ERGODIC THEORY OF RATIONAL BILLIARDS AND APPLICATIONS TO THE EQUILATERAL TRIANGLE

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Abstract

In this text, we are interested in studying the distribution of eigenfunctions on rational polygons, with special attention paid to the equilateral triangle. We attempt to provide the reader with sufficient context on spectral theory, semiclassical analysis and ergodic theory. We then prove a result due to Marklof and Rudnick [MR11] which states that "most" Laplace eigenfunctions, with either Dirichlet or Neumann boundary conditions, on a rational polygon equidistribute. We also identify exceptional subsequences for an orthonormal basis with either Dirichlet or Neumann boundary conditions on the equilateral triangle. Finally, we discuss the limiting behaviour exhibited by those same sequences.

Abrégé

Dans ce document, nous étudions la distribution des fonctions propres sur des polygones rationnels, payant une attention particulière au triangle équilatéral. Nous tentons d'offrir au lecteur le contexte nécessaire au sujet de la théorie spectrale, de la théorie semi-classique et de la théorie ergodique. Ensuite, on établit un résultat de Marklof et Rudnick [MR11] qui stipule que la "majorité" des fonctions de Laplacien sur un polygone rationnel, avec des conditions aux limites de Dirichlet ou avec des conditions aux limites de Neumann, vont équidistribuer. De plus, on identifie les sous-suites exceptionnelles pour une base orthonormale soit avec des conditions aux limites de Dirichlet ou avec des conditions aux limites de Neumann sur le triangle équilatéral. Finalement, nous discutons le comportement à l'infini présenté par ces mêmes séquences.

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Contents

1 Introduction and Setup

The behaviour of a microscopic particle can be best described by the (time-dependent) Schrödinger equation. Solving this equation provides us with the wave function of this given particle, as well as the energy spectrum. The most basic setting is possibly the problem of a particle in a box, otherwise known as the infinite potential well. That is, the problem of a particle that moves freely in a domain D. In this case, finding the wave functions and energies given by the Schrödinger equation amounts to solving the Helmholtz equation

$$\Delta u + \lambda u = 0$$

with appropriate boundary conditions (see [KKS99] and the references therein). Equivalently, we wish to find the eigenfunctions of the Laplacian on D with appropriate boundary conditions. There is a great range of applications for this problem, from physics to electronics and nanodevices (see, once again, [KKS99]).

Throughout this document, we focus primarily on the classification of quantum limits of eigenfunctions for the Laplacian on equilateral triangles in $\mathbb{R}^2$. More precisely, given a countable orthonormal L^2-basis $(\psi_n)_{n=1}^{\infty}$ consisting of eigenfunctions on a triangle ordered such that their respective eigenvalues are increasing, we ask how the eigenfunctions concentrate as n tends to infinity. Formally, we study the weak* limits of the probability measures $\left|\psi_{n_j}\right|^2 \mathrm{d}x$ as j tends to infinity where (ψ_{n_j}) is a subsequence of (ψ_n). We henceforth refer to these limits as *quantum limits*.

Such questions are largely motivated by a recent result of Marklof-Rudnick (see Theorem 1 in [MR11]) which addresses the concentration of eigenfunctions on rational polygons. Informally, their result states that almost all quantum limits of this sequence must be the Liouville measure. In other words, there is a density-one sequence of eigenfunctions which equidistributes on the polygon. More formally, our work is motivated by the following theorem, which suggests the plausibility of a complete classification of quantum limits. In this next statement, we denote by ∂A the topological boundary of a set A. Equivalently, ∂A is the closure of A minus the interior of A.

Theorem 1.1. *[MR11] Let D be a rational polygon and fix an orthonormal basis $(\varphi_n)_{n=1}^{\infty}$ of the Dirichlet Laplacian on D. Then, there exists a sequence of natural numbers (n_j) such that*

$$\lim_{j\to\infty} \int_A \left|\varphi_{n_j}(x)\right|^2 \mathrm{d}x = \frac{\mathrm{area}(A)}{\mathrm{area}(D)} \tag{1.1}$$

for all measurable sets $A \subseteq D$ with boundary ∂A having Lebesgue measure 0. Furthermore, we have

$$\lim_{N\to\infty} \frac{\#\left\{j : n_j \leq N\right\}}{N} = 1. \tag{1.2}$$

The conclusion drawn in (1.2) can be interpreted as saying that the subsequence $\left(\varphi_{n_j}\right)$ contains almost all eigenfunctions, or, even better, consists of a density one subfamily of (φ_n). In particular, we see that almost all quantum limits are simply a normalized Lebesgue measure. Consequently, a complete classification of the exceptional subsequences (that is, subsequences that do not obey (1.1)) would yield a classification of all possible quantum limits associated to the Dirichlet problem for the Laplacian on D.

In the context of the equilateral triangle, we expect all quantum limits to be absolutely continuous with respect to the Liouville measure. This hypothesis is supported by the following result of Jakobson [Jak96].

Theorem 1.2. *Fix a dimension $d \geq 1$. Every quantum limit of the torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ is absolutely continuous (with respect to the natural measure on the torus).*

In addition, this story begs the natural question of what can be said about the quantum limits associated to the same problem, but with Neumann boundary data. As with the Dirichlet case, this analysis is supported by the fact that Theorem 1.1 continues to hold with Neumann boundary conditions (see [MR11]). Furthermore, Theorem 1.1 extends to translation surfaces, although we will not discuss this here.

We also point out that the analysis of eigenfunctions (in both the Dirichlet and Neumann settings) has many applications within the sciences, especially in physics. Indeed, the study of eigenfunctions of the Laplacian originates from the study of vibrating membranes and plates (see [Zel17]). Let u be an eigenfunction of the Laplacian on a domain D satisfying appropriate boundary conditions with eigenvalue λ. Then u satisfies the equation

$$-\Delta u = \lambda u \quad \text{in } D.$$

Here, u is interpreted as the *profile* of vibrations whereas λ can be thought of as the corresponding energy and $\sqrt{\lambda}$ represents the frequency parameter. When studying diffusion, Grebenkov and Nguyen explain in [GN12, page 603] that the first eigenfunction describes the asymptotic spatial distribution, for long time, of particles in the given domain. However, other eigenfunctions lack a straightforward physical interpretation (see [GN12] and the references therein). We should also observe that the authors of this last paper point out applications to stochastic processes.

The study of eigenfunctions has also proven to be of use in probabilistic and statistical contexts. In particular, eigenfunctions have applications to the study of localization properties of disordered metals and carry information about atomic spectra. Furthermore, according to Samajdar and Jain in [SJ18, page 2], the distribution of their amplitudes is linked to the fluctuation of tunnelling conductance across quantum dots.

Finally, we would like to note that eigenvalues of the Laplacian are of particular importance in geometry. For instance, the eigenvalues of the Laplace-Beltrami operator on a compact manifold (M, g) contain enough geometric information about the manifold (M, g) to completely determine the Euler characteristic (see [Ros97]). Consequently, eigenvalues and eigenfunctions of the Laplacian capture geometric properties of the manifold. This point of view is partially justified by Selberg's trace formula which relates geodesic flows on hyperbolic manifolds to the spectrum of the Laplace-Beltrami operator (see [Mar04]).

In the subsection below, we take the time to formally recall some basic properties of eigenfunctions that will be freely relied upon throughout this text. In §2.1, we briefly discuss the billiard problem in the plane. In particular, in §2.2, we touch upon the relationship between billiards, geometry, and spectral theory. Following this, we devote §3 to the proof of Theorem 1.1 following the argument in [MR11].

In §4, we turn our attention to the Dirichlet and Neumann eigenvalue problems when the domain is an equilateral triangle. Using methods of reflection and identification and the literature regarding eigenfunctions on the parallelogram, we find an explicit L^2 orthonormal basis consisting of eigenfunctions. In Theorem 4.3, we fully classify the semiclassical quantum limits and, in particular, describe the measures associated to exceptional subsequences. Finally, in §4.4, we briefly discuss the billiard map on the equilateral triangle and find an operator that commutes with the Laplacian by identifying symmetries.

1.1 Prerequisites

We devote this section to a rapid overview of results that are well known but nonetheless essential to the analysis that follows. In §1.1.1, we provide an overview of results from spectral theory for the Laplacian. Then, in §1.1.2, we cover miscellaneous results from analysis that will be necessary in later parts of this document.

As a first step, we recall some elementary arithmetic results regarding sequences and series of real numbers. Although these are straightforward results, we provide their proofs since, as far as the author can tell, they have neither a name nor a common reference.

Lemma 1.3. *Suppose $a_1, \dots, a_n$ and $b_1, \dots, b_n$ are non-negative real numbers. If $b_1, \dots, b_n > 0$ then*

$$\frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} \geq \min_{1 \leq j \leq n} \frac{a_j}{b_j}.$$

Proof. Let j be such that a_j/b_j is minimized. Then

$$\frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} = \frac{b_1 \frac{a_1}{b_1} + \cdots + b_n \frac{a_n}{b_n}}{b_1 + \cdots + b_n} \geq \frac{b_1 \frac{a_j}{b_j} + \cdots + b_n \frac{a_j}{b_j}}{b_1 + \cdots + b_n} = \frac{a_j}{b_j}$$

as desired. □

The next arithmetic lemma is slightly more interesting, and somewhat harder to establish.

Lemma 1.4. *Suppose (a_n) is a non-negative sequence such that*

$$\sum_{n=1}^{\infty} a_n = \infty,$$

then

$$\lim_{N \to \infty} \frac{\sum_{n=1}^{N} 2^{-n} a_n}{\sum_{n=1}^{N} a_n} = 0.$$

Proof. Fix $\varepsilon > 0$ and let $L \in \mathbb{N}$ be such that

$$2^{-L} < \frac{\varepsilon}{2}.$$

Because $\sum_{n=1}^{\infty} a_n = \infty$, there exists $M > L$ such that

$$\frac{2}{\varepsilon} \sum_{n=1}^{L} a_n < \sum_{n=1}^{M} a_n.$$

Or, equivalently,

$$\sum_{n=1}^{L} a_n < \frac{\varepsilon}{2} \sum_{n=1}^{M} a_n.$$

Since the sequence (a_n) is non-negative, for any $N \geq M$,

$$\begin{aligned}\sum_{n=1}^{N} 2^{-n} a_n \leq \sum_{n=1}^{L} a_n + \sum_{n=L+1}^{N} 2^{-n} a_n &< \frac{\varepsilon}{2} \sum_{n=1}^{M} a_n + \frac{\varepsilon}{2} \sum_{n=L+1}^{N} a_n \\ &\leq \frac{\varepsilon}{2} \sum_{n=1}^{N} a_n + \frac{\varepsilon}{2} \sum_{n=1}^{N} a_n \\ &= \varepsilon \sum_{n=1}^{N} a_n.\end{aligned}$$

Consequently, for all $N \geq M$ we have

$$\frac{\sum_{n=1}^{N} 2^{-n} a_n}{\sum_{n=1}^{N} a_n} < \varepsilon.$$

Especially,

$$\limsup_{N \to \infty} \frac{\sum_{n=1}^{N} 2^{-n} a_n}{\sum_{n=1}^{N} a_n} \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the proof is complete. □

1.1.1 Spectral Theory of Laplace Eigenfunctions

Fix a compact Riemannian manifold (M, g) with (possibly empty) boundary Γ. For the sake of consistency with the literature, we formalize what we mean by the Dirichlet and Neumann problems. The Dirichlet spectral problem asks that we find all eigenvalues to the Dirichlet problem

$$\begin{cases} -\Delta_g u = \lambda u & \text{in } M \\ u = 0 & \text{on } \Gamma. \end{cases} \tag{1.3}$$

In the above, Δ_g is the Laplace-Beltrami operator on M, which is defined for all $u \in C^\infty(M)$. In a similar vein, the Neumann problem associated to the Laplace-Beltrami operator seeks a solution u to

$$\begin{cases} -\Delta_g u = \lambda u & \text{in } M \\ \nabla u \cdot \gamma = 0 & \text{on } \Gamma. \end{cases} \tag{1.4}$$

where γ denotes the unit inward normal vector field on Γ. In either setting, we shall denote by $E(\lambda)$ the eigenspace corresponding to λ. That is, $E(\lambda)$ is the collection of all functions $u \in C^\infty(M)$ such that $-\Delta_g u = \lambda u$ in M.

Before proceeding further, let us take a moment to recall a fundamental result addressing the eigenfunctions of either problem on (M, g). Albeit well known, this result is essential to the coherency of the results that will follow shortly.

Theorem 1.5. *[Lab15, Theorem 4.3.1] For the compact Riemannian manifold (M, g) with possibly empty boundary Γ, the following assertions hold true for the Dirichlet and Neumann spectral problems.*

1. *The collection of eigenvalues are real, non-negative numbers*

$$0 \leq \lambda_1 \leq \lambda_2 \leq \dots$$

   such that $\lambda_k \to \infty$ as $k \to \infty$.

2. *Each eigenvalue has finite multiplicity and the eigenspaces corresponding to distinct eigenvalues are orthogonal in $L^2(M)$.*

3. *Denoting by $E(\lambda_k)$ the eigenspace of each eigenvalue λ_k, we observe that the closure in $L^2(M)$ of*

$$\bigoplus_{k \geq 1} E(\lambda_k)$$

   is the entire space $L^2(M)$.

4. *Every eigenfunction is smooth.*

The result above remains valid for a mixed problem as well as the Steklov problem (see, for instance, [Lab15, §4.3] for more information). Item 3 in the theorem implies the existence of an orthogonal basis (u_k) of $L^2(M)$ composed entirely of Dirichlet (or Neumann) eigenfunctions. We note that by *basis*, we mean a Schauder basis. That is, for every $f \in L^2(M)$ there exists a sequence of scalars (a_k) such that

$$\lim_{K\to\infty} \left\| f - \sum_{k=1}^{K} a_k u_k \right\|_{L^2(M)} = 0.$$

We also note that Laplace eigenfunctions in open subsets of $\mathbb{R}^n$ are real analytic. That is, if $-\Delta u = \lambda u$ in some open set $\Omega \subseteq \mathbb{R}^n$, the u is analytic in Ω.

Let $D \subseteq \mathbb{R}^n$ be a bounded domain. By the previous theorem, there exists a countable orthonormal basis (u_k) of Dirichlet (resp. Neumann) eigenfunctions for the Laplacian. Suppose that f is a given Dirichlet (resp. Neumann) eigenfunction of the Laplacian with eigenvalue λ. Since (u_k) is an orthonormal basis of $L^2(D)$,

$$f = \sum_{k=1}^{\infty} \langle f, u_k \rangle_{L^2} u_k.$$

Noting that eigenspaces of different eigenvalues are orthogonal, we have

$$f = \sum_{\substack{k \\ -\Delta u_k = \lambda u_k}} \langle f, u_k \rangle_{L^2} u_k.$$

Since every eigenvalue has finite multiplicity, this sum is finite. We can therefore draw the following conclusion:

Proposition 1.6. *Given an orthonormal basis of Dirichlet (resp. Neumann) eigenfunctions on M, every Dirichlet (resp. Neumann) eigenfunction f is a finite linear combination of functions in this basis.*

1.1.2 Miscellaneous Results in Real Analysis

The next few standard results will be freely invoked throughout the remainder of this exposition. In particular, these will be greatly relied upon in §3 and in our discussion of Laplace eigenfunctions on the equilateral triangle.

Theorem 1.7. *For any bounded domain $D \subseteq \mathbb{R}^n$, the vector space $C_c^\infty(D)$ endowed with the uniform norm is separable.*

Proof. It follows from the Stone-Weierstrass Theorem (see [Fol99, Theorem 4.45]) that the collection of polynomials is dense in $C(\overline{D})$. Taking only the polynomials with rational coefficients, we obtain a sequence $(a_k) \in C^\infty(\overline{D})$ that is dense with respect to the uniform norm. We may then consider a collection of cut-off function $(\eta_k) \in C_c^\infty(D)$ such that

(1) $0 \leq \eta_k \leq 1$;

(2) For each $k \in \mathbb{N}$, the Lebesgue measure of the set $D \backslash \{\eta_k = 1\}$ is bounded above by $1/k$.

