Unipotents and Graph Manifold Groups

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To my father

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Abrégé Il a été observé par Button que les groupes de matrices de type fini ne contenant aucune matrice unipotente non triviale ont un comportement similaire aux groupes admettant une action propre par isométries semisimples sur un espace CAT(0) complet. Nous démontrons que tout groupe $\mathbb{C}$-linéaire de type fini admet une action sur un tel espace où, lorsque restreinte aux sous-groupes sans unipotents, l'action est en un certain sens docile. Ceci parfait un résultat de Button en caractéristique non nulle. En appliquant ce résultat aux représentations de $\pi_1(M)$ où M est une variété graphe n'admettant pas de métrique riemannienne de courbure négative ou nulle, nous démontrons que toute représentation de dimension finie de $\pi_1(M)$ envoie un élément non trivial de $\pi_1(M)$ sur une matrice unipotente. Lorsque jumelé à certains théorèmes notoires relatifs aux groupes fondamentaux des 3-variétés, il est possible d'en déduire la caractérisation suivante de la courbure négative ou nulle en dimension 3: une 3-variété fermé et asphérique admet une métrique riemannienne de courbure négative ou nulle si et seulement si son groupe fondamental se plonge dans un groupe de Lie compact. Pour certaines variétés M comme ci-dessus, nous démontrons qu'en fait il existe un élément non trivial de $\pi_1(M)$ dont l'image est virtuellement unipotente sous toute représentation de dimension finie de $\pi_1(M)$.

Abstract Button observed that finitely generated matrix groups containing no nontrivial unipotent matrices behave much like groups admitting proper actions by semisimple isometries on complete CAT(0) spaces. We show that any finitely generated $\mathbb{C}$-linear group possesses an action on such a space whose restrictions to unipotent-free subgroups are in some sense tame. This complements a result of Button in the positive-characteristic setting. As an application, we show that if M is a graph manifold that does not admit a nonpositively curved Riemannian metric, then any finite-dimensional linear representation of $\pi_1(M)$ maps a nontrivial element of $\pi_1(M)$ to a unipotent matrix. Together with existing knowledge of 3-manifold groups, this yields the following characterization of nonpositive curvature in dimension 3: a closed aspherical 3-manifold admits a nonpositively curved Riemannian metric if and only if its fundamental group embeds in a compact Lie group. For certain manifolds M as above, we show that in fact there is a

nontrivial element of $\pi_1(M)$ whose image under any finite-dimensional representation of $\pi_1(M)$ is virtually unipotent.

Preface

The body of this manuscript (Chapters 1, 2, and 3) draws from three self-authored articles ([Dou21b], [Dou21a], and [Dou21c], respectively). Theorem 1.0.1 is an unpublished result of Agol [Ago18], though the proof we present differs slightly. The main original contribution of Chapter 1 is Theorem 1.1.1. The main results of Chapters 2 and 3 are Theorems 2.0.2 and 3.0.7, respectively, each original.

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Chapter 0

Introduction

The main result of this thesis is a necessary and sufficient condition for the fundamental group of a closed 3-manifold to embed in a compact Lie group (Corollary 2.0.8). Since geometers tend to favour discrete subgroups among the countable subgroups of Lie groups, and since no infinite subgroup of a compact Lie group is discrete, it may come as a surprise that much of the thrust of this thesis is geometric. However, the following example already demonstrates that commutative algebra can provide a bridge between the theories of discrete and non-discrete groups.

Example 0.0.1. Consider the group $\Gamma = \mathrm{SO}_3(\mathbb{Z}[\sqrt[3]{2}])$ and let $\sigma : \mathbb{Q}(\sqrt[3]{2}) \to \mathbb{Q}(\omega\sqrt[3]{2})$ be a nontrivial field isomorphism, where $\omega \neq 1$ is a cube root of 1 in $\mathbb{C}$. Denote also by σ the map $\mathrm{SL}_3(\mathbb{Q}(\sqrt[3]{2})) \to \mathrm{SL}_3(\mathbb{Q}(\omega\sqrt[3]{2}))$ given by applying the field isomorphism σ entry by entry to the elements of $\mathrm{SL}_3(\mathbb{Q}(\sqrt[3]{2}))$. By the Borel–Harish-Chandra theorem [BHC62], the map $\mathrm{Id} \times \sigma : \Gamma \to \mathrm{SO}_3(\mathbb{R}) \times \mathrm{SO}_3(\mathbb{C})$ embeds Γ as a lattice in the latter product of Lie groups. Compactness of the factor $\mathrm{SO}_3(\mathbb{R})$ implies that the projection $\sigma(\Gamma)$ of Γ to the factor $\mathrm{SO}_3(\mathbb{C})$ remains a lattice in the latter. Moreover, since $\mathrm{SO}_3(\mathbb{R})$ contains no nontrivial unipotent elements (Definition 2.0.1), and since an element of $\mathrm{SL}_3(\mathbb{Q}(\sqrt[3]{2}))$ is unipotent if and only if the same is true of its image under σ, we obtain that the lattice $\sigma(\Gamma) \subset \mathrm{SO}_3(\mathbb{C})$ contains no nontrivial unipotents.

Now the adjoint representation $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}(\mathfrak{sl}_2(\mathbb{C}))$ has kernel $\{\pm\mathrm{Id}\}$ and preserves a nondegenerate symmetric bilinear form on the 3-dimensional complex Lie algebra $\mathfrak{sl}_2(\mathbb{C})$,

namely, that given by $(A, B) \to \mathrm{tr}(AB)$ for $A, B \in \mathfrak{sl}_2(\mathbb{C})$. Since such forms are all equivalent over $\mathbb{C}$, we obtain an embedding of the Lie group $\mathrm{PSL}_2(\mathbb{C})$ into $\mathrm{SO}_3(\mathbb{C})$ upon fixing an appropriate basis for $\mathfrak{sl}_2(\mathbb{C})$. By a dimensionality argument, this map is an isomorphism of Lie groups. Moreover, it sends parabolic elements of $\mathrm{PSL}_2(\mathbb{C})$ to nontrivial unipotents in $\mathrm{SO}_3(\mathbb{C})$. Thus, the preimage of $\sigma(\Gamma)$ under this map is a lattice in $\mathrm{PSL}_2(\mathbb{C})$ that lacks parabolics and is hence cocompact. Interpreting $\mathrm{PSL}_2(\mathbb{C})$ as the orientation-preserving isometry group of hyperbolic 3-space, we conclude that any torsion-free finite-index subgroup of Γ is isomorphic to the fundamental group of a closed hyperbolic 3-manifold (the existence of such finite-index subgroups is guaranteed by Selberg's lemma; see, for instance, [Cas86, Chapter 5, Theorem 4.1]).

The above example shows that the fundamental groups of some arithmetic hyperbolic 3-manifolds can be Galois conjugated into compact Lie groups. In Chapter 1, using much of what is known about fundamental groups of 3-manifolds, we conclude that in fact the fundamental group of any closed 3-manifold admitting a nonpositively curved Riemannian metric virtually embeds in a (typically higher-rank) arithmetic lattice that can be Galois conjugated into a compact Lie group.

Chapter 2 is devoted to proving that, among the closed aspherical 3-manifolds, those whose fundamental groups embed in compact Lie groups are precisely those admitting nonpositively curved Riemannian metrics (Corollary 2.0.7). The structure of the proof is as follows: On the one hand, the absence of a nonpositively curved metric on a closed aspherical 3-manifold M is an obstruction to the existence of a well-behaved action of $\pi_1(M)$ on any complete CAT(0) space. On the other hand, one can associate to any finitely generated subgroup $\Gamma < \mathrm{SL}_n(\mathbb{C})$ containing no nontrivial unipotent matrices (for instance, any precompact finitely generated subgroup) a complete CAT(0) space on which Γ acts in a well-behaved manner (Theorem 2.0.2). Thus, the fundamental group $\pi_1(M)$ of a manifold M as above cannot be realized as a unipotent-free group of matrices over a field of characteristic 0, let alone as a subgroup of a compact Lie group. In fact, we show that for such M any finite-dimensional linear representation of $\pi_1(M)$ maps some nontrivial element to a unipotent; this element a priori depends on the representation.

In Chapter 3, we strengthen the previous statement for some examples of closed aspherical 3-manifolds M admitting no nonpositively curved metrics. Namely, for these M, we show that there is a nontrivial element of $\pi_1(M)$ whose image under any finite-dimensional linear representation of $\pi_1(M)$ is virtually unipotent.

Literature Review

The proof of Theorem 1.0.1 is a slight variation on Agol's unpublished proof of the same fact [Ago18]. The advantage of our proof is that it leads to the discussion in Section 1.1, where we show that a right-angled Coxeter group associated to a connected finite graph embeds as a thin subgroup of a cocompact lattice in an indefinite orthogonal group. The question as to which groups arise as thin groups was raised by Ballas and Long in [BL20], where they show that many arithmetic lattices in $SO(n,1)$ virtually embed as thin subgroups of lattices in $SL_{n+1}(\mathbb{R})$. Prior to that, it was known by classical work of Borel [Bor60, Corollary 4.3] and Tits [Tit72, Theorem 3] that any lattice in the $\mathbb{R}$-points of a nontrivial connected semisimple algebraic $\mathbb{R}$-group without compact factors contains a thin nonabelian free subgroup, and by a famous construction of Kahn and Marković [KM12] that any cocompact lattice in $SO(3,1)$ contains a thin surface subgroup. For some other manifestations of surface groups as thin groups, see [Ham15], [LR16], [CF19], and [KLM18]. To the author's knowledge, Example 1.1.4 (with $n = 5$) is the first construction to appear in the literature of a thin surface subgroup of a cocompact lattice in the split Lie group $SO(2,3)$. The one-parameter family of representations σ_d described in the proof of Theorem 1.0.1 was also used by Danciger, Guéritaud, and Kassel [DGK20] to construct proper affine actions of finitely generated right-angled Coxeter groups on Euclidean spaces.

The discussion in Chapter 2 is intended to complement a series of papers by Button [But17b, But17a, But19] in which he examines finitely generated matrix groups containing no infinite-order unipotents. Button shows that such groups behave in some ways like finitely generated groups admitting proper actions by semisimple isometries on complete CAT(0) spaces; he terms groups admitting such actions *weak* CAT(0). Moreover, Button

proves that finitely generated matrix groups over fields of positive characteristic are indeed weak CAT(0), and asks whether the latter holds for all finitely generated matrix groups lacking infinite-order unipotents (note that unipotent matrices are torsion in positive characteristic; see, for instance, [But17a, Proposition 2.1]). This amounts to asking whether finitely generated subgroups of $SL_n(\mathbb{C})$ containing no nontrivial unipotents are weak CAT(0). While we do not answer this question, we do show that finitely generated subgroups of $SL_n(\mathbb{C})$ consisting entirely of diagonalizable matrices (which is a stronger condition than lacking nontrivial unipotents) are weak CAT(0) (Corollary 2.0.3), and Theorem 2.0.2 at the very least provides geometric reasons for Button's observations about finitely generated unipotent-free matrix groups.

Weak CAT(0) groups featured previously as the *Hadamard* groups studied by Kapovich and Leeb [KL96] in a paper of great relevance to this work. In fact, the group property of being Hadamard is stronger than that of being weak CAT(0), as Kapovich and Leeb adopt a notion of properness for isometric actions—namely, metric properness—that is more restrictive than that which appears here and in the work of Button. The latter notion of properness lies between metric properness and topological properness (for isometric actions) and is the one promoted by Bridson and Haefliger [BH99]. Nevertheless, all of the statements in [KL96] still hold when one replaces "Hadamard" with "weak CAT(0)." Among them is a result credited to Leeb [Lee92] stating that the fundamental group of a closed aspherical 3-manifold M is Hadamard if and only if M admits a nonpositively curved Riemannian metric (the statement in [KL96] includes the assumption that M is Haken, but Perelman's resolution of Thurston's geometrization conjecture [Per02, Per03a, Per03b] obviates this condition). It follows from this fact and Button's aforementioned result on finitely generated matrix groups in positive characteristic that the absence of a nonpositively curved metric on a closed aspherical 3-manifold M precludes linearity of $\pi_1(M)$ in positive characteristic; this does not seem to have been recorded anywhere in the literature.

Together with Corollary 2.0.3, Leeb's result also implies that fundamental groups of manifolds M as above cannot be realized as groups of diagonalizable matrices, already

ruling out embeddings of $\pi_1(M)$ in compact Lie groups, for instance. In order to rule out unipotent-free embeddings altogether, we adjust Leeb's argument to apply in a slightly more general setting (Lemma 2.2.7). Tools for doing so were contained in Duchesne's work [Duc15] on *ballistic* isometries of complete CAT(0) spaces; these are isometries with positive translation length that are not necessarily hyperbolic (to our knowledge, the term "ballistic" is due to Caprace and Monod [CM09]). We remark here that Theorem 1.4 in [Duc15] is false as stated; Lemma 2.2.5 is a substitute that suffices for our purposes.

Already visible in Example 0.0.1, the broad strategy at play in Chapter 2 is to extract information about a matrix group by varying the absolute value on its entry field. This technique was popularized by Tits [Tit72] via the proof of his alternative and was famously exploited by Margulis [Mar84] in the latter's proof of arithmeticity of higher-rank lattices. In our application of the above technique, the availability of "enough" absolute values is guaranteed by a result of Alperin and Shalen [AS82, Prop. 1.2], which they use to prove the following: a finitely generated subgroup $\Gamma < \mathrm{SL}_n(\mathbb{C})$ has finite virtual cohomological dimension provided there is some uniform $k \geq 0$ such that each finitely generated unipotent subgroup of Γ has Hirsch rank at most k. The case $k = 0$ can be viewed as a corollary of Theorem 2.0.2.

We remark that a result similar to Theorem 2.0.2 was announced by Matsnev [Mat07, Theorem 1.4]. However, the proof of [Mat07, Theorem 4.8], on which that result rests, contains an error; what is desired is a CAT(0) action of a finitely generated subgroup G of $\mathrm{SL}_n(\mathbb{C})$ whose restrictions to certain subgroups of G are proper, but what is provided is a proper CAT(0) action for each such subgroup of G.

The discussion in Chapter 3 is motivated by the observation that, often, a finitely generated group Γ admitting no faithful representation as a matrix group lacking infinite-order unipotents even contains an infinite-order element whose image under any finite-dimensional representation of Γ is virtually unipotent. We say an element of an abstract group Γ is *VU* if it possesses the latter property. The study of infinite-order VU elements can be traced at least as far back as a well-known paper of Lubotzky, Mozes, and Raghunathan [LMR00] where they show that an element γ of a finitely generated group Γ

generating a distorted subgroup of Γ is VU (Corollary 2.3.1, originally proved through different means by Button [But17b, Theorem 5.2], generalizes this fact). Other examples of infinite-order VU elements γ of finitely generated groups Γ were provided by Button, who showed that these include the Dehn twists in (most) mapping class groups [But17b] and a certain nontrivial element of Gersten's free-by-cyclic group [But17a]. In all the examples listed so far, it is even true that for any action of Γ on a complete π-visible CAT(0) space X, the element γ acts with translation length 0 on X (Section 4.0.4); for Γ a (generic) mapping class group and $\gamma \in \Gamma$ a Dehn twist, this is a result of Bridson [Bri10] (even without the π-visibility assumption) that preceded the aforementioned work of Button. That this is stronger than saying that γ is VU in Γ is implied for instance by Theorem 2.0.2 (see also Proposition 4.0.12), though we suspect it was previously known to experts.

Chapter 1

Precompact embeddings of right-angled Coxeter groups

The overarching goal of this chapter is to explain the relevance of the theory of virtually special groups, developed by Wise and collaborators, to the subject of precompact embeddings of 3-manifold groups.

To that end, let Σ_1 be a simplicial graph with vertex set $\{v_i\}_{i \in I}$; we think of Σ_1 as a Coxeter scheme in the sense of [VS93, pg. 201, Def. 1.7] all of whose edges are bold. Let W be the group given by the presentation with generators $\{\gamma_i\}_{i \in I}$ in bijection with the vertices of Σ_1 subject to the relations $\gamma_i^2 = 1$ for $i \in I$ and $[\gamma_i, \gamma_j] = 1$ for each distinct $i, j \in I$ such that v_i and v_j are *not* adjacent in Σ_1. The group W is the *(right-angled) Coxeter group* on the (or associated to the) graph Σ_1. (This convention will be convenient for our purposes; however, in the literature, the right-angled Coxeter group associated to a graph Σ is often defined as the right-angled Coxeter group associated to the *complement graph* of Σ in our sense.) We denote by W^+ the subgroup of W consisting of all elements that can be expressed as products of the γ_i of even length. In [Ago18], Agol observed the following.

Theorem 1.0.1. *If Σ_1 is a simplicial graph on n vertices, then the group W embeds in $\mathrm{O}(n)$.*

Throughout this chapter, if $M \in \mathrm{M}_n(\mathbb{R})$ is a symmetric matrix and A is a subdomain

of $\mathbb{C}$, we use the notation

$$\mathrm{O}(M; A) = \{g \in \mathrm{GL}_n(A) \ : \ g^T M g = M\},$$
$$\mathrm{SO}(M; A) = \{g \in \mathrm{SL}_n(A) \ : \ g^T M g = M\}.$$

Proof of Theorem 1.0.1. Fix an order $v_1, \ldots, v_n$ on the vertices of Σ_1. For $d \in \mathbb{R}$, let $M_d = (m_{ij}) \in \mathrm{M}_n(\mathbb{Z}[d])$ be the symmetric matrix given by

$$m_{ij} = \begin{cases} 1 & \text{if } i = j \\ -d & \text{if } i \neq j \text{ and } v_i, v_j \text{ are joined by an edge in } \Sigma_1 \\ 0 & \text{otherwise.} \end{cases}$$

(The matrix M_1 is the Gram matrix of the Coxeter scheme Σ_1 and the given order on the vertices of Σ_1. For $d > 1$, the matrix M_d is the Gram matrix of the Coxeter scheme Σ_d obtained from Σ_1 by replacing each edge with a dotted edge labeled by d. Here, we are again using the conventions employed by [VS93].)

For $d \geq 1$, let $\sigma_d : W \to \mathrm{GL}_n(\mathbb{R})$ be the Tits–Vinberg representation associated to the Coxeter scheme Σ_d and the given order on its vertices; this is the representation given by

$$\sigma_d(\gamma_i)(v) = v - 2(v^T M_d e_i)e_i$$

for $i = 1, \ldots, n$ and $v \in \mathbb{R}^n$, where $(e_1, \ldots, e_n)$ is the standard basis for $\mathbb{R}^n$. It follows from Vinberg's theory of reflection groups that the representations σ_d, $d \geq 1$, are faithful [Vin71, Theorem 5]. Note that we have $W_d := \sigma_d(W) \subset \mathrm{O}(M_d; \mathbb{Z}[d])$.

Now let $K \subset \mathbb{R}$ be a real quadratic extension of $\mathbb{Q}$, let $\tau : K \to K$ be the nontrivial element of $\mathrm{Gal}(K/\mathbb{Q})$, and let $\mathcal{O}_K$ be the ring of integers of K. Then by Dirichlet's unit theorem, there is a unit $\alpha \in \mathcal{O}_K^*$ such that $\alpha \geq \frac{1}{\epsilon}$, where $\epsilon \in (0, 1)$ is such that M_d is positive-definite for $d \in [-\epsilon, \epsilon]$. Thus, we have

$$\frac{|\tau(\alpha)|}{\epsilon} \leq \alpha|\tau(\alpha)| = |\alpha \cdot \tau(\alpha)| = 1,$$

where the final equality holds because $\alpha \in \mathcal{O}_K^*$. We conclude that $|\tau(\alpha)| \leq \epsilon$, and so $M_{\tau(\alpha)}$ is positive-definite. It follows that $\mathrm{O}(M_{\tau(\alpha)}; \mathbb{R})$ is a conjugate of $\mathrm{O}(n)$ in $\mathrm{GL}_n(\mathbb{R})$, so that

it suffices to produce an embedding of $W \cong W_\alpha$ in $\mathrm{O}(M_{\tau(\alpha)}; \mathbb{R})$. Such an embedding is provided by the map $W_\alpha \to \mathrm{O}(M_{\tau(\alpha)}; \mathbb{R})$ given by applying the Galois automorphism τ entry by entry to the elements of W_α. $\qquad\square$

1.1 Thin right-angled Coxeter groups

In this section, we digress slightly and discuss how the proof of Theorem 1.0.1 allows us to realize many right-angled Coxeter groups as thin subgroups of cocompact arithmetic lattices in indefinite orthogonal groups. We retain the notation used in that proof.

