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Crossing Symmetry and Chern-Simons Matter Theory

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Declaration of Authorship

I, Tal SHEAFFER, declare that this thesis titled, "Crossing Symmetry and Chern-Simons Matter Theory" and the work presented in it are my own. I confirm that:

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- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
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Abstract

Chern-Simons theory is a field theory in 2 spatial dimensions where particles interact by a generalized form of the Aharonov-Bohm effect. When particles circumnavigate one-another, their quantum-mechanical wavefunctions acquire a complex phase, imbuing them with anyonic statistics. It was recently discovered that in Chern-Simons theory crossing symmetry - a symmetry relating processes with an incoming particle to ones with an out-going anti-particle - takes on a modified form. In this thesis we investigate this modification by looking at $2 \to 2$ scattering of Bosons. We work in light-cone gauge and primarily at 1-loop in order to be able to probe the non-planar regime as well as to compute the scattering amplitude directly in various representations including the singlet-channel . We compute the 1-loop planar 4-point correlator and obtain gauge-dependent terms that survive the on-shell limit. This suggests that restoring gauge-invariance (possibly by dressing the amplitude with Wilson lines) might lead to the modified crossing relation. We perform a 1-loop calculation away from the planar limit. Finally, we demonstrate how the modified crossing relation is necessary to satisfy a relation between the phase of higher-spin form factors and the phase of the S-matrix.

La théorie de Chern-Simons décrit des particules se déplaçant dans un espace bi-dimensionel et interagissant par un effet d'Aharonov-Bohm généralisé. Quand une particule effectue une rotation autour d'une autre, sa fonction d'onde quantique acquiert une phase complexe, attribuant aux particules des statistiques dite anyoniques. Récemment, il a été découvert que la symmétrie d'échange - une relation entre des processus impliquant deux particules, et d'autres impliquant particule et anti-particule - existe sous une forme modifiée dans cette théorie. Cette thèse vise à clarifier cette modification en étudiant en détail l'amplitude décrivant les collisions $2 \to 2$ entre Bosons. Des calculs sont effectués dans une jauge axiale-nulle, et surtout à l'ordre de une boucle, afin d'explorer le régime non-planaire et calculer directement l'amplitude dans diverse représentations incluant la représentation triviale. La fonction de correlation est aussi étudiée pour des impulsions génériques (hors de la couche de masse) et il est démontré qu'un terme dépendant de jauge survit dans la limite de la couche de masse. Cela démontre que dans cette théorie, une définition plus précise de l'amplitude sera nécessaire pour restaurer son invariance de jauge (possiblement en incluant des lignes de Wilson), et suggére une origine microscopique de la modification de la symmétrie d'échange. On effectue aussi le calcul à une boucle hors de la limite planaire. Finalement, on montre que la modification proposée est compatible avec une identité reliant la phase de l'amplitude de diffusion et la phase de facteurs de forme.

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List of Abbreviations

QFT	**Q**uantum **F**ield **T**heory
CS	**C**hern-**S**imons
EM	**E**lectro-**M**agnetism

Dedicated to my parents Gad and Tamar

1 Preface

Chern-Simons theory is a 2+1 dimensional field theory that describes a "gauge field" - a field analogous to the familiar electromagnetic field - and its interactions with other matter particles. While the electromagnetic field in Maxwell's theory propagates (in the form of photons) between charged matter particles such as electrons , thereby mediating the exchange of momentum (force) between them, the Chern-Simons field is unable to propagate through space. Instead, it "sticks" to charged particles. Particles then interact when they come into contact with one another (and therefore with the Chern-Simons field carried by one another) or through the **Aharonov-Bohm** effect [1] at a distance. This latter - purely quantum mechanical - effect is present also in EM (electro-magnetism) and describes the interaction of a charged particle with a narrow tube of electromagnetic flux with which it doesn't come into direct contact.

The Aharonov-Bohm effect revolutionized physics in 1959 by demonstrating for the first time that the electromagnetic potential was an indispensable physical field, rather than a mere mathematical tool. The underpinning of this effect is best understood through Feynman's path-integral picture of quantum mechanics - the electron's motion is a sum (or integral) over all possible paths it could take, with paths on either side of the flux tube giving rise to different contributions to the final quantum mechanical transition-amplitude. This leads to a change in the interference pattern of an electron whose motion has been split into a superposition of such paths, as shown in figure 1. In our familiar 3+1 dimensional world, this is a feasible way for a particle to interact with a "tube" (or any "line-like" object), but in 2+1 dimensions, where particles are confined to a plane, there is also a meaningful sense in which a particle can circumnavigate another **particle**! This is where Chern-Simons theory comes in.

The effect of the Chern-Simons gauge field is sometimes described as imbuing particles with **anyonic statistics**. While the familiar Bosons and Fermions obeying the familiar Bosonic and Fermionic statistics acquire a phase of 1 (respectively -1) when identical quanta are exchanged, **anyons** acquire a more general complex phase. The phase acquired when circumnavigating the "flux" carried by a particle can be thought of as such anyonic statistics.

For more on Yang Mills and other Gauge theories we refer to [23, 27] or any quantum field theory textbook.

But what is the physical relevance of a theory that lives in 'flatland'? Chern-Simons theory has applications in Condensed matter theory, where one often deals with thin (effectively flat) objects or with excitations that are confined to the 2d boundary of a solid, such as a superconductor. In particular, it features heavily in our understanding of the quantum Hall

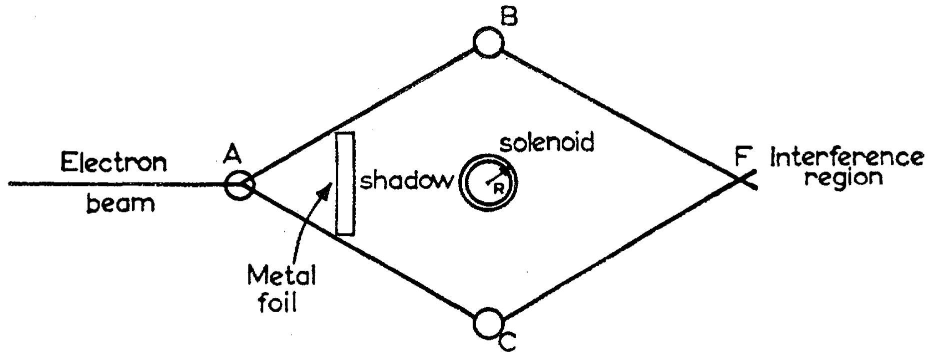


FIGURE 1: 1959 Experiment proposed by Yakir Aharonov and David Bohm[1]. A solenoid produces a tube of magnetic flux, and electrons passing on either side of it without coming into contact with it interfere in a way that depends on the flux. Picture credit: Y. Aharonov, D. Bohm [1].

effect[28, 14], where the "Hall conductivity" of a conductor becomes quantized. It is also of interest to physicists studying 2d rational conformal field theories[24] or the AdS-CFT correspondence[16], and to mathematicians studying knot theory[30]. We elaborate on this last connection in 2.3.

In one variety of Chern-Simons theory (the one we'll be studying in this thesis) the matter content of the theory is comprised of a single N-component massive Bosonic scalar field ϕ_i, $i = 1, \dots, N$, transforming in the fundamental representation of the gauge group of the theory $SU(N)$ - that is to say that it is "charged" in a sense that generalizes the electric charge, and that its charge is like that of the electron's - a fundamental building block for other charges. The (Euclidean) action of the theory (see the review in 2.1) is given by:

$$S = S_{\text{CS}} + S_{\text{Bose-matter}}, \tag{1.1}$$

$$S_{\text{CS}} = i\frac{k}{4\pi}\int\limits_{M} \text{Tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right) \tag{1.2}$$

$$= i\frac{N}{4\pi\lambda}\int d^3x \epsilon^{\mu\nu\rho}\text{Tr}\left(A_\mu \partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\right),$$

$$S_{\text{Bose-matter}} = \int d^3x D_\mu\bar{\phi}D^\mu\phi + m^2\bar{\phi}\phi + \frac{1}{2N}b_4\left(\bar{\phi}\phi\right)^2, \tag{1.3}$$

where A is the gauge field, and other symbols are defined in 2.4. There is evidence that this theory is **dual** to, meaning it is physically equivalent to, a similar theory where this matter field ϕ is replaced by a Fermionic field in the same representation[19, 18, 3]. The free energy (also known as the thermal partition function) has been computed[6, 4, 18, 3] in the 't Hooft limit, also known as the large N limit and as the planar limit (this could be thought of as an approximation where the number of particles in the theory is large, suppressing the contribution of certain processes in the computation of various quantities - see 2.1.5) and has been shown to match in the two theories.

Fermion-Boson dualities have been known to occur in 1+1 dimensional theories, and there they are best understood; explicit Bosonization (or Fermionization) maps are known that relate the fields in one theory to those in the other. Chern-Simons theory is the only known example so far in higher dimensions. In search of such a Bosonization (or Fermionization) map for Chern-Simons-matter theories, the authors in [19] opted to compute the S-matrix. The S-matrix is simply an object that "tabulates" the scattering amplitudes of the theory. Since the Chern-Simons field has no propagating modes, the matter particles are those that can be used in scattering experiments. The benefit of the S-matrix is its gauge invariance, and for this reason [19] chose it as their object of study. The fields themselves as well as their correlation functions are gauge dependent and as such aren't "real" but rather include redundant, unphysical information - in a similar vain to how the phase of the electron wavefunction isn't real.

The present project was initiated by one of the results in [19], which concerns the crossing-symmetry of the S-matrix. Crossing symmetry relates scattering processes involving an incoming (outgoing) particle with ones where it's been exchanged with an outgoing (incoming) anti-particle. It is one of the reasons why anti-particles are sometimes described as "particles moving back in time". This is a manifestation of CPT symmetry - a symmetry that simultaneously:

1. T - time reversal - reverses time,
2. C - charge conjugation - exchanges particles with anti-particles and
3. P - parity - inverts space ($\boldsymbol{x} \to -\boldsymbol{x}$) (in even dimensions) or reflects with respect to a spatial plane (in odd dimensions). In the latter case it is sometimes known as R - reflection, and is the one that is most relevant to us.

For further reading about the S-matrix, crossing symmetry and CPT symmetry we refer to [13, 25]. This symmetry could also be understood as analyticity of the S-matrix as a function the momenta of the participating particles. This is best understood in a tree-level perturbative calculation using Feynman diagrams (see 2.1.4), as computing the different processes translates immediately into evaluating the rational (and therefore meromorphic) functions for different values of external momenta. For instance, in $2 \to 2$ scattering of particles with mass m, if the incoming momenta are denoted p_1, p_2 we denote the center-of-mass energy squared as $s = (p_1 + p_2)^2 \geq 4m^2$. If we think of the $2 \to 2$ S-matrix as an

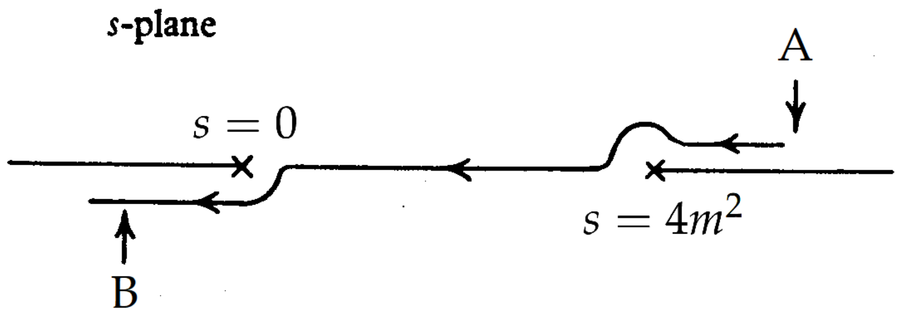


FIGURE 2: A trajectory in the complex s-plane connecting two scattering processes without crossing branch cuts. The process A stands for particle-particle scattering whereas the one at B stands for particle-anti particle scattering. Picture credit: Eden et. al. [13].

FIGURE 3: Branch cuts in the complex s-plane. Picture credit: Eden et. al. [13].

analytic function of s, we can think of analytically continuing it to the region $s < 0$ and then s can be reinterpreted as the momentum-transfer squared between an incoming particle and outgoing anti-particle. This analytic continuation in the complex s-plane is presented in figure 2. Standard unitarity arguments [13] show that the S-matrix has branch cuts in the complex s-plane originating in so-called "normal thresholds" like $s = 0, 4m^2, 9m^2, \ldots$ (see figure 3) where the incoming particles have enough energy to produce the particles in an intermediate process. Other branch cuts ("anomalous thresholds") may also exist. Figure 2 demonstrates how one can relate the process at A (particle - particle scattering) to one at B (particle - anti-particle scattering) without crossing any of the branch cuts. Note that we give s a small imaginary part (an $i\epsilon$ prescription) to pick the right branch of the S-matrix.

One remarkable result from [19] was the modified crossing relation (eq' (3.11) of [19]):

$$S_{\mathrm{S}} = \cos(\pi\lambda)\, I(p_1, p_2, p_3, p_4) + i\frac{\sin(\pi\lambda)}{\pi\lambda} T_{\mathrm{S}}^{\mathrm{naive}}, \tag{1.4}$$

where:

1. S_S is the **singlet-channel** S-matrix, found in the decomposition of the particle-antiparticle S-matrix into irreducible representations:

$$(S_\mathrm{PA})_{ik}^{jl} = \underbrace{\left(\delta_i^l \delta_k^j - \frac{\delta_i^j \delta_k^l}{N}\right) S_\mathrm{A}}_{\text{adjoint channel}} + \frac{\delta_i^j \delta_k^l}{N} S_\mathrm{S}. \tag{1.5}$$

2. $T_\mathrm{S}^\mathrm{naive}$ is the connected singlet-channel S-matrix naively expected to arise from the standard crossing relation (analytic continuation from the A (adjoint) channel or from the particle-particle S-matrix). In standard crossing the cos and sinc[1] are replaced with 1.

The meaning of these different channels is described in 2.5.2. Modified crossing implies that the analyticity properties of the S-matrix are unorthodox in this theory. The cos term is actually somewhat expected, as it reflects the contact interaction due to the trapped flux carried by each particle, and is present already in the standard quantum mechanical Aharonov-Bohm effect[1]. The magnitude of this flux is suppressed in the large N limit for the other channels of scattering, which is why it is absent there.

The authors conjectured this relation in order to satisfy **unitarity constraints** that they derived. This conjecture was shown to hold non trivially in the non-relativistic limit [12]. The authors believe the modification to the crossing relation follows from the **anyonic** statistics that the particles are imbued with by the Chern-Simons gauge field, and have given a heuristic argument based on knot invariants in the purely topological gauge sector of the theory.

Our central motivation is to better understand this modified crossing relation. Our main approach has been to compute the S-channel scattering equation directly. The S-matrix was successfully computed in [19] in the **other channels**. This computation was made possible in light cone gauge ($A_- = 0$) by use of a simplifying assumption about the external momenta that is invalid in the S-channel. We wish to relax this assumption and make perturbative calculations to see the emergence of this **non-analyticity** directly. We also wish to check this relation in the non-planar theory. Those are the main goals of the project.

1.1 Outline and Summary of Results

In section 2 we review the background material relevant to the project. We start with a quick review of gauge theory and field theory 2.1. Next 2.2 we review pure Chern-Simons theory

[1] $\mathrm{sinc}\,(x) := \frac{\sin(x)}{x}$

(the theory without matter fields). In this section we perform calculations determining the theory's Lagrangian, demonstrating its gauge invariance and the quantization of the Chern-Simons level k, and justifying the statement that the theory lacks propagating degrees of freedom. We then proceed to discuss the connection knot theory and knot invariants2.3, and illustrate said connection with explicit calculations in the Abelian $U(1)$ theory. In subsection 2.4 we discuss our variety of Chern-Simons matter theory by including the aforementioned fundamental scalar field ϕ. We reproduce the all-loop self-energy of scalar cited in [19] by solving an integral equation (see 2.4.2). The last part of the background section is a review of scattering kinematics and the S-matrix 2.5. We compute the tree-level S-matrix and demonstrate its gauge invariance.

In section 3 we discuss our attempts to compute a "color" factor of the 1-loop amplitude away from the planar limit. This color factor is in fact precisely the Abelian part of the 1-loop amplitude - in the Abelian theory it would constitute the entirety of the amplitude at that order in the coupling constants. We find that we are able to demonstrate gauge-invariance. Final results are forthcoming, but it appears that the analyticity properties of the result are not anomalous for a QFT in a way that would give rise to a modified crossing relation. We discuss the possible reasons for this.

In section 4 we discuss the planar connected 4-point correlator and its on-shell limit (the $2 \to 2$ S-matrix). We review the all-loop results from [19] and discuss the simplifying assumption $v \cdot s = 0$ used there and its implications. We reproduce the "effective exchange interaction" described in [19] and show how it is modified when one relaxes the assumption $v \cdot s = 0$ (see 4.1). We compute the off-shell 4-point correlator at 1-loop in generality (4.3) and find it includes terms that are gauge-dependent on shell. We discuss the possible reasons for this.

We proceed to section 5 where we compute the phase of the S-matrix and show how the modified crossing relation is necessary to satisfy predictions found in [9] regarding it and regarding the phase of form factors.

Finally we conclude the work so far and describe future work in section 6.

2 Background

In this section we review the background material relevant to the project. We start with a quick review of gauge theory and field theory in 2.1. Next, in 2.2, we review pure Chern-Simons theory (the theory without matter fields). We then proceed to discuss the connection knot theory and knot invariants2.3. In subsection 2.4 we discuss our variety of Chern-Simons matter theory by including the aforementioned fundamental scalar field ϕ. The last part of the background section is a review of scattering kinematics and the S-matrix2.5.

2.1 Gauge Theory and Field Theory

In this subsection we'll go through a short and basic review of quantum field theory and gauge theory. We will also describe the large N limit.

2.1.1 Classical Theory - KG and Maxwell Theory

A classical field theory describes the dynamics of a field by means of an **equation of motion** - a partial differential equation. For instance, Klein-Gordon theory describes a real or complex scalar field ϕ evolving according to the KG equation:

$$\partial^2\phi = \partial_t^2\phi - \nabla^2\phi = -m^2. \tag{2.1.1}$$

Wavepacket solutions to this equation behave like relativistic particles with mass m. This theory is **free** - there are no interactions between particles. This follows from the fact that (2.1.1) is **linear** in the field ϕ so solutions satisfy the **superposition principle**.

Another free theory is free Maxwell Electrodynamics (in 4 space-time dimensions), which describes a vector field A^μ satisfying **Maxwell's equations**:[2]

$$\begin{aligned} \partial^\mu F_{\mu\nu} &= 0, \\ \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} &= 0, \\ F_{\mu\nu} &\equiv (dA)_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \end{aligned} \tag{2.1.2}$$

This is also a **gauge theory**, as it satisfies a gauge symmetry:

$$A_\mu \to A_\mu + \partial_\mu\lambda, \tag{2.1.3}$$

[2]Note that (2.1.2) is known as Bianchi's identity and is trivially satisfied by virtue of the commutation of partial derivatives and the definition $F = dA$.

for some scalar field λ. Gauge symmetries are understood to be **redundancies** in the description of the system, as opposed to real, or **global** symmetries which relate physically distinguishable configurations. In a gauge theory one must often make a gauge choice - that is - exhaust the gauge symmetry by satisfying some **condition**, such as $\partial^\mu A_\mu = 0$ or $A_0 = 0$.

Most field theories can be described using the **action principle** - the E.O.M. is equivalent to the statement that the field extremizes the action functional S:

$$\text{E.O.M.}\,(\phi, A, \dots) \leftrightarrow 0 = \frac{\delta S}{\delta \phi} = \frac{\delta S}{\delta A} = \dots \tag{2.1.4}$$

For instance, Maxwell's equations can be obtained by varying the action:

$$S_{\text{Maxwell}} = -\frac{1}{4g^2}\int d^4x F^{\mu\nu}F_{\mu\nu}, \tag{2.1.5}$$

with respect to A_μ (g is a coupling constant). We can couple the **gauge field** A to a background current $J^\mu(x)$ by adding a term $\sim A_\mu J^\mu$ to the **Lagrangian** (the integrand of the action), which gives rise to an inhomogeneous ("source") term $\sim J_\mu$ for the E.O.M.

When the Lagrangian is at most quadratic in the fields, as is the case in (2.1.5), the theory is free. Otherwise, it is known as an **interacting** theory, as solutions no longer satisfy the superposition principle. An example is scalar QED, which couples Maxwell's gauge field A_μ to a complex scalar ϕ:

$$S_{\text{QED}} = \int d^4x \left(-\frac{1}{4g^2}F^{\mu\nu}F_{\mu\nu} + D_\mu\bar{\phi}D^\mu\phi + m^2\bar{\phi}\phi\right), \tag{2.1.6}$$

$$D_\mu = \partial_\mu - iA_\mu. \tag{2.1.7}$$

D_μ is known as the gauge-covariant derivative. This Lagrangian contains terms cubic ($\sim \bar{\phi}A\cdot\left(\overleftarrow{\partial} - \overrightarrow{\partial}\right)\phi$) and quartic ($\sim \bar{\phi}A^2\phi$) in the fields. These are known as interaction terms. The theory which describes the EM force in the real world (QED) is precisely this theory, except coupled to a Dirac spinor instead of a scalar.

2.1.2 Abelian Gauge Theory

The gauge symmetry that keeps the action (2.1.6) invariant is:

$$\phi \rightarrow e^{i\theta(x)}\phi, \tag{2.1.8}$$

$$A_\mu \rightarrow A_\mu + \partial_\mu\theta. \tag{2.1.9}$$

This symmetry can be identified with the group $U(1)$ of complex phases - θ parameterizes the gauge transformation by telling us what element of the **gauge group** ($U(1)$) is chosen at each point in space-time. Electrodynamics is known as an **Abelian** gauge theory because the gauge group is Abelian - all elements commute. From this point of view, the covariant derivative (2.1.7) could be understood as a derivative modified to take into account the fact that ϕ's phase is not an observable - its variation over space-time is therefore not entirely physical, but could be a reflection of our gauge choice.

The covariant derivative is also known a **connection** - it allows us to compare ϕ's phase at different positions in space-time. ϕ can be said to be covariantly constant on a curve $\gamma : [0,1] \to \mathcal{M}$ if:

$$\dot{\gamma}^{\mu} D_{\mu}\phi = 0, \tag{2.1.10}$$

where $\dot{\gamma} = \frac{d\gamma}{dt}$ is the tangent vector to the curve γ. The solution to this equation is:

$$\phi\left(\gamma\left(1\right)\right) = \phi\left(\gamma\left(0\right)\right) e^{i\int_{\gamma}\dot{\gamma}^{\mu}A_{\mu}dt}. \tag{2.1.11}$$

We can say that $\phi\left(\gamma\left(1\right)\right)$ is the result of parallel-transporting (moving while keeping covariantly-constant) ϕ along γ. The expression $W_{\gamma}\left(A\right) = \exp\left(i\int_{\gamma}\dot{\gamma}^{\mu}A_{\mu}dt\right)$ is known as a **Wilson line**. An important property of this object is how it transforms under gauge transformations:

$$W_{\gamma}\left(A\right) \to W_{\gamma}\left(A\right)\exp\left(i\int\limits_{\gamma}\dot{\gamma}^{\mu}\partial_{\mu}\theta dt\right) = e^{-i\theta(\gamma(0))}W_{\gamma}\left(A\right)e^{i\theta(\gamma(1))}. \tag{2.1.12}$$

In particular, this means that **Wilson loops**[3] ($\gamma\left(1\right) = \gamma\left(0\right)$) are gauge invariant. Much like the modulus of ϕ is its "physically observable" part, Wilson loops help us understand what the "physical" part of A is. To see this note that by Stoke's theorem, if Σ is a surface that satisfies $\partial\Sigma = \gamma$ ($\partial\Sigma$ means "the boundary of Σ"), then:

$$W_{\gamma}\left(A\right) = \exp\left(i\int\limits_{\Sigma}dA\right) = \exp\left(i\int\limits_{\Sigma}F\right). \tag{2.1.13}$$

This leads to the interpretation that $F_{\mu\nu}$, the field strength tensor, is the "locally observable part" of A. F is also known as the curvature of the connection, since it quantifies the change in a particle's phase as it's parallel transported in a small closed loop - an infinitesimal Wilson line.

[3] A Wilson loop is also known as the **holonomy** of the gauge field around the loop γ.

2.1.3 Non-Abelian Gauge Theory

The natural generalization of Maxwell theory is to pick a different gauge group G, and one that might not be Abelian. Fields will then transform in various **irreducible representations** ρ_R of G:

$$\phi_i \to \left(\rho_R\left(g\left(x\right)\right)\right)_i^j \phi_j = \left(\exp\left(i a^a T_R^a\right)\right)_i^j \phi_j, \tag{2.1.14}$$

where the T_R^a, $a = 1, \ldots, \dim G$ are the **generators** of the gauge group in the representation R, satisfying $\left[T_R^a, T_R^b\right] = i f^{abc} T_R^c$ with f^{abc} the structure constants, and a^a are parameters that take the place of θ. Such theories describe real-world forces, such as the strong interaction ($G = \mathrm{SU}(3)$) or the standard model ($G = \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$).

The gauge-covariant derivative has the same interpretation as before and takes the form:

$$D_\mu = \partial_\mu - i A_\mu, \tag{2.1.15}$$

$$A_\mu = A_\mu^a T^a, \tag{2.1.16}$$

so that now A is a Lie algebra valued vector field. The non-Abelian Wilson line arises in the same way as the Abelian one - it describes parallel transport. It is given by a path-ordered exponential:

$$\mathrm{P}\exp\left(i \int_\gamma A\right), \tag{2.1.17}$$

which means that in the expansion of the exponent the order of matrix products matches their ordering along γ. The field strength, or curvature, can again be defined as an "infinitesimal Wilson loop":

$$F_{\mu\nu} \equiv i\left[D_\mu, D_\nu\right] = \partial_\mu A_\nu - \partial_\nu A_\mu - i\left[A_\mu, A_\nu\right]. \tag{2.1.18}$$

For D_μ to make sense as a connection A must transform as:

$$A_\mu \to g A_\mu g^{-1} + g \partial_\mu g^{-1}, \tag{2.1.19}$$

$$\Rightarrow F_{\mu\nu} \to g F_{\mu\nu} g^{-1}, \tag{2.1.20}$$

under the gauge transformation $g\left(x\right)$. The classic example of non-Abelian gauge theory is Yang Mills theory:

$$S_{\mathrm{YM}} = -\frac{1}{4g^2} \int d^4x \operatorname{Tr}\left(F^{\mu\nu} F_{\mu\nu}\right). \tag{2.1.21}$$

The last term in (2.1.18) makes this theory an interacting theory.

2.1.4 Second Quantization

There are many ways to quantize a classical field theory. We will briefly and schematically review the path integral approach. In the quantum theory, rather than describe the evolution of the field configuration in time, we are interested in transition amplitudes:

$$\langle \phi_i (t_i) \mid \phi_f (t_f) \rangle . \tag{2.1.22}$$

The modulus-squared of these amplitudes give the transition probability - the probability of finding the field at the configuration ϕ_f at time t_f given that it was measured to be in a configuration ϕ_i at time t_i. In the path integral approach this is given by:

$$\langle \phi_i (t_i) \mid \phi_f (t_f) \rangle = \int_{\phi_i}^{\phi_f} D\phi \exp (iS [\phi]) , \tag{2.1.23}$$

where S is the action and $\int_{\phi_i}^{\phi_f} D\phi$ is an integration over all field configurations satisfying the boundary conditions:

$$\phi (t_i) = \phi_i, \, \phi (t_f) = \phi_f. \tag{2.1.24}$$

Thus the transition amplitude is the sum over paths consistent with the measurements, weighted by a complex phase determined by the action.

When the field configurations $\phi_{i,f}$ are large (or equivalently we rewrite $S \to S/\hbar$ and $\hbar$ is small) we can use the saddle point approximation where the integral gets contributions only from saddle points of the action - classical solution. This is the correspondence principle - at small $\hbar$ the theory becomes classical. Put differently, quantum corrections are $\hbar$-suppressed.

The path integral can be computed in free theories where the action is quadratic in the fields:

$$S = \int d^d x \frac{\phi \Delta \phi}{2}, \tag{2.1.25}$$

where Δ is some differential operator. Using an infinite dimensional generalization of the Gaussian integral:

$$\frac{1}{\sqrt{2\pi}} \int dx e^{-\frac{x^2}{2\sigma}} = \sqrt{\sigma}. \tag{2.1.26}$$

In interacting theories we write the action $S = S_{\text{kinetic}} + S_{\text{interaction}}$ where the kinetic part is quadratic in the fields and the interacting part is of higher order. When the coupling constants of the theory are small so that schematically $S_{\text{interaction}} \ll S_{\text{kinetic}}$ we can write:

$$\exp(iS) = \exp(iS_{\text{kinetic}}) \sum_{n=0}^{\infty} \frac{i^n S^n_{\text{interaction}}}{n!}. \tag{2.1.27}$$

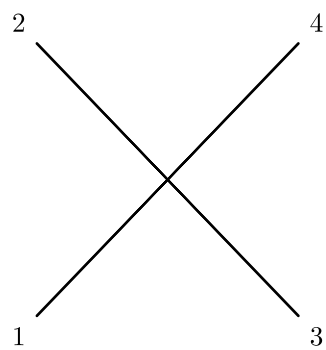
This is the key to computing quantities in **perturbation theory**. The higher the order in interaction terms, the more cluttered the path-integral becomes with powers of the fields. Integrating these monomials in the fields against $\exp(iS_{\text{kinetic}})$ gives rise to many integrals that are represented using Feynman diagrams. For instance, let's say we compute a **4-point correlator**:

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle = \int D\phi \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \exp(iS[\phi]), \tag{2.1.28}$$

and that $S_{\text{interaction}} = \int d^d x b_4 \frac{\phi^4}{4!}$. At first order in b_4 we have the tree-level correlator:

$$\int D\phi \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \exp(iS_{\text{kinetic}}[\phi]) \int d^d x b_4 \frac{\phi(x)^4}{4!}, \tag{2.1.29}$$

which gives rise to a number of diagrams. One such diagram is:



, (2.1.30)

Where the vertex in the center represents the monomial $\frac{\phi^4}{4!}$. For a more thorough review of quantization and Feynman perturbation theory, see [23].

As already mentioned, in quantum theory we are interested in transition amplitudes. A particularly useful subset is that of **scattering amplitudes** which encode the outcome of scattering experiments. The LSZ reduction formula[23] relates these to correlation functions like the 4-point correlator above. Its content is roughly that one must compute correlation functions in momentum space (the Fourier transforms of position space correlators) and then take a properly defined residue localizing them to the so-called "mass-shell condition" where the momenta satisfy Einstein's relation:

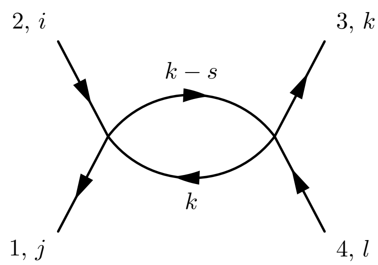
$$p_i^2 = m^2. \tag{2.1.31}$$

2.1.5 The Large N ('t Hooft) Limit

In much of this thesis we will be working in the Large N limit. A very cogent description of this limit can be found in chapter 8 of [11]. At its core the large N limit is an approximation assuming that the number of particles in the theory is large. For instance, consider the theory of N complex scalars ϕ^i, $i = 1, \ldots, N$ with action:

$$S = \int d^4x \partial_\mu \bar{\phi} \partial^\mu \phi + \frac{b_4}{2N} \left(\bar{\phi}\phi\right)^2, \tag{2.1.32}$$

where $\bar{\phi}\phi = \left(\phi^i\right)^* \phi_i$, b_4 is a finite constant and N is large. We see that, as that, as is often the case, interactions are $1/N$ suppressed, making the theory seemingly free. However, consider the diagram:



$$\propto \left(\frac{b_4}{N}\right)^2 \delta^n_m \delta^m_n = \frac{b_4^2}{N}. \tag{2.1.33}$$

We see that the large number of particles running in the "loop" enhances the interaction strength by $\delta^n_m \delta^m_n = N$, partially offsetting the $1/N$ suppression. However, not all diagrams contributing to this process at a given order in perturbation theory will have sufficient enhancements of this type to contribute to the leading $(O(1/N))$ part of the amplitude. Thus only diagrams with the maximal number of "index loops" contribute, as those are most enhanced by the multitude of particles in the theory. This tends to suppress diagrams with fewer faces for a given number of edges and vertices. Diagrams with the maximal number of faces are known as "planar" diagrams, as they can be drawn on a plane (or a sphere) without self-intersection.