Such a sequence can be obtained as follows. For $k \in \mathbb{N}$, we pick a set $V \Subset D$ such that $m(D \setminus V)$ is sufficiently small. Then, mollifying the function $\mathbb{1}_V$ appropriately we obtain a suitable function η_k. Finally, observe that the countable collection of functions

$$\{\eta_j a_k : j, k \in \mathbb{N}\}$$

is dense in $C_c^\infty(D)$. □

Theorem 1.8 (Identity Theorem)**.** *Let $D \subseteq \mathbb{R}^n$ be a non-empty domain and suppose that $f, g : D \to \mathbb{C}$ are analytic functions. If $f \equiv g$ in a neighbourhood of D, then $f \equiv g$ in all of D.*

The Identity Theorem is well-known for single-variable function, especially in the context of holomorphy. In order to conclude the result as stated above, we note that a given analytic function f must also be analytic with respect to each variable. Therefore, we obtain the result of this theorem by treating each variable independently.

Theorem 1.9 (The Riemann-Lebesgue Lemma). *[Fol99, Theorem 8.22] If (a_k) and (b_k) are two sequences of real numbers such that $(a_k^2 + b_k^2) \to \infty$ as $k \to \infty$, then*

$$\lim_{k\to\infty} \int_{\mathbb{R}^2} \chi(x,y) \exp\left(i\left[a_k x + b_k y\right]\right) \mathrm{d}x\mathrm{d}y = 0$$

for every $\chi \in L^1(\mathbb{R}^2)$.

The Riemann-Lebesgue Lemma is easily proven by first verifying the claim for step functions then using the density of these functions in $L^1(\mathbb{R}^2)$ to conclude the result. We also note that the Riemann Lebesgue Lemma remains valid in $\mathbb{R}^n$ for any positive integer n. This result is often stated in terms of the Fourier transform. That is, the Fourier transform of an integrable functions vanishes at infinity.

2 Background and Fundamentals

For the sake of clarity and completeness, we attempt to provide enough background for the uninitiated reader. More precisely, in this section, we state some of the basic results from semiclassical analysis and ergodic theory that give rise to much of the arguments that will follow. Additionally, we try to reiterate some of the most important definitions and structures used within these results. This includes, in no particular order, an overview of quantization, Egorov's theorem, Birkhoff's ergodic theorem, Weyl's law, and its local counter part.

Pseudodifferential operators are a natural generalization of differential operators. These linear operators can be defined using the Fourier transform.

In the simplest setting, we work with functions on $\mathbb{R}^n$. More specifically, we begin with the most natural setting for the Fourier transform: the Schwartz space. Given $n \geq 1$, the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ is defined as the collection of smooth function $u \in C^\infty(\mathbb{R}^n; \mathbb{C})$ such that for every pair of multi-indices α, β

$$\left\|x^\beta \partial^\alpha u(x)\right\|_\infty \leq C_{\alpha,\beta}$$

for some constant $C_{\alpha,\beta}$. In this expression, $\|\cdot\|_\infty$ denotes the L^∞-norm on $\mathbb{R}^n$. We topologize the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ by giving it the following countable family of semi-norms:

$$\|u\|_{\alpha,\beta} := \left\|x^\alpha \partial^\beta u(x)\right\|_\infty .$$

Equipped with these semi-norms, the Schwartz space forms a locally convex topological vector space over the field $\mathbb{C}$ of complex numbers. Furthermore, the space $\mathscr{S}(\mathbb{R}^n)$ is metrizable, first countable, and normal. Consequently, its topological properties are completely determined by its convergent sequences.

We now turn to a useful generalization of the classical Fourier transform, which is also an isomorphism of the Schwartz space $\mathscr{S}(\mathbb{R}^n)$. It should be noted that this semi-classical analogue of the Fourier transform is the main tool by which one can define quantization.

Definition 1. Given $h > 0$, the semiclassical Fourier transform is a map $\mathcal{F}_h : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ given by

$$\mathcal{F}_h u(\xi) = \int_{\mathbb{R}^n} e^{-\frac{i}{h}\langle x,\xi\rangle} u(x) \mathrm{d}x$$

for every $u \in \mathscr{S}(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$. Here, $\langle \cdot, * \rangle$ simply denotes the dot-product.

As for the classical Fourier transform, $\mathcal{F}_h$ can be naturally defined on the dual space $\mathscr{S}'(\mathbb{R}^n)$. Indeed, we may define $\mathcal{F}_h : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ by

$$\left(\mathcal{F}_h u\right) \varphi = u\left(\mathcal{F}_h \varphi\right)$$

for all $u \in \mathscr{S}'(\mathbb{R}^n)$ and $\varphi \in \mathscr{S}(\mathbb{R}^n)$. We accept the obvious abuse in notation by viewing $\mathscr{S}'(\mathbb{R}^n)$ as a function space containing $\mathscr{S}(\mathbb{R}^n)$. Then, the definition of $\mathcal{F}_h$ on the dual of the Schwartz space is simply an extension of Definition 1. In order to justify this approach, we note that each function in $v \in \mathscr{S}(\mathbb{R}^n)$ is entirely described by the values

$$\int_{\mathbb{R}^n} v(x)\varphi(x)\mathrm{d}x, \quad \varphi \in \mathscr{S}(\mathbb{R}^n).$$

We may therefore identify v with the functional $u \in \mathscr{S}'(\mathbb{R}^n)$ given by

$$u(\varphi) = \int_{\mathbb{R}^n} v(x)\varphi(x)\mathrm{d}x.$$

In this sense, $\mathscr{S}'(\mathbb{R}^n)$ is a generalized function space. We note that, with this approach, $\mathscr{S}'(\mathbb{R}^n)$ is also seen to contain $L^p(\mathbb{R}^n)$ for each $1 \leq p \leq \infty$.

Therefore, the semiclassical Fourier transform $\mathcal{F}_h$ is defined on $L^2(\mathbb{R}^n)$ and is, in fact, a Banach isomorphism of this space. Before moving forward, we provide without proof some properties of the Fourier transform that are of particular importance for our purposes.

(1) The Fourier transform has an inverse $\mathcal{F}_h^{-1}$. Furthermore, the Fourier inverse can be represented by the integral expression

$$\mathcal{F}_h^{-1}u(x) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x,\xi\rangle}u(\xi)\mathrm{d}\xi,$$

where the above exists in the classical sense for all $u \in \mathscr{S}(\mathbb{R}^n)$.

(2) For every multi-index α and $u \in \mathscr{S}'(\mathbb{R}^n)$,

$$\mathcal{F}_h\left((-x)^\alpha u(x)\right) = (hD_\xi)^\alpha\, \mathcal{F}_h u$$

Here, $D_\xi = -i\partial_\xi$.

(3) In a similar vein

$$\mathcal{F}_h\left((hD_x)^\alpha u\right) = \xi^\alpha \mathcal{F}_h u.$$

for every multi-index α and $u \in \mathscr{S}'(\mathbb{R}^n)$. In this last expression, we denote $D_x = -i\partial_x$.

We refer the interested reader to [Zwo12] for further information on the semiclassical Fourier transform.

Definition 2. Given $a \in \mathscr{S}(\mathbb{R}^{2n})$ and $h > 0$, the quantization of a is the operator $\mathrm{Op}_h(a) : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ defined by

$$\mathrm{Op}_h(a)u(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} a(x,\xi)u(y)\mathrm{d}y\mathrm{d}\xi.$$

This last expression is an iterated integral and cannot generally be interpreted as a double integral. Note that, equivalently,

$$\mathrm{Op}_h(a)u(x) = \mathcal{F}_h^{-1}\left(a(x,\cdot)\mathcal{F}_h u(\cdot)\right)(x).$$

As a result, we see that our definition of quantization remains valid for all $a \in \mathscr{S}'(\mathbb{R}^n)$. Furthermore, if

$$a(x,\xi) = \sum_{|\alpha|\leq k} a_\alpha(x)\xi^\alpha$$

then

$$\mathrm{Op}_h(a) = \sum_{|\alpha|\leq k} a_\alpha(x)\left(hD_x\right)^\alpha.$$

Thus, the class of pseudo-differential operators is indeed "larger" than that of differential operators and contains the latter as a proper subset.

We now turn our attention to *symbols*, a class of functions that will be of particular importance. More specifically, we will be interested in a class of functions known as Kohn-Nirenberg symbols, as these possess an invariance property that make it possible to extend their definition to manifolds (see

[Zwo12, §9.3]). In fact, this is precisely the space of functions which we will be interested in quantizing.

Definition 3. Given an integer m, the Kohn-Nirenberg symbol class $S^m(\mathbb{R}^{2n})$ is the collection of smooth function $a \in C^\infty(\mathbb{R}^{2n})$ such that,

$$\left|\partial_x^\alpha \partial_\xi^\beta a\right| \le C_{\alpha,\beta}\left(1+|\xi|^2\right)^{\frac{m-|\beta|}{2}}$$

for every pair of multi-indices α, β.

More generally, given an arbitrary set $V \subseteq \mathbb{R}^n$, the collection of functions $a \in C^\infty(V \times \mathbb{R}^n)$ satisfying this last expression on $V \times \mathbb{R}^n$ is denoted by $S^m(V \times \mathbb{R}^n)$. As briefly explained below, symbol classes can also be defined on a compact Riemannian manifold (M, g).

Definition 4. If a function $a \in C^\infty(T^*M)$ is such that for every coordinate chart $\gamma : U \to V$, the pullback of a is in $S^m(V \times \mathbb{R}^n)$, then we say that $a \in S^m(T^*M)$. We note that if $m = 0$, then we will simply write $S(T^*M)$ to denote the space $S^0(T^*M)$.

Remark 1. The pull back γ^*a of a is a map

$$V \times \mathbb{R}^n \xrightarrow{\gamma} T^*U \xrightarrow{a} \mathbb{C}.$$

In order for our definition to be consistent, it must be independent of our choice of coordinates. This is indeed the case since, for any integer m, the class of Kohn-Niremberg symbols is invariant under coordinate change (see [Zwo12, Theorem 9.4]). This is precisely the invariance property that was referred to in our earlier discussion.

Using partitions of unity (see [Zwo12, Chapter 14]), one can extend the concept of quantization to compact Riemannian manifolds. In this way, for $a \in S\left(T^*M\right)$ we obtain a linear operator

$$\mathrm{Op}_h(a) : C^\infty(M) \to C^\infty(M).$$

The quantization map Op_h satisfies the following properties:

(1) Op_h is a linear map on $S(T^*M)$;

(2) If $a \in S(T^*M)$ then $\mathrm{Op}_h(a) : L^2(M) \to L^2(M)$ is a bounded linear operator;

(3) For functions $a, b \in S(T^*M)$ there holds

$$\mathrm{Op}_h(a)\,\mathrm{Op}_h(b) = \mathrm{Op}_h(ab) + O_{L^2}(h).$$

(4) Given $a \in S(T^*M)$,

$$\mathrm{Op}_h(a)^* = \mathrm{Op}_h(\bar{a}) + O_{L^2}(h),$$

where $\mathrm{Op}_h(a)^*$ denotes the formal adjoint of $\mathrm{Op}_h(a)$ relative to the $L^2(M)$ inner product.

Remark 2. Combining (3) and (4), we see that

$$\mathrm{Op}_h(a)^*\,\mathrm{Op}_h(a) = \mathrm{Op}_h(|a|^2) + O_{L^2}(h).$$

2.1 The Billiard flow

We now discuss the billiard problem in the plane. Motivation for the study of billiards can be found in multiple fields of physics such as optics, mechanics and quantum systems. The billiard flow represents the free motion of a point mass within a given domain. At the boundary, the motion of a particle behaves according to the law of specular reflection. That is, the rule

the angle of incidence equal the angle of reflection

dictates the change in direction. If the boundary of D is described by the function $\gamma : [0, 1] \to \partial D$, then the billiard map can be visualized as follows.

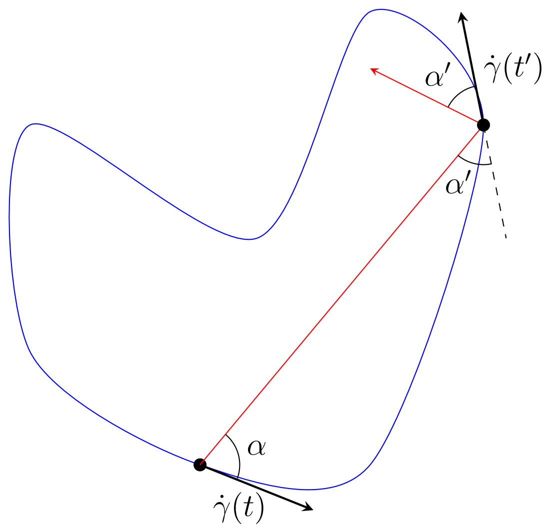


Figure 1: Billiard trajectory with base point $\gamma(t)$.

We now define the billiard flow in more mathematical terms. Given a closed non-empty connected set D with piece-wise smooth boundary, the billiard flow Φ_t is defined on the cotangent bundle

$$T^*D = D \times \mathbb{R}^2$$

and, as we will see, can be described by a dynamical system. We first note that a point $(x, \omega) \in T^*D$ represents the position x of a particle with velocity vector ω. Since Φ_t describes free motion, the speed must remain constant. Consequently, it will suffice to describe the billiard flow on unit cotangent bundle

$$S^*D = D \times S^1.$$

Then, for any $(x, \omega) \in S^*D$ and $r \in \mathbb{R}$ we define

$$\Phi_t\,(x, r\omega) = \Phi_{rt}(x, \omega), \quad t \in \mathbb{R}.$$

Now, given a point $(x, \omega) \in S^*D$, it will be convenient to write

$$(x(t), \omega(t)) := \Phi_t(x, \omega).$$

In the interior of D, we are in free motion;

$$\dot{x}(t) = \omega(t), \quad \dot{\omega}(t) = 0.$$

The Hamiltonian equation above can be more compactly represented by

$$\partial_t \Phi_t(x, \omega) = J\Phi_t(x, \omega) \tag{2.1}$$

where

$$J = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}.$$

On the boundary of D, the law of reflection states that $\omega(t)$ changes discontinuously. Despite this, $\Phi_t(x, \omega)$ is defined to be right-continuous with respect to t for each fixed point (x, ω). Given a direction vector ω, our new direction upon hitting the boundary will be

$$\omega - 2\langle \omega, \nu \rangle \nu$$

where ν is the inward unit vector at our position on the boundary of D. Note that we only assumed that the boundary was *piece-wise* smooth. Therefore, there are finitely many points where the normal vector is ill defined. At such points, the billiard flow remains undefined or can be defined arbitrarily.

We can also define the billiard map $\beta : \mathcal{N} \to \mathcal{N}$ where $\mathcal{N}$ is the subset of $\partial D \times S^1$ containing only the inward pointing vectors. This map is given by

$$\beta(x, \omega) = \Phi_{t(x,\omega)}(x, \omega)$$

where $t(x, \omega)$ is the smallest positive real number such that $\Phi_{t(x,\omega)}(x, \omega) \in \mathcal{N}$.

More precisely,

$$t(x, \omega) = \inf \{t > 0 : \Phi_t(x, \omega) \in \mathcal{N}\}.$$

2.2 Definitions and Results in Ergodic Theory

We now provide sufficient framework to properly discuss the ergodic properties of billiard flows on rational polygons. Although we assume that the reader is familiar with the setting of ergodic theory, we reiterate the more basic definitions for the sake of completeness.

Definition 5. Let $(X, \mathfrak{M}, \mu)$ be a probability space and fix a measurable function $\Phi : X \to X$. We say that Φ is μ-invariant if

$$\mu\left(\Phi^{-1}E\right) = \mu(E)$$

for every measurable set E.

Definition 6. Let $(X, \mathfrak{M}, \mu)$ be a probability space and suppose that we are given a measurable map $\Phi : X \to X$. If Φ is a μ-invariant transformation, we will call Φ ergodic whenever

$$\Phi^{-1}E = E,$$

implies $\mu(E) = 0$ or $\mu(E) = 1$. Or, equivalently, a μ-invariant transformation is said to be ergodic whenever the only invariant sets have either full or zero measure.

A natural counterpart of this definition is that of *unique ergodicity*.

Definition 7. Let $(X, \mathfrak{M})$ be a measurable space and fix a measurable function $\Phi : X \to X$. We say that Φ is uniquely ergodic if there exists a unique probability measure μ such that Φ is μ-invariant.

