Nodal Statistics for the Lamé Ensemble

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September, 2001

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Doctor of Philosophy

0-612-75610-6

Abstract

The Lamé polynomials naturally arise when separating variables in Laplace's equation in elliptic-spherical coordinates. The products of these polynomials form a class of spherical harmonics, which are the joint eigenfunctions of a quantum completely integrable system of commuting, second-order differential operators $P_0 = \Delta_{\mathbf{S}^N}, P_2, ..., P_{N-1}$ acting on $C^\infty(\mathbf{S}^N)$. These operators depend on parameters and thus constitute an ensemble.

In the main result presented in this thesis, we compute the limiting mean level spacings distribution for the zeroes of Lamé polynomials in various thermodynamic, asymptotic regimes. We give results both in the mean and pointwise, for an asymptotically full set of values of the parameters. As an application, we compute the limiting level spacings distribution of the zeroes of Van Vleck polynomials.

Résumé

Les polynômes de Lamé interviennent de façon naturelle lors de la séparation des variables, en coordonnées elliptiques-sphériques, de l'équation de Laplace. Les différents produits de ces polynômes forment ainsi une sous-classe d'harmoniques sphériques. De plus, ces produits peuvent être aussi utilisés pour décrire les fonctions propres d'un système quantique, complètement intégrable, d'opérateurs différentiels du second ordre $P_0 = \Delta_{\mathbf{S}^N}, P_2, ..., P_{N-1}$ définis sur $C^\infty(\mathbf{S}^N)$. Ces opérateurs dépendent intrinsèquement d'une certaine famille de paramètres, et constituent donc un ensemble, que l'on appelle ensemble de Lamé.

Les résultats principaux présentés dans cette thèse consistent à calculer le niveau moyen d'espacement de la distribution des zéros des polynômes de Lamé pour différents régimes thermodynamiques. Nous donnons les résultats sous forme de limite ponctuelle et de limite en moyenne, pour un sous-ensemble des paramètres de mesure asymptotiquement 1. Enfin, nous appliquons des techniques similaires afin de calculer le niveau moyen d'espacement limite pour la distribution des zéros des polynômes de Van Vleck.

Acknowledgements

First, and most of all, I wish to express my deepest gratitude to my supervisor Professor John A. Toth. Without his significant contributions, invaluable discussions and assistance, this thesis could not be presented as it is. His modern view of analysis has deeply influenced my work. Under his supervision and his enthusiasm. I have been introduced to many beautiful subjects of mathematics. Thanks for everything John !

I am also thankful to Professor D. Jakobson and Dr. Petridis for many useful discussions about this work and their precious advices for the unexperienced mathematician I am. For the same reasons and also his help in finding a post-doctoral position, I would like to thank Professor Min-Oo. Many thanks also go to Professor S. Zelditch for his numerous insightful comments about our paper [**BT**], and for giving me the opportunity to go to Johns Hopkins next year.

I will always remember the five years spent here at McGill as one of the most wonderful experience of my life. Over the years, I have met many people who became really good friends. So, I would like to take this opportunity to thanks all of them (hopefully !): Lassina, Ignazio, Gordon, Marc-Hubert, George, Isabelle, Javad, Cyr Émile, Dominic. But mostly, those years have given me the chance to find a wonderful person, my wife Gülhan Alpargu, to whom I would simply say: "Teşekkür ederim herşey için".

Also, many special thanks to all my family; my mother, my father, my sisters Michèle, Diane and Andrée who have been there for me over the last 31 years. Without them, this work could have never been accomplished.

Finally, I would like to thanks the I.S.M., the F.C.A.R. and McGill University for their financial support.

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acting on $C^\infty(\mathbf{S}^N)$. Here, S_k^{ij} denotes the k-th elementary symmetric polynomial in the α parameters with α_i and α_j deleted. It is easy to check that

$$(i) \quad P_0 = \Delta_{\mathbf{S}^N},$$
$$(ii) \quad [P_i, P_j] = 0 \quad \text{for all} \quad i, j = 0, ..., N-1.$$

Consequently, the P_j's form a QCI depending on the parameters $\alpha_0, ..., \alpha_N$ and thus constitute an ensemble.

Since the P_j's are jointly elliptic, they possess a Hilbert basis of joint eigenfunctions. Moreover, since P_0 is just the constant curvature spherical Laplacian, these eigenfunctions form a class of spherical harmonics, the so-called *generalized Lamé harmonics*. In our main result (Theorem 3.1 in Chapter 3), we derive asymptotic formulae for the level spacings distribution of the zeroes of these spherical harmonics.

To describe our results in more detail, we begin by noting that in terms of appropriate (see Chapter 1) parameterizing coordinates $(u_1, ..., u_N) \in (\alpha_0, \alpha_1) \times ... \times (\alpha_{N-1}, \alpha_N)$ on $\mathbf{S}^N$, and up to constant multiples, the joint eigenfunctions of $P_0, ..., P_{N-1}$ can be written in the form:

$$\psi(u_1, ..., u_N) = \prod_{j=1}^{N} \prod_{\nu=0}^{N} (u_j - \alpha_\nu)^{\beta_\nu/2} \cdot \phi(u_j).$$

Here, ϕ is a polynomial, and $\beta = (\beta_0, \dots, \beta_N)$ is a multi-index with $\beta_\nu \in \{0, 1\}$, $\nu = 0, ..., N$. Furthermore, the function $\psi(x) := \prod_{\nu=0}^{N} (x - \alpha_\nu)^{\beta_\nu/2} \cdot \phi(x)$ is a solution of the *generalized Lamé equation*

$$\prod_{\nu=0}^{N} (x - \alpha_\nu) \frac{d^2\psi}{dx^2} + \frac{1}{2} \sum_{\nu=0}^{N} \prod_{\lambda \neq \nu} (x - \alpha_\lambda) \frac{d\psi}{dx} + C(x)\psi = 0, \tag{0.3}$$

where, $C(x)$ is a polynomial of order $N-1$ depending linearly on the joint eigenvalues $\lambda_0, ..., \lambda_{N-1}$ of the operators $P_0, ..., P_{N-1}$. When $\beta = 0$, the solutions $\psi(x)$ are called *Lamé polynomials* and the corresponding $C(x)$ are called *Van Vleck polynomials*.

Consider

$$\mathcal{E}(k) := \left\{ \phi_1^{(k)}, ..., \phi_{j(k)}^{(k)} \right\},$$

the set of Lamé polynomials of degree k. By the standard theory of spherical harmonics **[WW]** and the fact that the corresponding *Lamé harmonics* form a Hilbert basis, we know that $j(k) = \sigma(N, k) := \frac{(N+k-1)!}{k!\,(N-1)!}$. Let $\theta_{i,1}^{(k)} < \cdots < \theta_{i,k}^{(k)}$ denote the (real) zeroes of the polynomial, $\phi_i^{(k)}$, where $i = 1, ..., \sigma(N, k)$.

theorems, namely the Heine-Stieltjes Theorem [**Sz**] concerning the distribution of the zeroes of Lamé polynomials, and an analogous result due to G. Shah [**Sh3**] about the distribution of the zeroes of Van Vleck polynomials.

In Chapter 4, we apply similar techniques to those in Chapter 3 to compute the limiting level spacings distribution of the zeroes of a Van Vleck polynomial. The results are presented in Theorem 4.1.

CHAPTER 1

Quantum integrable systems

In this first chapter, we introduce the classical C. Neumann problem on the N-sphere $\mathbf{S}^N$. We then show how this system can be integrated at the quantum level by exhibiting a family of commuting differential operators acting on $C^\infty(\mathbf{S}^N)$. We then apply similar arguments to integrate the motion of a free particle on $\mathbf{S}^N$ both at the classical and the quantum level.

1. Integrable Hamiltonian systems

In classical mechanics, one can describe the motion of a particle in the Euclidean space $\mathbb{R}^N$ in terms of the Hamiltonian equations:

$$\dot{x}_i = \frac{\partial H}{\partial \xi_i}, \quad \dot{\xi}_i = -\frac{\partial H}{\partial x_i},$$

where $x = (x_1, ..., x_N) \in \mathbb{R}^N$, $\xi = (\xi_1, ..., \xi_N) \in \mathbb{R}^N$. The smooth function $H = H(x, \xi)$ defined on some open domain in $\mathbb{R}^{2N}$ is called the Hamiltonian function.

More generally, let (M^{2N}, ω) be a symplectic manifold. That is, M is a smooth manifold of dimension $2N$ and, ω, a non-degenerate, closed 2-form on M. Under the identification

$$J : T^*M \longrightarrow TM,$$

defined by requiring that for all $v \in TM$, $\omega(J(\theta), v) = -\theta(v)$, one associates to a function $H \in C^\infty(M)$ the vector field $X_H := J(dH)$. In analogy with the local case above, we call H the Hamiltonian function and, X_H, the Hamiltonian vector field.

The Poisson bracket $\{F, G\}$ of two functions $F, G \in C^\infty(M)$ is defined to be:

$$\{F, G\} := \omega(J(dF), J(dG)).$$

Since the commutator $[X_F, X_G] := X_F X_G - X_G X_F$ is again a differential operator of the first order generated by the Hamiltonian $\{F, G\}$, i.e. $[X_F, X_G] = X_{\{F,G\}}$, the Hamiltonian vector fields form a Lie algebra. It is this Lie algebra which is the main object of study in classical mechanics.

In terms of local Darboux coordinates (see [**Ar**]), $x = (x_1, ..., x_N)$, $\xi = (\xi_1, ..., \xi_N)$, the symplectic form is just $\omega = \sum_{i=1}^N d\xi_i \wedge dx_i$, whereas the Poisson bracket and the Hamiltonian vector field are given by

$$\{F,G\} = \sum_{i=1}^{N} \left(\frac{\partial F}{\partial x_i}\frac{\partial G}{\partial \xi_i} - \frac{\partial F}{\partial \xi_i}\frac{\partial G}{\partial x_i} \right),$$

and,

$$X_H = \sum_{i=1}^{N} \left(\frac{\partial H}{\partial \xi_i}\frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i}\frac{\partial}{\partial \xi_i} \right).$$

Consequently, the flow of X_H is governed by the Hamiltonian equations

$$\dot{x}_i = \frac{\partial H}{\partial \xi_i}, \quad \dot{\xi}_i = -\frac{\partial H}{\partial x_i}, \quad \text{for } i = 1, ..., N.$$

A non-constant function $F \in C^\infty(M)$ is said to be an integral if $X_H(F) = \{F, H\} = 0$. This is equivalent to saying that $F(x(t), \xi(t))$ is independent of t, i.e. $F(x(t), \xi(t))$ is constant along the integral curves of X_H. In general, the existence of one or more integrals can be used to reduce the order of the system, and hence bring it to simpler one. Consequently, the knowledge of integrals is of interest.

Definition: A Hamiltonian system (M, ω, H) is said to be completely integrable if it possesses N integrals $F_1 = H, F_2, ..., F_N$ satisfying the two conditions

(*i*) $\{F_i, F_j\} = 0$ for $i, j = 1, ..., N$,

(*ii*) $dF_1, ..., dF_N$ are linearly independent on an open dense subset in M.

Functions satisfying (*i*) and (*ii*) are said to be in *involution*.

For an integrable Hamiltonian system (M, ω, H), the vector field X_H is clearly tangent to the submanifolds

$$M_c = \Big\{(x, \xi) \in M : F_1 = c_1, ..., F_N = c_N\Big\},$$

where $c = (c_1, ..., c_N)$ is a regular value of the function $F := (F_1, ..., F_N)$. In other words, these submanifolds are invariant under the flow generated by X_H. Thus, the fibers of F are foliated into N-dimensional invariant submanifolds. According to a theorem of V. I. Arnold [**Ar**], such a leaf is necessary a torus T^N provided it is compact and connected. Moreover, in a neighborhood of such a compact leaf the structure of an integrable system is particulary simple. Indeed, one can introduce symplectic action-angle variables $\theta_1, ..., \theta_N, I_1, ..., I_N$, in terms of which the Hamiltonian equations become

$$\dot{\theta}_i = \frac{\partial H}{\partial I_i}(I_1, ..., I_N), \quad \dot{I}_i = 0, \quad \text{for } i = 1, ..., N.$$

One can then solve this system of equations by simple quadrature.

Here, S denotes the generating function of the canonical transformation $(u,\xi) \mapsto (v,\eta)$ that satisfies the differential equations

$$\xi_j = \frac{\partial S}{\partial u_j}, \quad v_j = \frac{\partial S}{\partial \eta_j}, \quad j = 1, ..., N.$$

In fact, Neumann computed explicitly the expression of S (see [**Mo**]). He obtained

$$S(u,\eta) = \frac{1}{2}\sum_{j=1}^{N} \int_0^{u_j} \sqrt{-\frac{Q(z)}{A(z)}}dz,$$

where $Q(z) = z^N + 2\eta_0 z^{N-1} + \cdots + 2\eta_N$ and $A(z) = \prod_{\nu=0}^{N}(z-\alpha_\nu)$.