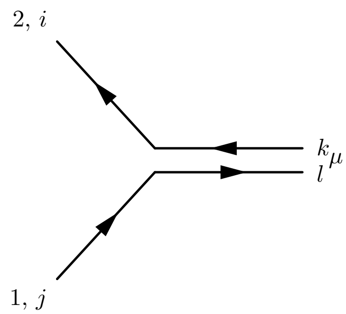
There are multiple benefits to calculations in the large N limit:

1. Fewer diagrams contribute to a given process.

2. Diagrams have fewer "topologies", meaning that the number of distinct propagators is lower. This makes integral reductions simpler and decreases the amount of momentum

shifts one must use in the course of the calculation. In some cases, such as $\mathcal{N} = 4$ SYM, this makes the notion of a Feynman "integrand" well defined.

3. Keeping track of "color factors" - factors involving indices like $i = 1, \ldots, N$. Away from the planar limit one must keep track of group generators in terms like $\left(T^a T^b\right)^i_j$ whereas in the planar limit one can use "double line notation" where the index is just kept constant along a line. For instance, in Chern-Simons matter theory we'll be using vertices like:



$$= (p_1 - p_2)_\mu \, \delta^i_k \delta^l_j. \tag{2.1.34}$$

The single lines stand for propagation of a particle in the fundamental representation of $SU(N)$, whereas the doubled lines represent the propagation of a gauge Boson (or particle in the adjoint representation).

4. In some cases, as we'll see in 2.4.2 and 4.2, one can obtain all-loop results by solving integral equations.

2.2 Pure Chern-Simons Theory

This is a topological theory of a single gauge field A, usually with gauge group $U(N)$ or $SU(N)$. By topological, we mean that it is equivalent in all conceivable coordinates. Therefore coordinate transformations are symmetries of the theory (see 2.2.3). To obtain a topological gauge invariant local action, we must construct an (up to total derivatives) gauge invariant 3-form without referencing a metric or a set of coordinates. We have a 1-form A and 2-form $F = DA = dA + i[A, A]$ so the 3-forms:

$$A \wedge dA,\; A \wedge A \wedge A, \tag{2.2.1}$$

form a basis we can work with. Note that the latter of these vanishes for Abelian theories. In abstract index notation we can write w.l.o.g. (without loss of generality):

$$\mathcal{L} = i \frac{k}{4\pi} \epsilon^{\mu\nu\rho} \mathrm{Tr} \left(A_\mu \left(\partial_\nu + b A_\nu\right) A_\rho\right). \tag{2.2.2}$$

Then given an infinitesimal gauge transformation:

$$\delta A_i = \partial_i \Lambda + i \left[\Lambda, A_i\right], \tag{2.2.3}$$

we need the Lagrangian to vary by a total derivative. We find:

$$\delta \mathcal{L} \propto 3b \left(1 + \frac{2i}{3b}\right) \epsilon^{ijk} \mathrm{Tr}\left(\partial_i \Lambda A_j A_k\right) + \partial_i \left(\epsilon^{ijk} \mathrm{Tr}\left(\Lambda \partial_j A_k\right)\right), \tag{2.2.4}$$

which gives a total derivative when:

$$b = -\frac{2i}{3}, \tag{2.2.5}$$

giving us the Chern-Simons Lagrangian:

$$\mathcal{L} = i \frac{k}{4\pi} \epsilon^{\mu\nu\rho} \mathrm{Tr}\left(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho\right). \tag{2.2.6}$$

Dimensional analysis gives the mass dimensions $[A] = 1$, $[k] = 0$. The constant k is known as the **level** and takes on integer values - as we'll prove in the next subsection in the non-Abelian case. In the Abelian case this quantization comes from considering the theory on a manifold with boundary.

Note that the ϵ symbol is real $\epsilon^{123} = 1$. This follows from unitarity - we need the Lagrangian to be invariant under simultaneous reversal of the time coordinate and complex conjugation. To see this, notice that:

$$\mathrm{Tr}\left(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho\right) = \frac{1}{2} A^a_\mu \partial_\nu A^a_\rho + \frac{1}{3} f^{abc} A^a_\mu A^b_\nu A^c_\rho \tag{2.2.7}$$

is clearly real, and under time reversal any term that is constructed by full contraction with the ϵ symbol gets a $-$ sign, unitarity amounts to:

$$i\epsilon^{\mu\nu\rho} = \left(i\left(-\epsilon^{\mu\nu\rho}\right)\right)^* = i\left(\epsilon^{\mu\nu\rho}\right)^*. \tag{2.2.8}$$

2.2.1 Gauge Invariance and Quantization of the Level k

Although invariance under infinitesimal gauge transformations was sufficient to fix the action, it doesn't guarantee that the theory is invariant under gauge transformations not connected to the identity - that is, maps $g : \mathcal{M} \to G$ with non-trivial homotopy.

Under a finite gauge transformation $g = e^{-i\Lambda}$ we have (see the appendix A):

$$\begin{aligned} A_\mu &\rightarrow gA_\mu g^{-1} + ig\partial_\mu g^{-1} && (2.2.9)\\ &= gA_\mu g^{-1} - i\partial_\mu g g^{-1} && (2.2.10)\\ &= g\left(A_\mu - \partial_\mu \Lambda\right) g^{-1}, && (2.2.11)\end{aligned}$$

$$\mathcal{L} \rightarrow \mathcal{L} + \mathrm{Tr}\left(\frac{k}{4\pi}\partial_\mu\left(\epsilon^{\mu\nu\rho}A_\nu\partial_\rho g g^{-1}\right) + i\frac{k}{12\pi}\epsilon^{\mu\nu\rho}g^{-1}\partial_\mu g g^{-1} g\partial_\nu g^{-1}\partial_\rho g\right). \tag{2.2.12}$$

The first term is a total derivative, as expected. The second, however, doesn't vanish. It is proportional [28] to the integrand of the "counting function":

$$w\left(g\right) = \frac{1}{24\pi^2}\int d^3x \epsilon^{\mu\nu\rho}g^{-1}\partial_\mu g g^{-1} g\partial_\nu g^{-1}\partial_\rho g \in \mathbb{Z}, \tag{2.2.13}$$

which counts the number of times g winds around the gauge group G. As expected: we found that gauge transformations connected to the identity leave the action invariant. More generally:

$$e^{-S} \rightarrow e^{-S-2\pi i k w(g)}. \tag{2.2.14}$$

Hence for gauge invariance we require $k \in \mathbb{Z}$. This term vanishes for Abelian theories and level-quantization is then instead related to the boundary term when the theory is considered on a manifold with boundary, or gauge transformations with non-trivial winding if the theory is considered on a manifold with compact dimensions (see, e.g. section 5.1.3 of [28]).

2.2.2 Equation of Motion

Varying A gives the E.O.M.:

$$\begin{aligned} \delta\mathcal{L} &\propto \delta\left(\epsilon^{\mu\nu\rho}\mathrm{Tr}\left(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\right)\right) && (2.2.15)\\ &= \frac{1}{2}\epsilon^{\mu\nu\rho}\delta\left(A^a_\mu\partial_\nu A^a_\rho\right) - \frac{2i}{3}\epsilon^{\mu\nu\rho}\mathrm{Tr}\left(\delta\left(A_\mu A_\nu A_\rho\right)\right) && (2.2.16)\\ &= \frac{1}{2}\partial_{[\mu}A^a_{\nu]}\epsilon^{\mu\nu\rho}\delta A^a_\rho - \frac{i}{2}\left[A_\mu, A_\nu\right]^a \epsilon^{\mu\nu\rho}\delta A^a_\rho + \text{total derivative}, && (2.2.17)\\ \Rightarrow 0 &= \partial_{[\mu}A^a_{\nu]} - i\left[A_\mu, A_\nu\right]^a, && (2.2.18)\\ \Rightarrow F^a_{\mu\nu} &= 0. && (2.2.19)\end{aligned}$$

Hence the space of solutions is the space of flat connections. Note that if we arbitrarily pick a "time" direction x_0, then we have:

$$F_{0i} = 0,\ i = 1,2\,(\text{first order in time}), \tag{2.2.20}$$

$$F_{12} = 0\,(\text{0-th order, a constraint on initial data}). \tag{2.2.21}$$

Let's specialize to a manifold $\mathcal{M} = \Sigma \times \mathbb{R} = \{(x_0, \bar{x}) \mid x_0 \in \mathbb{R} = \text{"time"},\ \bar{x} \in \Sigma\}$ so that we can pick the gauge $A_0 = 0$. Then:

$$0 = F_{0i} = \partial_0 A_i \Rightarrow A_i\,(t, \bar{x}) = A_i\,(\bar{x})\,. \tag{2.2.22}$$

Hence we see that there are **no propagating degrees of freedom**. Furthermore:

$$F_{ij} = 0. \tag{2.2.23}$$

Hence the space of solutions is precisely the **moduli space of flat connections** on Σ. If we couple the theory to matter by adding a source term:

$$-\,i\mathrm{Tr}\,(A_\mu J^\mu)\,, \tag{2.2.24}$$

we'll get an E.O.M.:

$$\frac{4\pi}{k}\epsilon^{\mu\nu\rho}F^a_{\nu\rho} = J^{\mu,a}. \tag{2.2.25}$$

Equation (2.2.25) tells us that matter charged under the gauge group traps flux. This is the main difference between Chern-Simons theory and Yang-Mills - In the latter charged particles interact with one another by acting as sources and sinks for field lines - by the exchange of gauge field quanta - whereas in the former all interaction is due to a generalized Aharonov Bohm effect [1].

2.2.3 Symmetries

In addition to the gauge symmetry, Chern-Simons theory possesses 2 notable global symmetries. The first is inherent in the moniker "topological" - diffeomorphism invariance. This is the infinite-dimensional group of coordinate transformations on the spacetime $\mathcal{M}$, sometimes denoted diff $(\mathcal{M})$. Invariance under it follows simply from the fact that the theory is formulated in the language of differential forms and without any reference to a particular set of coordinates. However, the above statement isn't entirely accurate - the symmetry actually only includes **orientation preserving** diffeomorphisms diff$^+$ $(\mathcal{M})$. Suppose we perform a change of coordinates $x'^{\mu} = x'^{\mu}(x^{\nu})$ then a 1-form transforms:

$$A_{\mu}(x) \to \frac{\partial x'^{\nu}}{\partial x^{\mu}} A_{\nu}(x'), \tag{2.2.26}$$

so one can work out:

$$\mathcal{L} \to \det\left(\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right)\mathcal{L}, \tag{2.2.27}$$

which is just the transformation law of a volume form. Of course, the integral measure transforms via the familiar Jacobian of the **inverse** transformation:

$$d^3x \to d^3x' \left|\det\left(\frac{\partial x^{\mu}}{\partial x'^{\nu}}\right)\right|, \tag{2.2.28}$$

so that in total we have:

$$S \to \text{sign}\left(\det\left(\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right)\right) S. \tag{2.2.29}$$

For orientation reversing transformations the above sign is negative. We can negate such a negative sign by simultaneously flipping the sign of k (or equivalently $\lambda = k/N$), which gives our second symmetry:

$$\mathbb{Z}_2 \approx \frac{\text{diff}(\mathcal{M})}{\text{diff}^{+}(\mathcal{M})}. \tag{2.2.30}$$

We can pick a representative element of this group. One such choice in $\mathbb{R}^3$ is the parity operation $x \to -x$, denoted P. Another is R - reflection of a single coordinate. In section 4 this $\mathbb{Z}_2$ symmetry will be used extensively to constrain Feynman integrals - for instance, by telling us which branch of a logarithm to pick.

2.2.4 Quantization

Let us rewrite the Lagrangian using integration by parts:

$$\begin{aligned} \mathcal{L} &= i\frac{k}{4\pi}\epsilon^{\mu\nu\rho}\text{Tr}\left(A_{\mu}\partial_{\nu}A_{\rho} - \frac{2i}{3}A_{\mu}A_{\nu}A_{\rho}\right) \\ &= -i\frac{k}{4\pi}\text{Tr}\left(\epsilon^{ij}A_i\partial_0 A_j\right) + i\frac{k}{2\pi}\text{Tr}\left(A_0\epsilon^{ij}F_{ij}\right). \end{aligned} \tag{2.2.31}$$

For our gauge $A_0 = 0$ we obtain a vanishing Hamiltonian and commutation relations:

$$\left[A_i(x), A_j(y)\right] \propto \frac{4\pi}{k}\epsilon_{ij}(2\pi)^3\delta^3(x-y). \tag{2.2.32}$$

But here we haven't enforced Gauss' law! Thankfully, we can show that the path integral localizes to the flat connections, thereby demonstrating the lack of propagating degrees of freedom. Consider shifting A by some vector ΔA. First, let us shift only $A_0 \to A_0 + \Delta A_0$. Now using (2.2.31) we get:

$$\mathcal{L} \to \mathcal{L} + i\frac{k}{2\pi}\mathrm{Tr}\left(\Delta A_0 \epsilon^{ij} F_{ij}\right). \tag{2.2.33}$$

One can now average over choices of ΔA_0:

$$\begin{aligned}\int DAe^{-S} &= \int DAD\left(\Delta A_0\right) e^{-S-i\int d^3x \epsilon^{ij}\Delta A_0^a F_{ij}^a} \\ &= \int DA\delta\left(F_{12}^a\right),\end{aligned}$$

where we have re-scaled ΔA_0 to absorb various constants. The theory has therefore reduced to **time-dependent** flat connections on the time-slice Σ. The remaining part of the action is linear in the components of either connection and an identical "trick" could be used to localize the remaining components of the field strength (note that one has use only shifts of localized integration variable - that is, shifts that leave $F_{1,2}$ invariant). Thus we get:

$$\begin{aligned}\int DAe^{-S} &\propto \int DAe^{-S}\delta\left(F\right) \\ &= \int DA\delta\left(F\right).\end{aligned}$$

Curiously, we have lost track of the level k. This is because in reality we want to have some insertion in the path integral. Does this localization still take place when gauge invariant observables are included in the integrand? We'll find that it does, by looking at the partition function $Z[J]$ rather then just the path integral $Z[0]$! To see how that works consider that we can repeat the above argument after coupling the field to an external "source" J by a term $-i\mathrm{Tr}\left(A_\mu J^\mu\right)$, where the "trace" stands in for any form of linear coupling, for instance $\bar{\psi}\not{A}\psi$. The result will simply be to modify the E.O.M to:

$$\epsilon^{\mu\nu\rho}F_{\nu\rho}^a = \frac{4\pi}{k}J^{\mu,a}, \tag{2.2.34}$$

and this is how the level k is relevant in the theory. Of course, in the pure CS theory, the observables are the Wilson loops:

$$\mathrm{Tr}_R \mathrm{P}\left[\exp\left(i\int_\gamma A\right)\right], \tag{2.2.35}$$

where $P[\cdot]$ stands for path-ordering and R is an irrep. By introducing auxiliary "worldline" fields living on the support of the Wilson loop, one can rewrite it as the exponent of such a linear coupling term with an appropriate current constructed with said fields . This is alluded to in [30] and described in David Tong's lecture notes [28] for gauge group $SU(N)$. Tong writes:

$$\begin{aligned} W_R(\gamma) &= \mathrm{Tr}_R\mathrm{P}\left[\exp\left(i\int_\gamma A\right)\right] && (2.2.36)\\ &= \int DwD\bar{w}D\alpha \exp\left(\int_\gamma dt\left(i\bar{w}D_t w-\kappa\alpha\right)\right) && (2.2.37)\\ &= \int DwD\bar{w}D\alpha \exp\left(\int_\gamma dt\left(i\bar{w}\partial_t w+\left(\bar{w}w-\kappa\right)\alpha+\bar{w}A_0 w\right)\right), && (2.2.38)\end{aligned}$$

where:

1. t is a parameter of the worldline (Wilson Loop).
2. $A_0(t)\equiv A(\dot{\gamma}(t))$ is the gauge field along the Wilson line.
3. w is the worldline field and takes values in $\mathbb{C}\mathrm{P}^{N-1}$.
4. α is a worldline gauge field associated with w's $U(1)$ phase ambiguity.
5. κ is a constant chosen so as to get the right irrep R.

α acts as a Lagrange multiplier enforcing a particular norm for $\bar{w}w=\kappa$ for the worldline field. $U(1)$ gauge invariance requires κ to be quantized to integer values (this is analogous to the quantization of the level k and in fact the term $\kappa\alpha$ is known as the Chern-Simons 1-form[4]), which correspond to various symmetric (anti symmetric) irreps for Bosonic (Fermionic) w. What will our equation of motion be in the presence of w? We can write:

$$\begin{aligned} \bar{w}A_0 w &= \left(w^*\right)^i A_i^j w_j && (2.2.39)\\ &= A^a\left(w^*\right)^i\left(T^a\right)_i^j w_j, && (2.2.40)\end{aligned}$$

[4]For further reading on Chern forms and Chern-Simons forms we refer to [7].

giving an E.O.M.:

$$\epsilon^{0\nu\rho}F^a_{\nu\rho} = \frac{8\pi}{k}\left(w^*\right)^i\left(T^a\right)_i^j w_j\delta^2\left(\bar{x}\right)$$
$$F^a_{i0} = 0,$$

where the δ-function localizes to the Wilson line, which we have taken to lie at the origin, and stretch along "time". Let us partially solve this equation. We can pick temporal gauge $A_0 = 0$, in which case the solutions are time-independent. We can also further specify the gauge so that at a specific time:

$$w = (\kappa, 0, \ldots, 0)\,. \tag{2.2.41}$$

Now at this time we can guess a solution where A_i is diagonal for all i (is in the Cartan subalgebra), in which case the field strength is just given by its Abelian variant. The solution then is the same as in the Abelian case, which we will deal with later (see 2.3.1). The important thing for us to note is that the magnitude of the holonomy of the gauge field now depends on the ratio $\frac{\kappa}{k}$. In fact, it will turn out to have the schematic form $\exp\left(2\pi i\frac{\kappa}{k}\right)$, indicating that there is an equivalence $\kappa \to \kappa + k$. This is, in fact, a manifestation of a more elaborate statement that when considering Wilson loops in Chern-Simons theory, it suffices to consider only those with so called **integrable representations**. More reading on that can be found in [30, 17].

What of "quadratic" coupling terms like $D_\mu\bar{\phi}D^\mu\phi = \cdots + \bar{\phi}A^2\phi$, with ϕ a scalar field? We can rewrite this as a "current" by introducing an auxiliary "Lagrange multiplier" field $\lambda_{\mu,i}$:

$$\mathcal{L} \to \mathcal{L} - \left(\bar{\lambda}_\mu + \bar{\phi}A_\mu\right)\left(\lambda^\mu + A^\mu\phi\right), \tag{2.2.42}$$

which transforms under a gauge transformation via:

$$\lambda_{\mu,i} \to g_i^j\lambda_{\mu,j} - i\left(g\partial_\mu g^{-1}g\right)_i^j\phi_j. \tag{2.2.43}$$

By introducing this spurious degree of freedom we linearize the interaction in terms of A.

While these considerations are useful for understanding the theory in general, and the statement that the gauge field lacks physical degrees of freedom in particular, in our work we ultimately didn't use this localization. More information about localization of path integrals can be found in [26].

2.2.5 Light-Cone Gauge

When we couple the theory to matter it will be useful to have the **gauge propagator** in our arsenal. To that end we must gauge-fix. Most (if not all) gauge choices break the diffeomorphism invariance down to Lorenz invariance by introducing a metric. Of course, this same breaking will happen explicitly when we couple to matter. We decided to work in light cone gauge, as it has the following advantages[5]:

1. The self-interaction vertices of the gauge field vanish.
2. A variety of additional diagrams in the matter-coupled theory vanish.
3. There is no need to introduce ghosts.
4. On-shell conditions, or the vanishing of propagators, can be solved without introducing branch-cuts (specifically square-roots):

$$p^2 = m^2 \Rightarrow p_{\pm} = \frac{m^2 + p_{\perp}^2}{2p_{\mp}}. \tag{2.2.44}$$

We found that it also has certain challenges:

1. Lorenz symmetry breaks down $SO\left(2,1\right) \to GL\left(1\right)$.
2. Demonstrating gauge invariance can be tricky as some of the gauge dependence is in the denominator of the gauge propagator. This stands in contrast to, for instance, the familiar ξ-gauge in YM theory:

$$\frac{g^{\mu\nu} - (1-\xi)\frac{p^{\mu\nu}}{p^2}}{p^2}, \tag{2.2.45}$$

   where dependence on gauge "parameter" ξ is localized to the numerator, making algebraic manipulations simpler.

Another property of this gauge choice is a mixed blessing - carrying out Feynman integrals is different. On the one hand one can avoid the hassle of using Schwinger / Feynman parameters, and instead use contour integration techniques that in a different gauge would introduce plenty of branch cuts and square-roots. On the other hand, some integrals become a bit ambiguous (see below), and relating their branch cut structure to kinematics becomes more difficult (due to the breakdown of Lorenz symmetry).

[5]Points 1 and 3 are also true for temporal gauge or any gauge that simply sets a component of the gauge field to 0.

Light-cone coordinates are related to Minkowski coordinates [6] as follows:

$$x^{\pm} = \frac{1}{\sqrt{2}}\left(x^1 \pm x^2\right),$$
$$x^{\perp} = x^3.$$

The square-roots are chosen so that the metric takes the simple form:

$$ds^2 = 2dx_+dx_- - dx_\perp^2, \tag{2.2.47}$$
$$g_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2.2.48}$$

We will often Wick-rotate into Euclidean space:

$$x_1 \rightarrow ix_1, \tag{2.2.49}$$
$$\Rightarrow x^{\pm} \rightarrow i\underbrace{\frac{1}{\sqrt{2}}\left(x^1 \mp ix^2\right)}_{x_{\mathrm{E}}^{\pm}}. \tag{2.2.50}$$

This entails redefining the metric:

$$ds^2 \rightarrow -ds_{\mathrm{E}}^2, \tag{2.2.51}$$
$$ds_{\mathrm{E}}^2 = dx_1^2 + dx_2^2 + dx_3^2 \tag{2.2.52}$$
$$= 2dx_+dx_- + dx_\perp^2. \tag{2.2.53}$$

We will suppress the the "E" from here on. We see that for real Euclidean coordinates $(x^+)^* = x^-$. Note that we'll often use the Levi civita tensor interpreted as $\epsilon^{\mu\nu\rho}$, $\mu, \nu, \rho = +, -, \perp$ with $\epsilon^{+-\perp} = 1$. For all vectors we have:

$$p_{\pm} = g_{\pm\nu}p^{\nu} = p^{\mp}, \tag{2.2.54}$$

so in particular:

$$\epsilon_{+-\perp} = \epsilon^{-+\perp} = -1. \tag{2.2.55}$$

[6] for which:

$$ds^2 = dx_1^2 - dx_2^2 - dx_3^2 \tag{2.2.46}$$

Light cone gauge is given by the condition:

$$A_{-} = A^{+} = 0. \tag{2.2.56}$$

In pure CS theory, this choice is indistinguishable from setting any component of A to 0, since the metric we introduced is arbitrary. Our gauge fixing function is:

$$G(A) = A_{-}. \tag{2.2.57}$$

Under a gauge transformation:

$$A_{\mu}^{\alpha} = e^{i\alpha^{a}T^{a}} \left(A_{\mu} - \partial_{\nu}\right) e^{-i\alpha^{a}T^{a}}, \tag{2.2.58}$$

$$G(A^{\alpha}) = e^{i\alpha^{a}T^{a}} \left(A_{-} - \partial_{-}\right) e^{-i\alpha^{a}T^{a}}. \tag{2.2.59}$$

Infinitesimally:

$$\frac{\delta G(A^{\alpha})}{\delta \alpha} = D_{-} = \partial_{-} - i\left[A_{-}^{\alpha}, \cdot\right], \tag{2.2.60}$$

hence in accordance with the Faddeev-Popov procedure we use the identity[23]:

$$1 = \int D\alpha \delta\left(G(A^{\alpha})\right) \det\left(\frac{\delta G(A^{\alpha})}{\delta\alpha}\right), \tag{2.2.61}$$

to write:

$$\int DAe^{iS[A]} = \int DAD\alpha\delta\left(G(A^{\alpha})\right)\det\left(\frac{\delta G(A^{\alpha})}{\delta\alpha}\right)e^{iS[A]} \tag{2.2.62}$$

$$= \int D\alpha \int DA\delta\left(G(A)\right)\det\left(\partial_{-} - i\left[A_{-}, \cdot\right]\right)e^{iS[A]}, \tag{2.2.63}$$

where in the last line we used gauge invariance:

$$DA \exp\left(iS[A]\right) = DA^{\alpha} \exp\left(iS[A^{\alpha}]\right), \tag{2.2.64}$$

and then renamed A^{α} to A. At this point one normally[23] deals with $\delta(G(A))$ by replacing $G(A) \to G(A, \omega)$ where ω parameterizes a family of gauge choices and then averages over choices of ω, so that ω, rather than A, is localized by $\delta(G(A))$. This is necessary if the gauge condition involves derivatives of A, as is common in covariant gauges, since localizing A isn't straightforward. With A not localized one must deal with $\det\left(\frac{\delta G(A^{\alpha})}{\delta\alpha}\right)$ by introducing Fermionic ghost fields. In our case, however, $\delta(G(A)) = \delta(A_{-})$ and so localizing A is straightforward. We can proceed:

$$\ldots = \int D\alpha \int DA\delta\left(A_{-}\right)\det\left(\partial_{-} - i\left[0,\cdot\right]\right)e^{iS[A]} \quad (2.2.65)$$

$$= \det\left(\partial_{-}\right)\int D\alpha \int DA\delta\left(A_{-}\right)e^{iS[A]}, \quad (2.2.66)$$

hence up to a constant:

$$\int DAe^{iS[A]} \propto \int DA_{+}DA_{\perp}e^{iS[A]}\Big|_{A_{-}=0}. \quad (2.2.67)$$

In other words, we can safely just set $A_{-} = 0$ and not worry about any ghosts! Note that one normally gets "Gauss' law" (in our case $F_{+\perp} = 0$) from varying A_{-} which we've just integrated out! Of course, it should still hold. To see this, note that Gauss' law generates perturbations in A_{-}, effectively coupling the system to a "background" A_{-}. This can the absorbed into $A_{+,\perp}$ via a gauge transformation, leaving the path-integral invariant. Invariance under an infinitesimal transformation then amounts to the statement of Gauss' law.

The Lagrangian becomes:

$$\mathcal{L} = i\frac{k}{8\pi}A_i^a\epsilon^{ij}\partial_{-}A_j^a, \quad (2.2.68)$$

with $i,j = +,\perp$ and $\epsilon^{+\perp} = 1$. In momentum space:

$$i\frac{k}{8\pi}A_i^a\epsilon^{ij}\partial_{-}A_j^a \rightarrow i\frac{k}{8\pi}A_{+}^a\left(-p\right)\left(ip_{-}\right)A_{\perp}^a\left(p\right) - \frac{k}{8\pi}A_{+}^a\left(-p\right)\left(-ip_{-}\right)A_{\perp}^a\left(p\right) \quad (2.2.69)$$

$$= -\frac{k}{4\pi}p^{+}A_{+}^a\left(-p\right)A_{\perp}^a\left(p\right), \quad (2.2.70)$$

so we get the propagator:

$$\left\langle A_i^a\left(-q\right)A_j^b\left(p\right)\right\rangle = -i\frac{4\pi}{k}\frac{1}{p^{+}}\left(2\pi\right)^3\delta^3\left(p-q\right)\delta^{ab}\epsilon_{ij}. \quad (2.2.71)$$

Another useful way of writing this is by defining the **vector gauge-parameter:**

$$v^{\mu} = \left(v^{-} = 1, v^{+} = 0, v^{\perp} = 0\right), \quad (2.2.72)$$

which satisfies:

$$v\cdot p = p^{+},\ v^2 = 0. \quad (2.2.73)$$

v is simply a basis vector in our basis choice:

$$p = p^+ e_+ + p^- e_- + p^\perp e_\perp, \tag{2.2.74}$$

$$v = e_+. \tag{2.2.75}$$

Now:

$$\left\langle A^a_\mu (-q) A^b_\nu (p) \right\rangle = -i \frac{4\pi}{k} (2\pi)^3 \delta^3 (p-q) \delta^{ab} \frac{v^\rho \epsilon_{\rho\mu\nu}}{v \cdot p}. \tag{2.2.76}$$

The color-conserving Kronecker delta ensures the equation of motion is satisfied:

$$\begin{aligned} \left\langle F^a_{\mu\nu} \right\rangle &= \partial_\mu \left\langle A^a_\nu \right\rangle - \partial_\nu \left\langle A^a_\mu \right\rangle - i \left\langle [A_\mu, A_\nu]^a \right\rangle & (2.2.77) \\ &\propto 0 - 0 + \delta^{bc} f^{bca} & (2.2.78) \\ &= 0, & (2.2.79) \end{aligned}$$

where in the last line we've used the anti-symmetry of the structure constants f. Note also that our gauge condition:

$$v \cdot A = 0, \tag{2.2.80}$$

commutes with a Lorenz boost along $\perp$:

$$p = \left(p^+, p^-, p^\perp\right) \to \left(e^\xi p^+, e^{-\xi} p^-, p^\perp\right). \tag{2.2.81}$$

Hence this is our unbroken Lorenz symmetry, which we'll refer to as $GL(1)_\mathrm{L}$. We can think of this as simply the scaling transformation for v, under which (2.2.76) is manifestly invariant. We will often regulate the propagator by (see [19, 18]):

$$\frac{1}{p^+} \to \frac{p^-}{p^+ p^- - i\epsilon} = \frac{2p^-}{p^2_\parallel - i\epsilon}. \tag{2.2.82}$$

This $i\epsilon$ prescription, known as the Leibbrandt–Mandelstam prescription, allows one to consistently Wick rotate, since the relative sign between the energy squared and ϵ is the same as for covariant propagators. We can also write $v \equiv v_+$ and define v_- as the "other" lightcone direction that satisfies $v_- \cdot v_+ = 1$ and has $GL(1)_\mathrm{L}$ weight -1, so the propagator gets rewritten:

$$\frac{v^\rho \epsilon_{\rho\mu\nu}}{v \cdot p} \to v^\rho_+ v^\sigma_- \frac{2p_\sigma \epsilon_{\rho\mu\nu}}{p^2_\parallel - i\epsilon}, \tag{2.2.83}$$

where the denominator is now $GL(1)_{\mathrm{L}}$ invariant. This allows us to think of our various Feynman integrals as Lorenz - covariant tensor integrals, albeit with a spatial split into $\perp, \|$.