As a sanity check, we ask if a uniquely ergodic function $\Phi : X \to X$ is also ergodic with respect to this unique measure μ. To see that this is indeed the case, suppose by way of contradiction that we can find a measurable set E such that $\Phi^{-1}E = E$ and $0 < \mu(E) < 1$. Then consider the measure ν on X given by

$$\nu(A) = \frac{1}{\mu(E)}\mu\left(A \cap E\right), \quad \forall A \in \mathfrak{M}.$$

Clearly, ν is a probability measure on X distinct from μ. Furthermore, to see that Φ is ν-invariant, observe that for any measurable set A there holds

$$\begin{aligned}
\nu\left(\Phi^{-1}(A)\right) = \frac{1}{\mu(E)}\mu\left(\Phi^{-1}(A) \cap E\right) &= \frac{1}{\mu(E)}\mu\left(\Phi^{-1}(A) \cap \Phi^{-1}(E)\right) \\
&= \frac{1}{\mu(E)}\mu\left(\Phi^{-1}(A \cap E)\right) \\
&= \frac{1}{\mu(E)}\mu(A \cap E) \\
&= \nu(A).
\end{aligned}$$

Thus, we have found another probability measure that makes Φ invariant, contradicting the assumption of unique ergodicity.

Definition 8. On a probability space $(X, \mathfrak{M}, \mu)$, a family $(\Phi_t)_{t\in\mathbb{R}}$ of bijective μ-invariant functions satisfying

(1) $\Phi_t \circ \Phi_s = \Phi_{t+s}$ for all $s, t \in \mathbb{R}$;

(2) For any measurable function $f : X \to \mathbb{C}$, the map $(t, x) \mapsto f\left(\Phi_t x\right)$ is $X \times \mathbb{R}$-measurable.

is called a flow.

The next result can be found in [KSF82].

Theorem 2.1 (The Birkhoff-Khinchin Ergodic Theorem)**.** *Suppose* $(X, \mathfrak{M}, \mu)$ *is a probability space with flow* Φ_t *and fix a function* $f \in L^1(X)$*. For almost every* $x \in X$ *the limit*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left[f \circ \Phi_t\right](x)\mathrm{d}t$$

exists. Furthermore, if Φ_t is ergodic for each $t \in \mathbb{R}$ then

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [f \circ \Phi_t](x)\mathrm{d}t = \int_X f\mathrm{d}\mu$$

for almost every $x \in X$.

In a later section, we will apply this result to a rational polygon. That is, a simple planar polygon D such that, at each vertex, the angle between two edges is a rational multiple of π. We take a moment to illustrate our setting. We can define the billiard flow

$$\Phi_t : S^*D \to S^*D$$

where t ranges in $\mathbb{R}$. One can readily verify that this family of functions is indeed a flow. Consider now a fixed direction $\omega \in S^1$ and observe that, since D is a rational polygon, the function $\Phi_t(x, \omega)$ alternates through finitely many angles. Indeed, the possible angles are generated by the possible reflections along the edges of D. More precisely, each edge of the polygon is associated with a linear map which takes a direction ω and maps it to the new direction after regular reflection at that edge. Define Γ to be the finite group generated under composition by these maps. We see that the billiard flow is invariant with respect to

$$D_\theta := D \times \bigcup_{\gamma\in\Gamma} \{\gamma\theta\}.$$

That is, given $t \in \mathbb{R}$ and $(x, \omega) \in D_\theta$ we have $\Phi_t(x, \omega) \in D_\theta$.

Remark 3. The Φ_t-invariant set D_θ can be visualized as the gluing of $\#\Gamma$ copies of D, with equivalent sides identified. It is thus seen that D_θ is an oriented compact surface (see [MT02]). In the case of the equilateral triangle, this process yields a surface of genus 1.

We denote the restriction of the billiard flow to D_θ by Φ_t^θ. Furthermore, on the set D_θ, the natural probability measure μ_θ is given by

$$\mathrm{d}\mu_\theta = \frac{1}{\#\Gamma}\frac{1}{m(D)}\mathrm{d}x\mathrm{d}\nu_\theta$$

where ν_θ is the counting measure on $\{\gamma\theta : \gamma \in \Gamma\}$. As seen in [KSF82], for every $t \in \mathbb{R}$ there holds that Φ_t^θ is μ_θ invariant. On the other hand, by Theorem 1 in [KMS86], we know that Φ_t^θ is uniquely ergodic for almost every θ. Combing these results, we conclude the following lemma.

Lemma 2.2. *For almost every θ, Φ_t^θ is ergodic.*

2.2.1 Egorov's Theorem and Weyl's Law

In this subsection, we discuss some central tools that will be needed for the proof of Theorem 1.1. The statement of this theorem involves *quantum limits*, which we now properly define.

Let (M, g) be a compact Riemannian manifold (possibly with boundary) having dimension $n \geq 1$. Now, we consider on (M, g) the eigenvalue problem

$$-\Delta_g u = \lambda^2 u \quad \text{in } M$$

where Δ_g denotes the Laplace-Beltrami operator.

Definition 9. Let ν be a measure on M. Suppose there exists a sequence (u_j) of $L^2(M)$-normalized eigenfunctions for Δ_g whose corresponding eigenvalues tend to positive infinity. Assume additionally that

$$\lim_{j\to\infty}\int_A |u_j(x)|^2\,\mathrm{d}x = \int_A \mathrm{d}\nu \tag{2.2}$$

for every measurable set $A \subseteq M$ such that ∂A has measure 0 with respect to $\mathrm{d}x$. Then ν is called a *quantum limit* on M. Furthermore, if ν is absolutely continuous with respect to the natural Riemann measure $\mathrm{d}x$ on M, then one has $\mathrm{d}\nu = f\mathrm{d}x$ for some function f known as the *density* of ν.

In this text, we will often be interested in quantum limits associated to a given sequence of eigenfunctions. Suppose that (u_j) is an orthonormal basis of $L^2(M)$ composed entirely of eigenfunctions for Δ_g such that their eigenvalues form an increasing sequence. Put otherwise, we assume that for each $j \in \mathbb{N}$ one has

$$-\Delta_g u_j = \lambda_j^2 u_j \quad \text{in } M$$

where (λ_j) is a non-negative, increasing sequence. If there exists a subsequence (u_{j_k}) of (u_j) satisfying

$$\lim_{k\to\infty} \int_A |u_{j_k}(x)|^2 \, \mathrm{d}x = \int_A \mathrm{d}\nu \tag{2.3}$$

for every measurable set $A \subseteq M$ with boundary of measure 0, then we say that ν is a quantum limit *associated* to (u_j).

As mentioned at the beginning of this section, we are interested in two useful tools that will be central to our analysis of quantum limits. The first of these is the standard Egorov's theorem, which establishes an important approximation relationship between quantum and classical time evolution. As our exposition of this topic is not meant to be exhaustive, we refer the reader to Chapters 11 and 15 of [Zwo12] for more information on this subject. Given a compact Riemannian (M, g), it is known (see [Zwo12, Theorem C.13]) that for each $t \in \mathbb{R}$ there exists a unitary operator on $L^2(M)$

$$U(t) = U(t; h) := e^{ith\Delta_g}$$

such that

$$\begin{cases} U(t)U(s) = U(t+s), \\ U(t)^* = U(-t) \\ \lim_{t\to 0} \|U(t)u - u\|_{L^2(M)} = 0 \quad \text{for all } u \in L^2(M). \end{cases}$$

Furthermore,

$$D_t\left(U(t)u\right) - hU(t)\Delta_g u = 0.$$

for all $u \in C^\infty(M)$ such that $-\Delta_g u \in L^2(M)$. Here, we adopt the convention $D_t = -i\partial_t$. Note that if u is an eigenfunction of $-\Delta_g$ with eigenvalue λ^2 then

$$D_t\left(U(t)u\right) + h\lambda^2 U(t)u = 0.$$

It follows that

$$U(t)u = e^{-ith\lambda^2}u.$$

Theorem 2.3 (Egorov's Theorem [Zwo12])**.** *Let $U(t)$ be as defined above, fix $T > 0$ and suppose that Φ_t solves the Hamiltonian equation (2.1). For any $a \in S(T^*M)$ there holds*

$$\left\|U(-t)\operatorname{Op}_h(a)U(t) - \operatorname{Op}_h\left(a \circ \Phi_t\right)\right\|_{L^2(M)\to L^2(M)} = O(h)$$

uniformly for $0 \leq t \leq T$.

Since a quantum limit is determined by the asymptotic behaviour of eigenfunctions, it is natural to seek a result that provides us with information about the eigenvalues and eigenfunctions in the limit. Thankfully, such a description is provided by Weyl's law (both global and local versions), for which we provide a precise formulation below. Although we shall not include proofs for these now standard results, having them stated formally will aid the reader in the arguments that follow. We quote these statements directly from [Dya16, §2.5].

Fix a sequence of eigenfunctions (u_j) forming an orthonormal basis of $L^2(M)$ with associated eigenvalues

$$0 \leq \lambda_1^2 \leq \lambda_2^2 \leq \dots$$

The Weyl law gives us insight on the asymptotic behaviour of the eigenvalues listed above.

Theorem 2.4 (Weyl law)**.** *As $R \to \infty$, there holds*

$$\left(\frac{2\pi}{R}\right)^n \#\left\{j : \lambda_j \leq R\right\} = \omega_n \operatorname{Vol}(M) + O\left(\frac{1}{R}\right)$$

Recall that our sequence of eigenvalues can be recovered from our eigenfunctions by "testing" these against the Laplace-Beltrami operator. In light of this, we ask if Weyl's law can therefore be extended in a way that reflects this phenomenon. Particularly, we can partially describe how the eigenfunctions behave when tested against a larger class of pseudodifferential operators on $L^2(M)$.

Theorem 2.5 (Local Weyl law)**.** *For any smooth compactly supported function $\chi : (0, \infty) \to \mathbb{R}$ and $a \in S(T^*M)$,*

$$\begin{aligned}
&\sum_{j=1}^{\infty} \chi\left(\frac{\lambda_j}{R}\right) \langle \operatorname{Op}_h(a) u_j, u_j \rangle_{L^2(M)} \\
&= \left(\frac{R}{2\pi}\right)^n \left[\int_{T^*M} \chi\left(|\xi|_g\right) a\left(x, \frac{\xi}{|\xi|_g}\right) \mathrm{d}x \mathrm{d}\xi + O\left(\frac{1}{R}\right)\right]
\end{aligned}$$

as $R \to \infty$

3 Eigenfunctions of a Rational Polygon

We dedicate this section to the proof of Theorem 1.1, following the argument put forth by Marklof and Rudnick in [MR11].

Let $D \subseteq \mathbb{R}^2$ be a rational polygon. Here, the polygon D includes its boundary. Recall that our phase space is the unit cotangent bundle, denoted

$$S^*D = D \times S^1.$$

The associated Liouville measure is given by

$$\mathrm{d}\mu(x,\omega) = \frac{1}{m(D)}\mathrm{d}x\mathrm{d}\phi$$

where $\omega = e^{2\pi i\phi}$ for $\phi \in \mathbb{R}/\mathbb{Z}$ and m denotes the Lebesgue measure on $\mathbb{R}^2$. A smooth function $a : S^*D \to \mathbb{C}$ is known as an observable. Furthermore, if there exists a function $a_0 : D \to \mathbb{C}$ such that

$$a(x,\omega) = a_0(x) \tag{3.1}$$

for all $\omega \in S^1$, the we say that a is an isotropic observable. In other words, an isotropic observable is a smooth function depending only in the position, or equivalently independent of the momentum. For any observable a, we define the time average by

$$a_T(x,\omega) := \frac{1}{2T}\int_{-T}^{T} a \circ \Phi_t(x,\omega)\mathrm{d}t.$$

Given $\theta \in S^1$, recall that we have defined

$$D_\theta := D \times \bigcup_{\gamma\in\Gamma} \{\gamma\theta\}.$$

as the subset of $D \times S^1$ containing all possible directions that can occur with initial direction θ after repeatedly reflecting on the sides of D. By Lemma 2.2, the restriction Φ_t^θ of Φ_t to D_θ is ergodic with respect to μ. This crucial result enables us to establish the quantum ergodic theorems that will be necessary in order to prove Theorem 1.1. In fact, from here on, our proof follows a now standard method that can be applied to obtain very general results on manifolds with ergodic flows (see for instance [Zwo12]). Nevertheless, following notes from Dyatlov [Dya16], we include a proof for the sake of completeness.

Returning to our argument, our last assertion was that Φ_t^θ is ergodic with respect to μ_θ for almost every θ. Then, the Birkhoff-Khinchin ergodic theorem (see Theorem 2.1) implies that for any observable a there holds

$$\lim_{T\to\infty} a_T(x,\theta) = \int_{D_\theta} a\mathrm{d}\mu_\theta$$

for a.e. $x \in D$. More compactly, we see that for almost every $(x,\omega) \in S^*D$ there holds

$$\lim_{T\to\infty} a_T(x,\omega) = \int_{D_\omega} a\mathrm{d}\mu_\omega = \frac{1}{\#\Gamma}\sum_{\gamma\in\Gamma}\frac{1}{m(D)}\int_D a(\tilde{x},\gamma\omega)\mathrm{d}\tilde{x}.$$

In particular, when a is isotropic the above reduces to

$$\lim_{T\to\infty} a_T(x,\omega) = \frac{1}{m(D)}\int_D a_0(\tilde{x})\mathrm{d}\tilde{x} =: \overline{a}$$

where a_0 is given in equation (3.1). Finally, by the dominated convergence theorem

$$\lim_{T\to\infty}\int_{S^*D} |a_T(x,\omega) - \overline{a}|^2 \,\mathrm{d}\mu = 0.$$

We formalize the above result in the form of a lemma.

Lemma 3.1. *[MR11, Lemma 2] For any isotropic observable*

$$a(x,\omega) = a_0(x),$$

we have

$$\lim_{T\to\infty}\int_{S^*D} |a_T - \overline{a}|^2 \,\mathrm{d}\mu = 0.$$

where $\overline{a} = \frac{1}{m(D)}\int_D a_0(x)\mathrm{d}x$.

We now fix an orthonormal basis of Dirichlet eigenfunctions

$$\psi_1, \psi_2, \psi_3, \ldots$$

with corresponding eigenvalues

$$0 \leq \lambda_1^2 \leq \lambda_2^2 \leq \lambda_3^2 \leq \dots$$

That is,

$$\begin{cases} -\Delta\psi_n = \lambda_n^2 \psi_n & \text{in } D^\circ \\ \psi_n = 0 & \text{on } \partial D \end{cases}$$

for each $n \in \mathbb{N}$. We also define the sequence

$$h_n = \frac{1}{\lambda_n}$$

which, as $n \to \infty$, concentrates near the origin. In Lemma 3.1, it was shown that the time average a_T of an isotropic observable a converges to the position average of a in $L^2(S^*D)$. We now ask if, for fixed T, one can compare the expectation of a with respect to the probability densities $|\psi_n|^2 \, \mathrm{d}x$ to that of a_T. More specifically, the following asymptotic result provides an accurate answer to this question.

Lemma 3.2. *For any $a \in S(T^*D)$ and $T > 0$,*

$$\left| \left\langle \mathrm{Op}_{h_n}(a)\psi_n, \psi_n \right\rangle_{L^2(D)} \right|^2 - \left| \left\langle \mathrm{Op}_{h_n}(a_T)\psi_n, \psi_n \right\rangle_{L^2(D)} \right|^2 \to 0$$

as n tends to infinity. In particular, if a is isotropic then $a(x,\omega) = a_0(x)$ for some function $a_0 : D \to \mathbb{C}$ and

$$\left| \int_D a_0(x) \left| \psi_n(x) \right|^2 \mathrm{d}x \right|^2 - \left| \left\langle \mathrm{Op}_{h_n}(a_T)\psi_n, \psi_n \right\rangle_{L^2(D)} \right|^2 \to 0$$

as $n \to \infty$.

