{*\alpha})\}^{2} \Big] \qquad (2,7)$$

2.2 The Equations of Motion

From the Lagrangian (2,3) we derive the following equation of motion

$$i \not\partial \Psi^{\alpha} - g \sigma \Psi^{\alpha} = 0$$

$$\sigma = g \sum_{\alpha} \overline{\Psi}^{\alpha} \Psi^{\alpha} \qquad (2,8)$$

We shall work with these equations with the coupling constant scaled out

$$i \not\partial \Psi^{\alpha} - \sigma \Psi^{\alpha} = 0$$

$$\sigma = \sum_{\alpha} \overline{\Psi}^{\alpha} \Psi^{\alpha} \qquad (2,9)$$

For the Dirac matrices the representation that we used is

$$\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \gamma^1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad (2,10)$$

$$\left[i \begin{bmatrix} 0 & \partial_t + \partial_x \\ \partial_t - \partial_x & 0 \end{bmatrix} + \begin{bmatrix} -\sigma & 0 \\ 0 & -\sigma \end{bmatrix} \right] \cdot \begin{bmatrix} \psi^{\alpha} \\ \phi^{\alpha} \end{bmatrix} = 0 \qquad (2,11)$$

Written in full, the equations are:

$$i \partial_t \phi^{\alpha} + i \partial_x \phi^{\alpha} = \psi^{\alpha} \sum_{\beta} (\psi^{*\beta} \phi^{\beta} + \phi^{*\beta} \psi^{\beta}) \qquad (2,12)$$

$$i \partial_t \psi^{\alpha} - i \partial_x \psi^{\alpha} = \phi^{\alpha} \sum_{\beta} (\psi^{*\beta} \phi^{\beta} + \phi^{*\beta} \psi^{\beta})$$

which, in light-cone coordinates, take the form

$$\partial_\eta \phi^\alpha = -i \psi^\alpha \sum_\beta \left(\psi^{*\beta} \phi^\beta + \phi^{*\beta} \psi^\beta \right)$$

$$\partial_\xi \psi^\alpha = -i \phi^\alpha \sum_\beta \left(\psi^{*\beta} \phi^\beta + \phi^{*\beta} \psi^\beta \right)$$

3. The Gross-Neveu Model and a Linear Differential Matrix Equation.

In this section, we show how the Gross-Neveu model is related to a very general matrix problem whose solutions belong to $GL(n,C)$, the general linear group defined on the field of complex numbers **C**. We introduce the concepts of equivalence class, gauge group and reduction group. We show how interesting systems, whose solutions belong to a subgroup of $GL(n,C)$, are defined by specifying the equivalence class, the gauge group and the reduction group. In particular, we indicate the relation between a gauge group, a reduction group, and the most general internal symmetry group of the Gross-Neveu model: the symplectic group, $Sp(2N,R)$.

3.1 The Problem and its Compatibility Condition

Consider a linear system of $2N \times 2N$ matrices satisfying:

$$\Psi_\xi = U\Psi \qquad (3,1a)$$

$$\Psi_\eta = V\Psi \qquad (3,1b)$$

where $\eta = \frac{t+x}{2}$, $\xi = \frac{t-x}{2}$ are the light-cone coordinates.

We assume that this system is compatible. Hence the mixed partial derivatives $\Psi_{\xi\eta}$ and $\Psi_{\eta\xi}$ must be equal. This leads to the relation.

$$U_\eta - V_\xi + [U, V] = 0 \qquad (3,2)$$

If no restriction is imposed upon U and V, the most general solution is trivial [6]

$$U = \Psi_\xi \Psi^{-1} \quad , \quad V = \Psi_\eta \Psi^{-1} \tag{3,3}$$

where Ψ is arbitrary matrix function of ξ and η.

3.2 Matrices Represented as Rational Functions of a Parameter λ.

Demanding that the matrices U and V be rational functions of a complex parameter λ transforms (3,2) into a non-trivial non-linear system of equations. We also require that U and V have the same numbers of poles and that the U-poles be situated to the opposite of the V-poles.

$$U(\lambda, \xi, \eta) = U_0(\xi, \eta) + \sum_{n=1}^{k} \frac{U_n(\xi, \eta)}{\lambda - a_n} \qquad (3,4a)$$

$$V(\lambda, \xi, \eta) = V_0(\xi, \eta) + \sum_{n=1}^{k} \frac{V_n(\xi, \eta)}{\lambda + a_n} \qquad (3,4b)$$

The problem can be stated in a more general manner (for example one might want to include double poles). But with respect to the number of poles and their relative location in U and V, the conditions imposed are necessary for defining a relativistically invariant problem [6]. In any case, equation (3,4) as stated is sufficient for our purpose. U and V are now, in addition to being functions of ξ and η, matrix functions of the parameter λ. System (3,1) should be compatible for any value of this parameter. We substitute equation (3,4) into equation (3,2) and we require that the coefficients of $1/(\lambda + a_n)$ and $1/(\lambda - a_n)$ vanish.

Recalling that

$$[(\lambda - a_n)(\lambda + a_m)]^{-1} = [(a_n + a_m)(\lambda - a_n)]^{-1} - [(a_n + a_m)(\lambda + a_m)]^{-1} \qquad (3,5)$$

we easily obtain the compatibility condition.

$$U_{0_\eta} - V_{0_\xi} + [U_0, V_0] = 0 \qquad (3,6)$$

$$U_{n\eta} + [U_n, \Phi_n] = 0, \quad V_{n\xi} + [V_n, \Psi_n] = 0 \qquad (3,7)$$

and

$$\Phi_n \equiv V_0 + \sum_{m=1}^{k} \frac{V_m}{a_n + a_m}$$

$$\Psi_n \equiv U_0 - \sum_{m=1}^{k} \frac{U_m}{a_n + a_m} \qquad (3,8)$$

In the next sections, we shall frequently refer to system (3,6) and (3,7).

3.3 Gauge Transformation

2K + 2 matrix functions determine U and V. However only 2K + 1 equations were obtained from the compatibility condition: K equations from the matrix coefficients of $1/(\lambda + a_n)$, K from $1/(\lambda - a_n)$ and one equation at the point $\lambda = \infty$. Hence there is one unused degree of freedom and it corresponds to some intrinsic invariance of the system.

Consider the new matrix function

$$\tilde{\psi} = g\,\psi \tag{3,9}$$

where g is an arbitrary non-degenerate matrix function of ξ and η. Such a transformation is called a gauge transformation.

Using equation(3,9) we rewrite system(3,1) in term of $\tilde{\psi}$ and this gives

$$\begin{aligned} \tilde{\psi}_\xi &= (gUg^{-1} - gg_\xi^{-1})\,\tilde{\psi} \\ \tilde{\psi}_\eta &= (gVg^{-1} - gg_\eta^{-1})\,\tilde{\psi} \end{aligned} \tag{3,10}$$

It suffices to define $\tilde{U} \equiv gUg^{-1} + g_\xi g^{-1}$ and $\tilde{V} \equiv gVg^{-1} + g_\eta g^{-1}$ to bring system (3,10) in the same form as system (3,1)

$$\tilde{\Psi}_\xi = \tilde{U}\,\tilde{\Psi}\ , \quad \tilde{\Psi}_\eta = \tilde{V}\,\tilde{\Psi} \tag{3,11}$$

The compatibility condition of system (3,11) has the same form as equations (3,6) and (3,7). It is obvious that we can define matrices

$\tilde{U}_0$, $\tilde{U}_n$, $\tilde{V}_0$, $\tilde{V}_n$ satisfying eq. (3,6) and (3,7). These matrices are related to the original ones through:

$$\tilde{U}_0 = g\, U_0\, g^{-1} + g_\xi g^{-1} \quad , \quad \tilde{U}_n = gU_n g^{-1} \tag{3,12}$$

and

$$\tilde{V}_0 = gV_0\, g^{-1} + g_\eta g^{-1} \quad , \quad \tilde{V}_n = gU_n g^{-1} \tag{3,13}$$

Also we note that Φ_n and Ψ_n transform according to the relations.

$$\tilde{\Phi}_n = g\, \Phi_n\, g^{-1} + g_\eta\, g^{-1} \tag{3,14}$$

$$\tilde{\Psi}_n = g\, \Psi_n\, g^{-1} + g_\xi\, g^{-1} \tag{3,15}$$

Systems expressed in different gauges may have drastically different aspects. The interest of the gauge transformation concept resides in the fact that we can group systems into equivalence classes. One then needs only to study one member of the class (usually the simplest one which can be specified through a judicious choice of the gauge). The set of all possible gauge transformations relating a solution of an equivalence class to another of the same class forms a group called the gauge group. Many different classes may have the same gauge group. Hence specifying the gauge group is not sufficient for determining the class. One might ask: how do we specify an equivalence class? The answer is twofold:

a) The gauge transformation that we consider, being λ - independent, cannot change the location of the poles a_n of U and V. Therefore the different U's and V's corresponding to the differents solutions *of a class must have the same set of poles $\{a_n\}$. The location of the poles partly specifies a class.*

b) From equation (3,7), we deduce that

$$\partial_\eta \, (U_n)^m = 0 \quad \text{and} \quad \partial_\xi \, (V_n)^m = 0 \quad m=1,\ 2,\ \cdots \tag{3,16}$$

Hence the normal Jordan form $\hat{U}_n$ ($\hat{V}_n$) of matrices U_m (V_m) depends only on ξ (η). $\hat{U}_n$ and $\hat{V}_n$ are the first integral of system (3,6) and (3,7). Equation (3,16) is gauge invariant therefore so is the normal Jordan form of U_n and V_n. Determining the matrices U_n, V_n gives a complete specification of the equivalence class.

Some gauges have special names. The canonical gauge is the gauge in which $\tilde{U}_0 = \tilde{V}_0 = 0$. We go to this gauge using

a transformation matrix g^0 solving the equations

$$\partial_\xi g^0 + V^0 g^0 = 0 \ , \quad \partial_\eta g^0 + V^0 g^0 = 0 \tag{3,17}$$

As we shall see, system (3,6) and (3,7) will be studied in the canonical gauge.

3.4 The Relativistically Invariant Spinor Problem

$U_n(\xi, \eta)$ and $\hat{U}(\xi)$ are related by a similarity transformation

$$U_n = \phi_n \hat{U}_n(\xi) \phi_n^{-1} \tag{3,18}$$

We want to find out what equation describes ϕ_n. First, we take the derivative of equation (3,18) with respect to η and we get

$$U_{n\eta} = \phi_{n\eta} \hat{U}_n(\xi) \phi_n^{-1} - \phi_n \hat{U}_n(\xi) \phi_n^{-1} \phi_{n\eta} \phi_n^{-1} \tag{3, 19}$$

$$= [\phi_{n\eta} \phi_n^{-1}, U_n]$$

$$= [\Phi_n, U_n]$$

Equation (3,19) is deduced with the use of equation (3,18) while equation (3,20) is equation (3,7) rewritten. U_n being general it follows that

$$\phi_{n\eta} = \Phi_n \phi_n \tag{3,21}$$

Similarly, we define ψ_n as the similarity transformation which relates a matrix $V_n(\xi, \eta)$ to its Jordan normal form $V_n(\eta)$.

$$V_n = \psi_n \hat{V}_n(\eta) \psi_n^{-1} \tag{3,22}$$

The equation that describes ψ_n is

$$\psi_{n\xi} = \Psi_n \psi_n \tag{3,23}$$

Ψ_n (in equation (3,23)) and Φ_n (in equation (3,21)) are defined by equation (3,8). Equation (3,21) and (3,23) can be rewritten in the form.

$$\nabla_\eta \phi_n = \sum_{m=1}^{k} \frac{\psi_m \nabla_m \psi_m^{-1}}{a_n + a_m} \phi_n$$

$$\nabla_\xi \psi_n = \sum_{m=1}^{k} \frac{\phi_m U_m \phi_m^{-1}}{a_n + a_m} \psi_n \tag{3,25}$$

(3,24)

where we have used equations (3,8), (3,18), (3,22) and defined $\nabla_\eta \equiv \partial_\eta - V_0$, $\nabla_\xi \equiv \partial_\xi - U_0$. $[\nabla_\eta, \nabla_\xi] = 0$ (from (3,6)).

Equation (3,24) and (3,25) have the form of a classical spinor field and are relativistically invariant. We call this system of equations the relativisitically invariant spinor problem or, in short, the spinor problem.

Under the transformation $\tilde{\phi}_n = h\phi_n$, $\tilde{\psi} = h\psi_n$, the system (3,24), (3,25) is form invariant:

$$\tilde{\phi}_{n\eta} = \tilde{\Phi}_n \tilde{\phi}_n, \quad \tilde{\psi}_{n\xi} = \tilde{\Psi}_n \tilde{\psi}_n \tag{3,26}$$

and, Φ_n and $\tilde{\Phi}_n$ [Ψ_n and $\tilde{\Psi}_n$] are related to each other by equation (3,14) [(3,15)] with h replacing g.

3.5 The Reduction Problem

In this subsection we adapt a section of the article [9], The Reduction Problem and the Inverse Scattering Method by A.V. Mikhailov, to suit our needs.

A reduction is the operation through which one imposes on U and V algebraic or differential constraints compatible with system (3,6) and (3,7). The consequence of a reduction is that the solutions to equation (3,1) belong to a subgroup of GL(N,C). The smaller the subgroup is, the deeper the reduction. We define the operators L_1 and L_2 such that

$$L_1\psi = \psi_\xi - U\psi = 0$$
$$L_2\psi = \psi_\eta - V\psi = 0 \qquad (3,27)$$

and $\tilde{L}_1$, $\tilde{L}_2$ satisfying

$$\tilde{\psi}\tilde{L}_1 = \tilde{\psi}_\xi + \tilde{\psi}\,U = 0$$
$$\tilde{\psi}\tilde{L}_2 = \tilde{\psi}_\eta + \tilde{\psi}\,V = 0 \qquad (3,28)$$

If $\psi(\xi, \eta, \lambda)$ is a solution of equation (3,27) then $\tilde{\psi} = \psi^{-1}$ obeys equation (3,28). Also we denote by $\{\psi(\lambda)\}$ a set of solutions to equation (3,27). As an example, consider the following constraint imposed on U and V.

$$U = -U^{tr}, \quad V = -V^{tr} \qquad (3,29)$$

Consequently $\tilde{\Psi} = \Psi^{tr}$ *satisfies equation (3,28) and we have*

$$\{\psi^{tr}(\lambda)\}^{-1} \in \{\psi(\lambda)\} \qquad (3,30)$$

Constraint (3,29) has led to the existence of automorphism (3,30). It can easily be shown that automorphism (3,30) leads to constraint (3,29). We denote this automorphism by

$$t_1: \psi(\lambda) \to \{\psi^{tr}(\lambda)\}^{-1} \in \{\psi(\lambda)\} \qquad (3,31)$$

Assuming that the poles of U and V are located on the real axis of the λ-plane, we can define the following automorphisms

$$h: \psi(\lambda) \to \hat{h}\,[\psi^{+}(h(\bar{\lambda}))]^{-1} \in \{\psi(\lambda)\} \qquad (3,32)$$

$$r: \psi(\lambda) \to \hat{r}\,[\bar{\psi}(r(\bar{\lambda}))] \in \{\psi(\lambda)\} \qquad (3,33)$$

$$t: \psi(\lambda) \to \hat{t}\,[\psi^{tr}(t(\lambda))]^{-1} \in \{\psi(\lambda)\} \qquad (3,34)$$

$$g: \psi(\lambda) \to \hat{g}\,[\psi(g(\lambda))] \in \{\psi(\lambda)\} \qquad (3,35)$$

where $\hat{h}$, $\hat{r}$, $\hat{t}$, $\hat{g}$ *are non-degenerate complex matrices of* ξ, η, λ *and* $h(\lambda)$, $r(\lambda)$, $t(\lambda)$, $g(\lambda)$ *are conformal self-mapping of the plane* λ.

We remark that automorphism (3,34) represents a generalization of automorphism (3,31). Equations (3,32), (3,33) and (3,35) also represent generalizations of fundamental automorphisms. Mappings (3,32) - (3,35) are reversible and associative and hence any subset of them gives rise to a group called the reduction group (G_R).

Automorphism (3,32) means that

$$L_{1,2}(h(\overline{\lambda}))\ \psi(h(\overline{\lambda})) = L_{1,2}(\lambda)\ \hat{h}[\psi^{+}(h(\overline{\lambda}))]^{-1} = 0 \tag{3,36}$$

from which we deduce that

$$-\hat{h}^{-1}\hat{h}_{\xi} + \hat{h}^{-1}\ U(\lambda)\ \hat{h} = U(h(\lambda)) \tag{3,37}$$

An analogous equation holds for V

Mappings (3,33) - (3,35) imply

$$-\hat{r}^{-1}\hat{r}_{\xi} + \hat{r}^{-1}\ U(\lambda)\ \hat{r} = \overline{U}\ (r(\overline{\lambda})) \tag{3,38}$$

$$-\hat{t}^{-1}\hat{t}_{\xi} + \hat{t}^{-1}\ U(\lambda)\ \hat{t} = -\ U^{tr}\ (t(\lambda)) \tag{3,39}$$

$$-\hat{g}^{-1}\hat{g}_{\xi} + \hat{g}^{-1}\ U(\lambda)\ \hat{g} = U\ (g(\lambda)) \tag{3,40}$$

The representation of the automorphism is, in general, gauge dependent. We examine the relation between the mappings in a gauge and the same mappings in another gauge. We assume a gauge transformation $f(\lambda, \xi, \eta)$[1] which relates two-solutions ψ and χ.

$$\psi = f\chi \tag{3,41}$$

[1]*This gauge transformation is slightly more general than those previously considered (it is λ - dependent). Assume that such a gauge transformation does not change $\{a_n\}$.*

Substitution of equation (3,41) into equation (3,32) leads to

$$\hat{h}[f^+(h(\bar{\lambda}))]^{-1}\ [\chi^+(h(\bar{\lambda}))]^{-1}\ \varepsilon\ \{\psi(\lambda)\}$$

Defining $\{\chi\ (\lambda)\}_f = \{f^{-1}\psi(\lambda)\}$, *we require that* $\hat{h}_f$ *must satisfy equation (3,32) in the gauge f. We find*

$$\hat{h}_f = f^{-1}(\lambda).\hat{h}(\lambda)[f^+(h(\bar{\lambda}))]^{-1} \quad (3,42)$$

Analogously,

$$\hat{r}_f = f^{-1}(\lambda)\ \hat{r}(\lambda)[\bar{f}(r(\bar{\lambda}))] \quad (3,43)$$

$$\hat{t}_f = f^{-1}(\lambda)\ \hat{t}(\lambda)[f^{tr}(t(\lambda))]^{-1} \quad (3,44)$$

$$\hat{g}_f = f^{-1}(\lambda)\ \hat{g}(\lambda)\ f[g(\lambda))] \quad (3,45)$$

Equations (3,42) - (3,45) means that the mappings of a reduction group commute with the gauge transformations of a gauge group.

The gauge group which preserves the representation of the reduction group is called the intrinsic gauge group, G_g^1

$$\hat{h}_f=\hat{h},\ \hat{r}_f=\hat{r},\ \hat{t}_f=\hat{t},\ \hat{g}_f=\hat{g}$$

Equations (3,37) - (3,40) are form-invariant under a transformation belonging to G_g^1. *That is, if* $f\ \varepsilon\ G_g^1$ *then U and* $\tilde{U}$ *both satisfy equation (3,37)-(3,40).*

3.6 Symplectic Symmetry and the Gross-Neveu Equations of Motion.

In this section, we finally reach the Gross-Neveu model by imposing additional constraints upon system (3,24), (3,25).

First, assume that we work in the canonical gauge and that V has only one pole situated at $\lambda=1$. System (3,1) then becomes

$$\partial_\xi \Psi = \frac{U_1}{\lambda - 1} \Psi \tag{3,46}$$

$$\partial_\eta \Psi = \frac{V_1}{\lambda + 1} \Psi \tag{3,47}$$

and the spinor system is transformed into

$$\partial_\eta \phi_1 = \frac{\psi_1 \hat{V}_1 \psi_1^{-1}}{2} \phi_1 \tag{3,48}$$

$$\partial_\xi \psi_1 = \frac{\phi_1 \hat{U}_1 \phi_1^{-1}}{2} \psi_1 \tag{3,49}$$

Next, we choose the reduction group and its representation. We will not use automorphims (3,32) and (3,35). We define the automorphism r' obtained from r by letting

$$\hat{r} = I, \; r(\bar{\lambda}) = \bar{\lambda} \tag{3,50}$$

The resulting mapping is

$$r' \; : \psi(\lambda) \rightarrow \bar{\psi}(\bar{\lambda}) \;\varepsilon\; \{\psi\;(\lambda)\} \tag{3,51}$$

Next, we define t' with the following $\hat{t}$ and $t(\lambda)$

$$\hat{t}=J=\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \quad t(\lambda) = \lambda \tag{3,52}$$

where I is the N X N unit matrix. We now work in the 2N X 2N matrix space. $\hat{r} = I$ means that $\hat{r}$ is equal to the 2N X 2N unit matrix.

The automorphism so defined is

$$t': \psi(\lambda) \to J\ [\psi^{tr}(\lambda)]^{-1}\ \varepsilon\ \{\psi(\lambda)\} \tag{3,53}$$

We require that the reduction group be composed of the identity automorphism i.

$$i\ :\ \psi(\lambda) \to \psi(\lambda)\varepsilon\{\psi(\lambda)\} \tag{3,54}$$

and the mapping t', r'. $q \in G_R$ implies $q \in \{i, t', r'\}$. From the mapping r', we deduce that U and V are real at real values of the parameter λ. From the mapping t', we deduce that U and V belong to the Lie algebra of the symplectic group:

$$J^{-1}\ U\ J = -U^{tr}, \quad J^{-1}V\ J = -V^{tr} \tag{3,55}$$

We ask that the gauge group be the intrinsic gauge group and equation (3,55) is then valid in any gauge. The gauge group is the set of all real matrices satisfying

$$f^{-1} = -\ J\ f^{tr}J \tag{3,56}$$

We recognize the fact that the gauge group that we have defined is the symplectic group. It follows that the solutions to system (3,46), (3,47), [or (3,48), (3,49)][1] *will now belong to the symplectic group.*

The matrices U and V have the form [8]

$$\begin{bmatrix} A & B \\ C & -A^{tr} \end{bmatrix} \tag{3,57}$$

where A, B, C are real N X N matrices and $B^{tr}=B$, $C^{tr}=C$.

We want to significantly simplify system (3,24, (3,25) and we specify the equivalence class by demanding that the rank of $\hat{U}_1$, *and* $\hat{V}_1$ *be in unity*

Let $\hat{U}_1 = \hat{V}_1$ *and choose*

$$A = C = 0 \text{ and } B = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & 0 \end{bmatrix} \tag{3,58}$$

We now explicity display equation (3,48) and (3,49) that we relate to the Gross-Neveu equations of motion. We rewrite equation (3,48) as

$$\partial_\eta \phi_1 = - \frac{1}{2} \psi_1 \hat{V}_1 J \psi_1^{tr} J \phi_1 \tag{3,59}$$

First notice that

[1]*Note that equation (3,48) is equation (3,47) at λ=1 and equation . (3,49) is equation (3,46) at λ=1*

First notice that

$$(\hat{V}_1 J)_{11} = 1 \ , \quad (\hat{V}_1 J)_{\delta\sigma} = 0 \quad (\delta,\sigma) \neq (1,1) \tag{3,60}$$

Hence

$$(\hat{V}_1 J\, \psi_1^{tr})_{\alpha\beta} = \sum_\rho (\hat{V}_1 J)_{\alpha\rho} \, (\psi_1^{tr})_{\rho\beta} = \begin{cases} (\psi_1^{tr})_{1\beta} = (\psi_1)_{\beta 1} & * \\ 0 \ \text{Else} \end{cases}$$

$$* \quad \alpha = 1 \quad \beta = 1, \cdots, 2N \tag{3,61}$$

Also

$$\begin{aligned} (J\,\phi_1)_{\beta\delta} &= \sum_\rho (J)_{\beta\rho} \, (\phi_1)_{\rho\delta} \\ &= -(\phi_1)_{N+\beta,\delta} \qquad \begin{array}{l} \beta = 1,\cdots,N \\ \delta = 1,\cdots,2N \end{array} \\ &= +(\phi_1)_{\beta-N,\delta} \qquad \begin{array}{l} \beta = N+1,\cdots,2N \\ \delta = 1,\cdots,2N \end{array} \end{aligned} \tag{3,62}$$

The next step is multiplying $(\hat{V}_1 J\, \psi_1^{tr})$ *and* $(J\phi_1)$. *We get*

$$\begin{aligned} [(\hat{V}_1 J \psi_1^{tr})(J\phi_1)]_{1\delta} &= \sum_{\beta=1}^{2N} (\hat{V}_1 J \psi_1^{tr})_{1\beta} (J\phi_1)_{\beta\delta} \\ &= \sum_{\gamma=1}^{N} [(\psi_1)_{\gamma+N,1} (\phi_1)_{\gamma\delta} \\ &\qquad - (\psi_1)_{\gamma 1} (\phi_1)_{\gamma+N,\delta}] \\ &\delta = 1,\cdots,2N \end{aligned} \tag{3,63}$$

All other elements being zero

The non-linear equation arises only on the elements of the first column of equation (3,39) (and the associated equation for $\partial_\xi \psi_1$)

$$(\partial_\eta \phi_1)_{\alpha 1} = \frac{1}{2} (\psi_1)_{\alpha 1} \sum_{\gamma=1}^{N} \{(\psi_1)_{\gamma 1} (\phi_1)_{\gamma+N,1} - (\psi_1)_{\gamma+N,1} (\phi_1)_{\gamma 1}\}$$

$$\alpha = 1, \cdots, 2N \tag{3,64}$$

Similarly

$$(\partial_\xi \psi_1)_{\alpha 1} = \frac{1}{2} (\phi_1)_{\alpha 1} \sum_{\gamma=1}^{N} \{(\phi_1)_{\gamma 1} (\psi_1)_{\gamma+N,1} - (\phi_1)_{\gamma+N,1} (\psi_1)_{\gamma 1}\}$$

$$\alpha = 1, \cdots, 2N \tag{3,65}$$

The fields $(\phi_1)_{\alpha 1}$ and $(\psi_1)_{\alpha 1}$ are two sets of real fields and they prepresent Majorana spinors. They can be combined so as to form Dirac spinors. We form complex fields.

$$\psi^\alpha = \frac{1}{2} \{(\psi_1)_{\alpha 1} + i (\psi_1)_{\alpha+N,1}\} \ , \ \phi^\alpha = \frac{-i}{2} \{(\phi_1)_{\alpha 1} + i (\phi_1)_{\alpha+N,1}\} \tag{3,66}$$

It is eay to verify that ψ^α and ϕ^α are described by the relations

$$\partial_\eta \phi^\alpha = -i\psi^\alpha \sum_\beta (\psi^{*\beta} \phi^\beta + \phi^{*\beta} \psi^\beta)$$

$$\partial_\xi \psi^\alpha = -i\phi^\alpha \sum_\beta (\psi^{*\beta} \psi^\beta + \phi^{*\beta} \psi^\beta) \tag{3,67}$$

which are the Gross-Neveu model equations of motion.

3.7 Symplectic notation

In this section, we briefly describe a notation by Neveu and Papanicolaou that gives simple equations of motion. This notation is analogous to the tensorial notation.

We define two contravariant vectors.

$$\begin{bmatrix} u^1 \\ \cdot \\ \cdot \\ \cdot \\ u^{2N} \end{bmatrix} \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} (\psi_1)_{11} \\ \cdot \\ \cdot \\ \cdot \\ (\psi_1)_{2N,1} \end{bmatrix} = \sqrt{2} \begin{bmatrix} Re\{\psi^1\} \\ \cdot \\ \cdot \\ Re\{\psi^N\} \\ Im\{\psi^1\} \\ \cdot \\ \cdot \\ Im\{\psi^N\} \end{bmatrix} \tag{3,69}$$

$$\begin{bmatrix} v^1 \\ \cdot \\ \cdot \\ \cdot \\ v^{2N} \end{bmatrix} \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} -\phi_{1,1} \\ \cdot \\ \cdot \\ -\phi_{N,1} \\ \phi_{N-1,1} \\ \cdot \\ \cdot \\ \phi_{2N,1} \end{bmatrix}, \begin{bmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_{2N} \end{bmatrix} = \sqrt{2} \begin{bmatrix} Re\{\phi^1\} \\ \cdot \\ Re\{\phi^N\} \\ Im\{\phi^1\} \\ \cdot \\ \cdot \\ Im\{\phi^N\} \end{bmatrix} \tag{3,69}$$

The metric tensor is

$$(\varepsilon)_{\alpha\beta} = (J)_{\alpha\beta} \quad \alpha, \beta = 1, \cdots, 2N$$
$$(\varepsilon)^{\alpha\beta} = (J^{tr})^{\alpha\beta} \tag{3,70}$$

Raising and lowering of indices follow the usual tensorial rules. Observe that $u^2 = v^2 = 0$, $u^\alpha v_\alpha = - u_\alpha v^\alpha$. We define

an invariant field $\sigma = uv \equiv u^\alpha v_\alpha$ (3,71)

The equations of motion then become

$$u,_\xi = -\sigma v \text{ and } v,_\eta = \sigma u \qquad (3,72)$$

4. Bäcklund Transformation

4.1 Bäcklund Transformation for N=1

This section will be descriptive rather than deductive. We present a Bäcklund transformation for N=1 and we will be content with showing that indeed it is the correct Bäcklund transformation.

First, following Neveu and Papanicolaou [3], we derive a mapping of the Gross-Neveu Model (N=1 only) into the sinh-Gordon equation.

