Data Assimilation on the Navier-Stokes Equations in Two Dimensions : Convergence and Stability Results for Coupling Methods

Pauline Lelandais, Department of Mathematics

McGill University, Montreal

December, 2021

A thesis submitted to McGill University in partial fulfillment of the requirements of the degree of

Master of Theoretical Mathematics

Dec. 2021

Abstract

We introduce several data assimilation techniques for the Navier-Stokes equations and, in the last chapter of this thesis, focus on a coupling scheme for the Navier-Stokes equations in two dimensions, employing mesh measurements of only one component in the velocity field. To do so, we start by using classical physical laws to derive the equation itself in the case of incompressible flows. We then work within the n-dimensional torus to present in details some classical results related to Fourier spaces, which we then employ to discuss the Leray projector and how to recover a solution from within the divergence-free space. In a second chapter, we provide a detailed analysis of two classical data assimilation techniques on the Lorenz equation. Finally we present a thorough proof of the final result of this thesis in which we provide conditions on the resolution of the measured data which are sufficient for the coupling algorithm to converge to the unique exact unknown two dimensional Navier-Stokes system at an exponential rate asymptotically in time.

Abrégé

Nous introduisons plusieurs techniques d'assimilations de données pour les équations de Navier-Stokes et, au cours du dernier chapitre de cette thèse, nous nous concentrons sur une méthode de couplage pour les équations de Navier-Stokes en deux dimensions, utilisant des mesures de maillage pour une seul composante du champ de vélocité. Pour ce faire, nous commençons par dériver les équations de Navier-Stokes (en nous limitant aux fluides incompressibles) en utilisant des lois fondamentales de physique. Par la suite, nous nous déplaçons vers le torus de dimension n pour y présenter quelques résultats classiques ayant rapport aux espaces de Fourier. Nous employons ensuite ces outils pour étudier le projecteur de Leray nous permettant ainsi de récupérer une solution aux équations de Navier Stokes à partir de l'espace de zéro-divergence. Dans un second chapitre, nous fournissons une analyse détaillée de deux techniques majeures d'assimilations de données, que nous appliquons à l'équation de Lorenz. Pour conclure, nous présentons le résultat final de ce texte dans lequel nous développons des conditions sur le niveau de résolution des données mesurées qui sont suffisantes pour que l'algorithme de couplage converge à l'unique solution inconnue du système de Navier-Stokes en two dimensions, à un taux exponentiel dans le temps.

Acknowledgements

Avant de commencer, je voudrais remercier principalement ma mère, sans qui je ne serais vraiment pas arrivée jusqu'ici. Je remercie mon père et mes frères et soeurs pour m'avoir soutenue dans mes études malgrès nos différences. Je remercie mon oncle pour m'avoir toujours donné la possibilité d'apprendre, et ma tante pour avoir tant élargit mes horizons.

I would like to thank my supervisor, for all his support and for always answering my emails when I needed help. More generally, I'm thankful for all my teachers at McGill University, and also for my first mathematics teachers at Pasadena City College. I was lucky to be taught by so many talented and inspiring teachers.

I thank my friend, peer and colleague Karim for all the assignments and study sessions. I would truly not have made it through these last few semesters without your help. A special word for Yiming, Alex, Isabelle and Monika, who have played an important role in keeping my spirits up throughout this degree.

Pour finir je voudrais remercier mes très chères amies Loïse et Johanna, pour toujours m'accompagner et veiller à ma santé, autant physique que mentale. Je vous dois beaucoup les filles.

Table of Contents

Introduction

When studying partial differential equations, it becomes quickly clear that in some cases, one will not be able to find a closed form solution to a given problem. That is why the topic of numerical approximations gains all its importance when confronted with PDEs such as the Navier-Stokes equations, which to this day cannot always be solved in three dimensions. In this thesis, we will discuss the subject of data assimilation techniques, which differs from other types of numerical approximation schemes in that it relies on a mathematical model on top of measured data. These data assimilation schemes have been proven to perform better than others when modeling chaotic (Stochastic) PDEs. This is due to the fact that, for example when dealing with weather forecasting models, a slight error in measured data can lead to a much larger error in the predicted state of the system. Such methods are widely implemented in meteorology or oceaonography, for example ([7]).

We distinguish two main types of data assimilation schemes. First we have the probabilistic approach, which includes the well known Kalman filter and all subsequent filtering methods ([4]). These usually rely on the assumption that the noise observed in the measured data is normally distributed and a careful study of the covariance quantity is required to understand the efficiency of this technique. One of the downside of these filters is that for most relevant applications, the generated systems can be as big as $\mathcal{O}(19)$, in global weather forecasting for example. Thus these can be expensive methods to implement. On the other side of filtering methods, one finds variational techniques, such as coupling methods, which will be the focus of this thesis ([3]). Classical methods of continuous data assimilation work by directly inserting measured data into a mathematical model before integrating with respect to time ; then with the use of finite Fourier modes, one is able to compute a good approximated solution. However in this thesis, we will be leading towards a result on couplings methods, which differs from other variational techniques in the fact that it introduces a feedback control term that forces the model towards the reference solution. Because we do not currently have set conditions for a global attractor to exists for the Navier Stokes Equation in $n > 2$ dimensions, we will focus on the two dimensional case, where the Navier-Stokes equations are known to have solutions (global in time) and a finite-dimensional global attractor.

However since this thesis is also meant to introduce the Navier-Stokes equations and some general ideas in data assimilation, we will dedicate Chapter 2 to the study of projection techniques onto the divergence free space, which is one the classical tool used to show existence and uniqueness of solutions in the n-dimensional torus. Then in Chapter 3, we will spend some time explaining the inner workings of two common types of data assimilation techniques (chaos synchronisation and the Kalman filter), however for simplicity we will work on the Lorenz equation as to avoid the convection term present in the Navier-Stokes equation.

Chapter 1

Derivation of the Navier Stokes Equation and the Leray Projector

This first chapter is dedicated to setting up the framework within which much of this thesis operates. We will first derive the Navier Stokes Equation, in an effort to better understand the physical phenomenon it models. Then we will define appropriate notation for norms and inner products that will be used throughout this discussion. Finally we will introduce Leray projections and show what role it plays in solving the Navier-Stokes equation.

1.1 Derivation of the PDE

The Navier-Stokes equation describes conservation of mass and momentum for Newtonian fluids (ie. certain models of fluids that accounts for viscosity). In this derivation we will restrict our attention to incompressible and isotropic fluids, which lets us assume constant fluid volume and orientation independence. Recall the Cauchy Momentum Equation :

$$\frac{D}{Dt}u = \frac{1}{\rho}\nabla \cdot \sigma + f$$

where the operator $\frac{D}{Dt}$ is the material derivative, which leads to

$$\frac{D}{Dt}u = \frac{\partial}{\partial t}u + u \cdot \nabla u.$$

The fluid density is denoted by ρ, $\nabla \cdot \sigma$ is the stress tensor and f is the acceleration vector. The stress σ can be decomposed into the sum of the volumic and deviatoric stress :

$$\sigma = \tau - pI.$$

In the case of incompressible flows, the stress tensor is a variable of Δu only and the isotropy assumption lets us rewrite $\nabla \tau = \mu \Delta u$, hence

$$\frac{1}{\rho}\nabla \cdot \sigma = \frac{1}{\rho}\big(\mu \Delta u - \nabla p\big).$$

Therefore, letting $\nu = \frac{\mu}{\rho}$, we find that the Cauchy Momentum Equation leads to :

$$\frac{\partial}{\partial t}u + u \cdot \nabla u = -\frac{1}{\rho}\nabla p + \nu \Delta u + f.$$

For simplicity, we will assume $\nu = 1$ and $f \equiv 0$ throughout this chapter. The incompressible flow assumption also leads to u being divergence free. This constraint together with the previous equation and an IVP form the Navier-Stokes Equation PDE :

$$\begin{cases} \frac{\partial}{\partial t}u + u \cdot \nabla u = -\frac{1}{\rho}\nabla p + \Delta u \\ \nabla \cdot u = 0 \\ u(0,t) \equiv g \\ \nabla \cdot g = 0 \end{cases} \tag{1.1}$$

Of course solving this equation is much harder than deriving it, but some settings are well studied (see [2], [1]).

1.2 The Fourier Space and some Preliminary Results

We will use some Fourier Analysis results throughout this thesis, simply because the linear structure and invertibility properties it offers will simplify some arguments.

Let us start by defining $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ such that the points 0 and 2π are identified with one another in $\mathbb{T}$. Then by letting the default norm in this chapter be the L^2-norm, for some function v defined on $\mathbb{T}$ we have :

$$\|v\|^2 := \|v\|_{\mathbb{T}}^2 = \int_0^{2\pi} |v(x)|^2 dx.$$

Recall the associated inner product

$$\langle u, v\rangle := \langle u, v\rangle_{\mathbb{T}} = \int_0^{2\pi} u(x)\bar{v}(x)dx.$$

Then for a function $v \in L^2(\mathbb{T})$, define its Fourier coefficients as

$$\hat{v}(k) = \frac{1}{2\pi}\langle v, e_k\rangle = \frac{1}{2\pi}\int_0^{2\pi} v(x)e^{-ikx}dx, \quad k \in \mathbb{Z},$$

with the representation

$$v(x) = \sum_{k\in\mathbb{Z}} \hat{v}(k)e_k.$$

Then we define the Sobolev space $H^s(\mathbb{T})$:

$$H^s(\mathbb{T}) = \{v \in L^2(\mathbb{T}) : \hat{v} \in l_s^2\},$$

where

$$l_s^2 = \{v \in l_2 : (|k|^s\hat{v}_k)_k \in l_2\},$$

And the norm

$$\|(\hat{v}_k)_{k\in\mathbb{Z}}\|_{l_2} = \Big(\sum_{k\in\mathbb{Z}} |\hat{v}_k|^2\Big)^{\frac{1}{2}}.$$

Then for $f, g \in H^s$, we have

$$\langle f, g\rangle_s = 2\pi \sum_{k\in\mathbb{Z}^n} (1 + |k|^2 s)\hat{f}(k)\bar{\hat{g}}(k),$$

which is equivalent to

$$\langle f, g\rangle_s = \langle f, g\rangle + \left\langle f^{(s)}, g^{(s)}\right\rangle,$$

when s in an integer, where $f^{(s)}, g^{(s)}$ are the sth derivatives of f and g.

Now, let h be the weak derivative of g and let us compute

$$\hat{h}(k) = \frac{1}{2\pi}\langle h, e_k\rangle = \frac{1}{2\pi}\int_{\mathbb{T}} h(x)e^{-ikx}dx.$$

By integration by parts, we get

$$= \frac{1}{2\pi}\Big(g(x)e^{-ikx}\Big)\Big|_{\mathbb{T}} + \frac{1}{2\pi}\int_{\mathbb{T}} g(x)ike^{-ikx}dx$$

And by our definition of $\mathbb{T}$ as an equivalence class, the first term becomes

$$g(0)\times 1 - g(0)\times 1 = 0$$

Hence we have

$$\hat{h}(k) = ik\frac{1}{2\pi}\int_{\mathbb{T}} g(x)e^{-ikx}dx = ik\langle g, e_k\rangle = ik\hat{g}(k),$$

And so

$$g^{(1)}(x) = \sum_{k\in\mathbb{Z}} \hat{g^{(1)}}(x)e^{kx} = \sum_{k\in\mathbb{Z}} \hat{h}(x)e^{kx} = \sum_{k\in\mathbb{Z}} ik\hat{g}(x)e^{kx}.$$

Thus for $l \in \mathbb{N}$, we find

$$g^{(l)}(x) = \sum_{k \in \mathbb{Z}} (ik)^l \hat{g}(x) e^{kx},$$

which leads to the inequality :

$$|g^{(l)}(x)| = \big| \sum_k (ik)^l \hat{g}(k) e^{ikx} \big| \leq \sum_k |k^l \hat{g}(k)|$$

Finally we will state Bernstein's Theorem as it will be useful in the final sections of this thesis.

Theorem 1. *(Bernstein) Let $s \in \mathbb{R}$ and $r \in \mathbb{N}_0$ be such that $s > r + \frac{n}{2}$. Then $H^s(\mathbb{T}^n)$ is continuously embedded in $C^r(\mathbb{T}^n)$. That is, we have $H^s(\mathbb{T}^n) \subset C^r(\mathbb{T}^n)$ and there exists a constant c such that $\|g\|_{C^r} \leq c \|g\|_s$ for all $g \in H^s(\mathbb{T}^n)$.*

A complete proof can be found in ([13]).

1.3 The Leray Projector

The goal of this section is to establish conditions on u, p and n such that the existence of a global solution to (3.1) is guaranteed on $\mathbb{T}^n$. More precisely, we will now develop a strategy to use global existence results from the Heat Equation in order to solve the Navier-Stokes equations problem. Let $u \in \mathbb{T}^n \times [0, T) \mapsto \mathbb{R}^n$ and $p :\in \mathbb{T}^n \times [0, T) \mapsto \mathbb{R}$, $0 < T \leq \infty$.

Consider the following quantity :

$$\hat{\partial_j u_k}(\xi) = \mathcal{F}(\partial_j u)_k(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \partial_j u(\xi) e^{-ik \cdot \xi} d\xi$$

$$= (2\pi)^{-n} \Big(u(\xi) e^{-ik \cdot \xi} \Big) \Big|_{\mathbb{T}^n} - \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} u(\xi) \partial_j e^{-ik \cdot \xi} d\xi$$

$$= 0 - \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} u(\xi)(-i\xi_j) e^{-ik\cdot\xi} d\xi = (i\xi_j)\hat{u_k}(\xi),$$

where the second line simply comes from integration by parts, using the periodicity conditions on u together with the Dirichlet initial value problem. We thus find, in vector notation :

$$\widehat{\nabla \cdot u}(\xi) = i\xi \cdot \hat{u}(\xi)$$

So in H^s, the divergence is component wise proportional to ξ, which is the radial component of the vector field. Hence if a vector field is divergence free then $\hat{u} \cdot \xi = 0$. For a scalar field we find :

$$\widehat{\nabla p}(\xi) = i\xi\hat{p}(\xi)$$

So the gradient is purely radial. Keeping these observations in mind, let us now define the Leray Projector [6] $\mathbb{P} : H^s \mapsto H^s$:

$$\widehat{\mathbb{P}u}(\xi) = \Big(I - \frac{\xi \otimes \xi}{|\xi|^2}\Big)\hat{u}(\xi)$$

Where $\xi \otimes \xi$ is notation for the matrix with entries $\{\xi_i \xi_j\}_{i,j\leq n}$. We claim that this operator acts as a projection onto the divergence free space . Indeed, if we assume u is divergence free except for its kth component, then

$$\widehat{\mathbb{P}u}(\xi) = \Big(I - \frac{\xi \otimes \xi}{|\xi|^2}\Big) \langle \hat{u}_j(\xi)\rangle_{j\leq n}$$

Where the radii $\xi_i = 0$ for all $i \neq k$ by the previous observations, so the divergent free components of u are only hit by the identity operator and thus remain unchanged. For $i = k$ on the other hand, the tensor term clearly cancels out with the identity operator thus removing the purely divergent component of u.