Proof. Consider the propagator

$$U(t) = U(t; n) = \exp\left(ith_n\Delta\right).$$

Since

$$U(t)\psi_n = e^{-it\lambda_n}\psi_n,$$

we see that

$$\begin{aligned}\langle \mathrm{Op}_h(a)\psi_n, \psi_n\rangle_{L^2(D)} &= \langle \mathrm{Op}_h(a)U(t)\psi_n, U(t)\psi_n\rangle_{L^2(D)} \\ &= \langle U(-t)\,\mathrm{Op}_h(a)U(t)\psi_n, \psi_n\rangle_{L^2(D)}\end{aligned}$$

where in the last step we have used that $U(t)^* = U(-t)$. It then follows from Egorov's Theorem, that

$$\left\langle \mathrm{Op}_{h_n}(a)\psi_n, \psi_n\right\rangle_{L^2(D)} - \left\langle \mathrm{Op}_{h_n}(a\circ\Phi_t)\psi_n, \psi_n\right\rangle_{L^2(D)} = O(h) \tag{3.2}$$

uniformly for $0 \le t \le T$. Therefore, taking the average integral from $-T$ to T on either side of (3.2), we infer that

$$\left|\left\langle \mathrm{Op}_{h_n}(a)\psi_n, \psi_n\right\rangle_{L^2(D)} - \left\langle \mathrm{Op}_{h_n}(a_T)\psi_n, \psi_n\right\rangle_{L^2(D)}\right| \to 0.$$

In order to conclude the result as stated in the lemma, we note that $\mathrm{Op}_{h_n}(a)$ and $\mathrm{Op}_{h_n}(a_T)$ are bounded linear operators on $L^2(D)$. Furthermore, this bound is uniform for all h_n sufficiently small, or rather for all n large. Hence,

$$\left\langle \mathrm{Op}_{h_n}(a)\psi_n, \psi_n\right\rangle_{L^2(D)}, \quad \left\langle \mathrm{Op}_{h_n}(a_T)\psi_n, \psi_n\right\rangle_{L^2(D)}$$

are uniformly bounded in n. Finally, the result follows from the fact that the mapping $x \mapsto |x|^2$ is uniformly continuous on bounded sets. □

In these two previous results, we carried out comparisons of a with respect to its time averages; first with respect to the Liouville measure on the unit cotangent bundle, and then with respect to the probability densities $|\psi_n|^2\,\mathrm{d}x$. In order to relate these results, we can compare how the function a averages against both the Liouville measure and the probability densities given by the eigenfunctions. In more precise terms, we postulate that the expectation of a function $a \in S(T^*D)$, taken with respect to the probability densities $|\psi_n|^2 \,\mathrm{d}x$, is comparable to the average of a on the unit cotangent bundle S^*D. Specifically, the expectation of a is eventually almost always bounded above by a constant multiple of

$$\int_{S^*D} |a(x,\omega)|^2 \,\mathrm{d}\mu.$$

In fact, we assert that there exists a constant C independent of a and R such that, as $R \to \infty$,

$$\frac{1}{R^2} \sum_{\lambda_n \in [R,2R]} \left| \left\langle \mathrm{Op}_{h_n}(a)\psi_n, \psi_n \right\rangle_{L^2(D)} \right|^2 \leq C \int_{S^*D} |a(x,\omega)|^2 \,\mathrm{d}\mu + O\left(\frac{1}{R}\right).$$

However, we first note that by Weyl's law, $\#\{n : \lambda_n \in [R,2R]\} \sim R^2$. Therefore, this inequality tells us that, eventually, most terms

$$\left\langle \mathrm{Op}_{h_n}(a)\psi_n, \psi_n \right\rangle_{L^2(D)}$$

are at most a constant multiple of the L^2-average of a on S^*D.

Lemma 3.3. *For any $a \in S(T^*D)$,*

$$\frac{1}{R^2} \sum_{\lambda_n \in [R,2R]} \left| \left\langle \mathrm{Op}_{h_n}(a)\psi_n, \psi_n \right\rangle_{L^2(D)} \right|^2 \leq C \int_{S^*D} |a(x,\omega)|^2 \,\mathrm{d}\mu + O\left(\frac{1}{R}\right)$$

for some constant C independent of a and R.

Proof. By the Cauchy-Schwartz inequality,

$$\begin{aligned} \sum_{\lambda_n \in [R,2R]} \left| \left\langle \mathrm{Op}_{h_n}(a)\psi_n, \psi_n \right\rangle_{L^2(D)} \right|^2 &\leq \sum_{\lambda_n \in [R,2R]} \left\| \mathrm{Op}_{h_n}(a)\psi_n \right\|_{L^2(D)}^2 &(3.3)\\ &\leq \sum_{n=1}^{\infty} \chi\left(\frac{\lambda_n}{R}\right) \left\| \mathrm{Op}_{h_n}(a)\psi_n \right\|_{L^2(D)}^2 . &(3.4) \end{aligned}$$

for any non-negative cut-off function $\chi \in C_c^\infty((0,\infty))$ whose restriction to the interval $[1,2]$ is the constant function 1. Now, observe that

$$\begin{aligned}\left\|\operatorname{Op}_{h_n}(a)\psi_n\right\|_{L^2(D)}^2 &= \left\langle \operatorname{Op}_{h_n}(a)\psi_n, \operatorname{Op}_{h_n}(a)\psi_n\right\rangle_{L^2(D)} \\ &= \left\langle \operatorname{Op}_{h_n}(a)^* \operatorname{Op}_{h_n}(a)\psi_n, \psi_n\right\rangle_{L^2(D)} \\ &= \left\langle \operatorname{Op}_{h_n}(|a|^2)\psi_n, \psi_n\right\rangle_{L^2(D)} + O(h_n).\end{aligned}$$

Using that $h_n = 1/\lambda_n$, and the asymptotic expansion above in (3.3)-(3.4), it follows that

$$\begin{aligned}&\sum_{\lambda_n \in [R,2R]} \left|\left\langle \operatorname{Op}_{h_n}(a)\psi_n, \psi_n\right\rangle_{L^2(D)}\right|^2 \\ &\leq \sum_{\lambda_n \in [R,2R]} \chi\left(\frac{\lambda_n}{R}\right)\left(\left\langle \operatorname{Op}_{h_n}(|a|^2)\psi_n, \psi_n\right\rangle_{L^2(D)} + O(h_n)\right) \\ &= \sum_{\lambda_n \in [R,2R]} \chi\left(\frac{\lambda_n}{R}\right)\left(\left\langle \operatorname{Op}_{h_n}(|a|^2)\psi_n, \psi_n\right\rangle_{L^2(D)} + O\left(\frac{1}{R}\right)\right).\end{aligned}$$

Finally, by Weyl's law and the local Weyl law (Theorems 2.4 and 2.5)

$$\frac{1}{R^2}\sum_{\lambda_n \in [R,2R]} \left|\left\langle \operatorname{Op}_{h_n}(a)\psi_n, \psi_n\right\rangle_{L^2(D)}\right|^2 \leq C\int_{S^*D} |a(x,\omega)|^2 \,\mathrm{d}\mu + O\left(\frac{1}{R}\right)$$

where C is independent on a and R. □

Finally, by combining the conclusions of Lemmas 3.1, 3.2 and 3.3, we can show that a fixed function a eventually "almost always" equidistributes with respect to the probability densities $|\psi_n|^2 \,\mathrm{d}x$. This is, of course, made formal below.

Lemma 3.4. *Let $a_0 \in C_c^\infty(D)$ and define $a \in S(T^*D)$ by $a(x,\omega) = a_0(x)$. Then*

$$\lim_{R\to\infty} \frac{1}{R^2}\sum_{\lambda_j \in [R,2R]} \left|\left\langle \operatorname{Op}_{h_n}(a)\psi_n, \psi_n\right\rangle_{L^2(D)} - \overline{a}\right|^2 = 0$$

where
$$\overline{a} = \int_{S^*D} a(x,\omega)\mathrm{d}\mu = \frac{1}{m(D)}\int_D a_0(x)\mathrm{d}x.$$

Proof. Let $T > 0$ be given. Note that, by applying Lemma 3.2,

$$\begin{aligned}
&\left|\left\langle \mathrm{Op}_{h_n}(a)\psi_n, \psi_n\right\rangle_{L^2(D)} - \overline{a}\right|^2 - \left|\left\langle \mathrm{Op}_{h_n}(a_T)\psi_n, \psi_n\right\rangle_{L^2(D)} - \overline{a}\right|^2 \\
&= \left|\left\langle \mathrm{Op}_{h_n}(a-\overline{a})\psi_n, \psi_n\right\rangle_{L^2(D)}\right|^2 - \left|\left\langle \mathrm{Op}_{h_n}((a-\overline{a})_T)\psi_n, \psi_n\right\rangle_{L^2(D)}\right|^2
\end{aligned}$$

is $o(1)$ as $n \to \infty$. Subsequently, invoking Weyl's law (see Theorem 2.4), it follows that

$$\frac{1}{R^2}\sum_{\lambda_j \in [R,2R]}\left(\left|\left\langle \mathrm{Op}_{h_n}(a-\overline{a})\psi_n, \psi_n\right\rangle_{L^2(D)}\right|^2 - \left|\left\langle \mathrm{Op}_{h_n}((a-\overline{a})_T)\psi_n, \psi_n\right\rangle_{L^2(D)}\right|^2\right)$$

tends to 0 as $R \to \infty$. On the other hand, appealing to Lemma 3.3,

$$\frac{1}{R^2}\sum_{\lambda_j \in [R,2R]}\left|\left\langle \mathrm{Op}_{h_n}((a-\overline{a})_T)\psi_n, \psi_n\right\rangle_{L^2(D)}\right|^2 \leq C\int_{S^*D}\left|a_T - \overline{a}\right|^2 \mathrm{d}\mu + O\left(\frac{1}{R}\right).$$

Combining these last two equations, we see that

$$\limsup_{R\to\infty}\frac{1}{R^2}\sum_{\lambda_j \in [R,2R]}\left|\left\langle \mathrm{Op}_{h_n}(a-\overline{a})\psi_n, \psi_n\right\rangle_{L^2(D)}\right|^2 \leq C\int_{S^*D}\left|a_T - \overline{a}\right|^2 \mathrm{d}\mu.$$

Our assertion is verified by citing Lemma 3.1 and taking $T \to \infty$. □

We are now in a position to prove an analogue of Theorem 1.1. Informally, this can be thought of as a "cut off" version of the equidistribution theorem due to Marklof-Rudnick [MR11]. Using this slight analogue, we will later provide a proof of Theorem 1.1.

Theorem 3.5. *[MR11, Theorem 4] There is a density-one sequence $n_j \to \infty$ such that*
$$\lim_{j\to\infty}\int_D a_0(x)\left|\psi_{n_j}(x)\right|^2 \mathrm{d}x = \frac{1}{m(D)}\int_D a_0(x)\mathrm{d}x.$$

for any $a_0 \in C_c^\infty(D^\circ)$. Here, by a density-one sequence we mean that

$$\lim_{N\to\infty} \frac{\#\{j : n_j \leq N\}}{N} = 1.$$

Proof. By Theorem 1.7, we may pick a sequence of functions $(a_k)_{k=1}^\infty$ in $C_c^\infty(D^\circ)$ that are dense in this space with respect to the uniform norm. We will use a diagonal argument to construct a density-one sequence (n_j) such that

$$\lim_{j\to\infty} \int_D a_k(x) \left|\psi_{n_j}(x)\right|^2 \mathrm{d}x = \frac{1}{m(D)} \int_D a_k(x)\mathrm{d}x. \tag{3.5}$$

for every $k \in \mathbb{N}$. Then, we will show that the more general result follows by density.

Before moving further, we establish some notation. Given $r \in \mathbb{N}$, let

$$N_r := \#\left\{j : \lambda_j \in [2^r, 2^{r+1})\right\}.$$

Then, given $s \in \mathbb{N}$ we set

$$\varepsilon_{r,s} := \max_{k\leq s} \left(\frac{1}{N_r} \sum_{\lambda_n \in [2^r, 2^{r+1})} \left| \left\langle \mathrm{Op}_{h_n}(a_k)\psi_n, \psi_n \right\rangle_{L^2(D)} - \overline{a}_k \right|^2 \right).$$

By Weyl's law and Lemma 3.3, the above tends to 0 as $r \to \infty$. In particular, we may construct a strictly increasing sequence $(r_s)_{s\in\mathbb{N}}$ such that for each $s \in \mathbb{N}$

$$\varepsilon_{r,s} < 2^{-4s}$$

for all $r \geq r_s$. Especially,

$$2^{2s} s \varepsilon_{r,s} \leq 2^{3s} \varepsilon_{r,s} < 2^{-s}$$

for each $r \geq r_s$. Using this sequence, we partition the natural numbers, removing unwanted terms from each set in the partition. More precisely, we consider the set

$$J_s = \left\{ n : \lambda_n \in [2^{r_s}, 2^{r_{s+1}}) \, , \, \max_{k \leq s} \left| \left\langle \mathrm{Op}_{h_n}(a_k)\psi_n, \psi_n \right\rangle_{L^2(D)} - \overline{a}_k \right| < 2^{-s} \right\}.$$

We claim that for each $s \in \mathbb{N}$

$$\frac{\# J_s}{\# \left\{ n : \lambda_n \in [2^{r_s}, 2^{r_{s+1}}) \right\}} \geq 1 - 2^{-s}.$$

To see this, we further partition our sets;

$$\left\{ n : \lambda_n \in [2^{r_s}, 2^{r_{s+1}}) \right\} = \bigcup_{r_s \leq r < r_{s+1}} \left\{ n : \lambda_n \in \left[2^r, 2^{r+1}\right) \right\}.$$

Applying a similar rule to J_s, it follows from Lemma 1.3 that[1]

$$\begin{aligned} \frac{\# J_s}{\# \left\{ n : \lambda_n \in [2^{r_s}, 2^{r_{s+1}}) \right\}} &\geq \min_{r_s \leq r < r_{s+1}} \frac{\# \left(J_s \cap \left\{ n : \lambda_n \in [2^r, 2^{r+1}) \right\} \right)}{\# \left\{ n : \lambda_n \in [2^r, 2^{r+1}) \right\}} \\ &= \min_{r_s \leq r < r_{s+1}} \frac{\# \left(J_s \cap \left\{ n : \lambda_n \in [2^r, 2^{r+1}) \right\} \right)}{N_r} \end{aligned} \tag{3.6}$$

On the other hand, given $r \geq r_s$ there holds

$$\frac{\# \left(J_s \cap \left\{ n : \lambda_n \in [2^r, 2^{r+1}) \right\} \right)}{N_r} = 1 - \frac{\# \left(\left\{ n : \lambda_n \in [2^r, 2^{r+1}) \right\} \setminus J_s \right)}{N_r}. \tag{3.7}$$

In order to obtain a bound on the last term above, will use Chebyshev's inequality. To this end set

$$\Omega = \left\{ n : \lambda_n \in \left[2^r, 2^{r+1}\right) \right\}$$

and associate the uniform distribution $\mathbb{P}$. We then define the random variable

[1]Note that if $\# \left\{ n : \lambda_n \in \left[2^r, 2^{r+1}\right) \right\} = 0$, then $\# \left(J_s \cap \left\{ n : \lambda_n \in \left[2^r, 2^{r+1}\right) \right\} \right) = 0$. Hence, both terms may be ignored in this case and Lemma 1.3 applies.