Under the canonical transformation induced by S, the Hamiltonian function takes the elementary form $H = \eta_1$, whereas the equations of motions are now governed by the Hamiltonian system

$$\dot{v}_i = \delta_{i1}, \quad \dot{\eta}_i = 0 \quad \text{for } i = 1, ..., N.$$

Consequently, the functions $\eta_1, ..., \eta_N$ yield N integrals, which are clearly in involution. Although Neumann proved the complete integrability of the system, he did not arrive at the algebraic expressions in terms of x and ξ of the integrals $\eta_1, ..., \eta_N$. In order to do this, we will now consider the following approach of Moser [**Mo**].

To show that the C. Neumann is completely integrable, Moser considered the following extended Hamiltonian system on $T^*(\mathbb{R}^{N+1})$:

$$\begin{cases} H(x,\xi) = \frac{1}{2}\sum_{\nu=0}^{N} \alpha_\nu x_\nu^2 + \frac{1}{2}\left[|x|^2|\xi|^2 - (x,\xi)\right] \\ \dot{x}_i = H_{\xi_i}, \quad \dot{\xi}_i = -H_{x_i} \end{cases} \tag{1.5}$$

This system has the integral $|x|^2$ since it is invariant under the symplectic transformation $(x,\xi) \mapsto (x, \xi + 2xs)$ generated by $|x|^2$. Therefore, one can reduce (1.5) by this integral. We denote by G the isotropy group induced by the action $(x,\xi) \mapsto (x, \xi + 2xs)$. To form the quotient manifold

$$\widetilde{M} := \left\{(x,\xi) \in \mathbb{R}^{2N+2} : |x|^2 = 1\right\}/G,$$

we single out the point x on the line $x + t\xi$ for which $(x,\xi) = 0$. As a result, we obtain that $\widetilde{M}$ is diffeomorphic to the manifold M with

$$M = \left\{(x,\xi) \in \mathbb{R}^{2N+2} : |x|^2 = 1, \ (x,\xi) = 0\right\}.$$

Note that, M can naturally be identified with the cotangent bundle of the N-sphere, i.e. $M = T^*(\mathbb{S}^N)$. In order to derive the Hamiltonian system obtained by reducing (1.5) to M, we put

$$H_M = H - \lambda(|x|^2 - 1),$$

Here, $S_j^k(u)$ denotes the j-th elementary symmetric polynomial with u_k omitted. The potential functions $V_j \in C^\infty(\mathbf{S}^N)$ are given by the local formulae

$$V_j(x) = (-1)^j S_{j+1}(u_1, ..., u_N), \text{ for } j = 1, ..., N.$$

3. The quantum C. Neumann system

We follow here the presentations of D. Gurarie **[G]** and J. A. Toth **[T1]**, **[T2]** to integrate the C. Neumann system at the quantum level. That is, we construct N pairwise commuting, partial differential operators on $L^2(\mathbf{S}^N)$, which in turn commute with the respective quantum Hamiltonian.

The quantum Hamiltonian associated to H is given by

$$\mathcal{H} = -\Delta_0 + V_0(x), \qquad (\hbar = 1),$$

acting on the dense subset $C^\infty(\mathbf{S}^N)$ of $L^2(\mathbf{S}^N)$. Here, $V_0(x) := (Ax, x)$, and Δ_0 denotes the Laplacian associated to the metric g_0 as defined in previous section. Note that, Δ_0 is simply the constant curvature Laplacian on $\mathbf{S}^N$.

First, we present the approach of D. Gurarie **[G]**. His procedure is rather straightforward; it consists of associating to the classical observables $f_0, ..., f_N$ constructed by Moser, the partial differential operators

$$\mathcal{F}_\nu(x, i\partial) = x_\nu^2 + \sum_{\mu \neq \nu} \frac{(x_\nu \partial_\mu - x_\mu \partial_\nu)^2}{\alpha_\nu - \alpha_\mu}, \qquad \nu = 0, ..., N. \tag{1.10}$$

As for the classical case, the α- weighted sum of the operators yields the quantum Hamiltonian, i.e.

$$\mathcal{H} = \sum_{\nu=0}^{N} \alpha_\nu \mathcal{F}_\nu, \tag{1.11}$$

and also, $\sum_{\nu=0}^N \mathcal{F}_\nu = |x|^2 = 1$, for $x \in \mathbf{S}^N$. To see that this gives a complete integrable system, it then suffices to restrict the operators $\mathcal{F}_\nu$ originally defined on the extended Hilbert space $L^2(\mathbb{R}^{N+1})$ to $L^2(\mathbf{S}^N)$. The proper commutation relations

$$[\mathcal{F}_i, \mathcal{F}_j] = 0, \qquad \text{for } i, j = 0, ..., N.$$

are verified by direct computation.

Another approach, though less transparent, was given by J. A. Toth **[T1]**, **[T2]**. To show that the quantum C. Neumann on $\mathbf{S}^N$ is completely integrable, he first considers the functions

$$\rho_j = \left(\det(g_j^{kl})\right)^{1/2} \left(\det(g_0^{kl})\right)^{-1/2}, \tag{1.12}$$

where, the g_j denotes the underlying metric defined in (1.9). As a consequence of (1.7), the ρ_j extend to smooth, strictly positive functions on S^N. He then constructs the partial differential operators

$$\mathcal{P}_j = -\left(\Delta_j + \nabla_j \log(\rho_j)\right) + V_j(x), \qquad j = 0, ..., N-1. \tag{1.13}$$

for which he verifies the required commutation relations $[\mathcal{P}_i, \mathcal{P}_j] = 0$. Note that, in the special case where $j = 0$, $\rho_0 \equiv 1$ and hence, $\mathcal{P}_0 = \mathcal{H}$. Therefore, the given system is quantum integrable.

One of the most important features of this quantum system resides in its spectral aspects. Indeed, by seeking joint eigenfunctions of the form $\prod_{j=1}^{N} \psi(u_j)$, where $u_1, ..., u_N$ denote the elliptic-spherical coordinates on S^N, Toth obtained, after separating variables, the following remarkable property. Namely, such eigenfunctions may be found by solving the single ordinary differential equation

$$\prod_{\nu=0}^{N}(x-\alpha_\nu)\frac{d^2\psi}{dx^2} + \frac{1}{2}\sum_{\nu=0}^{N}\prod_{\mu\neq\nu}(x-\alpha_\mu)\frac{d\psi}{dx} + \left(\sum_{\nu=0}^{N-1}\lambda_{N-\nu-1}x^\nu\right)\psi = 0. \tag{1.14}$$

The separation constants $\lambda_0, ..., \lambda_{N-1}$ are exactly the joint eigenvalues of the operators $\mathcal{P}_0, ..., \mathcal{P}_{N-1}$. As we will see in the next chapter, the last equation is a natural generalization of Lamé differential equation.

4. The free particle on S^N

Let us now consider the motion of a particle on the sphere S^N subject to no force. This can be viewed as the motion of a "free" particle moving on S^N. The corresponding Hamiltonian system is then given by

$$\begin{cases} \dot{x}_\nu = H_{\xi_\nu}, \quad \dot{\xi} = -H_{x_\nu} \text{ for } \nu = 0, ..., N, \\ H(x,\xi) = \frac{1}{2}|\xi|_0^2, \quad x \in S^N, \quad \xi \in \mathbb{R}^{N+1}, \end{cases} \tag{1.15}$$

where, as before, $|\ |_0$ denotes the standard metric on $T^*(S^N)$. Using similar methods as those developed by Moser to integrate the C. Neumann system (see section 2), one can show that this system is completely integrable. Indeed, the following Hamiltonian system

$$\begin{cases} \dot{x}_\nu = H_{\xi_\nu}, \quad \dot{\xi} = -H_{x_\nu} \text{ for } \nu = 0, ..., N \\ H(x,\xi) = \frac{1}{2}\left(|x|^2|\xi|_0^2 - (x,\xi)^2\right), \quad x \in \mathbb{R}^{N+1}, \quad \xi \in \mathbb{R}^{N+1} \end{cases} \tag{1.16}$$

on $T^*(\mathbb{R}^{N+1})$ is, when restricted to $T^*(S^N)$, equivalent to the system (1.15) defined above. One can then construct, for given parameters $\alpha_0, ..., \alpha_N$ with $0 < \alpha_0 < \cdots < \alpha_N$, $N+1$ integrals given by

$$f_i(x,\xi) = \sum_{j\neq i} \frac{(x_i\xi_j - x_j\xi_i)^2}{\alpha_i - \alpha_j}, \quad i = 0, ..., N. \tag{1.17}$$

As in the case of the C. Neumann system, for $x \in S^N$, only N of the integrals in (1.17) are independent since they satisfy the equation $\sum_{\nu=0}^N f_i = 0$. Moreover, the Hamiltonian function H is given by

$$H = \frac{1}{2}\sum_{\nu=0}^N \alpha_\nu f_\nu.$$

It is easy to check that $f_0, ..., f_N$ are in involution, and so the system (1.15) is completely integrable.

Next, we integrate this system at the quantum level by considering the second-order partial differential operators given by

$$P_k := \sum_{i<j} S_k^{ij}(\alpha_0, ..., \alpha_N)(x_i\partial_j - x_j\partial_i)^2, \quad k = 0, ..., N-1, \tag{1.18}$$

acting on $C^\infty(S^N)$. Here, S_k^{ij} denotes the k-th elementary symmetric polynomial in the α parameters with α_i and α_j deleted. Direct computation shows that

$$\begin{aligned} &(i) \quad \mathcal{H} = \Delta_{S^N} = P_0, \\ &(ii) \quad [P_i, P_j] = 0, \text{ for all } i, j = 0, ..., N-1. \end{aligned}$$

In other words, the operators $P_0, ..., P_{N-1}$ form a QCI system.

Since the P_j's are jointly elliptic, they possess a Hilbert basis of joint eigenfunctions. Also, since P_0 is just the constant spherical curvature Laplacian on S^N, these eigenfunctions form a class of spherical harmonics, the so-called *generalized Lamé harmonics*. To describe these harmonics in more detail, we proceed in a similar fashion as in the case of the C. Neumann system. Let $u_1, ..., u_N$ be elliptic-spherical coordinates and eigenfunctions of the form $\varphi(u_1, ..., u_N) = \prod_{j=1}^N \psi(u_j)$. The eigenvalue equation for the Laplace operator becomes

$$\sum_{j=1}^N \frac{4}{\prod_{i\neq j}(u_j - u_i)} \left[\sqrt{U(u_j)}\frac{\partial}{\partial u_j}\left(\sqrt{U(u_j)}\,\frac{\partial\varphi}{\partial u_j}\right)\right] = -\lambda_0\varphi, \tag{1.19}$$

where $U(x) = \prod_{\nu=0}^N (x - \alpha_\nu)$.

Substituting the ansatz for φ into (1.19), we get that the function $\psi(x)$ must satisfy the ordinary differential equation

$$\prod_{\nu=0}^N (x-\alpha_\nu)\frac{d^2\psi}{dx^2} + \frac{1}{2}\sum_{\nu=0}^N \prod_{\mu\neq\nu}(x-\alpha_\mu)\frac{d\psi}{dx} + \frac{1}{4}\left(\sum_{j=0}^{N-1} \lambda_{N-j-1}x^j\right)\psi = 0, \tag{1.20}$$

where the separation constants $\lambda_0, ..., \lambda_{N-1}$ are the joint eigenvalues of the partial differential operators $P_0, ..., P_{N-1}$ defined above. Equation (1.20) is the well-known *generalized Lamé equation*. It will be the subject of the second chapter of this thesis, where we give a complete and detailed study of it.

CHAPTER 2

The Lamé differential equation

We now turn our attention to the study of the differential equation (1.20) satisfied by the joint eigenfunctions of the QCI system introduced in the first chapter. We follow the presentation of Whittaker and Watson [**WW**] where this equation is introduced via the theory of ellipsoidal harmonics. We then describe two important results about the zeroes distribution of two classes of polynomials associated to differential equation of Lamé's types, namely the Stieltjes and Van Vleck polynomials.

1. Introduction

In his classical treatise on heat conduction in an ellipsoidal body, G. Lamé [**L**] was led to consider the class of homogeneous, harmonic polynomials on $\mathbb{R}^{N+1}$ that vanish on a family of confocal quadrics, the so-called ellipsoidal harmonics. There is an analogous construction of spherical harmonics that we will now describe.

Pick a set $\{\alpha_0, \dots, \alpha_N\}$ of positive real constants, all distinct, and ordered in increasing order. Define, for some real parameter θ, the diagonal matrix $A_\theta = \operatorname{diag}\left((\theta-\alpha_0)^{-1}, \dots, (\theta-\alpha_N)^{-1}\right)$. The problem then reduces to finding, for any positive integer k and any multi-index $\beta = (\beta_0, \dots, \beta_N) \in \{0,1\}^{N+1}$, k real numbers $\theta_1, ..., \theta_k$ for which the Niven's functions

$$f_\beta(X) = X^\beta \prod_{j=1}^{k} (A_{\theta_j} X, X), \qquad X \in \mathbb{R}^{N+1}, \tag{2.1}$$

are solutions of Laplace's equation $\Delta(f_\beta) = 0$. The restriction of the f_β's to $\mathbb{S}^N$ yield an important class of spherical harmonics: the *generalized Lamé harmonics*. As we shall see later, they form a complete basis of $L^2(\mathbb{S}^N)$. In addition, these functions are, up to a constant, the joint eigenfunctions of the operators $P_0, ..., P_{N-1}$ considered in Chapter 1.