Suppose that $n \geq 3$ and that Σ_1 is connected. Under these assumptions, the group W is infinite (since, for instance, the subgroup $\langle \gamma_i, \gamma_j \rangle \subset W$ is an infinite dihedral group when v_i and v_j are adjacent in Σ_1); in particular, the form M_d is indefinite for $d \geq 1$, since W_d is an infinite discrete subgroup of $\mathrm{O}(M_d; \mathbb{R})$ for such d [Vin71]. Let $D \geq 1$ be such that M_d is nondegenerate and its signature constant as d varies within $[D, \infty)$. If in the proof of Theorem 1.0.1 we require that $\alpha \geq \max\{\frac{1}{\epsilon}, D\}$, then since $M_{\tau(\alpha)}$ is positive-definite, we have that $\mathrm{O}(M_\alpha; \mathcal{O}_K)$ is a cocompact arithmetic lattice in $\mathrm{O}(M_\alpha; \mathbb{R})$ (for a survey of the relevant facts, see, for instance, Section 2 of [GPS87]). Since W_α is contained in $\mathrm{O}(M_\alpha; \mathcal{O}_K)$, we obtain the following.

Theorem 1.1.1. *A right-angled Coxeter group associated to a connected graph on $n \geq 3$ vertices embeds in a cocompact arithmetic lattice in $\mathrm{O}(p, q)$ for some $p, q \geq 1$ satisfying $p + q = n$.*

Let $\mathbf{G}$ be a semisimple algebraic $\mathbb{R}$-group and Γ a lattice in $G := \mathbf{G}(\mathbb{R})$. A subgroup $\Delta \subset \Gamma$ is said to be *thin* if Δ is Zariski-dense in G but of infinite index in Γ. We observe the following.

Theorem 1.1.2. *In the above setting, we have that W_α is thin in $\mathrm{O}(M_\alpha; \mathcal{O}_K)$.*

We first justify Zariski-density of W_α.

Lemma 1.1.3. *The group W_d is Zariski-dense in $\mathrm{O}(M_d; \mathbb{R})$ for $d \geq D$.*

Proof. The proof of the main theorem in [BdlH04] applies here, so we only sketch the argument provided there. Let $d \geq D$ and let G_d be the Zariski-closure of W_d in $\mathrm{O}(M_d; \mathbb{R})$. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of $\mathrm{O}(M_d; \mathbb{R})$ and G_d, respectively. It is enough to show that $\mathfrak{g} = \mathfrak{h}$, since the Zariski-closure of $\mathrm{SO}(M_d; \mathbb{R})^\circ$ is $\mathrm{SO}(M_d; \mathbb{R})$ and since $W_d \not\subset \mathrm{SO}(M_d; \mathbb{R})$.

For each distinct pair $i, j \in \{1, \ldots, n\}$, let $E_{i,j}$ be the orthogonal complement of $\langle e_i, e_j \rangle$ in $\mathbb{R}^n$ with respect to M_d. The subgroup of $\mathrm{O}(M_d; \mathbb{R})$ consisting of all elements that fix each vector in $E_{i,j}$ is a 1-dimensional closed subgroup of $\mathrm{O}(M_d; \mathbb{R})$ whose identity component $T_{i,j}$ corresponds to a subspace $\langle X_{i,j} \rangle$ of $\mathfrak{g}$ for some $X_{i,j} \in \mathfrak{g}$. Since M_d is nondegenerate, the elements $X_{i,j}$ form a basis for $\mathfrak{g}$ as a vector space [BdlH04, Lemme 7]. Thus, to show $\mathfrak{g} = \mathfrak{h}$, it suffices to show that $X_{i,j} \in \mathfrak{h}$ for each distinct pair $i, j \in \{1, \ldots, n\}$.

To that end, let $i, j \in \{1, \ldots, n\}$, $i \neq j$, and suppose first that v_i and v_j are adjacent in Σ_1. Then $\sigma_d(\gamma_i \gamma_j)$ generates an infinite cyclic subgroup of $T_{i,j}$, so that $T_{i,j} \subset G_d$. It follows that $X_{i,j} \in \mathfrak{h}$ in this case. One now verifies that, since Σ_1 is connected, any Lie subalgebra of $\mathfrak{g}$ that contains $X_{i,j}$ for all i, j such that v_i, v_j are adjacent in fact contains $X_{i,j}$ for each distinct pair $i, j \in \{1, \ldots, n\}$ [BdlH04, Preuve du Théorème, second cas]. □

Before proceeding to the proof of Theorem 1.1.2, we recall that, given some group property, we say a group G *virtually* possesses that property if a finite-index subgroup of G possesses that property. Likewise, we say a space *virtually* possesses some property if a finite-degree cover of that space possesses that property.

Proof of Theorem 1.1.2. By Lemma 1.1.3, it suffices to show that W_α is of infinite index in $\mathrm{O}(M_\alpha; \mathcal{O}_K)$. Indeed, suppose otherwise. Then W_α is a cocompact lattice in $\mathrm{O}(M_\alpha; \mathbb{R})$. If $n = 3$, then immediately we obtain a contradiction, since in this case W_α is virtually a closed hyperbolic surface group, whereas W is virtually free. If M_α has signature $(2, 2)$ (the one case under consideration in which $\mathrm{SO}(M_\alpha; \mathbb{R})^\circ$ is not simple), then we again obtain a contradiction as W has virtual cohomological dimension at most 3 (for instance, since the latter is an upper bound for the dimension of the Davis complex associated to the infinite right-angled Coxeter group W; see [Dav08, Chapter 1]), while the symmetric space

associated to $O(M_\alpha; \mathbb{R})$ has dimension 4. Now suppose that $n > 3$ and that the signature of M_α is not $(2,2)$. There is some $\beta > \alpha$ and a path $[\alpha, \beta] \to \mathrm{GL}_n(\mathbb{R}), d \mapsto h_d$ such that $h_d^T M_d h_d = M_\alpha$ for all $d \in [\alpha, \beta]$ (this follows, for example, from the fact that $\mathrm{GL}_n(\mathbb{R})$ acts continuously and transitively on the set $\Omega \subset M_n(\mathbb{R})$ of symmetric matrices with the same signature as M_α, and so the orbit map $\mathrm{GL}_n(\mathbb{R}) \to \Omega, g \mapsto g^T M_\alpha g$ is a fiber bundle). Setting $g_d = h_d h_\alpha^{-1}$ for $d \in [\alpha, \beta]$, we have that $g_\alpha = I_n$ and $g_d^T M_d g_d = M_\alpha$ for $d \in [\alpha, \beta]$. For $d \in [\alpha, \beta]$, let $\rho_d = g_d^{-1} \sigma_d g_d$, and note

$$\rho_d(W) \subset g_d^{-1} O(M_d; \mathbb{R}) g_d = O(g_d^T M_d g_d; \mathbb{R}) = O(M_\alpha; \mathbb{R}).$$

Let $\rho_d^+ = \rho_d\big|_{W^+}$ and $\sigma_d^+ = \sigma_d\big|_{W^+}$ for $d \in [\alpha, \beta]$. Then $\rho_\alpha^+(W^+)$ is a cocompact lattice in the connected non-compact simple Lie group $\mathrm{SO}(M_\alpha; \mathbb{R})^\circ$, and the latter is not locally isomorphic to $\mathrm{SO}(2,1)^\circ$ by our assumption that $n > 3$. Thus, by Weil local rigidity [Wei60, Wei62], up to choosing β closer to α, we may assume that for each $d \in [\alpha, \beta]$ there is some $a_d \in \mathrm{SO}(M_\alpha; \mathbb{R})^\circ$ such that

$$\rho_d^+ = a_d \rho_\alpha^+ a_d^{-1} = a_d \sigma_\alpha^+ a_d^{-1}. \tag{1.1}$$

But $\rho_d^+ = g_d^{-1} \sigma_d^+ g_d$, so we obtain from (1.1) that the trace $\mathrm{tr}(\sigma_d(\gamma_i \gamma_j))$ remains constant as d varies within $[\alpha, \beta]$, where $i, j \in \{1, \ldots, n\}$ are chosen so that the vertices v_i, v_j are adjacent in Σ_1.

We claim, however, that $\mathrm{tr}(\sigma_d(\gamma_i \gamma_j)) = 4d^2 - 4 + n$ for $d \geq D$. Indeed, let $d \geq D$. Then M_d is nondegenerate, so that $\mathbb{R}^d$ splits as a direct sum of the 2-dimensional subspace $\langle e_i, e_j \rangle \subset \mathbb{R}^n$ and its orthogonal complement $E_{i,j}$ with respect to M_d. Each of γ_i and γ_j acts as the identity on $E_{i,j}$, so our claim is equivalent to the assertion that $\mathrm{tr}\left(\sigma_d(\gamma_i \gamma_j)\big|_{\langle e_i, e_j \rangle}\right) = 4d^2 - 2$, and the latter follows from the fact that, with respect to the basis (e_i, e_j) of $\langle e_i, e_j \rangle$, the matrices representing $\sigma_d(\gamma_i), \sigma_d(\gamma_j)$ are $\begin{pmatrix} -1 & 2d \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2d & -1 \end{pmatrix},$ respectively. $\square$

Example 1.1.4. We consider the case that $n \geq 5$ and the complement graph of Σ_1 is the cycle $v_1 v_2 \ldots v_n$. In this case, the group W may be realized as the subgroup of $\mathrm{Isom}(\mathbb{H}^2)$

generated by the reflections in the sides of a right-angled hyperbolic n-gon, so that W is virtually the fundamental group of a closed hyperbolic surface. We have

$$M_d = (1 + d)I_n + d(J_n + J_n^{n-1}) - d(I_n + J_n + \ldots + J_n^{n-1}), \tag{1.2}$$

where $J_n \in \mathrm{M}_n(\mathbb{C})$ is the matrix

$$J_n = \begin{pmatrix} e_2 & e_3 & \ldots & e_n & e_1 \end{pmatrix}.$$

There is some $C \in \mathrm{GL}_n(\mathbb{C})$ such that

$$CJ_nC^{-1} = \mathrm{diag}(1, \zeta_n, \zeta_n^2, \ldots, \zeta_n^{n-1}),$$

where $\zeta_n = e^{2\pi i/n}$. Observe that

$$C(I_n + J_n + \ldots + J_n^{n-1})C^{-1} = \mathrm{diag}(n, 0, \ldots, 0),$$
$$C(J_n + J_n^{n-1})C^{-1} = \mathrm{diag}\left(2, 2\cos\frac{2\pi}{n}, 2\cos\frac{2\pi \cdot 2}{n}, \ldots, 2\cos\frac{2\pi(n-1)}{n}\right).$$

It follows from (1.2) that, counted with multiplicity, the eigenvalues of M_d are $1 - d(n-3)$ and $1 + d\left(1 + 2\cos\frac{2\pi k}{n}\right)$, where $k = 1, \ldots, n-1$. Note that for d sufficiently large, we have that $1 - d(n-3) < 0$, and that $1 + d\left(1 + 2\cos\frac{2\pi k}{n}\right) \geq 0$ if and only if $\cos\frac{2\pi k}{n} \geq -\frac{1}{2}$. We conclude that the signature of M_d is $(2\lfloor\frac{n}{3}\rfloor, n - 2\lfloor\frac{n}{3}\rfloor)$ for all d sufficiently large. In particular, if $n = 3m$, $m \geq 2$, then the signature of M_d is $(2m, m)$ for all d sufficiently large. The above discussion yields thin surface subgroups of uniform arithmetic lattices in $\mathrm{SO}(2\lfloor\frac{n}{3}\rfloor, n - 2\lfloor\frac{n}{3}\rfloor)$ for each $n \geq 5$.

1.2 Precompact embeddings of 3-manifold groups

In this section, we explain how Theorem 1.0.1 allows us to embed many closed 3-manifold groups into compact Lie groups.

Definition 1.2.1. A closed 3-manifold is *nonpositively curved* (*NPC*) if it admits a Riemannian metric of everywhere nonpositive sectional curvature.

Definition 1.2.2. A finitely generated group is *C-special* if it embeds in a right-angled Coxeter group associated to a finite graph. We say a finitely generated group is *virtually special* if it is virtually C-special.

Remark 1.2.3. Haglund and Wise [HW08] define a notion of *A-specialness* (respectively, *C-specialness*) for cube complexes and show that the 2-skeleton of an A-special (respectively, C-special) cube complex admits a local isometry into a square complex whose fundamental group is a right-angled Artin group (resp., whose fundamental group is the subgroup W^+ of a right-angled Coxeter group W). Since any right-angled Artin group is the fundamental group of an A-special cube complex [HW08, Example 3.3(2)] and since A-specialness is inherited by covers [HW08, Corollary 3.8], it follows that the fundamental groups of A-special cube complexes are precisely the subgroups of right-angled Artin groups, and, in the literature, a group is usually defined to be *special* if it belongs to this class of groups. This gives a notion of virtual specialness for finitely generated groups that is consistent with Definition 1.2.2, since a right-angled Coxeter group on a finite graph is virtually the fundamental group of an A-special cube complex [HW10], since a finitely generated special group embeds in a right-angled Artin group *on a finite graph* (see the discussion following Corollary 1.3 in [PW18]), and since a group of the latter kind embeds in a right-angled Coxeter group on a finite graph [HW99, DJ00].

Note that a free product (respectively, direct product) of finitely many right-angled Coxeter groups, each associated to a finite graph, is the right-angled Coxeter group associated to the join (resp., disjoint union) of the graphs of the factors. It follows that a finite free (resp., direct) product of C-special groups is again C-special. We observe in the following lemma that the previous statement remains true when we replace "C-special" with "virtually special" (this is evident for direct products, so we only prove the free product case).

Lemma 1.2.4. *Suppose that each of Γ_1 and Γ_2 is virtually special. Then the same is true of $\Gamma := \Gamma_1 * \Gamma_2$.*

Proof. For $i = 1, 2$, let K_i be a finite-index normal subgroup of Γ_i that is C-special, and let $\varphi_i : \Gamma_i \to \Gamma_i/K_i$ be the quotient map. Let $\psi_i : \Gamma \to \Gamma_i/K_i$ be the composition of the

projection $\Gamma \to \Gamma_i$ with φ_i, and let K be the intersection of the kernels of the ψ_i. Then K is a finite-index normal subgroup of Γ whose intersection with the factor Γ_i is K_i. Thus, by the Kurosh subgroup theorem (see, for instance, [SW79, Section 3]), the subgroup K splits as a free product of infinite cyclic subgroups and conjugates of the K_i. Since each of the latter groups is C-special, so is K. □

Remark 1.2.5. Lemma 1.2.4 can also be deduced from the observation that a wedge of two cube orbicomplexes (see [AGM13, Definition 2.1] and the references therein) each of which is virtually an A-special cube complex is virtually an A-special cube complex. A variant of this observation is vastly generalized in [HW12].

The theory of virtually special groups developed by Wise and collaborators has demonstrated that this class of groups is quite large (for an exposition, we refer the reader to Wise's monograph [Wis21]). One culmination of this theory is the following result.

Theorem 1.2.6 ([PW18, Corollary 1.4])**.** *The fundamental group of a closed NPC 3-manifold is virtually special.*

Remark 1.2.7. We discuss briefly the history of Theorem 1.2.6. Together with previous knowledge about closed nonpositively curved Riemannian manifolds [GW71, Yau71, Ebe82], Perelman's resolution of the geometrization conjecture implied that a closed NPC 3-manifold either admits one of the nonpositively curved Thurston geometries—namely, the $\mathbb{R}^3$, $\mathbb{H}^2 \times \mathbb{R}$, or $\mathbb{H}^3$ geometries—or is (virtually) a graph manifold (Definition 2.0.5) or a so-called *mixed* 3-manifold. Virtual specialness of NPC graph manifolds is due to Liu [Liu13], while that of mixed 3-manifolds is due to Przytycki and Wise [PW18]. Virtual specialness of fundamental groups of closed 3-manifolds locally modeled on $\mathbb{R}^3$ or $\mathbb{H}^2 \times \mathbb{R}$ is immediate, for instance since the group $\mathbb{Z}$, closed surface groups, and hence products thereof are C-special. Bergeron and Wise [BW12] showed that closed hyperbolic 3-manifold groups act properly and cocompactly on CAT(0) cube complexes using a result of Kahn and Marković [KM12], and, using in an essential way Wise's work on Gromov hyperbolic groups with a quasiconvex virtual hierarchy [Wis21, Chapter 13], Agol [AGM13] showed that any hyperbolic group acting properly and cocompactly on a CAT(0) cube

complex, hence any closed hyperbolic 3-manifold group by the aforementioned result of
Bergeron and Wise, is virtually special. Virtual specialness of fundamental groups of
closed hyperbolic 3-manifolds containing embedded geometrically finite incompressible
surfaces had already been established by Wise [Wis21, Theorem 17.1].

Since the fundamental group of the connected sum of two 3-manifolds is the free prod-
uct of the fundamental groups of the summands, we obtain the following from Theorem
1.2.6. For details on the Kneser–Milnor prime decomposition of a closed (not necessarily
orientable) 3-manifold, see, for instance, [Hat07, Section 1.1].

Corollary 1.2.8. *Let M be a closed 3-manifold with the property that none of the sum-
mands in the prime decomposition of M is both aspherical and non-NPC. Then $\pi_1(M)$ is
virtually special.*

Proof. By Lemma 1.2.4, it suffices to show that $\pi_1(M_i)$ is virtually special for each sum-
mand M_i in the prime decomposition of M. By assumption (and by geometrization),
either M_i is NPC, in which case $\pi_1(M_i)$ is virtually special by Theorem 1.2.6, or M_i
is virtually $\mathbb{S}^3$ or $\mathbb{S}^2 \times \mathbb{S}^1$, in which case $\pi_1(M_i)$ is virtually cyclic and hence virtually
special. $\qquad\square$

Note that if H is an index-m subgroup of a group G and H embeds in $\mathrm{O}(n)$ for
some n, then G embeds in $\mathrm{O}(mn)$ via the induced representation. We thus conclude the
following.

Corollary 1.2.9. *Any virtually special group embeds in a compact Lie group, and hence
so does $\pi_1(M)$ for any 3-manifold M as in Corollary 1.2.8.*

Remark 1.2.10. We remark that the virtually special groups do not exhaust all the
finitely generated subgroups of compact Lie groups. For instance, let $n \geq 5$, and consider
the subgroup $\mathrm{SO}_n(\mathbb{Z}[\sqrt[3]{2}])$ of $\mathrm{SO}(n)$. As in Example 0.0.1, we may embed $\mathrm{SO}_n(\mathbb{Z}[\sqrt[3]{2}])$
as a lattice in $\mathrm{SO}_n(\mathbb{C})$. In particular, the former group is infinite since the latter is
not compact. Since $n \geq 5$, the Lie group $\mathrm{SO}_n(\mathbb{C})$ has Kazhdan's property (T), and
so $\mathrm{SO}_n(\mathbb{Z}[\sqrt[3]{2}])$ inherits this property when viewed as a discrete group (see, for instance,

[BdlHV08, Chapter I.1] and the references therein). It follows that $\mathrm{SO}_n(\mathbb{Z}[\sqrt[3]{2}])$ is finitely generated and is not virtually special [NR97, Theorem B]. A similar argument, where $\mathbb{C}$ is replaced with the nonarchimedean local field $\mathbb{Q}_5$, shows that the subgroup $\mathrm{SO}_n(\mathbb{Z}[\frac{1}{5}])$ of $\mathrm{SO}(n)$ is infinite and has property (T) as a discrete group when $n \geq 5$ (see the proof of Proposition 5 in [Mar80]).

