# Equilibrium Asset Pricing with Heterogeneous Agents and Interdependent Habit Formation

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# Abstract

We study asset prices in a continuous time, complete market, pure exchange economy with the addition of an interdependent habit formation mechanism, linking individual agents' optimization problems and resulting in a Nash-like equilibrium. We first analyze the model analytically, establishing the existence and uniqueness of optimal portfolio/consumption policies, which are not, in general, the dominant strategies one usually obtains from the standard model. We also demonstrate the existence and uniqueness of equilibrium. The expressions for equilibrium quantities are too complex to interpret directly so we then proceed with a simulation study to explore the properties of the model's predictions. Our preliminary simulation results are broadly consistent with empirical facts and indicate that agent interactions may play a role in understanding those empirical anomalies not yet satisfactorily explained by the standard model.



# Résumé

Nous examinons le prix des biens dans un marché complet à temps continu au sein d'une économie d'échanges purs auquel nous avons ajouté un mécanisme interdépendant qui forme des habitudes. Cela nous aide à faire le lien entre les problèmes d'optimisation des agents individuels, ce qui résulte en un équilibre tel que Nash. Tout d'abord, nous portons un regard analytique sur le modèle, prenant soin d'établir l'existence et le caractère unique des recettes optimales de portefeuille/consommation qui, en général, ne sont pas les stratégies dominantes préconisées par le modèle de base. Nous démontrons également l'existence et le caractère unique de l'équilibre. Comme les équations obtenues pour les quantités d'équilibre sont trop complexes pour être interprétées explicitement, nous procédons à une étude simulée afin d'explorer les propriétés des prédictions du modèle. Les résultats préliminaires de la simulation sont, en règle générale, consistants avec les faits empiriques et indiquent que les interactions entre agents peuvent jouer un rôle dans la compréhension de ces anomalies empiriques qui ne sont pas encore expliquées de façon satisfaisante par le modèle de base.

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# Chapter 1

# Introduction

## 1.1 Motivation

In the development of any body of knowledge aiming to explain observable phenomena, there is a necessary and ongoing tension between theoretical advances and experimental observations: theorists challenge empiricists to test the implications of their theories and empiricists challenge theorists by providing evidence of phenomena not satisfactorily explained by existing models. So it is with financial economics; a number of empirically observed dynamical properties of financial asset returns have not yet been satisfactorily reconciled within the predominant modeling paradigm and have consequently been grouped under the heading of "asset pricing puzzles", indicating their importance in stimulating an intense examination of the standard model. These asset pricing puzzles have generated a huge amount of research activity in financial economics over the last two decades, led perhaps by three of the most notable examples: the "equity premium puzzle" (Mehra & Prescott (1985)), the "risk free rate puzzle" (Weil (1989)), and the "equity volatility puzzle" (Shiller (1981)). To this day, the discrepancy between theory and evidence highlighted by these and other puzzles continues to focus researchers' attention on the deficiencies of the standard model and this allocation of effort shows little sign of diminishing (see Campbell (2003)) and Mehra (2003)). With the increasing availability of financial market data and computing power, the standard model is being pushed more vigorously and in more directions than ever before.

The current, widely accepted paradigm for the valuation of financial assets has its conceptual origins in two fundamental areas of economics. In the first area, Wald's work in the 1930's on consolidating the earlier equilibrium ideas of Walras, was significantly extended by Arrow & Debreu (1954) who rigorously established the existence of general economic equilibrium with complete commodity markets. In Arrow (1953)[1] these results were adapted to the case of general equilibrium for complete financial markets and the ideas of "state prices" and "dynamic completeness" were introduced. In the second area, Lucas (1978), building on the ideas of Muth (1961), further introduced "rational expectations" into the general equilibrium modeling of complete financial markets, representing a spill-over effect from the "rational expectations revolution" occurring in macroeconomics at the time (see Snowdon & Vane (1997)). Through the foundational work of Merton (1971), Merton (1973), Radner (1968), Radner (1972), Radner (1979), Breeden (1979), Harrison & Kreps (1979), Harrison & Pliska (1981), Duffie & Huang (1985), Duffie (1986), Karatzas, Lehoczky & Shreve (1987), Cox & Huang (1989), Karatzas, Lehoczky & Shreve (1990), Karatzas, Lakner, Lehoczky & Shreve (1991), and Cox & Huang (1991), among many others, this model has been carefully formulated and extended to a continuous time setting, with many important results now established.

Over the last decade, many modifications and refinements of this basic complete markets model have been considered to answer various questions, including those surrounding asset pricing puzzles. Some modifications have retained the completeness of markets while others, by introducing market frictions such as (i) constraints on borrowing, liquidity, portfolios, and ability to trade (ii) transaction costs and taxes (iii) limited, costly, or asymmetrically held information and (iv) non-tradeable assets, have led to the study of incomplete markets. Although incomplete markets are certainly more realistic, we do not consider a model of this type for two reasons. First, there is some evidence that the incompleteness of markets does not significantly hinder the ability of agents to reduce risk; optimal risk sharing rules and equilibrium appear little changed (see for example Telmer (1993), Lucas (1994), Heaton & Lucas (1996)). An interesting exception is seen in Basak & Cuoco (1998) who provide a possible resolution to the equity premium puzzle using an incomplete market model

[1] See Arrow (1964) for an updated version in English.

of restricted stock market participation. Secondly, the mathematical problems associated with identifying optimal consumption and portfolio policies in incomplete market models, let alone with establishing the existence and uniqueness of equilibrium, are not satisfactorily understood. Since market completeness has not yet been clearly demonstrated to be irrelevant and also because of this assumption leads to mathematically tractable results, we opt to work in model of complete markets.

A large majority of the models already discussed in the literature, either with complete or incomplete markets, typically assume that all agents are identical or that a representative agent exists. Quite often, the existence and uniqueness of an equilibrium is assumed as well, rather than established. In this thesis, we consider a model with multiple, heterogeneous agents and we enrich price dynamics with a new game-theoretic feature: agents interact in the process of forming their individual consumption habits and in choosing their optimal portfolio/consumption strategies. The mathematical framework we use for studying heterogeneous agents and the existence and uniqueness of equilibrium closely follows that of Karatzas et al. (1990) and Karatzas et al. (1991), whose work was motivated by the ideas in Duffie & Huang (1985). This framework has the distinct advantage that one can prove the existence and uniqueness of an equilibrium as well as a representative agent rather than assuming them as given, and, it permits agents to be highly heterogeneous.

The model studied here leads to interesting questions about the aggregation of preferences into a representative agent, an issue that surrounds any model having multiple agents. In a complete market framework, Constantinides & Duffie (1996) state that all asset pricing implications of a heterogeneous agent economy are "isomorphic" to those of a representative agent economy. Similar aggregation results can be seen in Constantinides (1982) and Rubinstein (1974). It would therefore be interesting to know whether or not the complete market framework is capable of producing a richer, more complex representative agent. Investigations into the structure of a representative agent in incomplete markets conducted by Cuoco & He (1994), Basak & Cuoco (1998), Kraus & Sagi (2000), and Cuoco & He (2001) certainly suggest that this is true in the case of incomplete markets. Our investigations here of interdependent habit formation suggest that even in a complete market model, the representative agent has the potential to link aggregate quantities, such as aggregate consumption, to the behavior of asset prices in new ways.

The potential economic importance of heterogeneity for incomplete market asset pricing models was studied by Den Haan (2001) who showed how the behavior of interest rates can be quite dependent on the number of agents present. In the theoretical development of our model, we take advantage of the flexibility in our framework and allow very general heterogeneity in agents' characteristics. However, for the purposes of our preliminary simulation study, we later confine agents to have power functions for instantaneous utility but within this class of models we allow heterogeneity in the following characteristics: (i) subjective patience factors for discounting future utility can be processes (ii) relative risk aversions can be processes (iii) internal and external habit formation intensities can be processes. Our justification for (i) is the recent result of Gollier & Zeckhauser (2003) that if individuals have heterogeneous constant rates of impatience, the representative agent will not generally use a constant rate to discount future utility. Also, under the assumption of decreasing absolute risk aversion, they show that heterogeneous individual exponential discounting leads to collective hyperbolic discounting, meaning that the representative agent's impatience decreases with time. They calibrate their model to data and show how decision making is inappropriately biased toward the short term using aggregate exponential discounting with a constant rate and stress that "The effect of heterogeneous time preferences should be taken seriously" in the context of public financial policy. We conclude that if constant but heterogeneous rates of patience can yield unusual results for the representative agent then allowing heterogeneous processes for individual patience plus exponential discounting ought to enrich the structure of the representative agent further. In order to justify (ii), we point to the successful calibration exercise of Campbell & Cochrane (1999) in which the key structural component driving their results is a slowly varying aversion to risk. By allowing relative risk aversion to be a process, instantaneous utility becomes state dependent; such generalizations are attracting quite a bit of attention recently: Gordon & St-Amour (2003), Melino & Yang (2003), Danthine, Donaldson, Gianniko & Guirguis (2003), and Kraus & Sagi (2004). To justify (iii) one only has to look at the steadily increasing level of interest in habit formation as a means of explaining the extra variability in state prices: Sundaresan (1989), Constantinides (1990), Abel (1990), Detemple & Zapatero (1991), Detemple & Zapatero (1992), Ingersoll (1992), Heaton (1993), Gali (1994), Heaton (1995), Boldrin, Christiano & Fisher (1997), Chapman (1998), Abel (1999), Campbell & Cochrane (1999), Ljungqvist & Uhlig (1999), Chan & Kogan (2002), Schroder & Skiadas (2002), Wachter (2002), Bodie, Detemple, Otruba & Walter (2003), Polkovnichenko (2003). Early discussions and analyses of habit formation can be traced back to the works of Marshall (1920), Fisher (1930), Duesenberry (1949), Hicks (1965), Pollak (1970), and Ryder & Heal (1973).

On page 1 of their article, Ryder & Heal (1973) introduced a new variable, consumption habit, and commented that it

> ... may be interpreted either as the customary level of consumption, or as the expected level of consumption. Instantaneous satisfaction then depends both on instantaneous consumption and on the customary or expected consumption level. The justification for including such a variable is obvious: it is that *the amount of satisfaction that a man derives from consuming a given bundle of goods depends not only on that bundle, but also on his past consumption and on his general social environment.*

They emphasized again on page 2 that

> In setting up a formal model for the study of individual behavior, it would clearly be desirable to make [habits] *depend not only on the individual's past experience, but also on the consumption habits of those with whom he might compare himself.*

It is interesting to observe that the substantial amount of work on habit formation following this article has been limited in its attempts to explicitly model habits based on the consumption habits of those in an individual's general social environment with whom he might compare himself. At most, one sees in the "Catching Up With The Joneses" models of habit formation, as discussed in Abel (1990), Abel (1999), Campbell & Cochrane (1999), and Chan & Kogan (2002) for example, individual agents are unable to directly influence another agents' consumption and investment choices;

they are able to do so only indirectly through the adjustment of equilibrium market prices. These are models in which habit formation is purely "external" and cannot be influenced by any individual's consumption choices. However, there are hints in the literature that such direct consumption interactions are being considered more carefully by financial economists. Grinblatt, Keloharju & Ikaheimo (2003) conduct an empirical study on automobile sales attempting to identify the magnitude of direct consumption interactions. They conclude that the effect is significant and primarily due to geographical proximity. On the theoretical side, Samuelson (2004) studies a game theoretic model in which agents base their consumption choices on those made by others as a way to compensate for their lack of information about the consumption good and/or other economic factors relevant to decision making.

In this thesis, we take the standard complete markets model and augment it with direct consumption interactions through a form of habit formation. The formulation here was motivated by initially considering the use of mean-field techniques from statistical physics to model the aggregate effect of a large number of pairwise interactions between agents. To put our formulation into context, we outline the standard model in the next section. The section following this discussion of the standard model is then followed by a brief discussion of the habit formation mechanism studied in this thesis. Chapter 2 is a theoretical analysis of our model and Chapter 3 supplements the theoretical work with an initial exploration of the model's behavior.

## 1.2 Standard Model Overview

In this section, we present a brief and heuristic outline of the key structural features in the standard complete markets equilibrium asset pricing model, as formulated in continuous time with what are referred to as "time-additive" or "time-separable" utility functions. In the next section, we then indicate the structural changes that result from the introduction of the form of non-separability in utility discussed here, habit formation, as well as the new ingredient, interaction.

Each agent $a$ derives instantaneous utility $u_t^a(C_t^a)$ from some chosen time $t$ consumption rate $C_t^a$. Over a lifetime $[0, T]$, agent $a$ derives a total utility $\mathcal{U}^a(C^a)$ from his consumption choice $C^a$ given by

$$\mathcal{U}^a(C^a) = \mathbf{E}\left[\int_0^T u_\nu^a(C_\nu^a)d\nu\right]$$

Agent $a$ receives endowment at some exogenously specified time $t$ rate $E_t^a$ and wishes to maximize total utility over affordable consumption rates:

$$\mathbf{E}\left[\int_0^T \xi_\nu C_\nu^a d\nu\right] \leq \mathbf{E}\left[\int_0^T \xi_\nu E_\nu^a d\nu\right] \tag{1.2.1}$$

Here, $\xi$ is the state price density associated with the financial assets, and is an Itô process satisfying

$$d\xi_t = -\xi_t r_t dt - \xi_t \theta_t^{\top} dZ_t \tag{1.2.2}$$

where $r$ is the risk free rate and $\theta$ is the market price of risk. One can show that agent $a$'s optimal consumption rate process is characterized by the first order condition

$$u_t^{a'}(C_t^a) \;\; = \;\; y^a \xi_t, \quad a = 1, \ldots, A \tag{1.2.3}$$

where $y^a > 0$ is a Lagrange multiplier to be determined and differentiation is with respect to the consumption rate variable. Inverting (1.2.3), using $I^a$ to represent the inverse of $u^{a'}$, one obtains the form of optimal consumption in terms of state prices

$$C_t^a = I_t^a(y^a \xi_t), \quad a = 1, \ldots, A \tag{1.2.4}$$

For any given $\xi$, $y^a$ must be chosen so that $C^a$ is affordable and optimal. To achieve affordability and optimality, one can show that it is sufficient to choose $y^a$ so that the budget constraint (1.2.1) for each agent binds:

$$\mathbf{E}\left[\int_0^T \xi_\nu I_\nu^a(y^a \xi_\nu)d\nu\right] \;\; = \;\; \mathbf{E}\left[\int_0^T \xi_\nu E_\nu^a d\nu\right], \quad a = 1, \ldots, A \tag{1.2.5}$$

It is important to note here that for any given $\xi$ the determination of the Lagrange multipliers can be done agent by agent, independently, since $y^a$ enters into one and only one budget constraint, namely, that of agent $a$: there is no direct interdependence in the agents' optimization problems.

To achieve market clearing, the sum of the optimal consumption rates must equal the sum of the endowment rates:

$$E_t = \sum_{a=1}^{A} I_t^a(y^a \xi_t) \tag{1.2.6}$$

where $E_t = \sum_{a=1}^{A} E_t^a$ is the aggregate endowment rate. To determine an equilibrium, which requires affordable, optimal as well as market clearing consumption rates, one is then required to solve the equations (1.2.5) and (1.2.6) simultaneously for both $\xi$ and $\Upsilon = (y^1, \ldots, y^A)$. Defining the function $I_t(\cdot; \Upsilon)$ for each $\Upsilon$ by

$$I_t(y; \Upsilon) = \sum_{a=1}^{A} I_t^a(y^a y)$$

(1.2.6) then becomes

$$E_t = I_t(\xi_t; \Upsilon) \tag{1.2.7}$$

which can be inverted to obtain

$$\xi_t = J_t(E_t; \Upsilon) \tag{1.2.8}$$

where we use $J$ to denote the inverse of $I$:

$$I_t(J_t(x; \Upsilon); \Upsilon) = x \quad \textit{and} \quad J_t(I_t(y; \Upsilon); \Upsilon) = y$$

For each choice of $\Upsilon$, the market clearing state price density is given by (1.2.8). Using (1.2.8) to eliminate $\xi$ from (1.2.5), one can then solve the $A$ equations in the $A$ unknowns in (1.2.5) for the equilibrium multipliers $\Upsilon^*$ that ensure affordability and optimality. Once one has found the equilibrium multipliers $\Upsilon^*$, (1.2.8) gives the equilibrium state price density $\xi^* = J(E; \Upsilon^*)$. Applying Itô's lemma to $\xi^* = J(E; \Upsilon^*)$ and comparing drift and diffusion terms with those of (1.2.2) then enables one to identify the equilibrium $r^*$ and $\theta^*$. Also, from (1.2.4), one can compute optimal consumption.