Clearly, the $f_\beta(X)$ vanish on a family of confocal cones. Moreover, after the substitution of the ansatz into Laplace's equation, a straightforward computation shows that the relevant θ_j are obtained as solutions of the equations

$$\sum_{\nu=0}^{N}\frac{1}{\theta_j-\alpha_\nu}+\sum_{\nu=0}^{N}\frac{2\beta_\nu}{\theta_j-\alpha_\nu}+\sum_{i\neq j}\frac{4}{\theta_j-\theta_i}=0 \quad \text{for} \quad j=1,...,k. \tag{2.2}$$

In the literature, these equations are commonly referred as the Niven's equations (see [**KM**], [**WW**]).

Consequently, if we denote the solutions of (2.2) by $\theta_1,...,\theta_k$, it is not hard to see that the functions

$$\psi(x)=\prod_{\nu=0}^{N}(x-\alpha_\nu)^{\beta_\nu/2}\prod_{j=1}^{k}(x-\theta_j), \quad \beta_\nu\in\{0,1\}, \tag{2.3}$$

are solution of the second order differential equation

$$\prod_{\nu=0}^{N}(x-\alpha_\nu)\frac{d^2\psi}{dx^2}+\frac{1}{2}\sum_{\nu=0}^{N}\prod_{\mu\neq\nu}(x-\alpha_\mu)\frac{d\psi}{dx}+C(x)\psi=0, \tag{2.4}$$

where $C(x)$ is a polynomial of degree $N-1$ that we compute explicitly in the next section. This equation is known as the *generalized Lamé differential equation.*

In the special case where the multi-index $\beta=0$, it follows that the k-th degree polynomial $\phi(x)=\prod_{j=1}^{k}(x-\theta_j)$ is a solution of Lamé equation. These are known as *Lamé polynomials.* In our main result (Theorem 3.1 in Chapter 3), we compute the limiting level spacings distribution of the zeroes of these polynomials in various thermodynamic, asymptotic regimes.

2. Separation of variables and eigenvalues

We now restrict our attention to the N-sphere $\mathbf{S}^N$. As we mentioned in Chapter 1, the Cartesian coordinates $x_0,...,x_N$ of $\mathbb{R}^{N+1}$ are given in terms of the elliptic-spherical coordinates $u=(u_1,...,u_N)\in(\alpha_0,\alpha_1)\times\cdots\times(\alpha_{N-1},\alpha_N)$, by the relations

$$x_\nu^2=\frac{\prod_{j=1}^{N}(u_j-\alpha_\nu)}{\prod_{\mu\neq\nu}(\alpha_\mu-\alpha_\nu)}, \quad \nu=0,1,...,N. \tag{2.5}$$

One can use these expressions to rewrite the Niven's function (2.1) in the form,

$$f_\beta(u)=c\prod_{\nu=0}^{N}\prod_{j=1}^{N}(u_j-\alpha_\nu)^{\beta_\nu/2}\phi(u_j), \tag{2.6}$$

where c is some real constant depending only on $\alpha_0,...,\alpha_N$ and $\phi(x)$ is the k-th degree polynomial $\prod_{j=1}^{k}(x-\theta_j)$ defined above. By construction, the functions f_β are solutions of the eigenvalue problem for the Laplace operator on $\mathbf{S}^N$. Recall from Chapter 1 that the advantage of writing

Niven's function in terms of elliptic-spherical coordinates is that one can easily separate variables. Indeed, the separated equations have the form

$$\prod_{\nu=0}^{N}(x-\alpha_\nu)\frac{d^2\psi}{dx^2}+\frac{1}{2}\sum_{\nu=0}^{N}\prod_{\mu\neq\nu}(x-\alpha_\mu)\frac{d\psi}{dx}+\frac{1}{4}\left(\sum_{j=0}^{N-1}\lambda_{N-j-1}x^j\right)\psi=0, \tag{2.7}$$

where the separation constants $\lambda_0, ..., \lambda_{N-1}$ are the joint eigenvalues of the partial differential operators $P_0, ..., P_{N-1}$ defined in Chapter 1. Equation (2.7) is exactly the generalized Lamé equation considered in the introduction of this chapter. Consequently, the solutions are given by

$$\psi(x)=\prod_{\nu=0}^{N}(x-\alpha_\nu)^{\beta/2}\phi(x).$$

The reader should note that we have given an explicit expression for the $N-1$ degree polynomial $C(x)$ of (2.4).

As the next proposition shows, there exists simple relation expressing the eigenvalues $\lambda_0, ..., \lambda_{N-1}$ as functions of the parameters $\alpha_0, ..., \alpha_N$ and the zeroes $\theta_1, ..., \theta_k$ of the Lamé polynomial $\phi(x)$.

Proposition 2.1. *For $i=0,...,N-1$, we have that:*

$$\lambda_{N-i-1}=\frac{(-1)^{N+i+1}}{2}\left(\sum_{j=0}^{N}\sum_{l\in S(i,j)}\prod_{m=1}^{N-\nu}\alpha_{l_m}\right)\left(\sum_{j=0}^{N}\sum_{l=1}^{k}\frac{1}{\alpha_j-\theta_l}\right),$$

where $S(i,j):=\Big\{l=(l_1,...,l_{N-i})\ :\ 0\le l_1\le\cdots\le l_{N-i}\le N,\ l_m\neq j\quad \text{for}\quad m=1,\ldots,N-i\Big\}.$

Proof: The proof is a simple application of the residue calculus and the theory of Vandermonde matrices. For $\beta=0$ and $x\neq\theta_1,...,\theta_k$, divide each single term of the generalized Lamé equation (2.4) by $\prod_{\nu=0}^{N}(x-\alpha_\nu)\,\phi(x)$ to get:

$$-\frac{1}{4}\frac{\sum_{j=0}^{N-1}\lambda_{N-j-1}x^j}{\prod_{\nu=0}^{N}(x-\alpha_\nu)}=\frac{1}{\phi(x)}\left[\frac{d^2\phi}{dx^2}+\frac{1}{2}\left(\sum_{\mu=0}^{N}\frac{1}{x-\alpha_\mu}\right)\frac{d\phi}{dx}\right]. \tag{2.8}$$

Clearly, the LHS has simple poles at $x=\alpha_\nu$ for $\nu=0,1,...,N$. Therefore, if we compute the contour integral of both sides of (2.8) on a circle Γ_{α_ν} centered at α_ν, with Γ_α small enough so that it contains no other singularity than α_ν, we get on the one hand:

$$\begin{aligned}\oint_{\Gamma_{\alpha_\nu}}\frac{\sum_{j=0}^{N-1}\lambda_{N-j-1}x^j}{\prod_{\nu=0}^{N}(x-\alpha_\nu)}dx &= 2\pi i\ \mathrm{Res}\left[\frac{\sum_{j=0}^{N-1}\lambda_{N-j-1}x^j}{\prod_{\nu=0}^{N}(x-\alpha_\nu)},\alpha_\nu\right]\\ &= 2\pi i\left[(x-\alpha_\nu)\frac{\sum_{j=0}^{N-1}\lambda_{N-j-1}x^j}{\prod_{\nu=0}^{N}(x-\alpha_\nu)}\right]_{x=\alpha_\nu}\\ &= 2\pi i\frac{\sum_{j=0}^{N-1}\lambda_{N-j-1}\alpha_\nu^j}{\prod_{\mu\neq\nu}(\alpha_\nu-\alpha_\mu)}. \end{aligned}\tag{2.9}$$

On the other hand,

$$\begin{aligned}
\oint_{\Gamma_{\alpha_\nu}} \frac{\sum_{j=0}^{N-1} \lambda_{N-j-1} x^j}{\prod_{\nu=0}^{N}(x-\alpha_\nu)} dx &= 2\pi i \operatorname{Res}\left[\frac{1}{\phi(x)}\left(\frac{d^2\phi}{dx^2} + \frac{1}{2}\left(\sum_{\mu=0}^{N} \frac{1}{x-\alpha_\mu}\right)\frac{d\phi}{dx}\right), \alpha_\nu\right] \\
&= \pi i \left[(x-\alpha_\nu)\frac{1}{\phi(x)}\left(\sum_{\mu=0}^{N}\frac{1}{x-\alpha_\mu}\right)\frac{d\phi}{dx}\right]_{x=\alpha_\nu} \\
&= \pi i \left[(x-\alpha_\nu)\left(\sum_{\mu=0}^{N}\frac{1}{x-\alpha_\mu}\right)\left(\sum_{l=1}^{k}\frac{1}{x-\theta_l}\right)\right]_{x=\alpha_\nu} \\
&= \pi i \sum_{l=1}^{k} \frac{1}{\alpha_\nu - \theta_l}. \qquad (2.10)
\end{aligned}$$

By equating (2.9) and (2.10), we conclude that

$$\sum_{j=0}^{N-1} \lambda_{N-j-1}\alpha_\nu^j = \frac{1}{2}\prod_{\mu\neq\nu}(\alpha_\nu - \alpha_\mu)\left(\sum_{l=1}^{k}\frac{1}{\alpha_\nu-\theta_l}\right). \qquad (2.11)$$

The coefficients matrix for the eigenvalues $\lambda_0, \dots, \lambda_{N-1}$ is a Vandermonde matrix in the variables $\alpha_0, \dots, \alpha_N$. Thus, we can write equation (2.11) in matrix form $V\vec{\lambda} = \vec{b}$ with

$$V = \begin{pmatrix} 1 & \alpha_0 & \alpha_0^2 & \cdots & \alpha_0^N \\ 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^N \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \alpha_N & \alpha_N^2 & \cdots & \alpha_N^N \end{pmatrix}, \quad \vec{\lambda} = \begin{pmatrix} \lambda_{N-1} \\ \vdots \\ \lambda_0 \\ 0 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} \frac{1}{2}\prod_{\mu\neq 0}(\alpha_0-\alpha_\mu)\sum_{l=1}^{k}\frac{1}{\alpha_0-\theta_l} \\ \vdots \\ \frac{1}{2}\prod_{\mu\neq N}(\alpha_N-\alpha_\mu)\sum_{l=1}^{k}\frac{1}{\alpha_N-\theta_l} \end{pmatrix}.$$

Using the results of [Kl], we invert the matrix V and get:

$$V^{-1} := (v^{ij}) = \left(\frac{(-1)^{i+1}\sum_{l\in S(i,j)}\prod_{m=1}^{N-i}\alpha_{l_m}}{\prod_{l\neq j}(\alpha_l - \alpha_j)}\right),$$

with $S(i, j)$ as defined in the statement of the proposition. Consequently, the eigenvalues have the expressions

$$\lambda_{N-i-1} = \frac{(-1)^{N+i+1}}{2}\left(\sum_{j=0}^{N}\sum_{l\in S(i,j)}\prod_{m=1}^{N-i}\alpha_{l_m}\right)\left(\sum_{j=0}^{N}\sum_{l=1}^{k}\frac{1}{\alpha_j-\theta_l}\right) \qquad (2.12)$$

as desired. □

3. The distribution of zeroes of Stieltjes polynomials

In order to present the results contained in the remainder of the chapter, it is convenient to consider the generalization version of the Lamé equation given by

$$\prod_{\nu=0}^{N}(x-\alpha_\nu)\frac{d^2S}{dx^2} + 2\sum_{\nu=0}^{N}\rho_\nu\prod_{\mu\neq\nu}(x-\alpha_\mu)\frac{dS}{dx} + V(x)S = 0, \qquad (2.13)$$

where $V(x)$ is a polynomial of degree $N-1$. Here, the parameters ρ_ν are arbitrary, positive, real numbers. Of course, in the particular case where $\rho_\nu = 1/4$ for all $\nu = 0, ..., N$, we recover the generalized Lamé equation of the introduction.

Definition: $S(x)$ is called a Stieltjes polynomial, if it is a polynomial solution of the differential equation (2.13). The corresponding polynomial $V(x)$ is called a Van Vleck polynomial.

Note that in the case of the generalized Lamé differential equation, the Stieltjes polynomials are exactly the Lamé polynomials defined previously.

The next two sections will be devoted to the study of the distribution of zeroes for the Stieltjes and Van Vleck polynomials. The two main results are presented in Theorems 2.1 and 2.4; they will play an important role in the proof of the new results presented in this thesis (see Chapters 3 and 4).

One of the most elegant results concerning the Lamé equation is certainly the Heine-Stieltjes Theorem. In his two volumes "Handbuch der Kugelfunctionen", Heine **[H]** considered the general equation

$$A(x)\frac{d^2\phi}{dx^2} + 2B(x)\frac{d\phi}{dx} + C(x)\phi = 0, \tag{2.14}$$

with $A(x)$ and $B(x)$ two arbitrary polynomials of degree $N+1$ and N respectively. Heine proved that there are at most

$$\sigma(N,k) := \frac{(N+k-1)!}{k!\,(N-1)!}$$

polynomials $C(x)$ of degree $N-1$ for which the differential equation (2.14) has a polynomial solution $\phi(x)$ of preassigned degree k.