2.3 Connection to Knot Polynomials

In this subsection we'll discuss the relation between Chern-Simons theory and Knot invariants discovered by Witten in his paper "Quantum Field Theory and the Jones Polynomial"[30]. We'll start by motivating this connection with a $U(1)$ calculation, then go through a quick review of knot theory before describing Witten's results and connecting the question to crossing-symmetry. A useful reference on these matters is chapter 5 from Baez and Muniain's book [7]. Another is Tong's lecture notes [28].

2.3.1 Abelian Calculation

Let's begin by computing the correlation function of two Wilson loops on the curves $\gamma_{1,2}$ in representations $n_{1,2}$ of the gauge group $U(1)$:

$$W_{n_{1,2}}(\gamma_{1,2}) = \exp\left(n_{1,2} i \oint_{\gamma_{1,2}} A\right). \tag{2.3.1}$$

From the discussion in 2.2.4 we know this is just the partition function:

$$Z[J] = \int DA \exp\left(i\frac{k}{4\pi}\int d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \sum_{i=1}^{2} n_i i \oint_{\gamma_i} A\right), \tag{2.3.2}$$

where the Wilson line exponents act as the current, and that the path integral should localize to the classical solution. The Abelian theory is free and so this can be seen more directly by "completing the square" in terms of A and performing a Gaussian integral over A. This is equivalent to just solving the equation of motion:

$$\epsilon^{\mu\nu\rho} F_{\nu\rho} = -\frac{4\pi}{k} J^\mu, \tag{2.3.3}$$

or:

$$F_{\nu\rho} = -\frac{2\pi}{k} J^\mu \epsilon_{\mu\nu\rho} \equiv -\frac{2\pi}{k} J_{\nu\rho}, \tag{2.3.4}$$

where we interpret J as a 2-form. From linearity we know the solution will be a sum of terms $A_1 + A_2$, each sourced by one of the currents. Hence we get:

$$\exp\left(i \sum_i \oint_{\gamma_i} A_i + \sum_{i \neq j} i \oint_{\gamma_i} A_j \right). \tag{2.3.5}$$

Let's assume we are working in a simply connected space so that:

$$\exists \Sigma_i : \gamma_i = \partial \Sigma_i, \; i = 1, 2. \tag{2.3.6}$$

Then we get for the cross terms:

$$\sum_{i \neq j} \oint_{\gamma_i} A_j = \sum_{i \neq j} \int_{\Sigma_i} F_j = - \sum_{i \neq j} \frac{2\pi}{k} \oint_{\Sigma_i} J_j, \tag{2.3.7}$$

where in the last equality we used the E.O.M. (2.3.3). In other words, the Wilson line induced on one curve by the other is proportional to the charge flow through the surface obtained by shrinking it. Since in our case the current is localized to the support of the Wilson loop:

$$\int_{\Sigma_i} J_j = n_j L_{ji} \,(\text{no summation over } j), \tag{2.3.8}$$

where L_{ji} is the **linking number** of γ_j with γ_i, meaning the number of times γ_i intersects Σ_j with a positive orientation minus the number of intersections with a negative orientation. It's easy to see that $L_{ij} = L_{ji}$. An example of a configuration with $L_{12} = 0$ is shown in figure 4 and the link known as the "Hopf link" in knot theory, satisfying $L_{12} = \pm 1$ is displayed in figure 5.

What about the "self-interaction" terms? Those are given by:

$$\sum_i \int_{\gamma_i} A_i. \tag{2.3.9}$$

We will return to those after discussing some knot theory. Let us only remark that the discreteness of our results so far appear to be consistent with the topological nature of the theory - the correlation function is insensitive to deformations of the Wilson lines, as long as the curves don't "pass through" one-another (or themselves). Furthermore, since we seem to be getting something of the form:

$$e^{\frac{2\pi i}{k} n}, \; n \in \mathbb{Z}. \tag{2.3.10}$$

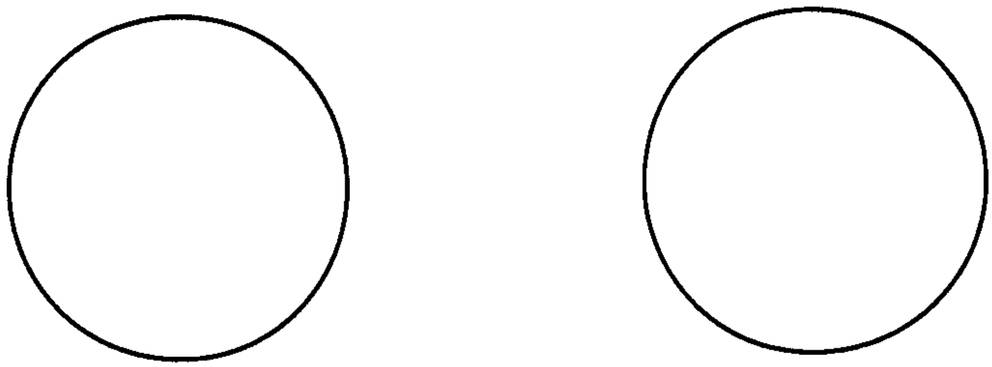


FIGURE 4: Two unlinked knots. Credit: John Baez and Javier P Muniain [7]

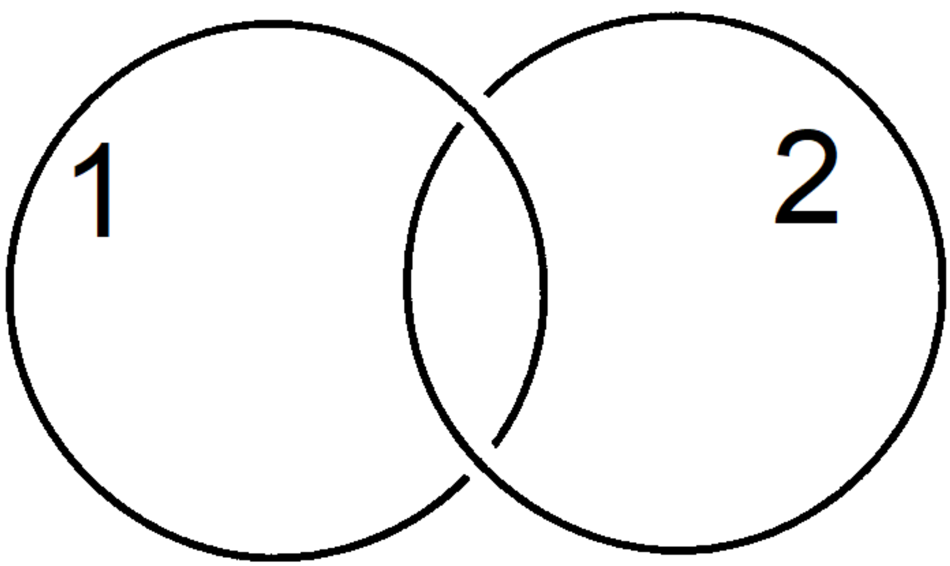


FIGURE 5: The Hopf link. Credit: John Baez and Javier P Muniain [7]

It's clear that integers like n are only observable $\mod k$ this is similar to our result in the non-Abelian case from 2.2.4. Let us briefly compute the gauge field sourced by a single Wilson loop with $n = 1$ on S^3 (spacetime with infinity identified as a point). Using our diffeomorphism invariance we can align the Wilson line W_γ along the "time" axis such that $\gamma(t) = (t, 0, 0)$. We have $J^\mu = (\delta^2(\bar{x}), 0, 0)$. In temporal gauge we have $A_0 = 0$ and the E.O.M. is:

$$0 = J^i \propto \epsilon^{ij} F_{0j} = \partial_0 \epsilon^{ij} A_j, \tag{2.3.11}$$

making the solution time-independent, and:

$$F_{12}(\bar{x}) = -\frac{2\pi}{k} \delta^2(\bar{x}). \tag{2.3.12}$$

We can change to polar coordinates:

$$\partial_r (r A_\theta) = -\frac{2}{kr} \delta(r). \tag{2.3.13}$$

A natural guess is now:

$$A_\theta(r, \theta) = -\frac{2\pi}{kr}, \tag{2.3.14}$$

Which gives rise to a holonomy:

$$\exp\left(i \oint_\gamma A\right) = \begin{cases} -\frac{2\pi i}{k} & |\, \gamma \text{ circumnavigates the origin} \\ 0 & |\, \text{otherwise} \end{cases}, \tag{2.3.15}$$

as expected.

2.3.2 Knot Theory

The linking number we've found in $U(1)$ Chern-Simons theory is known in knot theory as a **link invariant**. Knot theory is the mathematical study of knots and links - where a "knot" usually refers to a single "loop" (an embedding $\gamma : S^1 \to \mathcal{M}$ of the circle into a real 3-manifold $\mathcal{M}$) and a "link" is just a bunch of different knots. Sometimes the knots are dressed with extra structure - such as an orientation (a nowhere vanishing vector field tangent to the knot) or a framing (like an orientation but never tangent to the knot). Most importantly, knots are identified with one another when they can be related by **ambient isotopy** which is simply a transformation on $\mathcal{M}$ that is connected to the identity. In other words, knots are identified when they can be deformed into one-another with intersecting themselves along the way.

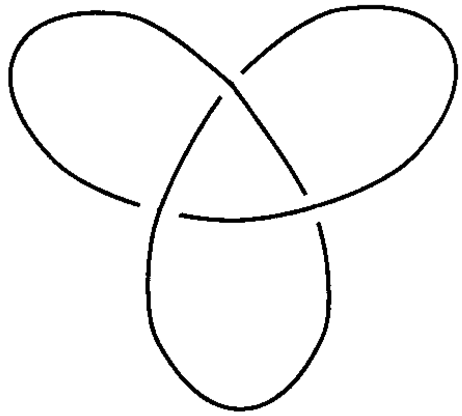


FIGURE 6: The trefoil knot. Credit: John Baez and Javier P Muniain [7]

Knots are often visualized by projecting them onto a plane. We demonstrate this in figure 6 using the “trefoil” knot.

It is not usually obvious whether two such projections represent the same knot. For this reason, knot theorists, in their efforts to classify all existing knots, are interested in knot invariants - numbers that characterize a knot independently of its projection. The linking number we saw in 2.3.1 is one such knot invariant. Another example is the Jones Polynomial[30].

2.3.3 Witten's Knot Invariants

Witten's insight in [30] was that correlation functions of Wilson loops in various representations in a **topological** theory should give rise to knot (or link) invariants. We've seen this in 2.3.1 in the Abelian case, where the correlation functions evaluate to linking numbers. Witten showed, among other things, that for gauge group $SU(2)$ and Wilson lines in the fundamental representation embedded in S^3, the correlation functions evaluate to the Jones polynomial. This was the first time that an inherently 3 dimensional definition of invariant derived. Historically, all invariants were defined using projections of knots, and then shown to be invariant under the 3 Reidemeister moves [30, 7] that are the building blocks of all ambient isotopies.

More precisely, Witten discovered **framed**-knot invariants. The correlation functions depend on a framing chosen for the knots. This is most easily seen when we return our attention to self-interaction terms (2.3.9). These are naively divergent. However, one can regulate them by a so-called point-splitting regularization. Given some framing $f^\mu(t)$ of the loop $\gamma(t)$ one can rewrite:

$$\int_{\gamma} A \to \int_{\gamma+\epsilon f} A, \tag{2.3.16}$$

where A is sourced by γ but is integrated along the slightly shifted contour $\gamma + \epsilon f$ where ϵ is a small real parameter. This now should evaluate to the linking of the shifted contour with the original contour - effectively counting the number of times the framing f winds around γ.

2.3.4 Possible Connection to Crossing Symmetry

Recall the crossing phase from (1.4):

$$T_S^{\text{naive}} \to \frac{\sin(\pi\lambda)}{\pi\lambda} T_S^{\text{naive}} = N \frac{\sin(\pi\lambda)}{\pi\lambda} T_{\text{particle-particle}} \tag{2.3.17}$$

(the extra factor of N is explained in 2.5.3). It was observed in [19] that this factor is exactly the relative factor (found by Witten in [30]) between the expectation value of a single fundamental $SU(N)$ Wilson loop in S^3 and that of 2 unlinked Wilson loops. In 7.4 of [19], the authors conjecture that those Wilson loops arise in the following heuristic way: to compute a truly gauge invariant quantity, one must dress the 4 field insertions in the correlation function with Wilson lines:

$$\begin{aligned} C(x_i) &= \delta^i_j \delta^k_l \left\langle \bar{\phi}^j(x_2)\, \phi_i(x_1)\, \bar{\phi}^l(x_4)\, \phi_k(x_3) \right\rangle & (2.3.18) \\ &\to \left\langle \bar{\phi}^j(x_2)\, W_{\text{F}}(\gamma_{21})^i_j\, \phi_i(x_1)\, \bar{\phi}^l(x_4)\, W_{\text{F}}(\gamma_{43})^k_l\, \phi_k(x_3) \right\rangle, & (2.3.19) \end{aligned}$$

where W_{F} is a Wilson loop in the fundamental representation and γ_{ij} is a contour connecting x_i to x_j. This is depicted in figure 7 where the points x_i are taken to lie at a sphere S^2_∞ at infinity of $\mathbb{R}^3$. The idea is that the scattering particles' motions together with the Wilson lines (which can be thought of as heavy "probe particles") close to form gauge invariant Wilson loops. The relative factor between the link invariants in the different channels produces the crossing phase. This difference of link topologies between particle-particle scattering and particle-antiparticle scattering doesn't arise when relating the exchange and adjoint channels. See 2.5.3 for a description of these different channels.

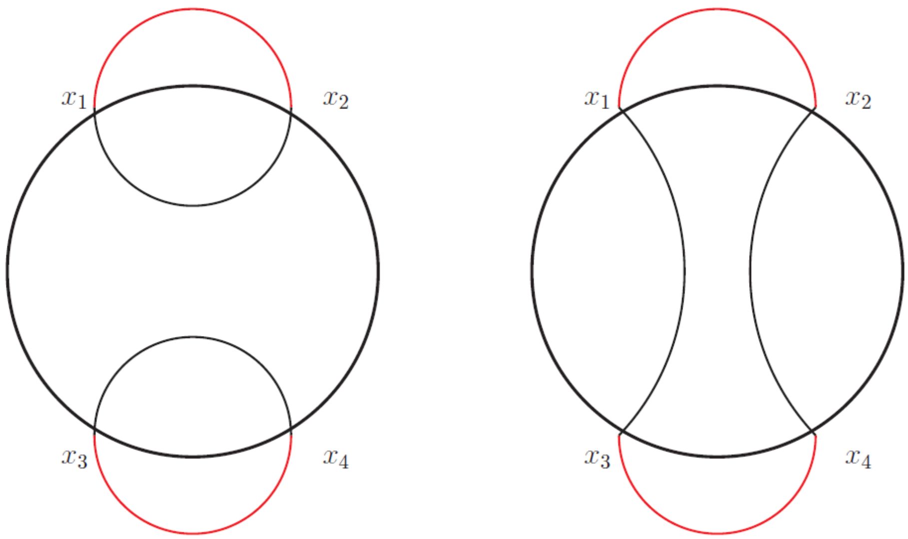


FIGURE 7: Scattering processes in the direct (particle-particle) channel (left) and in the singlet (particle-antiparticle) channel (right). The solid black circles represent the sphere S^2_∞ at infinity of $\mathbb{R}^3$ at which particles begin and end their motions. The red lines represent Wilson lines dressing the amplitude. The black lines represent the particles' trajectories. On the left, time flows to the right, so that the depicted process is the creation of particles in positions x_1, x_3 and their scattering to positions x_2 and x_4 respectively. On the right, time flows downwards, hence a particle and antiparticle are created at x_1 and respectively x_2 and then scatter to final positions x_3, x_4. The overall motion of the scattering particles as well as the "probe" particles (the Wilson lines) trace a single knot on the right but two knots on the left. Credit: Sachin Jain et. al.[19]

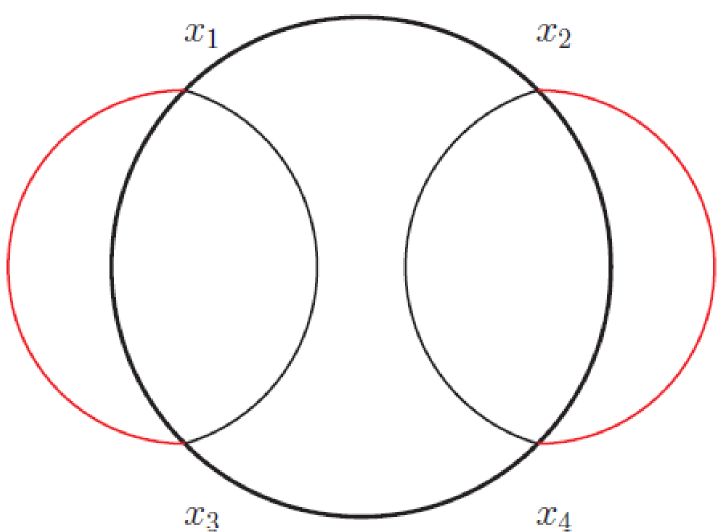


FIGURE 8: Scattering in the adjoint channel. Time flows downwards and the particles' motions are like in the singlet channel in figure 7. However, the Wilson lines dressing the amplitude give rise to an overall link topology that is the same as that in particle-particle scattering, which is why the crossing relation isn't modified for this channel. Credit: Sachin Jain et. al.[19]

2.4 Chern-Simons Matter Theory

In this section we'll describe the theory we'll be working with, where the CS gauge field is coupled to fundamental Bosonic matter.

Let's return to the full action (1.1), this time written in our chosen gauge 2.2.5:

$$S = \int d^3x \left(i\frac{k}{8\pi} A_i^a \epsilon^{ij} \partial_- A_j^a + D_\mu \bar{\phi} D^\mu \phi + m^2 \bar{\phi}\phi + \frac{1}{2N} b_4 \left(\bar{\phi}\phi\right)^2 \right), \tag{2.4.1}$$

$$D_\mu = \partial_\mu + iA_\mu, \tag{2.4.2}$$

$$\lambda = \frac{N}{k}, \tag{2.4.3}$$

$$A_\mu = A_\mu^a T^a, \tag{2.4.4}$$

$$\mathrm{Tr}\left(T^a T^b\right) = \frac{1}{2}\delta^{ab}, \tag{2.4.5}$$

where ϕ is a scalar in the fundamental representation of $SU(N)$. In [19] the authors use $U(N)$ but work in the large N ('t Hooft) limit, where the distinction is inconsequential. Let's take a closer look at the gauge-matter coupling term:

$$D_\mu \bar{\phi} D^\mu \phi = \overline{\partial_\mu \phi + iA_\mu \phi} \left(\partial^\mu \phi + iA^\mu \phi\right) \tag{2.4.6}$$

$$= \partial_\mu \bar{\phi} \partial^\mu \phi + i\bar{\phi}\left(A \cdot \left(\overleftarrow{\partial} - \overrightarrow{\partial}\right)\right)\phi + \bar{\phi}A^2\phi. \tag{2.4.7}$$

Note that in our gauge choice the quartic vertex looks like:

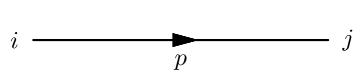
$$\bar{\phi}A^2\phi = \bar{\phi}A_3^2\phi. \tag{2.4.8}$$

Since an A_3 insertion can only Wick-contract with A_+ we learn that in a Feynman graph two quartic vertices cannot connect to one another via a gauge-Boson propagator. This is a source of much simplification when one enumerates Feynman diagrams, and plays an important role in the computability of all-loop quantities in the 't Hooft limit.

Even without the chosen gauge, the diffeomorphism symmetry of the theory is broken down to the much smaller (orientation preserving part of -) Poincaré symmetry in 3d. We retain the symmetry under simultaneous parity (or reflection) and negation of k (or λ).

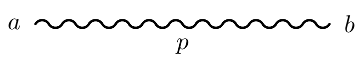
2.4.1 Feynman Rules

The following is written in Euclidean signature. The propagators are given by:



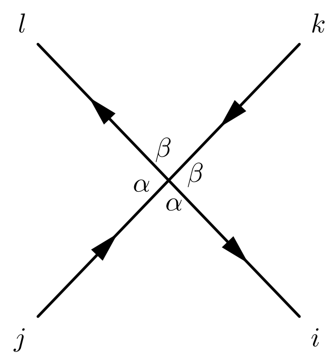
$$= \frac{\delta^i_j}{p^2 + m^2 - i\epsilon} \propto \left\langle \bar{\phi}_j \left(-p\right) \phi^i \left(p\right) \right\rangle \tag{2.4.9}$$

and:



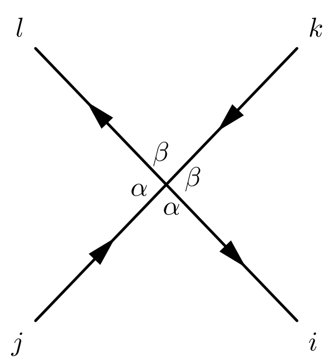
$$= -i \frac{4\pi}{k} \delta^{ab} \frac{v^\rho \epsilon_{\rho\mu\nu}}{v \cdot p} \propto \left\langle A^a_\mu \left(-p\right) A^b_\nu \left(p\right) \right\rangle . \tag{2.4.10}$$

The quartic scalar self-interaction can be written:

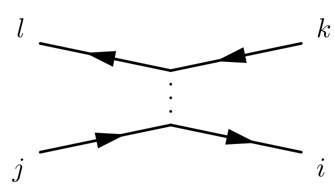


$$= -\frac{b_4}{N} \delta^i_j \delta^l_k . \tag{2.4.11}$$

Since there are 2 possible contractions it will be useful to "split" the diagram as:

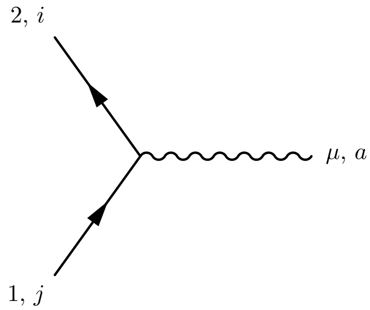


$$\equiv$$

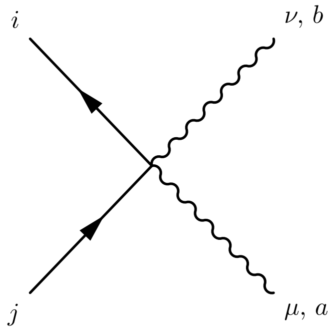


$$, \tag{2.4.12}$$

so that the "decorations" α, β are unnecessary. [7]The gauge interaction vertices are given by:



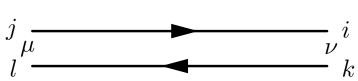
$$= \quad (p_1 - p_2)_\mu \left(T^a\right)^i_j, \text{ (all momenta outgoing)} \tag{2.4.14}$$



$$= \quad -g_{\mu\nu} \left\{T^a, T^b\right\}^i_j, \tag{2.4.15}$$

where $\{\cdot,\cdot\}$ is the anti-commutator.

Let us further specialize to the 't Hooft limit. Here we take $N, k \to \infty$ with $\lambda = \frac{N}{k}$ held constant. In any computed quantity, we must keep the leading terms in N. Our various interactions carry factors of $\frac{1}{N}$, but those are offset by terms of the type $\delta^i_i = N$ which arise in "color loops". This leads to the famous criterion that the relevant diagrams are the **planar diagrams**. The theory can now be treated as $U(N)$ and we can parameterize A by its matrix-indices instead of its generator indices and represent it using double-line notation:

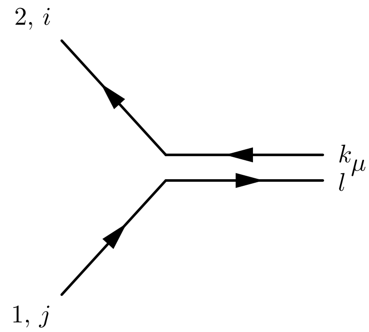


$$= -i\frac{2\pi\lambda}{N}\delta^i_k\delta^l_j\frac{\upsilon^\rho\epsilon_{\rho\mu\nu}}{\upsilon\cdot p} \propto \left\langle \left(A_\mu\right)^i_j(-p)\left(A_\nu\right)^l_k(p)\right\rangle. \tag{2.4.16}$$

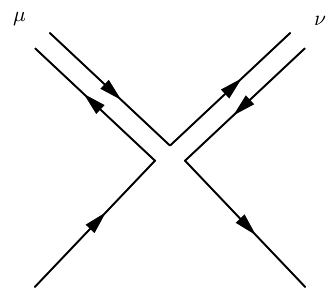
[7]This can be done formally, as in [19], by integrating out an auxiliary "heavy" field σ to obtain the 4-point interaction:

$$\sigma\bar{\phi}\phi - \frac{N}{2b_4}\sigma^2 = -\left(\sqrt{\frac{N}{2b_4}}\sigma - \sqrt{\frac{b_4}{2N}}\bar{\phi}\phi\right)^2 + \frac{b_4}{2N}\left(\bar{\phi}\phi\right)^2 \tag{2.4.13}$$

This way we need not keep track of generators. Our vertices become:



$$= \quad (p_1 - p_2)_\mu \, \delta^i_k \delta^l_j, \tag{2.4.17}$$



$$= \quad -g_{\mu\nu}, \tag{2.4.18}$$

where in the last graph we have suppressed the color structure, as we will continue to do going forward since it is trivially represented by the edges of the graph. Note that the two terms in the anti-commutator $\left\{T^a, T^b\right\}^i_j$ now appear as two distinct diagrams.

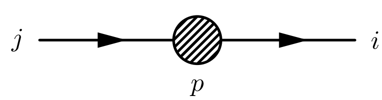
2.4.2 The Interacting Planar Scalar Propagator

Minwalla et. al. [19] state that the self energy Σ is momentum-independent and that the pole mass c of the scalar propagator is given by:

$$c^2 = \frac{\lambda^2}{4} c^2 - \frac{b_4}{4\pi} |c| + m^2. \tag{2.4.19}$$

We will now reproduce this by re-summing all planar 1PI graphs. Note that the manifest off-shell Lorenz and Gauge invariance of this result is unexpected. Some basic diagramatics indicate that the gauge propagator gets no corrections at leading order in N, so we can ignore such corrections in writing the diagrams below.

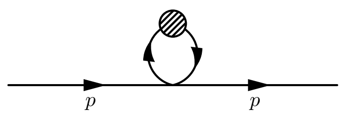
Denoting the interacting propagator by:



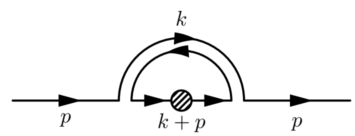
$$= \frac{\delta^i_j}{p^2 + m^2 + \Sigma(p)}. \tag{2.4.20}$$

We have:

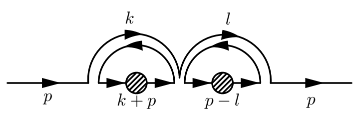
$$-\Sigma \quad = \tag{2.4.21}$$



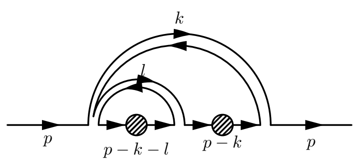
$$+$$



$$+ \tag{2.4.22}$$



$$+ \quad + \text{reflection}, \tag{2.4.23}$$

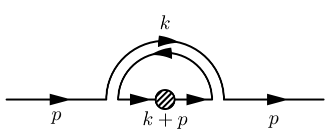


where we have organized all the diagrams participating in the process. This amounts to a recursive definition of Σ - an integral equation. Solving it will give the all-loop self energy. A few comments are in order:

1. The gauge propagator receives no corrections at leading order in $\frac{1}{N}$.

2. The diagrams are all $O(N^0)$.

3. The rainbow diagram vanishes:



$$\propto v^\mu \epsilon_{\mu\nu\rho} (k + 2p)^\nu (k + 2p)^\rho = 0. \tag{2.4.24}$$

We therefore find:

$$\Sigma(p) = b_4 I_1(\Sigma) + (2\pi\lambda)^2 \left((I_2(p, \Sigma))^2 - 2I_3(p, \Sigma) \right), \tag{2.4.25}$$

with:

$$I_1(\Sigma) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2+m^2+\Sigma(k^2)}, \tag{2.4.26}$$

$$I_2(p,\Sigma) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^+} \frac{2p^+ + k^+}{(p+k)^2+m^2+\Sigma\left((p+k)^2\right)}, \tag{2.4.27}$$

$$I_3(p,\Sigma) = \int \frac{d^3k}{(2\pi)^3} \frac{d^3l}{(2\pi)^3} \frac{1}{k^+} \frac{1}{l^+} \frac{2p^+ + 2k^+ + l^+}{(p+k+l)^2+m^2+\Sigma\left((p+k+l)^2\right)} \tag{2.4.28}$$

$$\times \frac{2p^+ + k^+}{(p+k)^2+m^2+\Sigma\left((p+k)^2\right)}. \tag{2.4.29}$$

After straightforward algebra we find:

$$I_2^2 - 2I_3 = I_1^2. \tag{2.4.30}$$

Hence all integrals have been reduced to Lorenz-invariants and therefore all gauge dependence is gone. Furthermore, all dependence on p vanishes. We conclude that:

$$\Sigma(p) = \text{const} = b_4 I_1(\Sigma) + \lambda^2 (2\pi I_1(\Sigma))^2, \tag{2.4.31}$$

$$I_1(\Sigma) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2+m^2+\Sigma}.$$

Using dimensional regularization we obtain:

$$I_1 = -\frac{1}{4\pi}\sqrt{m^2+\Sigma} = -\frac{|c|}{4\pi}, \tag{2.4.32}$$

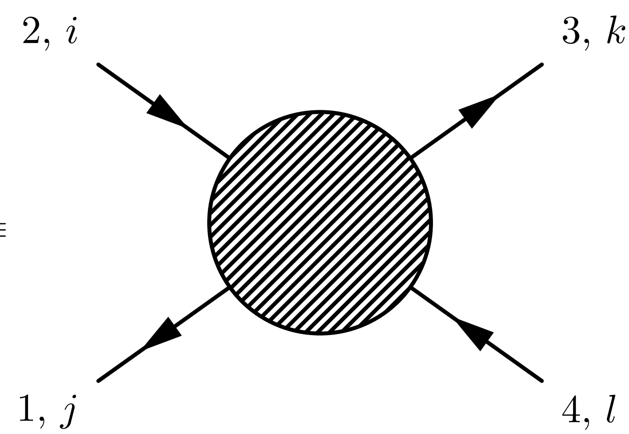
which when plugged into (2.4.31) gives (2.4.19). Going forward we will denote $c = m$ and forget about the original value.

2.5 Kinematics and Color

In this section we'll discuss the kinematics of scattering - the participating momenta and variables derived from them, choices of notation, the on-shell condition, etc. - and color - that is, what kind of tensor structures we expect to see that involve the fundamental representation indices $i = 1, \ldots, N$ and generator indices $a = 1, \ldots, \dim G$.