From (1.2.7), it appears that the properties of equilibrium are primarily determined by the aggregates $E$ and $I$. The idea of the representative agent is that a single agent with utility

$$\mathcal{U}(C) = \mathbf{E}\left[\int_0^T u_\nu(C_\nu) d\nu\right]$$

for some instantaneous utility $u$, will, given $\xi^*$, choose $E$ as his optimal consumption. This $u$ is some sort of aggregation of the preferences of individual agents as described by $u^a$. One then expects the first order condition at $E$ for the representative agent to be $u_t'(E_t; \Upsilon^*) = y\xi_t^*$. The multiplier $y$ can be absorbed into $\xi^*$ for simplicity to obtain

$$\xi_t^* = u_t'(E_t; \Upsilon^*) \tag{1.2.9}$$

Thus, comparing (1.2.7) and (1.2.9), it appears that the aggregate $I$ is indeed the inverse marginal utility of some aggregate instantaneous utility function $u$. It turns out that there does exist such a representative instantaneous utility function $u$; it is given by

$$u_t(x; \Upsilon) \;=\; \sup\left\{\sum_{a=1}^{A} \frac{1}{y^a} u_t^a(x^a) \;:\; \forall a, x^a \in (0,\infty) \textit{ and } x^1 + \cdots + x^A \leq x\right\}$$

and reflects a Pareto optimal sharing of total resources $x$. To see that this does represent all agents in the above sense, set $x^a = I_t^a(y^a J_t(x; \Upsilon))$ and note that

$$\sum_{a=1}^{A} x^a = \sum_{a=1}^{A} I_t^a(y^a J_t(x;\Upsilon)) = I_t(J_t(x;\Upsilon);\Upsilon) = x$$

Now, let $w^1, \ldots, w^A$ be any other choice such that $\sum_{a=1}^{A} w^a \leq x$. Hence, for at least one $a$, we have $w^a \neq x^a$. For all such $a$ we have, by the strict concavity of $u^a$,

$$u_t^a(w^a) < u_t^a(x^a) + (w^a - x^a) u_t^{a'}(x^a)$$

and so, using that $x^a = I_t^a(y^a J_t(x; \Upsilon))$ gives $\frac{1}{y^a} u_t^{a'}(x^a) = J_t(x;\Upsilon)$, we then have

$$\begin{aligned}
\sum_{a=1}^{A} \frac{1}{y^a} u_t^a(w^a) \;&<\; \sum_{a=1}^{A} \frac{1}{y^a} u_t^a(x^a) + \sum_{a=1}^{A} (w^a - x^a) \frac{1}{y^a} u_t^{a'}(x^a) \\
&=\; \sum_{a=1}^{A} \frac{1}{y^a} u_t^a(x^a) + \sum_{a=1}^{A} (w^a - x^a) J_t(x;\Upsilon) \\
&\leq\; \sum_{a=1}^{A} \frac{1}{y^a} u_t^a(x^a)
\end{aligned}$$

showing that the supremum is achieved and is unique. We can therefore write

$$u_t(x;\Upsilon) = \sum_{a=1}^{A} \frac{1}{y^a} u_t^a(x^a) = \sum_{a=1}^{A} \frac{1}{y^a} u_t^a(I_t^a(y^a J_t(x;\Upsilon)))$$

Differentiating $u$, we obtain

$$\begin{aligned} u_t'(x;\Upsilon) &= \sum_{a=1}^{A} \frac{1}{y^a} u_t^{a'}(I_t^a(y^a J_t(x;\Upsilon))) I_t^{a'}(y^a J_t(x;\Upsilon)) y^a J_t'(x;\Upsilon) \\ &= \sum_{a=1}^{A} y^a J_t(x;\Upsilon) I_t^{a'}(y^a J_t(x;\Upsilon)) J_t'(x;\Upsilon) \\ &= J_t(x;\Upsilon) J_t'(x;\Upsilon) \sum_{a=1}^{A} y^a I_t^{a'}(y^a J_t(x;\Upsilon)) \end{aligned}$$

Now, differentiating $I_t(J_t(x;\Upsilon);\Upsilon) = x$ and $I_t(y;\Upsilon) = \sum_{a=1}^{A} I_t^a(y^a y)$ we obtain the relationships

$$I_t'(J_t(x;\Upsilon);\Upsilon) J_t'(x;\Upsilon) = 1 \qquad \text{and} \qquad I_t'(y;\Upsilon) = \sum_{a=1}^{A} y^a I_t^{a'}(y^a y)$$

Thus, the derivative $u'$ takes the form

$$u_t'(x;\Upsilon) = J_t(x;\Upsilon) J_t'(x;\Upsilon) I_t'(J_t(x;\Upsilon);\Upsilon) = J_t(x;\Upsilon)$$

and so

$$(u_t')^{-1}(x;\Upsilon) = (J_t(x;\Upsilon))^{-1} = I_t(x;\Upsilon)$$

Thus, for this representative utility function, equilibrium can be characterized by

$$\xi_t^* = u_t'(E_t;\Upsilon^*)$$

For the specific form $u_t(x;\Upsilon) = e^{-\int_0^t \beta_\nu d\nu} v(x;\Upsilon)$ where $v$ is some time independent function and $\beta$ is a patience process, then Itô's rule can be applied quite easily to obtain $r^*$ and $\theta^*$. Dropping the reference to $\Upsilon^*$ for simplicity, and assuming that aggregate endowment satisfies

$$dE_t = E_t g_t dt + E_t \varrho_t dZ_t$$

we compute

$$\begin{aligned}
d\xi_t^* &= d(u_t'(E_t)) = d(e^{-\int_0^t \beta_{nu} d\nu} v'(E_t)) \\
&= -\beta_t e^{-\int_0^t \beta_\nu d\nu} v'(E_t)dt + e^{-\int_0^t \beta_\nu d\nu} d(v'(E_t)) \\
&= -\beta_t e^{-\int_0^t \beta_\nu d\nu} v'(E_t)dt + e^{-\int_0^t \beta_\nu d\nu}\Big\{v''(E_t)E_t g_t + \frac{1}{2}v'''(E_t)E_t^2\varrho_t^2\Big\}dt \\
&\qquad + e^{-\int_0^t \beta_\nu d\nu}\Big\{v''(E_t)E_t\varrho_t\Big\}dZ_t \\
&= \Big\{-\beta_t u_t'(E_t) + u_t''(E_t)E_t g_t + \frac{1}{2}u_t'''(E_t)E_t^2\varrho_t^2\Big\}dt + \Big\{u_t''(E_t)E_t\varrho_t\Big\}dZ_t
\end{aligned}$$

Comparing the drift and diffusion terms with those of (1.2.2), we then have, using $\xi_t^* = u_t'(E_t)$, the expressions

$$\begin{aligned}
r_t^* &= \beta_t - \frac{1}{u'(E_t)}\Big\{u_t''(E_t)E_t g_t + \frac{1}{2}u_t'''(E_t)E_t^2\varrho_t^2\Big\} \\
\theta^* &= -\frac{E_t u''(E_t)}{u'(E_t)}\varrho_t
\end{aligned}$$

Defining

$$R_t(E_t) = -\frac{E_t u_t''(E_t)}{u'(E_t)} \qquad and \qquad P_t(E_t) = \frac{u_t'''(E_t)E_t^2}{2u_t'(E_t)}$$

as the relative risk aversion and "prudence" processes of the representative agent, we then have

$$\begin{aligned}
\xi_t^* &= u_t'(E_t) \\
r_t^* &= \beta_t + g_t \cdot R_t(E_t) + \varrho_t^2 \cdot P_t(E_t) \\
\theta_t^* &= \varrho_t \cdot R_t(E_t)
\end{aligned}$$

which is a very nice example of the aggregation of microeconomic structure to macroeconomic structure. The macro variables $\xi^*$, $r^*$ and $\theta^*$ are themselves defined in terms of macro variables $(\beta, g, \varrho, E, u)$. All of the necessary microeconomic information is, of course, contained in $R$ and $P$ through their dependence on $u$, which is ultimately an aggregation of individual preferences. Many asset pricing puzzles concern the behavior of the first and second moments of these three quantities.

## 1.3 Standard Model with Interdependent Habits

One general question addressed in this thesis is to what extent can the prior program be extended, with a view toward explaining valuation puzzles. To this end, we consider a generalization of a particular linear "difference" model of habit formation. To our knowledge, the most general theoretical work to date on this model of habit formation can be found in Detemple & Zapatero (1991), Detemple & Zapatero (1992), and Detemple & Karatzas (2003) in which martingale methods are employed, although the models are restricted to a representative agent framework. At this level of model generality, one sees an interesting application of habit formation to optimal portfolio/consumption choices in the context of retirement planning in Bodie et al. (2003).

The key generalization we make is to allow direct agent interaction by having each agent $a$ form consumption habits on a generalized moving average of all other agents' consumption choices, including his own:

$$H_t^a = \int_0^t \sum_{b=1}^{A} \chi_{\nu t}^{ab} C_\nu^b d\nu$$

Agent $a$'s lifetime utility is assumed to take the form

$$\mathcal{U}^a(\mathbf{C}) = \mathbf{E}\left[\int_0^T u_\nu^a(C_\nu^a - H_\nu^a)d\nu\right]$$

where we use the difference habit model $u_t^a(C_t^a - H_t^a)$ and where $\mathbf{C}$ is a vector of consumption choices of all agents; this is required since agent $a$ must consider all agents' consumption choices when forming his own habits. The inclusion of habits introduces time non-separability as the the utility derived from consumption at any time depends on past consumption via habits. We expect that our results hold not only for the difference model of habits but, under assumptions similar to those in Detemple & Zapatero (1991) and Detemple & Zapatero (1992), we expect them to also hold more generally for total utilities

$$\mathcal{U}^a(\mathbf{C}) = \mathbf{E}\left[\int_0^T u_\nu^a(C_\nu^a, H_\nu^a)d\nu\right]$$

We leave this generalization for future work.

We shall see that the first order condition for optimal consumption takes the form

$$u_t^{a'}(C_t^a - H_t^a) - \mathbf{E}_t\left[\int_t^T u_\nu^{a'}(C_\nu^a - H_\nu^a)\left(\frac{\partial H_\nu^a}{\partial C_t^a}\right)d\nu\right] = y^a \xi_t$$

Interestingly, by allowing heterogeneity in the derivative $\frac{\partial H^a_\nu}{\partial C^a_t}$ across agents, several interesting problems arise not present with identical agents or a single representative agent: Defining a process $\gamma^a$ by

$$u^{a'}_t(C^a_t - H^a_t) = y^a \gamma^a_t$$

we then obtain

$$\gamma^a_t - \mathbf{E}_t \left[ \int_t^T \chi^{aa}_{t\nu} \gamma^a_\nu d\nu \right] = \xi_t$$

where $\chi^{aa}$ turns out to be the derivative of future habits with respect to current consumption. We prove in Theorem (2.1.4) that one can solve for $\gamma^a$ uniquely:

$$\gamma^a_t = \varphi^a(\xi)_t$$

for a known functional $\varphi^a$ of $\xi$. As a result, we then invert the first order condition $u^{a'}_t(C^a_t - H^a_t) = y^a\gamma^a_t = y^a\varphi^a(\xi)_t$ to obtain

$$C^a_t = H^a_t + I^a_t(y^a\varphi^a(\xi)_t) = \int_0^t \sum_{b=1}^{A} \chi^{ab}_{\nu t} C^b_\nu d\nu + I^a_t(y^a\varphi^a(\xi)_t)$$

Writing this in vector form we have a vector Volterra equation

$$\mathbf{C}_t = \int_0^t \boldsymbol{\chi}_{\nu t} \mathbf{C}_\nu d\nu + \mathbf{I}_t$$

which has a solution, giving the necessary form for optimal consumption

$$\mathbf{C}_t = \int_0^t \overleftarrow{\boldsymbol{\chi}}_{\nu t} \mathbf{I}_\nu d\nu + \mathbf{I}_t$$

In Proposition (2.6.5), we show that well known martingale methods used to establish the sufficiency of this form of consumption for individual optimality extend to the case of the type of interdependent portfolio/consumption optimization problem we encounter. We explore the requirements for mutual optimality in Propositions (2.7.7), (2.7.12), (2.7.14) and (2.7.15).

To achieve market clearing, we must have

$$E_t = \sum_{a=1}^{A} C^a_t = \sum_{a=1}^{A} I^a_t(y^a\varphi^a(\xi)_t) + \int_0^t \sum_{a,b=1}^{A} \chi^{ab}_\nu I^b_\nu(y^b\varphi^b(\xi)_\nu) d\nu$$

which clearly indicates that the dependence of state prices $\xi$ on aggregate endowment $E$ is significantly more complicated than in the time-separable case as well as in the representative agent case with habits. For instance, in the case that $\chi^{aa}$ is the same for all $a$, we then have a common process $\gamma = \varphi(\xi)$. As a result, the market clearing equation simplifies to

$$E_t = \sum_{a=1}^{A} C_t^a = \sum_{a=1}^{A} I_t^a(y^a \varphi(\xi)_t) + \int_0^t \sum_{a,b=1}^{A} \chi_\nu^{ab} I_\nu^b(y^b \varphi(\xi)_\nu) d\nu$$

which we show, using Theorem (2.1.4), can be solved uniquely for $I_t(\varphi(\xi)_t; \Upsilon) = \sum_{a=1}^{A} I_t^a(y^a \varphi(\xi)_t)$. Due to the invertibility of $I$ and $\varphi$, the state price density $\xi$ can then be solved for uniquely. Even with this simplification, the presence of $\varphi$ makes it unclear if the representative instantaneous utility function still has the form

$$u_t(x; \Upsilon) = \sup \left\{ \sum_{a=1}^{A} \frac{1}{y^a} u_t^a(x^a) : \forall a, x^a \in (0, \infty) \text{ and } x^1 + \cdots + x^A \leq x \right\}$$

as it does in the time-separable case. The general situation in which the $\varphi^a$ vary from agent to agent is even more complex.

It turns out that we cannot clearly establish the existence of a mutually optimal consumption configuration *before* considering the existence and uniqueness of equilibrium, for which mutually optimality is a necessary condition. In Theorem (2.8.17), we extend the existence and uniqueness proof of Karatzas et al. (1991) to include our model of habit interdependencies, thereby obtaining the existence of mutually optimal consumption as a by-product.

The issue of being able to solve the market clearing equation for $\xi$ has only partially been solved in this thesis; more remains to be done. In Proposition (2.8.21), we demonstrate the existence and uniqueness of a solution to the market clearing equation under certain assumptions on $\chi$ which are somewhat restrictive but far more general than any similar form of habits currently in the literature. In Proposition (2.8.22), we weaken the restrictions on $\chi$ and are still able to establish the existence, but unfortunately, not the uniqueness of a solution to the market clearing equation.

Also, the question of what form the representative utility function takes remains an open one. Even when invertibility can be established, it renders a description of the state price density $\xi$ that is far too complex to apply Itô's lemma so as to identify the equilibrium $r^*$ and $\theta^*$. The lack of an explicit inverse and the inability to derive tractable expressions for $r^*$ and $\theta^*$ leads us to a simulation study of the model. From our simulation results, we can see that our model is able to generate a high equity premium at the same time as a low real interest rate, potentially offering a solution to the equity premium and risk-free rate puzzles. One can also see from our results that the standard deviation of the state price density can be made quite high without introducing too much variation in the interest rate, indicating a possible explanation of the equity volatility puzzle. While embarking on a thorough explanation of these puzzles was not our primary aim, we do think it is important to at least demonstrate that our model is broadly consistent with these empirical facts. More interestingly, certain cases of the model exhibit a cyclical behavior in the price of risk; this definitely warrants further study as it may provide an alternative formulation to the highly successful model studied in Campbell & Cochrane (1999). It is hoped that with additional analysis, a fully general statement on invertibility and on representing equilibrium prices via a representative agent will emerge. It is also hoped that with additional simulations and calibrations of the model to data, we will be able to find a specification of the habit kernel $\chi$ that has a sound economic interpretation and that results in model predictions that closely fit observed asset price behavior.

# Chapter 2

# Model Analysis

## 2.1 Preliminaries

We place ourselves in the context of a frictionless, zero net-trade, pure exchange economy consisting of heterogeneous agents, each with zero initial wealth and each receiving endowment in the form of a single representative consumption good once economic activity commences. We take the representative consumption good as the numeraire so that all quantities are expressed in real terms rather than nominal terms. The investment opportunities available in this economy consist of a locally riskless asset and a collection of risky assets. Of course, one possible course of action open to all agents is to ignore all investment opportunities and simply consume their endowment as it arrives, not storing any wealth in the financial assets. However, the presence of these investments enables agents to strategically transfer their endowment wealth across time and states of uncertainty, offering the possibility for increasing utility over lifetime consumption.

Let the number of agents $A \geq 1$ be labeled by $a = 1, \ldots, A$. These agents make their investment and consumption decisions over a finite time horizon $[0, T]$ for some fixed $T \in (0, \infty)$. The underlying model of uncertainty is a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$[1] and we assume there are $N$ fundamental sources of economic uncertainty, modeled by an $\mathbb{R}^N$-valued, standard Wiener process $Z = (Z^1, \ldots, Z^N)^\top$ on $(\Omega, \mathcal{F}, \mathbf{P})$. Each source is indexed by $n = 1, \ldots, N$. All random economic quantities are to be

[1] Completeness is required since indistinguishable processes differ on $\mathbf{P}$-null sets, which need to be included in $\mathcal{F}$ so that one can define the stochastic integral and strong uniqueness of solutions to SDE's (Sections 3.2, 5.2 Karatzas & Shreve (1991)).

defined as non-anticipative functionals of $Z$ so that the temporal resolution of economic uncertainty is therefore given by the natural filtration $\mathbb{F}^Z$ generated by $Z$. We augment $\mathbb{F}^Z$ by the $\mathbf{P}$-null sets of $\mathcal{F}$ to obtain the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$.[2] For each $t$, $\mathcal{F}_t^Z$ and $\mathcal{F}_t$ differ, essentially, by $\mathbf{P}$-null sets and so the resolution of uncertainty continues to be described by $\mathbb{F}$. Since $Z$ is standard, the initial filtration $\mathcal{F}_0$ consists of sets of probability zero or one so that $\mathcal{F}_0$-measurable random variables are almost surely constant and $\mathbb{F}$-adapted processes have almost surely constant initial conditions. We also assume that $\mathcal{F} = \mathcal{F}_T$ so that all economic uncertainty is resolved by time $T$; the observation of $Z_t$ for all $t \in [0, T]$ is sufficient to completely specify the evolution of all economic quantities. All processes to follow are assumed to be $\mathbb{F}$-progressive [3] and all statements involving random quantities are assumed to hold $\mathbf{P}$-almost surely, unless otherwise stated. We use the notation $\|\cdot\|$ to denote the usual Euclidean norm defined by $\|D\| = \sqrt{\operatorname{tr}(D^\top D)}$ where $D$ is a matrix of any dimension. We also use $\boldsymbol{\lambda}$ to denote Lebesgue measure on $\mathbb{R}$.