Shortly afterwards, Stieltjes **[St]** considered the particular case where the two polynomials $A(x)$ and $B(x)$ are of the form:

$$\begin{aligned} A(x) &= \textstyle\prod_{\nu=0}^{N}(x-\alpha_\nu), \\ B(x) &= \textstyle\sum_{\nu=0}^{N}\rho_\nu\prod_{\mu\neq\nu}(x-\alpha_\mu), \qquad \rho_\nu > 0. \end{aligned}$$

This particular situation corresponds to the one described above, since in that case, equation (2.14) reduces to equation (2.13). He obtained the following remarkable result.

Theorem 2.1. ***(Heine-Stieltjes)*** *There are exactly $\sigma(N,k)$ polynomials $V(x)$ of degree $N-1$ for which the differential equation (2.13) has a polynomial solution of degree k. In addition, for each of the $\sigma(N,k)$ solutions, $S(x)$, the zeroes are simple and uniquely distributed in the intervals $(\alpha_0, \alpha_1), ..., (\alpha_{N-1}, \alpha_N)$.*

Therefore, the $(N+k-1)$-th degree polynomial

$$\prod_{\nu=0}^{N}(x-\alpha_\nu)S''(x)+2\sum_{\nu=0}^{N}\rho_\nu\prod_{\mu\neq\nu}(x-\alpha_\mu)S'(x) \tag{2.18}$$

vanishes at the points $x=\theta_j$, $j=1,...,k$, and so, it must be divisible by $S(x)$. If we denote by $-V(x)$ the quotient resulting from the division of equation (2.18) by $S(x)$, it follows that $V(x)$ is a polynomial of degree $N-1$ such that $S(x)$ satisfies the differential equations (2.13), i.e.

$$\prod_{\nu=0}^{N}(x-\alpha_\nu)\frac{d^2S}{dx^2}+2\sum_{\nu=0}^{N}\rho_\nu\prod_{\mu\neq\nu}(x-\alpha_\mu)\frac{dS}{dx}+V(x)S=0.$$

Furthermore, for any given Van Vleck polynomial $V(x)$, the point $(\theta_1,...,\theta_k)$ is unique. Otherwise, for $x\neq\alpha_\nu$, $\nu=0,...,N$, we would have two linearly independent solutions $S_1(x)$ and $S_2(x)$ that satisfy the relation

$$\prod_{\nu=0}^{N}(x-\alpha_\nu)\left(S_1'S_2-S_1S_2'\right)'+2\sum_{\nu=0}^{N}\rho_\nu\prod_{\mu\neq\nu}(x-\alpha_\mu)\left(S_1'S_2-S_1S_2'\right)=0.$$

This would imply, for some constant of integration, c,

$$\begin{aligned}\left(S_1'S_2-S_1S_2'\right)(x) &= c\exp\left[-2\int\sum_{\nu=0}^{N}\frac{\rho_\nu}{(x-\alpha_\nu)}dx\right]\\ &= c\prod_{\nu=0}^{N}|x-\alpha_\nu|^{-2\rho_\nu}.\end{aligned}$$

However, last equation yields a contradiction, for if $x\to\alpha_0$ or $x\to\alpha_1$, the product on the RHS goes to ∞.

In the argument above, we have assumed that all the zeroes lies in the interval (α_0,α_1). The same argument can be applied to any possible configurations of the zeroes of $S(x)$. Consequently, this completes the proof and shows that each polynomial solution of (2.13) is uniquely characterized by the distribution of its zeroes in the intervals $(\alpha_0,\alpha_1),...,(\alpha_{N-1},\alpha_N)$.

Remark: As we mentioned in section 1 of the present chapter, the Niven functions f_β form a complete basis of $L^2(\mathbb{S}^N)$. We are now in position to prove this important fact. As before, we denote by

$$f_\beta^{(n)}(X)=X^\beta\prod_{j=1}^{k}(A_{\theta_j}X,X),\quad X=(x_0,\dots,x_N)\in\mathbb{R}^{N+1},$$

the Niven functions of degree n, with $n=|\beta|+2k$. According to the Heine-Stieltjes Theorem, there exists $\sigma(N,k)$ independent Lamé polynomials $\prod_{j=1}^{k}(A_{\theta_j}X,X)$ of degree $2k$ corresponding to the multi-index $\beta = 0$. Similarly, there exists $\sigma(N, k-1)$ independent Lamé polynomials of degree $2k-2$ corresponding to $|\beta| = 2$, and more generally, for $|\beta| = 2j$ with $j \leq k$ correspond $\sigma(N, k-j)$ independent Lamé polynomials of degree $2k-2j$. The number of multi-index $\beta \in \{0,1\}^{N+1}$ for which $|\beta| = 2j$ is obviously given by $\binom{N+1}{2j}$. Consequently, the total number of independent Niven's functions of degree n is given by

$$S(N,n) = \sum_{j=0}^{k} \binom{N+1}{2j} \sigma(N, k-j). \tag{2.19}$$

A direct, but long and tedious computation shows that

$$S(N,n) = (2n+N)\frac{(N+n-2)!}{n!\,N!}.$$

This is exactly the number of independent spherical harmonics of degree n ([**Fo**]). Therefore, the Niven functions form a complete basis of $L^2(\mathbf{S}^N)$.

We conclude this section with an application of the Niven's equations (2.16). In order to make the measure $d\mu_{LS}$ of Theorem 3.1 well defined, the zeroes $\theta_1(\alpha,\rho), \ldots, \theta_k(\alpha,\rho)$ must be integrable functions of the parameters α. As the next proposition shows, a stronger conclusion actually holds.

Proposition 2.2. *The zeroes $\theta_1(\alpha), \ldots, \theta_k(\alpha)$ of any given Stieltjes polynomial are differentiable functions of the parameters $(\alpha_0, \ldots, \alpha_N) \in \Lambda^N$.*

Proof: In his paper, Shah [**Sh1**] gives an argument which appears to be incorrect. So, we present a slightly modified argument. Differentiating the Niven's equations with respect to the θ variables, we form the Jacobian matrix $B = (b_{ij})$ given by

$$b_{ij} = \begin{cases} -\sum_{\nu=0}^{N} \frac{\rho_\nu}{(\theta_j-\alpha_\nu)^2} - \sum_{m\neq i} \frac{1}{(\theta_j-\theta_m)^2} & \text{if } i = j, \\ \frac{1}{(\theta_j-\theta_i)^2} & \text{if } i \neq j. \end{cases}$$

By a standard result in matrix theory (Geršgorin's Theorem), it follows that all the eigenvalues of B are strictly negative, since if λ is an eigenvalue of B, then for some $j \in \{1, \ldots, k\}$:

$$\lambda \leq b_{jj} + \sum_{i\neq j} |b_{ij}| = -\sum_{\nu=0}^{N} \frac{\rho_\nu}{(\theta_j-\alpha_\nu)^2} < 0.$$

Hence, the determinant of B is also nonzero. The conclusion of the proposition follows from an easy application of the Implicit Function Theorem. □

4. The distribution of zeroes of Van Vleck polynomials

We now describe the configuration of the zeroes of the Van Vleck polynomials. Recall, $V(x)$ is a polynomial of degree $N-1$ for which the differential equation

$$\prod_{\nu=0}^{N}(x-\alpha_\nu)\frac{d^2S}{dx^2}+2\sum_{\nu=0}^{N}\rho_\nu\prod_{\mu\neq\nu}(x-\alpha_\mu)\frac{dS}{dx}+V(x)S=0,$$

has a polynomial solution $S(x)$ of preassigned degree k. As in previous section, we would like to know how the zeroes of any $V(x)$ distribute with respect to the parameters $\alpha_0, ..., \alpha_N$.

The first result in that direction was given by E. B. Van Vleck [**V**] himself. As in the case of the Stieltjes polynomials, he showed that all the zeroes of $V(x)$ lie inside the interval (α_0, α_N). The proof he gives is rather complicated and hard to read, so we prefer to present the simple algebraic argument due to M. Bôcher [**B**].

Let $S(x)$ denotes a Stieltjes polynomial and $v_1(\alpha), \ldots, v_{N-1}(\alpha)$ the zeroes of the corresponding Van Vleck polynomial $V(x)$. By (2.13), we have that

$$\frac{d^2S}{dx^2}(v_j)+2\left[\sum_{\nu=0}^{N}\frac{\rho_\nu}{v_j-\alpha_\nu}\right]\frac{dS}{dx}(v_j)=0, \qquad j=1,...,N-1.$$

On one hand, if $S'(v_j)=0$, the zero v_j would then coincide with an α_ν. Otherwise, $S''(v_j)=0$ and so $S^{(m)}(v_j)=0$ for all m; this implies that $S(x)\equiv$ *constant*, a contradiction. On the other hand , if $S'(v_j)\neq 0$, we then have

$$2\sum_{\nu=0}^{N}\frac{\rho_\nu}{v_j-\alpha_\nu}=\frac{S''(v_j)}{S'(v_j)}=2\sum_{l=1}^{k-1}\frac{1}{v_j-\theta'_l}, \qquad j=1,...,N-1,$$

where $\theta'_1, ..., \theta'_{k-1}$ are the zeroes of $S'(x)$. Consequently, the zeroes of $V(x)$ are either one of the points α_ν or satisfy

$$\sum_{\nu=0}^{N}\frac{\rho_\nu}{x-\alpha_\nu}+\sum_{i=1}^{k-1}\frac{1}{x-\theta'_i}=0. \qquad (2.20)$$

Now, assume that $V(x)$ has zeroes with positive imaginary part, and consider the one whose imaginary part is the greatest. Then equation (2.20) yields a contradiction, since the imaginary part of each term is less or equal to zero, and not all of them vanish since the α's are all positive. In a similar fashion, one can show that $V(x)$ cannot have a complex root with negative imaginary part. Finally, suppose that one of the zeroes of $V(x)$ is real and greater than α_N. Then, equation (2.20) yields once again a contradiction since no term is negative or zero. In the same way, one can show that $V(x)$ has no root less than α_0. This proves that the zeroes of $V(x)$ lie in the interval (α_0, α_N).

CHAPTER 3

Asymptotic statistics for the Lamé ensemble

We compute the limiting level spacings distributions for the Lamé polynomials in various thermodynamic, asymptotic regimes. We give both results in the mean and pointwise, for an asymptotically full set of values of the parameters $\alpha_0, ..., \alpha_N$.

1. Introduction

In Chapter 1, we constructed a quantum completely integrable system of commuting, second order differential operators acting on $C^\infty(\mathbf{S}^N)$ given by

$$P_k = \sum_{i<j} S_k^{ij}(\alpha_0, ..., \alpha_N)\,(x_i\partial_j - x_j\partial_i)^2, \quad k = 0, ..., N-1.$$

These operators naturally depend on the parameters $\alpha_0, ..., \alpha_N$ with $0 < \alpha_0 < \cdots < \alpha_N$, and thus constitute an ensemble.

As we have seen in Chapter 2, the joint eigenfunctions of $P_0, ..., P_{N-1}$ form a class of spherical harmonics known as the generalized Lamé harmonics. Recall, in terms of elliptic-spherical coordinates $u_1, ..., u_N$ on $\mathbf{S}^N$, these are of the form

$$f_\beta(u_1, ..., u_N) = \prod_{j=1}^{N}\prod_{\nu=0}^{N}(u_j - \alpha_\nu)^{\beta_\nu/2}\phi(u_j),$$

where, $\beta = (\beta_1, ..., \beta_N) \in \{0,1\}^{N+1}$, and ϕ is a Lamé polynomial.

Furthermore, the function $\psi(x) = \prod_{\nu=0}^{N}(x-\alpha_\nu)^{\beta_\nu/2}\phi(x)$ is a solution of the generalized Lamé differential equation

$$\prod_{\nu=0}^{N}(x-\alpha_\nu)\frac{d^2\psi}{dx^2} + \frac{1}{2}\sum_{\nu=0}^{N}\prod_{\lambda\neq\nu}(x-\alpha_\lambda)\frac{d\psi}{dx} + C(x)\psi = 0, \tag{3.1}$$

where, $C(x)$ is a polynomial of order $N-1$ depending linearly on the joint eigenvalues $(\lambda_0, ..., \lambda_{N-1}) \in$ Spec $(P_0, ..., P_{N-1})$. Although for simplicity, we only consider here the case where the multi-index $\beta = 0$, our main result (Theorem 3.1) can be proved for the other cases corresponding to $\beta \neq 0$ in a similar fashion.

In order to state our main theorem, we first consider

$$\mathcal{E}(k) := \left\{ \phi_1^{(k)}, ..., \phi_{j(k)}^{(k)} \right\},$$

the set of Lamé polynomials of degree k. By the standard theory of spherical harmonics [WW] and the fact that the corresponding generalized Lamé harmonics form a Hilbert basis, we know that

$$j(k) = \sigma(N,k) := \frac{(N+k-1)!}{k!\,(N-1)!}.$$

Let $\theta_{i,1}^{(k)}(\alpha) < \cdots < \theta_{i,k}^{(k)}(\alpha)$ denote the (real) zeroes of the polynomial, $\phi_i^{(k)}$, where $i = 1, ..., \sigma(N,k)$.