We close this section by observing that, in fact, the free product of any two countable subgroups of $\mathrm{O}(n)$ again embeds in a compact Lie group. The proof is essentially Shalen's [Sha79], but we have adjusted some of the arguments to produce the desired result.

Theorem 1.2.11. *Let Γ_1, Γ_2 be countable subgroups of $\mathrm{O}(n)$. Then $\Gamma_1 * \Gamma_2$ embeds in $\mathrm{U}(n+1)$.*

We first prove the following lemma.

Lemma 1.2.12. *Let Γ be a countable subgroup of $\mathrm{GL}_n(\mathbb{R})$ with the property that no nontrivial element of Γ is a scalar matrix. Then there is some $g \in \mathrm{O}(n)$ such that for any nontrivial $\gamma \in \Gamma$, the bottom left entry of $g\gamma g^{-1}$ is nonzero.*

Proof. Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{R}^n$. Since Γ is countable and no nontrivial element of Γ is a scalar matrix, the union of all the eigenspaces of nontrivial elements of Γ has Lebesgue measure 0, so there is some unit vector in $\mathbb{R}^n$ that is not an eigenvector of any nontrivial element of Γ. By transitivity of the action of $\mathrm{O}(n)$ on $\mathbb{S}^{n-1}$, we may assume this vector is e_1.

Since $\mathrm{O}(n)$ acts transitively on the set of pairs (v, H), where $v \in \mathbb{R}^n$ and H is a hyperplane of $\mathbb{R}^n$ containing v, it now suffices to find a hyperplane $H \subset \mathbb{R}^n$ containing e_1 such that $\gamma e_1 \notin H$ for any nontrivial $\gamma \in \Gamma$. We can take H to be the orthogonal complement in $\mathbb{R}^n$ of the subspace spanned by $(0, x_2, \ldots, x_n) \in \mathbb{R}^n$, where $x_2, \ldots, x_n \in \mathbb{R}$ are linearly independent over the entry field of Γ; since the latter field is countable, we can always find such x_i. $\qquad\square$

Proof of Theorem 1.2.11. Suppose first that no nontrivial element of either of the Γ_i is a scalar matrix (i.e., is equal to $-\mathrm{Id}$). In this case, we show that $\Gamma_1 * \Gamma_2$ embeds in $\mathrm{U}(n)$.

By Lemma 1.2.12, we may assume that all nontrivial elements of Γ_1 and Γ_2 have nonzero bottom left entry. Let Γ_2' be the group obtained from Γ_2 by transposing each of the latter's elements. Pick $t \in \mathbb{C}$ be such that $|t| = 1$ and t is transcendental over the entry field of $\langle \Gamma_1, \Gamma_2 \rangle < O(n)$, and let $s = (s_{ij}) \in U(n)$ be given by $s_{ij} = t^i \delta_{ij}$. Then, by the proof of [Sha79, Proposition 1.3], the subgroup $\langle \Gamma_1, s\Gamma_2's^{-1} \rangle < U(n)$ decomposes as the free product of its subgroups $\Gamma_1, s\Gamma_2's^{-1}$.

Theorem 1.2.11 now follows from the fact that the embedding of $O(n)$ in $O(n+1)$ given by extending by the identity on the orthogonal complement of $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$ takes any nontrivial element of $O(n)$ to a nonscalar element of $O(n+1)$. $\qquad\square$

Chapter 2

Proper $\mathrm{CAT}(0)$ actions of unipotent-free matrix groups

Let $\mathbb{F}$ be a field and n a positive integer.

Definition 2.0.1. An element of $\mathrm{SL}_n(\mathbb{F})$ is *unipotent* if it has the same characteristic polynomial as the identity matrix.

In [But17b, But19], Button demonstrated that finitely generated subgroups of $\mathrm{SL}_n(\mathbb{F})$ containing no infinite-order unipotent elements share some properties with groups acting properly by semisimple isometries on complete $\mathrm{CAT}(0)$ spaces. Indeed, Button showed that if $\mathbb{F}$ has positive characteristic (in which case any unipotent element of $\mathrm{SL}_n(\mathbb{F})$ has finite order), then any finitely generated subgroup of $\mathrm{SL}_n(\mathbb{F})$ admits such an action [But19, Theorem 2.3]. The following theorem, to whose proof this chapter is devoted, is intended to serve as an analogue of the latter result in the characteristic-0 setting.

Theorem 2.0.2. *Let Γ be a finitely generated subgroup of $\mathrm{SL}_n(\mathbb{C})$, $n > 0$. Then Γ acts on a complete π-visible $\mathrm{CAT}(0)$ space X such that*

(i) *for any subgroup $H < \Gamma$ containing no nontrivial unipotent matrices, the induced action of H on X is proper;*

(ii) *if such a subgroup H is free abelian of finite rank, then H preserves and acts as a*

lattice of translations on a thick flat in X; in particular, any infinite-order element of such a subgroup H acts ballistically on X;

(iii) if $g \in \Gamma$ is a diagonalizable, then g acts as a semisimple isometry of X.

See Section 2.1 for the relevant definitions. The space X is a finite product $\prod_i X_i$ of symmetric spaces of non-compact type and (possibly locally infinite) Euclidean buildings, and Γ acts on X via a product $\prod_i \mathrm{SL}_n(K_i)$, where the K_i are completions of the entry field E of Γ with respect to various absolute values on E.

Since an element of $\mathrm{SL}_n(\mathbb{C})$ that is both diagonalizable and unipotent must be trivial, the following corollary is immediate.

Corollary 2.0.3. *Any finitely generated subgroup of $\mathrm{SL}_n(\mathbb{C})$ consisting entirely of diagonalizable matrices acts properly by semisimple isometries on a complete $\mathrm{CAT}(0)$ space.*

Remark 2.0.4. Precompact subgroups of $\mathrm{SL}_n(\mathbb{C})$ are conjugate into $\mathrm{SU}(n)$ and thus consist entirely of diagonalizable matrices. Furthermore, by the Peter–Weyl theorem, any compact Lie group can be realized as a compact subgroup of $\mathrm{SL}_n(\mathbb{C})$ for some n [BTD85, Theorem III.4.1]. Thus, by Corollary 2.0.3, any finitely generated subgroup of a compact Lie group admits a proper action by semisimple isometries on a complete $\mathrm{CAT}(0)$ space.

Definition 2.0.5. A *graph manifold* is a connected closed orientable irreducible non-Seifert 3-manifold all of whose JSJ blocks are Seifert.

Property (ii) of the action described in Theorem 2.0.2 allows us to conclude the following fact about representations of fundamental groups of graph manifolds.

Theorem 2.0.6. *Let M be a graph manifold and let $\rho : \pi_1(M) \to \mathrm{SL}_n(\mathbb{C})$ be any representation. If M does not admit a nonpositively curved Riemannian metric, then there is a JSJ torus S of M and a nontrivial element $h \in \pi_1(S) < \pi_1(M)$ such that $\rho(h)$ is unipotent.*

Recall from Section 1.2 that fundamental groups of closed NPC 3-manifolds are virtually special and hence embed in compact Lie groups. On the other hand, if M is a closed

aspherical non-NPC 3-manifold, then either M is Seifert, in which case there is a non-trivial (hence infinite-order) element of $\pi_1(M)$ that gets mapped to a virtually unipotent matrix under any finite-dimensional linear representation of $\pi_1(M)$ (see, for example, the discussion in the introduction of Chapter 3), or the orientation cover of M is a non-NPC graph manifold. Thus, we obtain from Theorem 2.0.6 the following extension of Theorem 1.2.6.

Corollary 2.0.7. *Let M be a closed aspherical 3-manifold. Then the following are equivalent:*

(i) the manifold M is nonpositively curved;

(ii) the group $\pi_1(M)$ is virtually special;

(iii) the group $\pi_1(M)$ embeds in a compact Lie group;

(iv) there is a faithful finite-dimensional $\mathbb{C}$-linear representation of $\pi_1(M)$ whose image consists entirely of diagonalizable matrices;

(v) there is a finite-dimensional $\mathbb{C}$-linear representation of $\pi_1(M)$ mapping no nontrivial element of $\pi_1(M)$ to a unipotent matrix.

Corollary 2.0.7 tells us that the sufficient condition given in Corollary 1.2.8 for the fundamental group of a closed 3-manifold to embed in a compact Lie group is also necessary.

Corollary 2.0.8. *The fundamental group of a closed 3-manifold M embeds in a compact Lie group if and only if none of the summands in the prime decomposition of M is both aspherical and non-NPC.*

Organization

In Section 2.1, we define the relevant objects, discuss briefly some properties of ballistic isometries of complete CAT(0) spaces, and introduce the notion of a "thick flat" in such a space. In Section 2.2, we prove several lemmas used in the proofs of Theorems 2.0.2 and 2.0.6. The latter proofs are contained in Section 2.3.

2.1   Preliminaries

2.1.1   Complete CAT(0) spaces

We recall some definitions and facts from CAT(0) geometry. These are drawn mainly from [BH99].

A geodesic metric space X is said to be CAT(0) if any two points on a geodesic triangle in X are at most as distant as their analogues on a Euclidean triangle with the same side lengths. Any two points in such a space are in fact joined by a unique geodesic segment. Associated to a CAT(0) space is its *visual boundary* ∂X, the space of geodesic rays in X up to asymptoticity.

Let X be a complete CAT(0) space and ∂X its visual boundary. We will make references to the cone topology on $\overline{X} := X \cup \partial X$, described in [BH99]. Under this topology, a sequence of points $x_n \in X$ converges to $\xi \in \partial X$ if and only if for some (hence any) point $x_0 \in X$, the geodesics joining x_0 to x_n converge uniformly on compact intervals to the unique geodesic ray emanating from x_0 in the class of ξ. In addition, we will use the angular metric $\angle$ on ∂X, also described in [BH99]. The topology on ∂X induced by the angular metric is in general finer than the cone topology on ∂X.

An r-dimensional *flat* in X is an isometrically embedded copy of $\mathbb{R}^r$ in X. We say X is π-*visible* if for any $\xi, \eta \in \partial X$ satisfying $\angle(\xi, \eta) = \pi$, there is a geodesic line in X whose endpoints on ∂X are ξ and η. Since Euclidean spaces are π-visible, a complete CAT(0) space X with the property that any two points on ∂X lie on the boundary of a common flat in X is also π-visible. Note that if X is a Euclidean building, a symmetric space of non-compact type, or a product of such spaces, then X possesses the latter property by the building structure on ∂X, so that X is π-visible. For more information on symmetric spaces, we refer the reader to the monograph [Ebe96].

2.1.2   Isometries of complete CAT(0) spaces

Let (X, d_X) be a metric space and let $g \in \text{Isom}(X)$. The *translation length* of g on X is the quantity $|g|_X := \inf_{x \in X} d_X(x, gx)$.

For any $x \in X$, the limit $\lim_n \frac{d(x,g^n x)}{n}$ exists and is independent of the choice of $x \in X$. By the triangle inequality, we have $\frac{d(x,g^n x)}{n} \leq d(x,gx)$ for each positive integer n and each $x \in X$ (see Exercise 1, Section II.6.6 of [BH99]), so that $\lim_n \frac{d(x,g^n x)}{n} \leq |g|_X$. If X is a CAT(0) space, then this inequality is in fact an identity. This fact is well known, but we could not find a reference for it, so we have included a proof. (We in fact only use the inequality in this chapter, but the identity will be useful in Section 4.0.4.)

Lemma 2.1.1. *Let X be a* CAT(0) *space and let $g \in \mathrm{Isom}(X)$. Then $|g|_X = \lim_n \frac{d(x,g^n x)}{n}$ for each $x \in X$.*

Proof. It suffices to show that for each nonnegative integer n, we have $|g|_X \leq \frac{d(x,g^{2^n} x)}{2^n}$ for all $x \in X$. We prove this by induction on n. The base case $n = 0$ is evident. Now suppose the claim is true for some particular $n \geq 0$, let $x \in X$, and let y be the midpoint of the geodesic segment in X joining x and $g^{2^n} x$. Then $g^{2^n} y$ is the midpoint of the geodesic segment in X joining $g^{2^n} x$ and $g^{2^{n+1}} x$. Thus we have

$$d(y, g^{2^n} y) \leq \frac{d(x, g^{2^{n+1}} x)}{2} \tag{2.1}$$

by the midsegment theorem in Euclidean geometry and the CAT(0) property applied to the geodesic triangle in X with vertices x, $g^{2^n} x$, and $g^{2^{n+1}} x$. By the induction hypothesis, we have

$$|g|_X \leq \frac{d(y, g^{2^n} y)}{2^n} \leq \frac{d(x, g^{2^{n+1}} x)}{2^{n+1}},$$

where the second inequality follows from (2.1). □

Now suppose X is a complete CAT(0) space. The isometry g is *semisimple* if $|g|_X = d_X(x_0, gx_0)$ for some $x_0 \in X$. We say g is *ballistic* (resp., *neutral*) if $|g|_X > 0$ (resp., if $|g|_X = 0$), and *hyperbolic* if g is both ballistic and semisimple. A subgroup $H < \mathrm{Isom}(X)$ acts *neutrally* on X if each $h \in H$ is neutral.

If $g \in \mathrm{Isom}(X)$ is ballistic, then there is a point $\omega_g \in \partial X$ such that for any $x \in X$, we have $g^n x \to \omega_g$ as $n \to \infty$ with respect to the cone topology on $\overline{X}$ [CM09]; we call ω_g the *canonical attracting fixed point* of g. We use repeatedly the following fact, due to

Duchesne [Duc15, Prop. 6.2]. For an arbitrary group G and $g_1, \ldots, g_m \in G$, we denote by $\mathcal{Z}_G(g_1, \ldots, g_m)$ the centralizer of $g_1, \ldots, g_m$ in G.

Theorem 2.1.2. *Let X be a complete π-visible CAT(0) space and suppose $g \in \mathrm{Isom}(X)$ is ballistic. Then there is a closed convex subspace $Y \subset X$ and a metric decomposition $Y = Z \times \mathbb{R}$ such that*

- *$\mathcal{Z}_{\mathrm{Isom}(X)}(g)$ preserves Y and acts diagonally with respect to the decomposition $Y = Z \times \mathbb{R}$, acting by translations on the second factor;*

- *the isometry g acts neutrally on the factor Z.*

In accordance with [BH99], we define an isometric action of a group H on a metric space X to be *proper* if for any point $x \in X$, there is a neighborhood $U \subset X$ of x such that $\{h \in H : U \cap hU \neq \infty\}$ is finite. In this case, the set $\{h \in H : K \cap hK \neq \infty\}$ is finite for any compact subset $K \subset X$ (see, for example, [BH99, Remark I.8.3(1)]). Note, however, that if the metric space X is not proper, then X may contain balls B such that $\{h \in H : B \cap hB \neq \infty\}$ is infinite; that is, the notion of properness for isometric actions used here is strictly weaker than *metric properness*.

We will make use of the following well-known theorem [BH99, Theorem II.7.1].

Theorem 2.1.3. *Let H be a free abelian group of rank r acting properly by semisimple isometries on a complete CAT(0) space X. Then H preserves and acts as a lattice of translations on an r-dimensional flat in X.*

2.1.3   Thick flats

A closed convex subspace $Y \subset X$ together with an isometry $\varphi : Y \to Z \times \mathbb{R}^r$, where $r \geq 0$ and Z is some complete CAT(0) space, is called a *thick flat* of dimension r in X. We say a group H acting isometrically on X *preserves* the thick flat (Y, φ) if H preserves Y. Such a group H *acts as a lattice of translations* on the thick flat (Y, φ) if H acts diagonally with respect to the decomposition $Z \times \mathbb{R}^r$, acting neutrally on the first factor and by translations on the second, so that the induced map $H \to \mathbb{R}^r$ embeds H as a lattice of $\mathbb{R}^r$.

2.2   Lemmata

Lemmas 2.2.1 and 2.2.2 are probably well known, but we include their proofs for completeness. The objective is to determine the canonical attracting fixed point of a ballistic isometry acting diagonally on a product.

Lemma 2.2.1. *Let Y, Z be complete $\mathrm{CAT}(0)$ spaces and $X = Y \times Z$. Suppose $g_Y \in \mathrm{Isom}(Y)$ is neutral and $g_Z \in \mathrm{Isom}(Z)$ is hyperbolic, and let $g, g' \in \mathrm{Isom}(X)$ be the isometries $g_Y \times g_Z$, $\mathrm{Id}_Y \times g_Z$ of X, respectively. Then $\omega_g = \omega_{g'}$.*

Proof. There exist a geodesic line $\gamma_Z : \mathbb{R} \to Z$ in Z and a positive number ℓ such that $g_Z(\gamma_Z(t)) = \gamma_Z(t + \ell)$ for any $t \in \mathbb{R}$. The point $\omega_{g'} \in \partial X$ is represented by a geodesic ray of the form $(y_0, \gamma_Z(t))$, $t \geq 0$, $y_0 \in Y$. Thus, we reduce to the case that $Z = \mathbb{R}$ and g_Z is a translation by $\ell > 0$. Setting $x_0 = (y_0, 0)$, we show that the geodesics $\gamma^{(n)}$ in X joining x_0 to $g^n x_0$ converge uniformly on compact subsets as $n \to \infty$ to the geodesic ray $\gamma : [0, \infty) \to X$ given by $t \mapsto (y_0, t)$.

To that end, write $\gamma^{(n)}(t) = (\gamma_Y^{(n)}(t), \alpha_n t)$, where $\alpha_n > 0$ and $\gamma_Y^{(n)}$ is a linearly reparameterized geodesic in Y joining y_0 to $g_Y^n y_0$, and let $R > 0$. Note that the maximum value of $d_X(\gamma(t), \gamma^{(n)}(t))$ on $[0, R]$ is attained at $t = R$; indeed, for $0 \leq t \leq R$, we have

$$d_X(\gamma(t), \gamma^{(n)}(t))^2 = d_Y(y_0, \gamma_Y^{(n)}(t))^2 + t^2(1 - \alpha_n)^2.$$

Thus, it suffices to show that $d_X(\gamma(R), \gamma^{(n)}(R)) \to 0$. This will follow if we can show that $d_Y(y_0, \gamma_Y^{(n)}(R)) \to 0$ since

$$R^2 = d_X(x_0, \gamma^{(n)}(R))^2 = d_Y(y_0, \gamma_Y^{(n)}(R))^2 + \alpha_n^2 R^2.$$

To see that $d_Y(y_0, \gamma_Y^{(n)}(R)) \to 0$, note that since $\gamma_Y^{(n)}$ is a linearly reparameterized geodesic, we have

$$\frac{d_Y(y_0, \gamma_Y^{(n)}(R))}{d_Y(y_0, g_Y^n y_0)} = \frac{R}{d_X(x_0, g^n x_0)}$$

and so

$$
\begin{aligned}
d_Y(y_0, \gamma_Y^{(n)}(R))^2 &= R^2 \frac{d_Y(y_0, g_Y^n y_0)^2}{d_X(x_0, g^n x_0)^2} \\
&= R^2 \frac{d_Y(y_0, g_Y^n y_0)^2}{d_Y(y_0, g_Y^n y_0)^2 + n^2 \ell^2} \\
&= R^2 \frac{\left(\frac{d_Y(y_0, g_Y^n y_0)}{n}\right)^2}{\left(\frac{d_Y(y_0, g_Y^n y_0)}{n}\right)^2 + \ell^2}.
\end{aligned}
$$

Now the latter approaches 0 as $n \to 0$ since

$$
\lim_{n \to \infty} \frac{d_Y(y_0, g_Y^n y_0)}{n} = |g_Y|_Y
$$

and $|g_Y|_Y = 0$ by assumption. $\square$

Lemma 2.2.2. *Let X_1, X_2 be complete π-visible CAT(0) spaces, let $g_i \in \mathrm{Isom}(X_i)$ for $i = 1, 2$, and suppose g_1 is ballistic. Let $X = X_1 \times X_2$ and let $g = g_1 \times g_2 \in \mathrm{Isom}(X)$. Then g acts ballistically on X and*

$$
\omega_g = (\arctan(|g_2|/|g_1|), \omega_{g_1}, \omega_{g_2})
$$

in the spherical join $\partial X_1 * \partial X_2 = \partial X$.