We calculate an equation for $\sigma,_{\eta\xi}$

$$\sigma,_{\eta\xi} = (u,_{\eta}\nu),_{\xi} = u,_{\eta}\ \nu,_{\xi} + u,_{\eta\xi}\nu \tag{4,1}$$

Using the equations of motion, we obtain

$$u,_{\xi\eta} = -\sigma^2 u \tag{4,2}$$

Hence,

$$\sigma,_{\eta\xi} = u,_{\eta}\nu,_{\xi} - \sigma^3 \tag{4,3}$$

We define $\omega \equiv u,_{\eta}\nu,_{\xi}$. This is an invariant. For N=1, u and ν can be used as a basis in which other quantities can be expanded. We expand $u,_{\eta}$ and $\nu,_{\xi}$ in this basis.

Writing

$$u,_{\eta} = Au + B\nu \text{ and } u,_{\xi} = Cu + D\nu, \tag{4,4}$$

We easily find, successively taking "scalar"-products of equation (4,4) with u and ν, that

$$A = \frac{u,_{\eta}\nu}{\sigma} = \frac{\sigma,_{\eta}}{\sigma} \qquad B = \frac{uu,_{\eta}}{\sigma} = \frac{-h_1}{\sigma}$$
$$C = \frac{h_2}{\sigma} \qquad D = \frac{\sigma,_{\xi}}{\sigma} \tag{4,5}$$

$h_1 \equiv -uu,_{\eta}$ and $h_2 \equiv -\nu\nu,_{\xi}$ are the conserved energy-momentum densities: $h_1,_{\xi} = h_2,_{\eta} = 0.$

It follows from equation (4,4) and (4,5) that

$$\omega \equiv u,_{\eta}\nu,_{\xi} = \left(\sigma,_{\eta}\sigma,_{\xi} + h_1h_2\right)/\sigma \tag{4,6}$$

which if substituted into equation (4,1) leads to

$$\sigma\sigma,_{\eta\xi} - \sigma,_{\eta}\sigma,_{\xi} = h_1h_2 - \sigma^4 \tag{4,7}$$

If we introduce a new quantity Θ, $\sigma \equiv \exp \Theta$, then we have succeeded in mapping the Gross-Neveu model (N=1) into the sinh-Gordon equation since Θ satisfies

$$\Theta,_{\eta\xi} = -2 \sinh 2\Theta \; . \tag{4,8}$$

provided that we choose the conformal frame $h_1h_2=1$

The Bäcklund transformation for the sinh-Gordon equation is well known [5]. The integrability condition of the system.

$$\frac{1}{2}\left(\theta_\xi + \theta'_\xi\right) = -\alpha \sinh\left(\theta - \theta'\right)$$

$$\frac{1}{2}\left(\theta_\eta - \theta'_\eta\right) = \alpha^{-1} \sinh\left(\theta + \theta'\right) \qquad (4.9)$$

implies that both θ and θ' satisfy equation (4,8).

Substituting $\theta = \ln\sigma$ into equation (4,9), we obtain the Bäcklund transformation (B.T.) for the field σ.

$$\left[\frac{\sigma'}{\sigma}\right]_{,\xi} = \alpha\left(\sigma'^{2} - \sigma^{-2}\right) \qquad (4,10a)$$

$$(\sigma'\sigma)_{,\eta} = \frac{1}{\alpha}\left(\sigma^2 - \sigma'^{+2}\right) \qquad (4,10b)$$

Now consider the transformation

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \frac{1}{\sqrt{1+\gamma\sigma^2}} \begin{bmatrix} \sigma'/\sigma & \gamma\alpha/\sigma \\ -\alpha\sigma' & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \qquad (4,11)$$

where we have set $h_1 = \gamma$, $h_2 = 1/\gamma$

We will show that if σ and σ' satisfy the B.T. displayed above then equation (4,11) represents the B.T. in terms of the fundamental fields.

We take the ξ-derivative of u' and make use of equation (4,11)

$$u'_{\xi} = \frac{1}{\sqrt{1+\gamma\alpha^2}} \left[\left[\frac{\sigma'}{\sigma}\right]_{,\xi} u + \left[\frac{\sigma'}{\sigma}\right] u_{,\xi} \frac{-\gamma\alpha\sigma_{,\xi}\nu}{\sigma^2} + \frac{\gamma\alpha\,\nu}{\sigma} \right] \tag{4,12}$$

We introduce the B.T. for σ into (4,12) and, rearranging terms, we get

$$u'_{\xi} = \sigma' \left\{ \frac{(\alpha\sigma' u - \nu)}{\sqrt{1+\gamma\alpha^2}} + \frac{1}{\sqrt{1+\gamma\alpha^2}} \frac{\alpha}{\sigma^2} \left(-u + \gamma\sigma\nu_{\xi} - \gamma\,\nu(u\nu_{\xi})\right) \right\} \tag{4.13}$$

The factor in the large parenthesis belonging to the second term is equal to zero. Calling it A, it is very easy to show that $Au=A\nu=0$. Therefore A maps the basis $\{u,\nu\}$ into the null space and can only be equal to zero.

We are left with

$$u_{,\xi} = \sigma' \ \nu' \tag{4,14}$$

Similarly, it is possible to show $\nu_{,\eta} = -\sigma' u$, where $\sigma' = u'\nu'$. The new solution is defined in the same frame as the previous one. That is $h_1' = h_1 = \gamma, h_2' = h_2 = 1/\gamma$

4.2 Algebraic Bäcklund Transformation

We derive a purely algebraic Bäcklund transformation for the case N=1. We rewrite the B.T. in terms of the fundamental fields $\psi^{\alpha}, \phi^{\alpha}$

$$\begin{bmatrix} \psi^{\alpha'} \\ \phi^{\alpha'} \end{bmatrix} = \begin{bmatrix} (\sigma'/\sigma) & (i\gamma\alpha/\sigma) \\ (i\alpha\sigma') & (1) \end{bmatrix} \begin{bmatrix} \psi^{\alpha} \\ \phi^{\alpha} \end{bmatrix} \tag{4,15}$$

Next, we define a new field χ *through the change of variable* $\chi = C\psi$

$$\begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} (\gamma^{-1/4}) & (\gamma^{1/4}\,\sigma^{-1}) \\ (\gamma^{-1/4}) & (-\gamma^{1/4}\,\sigma^{-1}) \end{bmatrix} \begin{bmatrix} \psi^{\alpha} \\ \phi^{\alpha} \end{bmatrix} \tag{4,16}$$

Hence if we denote this ψ*-field B.T. by* $\psi' = B\psi$*, the B.T. for the field* χ *will be*

$$\chi' = [\,C'\;B\;C^{-1}]\;\chi = \tilde{B}\,\chi$$

where C' *relates* χ' *and* ψ' : $\chi' = C'\,\psi'$

$$\tilde{B} = \frac{1}{2\sqrt{1+\gamma\,\alpha^2}} \begin{bmatrix} \gamma^{-1/4} & \gamma^{1/4}\;\sigma'^{-1} \\ \gamma^{-1/4} & \gamma^{1/4}\;\sigma'^{-1} \end{bmatrix} *$$

$$* \begin{bmatrix} \sigma'/\sigma & i\,\gamma\alpha/\sigma \\ i\alpha\sigma' & 1 \end{bmatrix} \begin{bmatrix} \gamma^{1/4} & \gamma^{1/4} \\ \gamma^{-1/4}\sigma & -\gamma^{-1/4}\sigma \end{bmatrix}$$

$$= \frac{1}{\sqrt{1+\gamma\alpha^2}} \begin{bmatrix} \sigma'/2\sigma + \sigma/2\sigma' + i\gamma^{1/2}\alpha, & \sigma'/2\sigma - \sigma/2\sigma' \\ \sigma'/2\sigma - \sigma/2\sigma' & , \sigma'/2\sigma + \sigma/2\sigma' - i\,\alpha\gamma^{1/2} \end{bmatrix} - \alpha\gamma^{1/2} \quad (4,17)$$

We define $\tilde{\tilde{B}} = \sqrt{1+\gamma\alpha^2}\ \tilde{B}$ and we observe the system

$$\tilde{\tilde{B}}\chi = \sqrt{1+\gamma\alpha^2}\ \chi' \quad (4,18)$$

At $\gamma\alpha = i/\chi$, $\tilde{\tilde{B}}\chi = 0$ defines a non-trivial system since the determinant of $\tilde{\tilde{B}}$ is zero

It follows that

$$Cosh(\Theta'-\Theta) = \frac{1+\lambda^2}{1-\lambda^2} = \frac{\sigma'}{\sigma} + \frac{\sigma}{\sigma'}$$

$$Sinh(\Theta'-\Theta) = \frac{-2\lambda}{1-\lambda^2} = \frac{\sigma'}{\sigma} - \frac{\sigma}{\sigma'} \quad (4,11)$$

which is the algebraic Bäcklund transformation.

This algebraic B.T. can be written in a simple form in terms of σ and σ':

$$\sigma' = \frac{(1-\lambda)}{(1+\lambda)}\,\sigma \quad (4,12)$$

5. The Infinite Set of Conservation Laws

5.1 A Fundamental Conservation Law

We also opt for a descriptive approach in this section. We will show that the conservation law.

$$(u'u,_{\eta}),_{\xi} = \frac{\gamma\alpha^2}{\sqrt{1+\gamma\alpha^2}}(\sigma'\sigma),_{\eta}$$

(5, 1)

is compatible with the non-algebric Bäcklund transformation displayed in the preceding section. Using equation (3,11), we form the product $(u'u,_{\eta})$:

$$(u'u,_{\eta}) = \frac{1}{\sqrt{1+\gamma\alpha^2}}\left[\frac{-\sigma'\gamma}{\sigma} - \frac{\gamma\alpha\sigma,_{\eta}}{\sigma}\right]$$

(5, 2)

We take the ξ-derivative of equation (5,2) and we use equation (4,10a) for eliminating $(\sigma'/\sigma),_{\xi}$

$$(u'u,_{\eta}),_{\xi} = \frac{\alpha\gamma}{\sqrt{1+\gamma\alpha^2}}\left[-(\sigma'^2 - \sigma^{-2}) - \frac{\sigma,_{\xi\eta}}{\sigma} + \frac{\sigma,_{\eta}\sigma,_{\xi}}{\sigma^2}\right]$$

(5,3)

Then we make use of equation (4,7) to get rid of $\sigma\,\sigma,_{\eta\xi} - \sigma,_{\eta}\,\sigma,_{\xi}$. It should be kept in mind that equation (3,7) is valid only for N=1. We also set $h_1 h_2 = 1$.

It follows that

$$(u^1 u_{,\eta})_{,\xi} = \frac{\alpha\gamma}{\sqrt{1+\gamma\alpha^2}} \left[\sigma^2 - \sigma'^2\right]$$

$$= \frac{\alpha\gamma^2}{\sqrt{1+\gamma\alpha^2}} (\sigma' \sigma)_{,\eta} \qquad (5,4)$$

where the last step was obtained using equation (3,10b)

5.2 The Infinite Set of Conservation Laws

In this subsection, we derive an infinite set of conservations laws making use of equation (5,4) and the Bäcklund transformation (4,10 and (4,11). The Bäcklund transformation introduces a free parameter α into the new solution u', v', σ'. Since these relations are valid for all values of the parameter α, we can expand these functions in powers of α. Substituting the expressions thus obtained into equation (5,4), we obtain the infinite set of conservation laws by equating terms with the same powers of α as $\alpha \to 0$.

We assume

$$\sigma' = \sum_{m=0}^{\infty} \sigma'_m \, \alpha^m \tag{5,5}$$

and substitute this expression into (4,10b)

$$\sum_{m=1}^{\infty} (\sigma'_m \sigma)_{,\eta} \; \alpha^m = \sigma^2 - \{ \sum_{m=0}^{\infty} (\sum_{n=0}^{m} \sigma'_{m-n} \, \sigma'_n) \; \alpha^m \} \tag{5,6}$$

The zeroth order term immediately yields

$$\sigma'_0 = \sigma \tag{5,7}$$

while we get from the first order term

$$\sigma'_1 = - \frac{1}{2\sigma'_0} \quad (\sigma'_0 \, \sigma)_{,\eta} = - \sigma_{,\eta} \tag{5,8}$$

The second and third order terms are

$$\sigma'_2 = \frac{1}{2\sigma'_0}\left\{ -(\sigma'_1\,\sigma),_n - (\sigma'_1)^2 \right\} = \frac{\sigma,_{nn}}{2} \qquad (5,9)$$

$$\sigma'_3 = \frac{1}{2\sigma'_0}\left\{ -(\sigma'_2\,\sigma),_n - 2\sigma'_2\,\sigma'_1 \right\}$$

$$= -\frac{1}{4}\,\sigma,_{nnn} + \frac{\sigma,_n\,\sigma,_{nn}}{4\sigma} \qquad (5,10)$$

The recursion relation is:

$$\sigma'_m = \frac{1}{2\sigma'_0}\left\{ -(\sigma'_{m-1}\,\sigma),_n - \sum_{n=1}^{m}(\sigma'_{m-n}\,\sigma'_n) \right\} \qquad (5,11)$$

The next step on our way is the derivation of the Taylor series for u'.

We define

$$u' = \sum_{m=0}^{\infty} u'_m\alpha^m \text{ and } u'u,_n = \sum_{m=0}^{\infty}\Omega_m\alpha^m \qquad (5,12)$$

From equation (4,11) that we use once again we have.

$$u' = \sum_{m=0}^{\infty} u'_m\alpha^m = \frac{1}{\sqrt{1+\gamma\alpha^2}}\left[\left(\sum_m \sigma'_m\alpha^m\right)\frac{u}{\sigma} + \frac{\gamma\alpha v}{\sigma}\right] \qquad (5,13)$$

We expand $\frac{1}{\sqrt{1+\gamma\alpha^2}}$ in powers of $\gamma\alpha^2$

$$\left(1+\gamma\alpha^2\right)^{1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{(2n)!\; 2^n} \left(\gamma\alpha^2\right)^n$$
$$\equiv \sum_{n=0}^{\infty} A_n \left(\gamma\alpha^2\right)^n \qquad (5,14)$$

where $(2n-1)!! = (2n-1)(2n-3) \cdots 5\cdot 3\cdot 1$

and $A_0 = 1$, $A_1 = -\frac{1}{2}$, $A_2 = \frac{3}{8}$

Inserting equation (5,14) into equation (5,13) leads to

$$\sum_n u'_n \alpha^n = \sum_{p=0}^{\infty} \left(\sum_{m=0}^{[p/2]} \gamma^m A_m \sigma'_{p-m} \frac{u}{\sigma} \right) \alpha^p$$
$$+ \sum_{m=0}^{\infty} \left(A_m \gamma^{m+1} \frac{\nu}{\sigma} \right) \alpha^{2m+1} \qquad (5,15)$$

where $[p/2]$ means the largest integer smaller or equal to $P/2$: $p-2m \geq 0$

The zeroth order term yields

$$u'_0 = \frac{\sigma'_0}{\sigma} u = u \;, \quad \Omega_0 = u'_0 \, u,_n = h_1 \qquad (5,16)$$

The first order term gives

$$u'_1 = \frac{\sigma'_1}{\sigma} u + \frac{\gamma\nu}{\sigma} = -\frac{\sigma,_\eta}{\sigma} u + \frac{\gamma\nu}{\sigma}, \quad \Omega_1 = \frac{\sigma,_\eta \gamma}{\sigma} - \frac{\gamma\sigma,_\eta}{\sigma} = 0 \tag{5,17}$$

The second order term is

$$u'_2 = A_0 \sigma'_2 \frac{u}{\sigma} + \gamma A_1 \sigma'_0 \frac{u}{\sigma} = \frac{\sigma,_{\eta\eta}}{2} \frac{u}{\sigma} - \frac{\gamma\sigma u}{2\sigma} \tag{5,18}$$

$$\Omega_2 = -\frac{\gamma\sigma,_{\eta\eta}}{2\sigma} + \frac{\gamma^2}{2}$$

The third

$$u'_3 = A_0\sigma'_3 \frac{u}{\sigma} + \gamma A_1\sigma'_1 \frac{u}{\sigma} + \gamma^2 A_1 \frac{\nu}{\sigma}$$

$$= -\frac{1}{4} \sigma,_{\eta\eta\eta} \frac{u}{\sigma} + \frac{1}{4} \frac{\sigma,_{\eta\eta}\sigma,_\eta u}{\sigma^2} + \frac{\gamma \sigma,_\eta u}{2\sigma} - \frac{\gamma^2\nu}{2\sigma}.$$

$$\Omega_3 = \frac{\gamma\sigma,_{\eta\eta\eta}}{4\sigma} - \frac{\gamma\sigma,_{\eta\eta}\sigma,_\eta}{4\sigma^2} - \frac{\gamma^2\sigma,_\eta}{2\sigma} + \frac{\gamma^2\sigma,_\eta}{2\sigma}.$$

$$= \frac{\gamma}{4} \left\{ \frac{\sigma,_{\eta\eta\eta}}{\sigma} - \frac{\sigma,_{\eta\eta}\sigma,_\eta}{\sigma^2} \right\} \tag{5,19}$$

We next substitute equation (5,5) and (5,12) into equation (5,4) and we obtain.

$$\sum_{m=0}^{\infty} (\Omega_m)_{,\xi}\, \alpha^m = \gamma \sum_{p=0}^{\infty} \{ \sum_{q=0}^{[p/2]} \gamma^q A_q \,(\sigma'_{p-q}\, \sigma)_{,\eta} \}\, \alpha^{p+2} \qquad (5,20)$$

Equating equal powers of α on each side of equation (5,20) gives us an infinite set of conservation laws.

For $m=0$, we have

$$(\Omega_0)_{,\xi} = h_{1,\xi} = 0 \qquad (5,21)$$

which expresses conservation of energy - momentum (along with $h_{\overline{2},\eta} = 0$)

For $m = 1$, $(\Omega_1)_{,\xi} = 0$ is trivially satisfied since $\Omega_1 = 0$

For $m>2$

$$(\Omega_m)_{,\xi} = \{ \sum_{q=0}^{[m/2-1]} \gamma^{q+1} A_q\, \sigma'_{m-q-2}\sigma \}_{,\eta} \qquad (5,22)$$

which are the non-trivial conservation laws

For $m=2$

$$(\Omega_2)_{,\xi} = (-\frac{\gamma\sigma_{,\eta\eta}}{2\sigma})_{,\xi} = \gamma\, (\sigma^2)_{,\eta}\cdot(\frac{\sigma_{,\eta\eta}}{\sigma})_{,\xi} = -\,(2\sigma^2)_{,\eta} \qquad (5,23)$$

For m=2

$$(\Omega_2)_{,\xi} = \left(-\frac{\gamma\sigma_{,\eta\eta}}{2\sigma}\right)_{,\xi} = \gamma\,(\sigma^2)_{,\eta} \quad \left(\frac{\sigma_{,\eta\eta}}{\sigma}\right)_{,\xi} = -\,(2\sigma^2)_{,\eta} \tag{5,23}$$

and for m=3

$$\left(\frac{\sigma_{,\eta\eta\eta}}{4\sigma} - \frac{\sigma_{,\eta\eta}\sigma_{,\eta}}{4\sigma^2}\right)_{,\xi} = \left(-\sigma_{,\eta}\sigma - \frac{\gamma\sigma^2}{2}\right)_{,\eta} \tag{5,24}$$

We note the fact that similar relations hold with $u \leftrightarrow v$, $\xi \leftrightarrow \eta$.

Even though we relied heavily throughout this section on results valid only for N=1, the conservation laws that we have found hold for any N. For more information on this topic we refer the reader to Neveu and Papanicolaou's article [5].

6. Integrability of the Matrix Problem

6.1 The Local Vesture Method

In this section we describe a general method for solving system (3,6), (3,7). It was first introduced by Zakharov and Shabat [7]. Contrary to the inverse scattering method which is non-local (the Gel'fand-Levitan equation, an integral equation, is solved) this method is local. Zakharov and Mikhailov have shown the equivalence to the inverse scattering method for some cases [6].

Knowing one particular solution (usually chosen to be the vacuum solution) we obtain non-trivial solutions from it. The vacuum solution is said to be vested hence the name vesture method.

The solution is obtained by solving the Riemann problem that we describe in the next sections.

6.1.1 The Regular Riemann Problem

We quote Zakharov and Mikhailov who give a concise description of the problem [6]:

"Assume that in the complex plane of the variable λ there is given a contour Γ and on it an $N{\times}N$ matrix-function $G(\lambda)$ without singularities, but which in general does not admit an analytic continuation off the contour. We are required to find two matrix functions $\chi_1(\lambda)$, analytic inside the contour, and $\chi_2(\lambda)$, analytic outside the contour such that on the contour.

$$X_2(\lambda)\ X_1(\lambda) = G(\lambda) \qquad (6,1)$$

Here X_2, X_1 and G also depend on ξ, n.

Under the transformation $X_1 \to g^{-1}X$, $X_2 \to X_2 g$, equation (6,1) remains valid. The Riemann problem is said to be regular if $\det X_{1,2} \neq 0$ within their domain of analyticity. To obtain a unique solution, we must set the normalization, that is, the value of X_1 or X_2 at one point in the λ plane. When $X_1(\infty) = 1$, the normalization is said to be canonical.

In section (6.2) we shall show how to solve the Riemann problem for the case that concerns us.

6.1.2 Proliferation of solutions;

We show how to obtain solutions for the system under study with the help of the solution to the Riemann problem. Given a function $G_0(\lambda)$ defined on a contour Γ, we form the new function

$$G(\xi, n, \lambda) = \psi^0(\xi, n, \lambda)\, G_0(\lambda)\, \{\psi^0(\xi, n, \lambda)\}^{-1} \qquad (6,2a)$$

and ψ satisfies

$$\psi^0_\xi = U^0\psi^0\ ,\ \psi^0_\eta = V^0\psi^0 \qquad (6,2b)$$

and $G_0(\lambda)$ is independent of ξ, η.

Assume that the Riemann problem for G is solved. We differentiate relation [2] with respect to ξ and η, and we obtain

$$G_\xi(\lambda) = X_{2\xi} X_1 + X_2 X_{1\xi} = \{U^0\psi^0\} G_0(\lambda) \{\psi^0\}^{-1} - \psi^0 G_0(\lambda) \{\psi^0\}^{-1} U$$

$$= U^0 X_2 X_1 - X_2 X_1 U \tag{6,3}$$

$$G_\eta(\lambda) = X_{2\eta} X_1 + X_2 X_{1\eta} = V^0 X_2 X_1 - X_2 X_1 V \tag{6,4}$$

It is possible to define two functions U and V analytically continued from the contour Γ onto the entire complex λ-plane.

Let

$$U \equiv (X_{1\xi} + X_1 U^0) \{X_1\}^{-1} = - X_1 (\partial_\xi - U^0) \{X_1\}^{-1}$$

$$= \{X_2\}^{-1} (X_{2\xi} - U X_2) \tag{6,5}$$

$$V \equiv (X_{1\eta} + X_1 V^0) \{X_1\}^{-1} = - X_1 (\partial_\eta - V^0) \{X_1\}^{-1}$$

$$= X_2 (X_{2\eta} - V^0 \cdot X_2) \tag{6,6}$$

The second line of equations (6,5) and (6,6) were obtained using equations (6,3) and (6,4). No poles other than those of U_0 [V_0] are *present in (6,5) [(6,6)] since we deal with a regular Riemann problem*[1].

From (6,5) and (6,6), we observe that X_1 *is subject to*

$$X_{1_\xi} = U X_1 - X_1 U^0 \tag{6,7a}$$

$$X_{1_\eta} = V X_1 - X_1 V^0 \tag{6,7b}$$

If we set $X_1 \equiv \psi \{\psi^0\}^{-1}$, *we define a function* ψ *obeying the system.*

$$\psi_\xi = U\psi \qquad \psi_\eta = V\psi \tag{6,8}$$

Hence solving the Riemann problem allows us to find a new solution to system (3,1). We assume that U, V, U^0, V^0 *are rational functions of the parameter* λ *(ref: eq (3,4))*

Substituting the explicit formula for U, U_0 *into equation (6,7), we get*

$$\partial_\xi X_1 = \left[U_0 + \sum_{n=1}^{K} \frac{U_n}{\lambda - a_n} \right] X_1 - X_1 \left[U_0^0 + \sum_{n=1}^{K} \frac{U_n^0}{\lambda - a_n} \right] \tag{6,9}$$

Defining $g \equiv X_1(\lambda = \infty)$, $X_n \equiv X_1(\lambda = a_n)$, $\tilde{X}_n = \tilde{X}_1(\lambda = -a_n)$ *we easily derive*

[1]*For the Riemann problem with zeros (section 6.1.3) this is still true but it is an imposed condition.*

the relation between U_0^0, U_n^0 and U_0, U_n:

(1) Letting $\lambda \to \infty$ in (6,9) (and a similar equation for $\partial_\eta X_1$), we get

$$U_0 = g_\xi\, g^{-1} + g\, U_0^0\, g^{-1},\ V_0 = g_\eta\, g^{-1} + g V_0^0\, g^{-1} \tag{6,10}$$

(2) Let $\lambda \to a_n$ in (6,9). This gives

$$U_n = X_n\, U_n^0\, X_n^{-1}\ ,\ \partial_\eta\, X_n = \Phi_n X_n - X_n \Phi_n^0 \tag{6,11a}$$

where Φ_n, Φ_n^0 are defined in (2,8)

Similarly $\lambda \to -a_n$ in (6,7b) leads to

$$V_n = \tilde{X}_n\, V_n^0\, \tilde{X}_n^{-1}\ ,\quad \partial_\xi \tilde{X}_n = \Psi_n \tilde{X}_n - \tilde{X}_n \Psi_n^0 \tag{6,11b}$$

Usually the vacuum solution is defined in the canonical gauge. Canonical normalization defines a new solution also in the canonical gauge.

6.1.3 The Riemann Problem with Zeroes in the Case G=1.

Not all solutions of system (3,1) can be found by means of the regular Riemann problem. X_2 is said to have a simple zero at the point $\lambda=\lambda_n$ if X_1 has a simple pole at that point. Working in the canonical normalization X_1 takes the form

$$X_1 = I + \sum \frac{A_n}{\lambda - \lambda_n} \tag{6,12}$$

Equations (3,64) and (3,65) described real fields and we immediately impose upon X_1 that it be real at real λ

$$X_1(\lambda \in R) = \overline{X}_1(\lambda \in R)$$

X_1 is then restricted to have the form

$$X_1 = I + \sum_{n=1}^{n} \left(\frac{A_n}{\lambda - \lambda_n} + \frac{\overline{A}_n}{\lambda - \overline{\lambda}_n} \right) \tag{6,13}$$

where $\overline{A}$ denotes complex conjugation.

We will solve the Riemann problem with the assumption that $G \equiv 1$. Obviously, in such a case $X_2 = X_1^{-1}$. The solutions that we find will be soliton solution. We will also immediately demand that X_1

should belong to the symplectic group. It is required to leave invariant the form.

$$J \stackrel{\circ}{=} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

$$X_1^{tr} J X_1 = J \quad , \quad X^{-1} = J^{-1} X^{tr} J \tag{6,14}$$

The poles of X_1 should not coincide with the poles of U_0, V_0. Moreover the only poles of $U(V)$ should still be the poles of $U_0(V_0)$. The *substitution of (6,13) into equations (6,5) [(6,6)], defining $U(V)$, leads us to the conclusion that $U(V)$, in addition to the poles of $U_0[V_0]$, might have simple and double poles $\lambda=\lambda_n$, $\bar{\lambda}_n$.*

Requiring the absence of poles in $X^{-1}X = I$ allows to determine a system of equation for finding X. By demanding zero residue at the first as well as the second order poles of $U(V)$ at $\lambda=\lambda_n$, we can solve this system uniquely.

6.2 Solution to the Riemann Problem for the Symplectic Group (G≡1).

6.2.1 A System of Equation for X.

We now impose the absence-of-pole condition to $XX^{-1}=I$ and we obtain a set of matrix equations for An.

We first write $X^{-1}X=I$ in full.

$$\left[I+\sum_n\left(\frac{A_n}{\lambda-\lambda_n}+\frac{\bar{A}_n}{\lambda-\bar{\lambda}_n}\right)\right]J\left[I+\sum_n\left(\frac{A_n^{tr}}{\lambda-\lambda_n}+\frac{A_n^{+}}{\lambda-\bar{\lambda}_n}\right)\right]J^{-1}=$$

$$=I+\sum_n\frac{A_n}{(\lambda-\lambda_n)}J\left[I+\sum_{m\neq n}\frac{A_m^{tr}}{(\lambda-\lambda_m)}+\sum_m\frac{A_m^{+}}{(\lambda-\bar{\lambda}_m)}\right]J^{-1}$$

$$+\sum_n\frac{\bar{A}_n}{(\lambda-\bar{\lambda}_n)}J\left[I+\sum_m\frac{A_m^{tr}}{(\lambda-\lambda_m)}+\sum_{m\neq n}\frac{A_m^{+}}{(\lambda-\bar{\lambda}_m)}\right]J^{-1}$$

$$+\sum_n\frac{1}{(\lambda-\lambda_n)}\left[I+\sum_{m\neq n}\frac{A_m}{(\lambda-\lambda_m)}+\sum_m\frac{\bar{A}_m}{(\lambda-\bar{\lambda}_m)}\right]JA_n^{tr}J^{-1}+\ldots$$

$$\ldots + \sum_n \frac{1}{(\lambda - \bar{\lambda}_n)} \left[I + \sum_m \frac{A_m}{(\lambda - \lambda_m)} + \sum_{m \neq n} \frac{\bar{A}_m}{(\lambda - \bar{\lambda}_m)} \right] J A_n^{tr} J^{-1}$$

$$+ \sum_n \left[\frac{1}{(\lambda - \lambda_n)^2} A_n J A_n^{tr} J^{-1} + \frac{1}{(\lambda - \bar{\lambda}_m)^2} \bar{A}_n J A_n^{+} J^{-1} \right]$$

(6,15)

Then, we define

$$\tau_n \equiv I + \sum_{m \neq n} \frac{A_m}{(\lambda_n - \lambda_m)} + \sum_m \frac{\bar{A}_m}{(\lambda_n - \bar{\lambda}_m)}$$

(6,16)

Requiring the absence of simple and second order poles at $\lambda = \lambda_n$, we obtain (at $\lambda = \bar{\lambda}$ we simply get the complex conjugate equation).

$$A_n J A_n^{tr} = 0$$

(6,17a)

$$A_n J \tau_n^{tr} + \tau_n J A_n^{tr} = 0$$

(6,17b)

We want to find a solution to equation (6,17) consistent with the requirement that U and V should not have poles at $\lambda = \lambda_n$, $\bar{\lambda}_n$. We write A_n in a factorized form

$$A_n = M_n F_n^{+}$$

(6,18)

where M_n, F_n are rectangular matrices made up of 2N lines and K_n columns with $K_n < 2N$. For the sake of clarity assume that matrices F_n are known (we will show how to easily get a consistent solution) and M_n are unknown.