In our main result, we compute the asymptotic weak limit for the level spacings distribution averaged over the set, $\mathcal{E}(k)$, of k-th order Lamé polynomials. More precisely, consider

$$d\rho_{LS}^{AV}(x; N, k, \alpha) := \frac{1}{\sigma(N,k)} \sum_{l=1}^{\sigma(N,k)} \frac{1}{k-1} \sum_{j=1}^{k-1} \delta\Big(x - k \left(\theta_{l,j+1}^{(k)}(\alpha) - \theta_{l,j}^{(k)}(\alpha) \right) \Big) \tag{3.2}$$

where $\alpha \in \Lambda^N$ and

$$\Lambda^N := \left\{ (\alpha_0, ..., \alpha_N) \in [0,1]^{N+1};\ \alpha_0 < \alpha_1 < \cdots < \alpha_{N-1} < \alpha_N \right\}. \tag{3.3}$$

We henceforth put normalized Lebesgue measure $đ\alpha := (N+1)!\,d\alpha$ on Λ^N, so that meas $(\Lambda^N) = 1$. In order to state our first result, we will also need to introduce the integrated, averaged level spacings distribution:

$$d\mu_{LS}(x; N, k) := \int_{\Lambda^N} d\rho_{LS}^{AV}(x; N, k, \alpha)\ đ\alpha. \tag{3.4}$$

Theorem 3.1. [**BT**] *(i) Fix $0 < \epsilon < 1$ and assume that $k \sim N^{1-\epsilon}$ as $N \to \infty$. Then,*

$$w - \lim_{N\to\infty} d\mu_{LS}(x; N, k) = e^{-x}\,dx.$$

(ii) Suppose that $k(N)$ satisfies the hypotheses of part (i). Then, for any $0 < \delta < \epsilon$ there exist a measurable subset $J^N \subset \Lambda^N$ with meas $(J^N) \geq 1 - N^{-\delta}$, such that for any $\alpha \in J^N$,

$$w - \lim_{N\to\infty} d\rho_{LS}^{AV}(x; N, k, \alpha) = e^{-x}\,dx.$$

In both (i) and (ii), the weak-limit is taken in the dual space to $C_0^0([a,b])$, where $0 \leq a < b < \infty$.

We should point out that one can also easily determine the weak-limit of the level spacings measures $d\rho_{LS}^{AV}$ and $d\mu_{LS}$ before "unfolding" the zeroes, i.e. rescaling to unit mean level spacings. Indeed, by carrying out exactly the same analysis as for the proof of Theorem 3.1, one can deduce the following consequence.

Corollary 3.1. [**BT**] *For any $p \in [0,1)$, consider the following level spacings measure*

$$d\bar{\rho}_{LS}^{AV} = \frac{1}{\sigma(N,k)} \sum_{l=1}^{\sigma(N,k)} \frac{1}{k-1} \sum_{j=1}^{k-1} \delta\Big(x - k^p \left(\theta_{l,j+1}^{(k)}(\alpha) - \theta_{l,j}^{(k)}(\alpha)\right)\Big).$$

Then, under the same hypothesis as Theorem 3.1 above, we have that:

$$\begin{array}{ll} (i) & w\text{-}\lim\limits_{N\to\infty} \int_{\Lambda^N} d\bar{\rho}_{LS}(x;N,k,\alpha)\ d\!\!{}^-\alpha = \delta_0(x); \\ (ii) & w\text{-}\lim\limits_{N\to\infty} d\bar{\rho}_{LS}(x;N,k,\alpha) = \delta_0(x) \text{ for all } \alpha \in J^N. \end{array}$$

2. Preliminary results

By the Heine-Stieltjes Theorem, we know that zeroes of any Lamé polynomial $\phi(x)$ are simple and lie inside the interval (α_0, α_N). Moreover, each of the $\sigma(N,k)$ Lamé polynomial of degree k is uniquely characterized by its zeroes distribution among the intervals $(\alpha_0,\alpha_1), \ldots, (\alpha_{N-1},\alpha_N)$. According to this theorem, we denote the zeroes of $\phi(x)$ by $\theta_1(\alpha;l) < \cdots < \theta_k(\alpha;l)$, where $\alpha := (\alpha_0, \ldots, \alpha_N)$ whereas $l = (l_1, \ldots, l_k)$, $1 \le l_1 \le \cdots \le l_k \le N$, denotes the configuration of the zeroes. By this we mean that $\theta_1(\alpha;l)$ is the smallest zero lying in the interval $(\alpha_{l_1-1}, \alpha_{l_1})$, the next zero $\theta_2(\alpha;l)$ is contained in the interval $(\alpha_{l_2-1}, \alpha_{l_2})$ and so on.

Consequently, we can now rewrite $d\rho_{LS}^{AV}(x;N,k,\alpha)$ and $d\mu_{LS}(x;N,k)$ in the more convenient forms:

$$d\rho_{LS}^{AV}(x;N,k,\alpha) = \frac{1}{\sigma(N,k)} \sum_{1\le l_1\le\cdots\le l_k\le N} \frac{1}{k-1} \sum_{j=1}^{k-1} \delta\left(x - k\left(\theta_{j+1}(\alpha;l) - \theta_j(\alpha;l)\right)\right), \tag{3.5}$$

and so,

$$\begin{aligned} d\mu_{LS}(x;N,k) &:= \int_{\Lambda^N} d\rho_{LS}^{AV}(x;N,k,\alpha)\ d\!\!{}^-\alpha \\ &= \frac{1}{\sigma(N,k)} \sum_{1\le l_1\le\cdots\le l_k\le N} \frac{1}{k-1} \sum_{j=1}^{k-1} \int_{\Lambda^N} \delta\left(x - k\left(\theta_{j+1}(\alpha;l) - \theta_j(\alpha;l)\right)\right) d\!\!{}^-\alpha. \end{aligned} \tag{3.6}$$

The first result we need is a simple calculus lemma.

Lemma 3.1. *For any $0 \le i \le j \le N$ and multi-indices $\beta = (\beta_1, \beta_2) \in \mathbb{N}^2 \setminus \{(0,0)\}$, we have*

$$\int_{\Lambda^N} \alpha_i^{\beta_1} \alpha_j^{\beta_2}\ d\!\!{}^-\alpha = \frac{\prod_{l=1}^{\beta_1}(i+l)\ \prod_{l=1}^{\beta_2}(\beta_1+j+l)}{\prod_{l=1}^{|\beta|}(N+1+l)},$$

where, we define products of the form $\prod_{l=1}^{0}$ to be equal to 1 and $|\beta| := \beta_1 + \beta_2$.

Proof: A direct computation of the following iterated integrals gives

$$
\begin{aligned}
\int_{\Lambda^N} \alpha_i^{\beta_1}\alpha_j^{\beta_2}\, d\!\!\!{}^-\alpha &= \int_{0<\alpha_0<\cdots<\alpha_N<1} \alpha_i^{\beta_1}\alpha_j^{\beta_2}\, d\!\!\!{}^-\alpha \\
&= (N+1)! \int_0^1 \int_0^{\alpha_N} \cdots \int_0^{\alpha_1} \alpha_j^{\beta_2}\alpha_i^{\beta_1}\, d\alpha_0 \cdots d\alpha_N \\
&= \frac{(N+1)!}{i!} \int_0^1 \int_0^{\alpha_N} \cdots \int_0^{\alpha_{i+1}} \alpha_j^{\beta_2}\alpha_i^{\beta_1+i}\, d\alpha_i \cdots d\alpha_N \\
&= \frac{(N+1)!}{i!\,(\beta_1+i+1)\ldots(\beta_1+j)} \int_0^1 \int_0^{\alpha_N} \cdots \int_0^{\alpha_{j+1}} \alpha_j^{\beta_2+\beta_1+j}\, d\alpha_j \cdots d\alpha_N \\
&= \frac{(N+1)!}{i!\,(\beta_1+i+1)\ldots(\beta_1+j)\,(\beta_1+\beta_2+j+1)\ldots(\beta_1+\beta_2+N+1)} \\
&= \frac{\prod_{l=1}^{\beta_1}(i+l)\ \prod_{l=1}^{\beta_2}(\beta_1+j+l)}{\prod_{l=1}^{|\beta|}(N+1+l)}. \quad \square
\end{aligned}
$$

As a consequence of Lemma 3.1, we see that the integrals of consecutive monomials over the truncated positive Weyl chamber Λ^N are asymptotically equal as $N \to \infty$. Combined with the Heine-Stieltjes result, this fact leads to the following simple corollary.

Corollary 3.2. *For any configuration $l = (l_1, \ldots, l_k)$ and integer j satisfying $1 \le j \le k$, we have that*

$$\int_{\Lambda^N} \left| \theta_j(\alpha; l) - \alpha_{l_j} \right| \, d\!\!\!{}^-\alpha = \mathcal{O}(N^{-1}) \tag{3.7}$$

uniformly in k.

Proof: As a consequence of the Heine-Stieltjes Theorem, we know that given a configuration l, the j-th zero necessarily lies in the interval $(\alpha_{l_j-1}, \alpha_{l_j})$; that is,

$$\alpha_{l_j-1} \le \theta_j(\alpha; l) \le \alpha_{l_j}.$$

On the other hand, by Lemma 3.1, $\int_{\Lambda^N} \alpha_j \, d\!\!\!{}^-\alpha = \frac{j+1}{N+2}$. Thus,

$$
\begin{aligned}
\int_{\Lambda^N} \left| \theta_j(\alpha; l) - \alpha_{l_j} \right| \, d\!\!\!{}^-\alpha &\le \int_{\Lambda^N} (\alpha_{l_j} - \alpha_{l_j-1}) \, d\!\!\!{}^-\alpha \\
&= \frac{l_j+1}{N+2} - \frac{l_j}{N+2} \\
&= \mathcal{O}(N^{-1}). \quad \square
\end{aligned}
$$

Step 2: The next step involves computing the first term on the RHS of (3.8) explicitly. We claim that:

$$\frac{1}{\sigma(N,k)}\sum_{1\le l_1\le\cdots\le l_k\le N}\frac{1}{k-1}\sum_{j=1}^{k-1}\int_{\Lambda^N}\phi\left(k\left(\alpha_{l_{j+1}}-\alpha_{l_j}\right)\right)\,d\!\!{}^-\alpha$$
$$=\frac{k}{N+k-1}\,\phi(0)+\frac{N+1}{\sigma(N,k)}\sum_{m=0}^{N-2}\sigma(N-m-1,k-1)\int_0^1\phi(kx)\,\mathrm{binom}(N,m;x)\,dx,\quad(3.9)$$

where, $\mathrm{binom}(N,m;x):=\frac{N!}{m!\,(N-m)!}\,x^m\,(1-x)^{N-m}$ for $x\in[0,1]$.

In order to prove the identity in (3.9), we start with a simple lemma which involves a successive application of the Fubini Theorem.