We will be concerned with **connected, amputated** correlation functions:

$$\left\langle \phi^i(p_2)\,\bar{\phi}_j(p_1)\,\phi^l(p_4)\,\bar{\phi}_k(p_3)\right\rangle_{\text{C.A.}} \equiv$$

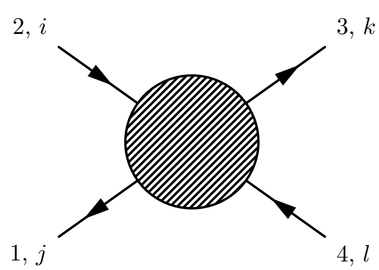


$$, \qquad (2.5.1)$$

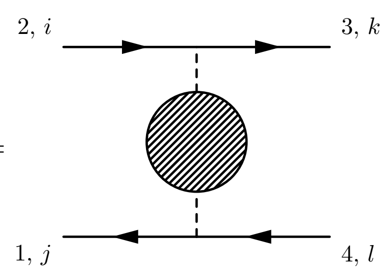
and their on-shell limits (the S-matrix).We'll think of all momenta as outgoing so that momentum conservation implies:

$$\sum_{i=1}^{4} p_i = 0. \qquad (2.5.2)$$

This (all-outgoing) approach is handy when one wishes to consider different channels of scattering, as opposed to just one. Bose symmetry is reflected by summing over 2 classes of diagrams:



$$=$$



$$+\left((2,i)\leftrightarrow(4,l)\right) \qquad (2.5.3)$$

Where the dashed line represents the most general exchange of any number of gauge Bosons and "heavy σ-s". We are therefore free to consider only the first term. In fact, had we included different "flavors" of scalar, and computed a "mixed" correlator involving two different flavors, only one of the terms in (2.5.3) would contribute.

We will define a basis of vectors:

$$\begin{aligned} s &= p_1 + p_2 && (2.5.4) \\ t &= p_1 + p_4 && (2.5.5) \\ u &= p_1 + p_3, && (2.5.6) \end{aligned}$$

which we'll refer to as the **"Mandelstam basis"**. The Mandelstam invariants are given in Euclidean signature by:

$$S = -s^2,\ T = -t^2,\ U = -u^2. \tag{2.5.7}$$

Due to momentum conservation, any 3 of the vectors $\{s, t, u, p_1, p_2, p_3, p_4\}$ forms a basis with which we can write all the external momenta in the problem. Of course, they also (generically) form a basis for 2+1 dimensional spacetime. The inverse transformation is given by:

$$\begin{aligned} p_1 &= \frac{1}{2}(s+t+u) && (2.5.8) \\ p_2 &= \frac{1}{2}(s-t-u) && (2.5.9) \\ p_3 &= \frac{1}{2}(-s-t+u) && (2.5.10) \\ p_4 &= \frac{1}{2}(-s+t-u). && (2.5.11) \end{aligned}$$

Given any 3 vectors v_i we will often use the notation:

$$\epsilon^{\mu\nu\rho}\left(v_1\right)_\mu\left(v_2\right)_\nu\left(v_3\right)_\rho \equiv \epsilon\left(v_1, v_2, v_3\right) \equiv v_1 \cdot \left(v_2 \times v_3\right), \tag{2.5.12}$$

and:

$$E\left(v_1, v_2, v_3\right) = \mathrm{sign}\left(\epsilon\left(v_1, v_2, v_3\right)\right). \tag{2.5.13}$$

2.5.1 On-Shell Kinematics

On shell we have:

$$p_i^2 = \begin{cases} 2p_i^+ p_i^- + \left(p_i^\perp\right)^2 = -m^2 & \mid \text{in Euclidean signature} \\ 2p_i^+ p_i^- - \left(p_i^\perp\right)^2 = m^2 & \mid \text{in Lorenzian signature} \end{cases}. \tag{2.5.14}$$

We can also write this as:

$$S + T + U = 4m^2, \tag{2.5.15}$$

$$s \cdot t = t \cdot u = u \cdot s = 0. \tag{2.5.16}$$

Hence the s, t, u basis is orthogonal on-shell! This also implies for any vector p:

$$\epsilon(p, s, t) = p \cdot u \frac{\epsilon(s, t, u)}{u^2} \text{ and cyclic rotations of } s, t, u, \tag{2.5.17}$$

$$\epsilon(s, t, u)^2 = STU. \tag{2.5.18}$$

2.5.2 The S-Matrix

The S-matrix tabulates the scattering amplitudes of the theory. Since those are observable, the S-matrix, if properly computed should be a gauge-invariant object. According to the LSZ reduction formula, the S-matrix is the on-shell limit of connected, amputated momentum-space correlation functions. Said correlation functions are not required to be gauge invariant, although one can modify them into gauge-invariant functions by dressing them with Wilson lines. Normally, in computing the S-matrix, such modification is unnecessary, and all gauge-dependence falls off as one approaches the mass-shell. Nevertheless, as we'll see in 4, this naive expectation appears not to be the case, at least for light-cone gauge, in Chern-Simons matter theory.

The S-matrix is a function of **on-shell** momenta, and can be written:

$$S\left(\{p_i, \alpha_i\}_{i=1,\dots,4}\right) = \delta^{\alpha_4}_{\alpha_1}\delta^{\alpha_2}_{\alpha_3} I(p_1, p_4; p_2, p_3) + \delta^{\alpha_2}_{\alpha_1}\delta^{\alpha_4}_{\alpha_3} I(p_1, p_2; p_3, p_4) + S_{\text{connected}}(p_i, \alpha_i), \tag{2.5.19}$$

where α-s are color-indices. The I-s correspond to free propagation and are given by:

$$I(p_1, p_2; p_3, p_4) = 2E_{\bar{p}_1}(2\pi)^2 \delta^2(\bar{p}_1 + \bar{p}_2) 2E_{\bar{p}_3}(2\pi)^2 \delta^2(\bar{p}_3 + \bar{p}_4),$$
$$E_{\bar{p}} = \sqrt{m^2 + \bar{p}^2},$$

while $S_{\text{connected}}$ is proportional to the **scattering amplitude** $\mathcal{M}$:

$$S_{\text{connected}}(p_i, \alpha_i) = i(2\pi)^3 \delta^3\left(\sum_i p_i\right) \mathcal{M}(p_i, \alpha_i). \tag{2.5.20}$$

Note that $I(p_1, p_4; p_2, p_3)$ can also be written in terms of the scattering angle θ between p_1 and p_4 and the center of mass energy $E = \sqrt{(p_1 + p_2)^2}$:

$$I(p_1, p_4; p_2, p_3) = (2\pi)^3 \delta^3\left(\sum_{i=1}^{4} p_i\right) 4\pi E \lim_{\epsilon \to 0}\left(\delta(\theta + \epsilon) + \delta(\theta - \epsilon)\right). \tag{2.5.21}$$

Our main interest is in:

$$\mathcal{M}\left(p_i,\alpha_i\right) \;=\; \mathcal{M}\left(p_1,j;p_2,i;p_3,k;p_4,l\right), \tag{2.5.22}$$

where we have placed color indices and momenta belonging to the same particle side by side. Since all generator indices are summed over, we can write:

$$\begin{aligned} & \mathcal{M}\left(p_1,j;p_2,i;p_3,k;p_4,l\right) \\ =\; & \underbrace{\delta^i_j\delta^l_k\mathcal{M}_{\rm D}\left(p_i\right)}_{\text{'direct'}}+\underbrace{\delta^i_k\delta^l_j\mathcal{M}_{\rm E}\left(p_i\right)}_{\text{'exchange'}} \\ =\; & \underbrace{\frac{1}{2}\left(\delta^i_j\delta^l_k+\delta^i_k\delta^l_j\right)\mathcal{M}_{\rm Sym}\left(p_i\right)}_{\text{symmetric}}+\underbrace{\frac{1}{2}\left(\delta^i_j\delta^l_k-\delta^i_k\delta^l_j\right)\mathcal{M}_{\rm ASym}\left(p_i\right)}_{\text{anti-symmetric}} \\ =\; & \underbrace{\left(\delta^i_k\delta^l_j-\frac{1}{N}\delta^i_j\delta^l_k\right)\mathcal{M}_{\rm A}\left(p_i\right)}_{\text{adjoint}}+\underbrace{\frac{1}{N}\delta^i_j\delta^l_k\mathcal{M}_{\rm S}\left(p_i\right)}_{\text{singlet}}. \end{aligned} \tag{2.5.23}$$

The (anti-)symmetric, adjoint and singlet "channels" correspond to different irreducible representations of $SU\left(N\right)$, and must be gauge-invariant (e.g. independent of the choice of null vector parameter v). A different way of organizing the color structure is to split $\mathcal{M}$ into "color factors" such as:

$$C_F\left(T^a\right)^i_j\left(T^a\right)^l_k,\;\left(T^aT^bT^a\right)^i_j\left(T^b\right)^l_k,\;\left(\left\{T^a,T^b\right\}\right)^i_j\left(\left\{T^a,T^b\right\}\right)^l_k,\;\ldots \tag{2.5.24}$$

This will be more useful in the non-planar regime, as we'll discuss in 3. There, we will also discuss in more detail what constitutes a basis of such color factors.

The various amplitudes $\mathcal{M}$ are functions of kinematic Lorenz invariants, which in 3d are the Mandelstam invariants S,T,U as well as $E\left(p_1,p_2,p_3\right)$ - the **handedness** of the triplet of vectors p_1,p_2,p_3. This is the only Lorenz invariant that **isn't** invariant under parity or reflection. In light of the $\mathbb{Z}_2$ symmetry described in 2.2.3, the handedness must enter the amplitude only through terms of odd power in λ.

2.5.3 Channels of Scattering

In our "all outgoing" convention, incoming particles will be represented by having negative energy. W.l.o.g. we can take $p_2^1<0$ so that we always have at least 1 incoming particle. There are then 3 configurations (consistent with both the mass-shell condition and momentum-conservation) we may consider:

1. $p_4^0 < 0$, $p_{1,3}^0 > 0 \Rightarrow U \geq 4m^2$, $T, S \leq 0$. This is **particle-particle** scattering which naturally decomposes into the symmetric and anti-symmetric representations.

2. $p_1^0 < 0$, $p_{3,4}^0 > 0 \Rightarrow S \geq 4m^2$, $T, U \leq 0$. This is **particle-antiparticle** scattering which naturally decomposes into the adjoint and singlet representations.

3. $p_3^0 < 0$, $p_{1,4}^0 > 0 \Rightarrow T \geq 4m^2$, $S, U \leq 0$. This too is **particle-antiparticle** scattering.

In the large N limit the Feynman diagrams split naturally into the "direct" and "exchange" color factors $\delta_j^i \delta_k^l$, $\delta_k^i \delta_j^l$ which are related by Bose symmetry. We will keep only those with factor $\delta_j^i \delta_k^l$ (and note that $\left(\delta_k^i \delta_j^l - \frac{1}{N}\delta_j^i \delta_k^l\right) \mathcal{M}_{\rm A}\left(p_i\right) \approx \delta_k^i \delta_j^l \mathcal{M}_{\rm A}\left(p_i\right)$) so that we have the following correspondence:

$$\begin{aligned} S &\geq 4m^2 \leftrightarrow \text{singlet channel}, \\ T &\geq 4m^2 \leftrightarrow \text{adjoint channel}, \\ U &\geq 4m^2 \leftrightarrow \text{direct channel}. \end{aligned} \tag{2.5.25}$$

That is to say that evaluating this set of diagrams with a particular choice of signage for the Mandelstam invariants computes the amplitude in a particular channel. Having chosen our color factor $\delta_j^i \delta_k^l$ the only question is which pair of indices corresponds to incoming particles. The Mandelstam invariant corresponding to this pair is then the one to satisfy $\geq 4m^2$ (for real, on-shell momenta). This leads to the following naive conjecture when $S \geq 4m^2$:

$$\frac{1}{N}\mathcal{M}_{\rm S}\left(S,T,U\right) \stackrel{?}{=} \mathcal{M}_{\rm D}\left(U,T,S\right) \stackrel{?}{=} \mathcal{M}_{\rm A}\left(T,S,U\right). \tag{2.5.26}$$

Note the factor of $1/N$ - this comes from the prefactor $\delta_j^i \delta_k^l / N$ in (2.5.23), and effectively "enhances" the singlet channel relative to the other channels. This, of course, is simply a way of stating **crossing symmetry**. Of course, as discussed in 1, the actual relationship is modified to 1.4. A crucial point is that for $S \geq 4m^2$ the vector s is timelike while t, u are spacelike so we can pick our vector gauge parameter to satisfy $v \cdot t = 0$ or $v \cdot u = 0$ but not $v \cdot s = 0$. These assumptions make the resummation in [19] possible. For this reason, the authors were only able to re-sum the S-matrix in the adjoint, direct and exchange channels but had to conjecture the form of the singlet channel. This suggests that the off-shell correlator has some form of non-analyticity in S. We wish to better understand how this arises.

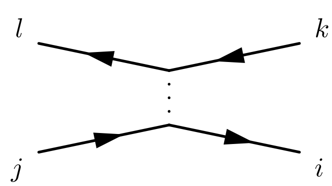
2.5.4 Gauge Invariance of the S-Matrix

Individual Feynman diagrams are often functions of the gauge parameter - whether it's v, as in our case, ξ in ξ-gauge or otherwise. This gauge dependence can remain even when we some the diagrams to form off-shell correlation functions. However, the on-shell scattering amplitudes must not depend on these parameters - they must be gauge invariant. Note that in our case, individual Feynman diagrams **are** invariant under **rescalings** of v - which amount to nothing more then our surviving boost symmetry. Hence our only actual parameter is the spatial direction of v. Checking gauge invariance provides us with a valuable sanity check for our calculations. The presence of v in expressions is also what breaks **Lorenz** invariance, by picking out a preferred spatial direction. Hence, the restoration of gauge invariance is equivalent to the restoration of Lorenz invariance.

Besides being a sanity check, we expect whatever unorthodox analyticity properties the S-matrix exhibits to be easiest to see once Lorenz invariance is restored. Hence naively, a crucial step in any calculation is to rid ourselves of the dependence on v. Our gauge choice is very useful in reducing the number of diagrams for us to consider, but, at least initially, hinders the consideration of analyticity properties. As we'll see in 4, some of the 1-loop quantities we compute are in fact gauge dependent.

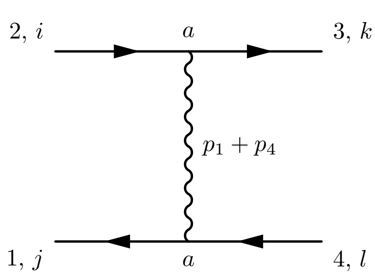
2.5.5 Example: Tree Level Gauge Invariance

At tree level the S-matrix is a meromorphic function (it contains only poles), so naive crossing should be satisfied. We will also see that it is gauge-invariant. There are 2 diagrams. The first:



$$\propto -\frac{b_4}{N}, \tag{2.5.27}$$

is trivially gauge-invariant. The second is simply:



$$= -i\frac{4\pi\lambda}{N}\left(T^a\right)^i_k\left(T^a\right)^l_j\frac{v\cdot(p_1-p_4)\times(p_3-p_2)}{v\cdot(p_1+p_4)}. \tag{2.5.28}$$

Let us motivate:

$$(p_1 + p_4) \cdot (p_1 - p_4) = p_1^2 - p_4^2, \tag{2.5.29}$$

$$(p_1 + p_4) \cdot (p_3 - p_2) = -(p_2 + p_3) \cdot (p_3 - p_2) \tag{2.5.30}$$

$$= p_2^2 - p_3^2. \tag{2.5.31}$$

Both of these vanish on shell! Hence subject to the on-shell condition we have:

$$p_1 + p_4 \perp p_1 - p_4,\, p_3 - p_2, \tag{2.5.32}$$

which, in a 2+1 d spacetime implies:

$$(p_1 - p_4) \times (p_3 - p_2) \propto p_1 + p_4, \tag{2.5.33}$$

making (2.5.28) independent of v. In fact we need not even choose v to be null, we may choose $v = p_1 + p_4$ to get:

$$\frac{v \cdot (p_1 - p_4) \times (p_3 - p_2)}{v \cdot (p_1 + p_4)} = \frac{(p_1 + p_4) \cdot (p_1 - p_4) \times (p_3 - p_2)}{(p_1 + p_4)^2} \tag{2.5.34}$$

$$= 4\frac{\epsilon\,(p_1, p_2, p_3)}{(p_2 + p_3)^2}. \tag{2.5.35}$$

What of the color factor?

$$(T^a)^i_k\,(T^a)^l_j = (T^a \otimes T^a)^{il}_{kj} \tag{2.5.36}$$

$$= \frac{1}{2}\begin{pmatrix} (T^a \otimes I + I \otimes T^a)\,(T^a \otimes I + I \otimes T^a) - \\ (T^a T^a \otimes I) - (I \otimes T^a T^a) \end{pmatrix}^{il}_{kj}. \tag{2.5.37}$$

This decomposition is an example of a more general formula - given irreps $R_{1,2}$ and R_i such that:

$$R_1 \otimes R_2 = \sum_i R_i, \tag{2.5.38}$$

we have in the R_i representation:

$$T^a_{R_1} \otimes T^a_{R_2} = \frac{1}{2} I_{R_i}\,(C\,(R_i) - C\,(R_1) - C\,(R_2))\,. \tag{2.5.39}$$

With $C\,(R)$ the quadratic Casimir invariant in the R irrep. We see that in each channel, at tree-level, the interaction induced by the CS gauge field is the same as that obtained in the Abelian theory, except that the flux carried by each particle, and therefore the anyonic statistics that the particles are imbued with, depend on the channel of scattering and are proportional to (2.5.39).

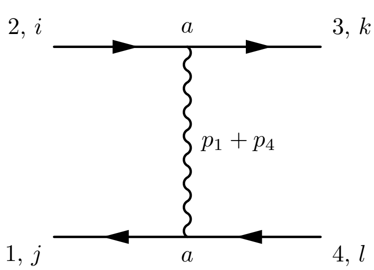
In our case, however, we are can just make an Ansatz based on the tracelessness of T^a to find:

$$\left(T^a\right)^i_k \left(T^a\right)^l_j = A\left(\delta^i_j\delta^l_k - \frac{\delta^i_k\delta^l_j}{N}\right), \tag{2.5.40}$$

This follows from the tracelessness of T^a. We only need to find the constant A. We can do this by contracting i with j and l with k:

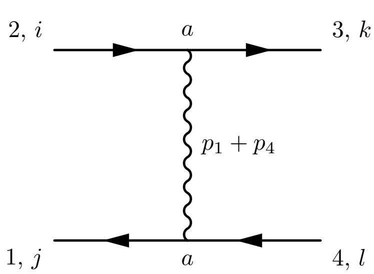
$$\frac{N^2-1}{2} = \mathrm{tr}\left(T^aT^a\right) \tag{2.5.41}$$
$$= \left(T^a\right)^i_l \left(T^a\right)^l_i \tag{2.5.42}$$
$$= A\left(N^2-1\right), \tag{2.5.43}$$
$$\Rightarrow A = \frac{1}{2}. \tag{2.5.44}$$

In total we obtain:



$$= -i\frac{8\pi\lambda}{N}\left(\delta^i_j\delta^l_k - \frac{\delta^i_k\delta^l_j}{N}\right)\frac{\epsilon\left(p_1,p_2,p_3\right)}{\left(p_2+p_3\right)^2}. \tag{2.5.45}$$

Note that in the large N limit we can write $\delta^i_j\delta^l_k - \frac{\delta^i_k\delta^l_j}{N} \approx \delta^i_j\delta^l_k$ so we get a result consistent with [19]. We can also write this in terms of Mandelstam invariants:



$$= -i\frac{4\pi\lambda}{N}\left(\delta^i_j\delta^l_k - \frac{\delta^i_k\delta^l_j}{N}\right)E\left(p_1,p_2,p_3\right)\sqrt{\frac{SU}{T}}. \tag{2.5.46}$$

3 The 1-loop Non-Planar Scattering Amplitude

The modified crossing relation (1.4) was obtained in the planar limit. It is therefore natural to ask whether it shows up away from the planar limit, or whether some other non-trivial analytic behavior emerges.

In this section we discuss our non-planar one loop computation of a particular gauge-invariant "color factor". In fact, this amounts to a calculation in the Abelian theory, where this color factor would correspond to the part of the amplitude of order $e_1^2 e_2^2$ (if the Bosons had different electric charges). We will start by discussing how we picked this color factor, then we present its covariantization (and thereby demonstrate its gauge invariance), and finally we discuss what form we expect the final result to take after the implementation of integral reduction techniques and what that implies for modified crossing. While this result is forthcoming, we indicate why it's unlikely to exhibit the anomalous analytic properties implicit in the modified crossing relation, and discuss the possible reasons for that.

3.1 The "Abelian" Color Factor

The 1-loop amplitude splits into 3 monomials in the coupling constants:

$$O\left(b_4^2\right),\, O\left(b_4\lambda\right),\, O\left(\lambda^2\right), \tag{3.1.1}$$

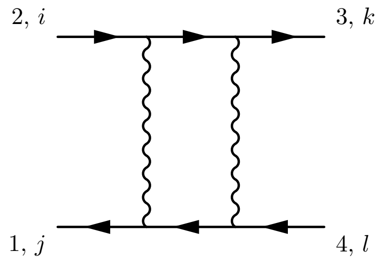
each of which must be separately gauge-invariant. Away from the planar limit we will focus on the λ^2 terms, or equivalently, set $b_4 = 0$. The modified crossing relation (1.4) doesn't depend on b_4 so we shouldn't miss anything substantial with this assumption. The relevant diagrams are given in eq' (3.1.2).

Note that (3.1.4) are gauge-propagator corrections that vanish in the planar limit. Similarly the second term in (3.1.2) (the "cross-box" diagram) as well as both diagrams in (3.1.3) are large-N suppressed, and their negligibility in the 't Hooft limit underlies the reduction of the all-loop 4-point function into a **resummable** sequence of ladder graphs.

We further wish to locate an even smaller gauge-invariant combination of diagrams. Suppose we considered the process with 2 different representations $R_{1,2}$, one for each of the scattering particles, then our diagrams will split schematically into "color factors":

$$\left(T_{R_1}^3\right)^i_k \left(T_{R_2}\right)^l_j,\, \left(T_{R_1}^2\right)^i_k \left(T_{R_2}^2\right)^l_j,\, \left(T_{R_1}\right)^i_k \left(T_{R_2}^3\right)^l_j, \tag{3.1.5}$$

$$i\mathcal{M}\Big|_{b_4=0} \;=$$



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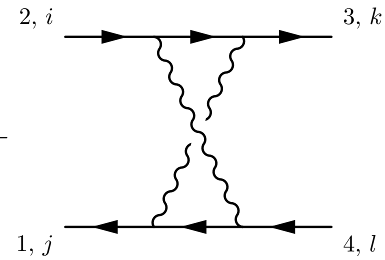

(3.1.2)

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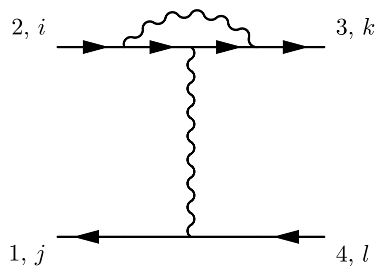

$+$ 1 reflection (3.1.3)

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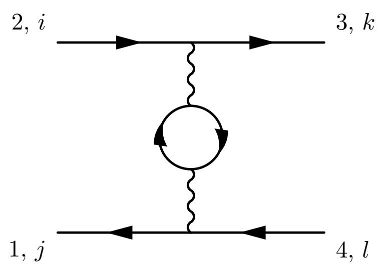

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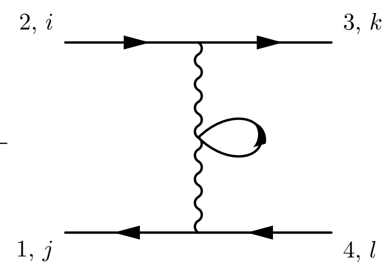

(3.1.4)

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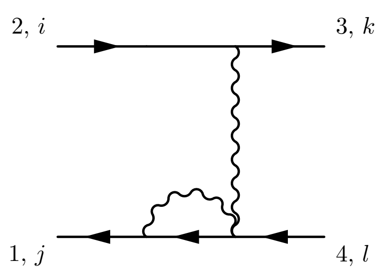

$+$ 3 reflections

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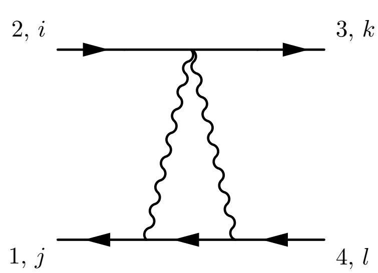

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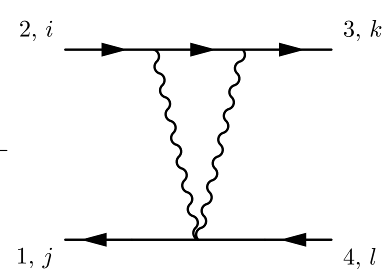


where the superscript is roughly an exponent. These are not, in general, independent. E.g.:

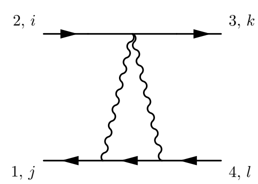
$$\begin{aligned}\left(\left[T^a_{R_1}, T^b_{R_1}\right] T^a_{R_1}\right)^i_k \left(T^b_{R_2}\right)^l_j &\propto f^{abc} f^{cad} \left(T^d_{R_1}\right)^i_k \left(T^b_{R_2}\right)^l_j \\ &\propto \left(T^a_{R_1}\right)^i_k \left(\left[T^a_{R_2}, T^b_{R_2}\right] T^b_{R_2}\right)^l_j .\end{aligned}$$

In other words, they are related via the representation-independence of the structure constants f^{abc}. We can therefore focus on the terms proportional to:

$$\left\{T^a_{R_1}, T^b_{R_1}\right\}^i_k \left\{T^a_{R_2}, T^b_{R_2}\right\}^l_j . \tag{3.1.6}$$

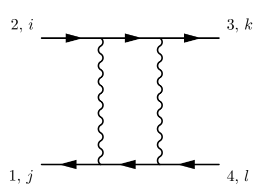
This "color factor" must satisfy gauge invariance by itself, as there is no prospect for a gauge-dependent part in it to cancel against other color factors. It is simply the "fully symmetrized" part of the middle term in (3.1.5). Of course, the mathematical manipulations by which this invariance is made manifest are blind to the choice of representations, so we can drop the generator subscripts going forward.

Which diagrams contribute to this color factor? First note that:



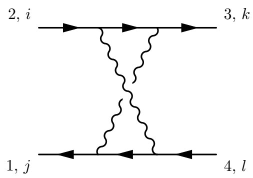
$$\propto \left\{T^a, T^b\right\}^i_k \left(T^a T^b\right)^l_j \tag{3.1.7}$$

$$= \frac{1}{2} \left\{T^a, T^b\right\}^i_k \left\{T^a, T^b\right\}^l_j , \tag{3.1.8}$$



$$\propto \left(T^b T^a\right)^i_k \left(T^a T^b\right)^l_j \tag{3.1.9}$$

$$= \frac{1}{4} \left\{T^a, T^b\right\}^i_k \left\{T^a, T^b\right\}^l_j - \frac{1}{4} \left[T^a, T^b\right]^i_k \left[T^a, T^b\right]^l_j , \tag{3.1.10}$$



$$\propto \left(T^aT^b\right)^i_k \left(T^aT^b\right)^l_j \tag{3.1.11}$$

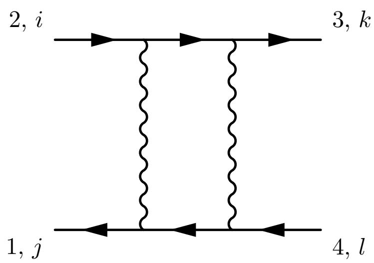
$$= \frac{1}{4}\left\{T^a,T^b\right\}^i_k\left\{T^a,T^b\right\}^l_j + \frac{1}{4}\left[T^a,T^b\right]^i_k\left[T^a,T^b\right]^l_j. \tag{3.1.12}$$

So to summarize we can write:

$$i\mathcal{M} = \frac{1}{4}\left(-i\frac{4\pi\lambda}{N}\right)^2\left\{T^a,T^b\right\}^i_k\left\{T^a,T^b\right\}^l_j i\mathcal{M}\Big|_{b_4=0,\,\mathrm{sym}} + \ldots \tag{3.1.13}$$

And **schematically**:

$$i\mathcal{M}\Big|_{b_4=0,\,\mathrm{sym}} =$$



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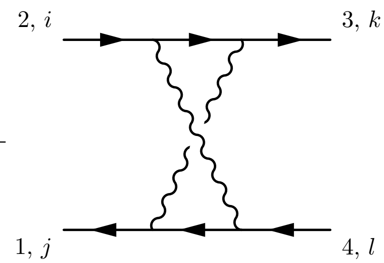
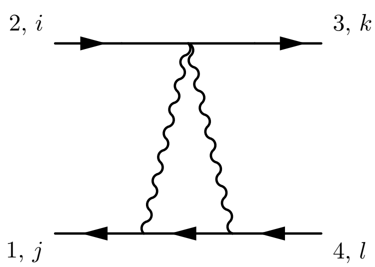


$+ \; 2\times$



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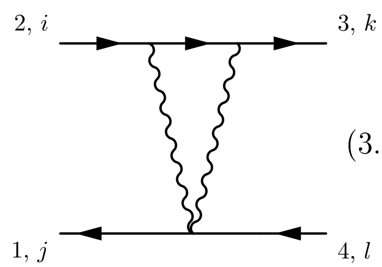


(3.1.14)

The presence of the non-planar cross-box diagram on an equal footing with the box diagram shows that this is an inherently non-planar quantity. This can also be understood in a different way: in the Abelian theory all commutators vanish and so this color factor is simply the $O\left(\lambda^2 e_1^2 e_2^2\right)$ part of the amplitude (with $e_{1,2}$ being the charges that take the place of $R_{1,2}$ in (3.1.5)). Hence we are simply considering the amplitude for $G = U(1)$, and nothing could be further from large N then $U(1)$!

We wish to rid ourselves of the v-dependence, subject to the on-shell condition. We find that we can accomplish this and thus bring the color factor into the form of a sum of covariant Feynman integrals. Using standard manipulations we will further reduce these into scalar "triangles", "bubbles" and "tadpoles" - that is - Feynman integrals involving at most 3 propagators and a numerator free of loop-momenta. These integrate into various transcendentality 1 functions (logarithms, arctangents, etc.). Does this square with the modified crossing relation (1.4)? Let us reproduce it here:

$$S_{\mathrm{S}} = \cos(\pi\lambda)\, I\,(p_1, p_4; p_2, p_3) + i\frac{\sin(\pi\lambda)}{\pi\lambda} T_{\mathrm{S}}^{\mathrm{naive}}. \tag{3.1.15}$$

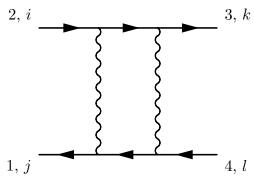
$T_{\mathrm{S}}^{\mathrm{naive}}$ is $O(\lambda)$ so the term we expect to see at $O(\lambda^2)$ comes only from the first term:

$$-(\pi\lambda)^2\, I\,(p_1, p_4; p_2, p_3)\,. \tag{3.1.16}$$

After extracting the overall momentum-conserving delta function this reduces to simply a δ-function at forward scattering $\delta(\theta)$. This doesn't seem to be captured by our result. A possible reason is that we use the Schouten identity to reduce box integrals into triangle integrals in a way that isn't valid at $\theta = 0$ (see below). Alternatively, it might be that we must compute the off-shell correlation function and carefully approach the mass-shell - this is the approach we take in section 4. Finally note that (1.4) is merely the result in the planar limit. For all we know the non-planar theory may have completely unexpected analyticity properties in the non-planar limit. It also could be that our specific form factor simply exhibits the standard crossing symmetry.