We therefore have the following properties :

$$\mathbb{P}^2 = \mathbb{P}, \quad \langle u - \mathbb{P}u, \mathbb{P}u\rangle = 0, \quad \nabla \cdot \mathbb{P}u = 0$$

$$\mathbb{P} : H^s \mapsto H^s \quad \text{is continuous and bounded.}$$

Let us now restrict our solution space to pairs (u, p) such that :

$$\begin{cases} u \in C^1(\mathbb{T}^n \times (0,T)) \text{ and } \partial_i \partial_j u \in C(\mathbb{T}^n \times (0,T)) \\ p \in C(\mathbb{T}^n \times (0,T)) \text{ and } \partial_i p \in C(\mathbb{T}^n \times (0,T)) \end{cases} \tag{1.2}$$

We will now apply $\mathbb{P}$ to the Navier-Stokes equations equation and show that solving the projected problem is equivalent to solving the Navier-Stokes equations (1.1).

$\underline{\Rightarrow}$: Let u and p, satisfying the assumptions above, be such that the Navier-Stokes equations hold true, ie :

$$\begin{cases} \frac{\partial}{\partial t} u + u \cdot \nabla u = -\frac{1}{\rho} \nabla p + \Delta u \\ \nabla \cdot u = 0 \end{cases}$$

Let us apply $\mathbb{P}$ to the Navier-Stokes equations :

$$\mathbb{P} \frac{\partial}{\partial t} u + \mathbb{P} u \cdot \nabla u = -\frac{1}{\rho} \mathbb{P} \nabla p + \mathbb{P} \Delta u$$

Clearly since $\nabla \cdot u = 0$, $\mathbb{P} u = u$. Then since the Laplacian commutes with isometries (ie; Fourier transforms), $\mathbb{P} \Delta u = \Delta \mathbb{P} u = \Delta u$. It is also straightforward to see that $\mathbb{P} \nabla p = 0$ since ∇p is a purely divergent term. Now for the time partial, consider

$$U(t) := u(\cdot, t) \quad \text{then} \quad \mathbb{P} \frac{\partial}{\partial t} u = \mathbb{P} U'(t)$$

And since we have assumed $u \in C^1$, by continuity we have

$$\mathbb{P} U'(t) = (\mathbb{P} U)'$$

$$\iff \mathbb{P} \frac{\partial}{\partial t} u = \frac{\partial}{\partial t} \mathbb{P} u = \frac{\partial}{\partial t} u$$

Finally we will rewrite the convection term as $\mathbb{P}u \cdot \nabla u := \mathbb{P}\text{div}(u \otimes u)$. Collecting terms, we find the projected NSE :

$$\frac{\partial}{\partial t}u + \mathbb{P}\text{div}(u \otimes u) = \Delta u \tag{1.3}$$

$\Leftarrow$: We will now show that solving the projected equation (1.3) is equivalent to solving the original Navier-Stokes equations (1.1). To this end let us consider $u : \mathbb{T}^n \times (0, T) \mapsto H^s$ such that the projected NSE is satisfied and $u(t) \xrightarrow[L^2]{t \longrightarrow 0} g$ for some $g \in L^2$ with $\mathbb{P}g = 0$. Let us apply $\mathbb{P}$ to (1.3) :

$$\mathbb{P}\frac{\partial}{\partial t}u + \mathbb{P}^2\text{div}(u \otimes u) = \mathbb{P}\Delta u$$

By the same arguments as before we find $\mathbb{P}\frac{\partial}{\partial t}u = \frac{\partial}{\partial t}\mathbb{P}u$ and $\mathbb{P}\Delta u = \Delta\mathbb{P}u$ (note that we have not assumed that $\nabla \cdot u = 0$). Also since $\mathbb{P}^2 = \mathbb{P}$ we find :

$$\frac{\partial}{\partial t}\mathbb{P}u + \mathbb{P}\text{div}(u \otimes u) = \Delta\mathbb{P}u \tag{1.4}$$

Now consider (1.3)-(1.4) :

$$(u - \mathbb{P}u)_T = \Delta(u - \mathbb{P}u)$$

which is the Heat Equation for $(u - \mathbb{P}u)$. The goal now is to apply the existence and uniqueness result we know hold for the Heat Equation. In particular since $u(t) \xrightarrow[L^2]{t \longrightarrow 0} g$ for some $g \in L^2$ with $\mathbb{P}g = 0$, then by uniqueness of the solutions of the Heat Equation, $u = \mathbb{P}u$ on $(0, T]$. This automatically gives us the divergence free constraint of the Navier-Stokes equations (ie : $\nabla \cdot u = 0$).

We are now left with showing that such a function u solves the original Navier-Stokes equations (1.1). To this end, let $v \in L^2$ and q a scalar field defined in the following way :

$$\hat{q}(\xi) = \begin{cases} -i\xi \cdot \frac{\hat{v}(\xi)}{|\xi|^2} & \xi \neq 0 \\ \hat{q}(0) = 0 & \text{else} \end{cases}$$

This function is designed to pick up the purely divergent components of v such that we find the following decomposition :

$$v = \mathbb{P} + \nabla q$$

If div$(u \otimes u) \in L^2$, then there exists a scalar field q such that

$$\text{div}(u \otimes u) = \mathbb{P}\text{div}(u \otimes u) + \nabla q$$

So going back to (1.3) :

$$u_t + \text{div}(u \otimes u) - \nabla q = \Delta u$$

And if we further assume that div$(u \otimes u)$ is smooth then we find :

$$\Rightarrow u_t + u \cdot \nabla u = \nabla q \Delta u$$

Thus the Navier-Stokes equations is satisfied. We have then showed that for some function such that $u(t) \xrightarrow[L^2]{t \longrightarrow 0} g$ for some $g \in L^2$ with $\mathbb{P}g = 0$ and div$(u \otimes u)$ is smooth then solving the Navier-Stokes equations (1.1) is equivalent to solving (1.3). Such a solution is called a strong solution to the Navier-Stokes equations.

On the other hand, one can define, using Duhamel's Principle (see appendix **5.1**), a mild solution to the NSE by :

$$u(t) = e^{t\Delta}g + \int_0^t e^{(t-\tau)\Delta} f(u(\tau)) d\tau \quad 0 < \tau < T$$

where $f(u) \equiv$ div$(u \otimes u)$. Therefore by Duhamel's Principle (part (a)), if $f(u) \in C(\mathbb{T}^n) \times (0, T]$ then u is a strong solution to the Navier-Stokes equations. Note that then if u is a continuous mild solution, then u satisfies the Navier-Stokes equations classically.

We note that this projection is the basis for many well-posedness proofs and enabled to show local existence of solutions of the Navier-Stokes equations. For example, the global well-posedness of solutions for small data and local well-posedness for large data was demonstrated in [9].

Chapter 2

Data Assimilation schemes for the Lorenz Equation

Now that we have some knowledge about the Navier-Stokes equations, we can start to ask ourselves how to implement data assimilation techniques on that system. However, since these schemes are non-trivial to understand and that the convection term in the Navier-Stokes equations complicates matters even more, we would like to start on a simpler model, the Lorenz Equations :

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = x(\rho - z) - y \\ \dot{z} = \beta z + xy \end{cases} \tag{2.1}$$

These equations were developed from the Navier-Stokes equations and the equation describing thermal energy diffusion, and they describe the motion of a fluid under Rayleigh-Bénard flow conditions. Thus this model has limitations and will not provide accurate results for all settings, it is however sufficient to get useful insight into fluid mechanics. Unfortunately, we will not be including a derivation of the Lorenz system in this thesis, as it involves a lot of additional concepts in fluid mechanics and physics. This chapter will be focused on studying the theory behind two of data assimilation methods : the Kalman Filter and Synchronisation of Chaos ; and we will end by an analysis of the results of an implementation of these techniques on the Lorenz equations.

2.1 The Kalman Filter

Originally more of a statistical tool rather than a numerical analysis method, the Kalman Filter is a predictor-corrector type estimator that optimises the estimated error covariance matrix at each time step of the scheme. It relies on properties of Gaußian random variables to understand the behavior of the noise generated from both the measurements and the process [14]. It has various practical applications in robotics, economics and trajectory estimations. In this section we'll study the basic principles behind the Kalman Filter in order to later apply it to a non-linear system.

2.1.1 The discrete Kalman Filter

For a system ruled by the matrix $A \in \mathbb{R}^{n \times n}$, let the vector $x_k \in \mathbb{R}^n$ represent the state of the system at time step k. We introduce the following system :

$$\begin{cases} x_{k+1} = Ax_k + Bu_k + \omega_{k+1} \\ z_k = Hx_k + v_k \end{cases} \tag{2.2}$$

Where $Bu_k \in \mathbb{R}^n$ is the control input (that we will take $\equiv 0$ going forward), $\omega_k \in \mathbb{R}^n$ is the process noise, $\nu_k \in \mathbb{R}^n$ is the measurement noise, $z_k \in \mathbb{R}^n$ is the "true" measured value at time step k and $H \in \mathbb{R}^{n \times n}$ represents how the measurements affect the state of the system. In practice the exact values of $\omega_k \in \mathbb{R}^n$ and $\nu_k \in \mathbb{R}^n$ are inaccessible, but we assume that they are both drawn from normally distributed random variables $W \sim N(0, Q)$ and $V \sim N(0, R)$, which are pairwise independent.

We define $\hat{x}_k$ and $\tilde{x}_k$ to be vectors in $\mathbb{R}^n$ that respectively describe the a priori estimate (given by the state update) and the a posteriori estimate (given by the measurement) at time step k. We also define the a priori and a posteriori errors as :

$$\hat{e}_k = x_k - \hat{x}_k \quad \text{and} \quad \tilde{e}_k = x_k - \tilde{x}_k$$

Then the a priori and a posteriori estimated covariance matrices become respectively :

$$\hat{P}_k = \mathbb{E}[\hat{e}_k \hat{e}_k^T] \quad \text{and} \quad \tilde{P}_k = \mathbb{E}[\tilde{e}_k \tilde{e}_k^T]$$

The goal of the Kalman filter is to compute an a posteriori estimate that minimizes the a posteriori estimated covariance matrix $\tilde{P}_k$, knowing both $\hat{x}_k$, $\tilde{x}_k$ and z_k. We therefore claim there exists a matrix $K \in \mathbb{R}^{n \times n}$, the *Kalman Gain*, such that the following iteration minimises $\tilde{P}_k$:

$$\tilde{x}_k = \hat{x}_k + K(z_k - H\hat{x}_k) \tag{2.3}$$

By taking the derivative with respect to K of $tr(\tilde{P}_k)$ one finds that

$$K = \tilde{P}_k H^T \big(H\tilde{P}_k H^T + R\big)^{-1} \tag{2.4}$$

Where we recall that R is the covariance matrix of the measurement noise. Observe that K has some interesting behaviors as R and $\tilde{P}_k$ vary. If R tends to 0 :

$$\lim_{R \longrightarrow 0} K = \tilde{P}_k H^T H \tilde{P}_k^T H^{-1} = H^{-1}$$

If we use $K = H^{-1}$ in (2.3) we see that the update schemes becomes

$$\tilde{x}_k = \hat{x}_k + H^{-1}(z_k - H\hat{x}_k) = H^{-1} z_k = x_k + H^{-1} v_k \quad \text{by (2.1) .}$$

We recall that H is the matrix that expresses how the measurements affect the state update and that v_k is a sample point drawn from the RV representing the measurement noise so this means that the gain is going to favor the experimental measurement input rather than the iterative method's prediction. On the other hand if $\tilde{P}_k$ tends to 0 we have :

$$\lim_{\tilde{P}_k \longrightarrow 0} K = 0$$

Therefore (2.3) becomes :

$$\tilde{x}_k = \hat{x}_k + K(z_k - H\hat{x}_k) = \hat{x}_k$$

We clearly see that in this case the filter trusts the expected value over the experimental input.
Another observation worth making is that the Kalman Filter maintains the expectation and variance of the state distribution :

$$\mathbb{E}[x_k] = \tilde{x}_k \quad \text{and} \quad \mathbb{E}[(x_k - \tilde{x}_k)(x_k - \tilde{x}_k)^T] = P_k$$

Note that in a situation where W and V are indeed centered random variables then $\mathbb{E}[x_k] = 0$. We conclude that the random variable $X_k|Z_k \sim N(\mathbb{E}[x_k], P_k) = N(\tilde{x}_k, P_k)$.

We are now ready to state the full iterative scheme. The Kalman filter works through feedback control and therefore it has a dual structure. The first part of the algorithm, the predictor, projects the process in time given knowledge of the state up to time step $k-1$ and the second part, the corrector, gives feedback on that estimate by means of noisy measurement in order to possibly refine the prediction. Given initial x_0 and P_0, the complete scheme can be summarized by the following two systems :

$$\begin{cases} \hat{x}_k = A\tilde{x}_k + Bu_k \\ \hat{P}_k = AP_{k-1}A^T + Q \end{cases} \tag{2.5}$$

$$\begin{cases} K_k = \hat{P}_k H^T (H \hat{P}_k H^T + R)^{-1} \\ \tilde{x}_k = \hat{x}_k + K_k (z_k - H \hat{x}_k) \\ P_k = (I - K_k H) \hat{P}_k \end{cases} \tag{2.6}$$

Where (2.5) would be the predictor and (2.6) the corrector. This recursion format is interesting for a number of reasons, one of them being that both loops run in simultaneously which makes it practical to implement, see Section 4 for more.