Substituting (6,18) into (6,17a)

$$M_n F_n^+ J \bar{F}_n M_n^{tr} = 0 \tag{6,19}$$

we observe that if

$$F_n^+ J \bar{F}_n = 0 \tag{6,20}$$

Then equation (6,17a) is satisfied independently of M_n. It is possible to determine F_n without knowing M_n. We now tackle equation (6,17b) with the help of equation (6,20). Substitution of the latter into the former leads to

$$M_n (F_n^+ J \tau_n^{tr}) + (\tau_n J \bar{F}_n) M_n^{tr} = 0 \tag{6,21}$$

Note that it suffices that $F_n^+ J F_n^{tr} = M_n^{tr}$ (since $J^{tr} = -J$) to satisfy the equation. However this may not be consistent with the absence-of-pole requirement or U and V. The most general solution is obtained by letting.

$$\tau_n J^{tr} \bar{F}_n = M_n \alpha_n \tag{6,22}$$

where α_n is some $K_n X K_n$ matrix to be determined consistently. For equation (6,21) to be satisfied, we must also have

$$\alpha_n^{tr} = \alpha_n \tag{6,23}$$

Writing equation (6,22) in full, we find that we have obtained a system of linear equations for M_n

$$J\bar{F}_n + \sum_{m \neq n} \frac{M_m}{(\lambda_n - \lambda_m)} (F_m^{+} J \bar{F}_n) + \sum_m \frac{\bar{M}_m (F_m^{tr} J \bar{F}_n)}{(\lambda_n - \bar{\lambda}_m)} = - M_n \alpha_n \tag{6,24}$$

6.2.2 Absence of double pole in U and V.

It is possible to find a solution F_n to equation (6,20) consistent with the absence of double pole in U and V at $\lambda = \lambda_n$
We recall that U satisfies

$$U = - X(\partial_\xi - U^0) X^{-1} \tag{6,25a}$$

and V, $$V = -X(\partial_\eta - V^0) X^{-1} \tag{6,25b}$$

We insert expression (6, 13) for X in (6,25a)

$$U = -\left[I + \sum_n \left(\frac{A_n}{(\lambda - \lambda_n)} + \frac{\bar{A}_n}{(\lambda - \bar{\lambda})} \right) \right] (\partial_\xi - U^0) *$$

$$* J \left[I + \sum_n \left(\frac{A_n^{tr}}{(\lambda - \lambda_n)} + \frac{A_n^+}{(\lambda - \bar{\lambda}_n)} \right) \right] J^{-1} \qquad (6,26)$$

Picking up terms in $1/(\lambda-\lambda_n)^2$ and requiring that the residue be zero at $\lambda=\lambda_n$, we obtain

$$A_n (\partial_\xi - U^0) \Big|_{\lambda=\lambda_n} J A_n^{tr} = 0$$

Define $D(\lambda_n) \equiv (\partial_\xi - U^0)\Big|_{\lambda = \lambda_n}$, $\tilde{D}(\lambda_n) = (\partial_n - V^0)\Big|_{\lambda = \lambda_n}$.

An being a factorized matrix, we require that

$$F_n^+ D(\lambda_n) J \bar{F}_n = 0 \qquad (6,28)$$

This represents an additional constraint on F_n and must be compatible with equation (6,20). Equation (6,20) and (6,28) are solved consistently if we assume

$$D(\lambda_n)\, J\, \overline{F}_n = J\, \overline{F}_n\, \beta_n \qquad (6,29)$$

where β_n is some ξ, η - dependent matrix. Substituting (6,29) into (6,28), the latter is identically satisfied due to (6,20).

Similarly one obtains

$$F_n^+\, \tilde{D}(\lambda_n)\, J\, F_{n'} = 0 \qquad (6,30)$$

which is solved by assuming

$$\tilde{D}(\lambda)\, J\, \overline{F}_n = J\, \overline{F}_n\, \tilde{\beta}_n \qquad (6,31)$$

Of course, equation (6,29) and (6,31) must satisfy the compatibility condition

$$\partial_{\xi\eta} J\, \overline{F}_n = \partial_{\eta\xi}\, J\, \overline{F}_n \qquad (6,32)$$

Which when written in full is

$$\partial_{\xi\eta} (J\, \bar{F}) = U^0_\eta\, J\, \bar{F}_n + U^0\, J\, \partial_\eta\, \bar{F}_n + J\, (\partial_\eta\, \bar{F}_n)\, \beta_n + J\, \bar{F}_n\, (\partial_\eta\, \beta_n)$$

$$= V^0_\xi\, J\, \bar{F}_n + V^0\, J\, \partial_\xi\, \bar{F}_n + J\, (\partial_\xi\, \bar{F}_n)\, \tilde{\beta}_n + J\, \bar{F}_n\, (\partial_\xi\, \tilde{\beta}_n) \qquad (6,33)$$

We use equations (6,29), (6,31) to eliminate $\partial_\eta \bar{F}_n$ and $\partial_\xi \bar{F}_n$ from equation (6,33). Two terms drop on each side due to the compatibility condition of the ψ^0 - system. Then $U^0\, J\, F_n\, \tilde{\beta}_n + V_0\, J\, \bar{F}_n\, \beta_n$ cancels on each side and we are left with

$$\partial_\xi\, \tilde{\beta}_m - \partial_\eta\, \beta_m + [\, \beta_m, \tilde{\beta}_m\,] = 0 \qquad (6,34)$$

Since no other constraint is imposed on this equation its solution is trivial [6]

$$\beta_m = g_{m\xi}\, g_m^{-1} \quad , \quad \tilde{\beta}_m = g_{m\eta}\, g_m^{-1} \qquad (6,35)$$

where g_m is any non-degenerate $K_n \times K_n$ matrix. Solving equation (6,29) and (6,31) is greatly simplified if we note that under the transformation

$$F_n \to F_n f_n^{+} \qquad M_n \to M_n f_n^{-1} \tag{6,36}$$

where f_n is an arbitrary non-degenerate matrix, A_n is not changed. This freedom in the choice of f_n reflects the ambiguity of the factorization of A_n. Hence choosing a particular β_n and $\tilde{\beta}_n$, which indirectly determines a given factorization, will not affect the final result as long as F_n satisfies eq. (6,29) and (6,31) with this choice[1]. We choose simple β_n, $\tilde{\beta}_n$ to simplify our task

$$\beta_n = \tilde{\beta}_n = 0 \tag{6,37}$$

We are left with the following equations for F_n

$$D(\lambda_n)\, J\, \overline{F}_n = \tilde{D}(\lambda_n)\, J\, \overline{F}_n = 0 \tag{6,38}$$

It suffices to note that

$$D(\lambda_n)\, \psi^0(\xi, \eta, \lambda_n) = 0$$
$$\tilde{D}(\lambda_n)\, \psi^0(\xi, \eta, \lambda_n) = 0 \tag{6,39}$$

To find a solution to equation (6,38): $\psi^0(\xi, \eta, \lambda_n)$ will give the ξ, η dependence of $J\, F_n$ and a multiplicative constant matrix will permit simultaneous solution with equation (6,20).

[1]See Appendix A for more details on this topic.

We let

$$J\bar{F}_n = \psi^0 (\xi, \eta, \lambda_n) J \bar{F}^0_n \qquad (6,40)$$

where $\bar{F}^0_n$ is some constant $2N \times K_n$ matrix. The presence of J on the right-hand side is a matter of convention and here we follow Zakharov and Mikhailov. To obtain a constraint on F^0_n, we must insert equation (6,40) into equation (6,20)

$$F^{tr}_n J F_n = (F^{0tr}_n J^{-1} \psi^{0tr} J) J (J^{-1} \psi^0 J F^0_n) = 0 \qquad (6,41)$$

Since ψ^0 belongs to the symplectic group it satisfies $\psi^{0tr} J = J \{\psi^0\}^{-1}$ and we are left with

$$F^{0tr}_n J \ F^0_n = 0 \qquad (6,42)$$

The set of matrices satisfying equation (6,42) forms a subspace of the $2N \times K_n$ matrix space. And in general, one must describe the basis spanning this subspace to obtain a complete description of equation (6,42).

6.2.3. Absence of Simple Poles in U and V

In (6.2.2), we showed that requiring absence of double pole in XX^{-1} and absence of double poles in U and V was consistent with assuming the factorization of A_m. Here we show that the absence of simple poles in

XX^{-1} is consistent with the absence of simple pole in U and V. However these results are not independent of section (6.2.2) since we will use equation (6,38). In this section we establish an equation for a_n, which is present in equation (6,22), that will eliminate the residue at $\lambda=\lambda_n$ in U and V if it is satisfied.

Going back to equation (6,15), we isolate the $1/(\lambda-\lambda_n)$ term and demand that its residue be zero. The result is

$$A_n D(\lambda_n)\, J\, \tau_n^{tr} J^{-1} + \tau_n D(\lambda_n)\, J\, A_n^{tr} J^{-1} - A_n \partial_\lambda U^0\big|_{\lambda=\lambda_n} J\, A_n^{tr} J^{-1} = 0 \qquad (6,43)$$

The last term of this equation is the contribution of U^0 to the residue. There is a term

$$1/(\lambda-\lambda_n)^2\, A_n\, U^0(\lambda)\, J\, A_n^{tr} J^{-1}$$

in the expression for U. Expanding $U^0(\lambda)$ in a Laurent series, we see that the $1/(\lambda-\lambda_n)$ contribution is exactly the last term of equation (6,43). There is a similar equation where $\tilde{D}(\lambda_n)$ replaces the operator $D(\lambda_n)$ in (6,43). Making use of equation (6,18), formula (6,43) is transformed into

$$M_n F_n^{+} D(\lambda_n)\, J\tau_n^{tr} + \tau_n D(\lambda_n) J F_n M_n^{tr} = M_n F_n^{+} \partial_\lambda U^0\big|_{\lambda=\lambda_n} J\, \bar{F}_n M^{tr} \qquad (6,44)$$

Since $D(\lambda_n)\, J\, \overline{F}_n = 0$ and $\tau_n\, J\, \overline{F}_n = -M_n\, \alpha_n$ we are left with

$$M_n F_n^{+}\, D(\lambda_n) J \tau_n^{tr} - M_n\, \alpha_n\, \partial_\xi \left(M_n^{tr}\right) = M_n\, F_n^{+}\, \partial_\lambda U^0 \Big|_{\lambda=\lambda_n} J\, \overline{F}_n\, M_n^{tr} \qquad (6,45)$$

Next we differentiate the transpose of equation (6,22) with respect to ξ. From this, we obtain an equation for $(\partial_\xi M_n^{tr})$ in terms of $(\partial_\xi \alpha_n)$

$$\alpha_n \left(\partial_\xi M_n^{tr}\right) + \left(\partial_\xi \alpha_n\right) M_n^{tr} = F_n^{+}\, J \left(\partial_\xi \tau_n^{tr}\right) + \left(\partial_\xi F_n^{+}\right) J \tau_n^{tr} \qquad (6,46)$$

We then eliminate $(\partial_\xi F_n^{+})$ from (6,46). Taking the transpose of equation (6,38) yields

$$\left(\partial_\xi F_n^{+}\right) J = F_n^{+}\, J\, U^{0tr} \Big|_{\lambda=\lambda_n} \qquad (6,47)$$

Since U^0 belongs to the Lie algebra of the symplectic group it satisfies [2]

$$U^0 J + J\, U^{0tr} = 0 \qquad (6,48)$$

It follows that

$$\left(\partial_\xi F_n^{+}\right) J = -F_n^{+} U^0 \Big|_{\lambda=\lambda_n} J \qquad (6,49)$$

and using this result in (6,46) we deduce

$$\alpha_n\left(\partial_\xi M_n\right)^{tr} = -\left(\partial_\xi \alpha_n\right) M_n^{tr} + F_n^+ D\left(\lambda_n\right) J\,\tau_n^{tr} \tag{6,50}$$

Substituting equation (6,50) into formula (6,45) gives the differential equation for α_n that we sought

$$\left(\partial_\xi \alpha_n\right) = F_n^+ \partial_\lambda U^0 \big|_{\lambda=\lambda_n} J\overline{F}_n \tag{6,51}$$

Similarly, we get

$$\left(\partial_\eta \alpha_n\right) = F_n^+ \partial_\lambda V^0 \big|_{\lambda=\lambda_n} J\overline{F}_n \tag{6,52}$$

Equations (6,51) and (6,52) allow us to determine an α_n such that there will be no first order pole in U and V at $\lambda_n, \overline{\lambda}_n$. Recapitulating, we recall that equations (6,40) and (6,42) determine F_n while equations (6,51) and (6,52) determine α_n. Substituting these results into equation (6,24), we obtain a system of linear equations for M_n that can be solved algebraically. Hence, using equation (6,18), the A_n's are determined and so is X. Since the vacuum solution ψ^0 is given we have found a non-trivial solution to our problem

$$\psi(\xi,\eta,\lambda) = X(\xi,\eta,\lambda)\,\psi^0(\xi,\eta,\lambda) \tag{6,53}$$

6.3 Solution to the Riemann Problem for the Symplectic Group (Part II)

6.3.1 A More General System of Equations for X

In their paper "On the Integrability of Classical Spinor Models in Two-Dimensional Space-Time", [7] Zakharov and Mikhailov not only studied symplectic symmetry but also unitary and orthogonal symmetry. For the unitary group, they presented a solution, with X having poles at conjugate points analogous to equation (6,13). However they also showed a solution with only one pole. We quote them:

"In the case of a unitary group there is a solution with only one pole:

$$X(\xi,\eta,\lambda) = I - \frac{\lambda_0 - \bar{\lambda}_0}{\lambda - \bar{\lambda}_0} F(F^+F)^{-1} F^+ \tag{6,54}$$

where $F(\xi,\eta) = \psi^0(\xi,\eta,\lambda) F^0$, F^0 is an arbitrary constant $N \times K$ matrix $(\det(F^{0+}F^0) \neq 0)$."

In the case of symplectic and orthogonal groups they didn't mention these one-pole solutions so that one was led to believe that they didn't exist. We ask ourselves: do these one-pole solutions exist or not?

Jumping to a seemingly unrelated problem, we look at equation (6,24) and observe that if λ_n is a real parameter, the third term of this equation goes to infinity. In that case, equation (6,24) ceases to be valid. Is there a solution when λ_n is real?

It just happens that the answers to these two questions are identical. The one-pole solution for the symplectic group is

$$X(\xi,\eta,\lambda) = I - \frac{(JF)a^{-I}\,F^{tr}}{(\lambda - \lambda_0)} \tag{6,55}$$

where $\lambda_0 \varepsilon$ **R** *and, F and a are real analogs of the matrices described in the previous subsections. In the following subsection these matrices will be described thoroughly.*

The one-pole solution is usually called a soliton solution while the conjugate-poles solution is called a doublet solution. The generic term for these two solutions is "soliton", which is a little confusing.

In this subsection and the two subsequent ones we plan to study the solution to the Riemann problem with X having P single poles and Q double poles. This X will yield the R-soliton solution with

$R = Q + P.$

Consider the following X matrix

$$X = I + \sum_{m=1}^{P} \frac{A_m}{(\lambda - \lambda_m)} + \sum_{q=P+1}^{R} \left(\frac{A_q}{(\lambda - \lambda_q)} + \frac{\bar{A}_q}{(\lambda - \bar{\lambda}_q)}\right) \tag{6,56}$$

where $R = P + Q$ and,

for $m = 1, \cdots, P$

$\lambda_m \in R$ and A_m is element of the 2N X 2N real matrix space,

for $m = P + 1, \cdots, P + Q$

$\lambda_m \in C$ and A_m is element of the 2N X 2N complex matrix space.

We ask, as in section (6.2.1), that poles be absent in $XX^{-1} = I$

$$XX^{-1} = I + \sum_{m=1}^{R} \frac{A_n}{(\lambda - \lambda_m)} J \tau_m^{tr}(\lambda) J^{-1} + \sum_{m=1}^{R} \frac{1}{(\lambda - \lambda_m)} \tau_m(\lambda) J A_m^{tr} J^{-1}$$

$$+ \sum_{q=P+1}^{R} \frac{\bar{A}_q}{(\lambda - \bar{\lambda}_q)} J \bar{\tau}_q(\lambda) J^{-1} + \sum_{q=P+1}^{R} \frac{1}{(\lambda - \bar{\lambda}_q)} \tau_q(\lambda) A_q^{tr} J^{-1}$$

$$+ \sum_{m=1}^{R} \frac{A_m J A_m^{tr} J^{-1}}{(\lambda - \lambda_m)^2} + \sum_{q=1}^{R} \frac{\bar{A}_q J A^{+} J^{-1}}{(\lambda - \bar{\lambda}_q)^2} \qquad (6,57)$$

where

$$\tau_m(\lambda) \equiv I + \sum_{\substack{p=1 \\ p\neq m}}^{P} \frac{A_p}{(\lambda-\lambda_p)} + \sum_{p=P+1}^{R} \left\{ \frac{A_p}{(\lambda-\lambda_p)} + \frac{\bar{A}_p}{(\lambda-\bar{\lambda}_p)} \right\}, \quad m=1,\cdots,P$$

$$\tau_m(\lambda) \equiv I +$$

$$+ \sum_{n=1}^{P} \frac{A_n}{(\lambda-\lambda_n)} + \sum_{\substack{n=P+1 \\ n\neq P}}^{R} \frac{A_n}{(\lambda-\lambda_n)} + \sum_{n=P+1}^{R} \frac{\bar{A}_n}{(\lambda-\bar{\lambda}_n)}, \quad m=P+1,\cdots,R \qquad (6,58)$$

The absence-of-simple-pole condition leads to

$$A_n J \tau_n^{tr} + \tau_n J A_n^{tr} = 0 \quad n=1,\cdots,R$$

$$\bar{A}_n J \tau_n^{+} + \bar{\tau}_n J A_n^{+} = 0 \quad n=P+1,\cdots,R \qquad (6,59)$$

where $\tau_n = \tau_n(\lambda_n)$ and $\tau_n(\lambda)$ is defined in equation (6,58).

And the absence-of-double-pole condition gives

$$A_m J A_m^{tr} J^{-1} = 0 \qquad m=1,\cdots,R$$

$$\bar{A}_m J A_m^{+} J^{-1} = 0 \qquad m=P+1,\cdots,R \qquad (6,60)$$

Equation (6,60) is solved assuming.

$$A_n \equiv M_n F_n^{tr} \qquad n=1,\cdots,R \tag{6,61}$$

where M_n and F_n are $2N \times K_n$ matrices which are real for $n < P$ and complex for $n > P$.

The matrices F_n satisfy

$$F_n J F_n^{tr} = 0 \qquad n=1,\cdots,R \tag{6,62}$$

The solution to equation (6,59) is

$$\tau_n J F_n = -M_n \alpha_n , \quad \alpha_n = \alpha_n^{tr} \tag{6,63}$$

where α_n is a $K_n \times K_n$ matrix function which is real for $n < P$ and complex for $n > P$.

For $n > P$, the accompanying complex equation is

$$\bar{\tau}_n J \bar{F}_n = -\bar{M}_n \bar{\alpha}_n \tag{6,64}$$

Writing equations (6,63) and (6,64) in full, we obtain

$$J F_n + \sum_{\substack{m=1 \\ m \neq n}}^{P} \frac{M_m (F_m^{tr} J F_n)}{(\lambda_n - \lambda_m)} + \sum_{q=P+1}^{P} \Big\{ \frac{M_q (F_q^{tr} J F_n)}{(\lambda_n - \lambda_q)}$$

$$+ \frac{\overline{M}_q (F_q^{+} J F_n)}{(\lambda_n - \overline{\lambda}_q)} \Big\} = - M_n \alpha_n \quad n=1,\cdots ,P \qquad (6,65a)$$

$$J F_n + \sum_{m=1}^{P} \frac{M_m (F_m^{tr} J F_n)}{(\lambda_n - \lambda_m)} + \sum_{\substack{q=P+1 \\ q \neq m}}^{R} \frac{M_q (F_q^{tr} J F_n)}{(\lambda_n - \lambda_q)}$$

$$+ \sum_{q=P+1}^{R} \frac{\overline{M}_q (F_q^{+} J F_n)}{(\lambda_n - \overline{\lambda}_q)} = - M_n \alpha_n \qquad n=P+1,\cdots ,R \qquad (6,65b)$$

$$J \overline{F}_n + \sum_{m=1}^{P} \frac{M_m (F_m^{tr} J \overline{F}_n)}{(\lambda_n - \lambda_m)} + \sum_{q=P+1}^{R} \frac{M_q (F_q^{tr} J \overline{F}_n)}{(\overline{\lambda}_n - \lambda_q)}$$

$$+ \sum_{\substack{q=P+1 \\ q \neq n}}^{R} \frac{\overline{M}_q (F_q^{+} J \overline{F}_n)}{(\overline{\lambda}_n - \overline{\lambda}_q)} = - \overline{M}_n \overline{\alpha}_n \qquad n=P+1,\cdots ,R \qquad (6,65c)$$

6.3.2 Absence of Double Pole in U and V.

In a way identical to that of section (6.2.2), we find a solution F_n to equation (6,62) consistent with the absence of double poles in U and V at $\lambda = \lambda_n$

$$U = U_0 + \sum_{m=1}^{R} \frac{[A_m D(\lambda)\ J\ \tau_m^{tr}(\lambda) + \tau_m(\lambda)\ D(\lambda)\ J\ A_m^{tr}]}{(\lambda - \lambda_m)} J^{-1}$$

$$+ \sum_{q=P+1}^{R} \frac{1}{(\lambda - \bar{\lambda}_q)} [\bar{A}_q D(\lambda)\ J\ \tau_q^{+}(\lambda) + \bar{\tau}_q(\lambda)\ D(\lambda)\ J\ A_q^{+}]\ J^{-1}$$

$$+ \sum_{m=1}^{R} \frac{1}{(\lambda - \lambda_m)^2} [A_m D(\lambda)\ J\ A_m^{tr}\ J^{-1}] + \sum_{m=P+1}^{R} \frac{\bar{A}_q}{(\lambda - \bar{\lambda}_q)^2} D(\lambda)\ JA^{+}\ J^{-1} \quad (6,66)$$

where $D(\lambda) = \partial_\xi - U^0(\lambda)$

We pick up terms in $1/(\lambda-\lambda_n)^2$ and we set the residue equal to zero at $\lambda = \lambda_n, \bar{\lambda}_n$.

$$A_n D(\lambda_n)\ J\ A_n^{tr} = 0 \qquad n=1,\cdots,R$$

$$A_n D(\bar{\lambda}_n)\ J\ A_n^{+} = 0 \qquad n=P+1,\cdots,R \quad (6,67)$$

An being a factorized matrix, we ask that

$$F_n^{tr}\ D(\lambda_n)\ J\ F_n = 0 \qquad n=1,\cdots,R \quad (6,68)$$

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The solution to equation (6,68) is

$$J F_n = \psi^0(\xi, \eta, \lambda_n)\, J F_n^0 \qquad n=1,\cdots,R \tag{6,69}$$

Since ψ^0 is real at real values of parameter λ_n it follows that F_n^0 is a real $2N \times K_n$ constant matrix for $N < P$. It satisfies

$$F_n^{0\,tr} J F_n^0 = 0 \tag{6,70}$$

In solving equation (6,68), we assumed

$$D(\lambda_n) J F_n = 0 \tag{6,71}$$

Similar equations hold for the V-case.

6.3.3 Absence of Simple Pole in U and V

The absence-of-simple-pole requirement leads to the equations.

$$A_n D(\lambda_n) J \tau_n^{tr} + \tau_n D(\lambda_n) J A_n^{tr} - A_n \partial_\lambda U^0 \big|_{\lambda=\lambda_n} J A_n^{tr} = 0 \qquad n=1,\cdots,R$$

$$\bar{A}_q D(\lambda_q) J \tau_q^{+} + \bar{\tau}_q D(\bar{\lambda}_q) J A_q^{+} - A_q \partial_\lambda U^0 \big|_{\lambda=\bar{\lambda}_q} J A_q^{+} = 0 \qquad q=P+1,\cdots,R$$

(6,72)

One easily deduces that the differential equations satisfied by α_m are

$$(\partial_\xi \alpha_m) = F_m^{tr} \, \partial_\lambda U^0 \big|_{\lambda = \lambda_m} J F_m, \quad m = 1, \cdots, R \tag{6,73}$$

$$(\partial_\eta \alpha_m) = F_m^{tr} \, \partial_\lambda U^0 \big|_{\lambda = \lambda_m} J F_m, \quad m = 1, \cdots, R \tag{6,74}$$

We have described the tools for obtaining the R-soliton solution. The non-trivial solution to our problem will again be

$$\psi(\xi,\eta,\lambda) = X(\xi,\eta,\lambda) \, \psi^0(\xi,\eta,\lambda) \tag{6,75}$$

We postpone the discussion of the Gross-Neveu fermion fields in terms of X and ψ^0 until section (6.2) since in order to do so we need to know a little bit more about the vacuum solution.

7. Integration of the Gross-Neveu Model

7.1 Vacuum Solution

Recall that we want to solve

$$U_{0_\eta} + V_{0_\xi} + [\,U_0, V_0\,] = 0$$

$$U_{n\xi} + [\,U_n, \Phi_n\,] = 0 \quad , \quad V_{n\eta} + [\,V_n, \Psi_n\,] = 0 \qquad (7,1)$$

The simplest solution that comes to mind is the one for which all partial derivatives and commutators present in system (7,1) are zero. Hence the matrices $U_0^0(\xi), V_0^0(\eta), U_n^0(\xi), V_n^0(\eta)$ *will be called a vacuum solution if they satisfy.*

$$[\,U_0^0, V_0^0\,] = 0, \quad [\,U_n^0, \Phi_n^0\,] = 0, \quad [\,V_n^0, \Psi_n^0\,] = 0 \qquad (7,2)$$

$$\Psi_n^0 = U_0^0 - \sum \frac{U_n^0}{(a_n + a_m)}, \quad \Phi_n^0 = V_0^0 + \sum_m \frac{V_m^0}{(a_n + a_m)} \qquad (7,3)$$

When the gauge of system (7,2) coincides with that of system (7,1) (the non-trivial system) we deal with a first order vacuum. If a gauge transformation is necessary to bring system (7,2) to the gauge of system (7,1) we shall obtain the matrices $\tilde{U}_0^0(\xi,\eta)$, $\tilde{V}_0^0(\xi,\eta)$, $\tilde{U}_n^0(\xi,\eta)$, $\tilde{V}_n^0(\xi,\eta)$ *called a second order vaccuum solution.*

The Gross-Neveu model was deduced from the one-pole problem at $a_1 = 1$ in the canonical gauge. The compatibility conditions were

$$\partial_\eta U_1 = 1/2[\, U_1 , V_1 \,] = \partial_\xi V_1 \qquad (7,4)$$

The first order vacuum is then directly defined in the canonical gauge.

$$[\, U_1^0(\xi),\, V_1^0(\eta)] = 0 \qquad (7,5)$$

For the second order vacuum we must go to a gauge which is not canonical, and solve

$$[U_0^0(\xi), V_0^0(\eta)] = 0, [U_1^0(\xi), V_0^0(\eta) + \tfrac{1}{2} V_1^0(\eta)] = 0, [V_1^0(\eta), U_0^0(\xi) - \tfrac{1}{2} U_1^0(\xi)] = 0 \qquad (7,6)$$

The second order vacuum is obtained by gauge transforming this solution back to the canonical gauge.

For the Gross-Neveu model it happens that a nilpotent matrix describes the first order vacuum and non-physical results arise from it. We leave this issue aside and immediately consider the second order vacuum. It is easier to find the second order vacuum solution directly from the Gross-Neveu equations of motion. We then use equations (2,18) and (2,22) to determine the matrices $\tilde{U}_1^0(\xi,\eta)$ and $\tilde{V}_1^0(\xi,\eta)$. Using a matrix (g^{-1}) we perform a gauge transformation to the U_1^0 and V_1^0 system. It just happens that U_1^0 and V_1^0 are matrices with constant coefficients. Denoting Φ^0, the solution to this system, $\Psi^0 = g\Phi^0$ will be the second order vacuum solution in the canonical gauge. We carry out this progam now.