The representation of local martingales with respect to the augmented Wiener filtration $\mathbb{F}$ is a crucial result in the continuous-time, complete-market setting in which we shall be working so we state the theorem here for reference:

[2] $\mathbb{F}^Z$ is only left continuous (Problem (2.7.1), Karatzas & Shreve (1991)). The augmented filtration $\mathbb{F}$ is right continuous as well (Problem (2.7.6), Karatzas & Shreve (1991)), hence continuous, and $Z$ remains a Wiener process relative to $\mathbb{F}$ (Proposition (2.7.7), Karatzas & Shreve (1991)). The continuity of $\mathbb{F}$ is required for the use of the martingale representation theorem (Theorem (3.4.15), Karatzas & Shreve (1991)).

[3] Let $M$ be a continuous, square integrable martingale. If $t \mapsto \langle M\rangle_t$ is absolutely continuous and $X$ is an $\mathbb{F}$-adapted, measurable process satisfying $\int_0^T X_\nu^2 d\nu < \infty$ then, for each $t \in [0, T]$, $\int_0^t X_\nu dM_\nu$ is defined. However, if $t \mapsto \langle M\rangle_t$ is not absolutely continuous, $X$ must be further restricted to an $\mathbb{F}$-progressive process for $\int_0^t X_s dM_s$ to exist. In using $M = Z$, we have $\langle M\rangle_t = \langle Z\rangle_t = t$, which is absolutely continuous, and so we need only require integrands to be measurable and $\mathbb{F}$-adapted (Sections 3.2 and 3.4, Karatzas & Shreve (1991)). In applying the martingale representation theorem, we obtain $\mathbb{F}$-progressive processes for the integrand in the representation. Since we shall be combining these $\mathbb{F}$-progressive integrands with other processes, we assume that these other processes are $\mathbb{F}$-progressive so that their algebraic combinations are $\mathbb{F}$-progressive as well. In addition, we shall occasionally consider martingale integrators other than $Z$ and confining ourselves to $\mathbb{F}$-progressive integrands frees us from having to show the absolute continuity of the martingale's quadratic variation.

**Theorem 2.1.1 (Martingale Representation Theorem)** *Let $X$ be a cadlag local $\mathbb{F}$-martingale. Then*

*(1) $X$ is continuous.*

*(2) There exists an $\mathbb{F}$-progressive, $\mathbb{R}^N$-valued process $\phi$ such that the following conditions hold:*

*(i)* $\int_0^T \|\phi_\nu\|^2 d\nu < \infty$

*(ii)* $\forall t \in [0,T],\ X_t = X_0 + \int_0^t \phi_\nu^\top dZ_\nu$

*(iii) If $\tilde{\phi}$ is another such process then $\phi = \tilde{\phi}$ holds $\boldsymbol{\lambda} \otimes \mathbf{P}$-a.s.*

**Proof:** See Theorem (3.4.15) and Problem (3.4.16) in Karatzas & Shreve (1991).

■

Also, many of the SDE's we consider fall under the hypotheses of the following theorem:

**Theorem 2.1.2 (SDE Existence & Uniqueness Theorem)** *Suppose that the pair of coefficient functions $a : [0,T] \times \Omega \times \mathbb{R}^M \to \mathbb{R}^M$ and $b : [0,T] \times \Omega \times \mathbb{R}^M \to \mathbb{R}^{M \otimes N}$ satisfy the following conditions:*

*(1) For any $\mathbb{R}^M$-valued, $\mathbb{F}$-progressive process $X$, the processes defined by the compositions $a(t,\omega,X_t(\omega))$ and $b(t,\omega,X_t(\omega))$ are also $\mathbb{F}$-progressive.*

*(2) There exist constants $C_1, C_2$ such that for all $x, y \in \mathbb{R}^M$, $t \in [0,T]$, and $\mathbf{P}$-a.e. $\omega \in \Omega$ we have uniformly linear and Lipshitz bounds:*

$$(i) \qquad \|a(t,\omega,x)\|^2 + \|b(t,\omega,x)\|^2 \le C_1(1 + \|x\|^2)$$

$$(ii) \qquad \|a(t,\omega,x) - a(t,\omega,y)\|^2 + \|b(t,\omega,x) - b(t,\omega,y)\|^2 \le C_2\|x - y\|^2$$

*Then the stochastic differential equation $dX_t = a(t,\cdot,X_t)dt + b(t,\cdot,X_t)dZ_t$ with $X_0 \in \mathbb{R}^M$ has an $\mathbb{F}$-progressive solution that is continuous and unique up to indistinguishability. Moreover, the solution satisfies the bound:*

$$\forall t \in [0,T], \qquad \mathbf{E}\|X_t\|^2 \le C_3(1 + \|X_0\|^2)e^{C_3 t}$$

*where $C_3$ is a constant depending only on $C_1$ and $T$.*

**Proof:** The proofs of Theorems (5.2.5) and (5.2.9) in Karatzas & Shreve (1991) give this result, even though the proofs are formulated only for non-random coefficients $a$ and $b$. See also Gihman & Skorohod (1979), Theorem (3.3), with the appropriate simplifications of the jump terms, or the more general results in Protter (1990). ∎

In order to easily apply Theorem (2.1.2) to the specific class of coefficient functions we usually encounter, we establish a lemma:

**Lemma 2.1.3** *Let $Q$ and $R$ be uniformly bounded, $\mathbb{F}$-progressive processes taking values in $\mathbb{R}^K$ and $\mathbb{R}^L$. Also, let $f : \mathbb{R}^K \times \mathbb{R}^M \to \mathbb{R}^M$ and $g : \mathbb{R}^L \times \mathbb{R}^M \to \mathbb{R}^{M \otimes N}$ be Borel measurable functions satisfying the following properties:*

*(1) $\|f(q,x)\|^2 + \|g(r,x)\|^2 \leq K_1(q,r)(1+\|x\|^2)$ for all $x \in \mathbb{R}^M$, $q \in \mathbb{R}^K$, and $r \in \mathbb{R}^L$ where $K_1 : \mathbb{R}^K \times \mathbb{R}^L \to \mathbb{R}_+$ is a continuous function.*

*(2) $\|f(q,x) - f(q,y)\|^2 + \|g(r,x) - g(r,y)\|^2 \leq K_2(q,r)\|x-y\|^2$ for all $x, y \in \mathbb{R}^M$, $q \in \mathbb{R}^K$, and $r \in \mathbb{R}^L$ where $K_2 : \mathbb{R}^K \times \mathbb{R}^L \to \mathbb{R}_+$ is a continuous function.*

*Then the functions defined by $a(t,\omega,x) = f(Q_t(\omega),x)$ and $b(t,\omega,x) = g(R_t(\omega),x)$ satisfy conditions (1) and (2) of Theorem (2.1.2).*

**Proof:** First,

$$\begin{aligned} \|a(t,\omega,x)\|^2 + \|b(t,w,x)\|^2 &= \|f(Q_t(\omega),x)\|^2 + \|g(R_t(w),x)\|^2 \\ &\leq K_1(Q_t(\omega),R_t(\omega))(1+\|x\|^2) \\ &\leq C_1(1+\|x\|^2) \end{aligned}$$

where $C_1 = \sup\{K_1(Q_t(\omega),R_t(\omega)) : (t,\omega) \in [0,T] \times \Omega\} < \infty$, owing to the uniform bounds on $Q$ and $R$ as well as the continuity of $K_1$. Similarly,

$$\begin{aligned} &\|a(t,\omega,x) - a(t,\omega,y)\|^2 + \|b(t,w,x) - b(t,\omega,y)\|^2 \\ &\quad = \|f(Q_t(\omega),x) - f(Q_t(\omega),y)\|^2 + \|g(R_t(w),x) - g(R_t(w),y)\|^2 \\ &\quad \leq K_2(Q_t(\omega),R_t(\omega))\|x-y\|^2 \\ &\quad \leq C_2\|x-y\|^2 \end{aligned}$$

where $C_2 = \sup\{K_2(Q_t(\omega), R_t(\omega)) : (t,\omega) \in [0,T] \times \Omega\} < \infty$. Now, let $X$ be any $\mathbb{R}^M$-valued, $\mathbb{F}$-progressive process. Since $Q$ is $\mathbb{R}^K$-valued, $X$ is $\mathbb{R}^M$-valued, and both are $\mathbb{F}$-progressive, we have for all $t \in [0,T]$, $A \in \mathcal{B}(\mathbb{R}^K)$ and $B \in \mathcal{B}(\mathbb{R}^M)$ that

$$\begin{aligned}
\{(s,\omega) : s \in [0,t], Q_s(\omega) \in A\} &= ([0,t] \times \Omega) \cap Q^{-1}(A) \in \mathcal{B}([0,t]) \otimes \mathcal{F}_t \\
\{(s,\omega) : s \in [0,t], X_s(\omega) \in B\} &= ([0,t] \times \Omega) \cap X^{-1}(B) \in \mathcal{B}([0,t]) \otimes \mathcal{F}_t
\end{aligned}$$

Thus,

$$\begin{aligned}
&\{(s,\omega) : s \in [0,t], (Q_s(\omega), X_s(\omega)) \in A \times B\} \\
&\qquad = ([0,t] \times \Omega) \cap (Q,X)^{-1}(A \times B) \\
&\qquad = \Big[([0,t] \times \Omega) \cap Q^{-1}(A)\Big] \cap \Big[([0,t] \times \Omega) \cap X^{-1}(B)\Big] \\
&\qquad \in \mathcal{B}([0,t]) \otimes \mathcal{F}_t
\end{aligned}$$

Since $([0,t] \times \Omega) \cap (Q,X)^{-1}(A \times B) \in \mathcal{B}([0,t]) \otimes \mathcal{F}_t$ and $\mathcal{B}(\mathbb{R}^K) \times \mathcal{B}(\mathbb{R}^M)$ generates $\mathcal{B}(\mathbb{R}^{K+M}) \equiv \mathcal{B}(\mathbb{R}^K) \otimes \mathcal{B}(\mathbb{R}^M)$ we have that for all $C \in \mathcal{B}(\mathbb{R}^{K+M})$

$$\{(s,\omega) : s \in [0,t], (Q_s(\omega), X_s(\omega)) \in C\} = ([0,t] \times \Omega) \cap (Q,X)^{-1}(C) \in \mathcal{B}([0,t]) \otimes \mathcal{F}_t$$

Since $f$ is Borel measurable, $f^{-1}(D) \in \mathcal{B}(\mathbb{R}^{K+M})$ for all $D \in \mathcal{B}(\mathbb{R}^M)$. Hence,

$$\begin{aligned}
&\{(s,\omega) : s \in [0,t], f(Q_s(\omega), X_s(\omega)) \in D\} \\
&\qquad = \{(s,\omega) : s \in [0,t], (Q_s(\omega), X_s(\omega)) \in f^{-1}(D)\} \\
&\qquad \in \mathcal{B}([0,t]) \otimes \mathcal{F}_t
\end{aligned}$$

for all $D \in \mathcal{B}(\mathbb{R}^M)$ and so $a(t,\omega,X_t(\omega)) = f(Q_t(\omega), X_t(\omega))$ is $\mathbb{F}$-progressive. A similar argument shows that the process $b(t,\omega,X_t(\omega)) = g(R_t(\omega), X_t(\omega))$ is $\mathbb{F}$-progressive as well. ∎

The following theorem deals with two mappings that we will be using extensively in the sequel so we state and prove their relevant properties here.

**Theorem 2.1.4** *Let $\Xi$ denote the space of $\mathbb{F}$-progressive, continuous and uniformly bounded processes from $[0,T]\times\Omega$ to $\mathbb{R}^M$, identifying elements which are indistinguishable. Let $\chi$ be any $\mathbb{R}^{M\otimes M}$-valued, doubly time-indexed random field whose entries satisfy $|\chi^{ab}_{\nu t}| \leq K_\chi < \infty$ for all $\nu, t \in [0,T]$ and $1 \leq a, b \leq M$ where $K_\chi$ is a constant. Assume that for each $(t,\omega) \in [0,T] \times \Omega$, the maps $\nu \mapsto \chi_{\nu t}(\omega)$ and $\nu \mapsto \chi_{t\nu}(\omega)$ are continuous. Assume also that for all $t, \nu \in [0,T]$, $\chi_{\nu t}$ is $\mathcal{F}_{t\vee\nu}$-measurable. For each $X \in \Xi$, define the linear maps $\Psi$ and $\Phi$ for $t \in [0,T]$ by*

$$\begin{aligned}\Phi(X)_t &= X_t - \mathbf{E}_t\left[\int_t^T \chi_{t\nu} X_\nu d\nu\right]\\ \Psi(X)_t &= X_t - \int_0^t \chi_{\nu t} X_\nu d\nu\end{aligned}$$

*using only the cadlag modification of the conditional expectation process in the definition of $\Phi$. Then, $\Phi$ and $\Psi$ are bijections between $\Xi$ and itself. Moreover, for each $Y \in \Xi$, the inverses to $\Phi$ and $\Psi$ are given by*

$$\begin{aligned}\Phi^{-1}(Y)_{t_0} &= Y_{t_0} + \mathbf{E}_{t_0}\left[\sum_{k=0}^{\infty}\int_{t_0}^T \cdots \int_{t_k}^T \left\{\prod_{i=0}^k \chi_{t_i t_{i+1}}\right\} Y_{t_{k+1}} dt_{k+1} dt_k \ldots dt_1\right]\\ \Psi^{-1}(Y)_{t_0} &= Y_{t_0} + \sum_{k=0}^{\infty}\int_0^{t_0} \cdots \int_0^{t_k} \left\{\prod_{i=0}^k \chi_{t_{i+1} t_i}\right\} Y_{t_{k+1}} dt_{k+1} dt_k \ldots dt_1\end{aligned}$$

*where $\prod_{i=0}^k \chi_{t_i t_{i+1}}$ and $\prod_{i=0}^k \chi_{t_{i+1} t_i}$ denote the two matrix products $\chi_{t_0 t_1} \cdots \chi_{t_k t_{k+1}}$ and $\chi_{t_1 t_0} \cdots \chi_{t_{k+1} t_k}$. An alternate representation of the inverse maps is*

$$\begin{aligned}\Phi^{-1}(Y)_t &= Y_t + \mathbf{E}_t\left[\int_t^T \vec{\chi}_{t\nu} Y_\nu\right] d\nu\\ \Psi^{-1}(Y)_t &= Y_t + \int_0^t \overleftarrow{\chi}_{\nu t} Y_\nu d\nu\end{aligned}$$

*where $\overleftarrow{\chi}$ and $\vec{\chi}$ are defined by*

$$\begin{aligned}\vec{\chi}_{t\nu} &= \chi_{t\nu} + \int_t^\nu \chi_{tt_1}\chi_{t_1\nu} dt_1 + \sum_{k=2}^{\infty}\int_t^\nu \int_t^{t_1} \cdots \int_t^{t_{k-1}} \chi_{tt_k}\left\{\prod_{i=1}^{k-1} \chi_{t_{k+1-i} t_{k-i}}\right\}\chi_{t_1\nu} dt_k \ldots dt_1\\ \overleftarrow{\chi}_{\nu t} &= \chi_{\nu t} + \int_\nu^t \chi_{t_1 t}\chi_{\nu t_1} dt_1 + \sum_{k=2}^{\infty}\int_\nu^t \int_{t_1}^{t} \cdots \int_{t_{k-1}}^{t} \chi_{t_k t}\left\{\prod_{i=1}^{k-1} \chi_{t_{k-i} t_{k+1-i}}\right\}\chi_{\nu t_1} dt_k \ldots dt_1\end{aligned}$$

*and where* $\prod_{i=1}^{k-1} \chi_{t_{k+1-i}t_{k-i}}$ *and* $\prod_{i=1}^{k-1} \chi_{t_{k-i}t_{k+1-i}}$ *similarly denote the two matrix products* $\chi_{t_k t_{k-1}} \cdots \chi_{t_2 t_1}$ *and* $\chi_{t_{k-1}t_k} \cdots \chi_{t_1 t_2}$*. The matrix valued processes* $\overleftarrow{\chi}$ *and* $\overrightarrow{\chi}$ *satisfy the bounds*

$$\begin{aligned} \|\overrightarrow{\chi}_{t\nu}\| &\leq MK_\chi e^{MK_\chi(\nu-t)} \\ \|\overleftarrow{\chi}_{\nu t}\| &\leq MK_\chi e^{MK_\chi(t-\nu)} \end{aligned}$$

*Also, if* $\|Y\| \leq K$ *then*

$$\begin{aligned} \left\|\Phi^{-1}(Y)_t\right\| &\leq KM^{\frac{3}{2}} e^{MK_\chi(T-t)} \\ \left\|\Psi^{-1}(Y)_t\right\| &\leq KM^{\frac{3}{2}} e^{MK_\chi t} \end{aligned}$$

*Finally, if* $Y$ *and* $\chi$ *have non-negative components, so too do* $\Phi^{-1}(Y)$ *and* $\Psi^{-1}(Y)$*.*

**Proof:** $\Psi$ clearly maps $\Xi$ into $\Xi$. To show $\Phi$ maps $\Xi$ into $\Xi$, we need only check that $\Phi(X)$ is $\mathbb{F}$-progressive and continuous, the rest being obvious. Rearranging the expression for $\Phi(X)_t$, we have

$$\Phi(X)_t = X_t + \int_0^t \chi_{t\nu} X_\nu d\nu - \mathbf{E}_t \left[\int_0^T \chi_{t\nu} X_\nu d\nu\right]$$

so we see that $\Phi(X)$ is the sum of two $\mathbb{F}$-progressive, continuous processes and an $\mathbb{F}$-martingale. By Proposition (1.3.13) in Karatzas & Shreve (1991), this $\mathbb{F}$-martingale has a cadlag modification, which is the modification used in the definition of $\Phi$. Then, by Theorem (2.1.1), this cadlag $\mathbb{F}$-martingale is continuous, hence, $\Phi(X)$ is continuous and $\mathbb{F}$-progressive.