2.1.2 The extended Kalman Filter

Recall that the previous derivations held under the assumption that A was a linear system. However most dynamical systems are some shade of non-linear, therefore our theory requires adjustments. One simple way of solving this problem would be to linearise our dynamical system, for example by means of Taylor Expansion. Thus consider the following system :

$$\begin{cases} x_k = f(x_{k-1}, u_k, \omega_k) \\ z_k = h(x_k, v_k) \end{cases} \tag{2.7}$$

Which is the equivalent of 3.1 for general f, h that we will assume non-linear going forward. We can then define the a priori estimates as :

$$\hat{x}_k = f(\tilde{x}_k, u_k, 0) \quad \text{and} \quad \hat{z} = h(\hat{x}, 0)$$

And we can now use these to Taylor expand (2.7) around the current estimate x_{k-1} :

$$\begin{cases} x_k \approx \hat{x}_k + A(x_{k-1} - \tilde{x}_{k-1}) + W\omega_k \\ z_k \approx \hat{z}_k + H(x_k - \hat{x}_k) + V\nu_k \end{cases} \tag{2.8}$$

Where x_k and z_k represent the *truth* of the system and of the measurement. We also have $A \in \mathbb{R}^{n\times n}$ such that $\forall i, j \leq n$:

$$A_{(i,j)} = \frac{\partial}{\partial x_j} f_i(\tilde{x}_k, u_k, 0) \quad \text{the Jacobian of } f$$

and $H \in \mathbb{R}^{n\times n}$ satisfies :

$$H_{(i,j)} = \frac{\partial}{\partial x_j} h_i(\hat{x}, 0)$$

The noise sample points ω_k and ν_k are as in (2.2) and $W, V \in \mathbb{R}^{n\times n}$ are such that :

$$W_{(i,j)} = \frac{\partial}{\partial \omega_j} f_i(\tilde{x}_k, u_k, 0) \quad \text{and} \quad V_{(i,j)} = \frac{\partial}{\partial \nu_j} h_i(\hat{x}, 0)$$

Note that the expression for the noises is not as nice as in the linear case because the Taylor expansion does not allow us to recover the covariance matrices.

We define the new estimated errors :

$$e_{x_k} \approx x_k - \hat{x}_k \quad \text{and} \quad e_{z_k} \approx z_k - \hat{z}_k$$

In practice we do not have access to x_k, but we can get our hands on z_k. Using (2.8), these can be rewritten as :

$$\begin{cases} e_{x_k} \approx A(x_{k-1} - \tilde{x}_{k-1}) + \epsilon_k \\ e_{z_k} \approx H(e_{x_k}) + \eta_k \end{cases}$$

Where ϵ_k and η_k are sample points drawn from new centered RVs with covariance matrices WQW^T and VRV^T respectively. Note that these expressions are linear, and that one can write :

$$x_k \approx \hat{x}_k + e_{x_k} = \tilde{x}_k \tag{2.9}$$

So now going back to the linear Kalman Filter scheme, we can use the second equation in (2.6) : $\tilde{x}_k = \hat{x}_k + K_k(z_k - H\hat{k})$ and write :

$$e_{x_k} = K_k e_{z_k}$$

Then using (2.9), the first equation of that system becomes :

$$K_k e_{z_k} + \hat{x}_k = \tilde{x}_k$$

And the last line reads :

$$\tilde{x} = K_k(z_k - \hat{z}_k) + \hat{x}_k$$

We now have our full non-linear scheme :

$$\begin{cases} \hat{x}_k = f(\tilde{x}_{k-1}, u_k, 0) \\ \hat{P}_k = AP_{k-1}A^T + WQW^T \end{cases} \tag{2.10}$$

$$\begin{cases} K_k = \hat{P}_k H^T (H\hat{P}_k H^T + VRV^T)^{-1} \\ \tilde{x}_k = \hat{x}_k + K_k(z_k - h(\hat{x}_k, 0)) \\ P_k = (I - K_k H)\hat{P}_k \end{cases} \tag{2.11}$$

We note that these need initial $\tilde{x}_0$ and P_0. We also mention that A, W, V, Q, R and H could change at each time step k, but the subscript is dropped here for lighter notation.

2.2 Synchronisation of Chaos

Synchronisation expresses a notion of strong correlation between coupled systems. Chaos normally arises when dynamical behaviors have locally dispersing characteristics. One tool for better understanding how chaotic of a particular system is called the Lyapunov exponent. In this section we will derive a method for coupling two dynamical systems and see that synchronisation is guaranteed provided the coupling strength is chosen above a certain threshold [5].

2.2.1 Synchronisation of Linear Systems

We start this exercise by considering two linear systems, with the elementary $y' = \lambda y$ example in mind. Let us define for some constant $a \neq 0$:

$$\begin{cases} \dot{x}_1 = a x_1 \\ \dot{x}_2 = a x_2 \end{cases}$$

such that these have solution : $x_i(t) = x_i(0)e^{at}$. Now let α be the coupling force, consider :

$$\begin{cases} \dot{x}_1 = a x_1 + \alpha(x_2 - x_1) \\ \dot{x}_2 = a x_2 + \alpha(x_1 - x_2) \end{cases} \tag{2.12}$$

Our goal will be to find values of α such that $z := x_1 - x_2$ converges to zero as $t \to \infty$. Using (2.12) we get :

$$\dot{z} = a x_1 - a x_2 + 2\alpha(x_2 - x_1) = z(a - 2\alpha)$$

Which has solution $z(t) = z(0)e^{(a-2\alpha)t}$. Let us evaluate its behavior at infinity:

$$\lim_{t \longrightarrow \infty} z(0)e^{(a-2\alpha)t} = 0 \iff a - 2\alpha < 0 \iff \alpha > \frac{a}{2}$$

Hence we define $\alpha_c := \frac{a}{2}$ to be the critical coupling value such that synchronisation is guaranteed provided we use a coupling strength $\alpha > \alpha_c$. This method is straight forward for linear systems, but will not be adaptable to the Lorenz system. In order to find a more robust model, let us rewrite the previous problem in vector form. If $\mathbf{x} = (x_1, x_2)^T$ then

(2.12) becomes :

$$\dot{\mathbf{x}} = [aI - \alpha L]\mathbf{x} \tag{2.13}$$

Where L is the Laplacian operator. Then the solution to (2.13) is $\mathbf{x}(t) = \exp\{(aI - \alpha L)t\}\dot{\mathbf{x}}(0)$. In this simple example, I the identity matrix and L commute, so $\exp\{(aI - \alpha L)t\} = e^{at}Ie^{-\alpha Lt}$. To fully use the underlying linear structure of this problem we compute the eigenvalues and corresponding eigenvectors of

$$L = \begin{bmatrix} 1-1 \\ \\ -1, 1 \end{bmatrix} \qquad \det(|\lambda I - L|) = (\lambda - 1)^2 - 1 = \lambda(\lambda - 2)$$

So the eigenvalues of L are $\lambda_1 = 0$ and $\lambda_2 = 2$, with eigenvectors $v_1 = (1, 1)$ and $v_2 = (1, -1)$. Since $\{v_1, v_2\}$ forms a basis for $\mathbb{R}^2$, we can rewrite $\mathbf{x}(0) = c_1 v_1 + c_2 v_2$, $c_i \in \mathbb{R}$ and thus since $\lambda_i = Lv_i$ we can write :

$$\mathbf{x}(t)e^{-\alpha Lt} = c_1 v_1 + c_2 e^{-\alpha\lambda_2 t} v_2$$

Therefore the solution $\mathbf{x}$ can be expressed as :

$$\mathbf{x}(t) = e^{aI - \alpha Lt}\mathbf{x}(0) = c_1 e^{at} v_1 + c_2 e^{(a - \alpha\lambda_2)t} v_2 \tag{2.14}$$

Recall that our goal was to have the coupled systems converge to each other, so we want $\mathbf{x}$ to converge to the synchronisation space generated by v_1 alone. Since v_1 and v_2 are independent this only happens when

$$\lim_{t \longrightarrow \infty} c_2 e^{(a - \alpha\lambda_2)t} v_2 = 0 \iff \alpha > \frac{a}{\lambda_2}$$

Letting $\alpha_c := \frac{a}{\lambda_2} = \frac{a}{2}$, we see that we recover our previous solution. Having made this link to eigenvalues and synchronisation subspaces, we are now ready to study non-linear cases.

2.2.2 Complete Synchronisation of a non-linear system

Keeping in mind that the goal of this chapter is to study the Lorenz system, we consider f a non linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ and H a smooth coupling function also from $\mathbb{R}^n$ to $\mathbb{R}^n$ with $H(0) = 0$, then we study the following system :

$$\begin{cases} \dot{\mathbf{x}}_1 = f(\mathbf{x}_1) + \alpha H(\mathbf{x}_2 - \mathbf{x}_1) \\ \dot{\mathbf{x}}_2 = f(\mathbf{x}_2) + \alpha H(\mathbf{x}_1 - \mathbf{x}_2) \end{cases} \tag{2.15}$$

We want to find α_c such that any coupling strength $\alpha > \alpha_c$ will ensure $\lim_{t\to\infty} \mathbf{x}_1(t) - \mathbf{x}_2(t) = 0$. Consider $H = I$, then $\alpha H(\mathbf{x}_2 - \mathbf{x}_1) = \alpha(\mathbf{x}_2 - \mathbf{x}_1)$ so just as before we will let $\mathbf{z} := \mathbf{x}_1 - \mathbf{x}_2$. The system 3.17 implies :

$$\dot{\mathbf{z}(t)} = f(\mathbf{x}_1) + \alpha H(\mathbf{x}_2 - \mathbf{x}_1) - f(\mathbf{x}_2) - \alpha H(\mathbf{x}_1 - \mathbf{x}_2) = f(\mathbf{x}_1 - \mathbf{x}_2) - 2\alpha \mathbf{z} \tag{2.16}$$

Now in order to use some of the arguments from the previous section, we will need to linearise f. Similarly as in the 2.2, we will use a Taylor expansion on $f(\mathbf{x}_2(t))$ around $f(\mathbf{x}_1(t))$:

$$f(\mathbf{x}_2(t)) = f(\mathbf{x}_1(t)) - Df(\mathbf{x}_1(t))(\mathbf{x}_1 - \mathbf{x}_2) + O(\|\mathbf{x}_1 - \mathbf{x}_2\|^2) = f(\mathbf{x}_1(t)) - \mathbf{z}Df(\mathbf{x}_1(t)) + O(\|\mathbf{z}\|^2)$$

$$\iff 0 = f(\mathbf{x}_1(t)) - f(\mathbf{x}_2(t)) - \mathbf{z}Df(\mathbf{x}_1(t)) + O(\|\mathbf{z}\|^2)$$

Now by (2.16) $\dot{\mathbf{z}} + 2\alpha\mathbf{z} = f(\mathbf{x}_1) - f(\mathbf{x}_2)$ therefore :

$$\Rightarrow 0 = \dot{\mathbf{z}} + 2\alpha\mathbf{z} - \mathbf{z}Df(\mathbf{x}_1(t)) + O(\|\mathbf{z}\|^2)$$

$$\iff \dot{\mathbf{z}} = \mathbf{z}(Df(\mathbf{x}_1 - 2\alpha I)) - O(\|\mathbf{z}\|^2) \tag{2.17}$$

After dropping the $O(\|\mathbf{z}\|^2)$ term, we will call (2.17) the first variational equation, and we note that unfortunately it is not autonomous. Thus to simplify analysis we will consider

$\mathbf{w}(t) = e^{2\alpha t}\mathbf{z}(t)$, then

$$\dot{\mathbf{w}}(t) = 2\alpha e^{2\alpha t}\mathbf{z}(t) + \dot{\mathbf{z}}(t)e^{2\alpha t} = [2\alpha\mathbf{z}(t) + \dot{\mathbf{z}}(t)]e^{2\alpha t} = [2\alpha I\mathbf{z} + \mathbf{z}Df(\mathbf{x}_1) - 2\alpha I\mathbf{z}]e^{2\alpha t}$$

$$= \mathbf{z}e^{2\alpha t}df(\mathbf{x}_1) = \mathbf{w}(t)Df(\mathbf{x}_1)$$

Now it's clear that we can apply a similar argument as in 2.2 to $\mathbf{w}$. Let $\Phi(\mathbf{x}_1(t)) := \Phi_1$ be the fundamental matrix for (2.17), such that any solution to (2.17) can be written as $\Phi(\mathbf{x}_1(t))\mathbf{z}(0)$. Now let $\{\lambda_j(\mathbf{x}_1(t))\}_{j=1}^n$ be the set of positive square roots of eigenvalues of $\Phi_1^T\Phi_1$ and define :

$$\Lambda := \max_j \lim_{t\to\infty} \frac{1}{t}\lambda_j(x_1(t))$$

Then we call Λ the Lyapunov exponent of the orbit $\mathbf{x}_1(t)$. It measure the infinitesimal divergence rate near the trajectory $\mathbf{x}_1(t)$. Our claim is now that if $\mathbf{x}_1(t)$ has Lyapunov exponent Λ then there exists $C > 0$ such that $\|\mathbf{w}(t)\| \leq Ce^{\Lambda t}$. So if we substitute back in terms of $\mathbf{z}$:

$$\|\mathbf{z}(t)\| \leq Ce^{\Lambda - 2\alpha)t}$$

Therefore we can take a limit on both side to get our synchronisation condition :

$$\lim_{t\to\infty} \|\mathbf{z}(t)\| \leq \lim_{t\to\infty} Ce^{\Lambda-2\alpha)t} = 0 \iff \Lambda - 2\alpha < 0 \iff \alpha > \frac{\Lambda}{2}$$

So we conclude that for a non-linear system, we need to apply a coupling force stronger than $\frac{\Lambda}{2} := \alpha_c$ in order to get full synchronisation. Before moving on to the implementation part of this chapter, we'll note that we made the assumptions that Λ and C were constant with respect to $\mathbf{x}_1$. The first statement is often the case for almost every trajectory $\mathbf{x}_1$ by ergodicity. However C always depends on where the system starts. Thus we'll observe non-uniform convergence, which implies potentially large difference in synchronisation times when varying initial data.