$X : \Omega \to [0, \infty)$ by

$$X(n) = \max_{k \le s} \left| \left\langle \mathrm{Op}_{h_n}(a_k)\psi_n, \psi_n \right\rangle_{L^2(D)} - \overline{a}_k \right|.$$

Observe that

$$\mathbb{P}\left[X \ge 2^{-s}\right] = \frac{\#\left(\{n : \lambda_n \in [2^r, 2^{r+1})\} \setminus J_s\right)}{N_r}.$$

On the other hand, by Chebyshev's inequality,

$$\begin{aligned}
\mathbb{P}\left[X \ge 2^{-s}\right] &= \mathbb{P}\left[|X|^2 \ge 2^{-2s}\right] \\
&\le 2^{2s} \int_\Omega |X|^2 \, \mathrm{d}\mathbb{P} \\
&= 2^{2s} \frac{1}{N_r} \sum_{\lambda_n \in [2^r, 2^{r+1})} \max_{k \le s} \left| \left\langle \mathrm{Op}_{h_n}(a_k)\psi_n, \psi_n \right\rangle_{L^2(D)} - \overline{a}_k \right|^2 \\
&\le 2^{2s} \frac{1}{N_r} \sum_{\lambda_n \in [2^r, 2^{r+1})} \sum_{k=1}^{s} \left| \left\langle \mathrm{Op}_{h_n}(a_k)\psi_n, \psi_n \right\rangle_{L^2(D)} - \overline{a}_k \right|^2 \\
&= 2^{2s} \sum_{k=1}^{s} \left(\frac{1}{N_r} \sum_{\lambda_n \in [2^r, 2^{r+1})} \left| \left\langle \mathrm{Op}_{h_n}(a_k)\psi_n, \psi_n \right\rangle_{L^2(D)} - \overline{a}_k \right|^2 \right) \\
&\le 2^{2s} s \max_{k \le s} \left(\frac{1}{N_r} \sum_{\lambda_n \in [2^r, 2^{r+1})} \left| \left\langle \mathrm{Op}_{h_n}(a_k)\psi_n, \psi_n \right\rangle_{L^2(D)} - \overline{a}_k \right|^2 \right) \\
&= 2^{2s} s \varepsilon_{s,r} < 2^{-s}.
\end{aligned}$$

Returning to (3.7),

$$\begin{aligned}
\frac{\#\left(J_s \cap \{n : \lambda_n \in [2^r, 2^{r+1})\}\right)}{N_r} &= 1 - \frac{\#\left(\{n : \lambda_n \in [2^r, 2^{r+1})\} \setminus J_s\right)}{N_r} \\
&= 1 - \mathbb{P}\left[X \ge 2^{-s}\right] \ge 1 - 2^{-s}.
\end{aligned}$$

Since this holds for all $r \geq r_s$, applying this bound to equation (3.6) yields

$$\begin{aligned}\frac{\# J_s}{\#\left\{n : \lambda_n \in [2^{r_s}, 2^{r_{s+1}})\right\}} &\geq \min_{r_s \leq r < r_{s+1}} \frac{\#\left(J_s \cap \{n : \lambda_n \in [2^r, 2^{r+1})\}\right)}{N_r} \\ &\geq 1 - 2^{-s}.\end{aligned}$$

Using this last inequality, we can show that an increasing sequence (n_j) satisfying

$$\{n_j : j \in \mathbb{N}\} = \bigcup_{s \in \mathbb{N}} J_s$$

must have density-one. Indeed, using Lemma 1.4 we see that

$$\begin{aligned}\frac{\#\left\{\bigcup_{s=1}^{S} J_s\right\}}{\#\left\{n : \lambda_n < 2^{r_{S+1}}\right\}} &= \frac{\sum_{s=1}^{S} \# J_s}{\sum_{s=1}^{S} \#\left\{n : \lambda_n \in [2^{r_s}, 2^{r_{s+1}})\right\}} \\ &\geq \frac{\sum_{s=1}^{S} (1 - 2^{-s}) \#\left\{n : \lambda_n \in [2^{r_s}, 2^{r_{s+1}})\right\}}{\sum_{s=1}^{S} \#\left\{n : \lambda_n \in [2^{r_s}, 2^{r_{s+1}})\right\}} \\ &= 1 - \frac{\sum_{s=1}^{S} 2^{-s} \#\left\{n : \lambda_n \in [2^{r_s}, 2^{r_{s+1}})\right\}}{\sum_{s=1}^{S} \#\left\{n : \lambda_n \in [2^{r_s}, 2^{r_{s+1}})\right\}} \\ &\xrightarrow{S \to \infty} 1.\end{aligned}$$

Consequently, (n_j) is a density-one sequence.

We now show that this sequence satisfies equation (3.5). Indeed, one has by construction that

$$\left| \left\langle \mathrm{Op}_{h_{n_j}}(a_k) \psi_{n_j}, \psi_{n_j} \right\rangle_{L^2(D)} - \bar{a}_k \right| \to 0$$

for any $k \in \mathbb{N}$. Finally, pick an arbitrary function $a \in C_c^\infty(D^\circ)$ and let $\varepsilon > 0$ be given. We may find $k \in \mathbb{N}$ such that

$$\sup_{x \in D^\circ} |a(x) - a_k(x)| < \varepsilon.$$

It follows that

$$
\begin{aligned}
&\limsup_{j\to\infty}\left|\int_D a(x)\left|\psi_{n_j}\right|^2 \mathrm{d}x - \frac{1}{m(D)}\int_D a(x)\mathrm{d}x\right| \\
&\le \limsup_{j\to\infty}\left(2\varepsilon + \left|\int_D a_k(x)\left|\psi_{n_j}\right|^2 \mathrm{d}x - \frac{1}{m(D)}\int_D a_k(x)\mathrm{d}x\right|\right) \\
&= 2\varepsilon.
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, our result follows. □

Theorem 1.1, which we recall below, is now finally within reach.

Theorem 1.1. *[MR11] Let D be a rational polygon and fix an orthonormal basis $(\varphi_n)_{n=1}^{\infty}$ of the Dirichlet Laplacian on D. Then, there exists a sequence of natural numbers (n_j) such that*

$$\lim_{j\to\infty}\int_A \left|\varphi_{n_j}(x)\right|^2 \mathrm{d}x = \frac{\mathrm{area}(A)}{\mathrm{area}(D)} \tag{1.1}$$

for all measurable sets $A \subseteq D$ with boundary ∂A having Lebesgue measure 0. Furthermore, we have

$$\lim_{N\to\infty}\frac{\#\{j : n_j \le N\}}{N} = 1. \tag{1.2}$$

Given Theorem 3.5, the proof is now a straightforward density argument. The idea behind the proof is to sharpen the conclusions of Theorem 3.5 by approximating the indicator function $\mathbb{1}_A$ of a set A by a carefully chosen sequence of functions $(a_j) \subseteq C_c^{\infty}(D^{\circ})$. Repeating the same argument for $D \setminus A$, we thereby obtain two inequalities that, when combined, establish the theorem.

Proof of Theorem 1.1. We will prove our assertion for the density-one sequence (n_j) obtained in Theorem 3.5. Consider an arbitrary measurable set $A \subseteq D$ with boundary of measure 0. Without loss of generality, we may suppose that A is closed. We construct a sequence $(a_k) \subseteq C_c^\infty(D^\circ)$ approximating $\mathbb{1}_A$ in $L^1(D)$. Furthermore, we may suppose without loss of generality

$$0 \leq a_k \leq 1 \quad \text{and} \quad a_k \leq a_{k+1} \leq \mathbb{1}_A$$

for each $k \in \mathbb{N}$. Note that such a sequence is obtained by multiplying $\mathbb{1}_A$ by a cut-off function then mollifying. For any $k \in \mathbb{N}$,

$$\begin{aligned}
\liminf_{j\to\infty} \int_A a_j(x) \left|\psi_{n_j}(x)\right|^2 \mathrm{d}x &\geq \liminf_{j\to\infty} \int_A a_k(x) \left|\psi_{n_j}(x)\right|^2 \mathrm{d}x \\
&= \lim_{j\to\infty} \int_A a_k(x) \left|\psi_{n_j}(x)\right|^2 \mathrm{d}x \\
&= \frac{1}{m(D)} \int_D a_k(x) \mathrm{d}x.
\end{aligned}$$

Letting $k \nearrow \infty$ yields

$$\liminf_{j\to\infty} \int_A a_j(x) \left|\psi_{n_j}(x)\right|^2 \mathrm{d}x \geq \frac{m(A)}{m(D)}.$$

Combining our results, we conclude that

$$\liminf_{j\to\infty} \int_A \left|\psi_{n_k}(x)\right|^2 \mathrm{d}x \geq \liminf_{j\to\infty} \int_A a_j(x) \left|\psi_{n_j}(x)\right|^2 \mathrm{d}x \geq \frac{m(A)}{m(D)}$$

So far, we have shown that for any measurable set $A \subseteq D$ with boundary of measure 0, there holds

$$\liminf_{j\to\infty} \int_A \left|\psi_{n_k}(x)\right|^2 \mathrm{d}x \geq \frac{m(A)}{m(D)}.$$

Since $D \setminus A$ is also a measurable subset of D with boundary of measure 0,

$$\begin{aligned}\limsup_{j\to\infty} \int_A \left|\psi_{n_j}(x)\right|^2 \mathrm{d}x &= \limsup_{j\to\infty} \left(\int_D \left|\psi_{n_j}(x)\right|^2 \mathrm{d}x - \int_{D\setminus A} \left|\psi_{n_j}(x)\right|^2 \right) \\ &= 1 - \liminf_{j\to\infty} \int_{D\setminus A} \left|\psi_{n_j}(x)\right|^2 \mathrm{d}x \\ &\leq 1 - \frac{m(D \setminus A)}{m(D)} = \frac{m(A)}{m(D)}.\end{aligned}$$

Thus, we also have the converse inequality which concludes out proof. □

Remark 4. Recall that Theorem 1.1 applies only to measurable sets $A \subseteq D$ having a boundary of measure 0. Our proof actually illustrates why this last condition is necessary. More precisely, we have shown that

$$\liminf_{j\to\infty} \int_{\bar{A}} |\psi_{n_k}(x)|^2 \mathrm{d}x \geq \frac{m\left(\bar{A}\right)}{m(D)}$$

By the same argument, the above inequality remains true on the closure of $D \setminus A$. That is,

$$\liminf_{j\to\infty} \int_{\overline{D\setminus A}} |\psi_{n_k}(x)|^2 \mathrm{d}x \geq \frac{m\left(\overline{D \setminus A}\right)}{m(D)}$$

Finally, using the ∂A has measure 0, we see that A and $D \setminus A$ differ from their respective closures by a set of measure 0. Hence, combining these last two inequalities yields the statement.

4 The Equilateral Triangle

Throughout the remainder of this thesis, we will denote by T an equilateral triangle having sides of length 1 embedded in the $\mathbb{R}^2$-plane (see Figure 2). We note that, following convention from an earlier section, T includes it's

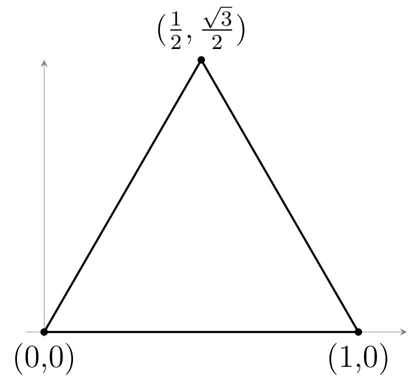


Figure 2: Equilateral Triangle T.

boundary. In this section, we ask whether it is possible to find an *explicit* countable orthonormal basis of $L^2(T)$ consisting purely of eigenfunctions on T satisfying either the Dirichlet or Neumann boundary conditions.

We now take a moment to outline the argument we will employ, which largely follows the approach used in Pinsky [Pin80]. Namely, we roughly describe how we will obtain an orthonormal basis of $L^2(T)$ consisting of Dirichlet (or Neumann) eigenfunctions of the Laplacian. Consider an eigenfunction f of the Dirichlet (resp. Neumann) Laplacian on the triangle T. By a reflection argument, we can extend f to be an eigenfunction of the Dirichlet Laplacian (resp. Neumann) on a parallelogram P.

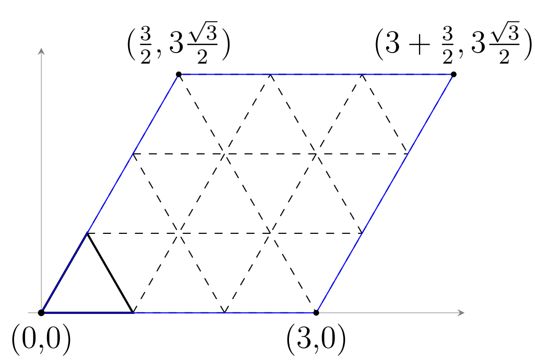


Figure 3: Parallelogram P.

It will be most convenient to also consider a parallelogram that contains 18 copies of the triangle T. In more precise terms, we want to consider a parallelogram P having end points $(0,0),(3,0),(3+\frac{3}{2},3\frac{\sqrt{3}}{2})$ and $(\frac{3}{2},3\frac{\sqrt{3}}{2})$; see the Figure 3 for more details.

Then, appealing to a now standard result, see [BGM71, page 148], we can write f as a series whose terms are of the form

$$\exp\left[\frac{2\pi i}{3}\left(\mu x+\frac{2\nu-\mu}{\sqrt{3}}y\right)\right] \tag{4.1}$$

where the indices (μ,ν) range over $\mathbb{Z}\times\mathbb{Z}$. Indeed, the above collection of Laplace eigenfunctions is known to be an orthogonal basis of $L^2(P)$. Now, observing patterns in the coefficients and grouping terms accordingly, we show that f can in fact be written as a linear combination of functions satisfying the Dirichlet (resp. Neumann) condition on the original triangle T. Since f was an arbitrary Dirichlet (resp. Neumann) eigenfunction on T, and these form a Schauder basis of $L^2(T)$, the functions obtained through this process must in turn form a basis $L^2(T)$ as well. However, we still want an orthonormal basis of $L^2(T)$. Luckily, given our convenient choice of parallelogram, it will turn out that these functions are already orthogonal.

We now formally state the results that we aim to establish within this section. They are completely analogous, one applying to Dirichlet boundary conditions and the other to Neumann conditions.

Theorem 4.1. *An orthonormal basis of Dirichlet eigenfunctions for $L^2(T)$ is given by*

$$\varphi_{m,n}(x,y)=3^{-1/4}\sqrt{\frac{2}{3}}\begin{pmatrix} e^{\frac{2\pi i}{3}\left((n-m)x+\sqrt{3}(m+n)y\right)}-e^{\frac{2\pi i}{3}\left((n-m)x-\sqrt{3}(m+n)y\right)} \\ +e^{\frac{2\pi i}{3}\left(-(2n+m)x-\sqrt{3}my\right)}-e^{\frac{2\pi i}{3}\left(-(2n+m)x+\sqrt{3}my\right)} \\ +e^{\frac{2\pi i}{3}\left((2m+n)x-\sqrt{3}ny\right)}-e^{\frac{2\pi i}{3}\left((2m+n)x+\sqrt{3}ny\right)} \end{pmatrix}$$

where (m,n) range over $\mathbb{N}\times\mathbb{N}$. The eigenvalue corresponding to $\varphi_{m,n}$ is

$$\frac{16\pi^2}{9}\left(m^2+mn+n^2\right).$$

Next we state the analogous Neumann result:

Theorem 4.2. *An orthonormal basis of Neumann eigenfunctions for $L^2(T)$ is given by*

$$\psi_{m,n}(x,y)=3^{-1/4}\sqrt{\frac{2}{3}}\begin{pmatrix} e^{\frac{2\pi i}{3}\left((n-m)x+\sqrt{3}(m+n)y\right)}+e^{\frac{2\pi i}{3}\left((n-m)x-\sqrt{3}(m+n)y\right)} \\ +e^{\frac{2\pi i}{3}\left(-(2n+m)x-\sqrt{3}my\right)}+e^{\frac{2\pi i}{3}\left(-(2n+m)x+\sqrt{3}my\right)} \\ +e^{\frac{2\pi i}{3}\left((2m+n)x-\sqrt{3}ny\right)}+e^{\frac{2\pi i}{3}\left((2m+n)x+\sqrt{3}ny\right)}\end{pmatrix}$$

where (m,n) range over $\mathbb{N}_0\times\mathbb{N}_0$. The eigenvalue corresponding to $\psi_{m,n}$ is

$$\frac{16\pi^2}{9}\left(m^2+mn+n^2\right).$$

4.1 Dirichlet Eigenfunctions

In this section, we establish the Dirichlet case of the previous two theorems, i.e. we give the proof of Theorem 4.1. Following the outline above, we begin with an arbitrary Dirichlet eigenfunction f on T. That is, $f\in C^2(\mathring{T})\cap C^0(T)$ is a solution to

$$\begin{cases}-\Delta f=\lambda f & \text{in } T\\ f=0 & \text{on } \partial T\end{cases}$$

for some $\lambda\geq 0$. We may then reflect f along each side of the triangle T (see [DL55]). In more precise terms, given a point $(x,y)\in T$, if (x',y') is the reflection of (x,y) along a side of the triangle, then we define

$$f(x',y')=-f(x,y).$$

Repeating this reflection process for each side of the triangle, it is easily verified that f can be extended to a smooth function on the entire plane. By construction, our function satisfies the equation

$$-\Delta f = \lambda f$$

on all in $\mathbb{R}^2$. Moreover,

$$f \circ R_i = -f \quad \text{on } \mathbb{R}^2 \tag{4.2}$$

for $i = 1, 2, 3$ where the reflection operators about each side of the equilateral triangle T are given by

$$\begin{cases} R_1(x,y) = (x,-y) \\ R_2(x,y) = \frac{1}{2}(-x+\sqrt{3}y, \sqrt{3}x+y) \\ R_3(x,y) = \frac{1}{2}(3-x-\sqrt{3}y, \sqrt{3}-\sqrt{3}x+y). \end{cases}$$