We now show that $\Phi$ and $\Psi$ are both 1-1. Suppose first that $\Phi(X) = \Phi(\tilde{X})$ for some $X, \tilde{X} \in \Xi$. Thus, for all $t \in [0, T]$,

$$0 = \Phi(X)_t - \Phi(\tilde{X})_t = X_t - \tilde{X}_t - \mathbf{E}_t \left[\int_t^T \chi_{t\nu}(X_\nu - \tilde{X}_\nu) d\nu\right]$$

and so

$$X_t - \tilde{X}_t = \mathbf{E}_t \left[\int_t^T \chi_{t\nu}(X_\nu - \tilde{X}_\nu) d\nu\right]$$

With $|\chi_{\nu t}^{ab}| \leq K_\chi$ we have $\|\chi_{\nu t}\| \leq MK_\chi$ and $\|\chi_{\nu t} x\| \leq M^{\frac{3}{2}} K_\chi \|x\|$ for all $\nu, t \in [0, T]$ and $x \in \mathbb{R}^M$. Taking expectations of norms, and setting $K_M = M^{\frac{3}{2}} K_\chi$, we obtain

$$\mathbf{E}\|X_t - \tilde{X}_t\| \leq \int_t^T \mathbf{E}\|\chi_{t\nu}(X_\nu - \tilde{X}_\nu)\| d\nu \leq K_M \int_t^T \mathbf{E}\|X_\nu - \tilde{X}_\nu\| d\nu$$

Letting $p_t = \mathbf{E}\|X_t - \tilde{X}_t\|$ we have $0 \leq p_t \leq K_M \int_t^T p_\nu d\nu$. Iterating this inequality, we have, for all $n \geq 1$,

$$0 \;\leq\; p_t \;\leq\; K_M \int_t^T p_\nu d\nu \;\leq\; K_M^2 \int_t^T (\nu - t) p_\nu d\nu \;\leq\; \cdots \;\leq\; K_M^n \int_t^T \frac{(\nu - t)^{n-1}}{(n-1)!} p_\nu d\nu$$

Since $X$ and $\tilde{X}$ are uniformly bounded, so is $p$, say, by a constant $K_p$. Thus,

$$0 \;\leq\; p_t \;\leq\; K_p K_M^n \int_t^T \frac{(\nu - t)^{n-1}}{(n-1)!} d\nu \;\leq\; K_p \frac{(K_M T)^n}{n!} \;\to 0 \text{ as } n \to \infty$$

Similarly, starting with $\Psi(X) = \Psi(\tilde{X})$, we have for all $t \in [0, T]$,

$$0 = \Psi(X)_t - \Psi(\tilde{X})_t = X_t - \tilde{X}_t - \int_0^t \chi_{\nu t}(X_\nu - \tilde{X}_\nu) d\nu$$

which leads to

$$\mathbf{E}\|X_t - \tilde{X}_t\| \;\leq K_M \int_0^t \mathbf{E}\|X_\nu - \tilde{X}_\nu\| d\nu$$

Iterating $0 \leq p_t \leq K_M \int_0^t p_\nu d\nu$ gives

$$0 \;\leq\; p_t \;\leq\; K_M \int_0^t p_\nu d\nu \;\leq\; \cdots \;\leq\; K_M^n \int_0^t \frac{(t - \nu)^{n-1}}{(n-1)!} p_\nu d\nu \;\leq\; K_p \frac{(K_M T)^n}{n!} \to 0$$

as $n \to \infty$. Hence, if either $\Phi(X) = \Phi(\tilde{X})$ or $\Psi(X) = \Psi(\tilde{X})$ then, for all $t$, $\mathbf{E}\|X_t - \tilde{X}_t\| = 0$ implying that $X$ and $\tilde{X}$ are modifications. As $X$ and $\tilde{X}$ are continuous, they are then necessarily indistinguishable, which shows that $\Phi$ and $\Psi$ are both 1-1.

To show $\Phi$ and $\Psi$ are onto, let $Y \in \Xi$ and show there exist $X$ and $\tilde{X}$ in $\Xi$ such that $Y = \Phi(X) = \Psi(\tilde{X})$. The equation $Y = \Psi(\tilde{X})$ defines a path-wise linear Volterra equation of the second kind with kernel $\chi$ for which we can construct a solution iteratively via a Liouville-Neumann series (this construction is similar to the Picard-Lindelöf iterations used to show existence of solutions to differential equations). Fortunately, this technique for linear operator equations can be applied to the equation $Y = \Phi(X)$ even though $\Phi$ involves the conditional expectation of an integral over future trajectories, as we now show.

Define the iterations

$$\begin{aligned}
X_t^{(0)} &= \tilde{X}_t^{(0)} = Y_t \\
X_t^{(n+1)} &= Y_t + \mathbf{E}_t\left[\int_t^T \chi_{t\nu} X_\nu^{(n)} d\nu\right] \qquad n \geq 0 \\
\tilde{X}_t^{(n+1)} &= Y_t + \int_0^t \chi_{\nu t} \tilde{X}_\nu^{(n)} d\nu \qquad n \geq 0
\end{aligned}$$

using only the cadlag modification of the conditional expectation process in defining $\{X^{(n)}\}$. The sequences $\{X^{(n)}\}$ and $\{\tilde{X}^{(n)}\}$ are clearly both in $\Xi$. From $\{X^{(n)}\}$, define a new sequence $\{f^{(n)}\}$ of processes in $\Xi$ by $f_t^{(n+1)} = \|X_t^{(n+1)} - X_t^{(n)}\|$, for $n \geq 0$, which is seen to satisfy the following inequality for $n \geq 1$:

$$\begin{aligned}
f_t^{(n+1)} &= \|X_t^{(n+1)} - X_t^{(n)}\| = \left\|\mathbf{E}_t\left[\int_t^T \chi_{t\nu}(X_\nu^{(n)} - X_\nu^{(n-1)}) d\nu\right]\right\| \\
&\leq K_M \mathbf{E}_t\left[\int_t^T f_\nu^{(n)} d\nu\right] \qquad (2.1.5)
\end{aligned}$$

Similarly, from $\{\tilde{X}^{(n)}\}$, define $\tilde{f}_t^{(n+1)} = \|\tilde{X}_t^{(n+1)} - \tilde{X}_t^{(n)}\|$ which then satisfies

$$\begin{aligned}
\tilde{f}_t^{(n+1)} &= \|\tilde{X}_t^{(n+1)} - \tilde{X}_t^{(n)}\| = \left\|\int_0^t \chi_{\nu t}(\tilde{X}_\nu^{(n)} - \tilde{X}_\nu^{(n-1)}) d\nu\right\| \\
&\leq K_M \int_0^t \tilde{f}_\nu^{(n)} d\nu \qquad (2.1.6)
\end{aligned}$$

Iterating (2.1.5) and (2.1.6),

$$\begin{aligned}
f_t^{(n+1)} &\leq K_M \mathbf{E}_t\left[\int_t^T f_\nu^{(n)} d\nu\right] \leq K_M^2 \mathbf{E}_t\left[\int_t^T (\nu - t) f_\nu^{(n-1)} d\nu\right] \\
&\leq \cdots \leq K_M^n \mathbf{E}_t\left[\int_t^T \frac{(\nu - t)^{n-1}}{(n-1)!} f_\nu^{(1)} d\nu\right] \\
\tilde{f}_t^{(n+1)} &\leq K_M \int_0^t \tilde{f}_\nu^{(n)} d\nu \leq K_M^2 \int_0^t (t - \nu) \tilde{f}_\nu^{(n-1)} d\nu \\
&\leq \cdots \leq K_M^n \int_0^t \frac{(t - \nu)^{n-1}}{(n-1)!} \tilde{f}_\nu^{(1)} d\nu
\end{aligned}$$

but, since $\|Y\| \leq K_Y < \infty$ for some constant $K_Y$, we have

$$\begin{aligned}
f_t^{(1)} &= \|X_t^{(1)} - X_t^{(0)}\| = \left\|\mathbf{E}_t\left[\int_t^T \chi_{t\nu} Y_\nu d\nu\right]\right\| \leq K_M K_Y T \\
\tilde{f}_t^{(1)} &= \|\tilde{X}_t^{(1)} - \tilde{X}_t^{(0)}\| = \left\|\int_0^t \chi_{\nu t} Y_\nu d\nu\right\| \leq K_M K_Y T
\end{aligned}$$

Thus, for all $n \geq 0$ we have the bounds:

$$\begin{aligned} f_t^{(n+1)} &\leq K_M^n \mathbf{E}_t \left[ \int_t^T \frac{(\nu - t)^{n-1}}{(n-1)!} f_\nu^{(1)} d\nu \right] \leq K_M K_Y T K_M^n \int_t^T \frac{(\nu - t)^{n-1}}{(n-1)!} d\nu \\ &\leq K_M T K_Y \frac{(K_M T)^n}{n!} \\ \tilde{f}_t^{(n+1)} &\leq K_M^n \int_0^t \frac{(\nu - t)^{n-1}}{(n-1)!} \tilde{f}_\nu^{(1)} d\nu \leq K_M K_Y T K_M^n \int_0^t \frac{(\nu - t)^{n-1}}{(n-1)!} d\nu \\ &\leq K_M T K_Y \frac{(K_M T)^n}{n!} \end{aligned}$$

For all $t$ and positive integers $m$ and $k$,

$$\begin{aligned} \|X_t^{(m+k)} - X_t^{(k)}\| &\leq \sum_{i=0}^{m-1} \|X_t^{(k+i+1)} - X_t^{(k+i)}\| = \sum_{i=0}^{m-1} f_t^{(k+i+1)} \\ &\leq K_M T K_Y \sum_{i=0}^{m-1} \frac{(K_M T)^{k+i}}{(k+i)!} \end{aligned} \tag{2.1.7}$$

Hence, for $\mathbf{P}$-a.e. $\omega \in \Omega$, $\{X^{(n)}(\omega)\}$ is a Cauchy sequence in the space continuous functions on $[0, T]$ equipped with the supremum norm. This space is complete so, $\mathbf{P}$-a.e., $X^{(n)} \to X$ in supremum norm for some $\mathbf{P}$-a.s. continuous $X$. Setting $k = 0$ in (2.1.7) yields, for all $m$,

$$\|X_t^{(m)} - X_t^{(0)}\| \leq K_M T K_Y \sum_{i=0}^{m-1} \frac{(K_M T)^i}{i!} < K_M T K_Y e^{K_M T}$$

Thus, recalling that $X^{(0)} = Y$,

$$\|X_t^{(m)}\| \leq \|X_t^{(0)}\| + K_M T K_Y e^{K_M T} \leq K_Y + K_M T K_Y e^{K_M T}$$

Taking limits, we see that $X$ is uniformly bounded:

$$\|X_t\| = \lim_{m \to \infty} \|X_t^{(m)}\| \leq K_Y + K_M T K_Y e^{K_M T}$$

Since $X_t^{(n)} \in \mathcal{F}_t$ for each $t$ and $n$, the limit $X_t$ is in $\mathcal{F}_t$, so is $\mathbb{F}$-adapted. Being continuous, $X$ is therefore also $\mathbb{F}$-progressive.

With $X$ and $X^{(n)}$ dominated by $K_Y + K_M T K_Y e^{K_M T}$ and the fact that $X^{(n)} \to X$ $\mathbf{P}$-a.s. in supremum norm, we can apply $\mathbf{P}$-a.s. point-wise as well as dominated convergence to the right side of

$$Y_t = X_t^{(n+1)} - \mathbf{E}_t \left[ \int_t^T \chi_{t\nu} X_\nu^{(n)} d\nu \right] \qquad \forall t \in [0, T]$$

to obtain

$$Y_t = X_t - \mathbf{E}_t\left[\int_t^T \chi_{t\nu}X_\nu d\nu\right] \qquad \forall t \in [0,T]$$

A similar argument with $\tilde{X}^{(n)}$ converging to $\tilde{X}$ results in

$$Y_t = \tilde{X}_t - \int_0^t \chi_{\nu t}\tilde{X}_\nu d\nu \qquad \forall t \in [0,T]$$

Thus, $\Phi$ and $\Psi$ are both onto and so are both bijections on $\Xi$.

To obtain the series representation, first construct the iterates for each $n$ and $t_0 \in [0,T]$:

$$\begin{aligned}
X_{t_0}^{(0)} &= Y_{t_0} \\
X_{t_0}^{(1)} &= Y_{t_0} + \mathbf{E}_{t_0}\left[\int_{t_0}^T \chi_{t_0t_1}Y_{t_1}dt_1\right] \\
X_{t_0}^{(2)} &= Y_{t_0} + \mathbf{E}_{t_0}\left[\int_{t_0}^T \chi_{t_0t_1}\left\{Y_{t_1} + \mathbf{E}_{t_1}\left[\int_{t_1}^T \chi_{t_1t_2}Y_{t_2}dt_2\right]\right\}dt_1\right] \\
&= Y_{t_0} + \mathbf{E}_{t_0}\left[\int_{t_0}^T \chi_{t_0t_1}Y_{t_1}dt_1\right] + \mathbf{E}_{t_0}\left[\int_{t_0}^T\int_{t_1}^T \chi_{t_0t_1}\chi_{t_1t_2}Y_{t_2}dt_2dt_1\right] \\
&\cdot \\
&\cdot \\
&\cdot \\
X_{t_0}^{(n)} &= Y_{t_0} + \mathbf{E}_{t_0}\left[\sum_{k=0}^{n-1}\int_{t_0}^T \cdots \int_{t_k}^T \left\{\prod_{i=0}^{k}\chi_{t_it_{i+1}}\right\}Y_{t_{k+1}}dt_{k+1}\cdots dt_1\right]
\end{aligned}$$

Taking the limit of these convergent partial sums gives a series representation of $X = \Phi^{-1}(Y)$. The series representation for $\tilde{X} = \Psi^{-1}(Y)$ is obtained similarly:

$$X_{t_0} = Y_{t_0} + \mathbf{E}_{t_0}\left[\sum_{k=0}^{\infty}\int_{t_0}^T \cdots \int_{t_k}^T \left\{\prod_{i=0}^{k}\chi_{t_it_{i+1}}\right\}Y_{t_{k+1}}dt_{k+1}dt_k\ldots dt_1\right] \qquad (2.1.8)$$

$$\tilde{X}_{t_0} = Y_{t_0} + \sum_{k=0}^{\infty}\int_0^{t_0} \cdots \int_0^{t_k} \left\{\prod_{i=0}^{k}\chi_{t_{i+1}t_i}\right\}Y_{t_{k+1}}dt_{k+1}dt_k\ldots dt_1 \qquad (2.1.9)$$

By reversing the orders of integration so that $t_{k+1}$ is integrated last, rearranging and relabeling, one obtains

$$\begin{aligned}
X_t &= \Phi^{-1}(Y)_t = Y_t + \mathbf{E}_t\left[\int_t^T \vec{\chi}_{t\nu}Y_\nu\right]d\nu \\
\tilde{X}_t &= \Psi^{-1}(Y)_t = Y_t + \int_0^t \overleftarrow{\chi}_{\nu t}Y_\nu d\nu
\end{aligned}$$

where $\overleftarrow{\chi}$ and $\overrightarrow{\chi}$ are as in the statement of the theorem. The notation $\leftharpoonup$ and $\rightharpoonup$ is to indicate the use of information: if the current time is $t$ then $\overleftarrow{\chi}_{\nu t}$ for $\nu \in [0,t]$ refers to historical information. $\overrightarrow{\chi}_{t\nu}$ for $\nu \in [t,T]$ refers to future information. For the bounds on $\overrightarrow{\chi}$ and $\overleftarrow{\chi}$, we have

$$
\begin{aligned}
&\|\overrightarrow{\chi}_{t\nu}\| \\
= &\left\| \chi_{t\nu} + \int_t^\nu \chi_{tt_1}\chi_{t_1\nu}dt_1 + \sum_{k=2}^\infty \int_t^\nu \int_t^{t_1} \cdots \int_t^{t_{k-1}} \chi_{tt_k} \left\{ \prod_{i=1}^{k-1} \chi_{t_{k+1-i}t_{k-i}} \right\} \chi_{t_1\nu} dt_k \ldots dt_1 \right\| \\
\leq &\ \|\chi_{t\nu}\| + \int_t^\nu \|\chi_{tt_1}\chi_{t_1\nu}\| dt_1 \\
&+ \sum_{k=2}^\infty \int_t^\nu \int_t^{t_1} \cdots \int_t^{t_{k-1}} \left\| \chi_{tt_k} \left\{ \prod_{i=1}^{k-1} \chi_{t_{k+1-i}t_{k-i}} \right\} \chi_{t_1\nu} \right\| dt_k \ldots dt_1 \\
\leq &\ MK_\chi + \int_t^\nu M^2K_\chi^2 dt_1 + \sum_{k=2}^\infty \int_t^\nu \int_t^{t_1} \cdots \int_t^{t_{k-1}} M^{k+1}K_\chi^{k+1} dt_k \ldots dt_1 \\
= &\ MK_\chi \left( 1 + MK_\chi(\nu - t) + \sum_{k=2}^\infty \frac{(MK_\chi(\nu-t))^k}{k!} \right) \\
= &\ MK_\chi e^{MK_\chi(\nu - t)}
\end{aligned}
$$