Lemma 3.2. *For any integers i,j with $0\le i<j\le N$, we have that*

$$\int_{\Lambda^N}\phi\left(k\left(\alpha_j-\alpha_i\right)\right)\,d\!\!{}^-\alpha=(N+1)\int_0^1\phi(kx)\,\mathrm{binom}(N,j-i-1;x)\,dx.\qquad(3.10)$$

Proof: Given the definition of Λ^N, it is clear that

$$\int_{\Lambda^N}\phi\left(k\left(\alpha_j-\alpha_i\right)\right)\,d\!\!{}^-\alpha=(N+1)!\int_0^1\int_0^{\alpha_N}\cdots\int_0^{\alpha_1}\phi\left(k\left(\alpha_j-\alpha_i\right)\right)\,d\alpha_0\cdots d\alpha_N.$$

By repeated application of Fubini's Theorem, we can ensure that the iterated integrals with respect to α_i and α_j are carried out last. More precisely, we apply Fubini's Theorem to the double integral with respect to α_j and α_{j+1} to reverse the order of integration. We then repeat the same procedure for the double integral with respect to α_j and α_{j+2} and so on, until the last integral involves the α_j variable. This gives

$$\int_{\Lambda^N}d\alpha=\int_0^1\int_{\alpha_j}^1\int_{\alpha_j}^{\alpha_N}\cdots\int_{\alpha_j}^{\alpha_{j+2}}\int_0^{\alpha_j}\cdots\int_0^{\alpha_1}d\alpha_0...d\alpha_{j-1}d\alpha_{j+1}...d\alpha_{N-1}d\alpha_N d\alpha_j.$$

We proceed in similar manner for α_i to finally obtain

$$\int_{\Lambda^N}d\alpha\;=\;\int_0^1\int_0^{\alpha_j}\int_{\alpha_j}^1\cdots\int_{\alpha_j}^{\alpha_{j+2}}\int_{\alpha_i}^{\alpha_j}\cdots\int_{\alpha_i}^{\alpha_{i+2}}\int_0^{\alpha_i}\cdots\int_0^{\alpha_1}d\alpha_0\ldots\widehat{d\alpha_i}\ldots\widehat{d\alpha_j}\ldots d\alpha_N d\alpha_i d\alpha_j,$$

where $\widehat{d\alpha_i}$, $\widehat{d\alpha_j}$ means that these variables are omitted in the product measure $d\alpha_0\cdots d\alpha_N$. We then carry out the iterated integration over the first $N-2$ variables $\alpha_0<\alpha_1<...<\alpha_{i-1}<\alpha_{i+1}<...<\alpha_{j-1}<\alpha_{j+1}<...<\alpha_N$ to get

$$\int_{\Lambda^N}d\!\!{}^-\alpha=(N+1)!\int_0^1\int_0^{\alpha_j}\frac{\alpha_i^i}{i!}\,\frac{(\alpha_j-\alpha_i)^{j-i-1}}{(j-i-1)!}\,\frac{(1-\alpha_j)^{N-j}}{(N-j)!}\,d\alpha_i d\alpha_j.$$

Finally, we make the change of variables $x = \alpha_j - \alpha_i$, $y = \alpha_j$ and integrate by parts i times with respect to y. It follows that

$$\begin{aligned}\int_{\Lambda^N} \phi\left(k\left(\alpha_j - \alpha_i\right)\right)\, d\!\!\!{}^-\alpha &= (N+1)! \int_0^1 \phi(kx) \frac{x^{j-i-1}}{(j-i-1)!} \frac{(1-x)^{N-(j-i-1)}}{(N-(j-i-1))!}\, dx \\ &= (N+1) \int_0^1 \phi(kx)\, \mathrm{binom}(N, j-i-1; x)\, dx.\end{aligned}$$

This completes the proof of Lemma 3.3. □

To complete Step 2, we need to compute the asymptotic averages of the integrals $\int_{\Lambda^N} \phi(k(\alpha_{l_{j+1}} - \alpha_{l_j}))\, d\!\!\!{}^-\alpha$. First, we start with a simple combinatorial lemma. In order to state the lemma, it is useful to introduce some notation at this point: We denote by $S_j(m)$ the set of all configurations $l = (l_1, ..., l_k)$ for which $l_{j+1} - l_j = m$. As the following result shows, the cardinality of $S_j(m)$ is independent of j.

Lemma 3.3. *For each $m = 0, ..., N-1$ and each $j = 1, \dots, k$, the number of k-tuples $(l_1, ..., l_k)$ with $1 \le l_1 \le \cdots \le l_k \le N$ and satisfying $l_{j+1} - l_j = m$ is given by*

$$\sigma(N-m, k-1) := \frac{(N-m+k-2)!}{(k-1)! \cdot (N-m-1)!}.$$

Proof: By identifying the j-th and the $j+1$-st zeroes, we are reduced to the problem of distributing $k-1$ zeroes amongst the remaining $N-m$ slots. This is clearly equal to $\sigma(N-m, k-1)$. □

As a consequence of Lemma 3.4 ,

$$\begin{aligned}&\frac{1}{\sigma(N,k)} \sum_{1 \le l_1 \le \cdots \le l_k \le N} \frac{1}{k-1} \sum_{j=1}^{k-1} \int_{\Lambda^N} \phi\left(k\left(\alpha_{l_{j+1}} - \alpha_{l_j}\right)\right)\, d\!\!\!{}^-\alpha \\ &\qquad = \frac{1}{(k-1)} \frac{1}{\sigma(N,k)} \sum_{m=0}^{N-1} \sum_{j=1}^{k-1} \sum_{l \in S_j(m)} \int_{\Lambda^N} \phi\left(k\left(\alpha_{l_{j+1}} - \alpha_{l_j}\right)\right)\, d\!\!\!{}^-\alpha. \qquad (3.11)\end{aligned}$$

In order to apply Lemma 3.3, we need to treat the two cases where $m = 0$ and $m > 0$ separately. Since by Lemma 3.4, we know that $S_j(0) = \sigma(N, k-1)$, it is clear that

$$\begin{aligned}&\frac{1}{\sigma(N,k)} \sum_{1 \le l_1 \le \cdots \le l_k \le N} \frac{1}{k-1} \sum_{j=1}^{k-1} \int_{\Lambda^N} \phi\left(k\left(\alpha_{l_{j+1}} - \alpha_{l_j}\right)\right)\, d\!\!\!{}^-\alpha \\ &= \frac{\sigma(N,k-1)}{\sigma(N,k)} \phi(0) + \frac{1}{(k-1)} \frac{1}{\sigma(N,k)} \sum_{m=1}^{N-1} \sum_{j=1}^{k-1} \sum_{l \in S_j(m)} \int_{\Lambda^N} \phi\left(k\left(\alpha_{l_{j+1}} - \alpha_{l_j}\right)\right)\, d\!\!\!{}^-\alpha. \qquad (3.12)\end{aligned}$$

After some further simplification involving Taylor expansions, we get that

$$\begin{aligned}\frac{\sigma(N-m-1,k-1)}{\sigma(N,k)} &= \frac{k}{N+k-1}\exp\left[-\sum_{j=0}^{k-2}\frac{m+1}{N}\left(1+\mathcal{O}\left(\frac{j}{N}\right)\right)+\mathcal{O}\left(\frac{km^2}{N^2}\right)\right] \\ &= \frac{k}{N+k-1}\exp\left[-\frac{(k-2)(m+1)}{N}+\mathcal{O}\left(\frac{mk^2}{N^2}\right)+\mathcal{O}\left(\frac{km^2}{N^2}\right)\right]. \\ &= \frac{k}{N+k-1}\exp\left(\frac{-mk}{N}\right)\left(1+\mathcal{O}\left(\frac{mk^2}{N^2}\right)+\mathcal{O}\left(\frac{km^2}{N^2}\right)\right).\end{aligned}$$

Finally, using the fact that $x^p e^{-x} = \mathcal{O}_p(1)$; for all $x \geq 0$, we get

$$\frac{\sigma(N-m-1,k-1)}{\sigma(N,k)} = \frac{k}{N+k-1}\exp\left(\frac{-mk}{N}\right)+\mathcal{O}\left(\frac{k^2}{N^2}\right)+\mathcal{O}\left(\frac{1}{N}\right). \tag{3.17}$$

Case 2: $(m >> (N/k)^{1+\beta})$. In this case, we can choose $0 < \beta < \frac{1-\epsilon}{\epsilon}$ so that with appropriate constants $C_1, C_2 > 0$,

$$\begin{aligned}\frac{\sigma(N-m-1,k-1)}{\sigma(N,k)} &= \frac{k}{N+k-1}\prod_{j=0}^{k-2}\left(1-\frac{m+1}{N+j}\right) \\ &\leq \frac{k}{N+k-1}\prod_{j=0}^{k-2}\left(1-C_1\frac{N^\beta}{k^{1+\beta}}\right) \\ &= \mathcal{O}\left(e^{-C_2(N/k)^\beta}\right). \end{aligned}\tag{3.18}$$

If we substitute the estimates obtained in (3.17) and (3.18) into (3.15) and use the facts that $\sum_{m=0}^{N}\text{binom}(N,m;x) = 1$ and $\int_0^1 \phi(kx)dx = \mathcal{O}(k^{-1})$, we finally obtain

$$\begin{aligned}d\mu_{LS}(x;N,K)(\phi) &= \frac{k(N+1)}{N+k-1}\sum_{m=0}^{N-2}e^{-\frac{mk}{N}}\int_0^1\phi(kx)\,\text{binom}(N,m;x)\,dx \\ &\quad +\mathcal{O}\left(\frac{k}{N}\right)+\mathcal{O}\left(\frac{1}{k}\right) \\ &= \frac{k(N+1)}{N+k-1}\sum_{m=0}^{N}e^{\frac{-mk}{N}}\int_0^1\phi(kx)\,\text{binom}(N,m;x)\,dx \\ &\quad +\mathcal{O}\left(\frac{k}{N}\right)+\mathcal{O}\left(\frac{1}{k}\right).\end{aligned}\tag{3.19}$$

since the terms for $m = N-1$ and $m = N$ are bounded by $1/N$.

Now, recall that for a function $f(x)$ defined on $[0,1]$, the N-th degree Bernstein polynomial of $f(x)$ is defined to be [**D**]:

$$B_N(f;x) = \sum_{m=0}^{N} f\left(\frac{m}{N}\right)\text{binom}(N,m;x).$$

It is easy to see that in the special case where $\exp_{-k}(x) := e^{-kx}$, there is a concise closed-form expression for $B_N(\exp_{-k}; x)$; indeed,

$$B_N(\exp_{-k}; x) = \left(x\, e^{-\frac{k}{N}} + (1-x)\right)^N . \tag{3.20}$$

From last identity, we easily derive:

Lemma 3.4. *For $x \geq 0$, we have that*

$$B_N(\exp_{-k}; x) = e^{-kx} + \mathcal{O}\left(\frac{k}{N}\right). \tag{3.21}$$

Proof: Expand $e^{-\frac{k}{N}}$ in a second-order Taylor series and use the identity (3.20) directly to get

$$\begin{aligned} B_N(\exp_{-k}; x) &= \left[1 + x\left(e^{-\frac{k}{N}} - 1\right)\right]^N \\ &= \left[1 - \frac{kx}{N}\left(1 + \mathcal{O}\left(\frac{k}{N}\right)\right)\right]^N . \end{aligned}$$

From the inequality

$$0 \leq e^{-x} - \left(1 - \frac{x}{N}\right)^N \leq \frac{x^2 e^{-x}}{N},$$

and the fact that $x^p e^{-x} = \mathcal{O}_p(1)$ for all $x \geq 0$, it follows that

$$\begin{aligned} B_N(e^{-kx}; x) &= e^{-kx}\left(1 + \mathcal{O}\left(\frac{k^2 x}{N}\right)\right) + \mathcal{O}(N^{-1}) \\ &= e^{-kx} + \mathcal{O}\left(\frac{k}{N}\right). \end{aligned}$$

This completes the proof of the lemma. □

Substituting (3.21) into (3.19), we finally obtain

$$\begin{aligned} d\mu_{LS}(x; N, K)(\phi) &= \frac{k(N+1)}{N+k-1}\int_0^1 \phi(kx)\; B_N(\exp_{-k}; x)\; dx + \mathcal{O}\left(\frac{k}{N}\right) + \mathcal{O}\left(\frac{1}{k}\right) \\ &= \frac{k(N+1)}{N+k-1}\int_0^1 \phi(kx)\, e^{-kx}\; dx + \mathcal{O}\left(\frac{k}{N}\right) + \mathcal{O}\left(\frac{1}{k}\right) \\ &= \int_0^1 \phi(x)\, e^{-x}\; dx + \mathcal{O}\left(\frac{k}{N}\right) + \mathcal{O}\left(\frac{1}{k}\right). \end{aligned} \tag{3.22}$$

By noting that $C_0^1([0,1])$ is dense in $C_0^0([0,1])$, this completes the proof of Theorem 3.1 (i). □

Lemma 3.5. *(i) For any $x, y \in [0,1]$,*

$$\sum_{m'=0}^{N-2}\sum_{m=0}^{N-2-m'} e^{-\frac{km}{N}} e^{-\frac{km'}{N}} \operatorname{multi}(N-1, m', m; x, y) = e^{-kx-ky} + \mathcal{O}\left(\frac{k}{N}\right). \tag{3.25}$$

(ii) Also, for $0 \le x \le y \le \frac{1}{k}$,

$$\sum_{m=0}^{N-2}\sum_{m'=0}^{m} e^{-\frac{km}{N}} e^{-\frac{km'}{N}} \operatorname{multi}(N-1, m', N-m; x, 1-y) = e^{-kx-ky} + \mathcal{O}\left(\frac{k}{N}\right). \tag{3.26}$$

Proof: (i) As in the proof of the first part of the theorem, modulo $\mathcal{O}(N^{-1})$ errors, we can replace $N-2$ by $N-1$ in the upper limit of both summations.