The vacuum solution of the Gross-Neveu model corresponds to a constant σ field indicating absence of solitions. The equations of motion are then

$$\partial_\eta \phi^\alpha = -i\sigma_0 \psi^\alpha , \quad \partial_\xi \psi^\alpha = -i\sigma_0 \phi^\alpha \tag{7,7}$$

This can be written as

$$\frac{\partial_\eta \phi^\alpha}{\psi^\alpha} = \frac{\partial_\xi \psi^\alpha}{\psi^\alpha} = i\sigma_0 = \text{constant} \tag{7,8}$$

From equation (7,8), we deduce that ψ^α, ϕ^α can be written as a product of two-functions depending only on ξ and η respectively. Also ψ^α and ϕ^α differ at most by a constant. The final result can be put in the form

$$\psi^\alpha = \beta_\alpha^{-1/2} A_\alpha e^{i\Theta_\alpha}, \quad \phi^\alpha = \beta^{1/2} A_\alpha e^{i\Theta_\alpha} \tag{7,9}$$

where

$$\Theta_\alpha = -\beta_\alpha \sigma_0 \xi - \sigma_0 \beta_\alpha^{-1} \eta + \theta_\alpha^0 , \quad \sigma_0 = \frac{1}{2} \sum_{\alpha=1}^{N} A_\alpha^2 \tag{7,10}$$

A_α, B_α are arbitrary real constants[1]

$\tilde{U}^0_1$ and $\tilde{V}^0_1$ are determined from ϕ^α and ψ^α.

From equation (3,22) we have

$$\tilde{V}^0_1 = \psi_1 V_1 \psi_1^{-1} \tag{7,10}$$

$\hat{V}_1$ is defined by (3,58). Hence

$$(\tilde{V}^0_1)_{\alpha\beta} = (\psi_1)_{\alpha\gamma} (\hat{V}_1)_{\gamma\delta} (J)_{\delta\varepsilon} (\psi^{tr})_{\varepsilon\sigma} (J)_{\sigma\beta}$$

$$= -\psi_{\alpha,1} \psi_{\beta+N,1} \quad \text{when } \beta < N \tag{7,11}$$

$$= \psi_{\alpha,1} \psi_{\beta-N,1} \quad \text{when } \beta \quad N$$

Summation signs were omitted in the first line of equation (7,11)

[1]It is important to notice that the fields have been redefined so as to cancel a factor of four appearing in $\tilde{U}^0_1$ and $\tilde{V}^0_1$. The new definitions are $\psi^\alpha = \psi_{\alpha,1} + i\,\psi_{\alpha+N,i}$, $\phi^\alpha = \phi_{\alpha+N,1} - i\phi_{\alpha,1}$, $\sigma = \frac{1}{2} \sum_\alpha Re\, \psi^* \phi$

Since $\tilde{V}_1^0$ belongs to the symplectic group it has the form

$$\tilde{V}_1^0 = \begin{bmatrix} A_2 & B_2 \\ C_2 & -A_2^{tr} \end{bmatrix} \quad B_2 = B_2^{tr}, \; C_2 = C_2^{tr} \qquad (7,12)$$

It follows that

$$(A_2)_{ij} = -\mathrm{Re}\psi^i \,\mathrm{Im}\,\psi^j = A_i A_j (\beta_i\beta_j)^{-1/2} \mathrm{Sin}\,\Theta_j \,\mathrm{Cos}\,\Theta_i$$

$$(B_2)_{ij} = \mathrm{Re}\psi^i \,\mathrm{Re}\,\psi^j = A_i A_j (\beta_i\beta_j)^{-1/2} \mathrm{Cos}\,\Theta_j \,\mathrm{Cos}\,\Theta_i$$

$$(C_2)_{ij} = \mathrm{Im}\psi^i \,\mathrm{Im}\,\psi^j = -A_i A_j (\beta_i\beta_j)^{-1/2} \mathrm{Sin}\,\Theta_j \,\mathrm{Sin}\,\Theta_i \qquad (7,13)$$

A similar calculation must also be performed for $\tilde{U}_0^1$

$$\tilde{U}_1^0 = \begin{bmatrix} A_1 & B_1 \\ C_1 & -A_1^{tr} \end{bmatrix} \qquad (7,14)$$

The result is

$$(A_1)_{ij} = -A_i A_j (\beta_i \beta_j)^{1/2} \operatorname{Sin}\theta_i \operatorname{Cos}\theta_j$$

$$(B_1)_{ij} = -A_i A_j (\beta_i \beta_j)^{1/2} \operatorname{Sin}\theta_i \operatorname{Sin}\theta_j$$

$$(C_1)_{ij} = A_i A_j (\beta_i \beta_j)^{1/2} \operatorname{Cos}\theta_i \operatorname{Cos}\theta_j \qquad (7,15)$$

The gauge transformation $\phi^0 = \{g^0\}^{-1} \psi^0$, with the symplectic matrix of the form[1]

$$g^0 = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$

$$(g_{11})_{ij} = (g_{22})_{ij} = \delta_{ij} \operatorname{Sin}\theta_j$$

$$(g_{12})_{ij} = (g_{21})_{ij} = \delta_{ij} \operatorname{Cos}\theta_j \qquad (7,16)$$

transforms

$$\partial_\xi \psi^0 = \frac{\tilde{U}_1^0}{(\lambda-1)} \psi^0, \quad \partial_\eta \psi^0 = \frac{\tilde{V}_1^0}{(\lambda+1)} \psi^0 \qquad (7,17)$$

[1] The derivation of this result is not shown since it sheds no light on our main concern. Its validity can easily be verified.

into

$$\partial_\xi \phi^0 = W_1 \phi^0 \quad , \quad \partial_\eta \phi^0 = W_2 \phi^0 \tag{7,18}$$

where W_1 and W_2 are matrices with constant coefficients

$$W_{1,2} = \begin{bmatrix} 0 & \omega_{1,2} \\ \tilde{\omega}_{1,2} & 0 \end{bmatrix} \tag{7,19}$$

with

$$(\omega_1)_{ij} = -\sigma_o \beta_i \delta_{ij} - \frac{A_i \quad A_j \quad (\beta_i \; \beta_j)^{1/2}}{(\lambda - 1)}$$
$$(\tilde{\omega}_1)_{ij} = \sigma_o \beta_i \delta_{ij} \tag{7,20}$$

and

$$(\tilde{\omega}_2)_{ij} = \sigma_o \beta_i^{-1} \delta_{ij} - \frac{A_i \quad A_j \quad (\beta_i \; \beta_j)^{-1/2}}{(\lambda + 1)}$$
$$(\omega_2)_{ij} = -\sigma_o \beta_i^{-1} \delta_{ij} \tag{7,21}$$

We observe that W_1 and W_2 commute at all values of the parameter λ. Therefore Φ^0 can be sought in the form

$$\Phi^0(\xi,\eta,\lambda) = \Phi_1^0(\xi,\lambda)\ \ \Phi_2^0(\eta,\lambda) \tag{7,22}$$

where Φ_1^0 and Φ_2^0 satisfy

$$\partial_\xi \Phi_1^0 = W_1 \Phi_1^0 \quad , \quad \partial_\eta \Phi_2^0 = W_2 \Phi_2^0 \tag{7,23}$$

The formal solution to equation (7,23) is

$$\Phi_1^0(\xi,\lambda) = \exp W_1(\xi + \xi^0) \ , \ \Phi_2^0(\eta,\lambda) = \exp W_2(\eta + \eta^0) \tag{7,24}$$

7.2 Solution of the Gross-Neveu Model

7.2.1 Solution of the spinor problem

We remarked that the equation satisfied by ψ_n is equation (3,1a) at $\lambda = -a_n$ while ϕ_n satisfies equation (3,1b) at $\lambda = -a_n$. Since we have shown how to find a non-trivial solution to system (7,21) for any λ this suggests an easy way of solving the invarient problem ϕ_n and ψ_n are obtained from the non-trivial solution by setting λ equal to a_n and $-a_n$ in it.

Actually a subtlety arises. There exists an ambiguity in the definition of ψ_n [ϕ_n]. The matrix obtained by right-multiplying ψ_n [Φ_n] by an arbitrary matrix function of η [ξ] is also a solution of the spinor problem. The ambiguity is even more general and it is discussed in detail by the original authors [ZM]. Making use of this fact simplifies the determination of ψ_n. We assume a factorized vacuum solution and we set

$$\phi_n = X_n(\xi,n)\; g^0(\xi,n)\; \Phi_2^0(\eta,a_n) \qquad (7,25a)$$

$$\psi_n = \tilde{X}_n(\xi,n)\; g^0(\xi,\eta)\; \Phi_1^0(\xi,a_n) \qquad (7,25b)$$

Taking the derivative of (7,25a) with respect to η we easily obtain.

$$\phi_{n\eta} = X_{n\eta}\, g^0 \Phi_2^0(a_n) + X_n\left(g^0 \Phi_2^0(a_n)\right)_\eta = \Phi_n X_n g^0 \Phi_2^0$$

$$+ X_n\left[-\Phi_n^0 g^0 \Phi_2^0(a_n) + \left(g^0 \Phi_2^0(a_n)\right)_\eta\right] = \Phi_n \phi_n \qquad (7,26)$$

The last step is obtained with the help of equation (5,2b) that we use at

$\lambda = a_n$ and which causes cancellation of the term inside the brackets. Derivation of (7,25b) with respect to ξ gives equation (3,23).

Hence we can find a solution to the spinor problem using equation (7,25).

7.2.2 The Gross-Neveu Model Solution

The Gross-Neveu Model is obtained from the one-pole problem with $a_n = 1$. Taking into account the redefinition of the fields (section 7.1), the solution of the Gross-Neveu model is:

$$\psi^\alpha = \sum_{\beta,\gamma} \left(X_{\alpha,\beta}(\xi,\eta,-1) + i X_{\alpha+N,\beta}(\xi,\eta,-1) \right) g_{\beta\gamma}(\xi,\eta) \left(\phi_1^0\right)_{\gamma,1}(\xi,-1) \qquad (7,27a)$$

$$\phi^\alpha = \sum_{\beta,\gamma} \left(X_{\alpha+N,\beta}(\xi,\eta,+1) - i X_{\alpha,\beta}(\xi,\eta,+1) \right) g_{\beta\gamma}(\xi,\eta) \left(\phi_2^0\right)_{\gamma,1}(\eta,+1) \qquad (7,27b)$$

7.3 Single-Soliton Solution

Using equation (6,65) we find the matrix X with one soliton (P=1) and no doublet (Q=0) present Equation (6,65) has the form

$$J\,F = -\,M\,\alpha \tag{7,28}$$

where we have omitted the unnecessary index.

Substitution of equation (7,28) into equation (6,61) and (6,56) yields

$$X = I - \frac{J\;F\;\alpha^{-1}\;F^{tr}}{(\lambda - \lambda_0)} \tag{7,29}$$

which is the solution displayed in section (6.3.1)

7.4 Two-Soliton Solution

Using equation (6,65) we find the matrix X with two solitions ($P=2$) and no doublet ($Q=0$)

Equation (6,65) reads

$$J F_1 + \frac{M_2}{(\lambda_1 - \lambda_2)} (F_2^{tr} J F_1) = - M_1 \alpha_1$$
$$J F_2 + \frac{M_1}{(\lambda_2 - \lambda_1)} (F_1^{tr} J F_2) = - M_2 \alpha_2 \qquad (7,30)$$

We define

$$F_{mn} \equiv F_m^{tr} J F_n \quad , \quad F_{mn}^{tr} = - F_{nm} \qquad (7,31)$$

The matrix M_1 is

$$M_1 = \left[- (J F_1) \alpha_1^{-1} + \frac{(J F_2) \alpha_2^{-1} F_{21} \alpha_1^{-1}}{(\lambda_1 - \lambda_2)} \right] \left[1 + \frac{F_{12} \alpha_2^{-1} F_{21} \alpha_2^{-1}}{(\lambda_1 - \lambda_2)^2} \right] \qquad (7,32)$$

An analogous equation holds for M_2. For the sake of simplicity we set $K_n = 1$ ($n = 1, 2$) and α_1, α_2, F_{12}, F_{21} become functions (no longer matrices). M_1 is transformed into

$$M_1 = \left[\frac{-(\lambda_2-\lambda_1)^2 \, J F_1 \alpha_2 + (\lambda_1-\lambda_2)\,(J F_2)\, F_{21}}{(\lambda_1-\lambda_2)^2 \alpha_1 \alpha_2 + F_{21} F_{12}} \right] \tag{7,33}$$

The matrix X takes the form

$$X = I + \frac{1}{D_{12}} \left[\frac{-(\lambda_1-\lambda_2)^2 \alpha_2 J F_1 F_1^{tr} + (\lambda_1-\lambda_2) F_{21} J F_2 F_1^{tr}}{(\lambda-\lambda_1)} + \frac{-(\lambda_2-\lambda_1)^2 \alpha_1 J F_2 F_2^{tr} + (\lambda_2-\lambda_1) F_{12} J F_1 F_2^{tr}}{(\lambda-\lambda_2)} \right]. \tag{7,34}$$

where $D_{12} \equiv (\lambda_1-\lambda_2)^2 \alpha_1 \alpha_2 + F_{21} F_{12}$.

7.5 Doublet Solution

We find the matrix X with one doublet ($Q = 1$) and no soliton ($Q = 0$). While solutions (7,29) and (7,34) were not given by Zakharov and Mikhailov, the solution that we display here was found by them. It is a direct consequence of their fundamental work that we have exposed in chapter 3 and sections 6.1 and 6.2. Equation (6,65) becomes

$$J F + \frac{\overline{M} \; (F^{+} \; J \; F)}{(\lambda_0 - \overline{\lambda}_0)} = - M\alpha \tag{7,35}$$

The complex conjugate of equation (7,35) provides the second equation necessary to find M.

Substitution of $\overline{M}$ into (35) gives

$$J F - \frac{J \; \overline{F}(\overline{\alpha})^{-1} \; (F^{+}J \; F)}{(\lambda_0 - \overline{\lambda}_0)} + \frac{M \; (F^{tr}J \; \overline{F})(\overline{\alpha})^{-1} \; (F^{+}J \; F)}{(\lambda_0 - \overline{\lambda}_0)^2} = - M \alpha \tag{7,36}$$

From equation (7,36), we easily get M.

$$M = (\lambda_0 - \overline{\lambda}_0) [\; J \overline{F} - (\lambda_0 - \overline{\lambda}_0) J F (F^{+} J F)^{-1} \overline{\alpha} \;] * \tag{7,37}$$

$$* [\; F^{tr} J \overline{F} + | \lambda_0 - \overline{\lambda}_0 |^2 (\alpha^{+} (F^{tr} J \overline{F})^{-1} \alpha)^{tr} \;]^{-1}$$

where we used $\alpha^{tr} = \alpha$ to write

$$-\alpha\,(F^{+} J F)^{-1}\bar{\alpha} = (\alpha^{+}(F^{tr} J \bar{F})^{-1}\alpha)^{tr} \quad \text{in equation (6.37).} \tag{7,38}$$

The matrix X of equation (7,25) is then

$$X(\xi,\eta,\lambda) = I + \frac{(\lambda_0 - \bar{\lambda}_0)}{(\lambda - \lambda_0)}\,[J\bar{F} - (\lambda_0 - \bar{\lambda}_0)\,J F\,(F^{+} J F)^{-1}\bar{\alpha}]\,*$$

$$*\,[F^{tr} J \bar{F} + |\lambda_0 - \bar{\lambda}_0|^2\,(\alpha^{+}(F^{tr} J \bar{F})^{-1}\alpha)^{tr}]^{-1}\,F^{tr}$$

$$-\frac{(\lambda_0 - \bar{\lambda}_0)}{(\lambda - \lambda_0)}\,[J F - (\lambda_0 - \bar{\lambda}_0)\,J\bar{F}\,(F^{tr} J \bar{F})^{-1}\alpha]\,*$$

$$*\,[F^{+} J F + |\lambda_0 - \bar{\lambda}_0|^2\,(\alpha^{+}(F^{tr} J \bar{F})^{-1}\alpha)\,]^{-1}\,F^{+} \tag{7,39}$$

When $\alpha = 0$, the solution becomes

$$X(\xi,\eta,\lambda) = I + \frac{\lambda_0 - \bar{\lambda}_0}{(\lambda - \lambda_0)}\,J\bar{F}(F^{tr} J\bar{F})^{-1}F^{tr} - \frac{\lambda_0 - \bar{\lambda}_0}{(\lambda - \bar{\lambda}_0)}\,JF(F^{+} JF)^{-1}F^{+} \tag{7,40}$$

7.6 R–Soliton solution (P solitons and Q doublets)

In this section, we determine the form of X when solitons and doublets are present in arbitrary number. Throughout this section we assume that $K_n = 1$ and $n = 1,\cdots,R$. It follows that α_n and $F_m^{tr} J F_n$ are 1 X 1 matrices, that is ordinary functions.

We define a $(P + 2Q)$ X $(P+2Q)$ matrix function D_{st}:

– The diagonal elements of D_{st} are

$$D_{ss} \equiv \alpha_s \text{ for } s = 1,\cdots,R \qquad (7,41a)$$

$$D_{\bar{s}\bar{s}} \equiv \alpha_s \text{ for } s = R+1,\cdots,P+2Q$$

– For $s = 1,\cdots,P+Q$, $t = 1,\cdots,P+Q$, $s \neq t$

$$D_{st} \equiv \frac{F_t^{tr} J F_s}{(\lambda_n - \lambda_m)} \qquad (7,41b)$$

– For $s = 1,\cdots,P+Q$, $t = R+1,\cdots,R+Q$,

$$D_{st} \equiv \frac{F_{(t-Q)}^{+} J F_s}{(\lambda_s - \bar{\lambda}_{(t-Q)})} \qquad (7,41c)$$

- For $s = R + 1, \cdots, P + 2Q$, $t = t = 1, \cdots, P$

$$D_{st} \equiv \frac{F_t^{tr} J F_{(s-Q)}}{(\lambda_n - \lambda_{(s-Q)})} \qquad (7,41d)$$

- For $s = R + 1, \cdots, P + 2Q$, $t = P + 1, \cdots, P + Q$

$$D_{st} \equiv \frac{F_t^{tr} J \overline{F}_{(s-Q)}}{(\overline{\lambda}_n - \lambda_{s-Q})} , \qquad (7,41e)$$

- For $s = R + 1, \cdots, P + 2Q, t = R + 1, \cdots, P + 2Q, s \neq t$

$$D_{st} \equiv \frac{F^{+}_{(t-Q)} J \overline{F}_{(s-Q)}}{\overline{\lambda}_{(s-Q)} - \overline{\lambda}_{(t-Q)}} \qquad (7,41f)$$

We recall that $\{F_m\}$ is a set of $2N \times 1$, matrices defined only for $m = 1, \cdots, R$.

Next, we define a set of $2N$ matrices E_n of dimension $(P + 2Q) \times 1$

$$(E_n)_m = (F_m)_n \quad m = 1, \cdots, R, \quad n = 1, \cdots, 2N \tag{7,42a}$$

$$(E_n)_m = (F_{(m-Q)})_n \quad m = R+1, \cdots, R+Q, \quad n = 1, \cdots, 2N \tag{7,42b}$$

where $(E_n)_m$ means the m^{th} component of the n^{th} vector E and $(F_m)_n$, the n^{th} component of the m^{th} vector F.

We also create a set of $(P + 2Q) \times (P + 2Q)$ matrices C_{pr} ($p = 1, \cdots, 2N$, $r = 1, \cdots, P + 2Q$) where the r^{th} column of matrix D has been replaced by E_p. "p" and "r" in C_{pr} should not be confused with row and column indices. They are labels used to distinguish different matrices. Indeed, we have

$$\begin{aligned} (C_{pr})_{ij} &= D_{ij} \quad j \neq r \\ (C_{pr})_{ir} &= (E_p)_i \end{aligned} \tag{7,43}$$

where "i" is the row index of matrix C_{pr} and "j", the column index

We finally define the last object necessary to obtainning the R - soliton solution. We form P + 2Q vectors C_r of dimension $2N \times 1$.

$$(C_r)_p \equiv \det C_{pr} \quad r = 1, \cdots, P + 2Q \tag{7,44}$$

where $(C_r)_p$ means the p^{th} element of vector C_r.

Looking at equation (6,65), we observe that

$$M_n = -\frac{J \cdot C_n}{|\ D\ |} \qquad n = 1, \cdots, R$$

$$\overline{M}_n = -\frac{J\ C_{(n+Q)}}{|\ D\ |} \qquad n = P + 1, \cdots, R \qquad (7,45)$$

$$= -\frac{J\ \overline{C}_n}{|\ D\ |}$$

The R-soliton solution now takes the form

$$X = I - \sum_{n=1}^{P} \frac{J\ C_n\ F_n^{tr}}{|\ D\ |(\lambda - \lambda_n)} + \qquad (7,46)$$

$$+ \sum_{m=P+1}^{R} \left[-\frac{J\ C_n\ F_n^{tr}}{(\lambda - \lambda_n)|\ D\ |} - \frac{J\ \overline{C}_n\ F_n^{+}}{(\lambda - \overline{\lambda}_n)|\ \overline{D}\ |} \right]$$

One should be careful in interpreting X. One might be led to believe that there is no soliton–doublet interaction since we have separate summations for solitons and doublets. These interactions are contained in the matrices C_n which expresses interaction of a given soliton (or doublet) with other solitons and doublets. From equation (7,46), it is easy to obtain the two fundamental interactions that we left aside: the soliton-doublet interaction and the doublet-doublet interaction.

7.7 Soliton-doublet Solution

In this section we apply the machinery of section (6.5) to find the soliton-doublet solution.

The matrix D is

$$D = \begin{bmatrix} \alpha_1 & \dfrac{F_2^{tr} J F_1}{(\lambda_1 - \lambda_2)} & \dfrac{F_2^{+} J F_1}{(\lambda_1 - \bar{\lambda}_2)} \\ \dfrac{F_1^{tr} J F_2}{(\lambda_2 - \lambda_1)} & \alpha_2 & \dfrac{F_2^{+} J F_2}{(\lambda_2 - \bar{\lambda}_2)} \\ \dfrac{F_1^{tr} J \bar{F}_2}{(\lambda_1 - \lambda_2)} & \dfrac{F_2^{tr} J \bar{F}_2}{(\bar{\lambda}_2 - \lambda_2)} & \bar{\alpha}_2 \end{bmatrix} \tag{7,47}$$

We symbolically designate ~~the~~ set of matrices E_n by

$$E = \begin{bmatrix} F_1 \\ F_2 \\ \bar{F}_2 \end{bmatrix} \tag{7,48}$$

It follows that matrices C_{p1} *and* C_{p2} *are*

$$C_{p1} \quad \begin{bmatrix} (F_1)_p & \dfrac{F_2^{tr}\, J\, F_1}{(\lambda_1 - \lambda_2)} & \dfrac{F_2^{+}\, J\, F_1}{(\lambda_1 - \bar{\lambda}_2)} \\ (F_2)_p & \alpha_2 & \dfrac{F_2^{+}\, J\, F_2}{(\lambda_2 - \bar{\lambda}_2)} \\ (\bar{F}_2)_p & \dfrac{F_2^{tr}\, J\, \bar{F}_2}{(\lambda_2 - \lambda_2)} & \bar{\alpha}_2 \end{bmatrix} \tag{7,49}$$

and

$$C_{p2} = \begin{bmatrix} \alpha_1 & (F_1)_p & \dfrac{F_2^{+}\, J\, F_1}{(\lambda_1 - \bar{\lambda}_2)} \\ \dfrac{F_1^{tr}\, J\, F_2}{(\lambda_2 - \lambda_1)} & (F_2)_p & \dfrac{F_2^{+}\, J\, F_2}{(\lambda_2 - \bar{\lambda}_2)} \\ \dfrac{F_1^{tr}\, J\, \bar{F}_2}{(\lambda_1 - \lambda_2)} & (\bar{F}_2)_p & \bar{\alpha}_2 \end{bmatrix} \tag{7,50}$$

We do not need matrix C_{p3}.

The determinant of matrices D, C_{p1} and C_{p2} are

$$|D| = a_1 \left\{ |a_2|^2 - \left| \frac{F_2^+ J F_2}{\lambda_2 - \bar{\lambda}_2} \right|^2 \right\}$$

$$- 2\,\mathrm{Re} \left\{ \frac{F_2^{tr} J F_1}{(\lambda_1 - \lambda_2)} \left[\frac{F_1^{tr} J F_2}{(\lambda_2 - \lambda_1)} \bar{a}_2 - \frac{F_2^+ J F_2}{(\lambda_2 - \bar{\lambda}_2)} \frac{F_1^{tr} J \bar{F}_2}{(\lambda_1 - \lambda_2)} \right] \right\} \tag{7,51}$$

Obviously $|D|$ is a real function.

$$|C_{1p}| = (F_1)_p \left\{ |a_2|^2 - \left| \frac{F_2^+ J F_2}{(\bar{\lambda}_2 - \lambda_2)} \right|^2 \right\}$$

$$- 2\,\mathrm{Re} \left\{ \frac{F_2^{tr} J F_1}{(\lambda_1 - \lambda_2)} \left[(F_2)_p \bar{a}_2 - \frac{F_2^+ J F_2}{(\lambda_2 - \bar{\lambda}_2)} (\bar{F}_2)_p \right] \right\}$$

$$|C_{2p}| = -(F_1)_p \left\{ \frac{F_1^{tr} J F_2}{(\lambda_2 - \lambda_1)} \bar{\alpha}_2 - \frac{F_2^{+} J F_2}{(\lambda_2 - \bar{\lambda}_2)} \frac{F_1^{tr} J F_2}{(\lambda_1 - \lambda_2)} \right\}$$

$$+ (F_2)_p \left\{ \alpha_1 \bar{\alpha}_2 - \frac{F_2^{+} J F_1}{(\lambda_1 - \bar{\lambda}_2)} \frac{F_1^{tr} J \bar{F}_2}{(\lambda_1 - \lambda_2)} \right\}$$

$$- (\bar{F}_2)_p \left\{ \alpha_1 \frac{F_2^{+} J F_2}{(\lambda_2 - \bar{\lambda}_2)} - \frac{F_2^{+} J F_1}{(\lambda_1 - \bar{\lambda}_2)} \frac{F_1^{tr} J F_2}{(\lambda_2 - \lambda_1)} \right\} \tag{7,53}$$

The matrix X of the soliton-doublet solution is

$$X = I \frac{1}{[D]} \left\{ - \frac{J C_1 F_1^{tr}}{(\lambda - \lambda_1)} - \frac{J C_2 F_2^{tr}}{(\lambda - \lambda_2)} - \frac{J \bar{C}_2 F_2^{+}}{(\lambda - \bar{\lambda}_2)} \right\}$$

$$= I + \frac{1}{|D|} \left\{ - \frac{1}{(\lambda - \lambda_1)} \left\{ J F_1 F_1^{tr} \left[|\alpha_2|^2 - \left| \frac{F_2^{+} J F_2}{(\bar{\lambda}_2 - \lambda_2)} \right|^2 \right\} \right.$$

$$\left. - 2 Re \left\{ \frac{F_2^{tr} J F_1}{(\lambda_1 - \lambda_2)} \left[J F_2 F_1^{tr} \bar{\alpha}_2 - \frac{F_2^{+} J F_2}{(\bar{\lambda}_2 - \lambda_2)} J \bar{F}_2 F_1^{tr} \right] \right\} \right\} + \cdots$$

$$\cdots - \frac{1}{(\lambda - \lambda_2)} \{ -JF_1 F_2^{tr} [\frac{F_1^{tr} JF_2 \bar{\alpha}_2}{(\lambda_2 - \lambda_1)} - \frac{F_2^+ J F_2}{(\lambda_2 - \bar{\lambda}_2)} \frac{F_1^{tr} J F_2}{(\lambda_1 - \lambda_2)}]$$

$$+ J F_2 F_2^{tr} [\alpha_1 \bar{\alpha}_2 - \frac{F_2^+ J F_1}{(\lambda_1 - \bar{\lambda}_2)} \frac{F_1^{tr} J \bar{F}_2}{(\lambda_1 - \lambda_2)}]$$

$$- J \bar{F}_2 F_2^{tr} [\frac{\alpha_1 \quad F_2^+ J F_2}{(\lambda_2 - \bar{\lambda}_2)} - \frac{F_2^+ J F_1}{(\lambda_1 - \bar{\lambda}_2)} \frac{F_1^{tr} J F_2}{(\lambda_2 - \lambda_1)}] \} + \bar{W} (\bar{\lambda})] \} \tag{7,54}$$

where $W(\lambda) = - J \dfrac{C_2 \; F_2^{tr}}{(\lambda - \lambda_2)}$

One can proceed in the same way and find the doublet-doublet solution which we do not display since it is too cumbersome.

8 Explicit Calculations for the Case $N = 1$

We will explicitly perform all the steps necessary to obtain soliton solutions within the framework of the theory developped in section (6.3).

First we list all the steps necessary to get the solution so that the reader can follow the calculation closely and easily. The following objects must be calculated.

(1) The Vacuum solution $\psi^0(\xi,\eta,\lambda)$ (2 X 2)

(2) The Functions F_n (2 X 1)

(3) The Functions α_n (1 X 1)

(4) The Matrices M_n (2 X 1)

(5) The Matrix X (2 X 2)

(6) The Spinor field $\begin{pmatrix} \psi(\xi,\eta) \\ \phi(\xi,\eta) \end{pmatrix}$

(7) The Scalar field $\sigma(\xi,\eta)$

Steps (4) to (7) must be performed separately for each type of soliton solution.