2.3 Implementation and study of the Lorenz System

In this section we'll review implementation techniques for both the Kalman Filter and Synchronisation of Chaos applied to a fully non-linear chaotic ODE : the Lorenz System ([12]). Thus let us start by defining the problem, let $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ and $\sigma, \beta, \rho \in \mathbb{R}$ such that :

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = x(\rho - z) - y \\ \dot{z} = \beta z + xy \end{cases} \tag{2.18}$$

For consistency we fixed $\sigma = 10$, $\beta = 28$ and $\rho = \frac{8}{3}$, which are parameters under which the system has been extensively studied and is known to show chaotic behavior [15]. In the next two sections we will discuss some python codes taken from my GitHub : https://github.com/LelandaisPauline/MATH578 in the Data Assimilation notebook.

2.3.1 The Kalman filter for sparse measurements

Recall from section 2.2, (2.9) and (2.10), that we want to implement :

$$\begin{cases} \hat{x}_k = f(\tilde{x}_{k-1}, 0, 0) \\ \hat{P}_k = AP_{k-1}A^T + WQW^T \end{cases}$$

$$\begin{cases} K_k = \hat{P}_k H^T (H\hat{P}_k H^T + VRV^T)^{-1} \\ \tilde{x}_k = \hat{x}_k + K_k(z_k - h(\hat{x}_k, 0)) \\ P_k = (I - K_k H)\hat{P}_k \end{cases}$$

In order to study the efficiency of the Kalman Filter, one needs to generate the random measurements that will be fed in the filter. However we note that in section 2 we have assumed that measurements were taken at every time step. In practice, it would be more convenient if the measurements were taken sparsely. With this argument in mind we set up our filter to run over 2000 time steps and generate a "truth" array using a fourth order Runge-Kutta Method. Then we randomly chose 50 of these points and added normally distributed perturbations to them in order to simulate an array of 50 noisy measurements that we will later feed our filter. The following function takes in nombre=50, length=2000 and X_list, the array containing the "true" state of the system. It outputs the list of $(t_i, \tilde{z}_i)$, the noisy measurements and their time stamps [11].

```
def gen_meas(nombre, length, X_list):
    meas_list=[rd.randint(0, length) for i in range(nombre)]
    meas_list.sort()
    X2_list=[]
    t2_list=[i*h for i in meas_list]
    for i in range(len(meas_list)):
        temp=[]
        for j in range(3):
            x=gauss(mu, sigma)
            temp.append(X_list[meas_list[i]][j]+x)
        X2_list.append(temp)
    return [t2_list, X2_list]
```

Next, we'll recall that we need to find the matrix A such that :

$$A_{(i,j)} = \frac{\partial}{\partial x_j} f_i(\tilde{x}_k, 0, 0)$$

However for faster convergence we actually implemented a three dimensional Forward Euler method :

```
def FE(X):
    matrix=[[1-h*param[0], h*param[0],           0],
            [h*param[1]   , 1-h          , -h*X[0]],
            [h/2*X[1]        , h/2*X[0]      , 1-h*param[2]]]
```

```
    return np.array(matrix)
```

Where h is the time step ($h = 0.01$ throughout the notebook), param is an array containing σ, β and ρ and X is the state of the system at time step k. Note that this returns a map, not a vector. We are now ready to implement the filter itself. Because we are dealing with sparse measurements, we implemented an extra if-statement to only run the measurement update part of the filter if that particular time step has a measurement input. The function reads :

```
for i in range(n):
    EKF=0
    index=-2
    for j in range(len(Z_list)):
        if i*h == Z_list[j][0]:
            EKF=1
            index=j
    omega=np.array([gauss(0, Q_eps) for i in range(3)])
    Xnxt_time=RK4_mat(X_list[i])+omega
    a=FE(Xnxt_time)
    Pnxt_time=np.matmul(np.matmul(a, P), np.transpose(a))+Q

    if EKF:
        #meas update
        K=np.matmul(Pnxt_time, np.linalg.inv(Pnxt_time+R))
        Xnxt_meas=Xnxt_time+np.matmul(K, Z_list[index][1]-Xnxt_time)
        Pnxt_meas=np.matmul(np.identity(3)-np.matmul(K, H), Pnxt_time)

        X_list.append(Xnxt_meas)
        P=Pnxt_meas
```

```
    else :
        X_list.append(Xnxt_time)
        P=Pnxt_time
```

Where the first for-loop is just assessing whether or not to run the measurement update. The resulting graphs showed very fast convergence (full convergence after 250 time steps). Occasionally however the model system started to diverge from the truth, but got redirected towards the truth by the measurement input sufficiently fast to keep a satisfactory convergence behavior

2.3.2 One-way Coupling

Finally we used the theory derived in section 3 to implement a one way coupling scheme. The system we are considering is a variation of (2.15) :

$$\begin{cases} \dot{\mathbf{x}}_1 = f(\mathbf{x}_1) \\ \dot{\mathbf{x}}_2 = f(\mathbf{x}_2) + \alpha H(\mathbf{x}_1 - \mathbf{x}_2) \end{cases}$$

In practice we will study a model system that we have access to (here $\mathbf{x}_2$), and try to synchronise it with a ”true” dynamical system that is out of our control (here $\mathbf{x}_1$). One-way coupling is more useful than the two-way coupling approach studied earlier.

Our first goal is to estimate the Lyapunov exponent of the Lorenz system for our choice of parameter. Now recall that the definition reads :

$$\Lambda = \max_j \lim_{t \longrightarrow \infty} \frac{1}{t} \lambda_j(x_1(t))$$

Which is not very practical in terms of numerical approximation. Instead of using this, we recalled that the Lyapunov exponent measured the change in divergence around trajectories $\mathbf{x}$. So we generated 30 random (normally distributed around $\mathbf{x}_0 = (-10, 10, 25)$) initial conditions and plotted the trajectories. Then we computed $\log(\|\mathbf{x}_0 - \tilde{\mathbf{x}}\|)$ for each sample and at each time step. Then by taking an average and using a least squares approximation, we estimated the slope of that change to be $\Lambda = 0.9108$. This is consistent with the known value of the Lyapunov exponent for this particular choice of parameters.

Finally in the last few cells of this notebook we implemented one-way coupling for two values of α, the coupling parameter, first for $\alpha = 0.4 < \frac{0.9108}{2}$ which did not lead to synchronisation, and then for $\alpha = 7.5$, which did lead to full synchronisation. Weaker coupling forces also worked but due to the non-uniform property of this system, they converged only after a long amount of time for this initial value.

We used a 4-5 Runge-Kutta scheme [10] on the following problem :

```
def Coupled (t, X):
    x1=param[0]*(X[1]-X[0])
    y1=X[0]*(param[1]-X[2])-X[1]
    z1=-1*param[2]*X[2]+X[0]*X[1]
    x2=param[0]*(X[4]-X[3])+alpha*(X[0]-X[3])
    y2=X[3]*(param[1]-X[5])-X[4]
    z2=-1*param[2]*X[5]+X[3]*X[4]
    return np.array([x1, y1, z1, x2, y2, z2])
```

Chapter 3

Convergence Analysis for Coupling Methods

We will now move from the simplified ODE case to the more challenging Navier-Stokes equations. The goal of this chapter is to establish bounds on the coupling parameter μ such that the slave system is guaranteed to synchronise with the master state.

3.1 Preliminaries

In this chapter we work within $\Omega \subset \mathbb{R}^2$ an open, bounded and connected set with a C^2 boundary. Let us denote by $\mathcal{V}$ the set of all divergence free and compactly supported C^∞ vector fields $F : \Omega \mapsto \mathbb{R}^2$. We recall the Sobolev space $H^s(\Omega) := W^{s,2}(\Omega)$ and denote $H := \bar{\mathcal{V}}_{L^2(\Omega)}$, the closure of $\mathcal{V}$ with respect to the L^2 norm , and $V := \bar{\mathcal{V}}_{H^1(\Omega)}$, the closure of $\mathcal{V}$ with respect to the H^1 norm.

Recall the Navier-Stokes equations in 2 dimensions, coordinate-wise :

$$\begin{cases} \frac{\partial}{\partial t}u_1 - \nu\Delta u_1 + u_1\partial_x u_1 + u_2\partial_y u_1 + \partial_x p = f_1 \\ \frac{\partial}{\partial t}u_2 - \nu\Delta u_2 + u_1\partial_x u_2 + u_2\partial_y u_2 + \partial_y p = f_2 \\ \partial_x u_1 + \partial_y u_2 = 0 \\ u_1(0,x,y) = u_1^0(x,y), \text{ and } u_2(0,x,y) = u_2^0(x,y) \end{cases} \tag{3.1}$$

Where $\nu > 0$ the kinematic viscosity is a positive constant determined by the target fluid.

Suppose that we can construct an approximated solution $U(t,x,y)$ from some interpolation operator $I_h(u_2(t))$ then we find :

$$\begin{cases} \frac{\partial}{\partial t}U_1 - \nu\Delta U_1 + U_1\partial_x U_1 + U_2\partial_y U_1 + \partial_x P = f_1 \\ \frac{\partial}{\partial t}U_2 - \nu\Delta U_2 + U_1\partial_x U_2 + U_2\partial_y U_2 + \partial_y P = f_2 - \mu\big(I_h(U_2) - I_h(u_2)\big) \\ \partial_x U_1 + \partial_y U_2 = 0 \\ U_1(0,x,y) = U_1^0(x,y), \text{ and } U_2(0,x,y) = U_2^0(x,y) \end{cases} \tag{3.2}$$

Where μ is the nudging parameter, which has the same role as α is Section 2.2.

As in Chapter 1, we work with the usual inner product on $L^2(\Omega)$ and H as

$$\langle u, v\rangle = \sum_{i=1}^{2}\int_\Omega u_i w_i dxdy$$

And we denote by $\|\cdot\|$ the naturally induced norm

$$\|u\|_{L^2(\Omega)} = \sqrt{\langle u, u\rangle} = \sqrt{\sum_{i=1}^{2}\int_\Omega u_i^2 dxdy}$$

In this chapter, we will work with linear interpolants of $I_h : H^1(\Omega) \mapsto L^2(\Omega)$ that satisfy the following property :

$$\|\phi - I_h(\phi)\| \leq Ch \|\phi\|_{H^1(\Omega)}, \tag{3.3}$$

where C is a positive constant and $h > 0$ represents the spatial resolution of the observational measurements. We note that quite a few interpolation methods meet this requirement, a good example is found in finite elements methods ([8]).

We will also use the following logarithmic estimate for the convection term of the Navier-Stokes equations in two dimensions : For all $u, v, w \in H_0^1(\Omega)$ with $w \neq 0$, the following inequality holds true,

$$\left| \int_\Omega u \partial_i v w dx dy \right| \leq c_T \|\nabla u\| \|\nabla v\| \|w\| \left(1 + \log\left(\frac{\|\nabla w\|}{\lambda_1^{\frac{1}{2}} \|w\|}\right)\right)^{\frac{1}{2}}$$

Where ∂_i refers to differentiation with respect to the i-th component and c_T is a positive dimensionless constant.

Next recall the Poincaré Inequality for $u \in V$:

$$\|u\|^2 \leq \lambda_1^{-1} \|\nabla u\|^2$$

where λ_1 is the smallest eigenvalue of A, the stokes operator.
Additionally, we will need the Ladyzhenskaya inequality for $u \in V$:

$$\|u\|_{L^4}^2 \leq c_L \|u\| \|\nabla u\|$$

where c_L is also a positive dimensionless constant.

Finally, because they play a more structural role in this chapter's convergence analysis, we will state and prove the following three results.

Lemma 1. *Let* $\phi(r) = r - \gamma(1 + \log(r))$*, where* $\gamma > 0$*. Then*

$$\min\{\phi(r) : r \geq 1\} \geq -\gamma \log(\gamma)$$

Proof. We proceed by simply taking the derivative of ϕ with respect to r and setting it equal to zero :

$$\frac{d}{dr}\phi = 1 - \gamma\frac{1}{r} = 0 \iff r = \gamma$$

Thus

$$\min \phi(r) = \phi(\gamma) = -\gamma \log(\gamma)$$

And since the minimiser of the unconstrained problem is always smaller or equal to any constrained problem

$$\min_{r \geq 1} \phi(r) \geq \min \phi(r) = -\gamma \log(\gamma)$$

And we are done. □

Lemma 2. *(Uniform Gronwall's inequality) Let* $\tau > 0$ *be arbitrary but fixed. Suppose that* $Y(t)$ *is an absolutely continuous function which is locally integrable and that it satisfies the following :*

$$\frac{d}{dt}Y + \beta(t)Y \leq 0, \quad \text{almost everywhere on } (0, \infty)$$

And

$$\text{liminf}_{t\to\infty} \int_t^{t+\tau} \beta(s)ds \geq C, \quad \text{and} \quad \text{limsup}_{t\to\infty} \int_t^{t+\tau} \beta^-(s)ds < \infty$$

for some $C > 0$*, where* $\beta^- := \max\{-\beta, 0\}$*. Then* $Y(t) \to 0$ *at an exponential rate as* $t \to \infty$*.*

Proof. First note that if $\frac{d}{dt}Y + \beta(t)Y \leq 0$ then equivalently

$$Y'(t) \leq -\beta(t)Y(t)$$

Let $a \in \mathbb{R}$ and define $X(t) = \exp\left\{ \int_a^t -\beta(s)ds \right\}$ on $[a, t]$ then by the chain rule and differentiating under the integral sign

$$\frac{d}{dt}X(t) = -\beta(t) \exp\left\{ \int_a^t \beta(s)ds \right\} = -\beta(t)X(t)$$

Notice that $X(a) = 1$ and $X(t) > 0$. Now by the quotient rule

$$\frac{d}{dt}\frac{Y(t)}{X(t)} = \frac{Y'(t)X(t) - X'(t)Y(t)}{X^2(t)} = \frac{Y'(t)X(t) + \beta(t)X(t)Y(t)}{x^2(t)} = \frac{Y'(t) + \beta(t)Y(t)}{X(t)}$$

By assumption we have

$$\leq \frac{-\beta(t)Y(t)}{X(t)} = 0$$

Thus the function $\frac{Y}{X}$ is decreasing on its domain. Thus

$$\frac{Y(t)}{X(t)} \leq \frac{Y(a)}{X(a)} = Y(a) \iff Y(t) \leq Y(a)X(t)$$

Denote $Y(a) := k$ then

$$Y(t) \leq k \exp\left\{ \int_a^t -\beta(s)ds \right\}$$

And by the assumptions on the limsup and liminf of $\exp\left\{ \int_a^t -\beta(s)ds \right\}$ we have

$$\lim_{t \to \infty} \left| \exp\left\{ \int_a^t -\beta(s)ds \right\} \right| \leq ke^{\gamma}$$

So $Y(t)$ is bounded by an exponential thus it decays to zero at an exponential rate.