3.2 Gauge Invariance of the Color Factor

Let us first focus on the box diagram:[8]



$$= -\int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2+m^2} \frac{1}{(k-p_1-p_2)^2+m^2} \tag{3.2.2}$$

$$\times \quad \frac{\epsilon\,(v, k+p_1, k-p_1-2p_2)}{v\cdot(k-p_1)} \frac{\epsilon\,(v, k-p_4, k+p_4+2p_3)}{v\cdot(k+p_4)}. \tag{3.2.3}$$

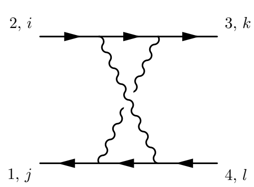
The box can be thought of as two tree level exchanges in sequence, each of which, as we saw in subsection 2.5.5 , is gauge invariant when the scalar legs leading to it are on-shell. Hence on the residue of both scalar propagators we should be able to write:

$$\frac{\epsilon\,(v, k+p_1, k-p_1-2p_2)}{v\cdot(k-p_1)} \to \frac{\epsilon\,(k-p_1, k+p_1, k-p_1-2p_2)}{(k-p_1)^2}. \tag{3.2.4}$$

We should be able to add-and-subtract this covariantized version of the box, and the difference will be proportional to the on-shell condition - that is - to inverse scalar propagators! We can see how that works using the Schouten identity: Using the on-shell condition and momentum conservation we can write:

$$\begin{aligned} k^2+m^2 &= k^2-p_{1,4}^2 = (k-p_{1,4})\cdot(k+p_{1,4})\,, \\ (k-p_1-p_2)^2+m^2 &= (k-p_1-2p_2)\cdot(k-p_1) \\ &= (k+p_4)\cdot(k+p_4+2p_3)\,. \end{aligned} \tag{3.2.5}$$

[8]The cross-box is related by a simple relabeling:



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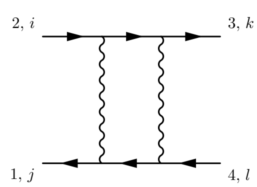


$$\Big|_{p_2\leftrightarrow p_3} \tag{3.2.1}$$

$$= -\int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2+m^2} \frac{1}{(k-p_1-p_3)^2+m^2}$$

$$\times \quad \frac{\epsilon\,(v, k+p_1, k-p_1-2p_3)}{v\cdot(k-p_1)} \frac{\epsilon\,(v, k-p_4, k+p_4+2p_2)}{v\cdot(k+p_4)}.$$

This suggests we can use the Schouten identity[9] to write:

$$\begin{aligned}
\frac{\epsilon\left(v, k+p_1, k-p_1-2p_2\right)}{v\cdot\left(k-p_1\right)} &= \frac{\left(k-p_1\right)^2 \epsilon\left(v, k+p_1, k-p_1-2p_2\right)}{v\cdot\left(k-p_1\right)\left(k-p_1\right)^2} \\
&= -\frac{\left(k-p_1\right)\cdot\left(k-p_1-2p_2\right)\epsilon\left(v, k-p_1, k+p_1\right)}{v\cdot\left(k-p_1\right)\left(k-p_1\right)^2} \\
&+ \frac{\left(k-p_1\right)\cdot\left(k+p_1\right)\epsilon\left(v, k-p_1, k-p_1-2p_2\right)}{v\cdot\left(k-p_1\right)\left(k-p_1\right)^2} \\
&+ \frac{\epsilon\left(k-p_1, k+p_1, k-p_1-2p_2\right)}{\left(k-p_1\right)^2}.
\end{aligned} \tag{3.2.7}$$

As expected, we obtain a covariantized version of the propagator along with terms proportional to inverse scalar propagators. We repeat this for the other gauge propagator, and for the cross box so we are left schematically with:

$$i\mathcal{M}\Big|_{b_4=0,\,\mathrm{sym}} = \text{covariant box} + \text{covariant cross-box} \tag{3.2.8}$$

$$+ \ \text{non-covariant triangles, bubbles, etc.} \tag{3.2.9}$$

We refer as triangles (bubbles) to terms where one (resp' two) scalar propagators have been canceled, as well as the actual triangle diagrams (3.1.14). We must deal with the non-covariant part. A useful hint as to how to proceed is to change the mass of particles 1,4 relative to particles 2,3:

$$p_1^2 = p_4^2 = -m_1^2 \neq -m_2^2 = p_2^2 = p_3^2, \tag{3.2.10}$$

$$\left(k^2+m^2\right)^{-1} \rightarrow \left(k^2+m_1^2\right)^{-1}, \tag{3.2.11}$$

$$\left(\left(k-p_1-p_2\right)^2+m^2\right)^{-1} \rightarrow \left(\left(k-p_1-p_2\right)^2+m_2^2\right)^{-1}, \tag{3.2.12}$$

$$\left(\left(k-p_1-p_3\right)^2+m^2\right)^{-1} \rightarrow \left(\left(k-p_1-p_3\right)^2+m_2^2\right)^{-1}. \tag{3.2.13}$$

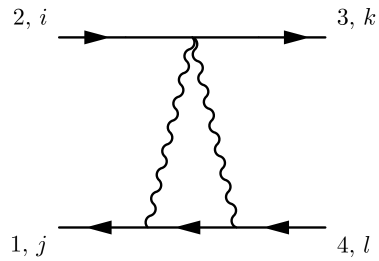
Earlier manipulations such as (3.2.5) carry through. The amplitude in this deformed 2-scalar theory must still be gauge invariant, and this tells us what terms we must combine - we may

[9] In our case this is simply:

$$0 = \sum_{i=1}^{4} \left(-1\right)^i p_{1+i}^{\mu} \epsilon\left(p_{2+i}, p_{3+i}, p_{4+i}\right), \tag{3.2.6}$$

where p_n, $n = 1, \ldots, 4$ are **any** 4 vectors and is understood mod 4.

focus only one class of triangle diagrams:



$$. \tag{3.2.14}$$

These should be gauge invariant up to bubble terms where the scalar propagator:

$$\left(k^2+m_1^2\right)^{-1}=\left(\left(k-p_1\right)\cdot\left(k+p_1\right)\right)^{-1}, \tag{3.2.15}$$

has canceled. Our expectation is to see the eventual cancellation of all the "spurious" poles $\left(v\cdot\left(k-p_1\right)\right)^{-1}$, $\left(v\cdot\left(k+p_4\right)\right)^{-1}$. Naively, we could try to just select an "algebraic basis" of inner products which trivializes the on-shell conditions and momentum conservation, and then see whether the poles cancel - this doesn't work! The reason is that the poles may cancel only when one remembers that the expression is under the integral sign - shifts and reflections of our loop momentum k can be applied to different terms before combination. Another issue is that our integrand isn't expressed purely in terms of inner products - there are also triple products - however, this is easily remedied by the identity:

$$\epsilon^{\mu_1\mu_2\mu_3}\epsilon_{\nu_1\nu_2\nu_3}=-\sum_{\sigma\in S(3)}\operatorname{sign}\left(\sigma\right)\prod_{i=1}^{3}\delta^{\mu_i}_{\nu_{\sigma(i)}}, \tag{3.2.16}$$

or equivalently:

$$\epsilon\left(x_1,\ldots,x_3\right)\epsilon\left(y_1,\ldots,y_3\right)=-\det\left(\left\{x_i\cdot y_j\right\}_{i,j=1,\ldots 3}\right). \tag{3.2.17}$$

As for momentum shifts - since there is a unique scalar propagator (with mass m_1) we expect them to be unnecessary until we cancel all terms that have **both** a scalar propagator and a spurious one. After choosing an appropriate basis we find we are left with only two terms involving the scalar propagator and **both** of the spurious poles:

$$\frac{\left(v\cdot\left(p_1-p_4\right)\right)^2}{\left(k^2+m^2\right)v\cdot\left(k-p_1\right)v\cdot\left(k+p_4\right)}-\frac{\left(v\cdot\left(p_1-p_4\right)\right)^2\left(\left(k-p_1\right)\cdot\left(k+p_4\right)\right)^2}{\left(k^2+m^2\right)v\cdot\left(k-p_1\right)\left(k-p_1\right)^2v\cdot\left(k+p_4\right)\left(k+p_4\right)^2}. \tag{3.2.18}$$

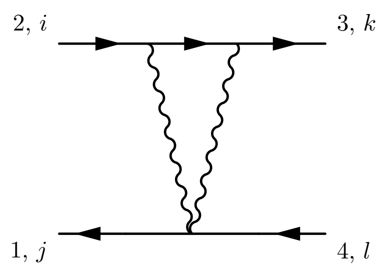
The reason is that we haven't used all constraints available to us - namely, we haven't used the dimensionality off spacetime. Linear dependencies among the vectors can be expressed using the Schouten identity, as we have done in (3.2.7). Another form of this identity is the vanishing of the Gram determinant of linearly dependent vectors:

$$G\left(x_1,\ldots,x_n\right)=\det\left(\left\{x_i\cdot x_j\right\}_{i,j=1,\ldots n}\right)=0\,\text{for}\;\;n>3, \tag{3.2.19}$$

$$\Rightarrow 0 \;\;=\;\; G\left(v,k-p_1,k+p_4,p_1-p_4\right), \tag{3.2.20}$$

$$\Rightarrow \left(v\cdot\left(p_1-p_4\right)\right)^2\left(\left(k-p_1\right)\cdot\left(k+p_4\right)\right)^2=\left(v\cdot\left(p_1-p_4\right)\right)^2\left(k-p_1\right)^2\left(k+p_4\right)^2+\ldots \tag{3.2.21}$$

Hence the terms in (3.2.18) cancel up to the terms in "..." - which all contain at least one inverse spurious propagator. At this point we again simplify using an algebraic basis and find the straightforward cancellation of all terms involving **both** scalar and spurious propagators. After repeating for the "inverted triangle" terms:



, (3.2.22)

and applying a few momentum-shifts, we arrive at a fully Lorenz invariant and gauge invariant integrand, involving no spurious poles. This expression is too long to be reproduced here.

3.3 Integral Reductions

The next step is to reduce our integrals to a basis of scalar integrals. In 3 dimensions, we expect all diagrams to reduce to ones having at most 3 propagators (triangle diagrams). We expect the final result to be a significantly simpler integrand satisfying the following:

1. No remaining gauge propagators. If any integrals remain that have a gauge propagator, then it can be "cut" (placed on-shell) along with the other propagators and the result should be the amplitude of a physical process involving a gauge Boson. Since Chern-Simons gauge Bosons do not propagate, we expect such a residue to be 0, and hence with sufficient algebra the pole should turn out to be spurious. An exception is

tadpole diagrams - those containing **only** a gauge propagator. However, those give no contribution in dimensional regularization and so can be ignored.

2. In the planar limit (see 4) the 1-loop amplitude reduces (roughly) to an integrand that corresponds to "pinching" all scalar propagators in the original diagrams derived from the Feynman rules - a few scalar tadpoles which integrate to constants along with a scalar bubble diagram which integrates to $\frac{1}{4\pi\sqrt{-S}}\arctan\left(\frac{\sqrt{-S}}{2m}\right)$ (see C). We thus expect a similar result here, except that the cross-box should give rise to a "crossed" bubble, with $S \to U$.

How does one perform this reduction? We'll describe it step by step.

3.3.1 Reduction of the Box Integrals

Our covariantized integrand is now given by:

$$i\mathcal{M}\Big|_{b_4=0,\,\mathrm{sym}} = \text{covariant box} + \text{covariant cross-box} \tag{3.3.1}$$

$$+ \text{covariant triangles, bubbles, etc.} \tag{3.3.2}$$

The box and cross-box have a "tensor" numerator, meaning it contains powers of the loop momentum k. Our first step is to express all such powers as inverse propagators, so that they cancel the propagators giving "lower order" integrals (triangles, bubbles, etc.). There are 3 independent external vectors, which we can choose to be any 3 of the vectors $\{s, t, u, p_1, p_2, p_3, p_4\}$, as described in 2.5. Hence we can write all the inner products in the numerator (slightly redundantly) as:

$$k^2,\ k \cdot p_i,\ i = 1, 2, 3. \tag{3.3.3}$$

Note that each box has 4 propagators. For instance, the (not-crossed) box has:

$$\left(k^2 + m^2\right)^{-1},\ \left((k-s)^2 + m^2\right)^{-1},\ (k-p_1)^{-2},\ (k+p_4)^{-2}. \tag{3.3.4}$$

Thus it is straightforward to solve for the inner products in (3.3.3) in terms of the inverse propagators and kinematic invariants. This needs to be done separately for the cross-box, although one can avoid this by using the relation (3.2.1).

Before long, we are left only with **scalar** box integrals. Reduction of these depends on the 3d nature of the problem and employs the Schouten identity. Specifically, we know that any 4 vectors are linearly dependent and so have a vanishing Gram determinant (determinant of the 4×4 matrix of inner products):

$$0 = G\left(k, p_1, p_2, p_3\right). \tag{3.3.5}$$

This expression can converted into a polynomial in inverse propagators and kinematic invariants S, T, U. We can solve (3.3.5) to write unity as:

$$1 = \frac{P_{\text{box}}\,(\text{inverse box propagators})}{\left(T+U\right)ST^2} = \frac{P_{\text{Xbox}}\,(\text{inverse cross-box propagators})}{\left(T+S\right)UT^2}, \tag{3.3.6}$$

where P stands for "polynomial". The denominators are simply the terms of order 0 in inverse propagators in (3.3.5). Now we can simply "multiply" our scalar box integrals with the appropriate expression for unity to obtain an integrand composed only of triangles and lower order integrals. Importantly, this reduction depends on the non-vanishing of the denominators in (3.3.6). Since the δ-function we expect from the crossing relation (1.4) is supported on $T = 0$, it is possible that by using this reduction we are missing something. Nevertheless, let us proceed.

3.3.2 Reduction of Triangles

Reducing the scalar box integrals depended on the Schouten identity in a way that cannot be used to reduce scalar triangles. However, we will see that tensor triangles can be reduced to scalar triangles and bubbles. The key is that powers of loop momentum can now be expressed as combinations of inverse propagators **and** numerators for which the integral vanishes! This can be seen clearly with an example. Let's consider the triangle integrand:

$$k^{-2}\left(k-p_1\right)^{-2}\left(k-p_2\right)^{-2} \times \text{numerator}. \tag{3.3.7}$$

We are ignoring mass terms and $p_{1,2}$ here have no relation to our actual scattering momenta. Now the inner products $k^2, k \cdot p_1, k \cdot p_2$ can be expressed in terms of inverse propagators. But what of other products? Consider the vector integral:

$$\int \frac{d^3k}{\left(2\pi\right)^3} \frac{k^\mu}{k^2\left(k-p_1\right)^2\left(k-p_2\right)^2}. \tag{3.3.8}$$

By symmetry considerations, it should evaluate to $A\left(p_1^\mu + p_2^\mu\right)$. The only important thing about this is that it should vanish when "dotted" into a vector **orthogonal** to both p_1 and

p_2, such as $p_1 \times p_2$. This means that:

$$\int \frac{d^3k}{(2\pi)^3} \frac{\epsilon(k, p_1, p_2)}{k^2 (k-p_1)^2 (k-p_2)^2} = 0. \tag{3.3.9}$$

In fact, we can write more generally:

$$\int \frac{d^3k}{(2\pi)^3} \frac{(\epsilon(k, p_1, p_2))^n}{k^2 (k-p_1)^2 (k-p_2)^2} N\left(k^2, k\cdot p_1, k\cdot p_2\right) = 0,\ n\,\text{odd}, \tag{3.3.10}$$

where N is some polynomial. That this integral vanishes follows from the existence of an isometry (reflection through the plane spanned by p_1, p_2) that negates $\epsilon(k, p_1, p_2)$ but not the other inner products.

Hence we can complete p_1, p_2 to a basis by adding the vector $p_1 \times p_2$! Finally, note that even powers of $\epsilon(k, p_1, p_2)$ can be re-expressed in terms of $k^2, k\cdot p_1, k\cdot p_2$ using (3.2.17). Hence all inner products in the numerator in (3.3.7) can be re-expressed as combinations of inverse propagators and kinematic invariants up to vanishing terms like (3.3.10), proving that we can reduce a general triangle to scalar triangles and bubbles.

3.3.3 Reduction of Bubbles

Reduction of bubble to scalar bubbles is very similar to the reduction of triangles. In our case, we are left only with **vector** bubbles. These are especially easy to reduce to scalars. Consider that:

$$\int \frac{d^3k}{(2\pi)^3} \frac{k^\mu}{k^2 (k-p)^2} = Ap^\mu. \tag{3.3.11}$$

Then we can write:

$$A = \frac{1}{p^2} \int \frac{d^3k}{(2\pi)^3} \frac{k\cdot p}{k^2 (k-p)^2} \tag{3.3.12}$$

$$= \frac{1}{2p^2} \int \frac{d^3k}{(2\pi)^3} \left(\frac{1}{(k-p)^2} - \frac{1}{k^2} + \frac{p^2}{k^2 (k-p)^2} \right). \tag{3.3.13}$$

So for a general vector numerator we can write:

$$\int \frac{d^3k}{(2\pi)^3} \frac{k\cdot l}{k^2 (k-p)^2} = Ap\cdot l, \tag{3.3.14}$$

completing the reduction.

3.4 Discussion

A trustworthy final result is forthcoming, so at present we cannot present a final expression. Nevertheless, we find with high confidence that this color factor is gauge invariant. This stands in contrast to our 1 loop results in the planar limit (see 3). Note that in the Abelian theory we nevertheless expect to see a δ function at forward momenta, so in the future it could be worthwhile look more carefully at how this color factor behaves near $\theta = 0$.

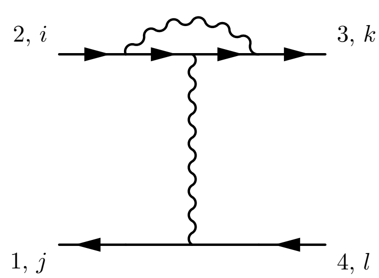
4 The Planar Scattering Amplitude

In much of the literature on the topic, including [19], computation is simplified greatly by the assumption $v \cdot s = 0$. For real v, s (in Minkowski space) this is only possible when s is space-like. Hence one can use this assumption to compute the 4-point correlation function and then go on-shell. Naively, there should be no problem in analytically continuing the result to time-like s. The modified crossing relation (1.4), however, casts doubt on that. Eq' (2.5.25) indicates that the results obtained for $v \cdot s = 0$ should hold in the adjoint and direct channels, but not necessarily the singlet. Our goal is then to compute the 1-loop amplitude off-shell and without any assumptions about v or s. This will constitute a check on [19]'s results, and perhaps show the emergence of (1.4) by direct calculation.

First, in 4.1 and 4.2, we'll review [19]'s results. We will then describe 1-loop our calculations and results. We find agreement with [19] up to some gauge-dependent corrections. In fact, we find that these corrections survive the on-shell limit, spoiling the amplitude's gauge invariance. However, these corrections exhibit some nontrivial structure - they take the form of a prefactor multiplying the **tree-level** amplitude. We believe this suggests that gauge invariance should be restored by dressing the amplitude with Wilson lines. Finally, we discuss our results in 4.4.

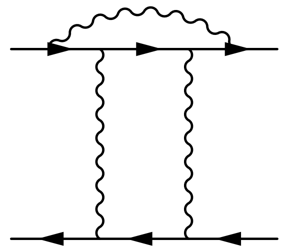
4.1 The Effective Exchange Interaction for $v \cdot s = 0$

In the planar limit, with fundamental matter, diagrams like:



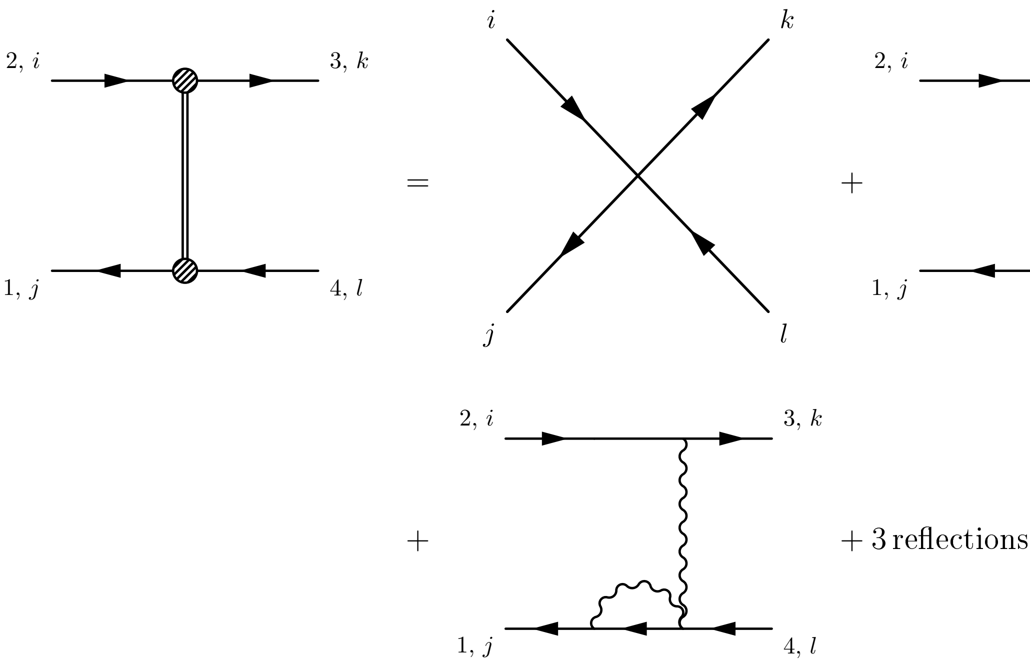
, (4.1.1)

are sub-leading in $\frac{1}{N}$. So are:

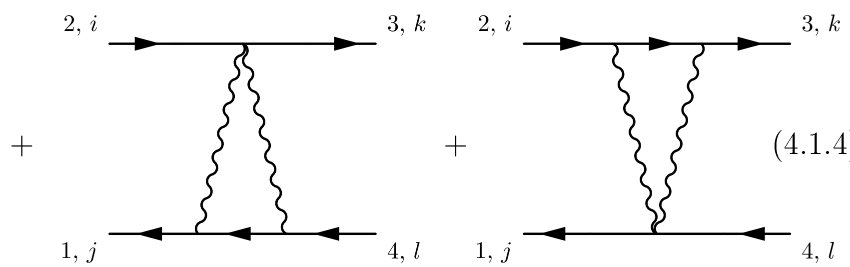


. (4.1.2)

In other words, if we think of time as flowing left-to-right, interactions "across time" are suppressed. To see this one can observe that most of those interactions, including (4.1.2) above, can be thought of for purposes of N power counting as gauge-propagator corrections (think of the 2 propagators leaving the lower line in the diagram as being one gauge propagator contributing to its self energy), which we have already observed are non-planar. This means that the interaction reduces to the sum of all ladder graphs. We follow [19] by looking first at one "rung" on the ladder, which we can think of as an effective exchange interaction. This is composed of the following diagrams:



(4.1.3)



(4.1.4)

It has both tree level and 1-loop contributions. The gauge dependence is, as usual, limited to the gauge propagators, and with an appropriate choice of shift for the loop momentum k the three propagator denominators in the problem can be written:

$$\frac{1}{k^+ - p_1^+}, \; \frac{1}{k^+ + p_4^+}, \; \frac{1}{p_1^+ + p_4^+} = \frac{1}{u^+}. \tag{4.1.5}$$

The linearity of these denominators means that partial fractioning can be used to write:

$$\frac{1}{k^+ - p_1^+}\frac{1}{k^+ + p_4^+} = -\frac{1}{p_1^+ + p_4^+}\left(\frac{1}{k^+ - p_1^+} - \frac{1}{k^+ + p_4^+}\right). \tag{4.1.6}$$

Also, with this choice we have 2 scalar propagators:

$$\frac{1}{k^2+m^2}, \frac{1}{\left(k-p_1-p_2\right)^2+m^2}=\frac{1}{\left(k-s\right)^2+m^2}. \tag{4.1.7}$$

With this in mind let us focus 2 of the propagator correction-like diagrams in (4.1.3). We will strip off the factor of $\left(-i\frac{2\pi\lambda}{N}\right)^2$ from the propagators, as well as the N from the color-loop in all of the following calculations. With that we have the integrand:

$$\begin{aligned}
& -\frac{\left(v\times\left(p_3-p_2\right)\right)^\mu}{u^+}\frac{1}{k^2+m^2}\left(\frac{\left(v\times\left(-k-p_1\right)\right)_\mu}{k^+-p_1^+}+\frac{-\left(v\times\left(p_4-k\right)\right)_\mu}{k^++p_4^+}\right) && (4.1.8)\\
= \; & \frac{p_3^+-p_2^+}{u^+}\frac{1}{k^2+m^2}\left(\frac{k^++p_1^+}{k^+-p_1^+}-\frac{k^+-p_4^+}{k^++p_4^+}\right) && (4.1.9)\\
= \; & 2\left(p_3^+-p_2^+\right)\frac{1}{k^2+m^2}\frac{k^+}{k^+-p_1^+}\frac{1}{k^++p_4^+}, && (4.1.10)
\end{aligned}$$

where in the second line we have used a variant of (3.2.17) which is simply:

$$\begin{aligned}
\left(v\times p_1\right)\cdot\left(v\times p_2\right) &= g^{\mu_1\mu_2}\epsilon_{\mu_1\nu_1\rho_1}v^{\nu_1}p_1^{\rho_1}\epsilon_{\mu_2\nu_2\rho_2}v^{\nu_2}p_2^{\rho_2} && (4.1.11)\\
&= g^{\perp\perp}\epsilon_{\perp j}p_1^j\epsilon_{\perp l}p_2^l && (4.1.12)\\
&= \left(-p_1^+\right)\left(-p_2^+\right) && (4.1.13)\\
&= v\cdot p_1 v\cdot p_2. && (4.1.14)
\end{aligned}$$

Hence these diagrams have combined to form triangle diagrams like those in (4.1.4). Let's examine one of those:

$$\begin{aligned}
& -\frac{1}{k^2+m^2}\frac{\left(v\times\left(-k-p_1\right)\right)_\mu}{k^+-p_1^+}\frac{-\left(v\times\left(p_4-k\right)\right)^\mu}{k^++p_4^+} && (4.1.15)\\
= \; & \frac{1}{k^2+m^2}\frac{k^++p_1^+}{k^+-p_1^+}\frac{k^+-p_4^+}{k^++p_4^+} && (4.1.16)\\
= \; & \frac{1}{k^2+m^2}+\frac{\left(k^++p_1^+\right)\left(k^+-p_4^+\right)-\left(k^+-p_1^+\right)\left(k^++p_4^+\right)}{\left(k^2+m^2\right)\left(k^+-p_1^+\right)\left(k^++p_4^+\right)} && (4.1.17)\\
= \; & \frac{1}{k^2+m^2}-2\left(p_1^+-p_4^+\right)\frac{k^+}{k^2+m^2}\frac{1}{k^+-p_1^+}\frac{1}{k^++p_4^+}. && (4.1.18)
\end{aligned}$$

We have extracted a "gauge independent part" $(k^2+m^2)^{-1}$. Let us combine the gauge dependent parts we have so far:

$$\frac{k^+}{k^2+m^2}\frac{1}{k^+-p_1^+}\frac{1}{k^++p_4^+}\left(2\left(p_3^+-p_2^+\right)-2\left(p_1^+-p_4^+\right)\right) \tag{4.1.19}$$

$$= \quad -4\frac{k^+}{k^2+m^2}\frac{1}{k^+-p_1^+}\frac{1}{k^++p_4^+}v\cdot s. \tag{4.1.20}$$

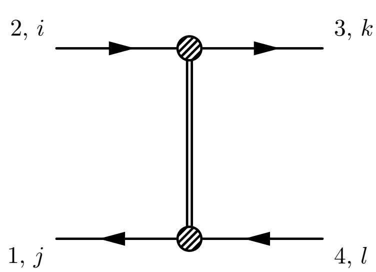
Hence we see that it falls off for $v\cdot s=0$. Meanwhile, the covariant part is:

$$\left(-i\frac{2\pi\lambda}{N}\right)^2 N\int\frac{d^3k}{(2\pi)^3}\frac{1}{k^2+m^2}=\left(\frac{2\pi\lambda}{N}\right)^2 N\frac{|m|}{4\pi}=\frac{\pi\lambda^2\,|m|}{N}. \tag{4.1.21}$$

The remaining 1-loop diagrams in (4.1.3), (4.1.4) are simply a reflection of those we dealt with so far and are identical with the replacement $(k^2+m^2)^{-1}\to\left((k-s)^2+m^2\right)^{-1}$. The 1-loop part of the effective exchange interaction, therefore, does not depend on external momenta at all, and is gauge invariant off-shell. It effectively corrects the contact interaction:

$$b_4\to b_4-2\pi\lambda^2\,|m|\equiv-\tilde{b}_4. \tag{4.1.22}$$

Thus we find:



$$= \quad \tilde{b}_4-i\frac{2\pi\lambda}{N}\frac{v\cdot(p_1-p_4)\times(p_3-p_2)}{v\cdot(p_1+p_4)} \tag{4.1.23}$$

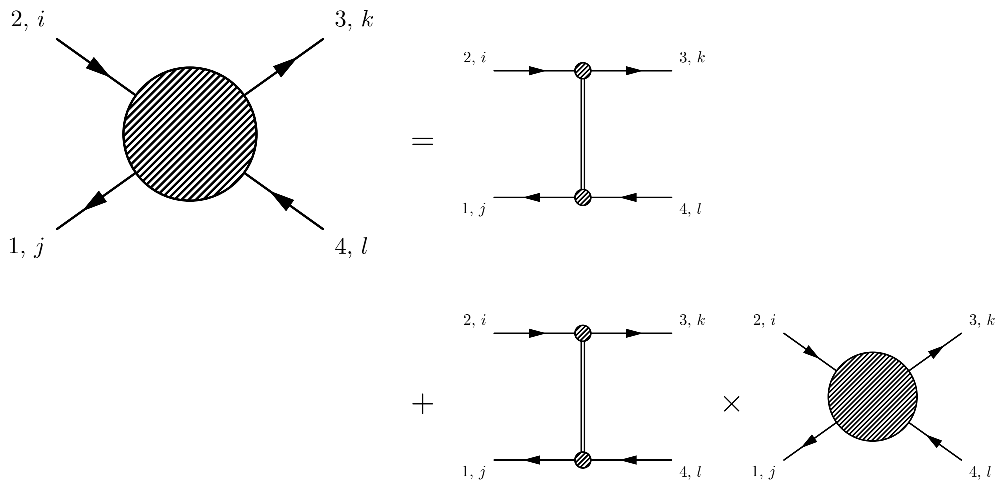
up to 2 additional terms that fall off under our assumption $v\cdot s=0$:

$$4\frac{(2\pi\lambda)^2}{N}\left(-\frac{k^+}{k^2+m^2}+\frac{k^+-s^+}{(k-s)^2+m^2}\right)\frac{1}{k^+-p_1^+}\frac{1}{k^++p_4^+}v\cdot s. \tag{4.1.24}$$

These will be shown to play a role when we compute the 1-loop correlator in 4.3.