8.1 The Vacuum Solution

We first solve two first order differential matrix equations with constant co-efficients to obtain $\phi^0_{(\xi,\eta,\lambda)}$

$$\partial_\xi \phi^0_1 = \begin{bmatrix} 0 & \omega_1 \\ \tilde{\omega}_1 & 0 \end{bmatrix} \phi^0_1 \;, \quad \partial_\eta \phi^0_2 = \begin{bmatrix} 0 & \omega_2 \\ \tilde{\omega}_2 & 0 \end{bmatrix} \phi^0_2$$

where $\omega_1 = -\sigma_0\beta \dfrac{(\lambda + 1)}{(\lambda - 1)}$, $\tilde{\omega}_1 = \sigma_0\beta$

$$\tilde{\omega}_2 = \frac{\sigma_0 \, (\lambda - 1)}{\beta \, (\lambda + 1)} \;, \quad \omega_2 = -\frac{\sigma_0}{\beta} \tag{8,1}$$

The most general solution to this system is

$$\phi^0_i = \begin{bmatrix} C^i_1 Sinh\,\mu_i + C^i_2 Cosh\,\mu_i & \nu(C^i_3 Cosh\,\mu_i + C^i_4 Sinh\,\mu_i) \\ \nu^{-1}(C^i_1 Cosh\,\mu_i + C^i_2 Sinh\,\mu_i) & C^i_3 Sinh\,\mu_i + C^i_4 Cosh\,\mu_i \end{bmatrix} \tag{8,2}$$

$i = 1, 2$

and

$$\nu = \sqrt{\frac{\omega_1}{\tilde{\omega}_1}} = \sqrt{\frac{\omega_2}{\tilde{\omega}_2}} = \sqrt{-\frac{(\lambda + 1)}{(\lambda - 1)}} ,$$

$$\mu_1 = \sqrt{\omega_1 \tilde{\omega}_1}\, \xi \;, U_2 = \sqrt{\omega_2 \tilde{\omega}_2}\, \eta, \qquad (8,3)$$

When it will be more convenient, we will display the ξ or η dependence of the solution by letting $\mu_1 \to \mu_1 \xi$ and $\mu_2 \to \mu_2 \eta$ with obvious redefinitions of μ_1, μ_2

Also, when necessary we will abbreviate Sinh by Sh and Cosh by Ch. In dealing specifically with Φ_1^0 and Φ_2^0, we will let $C_j^1 \to H_j$ and $C_g^2 \to G_j$, $j = 1, \cdots, 4$.

Two critical and important values of ν are $\nu = 0$ and $\nu = \infty$ ($\lambda = -1$ and $\lambda = 1$). We directly solve equation (9,1) for these two values. We demand to be able to obtain the Gross-Neveu-field vacuum solution from the gauge-transformed matrix vacuum solution according to equation (6,27) with $X = I$. We display the Gross-Neveu field vacuum solution for $N = 1$.

$$\psi = A\beta^{-1/2}\, e^{i\theta}\ ,\ \phi = A\beta^{1/2}\, e^{i\theta}$$

$$\theta = -\sigma_0\beta\xi - \sigma_0\beta^{-1}\eta + \theta^0\ ,\ \sigma_0 = \frac{A^2}{2} \tag{8.4}$$

and the gauge transformation matrix

$$g = \begin{bmatrix} \sin\Theta & \cos\Theta \\ -\cos\Theta & \sin\Theta \end{bmatrix} \tag{8,5}$$

The relevant matrix solutions at $\lambda = \pm 1$ are then

$$\phi_1^0(\xi,-1) = \begin{bmatrix} 0 & -\beta^{1/2}A^{-1} \\ A\beta^{-1/2} & -2A\beta^{3/2}\xi \end{bmatrix}$$

$$\phi_2^0(\eta,1) = \begin{bmatrix} -A\beta^{1/2} & 2A\beta^{-3/2}\eta \\ 0 & -A^{-1}\beta^{-1/2} \end{bmatrix} \tag{8.6}$$

Note that we display symplectic matrices which in the case of a 2 X 2 matrix space is equivalent to showing unimodular matrices.

Three constraints must now be imposed upon solution (8,2).

(i) $\Phi_1^0(\xi,\lambda)$ and $\Phi_2^0(\eta,\lambda)$ must be symplectic.

(ii) $\Phi_1^0(\xi,\lambda)$ and $\Phi_2^0(\eta,\lambda)$ must be commutative

(iii) $\Phi_1^0(\xi,\lambda)$ must reduce to Φ_0^1 of equation (8,6) at $\lambda = -1$ and Φ_2^0 must decay to Φ_2^0 of equation (8,6) at $\lambda = 1$.

The most general matrices $\Phi_1^0(\xi,\lambda)$, $\Phi_2^0(\eta,\lambda)$, solutions of equation (8,1), that are symplectic and commutative are.

$$\Phi_1^0(\xi,\lambda) = \begin{bmatrix} \left(H\,Sh\,\mu_1 + \sqrt{1+H^2}\;Ch\,\mu_1\right), & \nu\left(H\,Ch\,\mu_1 + \sqrt{1+H^2}\;Sh\,\mu_1\right) \\ \nu^{-1}\left(H\,Ch\,\mu_1 + \sqrt{1+H^2}\;Sh\,\mu_1\right), & \left(H\,Sh\,\mu_1 + \sqrt{1+H^2}\;Ch\,\mu_1\right) \end{bmatrix} \qquad (8.7)$$

$$\Phi_2^0(\eta,\lambda) = \begin{bmatrix} \left(G\,Sh\,\mu_2 - \sqrt{1+G^2}\;Ch\,\mu_2\right), & \nu\left(H\,Ch\,\mu_2 - \sqrt{1+G^2}\;Sh\,\mu_2\right) \\ \nu^{-1}\left(G\,Ch\,\mu_2 - \sqrt{1+G^2}\;Sh\,\mu_1\right), & \left(G\,Sh\,\mu_1 - \sqrt{1+G^2}\;Ch\,\mu_2\right) \end{bmatrix}$$

(8,8)

If we let $H = Sh\tau_1$ and $G = Sh\tau_2$, then we can write these solutions in the form.

$$\phi_1^0(\xi,\lambda) = \begin{bmatrix} Ch(\mu_1+\tau_1) & \nu Sh(\mu_1+\tau_1) \\ \nu^{-1} Sh(\mu_1+\tau_1) & Ch(\mu_1+\tau_1) \end{bmatrix} \tag{8,9}$$

$$\phi_2^0(\eta,\lambda) = -\begin{bmatrix} Ch(\mu_2-\tau_2) & \nu Sh(\mu_2-\tau_2) \\ \nu^{-1} Sh(\mu_2-\tau_2) & Ch(\mu_2-\tau_2) \end{bmatrix} \tag{8,10}$$

Equations (8,7) and (8,8) were obtained in the following way: We start with $\Phi_1(H_1, H_2, H_3, H_4)$ an $\Phi_2(G_1, G_2, G_3, G_4)$, where the H_i's and G_i's are the coefficients in front of the hyperbolic functions present in equation (8,2). Sympletic matrices are obtained by requiring that.

$$H_2H_4 - H_1H_3 = 1 \quad , \quad G_2G_4 - G_1G_3 = 1 \tag{8,11}$$

Then, writing $[\,\phi_1^0\,,\,\phi_2^0\,] = 0$, we obtain four systems of four equations. Each system corresponds to an element of the 2 X 2 commutator . Each equation within each system is obtained by demanding that the coefficients of the four possible combinations of hyperbolic functions be zero. For example, the system resulting from the element on the first row and first column of the commutator is

$$\begin{bmatrix} H_3 & 0 & -H_1 & 0 \\ 0 & H_3 & 0 & -H_1 \\ H_4 & 0 & -H_2 & 0 \\ 0 & H_4 & 0 & -H_2 \end{bmatrix} \cdot \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix} = 0 \qquad (8,12)$$

Demanding that the determinant of the matrix be zero leads to $H_3 = H_1$, $H_4 = H_2$. Then solving the system gives $G_1 = G_3$, $G_2 = G_4$. Using equation (8,11), we get

$$H_3 = H_1 = \sqrt{1 + H^2}\,, \quad H_2 = H_4 = H$$

$$(8,13)$$

$$G_3 = G_1 = \sqrt{1 + G^2}\,, \quad G_2 = G_4 = G$$

which if substituted into equation (8,2) gives equation (8,7) and (8,8).

The other three systems yield no new constraints and are compatible with equation (8,12). In equation (3,8) the presence of the minus sign in front of $\sqrt{1 + G^2}$ is necessary and will be understood as we impose constraint (iii). This constraint will completely determine Φ_1^0 and Φ_2^0.

Dealing with Φ_2^0, constraint (iii) means that at $\lambda = 1$ we must have

$$\lim_{\nu \to \infty} G\, Sh\mu_2 - \sqrt{1 + G^2}\; Ch\mu_2 = -A\beta^{1/2}$$

$$\lim_{\nu \to \infty} \nu^{-1}\left(G\, Ch\mu_2 - \sqrt{1 + G^2}\; Sh\mu_2\right) = 0 \qquad (8,14)$$

Since $\mu_2 = \omega_2 \eta / \nu$, as $\nu \to \infty$, $Sh\,\mu_2 \to 0$ and $Ch\,\mu_2 \to 1$. We then easily deduce

$$G = \sqrt{A^2\beta - 1} \qquad (8,15)$$

This limiting procedure is rather tricky. If we look at $(\phi_2^0)_{22}$, we have

$$-A^{-1}\beta^{-1/2} \neq -A\beta^{1/2} \qquad (8,16)$$

The left-hand term is obtained from equation (8,6) while the right-hand term is deduced from equation (8,8) using equation (8,15). It follows that the limit of equation at $\lambda = 1$ cannot be equal to equation (8,6). Equation (8,15) is right indeed and the paradox is solved as follows

(i) Since the fields arise on the first column of ϕ_2^0, it is important that they can be obtained from equation (8,8) at $\lambda = 1$ Equation (8,15) allows such a task to be performed.

(ii) Taking the limit of equation (8,8) is not straightforward consider the identity

$$\nu^2 \{ Ch^2 (\frac{\sigma_0 \eta}{\beta \nu}) - Sh^2 (\frac{\sigma_0 \eta}{\beta \nu}) \} = \nu^2 \tag{8,17}$$

As $\nu \to \infty$, $ch \to 1$, $sh \to 0$. If we proceeded in a naive manner, we would use L'Hôspital's rule to obtain the limit of $\nu^2 Sh^2 (\frac{\sigma_0 \eta}{\beta \nu})$. This would lead to

$$\nu^2 - (\frac{\sigma_0 \eta}{\beta \nu})^2 = \nu^2 \tag{8,18}$$

which is obviously wrong. Therefore

$$\lim_{\nu \to \infty} \nu^2 S h^2 (\frac{\sigma_0 \eta}{\beta \nu}) \to 0$$

is the correct answer.

The limit of equation (8,8) at $\lambda = 1$ is then

$$\phi_2^0(\eta,1) \quad \begin{bmatrix} \sqrt{1+G^2} & \infty \\ 0 & -\sqrt{1+G^2} \end{bmatrix} \qquad (8,20)$$

and we imply that $0 \cdot \infty = \frac{G}{\nu}$. $\nu G = G^2$

Demanding that we obtain the Gross-Neveu fermion fields from equation (8,20) leads to equation (8,15). Equation (8,6) is simply discarded since it is now irrelevant to our problem.

For ϕ_1, the situation is even more tricky. As $\lambda \to -1$, $\nu \to 0$ and we must have

$$H\, Sh\,\mu_1 + \sqrt{1+H^2}\; Ch\,\mu_1 \to 0$$

$$\nu^{-1}\left(H\, Ch\,\mu_1 + \sqrt{1+H^2}\; Sh\,\mu_1\right) \to A\,\beta^{-1/2} \qquad (8,21)$$

$\mu_1 = \tilde{\omega}_1 \nu \xi$ *so that as* $\lambda \to -1$, $Sh\,\mu_1 \to 0$ *and* $Ch\,\mu_1 \to 1$.
It is obvious that for finite H, equation (8,12) cannot be solved. We make the change of variable $H = \sqrt{(\nu K)^2 - 1}$ *and equation (8,12) becomes*

$$\begin{aligned} &\sqrt{(\nu K)^2-1}\ Sh\,\mu_1 + \nu K\, Ch\,\mu_1 \to 0 \\ &\nu^{-1}\sqrt{(\nu K)^2-1}\ Ch\,\mu_1 + K\, Sh\,\mu_1 \to A\,\beta^{-1/2} \end{aligned} \tag{8,22}$$

For finite K, this is easily solved and we obtain

$$K = \nu^{-1}\sqrt{1 + \frac{\nu^2 A^2}{\beta}} \tag{8,23}$$

But since $\nu \to 0$, K is not finite! Since we know that the limit exists, we define a new K to obtain the correct limit

$$K \equiv \frac{1}{(\nu + \varepsilon)}\sqrt{1 + \frac{\nu^2 A^2}{\beta}} \tag{8,24}$$

where ε is a small parameter important only when $\nu \to 0$.

In what proceeds we assume that $\nu \neq 0$ and we drop ε.

Hence, we obtain

$$H = \frac{\nu A}{\beta^{1/2}} , \tag{8,25}$$

The vacuum solution is now completely determined. It has the form.

$$\phi^{0}(\xi,\eta,\lambda) = - \begin{bmatrix} \delta & \nu\rho \\ \nu^{-1}\rho & \delta \end{bmatrix} \tag{8,26}$$

where $\delta^2 - \rho^2 = 1$

$$\delta = Ch\left(\mu_1 + \mu_2 + (\tau_1 - \tau_2)\right) = C\,h\Delta$$

$$\Delta = \mu_1 + \mu_2 + (\tau_1 - \tau_2)$$

We then have to multiply ϕ^0 and the matrix g to get the vacuum solution in the canonical gauge.

The result is

$$\phi^{0}(\xi,\eta,\lambda) = - \begin{bmatrix} \operatorname{Sin}\theta\,\delta + \nu^{-1}\operatorname{Cos}\theta\,\rho & \nu\operatorname{Sin}\theta\,\rho + \operatorname{Cos}\theta\,\delta \\ -\operatorname{Cos}\theta\,\delta + \nu^{-1}\operatorname{Sin}\theta\,\rho & -\nu\operatorname{Cos}\theta\,\rho + \operatorname{Sin}\theta\,\delta \end{bmatrix} \tag{8,27}$$

8.2 The Matrices F_n

The matrices F_n satisfy

$$F_n = J^{-1}\psi^0(\xi, \lambda_n)\, J\, F_n^0 \qquad n = 1,\cdots,R \tag{8,28}$$

The matrices F_n^0 satisfy equation (6,70) identically.

Let F_n^0 conveniently be of the form

$$F_n^0 = \begin{bmatrix} \nu^{-1/2}\, a_n \\ \nu^{1/2}\, b_n \end{bmatrix}$$ where a_n and b_n

are arbitrary constants which are real for $n < P$ and complex for $n > P$

The matrices F_n are then

$$F_n = \begin{bmatrix} +\psi_{21}(\lambda_n)\,\nu^{1/2}\, b_n - \psi_{22}(\lambda_n)\,\nu^{-1/2}\, a_n \\ -\psi_{11}(\lambda_n)\,\nu^{1/2}\, b_n + \psi_{12}(\lambda_n)\,\nu^{-1/2}\, a_n \end{bmatrix} \tag{8,29}$$

$$= \begin{bmatrix} \cos\theta\, Y_n + \sin\theta\, Z_n \\ -\cos\theta\, Z_n + \sin\theta\, Y_n \end{bmatrix}$$

where

$$Y_n = \nu^{1/2} (a_n \rho_n - b_n \delta_n)$$

$$Z_n = - \nu^{-1/2} (a_n \delta_n - b_n \rho_n)$$

$$\rho_n = \rho (\nu_n) \ , \ \delta_n = \delta (\nu_n)$$

$$\nu_n = \sqrt{\frac{1 + \lambda_n}{1 - \lambda_n}}$$

8.3 The Functions α_n

To find α_n we must solve equation (6,73) and equation (6,74). We have

$$\alpha_n = \alpha(\lambda_n) \tag{8,30}$$

If we let $F^0_n = F^0(\lambda_n)$, then we have

$$\partial_\xi \alpha(\lambda) = F^{0^{tr}}(\lambda)\, \Omega(\lambda)\; J\, F^0(\lambda) \tag{8,31a}$$

$$\partial_\eta \alpha(\lambda) = F^{0^{tr}}(\lambda)\, \Sigma(\lambda)\; J\, F^0(\lambda) \tag{8,31b}$$

$$\text{where } \Omega(\lambda) = -[\psi^0(\lambda)]^{-1} \frac{\tilde{U}^0_1}{(\lambda - 1)^2}\, \psi^0(\lambda)$$

$$\text{where } \Sigma(\lambda) = -[\psi^0(\lambda)]^{-1} \frac{\tilde{V}^0_1}{(\lambda + 1)^2}\, \psi^0(\lambda)$$

We will show that Ω and Σ can be related to the vacuum solution $\phi^0(\lambda)$ and that they do not depend upon the vacuum solution parameter θ. Using equation (3,32) and (3,33), we rewrite Ω and Σ

$$\Omega = -\frac{(\psi^0)^{-1}\psi^0_\xi}{(\lambda - 1)}, \qquad \Sigma = -\frac{(\psi^0)^{-1}\psi^0_\eta}{(\lambda + 1)} \tag{8,32}$$

Since $\psi^0 = g\,\phi^0$, $\psi_\xi = g_\xi\,\phi^0 + g\,\phi^0_\xi$ (similarly for η)

and

$$\Omega = -[\,(\phi^0)^{-1} g^{-1} g_\xi\,\phi^0 + (\phi^0)^{-1}\phi^0_\xi\,]\,(\lambda - 1)^{-1}$$
$$\Sigma = -[\,(\phi^0)^{-1} g^{-1} g_\eta\,\phi^0 + (\phi^0)^{-1}\phi^0_\eta\,]\,(\lambda - 1)^{-1} \tag{8,33}$$

The matrices $g^{-1} g_\xi$ and $g^{-1} g_\eta$ do not depend upon Θ

$$g^{-1} g_\xi = -\tilde{\omega}_1 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -\tilde{\omega}_1\, J$$
$$g^{-1} g_\eta = \omega_2\, J \tag{8,34}$$

$$\Omega = -[\,\tilde{\omega}_1\, J^{-1} (\phi^0)^{tr}\,\phi^0 + (\phi^0)^{-1}\phi^0_\xi\,]\,(\lambda - 1)^{-1}$$
$$\Sigma = -[\,\omega_2\, J\,(\phi^0)^{tr}\,(\phi^0) + (\phi^0)^{-1}\phi^0_\eta\,]\,(\lambda + 1)^{-1} \tag{8,35}$$

Recalling the form of ϕ^0 (equation 8,17), we obtain

$$J(\phi^0)^{tr}\phi = \begin{bmatrix} -\delta\rho\left(\nu+\nu^{-1}\right) & -\left(1+\nu^2\right)\delta^2+\nu^2 \\ -\nu^{-2}+\delta^2\left(1+\nu^{-2}\right) & \sigma\rho\left(\nu^{-1}+\nu\right) \end{bmatrix} \qquad (8,36)$$

$$\frac{(\phi^0)^{-1}\phi^0_\xi}{\tilde{\omega}_1\ \nu} = \frac{\nu}{\omega_2}(\phi^0)^{-1}\phi^0_\eta = \begin{bmatrix} 0 & \nu \\ \nu^{-1} & 0 \end{bmatrix} \qquad (8,37)$$

The matrices Ω and Σ are then

$$\Omega = -\frac{\tilde{\omega}_1}{(\lambda-1)} \begin{bmatrix} \delta\rho\left(\nu+\nu^{-1}\right) & \left(1+\nu^2\right)\delta^2 \\ -\left(1+\nu^{-2}\right)\rho^2 & -\delta\rho\left(\nu^{-1}+\nu\right) \end{bmatrix} \qquad (8,38)$$

$$\Sigma = -\frac{\omega_2}{(\lambda+1)} \begin{bmatrix} -\delta\rho\left(\nu+\nu^{-1}\right) & -\left(1+\nu^2\right)\rho^2 \\ \delta^2\left(1+\nu^{-2}\right) & \delta\rho\left(\nu^{-1}+\nu\right) \end{bmatrix} \qquad (8,39)$$

These two matrices are compatible in the sense that if they are inserted into equation (8,31), they yield a unique α.

$$\partial_\xi \alpha = -\frac{\tilde{\omega}_1}{(\lambda-1)} \left(\nu^{-1/2} a \quad \nu^{1/2} b\right) \begin{bmatrix} \delta\,\rho\,(\nu+\nu^{-1}) & (1+\nu^2)\,\delta^2 \\ -(1+\nu^{-2})\,\rho^2 & -\delta\,\rho\,(\nu+\nu^{-1}) \end{bmatrix} \begin{bmatrix} -\nu^{1/2} b \\ \nu^{-1/2} a \end{bmatrix}$$

$$= -\frac{\tilde{\omega}_1}{(\lambda-1)} \left(\nu + \nu^{-1}\right) \left(b\,\rho - a\,\delta\right)^2 \qquad (8,40a)$$

$$= -\frac{\nu^2}{2} \left(\nu + \nu^{-1}\right)^2 \tilde{\omega}_1\, Z^2 \qquad (8,40b)$$

$$\partial_\eta \alpha = \frac{\omega_2}{2\nu} \left(\nu + \nu^{-1}\right)^2 \left(b\,\delta - a\,\rho\right)^2 \qquad (8,41a)$$

$$= \frac{\omega_2}{2\nu^2} \left(\nu + \nu^{-1}\right)^2 \quad Y^2 \qquad (8,41b)$$

where we used

$$(\lambda + 1)^{-1} = 1/2\left(1 + \nu^{-2}\right), \quad (\lambda - 1)^{-1} = -\,1/2\,(1 + \nu^2)$$

Part (a) of equations (8,40) and (8,41) is more useful for actually doing the integration.

We have

$$\int \delta^2 \, d\mu_i = \frac{1}{4} \, Sh \; 2\Delta + \frac{\mu_i^2}{2} + d \qquad i = 1,2$$

$$\int \rho^2 \, d\mu_i = \frac{1}{4} \, Sh \; 2\Delta - \frac{\mu_i^2}{2} + r \qquad i = 1,2$$

$$\int \delta\rho \, d\mu_i = \frac{1}{4} \, Ch \; 2\Delta + s \qquad i = 1,2$$

where $\mu_1 = \tilde{\omega}_1 \nu \xi$, $u_2 = \frac{\omega_2 \eta}{\nu}$

It follows that

$$a_1 = \int (\partial_\xi \, \alpha) \; d\xi$$

$$= \frac{(\nu + \nu^{-1})^2}{4} \Big\{ \frac{b_2}{2} \left(Sh \; 2\Delta - 2\mu_1' + F_1(\eta) \right)$$

$$- \, b \, a \left(Ch \; 2\Delta + F_2(\eta) \right)$$

$$+ \frac{a}{2} \left(Sh \; 2\Delta + 2\mu_1 + F_3(\eta) \right) \Big\} \qquad (8,42)$$

and

$$\alpha_2 = \int (\partial_\eta \alpha)\, d\eta$$

$$= \frac{(\nu + \nu^{-1})^2}{4} \left\{ \frac{b^2}{2} \left(Sh\ 2\Delta + 2\mu_2 + G_1(\xi)\right) \right. \tag{8,43}$$

$$\left. - b\, a \left(Ch\ 2\Delta + G_2(\xi)\right) + \frac{a^2}{2} \left(Sh\ 2\Delta - 2\mu_2 + G_3(\xi)\right) \right\}$$

Requiring that $\alpha_1 = \alpha_2 \equiv \alpha$ gives

$$\alpha = L\ \alpha' \tag{8,44}$$

where $L \equiv \dfrac{(\nu + \nu^{-1})^2}{4}$

and

$$\alpha' = \frac{(b^2 + a^2)}{2} Sh\ 2\Delta - b\, a\ Ch\ 2\Delta + (\mu_1 - \mu_2)(a^2 - b^2) + \alpha_0 \tag{8,45}$$

To get α_n, we evaluate α at $\lambda = \lambda_n$. Of course the result is very similar to equations (8,44) and (8,45)

$$\alpha_n = L_n \alpha'_n \tag{8,46}$$

and

$$L_n \equiv \frac{(\nu_n + \nu_n^{-1})^2}{4}$$

$$\alpha'_n = \left(\frac{a_n^2 + b_n^2}{2}\right) Sh\, 2\Delta_n - a_n b_n Ch\, 2\Delta_n +$$

$$+ (a_n^2 - b_n^2)(\mu_{1n} - \mu_{2n}) + \alpha_{0n}$$

$$\text{where } \mu_{1n} = \tilde{\omega}_1 \nu_n \xi \qquad \mu_{2n} = \frac{\omega_2 \eta}{\nu_n}$$

$$\alpha_{0n} = \text{constant}$$

8.4 Single – soliton solution

8.4.1 The Matrix X

We have already found an expression for X and need not be concerned with the calculation of M.

We found

$$X = I - \frac{J\ F\ F^{tr}}{(\lambda - \lambda_0)\ \alpha_0} \tag{8,47}$$

Hence.

$$X(\lambda) = \begin{bmatrix} 1 + \dfrac{F_1 F_2}{(\lambda - \lambda_0)\ \alpha_0} & \dfrac{F_2^2}{(\lambda - \lambda_0)\ \alpha_0} \\ \dfrac{-F_1^2}{(\lambda - \lambda_0)\ \alpha_0} & 1 - \dfrac{F_1 F_2}{(\lambda - \lambda_0)\ \alpha_0} \end{bmatrix} \tag{8,48}$$

where the 2 X 1 matrix F is $F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$

given in equation (8,29):

$$F_1 = \cos\theta\ Y_0 + \sin\theta\ Z_0$$

$$F_2 = -\cos\theta\ Z_0 + \sin\theta\ Y_0$$

Using equation (8,48), we define four column vectors $\tilde{X}_1$, $\tilde{X}_2$, $\hat{X}_1$, $\hat{X}_2$

$$(\tilde{X}_1)_i \equiv X_{1i}(\lambda = -1) \;, \quad (\tilde{X}_2)_i \equiv X_{2i}(\lambda = -1)$$

$$(\hat{X}_1)_i \equiv X_{1i}(\lambda = 1) \;, \quad (\hat{X}_2)_i \equiv X_{2i}(\lambda = 1)$$

$$\tilde{X}_1 = \begin{bmatrix} 1 - \dfrac{F_1 F_2}{(1+\lambda_0)\,\alpha_0} \\[2ex] \dfrac{-\;F_2^2}{(1+\lambda_0)\,\alpha_0} \end{bmatrix} \qquad \tilde{X}_2 = \begin{bmatrix} \dfrac{F_1^2}{(1+\lambda_0)\,\alpha_0} \\[2ex] 1 + \dfrac{F_1 F_2}{(1+\lambda_0)\,\alpha_0} \end{bmatrix} \tag{8,49}$$

$$\hat{X}_1 = \begin{bmatrix} 1 + \dfrac{F_1 F_2}{(1-\lambda_0)\,\alpha_0} \\[2ex] \dfrac{F_2^2}{(1-\lambda_0)\,\alpha_0} \end{bmatrix} \qquad \hat{X}_2 = \begin{bmatrix} \dfrac{-\;F_1^2}{(1-\lambda_0)\,\alpha_0} \\[2ex] 1 - \dfrac{F_1 F_2}{(1-\lambda_0)\,\alpha_0} \end{bmatrix} \tag{8,50}$$

These vectors will be useful to find ψ and ϕ.

8.4.2 The Spinor Field $\begin{bmatrix} \psi \\ \phi \end{bmatrix}$

We define two new column sectors $\tilde{\psi}^0$, $\hat{\psi}^0$

$$\tilde{\psi}^0 \equiv \begin{bmatrix} \sum_\gamma g_{1\gamma}\, \phi_{1\gamma 1}(\xi,-1) \\ \sum_\gamma g_{2\gamma}\, \phi_{1\gamma 1}(\xi,-1) \end{bmatrix} = \begin{bmatrix} A\, \beta^{-1/2} \, Cos\theta \\ A\, \beta^{-1/2} \, Sin\theta \end{bmatrix} \tag{8,51}$$

and

$$\hat{\psi}^0 \equiv \begin{bmatrix} \sum_\gamma g_{1\gamma}\, \phi_{2\gamma 1}(\eta,-1) \\ \sum_\gamma g_{2\gamma}\, \phi_{2\gamma 1}(\eta,-1) \end{bmatrix} = \begin{bmatrix} -A\, \beta^{+1/2} \, Sin\theta \\ A\, \beta^{+1/2} \, Cos\theta \end{bmatrix} \tag{8,52}$$

From equations (8,49) - (8,52), we easily find the fields ψ and ϕ. We get

$$\psi = (\tilde{X}_1^{tr} + i\, \tilde{X}_2^{tr})\, \tilde{\psi}^0$$

$$= A\,\beta^{-1/2}\{\, e^{i\Theta} - \frac{F_1 F_2}{(1+\lambda_0)\,\alpha_0}\, e^{-i\Theta} + i\,\frac{F_1^2 \operatorname{Cos}\theta}{(1+\lambda_0)\alpha_0} - \frac{F_2^2 \operatorname{Sin}\theta}{(1+\lambda_0)\alpha_0}\}$$

$$= A\,\beta^{-1/2} e^{i\Theta}\{\, 1 + \frac{Y_0\,(Z_0 + i\,Y_0)}{(1+\lambda_0)\,L_0\,\alpha'_0}\}$$

$$= A\,\beta^{-1/2} e^{i\Theta}\{\, 1 + \frac{2\,Y_0(Z_0 + i\,Y_0)}{\nu_0\,(\nu_0 + \nu_0^{-1})\,\alpha'_0}\} \qquad (8,53)$$

where

$$Y_0 = \nu^{1/2}(\, b_0\, Ch\Delta_0 - a_0\, Sh\Delta_0)$$

$$Z_0 = \nu^{-1/2}(\, a_0\, Ch\Delta_0 - b_0\, Sh\Delta_0)$$

$$\nu_0 = \sqrt{\frac{(1+\lambda_0)}{(1-\lambda_0)}}$$

$$L_0 = \frac{(\nu - \nu^{-1})^2}{4}$$

$$\alpha'_0 = \left(\frac{a_0^2 + b_0^2}{2}\right) Sh2\Delta_0 - a_0 b_0\, Ch2\Delta_0 + (a_0^2 + b_0^2)(\mu_{10} - \mu_{20}) + \alpha_{00}$$

Similarly

$$\phi = (\hat{X}_2^{tr} - i \hat{X}_1^{tr}) \hat{\Psi}_0$$

$$= A \beta^{1/2} \{ e^{i\theta} - \frac{F_1 F_2 e^{-i\theta}}{(1-\lambda_0)\alpha_0} e^{-i\theta} + \frac{F_1^2 \sin\theta}{(1-\lambda_0)\alpha_0} - \frac{iF_2^2 \cos\theta}{(1-\lambda_0)\alpha_0} \}$$

$$= A \beta^{1/2} e^{i\theta} \{ 1 + \frac{2\nu_0 Z_0 (Y_0 - iZ_0)}{(\nu_0 + \nu_0^{-1})\alpha'_0} \} \qquad (8,54)$$

where we used $L_0 (1 - \lambda_0) = \frac{(\nu_0 + \nu_0^{-1})}{2\nu_0}$

The functions present in equation (8,54) are given under equation (8,53).