$\square$

To conclude we will state the following bounds on solutions u of the Navier-Stokes equations in two dimensions. First we denote by G the Grashof number in two dimension i.e. :

$$G = \frac{1}{\nu^2 \lambda_1} \|f\|$$

Now, let $\tau > 0$ and suppose that u is a solution to (3.1) subject to no-slip Dirichlet boundary conditions, then there exists a time $t_0 > 0$ such that for all $t \geq t_0$:

$$\int_t^{t+\tau} \|\nabla u(s)\|^2 \, ds \leq 2(1 + \tau\nu\lambda_1)\nu G^2, \quad \text{and} \quad \|\nabla u(t)\|^2 \leq \tilde{c}\nu^2\lambda_1 G^2 e^{G^4}$$

where $\tilde{c}$ is some positive dimensionless constant.

3.2 Convergence Analysis

We now state and prove the main of this thesis :

Theorem 2. *Suppose I_h satisfies (3.3), let $u(t,x,y) = \big(u_1(t,x,y), u_2(t,x,y)\big)$ be a strong solution to the two dimensional Navier-Stokes Equations with Dirichlet boundary conditions. Let $U(t,x,y)$ be a strong solution to ((3.2)), also with Dirichlet boundary conditions. If $\mu > 0$ is chosen such that*

$$\mu \geq 8c\nu\lambda_1(1 + \log(G) + G^4)G^2$$

and $h > 0$ is chosen such that $\mu c_0^2 h^2 \leq \nu$ then $\|u(t) - U(t)\|_2^2 \to 0$ as $t \to \infty$ at an exponential rate.

Proof. Let us define $\tilde{u} = u - U$ and $\tilde{p} = p - P$ then we find that

$$\begin{cases} \frac{\partial}{\partial t}u_1 - \frac{\partial}{\partial t}U_1 - \nu\Delta u_1 + \nu\Delta U_1 + u_1\partial_x u_1 + u_2\partial_y u_1 - U_1\partial_x U_1 - U_2\partial_y U_1 + \partial_x p - \partial_x P = 0 \\ \frac{\partial}{\partial t}u_2 - \frac{\partial}{\partial t}U_2 - \nu\Delta u_2 + \nu\Delta U_2 + u_1\partial_x u_2 + u_2\partial_y u_2 - U_1\partial_x U_2 - U_2\partial_y U_2 + \partial_y p - \partial_y P = \mu\big(I_h(U_2) - I_h(u_2)\big) \\ \partial_x u_1 + \partial_y u_2 - \partial_x U_1 - \partial_y U_2 = 0 \\ u_1(0,x,y) - U_1(0,x,y) = u_1^0(x,y) - U_1^0(x,y), \text{ and } u_2(0,x,y) - U_2(0,x,y) = u_2^0(x,y) - U_2^0(x,y) \end{cases}$$

Which reduces to

$$\begin{cases} \frac{\partial}{\partial t}\tilde{u}_1 - \nu\Delta\tilde{u}_1 + u_1\partial_x u_1 + u_2\partial_y u_1 - U_1\partial_x U_1 - U_2\partial_y U_1 + \partial_x\tilde{p} = 0 \\ \frac{\partial}{\partial t}\tilde{u}_2 - \nu\Delta\tilde{u}_2 + u_1\partial_x u_2 + u_2\partial_y u_2 - U_1\partial_x U_2 - U_2\partial_y U_2 + \partial_y\tilde{p} = \mu\big(I_h(U_2) - I_h(u_2)\big) \\ \partial_x\tilde{u}_1 + \partial_y\tilde{u}_2 = 0 \\ \tilde{u}_1(0,x,y) = \tilde{u}_1^0(x,y), \text{ and } \tilde{u}_2(0,x,y) = \tilde{u}_2^0(x,y) \end{cases} \tag{3.4}$$

We want to work on the first component equation and rewrite the mixed partial terms. Note that :

$$u_1\partial_x u_1 + u_2\partial_y u_1 - U_1\partial_x U_1 - U_2\partial_y U_1 = u_1\partial_x u_1 + u_2\partial_y u_1 - U_1\partial_x U_1 - U_2\partial_y U_1 + \Big(U_1\partial_x u_1 - U_1\partial_x u_1 + U_2\partial_y u_1 - U_2\partial_y u_1\Big)$$

$$= u_1\partial_x u_1 - U_1\partial_x u_1 + u_2\partial_y u_1 - U_2\partial_y u_1 + U_1\partial_x u_1 - U_1\partial_x U_1 + U_2\partial_y u_1 - U_2\partial_y U_1$$

$$= \tilde{u}_1\partial_x u_1 + \tilde{u}_2\partial_y u_1 + U_1\partial_x\tilde{u}_1 + U_2\partial_y\tilde{u}_1$$

The first line of (3.4) becomes

$$\frac{\partial}{\partial t}\tilde{u}_1 - \nu\Delta\tilde{u}_1 + \tilde{u}_1\partial_x u_1 + \tilde{u}_2\partial_y u_1 + U_1\partial_x\tilde{u}_1 + U_2\partial_y\tilde{u}_1 + \partial_x\tilde{p} = 0 \tag{3.5}$$

Now consider the inner product of $\tilde{u}_1$ and (3.5) :

$$\left\langle \frac{\partial}{\partial t}\tilde{u}_1, \tilde{u}_1 \right\rangle - \nu\left\langle \Delta\tilde{u}_1, \tilde{u}_1\right\rangle + \langle\tilde{u}_1\partial_x u_1, \tilde{u}_1\rangle + \langle\tilde{u}_2\partial_y u_1, \tilde{u}_1\rangle + \langle U_1\partial_x\tilde{u}_1, \tilde{u}_1\rangle + \langle U_2\partial_y\tilde{u}_1, \tilde{u}_1\rangle + \langle\partial_x\tilde{p}, \tilde{u}_1\rangle = 0 \tag{3.6}$$

We use integration by part and the divergence free condition of u, U and $\tilde{u}$ to find bounds for each of these terms.

First consider

$$\left\langle \frac{\partial}{\partial t}\tilde{u}_1, \tilde{u}_1 \right\rangle = \int_\Omega \big(\frac{\partial}{\partial t}\tilde{u}_1\big)\tilde{u}_1 dxdy$$

Note that $\frac{\partial}{\partial t}\tilde{u}_1^2 = 2\tilde{u}_1\frac{\partial}{\partial t}\tilde{u}_1$. Hence

$$\left\langle \frac{\partial}{\partial t}\tilde{u}_1, \tilde{u}_1 \right\rangle = \frac{1}{2}\int_\Omega \frac{\partial}{\partial t}\big(\tilde{u}_1^2\big)\tilde{u}_1 dxdy = \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega \big(\tilde{u}_1^2\big)\tilde{u}_1 dxdy = \frac{1}{2}\frac{\partial}{\partial t}\langle \tilde{u}_1, \tilde{u}_1\rangle = \frac{1}{2}\frac{\partial}{\partial t}\|\tilde{u}_1\|^2 \quad (3.7)$$

Next we have

$$\langle \Delta\tilde{u}_1, \tilde{u}_1\rangle = \int_\Omega \big(\Delta\tilde{u}_1\big)\tilde{u}_1 dxdy = (\nabla\tilde{u}_1)\tilde{u}_1\Big|_{\partial\Omega} - \int_\Omega \nabla\tilde{u}_1\nabla\tilde{u}_1 dxdy$$

By integration by parts, now by the Dirichlet boundary conditions, the first term is zero. Thus we are left with

$$\langle \Delta\tilde{u}_1, \tilde{u}_1\rangle = -\|\nabla\tilde{u}_1\|^2 \quad (3.8)$$

We deal with the mixed partials next, consider

$$\langle U_1\partial_x\tilde{u}_1, \tilde{u}_1\rangle + \langle U_2\partial_y\tilde{u}_1, \tilde{u}_1\rangle = \int_\Omega U_1(\partial_x\tilde{u}_1)\tilde{u}_1 dxdy + \int_\Omega U_2(\partial_y\tilde{u}_1)\tilde{u}_1 dxdy$$

Using integration by parts in the x component for the first term, one finds

$$\int_\Omega U_1(\partial_x\tilde{u}_1)\tilde{u}_1 dxdy = \int_{\Omega_y}\left(U_1\Big(\int_{\Omega_x}\partial_x\tilde{u}_1\tilde{u}_1 dx\Big)\bigg|_{\partial\Omega_x} - \int_{\Omega_x}\partial_x U_1\Big(\int_{\Omega_x}\partial_x\tilde{u}_1\tilde{u}_1 dx\Big)dx\right)dy$$

Because U_1 is assumed to satisfy Dirichlet boundary conditions, the first term is zero, hence

$$= -\frac{1}{2}\int_{\Omega_y}\left(\int_{\Omega_x}\partial_x U_1\tilde{u}_1^2 dx\right)dy = -\frac{1}{2}\int_\Omega \partial_x U_1\tilde{u}_1^2 dxdy$$

Similarly, using integration by parts in the y component on the second term, we compute

$$\begin{aligned}\int_\Omega U_2(\partial_y\tilde{u}_1)\tilde{u}_1 dxdy &= \int_{\Omega_x}\left(U_2\Big(\int_{\Omega_y}\partial_y\tilde{u}_1\tilde{u}_1 dy\Big)\bigg|_{\partial\Omega_y} - \int_{\Omega_y}\partial_y U_2\Big(\int_{\Omega_y}\partial_y\tilde{u}_1\tilde{u}_1 dy\Big)dy\right)dx \\ &= 0 - \frac{1}{2}\int_{\Omega_x}\left(\int_{\Omega_y}\partial_y U_2\tilde{u}_1^2 dy\right)dx = \int_\Omega \partial_y U_2\tilde{u}_1^2 dxdy\end{aligned}$$

Therefore by adding both quantities together we conclude

$$\langle U_1\partial_x\tilde{u}_1,\tilde{u}_1\rangle+\langle U_2\partial_y\tilde{u}_1,\tilde{u}_1\rangle=-\frac{1}{2}\int_\Omega(\partial_xU_1+\partial_yU_2)\tilde{u}_1^2dxdy$$

And by the divergence free condition imposed on U, we know $\partial_xU_1+\partial_yU_2=0$ thus

$$\langle U_1\partial_x\tilde{u}_1,\tilde{u}_1\rangle+\langle U_2\partial_y\tilde{u}_1,\tilde{u}_1\rangle=0 \tag{3.9}$$

We still have two terms to work on, denote

$$J_{1a}:=\langle\tilde{u}_1\partial_xu_1,\tilde{u}_1\rangle\quad\text{and}\quad J_{1b}:=\langle\tilde{u}_2\partial_yu_1,\tilde{u}_1\rangle$$

Using integration by parts on J_{1a}, we find

$$J_{1a}=\int_\Omega\tilde{u}_1^2\partial_xu_1dxdy=\tilde{u}_1^2u_1\Big|_{\partial\Omega}-\int_\Omega u_1(2\tilde{u}_1\partial_x\tilde{u}_1)dxdy=-2\left\langle u_1\tilde{u}_1,\partial_x\tilde{u}_1\right\rangle$$

By the divergence free condition assumed on $\tilde{u}$, we find that $-\partial_x\tilde{u}_1=\partial_y\tilde{u}_2$ thus

$$=2\left\langle u_1\tilde{u}_1,\partial_y\tilde{u}_2\right\rangle$$

And using integration by parts again

$$=2\int_\Omega u_1\tilde{u}_1\partial_y\tilde{u}_2dxdy=2(u_1\tilde{u}_1\tilde{u}_2)\Big|_{\partial\Omega}-2\int_\Omega\tilde{u}_2\partial_y(u_1\tilde{u}_1)dxdy=0-2\int_\Omega\tilde{u}_2\big(\tilde{u}_1\partial_yu_1+u_1\partial_y\tilde{u}_1\big)dxdy$$

$$=2\int_\Omega u_1\tilde{u}_1\partial_y\tilde{u}_2dxdy=2(u_1\tilde{u}_1\tilde{u}_2)\Big|_{\partial\Omega}-2\int_\Omega\tilde{u}_2\partial_y(u_1\tilde{u}_1)dxdy=0-2\int_\Omega\tilde{u}_2\big(\tilde{u}_1\partial_yu_1+u_1\partial_y\tilde{u}_1\big)dxdy$$

Where the first term disappears due to the Dirichlet boundary conditions. We may split the integral into two terms :

$$J_{1a}=-2\int_\Omega\tilde{u}_2\tilde{u}_1\partial_yu_1dxdy-2\int_\Omega\tilde{u}_2u_1\partial_y\tilde{u}_1dxdy=-2\left\langle\tilde{u}_2,\tilde{u}_1\partial_yu_1\right\rangle-2\left\langle\tilde{u}_2,u_1\partial_y\tilde{u}_1\right\rangle:=-2\big(J_{1a_1}+J_{1a_2}\big)$$

Now recall the log estimates and apply to $|J_{1a_1}|$:

$$|J_{1a_1}| = | -2 \int_\Omega \tilde{u}_1 \partial_y u_1 \tilde{u}_2 dxdy| \leq c_T \|\nabla \tilde{u}_1\| \|\nabla u_1\| \|\tilde{u}_2\| \left(1 + \log\left(\frac{\|\nabla \tilde{u}_2\|}{\lambda_1^{\frac{1}{2}} \|\tilde{u}_2\|}\right)\right)^{\frac{1}{2}}$$

Now by Young's inequality :

$$|J_{1a_1}| \leq \frac{c_T^2}{2} \|\nabla \tilde{u}_1\|^2 + \frac{1}{2} \|\nabla u_1\|^2 \|\tilde{u}_2\|^2 \left(1 + \log\left(\frac{\|\nabla \tilde{u}_2\|}{\lambda_1^{\frac{1}{2}} \|\tilde{u}_2\|}\right)\right)$$