A similar argument yields

$$
\|\overleftarrow{\chi}_{\nu t}\| \leq MK_\chi e^{MK_\chi(t-\nu)}
$$

To obtain the bounds on $\Phi^{-1}(Y)_t$ and $\Psi^{-1}(Y)_t$,

$$
\begin{aligned}
\|\Phi^{-1}(Y)_t\| &= \left\| Y_t + \mathbf{E}_t \left[ \int_t^T \overrightarrow{\chi}_{t\nu} Y_\nu \right] d\nu \right\| \\
&\leq \|Y_t\| + \mathbf{E}_t \left[ \int_t^T \|\overrightarrow{\chi}_{t\nu} Y_\nu\| \right] d\nu \\
&\leq K_Y + \int_t^T M^{\frac{3}{2}} K_Y M K_\chi e^{MK_\chi(\nu-t)} d\nu \\
&= K_Y M^{\frac{3}{2}} e^{MK_\chi(T-t)}
\end{aligned}
$$

and similarly for $\Psi^{-1}(Y)_t$. The final statement about non-negativity is clear from the series representations. ■

In Detemple & Karatzas (2003), a fairly explicit inverse for $\Phi$ is obtained in a direct, non-iterative manner. We demonstrate and somewhat generalize their method in two lemmas:

**Lemma 2.1.10** *If the kernel $\chi$ in Proposition (2.1.4) satisfies*

$$\forall \nu, t \in [0,T], \qquad \chi_{\nu t} = \mathbf{B}_\nu \mathbf{B}_t^{-1} = \mathbf{B}_t^{-1}\mathbf{B}_\nu$$

*for some invertible matrix valued process* $\mathbf{B}$ *then the inverse maps are given by*

$$\Phi^{-1}(Y)_t = Y_t + \mathbf{E}_t\left[\int_t^T e^{\nu - t}\chi_{t\nu} Y_\nu d\nu\right] \tag{2.1.11}$$

$$\Psi^{-1}(Y)_t = Y_t + \int_0^t e^{t-\nu}\chi_{\nu t} Y_\nu d\nu \tag{2.1.12}$$

**Proof:** Setting $\Psi(X)_t = Y_t$ and using integration by parts and the fact that $X_0 = Y_0$, we obtain

$$\begin{aligned}
X_t - \int_0^t \chi_{\nu t} X_\nu d\nu Y_t &= Y_t \\
X_t - \int_0^t \mathbf{B}_t^{-1}\mathbf{B}_\nu X_\nu d\nu &= Y_t \\
\mathbf{B}_t X_t - \int_0^t \mathbf{B}_\nu X_\nu d\nu &= \mathbf{B}_t Y_t \\
d(\mathbf{B}_t X_t) - \mathbf{B}_t X_t dt &= d(\mathbf{B}_t Y_t) \\
d(e^{-t}\mathbf{B}_t X_t) &= e^{-t} d(\mathbf{B}_t Y_t) \\
e^{-t}\mathbf{B}_t X_t - \mathbf{B}_0 X_0 &= e^{-t}\mathbf{B}_t Y_t - \mathbf{B}_0 Y_0 + \int_0^t e^{-\nu}\mathbf{B}_\nu Y_\nu d\nu \\
e^{-t}\mathbf{B}_t X_t &= e^{-t}\mathbf{B}_t Y_t + \int_0^t e^{-\nu}\mathbf{B}_\nu Y_\nu d\nu \\
X_t &= Y_t + \int_0^t e^{t-\nu}\mathbf{B}_t^{-1}\mathbf{B}_\nu Y_\nu d\nu \\
X_t &= Y_t + \int_0^t e^{t-\nu}\chi_{\nu t} Y_\nu d\nu
\end{aligned}$$

Proceeding similarly for $\Phi(X)_t = Y_t$:

$$\begin{aligned}
X_t - \mathbf{E}_t\left[\int_t^T \chi_{t\nu} X_\nu d\nu\right] &= Y_t \\
X_t - \mathbf{E}_t\left[\int_t^T \mathbf{B}_t \mathbf{B}_\nu^{-1} X_\nu d\nu\right] &= Y_t \\
\mathbf{B}_t^{-1} X_t - \mathbf{E}_t\left[\int_t^T \mathbf{B}_\nu^{-1} X_\nu d\nu\right] &= \mathbf{B}_t^{-1} Y_t \\
\mathbf{B}_t^{-1} X_t - \mathbf{E}_t\left[\int_0^T \mathbf{B}_\nu^{-1} X_\nu d\nu\right] + \int_0^t \mathbf{B}_\nu^{-1} X_\nu d\nu &= \mathbf{B}_t^{-1} Y_t \\
\mathbf{B}_t^{-1} X_t - M_t + \int_0^t \mathbf{B}_\nu^{-1} X_\nu d\nu &= \mathbf{B}_t^{-1} Y_t \\
\mathbf{B}_t^{-1} X_t + \int_0^t \mathbf{B}_\nu^{-1} X_\nu d\nu &= \mathbf{B}_t^{-1} Y_t + M_t
\end{aligned}$$

where $M_t$ denotes the martingale $\mathbf{E}_t\left[\int_0^T \mathbf{B}_\nu^{-1} X_\nu d\nu\right]$. Continuing, noting that $X_T = Y_T$, we have

$$\begin{aligned}
d(\mathbf{B}_t^{-1} X_t) + \mathbf{B}_t^{-1} X_t dt &= d(\mathbf{B}_t^{-1} Y_t) + dM_t \\
d(e^t \mathbf{B}_t^{-1} X_t) &= e^t d(\mathbf{B}_t^{-1} Y_t) + e^t dM_t \\
e^T \mathbf{B}_T^{-1} X_T - e^t \mathbf{B}_t^{-1} X_t &= e^T \mathbf{B}_T^{-1} Y_T - e^t \mathbf{B}_t^{-1} Y_t - \int_t^T e^\nu \mathbf{B}_\nu^{-1} Y_\nu d\nu + \int_t^T e^\nu dM_\nu \\
X_t &= Y_t + \int_t^T e^{\nu-t} \mathbf{B}_t \mathbf{B}_\nu^{-1} Y_\nu d\nu + e^{-t} \mathbf{B}_t \int_t^T e^\nu dM_\nu \\
X_t &= Y_t + \mathbf{E}_t\left[\int_t^T e^{\nu-t} \mathbf{B}_t \mathbf{B}_\nu^{-1} Y_\nu d\nu\right] \\
X_t &= Y_t + \mathbf{E}_t\left[\int_t^T e^{\nu-t} \chi_{t\nu} Y_\nu d\nu\right]
\end{aligned}$$

since, upon taking conditional expectations, $\mathbf{E}_t\left[\int_t^T e^\nu dM_\nu\right] = 0$. ■

We also give the special case of a constant kernel as this will be used in diagnosing the accuracy of the simulation procedure as well as developing intuition about how changes in the kernel will impact upon equilibrium:

**Lemma 2.1.13** *If the kernel $\chi$ in Proposition (2.1.4) is a constant matrix $\mathbf{B}$, not necessarily invertible,*

$$\forall \nu, t \in [0, T], \qquad \chi_{\nu t} = \mathbf{B}$$

*then the inverse maps are given by*

$$\Phi^{-1}(Y)_t = Y_t + \mathbf{E}_t\left[\int_t^T \mathbf{B}e^{\mathbf{B}(\nu-t)}Y_\nu d\nu\right] \quad (2.1.14)$$

$$\Psi^{-1}(Y)_t = Y_t + \int_0^t \mathbf{B}e^{\mathbf{B}(t-\nu)}Y_\nu d\nu \quad (2.1.15)$$

**Proof:** Setting $\Psi(X)_t = Y_t$ and proceeding as in Lemma (2.1.10),

$$\begin{aligned}
X_t - \int_0^t \mathbf{B}X_\nu d\nu &= Y_t \\
dX_t - \mathbf{B}X_t dt &= dY_t \\
d(e^{-\mathbf{B}t}X_t) &= e^{-\mathbf{B}t}dY_t \\
e^{-\mathbf{B}t}X_t - X_0 &= e^{-\mathbf{B}t}Y_t - Y_0 + \int_0^t \mathbf{B}e^{-\mathbf{B}\nu}Y_\nu d\nu \\
X_t &= Y_t + \int_0^t \mathbf{B}e^{\mathbf{B}(t-\nu)}Y_\nu d\nu
\end{aligned}$$

and

$$\begin{aligned}
X_t - \mathbf{E}_t\left[\int_t^T \mathbf{B}X_\nu d\nu\right] &= Y_t \\
X_t + \int_0^t \mathbf{B}X_\nu d\nu - \mathbf{E}_t\left[\int_0^T \mathbf{B}X_\nu d\nu\right] &= Y_t \\
dX_t + \mathbf{B}X_t dt &= dY_t + dM_t \\
d(e^{\mathbf{B}t}X_t) &= e^{\mathbf{B}t}dY_t + e^{\mathbf{B}t}dM_t \\
e^{\mathbf{B}T}X_T - e^{\mathbf{B}t}X_t &= e^{\mathbf{B}T}Y_T - e^{\mathbf{B}t}Y_t - \int_t^T \mathbf{B}e^{\mathbf{B}\nu}Y_\nu d\nu + \int_t^T e^{\mathbf{B}\nu}dM_\nu \\
X_t &= Y_t + e^{-\mathbf{B}t}\int_t^T \mathbf{B}e^{\mathbf{B}\nu}Y_\nu d\nu - e^{-\mathbf{B}t}\int_t^T e^{\mathbf{B}\nu}dM_\nu \\
X_t &= Y_t + \mathbf{E}_t\left[\int_t^T \mathbf{B}e^{\mathbf{B}(\nu-t)}Y_\nu d\nu\right]
\end{aligned}$$

■

The explicit form obtained in Detemple & Karatzas (2003) is for a constant kernel $\chi$ in one dimension and uniqueness was not established. However, uniqueness up to indistinguishability follows from Theorem (2.1.4). The explicit solutions in Lemmas (2.1.10) and (2.1.13) can also be obtained as special cases of the series expansions (2.1.8) and (2.1.9) in Theorem (2.1.4): Under the assumptions of Lemma (2.1.10), the matrix products simplify to

$$\begin{aligned}
\prod_{i=0}^{k} \chi_{t_i t_{i+1}} &= \prod_{i=0}^{k} \mathbf{B}_{t_i}\mathbf{B}_{t_{i+1}}^{-1} = \mathbf{B}_{t_0}\mathbf{B}_{t_{k+1}}^{-1} = \chi_{t_0 t_{k+1}} \\
\prod_{i=0}^{k} \chi_{t_{i+1} t_i} &= \prod_{i=0}^{k} \mathbf{B}_{t_{i+1}}\mathbf{B}_{t_i}^{-1} = \mathbf{B}_{t_{k+1}}\mathbf{B}_{t_0}^{-1} = \chi_{t_{k+1} t_0}
\end{aligned}$$

Then, integrating by parts, and recognizing series expansions for the exponential functions $e^{t-\nu}$ and $e^{\nu-t}$, one explicitly computes that $\overleftarrow{\chi}_{\nu t} = e^{t-\nu}\chi_{\nu t}$ and $\overrightarrow{\chi}_{t\nu} = e^{\nu-t}\chi_{t\nu}$ and so (2.1.11) and (2.1.12) follow. Under the assumptions of Lemma (2.1.13), the matrix products are

$$\prod_{i=0}^{k} \chi_{t_i t_{i+1}} = \prod_{i=0}^{k} \chi_{t_{i+1} t_i} = \prod_{i=0}^{k} \mathbf{B} = \mathbf{B}^{k+1}$$

Again, integrating by parts, and recognizing series expansions for the matrix exponential functions $e^{\mathbf{B}(t-\nu)}$ and $e^{\mathbf{B}(\nu-t)}$, one obtains that $\overleftarrow{\chi}_{\nu t} = \mathbf{B}e^{\mathbf{B}(t-\nu)}$ and $\overrightarrow{\chi}_{t\nu} = \mathbf{B}e^{\mathbf{B}(\nu-t)}$ and hence (2.1.14) and (2.1.15).

A quick remark on notation. $\boldsymbol{\chi}$ is an $\mathbb{R}^{M\otimes M}$-valued process and individual elements of $\boldsymbol{\chi}$ will be denoted by either $\chi^{ab}$ or $\boldsymbol{\chi}^{ab}$ or even $(\boldsymbol{\chi})^{ab}$ for $1 \le a, b \le M$. From $\boldsymbol{\chi}$ we can construct the $\mathbb{R}^{M\otimes M}$-valued processes $\overleftarrow{\boldsymbol{\chi}}$ and $\overrightarrow{\boldsymbol{\chi}}$. The individual elements of $\overleftarrow{\boldsymbol{\chi}}$ and $\overrightarrow{\boldsymbol{\chi}}$ will similarly be denoted by $\overleftarrow{\chi}^{ab}$, $\overleftarrow{\boldsymbol{\chi}}^{ab}$, or $(\overleftarrow{\boldsymbol{\chi}})^{ab}$ and $\overrightarrow{\chi}^{ab}$, $\overrightarrow{\boldsymbol{\chi}}^{ab}$ or $(\overrightarrow{\boldsymbol{\chi}})^{ab}$. On occasion, we will need to take the $ab$ element $\chi^{ab}$ of $\boldsymbol{\chi}$ and construct scalar versions of $\overleftarrow{\boldsymbol{\chi}}$ and $\overrightarrow{\boldsymbol{\chi}}$ from $\chi^{ab}$ as was done in Theorem (2.1.4) . For these scalar versions we use the notation $\overrightarrow{\chi_{\nu t}^{ab}}$ and $\overleftarrow{\chi_{\nu t}^{ab}}$. Lastly, a lemma which shall be needed later:

**Lemma 2.1.16** *Let $\{\mathbf{A}^{(m)} = \{a_{ij}^{(m)}\}\}_{m=1}^{M}$ be any family of $N \times N$ matrices, each having constant row sums:*

$$\forall m, \quad \forall i, \quad \sum_{j=1}^{N} a_{ij}^{(m)} = R^{(m)}$$

*for some constants $R^{(m)}$. Then, the product of matrices has constant row sums $\prod_{m=1}^{M} R^{(m)}$. Suppose instead that the matrices $\mathbf{A}^{(m)}$ have constant column sums:*

$$\forall m, \quad \forall j, \quad \sum_{i=1}^{N} a_{ij}^{(m)} = C^{(m)}$$

*for some constants $C^{(m)}$. Then, the product of matrices has constant column sums $\prod_{m=1}^{M} C^{(m)}$.*

**Proof:** Start with the first two matrices $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$. Under the first assumption, the row sums for $\mathbf{A}^{(1)}\mathbf{A}^{(2)}$ are, for each row $i$,

$$\sum_{j=1}^{N}(\mathbf{A}^{(1)}\mathbf{A}^{(2)})_{ij} = \sum_{j=1}^{N}\sum_{k=1}^{N} a_{ik}^{(1)} a_{kj}^{(2)} = \sum_{k=1}^{N} a_{ik}^{(1)} \sum_{j=1}^{N} a_{kj}^{(2)} = \sum_{k=1}^{N} a_{ik}^{(1)} R^{(2)} = R^{(1)}R^{(2)}$$

Similarly, under the second assumption, the column sum for each column $j$ of $\mathbf{A}^{(1)}\mathbf{A}^{(2)}$ is

$$\sum_{i=1}^{N}(\mathbf{A}^{(1)}\mathbf{A}^{(2)})_{ij} = \sum_{i=1}^{N}\sum_{k=1}^{N} a_{ik}^{(1)} a_{kj}^{(2)} = \sum_{k=1}^{N}\sum_{i=1}^{N} a_{ik}^{(1)} a_{kj}^{(2)} = \sum_{k=1}^{N} C^{(1)} a_{kj}^{(2)} = C^{(1)}C^{(2)}$$

Proceeding by induction, one can show easily that under the first assumption the product of all matrices has constant row sums $\prod_{m=1}^{M} R^{(m)}$. Similarly, under the second assumption, the product of all matrices has constant column sums $\prod_{m=1}^{M} C^{(m)}$. ■

## 2.2 Financial Assets & Discount Factors

**Riskless Asset:** Let the bond have an instantaneous relative net rate of return given by some $\mathbb{R}_+$-valued, $\mathbb{F}$-progressive process $r$, uniformly bounded by some constant $K_r$:

$$\forall t \in [0,T], \qquad |r_t| \leq K_r < \infty$$

The bond is therefore an investment which yields the gross net instantaneous return:

$$\forall t \in [0,T], \qquad dB_t = r_t B_t dt, \qquad B_0 > 0 \text{ is constant}$$

Theorem (2.1.2) and Lemma (2.1.3) apply with $f(q,x) = qx$ and $g(r,x) = 0$, giving us the existence of an indistinguishably unique solution. By applying Itô's lemma, this solution can be verified to be

$$\forall t \in [0,T], \qquad B_t = B_0 \exp\left[\int_0^t r_\nu d\nu\right] > 0$$

Because the solution is strictly positive we see that

$$\frac{dB_t}{B_t} = r_t dt$$

represents the relative net return over the infinitesimal time interval $[t, t+dt]$. Since the relative return over the interval $[t, t+dt]$ is $r_t dt$, which is known with certainty at time $t$, this return is "locally riskless". Over a finite time interval $[t, t+\Delta t]$, the relative return is not known with certainty at time $t$ for the entire interval and so this investment is "globally risky", leading to the term structure of interest rates. We refer to the bond as a "riskless asset" even though it is only instantaneously riskless.