It should be noted that the only effect of the gauge transformation g on the field ϕ and ψ is to introduce the space-time dependent phase factor $e^{i\theta}$.

8.4.3 The Field σ

We first calculate the complex scalar field $\sigma' \equiv \frac{1}{2}\psi^* \phi$

The result is

$$\sigma' = \sigma_0 \{ 1 + \frac{2}{\alpha'_0} [Y_0 Z_0 - i \frac{(\nu_0^{-1} Y_0^2 + \nu_0 Z_0^2)}{(\nu_0 + \nu_0^{-1})}] - \frac{4i\, Y_0 Z_0 (Y_0^2 + Z_0^2)}{(\nu_0 + \nu_0^{-1})^2 \, (\alpha'_0)^2} \} \qquad (8,57)$$

The field σ that we are looking for is then

$$\sigma = R\, e(\sigma') = \sigma_0 \{ 1 + \frac{2\, Y_0 Z_0}{\alpha'_0} \} \qquad (8,58)$$

8.4.4 Verification

Using

$$Y_{0\xi} = (- \sigma_0 \beta) \nu_0^2 \; Z_0, \quad Z_{0\xi} = (- \sigma_0 \beta) \, Y_0$$

$$Y_{0\eta} = (\frac{\sigma_0 Z_0}{\beta}), \qquad Z_{0\eta} = (\frac{\sigma_0 Y_0}{\beta \, \nu_0^2})$$

$$\alpha'_{0\xi} = 2\, \nu_0^2 \, \tilde{\omega}_1 \, Z_0^2 \qquad (8,59)$$

$$\alpha'_{0\eta} = 2 \frac{\omega_2}{\nu_0^2} Y_0^2$$

we can easily check that indeed we found a solution.

$$\psi_\xi = - i\, \sigma\, \phi$$

$$= \sigma_0 A\, \beta^{1/2} e^{i\theta} \left[-i - \frac{2\, i\, Y_0\, Z_0}{\alpha'_0} - \frac{2\, Z_0\, (Z_0 + iY_0)\, \nu_0}{(\nu_0 + \nu_0^{-1})\, \alpha'_0} - \frac{4\, \nu_0\, Y_0\, Z_0^2\, (Z_0 + i\, Y_0)}{\alpha_0'^2} \right] \tag{8,59}$$

Similarly

$$\phi_\eta = -i\, \sigma\, \psi$$

$$= \sigma_0 A\, \beta^{-1/2} e^{i\theta} \left[-i\, \frac{-2i\, Z_0\, Y_0}{\alpha'_0} + \frac{2\, Y_0\, (Y_0 - iZ_0)}{\nu_0\, (\nu_0 + \nu_0^{-1})\, \alpha'_0} + \frac{4\, Z_0\, Y_0^2\, (Y_0 - i\, Z_0)}{\nu^{-1}\, (\alpha'_0)^2} \right] \tag{8.60}$$

We shall wait until chapter 9 and 10 for an extensive discussion of this solution.

8.5 Two-Soliton Solution

8.5.1 The Matrix X

We rewrite equation (7,34) in the form

$$X = I + \frac{1}{D_{12}} \sum_{\substack{i,j=1 \\ j\neq j}}^{2} \frac{-(\lambda_i-\lambda_j)^2 \alpha_j J F_i F_i^{tr} + (\lambda_i-\lambda_j) F_{ji} J F_j F_i^{tr}}{(\lambda-\lambda_i)} \tag{8,61}$$

The solution will be defined in the center-of-mass system and can be obtained in any frame by a Lorentz boost. Set $\lambda_1 = -\lambda_2 \equiv \lambda_0$. A slightly different notation than that of section (8,4) will be used for the matrices F_i.

$$F_i \equiv \begin{bmatrix} F_a^i \\ F_b^i \end{bmatrix} \qquad i = 1,2 \tag{8,62}$$

It follows that

$$J F_j F_i^{tr} \equiv \begin{bmatrix} (-F_a^i F_b^j) & (-F_b^i F_b^j) \\ (F_a^i F_a^j) & (F_b^i F_a^j) \end{bmatrix} \qquad i,j = 1,2 \tag{8,63}$$

$$F_{ij} = F_i^{tr} J F_j = F_a^j F_b^i - F_b^j F_a^i, \tag{8,64}$$

The matrix X's elements are then

$$X_{11} = 1 + \sum_{i,j}{}' \frac{F_a^i \, \Delta_b^{ij}}{(\lambda - \lambda_i)} \qquad (8,65a)$$

$$X_{12} = \sum_{i,j}{}' \frac{F_b^i \, \Delta_b^{ij}}{(\lambda - \lambda_i)} \qquad (8,65b)$$

$$X_{21} = -\sum_{i,j}{}' \frac{F_a^i \, \Delta_a^{ij}}{(\lambda - \lambda_i)} \qquad (8,65c)$$

$$X_{22} = 1 - \sum_{i,j}{}' \frac{F_b^i \, \Delta_a^{ij}}{(\lambda - \lambda_i)} \qquad (8,65d)$$

where

$$\Delta_a^{ij} \equiv \frac{4 \lambda_0^2 \, \alpha_j \, F_a^i - H \, F_a^j}{D_{12}} \qquad (8,65e)$$

$$\Delta_b^{ij} \equiv \frac{4 \lambda_0^2 \, \alpha_j \, F_b^i - H \, F_b^j}{D_{12}} \qquad (8,65f)$$

$$H \equiv (\lambda_i - \lambda_j)\, F_{ji} = 2\lambda_0 (Z_1 Y_2 - Y_1 Z_2) \tag{8,65g}$$

$$F_a^i \equiv \mathrm{Cos}\Theta\, Y_i + \mathrm{Sin}\Theta\, Z_i \tag{8,65h}$$

$$F_o^i \equiv -\mathrm{Cos}\Theta\, Z_i + \mathrm{Sin}\Theta\, Y_i \tag{8,65i}$$

and the prime in Σ' means that elements with $i = j$ are forbidden.

Next, the convenient column vector X' is defined

$$X'(\lambda) \equiv \begin{bmatrix} X_{11} + i X_{21} \\ X_{12} + i X_{22} \end{bmatrix} \tag{8,66}$$

For example, at $\lambda = -1$, X' takes the form.

$$X'(-1) = \begin{bmatrix} \dfrac{1 - \sum' F_a^i (\Delta_b^{ij} - i\, \Delta_a^{ij})}{(1+\lambda_i)} \\ \dfrac{i - \sum' F_a^i (\Delta_b^{ij} - i\, \Delta_a^{ij})}{(1+\lambda_i)} \end{bmatrix} \tag{8,67}$$

8.5.2 The Spinor Field

Using Equations (8,51) and (8,52), the spinor field is calculated

$$\psi = X'^{tr}(-1)\,\tilde{\psi}^0$$

$$= A\beta^{-1/2}\left\{ e^{i\theta} - \sum{}' \frac{(\Delta_b^{ij} - i\Delta_a^{ij})}{(1+\lambda_i)} (F_a^i \operatorname{Cos}\theta + F_b^i \operatorname{Sin}\theta)\right\} \qquad (8,68a)$$

$$\phi = X'^{tr}(1)\,\hat{\psi}^0$$

$$= A\beta^{-1/2}\left\{ e^{i\theta} - i\sum{}' \frac{(\Delta_b^{ij} - i\Delta_a^{ij})}{(1-\lambda_i)} (-F_a^i \operatorname{Sin}\theta + F_b^i \operatorname{Cos}\theta) \right. \qquad (8,68b)$$

Using equations (65h,i), we have

$$F_a^i \operatorname{Cos}\theta + F_b^i \operatorname{Sin}\theta = Y_i' \qquad (8,69a)$$

$$-F_a^i \operatorname{Sin}\theta + F_b^i \operatorname{Cos}\theta = -Z_i \qquad (8,69b)$$

Also

$$\Delta_b^{ij} - i\Delta_a^{ij} = \frac{1}{D_{12}}\left(4\lambda_0^2 \alpha_j (F_b^i - iF_a^i) - H(F_b^j - iF_a^j)\right) \qquad (8,70a)$$

and

$$F_b^i - i\,F_a^i = -e^{i\theta}(Z_i + i\,Y_i) \qquad (8,70b)$$

Substituting equations (8,69) and (8,70) into equation (8,68), the spinor *field is obtained at a mathematical level equivalent to that of equations (8,53) and (8,54) (which defined the single-soliton solution).*

$$\psi = A\beta^{-1/2} e^{i\theta} \left\{ 1 + \frac{1}{D_{12}} \sum_{i,j}{}' \left[\frac{4\lambda_0^2 \alpha_j (Z_i + iY_i) - H(Z_j + iY_j)}{(1+\lambda_i)} \right] Y_i \right\}$$

(8,71a)

$$\phi = A\beta^{1/2} e^{i\theta} \left\{ 1 - \frac{i}{D_{12}} \sum_{i,j}{}' \left[\frac{4\lambda_0^2 \alpha_j (Z_i + iY_i) - H(Z_j + iY_j)}{(1-\lambda_i)} \right] Z_i \right\}$$

$$= A\beta^{1/2} e^{i\theta} \left\{ 1 + \frac{1}{D_{12}} \sum_{i,j}{}' \left[\frac{4\lambda_0^2 \alpha_i (Y_j + iZ_j) - H(Y_i - iZ_j)}{(1+\lambda_i)} \right] Z_j \right\}$$

(8,71b)

8.5.3 The Field σ

If equations (8,71a) and (8,71b) are substituted into $\sigma = 1/2\, Re\,\{\psi^*\phi\}$, it is found that, at first sight, σ will have terms proportional to $(1/D_{12})$ and $(1/D_{12})^2$. It easy to trace back the origin of these two terms by looking at equation(8,71). The field σ will have the form

$$\sigma = \sigma_0 \left(1 + \frac{\sigma_1}{D_{12}} + \frac{\sigma_2}{D_{12}^2} \right)$$

(8,72)

We believe that the definition of σ_1 and σ_2 is obvious

$$\sigma_1 = \sum_{i,j}{}' \left\{ \frac{4\lambda_0^2 \left(\Sigma'' \alpha_l Y_m Z_m\right) - 2H\, Y_i Z_j}{(1+\lambda_i)} \right\} \tag{7,73}$$

where $l = i,j$, $m = i,j$, $l \neq m$. The second prime in Σ'' is to remind us that m and l are meta–indices. That is, indices whose values they take are themselves indices.

And

$$\sigma_2 = \sum{}' \frac{Y_i Z_i\, 4\lambda_0^2 (Z_i Y_i - Y_i Z_j)\, H(\alpha_j - \alpha_j)}{(1+\lambda_i)(1-\lambda_i)}$$

$$+ \sum{}' \frac{Y_i Z_j \left(16\lambda_0^4 \alpha_i \alpha_j - H^2\right)(Z_i Y_j - Y_i Z_j)}{(1+\lambda_i)^2} \tag{8,74}$$

where we used

$$(Z_i - iY_i)(Y_i - iZ_i) = -i(Y_i^2 + Z_i^2)$$

$$(Z_i - iY_i)(Y_j - iZ_j) = (Z_i Y_j + Y_i Z_j) - i(Y_i Y_i + Z_i Z_i)$$

Notice that the first term of equation (8,74) is zero (of course!). We left it there so that the reader can retrace the origin of the calculation more easily. Using

$$4\lambda_0^2 D_{12} = 16\lambda_0^4 \alpha_i \alpha_j - H^2 \qquad (8,75)$$

together with equations (8,73) and (8,74) in equation (8,72), the field σ is deduced.

$$\sigma + \sigma_0 \{ 1 + \frac{1}{D_{12}} \sum_{i,j} [\frac{4\lambda_0^2 (\sum' \alpha_l Y_m Z_m)}{(1+\lambda_i)} - \frac{2 Y_i Z_j H}{(1+\lambda_l)^2}] \} \qquad (8,76)$$

8.5.4 Verification

Since equations (8,71) anb (8,76) are displayed for the first time, we believe that it is important to show explicityly that they satisfy the Gross–Neveu equations of motion.

$(D_{12})_\xi$ and $(H)_\xi$ are found using equation (8,58) and

$$(\alpha_l)_\xi = \frac{2\sigma_0 \beta Z_l^2}{(1-\lambda_l)^2} \qquad (8,77)$$

We find

$$(D_{12})_\xi = 2\sigma_0 \beta \sum{}' \frac{Z_j}{(1+\lambda_i)} \left[\frac{4\lambda_0^2 \alpha_i Z_j}{(1+\lambda_i)} - HZ_i \right]$$

(8,78a)

$$(H)_\xi = 4\lambda_0^2 \sigma_0 \beta \sum{}' \frac{Z_i Z_j}{(1+\lambda_i)} \quad (8,79b)$$

Let's calculate $\sigma\phi$. It takes the form

$$\sigma\phi \equiv \sigma_0 A\beta^{1/2} e^{i\theta} \left\{ 1 + \frac{(\sigma\phi)_1}{D_{12}} + \frac{(\sigma\phi)_2}{D_{12}^2} \right\}$$

(8,79)

Again, $(\sigma\phi)_1$ and $(\sigma\phi)_2$ are defined by isolating the terms proportional to $(D_{12})^{-1}$ and $(D_{12})^{-2}$ immediately after the multiplication of ϕ an σ. One finds that $(\sigma\phi)_2$ contains a term proportional to D_{12}. Define

$$(\sigma\phi)_2 \equiv (\sigma\phi)_3 D_{12} + (\sigma\phi)_4$$

(8,80)

The final result is:

$$(\sigma\,\phi)_1 = {\sum_{i,j}}'\Big\{ \frac{4\,\lambda_0^2\,[\alpha_i\,(2Y_j - i\,Z_j)\,Z_j + \alpha_j\,Y_i\,Z_i]}{(1+\lambda_i)} - \frac{2\,Y_i\,Z_j\,H}{(1+\lambda_i)^2} - \frac{H\,Z_j\,(Y_i - iZ_i)}{(1+\lambda_i)} \qquad (8,81a)$$

$$(\sigma\,\phi)_3 = -{\sum_{i,j}}'\Big\{\frac{8\,\lambda_0^2\,Y_j\,Z_i\,Z_j\,(Y_i - iZ_i)}{(1+\lambda_i)\,(1-\lambda_i)^2}\Big\} \qquad (8,81b)$$

$$\underline{(\sigma\,\phi)_4} = {\sum_{i,j}}'\Big\{\frac{64\,\lambda_0^4\,Y_i\,Z_i\,Z_j\,(Y_j - iZ_j)\,\alpha_i\,\alpha_j}{(1+\lambda_i)^2\,(1-\lambda_i)}\Big\}$$

$$+\frac{32\,\lambda_0^4\,Y_j\,Z_j^2\,(Y_j - iZ_j)\,\alpha_i^2}{(1+\lambda_i)^2\,(1-\lambda_i)}$$

$$-\frac{16\,\lambda_0^2\,Y_j\,Z_i\,Z_j\,(Y_j - iZ_j)\,\alpha_i\,H}{(1+\lambda_i)\,(1-\lambda_i)^2}$$

$$-\frac{8\,\lambda_0^2\,Y_i\,Z_j^2\,(Y_j - iZ_j)\,\alpha_i\,H}{(1+\lambda_i)^3}$$

$$+\ldots$$

$$\ldots - \frac{8 \lambda_0^2 Y_i Z_i^2 (Y_i - iZ_i) \alpha_i H}{(1+\lambda_i)^2 (1-\lambda_i)} + \frac{2 Y_i Z_i^2 (Y_i - iZ_i) H^2}{(1+\lambda_i)^3} \}$$

(8,81c)

The next step is the calculation of $i\psi_\xi$.

Its form is

$$i\psi_\xi = \sigma_0 A \beta^{1/2} e^{i\Theta} \{ 1 + \frac{\psi_1}{D_{12}} + \frac{\psi_2}{D_{12}^2} \}.$$

(8,82)

Again ψ_2 contains a term proportional to D_{12} and we define

$$\psi_2 = \psi_3 D_{12} + \psi_4$$

(8,83)

The calculation yields

$$\psi_1 = \sum_{i,j}{}' \left\{ \frac{4\lambda_0^2 [\alpha_i (2Y_j - iZ_j) Z_j + \alpha_j Y_i Z_i]}{(1+\lambda_i)} \right\}$$

$$- \frac{2 Y_i Z_j H}{(1+\lambda_i)^2} - \frac{H Z_j (Y_i - iZ_i)}{(1+\lambda_i)}$$

(8,84a)

$$- \frac{8\lambda_0^2 Y_i Z_j^2 (Y_i - iZ_i)}{(1+\lambda_i)^3} + \frac{8\lambda_0^2 Y_i Z_i Z_j (Y_j - iZ_j)}{(1+\lambda_i)^2 (1-\lambda_i)} \}$$

$$\psi_3 = \sum_{i,j}{}' \frac{8\lambda_0^2 Y_i Z_j^2 (Y_i - iZ_i)}{(1+\lambda_i)^3} - \frac{16\lambda_0^2 Y_i Z_i Z_j (Y_i - iZ_j)}{(1+\lambda_i)^2 (1-\lambda_i)}$$

(8,84b)

$$\psi_4 = \sum_{i,j}{}' \left\{ \frac{64\lambda_0^4 Y_i Z_i Z_j (Y_j - iZ_j) \alpha_i \alpha_j}{(1+\lambda_i)^2 (1-\lambda_i)} \right\}$$

$$+ \frac{32\lambda_0^4 Y_i Z_i^2 (Y_i - iZ_i) \alpha_j^2}{(1+\lambda_i)(1-\lambda_i)^2}$$

$+ \ldots$

$$\ldots - \frac{16\,\lambda_0^2\, Y_j\, Z_i\, Z_j (Y_i - iZ_i)\, \alpha_j\, H}{(1-\lambda_i)\qquad (1+\lambda_i)^2}$$

$$- \frac{8\,\lambda_0^2\, Y_i\, Z_j^2\, (Y_j - iZ_j)\, \alpha_i\, H}{(1+\lambda_i)^3}$$

$$- \frac{8\,\lambda_0^2\, Y_i\, Z_i^2\, (Y_j - iZ_j)\, \alpha_j\, H}{(1+\lambda_i)\qquad (1-\lambda_i)^2}$$

$$+ \frac{2\, Y_i\, Z_j^2 (Y_i - iZ_i)\, H^2}{(1+\lambda_i)^3}$$

(8,84c)

The insertion of equations (8,81) and (8,84) into equations (8,79) and (8,82) respectively yields.

$$\psi_\xi = -i\,\sigma\,\psi \qquad (8,85)$$

which is the desired result.

One can similarly deduce that $\psi_\eta = -i\,\sigma\,\psi$. As in the one-soliton solution case, we wait until chapters 9 and 10 for further comments.

8.6 Doublet Solution

8.6.1 From the Two - Soliton Solution to the Doublet Solution.

Doublets are generally described as two bound oscillating solitons. The frequency of oscillation is constant. Apart from their mass, two parameters are needed to describe doublets: their translation velocity and their frequency of oscillation.

The matrix X from which the doublet is obtained is defined in equation (7,39). This equation can be cast into the form.

$$X = I + \frac{1}{\tilde{D}_{12}} \tilde{\sum} \left\{ \frac{-(\lambda_0-\bar{\lambda}_0)^2 \; J \; \bar{F} \; F^{tr} + (\lambda_0-\bar{\lambda}_0) \; \tilde{F} \; J \; \bar{F} \; F^{tr}}{(\lambda - \lambda_0)} \right\} \qquad (8,86a)$$

where

$$\tilde{D}_{12} \equiv |\lambda_0 - \bar{\lambda}_0|^2 \, |a|^2 - |\tilde{F}|^2 \qquad (8,86b)$$

$$\tilde{F} \equiv F^{+} J F \qquad (8,86c)$$

$\tilde{\Sigma}$ *means that, to the term in curly brackets, we must add a term which is the complex-conjugate of the curly-bracket term evaluated at* $\lambda = \lambda$.

λ_0 *is, in general, a complex number. Set*

$$\lambda_0 = \bar{\lambda}_0 + i\,\tilde{\lambda}_0 \qquad (8,87)$$

$\bar{\lambda}_0$ is the velocity of the center of mass and $\tilde{\lambda}_0$ is related to the oscillation frequency of the doublet (In a similar way λ_1 in section (8.5) was related to the velocity at which the two solitons were getting closer (away) to (from) each other. We shall describe doublets at rest so that we set $\lambda_0 = i\,\tilde{\lambda}_0$. It follows that $\bar{\lambda}_0 = -\lambda_0$. Comparing equation (8,86) and (8,61) in the center-of-mass system shows that they are almost identical. Evaluating an element corresponding to the second soliton in equation (8,61) is equivalent to taking the complex conjugate of an element in equation (8,86). This is due to the fact that $\bar{\lambda}_0 = -\lambda_0$ corresponds to $\lambda_2 = -\lambda_1$. Consequently, the doublet solution is obtained by analytically continuing the two-soliton solution to complex values of the parameter λ_0. The doublet solution shall be given explicity only in chapter 9.

8.7 General Remark

In this chapter we have developped the tools necessary to obtain explicit solutions. We have found the single–soliton solution, the two–soliton solution and from it the doublet solution. All these calculation were described in great detail. Due to lack of space we do not give the soliton–doublet solution. After what was done, it is obvious that it represents a straightforward application of equation (7,54). However the algebra should be quite complicated. Finding this solution would not bring forth any new theoretical point. Instead, our energies shall be spent on calculating solutions when an arbitrary number of fermions are present. This is the aim of chapter 9.

9. Soliton Solution for Arbitrary N.

9.1 The Vacum Solution

In this section, we shall solve equation (7,18) for an arbitrary number of fermions N. We shall find two matrices $S(\xi,\lambda)$ and $T(\eta,\lambda)$ satisfying

$$\partial_\xi S = W_1 S \qquad (9,1)$$

$$\partial_\eta T = W_2 T \qquad (9,2)$$

where S and T are 2N X 2N matrices which are symplectic and commutative. W_1 and W_2 are defined in equation (7,19), (7,20) and (7,21).

The vacuum solution in the non-canonical gauge will be.

$$\phi^0(\xi,\eta,\lambda) = S(\xi,\lambda)\, T(\eta,\lambda) \qquad (9,3)$$

Throughout this section we shall implicitly rely upon section 8 to make symplifying assumptions.

9.1.1 The Matrix $S(\xi,\lambda)$

In this section we shall find the matrix S.

We write S in the form

$$S = \begin{bmatrix} S^1 & S^2 \\ S^3 & S^4 \end{bmatrix} \tag{9,4}$$

The matrices S^i are N X N matrices.

Three vectors r, s, t are defined through the relations

$$r_i \equiv (\sigma_i \beta_i)^2 , \quad s_i \equiv \frac{\sigma_0 A_1 \beta_i^{1/2}}{(\lambda - 1)} , \quad t_i \equiv A_i \beta_i^{3/2} \tag{9,5}$$

Equation (9,1) can be expressed as four systems of equations:

$$(S^1_{ij})_\xi = - r_i^{1/2} S^3_{ij} - \frac{s_i}{\sigma_0} \sum_r \frac{t_r}{\beta_r} S^3_{rj} \tag{9,6}$$

$$(S^2_{ij})_\xi = - r_i^{1/2} S^4_{ij} - \frac{s_i}{\sigma_0} \sum_r \frac{t_r}{\beta_r} S^4_{rj} \tag{9,7}$$

$$(S^3_{ij})_\xi = r_i^{1/2} \; S^1_{ij} \tag{9,8}$$

$$(S^4_{ij})_\xi = r_i^{1/2} \; S^2_{ij} \tag{9,9}$$

These systems can be decoupled. For example, if we take the ξ-derivative of equation (9,8) and use equation (9,6), we get a second order differential equation for S^3_{ij}

The decoupled system is

$$(S^1_{ij})_{\xi\xi} = - r_i \, S^1_{ij} - s_i \sum_r t_r \; S^1_{rj} \tag{9,10}$$

$$(S^3_{ij})_{\xi\xi} = - r_i \; S^3_{ij} - s'_i \sum_r t'_r \; S^1_{rj} \tag{9,11}$$

where $s'_i = \dfrac{\sigma_o \; A_i \; \beta^{3/2}}{(\lambda - 1)}$, $t'_i = A_i \, \beta_i^{1/2}$

Identical equations are obtained if we let 1 → 2 in equation (9,10) and 3 → 4 in equation (9,11).

Now, we shall solve equation (9,10) for $j = 1$. Define

$$S_i \equiv S^1_{i1} \qquad i = 1,\cdots, N$$

Next, we assume that

$$S_i = \sum_p C_{ip}\, Ch\,(A_p^{1/2}\xi + c_p) \tag{9,12}$$

where, as of now, C_{ip}, A_p and c_p are arbitrary parameters.

Substituting equation (9,12) into equation (9,10) leads to

$$\sum_p [\, C_{ip} a_p + r_i C_{ip} + s_i \sum_r t_r C_{rp} \,]\, Ch\,(a_p^{1/2}\xi + c_p) = 0 \tag{9,13}$$

The $Ch(\,a_p^{1/2}\ \xi + \nu_p)$'s being independent functions, every coefficient in the summation must vanish. We can form a system of linear equations for the C_{ip}'s

$$\begin{bmatrix} a_p + r_1 + a_1 t_1 , & s_1 t_2 \cdots & s_1 t_N \\ \vdots & \ddots & \vdots \\ l_N m_1 & & a_p + r_p + s_N t_N \end{bmatrix} \begin{bmatrix} C_{1p} \\ \vdots \\ C_{Np} \end{bmatrix} = 0 \tag{9,14}$$

$p = 1,\ldots, N$

To obtain a non-trivial solution to equation (9,14), the determinant of the matrix must vanish. This yields a N^{th} order polynomial equation for the a_p's which allows their determination. After a few simple operations we rewrite equation (9,14) as.

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & -\frac{(a_p+r_N)}{(A_p+r_1)}\frac{s_1 t_1}{s_N t_N} \\ 0 & 1 & & & \\ \vdots & & \ddots & & \vdots \\ & & & 1 & \\ 0 & & \cdots & & \frac{(a_p+r_n)}{(s_N t_N)}\left[\sum_{i=1}^{N}\frac{s_i t_i}{a_p+r_i}+1\right] \end{bmatrix} \begin{bmatrix} t_1 C_{1p} \\ \vdots \\ t_N C_{Np} \end{bmatrix}_{p=1,\cdots,N} = 0 \qquad (9,15)$$

The first $N - 1$ diagonal terms are equal to one. The last diagonal term is $\left(\frac{a_p + r_N}{s_N t_N}\right)\left(\sum_{i=1}^{N}\frac{s_i t_i}{a_p + r_i} + 1\right)$. The last column's nth element is $-\left(\frac{a_p + r_N}{a_p + r_n}\right)\left(\frac{s_n t_n}{s_N t_N}\right), n = 1,\cdots, N - 1$. All other elements are zero.

Obviously, the determinant is equal to zero if and only if

$$\sum_{i=1}^{n} \frac{s_i t_i}{a_p + r_i} + 1 = 0 \qquad (9,16)$$

This is equivalent to the polynomial of order N previously anticipated. C_{Np} is now considered a free parameter and we have

$$C_{ip} = \left(\frac{a_p + r_N}{a_p + r_i} \right) \left(\frac{s_i}{s_N} \right) C_{Np} \qquad (9,17)$$

It follows that

$$S_i = S^1_{i1} = \sum_{p=1}^{n} C_{Np} \left(\frac{a_p + r_n}{a_p + r_i} \right) \left(\frac{s_i}{s_n} \right) Ch \left(a_p^{1/2} \xi + c_p \right) \qquad (9,18)$$

We omit the index N in C_{Np} which has become irrelevant and replace it by an index j which corresponds to the column index of S^1_{ij}. We also add an index j to c_p. The matrix Φ^1 is then

$$S^1 = M^1 P^1 \qquad (9,19)$$

where M and P are N X N matrices

$$M^1_{nm} = \left(\frac{a_m + r_N}{a_m + r_n}\right)\left(\frac{s_n}{s_N}\right) \qquad (9,20)$$

$$P^1_{nm} = C_{nm}\, Ch\, (a_n^{1/2}\,\xi + c_{nm}) \qquad (9,21)$$

and C_{nm} and c_{nm} are arbitrary constants.