Fix c_T and c such that

$$\frac{c_T^2}{2} \leq \frac{\nu}{64} \quad \text{and} \quad 1 \leq \frac{c}{8\nu}$$

Then

$$|J_{1a_1}| \leq \frac{\nu}{64} \|\nabla \tilde{u}_1\|^2 + \frac{c}{16\nu} \|\nabla u_1\|^2 \|\tilde{u}_2\|^2 \left(1 + \log\left(\frac{\|\nabla \tilde{u}_2\|}{\lambda_1^{\frac{1}{2}} \|\tilde{u}_2\|}\right)\right)$$

Similarly for $|J_{1a_2}|$:

$$\begin{aligned} | \langle u_1 \partial_y \tilde{u}_1, \tilde{u}_2 \rangle | &\leq c_T \|\nabla u_1\| \|\nabla \tilde{u}_1\| \|\tilde{u}_2\| \left(1 + \log\left(\frac{\|\nabla \tilde{u}_2\|}{\lambda_1^{\frac{1}{2}} \|\tilde{u}_2\|}\right)\right)^{\frac{1}{2}} \\ &\leq \frac{\nu}{64} \|\nabla \tilde{u}_1\|^2 + \frac{c}{16\nu} \|\nabla u_1\|^2 \|\tilde{u}_2\|^2 \left(1 + \log\left(\frac{\|\nabla \tilde{u}_2\|}{\lambda_1^{\frac{1}{2}} \|\tilde{u}_2\|}\right)\right) \end{aligned}$$

Therefore

$$\begin{aligned} |J_{1a}| &= 2\big|J_{1a_1} + J_{1a_2}\big| \\ &\leq 2|J_{1a_1}| + 2|J_{1a_2}| \\ &\leq \frac{\nu}{16} \|\nabla \tilde{u}_1\|^2 + \frac{c}{4\nu} \|\nabla u_1\|^2 \|\tilde{u}_2\|^2 \left(1 + \log\left(\frac{\|\nabla \tilde{u}_2\|}{\lambda_1^{\frac{1}{2}} \|\tilde{u}_2\|}\right)\right) \end{aligned} \tag{3.10}$$

The J_{1b} term is a little easier to estimate since it is already written in terms of the targeted quantities $(\tilde{u}_1, \tilde{u}_2)$. We use the log estimates directly :

$$|J_{1b}| = | \langle \partial_y u_1 \tilde{u}_1, \tilde{u}_2 \rangle | \leq c_T \|\nabla \tilde{u}_1\| \|\nabla u_1\| \|\tilde{u}_2\| \left(1 + \log\left(\frac{\|\nabla \tilde{u}_2\|}{\lambda_1^{\frac{1}{2}} \|\tilde{u}_2\|}\right)\right)^{\frac{1}{2}}$$

By Young's inequality again :

$$\leq \frac{c_T^2}{2}\left\|\nabla \tilde{u}_1\right\|^2+\frac{1}{2}\left\|\nabla u_1\right\|^2\left\|\tilde{u}_2\right\|^2\left(1+\log \Big(\frac{\left\|\nabla \tilde{u}_2\right\|}{\lambda_1^{\frac{1}{2}}\left\|\tilde{u}_2\right\|}\Big)\right)$$

Note that :

$$\frac{c_T^2}{2} \leq \frac{\nu}{32} \leq \frac{\nu}{16}, \quad \text{and} \quad 1 \leq \frac{c}{8\nu} \leq \frac{c}{4\nu}$$

Thus :

$$\left|J_{1b}\right| \leq \frac{\nu}{16}\left\|\nabla \tilde{u}_1\right\|^2+\frac{c}{4\nu}\left\|\nabla u_1\right\|^2\left\|\tilde{u}_2\right\|^2\left(1+\log \Big(\frac{\left\|\nabla \tilde{u}_2\right\|}{\lambda_1^{\frac{1}{2}}\left\|\tilde{u}_2\right\|}\Big)\right) \tag{3.11}$$

Therefore using (3.6) we can establish the following estimate :

$$\begin{aligned}
&\left\langle\frac{\partial}{\partial t}\tilde{u}_1, \tilde{u}_1\right\rangle-\nu\left\langle\Delta \tilde{u}_1, \tilde{u}_1\right\rangle+J_{1a}+J_{1b}+0+\left\langle\partial_x \tilde{p}, \tilde{u}_1\right\rangle=0 \\
&\left\langle\frac{\partial}{\partial t}\tilde{u}_1, \tilde{u}_1\right\rangle-\nu\left\langle\Delta \tilde{u}_1, \tilde{u}_1\right\rangle \leq -J_{1a}-J_{1b}-\left\langle\partial_x \tilde{p}, \tilde{u}_1\right\rangle \\
&\quad\frac{1}{2}\frac{\partial}{\partial t}\left\|\tilde{u}_1\right\|^2+\nu\left\|\nabla \tilde{u}_1\right\|^2 \leq \left|J_{1a}\right|+\left|J_{1b}\right|-\left\langle\partial_x \tilde{p}, \tilde{u}_1\right\rangle
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
\Longleftrightarrow \quad \frac{1}{2}\frac{\partial}{\partial t}\left\|\tilde{u}_1\right\|^2+\nu\left\|\nabla \tilde{u}_1\right\|^2 \leq{}& \frac{\nu}{8}\left\|\nabla \tilde{u}_1\right\|^2 \\
&+\frac{c}{2\nu}\left\|\nabla u_1\right\|^2\left\|\tilde{u}_2\right\|^2\left(1+\log \Big(\frac{\left\|\nabla \tilde{u}_2\right\|}{\lambda_1^{\frac{1}{2}}\left\|\tilde{u}_2\right\|}\Big)\right)-\left\langle\partial_x \tilde{p}, \tilde{u}_1\right\rangle
\end{aligned} \tag{3.13}$$

Now we need to establish a similar result for the second component of $\tilde{u}=(\tilde{u}_1, \tilde{u}_2)$. Recall from (3.4) :

$$\frac{\partial}{\partial t}\tilde{u}_2-\nu \Delta \tilde{u}_2+u_1 \partial_x u_2+u_2 \partial_y u_2-U_1 \partial_x U_2-U_2 \partial_y U_2+\partial_y \tilde{p}=\mu\big(I_h(U_2)-I_h(u_2)\big)$$

We add and subtract the relevant quantities to rewrite the mixed partials :

$$u_1 \partial_x u_2+u_2 \partial_y u_2-U_1 \partial_x U_2-U_2 \partial_y U_2+U_2 \partial_y u_2-U_2 \partial_y u_2+U_1 \partial_x u_2-U_1 \partial_x u_2$$

$$= \tilde{u}_2 \partial_y u_2 + \tilde{u}_1 \partial_x u_2 + U_1 \partial_x \tilde{u}_2 + U_2 \partial_y \tilde{u}_2$$

Hence $\tilde{u}_2$ satisfies

$$\frac{\partial}{\partial t}\tilde{u}_2 - \nu \Delta \tilde{u}_2 + \tilde{u}_2 \partial_y u_2 + \tilde{u}_1 \partial_x u_2 + U_1 \partial_x \tilde{u}_2 + U_2 \partial_y \tilde{u}_2 + \partial_y \tilde{p} = -\mu\big(I_h(\tilde{u}_2)\big) \tag{3.14}$$

Now we consider the inner product of (3.14) and $\tilde{u}_2$:

$$\left\langle \frac{\partial}{\partial t}\tilde{u}_2, \tilde{u}_2 \right\rangle - \nu \left\langle \Delta \tilde{u}_2, \tilde{u}_2 \right\rangle + \left\langle \tilde{u}_2 \partial_y u_2, \tilde{u}_2 \right\rangle + \left\langle \tilde{u}_1 \partial_x u_2, \tilde{u}_2 \right\rangle + \left\langle U_1 \partial_x \tilde{u}_2, \tilde{u}_2 \right\rangle + \left\langle U_2 \partial_y \tilde{u}_2, \tilde{u}_2 \right\rangle + \left\langle \partial_y \tilde{p}, \tilde{u}_2 \right\rangle$$
$$= -\mu \left\langle I_h(\tilde{u}_2), \tilde{u}_2 \right\rangle \tag{3.15}$$

Using the same techniques as before we find

$$\left\langle \frac{\partial}{\partial t}\tilde{u}_2, \tilde{u}_2 \right\rangle = \frac{1}{2}\frac{\partial}{\partial t}\left\|\tilde{u}_2\right\|^2 \tag{3.16}$$

And

$$\left\langle \Delta \tilde{u}_2, \tilde{u}_2 \right\rangle = -\left\|\nabla \tilde{u}_2\right\|^2 \tag{3.17}$$

The $U, \tilde{u}$ cross term also vanishes, ie:

$$\left\langle U_1 \partial_x \tilde{u}_2, \tilde{u}_2 \right\rangle + \left\langle U_2 \partial_y \tilde{u}_2, \tilde{u}_2 \right\rangle = 0 \tag{3.18}$$

Denote

$$J_{2a} := \left\langle \tilde{u}_1 \partial_x u_2, \tilde{u}_2 \right\rangle \quad \text{and} \quad J_{2b} := \left\langle \tilde{u}_2 \partial_y u_2, \tilde{u}_2 \right\rangle$$

Since J_{2a} is already written in terms of the desired quantities, we directly apply the log estimates to its absolute value :

$$|J_{2a}| \leq c_T \left\|\nabla \tilde{u}_1\right\| \left\|\nabla u_2\right\| \left\|\tilde{u}_2\right\| \left(1 + \log\Big(\frac{\left\|\nabla \tilde{u}_2\right\|}{\lambda_1^{\frac{1}{2}}\left\|\tilde{u}_2\right\|}\Big)\right)^{\frac{1}{2}}$$

Then by applying Young's Inequality with $\frac{c_T^2}{2} \leq \frac{\nu}{64}$ and $1 \leq \frac{c}{8\nu}$:

$$\leq \frac{\nu}{64} \|\nabla \tilde{u}_1\|^2 + \frac{c}{16\nu} \|\nabla u_2\|^2 \|\tilde{u}_2\|^2 \left(1 + \log\left(\frac{\|\nabla \tilde{u}_2\|}{\lambda_1^{\frac{1}{2}} \|\tilde{u}_2\|}\right)\right)$$

Hence we can take a more convenient upper bound :

$$|J_{2a}| \leq \frac{\nu}{32} \|\nabla \tilde{u}_1\|^2 + \frac{c}{2\nu} \|\nabla u_2\|^2 \|\tilde{u}_2\|^2 \left(1 + \log\left(\frac{\|\nabla \tilde{u}_2\|}{\lambda_1^{\frac{1}{2}} \|\tilde{u}_2\|}\right)\right) \tag{3.19}$$

The J_{2b} term is more problematic because we are not able to make $\tilde{u}_1$ appear then. Instead we use Ladyzhenskaya's inequality to bound the inner product. We need to rewrite the term slightly first :

$$|J_{2b}| = |\left\langle \tilde{u}_2 \partial_y u_2, \tilde{u}_2 \right\rangle| = |\int_\Omega \tilde{u}_2^2 \partial_y u_2 dxdy| = |\left\langle \tilde{u}_2^2, \partial_y u_2 \right\rangle|$$

By Cauchy-Schwartz

$$|J_{2b}| \leq \|\tilde{u}_2^2\|_2 \|\partial_y u_2\|_2 = \left(\int_\Omega \tilde{u}_2^4 dxdy\right)^{\frac{1}{2}} \|\partial_y u_2\|_2 = \|\tilde{u}_2\|_4^2 \|\partial_y u_2\|_2$$

Thus by Ladyzhenskaya's inequality

$$|J_{2b}| \leq c_L \|\tilde{u}_2\|_2 \|\nabla \tilde{u}_2\|_2 \|\partial_y u_2\|_2$$

Now, applying Young's inequality yields

$$|J_{2b}| \leq \frac{c_L^2}{2} \|\nabla \tilde{u}_2\|^2 + \frac{1}{2} \|\tilde{u}_2\|^2 \|\partial_y u_2\|^2$$

Hence after defining the relevant constants

$$|J_{2b}| \leq \frac{\nu}{32} \|\nabla \tilde{u}_2\|^2 + \frac{c}{\nu} \|\tilde{u}_2\|^2 \|\partial_y u_2\|^2 \tag{3.20}$$

Finally, we need to estimate the term containing the interpolant : $\langle I_h(\tilde{u}_2), \tilde{u}_2\rangle$. Recall that by assumption $\mu c_0^2 h^2 \leq \nu$ and that I_h satisfies (3.3). We compute

$$-\mu \langle I_h(\tilde{u}_2), \tilde{u}_2\rangle = -\mu \int_\Omega I_h(\tilde{u}_2)\tilde{u}_2 dxdy = -\mu \int_\Omega I_h(\tilde{u}_2)\tilde{u}_2 + (\tilde{u}_2^2 - \tilde{u}_2^2) dxdy$$
$$= -\mu \int_\Omega \big(I_h(\tilde{u}_2) - \tilde{u}_2\big)\tilde{u}_2 dxdy - \mu \int_\Omega \tilde{u}_2^2 dxdy = -\mu \langle I_h(\tilde{u}_2) - \tilde{u}_2, \tilde{u}_2\rangle - \mu \|\tilde{u}_2\|^2$$
$$\leq \mu |\langle I_h(\tilde{u}_2) - \tilde{u}_2, \tilde{u}_2\rangle| - \mu \|\tilde{u}_2\|^2$$

Thus applying Cauchy-Schwartz

$$-\mu \langle I_h(\tilde{u}_2), \tilde{u}_2\rangle \leq \|I_h(\tilde{u}_2) - \tilde{u}_2\| \, \|\tilde{u}_2\| - \mu \|\tilde{u}_2\|^2$$

Thus clearly

$$-\mu \langle I_h(\tilde{u}_2), \tilde{u}_2\rangle \leq \|I_h(\tilde{u}_2) - \tilde{u}_2\|^{\frac{1}{2}\times 2} \|\tilde{u}_2\| - \mu \|\tilde{u}_2\|^2$$

Hence by applying (3.3),

$$\leq \mu c_0 h \|\nabla \tilde{u}_2\| \, \|\tilde{u}_2\| - \mu \|\tilde{u}_2\|^2$$