Since the collection of functions in equation (4.1) form an orthogonal basis of $L^2(P)$, the restriction of f to the parallelogram P can be expressed as a series of the form

$$f = \sum_{(\mu,\nu)} C_{\mu,\nu} \exp\left[\frac{2\pi i}{3}\left(\mu x + \frac{2\nu-\mu}{\sqrt{3}} y\right)\right]. \tag{4.3}$$

We now make use of equation (4.2) in order to establish a pattern in the coefficients $C_{\mu,\nu}$. First, taking $i = 1$, we see that

$$
\begin{aligned}
f \circ R_1(x,y) &= \sum_{(\mu,\nu)} C_{\mu,\nu} \exp\left[\frac{2\pi i}{3}\left(\mu x - \frac{2\nu-\mu}{\sqrt{3}} y\right)\right] \\
&= \sum_{(\mu,\nu)} C_{\mu,\mu-\nu} \exp\left[\frac{2\pi i}{3}\left(\mu x - \frac{2(\mu-\nu)-\mu}{\sqrt{3}} y\right)\right] \\
&= \sum_{(\mu,\nu)} C_{\mu,\mu-\nu} \exp\left[\frac{2\pi i}{3}\left(\mu x + \frac{2\nu-\mu}{\sqrt{3}} y\right)\right] = -f(x,y).
\end{aligned}
$$

Therefore, we have

$$
C_{\mu,\nu} = -C_{\mu,\mu-\nu} \tag{4.4}
$$

for all pairs (μ,ν). Similarly, for $i=2$ we obtain

$$
\begin{aligned}
f \circ R_2(x,y) &= \sum_{(\mu,\nu)} C_{\mu,\nu} \exp\left[\frac{2\pi i}{3}\left(\mu \frac{-x+\sqrt{3}y}{2} + \frac{2\nu-\mu}{\sqrt{3}} \frac{\sqrt{3}x+y}{2}\right)\right] \\
&= \sum_{(\mu,\nu)} C_{\mu,\nu} \exp\left[\frac{2\pi i}{3}\left((\nu-\mu)x + \frac{\mu+\nu}{\sqrt{3}} y\right)\right] \\
&= \sum_{(\mu,\nu)} C_{\nu-\mu,\nu} \exp\left[\frac{2\pi i}{3}\left(\mu x + \frac{2\nu-\mu}{\sqrt{3}} y\right)\right] = -f(x,y)
\end{aligned}
$$

whence

$$
C_{\mu,\nu} = -C_{\nu-\mu,\nu} \tag{4.5}
$$

for all pairs (μ,ν). Finally, we have

$$
\begin{aligned}
f \circ R_3(x,y) &= \sum_{(\mu,\nu)} C_{\mu,\nu} \exp\left[\frac{2\pi i}{3}\left(\mu \frac{3-x-\sqrt{3}y}{2} + \frac{2\nu-\mu}{\sqrt{3}} \frac{\sqrt{3}-\sqrt{3}x+y}{2}\right)\right] \\
&= \sum_{(\mu,\nu)} C_{-\nu,-\mu} \exp\left(-\frac{2\pi i}{3}(\mu+\nu)\right) \exp\left[\frac{2\pi i}{3}\left(\mu x + \frac{2\nu-\mu}{\sqrt{3}} y\right)\right] \\
&= -f(x,y)
\end{aligned}
$$

so

$$C_{\mu,\nu} = -C_{-\nu,-\mu} \exp\left(-\frac{2\pi i}{3}(\mu+\nu)\right).$$

On the other hand, using the information in (4.4) and (4.5) yields

$$C_{\mu,\nu} = -C_{\mu,\mu-\nu} = C_{-\nu,\mu-\nu} = -C_{-\nu,-\mu}.$$

Combining the last two equations we conclude that if $C_{\mu,\nu} \neq 0$ then

$$\exp\left(-\frac{2\pi i}{3}(\mu+\nu)\right) = 1. \tag{4.6}$$

Or, equivalently, $\mu + \nu \equiv 0 \bmod 3$. Finally, notice that if $\mu = 2\nu$ then

$$C_{\mu,\nu} = -C_{\mu,\mu-\nu} = -C_{\mu,\nu}$$

so $C_{\mu,\nu} = 0$. Therefore, if $C_{\mu,\nu} \neq 0$ then $\mu \neq 2\nu$ and, similarly, $\nu \neq 2\mu$. From our work thus far we conclude that if $C_{\mu,\nu} \neq 0$ then[2]

(1) $\mu + \nu \equiv 0 \bmod 3$,

(2) $\mu \neq 2\nu$,

(3) $\nu \neq 2\mu$,

Since $\mu + \nu \equiv 0 \bmod 3$, one may write

$$\mu = n - m \quad \text{and} \quad \nu = 2n + m$$

[2]One may observe that if the pair (μ, ν) satisfies conditions (1), (2) and (3) then so do the pair $(\mu, \mu - \nu)$ and $(\nu - \mu, \nu)$.

for integers m, n. With this notation, we see that the function f must be a linear combination of functions

$$\begin{aligned}\widetilde{\varphi}_{m,n}(x,y) = &\exp\left[\frac{2\pi i}{3}\left((n-m)x+\sqrt{3}(m+n)y\right)\right]-\exp\left[\frac{2\pi i}{3}\left((n-m)x-\sqrt{3}(m+n)y\right)\right]\\ &+\exp\left[\frac{2\pi i}{3}\left(-(2n+m)x-\sqrt{3}my\right)\right]-\exp\left[\frac{2\pi i}{3}\left(-(2n+m)x+\sqrt{3}my\right)\right]\\ &+\exp\left[\frac{2\pi i}{3}\left((2m+n)x-\sqrt{3}ny\right)\right]-\exp\left[\frac{2\pi i}{3}\left((2m+n)x+\sqrt{3}ny\right)\right]\end{aligned}$$

where (m, n) range over $\mathbb{Z}\times\mathbb{Z}$. Since f was an arbitrary Dirichlet eigenfunction on T, the above collection of functions consequently forms a (Schauder) basis of $L^2(T)$.

However, it turns out that this collection contains redundancies. More precisely, restricting ourselves to $(m, n) \in \mathbb{N} \times \mathbb{N}$, this collection remains a (Schauder) basis of $L^2(T)$. To see this, we make 3 important observations:

(1) If $n = 0$ or $m = 0$ then $\widetilde{\varphi}_{m,n} = 0$,

(2) $\widetilde{\varphi}_{m,n} = -\widetilde{\varphi}_{-n,-m}$,

(3) $\widetilde{\varphi}_{m,n} = -\widetilde{\varphi}_{m+n,-n}$.

From the above we see that all pairs $(m, n) \notin \mathbb{N} \times \mathbb{N}$ may be discarded. Indeed, if $m = 0$ or $n = 0$ then by (1) we have $\widetilde{\varphi}_{m,n} = 0$ and there is nothing to show. If m, n are both negative then we may apply (2) to conclude that $\widetilde{\varphi}_{m,n}$ is redundant given $\widetilde{\varphi}_{-n,-m}$. Suppose now that m is positive but n is negative. If $m + n \geq 0$ then (3) shows that $\widetilde{\varphi}_{m,n}$ is redundant. On the other hand, if $m + n < 0$ then (2) and (3) show that $\widetilde{\varphi}_{m,n} = -\widetilde{\varphi}_{-n,-m} = \widetilde{\varphi}_{-(m+n),m}$ so once again $\widetilde{\varphi}_{m,n}$ is seen to be redundant. Finally, we can handle the case where m is negative but n is positive with a similar argument.

Recalling that our goal was to find an orthonormal basis of Dirichlet eigenfunctions for $L^2(T)$, we ask whether $\widetilde{\varphi}_{m,n}$ is a Dirichlet eigenfunction for each index $(m, n) \in \mathbb{N} \times \mathbb{N}$. We first check that the $\widetilde{\varphi}_{m,n}$ indeed solve the

Helmholtz equation. Indeed, by a straightforward computation, we see that

$$-\Delta\widetilde{\varphi}_{m,n} = \frac{16\pi^2}{9}\left(m^2 + mn + n^2\right)\widetilde{\varphi}_{m,n}.$$

Furthermore, along the lines $y = 0, y = \sqrt{3}x$ and $y = \sqrt{3} - \sqrt{3}x$, one can readily verify that $\widetilde{\varphi}_{m,n}(x, y) = 0$. In particular, we have $\widetilde{\varphi}_{m,n} \mid_{\partial T} \equiv 0$ as desired.

In order to establish that $\widetilde{\varphi}_{m,n}$ are indeed eigenfunctions, it remains to check that they are non-trivial. It is easily established that cancellation occurs if and only if $n = 0$, $m = 0$ or $m = -n$. Therefore, these are the only cases where $\widetilde{\varphi}_{m,n} \equiv 0$. Since in all three cases, we cannot have that n and m are positive integers, the functions $\widetilde{\varphi}_{n,m}$ for $(n, m) \in \mathbb{N} \times \mathbb{N}$ are non-vanishing and thus eigenfunctions.

Remark 5. Notice that $\mu = 2\nu$ if and only if $n = -m$ and $\nu = 2\mu$ if and only if $m = 0$. In these cases, we have that $\widetilde{\varphi}_{m,n}$ is trivial.

We now show that the collection of functions $\widetilde{\varphi}_{m,n}$ is orthogonal with respect to the L^2-inner product on T. First, we make one more observation. Our expression for $\widetilde{\varphi}_{m,n}$ is valid on all of $\mathbb{R}^2$. On the entire plane, $\widetilde{\varphi}_{m,n}$ is a Laplace eigenfunction and, in particular, analytic. As mentioned previously, repeated reflection also allows us to extend $\widetilde{\varphi}_{m,n}$ to an eigenfunction on the entire plane. By the Identity Theorem (Theorem 1.8), this extension coincides with our expression for $\widetilde{\varphi}_{m,n}$. In particular, $\widetilde{\varphi}_{m,n}$ satisfies equation (4.2). That is,

$$\widetilde{\varphi}_{m,n} \circ R_i = -\widetilde{\varphi}_{m,n}$$

for each $i = 1, 2, 3$ and every $(m, n) \in \mathbb{N} \times \mathbb{N}$. It follows that for any two pairs $(m, n), (m', n') \in \mathbb{N} \times \mathbb{N}$,

$$\int_T \widetilde{\varphi}_{m,n}\overline{\widetilde{\varphi}_{m',n'}}\mathrm{d}x\mathrm{d}y = \frac{1}{2}\int_0^{\sqrt{3}/2}\int_0^1 \widetilde{\varphi}_{m,n}\overline{\widetilde{\varphi}_{m',n'}}\mathrm{d}x\mathrm{d}y \tag{4.7}$$

Now, the product

$$\widetilde{\varphi}_{m,n}\overline{\widetilde{\varphi}_{m',n'}}.$$

is a linear combination of functions of the form

$$\exp\left(\frac{2\pi i}{3}(3ax+\sqrt{3}by)\right)$$

where $a, b \in \mathbb{Z}$. Furthermore, if the pair (m, n) is distinct from (m', n') then it cannot be the case that both a and b are zero. Finally, observe that if $a = 0$ then b must be even. Therefore,

$$\begin{aligned}
&\int_0^{\sqrt{3}/2}\int_0^1 \exp\left(\frac{2\pi i}{3}(3ax+\sqrt{3}by)\right)\mathrm{d}x\mathrm{d}y \\
&= \begin{cases} \frac{-\sqrt{3}}{4\pi^2 ab}\left(e^{2\pi i a+\pi i b}-e^{2\pi i a}-e^{\pi i b}+1\right) & \text{if } a, b \neq 0 \\ \frac{\sqrt{3}}{2\pi i b}\left(e^{\pi i b}-1\right) & \text{if } a = 0 \\ \frac{\sqrt{3}}{4\pi i}\left(e^{2\pi i a}-1\right) & \text{if } b = 0 \end{cases} \\
&= 0
\end{aligned} \tag{4.8}$$

and we conclude from equation (4.7) that the eigenfunctions are indeed orthogonal in $L^2(T)$.

It remains only to normalize our orthogonal eigenfunctions. We therefore compute the L^2 norm of $\widetilde{\varphi}_{m,n}$. That is, we repeat our last computation but for $(m, n) = (m', n')$. In this case, the product

$$|\widetilde{\varphi}_{m,n}|^2 = \widetilde{\varphi}_{m,n}\overline{\widetilde{\varphi}_{m,n}}.$$

is precisely equal to 6 plus a linear combination of functions of the form

$$\exp\left(\frac{2\pi i}{3}(3ax+\sqrt{3}by)\right)$$

where $a, b \in \mathbb{Z}$ where

(1) at most one of a and b can be zero;

(2) if $a = 0$ then b must be even.

We refer the interested reader to the appendix for an explicit computation of $|\widetilde{\varphi}_{m,n}|^2$. By equation (4.8), the integral of $|\widetilde{\varphi}_{m,n}|^2$ is entirely determined by the constant term. That is,

$$\int_T |\widetilde{\varphi}_{m,n}|^2 = \frac{1}{2}\int_0^{\sqrt{3}/2}\int_0^1 6\,\mathrm{d}x\mathrm{d}y = \frac{3}{2}\sqrt{3}.$$

Or, rather, $\|\widetilde{\varphi}_{m,n}\|_{L^2(T)} = 3^{1/4}\sqrt{3/2}$ and we see that

$$\varphi_{m,n} := 3^{-1/4}\sqrt{\frac{2}{3}}\widetilde{\varphi}_{m,n}.$$

is the desired orthonormal basis of $L^2(T)$ as described in Theorem 4.1.

4.2 Neumann Eigenfunctions

We now treat the Neumann analogue of Theorem 4.1. In particular, we prove Theorem 4.2. Consider an eigenfunction f satisfying Neumann boundary conditions, i.e. $f \in C^2(\mathring{T}) \cap C^1(T)$ and $\partial_\nu f \equiv 0$ on ∂T where ν denotes the outward normal vector field on ∂T.

As in the previous section, we want to write f as a linear combination of "well understood" functions. This is also done by way of a reflection-type argument. However, we will now be using positive reflections to achieve this. More precisely, we can extend f to a Laplace eigenfunction on all of $\mathbb{R}^2$ in a way that satisfies (see [DL55])

$$f \circ R_i = f \quad \text{on } \mathbb{R}^2 \tag{4.9}$$

for $i = 1, 2, 3$. Here, we are once again R_i to denote the reflection operations about each side of the equilateral triangle. More specifically, we have

$$\begin{cases} R_1(x,y) = (x,-y) \\ R_2(x,y) = \frac{1}{2}(-x+\sqrt{3}y, \sqrt{3}x+y) \\ R_3(x,y) = \frac{1}{2}(3-x-\sqrt{3}y, \sqrt{3}-\sqrt{3}x+y). \end{cases}$$

Then, since the functions in (4.1) form an orthogonal basis of $L^2(P)$, the restriction of f to the parallelogram P may be expressed as a series

$$f = \sum_{(\mu,\nu)} C_{\mu,\nu} \exp\left[\frac{2\pi i}{3}\left(\mu x + \frac{2\nu-\mu}{\sqrt{3}}y\right)\right]$$

Now, taking $i=1$ in (4.9) we obtain

$$\begin{aligned} f \circ R_1(x,y) &= \sum_{(\mu,\nu)} C_{\mu,\nu} \exp\left[\frac{2\pi i}{3}\left(\mu x - \frac{2\nu-\mu}{\sqrt{3}}y\right)\right] \\ &= \sum_{(\mu,\nu)} C_{\mu,\mu-\nu} \exp\left[\frac{2\pi i}{3}\left(\mu x + \frac{2\nu-\mu}{\sqrt{3}}y\right)\right] = f(x,y). \end{aligned}$$

Therefore,

$$C_{\mu,\nu} = C_{\mu,\mu-\nu} \tag{4.10}$$

for all pairs (μ,ν). Similarly, using (4.9) with $i=2$ we see that

$$C_{\mu,\nu} = C_{\nu-\mu,\nu} \tag{4.11}$$

for all pairs (μ,ν). Finally, considering $i=3$, we can derive the equation

$$C_{\mu,\nu} = C_{-\nu,-\mu} \exp\left(-\frac{2\pi i}{3}(\mu+\nu)\right).$$

On the other hand, combining equations (4.10) and (4.11), we also have

$$C_{\mu,\nu} = C_{\mu,\mu-\nu} = C_{-\nu,\mu-\nu} = C_{-\nu,-\mu}.$$

Therefore, we conclude that if $C_{\mu,\nu} \neq 0$ then

$$\exp\left(-\frac{2\pi i}{3}(\mu+\nu)\right) = 1. \tag{4.12}$$

Or, equivalently, $\mu + \nu \equiv 0 \bmod 3$. Hence, one may write

$$\mu = n - m \quad \text{and} \quad \nu = 2n + m$$

for integers m, n. It follows that f can be represented as a linear combination of functions taking the form

$$\begin{aligned}
\widetilde{\psi}_{m,n}(x,y) = &\exp\left[\frac{2\pi i}{3}\left((n-m)x + \sqrt{3}(m+n)y\right)\right] + \exp\left[\frac{2\pi i}{3}\left((n-m)x - \sqrt{3}(m+n)y\right)\right] \\
&+ \exp\left[\frac{2\pi i}{3}\left(-(2n+m)x - \sqrt{3}my\right)\right] + \exp\left[\frac{2\pi i}{3}\left(-(2n+m)x + \sqrt{3}my\right)\right] \\
&+ \exp\left[\frac{2\pi i}{3}\left((2m+n)x - \sqrt{3}ny\right)\right] + \exp\left[\frac{2\pi i}{3}\left((2m+n)x + \sqrt{3}ny\right)\right]
\end{aligned}$$

where (m,n) range over $\mathbb{Z} \times \mathbb{Z}$. As f was taken to be an arbitrary eigenfunction of the Laplacian on T with Neumann boundary conditions, it follows that every Neumann eigenfunction is a linear combination of the functions above.