The bond corresponds to borrowing and lending at the instantaneous rate $r$. For a zero net trade, pure exchange, frictionless economy in equilibrium, each amount loaned is associated with an equal amount borrowed. Thus, the net aggregate wealth held in the bond at any time in equilibrium is zero.

**Risky Assets:** There are also $N \geq 1$ risky assets available for investment, which we index by $k = 1, \ldots, N$. Let $\mu = (\mu^1, \ldots, \mu^N)^\top$ be any $\mathbb{R}^N$-valued, $\mathbb{F}$-progressive process such that there exists a constant $K_\mu$ uniformly bounding $\mu$ in Euclidean norm:

$$\forall t \in [0,T], \qquad \|\mu_t\| \leq K_\mu < \infty$$

Also, let $\sigma = (\sigma^{kn})_{1 \leq k,n \leq N}$ be any $\mathbb{R}^{N \otimes N}$-valued, $\mathbb{F}$-progressive process such that there exists a constant $K_\sigma$ uniformly bounding $\sigma$ in Euclidean norm:

$$\forall t \in [0,T], \qquad \|\sigma_t\| \leq K_\sigma < \infty$$

Denote the $k^{th}$ row and the $n^{th}$ column of $\sigma_t$ by $\sigma_t^{k\bullet}$ and $\sigma_t^{\bullet n}$, respectively. The process $\sigma$ is also assumed to be non-degenerate: there exists an $\epsilon_\sigma > 0$ such that

$$\forall x \in \mathbb{R}^N, \forall t \in [0,T], \forall \omega \in \Omega \qquad x^\top \sigma_t(\omega)\sigma_t^\top(\omega)x \geq \epsilon_\sigma \|x\|^2$$

The prices $S = (S^1, \ldots, S^N)^\top$ of the $N$ risky assets are assumed to yield gross returns over $[t, t+dt]$ (in terms of the consumption good) according to

$$\forall t \in [0,T], \qquad dS_t^k \;=\; S_t^k \left[\mu_t^k dt + \sigma_t^{k\bullet} dZ_t\right], \qquad S_0^k > 0 \text{ is constant}, \quad k = 1, \ldots, N$$

Theorem (2.1.2) and Lemma (2.1.3) apply to the SDE's with $f(q,x) = xq$ and $g(r,x) = xr$ so there exist indistinguishably unique, continuous solutions for the initial conditions $S_0^k > 0, k = 1, \ldots, N$. The component-wise explicit forms of these solutions are given by

$$\forall t \in [0,T], \qquad S_t^k \;=\; S_0^k \exp\left[\int_0^t \left(\mu_\nu^k - \frac{1}{2}\|\sigma_\nu^{k\bullet}\|^2\right) d\nu + \int_0^t \sigma_\nu^{k\bullet} dZ_\nu\right] > 0$$

as can be verified by an application of Itô's lemma. Note that Theorem (2.1.2) also gives us the bounds:

$$\forall t \in [0,T], \qquad \mathbf{E}[(S_t^k)^2] \leq C_3^k(1+(S_0^k)^2)e^{C_3^k t} \quad \text{and} \quad \mathbf{E}[\|S_t\|^2] \leq C_3(1+\|S_0\|^2)e^{C_3 t}$$

for appropriate constants $C_3^k, C_3$. Since $S_t^k > 0$ we see that the relative returns over $[t, t+dt]$ are

$$\frac{dS_t^k}{S_t^k} = \mu_t^k dt + \sigma_t^{k\bullet} dZ_t \qquad k = 1, \ldots, N$$

The innovations "$dZ_t$" in these relative returns are not known with certainty at time $t$ and so introduce "local" risk which has conditional mean zero. Thus, $\mu_t^k dt$ represents the conditional mean relative return on asset $k$ over the time interval $[t, t+dt]$. These assets are locally risky, in contrast to the bond, and are globally risky as well. The process $\sigma$ determines the intensity of the innovations "$dZ_t$"and is often referred to as the volatility process; in fact $\sigma$ is a standard deviation process in the sense that the conditional variance-covariance process for $S$ is given by $\sigma\sigma^\top$.

In our pure exchange setting, risky assets are essentially borrowing and lending instruments, just as is the bond. Since we are assuming that initial wealth is zero, so that we focus on exchanges being made between agents, these risky assets are also in zero net supply in equilibrium. If we were to assume a non-zero initial wealth then we would require these assets to be in positive net supply to hold this wealth.

**Discount Factors:** We associate with the bond a family of present value discounting factors:

$$\forall s, t \in [0,T], \qquad \mathcal{D}_{st}^r := \frac{B_s}{B_t} = \exp\left[-\int_s^t r_\nu d\nu\right]$$

If $s < t$ then one unit of the consumption good at time $t$ is worth $\mathcal{D}_{st}^r$ units at time $s$ in the sense that one can invest $\mathcal{D}_{st}^r$ units in the riskless asset at time $s$, and watch it grow to one unit by time $t$. As we will be using many discounting factors of a similar form, with $r$ replaced by some other process, we point out the useful algebra of this notation. Let $X$ and $Y$ be any $\mathbb{F}$-progressive, $\mathbb{R}$-valued processes satisfying $\int_0^T |X_\nu| d\nu < \infty$ and $\int_0^T |Y_\nu| d\nu < \infty$. Define the strictly positive discount factors associated with $X$ and $Y$ by

$$\forall s, t \in [0,T], \qquad \mathcal{D}^X_{st} := \exp\left[-\int_s^t X_\nu d\nu\right] \qquad \mathcal{D}^Y_{st} := \exp\left[-\int_s^t Y_\nu d\nu\right]$$

For all $s, t, u \in [0,T]$ we then have the algebra

$$\mathcal{D}^X_{st}\mathcal{D}^X_{tu} = \mathcal{D}^X_{tu}\mathcal{D}^X_{st} = \mathcal{D}^X_{su} = \mathcal{D}^{-X}_{us} \qquad \text{and} \qquad \mathcal{D}^X_{st}\mathcal{D}^Y_{st} = \mathcal{D}^{X+Y}_{st}$$

**State Price Density:** The state price density is a special discounting factor with a different form than $\mathcal{D}^X$, and also goes by the names "stochastic discount factor", or "pricing kernel". In some circumstances, such as equilibrium with time-separable utility specifications, the state price density is the same as the "inter-temporal marginal rate of substitution". To define the state price density, we apply the following lemma to $\sigma$, which we recall is assumed to be non-degenerate.

**Lemma 2.2.1** *Suppose $M \in \mathbb{R}^{N\otimes N}$ is a matrix for which there exists $\epsilon > 0$ such that for all $x \in \mathbb{R}^N$ we have $x^\top M M^\top x \geq \epsilon \|x\|^2$. Then, $M^\top$ and hence $M$ are non-singular and both $\|M^{-1}x\| \leq \frac{1}{\sqrt{\epsilon}}\|x\|$ and $\|(M^\top)^{-1}x\| \leq \frac{1}{\sqrt{\epsilon}}\|x\|$ hold for all $x \in \mathbb{R}^N$.*

**Proof:** See Problem (5.8.1) in Karatzas & Shreve (1991). ■

Since $\sigma$ is non-degenerate, $\sigma$ and $\sigma^\top$ are therefore non-singular, allowing us to define the market price of risk process by

$$\theta_t := \sigma_t^{-1}(\mu_t - r_t\mathbf{1})$$

which is $\mathbb{F}$-progressive. The process $\theta$ measures the relative returns in excess of the riskless rate, known as the equity premium, relative to the standard deviation process; this is a dynamic, multi-dimensional analogue of the one-dimensional Sharpe ratio which measures the excess return per unit of standard deviation, giving the additional returns the market rewards an investor for bearing an additional standard deviation of risk. Under our assumptions of uniform boundedness of $r, \mu, \sigma$ as well as non-degeneracy of $\sigma$, $\theta$ is also uniformly bounded by some constant $K_\theta$:

$$\|\theta_t\| = \|\sigma_t^{-1}(\mu_t - r_t\mathbf{1})\| \leq \frac{1}{\sqrt{\epsilon_\sigma}}\|\mu_t - r_t\mathbf{1}\| \leq \frac{1}{\sqrt{\epsilon_\sigma}}(K_\mu + NK_r) =: K_\theta < \infty$$

As a result, we have that $\mathbf{E}\Big[\exp\Big(\frac{1}{2}\int_0^T \|\theta_\nu\|^2 d\nu\Big)\Big] < \infty$ and so Novikov's Theorem (Corollary (3.5.13) in Karatzas & Shreve (1991)) ensures that

$$\eta_t := \exp\left[-\frac{1}{2}\int_0^t \|\theta_\nu\|^2 d\nu - \int_0^t \theta_\nu^\top dZ_\nu\right]$$

defines a $\mathbb{R}_{++}$-valued, continuous $\mathbb{F}$-martingale (hence is $\mathbb{F}$-progressive) with $\mathbf{E}[\eta_t] = 1$ for all $t \in [0, T]$. The process $\eta$ is associated with an absolutely continuous change of probability measure from $\mathbf{P}$ to the so-called "risk-neutral" measure. The state price density is then defined by

$$\xi_t := \mathcal{D}_{0t}^r \eta_t = \exp\left[-\int_0^t \left(r_\nu + \frac{1}{2}\|\theta_\nu\|^2\right) d\nu - \int_0^t \theta_\nu^\top dZ_\nu\right]$$

which incorporates both present value discounting and risk neutralization; the state price density is a discount factor such that the discounted price vector $\xi S$ is a martingale. Note that $\xi$ is also continuous, $\mathbb{R}_{++}$-valued, and $\mathbb{F}$-progressive. Although it appears restrictive, we narrow our attention to only those models for which

$$0 < k_\xi \le \xi_t \le K_\xi < \infty$$

holds for some constants $k_\xi, K_\xi$. Since $r$ is bounded and $\eta_t = \mathcal{D}_{t0}^r \xi_t$, we have that $\eta$ is also uniformly bounded. This restriction on $\xi$ means that $\theta_t = \sigma_t^{-1}(\mu_t - r_t \mathbf{1})$ must be quite a special process: the large deviations of the stochastic integral $\int_0^t \theta_\nu^\top dZ_\nu$ appearing in the definitions of $\eta_t$ and $\xi_t$ must be tightly controlled by $\theta$ so that the bounds on $\eta$ and $\xi$ are satisfied. We are therefore restricting our attention to models for which the processes $r, \mu$, and $\sigma$ must be closely related. Economically, one would expect risk and return to be set in equilibrium, and, as it turns out, the equilibrium state price density we eventually find is indeed bounded, showing that $r, \mu$ and $\sigma$ are tightly linked, as expected.

## 2.3 Endowment, Consumption, Portfolios, Wealth & Credit

**Consumption:** Our agents derive utility from the consumption of non-durable goods and services, all aggregated into a single representative non-durable consumption good. Non durable goods and services once consumed, cannot be retrieved. Durable goods, on the other hand, can be stored and resold. In our pure exchange setting, the presence of durable goods as a means of storing wealth is implicitly included in the economy via the presence of investment opportunities. The rates of consuming non-durable goods and services must be non-negative, as a negative consumption rate indicates that an agent is able to retrieve a non-durable good or service from storage and reintroduce it into the economy. We define the set of all physically possible representative non-durable consumption rate processes by $\mathcal{C}$:

$$\mathcal{C} := \left\{ \mathbb{F} - \text{progressive, continuous processes } C : \exists\, K_C,\ \forall t \in [0,T],\ 0 \leq C_t \leq K_C < \infty \right\}$$

Note if $C \in \mathcal{C}$ then the rate of consumption at any point in time is bounded and hence cumulative consumption $\int_0^T C_\nu d\nu$ is finite. Now, each agent $a$ can physically consume at any rate $C^a \in \mathcal{C}$. However, not all physically possible consumption rates are economically possible; agents are economically restricted to consumption rates in $\mathcal{C}$ that must satisfy further feasibility conditions, discussed below.

**Endowments:** Each agent $a$ is endowed in the single non-durable consumption good and receives this "income" at a rate given by a $\mathbb{F}$-progressive, continuous process $E^a$. We assume that there exist constants $k_E, K_E$ such that for all agents $a$ and $\forall t \in [0,T]$, $0 < k_E \leq E_t^a \leq K_E < \infty$. Endowments are exogenously specified.

**Portfolios & Wealth:** At any time $t$, agent $a$'s total "stored" wealth, denoted by $W_t^a$, is divided between the riskless asset and the $N$ risky assets. Agent $a$'s portfolio in the $N$ risky assets is assumed to be a $\mathbb{R}^N$-valued, $\mathbb{F}$-progressive process $\pi^a = (\pi^{a1}, \ldots, \pi^{aN})^\top$. The amount $\pi_t^{ak}$ is the wealth at time $t$, measured in terms of the consumption good, that agent $a$ has invested in risky asset $k$. Over the interval $[t, t+dt]$ this investment yields a relative return of $\frac{dS_t^k}{S_t^k}$. The quantity $\sum_{k=1}^N \pi_t^{ak}$ is agent $a$'s total wealth placed in the risky assets so the remainder, $W_t^a - \sum_{k=1}^N \pi_t^{ak}$, is agent $a$'s wealth in the riskless asset, which earns a relative return of $\frac{dB_t}{B_t}$ over the interval $[t, t+dt]$. If agent $a$ follows the portfolio/consumption policy $(\pi^a, C^a)$ then we model agent $a$'s resulting wealth dynamics by [4]

$$\begin{aligned} dW_t^a &= \left(W_t^a - \sum_{k=1}^N \pi_t^{ak}\right)\frac{dB_t}{B_t} + \sum_{k=1}^N \pi_t^{ak}\frac{dS_t^k}{S_t^k} + (E_t^a - C_t^a)dt \\ &= W_t^a \frac{dB_t}{B_t} + \sum_{k=1}^N \pi_t^{ak}\left(\frac{dS_t^k}{S_t^k} - \frac{dB_t}{B_t}\right) + (E_t^a - C_t^a)dt \\ &= W_t^a r_t dt + \sum_{k=1}^N \pi_t^{ak}\left((\mu_t^k - r_t)dt + \sigma_t^{k\bullet} dZ_t\right) + (E_t^a - C_t^a)dt \\ &= W_t^a r_t dt + \pi_t^{a\top}\Big((\mu_t - r_t\mathbf{1})dt + \sigma_t dZ_t\Big) + (E_t^a - C_t^a)dt \qquad (2.3.1) \end{aligned}$$

where $\mathbf{1} = (1, \ldots, 1)^\top$ is an $A \times 1$ vector of $1's$. A minimal mathematical requirement for the SDE (2.3.1) to have a solution is that the integrand of the Itô integral be square integrable in time. Thus, knowing that $\sigma$ is bounded, we define the set of physically possible portfolio processes by

$$\mathcal{P} := \left\{ \mathbb{F} - \text{progressive processes } \pi \ : \ \int_0^T \|\pi_\nu\|^2 d\nu < \infty \right\}$$

and restrict agents' portfolio choices to $\mathcal{P}$. However, as with consumption, there will be additional economic restrictions on portfolios which we discuss below.

Note that Theorem (2.1.2) and Lemma (2.1.3) cannot be applied to (2.3.1) since both coefficient functions

$$\begin{aligned} a(t,\omega,x) &= x r_t(\omega) + \pi_t^a(\omega)^\top(\mu_t(\omega) - r_t(\omega)\mathbf{1}) + E_t^a(\omega) - C_t^a(\omega) \\ b(t,\omega,x) &= \pi_t^a(\omega)^\top \sigma_t(\omega) \end{aligned}$$

involve $\pi^a$ which may not be uniformly bounded. However, (2.3.1) can still be solved:

**Proposition 2.3.2** *Under the preceding assumptions, the SDE (2.3.1) for the wealth process*

$$dW_t^a = W_t^a r_t dt + \pi_t^{a\top}\Big((\mu_t - r_t\mathbf{1})dt + \sigma_t dZ_t\Big) + (E_t^a - C_t^a)dt$$

*resulting from the policy* $(\pi^a, C^a) \in \mathcal{P} \times \mathcal{C}$ *and initial wealth* $W_0^a \in \mathbb{R}$ *has the* $\mathbb{F}$-*progressive solution*

$$W_t^a = \mathcal{D}_{t0}^r W_0^a + \int_0^t \mathcal{D}_{t\nu}^r \pi_\nu^{a\top}(\mu_\nu - r_\nu\mathbf{1})d\nu + \int_0^t \mathcal{D}_{t\nu}^r \pi_\nu^{a\top}\sigma_\nu dZ_\nu + \int_0^t \mathcal{D}_{t\nu}^r (E_\nu^a - C_\nu^a)d\nu$$

[4]See Merton (1992), Chapter 3, for a discussion of the economic assumptions implicit in a continuous time formulation.