Now by Young's inequality

$$\leq \frac{\mu c_0^2 h^2}{2} \|\nabla \tilde{u}_2\|^2 + \frac{\mu}{2} \|\tilde{u}_2\|^2 - \mu \|\tilde{u}_2\|^2$$

And finally by the assumptions on μ :

$$-\mu \langle I_h(\tilde{u}_2), \tilde{u}_2\rangle \leq \frac{\nu}{2} \|\nabla \tilde{u}_2\|^2 - \frac{\mu}{2} \|\tilde{u}_2\|^2 \tag{3.21}$$

Thus (3.15) yields

$$\frac{1}{2}\frac{\partial}{\partial t} \|\tilde{u}_2\|^2 + \nu \|\nabla \tilde{u}_2\|^2 + J_{2a} + J_{2b} + 0 + \langle \partial_y \tilde{p}, \tilde{u}_2\rangle = -\mu \langle I_h(\tilde{u}_2), \tilde{u}_2\rangle$$

$$\iff \frac{1}{2}\frac{\partial}{\partial t} \|\tilde{u}_2\|^2 + \nu \|\nabla \tilde{u}_2\|^2 \leq |J_{2a}| + |J_{2b}| - \langle \partial_y \tilde{p}, \tilde{u}_2\rangle - \mu \langle I_h(\tilde{u}_2), \tilde{u}_2\rangle \tag{3.22}$$

We are now ready to add (3.12) and (3.22) together :

$$\frac{1}{2}\frac{\partial}{\partial t}\big(\,\|\tilde{u}_1\|^2+\|\tilde{u}_2\|^2\,\big)+\nu\big(\,\|\nabla\tilde{u}_1\|^2+\|\nabla\tilde{u}_2\|^2\,\big)\leq|J_{1a}|+|J_{1b}|+|J_{2a}|+|J_{2b}|-\langle\partial_x\tilde{p}+\partial_y\tilde{p},\tilde{u}_1\rangle-\mu\,\langle I_h(\tilde{u}_2),\tilde{u}_2\rangle$$

$$\iff\ \frac{1}{2}\frac{\partial}{\partial t}\|\tilde{u}\|^2+\nu\,\|\nabla\tilde{u}\|^2\leq|J_{1a}|+|J_{1b}|+|J_{2a}|+|J_{2b}|-\langle\nabla\cdot\tilde{p},\tilde{u}_1\rangle-\mu\,\langle I_h(\tilde{u}_2),\tilde{u}_2\rangle \quad (3.23)$$

Recall that the pressure $\tilde{p}$ is assumed to be divergence free, hence $\langle\nabla\cdot\tilde{p},\tilde{u}_1\rangle=0$.

Using (3.10) and (3.11) we find

$$|J_{1a}|+|J_{1b}|\leq\frac{\nu}{8}\|\nabla\tilde{u}_1\|^2+\frac{c}{2\nu}\|\nabla u_1\|^2\|\tilde{u}_2\|^2\left(1+\log\Big(\frac{\|\nabla\tilde{u}_2\|}{\lambda_1^{\frac{1}{2}}\|\tilde{u}_2\|}\Big)\right)$$

Using (3.19) and (3.20) :

$$\begin{aligned}|J_{2a}|+|J_{2b}|&\leq\frac{\nu}{32}\|\nabla\tilde{u}_1\|^2+\frac{c}{2\nu}\|\nabla u_2\|^2\|\tilde{u}_2\|^2\left(1+\log\Big(\frac{\|\nabla\tilde{u}_2\|}{\lambda_1^{\frac{1}{2}}\|\tilde{u}_2\|}\Big)\right)+\frac{\nu}{32}\|\nabla\tilde{u}_2\|^2+\frac{c}{\nu}\|\tilde{u}_2\|^2\|\partial_y u_2\|^2\\&\leq\frac{\nu}{32}\|\nabla\tilde{u}\|^2+\frac{c}{2\nu}\|\nabla u_2\|^2\|\tilde{u}_2\|^2\left(1+\log\Big(\frac{\|\nabla\tilde{u}_2\|}{\lambda_1^{\frac{1}{2}}\|\tilde{u}_2\|}\Big)\right)+\frac{c}{\nu}\|\tilde{u}_2\|^2\|\partial_y u_2\|^2\end{aligned}$$

Therefore

$$\begin{aligned}|J_{1a}|+|J_{1b}|+|J_{2a}|+|J_{2b}|-\mu\,\langle I_h(\tilde{u}_2),\tilde{u}_2\rangle&\leq\frac{\nu}{8}\|\nabla\tilde{u}_1\|^2+\frac{\nu}{32}\|\nabla\tilde{u}\|^2+\frac{\nu}{2}\|\nabla\tilde{u}_2\|^2\\&\quad+\|\tilde{u}_2\|^2\left[\frac{c}{2\nu}\left(\|\nabla u\|^2\left(1+\log\Big(\frac{\|\nabla\tilde{u}_2\|}{\lambda_1^{\frac{1}{2}}\|\tilde{u}_2\|}\Big)\right)+\|\partial_y u_2\|^2\right)-\frac{\mu}{2}\right]\end{aligned}$$

Now, the first term can be bounded as follows :

$$\begin{aligned}\frac{\nu}{8}\|\nabla\tilde{u}_1\|^2+\frac{\nu}{32}\|\nabla\tilde{u}\|^2+\frac{\nu}{2}\|\nabla\tilde{u}_2\|^2&=\frac{\nu}{8}\|\nabla\tilde{u}\|^2+\frac{\nu}{32}\|\nabla\tilde{u}\|^2+\frac{\nu}{2}\|\nabla\tilde{u}\|^2-\frac{\nu}{8}\|\nabla\tilde{u}_2\|^2-\frac{\nu}{2}\|\nabla\tilde{u}_1\|^2\\&=\frac{21\nu}{32}\|\nabla\tilde{u}\|^2-\frac{\nu}{8}\|\nabla\tilde{u}_2\|^2-\frac{\nu}{2}\|\nabla\tilde{u}_1\|^2\\&\leq\frac{21\nu}{32}\|\nabla\tilde{u}\|^2\end{aligned}$$

Also notice that since $\|\partial_y u_2\|^2 \geq 0$, we can bound the $\log$ term :

$$\|\tilde{u}_2\|^2 \left[\frac{c}{2\nu}\left(\|\nabla u\|^2 \left(1+\log\left(\frac{\|\nabla \tilde{u}_2\|}{\lambda_1^{\frac{1}{2}} \|\tilde{u}_2\|}\right)\right) + \|\partial_y u_2\|^2 \right) - \frac{\mu}{2}\right]$$
$$\leq \|\tilde{u}_2\|^2 \left[\frac{c}{2\nu} \|\nabla u\|^2 \left(1+\log\left(\frac{\|\nabla \tilde{u}_2\|}{\lambda_1^{\frac{1}{2}} \|\tilde{u}_2\|}\right)\right) - \frac{\mu}{2}\right]$$

Therefore (3.23) leads to

$$\frac{1}{2}\frac{\partial}{\partial t}\|\tilde{u}\|^2 + \nu\|\nabla\tilde{u}\|^2 \leq \frac{21\nu}{32}\|\nabla\tilde{u}\|^2 + \|\tilde{u}_2\|^2\left[\frac{c}{2\nu}\|\nabla u\|^2\|\tilde{u}_2\|^2\left(1+\log\left(\frac{\|\nabla \tilde{u}_2\|}{\lambda_1^{\frac{1}{2}} \|\tilde{u}_2\|}\right)\right) - \frac{\mu}{2}\right]$$
$$\iff \frac{1}{2}\frac{\partial}{\partial t}\|\tilde{u}\|^2 + \frac{\nu}{4}\|\nabla\tilde{u}\|^2 \leq \frac{1}{2}\frac{\partial}{\partial t}\|\tilde{u}\|^2 + \frac{11}{32}\nu\|\nabla\tilde{u}\|^2 \leq \|\tilde{u}_2\|^2\left[\frac{c}{2\nu}\|\nabla u\|^2\left(1+\log\left(\frac{\|\nabla \tilde{u}_2\|}{\lambda_1^{\frac{1}{2}} \|\tilde{u}_2\|}\right)\right) - \frac{\mu}{2}\right]$$

Thus we reach

$$\frac{\partial}{\partial t}\|\tilde{u}\|^2 + \frac{\nu}{2}\|\nabla\tilde{u}\|^2 \leq \|\tilde{u}_2\|^2\left[\frac{c}{\nu}\|\nabla u\|^2\left(1+\log\left(\frac{\|\nabla \tilde{u}_2\|}{\lambda_1^{\frac{1}{2}} \|\tilde{u}_2\|}\right)\right) - \mu\right]$$

Adding and subtracting positive quantities from the right and the left respectively yields

$$\frac{\partial}{\partial t}\|\tilde{u}\|^2 + \frac{\nu}{4}\|\nabla\tilde{u}\|^2 + \frac{\nu}{4}\|\nabla\tilde{u}_2\|^2 \leq \|\tilde{u}_2\|^2\left[\frac{c}{\nu}\|\nabla u\|^2\left(1+\log\left(\frac{\|\nabla \tilde{u}_2\|^2}{\lambda_1 \|\tilde{u}_2\|^2}\right)\right) - \mu\right] \tag{3.24}$$

Applying the Poincaré inequality on $\nabla\tilde{u}$ next leads to :

$$\frac{\nu}{4}\|\nabla\tilde{u}\|^2 \geq \frac{\nu\lambda_1}{4}\|\tilde{u}\|^2$$

Thus (3.24) is equivalent to

$$\frac{\partial}{\partial t}\|\tilde{u}\|^2 + \frac{\nu\lambda_1}{4}\|\tilde{u}\|^2 + \frac{\nu\lambda_1}{4}\frac{\|\nabla\tilde{u}_2\|^2}{\lambda_1\|\tilde{u}_2\|^2}\|\tilde{u}_2\|^2 \leq \|\tilde{u}_2\|^2\left[\frac{c}{\nu}\|\nabla u\|^2\left(1+\log\left(\frac{\|\nabla \tilde{u}_2\|^2}{\lambda_1 \|\tilde{u}_2\|^2}\right)\right) - \mu\right] \tag{3.25}$$

Let us define the following quantities

$$r(t) := \frac{\|\nabla \tilde{u}_2\|^2}{\lambda_1 \|\tilde{u}_2\|^2}, \quad \gamma(t) := 4\frac{c}{\nu^2\lambda_1}\|\nabla u\|^2, \quad \text{and} \quad \phi(r(t)) := r(t) - \gamma(t)\Big(1 + \log(r(t))\Big)$$

We are now able to rewrite (3.25) in terms of these new quantities :

$$\begin{aligned}
&\frac{\partial}{\partial t}\|\tilde{u}\|^2 + \frac{\nu\lambda_1}{4}\|\tilde{u}\|^2 + \frac{\nu\lambda_1}{4} r(t)\|\tilde{u}_2\|^2 \leq \|\tilde{u}_2\|^2\left[\frac{\nu\lambda_1}{4}\gamma(t)\Big(1 + \log\big(r(t)\big)\Big) - \mu\right] \\
\iff &\frac{\partial}{\partial t}\|\tilde{u}\|^2 + \frac{\nu\lambda_1}{4}\|\tilde{u}\|^2 + \frac{\nu\lambda_1}{4}\|\tilde{u}_2\|^2\left(r(t) - \gamma(t)\Big(1 + \log\big(r(t)\big)\Big) + \frac{4}{\nu\lambda_1}\mu\right) \leq 0
\end{aligned}$$

$$\Rightarrow \frac{\partial}{\partial t}\|\tilde{u}\|^2 + \frac{\nu\lambda_1}{4}\|\tilde{u}\|^2 + \frac{\nu\lambda_1}{4}\|\tilde{u}_2\|^2\big(\phi(r(t)) + \frac{4}{\nu\lambda_1}\mu\big) \leq 0 \tag{3.26}$$

Recall lemma 2.3, which applies since $\gamma(t) = \frac{c}{\nu}\|\nabla u\|^2 > 0$ ($\lambda_1, c > 0$) then

$$\min\{\phi(r(t)) : r(t) \geq 1\} \geq -\gamma\log(\gamma)$$

Hence

$$\phi(r(t)) \geq -\gamma\log(\gamma) = -\frac{4c}{\nu^2\lambda_1}\|\nabla u\|^2 \log\left(\frac{4c}{\nu^2\lambda_1}\|\nabla u\|^2\right)$$

Now let us define

$$\beta(t) := \mu - \frac{c}{\nu}\|\nabla u\|^2 \log\left(\frac{4c}{\nu^2\lambda_1}\|\nabla u\|^2\right) \tag{3.27}$$

Thus we have

$$\frac{\nu\lambda_1}{4}\phi(r(t)) + \mu \geq \beta(t)$$

We'll apply Gronwall's lemma to this quantity. Substituting β in (3.26) leads to

$$\begin{aligned}
&\frac{\partial}{\partial t}\|\tilde{u}\|^2 + \frac{\nu\lambda_1}{4}\|\tilde{u}\|^2 + \|\tilde{u}_2\|^2\beta(t) \leq 0 \\
\iff &\frac{\partial}{\partial t}\|\tilde{u}\|^2 + \frac{\nu\lambda_1}{4}\|\tilde{u}_1\|^2 + \frac{\nu\lambda_1}{4}\|\tilde{u}_2\|^2 + \|\tilde{u}_2\|^2\beta(t) \leq 0
\end{aligned}$$

And note that

$$\min\{\frac{\nu\lambda_1}{4},\beta(t)\}\leq 2\min\{\frac{\nu\lambda_1}{4},\beta(t)\}\leq\frac{\nu\lambda_1}{4}\beta(t)$$

Thus we find

$$\frac{\partial}{\partial t}\left\|\tilde{u}\right\|^2+\frac{\nu\lambda_1}{4}\left\|\tilde{u}_1\right\|^2+\min\{\frac{\nu\lambda_1}{4},\beta(t)\}\left\|\tilde{u}_2\right\|^2\leq 0 \tag{3.28}$$

Now we set $\tau := \frac{1}{\nu\lambda_1}$ and consider

$$\int_t^{t+\tau}\frac{c}{\nu}\left\|\nabla u(s)\right\|^2\log\Big(\frac{4c}{\nu^2\lambda_1}\left\|\nabla u(s)\right\|^2\Big)ds$$