We now assert that it is enough to consider $\widetilde{\psi}_{m,n}$ where (m,n) range over $\mathbb{N}_0 \times \mathbb{N}_0$. As in the previous section, this fact follows from the following observations;

(1) $\widetilde{\psi}_{m,n} = \widetilde{\psi}_{-n,-m}$,

(2) $\widetilde{\psi}_{m,n} = \widetilde{\psi}_{m+n,-n}$.

To summarize, we have obtained a Schauder basis of $L^2(T)$. It remains to verify that these are orthogonal Neumann eigenfunctions of the Laplacian on T. A straightforward calculation gives

$$-\Delta\widetilde{\psi}_{m,n} = \frac{16\pi^2}{9}\left(m^2+mn+n^2\right)\widetilde{\psi}_{m,n} \quad \text{in } \mathbb{R}^2.$$

Therefore, the functions $\widetilde{\psi}_{n,m}$ solve the Helmholtz equation in $\mathbb{R}^2$. It is also clear that $\widetilde{\psi}_{n,m} \not\equiv 0$ for each $n,m \in \mathbb{N}_0$. Hence, $\widetilde{\psi}_{n,m}$ is indeed an eigenfunction for the Laplacian for each $n,m \in \mathbb{N}_0$. Along the line $y=0$, we see that

$$\begin{aligned}
&\partial_\nu\widetilde{\psi}_{m,n}(x,0)\\
&= -\frac{2\pi i}{3}\Bigg(\sqrt{3}(m+n)\exp\left[\frac{2\pi i}{3}(n-m)x\right] - \sqrt{3}(m+n)\exp\left[\frac{2\pi i}{3}(n-m)x\right]\\
&\quad - \sqrt{3}m\exp\left[-\frac{2\pi i}{3}(2n+m)x\right] + \sqrt{3}m\exp\left[-\frac{2\pi i}{3}(2n+m)x\right]\\
&\quad -\sqrt{3}n\exp\left[\frac{2\pi i}{3}(2m+n)x\right] + \sqrt{3}n\exp\left[\frac{2\pi i}{3}(2m+n)x\right]\Bigg) = 0.
\end{aligned}$$

Similarly, we see that $\partial_\nu\widetilde{\psi}_{m,n} = 0$ along the line $y=\sqrt{3}x$ and $y=\sqrt{3}-\sqrt{3}x$. In particular, we have

$$\partial_\nu\widetilde{\psi}_{m,n}\mid_{\partial T}\equiv 0$$

so the eigenfunctions satisfy the Neumann boundary condition.

Finally, carrying out the same computations as in the previous section shows that the collection $\widetilde{\psi}_{n,m}$ over $(m,n) \in \mathbb{N}_0 \times \mathbb{N}_0$ is orthogonal with respect to the $L^2(T)$ inner product. Normalizing this functions, we infer that

$$\psi_{m,n} := 3^{-1/4}\sqrt{\frac{2}{3}}\widetilde{\psi}_{m,n},$$

for $(m,n) \in \mathbb{N}_0 \times \mathbb{N}_0$, is the desired orthonormal basis of Neumann eigenfunctions as in Theorem 4.2.

4.3 Quantum Limits

Having obtained an orthonormal sequence of Dirichlet (resp. Neumann) eigenfunctions, we now describe their asymptotic behaviour. That is, we ask how these concentrate as the eigenvalues tend to infinity. By Theorem 1.1, we know that "most" subsequences of eigenfunctions will equidistribute. In this section, we explicitly find all quantum limits associated to our sequences of eigenfunctions. As far as the author can tell, this result was not previously available in the literature.

Theorem 4.3. *For the orthonormal basis of Dirichlet (resp. Neumann) eigenfunctions on T given by Theorem 4.1 (resp. Theorem 4.2) the possible quantum limits are precisely the weighted Lebesgue measures on T with density given by*

$$f_n := \frac{2}{3\sqrt{3}} \sum_{j=1}^{6} f_{n,j},$$

where $n \in \mathbb{N}_0$ and

$$\begin{aligned}
f_{n,1}(x, y) &= 1 - \exp\left(\frac{2\pi i}{3} n \left(3x + \sqrt{3}y\right)\right), \\
f_{n,2}(x, y) &= 1 - \exp\left(\frac{2\pi i}{3} n \left(3x - \sqrt{3}y\right)\right), \\
f_{n,3}(x, y) &= 1 - \exp\left(\frac{2\pi i}{3} n \left(-3x + \sqrt{3}y\right)\right), \\
f_{n,4}(x, y) &= 1 - \exp\left(\frac{2\pi i}{3} n \left(-3x - \sqrt{3}y\right)\right), \\
f_{n,5}(x, y) &= 1 - \exp\left(\frac{2\pi i}{3} n \left(2\sqrt{3}y\right)\right), \\
f_{n,6}(x, y) &= 1 - \exp\left(\frac{2\pi i}{3} n \left(-2\sqrt{3}y\right)\right).
\end{aligned}$$

Note that for $n = 0$, this is simply the Liouville measure for the triangle T.

Proof. We first handle the Dirichlet case. In order to identify all the possible quantum limits associated to $(\varphi_{m,n})$, let us fix an arbitrary subsequence $\left(\varphi_{m_j,n_j}\right)_{j\in\mathbb{N}}$ whose associated eigenvalues are increasing, such that

$$\lim_{j\to\infty}\int_A \left|\varphi_{m_j,n_j}(x,y)\right|^2 \to \int_A \mathrm{d}\nu$$

for all $A \subseteq T$ with negligible boundary. We must show that there exists $n_0 \in \mathbb{N}_0$ such that

$$\mathrm{d}\nu = f_{n_0}\mathrm{d}x\mathrm{d}y$$

where f_{n_0} is as in the statement of this theorem. We begin with a simple observation: note that the corresponding sequence of eigenvalues is given by

$$\frac{16\pi^2}{9}\left(m_j^2 + m_j n_j + n_j^2\right).$$

Since the above must tend to infinity as $j \to \infty$, one of (m_j) or (n_j) must also diverge to infinity. More precisely, we must be in one of the following three cases as $j \to \infty$;

(1) Both n_j and m_j tend to infinity;

(2) n_j tends to infinity but m_j does not;

(3) m_j tends to infinity but n_j does not.

In the first case, we claim that $\mathrm{d}\nu = f_0\mathrm{d}x\mathrm{d}y$. That is, we show that

$$\int_A \left|\varphi_{m_j,n_j}(x,y)\right|^2 \mathrm{d}x\mathrm{d}y \to \frac{4}{\sqrt{3}}\int_A \mathrm{d}x\mathrm{d}y$$

for all $A \subseteq T$ such that the boundary of A has measure 0. In fact, we are able to show that

$$\int_T \chi(x,y)\left|\varphi_{m_j,n_j}(x,y)\right|^2 \mathrm{d}x\mathrm{d}y \to \frac{4}{\sqrt{3}}\int_T \chi(x,y)\mathrm{d}x\mathrm{d}y$$

for all $\chi \in L^1(T)$. After inspecting our expression (†) for $|\varphi_{m,n}|^2$ (see Appendix), we see that the above follows at once from the Riemann-Lebesgue Lemma 1.9.

Suppose now that $m_j \to \infty$ but n_j does not tend to infinity. Then, taking a subsequence if necessary, we may suppose without loss of generality that $n_j = n_0$ is constant. Once again, after inspecting our expression (†) for $|\varphi_{m,n}|^2$ and applying the Riemann-Lebesgue Lemma 1.9, we conclude that

$$\int_T \chi(x,y) \left|\varphi_{m_j,n_j}(x,y)\right|^2 \mathrm{d}x\mathrm{d}y \to \int_T \chi(x,y) f_{n_0} \mathrm{d}x\mathrm{d}y$$

Similarly, if $m \to m_0$ and $n \to \infty$ then

$$\int_T \chi(x,y) \left|\varphi_{m_j,n_j}(x,y)\right|^2 \mathrm{d}x\mathrm{d}y \to \int_T \chi(x,y) f_{m_0} \mathrm{d}x\mathrm{d}y$$

In the Neumann case, we proceed in an identical manner. That is, we begin with arbitrary subsequence $\left(\psi_{m_j,n_j}\right)_{j\in\mathbb{N}}$ whose associated eigenvalues are increasing, such that

$$\lim_{j\to\infty} \int_T \chi(x,y) \left|\psi_{m_j,n_j}(x,y)\right|^2 \to \int_T \chi \mathrm{d}\nu$$

for all $A \subseteq T$ such that the boundary of A is negligible. Once again, we consider the cases (1), (2) and (3). After inspecting our expression (‡) for $|\psi_{m,n}|^2$ (see Appendix), an application of the Riemann-Lebesgue Lemma 1.9 yields the desired results. □

4.3.1 Frequencies

Having found quantum limits associated to given sequences of eigenvalues, we ask what is known about other quantum limits on T. A Theorem by Jakobson [Jak96, Theorem 1.2] gives us information about the quantum limits on the torus. More specifically, we have the following result.

Theorem 4.4. *[Jak96, Theorem 1.3] The density of every quantum limit on the torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ is a trigonometric polynomial whose frequencies all lie on at most two circles centered at the origin.*

Let us take a moment to recall some definitions and explain the meaning of this result. Suppose we are given an $L^2(\mathbb{T})$-normalized sequence composed of Laplace eigenfunctions with corresponding eigenvalues

$$0 \leq \lambda_1^2 \leq \lambda_2^2 \leq \dots$$

such that $\lambda_j \to \infty$. Recall that if

$$\int_A |u_j(x)|^2 \,\mathrm{d}x \to \int_A \mathrm{d}\nu$$

for every subset $A \subseteq \mathbb{T}^2$ with boundary of measure 0, then ν is a quantum limit on $\mathbb{T}^2$. Then, Theorem 4.4 states that ν is absolutely continuous with respect to the natural measure on $\mathbb{T}^2$ and the density of ν is given by

$$\sum_{\tau \in \mathbb{Z}^2} c_\tau e^{i\langle \tau, x\rangle},$$

where the above is a finite sum. Furthermore, there exist two positive numbers r_1, r_2 such that

$$|\tau| = r_1 \quad \text{or} \quad |\tau| = r_2$$

whenever $c_\tau \neq 0$ and $\tau \neq 0$. These vectors τ are known as the *frequencies*.

On the equilateral triangle, we observe that all the quantum limits we have obtained in Theorem 4.3 satisfy the result of Theorem 4.4. Even more so, for every quantum limit we have found, the frequencies lied on a single circle centered at the origin. We expect the argument put forth by Jakobson in [Jak96] to extend to the equilateral triangle. In fact given that the frequencies of every quantum limit that we have found lied on a single circle, we formulate the following conjecture.

Conjecture. *For every quantum limit on the equilateral triangle associated to an orthonormal basis of Dirichlet (resp. Neumann) eigenfunctions, the frequencies lie on a single circle centered at the origin.*

4.4 The Billiard Map

We seek a function p which is β-invariant where β is the billiard map on the triangle T. That is, such that

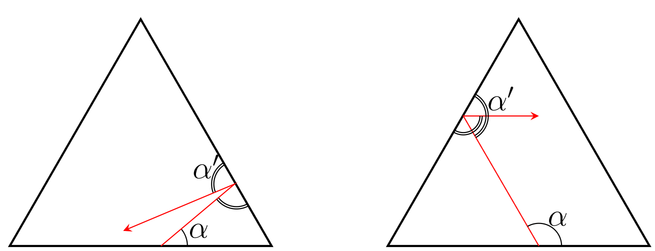
$$p \circ \beta = p.$$

Denote by M the collection of inward pointing unit vectors based on ∂T. That is, every point in M can be described with coordinates $((x_1, x_2), \alpha)$, where $(x_1, x_2) \in \partial T$ denotes the base position of the vector and $\alpha \in (0, \pi)$ denotes the angle between ∂T and our unit vector. Now, suppose

$$\beta((x_1, x_2), \alpha) = ((x_1', x_2'), \alpha')$$

and observe that either

$$\alpha + \alpha' = \frac{2\pi}{3} \quad \text{or} \quad (\pi - \alpha) + (\pi - \alpha') = \frac{2\pi}{3}.$$



In either case,

$$6\alpha' \equiv 6\alpha \mod 2\pi. \tag{4.13}$$

We therefore consider the function $p : M \to \mathbb{C}$ given by

$$((x_1, x_2), \alpha) \mapsto \cos(6\alpha).$$

It is clear from equation (4.13) that

$$p \circ \beta = p,$$

as desired. A point in M can also be represented by pair $((x_1, x_2), (\xi_1, \xi_2))$ where $\xi_1 = \cos\alpha$ and $\xi_2 = \sin\alpha$. We therefore re-define

$$p : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{C}, \quad ((x_1, x_2), (\xi_1, \xi_2)) \mapsto \cos\left(6 \arctan(\xi_2/\xi_1)\right).$$

Since p only depends on it's second variable, we will simply write $p(\xi)$ from here on. Furthermore, if $\xi_1 = 0$ then we define

$$\arctan(\xi_2/\xi_1) = \operatorname{sign}(\xi_2)\frac{\pi}{2}$$

We now consider the pseudodifferential operator $P_h : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ given by

$$\begin{aligned} P_h u := p^W(hD)u &= \mathcal{F}_h^{-1}\left(p(\cdot)(\mathcal{F}_h u)(\cdot)\right) \\ &= \frac{1}{(2\pi h)^2} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} p(\xi)u(y)\mathrm{d}y \right) \mathrm{d}\xi \end{aligned}$$

4.4.1 Joint Eigenfunctions

Recall that we have obtained two distinct orthonormal bases for $L^2(T)$ composed of Laplace eigenfunctions. More specifically, we have the orthonormal basis $(\varphi_{m,n})_{m,n\in\mathbb{N}}$ of Dirichlet eigenfunctions and the orthonormal basis $(\psi_{m,n})_{m,n\in\mathbb{N}_0}$ of Neumann eigenfunctions. We ask if these same eigenfunctions also solve the eigenproblem

$$P_h u = \lambda u.$$

In order to answer this question, we simply fix $m, n \in \mathbb{N}$ and compute

$$P_h \varphi_{m,n}.$$

Since $\varphi_{m,n}$ is a linear combination of functions of the form $u = e^{i(ax_1+bx_2)}$, we first evaluate $P_h u$. This yields

$$\begin{aligned} P_h u = \mathcal{F}_h^{-1}\left(p(\cdot)(\mathcal{F}_h u)(\cdot)\right) &= \mathcal{F}_h^{-1}\left(p(\cdot)(2\pi h)^2 \delta_{(a,b)}(\cdot)\right) \\ &= \mathcal{F}_h^{-1}\left(p(a,b)(2\pi h)^2 \delta_{(a,b)}(\cdot)\right) \\ &= p(a,b)u. \end{aligned} \tag{4.14}$$