4.2 The All-Loop Planar Amplitude

Using the effective interaction we can write an integral equation for the amplitude:



(4.2.1)

(4.2.2)

Which is solved in [19] to give the all-loop planar scattering amplitude:

$$
\begin{aligned}
i\mathcal{M} &= i\frac{4\pi\lambda}{N}E\left(p_1,p_2,p_3\right)\sqrt{\frac{SU}{T}} && (4.2.3)\\
&- i\frac{4\pi\lambda}{N}\sqrt{-S}\frac{\left(\tilde{b}_4-4\pi i\lambda\sqrt{-S}\right)+\left(\tilde{b}_4+4\pi i\lambda\sqrt{-S}\right)e^{-2i\lambda\arctan\left(\frac{\sqrt{-S}}{2m}\right)}}{-\left(\tilde{b}_4-4\pi i\lambda\sqrt{-S}\right)+\left(\tilde{b}_4+4\pi i\lambda\sqrt{-S}\right)e^{-2i\lambda\arctan\left(\frac{\sqrt{-S}}{2m}\right)}}. && (4.2.4)
\end{aligned}
$$

This result was obtained for $v\cdot s=0$ so s is spacelike, which means S is negative. The first term is simply the tree level contribution and is the only part of the amplitude odd in powers of λ. As discussed in 2.5.2, it enters the amplitude with a factor of the handedness $E\left(p_1,p_2,p_3\right)$, consistent with $\mathbb{Z}_2$ symmetry. That the second term is even in λ can be seen by negating λ and then multiplying numerator and denominator by $\exp\left(-2i\lambda\arctan\left(\frac{\sqrt{-S}}{2m}\right)\right)$. This also means that the analytic continuation of the second term to positive S is unambiguous, since we get $\sqrt{-S}\to\pm i\sqrt{S}$ but one can simultaneously replace $\lambda\to\mp\lambda$, canceling the ambiguity that under normal circumstances would be resolved by the $i\epsilon$ prescription.

Let's consider the $\lambda \to 0$ limit:

$$\ldots = \frac{1}{N}\frac{b_4}{1+b_4\frac{\arctan\left(\frac{\sqrt{-S}}{2m}\right)}{4\pi\sqrt{-S}}} \tag{4.2.5}$$

$$= \frac{1}{N}\sum_{n=0}^{\infty} b_4^{n+1}\left(-\frac{\arctan\left(\frac{\sqrt{-S}}{2m}\right)}{4\pi\sqrt{-S}}\right)^n . \tag{4.2.6}$$

The result matches our expectation from a theory with only a quartic self-interaction in the planar limit - a geometric sum of bubble diagrams.

The expression in (4.2.3) is merely the on-shell result. The full off-shell correlation function found in [19] is given by:

$$i\mathcal{M} = \exp\left(-2i\lambda\left(\arctan\left(\frac{2\sqrt{2p_1^+p_1^-+m^2}}{s_\perp}\right)-\arctan\left(\frac{2\sqrt{2p_4^+p_4^-+m^2}}{s_\perp}\right)\right)\right) \tag{4.2.7}$$

$$\times \left(4\pi i\lambda s_\perp\frac{p_1^+-p_4^+}{p_1^++p_4^+}+j\left(\left|s_\perp\right|,\lambda\right)\right), \tag{4.2.8}$$

$$j\left(\left|s_\perp\right|,\lambda\right) = 4\pi i\lambda\left|s_\perp\right|\frac{\left(\tilde{b}_4-4\pi i\lambda\left|s_\perp\right|\right)+\left(\tilde{b}_4+4\pi i\lambda\left|s_\perp\right|\right)e^{-2i\lambda\arctan\left(\frac{\left|s_\perp\right|}{2m}\right)}}{-\left(\tilde{b}_4-4\pi i\lambda\left|s_\perp\right|\right)+\left(\tilde{b}_4+4\pi i\lambda\left|s_\perp\right|\right)e^{-2i\lambda\arctan\left(\frac{\left|s_\perp\right|}{2m}\right)}}.$$

Note that since $s^+=0$:

$$\left|s_\perp\right| = \sqrt{s^2}=\sqrt{-S}, \tag{4.2.9}$$

and:

$$s_\perp\frac{p_1^+-p_4^+}{p_1^++p_4^+} = \frac{s_\perp\left(p_1^+-p_4^+\right)-s^+\left(p_1^\perp-p_4^\perp\right)}{p_1^++p_4^+} = \frac{\epsilon\left(v,p_1-p_4,s\right)}{v\cdot t}, \tag{4.2.10}$$

making the expression in (4.2.8) match (4.2.3). But what of of the prefactor (4.2.7)? It's important that in [19] the authors treat s as a real vector in Euclidean space, meaning that:

$$s^- = \left(s^+\right)^* = \left(v\cdot s\right)^* = 0, \tag{4.2.11}$$

so:

$$s = \left(0,0,s_\perp\right) = \left(0,0,\pm\sqrt{-S}\right). \tag{4.2.12}$$

When on-shell this gives:

$$\begin{aligned} 0 &= m^2 - m^2 && (4.2.13) \\ &= p_1^2 - p_2^2 && (4.2.14) \\ &= p_1^2 - (p_1 - s)^2 && (4.2.15) \\ &= 2p_1 \cdot s - s^2 && (4.2.16) \\ &= 2p_1^\perp \left(\pm\sqrt{-S}\right) + S && (4.2.17) \\ \Rightarrow \left(p_1^\perp\right)^2 &= \frac{S}{4}. && (4.2.18) \end{aligned}$$

A similar calculation leads to:

$$\begin{aligned} \Rightarrow \left(p_i^\perp\right)^2 &= \frac{S}{4},\ i = 1, \ldots, 4, && (4.2.19) \\ \Rightarrow \frac{2\sqrt{2p_i^+ p_i^- + m^2}}{s^\perp} &= i\mathrm{Sign}\left(s^\perp\right). && (4.2.20) \end{aligned}$$

Hence the prefactor (4.2.7) naively goes to:

$$\exp\left(i\mathrm{Sign}\left(s^\perp\right)\lambda\left(\arctan(i) - \arctan(i)\right)\right). \quad (4.2.21)$$

Although $\arctan(i)$ is divergent, the authors argue that we should interpret this factor as 1. Note that the function arctan also has a πn ambiguity, so hypothetically one could interpret the limit as:

$$e^{2\pi i n\lambda}, \quad (4.2.22)$$

for some integer n. Of course this isn't $\mathbb{Z}_2$ symmetric, but it resembles the trigonometric factors in the modified crossing relation (1.4). One could think of $\mathbb{Z}_2$ symmetrizing by averaging over $\lambda = \lambda, -\lambda$, to get:

$$\cos(2\pi n\lambda), \quad (4.2.23)$$

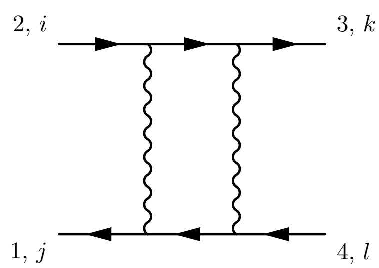
but this would only match the cosine in (1.4) for non-integer $n = \frac{1}{2}$ and it wouldn't be multiplying the predicted δ-function, but rather taking the place of $\frac{\sin \pi\lambda}{\pi\lambda}$. Hence the relation between (4.2.7) and modified crossing, if there is one, is unclear. However, the presence of such factors motivates us to carefully compute the off-shell 1-loop 4-point correlator away from the $v \cdot s = 0$ assumption, to which we currently proceed.

4.3 The 1-loop Planar Amplitude

To get an idea of what to expect, let us expand [19]'s to orders b_4^2, λ^2 (we know that terms odd in λ like $b_4\lambda$ will not be present):

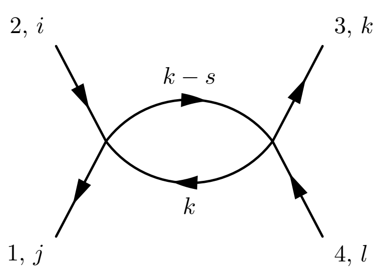
$$
\begin{aligned}
i\mathcal{M} &= \cdots + \frac{1}{N}\frac{\arctan\left(\frac{\sqrt{-S}}{2m}\right)}{4\pi\sqrt{-S}}b_4^2 + \frac{1}{N}\left(\frac{1}{\pi}\sqrt{-S}\arctan\left(\frac{\sqrt{-S}}{2m}\right)+\frac{m}{2\pi}\right)(2\pi\lambda)^2 && (4.3.1)\\
&+ \ \ldots && (4.3.2)
\end{aligned}
$$

We recognize the term $2\pi m\lambda^2$ as the 1-loop part of the effective exchange interaction (4.1.23). The missing $O\left(\lambda^2\right)$ diagram at 1-loop is the planar box diagram:



$$. \qquad (4.3.3)$$

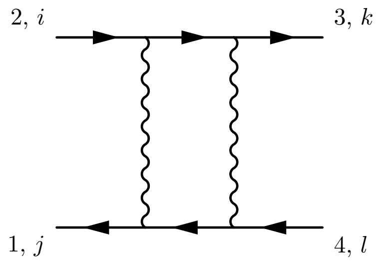
The $O\left(b_4^2\right)$ is also not very surprising, as it is simply the bubble integral:



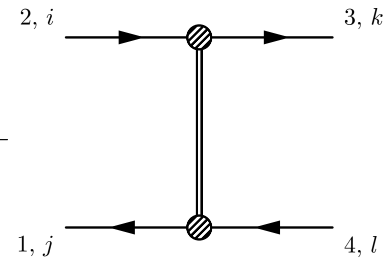
$$= \frac{b_4^2}{N}\frac{1}{4\pi\sqrt{-S}}\arctan\left(\frac{\sqrt{-S}}{2m}\right), \qquad (4.3.4)$$

which we compute in the appendix C. The remaining diagrams are:

$iM = \cdots +$

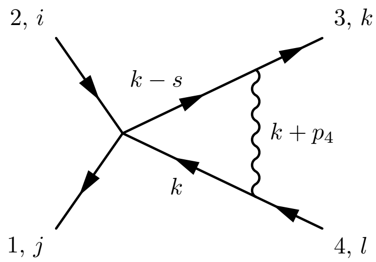


$+$

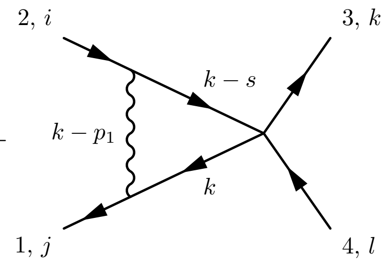


$\Big|_{\text{1-loop}}$

$+$



$+$

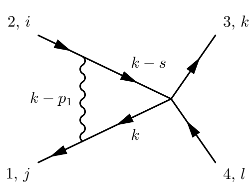


, (4.3.5)

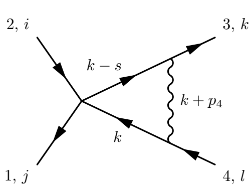
where in the first line we keep only the 1-loop order terms in the effective exchange interaction. Interestingly, the $O(\lambda^2)$ terms in (4.3.1) look roughly like what one would expect to obtain from shrinking all gauge propagators.

4.3.1 $O(b_4\lambda)$ integrals

The diagrams in (4.3.5) are $O(b_4\lambda)$ and should vanish, at least when $v \cdot s = 0$. Let's focus our attention on them[10]:



$$\propto I(p_1,p_2) = \begin{array}{l}\displaystyle\int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2+m^2}\frac{1}{(k-s)^2+m^2} \\ \displaystyle\times\frac{\epsilon(v,k+p_1,k-p_1-2p_2)}{v\cdot(k-p_1)}\end{array}, \tag{4.3.6}$$



$$\propto -I(-p_4,-p_3) = -\left(\begin{array}{l}\displaystyle\int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2+m^2}\frac{1}{(k-s)^2+m^2} \\ \displaystyle\times\frac{\epsilon(v,k-p_4,k+p_4+2p_3)}{v\cdot(k+p_4)}\end{array}\right). \tag{4.3.7}$$

[10] Recall that $s = p_1 + p_2 = -p_3 - p_4$

Since the prefactor is $(-b_4)(-2\pi i\lambda)\frac{1}{N}$, $\mathbb{Z}_2$ symmetry requires the integral to be parity-odd, which it indeed is. We perform these integrals in the appendix B.1 in Lorenzian signature,[11] where we find:

$$I=-\frac{\epsilon(v,p_1,p_2)}{2\pi\sqrt{A(p_1,p_2,m)}}\arctan\left(\frac{\sqrt{A(p_1,p_2,m)}}{m\left(p_1^+-p_2^+\right)}\right), \tag{4.3.10}$$

with:

$$A(p_1,p_2,m)=\epsilon(v,p_1,p_2)^2-s^+\left(\left(m^2+p_2^2\right)p_1^++\left(m^2+p_1^2\right)p_2^+\right). \tag{4.3.11}$$

It can easily be seen that either the on-shell condition **or** $s^+=0$ are sufficient to obtain:

$$A\to(\epsilon(v,p_1,p_2))^2, \tag{4.3.12}$$

giving:

$$I=-\frac{1}{2\pi}\arctan\left(\frac{\epsilon(v,p_1,p_2)}{mv\cdot(p_1-p_2)}\right). \tag{4.3.13}$$

We find that for $v\cdot s=0$ this reduces to:

$$I=\operatorname{sign}\left(s^\perp\right)\frac{\arctan\left(\frac{\sqrt{-S}}{2m}\right)}{2\pi}. \tag{4.3.14}$$

The integral loses its p_1-dependence and becomes almost Lorenz-invariant, except for the pre-factor which is necessary for parity-oddness. The other integral (when expressed using p_4,s) is simply obtained by replacing $p_1\to-p_4$, but since in the $v\cdot s$ case p_1 is absent, the two integrals match. Then the overall minus sign in (4.3.7) ensures the 2 diagrams cancel.

In general these terms do not cancel one another giving a final answer (reintroducing the coefficients):

$$i\mathcal{M}\Big|_{O(b_4\lambda)}=i\frac{b_4\lambda}{N}\left(\arctan\left(\frac{\epsilon(v,p_1,p_2)}{mv\cdot(p_1-p_2)}\right)+\arctan\left(\frac{\epsilon(v,p_3,p_4)}{mv\cdot(p_4-p_3)}\right)\right). \tag{4.3.15}$$

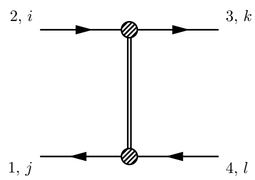
[11] There is a relative factor of -2 since (using anti-symmetry):

$$\epsilon(v,k+p_1,k-p_1-2p_2)=-2\epsilon(v,k+p_1,p_1+p_2)=-2\epsilon(v,k+p_1,s) \tag{4.3.8}$$

$$\epsilon(v,k-p_4,k+p_4+2p_3)=2\epsilon(v,k-p_4,p_4+p_3)=-2\epsilon(v,k-p_4,s) \tag{4.3.9}$$

4.3.2 $O\left(\lambda^2\right)$ integrals

Let's move on to the $O\left(\lambda^2\right)$ part of the correlator. We know from 4.1 that the 1-loop part of the effective exchange interaction evaluates to:

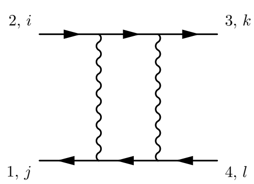


$$\Big|_{\text{1-loop}} = \frac{2\pi\lambda^2\left|m\right|}{N}\left(\text{gauge invariant part}\right)$$

$$+ \quad 4\frac{\left(2\pi\lambda\right)^2}{N}\left(-\frac{k^+}{k^2+m^2}+\frac{k^+-s^+}{\left(k-s\right)^2+m^2}\right) \tag{4.3.16}$$

$$\times \quad \frac{1}{k^+-p_1^+}\frac{1}{k^++p_4^+}v\cdot s. \tag{4.3.17}$$

The "gauge dependent" parts proportional to $v\cdot s$ are integrated in the appendix B.1.1 but we won't use that result here. We'll keep them in mind as we look at the remaining diagram, the box:



$$= -\frac{\left(2\pi\lambda\right)^2}{N}\int\frac{d^3k}{\left(2\pi^3\right)}\frac{1}{k^2+m^2}\frac{1}{\left(k-s\right)^2+m^2} \tag{4.3.18}$$

$$\times \quad \frac{\epsilon\left(v,k+p_1,k-p_1-2p_2\right)}{k^+-p_1^+}\frac{\epsilon\left(v,k+2p_3+p_4,k-p_4\right)}{k^++p_4^+} \tag{4.3.19}$$

$$= 4\frac{\left(2\pi\lambda\right)^2}{N}\int\frac{d^3k}{\left(2\pi^3\right)}\frac{1}{k^2+m^2}\frac{1}{\left(k-s\right)^2+m^2} \tag{4.3.20}$$

$$\times \quad \frac{\epsilon\left(v,k+p_1,s\right)}{k^+-p_1^+}\frac{\epsilon\left(v,k-p_4,s\right)}{k^++p_4^+}. \tag{4.3.21}$$

Note that for $v\cdot s=0$ we have:

$$\frac{\epsilon\left(v,k+p_1,s\right)}{k^+-p_1^+}\frac{\epsilon\left(v,k-p_4,s\right)}{k^++p_4^+} = s_\perp^2\frac{k^++p_1^+}{k^+-p_1^+}\frac{k^+-p_4^+}{k^++p_4^+} \tag{4.3.22}$$

$$= -2S\frac{k^+u^+}{\left(k^+-p_1^+\right)\left(k^++p_4^+\right)}-S, \tag{4.3.23}$$

where we have used the fact that $-S = s^2 = s_\perp^2$. The gauge-invariant term $-S$ when plugged back into the integrand gives rise to another bubble diagram which integrates to:

$$4\frac{(2\pi\lambda)^2}{N}(-S)\frac{1}{4\pi\sqrt{-S}}\arctan\left(\frac{\sqrt{-S}}{2m}\right) = \frac{(2\pi\lambda)^2}{N}\frac{1}{\pi}\sqrt{-S}\arctan\left(\frac{\sqrt{-S}}{2m}\right), \tag{4.3.24}$$

consistent with (4.3.1). We expect the gauge dependent term to integrate to 0. One can see that it does by integrating out k^-. As discussed in the computation of the triangle diagram in B.1.2, the k^- integration localizes the range of k^+ to $(0, s^+)$. As $s^+ \to 0$ the range shrinks to 0, but there is a non-zero contribution nevertheless since for $k^+ = 0$ the integrand becomes finite and k^- independent, leading to a $\delta(0)$. In the case of the box, the "gauge-dependent" part of the integrand is proportional to k^+u^+ and so vanishes. However, we will be more thorough and fully integrate the box for $s^+ \neq 0$.

It will be useful to add and subtract the "bubble":

$$-4\frac{(2\pi\lambda)^2}{N}S\int\frac{d^3k}{(2\pi^3)}\frac{1}{k^2+m^2}\frac{1}{(k-s)^2+m^2}, \tag{4.3.25}$$

in anticipation of its "popping out" anyway. Thus we are led to consider the integral:

$$4\frac{(2\pi\lambda)^2}{N}\int\frac{d^3k}{(2\pi^3)}\frac{1}{k^2+m^2}\frac{1}{(k-s)^2+m^2} \tag{4.3.26}$$

$$\times\left(\frac{\epsilon(v, k+p_1, s)}{k^+-p_1^+}\frac{\epsilon(v, k-p_4, s)}{k^++p_4^+}+S\right). \tag{4.3.27}$$

The integrand is $O\left(k_\perp^{-2}\right), O\left((k^-)^{-2}\right)$ and $O\left((k^+)^{-3}\right)$ and so is naively UV convergent. However, a subtlety arises if one integrates out k^- using contour integration. One of the scalar propagators "shrinks" in the residue, making it so that the integrand becomes $O\left(k_\perp^0\right)$ - linearly divergent. We wish to find a simpler integral to subtract from the box in order to cancel the $\sim k_\perp^2(s^+)^2$ term in the numerator. Since the scalar propagators contain terms of order $k_\perp^2$, one can imagine subtracting a "triangle" like:

$$\sim\int\frac{d^3k}{(2\pi^3)}\frac{(s^+)^2}{k^2+m^2}\frac{1}{k^+-p_1^+}\frac{1}{k^++p_4^+}\text{ or }\sim\int\frac{d^3k}{(2\pi^3)}\frac{(s^+)^2}{(k-s)^2+m^2}\frac{1}{k^+-p_1^+}\frac{1}{k^++p_4^+}. \tag{4.3.28}$$

However, this introduces powers of k^- into the numerator, which upon integration are evaluated at the residue $k^- = O\left(k_\perp^2\right)$, so the integrand is still UV divergent. It turns out that a unique combination triangle diagrams cancels the $k_\perp^2(s^+)^2$ term in the numerator **without** introducing powers of k^-, and that combination is precisely the gauge dependent terms

(4.3.16) that we said we should keep in mind. Thus we are led to consider the integral:

$$\int \frac{d^3k}{(2\pi^3)} \frac{1}{k^2+m^2} \frac{1}{(k-s)^2+m^2} \tag{4.3.29}$$

$$\times \left(\frac{\epsilon(v,k+p_1,s)}{k^+-p_1^+} \frac{\epsilon(v,k-p_4,s)}{k^++p_4^+} - s^2 \right) \tag{4.3.30}$$

$$+ \int \frac{d^3k}{(2\pi^3)} \left(-\frac{k^+}{k^2+m^2} + \frac{k^+-s^+}{(k-s)^2+m^2} \right) \frac{1}{k^+-p_1^+} \frac{1}{k^++p_4^+} s^+, \tag{4.3.31}$$

where we have divided out the prefactor $4\frac{(2\pi\lambda)^2}{N}$. Now integration of k^- only leads to a logarithmic divergence $O\left(k_\perp^{-1}\right)$ which gives rise to a finite "arc at infinity" contribution. This integral should account for any difference between our result and the gauge invariant result (4.3.1).

One can integrate this in exactly the same way as in B.1.2 (in Lorenzian signature). Note also that thanks to the partial fraction relation:

$$\frac{1}{k^+-p_1^+} \frac{1}{k^++p_4^+} = \frac{1}{t^+} \left(\frac{1}{k^+-p_1^+} - \frac{1}{k^++p_4^+} \right), \tag{4.3.32}$$

the integral can be thought of as a sum of two triangle integrals, one related to the other by a simple relabeling of momenta. Indeed, upon integration (in Lorenzian signature) we get:

$$\ldots = \frac{\arctan\left(\frac{\sqrt{A(p_1,p_2,m)+i\epsilon}}{mv\cdot(p_2-p_1)} \right)}{4\pi t^+ \sqrt{A(p_1,p_2,m)+i\epsilon}} \left(A(p_1,p_2,m) - \epsilon(v,p_1,p_2)\,\epsilon(v,p_3,p_4) \right) \tag{4.3.33}$$

$$+ \left((p_1,p_2) \leftrightarrow (-p_4,-p_3) \right), \tag{4.3.34}$$

where:

$$A(p_1,p_2,m) = (\epsilon(v,p_1,p_2))^2 - s^+\left(\left(m^2+p_2^2\right)p_1^+ + \left(m^2+p_1^2\right)p_2^+ \right). \tag{4.3.35}$$

Note the similarity of this result to the one obtained at $O(b_4\lambda)$ - (4.3.10). This similarity will become even more striking on-shell and for $v\cdot s=0$. As in 4.3.1 either the on-shell condition **or** $s^+=0$ are sufficient to obtain:

$$A \to (\epsilon(v,p_1,p_2))^2, \tag{4.3.36}$$

giving:

$$\ldots = \frac{4\pi\lambda^2}{N}\left(\arctan\left(\frac{\epsilon\left(v,p_1,p_2\right)}{mv\cdot\left(p_2-p_1\right)}\right)+\arctan\left(\frac{\epsilon\left(v,p_3,p_4\right)}{mv\cdot\left(p_4-p_3\right)}\right)\right)\frac{\epsilon\left(s,v,u\right)}{v\cdot t} \quad (4.3.37)$$

On shell this changes slightly:

$$\frac{\epsilon\left(s,v,u\right)}{v\cdot t}\to\frac{\epsilon\left(s,t,u\right)}{T}, \quad (4.3.38)$$

and for $v\cdot s=0$ (equivalently $p_2^+=-p_1^+$) it is:

$$\frac{\epsilon\left(v,p_1,p_2\right)}{mv\cdot\left(p_2-p_1\right)} = \frac{p_1^+p_2^\perp+p_1^+p_1^\perp}{m\left(-2p_1^+\right)} \quad (4.3.39)$$

$$= \frac{s^\perp}{2m} \quad (4.3.40)$$

$$= -\text{sign}\left(s^\perp\right)\frac{\sqrt{-S}}{2m}, \quad (4.3.41)$$

but:

$$\frac{\epsilon\left(v,p_3,p_4\right)}{mv\cdot\left(p_4-p_3\right)}=\text{sign}\left(s^\perp\right)\frac{\sqrt{-S}}{2m}, \quad (4.3.42)$$

making it so that the arctangents cancel.

4.3.3 Final on-shell result

Putting it all together we get:

$$\begin{aligned} i\mathcal{M}_1 &= \frac{1}{N}\frac{\arctan\left(\frac{\sqrt{-S}}{2m}\right)}{4\pi\sqrt{-S}}\left(b_4^2-4\pi\lambda^2S\right)+2\pi\lambda^2m \\ &- i\lambda\left(\arctan\left(\frac{\epsilon\left(v,p_1,p_2\right)}{mv\cdot\left(p_2-p_1\right)}\right)+\arctan\left(\frac{\epsilon\left(v,p_3,p_4\right)}{mv\cdot\left(p_4-p_3\right)}\right)\right)i\mathcal{M}_0, \end{aligned} \quad (4.3.43)$$

where $i\mathcal{M}_1$ is the 1-loop amplitude and $i\mathcal{M}_0$ is the tree-level amplitude found in 2.5.5:

$$i\mathcal{M}_0=\frac{4\pi i\lambda}{N}\frac{\epsilon\left(s,t,u\right)}{T}-\frac{b_4}{N}. \quad (4.3.44)$$

We've obtained a result that matches [19] up to gauge dependent terms. The form of said terms indicates some structure. In particular, it suggests that to restore gauge invariance one must dress the correlation functions with Wilson lines connecting particles 1 to 2 and 3 to 4. It is possible that that the interaction of these Wilson lines with the scattering process will give rise to a term to cancel (4.3.43). We are also missing a gauge dependent term that vanishes on shell but is present in the expansion of exponential prefactor in [19]'s off-shell result (4.2.7):

$$-2i\lambda\left(\arctan\left(\frac{2\sqrt{2p_1^+p_1^-+m^2}}{s_\perp}\right)-\arctan\left(\frac{2\sqrt{2p_4^+p_4^-+m^2}}{s_\perp}\right)\right)i\mathcal{M}_0. \tag{4.3.45}$$

Although this has a similar form to (4.3.43), it is quite distinct. Most significantly, it vanishes an shell, whereas (4.3.43) vanishes for $s^+=0$. The reason for this discrepancy likely has to do with **sub-gauge conditions**, that is - residual gauge freedom in light-cone gauge. The Leibbrandt-Mandelstam propagator prescription corresponds to a particular sub-gauge. For a discussion on various prescriptions and their relation to sub-gauge conditions we refer to [10]. To obtain (4.3.43) we used an integration procedure (see appendix B.1.2) in which the lightcone "energy" k^- is integrated first. Although naively we have used the Leibbrandt-Mandelstam prescription, it is claimed in [8] that this integration procedure is equivalent to the use of a prescription not involving k^-, e.g. the **principal value** (PV) prescription which in Lorenzian signature has the form:

$$\frac{1}{k^+}\to\frac{1}{2}\frac{1}{k^++i\epsilon}+\frac{1}{2}\frac{1}{k^+-i\epsilon}=\frac{k^+}{(k^+)^2+\epsilon^2}. \tag{4.3.46}$$

Hence we have used a different sub-gauge than that used in [19]. The discrepancy therefore does not contradict the calculations in [19] since it vanishes in the combined on-shell and $v\cdot s=0$ limits. However, it is not consistent with the continuation of [19] for $v\cdot s\neq 0$, indicating a breakdown of gauge-invariance.

We can investigate the discrepancy by focusing on the integral that showed up in 4.3.1:

$$\int\frac{d^3k}{(2\pi)^3}\frac{1}{k^2+m^2}\frac{1}{(k-s)^2+m^2}\frac{\epsilon\left(v,k+p_1,k-p_1-2p_2\right)}{v\cdot(k-p_1)}. \tag{4.3.47}$$

This can be thought of as the 1-loop form factor for the spin-0 current $J_0=\bar{\phi}\phi$ to create a particle-antiparticle pair with combined momentum s, and so should be gauge-invariant on-shell. If evaluated in a covariant gauge it has the form:

$$\sim\int\frac{d^3k}{(2\pi)^3}\frac{1}{k^2+m^2}\frac{1}{(k-p_1-p_2)^2+m^2}\frac{\epsilon\left(k,p_1,p_2\right)}{(k-p_1)^2}, \tag{4.3.48}$$

and so vanishes by Lorenz symmetry (the integrand is odd in the component of k along $p_1\times p_2$). Hence if we subtract 0 in this form from (4.3.47), and use the Schouten identity and the on-shell condition, (4.3.47) reduces to:

$$\sim \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2+m^2} \frac{1}{(k-p_1)^2} \frac{\epsilon(v,k,p_1)}{v\cdot(k-p_1)} + (p_1 \to p_2) \,. \tag{4.3.49}$$

These integrals are much simpler to integrate in the Leibbrandt prescription and together yield 0 on-shell. The integrals vanish individually when k^- is integrated first (PV prescription). This means that the on-shell arc-tangents in (4.3.43) are not reproduced if one uses the Schouten identity prior to integration, leading to a contradiction.

How can this be? Leibbrandt claims [21, 20] that the PV prescription for light-cone gauge integrals is inconsistent and demonstrates that they give inconsistent results and violate the Ward and BRS identities in [21]. However, Capper claims that consistent results can be obtained if one uses the "method of exponentiation of propagators" - a slight variation on the Feynman trick [8]. We find that this method of integration likewise produces the arc-tangent in (4.3.43), and so likewise gives inconsistent results.

Furthermore, we find that if the covariant integral (4.3.48) is integrated in light-cone coordinates, then it doesn't vanish as expected, but rather evaluates to an expression similar to the arctangent in (4.3.43). In summary, it appears that integration in light-cone coordinates has many subtleties and one has to work more carefully to make them well-defined.

What of the modified crossing relation (1.4)? At 1-loop we expect a term:

$$-\frac{(\pi\lambda)^2}{2N} 8\pi E \delta(\theta) \,, \tag{4.3.50}$$

where E is the energy and we have used the expression (2.5.21) for the identity matrix. Could this δ function be hiding somewhere in our result? Note that arctan has an $in\pi$ ambiguity (for integer n). The "branch" we implicitly chose in (4.3.43) is ostensibly the unique branch satisfying $\mathbb{Z}_2$ symmetry, since:

$$i\pi n \frac{4\pi\lambda^2}{N} \frac{\epsilon(s,t,u)}{T}, \tag{4.3.51}$$

is $\mathbb{Z}_2$-odd. However, we can rewrite this for small scattering angle as:

$$\frac{\epsilon(s,t,u)}{T} = E\cot\left(\frac{\theta}{2}\right) \approx 2E\frac{1}{\theta}. \tag{4.3.52}$$

This can be made $\mathbb{Z}_2$ symmetric by writing:

$$i\pi n\frac{8\pi\lambda^2}{N}E\left(\frac{1}{\theta+i\epsilon}-\frac{1}{\theta-i\epsilon}\right) \tag{4.3.53}$$

$$= \quad 4n\frac{(\pi\lambda)^2}{2N}8\pi E\delta\left(\theta\right). \tag{4.3.54}$$

Although the coefficient isn't quite right for any choice of n, it's possible that a careful calculation paying more attention to the $i\epsilon$ prescription would give rise to such a term. However, this is just a possible scenario.