By Prop 2.5, $\left\|\nabla u\right\|^2\leq\tilde{c}\nu^2\lambda_1 g^2 e^{G^4}$ thus

$$\begin{aligned}\frac{c}{\nu}\int_t^{t+\tau}\left\|\nabla u(s)\right\|^2\log\Big(\frac{4c}{\nu^2\lambda_1}\left\|\nabla u(s)\right\|^2\Big)ds &\leq\frac{c}{\nu}\int_t^{t+\tau}\left\|\nabla u(s)\right\|^2\log\Big(G^2e^{G^4}\Big)ds\\ &=\frac{c}{\nu}\big(1+\log(G)+G^4\big)\int_t^{t+\tau}\left\|\nabla u(s)\right\|^2ds\end{aligned}$$

Still by Prop. 2.5, we know that

$$\int_t^{t+\tau}\left\|\nabla u(s)\right\|^2ds\leq 2(1+\tau\nu\lambda_1)\nu G^2$$

Thus

$$\Rightarrow\frac{c}{\nu}\big(1+\log(G)+G^4\big)\int_t^{t+\tau}\left\|\nabla u(s)\right\|^2dd\leq\frac{c}{\nu}\big(1+\log(G)+G^4\big)\cdot 2(1+1)\nu G^2$$

Hence we found that

$$\frac{c}{\nu}\int_t^{t+\tau}\left\|\nabla u(s)\right\|^2\log\Big(\frac{4c}{\nu^2\lambda_1}\left\|\nabla u(s)\right\|^2\Big)ds\leq 4c\big(1+\log(G)+G^4\big)G^2 \tag{3.29}$$

Recall that by assumption $\mu\geq 8c\nu\lambda_1\big(1+\log(G)+G^4\big)G^2$ which implies

$$\frac{\mu}{2\nu\lambda_1}\geq 4c\big(1+\log(G)+G^4\big)G^2$$

Thus we are ready to check that Gronwall's lemma indeed applies :

$$\begin{aligned}
\text{liminf}_{t\to\infty}\int_t^{t+\tau}\beta(s)ds &= \text{liminf}_{t\to\infty}\int_t^{t+\tau}\mu-\frac{c}{\nu}\int_t^{t+\tau}\|\nabla u(s)\|^2\log\Big(\frac{4c}{\nu^2\lambda_1}\|\nabla u(s)\|^2\Big)ds\\
&\geq \int_t^{t+\tau}\mu ds - 4c\big(1+\log(G)+G^4\big)G^2 = \frac{\mu}{\nu\lambda_1}-4c\big(1+\log(G)+G^4\big)G^2\\
&\geq \frac{\mu}{\nu\lambda_1}-\frac{\mu}{2\nu\lambda_1}=\frac{\mu}{2\nu\lambda_1}>0
\end{aligned}$$

We also need to check that the limsup is finite :

$$\begin{aligned}
\text{limsup}_{t\to\infty}\int_t^{t+\tau}\beta(s)ds &= \text{limsup}_{t\to\infty}\int_t^{t+\tau}\mu-\frac{c}{\nu}\int_t^{t+\tau}\|\nabla u(s)\|^2\log\Big(\frac{4c}{\nu^2\lambda_1}\|\nabla u(s)\|^2\Big)ds\\
&\leq \int_t^{t+\tau}|\mu|ds+\frac{c}{\nu}\int_t^{t+\tau}\|\nabla u(s)\|^2\log\Big(\frac{4c}{\nu^2\lambda_1}\|\nabla u(s)\|^2\Big)ds\\
&\leq \frac{\mu}{\nu\lambda_1}+4c\big(1+\log(G)+G^4\big)G^2\leq\frac{\mu}{\nu\lambda_1}+\frac{\mu}{2\nu\lambda_1}=\frac{3\mu}{\nu\lambda_1}<\infty
\end{aligned}$$

Therefore if we define $\tilde{\beta}(t) := \min\{\frac{\nu\lambda_1}{4},\beta(t)\}$, we see that $\tilde{\beta}(t)$ satisfies the conditions of Gronwall's lemma, thus

$$\|\tilde{u}\|^2=\|u-U\|^2\to 0 \quad \text{at an exponential rate as } t\to\infty.$$

$\square$

Appendix

4.1 Duhamel's Principle

The following Theorem considers u, f such that

$$u'(t) = \nabla u(t) + f(t), \quad 0 < t < T \tag{4.1}$$

and is therefore not specific to the Navier Stokes Equation.

Theorem 3. *(Duhamel's Principle)*

1. *Let $u \in C([0,T), L^2(\mathbb{T}^n))$ and $f \in C((0,T), L^2(\mathbb{T}^n))$ satisfy (3.27) and $u(0) = g$. Assume that for each each $t \in (0,T)$, $u'(t)$ and $\Delta u(t)$ both exist in $L^2(\mathbb{T}^n)$, also assume that*

$$\lim_{\epsilon \longrightarrow 0} \int_\epsilon^a |f(t)|\, dt < \infty, \quad \textit{for some } 0 < a < T \tag{4.2}$$

Then we have

$$u(t) = e^{t\Delta} g + \lim_{\epsilon \longrightarrow 0} \int_\epsilon^t e^{(t-\tau)\Delta} f(\tau) d\tau, \quad 0 < t < T,$$

Where the limit $\epsilon \to 0$ can be replaced by the evaluation $\epsilon = 0$ if $f \in C([0,T), L^2(\mathbb{T}^n))$. We also have that the solution to (3.27) is unique.

2. *Let $g \in H^\alpha(\mathbb{T}^n)$ and let $f \in C((0,T), H^\sigma(\mathbb{T}^n))$ with $0 \leq \alpha \leq \sigma$. Also assume (3.26). Then the function*

$$u(t) = e^{t\Delta} g + \lim_{\epsilon \longrightarrow 0} \int_\epsilon^{t-\epsilon} e^{(t-\tau)\Delta} f(\tau) d\tau, \quad 0 < t < T,$$

is in $C((0,T), H^s(\mathbb{T}^n))$ for all $\alpha \leq s < \sigma + 2$, and satisfies the estimate :

$$\left|u(t)\right|_s \leq C(1 + t^{-\frac{s+\alpha}{2}}) \left|g\right|_\alpha + C\beta(t), \quad 0 < t < T,$$

Where the constant C depends only on $s, \alpha and \sigma$ and

$$\beta(t) = \lim_{\epsilon \longrightarrow 0} \int_\epsilon^{t-\epsilon} (1 + |t-\tau|^{-\frac{s+\sigma}{2}}) \left|f(t)\right|_\sigma d\tau < \infty, \quad 0 < t < T$$

Finally we have that $\lim_{t \longrightarrow 0} u(t) = g$ in H^α, and u is a strong H^s solution of (3.27) for $s < \sigma$.

4.2 Local Well Posedness

In the following theorem we consider X a Banach space and the following operators :

$$L : X \mapsto \Phi_\infty, \quad N_T : \Phi_T \mapsto \Phi_T$$

with $\Phi_T = C_b([0,T), X)$ for $0 < T \leq \infty$. We consider the IVP :

$$u = Lg + N_T u \tag{4.3}$$

And make the following assumptions :

$$Lg(0) = g \text{ for } g \in X, \quad \text{and } N_T(u)(0) = 0 \text{ for } u \in \Phi_T, \tag{4.4}$$

$$\|Lg\|_{\Phi_\infty} \leq \|g\|_X \text{ for } g \in X, \tag{4.5}$$

And finally assume that there exists $C_{R,T} \xrightarrow{T\longrightarrow 0} 0$ for any fixed $R > 0$ such that

$$\|N_T(u) - N_T(v)\|_{\Phi_T} \leq C_{R,T} \|u - v\|_{\Phi_T} \text{ for all } u, v \in \Phi_T \text{ s.t } \|u\|_{\Phi_T}, \|v\|_{\Phi_T} \leq R. \tag{4.6}$$

We are now ready to state the Local Well posedness Theorems for IVPs of the type (3.24).

Theorem 4. *Local Well posedness For any $r > 0$ there exists $T > 0$ such that as long as the initial datum satisfies $\|g\|_X \leq r$, the IVP (3.24) has a solution in Φ_T. Moreover, this solution is unique in Φ_t and the solution map is locally Lipschitz in the sense that if $u_i \in \Phi_T$ is the solution with $u_i(0) = g_i$ and $\|g_i\|_X \leq r$, $i = 1, 2$ then there exists a constant c such that*

$$\|u_1 - u_2\|_{\Phi_T} \leq c \|g_1 - g_2\|_X$$

4.3 Uniqueness Result

In order to have uniqueness for the problem (3.24), we need to make a few more assumptions on the operators N_T and L :

$$N_T(u\big|_{[0,T]}) = N_{T'}(u)\big|_{[0,T]} \text{ for any } u \in \Phi_{T'} \text{ and } 0 < T < T'. \tag{4.7}$$

$$L^t L^s g = L^{t+s} g \text{ for } g \in X \text{ and } s, t \geq 0, \tag{4.8}$$

and finally

$$N^{T+t}(u) = L^t N^T(u) + N^t(u(\cdot + T)), \text{ for any } u \in \Phi_{T'} \text{ with } 0 < T < T' \text{ and } 0 \leq t < T' - T. \tag{4.9}$$

Then we find that the following holds :

Theorem 5. *Uniqueness If $u_1 \in \Phi_T$ and $u_2 \in \Phi_{T'}$ are solutions of (3.24) with $0 < T < T'$, then $u_1 = u_2$ on $[0, T)$.*

4.4 Blow-up Criterion

Let us first define the *maximal interval of existence* :

Definition : For some fixed initial data $g \in X$, let us collect all $T_\alpha > 0$ such that there is a solution $u_\alpha \in \Phi_{T_\alpha}$ for the IVP (3.24). Then the *maximal interval of existence* of (3.24) is defined by $I = I(g) = \cup_{\alpha[0,T_\alpha)}$ and the maximal solution is given by

$$u(t) = u_{\alpha_t}(t), \quad t \in I,$$

where $\alpha_t \in \{\alpha : t \in I_\alpha\}$ for each $t \in I$.

Then the following holds true :

Theorem 6. *Blow-up Criterion For an IVP of the type (3.24), the maximal interval of existence is necessarily of the form $I = [0, T)$ for some $0 < T \leq \infty$. If $T < \infty$ then*

$$\|u(t)\|_X \xrightarrow{t \longrightarrow T} \infty$$

4.5 Existence and Uniqueness of Mild Solutions

In this final section of the appendix, we will state the follwing existence and uniqueness result ; under the assumption that $f(u)$ is locally Lipschitz, then it is a mild unique solution to the NSE.

Consider the IVP :

$$\begin{cases} u(t) = e^{t\Delta}g + \int_0^t e^{(t-\tau)\Delta} f(u(\tau))d\tau \\ u(0,t) \equiv g \end{cases} \tag{4.10}$$

Theorem 7. *Let $s \geq 0$, $\delta < 2$, $f : H^s \mapsto H^{s-\delta}$ be locally Lipschitz, then the IVP (4.10) with $g \in H^s$ has a unique maximal solution $u \in C((0,T], H^s)$ for $0 < T \leq \infty$.*

In addition if $T < \infty$ *then* $\left|u(t)\right|_s \to \infty$ *as* $t \to T$.

Finally if $f : H^{s'} \mapsto H^{s'-\delta}$ *is locally Lipschitz for all* $s' > s$ *then* $u \in C^\infty(\mathbb{T}^n) \times C^1((0,T))$.

The proof relies heavily on the Duhamel Principle and the Local Well-Posedness principle presented in the previous sections.

Bibliography

[1] AMANN, H. *On the Strong Solvability of the Navier-Stokes Equations*. Journal of Mathematical Fluid Mechanics, 2020.

[2] BEN-ARTZI, M. *Global Solutions of two-dimensional Navier-Stokes and Euler Equations*. Archives for Rational Mechanics and Analysis, 1994.

[3] BENRA, F., DOHMEN, H., ANDSEBASTIAN SCHUSTER, J. P., AND WAN, B. A comparison of one-way and two-way coupling methods for numerical analysis of fluid-structure interactions. *Journal of Applied Mathematics* (2011).

[4] BRETT, C., LAM, K., LAW, K., MCCORMICK, D., SCOTT, M., AND STUART, A. Stabilities of filter for the navier stokes equations. *Warwick Mathematics Institute* (2011).

[5] EROGLU, D., LAMB, J., AND PEREIRA, T. *Synchronisation of chaos and its applications*. Contemporary Physics, 2017.

[6] FARWIG, R. *From Jean Leray to the millenium problem : the Navier Stokes equations*. Journal of Evoution Equations, 2020.

[7] GAO, X., GAO, S., AND YANG, Y. A comparison between 3dvar and enkf for data assimilation effects on the yellow sea fog forecast. *Atmosphere* (2018).

[8] GHADIMI, P., AND DASHTIMANESH, A. Solution of 2d-navier-stokes equation by couple finite difference-dual reciprocity boundary element method. *Applied Mathematical Modelling* (2010).

[9] KOCH, H., AND TATARU, D. *Well-Posedness for the Navier-Stokes Equations*. Northwestern University Journal, 2007.

[10] MATHEWS, J., AND FINK, K. *Numerical Methods using Matlab, 4th Edition*. Prentice-Hall Inc., 2004.

[11] MOMBEINI, M., AND KHALOOZADEH, H. *State and Parameter Estimation of The Lorenz System in Existence of Colored Noise*. IPM, Teheran, Iran, 2010.

[12] S.E., S. A., SURAJ, P., AND SAN, O. *PyDA : A Hands-On Introduction to Dynamical Data Assimilation with Python*. Fluids, 2020.

[13] TSOGTGEREL, G. A gentle introduction to the regularity theory of the navier stokes equations, 2013.

[14] WELCH, G., AND BISHOP, G. *An introduction to the Kalman Filter*. ACM SIGGRAPH, 2007.

[15] YANG, BAKER, LI, CORDES, HUFF, NAGPAL, OKEREKE, VILLAFAÑE, KALNAY, AND DUANE. *Data Assimilation as Synchronisation of Truth and Model : Experiments with Three-Variable Lorenz System*. American Meteorological Society, 2006.