Define $\exp_{-k}(x) := \exp(-kx)$. Then, as a consequence of Lemma 3.5, we have that

$$\begin{aligned}
&\sum_{m'=0}^{N-2}\sum_{m=0}^{N-2-m'} e^{-\frac{km}{N}} e^{-\frac{km'}{N}} \operatorname{multi}(N-1, m', m; x, y) \\
&= \sum_{m'=0}^{N-1}\sum_{m=0}^{N-1-m'} \operatorname{multi}(N-1, m', m; xe^{-\frac{k}{N}}, ye^{-\frac{k}{N}}) + \mathcal{O}(N^{-1}) \\
&= \left(1-(x+y)+(x+y)e^{-\frac{k}{N}}\right)^{N-1} + \mathcal{O}(N^{-1}) \\
&= B_N(\exp_{-k}; x+y) + \mathcal{O}\left(\frac{k}{N}\right) \\
&= \exp(-kx-ky) + \mathcal{O}\left(\frac{k}{N}\right).
\end{aligned}$$

(ii) Once again, we can replace $N-2$ by $N-1$ in the upper limit of the first sum. We make successive applications of the Binomial Theorem to get:

$$\begin{aligned}
&\sum_{m=0}^{N-2}\sum_{m'=0}^{m} e^{-\frac{km}{N}} e^{-\frac{km'}{N}} \operatorname{multi}(N-1, m', N-m; x, 1-y) \\
&= (N-1)! \sum_{m=0}^{N-1} e^{-\frac{km}{N}} \frac{\left(x-y+ye^{-\frac{k}{N}}\right)^{m-1}}{(m-1)!} \frac{(1-x)^{N-m}}{(N-m)!} + \mathcal{O}(N^{-1}) \\
&= e^{-\frac{k}{N}}\left(1-(x+ye^{-\frac{k}{N}})+(x+ye^{-\frac{k}{N}})e^{-\frac{k}{N}}\right)^{N-1} + \mathcal{O}(N^{-1}) \\
&= e^{-\frac{k}{N}} B_{N-1}(\exp_{-k}; x+e^{-\frac{k}{N}}y) + \mathcal{O}(N^{-1}) \\
&= e^{-kx-ky} + \mathcal{O}\left(\frac{k}{N}\right). \quad \square
\end{aligned}$$

We now use the combinatorial identities in Lemma 3.6 to estimate the variance of the averaged level spacings measure $d\rho_{LS}^{AV}$.

$0 < \alpha_0 < \ldots < \alpha_N < 1$, we consider the following two subsets of k elements given by

$$\alpha_{l_1} \leq \ldots \leq \alpha_{l_k} \text{ and } \alpha_{l'_1} \leq \ldots \leq \alpha_{l'_k}. \tag{3.28}$$

For each of the subsets above, there are $k-1$ pairs of the form $(\alpha_{l_j}, \alpha_{l_{j+1}})$ and $(\alpha_{l'_i}, \alpha_{l'_{i+1}})$. From (3.28) it follows that for any fixed pair $(\alpha_{l_j}, \alpha_{l_{j+1}})$, there is at most one pair $(\alpha_{l'_i}, \alpha_{l'_{i+1}})$ for which Case 2 is possible.

Case 3: $\alpha_{l_j} < \alpha_{l'_i} < \alpha_{l'_{i+1}} < \alpha_{l_{j+1}}$ (or equivalently, $\alpha_{l'_i} < \alpha_{l_j} < \alpha_{l_{j+1}} < \alpha_{l'_{i+1}}$).

This case can be dealt with in a similar fashion to Case 1. That is, we apply the Fubini Theorem repeatedly to ensure that the last four iterated integrals involve $\alpha_{l'_i}, \alpha_{l_j}, \alpha_{l_{j+1}}$ and $\alpha_{l'_{i+1}}$. Then, we integrate by parts with respect to the remaining α's. Finally, we make the change of variables $x = \alpha_{l'_{i+1}} - \alpha_{l'_i}$ and $y = \alpha_{l_{j+1}} - \alpha_{l_j}$ and integrate by parts again with respect to $\alpha_{l'_{i+1}}$ and $\alpha_{l_{j+1}}$ to get

$$\begin{aligned}
&\int_{\Lambda^N} \phi_k(\alpha_{l_{j+1}} - \alpha_{l_j})\, \phi_k(\alpha_{l'_{i+1}} - \alpha_{l'_i})\, d\alpha \\
&= (N+1)N \int_0^1 \int_x^1 \phi_k(x)\phi_k(y) \operatorname{multi}(N-1, l'_{i+1} - l'_i - 1, N - l_{j+1} - l_j + 1; x, 1-y)\, dydx \\
&= (N+1)N \int_0^1 \int_0^1 \phi_k(x)\phi_k(y) \operatorname{multi}(N-1, l'_{i+1} - l'_i - 1, N - l_{j+1} - l_j + 1; x, 1-y)\, dydx \\
&\quad + \mathcal{O}(k^{-1}).
\end{aligned}$$

As in the proof of part (i) of Theorem 3.1, we make the substitution $m = l_{j+1} - l_j - 1$ and $m' = l'_{i+1} - l'_i - 1$ in order to apply Lemma 3.6. From the estimate in (3.23) and the analysis of Cases 1-3 above, we deduce that

$$\begin{aligned}
&\int_{\Lambda^N} \left| d\rho_{LS}^{AV}(x; N, k, \alpha)(\phi) \right|^2 d\alpha \\
&= \frac{k^2(N+1)N}{(N+k-1)^2} \Bigg[\sum_{m'=0}^{N-2} \sum_{m=0}^{N-2-m'} e^{-\frac{mk}{N}} e^{-\frac{m'k}{N}} \int_0^1 \int_0^1 \phi_k(x)\phi_k(y) \operatorname{multi}(N-1, m', m; x, y)\, dxdy \\
&\quad + \sum_{m'=0}^{N-2} \sum_{m=0}^{m'} e^{-\frac{mk}{N}} e^{-\frac{m'k}{N}} \int_0^1 \int_0^1 \phi_k(x)\phi_k(y) \operatorname{multi}(N-1, m', N-m; x, 1-y)\, dydx \Bigg] \\
&\quad + \mathcal{O}\left(\frac{k}{N}\right) + \mathcal{O}\left(\frac{1}{k}\right).
\end{aligned}$$

By Lemma 3.6, we finally conclude that

$$\begin{aligned}\int_{\Lambda^N} \left|d\rho_{LS}^{AV}(x;N,k,\alpha)(\phi)\right|^2 \, d\!\!{}^-\alpha &= \frac{k^2(N+1)N}{(N+k-1)^2}\left(\int_0^1 \phi_k(x)\, e^{-kx}\, dx\right)^2 + \mathcal{O}\left(\frac{k}{N}\right) + \mathcal{O}\left(\frac{1}{k}\right) \\ &= \left(\int_0^1 \phi(x)\, e^{-x}\, dx\right)^2 + \mathcal{O}\left(\frac{k}{N}\right) + \mathcal{O}\left(\frac{1}{k}\right). \quad \square\end{aligned}$$

Theorem 3.1 (ii) is then an immediate consequence of the Chebyshev inequality and the following corollary of Proposition 3.1:

Corollary 3.3. *For any $\phi \in C_0^1([0,1])$, we have*

$$\int_{\Lambda^N} \left(d\rho_{LS}^{AV}(x;N,k,\alpha)(\phi) - \int_0^1 e^{-x}\phi(x)\, dx\right)^2 \, d\!\!{}^-\alpha = \mathcal{O}\left(\frac{k}{N}\right) + \mathcal{O}\left(\frac{1}{k}\right). \tag{3.29}$$

Proof: The corollary follows directly from Proposition 3.1 and the estimate for the convergence of the integrated, averaged level spacings measure in (3.23).

CHAPTER 4

Level spacings distribution of Van Vleck polynomials

Here, we give another application of the methods of Chapter 3 and compute the asymptotic mean level spacings distribution of the zeroes for any Van Vleck polynomial.

1. Introduction

In Chapter 2, we defined the notion of a Van Vleck polynomial $V(x)$ as the polynomial of degree $N-1$ for which the *generalized Lamé equation*

$$\prod_{\nu=0}^{N}(x-\alpha_\nu)\frac{d^2S}{dx^2}+\sum_{\nu=0}^{N}\rho_\nu\prod_{\mu\neq\nu}(x-\alpha_\mu)\frac{dS}{dx}+V(x)S=0,$$

has a polynomial solution $S(x)$ of preassigned degree k. As a consequence of Heine-Stieltjes Theorem, we know that there exists exactly $\sigma(N,k)$ distinct polynomials $V(x)$.

Let $v_1(\alpha)<\cdots<v_{N-1}(\alpha)$ denote the $N-1$ ordered zeroes of a given Van Vleck polynomial. Recall, these zeroes are all simple and lie inside the interval (α_0,α_N). In analogy with the case of Lamé (Stieltjes) polynomials treated in Chapter 3, we compute the asymptotic level spacings distribution for the zeroes of $V(x)$.

Define for $p\in[0,1)$, the probability measures

$$d\zeta_{LS}^{AV}(x;N,\alpha):=\frac{1}{N-2}\sum_{j=1}^{N-2}\delta(x-N^p(v_{j+1}(\alpha)-v_j(\alpha))), \tag{4.1}$$

where $\alpha\in\Lambda^N$, and Λ^N is the positive truncated Weyl chamber defined in the introduction of Chapter 3. As before, we let $\bar{d}\alpha:=(N+1)!d\alpha$ denotes the normalized Lebesgue measure on Λ^N. To state our main result, we also need to introduce the integrated, averaged level spacings distribution of $d\zeta_{LS}^{AV}$, i.e.

$$d\nu_{LS}(x;N)=\int_{\Lambda^N}d\zeta_{LS}^{AV}(x;N,\alpha)\ \bar{d}\alpha. \tag{4.2}$$

The following theorem is the analogue of Theorem 3.1 and its corollary:

Theorem 4.1. [Bo] *(i) The weak-limit of the measures $d\nu_{LS}(x;N)$ is given by*

$$w-\lim_{N\to\infty}d\nu_{LS}(x;N)=\delta_0(x).$$

(ii) For any $0 < \delta < (1-p)/2$, there exists a measurable subset $J^N \subseteq \Lambda^N$ with $\text{meas}(J^N) >$ $1 - N^{-1+p+2\delta}$*, such that for any $x \in J^N$,*

$$w-\lim_{N\to\infty} d\zeta_{LS}^{AV}(x; N, \alpha) = \delta_0(x).$$

In both (i) and (ii), the weak-limit is taken in the dual space to $C_0^0([a,b])$, where $0 \leq a < b\infty$.

2. Preliminary results

A look at Step 1 of the proof of Theorem 3.1 shows that the argument relies on the fact that one has, thanks to the Heine-Stieltjes Theorem, a precise description of the location of the zeroes of any Lamé polynomial in terms of the parameters $\alpha_0, ..., \alpha_N$.

Unfortunately, no result like this is known for the zeroes of Van Vleck polynomials. However, one can prove the following simple corollary of Theorem 2.2. It turns out that the next proposition will play exactly the same role as the one played by the Heine-Stieltjes Theorem in the results of Chapter 3.

Proposition 4.1. *Let $V(x)$ be any Van Vleck polynomial of degree N-1, and denote its ordered zeroes by $v_1(\alpha) < \cdots < v_{N-1}(\alpha)$. Then, the following conclusions hold: $v_1(\alpha) \in (\alpha_0, \alpha_2]$, $v_{N-1}(\alpha) \in [\alpha_{N-2}, \alpha_N)$, and $v_j(\alpha) \in [\alpha_{j-1}, \alpha_{j+1}]$ for $j = 2, ..., N-2$.*

Proof: By Theorem 2.2, we know that $(\alpha_0, \alpha_{j+1}]$ contains at least the first j zeroes of $V(x)$, so we must have $v_j(\alpha) \leq \alpha_{j+1}$. On the other hand, since $(\alpha_0, \alpha_{j-1}]$ contains at most the first $j-1$ zeroes of $V(x)$, hence $v_j(\alpha) \geq \alpha_{j-1}$. □

We are now in position to prove the following useful lemma. As the reader will notice, the conclusion of this lemma is analogous to Lemma 3.2 about zeroes of Lamé polynomials.

Lemma 4.1. *For any integer j with $1 \leq j \leq N-1$, we have that*

$$\int_{\Lambda^N} |v_j(\alpha) - \alpha_j| \ d\!\!{}^-\alpha = \mathcal{O}\left(N^{-1}\right) \tag{4.3}$$

uniformly in j.