Using equation (9,8), we obtain $\bar{S}^3$

$$\begin{aligned}(\underline{S^3_{ij}}) &= \sigma_0\,\beta_i \int S^1_{ij}\, d\xi \\ &= (M^1 \sigma_0 \beta_i \int P^1 d\xi)_{ij} \\ &= M^3\, P^3\end{aligned} \qquad (9,22)$$

where

$$M^3_{nm} = \sigma_0\,\beta_n\, M^1_{nm}$$

$$P^3_{nm} = \frac{C_{nm}}{a_n^{1/2}}\; Sh\,(a_n^{1/2}\,\xi + c_{nm})$$

For S^2, we start with a Sh series and find

$$S^2 = M^1 P^2 \tag{9,23}$$

where $P^2_{nm} = E_{nm}\, Sh\,(a_n^{1/2}\, \xi + e_{nm})$ and M^1 is defined in equation (9,20). E_{nm} and e_{nm} are arbitrary constants.

Similarly,

$$S^4 = M^3 P^4 \tag{9,24}$$

where $P^4_{nm} = \dfrac{E_{nm}}{a_n^{1/2}}\, Ch\,(a_n^{1/2}\, \xi + e_{nm})$ and M^3 is defined in equation (9,22).

Thus, we found the most general solution to equation (9.1). It is a first order $2N \times 2N$ matrix differential equation and should contain $4N^2$ free parameters. These are C_{nm}, c_{nm}, E_{nm} and e_{nm}. S is cast in a convenient form.

9.1.2 The Matrix $T(n,\lambda)$

To find the matrix $T(n,\lambda)$, we proceed in the same fashion. We state the result. T is written in the form

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$$

(9,25)

The differential equation satisfied by the T^i's are

$$(T^1_{ij})_\eta = -u_i^{1/2} T^3_{ij}$$

$$(T^2_{ij})_\eta = -u_i^{1/2} T^4_{ij}$$

$$(T^3_{ij})_\eta = u_i^{1/2} T^1_{ij} - \frac{\nu_i}{\sigma_0} \sum_k w_k \beta_k T^3_{ij}$$

$$(T^4_{ij})_\eta = u_i^{1/2} T^2_{ij} - \frac{\nu_i}{\sigma_0} \sum_k w_k \beta_k T^4_{ij}$$

(9,26)

where $u_i = \left(\frac{\sigma_0}{\beta_i}\right)^2$, $\nu_i = \frac{\sigma_0 A_i}{(\lambda+1)\beta_i^{1/2}}$, $w_i = \frac{A_i}{\beta_i^{3/2}}$,

The solution is

$$T^3 = -N^3 R^3$$

(9,27)

where $N^3_{nm} \equiv \left(\frac{b_n + u_N}{b_n + u_m} \right) \left(\frac{\nu_n}{\nu_N} \right)$

and $R^3_{nm} \equiv D_{nm} Sh (b_n^{1/2} \eta + d_{nm})$

$$T^1 = - N^1 R^1 \qquad (9,28)$$

where $N^1_{nm} \equiv - \frac{\sigma_0}{\beta_n} N^3_{nm}$

$R^1_{nm} \equiv \frac{D_{nm}}{b_n^{1/2}} Ch (b_n^{1/2} \eta + d_{nm})$

$$T^4 = - N^3 R^4 \qquad (9,29)$$

where $R^4_{nm} = F_{nm} Ch (b_n^{1/2} \eta + f_{nm})$. N^3 is defined in equation (9,27)

$T^2 \equiv - N^1 R^1$

where $R^1_{nm} \equiv \frac{F_{nm}}{B_n^{1/2}} Sh (b_n^{1/2} \eta + f_{nm})$ N' is defined in equation (9,28).

The parameters b_n are determined by the formula

$$\sum_{i=1}^{n} \frac{v_i \, w_i}{b_n + u_i} - 1 = 0 \quad n = 1, \cdots, N \tag{9,30}$$

D_{nm}, F_{nm}, d_{nm}, f_{nm} are the free parameters.

9.1.3 Symplectic Solutions

The conditions that the matrices S and T must satisfy to be symplectic are given.

Symplectic matrices satisfy.

$$A^T J A = J \tag{9,31a}$$

where J is the symplectic form that we have often encountered.
We first consider S. Equation (9,31) in terms of the matrices S^i becomes

$$(S^1)^{tr} S^3 = (S^3)^{tr} S^1 \quad (9,31b)$$

$$(S^2)^{tr} S^4 = (S^4)^{tr} S^2 \quad (9,31b)$$

$$(S^1)^{tr} S^4 - (S^3)^{tr} S^2 = 1 \quad (9,31c)$$

Lets define

$$K_{nm} \equiv (M^{1tr} M^3)_{nm} \quad (9,32)$$

Substituting equation (9,20) and equation (9,22) into equation (9,32) gives

$$K_{nm} = \frac{\sigma_0^2}{(\lambda - 1)} \left(\frac{a_n + r_N}{s_N} \right) \left(\frac{a_m + r_N}{s_N} \right) \sum_l \frac{s_l \, t_l}{(a_n + r_l)\,(a_m + r_l)} \quad (9,33)$$

Substracting equation (9,16) from itself yields

$$\sum_l \frac{s_l \, t_l \, (a_m - a_n)}{(a_n + r_l)\,(a_m + r_l)} = 0 \tag{9,34}$$

That is

$$\sum_l \frac{s_l \, t_l}{(a_n + r_l)\,(a_m + r_l)} = \sum_l \frac{s_l \, t_l}{(a_n + r_l)^2} \, \delta_{nm} \tag{9,35}$$

It follows that

$$K_{nm} = K_n \, \delta_{nm} \tag{9,36}$$

where $K_n = \dfrac{\sigma_0^2}{(\lambda - 1)} \dfrac{(a_n + r_n)^2}{(s_n)^2} \displaystyle\sum_l \frac{s_l \, t_l}{(a_n + r_l)^2}$

We use this result into equation (31b) which written in full is

$$\left(P^{1tr}KP^{3}\right)_{nm} = \sum_{l} \frac{K_n}{a_n^{1/2}} C_{ln}C_{lm} Ch\left(A_l^{1/2}\xi + C_{ln}\right) Sh\left(a_l^{1/2}\xi + C_{lm}\right)$$

$$= \left(P^{1tr}K\,P^{3}\right)_{mn}$$

The last equality will be satisfied only if.

$$c_{ln} = c_{lm} = c_l \tag{9,37}$$

Similarly, equation (9,31c) yields

$$e_{ln} = e_{lm} = e_l \tag{9,38}$$

Now we get the constraint deriving from (9,31d):

$$\sum_{1} K_l \left(P^1_{lm} P^4_{ln} - P^3_{lm} P^2_{ln}\right) = \delta_{mn} \tag{9,39}$$

Inserting the values of the P^i's into (9,39), we obtain

$$\sum_l \frac{K_l}{a_l^{1/2}} C_{lm} E_{ln} Ch\,(e_l - c_l) = \delta_{mn} \qquad (9,40)$$

We want to decouple the value of the matrices C_{lm}, E_{lm} from e_l, c_l so that we let $e_l = c_l$. Consequently

$$\sum_l \frac{K_l}{a_l^{1/2}} C_{lm} E_{ln} = \delta_{mn} \qquad (9,41)$$

For $N = 1$, this reduces to the condition of unimodularity that we imposed.

The same constraints applied to T leads to

$$d_{nm} = f_{nm} = d_n \qquad (9,42)$$

and

$$\sum_l \frac{L_l}{b_l^{1/2}} D_{lm} F_{ln} = \delta_{mn} \qquad (9,43)$$

where

$$L_n = \frac{\sigma_0^2}{(\lambda - 1)} \left[\frac{b_n + u_n}{\nu_n} \right]^2 \sum_l \frac{\nu_l \, w_l}{(b_n + u_l)^2}$$

9.1.4 Commutative Solutions

In this section, it is required that the matrices $S(\xi,\lambda)$ and $T(\eta,\lambda)$ be commutative. Four systems of equations must be solved:

$$S_1 T_1 + S_2 T_3 = T_1 S_1 + T_2 S_3 \tag{9,44a}$$

$$S_3 T_2 + S_4 T_4 = T_3 S_2 + T_4 S_4 \tag{9,44b}$$

$$S_1 T_2 + S_2 T_4 = T_1 S_2 + T_2 S_4 \tag{9,44c}$$

$$S_3 T_1 + S_4 T_3 = T_3 S_1 + T_4 S_3 \tag{9,44d}$$

These equations will constraint the matrices C, D, E, F but not the matrices c, d, e, f. Hence $4N^2$ parameters at our disposal.

Written in full, equations (9,44a) to (9,44d) are at our disposal.

$$\sum_{m,n} \left[\frac{C_{mj} N^1_{in}}{b_n^{1/2}} \sum_l (D_{nl} M^1_{lm}) - \frac{M^1_{im} D_{nm}}{b_n^{1/2}} \sum_l (C_{ml} N^1_{ln}) \right] Ch\mu_n Ch\nu_m$$

$$+ \sum_{m,n} \left[\frac{N^1_{in} C_{mj}}{b_n^{1/2} a_m^{1/2}} \sum_l F_{nm} M^3_{lm} - M^1_{im} D_{nm} \sum_l E_{ml} N^3_{ln} \right] Sh\mu_n Sh\nu_m = 0 \qquad (9,45a)$$

where $\nu_m \equiv a_n^{1/2} \xi + c_n$

$\mu_m \equiv b_n^{1/2} \eta + d_n$

$$\sum_{m,n} \left[\frac{M^3_{im} D_{nj}}{a_m^{1/2}} \sum_l (E_{ml} N^3_{ln}) - \frac{N^3_{in} E_{mj}}{a_n^{1/2}} \sum_l (F_{nl} M^3_{lm}) \right] Ch\mu_n Ch\nu_m$$

$$+ \sum_{m,n} \left[\frac{M^3_{im} D_{nj}}{b_n^{1/2} a_m^{1/2}} \sum_l (C_{ml} N^1_{ln}) - N^3_{im} D_{mj} \sum_l (D_{nl} M^1_{lm}) \right] Sh\mu_n Sh\nu_m = 0 \qquad (9,45b)$$

$$\sum_{m,n}\Big[\frac{N^1_{in}E^1_{mj}}{b_n^{1/2}}\sum_l(D_{nl}M^1_{lm})-M^1_{im}F_{nj}\sum_l(E_{ml}N^3_{ln})\Big]\,Ch\mu_n\,Ch\nu_m$$

$$+\sum_{m,n}\Big[\frac{N^1_{in}}{b_n^{1/2}}\frac{E_{mj}}{a_m^{1/2}}\sum_l F_{nl}M^3_{lm}-\frac{M^1_{im}}{b_n^{1/2}}\frac{F_{nj}}{}\sum_l(C_{ml}N^1_{ln})\Big]\,Sh\mu_n Sh\nu_m=0 \qquad (9,45c)$$

$$\sum_{m,n}\Big[N^3_{in}C^1_{mj}\sum_l(D_{nl}M^1_{lm})-\frac{M^3_{im}D_{nj}}{a_m^{1/2}}\sum_l(C_{ml}N^3_{ln})\Big]\,Sh\mu_n\,Ch\nu_m$$

$$+\sum_{m,n}\Big[\frac{N^3_{in}C_{mj}}{a_m^{1/2}}\sum_l(F_{nl}M^3_{lm})-\frac{M^3_{lm}}{a_m^{1/2}}\frac{D_{nj}}{b_n^{1/2}}\sum_l(C_{ml}N^1_{ln})\Big]\,Sh\mu_n\,Sh\nu_m=0 \qquad (9,45d)$$

Next, define

$$V_{nm}\equiv\sum_l C_{nl}N^1_{lm}\quad,\quad W_{nm}\equiv\sum_l E_{nl}N^3_{lm} \qquad (9,46a)$$

$$V_{nm}\equiv\sum_l D_{nl}M^1_{lm}\quad,\quad Z_{nm}\equiv\sum_l F_{nl}M^3_{lm} \qquad (9,46b)$$

Back to equation (9,45), we find that the coefficients of the "composite" hyperbolic functions (e.g. $Sh\mu_n\ Ch\nu_m$) must vanish since the functions are independent from one antoher for any value of m and n. A set of *eight systems of equations is then obtained*[1]. *These eight systems of equations give us* $4N^4$ *equations to solve but, as we said, only* $4N^2$ *parameters are available. Unless strong symmetries exist or can be imposed upon the system, in general it will not be solvable (except for* $N = 1$*). Using equation (9,46) and performing simple algebraic manipulations upon the coefficients of equation (9,45), we obtain.*

$$M^{1}_{im} D_{nj} V_{mn} + \left(\frac{\sigma_0}{\beta_i}\right) N^{3}_{in} C_{mj} Y_{nm} = 0 \tag{9,47a}$$

$$M^{1}_{im} F_{nj} V_{mn} - \left(\frac{b_n^{1/2}\, a_m^{1/2}}{\sigma_i\, \beta_i}\right) N^{3}_{in} E_{mj} Y_{nm} = 0 \tag{9,47b}$$

$$M^{1}_{im} D_{nj} W_{mn} + \left(\frac{\sigma_0}{\beta_i\, b_n^{1/2}\, a_m^{1/2}}\right) N^{3}_{in} C_{mj} Z_{nm} = 0 \tag{9,47c}$$

[1] *But as we shall see, only four are independent.*

$$M^1_{im} F_{nj} W_{mn} - \left(\frac{1}{\sigma_0 \beta_i}\right) N^3_{in} E_{mj} Z_{nm} = 0 \tag{9,47d}$$

$$M^1_{im} D_{nj} W_{mn} - \left(\frac{a_m^{1/2}}{\sigma_0 \; \beta_i}\right) N^3_{in} C_{mj} Y_{nm} = 0 \tag{9,47e}$$

$$M^1_{im} F_{nj} W_{mn} + \left(\frac{\sigma_0}{\beta_i \; b_n^{1/2}}\right) N^3_{in} E_{mj} Y_{nm} = 0 \tag{9,47f}$$

$$M^1_{im} D_{nj} V_{mn} - \left(\frac{b_n^{1/2}}{\sigma_0 \; b_i}\right) N^3_{in} C_{mj} Y_{nm} = 0 \tag{9,47g}$$

$$M^1_{im} F_{nj} V_{mn} + \left(\frac{\sigma_0}{\beta_i \; a_m^{1/2}}\right) N^3_{in} E_{mj} Z_{nm} = 0 \tag{9,47h}$$

From the operation (47d) minus (47f), we get

$$Y_{nm} + \frac{b_n^{1/2}}{\sigma_0^2} Z_{nm} = 0 \tag{9,48a}$$

An identical equation is the result of the operation (47a) - (47g). From (47c) - (47e) or (47b) - (47h), we have

$$Y_{nm} + \frac{\sigma_0^2}{b_n^{1/2}\ a_m} Z_{nm} = 0 \tag{9,48b}$$

Solving equation (47) in terms of V and W instead of Y and Z yields.

$$V_{mn} + \frac{\sigma_0^2\ W_{mn}}{a_m^{1/2}} = 0 \tag{9,49a}$$

and

$$V_{mn} + \frac{b_n\ a_m^{1/2}}{\sigma_0^2} W_{mn} = 0 \tag{9,49b}$$

Equations (9,48) and (9,49) can be solved if and only if

$$b_n a_m = \sigma_0^4 \tag{9,50}$$

From which it follows that

$$b_n = b,\ a_m = a,\ ba = \sigma_0^4 \tag{9,51}$$

The roots of equations (9,16) and (9,30) are N times degenerate. Solving equation (9,16) and (9,30) in such a case yields.

$$\beta_i = \beta\ ,\ A_i = A\ ,\ \sigma_0 = NA^2 \tag{9,52}$$

and

$$a = -r - Nst = +\sigma_0^2\,\beta^2\,\frac{(1+\lambda)}{(1-\lambda)} = \sigma_0^2\,\beta^2\,\nu^2 \tag{9,53}$$

where $\nu \equiv \sqrt{\frac{1+\lambda}{1-\lambda}}$ and

$$b = -u + N\nu w = \frac{\sigma_0^2}{\beta^2}\left(\frac{1-\lambda}{1+\lambda}\right) = \frac{\sigma_0^2}{\beta^2\,\nu^2} \tag{9,54}$$

For convenience, choose

$$a^{1/2} = \sigma_0 \beta \nu \qquad , \qquad \beta^{1/2} = -\frac{\sigma_0}{\beta\nu} \tag{9,55}$$

Now, we examine the consequence of the degenaracy of the a_m's and b_n's. μ_m and ν_n look like

$$\mu_m = a^{1/2}\xi + c_m \, , \, \nu_n = b^{1/2}\eta + d_n \tag{9,56}$$

The role of the c_m's and d_n's is to define the vacuum solution in such a way that at $\lambda = 1$ and $\lambda = -1$, the Gross-Neveu vacuum fermion fields are obtained. Since the vacuum fermions are all identical (see equation (9,52)), so are the c_m's and the d_n's. We let $c_m = c$ and $d_n = d$. It follows that

$$\mu_m = a^{1/2}\xi + c \equiv u, \, \nu_n = b^{1/2}\eta + d \equiv \nu^1 \tag{9,57}$$

Products of hyperbolic functions are no longer independent. Indeed, we have $Ch\mu_n \, Ch\nu_m = Ch\mu \, Ch\nu^1$, $Ch\mu_n \, Sh\nu_m = Ch\mu \, Sh\nu^1$ and so on. Therefore, equation (9,47a) to equation (9,47h) must be summed over the indices m and n.

[1]*Do not confuse this ν with the one defined in equation (9,53)*

For example equation (9,47a) and equation (9,47d) become

$$\sum_{l} \{ \sum_{n} D_{nj} \sum_{m} C_{ml} - \sum_{m} C_{mj} \sum_{n} D_{nl} \} = 0 \qquad (9,58a)$$

$$\sum_{l} \{ \sum_{n} F_{nj} \sum_{m} E_{ml} - \sum_{n} E_{mj} \sum_{n} F_{nl} \} = 0 \qquad (9,58b)$$

If we define

$$\sum_{m,l} C_{ml} \equiv C \;, \sum_{m,l} D_{ml} \equiv D \,, \sum_{n} D_{nj} \equiv D_j \,, \sum_{m} C_{mj} \equiv C_j \qquad (9,59a)$$

$$\sum_{m,l} E_{ml} \equiv E \;, \sum_{m,l} F_{ml} \equiv F \,, \sum_{m} E_{mj} \equiv E_j \,, \sum_{m} F_{mj} \equiv F_j \qquad (9,59b)$$

then equation (9,58) can be rewritten as

$$D_j \, C - C_j \, D = 0 \qquad j = 1, \cdots, N \qquad (9,60a)$$

$$F_j \, E - E_j \, F = 0 \qquad j = 1, \cdots, N \qquad (9,60b)$$

The sum over l is omitted since it is completely useless.

The remaining six equations are

$$F_j C - E_j D = 0 \quad (9,60c)$$

$$D_j E - C_j F = 0 \quad (9,60d)$$

$$D_j E - \nu C_j D = 0 \quad (9,60e)$$

$$F_j E - \nu E_j D = 0 \quad (9,60f)$$

$$D_j C - \nu^{-1} C_j F = 0 \quad (9,60g)$$

$$F_j C - \nu^{-1} E_j F = 0 \quad (9,60h)$$

This system is contradiction-free if

$$\nu D = F \quad , \nu C = E \quad (9,61)$$

Substitution of equation (9,61) into equation (9,60c–h) yields equation (9,60a) and equation (9,60b). Hence the system involves (at most) 2N + 2 independent constraints (equation (9,60) and equation (9,61).

9.1.5 Symplectic and Commutative Solutions

In this section, we combine the constraints of sections (9.1.3) and (9.1.4) in a contradiction-free fashion. The constraints that we combine are equations (9,41), (9,43), (9,60) and (9,61) along with minor conditions defined throughout the two preceding sections.

Substituting equation (9,52) into equation (9,41) and equation (9,43), we get

$$\sum_l D_{lm} F_{ln} = \frac{\nu}{N} \delta_{mn} \qquad (9,62)$$

$$\sum_l C_{lm} E_{ln} = \frac{\nu}{N} \delta_{mn} \qquad (9,63)$$

An important point to notice is that, due to the strong symmetries imposed by equation (9,52), the individual matrix elements C_{lm}, D_{lm}, E_{lm}, F_{lm} no longer appear in the vacuum solution but only the sums of these elements over their line-index do. These objects (C_j, D_j, E_j, F_j) are defined in equation (9,59). Hence the individual matrix elements have become irrelevant only their sum is relevant.

To solve equations (9,62), (9,63), we assume that

$$C_{lm} = C_m \delta_{lm},\ D_{lm} = D_m \delta_{lm},\ E_{lm} = E_m \delta_{lm}, F_{lm} = F_m \delta_{lm} \qquad (9,64)$$

and we get

$$D_m F_n \delta_{mn} = \frac{\nu}{N} \delta_{mn} \tag{9,65}$$

$$C_m E_n \delta_{mn} = \frac{\nu}{N} \delta_{mn} \tag{9,66}$$

From equations (9,65) and (9,66), we deduce

$$D_m \equiv \tilde{D} \quad , \quad F_n \equiv \tilde{F} \tag{9,67a}$$

$$C_m \equiv \tilde{F} \quad , \quad E_n \equiv \tilde{E} \tag{9,67b}$$

$$\tilde{D} = \frac{\nu}{N} (\tilde{F})^{-1} \quad , \quad \tilde{C} \equiv \frac{\nu}{N} (\tilde{E})^{1} \quad , \tag{9,67c}$$

Equation (9,60) is automatically satisfied while equation (9,61) yields

$$\tilde{F} = \nu \tilde{D} \quad , \quad \tilde{E} = \nu \tilde{C} \tag{9,68}$$

Simultaneously solving equation (9,68) and equation (9,67) gives

$$\tilde{D} = \frac{1}{N^{1/2}}, \quad \tilde{F} = \frac{\nu}{N^{1/2}} , \quad \tilde{C} = \frac{1}{N^{1/2}} , \quad \tilde{E} = \frac{\nu}{N^{1/2}} \tag{9,69}$$

9.1.6 The Vacuum Solution

From section 8, we infer that the parameters c and e are

$$c = Sh^{-1} \left(\frac{N^{1/2} \nu A}{\beta^{1/2}} \right) , \quad e = - Sh^{-1} \left(N^{1/2} \sqrt{A^2 \beta^{-1}} \right) \tag{9,70}$$

The matrices $S(\xi,\lambda)$ and $T(\eta,\lambda)$ take the form

$$S(\xi,\lambda) = \begin{bmatrix} S^1 & S^2 \\ S^3 & S^4 \end{bmatrix} \tag{9,71}$$

where

$$(S^1)_{ij} = \frac{1}{N^{1/2}} Ch(\sigma_0 \beta\nu\xi + c) \quad i=1,\cdots,N \quad j=1,\cdots,N$$

$$(S^2)_{ij} = \frac{\nu}{N^{1/2}} Sh(\sigma_0 \beta\nu\xi + c) \quad i=1,\cdots,N \quad j=1,\cdots,N$$

$$(S^3)_{ij} = \frac{\nu^{-1}}{N^{1/2}} Sh(\sigma_0 \beta\nu\xi + c) \quad i=1,\cdots,N \quad j=1,\cdots,N$$

$$(S^4)_{ij} = \frac{1}{N^{1/2}} Ch(\sigma_0 \beta\nu\xi + c) \quad i=1,\cdots,N \quad j=1,\cdots,N$$

and $\nu \equiv \sqrt{\frac{1+\lambda}{1-\lambda}}$.

And

$$T(\eta,\lambda) = \begin{bmatrix} T^1 & T^2 \\ T^3 & T^4 \end{bmatrix} \tag{9,72}$$

where

$$(T^1)_{ij} = -\frac{1}{N^{1/2}} Ch\left(-\frac{\sigma_0 \eta}{\beta\nu} + e\right)$$

$$(T^2)_{ij} = -\frac{\nu}{N^{1/2}} Sh\left(-\frac{\sigma_0 \eta}{\beta\nu} + e\right)$$

$$(T^3)_{ij} = -\frac{\nu^{-1}}{N^{1/2}} Sh\left(-\frac{\sigma_0 \eta}{\beta\nu} + e\right)$$

$$(T^4)_{ij} = -\frac{1}{N^{1/2}} Ch\left(-\frac{\sigma_0 \eta}{\beta\nu} + e\right)$$

$$i = 1,\cdots,N \qquad j = 1,\cdots,N$$

The vacuum solution $\Phi^0(\xi,\eta;\lambda)$ is

$$\Phi^0(\xi,\eta,\lambda) = -\begin{bmatrix} Ch\,\Delta & \nu\, Sh\,\Delta \\ \nu^{-1} Sh\,\Delta & Ch\,\Delta \end{bmatrix} \qquad (9,73)$$

where $\Delta = \sigma_0 \beta \nu \xi - \frac{\sigma_0 \eta}{\beta} + c + e$ and $Ch\,\Delta$ represents a $N \times N$ matrix whose elements are all identical to $Ch\,\Delta$. Similarly, $Sh\,\Delta$ is a $N \times N$ matrix whose elements are all identical to $Sh\,\Delta$.

It follows that the vacuum solution in the canonical gauge will be very similar to equation (8,27). the only difference is that the elements of equation (8,27) ($N = 1$) must now be interpreted as $N \times N$ matrices whose elements are all identical[1].

[1] To symplify, we assume that $\Theta_\alpha^0 = \Theta^0$ so that $\Theta_\alpha = \Theta$

9.2 The Matrices F_n

Equation (8,28) must now be given for arbitrary N. In this section, results are almost identical to those of section (8.2) except that we state them using a different parametrization for the free parameters a and b.

Due to the simple form of the vacuum solution, F_n^0, which in principle contains 2N free parameters, can be described using only two free parameters. We have

$$(F_n^0)_i \equiv \frac{\nu^{-1/2} a_n}{N^{1/2}} \quad i=1,\cdots,N \tag{9,74a}$$

$$(F_n^0)_j \equiv \frac{\nu^{1/2} b_n}{N^{1/2}} \quad j=N+1,\cdots,2N \tag{9,74b}$$

The matrices F_n have the elements

$$(F_n)_i = \cos\theta\, Y_n + \sin\theta\, Z_n \qquad i=1,\cdots,N \tag{9,75a}$$

$$(F_n)_j = -\cos\theta\, Z_n + \sin\theta\, Y_n \qquad i=N+1,\cdots,2N \tag{9,75b}$$

where

$$Y_n = \frac{\nu^{1/2}}{N^{1/2}}\,(a_n \rho_n - b_n \delta_n)$$

$$Z_n = -\frac{\nu^{1/2}}{N^{1/2}}\,(a_n \delta_n - b_n \rho_n)$$

Instead of using the "cartesian coordinates" a_n and b_n, we introduce "hyperbolic coordinates" R_n and χ_n through

$$b_n^2 - a_n^2 \equiv R_m^2 \tag{9,76a}$$

$$b_n \equiv R_n \operatorname{Ch}\chi_n\,, \quad a_n \equiv R_n \operatorname{Sh}\chi_n \tag{9,76b}$$

Hence Y_n and Z_n become

$$Y_n = -\frac{\nu^{1/2}}{N^{1/2}} R_n \, Ch(\tilde{\Delta}_n) \tag{9,77a}$$

$$Z_n = \frac{\nu^{-1/2}}{N^{1/2}} R_n \, Sh(\tilde{\Delta}_n) \tag{9,77b}$$

where $\Delta_n = \Delta(\nu_n)$, $\nu_n = \sqrt{\frac{1+\lambda}{1-\lambda}}$

$$\tilde{\Delta}_n \equiv \Delta_n - \chi_n$$

Hence the matrix elements $(F_n)_i$ take the simpler form

$$(F_n)_i = -\frac{\nu^{1/2}}{N^{1/2}} R_n \left(Cos\Theta \, Ch\tilde{\Delta} - Sin\Theta \, Sh\tilde{\Delta}_n \right) \quad i=1,\cdots,N \tag{9,78a}$$

$$(F_n)_j = -\frac{\nu^{1/2}}{N^{1/2}} R_n \left(Cos\Theta \, Sh\tilde{\Delta} + Sin\Theta \, Ch\tilde{\Delta}_n \right) \quad i=1,\cdots,N \tag{9,78b}$$

9.3 The Functions α_n

The functions α_n are independent of the number of fermions N. We have

$$\alpha_n = L_n \alpha_n \tag{9,79}$$

where $L_n \equiv \dfrac{(\nu_n + \nu_n^{-1})^2}{4}$

$$\alpha'_n = \frac{R_n^2}{2} \left(Ch\, 2\chi_n\, Sh\, 2\Delta_n - Sh\, 2\chi_n\, Ch\, 2\Delta_n + 2(\mu_{2n} - \mu_{1n}\right) + \alpha_{0n} \tag{9,80}$$

$$= \frac{R_n^2}{2} \left[Sh\, 2\tilde{\Delta}_n + 2(\mu_{2n} - \mu_{1n}) \right] + \alpha_{0n}$$

9.4 Single - Soliton Solution

9.4.1 The matrix X

We easily generalize equation (8,48) and get

$$\chi(\lambda) = \begin{bmatrix} I + \dfrac{F_2 F_1^{tr}}{(\lambda-\lambda_0)\alpha_0} & \dfrac{F_2 F_2^{tr}}{(\lambda-\lambda_0)\alpha_0} \\ -\dfrac{F_1 F_1^{tr}}{(\lambda-\lambda_0)\alpha_0} & I - \dfrac{F_1 F_2^{tr}}{(\lambda-\lambda_0)\alpha_0} \end{bmatrix}$$

Each "element" of this matrix represents a N X N matrix. I is the N X N identity matrix.