Proof. We suppose first that g_1, g_2 are both ballistic, so that we may assume that X_i admits a decomposition $X_i = Y_i \times Z_i$ with respect to which g_i acts diagonally, where Z_i is isometric to $\mathbb{R}$, and where g_i acts neutrally on the first factor and acts by a translation of $|g_i|$ on the second factor. Let $g_i' \in \mathrm{Isom}(X_i)$ be the product of the identity on Y_i with the translation by $|g_i|$ on Z_i, and let $g' = g_1' \times g_2' \in \mathrm{Isom}(X)$. Note we have $|g_i| = |g_i'|$, and by Lemma 2.2.1, we have $\omega_{g_i} = \omega_{g_i'}$. Moreover, by viewing X as the product $X = (Y_1 \times Y_2) \times (Z_1 \times Z_2)$, we also have $\omega_g = \omega_{g'}$ by Lemma 2.2.1. Thus, to establish the lemma, it suffices to show

$$
\omega_{g'} = (\arctan(|g_1'|/|g_2'|), \omega_{g_1'}, \omega_{g_2'}),
$$

but this follows from plane geometry since g_1', g_2' preserve and act as translations on the 2-dimensional flat $\{(y_1, y_2)\} \times (Z_1 \times Z_2) \subset X$, where y_i is any point in Y_i.

If g_2 is neutral, then we may only assume that X_1 admits a decomposition $X_1 = Y_1 \times Z_1$ as above, and now the lemma follows immediately from Lemma 2.2.1 by viewing X as the product $X = (Y_1 \times X_2) \times Z_1$. $\hfill\square$

We apply Lemma 2.2.2 to the special case of matrices acting on symmetric spaces.

Lemma 2.2.3. *Let M be a symmetric space associated to $\mathrm{GL}_n(\mathbb{C})$ and let $g \in \mathrm{GL}_n(\mathbb{C})$ be of the form*

$$g = \mathrm{diag}(\lambda_1 U_1, \ldots, \lambda_m U_m)$$

where $\lambda_1, \ldots, \lambda_m \in \mathbb{C}^*$ with $|\lambda_k| \neq 1$ for at least one $k \in \{1, \ldots, m\}$, and $U_k \in \mathrm{SL}_{n_k}(\mathbb{C})$ is an upper unitriangular matrix for $k \in \{1, \ldots, m\}$. Then g acts ballistically on M and has the same canonical attracting fixed point as

$$g' := \mathrm{diag}(\lambda_1 I_{n_1}, \ldots, \lambda_m I_{n_m})$$

on ∂M. The same statement holds when $\mathrm{GL}_n(\mathbb{C})$ is replaced with $\mathrm{SL}_n(\mathbb{C})$.

Proof. For $k = 1, \ldots, m$, let X, X_k, Y_k, Z_k be the projections of the subgroups

$$\{\mathrm{diag}(h_1, \ldots, h_m) \ : \ h_k \in \mathrm{GL}_{n_k}(\mathbb{C})\}$$
$$\{\mathrm{diag}(I_{n_1}, \ldots, I_{n_{k-1}}, \ h, \ I_{n_{k+1}}, \ldots, I_{n_m}) \ : \ h \in \mathrm{GL}_{n_k}(\mathbb{C})\}$$
$$\{\mathrm{diag}(I_{n_1}, \ldots, I_{n_{k-1}}, \ h, \ I_{n_{k+1}}, \ldots, I_{n_m}) \ : \ h \in \mathrm{SL}_{n_k}(\mathbb{C})\}$$
$$\{\mathrm{diag}(I_{n_1}, \ldots, I_{n_{k-1}}, \ e^t I_{n_k}, \ I_{n_{k+1}}, \ldots, I_{n_m}) \ : \ t \in \mathbb{R}\}$$

of $\mathrm{GL}_n(\mathbb{C})$ to M under the quotient map $\mathrm{GL}_n(\mathbb{C}) \to M = \mathrm{GL}_n(\mathbb{C})/\mathrm{U}(n)$, respectively. Then X is a closed convex subspace of M admitting a decomposition $X = \prod_{k=1}^m X_k$. The subspace X_k in turn admits a decomposition $X_k = Y_k \times Z_k$, and the factor Z_k is isometric to $\mathbb{R}$. Each of the isometries g, g' preserves X and acts diagonally with respect to the decomposition $X = \prod_{k=1}^m X_k$. On each factor X_k, each of g, g' also acts diagonally with respect to the decomposition $X_k = Y_k \times Z_k$, acting neutrally on the first factor and as a translation by $\alpha_k \ln|\lambda_k|$ on the second for some $\alpha_k > 0$. Thus, the lemma follows from a repeated application of Lemma 2.2.2.

To see that the lemma remains true when $\mathrm{GL}_n(\mathbb{C})$ is replaced with $\mathrm{SL}_n(\mathbb{C})$, note that a symmetric space for $\mathrm{SL}_n(\mathbb{C})$ embeds as a closed convex $\mathrm{SL}_n(\mathbb{C})$-invariant subspace of a symmetric space for $\mathrm{GL}_n(\mathbb{C})$. $\qquad\square$

We now observe that a collection of pairwise commuting matrices over $\mathbb{C}$ can be simultaneously put into the form described in Lemma 3.1.3.

Lemma 2.2.4. *Let K be an algebraically closed field and let $h_\alpha \in \mathrm{M}_n(K)$ be a collection of pairwise commuting matrices. Then there are $s \in \mathbb{N}$ and $C \in \mathrm{SL}_n(K)$ such that*

$$Ch_\alpha C^{-1} = \mathrm{diag}(h_{\alpha,1}, \ldots, h_{\alpha,s})$$

where $h_{\alpha,\ell} \in \mathrm{M}_{n_\ell}(K)$ is upper triangular and has a single eigenvalue for $\ell = 1, \ldots, s$.

Proof. Since K is algebraically closed, it suffices to find such $C \in \mathrm{GL}_n(K)$; indeed, we may ultimately replace C with μC, where μ is an nth root of $1/\det(C)$. We now proceed by induction on n. The case $n = 1$ is trivial. Now let $n > 1$ and suppose the above claim has been established for matrices of smaller dimension. If each of the h_α has a single eigenvalue, then the statement follows from the fact that any collection of pairwise commuting elements of $\mathrm{M}_n(K)$ are simultaneously upper triangularizable [RR00, Theorem 1.1.5]. Now suppose a matrix $h \in \{h_\alpha\}_\alpha$ has more than one eigenvalue. By putting h into Jordan canonical form, for instance, we may assume h is of the form

$$h = \mathrm{diag}(h_1, h_2),$$

where $h_i \in \mathrm{M}_{n_i}(K)$ for $i = 1, 2$ and h_1, h_2 do not share an eigenvalue. Since the h_α commute with h, they preserve the generalized eigenspaces of h, and so h_α also has a block-diagonal structure

$$h_\alpha = \mathrm{diag}(h_{\alpha,1}, h_{\alpha,2}),$$

where $h_{\alpha,i} \in \mathrm{M}_{n_i}(K)$ for $i = 1, 2$. The lemma now follows by applying the induction hypothesis to the collections $\{h_{\alpha,i}\}_\alpha$, $i = 1, 2$. $\qquad\square$

We now prove what one might call a "thick flat torus theorem."

Lemma 2.2.5. *Suppose X is a complete π-visible CAT(0) space and H is a free abelian subgroup of $\mathrm{Isom}(X)$ with a basis $h_1, \ldots, h_r \in H$ consisting of ballistic isometries such that for each $m \in \{1, \ldots, r\}$, there is no $(m-1)$-dimensional flat in X whose boundary contains the canonical attracting fixed points $\omega_{h_1}, \ldots, \omega_{h_m}$. Then H preserves and acts as a lattice of translations on a thick flat of dimension r in X.*

Proof. We prove by induction the following statement: for $m \in \{1, \ldots, r\}$, there is a closed convex subspace Y_m of X and a decomposition $Y_m = Z_m \times \mathbb{R}^m$ such that

- $\mathcal{Z}_{\mathrm{Isom}(X)}(h_1, \ldots, h_m)$ preserves Y_m and acts diagonally with respect to the decomposition $Y_m = Z_m \times \mathbb{R}^m$, acting by translations on the second factor;

- the subgroup $\langle h_1, \ldots, h_m \rangle$ acts neutrally on the first factor and as a lattice of translations on the second.

The base case $m = 1$ is given by Theorem 2.1.2. Now suppose the above holds for $m - 1$, where $m \in \{2, \ldots, r\}$. Then h_m must act ballistically on the factor Z_{m-1}, since otherwise $\omega_{h_1}, \ldots, \omega_{h_m}$ would be contained in the boundary of $\{z\} \times \mathbb{R}^{m-1}$ by Lemma 2.2.1, where z is any point in Z_{m-1}. Now Z_{m-1} is a complete π-visible CAT(0) space, so that by Theorem 2.1.2 there is a closed convex subspace Y of Z_{m-1} and a decomposition $Y = Z \times \mathbb{R}$ satisfying

- $\mathcal{Z}_{\mathrm{Isom}(Z_{m-1})}(h_m)$ preserves Y and acts diagonally with respect to the decomposition $Y = Z \times \mathbb{R}$, acting by translations on the second factor;

- the action of h_m on the first factor Z is neutral.

Then the subspace $Y_m := Y \times \mathbb{R}^{m-1} \subset Z_{m-1} \times \mathbb{R}^{m-1}$ has the desired properties. $\square$

The following observation is used in the proof of Lemma 2.2.7.

Lemma 2.2.6. *Let X be a complete CAT(0) space and suppose $H < \mathrm{Isom}(X)$ is a free abelian subgroup with a basis $h_1, \ldots, h_r \in H$. Suppose H preserves and acts as a lattice of translations on thick flats Y, Y' in X, and let ϕ, ϕ' be the maps $H \to \mathbb{R}^r$ induced by the actions of H by translations on the Euclidean factors of Y, Y', respectively. Then the unique linear map $T : \mathbb{R}^r \to \mathbb{R}^r$ satisfying $T(\phi(h_i)) = \phi'(h_i)$ for $i = 1, \ldots, r$ is orthogonal.*

Proof. We wish to show that T preserves the standard inner product on $\mathbb{R}^r$. Since the $\phi(h_i)$ constitute a basis for $\mathbb{R}^r$, it suffices to show that $\langle \phi'(h_i), \phi'(h_j) \rangle = \langle \phi(h_i), \phi(h_j) \rangle$ for $i, j \in \{1, \ldots, r\}$. This is equivalent to saying that for $i, j \in \{1, \ldots, r\}$, we have $\|\phi(h_i)\| = \|\phi'(h_i)\|$ and $\angle(\phi(h_i), \phi(h_j)) = \angle(\phi'(h_i), \phi'(h_j))$. The former is true since

$$\|\phi(h_i)\| = |h_i|_X = \|\phi'(h_i)\|$$

and the latter is true since $\angle(\phi(h_i), \phi(h_j)) = \angle(\phi'(h_i), \phi'(h_j)) = \angle(\omega_{h_i}, \omega_{h_j})$ by Lemma 2.2.1. $\qquad\square$

The proof of the following lemma borrows heavily from an argument of Leeb; see the proof of Theorem 2.4 in [KL96]. Note that we work with the JSJ decomposition of a graph manifold as opposed to its geometric decomposition, so that, for example, the twisted circle bundle over the Möbius band may appear as a JSJ block of a graph manifold.

Lemma 2.2.7. *Let M be a graph manifold and suppose $\pi_1(M)$ acts by isometries on a complete π-visible CAT(0) space X such that for each JSJ torus S of M, the subgroup $\pi_1(S) < \pi_1(M)$ preserves and acts as a lattice of translations on a thick flat in X. Then M admits a nonpositively curved Riemannian metric.*

Proof. Let B be a JSJ block of M, and let $f \in \pi_1(B)$ be an element representing a generic fiber of B. The element f acts ballistically on X since f is a nontrivial element of $\pi_1(S)$, where S is a torus boundary component of B, and $\pi_1(S)$ preserves and acts as a lattice of translations on a thick flat in X by assumption. By Theorem 2.1.2, there is a closed convex subspace $Y \subset X$ with a metric decomposition $Y = Z \times \mathbb{R}$ such that

- any element of $\pi_1(B)$ preserves Y and acts diagonally with respect to the decomposition $Y = Z \times \mathbb{R}$, acting as a translation on the second factor;

- the action of f on the first factor Z is neutral.

Moreover, for each element $z \in \pi_1(B)$ representing a boundary component of the base orbifold O of B, the action of z on Z is ballistic since the subgroup $\langle f, z \rangle < \pi_1(B)$ preserves and acts as a lattice of translations on a thick flat in X.

We now realize B as a nonpositively curved Riemannian manifold with totally geodesic flat boundary as follows. Endow the orbifold O with a nonpositively curved Riemannian metric that is flat near the boundary so that the length of each boundary component c of O is equal to the translation length on Z of an element in $\pi_1(B)$ representing c. We let $\pi_1(B)$ act on the universal cover $\tilde{O}$ of O via the projection $\pi_1(B) \to \pi_1(O)$, where $\pi_1(O)$ acts on $\tilde{O}$ by deck transformations. The product of this action with the action of $\pi_1(B)$ on $\mathbb{R}$ coming from the decomposition $Y = Z \times \mathbb{R}$ yields a covering space action of $\pi_1(B)$ on $\tilde{O} \times \mathbb{R}$. The quotient of $\tilde{O} \times \mathbb{R}$ by this action is the desired geometric realization of B. We may do this for each Seifert component of M; the flat metrics on any pair of boundary tori that are matched in M will coincide by Lemma 2.2.6, so that we may glue the metrics on the Seifert components to obtain a smooth nonpositively curved metric on M. $\qquad\square$

The following lemma will not be used in the proofs of Theorems 2.0.2 or 2.0.6, but will be applied to derive Corollary 2.3.1 from Theorem 2.0.2.

Lemma 2.2.8. *Let Γ be a finitely generated group and H_0 a free abelian subgroup of Γ of rank $r \geq 0$. Suppose Γ acts on a complete $\mathrm{CAT}(0)$ space X such that H_0 preserves and acts as a lattice of translations on a thick flat in X. Then H_0 is undistorted in Γ.*

Proof. Let $\mathcal{B} = \{h_1, \ldots, h_r\} \subset H_0$ be a basis for H_0, and let $|\cdot|_{\mathcal{B}}$ be the word metric on H_0 with respect to $\mathcal{B}$. Let $\mathcal{S} \subset \Gamma$ be a finite generating set for Γ and let $|\cdot|_{\mathcal{S}}$ be the word metric on Γ with respect to $\mathcal{S}$. Let $\phi : H_0 \to \mathbb{R}^r$ be the homomorphism to $\mathbb{R}^r$ induced by the action of H_0 on a thick flat in X, and let $y_0 \in Y$, $K = \max_{s \in \mathcal{S} \cup \mathcal{S}^{-1}} d_X(y_0, sy_0)$. Since any two norms on $\mathbb{R}^r$ are equivalent, there is some $C > 0$ such that $\|\phi(h)\| \geq C|h|_{\mathcal{B}}$ for any $h \in H_0$. Thus, for $h \in H_0$, we have

$$K|h|_{\mathcal{S}} \geq d_X(y_0, hy_0) \geq \|\phi(h)\| \geq C|h|_{\mathcal{B}}$$

where the first inequality follows from the triangle inequality. $\qquad\square$

2.3   Proofs of Theorems 2.0.2 and 2.0.6

Proof of Theorem 2.0.2. (i) Since Γ is finitely generated, we have that $\Gamma \subset \mathrm{SL}_n(A)$ for some finitely generated subdomain $A \subset \mathbb{C}$. Let $E = \mathbb{Q}(A) \subset \mathbb{C}$, so that E is a finitely generated field extension of $\mathbb{Q}$. The extension $E/\mathbb{Q}$ has the structure $\mathbb{Q} \subset F \subset F(T) \subset E$, where F is the algebraic closure of $\mathbb{Q}$ in E, and T is a (possibly empty) transcendence basis for E over F. Since the extension $E/\mathbb{Q}$ is finitely generated, the set T is finite and the extensions $F/\mathbb{Q}$ and $E/F(T)$ are of finite degree.

Let $d = \deg(F/\mathbb{Q})$, and let $\sigma_1, \ldots, \sigma_d$ be the embeddings of F in $\mathbb{C}$. Since $\sigma_j(F)$ is countable but $\mathbb{C}$ is not, the extension $\mathbb{C}/\sigma_j(F)$ has infinite transcendence degree, and hence we may extend σ_j to an embedding $\sigma_j : F(T) \to \mathbb{C}$. The latter may in turn be extended to an embedding $\sigma_j : E \to \mathbb{C}$ since $E/F(T)$ is algebraic and $\mathbb{C}$ is algebraically closed. The embedding $\sigma_j : E \to \mathbb{C}$ induces an embedding $\sigma_j : \mathrm{SL}_n(E) \to \mathrm{SL}_n(\mathbb{C})$. Let

$$\sigma : \mathrm{SL}_n(E) \to G_1 := \prod_{j=1}^{d} \mathrm{SL}_n(\mathbb{C})$$

be the diagonal embedding induced by the maps $\sigma_j : \mathrm{SL}_n(E) \to \mathrm{SL}_n(\mathbb{C})$. Then $\mathrm{SL}_n(E)$ acts by isometries on the Hadamard manifold $X_1 := \prod_{j=1}^{d} M_j$ via the embedding σ, where each M_j is a copy of the symmetric space (unique up to scaling of the Riemannian metric) associated to the simple Lie group $\mathrm{SL}_n(\mathbb{C})$.

By [AS82, Prop. 1.2], there are finitely many discrete valuations $\nu_1, \ldots, \nu_m$ on E such that $A \cap \bigcap_{i=1}^{m} \mathcal{O}_i \subset \mathcal{O}$, where $\mathcal{O}$ is the ring of integers of F and $\mathcal{O}_i$ is the valuation ring of ν_i. Let B_i be the Bruhat–Tits building associated to $\mathrm{SL}_n(E_{\nu_i})$, where E_{ν_i} is the completion of E with respect to ν_i; let $X_2 = \prod_{i=1}^{m} B_i$; and let $\tau : \mathrm{SL}_n(E) \to G_2 := \prod_{i=1}^{m} \mathrm{SL}_n(E_{\nu_i})$ be the diagonal embedding. Then $\mathrm{SL}_n(E)$ acts by automorphisms on X_2 via the embedding τ. We claim that the diagonal action of Γ on $X := X_1 \times X_2$ via $\sigma \times \tau : \mathrm{SL}_n(E) \to G_1 \times G_2$ has the desired properties.