*which is indistinguishably unique, continuous, and square integrable in time.*

**Proof:** Proceeding formally, and recalling that $\mathcal{D}^r_{0t} = \exp\left[-\int_0^t r_\nu d\nu\right]$, we obtain

$$\begin{aligned}
dW^a_t - W^a_t r_t dt &= \pi^{a\top}_t(\mu_t - r_t\mathbf{1})dt + \pi^{a\top}_t \sigma_t dZ_t + (E^a_t - C^a_t)dt \\
d(\mathcal{D}^r_{0t}W^a_t) &= \mathcal{D}^r_{0t}\pi^{a\top}_t(\mu_t - r_t\mathbf{1})dt + \mathcal{D}^r_{0t}\pi^{a\top}_t\sigma_t dZ_t + \mathcal{D}^r_{0t}(E^a_t - C^a_t)dt \\
\mathcal{D}^r_{0t}W^a_t &= W^a_0 + \int_0^t \mathcal{D}^r_{0\nu}\pi^{a\top}_\nu(\mu_\nu - r_\nu\mathbf{1})d\nu + \int_0^t \mathcal{D}^r_{0\nu}\pi^{a\top}_\nu\sigma_\nu dZ_\nu \\
&\quad + \int_0^t \mathcal{D}^r_{0\nu}(E^a_\nu - C^a_\nu)d\nu \\
W^a_t &= \mathcal{D}^r_{t0}W^a_0 + \int_0^t \mathcal{D}^r_{t\nu}\pi^{a\top}_\nu(\mu_\nu - r_\nu\mathbf{1})d\nu + \int_0^t \mathcal{D}^r_{t\nu}\pi^{a\top}_\nu\sigma_\nu dZ_\nu \\
&\quad + \int_0^t \mathcal{D}^r_{t\nu}(E^a_\nu - C^a_\nu)d\nu
\end{aligned}$$

All of the integrals in this last equation exist: $\mu$ and $r$ are bounded and $\pi^a$ is square integrable in time so the left hand integrand is also square integrable in time, hence integrable in time, implying this integral exists at each $t \in [0,T]$. Since $\sigma$ is bounded, the integrand of the Itô integral is square integrable in time and so the Itô integral exists. Finally, $E^a$, $C^a$ and $r$ are bounded so the right hand integral exists. This expression for $W^a$ is clearly $\mathbb{F}$-adapted and has almost surely continuous sample paths so is $\mathbb{F}$-progressive. To show square integrability in time, use the bounds on $r, \mu, \sigma, E^a, C^a$ as well as the Cauchy-Schwartz inequality to obtain

$$\begin{aligned}
|W^a_t| &\leq |W^a_0| + e^{K_rT}(K_\mu + NK_r)\int_0^T \|\pi^a_\nu\| d\nu \\
&\quad + \sup_{t\in[0,T]} \int_0^t \mathcal{D}^r_{t\nu}\pi^{a\top}_\nu\sigma_\nu dZ_\nu + e^{K_rT}(K_E + K_C)T \\
&=: X < \infty
\end{aligned}$$

Thus, we have $\int_0^T (W^a_\nu)^2 d\nu \leq \int_0^T X^2 d\nu = X^2T < \infty$ so $W^a$ is square integrable in time. Applying integration by parts to $\mathcal{D}^r_{0t}W^a_t$, one sees that our formally derived expression for $W^a$ solves the wealth SDE (2.3.1). To show that this is the unique solution on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$, suppose that $\tilde{W}^a$ also solves (2.3.1). Hence, we have $dW^a_t - W^a_t r_t dt = d\tilde{W}^a_t - \tilde{W}^a_t r_t dt$ so that $d(W^a_t - \tilde{W}^a_t) = (W^a_t - \tilde{W}^a_t)r_t dt$ which, by Theorem (2.1.2) and Lemma (2.1.3), has the indistinguishably unique, continuous solution $W^a_t - \tilde{W}^a_t = (W^a_0 - \tilde{W}^a_0)\mathcal{D}^r_{t0}$. For a given common initial condition, $W^a_0 = \tilde{W}^a_0$, this results in

$W_t^a - \tilde{W}_t^a = 0$ for all $t \in [0, T]$, establishing the indistinguishability of $W^a$ and $\tilde{W}^a$.

■

**Credit Restrictions:** We set $W_0^a = 0$ for all $a = 1, \dots, A$ which assumes that agents are born with no wealth.[5] Secondly, we require that all debts must be settled by time $T$, and, thirdly, debts at intermediate times cannot become arbitrarily large. On these economic grounds, we therefore define the set of credit restricted portfolio/consumption policies by

$$\mathcal{CR}^a := \left\{ (\pi^a, C^a) \in \mathcal{P} \times \mathcal{C} \ : \ W_T^a \geq 0 \ \& \ \exists K_W \geq 0, \ \forall t \in [0, T], \ W_t^a \geq -K_W > -\infty \right\}$$

It is clear that $\mathcal{CR}^a \neq \emptyset$; $0 \in \mathcal{P}$ and $E^a \in \mathcal{C}$ so $(0, E^a) \in \mathcal{P} \times \mathcal{C}$. Using the policy $(\pi^a, C^a) := (0, E^a)$ in the wealth equation just derived, and recalling that $W_0^a = 0$, gives

$$\begin{aligned} W_t^a &= \mathcal{D}_{t0}^r W_0^a + \int_0^t \mathcal{D}_{t\nu}^r \pi_\nu^{a\top} (\mu_\nu - r_\nu \mathbf{1}) d\nu + \int_0^t \mathcal{D}_{t\nu}^r \pi_\nu^{a\top} \sigma_\nu dZ_\nu + \int_0^t \mathcal{D}_{t\nu}^r (E_\nu^a - C_\nu^a) d\nu \\ &= 0 \end{aligned}$$

which implies that $(0, E^a) \in \mathcal{CR}^a$. In fact, in all that follows, we shall impose the reasonable economic requirement that it always be possible for some or all agents to completely avoid using the financial assets and simply consume their endowment as it arrives; that is, the policy $(0, E^a)$ shall always be economically possible for every agent.

By discounting wealth instead by the state price density, these credit restrictions can be reformulated. Using $d\xi_t = -\xi_t r_t dt - \xi_t \theta_t^\top dZ_t$ and the SDE (2.3.1) for $W^a$, integration by parts then yields

$$d(\xi_t W_t^a) = \xi_t (E_t^a - C_t^a) dt + \xi_t (\pi_t^{a\top} \sigma_t - W_t^a \theta_t^\top) dZ_t$$

and we obtain a form of the wealth equation that will be used most often:

$$\xi_t W_t^a + \int_0^t \xi_\nu C_\nu^a d\nu = \int_0^t \xi_\nu E_\nu^a d\nu + \int_0^t \xi_\nu (\pi_\nu^{a\top} \sigma_\nu - W_\nu^a \theta_\nu^\top) dZ_\nu \qquad (2.3.3)$$

[5]This does not preclude the possibility of a large inheritance shortly after birth which can be modeled by large values of $E_t^a$ at early times $t$ for a short duration.

Define now the set $\mathcal{A}^a$ of admissible portfolio/consumption policies for agent $a$ by

$$\mathcal{A}^a := \left\{(\pi^a, C^a) \in \mathcal{P} \times \mathcal{C} : \forall t \in [0,T], \quad \xi_t W_t^a \geq -\mathbf{E}_t\left[\int_t^T \xi_\nu E_\nu^a d\nu\right]\right\}$$

The key to relating $\mathcal{CR}^a$ and $\mathcal{A}^a$ are the following lemma and proposition:

**Lemma 2.3.4** *If $X$ is a local martingale uniformly bounded below then $X$ is a supermartingale.*

**Proof:** Let $K$ be the lower bound on $X$: $\forall t \in [0,T]$, $X_t \geq K$. Define $Y_t = X_t - K$ which then has the lower bound 0. $X$ is a local martingale and hence so is $Y$. Now, by the definition of a local martingale, there exists a sequence of $\mathbb{F}$-stopping times $\{\tau_n\}_{n=1}^{\infty}$ such that $\tau_n \uparrow \infty$ as $n \to \infty$ and for each $n$ the stopped process $\{Y_{t\wedge\tau_n}\}_{t\in[0,T]}$ is a martingale. So, for all $0 \leq t \leq s \leq T$ and $n$ we have $\mathbf{E}_t[Y_{s\wedge\tau_n}] = Y_{t\wedge\tau_n}$. Applying Fatou's lemma for conditional expectations to the sequence of non-negative random variables $\{Y_{s\wedge\tau_n}\}_{n=1}^{\infty}$ we obtain

$$\mathbf{E}_t[Y_s] = \mathbf{E}_t[\lim_{n\to\infty} Y_{s\wedge\tau_n}] = \mathbf{E}_t[\liminf_{n\to\infty} Y_{s\wedge\tau_n}] \leq \liminf_{n\to\infty} \mathbf{E}_t[Y_{s\wedge\tau_n}] = \liminf_{n\to\infty} Y_{t\wedge\tau_n} = Y_t$$

so $Y$ is a supermartingale. Adding $K$ to both sides yields that $X$ is a supermartingale. ■

**Proposition 2.3.5** *Under our assumptions, we have $\mathcal{CR}^a = \mathcal{A}^a$ and hence the no-investment consumption policy $(0, E^a)$ is in $\mathcal{A}^a$.*

**Proof:** Let $(\pi^a, C^a) \in \mathcal{CR}^a$. Thus, $(\pi^a, C^a) \in \mathcal{P} \times \mathcal{C}$, $W_T^a \geq 0$ and for some constant $K_W \geq 0$, we have that for all $t \in [0,T]$, $W_t^a \geq -K_W$. Rearranging the wealth equation (2.3.3), and using assumed bounds, we obtain

$$\int_0^t \xi_\nu({\pi_\nu^a}^\top \sigma_\nu - W_\nu^a \theta_\nu^\top) dZ_\nu = \xi_t W_t^a + \int_0^t \xi_\nu (C_\nu^a - E_\nu^a) d\nu \geq -K_\xi K_W - TK_\xi K_E$$

showing that the Itô integral is uniformly bounded below. Since $\int_0^T \|\pi_\nu^a\|^2 d\nu < \infty$, $\int_0^T (W_\nu^a)^2 d\nu < \infty$, and $\xi, \sigma, \theta$ are bounded, $\int_0^T \|\xi_\nu({\pi_\nu^a}^\top \sigma_\nu - W_\nu^a \theta_\nu^\top)\|^2 d\nu < \infty$. Thus, the Itô integral is at least a local martingale. Since it is also uniformly bounded below it is therefore a supermartingale: for all times $t, s$ such that $0 \leq t \leq s \leq T$ we have

$$\mathbf{E}_t\left[\int_t^s \xi_\nu({\pi_\nu^a}^\top \sigma_\nu - W_\nu^a \theta_\nu^\top) dZ_\nu\right] \leq 0$$

Evaluating (2.3.3) at times $t$ and $T$ then taking the difference of $\xi_t W_t^a$ and $\xi_T W_T^a$ yields

$$\xi_t W_t^a = \xi_T W_T^a + \int_t^T \xi_\nu (C_\nu^a - E_\nu^a) d\nu - \int_t^T \xi_\nu ({\pi_\nu^a}^\top \sigma_\nu - W_\nu^a \theta_\nu^\top) dZ_\nu$$

Then, applying the conditional expectation operator at time $t$ gives

$$\begin{aligned} \xi_t W_t^a &\geq \mathbf{E}_t \left[ \xi_T W_T^a + \int_t^T \xi_\nu (C_\nu^a - E_\nu^a) d\nu \right] \\ &\geq \mathbf{E}_t \left[ \int_t^T \xi_\nu (C_\nu^a - E_\nu^a) d\nu \right] \geq -\mathbf{E}_t \left[ \int_t^T \xi_\nu E_\nu^a d\nu \right] \end{aligned} \tag{2.3.6}$$

using the non-negativity of both $\xi_T W_T^a$ and $C^a$ as well as the supermartingale property of the Itô integral. The inequality (2.3.6) shows that $(\pi^a, C^a) \in \mathcal{A}^a$. Now, let $(\pi^a, C^a) \in \mathcal{A}^a$. Hence, $(\pi^a, C^a) \in \mathcal{P} \times \mathcal{C}$ and $\xi_t W_t^a \geq -\mathbf{E}_t \left[ \int_t^T \xi_\nu E_\nu^a d\nu \right] \geq -T K_\xi K_E$ which yields $W_t^a \geq -T \frac{K_\xi}{\xi_t} K_E \geq -T \frac{K_\xi}{k_\xi} K_E$. Thus, there exists a $K_W \geq 0$ such that $W_t^a \geq -K_W$ for all $t \in [0, T]$. Moreover, the admissibility constraint at time $T$ gives $\xi_T W_T^a \geq 0$ so $W_T^a \geq 0$ since $\xi_T > 0$ so $(\pi^a, C^a) \in \mathcal{CR}^a$.

This shows $\mathcal{CR}^a = \mathcal{A}^a$ and since $(0, E^a) \in \mathcal{CR}^a$ we therefore have that $(0, E^a) \in \mathcal{A}^a$.

■

As a result of Proposition (2.3.5), we can interchangeably use the properties of policies from $\mathcal{CR}^a$ and $\mathcal{A}^a$. Note that within (2.3.6) we have the inequality

$$\xi_t W_t^a \geq \mathbf{E}_t \left[ \int_t^T \xi_\nu (C_\nu^a - E_\nu^a) d\nu \right] \tag{2.3.7}$$

which, evaluated at time 0, suggests an additional characterization of economic feasibility that facilitates the determination of optimal policies:

$$\mathbf{E} \left[ \int_0^T \xi_\nu (C_\nu^a - E_\nu^a) d\nu \right] \leq W_0^a = 0 \tag{2.3.8}$$

We therefore define the set of budget feasible consumption processes for agent $a$ by

$$\mathcal{B}^a := \left\{ C^a \in \mathcal{C} : \mathbf{E} \left[ \int_0^T \xi_\nu C_\nu^a d\nu \right] \leq \mathbf{E} \left[ \int_0^T \xi_\nu E_\nu^a d\nu \right] \right\}$$

This characterization is useful since it does not explicitly refer to any portfolio used to implement a given consumption policy. The relationship between $\mathcal{A}^a$ and $\mathcal{B}^a$ is given by the following proposition:

**Proposition 2.3.9** *Considering the sets $\mathcal{A}^a$ and $\mathcal{B}^a$ as defined above, we have:*

*(1) If $(\pi^a, C^a) \in \mathcal{A}^a$ then $C^a \in \mathcal{B}^a$. In particular, $E^a \in \mathcal{B}^a$.*

*(2) If $C^a \in \mathcal{B}^a$ then there exists a portfolio $\pi^a \in \mathcal{P}$ such that $(\pi^a, C^a) \in \mathcal{A}^a$. One such portfolio is given by $\pi_t^a = (\sigma_t^\top)^{-1}\left(\xi_t^{-1}\phi_t^a + X_t^a\theta_t\right)$ where $X^a$ is the process defined by*

$$\xi_t X_t^a = \mathbf{E}_t\left[\int_t^T \xi_\nu(C_\nu^a - E_\nu^a)d\nu\right] - \mathbf{E}\left[\int_0^T \xi_\nu(C_\nu^a - E_\nu^a)d\nu\right]$$

*and $\phi^a$ is the $\boldsymbol{\lambda} \otimes \mathbf{P}$-a.e. unique $\mathbb{R}^N$-valued, $\mathbb{F}$-progressive process such that $\int_0^T \|\phi_\nu^a\|^2 d\nu < \infty$ and which represents the martingale*

$$\int_0^t \phi_\nu^{a\top} dZ_\nu = \mathbf{E}_t\left[\int_0^T \xi_\nu(C_\nu^a - E_\nu^a)d\nu\right] - \mathbf{E}\left[\int_0^T \xi_\nu(C_\nu^a - E_\nu^a)d\nu\right]$$

*The wealth process $W^a$ resulting from this particular $(\pi^a, C^a)$ policy is given by $W^a = X^a$*

*(3) If $C^a \in \mathcal{B}^a$ binds the constraint in $\mathcal{B}^a$ then the portfolio $\pi^a$ identified in (2) is the $\boldsymbol{\lambda} \otimes \mathbf{P}$-a.s. unique one such that $(\pi^a, C^a) \in \mathcal{A}^a$ and we have*

$$\begin{aligned}\int_0^t \phi_\nu^{a\top} dZ_\nu &= \mathbf{E}_t\left[\int_0^T \xi_\nu(C_\nu^a - E_\nu^a)d\nu\right] \\ \xi_t W_t^a &= \mathbf{E}_t\left[\int_t^T \xi_\nu(C_\nu^a - E_\nu^a)d\nu\right]\end{aligned}$$

**Proof:** To prove (1), let $(\pi^a, C^a) \in \mathcal{A}^a$. Thus, $C^a \in \mathcal{C}$ and with (2.3.8) we have $C^a \in \mathcal{B}^a$. And $(0, E^a) \in \mathcal{A}^a$ implies $E^a \in \mathcal{B}^a$, which also follows directly from (2.3.8). To prove (2), let $C^a \in \mathcal{B}^a$. Consider an arbitrary $\pi^a \in \mathcal{P}$ so that $(\pi^a, C^a) \in \mathcal{P} \times \mathcal{C}$. Now, $(\pi^a, C^a) \in \mathcal{A}^a$ if and only if

$$\forall t \in [0,T], \qquad \xi_t W_t^a \geq -\mathbf{E}_t\left[\int_t^T \xi_\nu E_\nu^a d\nu\right] \tag{2.3.10}$$