A clue as to how this might happen comes from looking at the Wilson line dressed version of the form factor (4.3.47). If one attaches 2 Wilson lines, one to each produced particles, that head off to infinity in a specified direction n, the resulting 1-loop graphs have "Eikonal" factors of the form:

$$\frac{1}{n\cdot k}, \tag{4.3.55}$$

where k is the loop momentum. Note that if one takes $n=v$ then the Wilson lines are trivial (by virtue of the choice of gauge $v\cdot A=0$) and so what remains is simply the "undressed" amplitude! In this case the dependence of the "undressed" amplitude on v no longer reflects gauge-dependence but rather dependence on the Wilson lines. This gives us a hint as to what the a Wilson line dressed amplitude of this type (where all Wilson lines are infinite and parallel) should look like in general - it should have the same form as the undressed amplitude in light-cone gauge (or more generally axial gauge $v^2\neq 0$) but with v replaced by n. Indeed, the Eikonal factor has a form similar to the spurious poles of the gauge propagator in axial gauge (in Chern-Simons as well Yang-Mills theory).

We find that indeed this is the case for the form factor (4.3.47). The diagrams involving interaction between the Wilson lines and the propagating particles combine with the single triangle diagram (4.3.47) present in the undressed amplitude in such a way that by the Schouten identity one remains with:

$$\int\frac{d^3k}{(2\pi)^3}\frac{1}{k^2+m^2}\frac{1}{(k-s)^2+m^2}\frac{\epsilon\left(n,k+p_1,k-p_1-2p_2\right)}{n\cdot(k-p_1)}. \tag{4.3.56}$$

This "replacement" carries through if one works in covariant gauge or in any other axial gauge.

However, now one must find the regularization prescription for the spurious pole due to the Eikonal factor. Requiring the Wilson lines to be invariant under gauge transformations at infinity gives the prescription:

$$\frac{1}{n\cdot k}\to\frac{1}{n\cdot k-i\epsilon} \tag{4.3.57}$$

This prescription can be decomposed into "Eigenvectors" of parity:

$$\frac{1}{n\cdot k-i\epsilon} = \frac{1}{2}\frac{\mathrm{PV}}{n\cdot k}+\frac{1}{2}\left(\frac{1}{n\cdot k-i\epsilon}-\frac{1}{n\cdot k+i\epsilon}\right) \tag{4.3.58}$$

$$= \frac{1}{2}\frac{\mathrm{PV}}{n\cdot k}-i\delta\left(n\cdot k\right) \tag{4.3.59}$$

And this is precisely the kind of δ-function we need to produce the contact interaction term in the modified crossing relation. More work is needed to understand this.

Although this is a possible scenario by which the δ-function term might arise, we stress that it is currently speculation and that the above discussion is not material to the rest of this thesis.

4.4 Discussion

We have found that the naive scattering amplitude contains gauge-dependent terms, which take the form of a factor:

$$-i\lambda\left(\arctan\left(\frac{\epsilon\left(v,p_1,p_2\right)}{mv\cdot\left(p_2-p_1\right)}\right)+\arctan\left(\frac{\epsilon\left(v,p_3,p_4\right)}{mv\cdot\left(p_4-p_3\right)}\right)\right), \tag{4.4.1}$$

multiplying the tree-level amplitude. Scattering amplitudes are observables, and hence one normally assumes they should be gauge invariant, so what happened? The amplitude roughly corresponds to a correlation function:

$$\delta^i_j\delta^k_l\left\langle\bar{\phi}^j\left(x_2\right)\phi_i\left(x_1\right)\bar{\phi}^l\left(x_4\right)\phi_k\left(x_3\right)\right\rangle, \tag{4.4.2}$$

where the positions are taken to be "at infinity" (far removed from one another). Under a gauge transformation $\exp\left(i\Lambda\left(x\right)\right)$, this will transform non-trivially, since the particles are at different positions, unless Λ falls off at ∞. To make this gauge invariant, one must take a closer look at the Kronecker δ-s in this expression. They project onto the case that the particles are of the same color. Of course, the only real way to compare the color structures of particles that don't coincide, is to parallel transport one to the other, and only then project!

This is precisely the purpose of Wilson lines:

$$\delta^i_j \bar{\phi}^j (x_2) \phi_i (x_1) \to W_{\mathrm{F}} (\gamma_{21})^i_{Wj} \bar{\phi}^j (x_2) \phi_i (x_1) \tag{4.4.3}$$

This leads us to consider Wilson-line dressed correlation functions, as described in 2.3.4. Usually in gauge theories, one assumes that the connection is flat at infinity so that one can pick a gauge where all Wilson lines trivialize. However, the anyonic statistics induced by the CS gauge field amount to a topological interaction and as such don't "care" how far apart we take the particles!

Indeed, the form of (4.3.43) suggests that it could be obtained from the Wilson lines $W(\gamma_{21}), W(\gamma_{43})$, given that is a sum of terms where one depends on $p_{1,2}$ and the other on $p_{3,4}$.

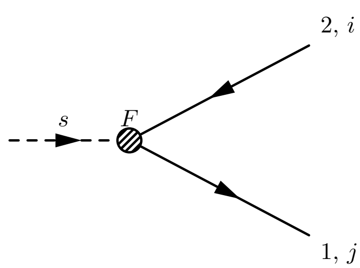
Another mystery that remains is the discrepancy between our result and that of [19], although that vanishes in the combined on-shell and $v \cdot s = 0$ limit.

5 Unitarity and Form Factors

In this section we'll show that the modified crossing relation (1.4) is crucial for consistency with some previous results regarding the phase of the S-matrix[9] and certain form factors. The motivation for derives from the phase $\exp\left(i\pi\lambda/2\right)$ of the form factors computed in [3], and their similarity to trigonometric factors $\cos/\mathrm{sinc}\left(\pi\lambda\right)$ found in (1.4).

Let us recall what form factors are. Given some local operator $\mathcal{F}$ on the theory's Hilbert space, a **form factor** is simply the matrix element of this operator between some in-state and out-state (scattering states). We will focus on vacuum-to-out-state form factors, which can be thought of as the amplitude with which the operator creates a particular out-state. For instance, to make contact with our scattering notation, the form factor to create a particle-antiparticle state with center of mass energy squared $S = -s^2 = E^2$ (in Euclidean or mostly-plus signatures) is given by:

$F \equiv$



$$= {}_{\text{out}}\left\langle P\left(p_1\right)_j, A\left(p_2\right)^i \mid \mathcal{F}\left(s\right) \mid 0\right\rangle . \tag{5.1}$$

These form factors have been shown in [9] to satisfy:[12]

$$F\left(p_i\right) = e^{-i\pi D} R F^*\left(-p_i\right) = e^{-i\pi\left(D-\Delta_0\right)} R F^*\left(p_i\right), \tag{5.2}$$

where R is a spatial reflection and $D = \sum_i p^\mu \frac{\partial}{\partial p^\mu}$ is the dilatation operator and Δ_0 is the mass-dimension of the operator (that is, scaling dimension without quantum corrections). $D - \Delta_0$ is the anomalous dimension. The operator $\exp\left(-i\pi D\right)$ has the effect of negating all the momenta. The form factor is computed using a small positive imaginary part which enforces time-ordering. A negative imaginary part would give anti-time-ordering, hence there is a branch cut on the real energy axis. For this reason $e^{-i\pi D}$ relates the anti-time-ordered process at negative momenta to the time ordered process at positive momenta, via analytic continuation (see figure 9).

[12] In [9] the operation R is missing. There is a remark that the identity should include CPT in order to cancel the flip of momenta from $e^{-i\pi D}$. In 2+1 dimensions, the CPT is not a symmetry and must be replaced by CRT, which motivates the extra R in the above.

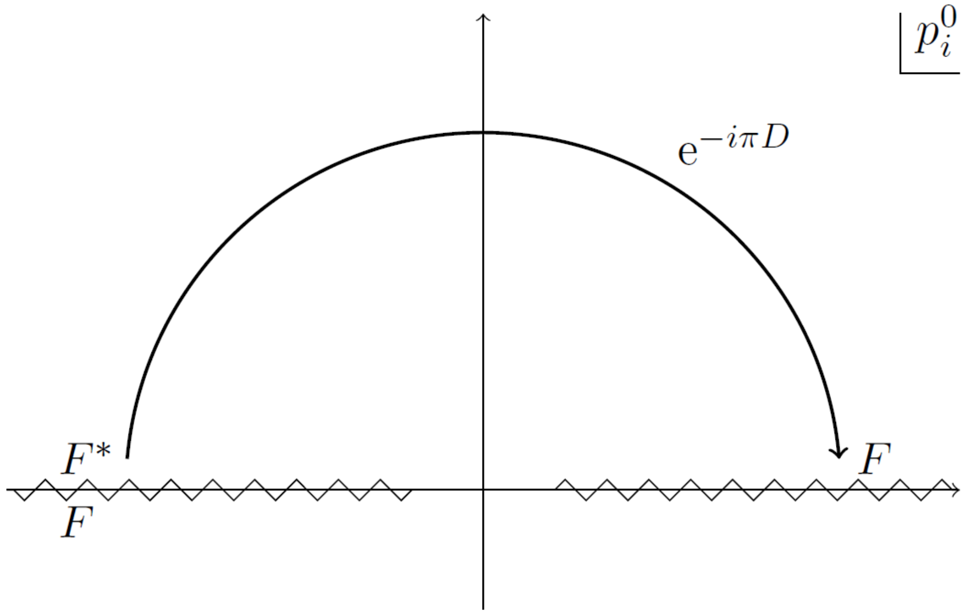


FIGURE 9: Analytic continuation induced by the operation $e^{-i\pi D}$ in the complex energy-plane of a participating particle. The form factor is computed using a small positive imaginary part which enforces time-ordering. A negative imaginary part would give anti-time-ordering, hence there is a branch cut on the real axis. For this reason $e^{-i\pi D}$ relates the anti-time-ordered process at negative momenta to the time ordered process at positive momenta. Picture credit: Caron-Huot & Wilhelm[9]

For the form factors J_n we'll be concerned with $D = n + d - 1$ and receives no quantum corrections ($D = \Delta_0$) at leading order in N[22]. Hence we should have:

$$J_n = RJ_n^*. \tag{5.3}$$

We'll see based on results from [3] that the form factor to emit particles with relative angle θ is:

$$J_n \propto e^{in\theta + i\frac{\pi\lambda}{2}}. \tag{5.4}$$

Since in Chern-Simons theory we must accompany R with a negation of λ (see 2.2.3), this appears to be consistent with (5.3). Eq' (5.4) also indicates that the form factor to emit particles with angular momentum n (see (5.2)) is:

$$J_n \propto e^{i\frac{\pi\lambda}{2}}. \tag{5.5}$$

This becomes relevant to another result from [9]. There, it was found that a variant of the optical theorem implies:

$$F = SF^*, \tag{5.6}$$

with S the S-matrix. Hence we expect the phase of the S-matrix θ (this notion will be made precise in 5.2) to satisfy:

$$e^{i\frac{\pi\lambda}{2}} = e^{i\theta - i\frac{\pi\lambda}{2}} \Rightarrow \theta = \pi\lambda. \tag{5.7}$$

This will turn out to depend on the modified crossing relation (1.4). In this chapter, we somewhat haphazardly analytically continue results obtained for $v \cdot s = 0$ to timelike s. This calls for a more careful 1-loop calculation, similar to the one we did in 4.3. Such a calculation will have the schematic form of a triangle diagram much like (4.3.6), and so we should expect the same kind of on-shell gauge dependent terms to arise. Potential future work could focus on such a form factor, to see what might restore gauge invariance for $v \cdot s \neq 0$.

5.1 Form Factors of Higher-Spin Currents

Higher-spin currents are an infinite set of traceless (w.r.t. the metric tensor) operators $\{J_n^{\mu_1,\mu_2,\ldots,\mu_n}\}_{n=0,1,\ldots}$ of increasing tensor rank - and therefore spin. They are described more thoroughly in Section 4 of [4] where they arise in the context of the large N limit of $\mathcal{N} = 2$ super-conformal Chern-Simons-matter theory. Importantly, they are color singlets (e.g. $J_0 =$

$\bar{\phi}\phi$) and therefore can only create particles in the singlet channel:

$$_{\text{out}}\left\langle P\left(p_1\right)_j, A\left(p_2\right)^i \mid J_n^{\mu_1,\mu_2,\ldots,\mu_n}\left(s\right) \mid 0\right\rangle \equiv \delta_j^i \delta^3\left(s-p_1-p_2\right) V_n^{\mu_1,\mu_2,\ldots,\mu_n}\left(p_1,s\right). \tag{5.1.1}$$

This form factor has been computed in [3] for the all-plus case $\mu_i = +$ together with the assumption $v \cdot s = s^+ = 0$. This was computed in a different theory then the one we've been working with:

1. The particles are massless.
2. The quartic interaction is missing $b_4 = 0$.
3. There is a sextic interaction $\frac{b_6}{3!N^2}\left(\bar{\phi}\phi\right)^3$.
4. The theory is **conformally** invariant.

Nevertheless the planar all-loop 4-point correlator is computed in [3] for $v \cdot s = 0$ and can be shown to match with [19]'s result for $m = b_4 = 0$. The form factor is given by:

$$V_n^{+,+,\ldots,+}\left(p_1,s\right) = \alpha_n\left(p_1^+\right)^n \exp\left(2i\lambda\left(\arctan\left(2\frac{\Lambda}{\sqrt{-S}}\right) - \arctan\left(2\sqrt{\frac{2p_1^+p_1^-}{-S}}\right)\right)\right), \tag{5.1.2}$$

where:

1. α_n is an overall normalization.
2. Λ is a UV cutoff.
3. $v \cdot s = 0$ implies that s is spacelike so $\sqrt{-S}$ is real.

If we take this on-shell we get:

$$\arctan\left(2\sqrt{\frac{2p_1^+p_1^-}{-S}}\right) \to \arctan\left(1\right) = \frac{\pi}{4}, \tag{5.1.3}$$

and if we take the cutoff to ∞ we get:

$$\arctan\left(2\frac{\Lambda}{\sqrt{-S}}\right) \to \frac{\pi}{2}. \tag{5.1.4}$$

Thus:

$$V_n^{+,+,\ldots,+}\left(p_1,s\right) \to \alpha_n\left(p_1^+\right)^n e^{i\frac{\pi\lambda}{2}}. \tag{5.1.5}$$

We can think of analytically continuing this result to timelike s and reinterpreting the $+$ direction as lying in the (spatial) plane perpendicular to s:

$$p = \left(p^0, p^1, p^2\right), \, p^{\pm} = p^1 \pm ip^2. \tag{5.1.6}$$

Then for massless on-shell p_1 we have $p_1^+ \propto Ee^{i\theta}$ where θ is the angle between p_1, p_2 in the spatial plane. Thus we have:

$$V_n^{+,+,\dots,+}\left(p_1, s\right) = V_n^{+,+,\dots,+}\left(E, \theta\right) \propto E^n e^{in\theta} e^{i\frac{\pi\lambda}{2}}, \tag{5.1.7}$$

as stated in (5.4). We see that the spin n all-plus current component produces a state with angular momentum n:

$$J_n^{++\cdots+}\left(s\right)\left|0\right\rangle \propto e^{\frac{i\pi\lambda}{2}}\left|E, n, \text{singlet}\right\rangle_{\text{out}}, \tag{5.1.8}$$

with an overall phase $\frac{\pi\lambda}{2}$, as stated in (5.5). This is exactly **half** the phase of the S-matrix (5.2.13) in the singlet channel at angular momentum $+n$, obtained using the modified crossing relation, as we shall see in 5.2.

5.2 The Phase of the S-matrix

The $2 \to 2$ S-matrix can be thought of as a function of $S = E^2$ - the center of mass energy squared - and θ - the angle of scattering. We can think of a scattering operator S satisfying:

$$S\left|E, \theta_{\text{in}}\right\rangle_{\text{in}} = \int \frac{d\theta_{\text{out}}}{8\pi E} S\left(E, \theta_{\text{in}} - \theta_{\text{out}}\right)\left|E, \theta_{\text{out}}\right\rangle_{\text{out}}. \tag{5.2.1}$$

Let us write an angular momentum eigenstate as:

$$\left|E, n\right\rangle = \int \frac{d\theta}{2\pi} e^{-in\theta}\left|E, \theta\right\rangle,$$
$$\left|E, \theta\right\rangle = \sum_{n\in\mathbb{Z}} e^{in\theta}\left|E, n\right\rangle.$$

We can similarly define the Fourier transform of the S-matrix:

$$S\left(E, \theta\right) = 4E \sum_n S_n\left(E\right) e^{in\theta}. \tag{5.2.2}$$

Angular momentum conservation implies that momentum eigenstates should also be S-matrix eigenstates, and indeed:

$$S\left|E,n_{\mathrm{in}}\right\rangle_{\mathrm{in}} = \int\frac{d\theta}{2\pi}e^{-in_{\mathrm{in}}\theta_{\mathrm{in}}}\int\frac{d\theta_{\mathrm{out}}}{8\pi E}4E\sum_{n}S_{n}\left(E\right)e^{in\left(\theta_{\mathrm{in}}-\theta_{\mathrm{out}}\right)}\left|E,\theta_{\mathrm{out}}\right\rangle_{\mathrm{out}} \tag{5.2.3}$$

$$= \int\frac{d\theta_{\mathrm{out}}}{2\pi}S_{n_{\mathrm{in}}}\left(E\right)e^{-in_{\mathrm{in}}\theta_{\mathrm{out}}}\left|E,\theta_{\mathrm{out}}\right\rangle_{\mathrm{out}} \tag{5.2.4}$$

$$= S_{n_{\mathrm{in}}}\left(E\right)\left|E,n_{\mathrm{in}}\right\rangle_{\mathrm{out}}, \tag{5.2.5}$$

where we have used the identity:

$$\int\frac{d\theta}{2\pi}e^{i\left(n_{1}-n_{2}\right)\theta}=\delta_{n_{1},n_{2}} \tag{5.2.6}$$

twice. We see that $S_{n}\left(E\right)$ is an eigenvalue of the S-matrix. Note that unitarity now implies that such eigenvalues be complex phases:

$$1={}_{\mathrm{in}}\left\langle E,n\right|S^{\dagger}S\left|E,n\right\rangle_{\mathrm{in}}=\left|S_{n}\left(E\right)\right|^{2}.$$

Let's see how that works for the identity S-matrix (see (2.5.21)):

$$I\left(E,\theta\right) = 4\pi E\lim_{\epsilon\to0}\left(\delta\left(\theta+\epsilon\right)+\delta\left(\theta-\epsilon\right)\right) \tag{5.2.7}$$

$$\Rightarrow I_{n}\left(E\right) = 1 \tag{5.2.8}$$

More generally we expect:

$$S_{n}\left(E\right)=e^{is_{n}}, \tag{5.2.9}$$

where s_{n} is what we will refer to as the phase of the S-matrix. In the following we'll always assume $n\neq0$. Now instead we consider the case:

$$\mathcal{M}_{0}\left(E,\theta\right)=-4\pi\lambda E\cot\left(\frac{\theta}{2}\right). \tag{5.2.10}$$

This function has a divergence at $\theta=0$ but its principle value can shown to be $\left(\mathcal{M}_{0}\right)_{n}\left(E\right)=\mathrm{sign}\left(n\right)\pi\lambda$. Clearly this has incorrect modulus to satisfy unitarity. Hence the "naive" S -matrix:

$$S_{0}=I+i\mathcal{M}_{0} \tag{5.2.11}$$

isn't unitary. On the other hand, if we mimic the modified crossing relation (1.4) and take:

$$S = \cos(\pi\lambda) I + i\frac{\sin(\pi\lambda)}{\pi\lambda}\mathcal{M}_0, \tag{5.2.12}$$

we get:

$$S_n(E) = \exp(i \mathrm{sign}(n)\pi\lambda). \tag{5.2.13}$$

Which has the correct modulus. This is part of what underlies (1.4). Of course, we see now that it also ensures consistency with (5.7), and therefore with the "optical theorem" (5.6).

We'll now show that (5.2.12) is precisely the S matrix relevant in our case, up to some irrelevant terms! We described it in 4.2, where its connected part $i\mathcal{M}$ was given by (4.2.3):

$$\begin{aligned} i\mathcal{M} &= i\frac{4\pi\lambda}{N}E(p_1,p_2,p_3)\sqrt{\frac{SU}{T}} & (5.2.14)\\ &- \underbrace{i\frac{4\pi\lambda}{N}\sqrt{-S}\frac{\left(\tilde{b}_4-4\pi i\lambda\sqrt{-S}\right)+\left(\tilde{b}_4+4\pi i\lambda\sqrt{-S}\right)e^{-2i\lambda\arctan\left(\frac{\sqrt{-S}}{2m}\right)}}{-\left(\tilde{b}_4-4\pi i\lambda\sqrt{-S}\right)+\left(\tilde{b}_4+4\pi i\lambda\sqrt{-S}\right)e^{-2i\lambda\arctan\left(\frac{\sqrt{-S}}{2m}\right)}}}_{j(\sqrt{S})}. & (5.2.15)\end{aligned}$$

However, since we are interested in the singlet channel, we must multiply this by N as per the naive crossing relation (2.5.26). The second term j depends only on $\sqrt{-S}\sim E$ and not on θ, and will get projected out by the angular integration when computing S_n for $n\neq 0$. Thus it doesn't matter that this term is evaluated for $m, b_4 \neq 0$. We can focus on the first term which is simply the tree level amplitude! We can pick the c.o.m. frame where $s = E(1,0,0)$, $t=\frac{1}{2}E(0,1-\cos\theta,-\sin\theta)$, $u=\frac{1}{2}E(0,1+\cos\theta,\sin\theta)$ and rewrite:

$$\begin{aligned} E(p_1,p_2,p_3)\sqrt{\frac{SU}{T}} &= \frac{\epsilon(s,t,u)}{t^2} & (5.2.16)\\ &= E\frac{2\sin\theta}{-(1-\cos\theta)^2-\sin^2\theta} & (5.2.17)\\ &= -E\frac{\sin\theta}{1-\cos\theta} & (5.2.18)\\ &= -E\frac{\sin(\theta/2)\cos(\theta/2)}{\sin^2(\theta/2)} & (5.2.19)\\ &= -E\cot\left(\frac{\theta}{2}\right). & (5.2.20)\end{aligned}$$

Plugging that back in we get:

$$i\mathcal{M} = -4\pi i\lambda E \cot\left(\frac{\theta}{2}\right) - j\left(E\right) = i\mathcal{M}_0 - j\left(E\right), \tag{5.2.21}$$

as required. Thus the modified crossing relation plays a role not just in the standard optical theorem (unitarity of the S-matrix), but also in its variant (5.6).

6 Conclusions and Future Work

We set out to investigate the analytic properties of the S-matrix in Chern-Simons theory coupled to Bosonic fundamental matter, specifically as it relates to crossing symmetry.

Working at 1-loop in the planar limit (see 4.3), we relaxed the assumption $v \cdot s = 0$, in light cone gauge $v \cdot A = 0$. Initially, we expected the amplitude to be gauge invariant, but that result eluded us for a while. We came to see that the naive amplitude is, in fact, gauge dependent. This substantiates the intuition in [19] that one must look at Wilson-line dressed amplitudes instead. Indeed, the gauge dependent terms we obtain have the suggestive form of a prefactor multiplying the tree-level amplitude.

Our 1-loop result agrees with [19] when both the on-shell condition and $v \cdot s = 0$ are satisfied. Apart from that, there are 2 discrepancies:

1. The all-loop result obtained in [19] has an exponential prefactor which evaluates to unity on-shell, but away from the mass shell has the form:

$$-2i\lambda \left(\arctan\left(\frac{2\sqrt{2p_1^+ p_1^- + m^2}}{s_\perp} \right) - \arctan\left(\frac{2\sqrt{2p_4^+ p_4^- + m^2}}{s_\perp} \right) \right) i\mathcal{M}_0, \tag{6.1}$$

   where $i\mathcal{M}_0$ is the tree-level amplitude. Our result instead has:

$$-i\lambda \left(\arctan\left(\frac{\epsilon\left(v, p_1, p_2\right)}{mv \cdot \left(p_2 - p_1\right)} \right) + \arctan\left(\frac{\epsilon\left(v, p_3, p_4\right)}{mv \cdot \left(p_4 - p_3\right)} \right) \right) i\mathcal{M}_0, \tag{6.2}$$

   and it actually **survives** on-shell but **not** for $v \cdot s = 0$. This discrepancy most likely has to do with sub-gauge conditions, but possibly also with problems in well-defining light-cone integrals, as discussed in 4.4.

2. While it is true that the choice $v \cdot s$ is **possible** in those channels where s is a space-like vector, we find that the gauge dependent part survives in those channels as well, indicating that there is a problem to be resolved in all channels, rather than only in the singlet channel.

The fact that both the $O\left(b_4\right)$ and $O\left(\lambda\right)$ terms in $\mathcal{M}_0$ emerged from the calculation dressed with the same overall factor is probably not coincidence. Hence, these results indicate that we should instead consider manifestly gauge-invariant objects. Possible such objects include:

1. Wilson-line dressed 4-point correlators, as already mentioned, but also

2. Form factors of gauge invariant operators such as higher-spin currents, where the emitted particles are dressed with Wilson lines, and

3. Correlation functions of higher spin currents.

2 (without the Wilson-line dressing) and 3 have been computed, as mentioned in 5, by Aharony et. al. in [3], in the $v \cdot s = 0$ regime. So a similar one-loop comparison can be made. We find that dressing the form factor with Wilson lines does in fact restore (off-shell) gauge invariance (if one can trust the Schouten identity for possibly poorly defined integrands).

In both the planar and non-planar 1-loop amplitudes, the terms one would expect to arise from modified crossing are localized at forward scattering. It could prove useful to revisit these calculations and pay special attention to the $\theta = 0$ limit. The contribution at 2-loops should have the form $\sim \lambda^2 i\mathcal{M}_0$ (up to constants), and so perhaps could be obtained directly from a 2-loop calculation. Our past attempts at a 2-loop calculation have focused on its covariantization (equivalently, gauge invariance). However, we now know not to expect a gauge invariant result to begin with, and are equipped with full off-shell 1-loop results that could significantly speed up such a calculation.

Our results at 1-loop away from the planar limit 3 appear to be gauge invariant. The form factor we computed corresponds to an Abelian calculation, but together with the planar result, the full non-planar 1-loop amplitude can be reconstructed. Hence one could say that whatever nontrivial analytic properties of the S-matrix arise, they will likely have to do with the "planar part" of the amplitude.

As an alternative to the covariantization and integral reductions described in 3, we can use our planar 1-loop results to integrate the various diagrams comprising it. Hypothetically, we might find that the various gauge dependent terms cancel.

In 5 we saw another way in which the modified crossing relation (1.4) plays a role in the theory - it is crucial in satisfying the relations regarding the phase of form factors and of the S-matrix derived in [9].

Another possible future direction is to study the S-matrix in the context of the brane construction of Chern-Simons theory [29, 15]. In this setting, N coincident D3 branes in type IIB superstring theory end on an NS5 brane. The low energy effective field theory on the D3 branes is $\mathcal{N} = 4$ SYM. The NS5 brane, in general breaks some of the SUSY. However, if one insists on the retention of certain subset of SUSY, the theory becomes topological. In particular, the boundary action associated to the 3 dimensional intersection of the D3 branes with the NS5 brane describes an $SU(N)$ Chern-Simons theory with a gauge field constructed out of the SYM fields. For details we refer to [29, 15]. In this configuration one can add super-Wilson lines confined to the 3D boundaries. This gives a 4-dimensional viewpoint on knot polynomials and is related to the study of Khovanov homology. One can try to couple such a theory to matter and study a 4d or higher dimensional realization of the S-matrix. Perhaps an entirely different perspective on the modified crossing relation can be achieved.

Furthermore, one can consider the scattering of particles in different representations of the gauge group. For instance, the scalars and spinors of ABJ(M) theory [5, 2] transform in the bi-fundamental of $SU(N) \times SU(M)$. This can be studied from the perspective of M-theory, or through its holographic correspondence with type IIA superstring theory on $\mathrm{AdS}_4 \times \mathrm{CP}^3$.

A Variation of The CS Action Under a Finite Gauge Transformation

In this appendix we'll show how the Lagrangian of pure Chern-Simons theory (2.2.6) transforms under a finite gauge transformation.

Under a finite gauge transformation $g = e^{-i\Lambda}$ we have:

$$\begin{aligned} A_\mu &\to gA_\mu g^{-1} + ig\partial_\mu g^{-1} && \text{(A.1)}\\ &= gA_\mu g^{-1} - i\partial_\mu g g^{-1} && \text{(A.2)}\\ &= g\left(A_\mu - \partial_\mu \Lambda\right) g^{-1}. && \text{(A.3)} \end{aligned}$$

It will be useful to rewrite the Lagrangian in terms of the field strength:

$$\begin{aligned} \epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho &= \frac{1}{2}\epsilon^{\mu\nu\rho}A_\mu\left(\partial_\nu A_\rho - \partial_\rho A_\nu\right) && \text{(A.4)}\\ &= \frac{1}{2}\epsilon^{\mu\nu\rho}A_\mu\left(F_{\nu\rho} + i\left[A_\nu, A_\rho\right]\right) && \text{(A.5)}\\ &= \frac{1}{2}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} + i\epsilon^{\mu\nu\rho}A_\mu A_\nu A_\rho, && \text{(A.6)} \end{aligned}$$

$$\Rightarrow \mathcal{L} = i\frac{k}{4\pi}\epsilon^{\mu\nu\rho}\mathrm{Tr}\left(\frac{1}{2}A_\mu F_{\nu\rho} + \frac{1}{3}iA_\mu A_\nu A_\rho\right). \quad \text{(A.7)}$$

We now transform the Lagrangian:

$$\begin{aligned} \mathcal{L} &\to \mathcal{L} + i\frac{k}{4\pi}\epsilon^{\mu\nu\rho}\mathrm{Tr}\left(-\frac{1}{2}\partial_\mu\Lambda F_{\nu\rho} + iA_\mu\partial_\nu\Lambda\partial_\rho\Lambda - iA_\mu A_\nu\partial_\rho\Lambda - \frac{1}{3}i\partial_\mu\Lambda\partial_\nu + \Lambda\partial_\rho\Lambda\right)\\ &= \mathcal{L} + i\frac{k}{4\pi}\epsilon^{\mu\nu\rho}\mathrm{Tr}\left(-\partial_\mu A_\nu\partial_\rho\Lambda - iA_\mu\partial_\nu\Lambda\partial_\rho\Lambda - \frac{1}{3}i\partial_\mu\Lambda\partial_\nu\Lambda\partial_\rho\Lambda\right). && \text{(A.8)} \end{aligned}$$

We focus our attention on the first term:

$$\begin{aligned} \epsilon^{\mu\nu\rho}\partial_\mu A_\nu\partial_\rho\Lambda &= i\epsilon^{\mu\nu\rho}\partial_\mu A_\nu\partial_\rho g g^{-1} && \text{(A.9)}\\ &= i\partial_\mu\left(\epsilon^{\mu\nu\rho}A_\nu\partial_\rho g g^{-1}\right) - i\epsilon^{\mu\nu\rho}A_\nu\partial_\rho g\partial_\mu g^{-1} && \text{(A.10)}\\ &= i\partial_\mu\left(\epsilon^{\mu\nu\rho}A_\nu\partial_\rho g g^{-1}\right) - i\epsilon^{\mu\nu\rho}A_\mu\underbrace{\partial_\nu g g^{-1}}_{-i\partial_\nu\Lambda}\underbrace{g\partial_\rho g^{-1}}_{i\partial_\rho\Lambda} && \text{(A.11)}\\ &= i\partial_\mu\left(\epsilon^{\mu\nu\rho}A_\nu\partial_\rho g g^{-1}\right) - i\epsilon^{\mu\nu\rho}A_\mu\partial_\nu\Lambda\partial_\rho\Lambda. && \text{(A.12)} \end{aligned}$$

Plugging this back into (A.8) and rewriting the last term we get:

$$\ldots \;=\; \mathcal{L} + \mathrm{Tr}\left(\frac{k}{4\pi}\partial_\mu\left(\epsilon^{\mu\nu\rho}A_\nu\partial_\rho g g^{-1}\right) + i\frac{k}{12\pi}\epsilon^{\mu\nu\rho}g^{-1}\partial_\mu g g^{-1}g\partial_\nu g^{-1}\partial_\rho g\right). \quad \text{(A.13)}$$

B Integration in Light-Cone Coordinates

We will compute Feynman integrals in light-cone coordinates and discuss their technicalities.