To that end, let H be a subgroup of Γ containing no nontrivial unipotent elements. We first claim that for any vertex v of X_2, the subgroup $\sigma(H_v) < G_1$ is discrete, where H_v is the stabilizer of v in H. Indeed, let $h \in H_v$. Then for $i = 1, \ldots, m$, the element h

fixes a vertex of B_i and (since $\mathrm{GL}_n(E)$ acts transitively on the vertices of B_i) is thus conjugate within $\mathrm{GL}_n(E)$ into $\mathrm{SL}_n(\mathcal{O}_i)$; in particular, the coefficients of the characteristic polynomial χ_h of h lie in $\mathcal{O}_i$. Since this is true for each $i \in \{1, \ldots, m\}$ and since $h \in \mathrm{SL}_n(A)$, we have that the coefficients of χ_h lie in $A \cap \bigcap_{i=1}^{m} \mathcal{O}_i$ and hence in $\mathcal{O}$. We thus have a commutative diagram

$$
\begin{array}{ccc}
G_1 = \prod_{j=1}^{d} \mathrm{SL}_n(\mathbb{C}) & \xrightarrow{\ P\ } & \prod_{j=1}^{d} \mathbb{C}^n \\
\sigma \uparrow & & \hat{\sigma} \uparrow \\
H_v & \xrightarrow{\ \ p\ \ } & \mathcal{O}^n
\end{array}
\tag{2.2}
$$

where the function p maps an element $h \in H_v$ to the n-tuple whose entries are the non-leading coefficients of χ_h, the function P is the d-fold product of the analogous map $\mathrm{SL}_n(\mathbb{C}) \to \mathbb{C}^n$, and the function $\hat{\sigma}$ is given by

$$
\hat{\sigma}(\alpha_1, \ldots, \alpha_n) = (\sigma_1(\alpha_1), \ldots, \sigma_1(\alpha_n), \ldots, \sigma_d(\alpha_1), \ldots, \sigma_d(\alpha_n))
$$

for $\alpha_1, \ldots, \alpha_n \in \mathcal{O}$. Since $\hat{\sigma}$ has discrete image (see, for example, Lemma 25.1.10 in [KM79]) and the diagram (2.2) is commutative, it follows that $P(\sigma(H_v))$ is discrete in $\prod_{j=1}^{d} \mathbb{C}^n$. Now suppose we have a sequence $(h_k)_{k \in \mathbb{N}}$ in H_v such that $\sigma(h_k) \to 1$ in G_1. Then, by continuity of the function P, we have $P(\sigma(h_k)) \to P(1)$. By discreteness of $P(\sigma(H_v))$, this implies that $P(\sigma(h_k)) = P(1)$ for k sufficiently large. It follows that for such k the matrix h_k is unipotent and hence trivial by our assumption that H contains no nontrivial unipotent elements. We conclude that $\sigma(H_v)$ is indeed discrete in G_1.

We now argue that for any $x \in X_2$, there is a neighborhood V of x in X_2 such that $H_V \subset H_v$ for some vertex v of X_2, where

$$
H_V = \{h \in H \ : \ V \cap hV \neq \emptyset\}.
$$

Let c be the cell of X_2 containing x and let ℓ be the dimension of c. Let $\epsilon > 0$ be such that the intersection of the ball $B_{X_2}(x, \epsilon)$ with the ℓ-skeleton X_2^ℓ of X_2 is contained in c. Then we may take $V = B_{X_2}(x, \epsilon/2)$. Indeed, if $h \in H_V$, then $hx \in X_2^\ell \cap B_{X_2}(x, \epsilon) \subset c$, and so $hc = c$. Since $\mathrm{SL}_n(E)$ acts on B_i without permutations, it follows that $h \in H_v$ for any vertex v of c.

Now, to see that H acts properly on X, we observe that for any point $x \in X_2$ and any ball $B \subset X_1$, the set $U := B \times V \subset X$ has the property that $\{h \in H : U \cap hU \neq \emptyset\}$ is finite, where $V \subset X_2$ is as in the preceding paragraph. Indeed, we have $H_V \subset H_v$ for some vertex v of X_2, and H_v acts properly on X_1 since σ embeds H_v discretely in G_1.

(ii) Suppose H is free abelian with a basis $h_1, \ldots, h_r \in H$. We show that this basis is as in the statement of Lemma 2.2.5, so that H preserves and acts as a lattice of translations on a thick flat in X. Indeed, by Lemma 2.2.4, we may assume that for $j \in \{1, \ldots, d\}$, $k \in \{1, \ldots, r\}$, we have

$$\sigma_j(h_k) = \mathrm{diag}(h_{j,k,1}, \ldots, h_{j,k,s})$$

where $h_{j,k,\ell} \in \mathrm{GL}_{n_\ell}(\mathbb{C})$ is upper triangular with a single eigenvalue for $\ell \in \{1, \ldots s\}$. We now have a homomorphism $\Delta_j : H \to \mathrm{SL}_n(\mathbb{C})$ that maps $h \in H$ to the diagonal part of $\sigma_j(h)$; note that Δ_j is injective since H contains no nontrivial unipotent matrices. The embeddings Δ_j produce a diagonal embedding $\Delta : H \to G_1$. Now let $\Delta' : H \to G_1 \times G_2$ be the product of Δ with $\tau|_H : H \to G_2$. Then, since $\Delta_j(h)$ has the same characteristic polynomial as $\sigma_j(h)$ for each $h \in H$, and since $\Delta_j(H)$ contains no nontrivial unipotent matrices, the action of $\Delta'(H)$ on X is proper by the above arguments. Since the latter action is by semisimple isometries, by Theorem 2.1.3 there is a genuine r-dimensional flat in X preserved by $\Delta'(H)$ on which $\Delta'(H)$ acts as a lattice of translations. Thus, by Lemmas 2.2.2 and 3.1.3, each nontrivial $h \in H$ acts ballistically on X and the canonical attracting fixed point of h on ∂X is equal to that of $\Delta'(h)$; in particular, $\omega_{h_1}, \ldots, \omega_{h_r}$ must be of the desired form.

(iii) Suppose $g \in \Gamma$ is diagonalizable (over $\mathbb{C}$). Since any isometry of X_2 is semisimple, to show that g acts as a semisimple isometry of X, it suffices to show that $\sigma_j(g)$ is a semisimple isometry of M_j for $j = 1, \ldots, d$. To that end, we show that $\sigma_j(g)$ is diagonalizable. Indeed, since a diagonalization of g has entries in the splitting field $\tilde{E} \subset \mathbb{C}$ of χ_g over E, we in fact have $g = CDC^{-1}$ for some $C, D \in \mathrm{SL}_n(\tilde{E})$ with D diagonal (see, for example, [Rom13, Theorem 8.11]). Since $\mathbb{C}$ is algebraically closed, we may extend σ_j

to an embedding $\tilde{\sigma}_j : \tilde{E} \to \mathbb{C}$. Now

$$\sigma_j(g) = \tilde{\sigma}_j(g) = \tilde{\sigma}_j(C)\,\tilde{\sigma}_j(D)\,\tilde{\sigma}_j(C)^{-1}$$

and $\tilde{\sigma}_j(D)$ is diagonal.

This completes the proof of Theorem 2.0.2.   □

We recover the following result, due to Button [But17b, Theorem 5.2].

Corollary 2.3.1. *Let Γ be a finitely generated group and H a distorted finitely generated abelian subgroup of Γ. Then for any representation $\rho : \Gamma \to \mathrm{SL}_n(\mathbb{C})$, there is an infinite-order element $h \in H$ such that $\rho(h)$ is unipotent.*

Proof. Let $H_0 < H$ be a free abelian subgroup of finite-index, and suppose there is a representation $\rho_0 : \Gamma \to \mathrm{SL}_n(\mathbb{C})$ that does not map any nontrivial element of H_0 to a unipotent matrix (in particular, ρ is faithful on H_0). Then, by Theorem 2.0.2, there is an action of Γ via ρ on a complete CAT(0) space X such that H_0 preserves and acts by translations on a thick flat in X. By Lemma 2.2.8, it follows that H_0 is undistorted in Γ, and hence the same is true of H.   □

Proof of Theorem 2.0.6. Suppose otherwise, so that for each JSJ torus S of M, the representation ρ is faithful on $\pi_1(S) < \pi_1(M)$ and the image $\rho(\pi_1(S))$ contains no nontrivial unipotent matrices. Then, by Theorem 2.0.2, there is an action of $\pi_1(M)$ via ρ on a complete CAT(0) space X such that for each JSJ torus S of M, the subgroup $\pi_1(S)$ preserves and acts as a lattice of translations on a thick flat in X. Thus, M admits a nonpositively curved metric by Lemma 2.2.7.   □

Chapter 3

VU curves in some non-NPC graph manifolds

We say a matrix $P \in \mathrm{GL}_n(\mathbb{F})$, where $\mathbb{F}$ is a field, is *virtually unipotent* if P^m is unipotent for some positive integer m, that is, if the eigenvalues of P are all roots of 1 in the algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$. Note that a matrix in $\mathrm{GL}_n(\overline{\mathbb{F}})$ that is both virtually unipotent and diagonalizable has finite order.

We begin this chapter with an observation about the integral Heisenberg group H, defined as the subgroup of $\mathrm{GL}_3(\mathbb{R})$ consisting of the upper unitriangular integer matrices.

Remark 3.0.1. Let $\mathbb{F}$ be a field, let $\rho : H \to \mathrm{GL}_n(\mathbb{F})$ be any representation, and let $x, y, z \in H$ be the matrices

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Up to replacing $\mathbb{F}$ with its algebraic closure and postconjugating ρ, we may assume that $\rho(z)$ has a block-diagonal structure

$$\rho(z) = \mathrm{diag}(Z_1, \ldots, Z_k)$$

where $Z_r \in \mathrm{GL}_{n_r}(\mathbb{F})$ is upper triangular with a unique eigenvalue $\lambda_r \in \mathbb{F}^*$, and that $\lambda_1, \ldots, \lambda_k$ are distinct. Since $\rho(x), \rho(y)$ commute with $\rho(z)$, each of the former preserves

the generalized eigenspaces of $\rho(z)$ and thus has a block-diagonal structure

$$\rho(x) = \mathrm{diag}(X_1, \ldots, X_k)$$
$$\rho(y) = \mathrm{diag}(Y_1, \ldots, Y_k)$$

where $X_r, Y_r \in \mathrm{GL}_{n_r}(\mathbb{F})$. Since $z = [x, y]$, we have $Z_r = [X_r, Y_r]$, and so

$$\lambda_r^{n_r} = \det Z_r = 1$$

for $r = 1, \ldots, k$. We conclude that $\rho(z)$ is a virtually unipotent matrix.

Remark 3.0.1 motivates the following definition.

Definition 3.0.2. An element γ of an arbitrary group Γ is *VU* if any finite-dimensional linear representation of Γ maps γ to a virtually unipotent matrix.

Remark 3.0.3. If Γ is a residually finite group, as are many groups of interest and, in particular, as is the fundamental group of any closed 3-manifold [Hem16], then for any nontrivial element $\gamma \in \Gamma$, there is a finite-dimensional unitary representation ρ of Γ such that $\rho(\gamma)$ is nontrivial and hence, by diagonalizability of unitary matrices, not unipotent. Thus, for our purposes, it is not sensible to omit the word "virtually" in Definition 3.0.2.

Remark 3.0.4. If γ is a VU element of a group Γ, then any element in the conjugacy class of γ is VU in Γ. Moreover, if Γ_0 is an abelian subgroup of Γ generated by VU elements of Γ, then any element of Γ_0 is VU in Γ. The latter follows from the previously used fact that commuting matrices over an algebraically closed field are simultaneously triangularizable [RR00, Theorem 1.1.5].

Remark 3.0.5. Suppose Γ_0 is a finite-index normal subgroup of a group Γ, and that γ is a VU element of Γ. Then a generator γ_0 of $\langle \gamma \rangle \cap \Gamma_0$ is a VU element of Γ_0. Indeed, let ρ_0 be a finite-dimensional linear representation of Γ_0. Then ρ_0 is a direct summand of the restriction $\rho\big|_{\Gamma_0}$, where ρ is the representation induced by ρ_0 on Γ. Since $\rho(\gamma_0)$ is a virtually unipotent matrix, it follows that the same is true for $\rho_0(\gamma_0)$.

Remark 3.0.6. Lubotzky, Mozes, and Raghunathan [LMR00, Prop. 2.4] showed that an element generating a distorted cyclic subgroup of a finitely generated group is VU. This fact can also be seen as a special case of Corollary 2.3.1.

Note that a finite-order element of any group is VU. From Remark 3.0.1 (or Remark 3.0.6), one observes that the integer Heisenberg group H, viewed as an abstract group, contains an *infinite-order* VU element (namely, a generator of the center of H), and hence by Remark 3.0.5 so does the fundamental group of any closed 3-manifold with Nil geometry. In fact, the argument in Remark 3.0.1 shows that if an element γ of a group Γ is contained in $[C_\Gamma(\gamma), C_\Gamma(\gamma)]$, where $C_\Gamma(\gamma)$ is the centralizer of γ in Γ, then γ is a VU element of Γ. Thus, for example, an element of $\pi_1(M)$ representing a Seifert fiber of a closed 3-manifold M with $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ geometry is VU in $\pi_1(M)$.

Closed 3-manifolds locally modeled on Nil or $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ are not NPC [GW71, Yau71, Ebe82]. The goal of this chapter is to exhibit nontrivial VU elements within fundamental groups of non-NPC 3-manifolds of a different nature.

Theorem 3.0.7. *Let M be a connected closed orientable irreducible 3-manifold containing exactly one JSJ torus, and each of whose JSJ blocks is a product of S^1 with a surface. If M is not NPC, then $\pi_1(M)$ contains a nontrivial VU element.*

We use a necessary and sufficient condition (Theorem 3.1.2) for such 3-manifolds M to be NPC due to Buyalo and Kobel'skii [BK95], and independently Kapovich and Leeb [KL96] in the case that M has two JSJ blocks. Our argument is similar to Button's proof that Gersten's free-by-cyclic group contains a nontrivial VU element [But17a, Theorem 4.5]. We remark that if M is a 3-manifold as in the statement of Theorem 3.0.7 that is not the mapping torus of an Anosov homeomorphism of the 2-torus, then it follows from [KL98] that all cyclic subgroups of $\pi_1(M)$ are undistorted.

An example of a 3-manifold M as in the statement of Theorem 3.0.7 is the mapping torus of a Dehn twist about an essential simple closed curve on a closed orientable surface of genus at least 2 [KL96, Theorem 3.7]. In this case, our proof in fact shows that an element of $\pi_1(M)$ representing that curve is VU.

Remark 3.0.8. If $\mathbb{F}$ is a field of positive characteristic, then any unipotent matrix over $\mathbb{F}$ is torsion [But17a, Proposition 2.1], and hence no group containing an infinite-order VU element is linear over $\mathbb{F}$. Thus, a consequence of Theorem 3.0.7 is that the fundamental group of a non-NPC 3-manifold M as in Theorem 3.0.7 is not linear over a field of positive

characteristic. In fact, the same can be said about the fundamental group of any closed aspherical non-NPC 3-manifold by [KL96, Theorem 2.4] and [But19, Theorem 2.3].

3.1 Preliminaries

3.1.1 Definitions

If S is a (not necessarily connected) closed surface embedded in a 3-manifold M, we denote by $M \big| S$ the complement in M of a small open tubular neighborhood of S. If M is a connected closed orientable irreducible 3-manifold, then there is, up to isotopy, a unique minimal collection $\mathcal{E}$ of disjoint embedded incompressible tori such that each component of $M \big| \bigcup \mathcal{E}$ is either Seifert or atoroidal (see, for example, [Kap01, Thm 1.41] and the references therein). The decomposition of M into the components of $M \big| \bigcup \mathcal{E}$ is called the *Jaco–Shalen–Johannson (JSJ) decomposition* of M. If $\mathcal{E} = \emptyset$, we say M has *trivial* JSJ decomposition. Note that if M is the mapping torus of an Anosov homeomorphism of the 2-torus, then M has nontrivial JSJ decomposition.

Let $\mathfrak{G}$ denote the class of all connected closed orientable irreducible non-Seifert 3-manifolds M such that each component M_v of $M \big| \bigcup \mathcal{E}$, where $\mathcal{E}$ is the collection of JSJ tori in M, is a trivial S^1-bundle over a compact orientable surface Σ_v with boundary. The manifolds M_v are the *blocks* of M. The *underlying graph* $\mathcal{G} = \mathcal{G}(M)$ of M is the graph dual to the JSJ decomposition of M; the graph $\mathcal{G}$ is well-defined since the collection $\mathcal{E}$ is unique up to isotopy. We identify the vertex set $\mathcal{V}$ of $\mathcal{G}$ with the set of blocks of M, and the set of unoriented edges of $\mathcal{G}$ with $\mathcal{E}$. Denote by $\mathcal{W}$ the set of oriented edges of $\mathcal{G}$. We identify $\mathcal{W}$ with the set of boundary components of $M \big| \bigcup \mathcal{E}$ by assigning to each oriented edge $w \in \partial v \subset \mathcal{W}$ the corresponding boundary component T_w of M_v.

Choose an orientation of M, thereby inducing an orientation on each block M_v of M, and hence on each component of ∂M_v. For each $v \in \mathcal{V}$, choose an orientation of the fibers in M_v, as well as a *Waldhausen basis* for $H_1(\partial M_v; \mathbb{Z})$; that is, a basis $\{(f_w, z_w) \mid w \in \partial v\}$ for $H_1(\partial M_v; \mathbb{Z}) = \bigoplus_{w \in \partial v} H_1(T_w; \mathbb{Z})$ such that the elements f_w represent oriented fibers, the algebraic intersection number $\hat{i}(z_w, f_w)$ on T_w is $+1$, and the sum $\oplus_{w \in \partial v} z_w$ lies in the

kernel of the map $H_1(\partial M_v; \mathbb{Z}) \to H_1(M_v; \mathbb{Z})$ induced by inclusion. We call the additional structure on M given by the choices made in this paragraph a *framing* of M.

An oriented edge $w \in \mathcal{W}$ corresponds to a gluing homeomorphism $T_{-w} \to T_w$, which induces an isomorphism $\phi_w : H_1(T_{-w}; \mathbb{Z}) \to H_1(T_w; \mathbb{Z})$. Define $B_w = \begin{pmatrix} a_w & b_w \\ c_w & d_w \end{pmatrix} \in$ $\mathrm{GL}_2(\mathbb{Z})$ to be the matrix whose entries satisfy

$$\phi_w(f_{-w}) = a_w f_w + b_w z_w$$
$$\phi_w(z_{-w}) = c_w f_w + d_w z_w$$

Note that $\det B_w = -1$ since M is orientable, that $B_{-w} = B_w^{-1}$, and that $b_w \neq 0$ by minimality of $\mathcal{E}$.

This chapter is concerned with the subclasses $\mathfrak{E}, \mathfrak{L}$ of $\mathfrak{G}$ consisting of all manifolds M in $\mathfrak{G}$ whose underlying graph is a single edge (joining distinct vertices) or a loop, respectively. We call $B \in \mathrm{GL}_2(\mathbb{Z})$ a *gluing matrix* for such a manifold M if $B = B_w$ for an oriented edge w of $\mathcal{G}(M)$ with respect to some framing of M.