The matrix F has the form $F = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix}$ where F_1 and F_2 are

N X 1 matrice and

$$(F_1)_i = \cos\theta\, Y_0 + \sin\theta\, Z_0 \quad i=1,\cdots,N$$

$$(F_2)_i = -\cos\theta\, Z_0 + \sin\theta\, Y_0 \quad i=1,\cdots,N \qquad (9,82)$$

Y_0 and Z_0 are defined in equation (9,77).

Generalizing equation (8,49) to equation(8,52) is easy. Equation (8,49) and (8,50) become 2N X N matrices while equation (8,51) and (8,52) become 2N X 1 matrices. We shall not show the generalization of these equations explicitly.

9.4.2. The Spinor Field

The generalization of equation (8,53) is a N X 1 matrix. Each element of this N X 1 matrix represents one of the N ψ^{α}-fields that we are looking for. We have

$$\psi^{\alpha} = A\,\beta^{-1/2} \Big\{ e^{i\theta} - \frac{\sum_{\beta} \left(F_1 F_2^{tr} \right)_{\alpha\beta}}{\left(1+\lambda_0 \right) \alpha_0} + i\,\frac{\sum_{\beta} \left(F_1 F_1^{tr} \right)_{\alpha\beta} \operatorname{Cos}\theta}{\left(1+\lambda_0 \right) \alpha_0}$$

$$- \frac{\sum_{\beta} \left(F_2 F_2^{tr} \right)_{\alpha\beta} \operatorname{Sin}\theta}{\left(1+\lambda_0 \right) \alpha_0} \Big\}$$

$$= A\beta^{-1/2} e^{i\theta} \left\{ 1 + \frac{\tilde{Y}_0 \left(\tilde{Z}_0 + i\,\tilde{Y}_0 \right)}{\nu \left(\nu + \nu^{-1} \right) \alpha_0'} \right\} \quad \alpha = 1, \cdots, N \qquad (9,83)$$

where

$$\tilde{Y}_0 \equiv Y_0 \,(N = 1) = N^{1/2}\, Y_0 \,(N)$$

$$\tilde{Z}_0 \equiv Z_0 \,(N = 1) = N^{1/2}\, Z_0 \,(N)$$

Similarly, we find

$$\phi^{\alpha} = A\beta^{1/2} e^{i\theta} \{ 1 + \frac{2\nu}{(\nu+\nu^{-1})} \tilde{Z}_0 (\tilde{Y}_0 - i \tilde{Z}_0) \} \quad (9,84)$$

When $\theta_{\alpha}^{0} = \theta^{0}$ doesn't hold, the solution becomes

$$\psi^{\alpha} = A\beta^{-1/2} e^{i\theta} \{ e^{i\theta_{\alpha}^{0}} + \frac{e^{i\tilde{\theta}^{0}} \tilde{Y}_0 (\tilde{Z}_0 + i \tilde{Y}_0)}{\nu_{\alpha} (\nu+\nu^{-1}) \alpha_0'} \} \quad (9,85)$$

where

$$\theta = -\sigma_0 \beta \xi - \frac{\sigma_0 \eta}{\beta}$$

$$\tilde{\theta}^{0} = \sum_{\alpha} \theta_{\alpha}^{0}$$

A similar equation holds for ϕ^{α}

9.4.3 The Field σ

The Field σ is

$$\sigma = \frac{1}{2} \sum_{\alpha=1}^{N} Re\left((\phi^{\alpha*} \phi^{\alpha} \right) \tag{9,86}$$

Using equation (9,85) and the equivalent equation for ϕ^{α}, we find

$$\sigma = \sigma_0 \left\{ 1 + \frac{2\tilde{Y}_0 \tilde{Z}_0}{\alpha'_0} \right\} \quad n=1,\cdots \tag{9,87}$$

Thus, we have shown the field σ is independent of the number (N) of fermions present in the system. The result will generally be true for any type of solution of the Gross-Neveu model (two-soliton solution, doublet solution, etc).

Using the representation of σ in terms of R_0 and χ_0, we find.

$$\sigma = \sigma_0 \left\{ 1 - \frac{2R_0^2 \; Ch\,\tilde{\Delta}_0 \; Sh\,\tilde{\Delta}_0}{1/2\, R_0^2 \left[Sh\,2\tilde{\Delta}_0 + 2(\mu_{20} - \mu_{10}) \right] + \alpha_\infty} \right\} \tag{9,88a}$$

$$\sigma = \sigma_0 \left\{ 1 - \frac{2\,Sh\,2\tilde{\Delta}_0}{\left[Sh\,2\tilde{\Delta}_0 + 2\left(\mu_{20} - \mu_{10}\right)_\sigma \right] + \tilde{\alpha}_\infty} \right\} \tag{9,88b}$$

where $\tilde{\alpha}_\infty \equiv R_0^{-2}\ \alpha_\infty$ and $\tilde{\Delta}_0 = \Delta_0 - \chi_0$

Equation (9,88a) is not a pure soliton since it implicitly contains the two vacua of the Gross-Neveu model. Indeed at $R_0 = 0$ and $R_0 = \infty$, we have $\sigma = \sigma_0$ and $\sigma = -\sigma_0$ respectively. The fact that the other vacuum can be obtained as a particular case of equation (9,88a) is a little suprising since, up until now, no reference whatsoever has been made to this vacuum. Equation (9,88a) has the interesting feature of possessing a parameter which can be used to expand the solution. We assume that R_0^2 is infinitesimal and, up to first order in R_0^2, σ is

$$\sigma = \sigma_0 \left\{ 1 - \frac{2R_0^2\ Ch\ 2\ \tilde{\Delta}_0}{\alpha_\infty} \right\} \tag{9,89}$$

On the other hand, equation (9,88b) is a perfectly well-formed soliton and no reference is made to the parameter R_0. We also notice that since χ_0 is an arbitrary parameter, c and e in Δ_0 are useless. We redefine Δ_0 and χ_0 in such a way that

$$\tilde{\Delta}_0 \equiv \Delta_0 \equiv \mu_{10}\xi + \mu_{20}\eta - \chi_0 \tag{9,90}$$

The fact that the field σ is N-independent was conjectured before by N.P. [5] . However, we believe that this is shown explicitly for the first time. As we have seen, this is a consequence of the degeneracy of the a_m's and b_n's which was required by the commutativity condition.

But what is the physical basis of the commutativity condition? Nothing else than the relativistic invariance of the equation! Indeed, it is only in commutative vacuum solutions that ξ and η play a symmetric role.

Let's examine our soliton in the physical coordinates x and t. But first, we introduce a velocity parameter λ_ν related to β.

$$\beta \equiv \sqrt{\frac{1+\lambda_\nu}{1-\lambda_\nu}} \tag{9,91}$$

The "ν" in λ_ν stands for "vacuum" since λ_ν is the velocity of the vacuum fermions. The physically measurable velocity of the field σ is λ_σ and is related to λ_ν and λ_0 through

$$\lambda_\sigma = \frac{\lambda_\nu + \lambda_0}{1 + \lambda_0 \lambda_\nu} \tag{9,92}$$

which is the rule of velocity addition in classical relativistic mechanics. Therefore λ_0 is interpreted as the velocity of the field σ measured with respect to the vacuum fermions. The following relationships hold

$$\beta\nu - \frac{1}{\beta\nu} = \frac{2\lambda_\sigma}{\sqrt{1-\lambda_\sigma^2}} , \quad \beta\nu + \frac{1}{\beta\nu} = \frac{2}{\sqrt{1-\lambda_\sigma^2}} \tag{9,93}$$

Consequently, the field σ in physical coordinates is

$$\sigma = \sigma_0 \left[\frac{-Sh\left[\frac{2\sigma_0\,(x-\lambda_\sigma t+\chi_0)}{\sqrt{1-\lambda_\sigma^2}}\right] - 2\sigma_0(\lambda_\sigma x + t) + \tilde{\alpha}}{Sh\left[\frac{2\sigma_0\,(x-\lambda_\sigma t+\chi_0)}{\sqrt{1-\lambda_\sigma^2}}\right] - 2\sigma_0(\lambda_\sigma x + t) + \tilde{\alpha}} \right] \tag{9,94}$$

where $\tilde{\alpha}$ = constant.

Our soliton at rest is:

$$\sigma = \sigma_0 \left[\frac{-Sh(2\sigma_0 x) - 2\sigma_0 t + \tilde{\alpha}}{Sh(2\sigma_0 x) - 2\sigma_0 t + \tilde{\alpha}} \right] , \chi_0 = 0 \tag{9,95}$$

But what a strange soliton, indeed!

The usual type of soliton is described as a solitary wave propagating unchanged in shape with constant velocity. Equation (9,95) is quite different. The important, and tragic, point to notice is that it is singular at

$$t = \frac{Sh\left(2\sigma_0\, x\right) + \tilde{\alpha}}{-2\sigma_0}$$

So this soliton is not well-behaved and cannot serve as an elementary-particle-model solution. However it might have other applications. The second point to notice (which is obvious) is that contrary to the usual static solitons this soliton's shape is time-dependent.

This soliton is not a bag soliton because of its topological properties. A bag behaves as follows: as $t \to \pm\infty$ or $X \to \pm\infty$, $\sigma \to \sigma_0$. This solution behaves according to $\sigma \to \sigma_0$ as $t \to$ and $\sigma \to -\sigma_0$ and $x \to \pm\infty$.

We close this section with an example of possible application of this type of soliton. It is very elementary and we borrow it to the field of meteorology. This soliton might describe a pressure front density. The singularity simply means that the pressure is discontinuous at the border of the cold front and warm front. Measuring the pressure density with respect to $\sigma = \sigma_0$, we find that the pressure is

$$P(x,t) = \frac{2\tanh^{-1}}{\sqrt{a^2+1}}\left\{\frac{-a\tanh(2\sigma_0 x)+1}{\sqrt{a^2+1}}\right\}, \quad t < \left\{\frac{Sh(2\sigma_0 x)-\tilde{\alpha}}{2\sigma_0}\right\}$$

$$= \frac{1}{\sqrt{a^2+1}} \log\left[\frac{a\, th(2\sigma_0 x) - 1 - \sqrt{a^2+1}}{[a\, th(2\sigma_0 x) - 1] + \sqrt{a^2+1}}\right],$$

$$t > \left\{\frac{Sh\, 2\sigma_0 x - \tilde{\alpha}}{2\sigma_0}\right\} \qquad (9,96)$$

Where $a = -2\sigma_0 t + \tilde{\alpha}$

The analogy is not perfect but we only wanted to show that a singular soliton was not totally senseless from a physical point of view. Note the pressure discontinuity in equation (9,96)

9.5 Two-Soliton Solution

Our main task in this section will be to display the explicit analytic form of equation (8,76) using the notation developped earlier in this chapter. Equation (8,71) can be very easily generalized so that we will not display ψ and ϕ for arbitrary N. Written in full the field σ takes the form

$$\sigma = \sigma_0 \{ 1 + \frac{1}{D_{12}} [\frac{4 \lambda_0^2 (\alpha_1 Y_2 Z_2 + \alpha_2 Y_1 Z_1)}{(1 - \lambda_0^2)} - 2H \{ \frac{Y_1 Z_2}{(1 + \lambda_0)^2} + \frac{Y_2 Z_1}{(1 - \lambda_0)^2} \}] \} \tag{9,97}$$

$$\text{where } D_{12} = 4\lambda_0^2 \alpha_1 \alpha_2 - \frac{H^2}{4\lambda_0^2}$$

Define

$$L \equiv \frac{Y_1 Z_2}{(1 + \lambda_0)^2} + \frac{Y_2 Z_1}{(1 - \lambda_0)^2} \qquad (9,98)$$

$$K \equiv \frac{4 \lambda_0^2}{(1 - \lambda_0^2)} (\alpha_1 Y_2 Z_2 + \alpha_2 Y_1 Z_1) \qquad (9,99)$$

$$M \equiv K - 2 HL \qquad (9,100)$$

The field σ is $\sigma = 1 + \frac{M}{D_{12}}$

We have:

$$H = \frac{2 \lambda_0 R_1 R_2}{(1 - \lambda_0^2)^{1/2}} (Sh\, T + \lambda_0\, Sh\, X) \qquad (9,101a)$$

where $T = \frac{2 \sigma_0 \lambda_0 t}{\sqrt{1 + \lambda_0^2}}$, $X = \frac{2 \sigma_0 x}{\sqrt{1 - \lambda_0^2}}$,[1]

[1] χ_{01} and χ_{02} were set equal to zero for convenience.

$$L = -\frac{R_1 R_2}{(1-\lambda_0^2)^{3/2}}\,(\,Sh\,X - \lambda_0\,Sh\,T\,)$$

(9, 101b)

$$K = -\frac{\lambda_0^2 R_1^2 R_2^2}{(1-\lambda_0^2)^2}\{Ch\,2X - Ch\,2T - 4\sigma_0\lambda_0\,x\,Ch\,X\,Sh\,T - 4\sigma_0\,t\,Sh\,X\,Ch\,T$$

$$+\frac{\tilde{\alpha}_1}{R_1^2}\,Sh\,2\Delta_+ + \frac{\tilde{\alpha}_2}{R_2^2}\,Sh\,2\Delta_-\}$$

(9, 101c)

where $\Delta_+ = X + T$ and $\Delta_- = X - T$

$$M = \frac{R_1^2 R_2^2 \lambda_0}{(1-\lambda_0^2)^2}\{\lambda_0[\,Ch\,2X - Ch\,2T\,] + 4(1-\lambda_0^2)\,Sh\,X\,Sh\,T$$

$$+ 4\sigma_0\lambda_0^2\,x\,Ch\,X\,Sh\,T + 4\lambda_0\sigma_0\,t\,Sh\,X\,Ch\,T$$

$$-\frac{\tilde{\alpha}_1}{R_1^2}\,Sh\,2\Delta_+ - \frac{\tilde{\alpha}_2}{R_2^2}\,Sh\,2\Delta_-\}$$

(9, 101d)

$$D_{12} = \frac{R_1^2 R_2^2}{(1-\lambda_0^2)^2} \{ (-\frac{1}{2} + \lambda_0^2) \, Ch\, 2X + (-1 + \frac{\lambda_0^2}{2}) \lambda_0^2 \, Ch\, 2T$$

$$+ 4\sigma_0 \, t \, Sh\, X \, Ch\, T + 4\sigma_0 \lambda_0 \, x \, Ch\, X \, Sh\, T$$

$$+ \frac{\tilde{\alpha}_1}{R_1^2} \, Sh\, 2\Delta_+ + \frac{\tilde{\alpha}_2}{R_2^2} \, Sh\, 2\Delta + 4\sigma_0^2 (t^2 - \lambda_0^2 x^2)$$

$$+ 2\sigma_0 \lambda_0 \, x \left(\frac{\tilde{\alpha}_1}{R_1^2} - \frac{\tilde{\alpha}_2}{R_2^2} \right) - 2\sigma_0 \, t \left(\frac{\tilde{\alpha}_1}{R_1^2} + \frac{\tilde{\alpha}_2}{R_2^2} \right)$$

$$+ (1 - \lambda_0^4) \} \qquad (9,101e)$$

The two-soliton solution defined by equations (9,101d) and (9,101e) is not a pure solution. By playing with particular values of R_1 and R_2, namely ($R_1 = 0$ or $R_1 = \infty$) and/or ($R_2 = 0$ or $R_2 = \infty$). one can find:

(i) The two vacua $\sigma = \sigma_0$, $\sigma = -\sigma_0$

(ii) Solitons or antisolitons travelling in either directions.

We now assume that $R_1 \neq 0, \infty$ or $R_2 \neq 0, \infty$ and let

$\frac{\tilde{\alpha}_1}{R_1^2} \to \tilde{\alpha}_1, \frac{\tilde{\alpha}_2}{R_2^2} \to \tilde{\alpha}_2$ and we give the "pure" two-soliton solution.

$$\sigma = 1 + \frac{\tilde{M}}{\tilde{D}_{12}} \tag{9,102a}$$

where

$$\tilde{M} = \lambda_0 \{ \lambda_0 [Ch\, 2X - Ch\, 2T] + 4(1-\lambda_0^2) Sh\, X \cdot Sh T$$
$$+ 4\sigma_0 \lambda_0^2 X\, Ch\, X\;\; ShT + 4\sigma_0 \lambda_0 t\, Sh\, X\, Ch\, T$$
$$- \tilde{\alpha}_1 Sh\, 2\Delta_+ - \tilde{\alpha}_2 Sh\, 2\Delta_- \} \tag{9,102b}$$

$$\tilde{D}_{12} = \{ (-\frac{1}{2} + \lambda_0^2) Ch 2X + (-1 + \frac{\lambda_0^2}{2}) \lambda_0^2 Ch 2X$$
$$+ 4\sigma_0 t\;\; Sh\, X\, Ch\, T + 4\sigma_0 \lambda_0 x\, Ch\, X\;\; Sh\, T$$
$$+ \tilde{\alpha}_1 Sh\, 2\Delta + \tilde{\alpha}_2 Sh\, 2\Delta - + 4\sigma_0^2 (t^2 - \lambda_0^2 x^2)$$
$$+ 2\sigma_0 \lambda_0\, x(\tilde{\alpha}_1 - \tilde{\alpha}_2) - 2\sigma_0 t(\tilde{\alpha}_1 + \tilde{\alpha}_2) + (1 - \lambda_0^4) \} \tag{9,102c}$$

9.6 Doublet Solution.

The doublet is obtained by a very simple transformation of equation (102). Let $\tilde{\alpha}_2 \to \bar{\tilde{\alpha}}_1$ (the complex-conjugate of $\tilde{\alpha}_1$) and $\lambda_0 \to i\,\tilde{\lambda}$, $\tilde{\alpha}_1$ will, in general, be a complex number.

Define

$$\tilde{\alpha}_1 \equiv \tilde{\alpha} + i\,\tilde{\beta} \qquad (9,103)$$

It follows that $T = i\,\dfrac{2\sigma_0 \tilde{\lambda}\, t}{\sqrt{1+\lambda_0^2}}$ and $X = \dfrac{2\sigma_0\, x}{\sqrt{1+\lambda_0^2}}$.

Let $T \to -i\,T$, $X \to X$.

The doublet is described by the equation

$$\sigma = 1 + \frac{M}{D_{12}} \qquad (9,104a)$$

Where

$$M = \tilde{\lambda}\,\{\,\tilde{\lambda}\,[\,\mathrm{Cosh}\,2X - \mathrm{Cos}\,2T\,] + 4(1+\tilde{\lambda}^2)\,\mathrm{Sinh}\,X\,\mathrm{Sin}\,T$$

$$- 4\,\sigma_0\,\tilde{\lambda}^2\,\chi\,\mathrm{Cosh}\,X\,\mathrm{Sin}\,T + 4\,\sigma_0\,\tilde{\lambda}\,t\,\mathrm{Sinh}\,\chi\,\mathrm{Cos}\,T$$

$$+ 2\,\tilde{\alpha}\,\mathrm{Sinh}\,X\,\mathrm{Cos}\,T - 2\,\tilde{\beta}\,\mathrm{Cosh}\,X\,\mathrm{Sin}\,T\}$$

(9,104b)

$$D_{12} = \{\,(\frac{1}{2} + \tilde{\lambda}^2)\,\mathrm{Cosh}\,2X - (1 + \frac{\tilde{\lambda}^2}{2})\,\tilde{\lambda}^2\,\mathrm{Cos}\,2T$$

$$- 4\,\sigma_0\,t\,\mathrm{Sinh}\,X\,\mathrm{Cos}\,T + 4\,\sigma_0\,\tilde{\lambda}\,\chi\,\mathrm{Cosh}\,\chi\,\mathrm{Sinh}\,T$$

$$+ 2\,\tilde{\alpha}\,\mathrm{Sinh}\,X\,\mathrm{Cos}\,T - 2\,\tilde{\beta}\,\mathrm{Cosh}\,X\,\mathrm{Sin}\,T$$

$$+ 4\,\sigma_0^2\,(t^2 + \tilde{\lambda}^2\,\chi^2) + (1 + \tilde{\lambda}^4)$$

$$+ 4\,\sigma_0\,\tilde{\lambda}\,\chi\,\tilde{\beta} - 4\,\sigma_0\,t\,\tilde{\alpha}\,\}$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are arbitrary real parameters and

$$X = \frac{2\,\sigma_0\,x}{\sqrt{1+\tilde{\lambda}^2}}\;,\quad T = \frac{2\,\sigma_0\,\tilde{\lambda}\,t}{\sqrt{1+\tilde{\lambda}^2}}$$

10 Conclusion

The known non-singular (bounded) single soliton solution of the Gross-Neveu model is [5]

$$\psi = \left(\frac{\zeta}{2} \right)^{1/2} A \left[1 - \frac{i\alpha}{\zeta + i\alpha} (1 - \tanh \rho) \right] e^{-i\theta}$$

$$\phi = \pm \frac{A}{(2\zeta)^{1/2}} \left[\tanh\rho + \frac{i\alpha}{\zeta + i\alpha} (1 - \tanh \rho) \right] e^{-i\theta}$$

$$\sigma = \pm \sigma_0 \tanh \rho \qquad \sigma_0 = A^2$$

$$\theta \equiv \zeta\eta + \zeta^{-1} \xi \qquad \rho \equiv \alpha^{-1} \xi - \alpha \eta$$

The topological properties of this solution are very different from those of equation (9,95). This solution links the two vacua. As $\rho \to \infty$, $\sigma \to \pm \sigma_0$ and as $\sigma \to -\infty$, $\sigma \to \mp \sigma_0$.

We have tried to deduce the known soliton within the framework of the theory developped in the preceding chapters. But things behave as if the matrix structure of the solution (the X matrix) constrains the soliton to some well-defined topological properties (which are not those of the bounded solitons). The topological structure implied by a given X matrix cannot be predicted without further resort topology. No doubt another X matrix exists which would allow the derivation of this known soliton using the vesture method. It remains to be found (to our knowledge).

In the present work, we have.

(i) *Found the one-pole solution corresponding to the conjugate-pole solution given by Zakharov and Mikhailov.*

(ii) *Derived the associated solution containing an arbitrary number of solitons and doublets. (As an example, the soliton-doublet solution was displayed).*

(iii) *Calculated the vacuum solution when an arbitrary number of fermions are present. It was found that vacuum fermions were degenerate in amplitude and velocity. As a consequence any σ-field of the Gross-Neveu model is N-independent. The fermion fields in the case of arbitrary N are trivially related to the case $N = 1$. A practical consequence is that the worker is relieved from the burden of working with 2N X 2N matrices.*

(iv) *Found the analytic expression for the single-soliton, two-soliton and doublet solution. All these solutions have a singularity and a time-dependent amplitude.*

As a closing remark, we might say that it would be an important advance if the topological property of a solution could be predicted regardless of its analytic form (for example, if topological properties could be obtained from the X-matrix). Also one might contest the usefulness of searching for the tanhρ soliton since it has already been derived within the framework of the standard inverse scattering theory. It appears clear to us that the solution of this problem could be transposed to other equations (see articles [7] and [4]) and provide clues to the solution of the topological property problem, hence, it is a relevant question.

Appendix A: Factorization of the Matrix A

In this appendix, we solve equations (6,29) and (6,31) for arbitrary β *and* $\tilde{\beta}$. *We prove that the fields found are independent of a given factorization. We use a new procedure: all quantities are first worked out in the non-canonical gauge. The fields in this gauge are easily converted to the canonical gauge. This method, which is simpler that the one we used came up to our mind only after chapters 2 to 9 were written.*

So as not to confuse β *with the velocity parameter of the vacuum fermions (also* β*), we redefine the functions* β *and* $\tilde{\beta}$: $\beta \to \gamma$, $\tilde{\beta} \to \tilde{\gamma}$. *Equation (6,68) and its* η*-counterpart become*

$$D(\lambda)\, J\, F = \gamma\, J\, F \qquad (A,1a)$$

$$\tilde{D}(\lambda)\, J\, F = \tilde{\gamma}\, J\, F \qquad (A,1b)$$

while (6,51) and (6,52) are transformed into

$$\partial_\xi \alpha - \gamma\alpha = -\frac{F^{tr}\, \tilde{U}_1^0\, J\, F}{(1-\lambda)^2} \qquad (A,2a)$$

$$\partial_\eta \alpha - \tilde{\gamma}\alpha = -\frac{F^{tr}\, \tilde{V}_1^0\, J\, F}{(1+\lambda)^2} \qquad (A,2b)$$

We define a new 2 X 1 matrix[1] E

$E = g^{-1} F$

where g is given in equation (8,5). (A,3)

Equations (A,1) and (A,2) in the non-canonical gauge then become

$$(\partial_\xi - W_1)\, J\, E = \gamma\, J\, E \qquad (A,4a)$$

$$(\partial_\xi - W_2)\, J\, E = \tilde{\gamma}\, J\, E \qquad (A,4b)$$

$$\partial_\xi \alpha - \gamma\alpha = - \frac{E^{tr}\, U_1^0\, J\, E}{(1-\lambda)^2} \qquad (A,5a)$$

$$\partial_\eta \alpha - \tilde{\gamma}\alpha = - \frac{E^{tr}\, V_1^0\, J\, F}{(1+\lambda)^2} \qquad (A,5b)$$

where

$$W_1 = \sigma_0 \beta \begin{bmatrix} 0 & \nu^2 \\ 1 & 0 \end{bmatrix} \qquad W_2 = -\frac{\sigma_0}{\beta} \begin{bmatrix} 0 & 1 \\ \nu^2 & 1 \end{bmatrix} \qquad (A,6a)$$

[1]We no longer work in the arbitrary N case (see section 9.1)

$$U_1^0 = \sigma_0 \beta \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \qquad V_1 = \frac{\sigma_0}{\beta} \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix} \tag{A,6a}$$

Defining two new matrices $\tilde{E}_1$ and $\tilde{E}_2$,

$$\tilde{E}_1 = \nu^{1/2} E_1 \quad , \quad \tilde{E}_2 = \frac{E_2}{\nu^{1/2}} \tag{A,7}$$

we find that they obey

$$\partial_\xi \tilde{E}_1 = -\sigma_0 \beta\nu \; \tilde{E}_2 + \gamma \tilde{E}_1 \tag{A,8a}$$

$$\partial_\xi \tilde{E}_2 = -\sigma_0 \beta\nu \; \tilde{E}_1 + \gamma \tilde{E}_2 \tag{A,8b}$$

$$\partial_\eta \tilde{E}_1 = \frac{\sigma_0}{\beta\nu} \; \tilde{E}_2 + \gamma \tilde{E}_1 \tag{A,8c}$$

$$\partial_\eta \tilde{E}_2 = \frac{\sigma_0}{\beta\nu} \; \tilde{E}_1 + \gamma \tilde{E}_2 \tag{A,8d}$$

If we assume $\gamma = \frac{h_\xi}{h}$ and $\tilde{\gamma} = \frac{h_\eta}{h}$, then the solutions to equation (A,8) are

$$\tilde{E}_1 = h \{ A \exp(\Delta) + B \exp(-\Delta) \} \equiv h\ H_1 \qquad (A,9a)$$

$$\tilde{E}_2 = h \{ A \exp(\Delta) - B \exp(-\Delta) \} \equiv h\ H_2 \qquad (A,9b)$$

where A and B are arbitrary constants while

$$\Delta = \sigma_0 \left(B\nu\xi - \frac{\eta}{\beta\nu} \right)$$

To solve (A,2), assume that $h = f\alpha$ where f is an arbitrary function. We find

$$\partial_\xi \alpha - \gamma\alpha = -\frac{f_\xi \alpha}{f} \qquad (A,10a)$$

$$\partial_\eta \alpha - \tilde{\gamma}\alpha = -\frac{f_\eta \alpha}{f} \qquad (A,10b)$$

Therefore

$$f = \frac{\sqrt{1-\lambda^2}}{2G^{1/2}} \qquad (A,11)$$

Where $G_\xi = \sigma_0 \beta\nu\alpha \; H_1^2$ and $G_\eta = \sigma_0 \left(\frac{\beta\alpha}{\nu}\right) H_2^2$

Substituting these results into the one-soliton solution, we find

$$\psi = A\,\beta^{-1/2}\, e^{i\theta} \left\{ 1 + \frac{(1-\lambda_0)\alpha\, H_2\, (H_1 + i\nu\, H_2)}{4\, G} \right\} \qquad (A,12a)$$

$$\psi = A\,\beta^{1/2}\, e^{i\theta} \left\{ 1 + \frac{(1+\lambda_0)\alpha\, H_1\, (H_2 - i\nu^{-1}\, H_1)}{4\, G} \right\} \qquad (A,12b)$$

$$\sigma = \sigma_0 \left\{ 1 - \frac{\alpha\, H_1\, H_2}{2G} \right\} \qquad (A,12c)$$

If equation (A,12) is inserted into the equations of motion, it is found that the solution is one indeed only if $\alpha_\xi = 0$. Thus, we fall back on the previously known solution.

Useful relationships are

$$H_{1\xi} = H_2\,(\sigma_0 \beta\nu) \qquad H_{1\eta} = \frac{-\sigma_0\beta}{\nu}\, H_2 \qquad (A,13a)$$

$$H_{2\xi} = H_1\,(\sigma_0 \beta\nu) \qquad H_{2\eta} = \frac{-\sigma_0\beta}{\nu}\, H_1 \qquad (A,13b)$$

Survey of Some Developments in the Gross-Neveu Model

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NOTE 1: *Articles preceded by an Asterisk (*) treat directly of the Gross-Neveu Model. The others are complementary references.*

NOTE 2: *Book's titles are underlined while article's titles are stated within quotes.*

NOTE 3: *Within each class articles are placed in authors' alphabetical order. Classes may overlap.*