Proof: As a consequence of Proposition 4.1, we know that

$$\alpha_{j-1} \leq v_j(\alpha) \leq \alpha_{j+1}.$$

A simple application of Lemma 3.2 where the multi-index $|\beta| = 1$ gives

$$\int_{\Lambda^N} \alpha_j \ d\!\!{}^-\alpha = \frac{j+1}{N+2}.$$

Therefore, we finally obtain

$$\begin{aligned} \int_{\Lambda^N} |v_j(\alpha) - \alpha_j| \, d\alpha &\leq \int_{\Lambda^N} (\alpha_{j+1} - \alpha_{j-1}) \, d\alpha \\ &= \frac{j+2}{N+2} - \frac{j+1}{N+2} \\ &= \mathcal{O}\left(N^{-1}\right). \quad \square \end{aligned}$$

3. Proof of part (i) of Theorem 4.1

For notational simplicity, we assume once more that $[a, b] = [0, 1]$ and $\phi \in C_0^1([0,1])$. The argument for more general non-negative intervals $[a, b]$ follows in the same way. In addition, we henceforth define $\phi_{N^p}(x) := \phi(N^p x)$ for any function $\phi \in C_0^1([0,1])$. Our first task is to show that,

$$d\nu_{LS}(x; N)(\phi) = \frac{1}{N-2} \sum_{j=1}^{N-2} \int_{\Lambda^N} \phi_{N^p}\left(\alpha_{j+1} - \alpha_j\right) \, d\alpha + \mathcal{O}\left(N^{-1+p}\right). \tag{4.4}$$

First, we make a first-order Taylor expansion of ϕ_{N^p} around $(\alpha_{j+1} - \alpha_j)$ in (4.4). This gives:

$$d\nu_{LS}(x; N)(\phi) = \frac{1}{N-2} \sum_{j=1}^{N-2} \int_{\Lambda^N} \phi_{N^p}\left(\alpha_{j+1} - \alpha_j\right) \, d\alpha + E_1(N, \phi), \tag{4.5}$$

where the error term $E_1(N, \phi)$ is given by

$$E_1(N, \phi) = \frac{N}{N-2} \sum_{j=0}^{N-1} \int_{\Lambda^N} \phi'(N^p \xi_j(\alpha)) \Big[(v_{j+1}(\alpha) - \alpha_{j+1}) - (v_j(\alpha) - \alpha_j) \Big] \, d\alpha,$$

with some $\xi_j(\alpha) \in (0, 1)$. All we need to show is that $E_1(N, \phi) = \mathcal{O}(N^{-1+p})$. This is a simple consequence of Lemma 4.1, since

$$\begin{aligned} |E_1(N, \phi)| &\leq \frac{N^p}{N-2} \sum_{j=1}^{N-2} \int_{\Lambda^N} |\phi'(N^p \xi_j(\alpha))| \Big[|v_{j+1}(\alpha) - \alpha_{j+1}| + |v_j - \alpha_j| \Big] \, d\alpha \\ &\leq 2 \frac{N^p}{N-2} \|\phi\|_{C^1} \sum_{j=0}^{N-1} \int_{\Lambda^N} |v_j - \alpha_j| \, d\alpha \\ &= 2 \frac{N^p}{N-2} \|\phi\|_{C^1} \sum_{j=0}^{N-1} \mathcal{O}(N^{-1}) \\ &= \mathcal{O}(N^{-1+p}). \end{aligned}$$

Having completed the first step of the proof, our next task is to show that

$$d\nu_{LS}(x; N)(\phi) = \int_0^N \phi(N^{-1+p} x) \left(1 - \frac{x}{N}\right)^N dx + \mathcal{O}\left(N^{-1+p}\right). \tag{4.6}$$

This is an immediate consequence of Lemma 3.3 of the previous chapter. Indeed, this lemma asserts that for any integer i, j, with $0 \le i < j \le N$,

$$\int_{\Lambda^N} \phi\left(\alpha_j - \alpha_i\right)\, d\!\!\!{}^{-}\alpha = (N+1) \int_0^1 \phi(x)\ \mathrm{binom}(N, j-i-1; x)\, dx.$$

Thus,

$$\begin{aligned} d\nu_{LS}(x;N)(\phi) &= \frac{1}{N-2} \sum_{j=1}^{N-2} \int_{\Lambda^N} \phi_{N^p}\left(\alpha_{j+1} - \alpha_j\right)\, d\!\!\!{}^{-}\alpha + \mathcal{O}\left(N^{-1+p}\right) \\ &= \frac{N+1}{N-2} \sum_{j=1}^{N-2} \int_0^1 \phi_{N^p}(x)\,(1-x)^N\, dx + \mathcal{O}\left(N^{-1+p}\right) \\ &= \frac{(N+1)}{N} \int_0^N \phi(N^{-1+p}x)\left(1 - \frac{x}{N}\right)^N dx + \mathcal{O}\left(N^{-1+p}\right) \\ &= \int_0^N \phi(N^{-1+p}x)\left(1 - \frac{x}{N}\right)^N dx + \mathcal{O}\left(N^{-1+p}\right). \end{aligned}$$

Finally, from the basic inequality

$$0 \le e^{-x} - \left(1 - \frac{x}{N}\right)^N \le \frac{x^2 e^{-x}}{N}, \quad \text{for } x \ge 0,$$

and the fact that $x^p e^{-x} = \mathcal{O}_p(1)$, we conclude that

$$\begin{aligned} d\nu_{LS}(x;N)(\phi) &= \int_0^N \phi(N^{-1+p}x)\, e^{-x}\, dx + \mathcal{O}\left(N^{-1+p}\right). \\ &= \phi(0) + \mathcal{O}(N^{-1+p}). \end{aligned}$$

Since the compactly supported C^1 functions in $[0,1]$ are dense in $C_0^0([0,1]$, this completes the proof of part (i) of Theorem 4.1. □

4. Proof of part (ii) of Theorem 4.1

The first relation we need to prove is the following:

$$\begin{aligned} &\int_{\Lambda^N} \left|d\zeta_{LS}^{AV}(x;N,\alpha)(\phi)\right|^2\, d\!\!\!{}^{-}\alpha \\ &= \frac{1}{(N-2)^2} \sum_{j,j'=1}^{N-2} \int_{\Lambda^N} \phi_{N^p}(\alpha_{j+1} - \alpha_j)\, \phi_{N^p}(\alpha_{j'+1} - \alpha_{j'})\, d\!\!\!{}^{-}\alpha + \mathcal{O}\left(N^{-1+p}\right). \end{aligned} \tag{4.7}$$

To establish (4.7), we repeat the same sort of argument as in equation (4.4) in part (i). That is, we expand each of the functions $\phi_{N^p}(v_{j+1}(\alpha) - v_j(\alpha))$ and $\phi_{N^p}(v_{j'+1}(\alpha) - v_{j'}(\alpha))$ in a first-order

Taylor series around $(\alpha_{j+1}-\alpha_j)$ and $(\alpha_{j'+1}-\alpha_{j'})$ respectively. As a result, we obtain

$$\begin{aligned}&\int_{\Lambda^N}\left|d\zeta_{LS}^{AV}(x;N,\alpha)(\phi)\right|^2\, d\!\!\!{}^-\alpha\\ &= \frac{1}{(N-2)^2}\sum_{j,j'=1}^{N-2}\int_{\Lambda^N}\phi_{N^p}(\alpha_{j+1}-\alpha_j)\,\phi_{N^p}(\alpha_{j'+1}-\alpha_{j'})\, d\!\!\!{}^-\alpha + E_2(N,\phi),\end{aligned}$$

where $E_2(N,\phi)$ involves only terms with derivatives of ϕ. Since $E_1(N,\phi)=\mathcal{O}\left(N^{-1+p}\right)$, it is easy to see that $E_2(N,\phi)$ is bounded by

$$|E_2(N,\phi)| \le 2\frac{N^2}{(N-2)^2}\|\phi\|_{C^1}^2\sum_{j,j'=1}^{N-2}\int_{\Lambda^N}|v_j(\alpha)-\alpha_j|\,|v_{j'}(\alpha)-\alpha_{j'}|\, d\!\!\!{}^-\alpha + \mathcal{O}\left(N^{-1+p}\right). \tag{4.8}$$

Moreover, a similar computation as in Proposition 4.1 shows that

$$\begin{aligned}\int_{\Lambda^N}\left|v_j^l(\alpha)-\alpha_j\right|\left|v_{j'}^{l'}(\alpha)-\alpha_{j'}\right| d\!\!\!{}^-\alpha &\le \int_{\Lambda^N}(\alpha_{j+1}-\alpha_{j-1})(\alpha_{j'+1}-\alpha_{j'})\, d\!\!\!{}^-\alpha\\ &= \frac{(j+3)(j'+2)-(j+3)(j'+1)}{(N+2)(N+3)}\\ &\quad -\frac{(j+1)(j'+2)-(j+1)(j'+1)}{(N+2)(N+3)}\\ &= \frac{4}{(N+2)(N+3)}.\end{aligned} \tag{4.9}$$

Consequently, if we combine (4.9) with (4.8), we immediately obtain the estimate (4.7). We are in now in position to prove the following key proposition.

Proposition 4.2. *For any $\phi\in C_0^1([0,1])$, we have that*

$$\int_{\Lambda^N}\left|d\zeta_{LS}^{AV}(x;N,\alpha)(\phi)\right|^2\, d\!\!\!{}^-\alpha = \phi^2(0)+\mathcal{O}\left(N^{-1+p}\right).$$

Proof: As a consequence of the estimate (4.7), we simply need to show

$$\frac{1}{(N-2)^2}\sum_{j,j'=1}^{N-2}\int_{\Lambda^N}\phi_{N^p}(\alpha_{j+1}-\alpha_j)\,\phi_{N^p}(\alpha_{j'+1}-\alpha_{j'})\, d\!\!\!{}^-\alpha = \phi^2(0)+\mathcal{O}\left(N^{-1+p}\right). \tag{4.10}$$

In order to do this, it suffices to consider the case where $j'<j$ (or, equivalently, $j<j'$); for, when compared with all the possible pairs (α_j,α_{j+1}), the number of pairs $(\alpha_{j'},\alpha_{j'+1})$ for which $j=j'$ is like $\mathcal{O}\left(N^{-1}\right)$.

So, let's assume that $j'<j$. We apply Fubini's Theorem to ensure that the last four iterated integrals only involve $\alpha_{j+1},\alpha_j,\alpha_{j'+1},\alpha_{j'}$. We then perform the first $N-4$ integrals to get

$$\int_{\Lambda^N} d\!\!\!{}^-\alpha = (N+1)N\int_0^1\int_0^{\alpha_{j+1}}\int_0^{\alpha_j}\int_0^{\alpha_{j'+1}}\frac{(1-\alpha_{j+1})^{N-j-1}}{(N-j-1)!}\,\frac{(\alpha_j-\alpha_{j'+1})^{j-1}}{(j-1)!}\,\frac{(\alpha_{j'})^{j'}}{j'!}d\alpha_{j'}d\alpha_{j'+1}d\alpha_j d\alpha_{j+1}.$$

where we have omitted the integrand $\phi_{N^p}(\alpha_{j+1}-\alpha_j)\,\phi_{N^p}(\alpha_{j'+1}-\alpha_{j'})$ to simplify the writing. In order to reduce the last quadruple integral to a double integral, we make the change of variables

$$x = \alpha_{j+1} - \alpha_j, \quad y = \alpha_{j'+1} - \alpha_{j'},$$

and integrate by parts with respect to α_{j+1} and $\alpha_{j'+1}$. The end result is that

$$\begin{aligned}
&\int_{\Lambda^N} \phi_{N^p}(\alpha_{j+1}-\alpha_j)\,\phi_{N^p}(\alpha_{j'+1}-\alpha_{j'})\, d\!\!\!{}^-\alpha \\
&= (N+1)N\int_0^1\int_0^{1-y} \phi_{N^p}(x)\,\phi_{N^p}(y)\,(1-x-y)^{N-1}dxdy \\
&= \int_0^N\int_0^{N(1-y)} \phi(N^{-1+p}x)\,\phi(N^{-1+p}y)\left(1-\frac{x}{N}-\frac{y}{N}\right)^{N-1} dxdy \\
&= \int_0^N\int_0^{N(1-y)} \phi(N^{-1+p}x)\,\phi(N^{-1+p}y)\,e^{-x-y}dxdy + \mathcal{O}\left(N^{-1+p}\right) \\
&= \phi^2(0) + \mathcal{O}\left(N^{-1+p}\right).
\end{aligned}$$

The second equality follows from the fact that $\phi_{N^p}(y)$ is supported in $[0,1]$, so we must have $0 \le y \le 1/N^p$. This proves (4.10) and concludes the proof. □

The statement of part (ii) of Theorem 4.1 is now an immediate consequence of the previous proposition and Chebychev's inequality. Indeed,

$$\begin{aligned}
&\operatorname{meas}\left(\left\{\alpha\in\Lambda^N : \left|d\zeta_{LS}^{AV}(x;N,\alpha)(\phi)-\phi(0)\right| \ge N^{-\delta}\right\}\right) \\
&\qquad \le N^{2\delta}\int_{\Lambda^N}\left|d\zeta_{LS}^{AV}(x;N,\alpha)(\phi)-\phi(0)\right|^2 d\!\!\!{}^-\alpha \\
&\qquad = N^{2\delta}\left[\int_{\Lambda^N}\left|d\zeta_{LS}^{AV}(x;N,\alpha)(\phi)\right|^2 d\!\!\!{}^-\alpha - \phi^2(0) + \mathcal{O}\left(N^{-1+p}\right)\right] \\
&\qquad = \mathcal{O}\left(N^{-1+p+2\delta}\right).
\end{aligned}$$

The conclusion follows by complementarity.

Heine-Stieltjes Theorem has been extended by M. Marden **[Ma1]**, **[Ma2]**, **[Ma3]** to the complex case. However, the results he obtained do not yield precise location, as in the real case, of the zeroes of Lamé polynomials in terms of the parameters α. The natural question to ask is then: Can we apply similar methods as those developed in this thesis to get interesting results in that situation ?

(iii) It would be of considerable interest to determine how the actual zeroes of the Lamé harmonics are distributed in the sense of Riemann measure on $\mathbf{S}^N$ itself. A natural starting point would be to look at the density of states measures (see **[ShZ]**). In light of our results in this thesis, the zeroes should, at least on average, behave like random variables in the asymptotic regime where $k(N) \sim N^{1-\epsilon}$. Consequently, we believe that the density of states should on average tend to uniform measure on $\mathbf{S}^N$.

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