The fundamental group $\pi_1(M)$ of a manifold $M \in \mathfrak{E}$ with gluing matrix $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and whose blocks $M_v, M_{v'}$ have base surfaces Σ, Σ' of genus g, g', respectively, is isomorphic to the group $\Gamma_{g,g',B}^{\mathfrak{E}}$ given by the presentation with generators

$$x_1, y_1, \ldots, x_g, y_g, z, f,$$
$$x_1', y_1', \ldots, x_{g'}', y_{g'}', z', f'$$

subject to the relations

(I) $z = \prod_{i=1}^{g}[x_i, y_i]$,

(II) $[x_i, f] = [y_i, f] = 1$ for $i = 1, \ldots, g$,

(III) $z' = \prod_{i=1}^{g'}[x_i', y_i']$,

(IV) $[x_i', f'] = [y_i', f'] = 1$ for $i = 1, \ldots, g'$,

(V) $f' = f^a z^b$,

(VI) $z' = f^c z^d$,

where the subgroup $\langle x_1, y_1, \ldots, x_g, y_g \rangle$ (resp., $\langle x_1', y_1', \ldots, x_{g'}', y_{g'}' \rangle$) is the image of the map $\pi_1(\Sigma) \to \pi_1(M)$ (resp., $\pi_1(\Sigma') \to \pi_1(M)$) induced by the inclusions $\Sigma \subset M_v \subset M$ (resp., $\Sigma' \subset M_{v'} \subset M$), and the element f (resp., f') represents an oriented fiber of M_v (resp., $M_{v'}$).

Remark 3.1.1. Note that if C is obtained from B by negating a row or a column of B, then $\Gamma_{g,g',B}^{\mathfrak{E}} \cong \Gamma_{g,g',C}^{\mathfrak{E}}$. Note also that $\Gamma_{g,g',B}^{\mathfrak{E}} \cong \Gamma_{g',g,B^{-1}}^{\mathfrak{E}}$.

The fundamental group $\pi_1(M)$ of a manifold $M \in \mathfrak{L}$ with gluing matrix $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and the base surface Σ of whose unique block M_v has genus g is isomorphic to the group $\Gamma_{g,B}^{\mathfrak{L}}$ given by the presentation with generators

$$x_1, y_1, \ldots, x_g, y_g, z, z', f, t$$

subject to the relations

(1) $zz' = \prod_{i=1}^{g}[x_i, y_i]$,

(2) $[x_i, f] = [y_i, f] = [z, f] = 1$ for $i = 1, \ldots, g$,

(3) $tft^{-1} = f^a z^b$,

(4) $tz't^{-1} = f^c z^d$,

where the subgroup $\langle x_1, y_1, \ldots, x_g, y_g, z \rangle$ is the image of the map $\pi_1(\Sigma) \to \pi_1(M)$ induced by the inclusion $\Sigma \subset M_v \subset M$, and the element f represents an oriented fiber of M_v.

The following theorem is a special case of a result of Buyalo and Kobel'skii [BK95], and was proved independently by Kapovich and Leeb [KL96] in the case $M \in \mathfrak{E}$.

Theorem 3.1.2. *Let $M \in \mathfrak{E}$ (resp., $M \in \mathfrak{L}$) and let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$ be a gluing matrix for M. Then M is NPC if and only if $a = d = 0$ (resp., if and only if $|a - d| < 2$).*

3.1.2   Basic lemmas

The following lemma will allow us to conjugate a representation ρ of the appropriate group Γ in a manner that makes the interactions between generalized eigenspaces of certain elements of $\rho(\Gamma)$ more apparent.

Lemma 3.1.3. *Let $\mathbb{F}$ be an algebraically closed field, let $P, P', Q \in \mathrm{M}_n(\mathbb{F})$, and let $\lambda_1, \ldots, \lambda_k$ (resp. $\lambda'_1, \ldots, \lambda'_\ell$) be the distinct eigenvalues of P (resp. P'). If P, P', Q pairwise commute, then there is a matrix $C \in \mathrm{GL}_n(\mathbb{F})$ such that*

$$CPC^{-1} = \mathrm{diag}\left(P_{1,1}, \ldots, P_{1,\ell}, \ldots, P_{k,1}, \ldots, P_{k,\ell}\right),$$
$$CP'C^{-1} = \mathrm{diag}\left(P'_{1,1}, \ldots, P'_{1,\ell}, \ldots, P'_{k,1}, \ldots, P'_{k,\ell}\right),$$
$$CQC^{-1} = \mathrm{diag}\left(Q_{1,1}, \ldots, Q_{1,\ell}, \ldots, Q_{k,1}, \ldots, Q_{k,\ell}\right),$$

where $P_{r,s}, P'_{r,s}, Q_{r,s}$ are (possibly empty) upper triangular matrices and the only eigenvalue of $P_{r,s}$ (resp., $P'_{r,s}$) is λ_r (resp., λ'_s).

Proof. For $r = 1, \ldots, k$, let W_r be the generalized λ_r-eigenspace of P, and let $n_r = \dim W_r$. We index the standard ordered basis for $\mathbb{F}^n$ as follows:

$$(e_{1,1}, \ldots, e_{1,n_1}, \ldots, e_{k,1}, \ldots, e_{k,n_k}).$$

We may assume that $W_r = \mathrm{Span}(e_{r,1}, \ldots, e_{r,n_r})$. Since each of P', Q commutes with P, we have that P', Q preserve the generalized eigenspaces of P, so P, P', Q share a block-diagonal structure

$$P = \mathrm{diag}(P_1, \ldots, P_k),$$
$$P' = \mathrm{diag}(P'_1, \ldots, P'_k),$$
$$Q = \mathrm{diag}(Q_1, \ldots, Q_k)$$

where $P_r, P'_r \in \mathrm{M}_{n_r}(\mathbb{F})$. We may also assume that for some indexing

$$(e_{r,1,1}, \ldots, e_{r,1,n_{r,1}}, \ldots, e_{r,\ell,1}, \ldots, e_{r,\ell,n_{r,\ell}})$$

of the ordered basis $(e_{r,1}, \ldots, e_{r,n_r})$ for W_r, where the $n_{r,s}$ are nonnegative integers satisfying $\sum_{s=1}^{\ell} n_{r,s} = n_r$, we have that $\mathrm{Span}(e_{r,s,1}, \ldots, e_{r,s,n_{r,s}})$ is the generalized λ'_s-eigenspace

of P'_r. Since each of P_r, Q_r commutes with P'_r, we have that P_r, Q_r preserve the generalized eigenspaces of P'_r, so P_r, P'_r, Q_r share a block-diagonal structure

$$P_r = \mathrm{diag}(P_{r,1}, \ldots, P_{r,\ell}),$$
$$P'_r = \mathrm{diag}(P'_{r,1}, \ldots, P'_{r,\ell}),$$
$$Q_r = \mathrm{diag}(Q_{r,1}, \ldots, Q_{r,\ell}).$$

Now since $P_{r,s}, P'_{r,s}, Q_{r,s}$ pairwise commute, they are simultaneously upper triangularizable [RR00, Theorem 1.1.5], and Lemma 3.1.3 follows. $\qquad\square$

The following lemma will allow us to reduce systems of equations whose unknowns lie in $\mathbb{F}^*$, where $\mathbb{F}$ is some field, to systems of linear equations with integer unknowns. It is a step in the proof of Theorem 4.5 in [But17a]. We include Button's argument for the convenience of the reader.

Lemma 3.1.4. *Let M be an integer matrix with L columns and suppose there is a subset $I \subset \{1, \ldots, L\}$ such that for any $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_L)^T \in \mathbb{Z}^L$ satisfying $M\boldsymbol{\alpha} = 0$, we have $\alpha_i = 0$ for $i \in I$. Let A be a torsion-free abelian group, and suppose $\mathbf{a} = (a_1, \ldots, a_L)^T \in A^L$ satisfies $M\mathbf{a} = 0$. Then $a_i = 0$ for $i \in I$.*

Proof. Let $A_0 = \langle a_1, \ldots, a_L \rangle \subset A$. Then A_0 is a finitely generated torsion-free abelian group, so there is an isomorphism $\varphi : A_0 \to \mathbb{Z}^K$ for some K. For each $j = 1, \ldots, K$, we have

$$M(\varphi_j(a_1), \ldots, \varphi_j(a_L))^T = 0$$

where $\varphi_j = p_j \circ \varphi$ and $p_j : \mathbb{Z}^K \to \mathbb{Z}$ is the projection onto the jth coordinate, so that $\varphi_j(a_i) = 0$ for $i \in I$. We conclude that $\varphi(a_i) = 0$, and hence $a_i = 0$, for $i \in I$. $\qquad\square$

3.2 Proof of Theorem 3.0.7

We divide Theorem 3.0.7 into Theorem 3.2.1 (the loop case) and Theorem 3.2.2 (the edge case), and prove each separately.

Theorem 3.2.1. *Suppose $M \in \mathfrak{L}$ is not NPC, and let $\Gamma = \pi_1(M)$. Then Γ contains a nontrivial VU element.*

Proof. By Theorem 3.1.2, we have $\Gamma = \Gamma_{g,B}^{\mathfrak{L}}$ for some $g \geq 0$ and $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z})$ with $\det B = -1$, $b \neq 0$, and $|a - d| \geq 2$. We show that $f^{a-1}z^b \in \Gamma$ is VU if $a - d \geq 2$, and that $f^{a+1}z^b$ is VU if $a - d \leq -2$.

Let $\mathbb{F}$ be an algebraically closed field, $n \geq 1$, and $\rho : \Gamma \to \mathrm{GL}(n, \mathbb{F})$ any representation. We may assume that ρ is indecomposable. Let $\lambda_1, \ldots, \lambda_k \in \mathbb{F}^*$ be the distinct eigenvalues of $\rho(f)$, and let $f' = tft^{-1}$. By Lemma 3.1.3 and relation (3) in the presentation of Γ, we may assume further that

$$\rho(f) = \mathrm{diag}(F_{1,1}, \ldots, F_{1,k}, \ldots, F_{k,1}, \ldots, F_{k,k}),$$
$$\rho(f') = \mathrm{diag}(F'_{1,1}, \ldots, F'_{1,k}, \ldots, F'_{k,1}, \ldots, F'_{k,k}),$$
$$\rho(z) = \mathrm{diag}(D_{1,1}Z_{1,1}, \ldots, D_{1,k}Z_{1,k}, \ldots, D_{k,1}Z_{k,1}, \ldots, D_{k,k}Z_{k,k}),$$

where $F_{r,s}, F'_{r,s}, Z_{r,s} \in \mathrm{GL}(n_{r,s}, \mathbb{F})$ are (possibly empty) upper triangular matrices, $D_{r,s}$ is a (possibly empty) diagonal matrix in $\mathrm{GL}(n_{r,s}, \mathbb{F})$ whose diagonal entries are $|b|^{\mathrm{th}}$ roots of 1 in $\mathbb{F}$, and the only eigenvalue of $F_{r,s}$ (resp., $F'_{r,s}, Z_{r,s}$) is λ_r (resp., $\lambda_s, \mu_{r,s}$), with $\mu_{r,s} \in \mathbb{F}^*$ satisfying

$$\lambda_s = \lambda_r^a \mu_{r,s}^b. \tag{3.1}$$

Since, by relation (2) in the presentation of Γ, each of $\rho(z'), \rho(x_1), \rho(y_1), \ldots, \rho(x_g), \rho(y_g)$ commutes with $\rho(f)$, each preserves the generalized eigenspaces of $\rho(f)$, and so

$$\rho(z') = \mathrm{diag}(Z'_1, \ldots, Z'_k),$$
$$\rho(x_i) = \mathrm{diag}(X_1^{(i)}, \ldots, X_k^{(i)}),$$
$$\rho(y_i) = \mathrm{diag}(Y_1^{(i)}, \ldots, Y_k^{(i)})$$

for some $Z'_r, X_r^{(i)}, Y_r^{(i)} \in \mathrm{GL}(n_r, \mathbb{F})$, where $n_r = \sum_{s=1}^k n_{r,s}$ is the dimension of the generalized λ_r-eigenspace of $\rho(f)$.

Let V_r be the generalized λ_r-eigenspace of $\rho(f')$. Then $\rho(t)^{-1}V_r$ is the generalized λ_r-eigenspace of $\rho(t)^{-1}\rho(f')\rho(t) = \rho(f)$, and the characteristic polynomial of

$$\rho(z')\big|_{\rho(t)^{-1}V_r} = \rho(t)^{-1}\rho(f^c h^d)\rho(t)\big|_{\rho(t)^{-1}V_r}$$

coincides with the characteristic polynomial of $\rho(f^c z^d)\big|_{V_r}$. Thus, up to multiplying each root by a root of 1, the characteristic polynomial of the block Z'_r is

$$(x - \lambda_1^c \mu_{1,r}^d)^{n_{1,r}} \ldots (x - \lambda_k^c \mu_{k,r}^d)^{n_{k,r}}.$$

Now let $Z_r = \text{diag}(D_{r,1} Z_{r,1}, \ldots, D_{r,k} Z_{r,k})$. Then, by relation (1) in the presentation of Γ, we have $Z_r Z'_r = \prod_{i=1}^{g}[X_r^{(i)}, Y_r^{(i)}]$, so that $\det(Z_r Z'_r) = 1$. It follows that

$$\prod_{s=1}^{k} \mu_{r,s}^{n_{r,s}} (\lambda_s^c \mu_{s,r}^d)^{n_{s,r}} = 1 \tag{3.2}$$

in the quotient A of the group of units $\mathbb{F}^*$ by its torsion subgroup. Viewing (3.1) also as equations in A and switching to additive notation within A, we obtain the equations

$$\lambda_s = a\lambda_r + b\mu_{r,s}, \tag{3.3}$$

$$\sum_{s=1}^{k} (n_{r,s}\mu_{r,s} + n_{s,r}(c\lambda_s + d\mu_{s,r})) = 0. \tag{3.4}$$

Multiplying (3.4) by b and substituting $\lambda_s - a\lambda_r$ for $b\mu_{r,s}$, we have

$$\sum_{s=1}^{k} \left(n_{r,s}(\lambda_s - a\lambda_r) + n_{s,r}\big(bc\lambda_s + d(\lambda_r - a\lambda_s)\big) \right) = 0$$

and so

$$\sum_{s=1}^{k} \big(n_{r,s} + (bc - ad)n_{s,r}\big)\lambda_s = \lambda_r \sum_{s=1}^{k} (an_{r,s} - dn_{s,r}). \tag{3.5}$$

Since $bc - ad = -\det B = 1$, the left-hand side of (3.5) is equal to $\sum_{s=1}^{k}(n_{r,s} + n_{s,r})\lambda_s$. On the other hand, since $\sum_{s=1}^{k} n_{s,r} = \sum_{s=1}^{k} n_{r,s} = n_r$, the right-hand side of (3.5) is equal to $(a - d)n_r\lambda_r$.

In summary, $\lambda_1, \ldots, \lambda_k$ satisfy

$$\sum_{s=1}^{k} (n_{r,s} + n_{s,r})\lambda_s = (a - d)n_r\lambda_r \tag{3.6}$$

as elements of A.

Consider the case that $a - d \geq 2$. We show that, in this case, if we set $A = \mathbb{Z}$, then (3.6) implies $\lambda_1 = \ldots = \lambda_k$, so that

$$(a - 1)\lambda_r + b\mu_{r,s} = a\lambda_r - \lambda_s + b\mu_{r,s} = 0,$$

where the second equality follows from (3.3). By Lemma 3.1.4, it will follow that $(a-1)\lambda_r + b\mu_{r,s} = 0$ in the original torsion-free abelian group A, i.e., that $\rho(f^{a-1}z^b)$ is virtually unipotent, and thus completing the proof of the $a - d \geq 2$ case.

To that end, suppose for a contradiction that the integers $\lambda_1, \ldots, \lambda_k$ are not all equal. Then we may assume

$$\lambda_1 = \ldots = \lambda_{r_0} > \lambda_{r_0+1}, \ldots, \lambda_k$$

for some $r_0 \in \{1, \ldots, k-1\}$. Thus, for $r = 1, \ldots, r_0$, either we have $n_{r,s} + n_{s,r} = 0$ for all $s > r_0$, or we obtain the contradiction

$$2n_r\lambda_r = \sum_{s=1}^{k}(n_{r,s} + n_{s,r})\lambda_r > \sum_{s=1}^{k}(n_{r,s} + n_{s,r})\lambda_s = (a-d)n_r\lambda_r \geq 2n_r\lambda_r.$$

We conclude that $n_{r,s} = n_{s,r} = 0$ for $r \leq r_0$ and $s > r_0$, so that $\rho(t)$ preserves the span of the first $\sum_{r=1}^{r_0} n_r$ standard basis vectors and the span of the last $\sum_{r=r_0+1}^{k} n_r$ standard basis vectors of $\mathbb{F}^n$. But then $\rho(\Gamma)$ also preserves each of these subspaces, contradicting the indecomposability of ρ. This completes the proof for the case $a - d \geq 2$.

We now consider the case $a - d \leq -2$. We show in this case that, considered over $\mathbb{Z}$, equations (3.3) and (3.6) imply that $(a+1)\lambda_r + b\mu_{r,s} = 0$ for all r, s with $n_{r,s} \neq 0$. One then applies Lemma 3.1.4 to conclude that $\rho(f^{a+1}z^b)$ is virtually unipotent.

We first observe that, setting $A = \mathbb{Z}$, equation (3.6) implies that $|\lambda_1| = \ldots = |\lambda_k|$. Indeed, suppose otherwise. Then we may assume $|\lambda_1| = \ldots = |\lambda_{r_0}| > |\lambda_{r_0+1}|, \ldots, |\lambda_k|$ for some $r_0 \in \{1, \ldots, k-1\}$. Thus, for $r \leq r_0$, either we have $n_{r,s} + n_{s,r} = 0$ when $s > r_0$, or we obtain the contradiction

$$2n_r|\lambda_r| = \sum_{s=1}^{k}(n_{r,s} + n_{s,r})|\lambda_r| > \sum_{s=1}^{k}(n_{r,s} + n_{s,r})|\lambda_s| \geq \left|\sum_{s=1}^{k}(n_{r,s} + n_{s,r})\lambda_s\right| = |a-d|n_r|\lambda_r|$$

$$\geq 2n_r|\lambda_r|.$$

We conclude that indeed $n_{r,s} = n_{s,r} = 0$ for $r \leq r_0$ and $s > r_0$, so that $\rho(t)$ preserves the span of the first $\sum_{r=1}^{r_0} n_r$ standard basis vectors and that of the last $\sum_{r=r_0+1}^{k} n_r$ standard basis vectors of $\mathbb{F}^n$. But then $\rho(\Gamma)$ also preserves each of these subspaces, contradicting the indecomposability of ρ. This establishes our claim that $|\lambda_1| = \ldots = |\lambda_k|$.

Now suppose for a contradiction that there is a solution to equation (3.6) with $\lambda_1 = \ldots = \lambda_k = \lambda$, where λ is a nonzero integer. Then we have

$$2n_1\lambda = \sum_{s=1}^{k}(n_{1,s} + n_{s,1})\lambda = (a - d)n_1\lambda,$$

and hence $a - d = 2$, contradicting our assumption that $a - d \leq -2$.