It then follows from linearity that $P_h \varphi_{m,n}$ is precisely the function

$$\begin{aligned} & p\left(\frac{2\pi}{3}(n-m), \frac{2\pi}{3}\sqrt{3}(m+n)\right) \exp\left[\frac{2\pi i}{3}\left((n-m)x + \sqrt{3}(m+n)y\right)\right] \\ & - p\left(\frac{2\pi}{3}(n-m), -\frac{2\pi}{3}\sqrt{3}(m+n)\right) \exp\left[\frac{2\pi i}{3}\left((n-m)x - \sqrt{3}(m+n)y\right)\right] \\ & + p\left(-\frac{2\pi}{3}(2n+m), -\frac{2\pi}{3}\sqrt{3}m\right) \exp\left[\frac{2\pi i}{3}\left(-(2n+m)x - \sqrt{3}my\right)\right] \\ & - p\left(-\frac{2\pi}{3}(2n+m), \frac{2\pi}{3}\sqrt{3}m\right) \exp\left[\frac{2\pi i}{3}\left(-(2n+m)x + \sqrt{3}my\right)\right] \\ & + p\left(\frac{2\pi}{3}(2m+n), -\frac{2\pi}{3}\sqrt{3}n\right) \exp\left[\frac{2\pi i}{3}\left((2m+n)x - \sqrt{3}ny\right)\right] \\ & - p\left(\frac{2\pi}{3}(2m+n), \frac{2\pi}{3}\sqrt{3}n\right) \exp\left[\frac{2\pi i}{3}\left((2m+n)x + \sqrt{3}ny\right)\right] \end{aligned}$$

We claim that the above expression can be reduced to

$$P_h \varphi_{m,n}(x,y) = \lambda \varphi_{m,n}(x,y) \tag{4.15}$$

where

$$\lambda := p\left(\frac{2\pi}{3}(n-m), \frac{2\pi}{3}\sqrt{3}(m+n)\right)$$

It suffices to show that

$$p\left(\frac{2\pi}{3}a, \frac{2\pi}{3}\sqrt{3}b\right) = \lambda \quad \forall (a,b) \in \Lambda \tag{4.16}$$

where

$$\begin{aligned}\Lambda = \{&(n-m, m+n), (n-m, -m-n), (-2n-m, -m), \\ &(-2n-m, m), (2m+n, -n), (2m+n, n)\}.\end{aligned}$$

To see that this is indeed the case, we consider first the case $m = n$. Then, by inspection, we readily conclude that equation (4.15) holds with $\lambda = -1$. We may therefore move on to the more difficult setting where $m \neq n$. In order to simplify our computations, we make a simple observation; for every $\xi_1, \xi_2 \in \mathbb{R}^2$ there holds

$$\begin{aligned}p(\xi_1, \xi_2) = \cos\left(6\arctan(\xi_2/\xi_1)\right) &= \cos\left(-6\arctan(\xi_2/\xi_1)\right) \\ &= \cos\left(6\arctan(-\xi_2/\xi_1)\right) \\ &= p(\xi_1, -\xi_2).\end{aligned}$$

It therefore only remains to establish (4.16) for three different terms in Λ. Since the computations are not particularly informative, we only include one as an example.

$$\begin{aligned}p\left(\frac{2\pi}{3}(n-m), \frac{2\pi}{3}\sqrt{3}(m+n)\right) &= \cos\left(6\left[\arctan\left(\frac{\sqrt{3}(m+n)}{n-m}\right) + \arctan(\sqrt{3})\right]\right) \\ &= \cos\left(6\left[\arctan\left(\frac{\frac{\sqrt{3}(m+n)}{n-m} + \sqrt{3}}{1 - \sqrt{3}\frac{\sqrt{3}(m+n)}{n-m}}\right)\right]\right) \\ &= \cos\left(6\left[\arctan\left(-\frac{\sqrt{3}n}{2m+n}\right)\right]\right) \\ &= p\left(\frac{2\pi}{3}(2m+n), -\frac{2\pi}{3}\sqrt{3}n\right).\end{aligned}$$

Having established (4.16), we conclude that $\varphi_{m,n}$ solves the eigenproblem (4.15) for every $n, m \in \mathbb{N}$. That is, the collection $(\varphi_{m,n})$ is composed of joint eigenfunctions for both the Laplacian and the operator P_h. Similarly we see that for all $m, n \in \mathbb{N}_0$, the collection $(\psi_{m,n})$ consists of joint eigenfunctions for both the Laplacian and the operator P_h.

Remark 6. Not all Laplace eigenfunctions solve the eigenproblem (4.15). In particular, linear combinations of functions in the collection $(\varphi_{m,n})$ or $(\psi_{m,n})$ are not guaranteed guaranteed to solve this equation.

Having seen that P_h and $-\Delta$ share a basis of eigenfunctions, we also note that these operators commute. To see this, we first note that $-\Delta$ is a pseudodifferential operator. More specifically, we have

$$-\Delta u(x) = \mathcal{F}_h^{-1} \left(|\xi|^2 \mathcal{F}_h u(\xi) \right)(x).$$

We note that the above formula is valid for all $u \in L^2(D)$. Therefore, given $u \in L^2(D)$, we see that

$$\left[P_h \circ (-\Delta)\right] u = \mathcal{F}_h^{-1} \left(p(\xi) \, |\xi|^2 \mathcal{F}_h u(\xi) \right)(x) = \left[(-\Delta) \circ P_h\right] u.$$

Hence, these operators indeed commute. In general, quantizations of symbols depending only on the momentum commute.

5 Conclusion

Thus far, we have provided an L^2-complete collection of Laplace eigenfunctions (with either Dirichlet or Neumann boundary conditions) on the equilateral triangle and have also completely determined their associated quantum limits. Similar results are known to hold on the rectangle, but such questions remain open when working in more general classes of rational polygons (or, in domains with less symmetries). However, we expect the problem of finding an explicit form for a collection of Laplace eigenfunctions (in both Dirichlet and Neumann settings) to become difficult in these cases.

The classification theorem (see [McC08]) states that the only polygonal domains for which there exists an L^2-complete family consisting of trigonometric eigenfunctions are the rectangle, the square, the isosceles right triangle, the equilateral triangle, and the hemiequilateral triangle. A complete set of eigenfunctions is well known on the rectangle and, in this document, we have provided such a family for the equilateral triangle. As mentioned in [McC08], the eigenfunctions of the isosceles right triangle and the hemiequilateral triangle are a subset of those obtained for the square and the equilateral triangle, respectively. We therefore expect new methods to be needed if one is to extend our result to polygonal domains other than the ones we have listed. In light of this and the arguments used in Theorem 4.3, it reasons that the problem of fully classifying the quantum limits will also present new challenges when working outside of these "nice" domains.

Finally, let us consider once more Conjecture where we hypothesize that all frequencies of a quantum limit (on the equilateral triangle) lie on a single circle centered at the origin. As mentioned previously, a similar result limiting the frequencies to *two* circles centered at the origin is known to hold on the torus $\mathbb{T}$. It is our hope that in future works we can sharpen this conclusion in the setting of an equilateral triangle.

6 Appendix

The proof of Theorem 4.3 relied on inspecting the expressions for the squares of the Dirichlet eigenfunctions $\varphi_{m,n}$ and the Neumann eigenfunctions $\psi_{m,n}$. For the reader's convenience, we have included these expressions here.

The Dirichlet eigenfunctions are given by

$$\varphi_{m,n} = 3^{-1/4}\sqrt{\frac{2}{3}}\widetilde{\varphi}_{m,n}.$$

where $\widetilde{\varphi}_{m,n}$ are as in §4.1. Therefore,

$$|\varphi_{m,n}|^2 = \frac{2}{3}\sqrt{3}\,|\widetilde{\varphi}_{m,n}|^2$$

and we explicitly compute

$$\begin{aligned}
|\widetilde{\varphi}_{m,n}(x,y)|^2 &= \widetilde{\varphi}_{m,n}(x,y)\overline{\widetilde{\varphi}_{m,n}}(x,y) \qquad (\dagger)\\
&= 6 - \exp\left[\frac{2\pi i}{3}\left(2\sqrt{3}(m+n)y\right)\right]\\
&+ \exp\left[\frac{2\pi i}{3}\left(3nx+\sqrt{3}(2m+n)y\right)\right]\\
&- \exp\left[\frac{2\pi i}{3}\left(3nx+\sqrt{3}ny\right)\right]\\
&+ \exp\left[\frac{2\pi i}{3}\left(-3mx+\sqrt{3}(2n+m)y\right)\right]\\
&- \exp\left[\frac{2\pi i}{3}\left(-3mx+\sqrt{3}my\right)\right]\\
&- \exp\left[\frac{2\pi i}{3}\left(-2\sqrt{3}(m+n)y\right)\right]\\
&- \exp\left[\frac{2\pi i}{3}\left(3nx-\sqrt{3}ny\right)\right]\\
&+ \exp\left[\frac{2\pi i}{3}\left(3nx-\sqrt{3}(2m+n)y\right)\right]\\
&- \exp\left[\frac{2\pi i}{3}\left(-3mx-\sqrt{3}my\right)\right]\\
&+ \exp\left[\frac{2\pi i}{3}\left(-3mx-\sqrt{3}(2n+m)y\right)\right]\\
&+ \exp\left[\frac{2\pi i}{3}\left(-3nx-\sqrt{3}(2m+n)y\right)\right]\\
&- \exp\left[\frac{2\pi i}{3}\left(-3nx+\sqrt{3}ny\right)\right]\\
&- \exp\left[-\frac{2\pi i}{3}2\sqrt{3}my\right]
\end{aligned}$$

$$
\begin{aligned}
&+\exp\left[\frac{2\pi i}{3}\left(-3(m+n)x+\sqrt{3}(n-m)y\right)\right]\\
&-\exp\left[\frac{2\pi i}{3}\left(-3(m+n)x-\sqrt{3}(m+n)y\right)\right]\\
&-\exp\left[\frac{2\pi i}{3}\left(-3nx-\sqrt{3}ny\right)\right]\\
&+\exp\left[\frac{2\pi i}{3}\left(-3nx+\sqrt{3}(2m+n)y\right)\right]\\
&-\exp\left[\frac{2\pi i}{3}2\sqrt{3}my\right]\\
&-\exp\left[\frac{2\pi i}{3}\left(-3(m+n)x+\sqrt{3}(m+n)y\right)\right]\\
&+\exp\left[\frac{2\pi i}{3}\left(-3(m+n)x+\sqrt{3}(m-n)y\right)\right]\\
&+\exp\left[\frac{2\pi i}{3}\left(3mx-\sqrt{3}(2n+m)y\right)\right]\\
&-\exp\left[\frac{2\pi i}{3}\left(3mx+\sqrt{3}my\right)\right]\\
&+\exp\left[\frac{2\pi i}{3}\left(3(m+n)x-\sqrt{3}(n-m)y\right)\right]\\
&-\exp\left[\frac{2\pi i}{3}\left(3(m+n)x-\sqrt{3}(m+n)y\right)\right]\\
&-\exp\left[-\frac{2\pi i}{3}2\sqrt{3}ny\right]\\
&-\exp\left[\frac{2\pi i}{3}\left(3mx-\sqrt{3}my\right)\right]\\
&+\exp\left[\frac{2\pi i}{3}\left(3mx+\sqrt{3}(2n+m)y\right)\right]\\
&-\exp\left[\frac{2\pi i}{3}\left(3(m+n)x+\sqrt{3}(m+n)y\right)\right]\\
&+\exp\left[\frac{2\pi i}{3}\left(3(m+n)x-\sqrt{3}(m-n)y\right)\right]\\
&-\exp\left[\frac{2\pi i}{3}2\sqrt{3}ny\right].
\end{aligned}
$$

Similarly, the Neumann eigenfunctions are

$$\psi_{m,n} = 3^{-1/4}\sqrt{\frac{2}{3}}\widetilde{\psi}_{m,n}.$$

where the functions $\widetilde{\psi}_{m,n}$ are as in §4.2. Hence,

$$|\psi_{m,n}|^2 = \frac{2}{3}\sqrt{3}\left|\widetilde{\psi}_{m,n}\right|^2$$

and we explicitly compute

$$\begin{aligned}
\left|\widetilde{\psi}_{m,n}(x,y)\right|^2 &= \widetilde{\psi}_{m,n}(x,y)\overline{\widetilde{\psi}_{m,n}}(x,y) \qquad (\ddagger)\\
&= 6 + \exp\left[\frac{2\pi i}{3}\left(2\sqrt{3}(m+n)y\right)\right]\\
&+ \exp\left[\frac{2\pi i}{3}\left(3nx + \sqrt{3}(2m+n)y\right)\right]\\
&+ \exp\left[\frac{2\pi i}{3}\left(3nx + \sqrt{3}ny\right)\right]\\
&+ \exp\left[\frac{2\pi i}{3}\left(-3mx + \sqrt{3}(2n+m)y\right)\right]\\
&+ \exp\left[\frac{2\pi i}{3}\left(-3mx + \sqrt{3}my\right)\right]\\
&+ \exp\left[\frac{2\pi i}{3}\left(-2\sqrt{3}(m+n)y\right)\right]\\
&+ \exp\left[\frac{2\pi i}{3}\left(3nx - \sqrt{3}ny\right)\right]\\
&+ \exp\left[\frac{2\pi i}{3}\left(3nx - \sqrt{3}(2m+n)y\right)\right]\\
&+ \exp\left[\frac{2\pi i}{3}\left(-3mx - \sqrt{3}my\right)\right]\\
&+ \exp\left[\frac{2\pi i}{3}\left(-3mx - \sqrt{3}(2n+m)y\right)\right]
\end{aligned}$$

$$+\exp\left[\frac{2\pi i}{3}\left(-3nx-\sqrt{3}(2m+n)y\right)\right]$$
$$+\exp\left[\frac{2\pi i}{3}\left(-3nx+\sqrt{3}ny\right)\right]$$
$$+\exp\left[-\frac{2\pi i}{3}2\sqrt{3}my\right]$$
$$+\exp\left[\frac{2\pi i}{3}\left(-3(m+n)x+\sqrt{3}(n-m)y\right)\right]$$
$$+\exp\left[\frac{2\pi i}{3}\left(-3(m+n)x-\sqrt{3}(m+n)y\right)\right]$$
$$+\exp\left[\frac{2\pi i}{3}\left(-3nx-\sqrt{3}ny\right)\right]$$
$$+\exp\left[\frac{2\pi i}{3}\left(-3nx+\sqrt{3}(2m+n)y\right)\right]$$
$$+\exp\left[\frac{2\pi i}{3}2\sqrt{3}my\right]$$
$$+\exp\left[\frac{2\pi i}{3}\left(-3(m+n)x+\sqrt{3}(m+n)y\right)\right]$$
$$+\exp\left[\frac{2\pi i}{3}\left(-3(m+n)x+\sqrt{3}(m-n)y\right)\right]$$
$$+\exp\left[\frac{2\pi i}{3}\left(3mx-\sqrt{3}(2n+m)y\right)\right]$$
$$+\exp\left[\frac{2\pi i}{3}\left(3mx+\sqrt{3}my\right)\right]$$
$$+\exp\left[\frac{2\pi i}{3}\left(3(m+n)x-\sqrt{3}(n-m)y\right)\right]$$
$$+\exp\left[\frac{2\pi i}{3}\left(3(m+n)x-\sqrt{3}(m+n)y\right)\right]$$
$$+\exp\left[-\frac{2\pi i}{3}2\sqrt{3}ny\right]$$
$$+\exp\left[\frac{2\pi i}{3}\left(3mx-\sqrt{3}my\right)\right]$$
$$+\exp\left[\frac{2\pi i}{3}\left(3mx+\sqrt{3}(2n+m)y\right)\right]$$

$$
\begin{aligned}
&+ \exp\left[\frac{2\pi i}{3}\left(3(m+n)x + \sqrt{3}(m+n)y\right)\right] \\
&+ \exp\left[\frac{2\pi i}{3}\left(3(m+n)x - \sqrt{3}(m-n)y\right)\right] \\
&+ \exp\left[\frac{2\pi i}{3} 2\sqrt{3}ny\right].
\end{aligned}
$$

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