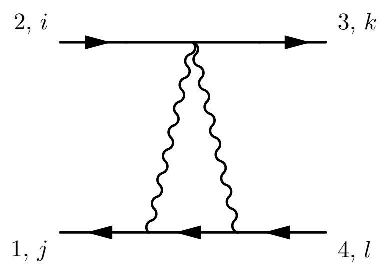
B.1 Lorenzian Signature

B.1.1 Triangle diagram with 2 gauge propagators

Let's consider the integral:[13]

$$I\left(p_1,p_2\right) \;=\; \int\frac{d^3k}{\left(2\pi\right)^3}\frac{k^+}{k^2-m^2}\frac{1}{k^+-p_1^+}\frac{1}{k^+-p_2^+} \quad \text{(B.1.1)}$$

$\propto$



.

Let us make manifest our $i\epsilon$ prescription (2.2.82):

$$\int\frac{d^3k}{\left(2\pi\right)^3}\frac{k^+}{2k^+k^- - \left(k^\perp\right)^2 - m^2 + i\epsilon}\frac{k^- - p_1^-}{\left(k^+-p_1^+\right)\left(k^--p_1^-\right)+i\epsilon}\frac{k^- - p_2^-}{\left(k^+-p_2^+\right)\left(k^--p_2^-\right)+i\epsilon}. \quad \text{(B.1.2)}$$

What are the convergence properties of this integral? We can think of it as an integration of 2 vectors $\left(k^\parallel, k^\perp\right) \in \mathbb{R}^2\times\mathbb{R}$:

$$\int\frac{d^3k}{\left(2\pi\right)^3}\frac{k_\parallel^\mu}{k_\parallel^2-\left(k^\perp\right)^2-m^2}\frac{k_\parallel^\nu - p_{1,\parallel}^\nu}{\frac{1}{2}\left(k_\parallel - p_{1,\parallel}\right)^2}\frac{k_\parallel^\rho - p_{2,\parallel}^\rho}{\frac{1}{2}\left(k_\parallel - p_{2,\parallel}\right)^2}. \quad \text{(B.1.3)}$$

In $k_\parallel$ it is an $O\left(k_\parallel^{-3}\right)$ UV-convergent rank 3 tensor integral. In $k^\perp$ it's an $O\left(k_\perp^{-2}\right)$ UV convergent scalar integral. We can use this form to carry out integration in more-or-less

[13] This integral, as written, has -1 charge under the $GL\left(1\right)_{\mathrm{L}}$ symmetry (2.2.81). Therefore it should come with some charge 1 prefactor like $p_1^+ + p_2^+$.

standard ways such as the Schwinger trick[23]:

$$\frac{1}{k_{\parallel}^2-\left(k^{\perp}\right)^2-m^2}\frac{1}{\left(k_{\parallel}-p_{1,\parallel}\right)^2}\frac{1}{\left(k_{\parallel}-p_{2,\parallel}\right)^2} \tag{B.1.4}$$

$$= \int\frac{dxdydz}{GL\left(1\right)}\frac{1}{\left(\left(k_{\parallel}^2-\left(k^{\perp}\right)^2-m^2\right)x+\left(k_{\parallel}-p_{1,\parallel}\right)^2y+\left(k_{\parallel}-p_{2,\parallel}\right)^2z\right)^3}. \tag{B.1.5}$$

However, in this case it will be simpler to work directly with the form (B.1.2). Interestingly, this form is $O\left(\left(k^{+}\right)^{-3}\right)$ but is **logarithmically** UV-divergent $O\left(\left(k^{-}\right)^{-1}\right)$. This is an artifact of this choice of coordinates and is one of the many subtleties that arise in light-cone integration. In this case, however, one can still integrate k^{-} using contour techniques, as long as one pays attention to the contribution from the "arc at infinity". To avoid this subtlety we can start by integrating out k^{+} or $k^{\perp}$. Integrating $k^{\perp}$ will introduce square-roots into the integrand and therefore complicate its analyticity properties. Hence we start by integrating k^{+} by contour methods. This will amount to summing a few residues to give an analytic integrand. We can deform the k^{+} contour into the upper- or lower-half of the complex k^{+} plane, but where are the poles located? It is here that a unique aspect of light-cone coordinates becomes relevant. The poles are located at:

$$\Im\left(k^{+}\right)=-\frac{\epsilon}{2k^{-}},\ -\frac{\epsilon}{2\left(k^{-}-p_1^{-}\right)},\ -\frac{\epsilon}{2\left(k^{-}-p_2^{-}\right)}. \tag{B.1.6}$$

When $k^{-}<0,p_1^{-},p_2^{-}$ all poles are in the upper half plane and so one can deform the contour downwards to get 0. Similarly when $k^{-}>0,p_1^{-},p_2^{-}$ one can deform upwards to get 0! Hence the k^{+} integral localizes k^{-} to a finite range! In general, we can choose to avoid one the three poles in (B.1.6). Let's choose to avoid the pole in the scalar propagator $\left(k^2-m^2\right)^{-1}$ because the residue there is the most "complicated". Hence we will deform downwards for $k^{-}<0$ and upwards for $k^{-}>0$. If $p_1^{-}>0$ we'll get a **counter-clockwise** residue:

$$\int_{-\infty}^{\infty}\frac{dk^{\perp}}{2\pi}\int_{0}^{p_1^{-}}\frac{dk^{-}}{2\pi}\frac{(+2\pi i)}{2\pi}\frac{p_1^{+}}{2p_1^{+}k^{-}-\left(k^{\perp}\right)^2-m^2+i\epsilon}\frac{k^{-}-p_2^{-}}{\left(p_1^{+}-p_2^{+}\right)\left(k^{-}-p_2^{-}\right)+i\epsilon}. \tag{B.1.7}$$

If $p_1^{-}<0$ we'll get the same thing since:

$$\int_{p_1^{-}}^{0}\frac{dk^{-}}{2\pi}\frac{(-2\pi i)}{2\pi}=\int_{0}^{p_1^{-}}\frac{dk^{-}}{2\pi}\frac{(+2\pi i)}{2\pi}, \tag{B.1.8}$$

and of course we'll get a similar term from the p_2 pole:

$$I = \frac{ip_1^+}{p_1^+ - p_2^+} \int_{-\infty}^{\infty} \frac{dk^\perp}{2\pi} \int_0^{p_1^-} \frac{dk^-}{2\pi} \frac{1}{2p_1^+ k^- - (k^\perp)^2 - m^2 + i\epsilon} + (1 \leftrightarrow 2)\,. \tag{B.1.9}$$

We are left with a finite-range integral and an infinite-range one. What is the preferred order? Integrating k^- will introduce transcendentality 1 functions (logarithms) into the integrand - functions that posses branch cuts and will interfere with contour methods for the $\perp$ integral. Integrating $k^\perp$ first will introduce square-roots, which won't significantly complicate the k^- integral. So let's start with $k^\perp$:

$$\begin{aligned} I &= -\frac{ip_1^+}{p_1^+ - p_2^+} \frac{2\pi i}{2\pi} \int_0^{p_1^-} \frac{dk^-}{2\pi} \frac{1}{2\sqrt{2p_1^+ k^- - m^2 + i\epsilon}} + (1 \leftrightarrow 2) \\ &= \frac{1}{2} \frac{p_1^+}{p_1^+ - p_2^+} \int_0^{p_1^-} \frac{dk^-}{2\pi} \frac{1}{\sqrt{2p_1^+ k^- - m^2 + i\epsilon}} + (1 \leftrightarrow 2) \\ &= \frac{1}{4\pi} \frac{1}{p_1^+ - p_2^+} \sqrt{2p_1^+ k^- - m^2}\Big|_0^{p_1^-} + (1 \leftrightarrow 2) \\ &= \frac{1}{4\pi} \frac{1}{p_1^+ - p_2^+} \left(\sqrt{2p_1^+ p_1^- - m^2} - im \right) + (1 \leftrightarrow 2) \\ &= \frac{1}{4\pi} \frac{1}{p_1^+ - p_2^+} \left(\sqrt{2p_1^+ p_1^- - m^2} - \sqrt{2p_2^+ p_2^- - m^2} \right). \end{aligned} \tag{B.1.10}$$

A comment is in order - the integral in its initial form(B.1.1) appears to only be "aware" of $p_{1,2}$ through their "+" components, yet the final answer depends also on their "$-$" components. These components enter the integral only through the $i\epsilon$ prescription.

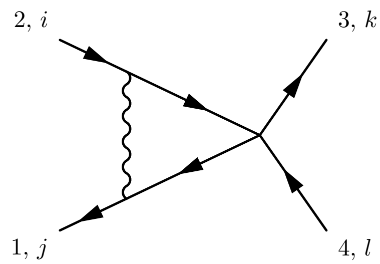
B.1.2 Triangle diagram with 1 gauge propagator

We'll consider the integral:

$$I\left(p_1, s\right) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 - m^2} \frac{1}{(k-s)^2 - m^2} \frac{\epsilon\left(v, k+p_1, s\right)}{v \cdot \left(k - p_1\right)} \tag{B.1.11}$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 - m^2} \frac{1}{(k-s)^2 - m^2} \frac{\left(k^+ + p_1^+\right) s^\perp - \left(k^\perp + p_1^\perp\right) s^+}{k^+ - p_1^+} \tag{B.1.12}$$

$$\propto \qquad . \tag{B.1.13}$$



As before, we can start by integrating k^+ and again we'll be forced to pick up 2 of the 3 poles. This time, however, at least one of these will be a scalar propagator and will complicate our integrand. However, unlike in (B.1.1) our integrand is $O\left(\left(k^-\right)^{-2}\right)$ - it is UV convergent in k^- - so we can start by integrating k^-. This will localize the k^+ integration to the range $(0, s^+)$. Note that in the case $s^+ = 0$ this range vanishes but at the same time the integrand becomes k^- - independent at $k^+ = 0$ so we should get something like $\delta(0)$. Although what this means is unclear, suffice it to say that we don't necessarily expect the integral to vanish for $s^+ = 0$, despite the empty range of integration. Note that with the $i\epsilon$ prescription we will have another pole:

$$\frac{1}{k^+ - p_1^+} = \frac{k^- - p_1^-}{\left(k^+ - p_1^+\right)\left(k^- - p_1^-\right) + i\epsilon}, \tag{B.1.14}$$

but the residue is proportional to ϵ and can be neglected. Thus we choose to take only the residue at $\left(k^2 - m^2\right)^{-1}$:

$$I = i \int_{-\infty}^{\infty} \frac{dk^\perp}{2\pi} \int_0^{s^+} \frac{dk^+}{2\pi} \frac{1}{2k^+} \frac{1}{(k-s)^2 - m^2} \tag{B.1.15}$$

$$\times \left. \frac{\left(k^+ + p_1^+\right) s^\perp - \left(k^\perp + p_1^\perp\right) s^+}{k^+ - p_1^+} \right|_{k^- = \frac{\left(k^\perp\right)^2 + m^2}{2k^+}} . \tag{B.1.16}$$

Interestingly, the new integrand is logarithmically divergent in $k^\perp$ - it is $O\left(\left(k^\perp\right)^{-1}\right)$. We will have to include the effect of the arc at infinity when deforming the $k^\perp$ contour. First note that:

$$\left((k-s)^2-m^2\right)2k^+\Big|_{k^-=\frac{(k^\perp)^2+m^2}{2k^+}} \tag{B.1.17}$$

$$= \cdots - 2s^+\left(m^2-i\epsilon\right), \tag{B.1.18}$$

where ... doesn't include m^2 and therefore doesn't include $i\epsilon$. Hence the sign of s^+ determines the sign of the imaginary part of the denominator. Note that $\text{sign}\,(s^+)$ is actually a $GL\,(1)_{\text{L}}$ - invariant since $\text{sign}\left(s^+e^{\xi}\right) = \text{sign}\,(s^+)$. Of the two poles in $k^\perp$, one is in the lower half $k^\perp$-plane, and the other is in the upper. We will decide to pick up a particular pole, but whether that corresponds to deforming the contour into the upper or lower half-plane will depend on sign (s$^+$). The arc at infinity contribution looks like (ignoring the numerator):

$$i\int_0^{s^+}\frac{dk^+}{2\pi}\int_{\text{arc}}\frac{dk^\perp}{2\pi}\frac{1}{2k^\perp}\frac{1}{k^+-p_1^+} = i\int_0^{s^+}\frac{dk^+}{2\pi}\int\frac{id\theta e^{i\theta}}{2\pi}\frac{1}{2e^{i\theta}}\frac{1}{k^+-p_1^+} \tag{B.1.19}$$

$$= -\frac{1}{4}\text{sign}\left(s^+\right)\int_0^{s^+}\frac{dk^+}{2\pi}\frac{1}{k^+-p_1^+}. \tag{B.1.20}$$

Combining this with the residue gives:

$$I = \frac{1}{4}\text{sign}\left(s^+\right)\int_0^{s^+}\frac{dk^+}{2\pi}\frac{1}{\sqrt{-S\left(k^+\right)^2+Sk^+s^+-m^2\left(s^+\right)^2}}\frac{s^+p_1^\perp-s^\perp p_1^+}{k^+-p_1^+}, \tag{B.1.21}$$

where $S=s^2$. Note that we've sloppily ignored the $i\epsilon$ prescription for the gauge-propagator pole, but we'll just get to the final answer and figure out how to resolve any ambiguity there. Upon integration we get:

$$I = \frac{\text{sign}\,(s^+)\left(s^+p_1^\perp-s^\perp p_1^+\right)}{8\pi\sqrt{A}}\log\left(\frac{s^+-p_1^+}{p_1^+}\frac{2m^2s^+-Sp_1^+-2im\,\text{sign}\,(s^+)\sqrt{A}}{-2m^2s^+-S\left(p_1^+-s^+\right)+2im\,\text{sign}\,(s^+)\sqrt{A}}\right), \tag{B.1.22}$$

$$A \equiv -S\left(p_1^+\right)^2+Ss^+p_1^+-m^2\left(s^+\right)^2. \tag{B.1.23}$$

Interestingly, we can set $\text{sign}\,(s^+)\to 1$, since the argument of the log, which we can denote $B\left(\text{sign}\,(s^+),\dots\right)$, can be shown to satisfy:

$$B\left(-\text{sign}\left(s^+\right),\dots\right) = \frac{1}{B\left(\text{sign}\left(s^+\right),\dots\right)}, \tag{B.1.24}$$

so that the combined expression:

$$\text{sign}\left(s^{+}\right)\log\left(B\left(\text{sign}\left(s^{+}\right),\ldots\right)\right) \tag{B.1.25}$$

is actually independent of sign (s^{+}). Hence we can rewrite:

$$I=\frac{s^{+}p_1^{\perp}-s^{\perp}p_1^{+}}{8\pi\sqrt{A}}\log\left(\frac{s^{+}-p_1^{+}}{p_1^{+}}\frac{2m^2s^{+}-Sp_1^{+}-2im\sqrt{A}}{-2m^2s^{+}-S\left(p_1^{+}-s^{+}\right)+2im\sqrt{A}}\right). \tag{B.1.26}$$

The same algebra shows that this expression is independent of the branch chosen for $\sqrt{A}$ when A is negative, and that the expression is parity-odd as expected (in 4.3 it comes with a factor $\sim b_4\lambda$ and so this parity-oddness is required to satisfy $\mathbb{Z}_2$ symmetry). This behavior under parity also tells us that we've chosen the right branch for the log - shifting it by $2\pi i$ would give rise to a term that is parity-even. Hence our sloppiness in disregarding the $i\epsilon$ prescription appears to be inconsequential. Since all of these operations (parity, branch choice for $\sqrt{A}$) amount to just complex conjugating the log's argument B (at least when A is positive, but one can analytically continue the resulting expressions to negative A), the fact that it inverts B indicates that B is a complex phase $B=e^{i\theta}$ so that:

$$\log B=i\theta=i\arctan\left(\frac{\Im\left[B\right]}{\Re\left[B\right]}\right). \tag{B.1.27}$$

Hence we rewrite the result in terms of and arctan (we'll also use variables $p_2\equiv s-p_1$, p_1 and S):

$$I=\frac{p_2^{\perp}p_1^{+}-p_2^{+}p_1^{\perp}}{8\pi\sqrt{-m^2\left(p_1^{+}+p_2^{+}\right)^2+Sp_1^{+}p_2^{+}}}i\arctan\left(\frac{2m\left(p_1^{+}-p_2^{+}\right)\sqrt{-m^2\left(p_1^{+}+p_2^{+}\right)^2+Sp_1^{+}p_2^{+}}}{2m^2\left(\left(p_1^{+}\right)^2+\left(p_2^{+}\right)^2\right)-Sp_1^{+}p_2^{+}}\right). \tag{B.1.28}$$

This can be further simplified. Note that for general x, y:

$$\arctan\left(x\right) = \frac{1}{2}i\log\left(\frac{1-ix}{1+ix}\right) \tag{B.1.29}$$

$$\arctan(x) + \arctan(y) = \frac{1}{2} i \log\left(\frac{1-ix}{1+ix}\frac{1-iy}{1+iy}\right) \tag{B.1.30}$$

$$= \frac{1}{2} i \log\left(\frac{1-i\frac{x+y}{1-xy}}{1+i\frac{x+y}{1-xy}}\right) \tag{B.1.31}$$

$$= \arctan\left(\frac{x+y}{1-xy}\right), \tag{B.1.32}$$

so in particular:

$$\arctan\left(\frac{2x}{1-x^2}\right) = 2\arctan(x). \tag{B.1.33}$$

Now note that we can rewrite:

$$\frac{2m\left(p_1^+ - p_2^+\right)\sqrt{-m^2\left(p_1^+ + p_2^+\right)^2 + S p_1^+ p_2^+}}{2m^2\left(\left(p_1^+\right)^2 + \left(p_2^+\right)^2\right) - S p_1^+ p_2^+} \tag{B.1.34}$$

$$= \frac{2\frac{\sqrt{-m^2\left(p_1^+ + p_2^+\right)^2 + S p_1^+ p_2^+}}{m\left(p_1^+ - p_2^+\right)}}{1 + \frac{2m^2\left(\left(p_1^+\right)^2 + \left(p_2^+\right)^2\right) - S p_1^+ p_2^+ - \left(m\left(p_1^+ - p_2^+\right)\right)^2}{\left(m\left(p_1^+ - p_2^+\right)\right)^2}} \tag{B.1.35}$$

$$= \frac{2\frac{\sqrt{-m^2\left(p_1^+ + p_2^+\right)^2 + S p_1^+ p_2^+}}{m\left(p_1^+ - p_2^+\right)}}{1 - \frac{-m^2\left(p_1^+ + p_2^+\right)^2 + S p_1^+ p_2^+}{\left(m\left(p_1^+ - p_2^+\right)\right)^2}}, \tag{B.1.36}$$

so we can finally write:

$$I = \frac{p_2^\perp p_1^+ - p_2^+ p_1^\perp}{4\pi\sqrt{-m^2\left(p_1^+ + p_2^+\right)^2 + S p_1^+ p_2^+}} i \arctan\left(\frac{\sqrt{-m^2\left(p_1^+ + p_2^+\right)^2 + S p_1^+ p_2^+}}{m\left(p_1^+ - p_2^+\right)}\right). \tag{B.1.37}$$

Furthermore, after some manipulation we find:

$$-m^2\left(p_1^+ + p_2^+\right)^2 + S p_1^+ p_2^+ = \epsilon\left(v, p_1, p_2\right)^2 \tag{B.1.38}$$

$$- s^+\left(\left(m^2 - p_2^2\right) p_1^+ + \left(m^2 - p_1^2\right) p_2^+\right), \tag{B.1.39}$$

and of course:

$$p_2^\perp p_1^+ - p_2^+ p_1^\perp = \epsilon\left(v, p_1, p_2\right), \tag{B.1.40}$$

so that when either the on-shell condition $p_1^2 = p_2^2 = m^2$ or the condition $v \cdot s = 0$ are satisfied we get:

$$-\frac{i}{4\pi}\arctan\left(\frac{\epsilon\left(v,p_1,p_2\right)}{mv\cdot\left(p_1-p_2\right)}\right). \tag{B.1.41}$$

When $s^+ \to 0$ this further simplifies to:[14]

$$I \;=\; -i\mathrm{sign}\left(s^\perp\right)\frac{\arctan\left(\frac{\sqrt{-S}}{2m}\right)}{4\pi}. \tag{B.1.43}$$

B.2 Euclidean Signature

Let's consider again the integral (B.1.1), but now in Euclidean signature:

$$I\left(p_1,p_2\right) = \int \frac{d^3k}{\left(2\pi\right)^3}\frac{k^+}{k^2+m^2}\frac{1}{k^+-p_1^+}\frac{1}{k^+-p_2^+}. \tag{B.2.1}$$

We'll see that this time around we can carry out the integral without the $i\epsilon$ prescription, and get the same answer (B.1.10) as in Lorenzian signature. The key is to write:

$$k^\pm = \left|k_\parallel\right|e^{\pm i\theta},\; dk^+dk^- = \left|k_\parallel\right|d\left|k_\parallel\right|d\theta, \tag{B.2.2}$$

[14] Note that dependence on p_1 is lost. This could have been foreseen from the fact that the k^+ integral localizes to 0 where:

$$\frac{\epsilon\left(v,k+p_1,s\right)}{k^+-p_1^+} = \frac{\left(k^++p_1^+\right)s^\perp}{k^+-p_1^+} = -s^\perp \tag{B.1.42}$$

and thus becomes p_1-independent.

$$\Rightarrow I = \int \frac{dk_\perp}{2\pi} \int \frac{\left|k_\parallel\right|^2 d\left|k_\parallel\right|}{2\pi} \int \frac{d\theta e^{i\theta}}{2\pi} \frac{1}{2k_\parallel^2 + k_\perp^2 + m^2} \tag{B.2.3}$$

$$\times \frac{1}{k^+ - p_1^+} \frac{1}{k^+ - p_2^+} \tag{B.2.4}$$

$$= -i \int \frac{dk_\perp}{2\pi} \int \frac{\left|k_\parallel\right|^2 d\left|k_\parallel\right|}{2\pi} \oint_\gamma \frac{d\Big(\underbrace{e^{i\theta}}_{\equiv z}\Big)}{2\pi} \frac{1}{2k_\parallel^2 + k_\perp^2 + m^2} \tag{B.2.5}$$

$$\times \frac{1}{\left|k_\parallel\right| e^{i\theta} - p_1^+} \frac{1}{\left|k_\parallel\right| e^{i\theta} - p_2^+} \tag{B.2.6}$$

$$= -i \int \frac{dk_\perp}{2\pi} \int \frac{\left|k_\parallel\right|^2 d\left|k_\parallel\right|}{2\pi} \oint_\gamma \frac{dz}{2\pi} \frac{1}{2k_\parallel^2 + k_\perp^2 + m^2} \tag{B.2.7}$$

$$\times \frac{1}{\left|k_\parallel\right| z - p_1^+} \frac{1}{\left|k_\parallel\right| z - p_2^+}. \tag{B.2.8}$$

We have turned the θ integral into a complex contour integral over the unit circle γ. The integrand is $O\left(z^{-2}\right)$ so we can safely deform the contour to infinity or shrink it to 0. If we shrink to 0 we'll pick up the pole at p_i when $\frac{\left|p_i^+\right|}{\left|k_\parallel\right|} < 1$, and when we expand the contour to infinity we'll pick up the same pole if $\frac{\left|p_i^+\right|}{\left|k_\parallel\right|} > 1$. Hence we can choose to avoid the p_2 pole, and we'll pick up the pole at p_1 when $\left|k_\parallel\right|$ is localized to a finite range, in a similar vain to the localization we saw in B.1. If $\left|p_1^+\right| < \left|p_2^+\right|$ we'll get:

$$I = -i \int \frac{dk_\perp}{2\pi} \int_{\left|p_1^+\right|}^{\left|p_2^+\right|} \frac{\left|k_\parallel\right|^2 d\left|k_\parallel\right|}{2\pi} \frac{2\pi i}{2\pi} \frac{1}{2k_\parallel^2 + k_\perp^2 + m^2} \frac{1}{\left|k_\parallel\right|} \frac{1}{p_1^+ - p_2^+}. \tag{B.2.9}$$

When $\left|p_1^+\right| > \left|p_2^+\right|$ we'll get the same since:

$$\int_{\left|p_2^+\right|}^{\left|p_1^+\right|} (-2\pi i) = \int_{\left|p_1^+\right|}^{\left|p_2^+\right|} 2\pi i, \tag{B.2.10}$$

so we get:

$$I = \frac{1}{p_1^+ - p_2^+} \int \frac{dk_\perp}{2\pi} \int_{|p_1^+|}^{|p_2^+|} \frac{|k_\parallel|\, d\,|k_\parallel|}{2\pi} \frac{1}{2k_\parallel^2 + k_\perp^2 + m^2} \tag{B.2.11}$$

$$= \frac{1}{p_1^+ - p_2^+} \int_{|p_1^+|}^{|p_2^+|} \frac{d\,|k_\parallel|}{2\pi} \frac{|k_\parallel|}{2\sqrt{2k_\parallel^2 + m^2}} \tag{B.2.12}$$

$$= \frac{1}{4\pi} \frac{1}{p_1^+ - p_2^+} \int_{\sqrt{2|p_1^+|^2+m^2}}^{\sqrt{2|p_2^+|^2+m^2}} d\left(\sqrt{2k_\parallel^2 + m^2}\right) \tag{B.2.13}$$

$$= \frac{1}{4\pi} \frac{1}{p_1^+ - p_2^+} \left(\sqrt{2\,|p_2^+|^2 + m^2} - \sqrt{2\,|p_1^+|^2 + m^2}\right). \tag{B.2.14}$$

If we further assume that that $p_{1,2}$ are real momenta we get:

$$I = \frac{1}{4\pi} \frac{1}{p_1^+ - p_2^+} \left(\sqrt{2p_2^+ p_2^- + m^2} - \sqrt{2p_1^+ p_1^- + m^2}\right). \tag{B.2.15}$$

Does this match the result (B.1.10)? We'll have to use $p_i^+ p_i^- \to -p_i^+ p_i^-$ and $m^2 \to m^2 - i\epsilon$ so that:

$$\sqrt{2p_i^+ p_i^- + m^2} \to -i\sqrt{2p_i^+ p_i^- - m^2}. \tag{B.2.16}$$

Also the integral measure transforms:

$$d^3k_{\mathrm{E}} \to -i d^3 k_{\mathrm{L}}, \tag{B.2.17}$$

so we really should have included this factor to begin with. We get:

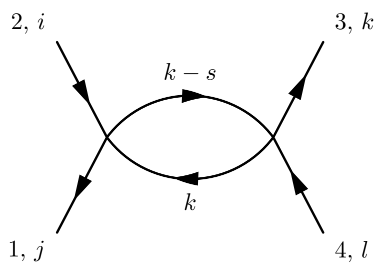
$$I \to (-i)^2 \frac{1}{4\pi} \frac{1}{p_1^+ - p_2^+} \left(\sqrt{2p_2^+ p_2^- - m^2} - \sqrt{2p_1^+ p_1^- - m^2}\right) \tag{B.2.18}$$

$$= \frac{1}{4\pi} \frac{1}{p_1^+ - p_2^+} \left(\sqrt{2p_1^+ p_1^- - m^2} - \sqrt{2p_2^+ p_2^- - m^2}\right), \tag{B.2.19}$$

which matches (B.1.10) as expected.

C The Bubble Integral

We wish to compute (ignoring coupling constants):



$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2+m^2} \frac{1}{(k-s)^2+m^2}. \tag{C.1}$$

We work in Euclidean coordinates and we can assume s lies entirely along the $\perp$ axis so we can write $k = \left(k_{\parallel} \equiv k, k_{\perp} \equiv l\right)$:

$$\begin{aligned} \dots &= \int \frac{d^2k dl}{(2\pi)^3} \frac{1}{k^2+l^2+m^2} \frac{1}{k^2+(l-s)^2+m^2} && \text{(C.2)} \\ &= \int_0^\infty \frac{kdkdl}{2\pi} \int_{-\infty}^{\infty} \frac{dl}{2\pi} \frac{1}{k^2+l^2+m^2} \frac{1}{k^2+(l-s)^2+m^2} && \text{(C.3)} \\ &= \frac{1}{8\pi s} \int_{-\infty}^{\infty} \frac{dl}{2\pi} \frac{\log\left(\frac{l^2+m^2}{l^2-2ls+m^2+s^2}\right)}{\left(l-\frac{1}{2}s\right)} && \text{(C.4)} \\ &= \frac{1}{8\pi s} \int_{-\infty}^{\infty} \frac{dl}{2\pi} \frac{\log\left(\frac{l^2+ls+m^2+\frac{s^2}{4}}{l^2-ls+m^2+\frac{s^2}{4}}\right)}{l}. && \text{(C.5)} \end{aligned}$$

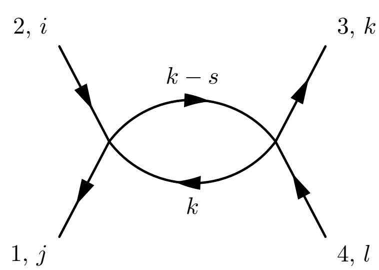
The remaining integral is UV (and IR) convergent since the argument of the log is unity for $l \to \infty$ (and $l \to 0$). Hence we can use contour integration. The numerator:

$$\log\left(l^2+ls+m^2+\frac{s^2}{4}\right) - \log\left(l^2-ls+m^2+\frac{s^2}{4}\right), \tag{C.6}$$

can be thought of as analytic in the upper half plane except on the line (branch cut) connecting $l = im \pm \frac{s}{2}$. Hence the integral localizes as:

$$\begin{aligned} \ldots &= \frac{1}{8\pi\,|s|} \int\limits_{im-|s|/2}^{im+|s|/2} \frac{dl}{2\pi} \frac{\text{Disc}\left[\log\left(\frac{l^2+ls+m^2+\frac{s^2}{4}}{l^2-ls+m^2+\frac{s^2}{4}}\right)\right]}{l} && \text{(C.7)} \\ &= \frac{1}{8\pi\,|s|} \int\limits_{|s|/2}^{|s|/2} \frac{dl}{2\pi} \frac{\text{Disc}\left[\log\left(\frac{(2l+s)(2l+4im+s)}{(2l-s))(2l+4im-s)}\right)\right]}{l+im} && \text{(C.8)} \\ &= \frac{1}{8\pi\,|s|} \int\limits_{|s|/2}^{|s|/2} \frac{dl}{2\pi} \frac{2\pi i}{l+im} && \text{(C.9)} \\ &= \frac{1}{4\pi\,|s|} \frac{i}{2} \log\left(\frac{|s|+2im}{-|s|+2im}\right) && \text{(C.10)} \\ &= \frac{1}{4\pi\,|s|} \frac{i}{2} \log\left(\frac{1-i\frac{|s|}{2m}}{1+i\frac{|s|}{2m}}\right) && \text{(C.11)} \\ &= \frac{1}{4\pi\,|s|} \arctan\left(\frac{|s|}{2m}\right). && \text{(C.12)} \end{aligned}$$

Finally we identify $|s| = \sqrt{-S}$ to get:



$$= \frac{1}{4\pi\sqrt{-S}} \arctan\left(\frac{\sqrt{-S}}{2m}\right). \tag{C.13}$$

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