Thus, if the λ_r are to satisfy equation (3.6), then either $\lambda_1 = \ldots = \lambda_k = 0$, or, up to reordering the λ_r, we have $\lambda_1 = \ldots = \lambda_{r_0} = -\lambda_{r_0+1} = \ldots = -\lambda_k = \lambda$ for some nonzero integer λ and some $r_0 \in \{1, \ldots, k-1\}$. In either case, we have by (3.3) that

$$(a + 1)\lambda_r + b\mu_{r,s} = a\lambda_r - \lambda_s + b\mu_{r,s} = 0$$

whenever $r \leq r_0$ and $s > r_0$, and whenever $r > r_0$ and $s \leq r_0$. This is in fact true for all r and s in the first case. In the second case, we have by (3.6) that for $r > r_0$,

$$(a - d)n_r(-\lambda) = \sum_{s=1}^{r_0}(n_{r,s} + n_{s,r})\lambda - \sum_{s=r_0+1}^{k}(n_{r,s} + n_{s,r})\lambda = 2n_r\lambda - 2\sum_{s=r_0+1}^{k}(n_{r,s} + n_{s,r})\lambda,$$

and so

$$2\sum_{s=r_0+1}^{k}(n_{r,s} + n_{s,r}) = (2 + (a - d))n_r \leq 0$$

since $a - d \leq -2$, from which we conclude that $n_{r,s} = 0$ for $r, s > r_0$, and hence also for $r, s \leq r_0$ (in other words, the matrix $\rho(t)$ interchanges the span of the first $\sum_{r=1}^{r_0} n_r$ standard basis vectors with that of the last $\sum_{r=r_0+1}^{k} n_r$ standard basis vectors of $\mathbb{F}^n$). This completes the proof for the case $a - d \leq -2$. $\square$

Theorem 3.2.2. *Suppose $M \in \mathfrak{E}$ is not NPC, and let $\Gamma = \pi_1(M)$. Then Γ contains a nontrivial VU element.*

Proof. We have $\Gamma = \Gamma_{g,g',B}^{\mathfrak{E}}$ for some $g, g' \geq 1$, where $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a gluing matrix for M. Note that $b \neq 0$, and that, by Theorem 3.1.2, one of a, d is nonzero. By Remark 3.1.1, up to replacing B with its inverse, we may assume $a \neq 0$. Furthermore, by Remark 3.1.1 and the fact that $|\det B| = 1$, up to negating rows and columns of B, we may

assume $a, b, c, d \geq 0$. We show that if $c = 0$ (resp., $c > 0$) then z (resp., f) is a VU element of Γ.

Let $\mathbb{F}$ be an algebraically closed field, $n \geq 1$, and $\rho : \Gamma \to \mathrm{GL}_n(\mathbb{F})$ any representation. Let $\lambda_1, \ldots, \lambda_k \in \mathbb{F}^*$ (resp., $\lambda'_1, \ldots, \lambda'_\ell \in \mathbb{F}^*$) be the distinct eigenvalues of $\rho(f)$ (resp., $\rho(f')$). By Lemma 3.1.3 and relation (V) in the presentation of Γ, we may assume that

$$\rho(f) = \mathrm{diag}(F_{1,1}, \ldots, F_{1,\ell}, \ldots, F_{k,1}, \ldots, F_{k,\ell}),$$
$$\rho(f') = \mathrm{diag}(F'_{1,1}, \ldots, F'_{1,\ell}, \ldots, F'_{k,1}, \ldots, F'_{k,\ell}),$$
$$\rho(z) = \mathrm{diag}(D_{1,1}Z_{1,1}, \ldots, D_{1,\ell}Z_{1,\ell}, \ldots, D_{k,1}Z_{k,1}, \ldots, D_{k,\ell}Z_{k,\ell}),$$

where $F_{r,s}, F'_{r,s}, Z_{r,s} \in \mathrm{GL}_{n_{r,s}}(\mathbb{F})$ are (possibly empty) upper triangular matrices, $D_{r,s}$ is a diagonal matrix in $\mathrm{GL}_{n_{r,s}}(\mathbb{F})$ whose diagonal entries are b^{th} roots of 1 in $\mathbb{F}$, and the only eigenvalue of $F_{r,s}$ (resp., $F'_{r,s}, Z_{r,s}$) is λ_r (resp., $\lambda'_s, \mu_{r,s}$), with $\mu_{r,s} \in \mathbb{F}^*$ satisying

$$\lambda'_s = \lambda_r^a \mu_{r,s}^b. \tag{3.7}$$

Since, by relation (II) in the presentation of Γ, the $\rho(x_i), \rho(y_i)$ commute with $\rho(f)$, each of the former preserves the generalized eigenspaces of $\rho(f)$. Thus, we have

$$\rho(x_i) = \mathrm{diag}(X_1^{(i)}, \ldots, X_k^{(i)}),$$
$$\rho(y_i) = \mathrm{diag}(Y_1^{(i)}, \ldots, Y_k^{(i)}),$$

for some $X_r^{(i)}, Y_r^{(i)} \in \mathrm{GL}_{n_r}(\mathbb{F})$, where $n_r = \sum_{s=1}^k n_{r,s}$ is the dimension of the generalized λ_r-eigenspace of $\rho(f)$. Letting $Z_r = \mathrm{diag}(D_{r,1}Z_{r,1}, \ldots, D_{r,\ell}Z_{r,\ell})$, we have by relation (I) in the presentation of Γ that

$$Z_r = \prod_{i=1}^g [X_r^{(i)}, Y_r^{(i)}]$$

for $r = 1, \ldots, k$. Thus, $\det Z_r = 1$, and so

$$\prod_{s=1}^\ell \mu_{r,s}^{n_{r,s}} = 1 \tag{3.8}$$

in the quotient A of $\mathbb{F}^*$ by its torsion subgroup.

Since, by relation (IV) in the presentation of Γ, the $\rho(x_i'), \rho(y_i')$ commute with $\rho(f')$, each of the former preserves the eigenspaces of $\rho(f')$. Thus, by a similar argument to the one given above, and by relation (VI) in the presentation of Γ, we have

$$\prod_{r=1}^{k} (\lambda_r^c \mu_{r,s}^d)^{n_{r,s}} = 1 \tag{3.9}$$

in A for $s = 1, \ldots, \ell$. Switching to additive notation within A, we obtain from (3.7), (3.8), (3.9) the equations

$$a\lambda_1 + b\mu_{1,s} = \ldots = a\lambda_k + b\mu_{k,s} \text{ for } s = 1, \ldots, \ell, \tag{3.10}$$

$$\sum_{s=1}^{\ell} n_{r,s}\mu_{r,s} = 0 \text{ for } r = 1, \ldots, k, \tag{3.11}$$

$$\sum_{r=1}^{k} n_{r,s}(c\lambda_r + d\mu_{r,s}) = 0 \text{ for } s = 1, \ldots, \ell. \tag{3.12}$$

We now set $A = \mathbb{Z}$ and show that, in this context, equations (3.11), (3.12), and (3.10) imply that if $c = 0$ (resp., $c > 0$) then $\mu_{r,s} = 0$ whenever $n_{r,s} > 0$ (resp., then $\lambda_r = 0$ for $r = 1, \ldots, k$). By Lemma 3.1.4, the same statements will hold in the original torsion-free abelian group A, thus completing the proof.

Suppose first that $c = 0$. Note that since $|\det B| = 1$, this implies that $a = d = 1$, so that equations (3.10), (3.12) are reduced to

$$\lambda_1 + b\mu_{1,s} = \ldots = \lambda_k + b\mu_{k,s} \text{ for } s = 1, \ldots, \ell, \tag{3.13}$$

$$\sum_{r=1}^{k} n_{r,s}\mu_{r,s} = 0 \text{ for } s = 1, \ldots, \ell. \tag{3.14}$$

We show by induction on $k + \ell$ that, in this case, $\mu_{r,s} = 0$ if $n_{r,s} > 0$. The base case $k + \ell = 2$ is trivial. By the symmetry of equations (3.11), (3.14), (3.13), we may assume that $\mu_{k,1} \geq \mu_{r,s}$ for all r and s, and that $\mu_{k,1} \geq \ldots \geq \mu_{k,\ell}$. Note that the former implies that in particular $\mu_{k,1} \geq \mu_{r,1}$, so we obtain from (3.13) that $\mu_{k,\ell} \geq \mu_{r,\ell}$ for $r = 1, \ldots, k$. If $\mu_{k,\ell} \geq 0$, then since $\sum_{s=1}^{\ell} n_{k,s}\mu_{k,s} = 0$, we must have

$$n_{k,1}\mu_{k,1} = \ldots = n_{k,\ell}\mu_{k,\ell} = 0.$$

This implies that $\mu_{k,s} = 0$ if $n_{k,s} > 0$, so we may apply the induction hypothesis to the system of equations

$$\lambda_1 + b\mu_{1,s} = \ldots = \lambda_{k-1} + b\mu_{k-1,s} \text{ for } s = 1, \ldots, \ell,$$

$$\sum_{s=1}^{\ell} n_{r,s}\mu_{r,s} = 0 \text{ for } r = 1, \ldots, k-1,$$

$$\sum_{r=1}^{k-1} n_{r,s}\mu_{r,s} = 0 \text{ for } s = 1, \ldots, \ell.$$

Now suppose that $\mu_{k,\ell} < 0$. Since $\mu_{k,\ell} \geq \mu_{r,\ell}$ for $r = 1, \ldots, k$ and $\sum_{r=1}^{k} n_{r,\ell}\mu_{r,\ell} = 0$, we have that $n_{1,\ell}\mu_{1,\ell} = \ldots = n_{k,\ell}\mu_{k,\ell} = 0$. This implies that $\mu_{r,\ell} = 0$ if $n_{r,\ell} > 0$, so we may apply the induction hypothesis to the system of equations

$$\lambda_1 + b\mu_{1,s} = \ldots = \lambda_k + b\mu_{k,s} \text{ for } s = 1, \ldots, \ell-1,$$

$$\sum_{s=1}^{\ell-1} n_{r,s}\mu_{r,s} = 0 \text{ for } r = 1, \ldots, k,$$

$$\sum_{r=1}^{k} n_{r,s}\mu_{r,s} = 0 \text{ for } s = 1, \ldots, \ell-1.$$

This completes the proof for the case $c = 0$.

We assume for the remainder of the proof that $c > 0$. Define

$$N = \begin{pmatrix} n_{1,1} & \ldots & n_{k,1} \\ \vdots & \ddots & \vdots \\ n_{1,\ell} & \ldots & n_{k,\ell} \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} a\lambda_1 + b\mu_{1,1} \\ \vdots \\ a\lambda_1 + b\mu_{1,\ell} \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} c\lambda_1 \\ \vdots \\ c\lambda_k \end{pmatrix},$$

and let N_r be the rth column of N. We have

$$\mathbf{u}^T N_r = (a\lambda_r + b\mu_{r,1}, \ldots, a\lambda_r + b\mu_{r,\ell})N_r$$

$$= (a\lambda_r, \ldots, a\lambda_r)N_r + b(\mu_{r,1}, \ldots, \mu_{r,\ell})N_r$$

$$= (a\lambda_r, \ldots, a\lambda_r)N_r$$

$$= \sum_{s=1}^{\ell} n_{r,s}a\lambda_r,$$

where the first equality follows from (3.10) and the third follows from (3.11). Thus,

$$\mathbf{u}^T N\mathbf{w} = \left(\sum_{s=1}^{\ell} n_{1,s}a\lambda_1, \ldots, \sum_{s=1}^{\ell} n_{k,s}a\lambda_k \right) \mathbf{w} = \sum_{r,s} n_{r,s}ac\lambda_r^2. \qquad (3.15)$$

On the other hand, we have

$$N\mathbf{w} = \begin{pmatrix} \sum_{r=1}^{k} n_{r,1} c\lambda_r \\ \vdots \\ \sum_{r=1}^{k} n_{r,\ell} c\lambda_r \end{pmatrix} = - \begin{pmatrix} \sum_{r=1}^{k} n_{r,1} d\mu_{r,1} \\ \vdots \\ \sum_{r=1}^{k} n_{r,\ell} d\mu_{r,\ell} \end{pmatrix},$$

where the second equality follows from (3.12). It follows that

$$-\mathbf{u}^T N\mathbf{w} = \mathbf{u}^T \begin{pmatrix} \sum_{r=1}^{k} n_{r,1} d\mu_{r,1} \\ \vdots \\ \sum_{r=1}^{k} n_{r,\ell} d\mu_{r,\ell} \end{pmatrix} = \sum_{r,s} n_{r,s}(a\lambda_1 + b\mu_{1,s})d\mu_{r,s} = \sum_{r,s} n_{r,s}(a\lambda_r + b\mu_{r,s})d\mu_{r,s},$$

(3.16)

where the last equality follows from (3.10). Combining (3.15) and (3.16), we obtain

$$0 = \sum_{r,s} n_{r,s} ac\lambda_r^2 + \sum_{r,s} n_{r,s}(a\lambda_r + b\mu_{r,s})d\mu_{r,s} = \sum_{r,s} n_{r,s}(bd\mu_{r,s}^2 + ad\lambda_r\mu_{r,s} + ac\lambda_r^2). \quad (3.17)$$

We claim that $bd\mu_{r,s}^2 + ad\lambda_r\mu_{r,s} + ac\lambda_r^2 \geq 0$ for any r and s. If $d = 0$, this is clear. Otherwise, we may view $bd\mu_{r,s}^2 + ad\lambda_r\mu_{r,s} + ac\lambda_r^2$ as a quadratic polynomial in $\mu_{r,s}$ with positive leading coefficient bd and discriminant

$$\Delta_r = (ad - 4bc)ad\lambda_r^2 = (\det B - 3bc)ad\lambda_r^2.$$

Since $|\det B| = 1$, we have that $\Delta_r \leq 0$, and so $bd\mu_{r,s}^2 + ad\lambda_r\mu_{r,s} + ac\lambda_r^2 \geq 0$.

Now let $r \in \{1, \ldots, k\}$. We show that $\lambda_r = 0$. Indeed, we have $n_{r,s} > 0$ for some s since $\sum_{s=1}^{k} n_{r,s} = n_r > 0$. Thus, by (3.17) and the previous paragraph, we have

$$bd\mu_{r,s}^2 + ad\lambda_r\mu_{r,s} + ac\lambda_r^2 = 0.$$

If $d = 0$, this immediately implies that $\lambda_r = 0$. Now suppose $d, \lambda_r > 0$. Then $\Delta_r < 0$ and so $bd\mu_{r,s}^2 + ad\lambda_r\mu_{r,s} + ac\lambda_r^2 > 0$, a contradiction. $\qquad\square$

Remark 3.2.3. Note that if $c, d > 0$, we also obtain that $\mu_{r,s} = 0$ whenever $n_{r,s} > 0$, so that $\rho(z)$ is also a virtually unipotent matrix. Thus, if all the entries of a gluing matrix for a manifold $M \in \mathfrak{E}$ are nonzero, then any element of $\pi_1(M)$ representing a curve on the JSJ torus of M is VU.

Chapter 4

Conclusion

To conclude, we summarize and contextualize the content of the previous chapters, and suggest some future directions.

4.0.1 Unipotent-free right-angled Coxeter groups

Let Σ_1 be a finite simplicial graph with vertex set S, thought of as a Coxeter scheme in the classical sense with only bold edges, let W be the right-angled Coxeter group on Σ_1, and let V be the free real vector space on S. For this discussion, we may assume that $|S| \geq 3$ and that Σ_1 is connected. In the proof of Theorem 1.0.1, we considered certain deformations $\sigma_d : W \to \mathrm{GL}(V)$ of the Tits canonical representation σ_1 of W; the σ_d were the Tits–Vinberg representations of W associated to the deformations Σ_d of Σ_1, where Σ_d is the Coxeter scheme obtained from Σ_1 by replacing each edge with a dotted edge labeled by $d \geq 1$. We showed that for certain $d > 1$, the discrete representation σ_d can be Galois conjugated to a precompact one. In particular, for such d, the image of σ_d contains no nontrivial unipotents. It appears the following question is open.

Question 4.0.1. *Does $W_d := \sigma_d(W)$ lack nontrivial unipotents for each $d > 1$?*

The set of all $d > 1$ for which the answer to Question 4.0.1 is positive is certainly dense in $(1, \infty)$, since σ_d can be "Galois conjugated" to a precompact representation for any transcendental $d > 1$ by an argument similar to the proof of Theorem 1.0.1 (indeed,

this was Agol's original argument [Ago18]). However, the trick of Galois conjugation does not apply in the case of interest where d is an integer larger than 1.

Let X be the symmetric space for $\mathrm{GL}(V)$ and let $\omega_d : W \to X$ be the composition of σ_d with some fixed orbit map $\mathrm{GL}(V) \to X$. Since all cyclic subgroups of W are undistorted (by Corollary 2.3.1, for instance), an affirmative answer to the following question would provide one for Question 4.0.1.

Question 4.0.2. *Is ω_d a quasi-isometric embedding for each $d > 1$?*

If it happens that the Gram matrix M_d of Σ_d has precisely one negative eigenvalue, then, for the correct choice of $\mathrm{GL}(V)$-invariant metric on X, the group W_d preserves an isometrically embedded copy of $\mathbb{H}^p$ in X (where p is the number of positive eigenvalues of M_d) on which W_d acts as a discrete reflection group with fundamental chamber C a right-angled polyhedron (see [Vin85] for a more general discussion). If moreover $d > 1$, then no two walls of C are asymptotic, so that the map ω_d is indeed a quasi-isometric embedding (see, for instance, [DH13, Theorem 4.7]; it is enough to check this condition only for the codimension-1 faces of C because C is right-angled). Since it admits a convex cocompact action on $\mathbb{H}^p$, the group W is necessarily Gromov hyperbolic in this case. More generally, Danciger, Guéritaud, and Kassel [DGK18] show that the representation σ_d is Anosov whenever $d > 1$ and W is Gromov hyperbolic; in particular, the answer to Question 4.0.2 is positive for such W.

A group is said to be *compact special* if it is the fundamental group of a compact nonpositively curved cube complex that is A-special in the sense of Haglund and Wise [HW08, Definition 3.2]. Haglund and Wise show that such a cube complex Y admits a local isometry $f : Y \to Y'$ into a compact nonpositively curved cube complex Y' whose fundamental group is a right-angled Artin group. The map f lifts to a $\pi_1(Y)$-equivariant embedding of the universal cover $\tilde{Y}$ of Y as a convex subcomplex of the universal cover $\tilde{Y}'$ of Y'. It thus follows from compactness of Y that the map $\pi_1(Y) \to \pi_1(Y')$ induced by f embeds $\pi_1(Y)$ as an *undistorted* (i.e., quasi-isometrically embedded) subgroup of the right-angled Artin group $\pi_1(Y')$. Since a finitely generated right-angled Artin group embeds as a finite-index, hence undistorted, subgroup of a finitely generated right-angled

Coxeter group [DJ00], and since for $n \geq 3$ any orbit map $\mathrm{GL}_n(\mathbb{Z}) \to X_n$ given by the action of $\mathrm{GL}_n(\mathbb{Z})$ on the symmetric space X_n of $\mathrm{GL}_n(\mathbb{R})$ is a quasi-isometric embedding [LMR00, Theorem A], we obtain that a positive answer to Question 4.0.2 (even just for *some* integer $d \geq 2$) implies one for the following question suggested to us by Konstantinos Tsouvalas.

Question 4.0.3. *Does every virtually compact special group embed as an* undistorted *subgroup of* $\mathrm{GL}_n(\mathbb{Z})$ *for some n?*

4.0.2 Precompact embeddings of rank-one lattices

We showed in Chapter 2 that a finitely generated subgroup Γ of a compact Lie group acts properly by semisimple isometries on a complete CAT(0) space (Corollary 2.0.3). The classical examples of groups possessing the latter property are fundamental groups of closed nonpositively curved Riemannian manifolds. However, even among such examples can be found groups that do not embed into compact Lie groups. Indeed, it is a consequence of Margulis superrigidity that there are closed nonpositively curved locally symmetric spaces of real rank at least 2 whose fundamental groups do not admit infinite-image morphisms into compact Lie groups (see, for instance, [Mor15, Warning 16.4.3]). However, the following question appears to be unresolved.

Question 4.0.4. *Is there an irreducible symmetric space X of real rank 1 and a cocompact lattice Γ in $\mathrm{Isom}(X)$ such that Γ does not embed in a compact Lie group?*

Note that if we drop the cocompactness assumption, then we may take X to be a complex hyperbolic space of complex dimension at least 2 and Γ to be any noncocompact lattice in $\mathrm{Isom}(X)$ since such Γ contain non-virtually-abelian nilpotent subgroups, whereas solvable subgroups of compact Lie groups are virtually abelian (the most efficient way to conclude the latter is to apply the Lie–Kolchin–Malcev theorem; see, for instance, [KM79, Theorem 21.1.5]).

We remark briefly on the case $X = \mathbb{H}^n$, where $\mathbb{H}^n$ is the real hyperbolic space of dimension n. Recall the remarkable fact that all cocompact lattices in $\mathrm{Isom}(\mathbb{H}^3)$ are

virtually special (see Remark 1.2.7) and hence embed in compact Lie groups (prior work of Wise [Wis09] showed that the noncocompact lattices are also virtually special). Bergeron and Wise [BW12] showed that many arithmetic lattices in $\mathrm{Isom}(\mathbb{H}^n)$ for arbitrary n are virtually special. Indeed, Wise has conjectured that all cocompact lattices in $\mathrm{Isom}(\mathbb{H}^n)$ for any n are cocompactly cubulated [Wis14, Conjecture 13.52], and hence virtually special by Agol's theorem [AGM13].

In the case that X is a quaternionic hyperbolic space of quaternionic dimension at least 2, all lattices in $\mathrm{Isom}(X)$ are arithmetic [Cor92, GS92] and all cocompact lattices in $\mathrm{Isom}(X)$ may be Galois conjugated into compact Lie groups [EK18, Prop. 2.8]. We expect the same is true in the case that X is the Cayley hyperbolic plane. This suggests that, if Question 4.0.4 has an affirmative answer, it would perhaps be useful to examine the case that X is a complex hyperbolic space $\mathbb{CH}^n$ of complex dimension $n \geq 2$. By a result of Py [Py13], cocompact lattices in $\mathrm{Isom}(\mathbb{CH}^n)$ are not virtually special. Perhaps there are cocompact lattices Γ in $\mathrm{Isom}(\mathbb{CH}^n)$ that are "superrigid" in a strong enough sense to imply that Γ admits no infinite-image morphisms to compact Lie groups, as is the case for some higher-rank lattices.

4.0.3 Unipotents and $\mathbb{Q}$-linearity

By Theorem 2.0.2, any finitely generated subgroup $\Gamma < \mathrm{SL}_n(\mathbb{C})$ containing no nontrivial unipotents acts properly on a finite product of Euclidean buildings B_i and symmetric spaces of noncompact type. In the case that the entry field of Γ is a number field, one can moreover choose each of the B_i to be locally finite via an embedding of Γ in an S-arithmetic lattice. That being said, we are not aware of a unipotent-free finitely generated subgroup of $\mathrm{SL}_n(\mathbb{C})$ that does not embed as a unipotent-free subgroup of $\mathrm{SL}_m(\mathbb{Q})$.

Question 4.0.5. *Let Γ be a finitely generated subgroup of $\mathrm{SL}_n(\mathbb{C})$ lacking nontrivial unipotents. Is Γ linear over $\mathbb{Q}$? Does Γ even embed as a subgroup of $\mathrm{SL}_m(\mathbb{Q})$ without nontrivial unipotents for some m?*

Remark 4.0.6. We remark that $\mathbb{Q}$-linearity is the same as $\overline{\mathbb{Q}}$-linearity for finitely generated groups. Indeed, if Γ is a finitely generated subgroup of $\mathrm{SL}_n(\overline{\mathbb{Q}})$ then the entry

field E of Γ is a number field of degree, say, d over $\mathbb{Q}$. We now have an embedding $\rho : \mathrm{SL}_n(E) \to \mathrm{GL}_{nd}(\mathbb{Q})$ given by restriction of scalars. This map can be chosen so that $\rho(\mathrm{SL}_n(\mathcal{O}_E)) \subset \mathrm{GL}_{nd}(\mathbb{Z})$, where $\mathcal{O}_E$ is the ring of integers of E. Moreover, given $g \in \mathrm{GL}_n(E)$, each eigenvalue of g appears as an eigenvalue of $\rho(g)$ (since, for instance, the minimal polynomial of $\rho(g)$ annihilates g), so that if g is not unipotent, neither is $\rho(g)$.

Question 4.0.5 is motivated in part by the observation that certain classical examples of finitely generated unipotent-free matrix groups satisfy some notion of *stability* that implies that they may be realized as unipotent-free matrix groups with algebraic entries. For instance, if $\Gamma < \mathrm{SL}_2(\mathbb{C})$ is convex cocompact, then there is a neighborhood $\mathcal{U}$ of the inclusion $\Gamma \to \mathrm{SL}_2(\mathbb{C})$ in $\mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ consisting entirely of faithful convex cocompact representations [Mar74, Sul85]. Now $\mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ may be viewed as an algebraic variety defined over $\mathbb{Q}$ whose $\overline{\mathbb{Q}}$-points correspond to those representations with image in $\mathrm{SL}_2(\overline{\mathbb{Q}})$, so that representations of the latter kind are topologically dense in $\mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ (see, for instance, [BG08, Lemma 3.2]). It follows that there is a representation $\rho \in \mathcal{U}$ such that $\rho(\Gamma) \subset \mathrm{SL}_2(\overline{\mathbb{Q}})$, and since $\rho \in \mathcal{U}$, we have that ρ maps no nontrivial element of Γ to a unipotent.

Remark 4.0.7. One can replace $\mathrm{SL}_2(\mathbb{C})$ in the previous paragraph with $\mathrm{SL}_n(\mathbb{C})$ and "convex cocompact" with "Anosov"; stability in this context was established by Guichard and Wienhard [GW12, Theorem 1.2]. (To be more precise, Guichard and Wienhard show that the subset of $\mathrm{Hom}(\Gamma, \mathrm{SL}_n(\mathbb{C}))$ consisting of the Anosov representations is open. Since Anosov representations have finite kernel, since finite subgroups of $\mathrm{SL}_n(\mathbb{C})$ are locally rigid, and since Gromov hyperbolic groups possess finitely many conjugacy classes of finite subgroups [BG95], we obtain that any representation $\Gamma \to \mathrm{SL}_n(\mathbb{C})$ sufficiently close to the inclusion is simultaneously faithful and Anosov.) The Anosov subgroups of $\mathrm{SL}_n(\mathbb{C})$ include, for instance, the convex cocompact subgroups of $\mathrm{SU}(n-1, 1)$ (this is apparent from, say, Corollary 2.20 in [GGKW17] together with characterization (4) of Anosovity in Theorem 1.3 of the same paper), which in turn include all convex cocompact subgroups of $\mathrm{SO}(n-1, 1)$.

Remark 4.0.8. It is worth pointing out that all convex cocompact—indeed, all geo-

metrically finite—subgroups of $SL_2(\mathbb{C})$ are virtually special; see [Wis21, Section 17.c] and Remark 1.2.7. Hence, such groups can even be realized as unipotent-free groups of *integer* matrices by the proof of Theorem 1.0.1 and Remark 4.0.6. (By the density conjecture [NS12], we in fact need not require geometric finiteness, only discreteness and finite generation.)

As an example of a finitely generated $\mathbb{C}$-linear group that is not $\mathbb{Q}$-linear, consider the subgroup $T < SL_2(\mathbb{C})$ generated by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix},$$

where $t \in \mathbb{C}$ is transcendental. The group T has infinite cohomological dimension as its upper unitriangular subgroup is free abelian of infinite rank, and hence T is not linear over $\mathbb{Q}$ [Ser71, Théorème 5]. The existence of finitely generated abelian unipotent subgroups of arbitrarily large rank is in fact the only obstruction to finite virtual cohomological dimension for finitely generated subgroups of $SL_n(\mathbb{C})$ [AS82].

Button observed that, by an argument of Shalen [Sha79], a free product of two unipotent-free subgroups of $SL_n(\mathbb{C})$ embeds as a unipotent-free subgroup of $SL_{n+1}(\mathbb{C})$ [But17a, Proposition 2.5(iv)] . However, this argument employs transcendentals. We propose the following subquestion of Question 4.0.5.

Question 4.0.9. *Is $SO_5(\mathbb{Z}[\frac{1}{5}]) * (\mathbb{Z}/2)$ linear over $\mathbb{Q}$? Does $SO_5(\mathbb{Z}[\frac{1}{5}]) * (\mathbb{Z}/2)$ embed as a unipotent-free subgroup of $SL_n(\mathbb{Q})$ for some n?*

We have chosen the factor $SO_5(\mathbb{Z}[\frac{1}{5}])$ because it is not virtually special (see Remark 1.2.10). Recall that a free product of two virtually special groups is again virtually special (Lemma 1.2.4) and hence can be realized as a unipotent-free group of integer matrices.

Question 4.0.9 may also be regarded as a subquestion of the following question.

Question 4.0.10. *Let K be a field and Γ_1, Γ_2 finitely generated subgroups of $SL_n(K)$. If K' is an extension of K and $m \geq n$, regard the Γ_i also as subgroups of $SL_m(K')$ by adding 1's on the diagonal, and given $g \in GL_m(K')$, let $\rho_g : \Gamma_1 * \Gamma_2 \to SL_m(K')$ be the*

representation whose restriction to the factor Γ_1 is the inclusion and whose restriction to the factor Γ_2 is the inclusion postconjugated by the element g. Is there an algebraic extension K' of K, some $m \geq n$, and an element $g \in \mathrm{GL}_m(K')$ such that $\rho_g(\gamma)$ is not unipotent for any element $\gamma \in \Gamma_1 * \Gamma_2$ not conjugate into either of the Γ_i?

4.0.4   Type-N elements

We suggest the following definition.

Definition 4.0.11. An element γ of a group Γ is of *type N* (for "neutral") if $|\gamma|_X = 0$ for every isometric action of Γ on a complete π-visible CAT(0) space X.

We will mainly be interested in the case where Γ is finitely generated.

We record several facts about type-N elements (compare the discussion on VU elements in the introduction of Chapter 3).

(1) If $\gamma \in \Gamma$ is of type N, then any element of Γ that is conjugate to γ is also of type N since translation length is constant on conjugacy classes. If M is a connected manifold and c is a closed curve in M, we will say c is of *type N* if some (hence every) element of $\pi_1(M)$ representing c is of type N.

(2) If $\gamma, \gamma' \in \Gamma$ are commensurable, then γ is of type N if and only if the same holds for γ'. This is a direct consequence of Lemma 2.1.1.

(3) An abelian subgroup of Γ generated by type-N elements consists entirely of type-N elements [Duc15, Lemma 6.3].

(4) If H is a finite-index subgroup of Γ and $\gamma \in \Gamma$ is of type N in Γ, then a generator γ_0 of $\langle \gamma \rangle \cap H$ is of type N in H. Indeed, assume first that H is normal in Γ, let $n = [\Gamma : H]$, and suppose H acts isometrically on a complete π-visible CAT(0) space X. We may induce the action of H on X to an action of Γ on X^n whose restriction to H is diagonal and contains the original action of H on X as a factor [KL96, Section 2.1]. Thus, if $\gamma \in \Gamma$ acts neutrally on X^n, then so does γ_0, and so γ_0 must act neutrally on the factor X. This justifies our claim in the case that H is

normal; for an arbitrary finite-index subgroup $H < \Gamma$, we may pass to the normal core of H in Γ to reduce to the above case, and then apply (2).

(5) Finite-order elements are of type N. This is clear from Lemma 2.1.1. Indeed, by the Bruhat–Tits fixed point theorem [Bro98, Section VI.4], it is even true that any finite-order isometry of a complete CAT(0) space fixes a point in the space.

(6) If Γ is finitely generated and $\gamma \in \Gamma$ generates a distorted subgroup of Γ, then γ is an infinite-order element of type N in Γ. This follows from Lemma 2.1.1 and the fact that if Γ acts on a metric space X, then any orbit map $\Gamma \to X$ is Lipschitz (here Γ is endowed with the word metric induced by any finite generating set for Γ).

We are interested in type-N elements because of the following observation.

Proposition 4.0.12. *A type-N element of a finitely generated group is VU.*

Proof. This follows immediately from Theorem 2.0.2, but here is a more direct argument. Let Γ be a finitely generated group and suppose $\gamma \in \Gamma$ is not VU. Then there is some algebraically closed field $\mathbb{F}$ and a representation $\rho : \Gamma \to \mathrm{SL}_n(\mathbb{F})$ such that $\rho(\gamma)$ has an eigenvalue λ of infinite order in $\mathbb{F}^*$. Since Γ is finitely generated, we have that $\rho(\Gamma) \subset \mathrm{SL}_n(\mathbb{F}')$ for some finitely generated subfield $\mathbb{F}' \subset \mathbb{F}$, and we may assume $\mathbb{F}'$ contains the eigenvalues of $\rho(\gamma)$. By [Tit72, Lemma 4.1], we can embed $\mathbb{F}'$ in a local field K endowed with an absolute value $|\cdot|$ such that $|\lambda| \neq 1$. Now Γ acts on the symmetric space or Bruhat–Tits building X associated to $\mathrm{SL}_n(K)$ via $\mathrm{SL}_n(\mathbb{F}') \subset \mathrm{SL}_n(K)$, and the element γ acts ballistically on X. We conclude that γ is not of type N. $\square$

In particular, a finitely generated group containing an infinite-order type-N element does not embed in a compact Lie group, nor is such a group linear over a field of positive characteristic (see Remarks 2.0.4 and 3.0.8, respectively).

Many of the examples of VU elements discussed in the introduction of Chapter 3 are in fact of type N. The following proposition is implicit in the proof of [Bri10, Theorem 2.6]; it implies for instance that a Seifert fiber of the unit tangent bundle of a closed hyperbolic surface, and Dehn twists in (most) mapping class groups, are of type N.

Proposition 4.0.13. *Let X be a complete* $\mathrm{CAT}(0)$ *space and suppose* $\gamma \in \mathrm{Isom}(X)$ *is contained in* $[\mathcal{Z}_{\mathrm{Isom}(X)}(\gamma), \mathcal{Z}_{\mathrm{Isom}(X)}(\gamma)]$. *Then* $|\gamma|_X = 0$.

Proof. Suppose $|\gamma|_X > 0$. Denote by ξ the canonical attracting fixed point of γ on ∂X, by $\mathrm{Isom}(X)_\xi$ the stabilizer of ξ in $\mathrm{Isom}(X)$, and by $\beta_\xi : \mathrm{Isom}(X)_\xi \to \mathbb{R}$ the Busemann character associated to ξ (see [CM09, page 673]). We have that $\mathcal{Z}_{\mathrm{Isom}(X)}(\gamma) \subset \mathrm{Isom}(X)_\xi$ and $|\beta_\xi(\gamma)| = |\gamma|_X \neq 0$. Since β_ξ is a homomorphism to the abelian group $\mathbb{R}$, this implies that $\gamma \notin [\mathcal{Z}_{\mathrm{Isom}(X)}(\gamma), \mathcal{Z}_{\mathrm{Isom}(X)}(\gamma)]$. $\square$

The preceding discussion implies that a closed aspherical 3-manifold M admitting a Thurston geometry is NPC if and only if M does not possess an essential type-N curve.

Question 4.0.14. *Are the VU elements identified in the proof of Theorem 3.0.7 in fact of type N?*

We can even ask the following bolder question.

Question 4.0.15. *Is a closed aspherical 3-manifold NPC if and only if it lacks an essential type-N curve?*

This amounts to asking whether each non-NPC graph manifold contains an essential type-N curve.

In [But17a, Theorem 4.5], Button shows that if g, h are commuting elements of a group Γ such that h, gh, and g^2h are all conjugate, then g is VU in Γ. He uses this to show that Gersten's free-by-cyclic group H contains a nontrivial (hence infinite-order) VU element. We prove that, under the same conditions, the element g is of type N in Γ. By Proposition 4.0.12, this implies Button's result in the case that Γ is finitely generated. Our argument resembles Gersten's proof that the group H does not act properly and cocompactly on a $\mathrm{CAT}(0)$ space [Ger94, Proposition 2.1].

Proposition 4.0.16. *Let X be a complete π-visible* $\mathrm{CAT}(0)$ *space and g, h commuting isometries of X. If h, gh, and g^2h are all conjugate within* $\mathrm{Isom}(X)$, *then* $|g|_X = 0$.

Lemma 4.0.17. *Let Z be a* $\mathrm{CAT}(0)$ *space and g, h commuting isometries of Z with $|g|_Z = 0$. Then $|h|_Z = |gh|_Z$.*

Proof. Fix $z \in Z$ and observe that for each positive integer n, we have

$$d((gh)^n z, z) = d(g^n h^n z, z) \leq d(g^n h^n, g^n z) + d(g^n z, z) = d(h^n z, z) + d(g^n z, z)$$

and

$$d(h^n z, z) = d(g^n h^n, g^n z) \leq d(g^n h^n, z) + d(z, g^n z) = d((gh)^n z, z) + d(z, g^n z)$$

so that

$$d(h^n z, z) - d(g^n z, z) \leq d((gh)^n z, z) \leq d(h^n z, z) + d(g^n z, z). \tag{4.1}$$

Dividing (4.1) by n and taking limits as $n \to \infty$ gives the desired identity by Lemma 2.1.1. $\square$

Proof of Proposition 4.0.16. Suppose $|g|_X > 0$. Then there is a closed convex subspace $Y \subset X$ and a decomposition $Y = Z \times \mathbb{R}$ as in the statement of Theorem 2.1.2, giving maps $\mathcal{Z}_{\mathrm{Isom}(X)}(g) \to \mathrm{Isom}(Z)$ and $\mathcal{Z}_{\mathrm{Isom}(X)}(g) \to \mathbb{R}$. For any $\gamma \in \mathcal{Z}_{\mathrm{Isom}(X)}(g)$, denote by γ_Z and $\gamma_{\mathbb{R}}$ the images of γ under the first and second maps, respectively. We have

$$|h|_X^2 = |h_Z|_Z^2 + h_{\mathbb{R}}^2,$$
$$|gh|_X^2 = |g_Z h_Z|_Z^2 + (g_{\mathbb{R}} + h_{\mathbb{R}})^2,$$
$$|g^2 h|_X^2 = |g_Z^2 h_Z|_Z^2 + (2g_{\mathbb{R}} + h_{\mathbb{R}})^2,$$

and these quantities are all equal since h, gh, and $g^2 h$ are all conjugate within $\mathrm{Isom}(X)$. Moreover, since g and h commute and $|g_Z|_Z = 0$, we have

$$|h_Z|_Z = |g_Z h_Z|_Z = |g_Z^2 h_Z|_Z$$

by Lemma 4.0.17. Thus, we have $h_{\mathbb{R}}^2 = (g_{\mathbb{R}} + h_{\mathbb{R}})^2 = (2g_{\mathbb{R}} + h_{\mathbb{R}})^2$, and so $0 = |g_{\mathbb{R}}| = |g|_X$, a contradiction. $\square$

Remark 4.0.18. Piotr Przytycki has suggested that it might be possible to dispense with the π-visibility assumption in Proposition 4.0.16 by replacing the action of $\mathcal{Z}_{\mathrm{Isom}(X)}(g)$ on Y with the so-called horoaction of $\mathrm{Isom}(X)_\xi \supset \mathcal{Z}_{\mathrm{Isom}(X)}(g)$, where $\xi \in \partial X$ is the canonical attracting fixed point of g (see [CM13, Section 3.1]). We have not pursued this here.

Type-N elements are not the only examples of VU elements of finitely generated groups. Indeed, if Γ is a Burger–Mozes group [BM97], then every element of Γ is VU since all finite-dimensional linear representations of Γ are trivial, but Γ is a torsion-free group acting properly and cocompactly on a product X of two trees, so that each nontrivial element of Γ acts ballistically (indeed, hyperbolically) on X. However, we are not aware of an answer to the following question.

Question 4.0.19. *Is there a finitely generated* linear *group containing a VU element that is not of type N?*

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