However, $(\pi^a, C^a)$ generates the wealth process (2.3.3). Combining (2.3.10) with (2.3.3), and rearranging, we then have that $(\pi^a, C^a) \in \mathcal{A}^a$ if and only if $\forall t \in [0,T]$

$$\begin{aligned}\int_0^t \xi_\nu\left(\pi_\nu^{a\top}\sigma_\nu - W_\nu^a\theta_\nu^\top\right)dZ_\nu &\geq -\mathbf{E}_t\left[\int_t^T \xi_\nu E_\nu^a d\nu\right] - \int_0^t \xi_\nu(E_\nu^a - C_\nu^a)d\nu \\ &= \mathbf{E}_t\left[\int_0^T \xi_\nu(C_\nu^a - E_\nu^a)d\nu\right] - \mathbf{E}_t\left[\int_t^T \xi_\nu C_\nu^a d\nu\right]\end{aligned} \tag{2.3.11}$$

The first conditional expectation in (2.3.11) can be taken as a cadlag $\mathbb{F}$-martingale (Proposition (1.3.13) in Karatzas & Shreve (1991)) and with the Martingale Representation Theorem (2.1.1) we conclude it is continuous and obtain

$$\mathbf{E}_t\left[\int_0^T \xi_\nu(C_\nu^a - E_\nu^a)d\nu\right] = \mathbf{E}\left[\int_0^T \xi_\nu(C_\nu^a - E_\nu^a)d\nu\right] + \int_0^t {\phi^a}_\nu^\top dZ_\nu \qquad (2.3.12)$$

for some $\mathbb{F}$-progressive, $\boldsymbol{\lambda}\otimes\mathbf{P}$-a.s. unique process $\phi^a$ such that $\int_0^T \|\phi_\nu^a\|^2 d\nu < \infty$. We therefore have from (2.3.11) that $(\pi^a, C^a) \in \mathcal{A}^a$ if and only if $\forall t \in [0,T]$

$$\int_0^t \left[\xi_\nu\left({\pi_\nu^a}^\top\sigma_\nu - W_\nu^a\theta_\nu^\top\right) - {\phi_\nu^a}^\top\right]dZ_\nu \;\geq\; \mathbf{E}\left[\int_0^T \xi_\nu(C_\nu^a - E_\nu^a)d\nu\right] - \mathbf{E}_t\left[\int_t^T \xi_\nu C_\nu^a d\nu\right] \qquad (2.3.13)$$

Since $C^a \in \mathcal{B}^a$, the right hand side of (2.3.13) is non-positive. Hence, finding a $\pi^a \in \mathcal{P}$ making the Itô integral zero ensures (2.3.13) holds. This $(\pi^a, C^a)$ and the wealth $W^a$ it generates ought to satisfy:

$$\begin{aligned} \xi_t\left({\pi_t^a}^\top\sigma_t - W_t^a\theta_t^\top\right) &= {\phi_t^a}^\top \\ \Longrightarrow \quad \pi_t^a &= (\sigma_t^\top)^{-1}\left(\xi_t^{-1}\phi_t^a + W_t^a\theta_t\right) \end{aligned} \qquad (2.3.14)$$

If such a $\pi^a$ exists, we can check that $(\pi^a, C^a) \in \mathcal{A}^a$ by observing that the wealth process generated by this $(\pi^a, C^a)$, together with the fact that $C^a \in \mathcal{B}^a$, yields

$$\begin{aligned} \xi_t W_t^a &= \int_0^t \xi_\nu(E_\nu^a - C_\nu^a)d\nu + \int_0^t \xi_\nu\left({\pi_\nu^a}^\top\sigma_\nu - W_\nu^a\theta_\nu^\top\right)dZ_\nu \\ &= \int_0^t \xi_\nu(E_\nu^a - C_\nu^a)d\nu + \int_0^t {\phi_\nu^a}^\top dZ_\nu \\ &= \int_0^t \xi_\nu(E_\nu^a - C_\nu^a)d\nu + \mathbf{E}_t\left[\int_0^T \xi_\nu(C_\nu^a - E_\nu^a)d\nu\right] - \mathbf{E}\left[\int_0^T \xi_\nu(C_\nu^a - E_\nu^a)d\nu\right] \\ &= \mathbf{E}_t\left[\int_t^T \xi_\nu(C_\nu^a - E_\nu^a)d\nu\right] - \mathbf{E}\left[\int_0^T \xi_\nu(C_\nu^a - E_\nu^a)d\nu\right] \qquad (2.3.15) \\ &\geq \mathbf{E}_t\left[\int_t^T \xi_\nu(C_\nu^a - E_\nu^a)d\nu\right] \;\geq\; -\mathbf{E}_t\left[\int_t^T \xi_\nu E_\nu^a d\nu\right] \qquad (2.3.16) \end{aligned}$$

(2.3.16) shows that $(\pi^a, C^a) \in \mathcal{A}^a$ and (2.3.15) motivates the choice of $X^a$. To prove that such a $\pi^a$ exists rigorously, let $C^a \in \mathcal{B}^a$ and *define* a process $X^a$ by

$$\xi_t X_t^a = \mathbf{E}_t\left[\int_t^T \xi_\nu(C_\nu^a - E_\nu^a)d\nu\right] - \mathbf{E}\left[\int_0^T \xi_\nu(C_\nu^a - E_\nu^a)d\nu\right]$$

Take $\phi^a$ as in (2.3.12) and $\pi^a = (\sigma_t{}^\top)^{-1}\left(\xi_t^{-1}\phi_t^a + X_t^a\theta_t\right)$ as suggested by (2.3.14). We must show (i) $\pi^a \in \mathcal{P}$ (ii) $(\pi^a, C^a) \in \mathcal{A}^a$ and (iii) $W^a$ generated by $(\pi^a, C^a)$ is given by $X^a$. To show (i), note that

$$\|\pi_t^a\| = \|(\sigma_t^\top)^{-1}(\xi_t^{-1}\phi_t^a + X_t^a\theta_t)\| \leq \frac{1}{\sqrt{\epsilon_\sigma}}\|\xi_t^{-1}\phi_t^a + X_t^a\theta_t\| \leq \frac{1}{\sqrt{\epsilon_\sigma}}\|k_\xi^{-1}\phi_t^a + X_t^a K_\theta\|$$

Since $\phi^a$ and $X^a$ are square integrable in time, and all processes are $\mathbb{F}$-progressive, we have $\pi^a \in \mathcal{P}$. This $(\pi^a, C^a)$ generates wealth $W^a$ which satisfies

$$\begin{aligned}
\xi_t W_t^a &= \int_0^t \xi_\nu(E_\nu^a - C_\nu^a)d\nu + \int_0^t \xi_\nu\left(\pi_\nu^{a\top}\sigma_\nu - X_\nu^a\theta_\nu^\top\right)dZ_\nu \\
&= \int_0^t \xi_\nu(E_\nu^a - C_\nu^a)d\nu + \int_0^t \phi_\nu^{a\top}dZ_\nu \\
&= \int_0^t \xi_\nu(E_\nu^a - C_\nu^a)d\nu + \mathbf{E}_t\left[\int_0^T \xi_\nu(C_\nu^a - E_\nu^a)d\nu\right] - \mathbf{E}\left[\int_0^T \xi_\nu(C_\nu^a - E_\nu^a)d\nu\right] \\
&= \mathbf{E}_t\left[\int_t^T \xi_\nu(C_\nu^a - E_\nu^a)d\nu\right] - \mathbf{E}\left[\int_0^T \xi_\nu(C_\nu^a - E_\nu^a)d\nu\right] \qquad (2.3.17) \\
&\geq \mathbf{E}_t\left[\int_t^T \xi_\nu(C_\nu^a - E_\nu^a)d\nu\right] \geq -\mathbf{E}_t\left[\int_t^T \xi_\nu E_\nu^a d\nu\right] \qquad (2.3.18)
\end{aligned}$$

(2.3.18) implies $(\pi^a, C^a) \in \mathcal{A}^a$. (2.3.17) implies $W^a = X^a$, proving (2).

To prove (3), suppose $C^a \in \mathcal{B}^a$ binds (2.3.8). Let $\pi^a \in \mathcal{P}$ be any portfolio such that $(\pi^a, C^a) \in \mathcal{A}^a$ ((2) shows at least one exists). Now, $(\pi^a, C^a) \in \mathcal{A}^a$ implies

$$\xi_t W_t^a \geq \mathbf{E}_t\left[\int_t^T \xi_\nu(C_\nu^a - E_\nu^a)d\nu\right]$$

Combining this with the wealth equation (2.3.3) we obtain

$$\begin{aligned}
0 &\leq \xi_t W_t^a + \mathbf{E}_t\left[\int_t^T \xi_\nu(E_\nu^a - C_\nu^a)d\nu\right] \\
&= \mathbf{E}_t\left[\int_0^T \xi_\nu(E_\nu^a - C_\nu^a)d\nu\right] + \int_0^t \xi_\nu(\pi_\nu^{a\top}\sigma_\nu - W_\nu^a\theta_\nu^\top)dZ_\nu \qquad (2.3.19)
\end{aligned}$$

Taking expectations of (2.3.19), using the supermartingale property and the fact that $C^a$ binds its constraint, we obtain

$$0 \leq \mathbf{E}\left[\xi_t W_t^a + \mathbf{E}_t\left[\int_t^T \xi_\nu(E_\nu^a - C_\nu^a)d\nu\right]\right] \leq \mathbf{E}\left[\int_0^T \xi_\nu(E_\nu^a - C_\nu^a)d\nu\right] = 0 \qquad (2.3.20)$$

Now, (2.3.19) and (2.3.20) together imply

$$\xi_t W_t^a + \mathbf{E}_t\left[\int_t^T \xi_\nu (E_\nu^a - C_\nu^a)d\nu\right] = 0 \tag{2.3.21}$$

showing that if $C^a$ is any process that binds the constraint and $\pi^a$ is any portfolio such that $(\pi^a, C^a) \in \mathcal{A}^a$ then the wealth process generated by $(\pi^a, C^a)$ *must* have the form given by (2.3.21). Inequality (2.3.19) then becomes an equality:

$$\mathbf{E}_t\left[\int_t^T \xi_\nu (C_\nu^a - E_\nu^a)d\nu\right] = \xi_t W_t^a = \int_0^t \xi_\nu (E_\nu^a - C_\nu^a)d\nu + \int_0^t \xi_\nu ({\pi_\nu^a}^\top \sigma_\nu - W_\nu^a \theta_\nu^\top)dZ_\nu$$

Rearranging, we obtain

$$\mathbf{E}_t\left[\int_0^T \xi_\nu (C_\nu^a - E_\nu^a)d\nu\right] = \int_0^t \xi_\nu ({\pi_\nu^a}^\top \sigma_\nu - W_\nu^a \theta_\nu^\top)dZ_\nu \tag{2.3.22}$$

From (2.3.12) and the fact that $C^a$ binds the constraint in $\mathcal{B}^a$ we have

$$\int_0^t {\phi_\nu^a}^\top = \mathbf{E}_t\left[\int_0^T \xi_\nu (C_\nu^a - E_\nu^a)d\nu\right]$$

and so (2.3.22) becomes

$$\forall t \in [0,T], \qquad \int_0^t {\phi_\nu^a}^\top dZ_\nu = \int_0^t \xi_\nu ({\pi_\nu^a}^\top \sigma_\nu - W_\nu^a \theta_\nu^\top)dZ_\nu \tag{2.3.23}$$

By the Martingale Representation Theorem (2.1.1), equation (2.3.23) implies that

$${\phi^a}^\top = \xi\left({\pi^a}^\top \sigma - W^a \theta^\top\right)$$

holds $\boldsymbol{\lambda} \otimes \mathbf{P}$-a.s. The invertibility of $\sigma$ and strict positivity of $\xi$ then enables one to $\boldsymbol{\lambda} \otimes \mathbf{P}$-a.s. uniquely determine the required portfolio as in (2.3.14). ∎

Proposition (2.3.9) shows that $\mathcal{B}^a$ is the Euclidean projection of $\mathcal{A}^a$ onto $\mathcal{C}$, and, on the set of consumption streams that bind the budget constraint (2.3.8), this projection is 1-1, identifying processes that are $\boldsymbol{\lambda} \otimes \mathbf{P}$-a.s. the same. This also shows that the portfolio component of any portfolio/consumption policy is essentially auxiliary if one is concerned primarily with consumption.

Much of the material just presented can be found in various forms and in various places in the literature, some starting with Pliska (1986), Karatzas et al. (1987), and Cox & Huang (1989). For a recent compilation, see Karatzas & Shreve (1998), as well as the references therein. We include the material just discussed to make the presentation self-contained as well as to introduce the notation and assumptions used later in this and the next chapter.

## 2.4 Consumption Configurations, Interdependent Habits & Utility

In this section, we introduce a fairly general mechanism by which agents form habits on past consumption and how they derive utility from consumption in excess of their consumption habits. We indicate how this model includes many models of habit formation already discussed in the literature as well as how our generalization introduces a new feature: interaction. In our model, interdependencies may exist between agents' consumption and habit levels, leading to so-called "consumption externalities". As a result of these externalities, we can no longer treat an agent as an isolated, independent agent, taking prices as given and responding only to prices with, in the language of game theory, a dominant optimal strategy. Instead, our agents must consider the economy-wide configuration of consumption in the determination of their optimal responses. To formalize this, it is necessary to introduce some notation for configurations:

The sets of physically possible consumption and portfolio configuration choices are defined by

$$\boldsymbol{\mathcal{C}} = \mathcal{C}^A \qquad \text{and} \qquad \boldsymbol{\mathcal{P}} = \mathcal{P}^A$$

whose elements are denoted by $\mathbf{C} = (C^1, \ldots, C^A)^\top$ and $\boldsymbol{\pi} = (\pi^1, \ldots, \pi^A)$, respectively. We denote the given endowment configuration similarly by $\mathbf{E} = (E^1, \ldots, E^A)^\top$. The set of budget feasible consumption configurations $\boldsymbol{\mathcal{B}}$ is specified by

$$\boldsymbol{\mathcal{B}} = \prod_{a=1}^{A} \mathcal{B}^a \tag{2.4.1}$$

with elements $\mathbf{C} = (C^1, \ldots, C^A)^\top \in \boldsymbol{\mathcal{B}}$. Similarly, the set $\boldsymbol{\mathcal{A}}$ of admissible portfolio/consumption policy configurations is given by

$$\boldsymbol{\mathcal{A}} = \prod_{a=1}^{A} \mathcal{A}^a \tag{2.4.2}$$

A policy element of $\mathcal{A}^a$ is denoted by $(\pi^a, C^a)$ and a policy configuration element of $\boldsymbol{\mathcal{A}}$ is denoted by $[\boldsymbol{\pi}, \mathbf{C}] = ((\pi^1, C^1), \ldots, (\pi^A, C^A))$. Denote by $H_t^a$ agent $a$'s consumption habit level at time $t$ and denote the full configuration of consumption habits by $\mathbf{H} =$

$(H^1, \ldots, H^A)^\top$. We assume that the full configuration of habits $\mathbf{H}$ is a generalized moving average over past consumption streams:

$$\mathbf{H}_t = \int_0^t \chi_{\nu t} \mathbf{C}_\nu d\nu \tag{2.4.3}$$

Here, $\chi = \{\chi^{ab}\}_{a,b=1}^A$ is an $\mathbb{R}_+^{A \otimes A}$-valued, doubly time-indexed random field such that

$$0 \le \chi_{\nu t}^{ab} \le K_\chi < \infty \qquad \forall t, \nu \in [0,T] \qquad \forall a, b$$

Since $\chi$ and $\mathbf{C}$ are bounded, this integral exists. Also, $\chi$ is assumed to be $\mathbb{F}$-progressive and continuous in the following senses: for all $t, \nu \in [0,T]$, $\chi_{t\nu} \in \mathcal{F}_{t \vee \nu}$, and, for all $(t, \omega) \in [0,T] \times \Omega$, the maps $\nu \mapsto \chi_{\nu t}(\omega)$ and $\nu \mapsto \chi_{t\nu}(\omega)$ are continuous. $\mathbf{H}$ is therefore also $\mathbb{F}$-progressive and continuous.

This linear habit formation mechanism allows for the possibility that (i) an agent's habits can be influenced by other agents' consumption, (ii) the degree of influence can vary with time and (iii) the degree of influence can vary with the state. As $\mathbf{C}$ will be endogenously determined in equilibrium through utility optimization, habits will also be endogenously formed. It is sometimes convenient to work with habits component-wise and so we note the form here for later use:

$$H_t^a = \int_0^t \sum_{b=1}^A \chi_{\nu t}^{ab} C_\nu^b d\nu = \psi_t^a + \int_0^t \chi_{\nu t}^{aa} C_\nu^a d\nu \tag{2.4.4}$$

where we define

$$\psi_t^a = \int_0^t \sum_{b \neq a} \chi_{\nu t}^{ab} C_\nu^b d\nu \tag{2.4.5}$$

The process $\psi^a$ represents the contribution to $H^a$ of all other agents' consumption choices. Note that $\psi^a$ does not change if only agent $a$'s consumption $C^a$ changes. In the case where $\chi$ is a diagonal matrix, $\psi^a = 0$, so agent $a$'s habits can only change through a change in his own consumption $C^a$; this habit formation structure is usually referred to as "internal"; an agent's consumption and habit processes do not directly influence nor are directly influenced by those of any other agent. If all the diagonal elements of $\chi$ are zero then agent $a$'s habits can only change through a change in the consumption choices of other agents. This case involves a complete absence of internal habit formation and is similar to, but not precisely like, what is referred to as "external habit formation". External habit formation is a term normally used to indicate that individual habits are formed on some aggregate of consumption, such as per capita consumption, rather than on an individual consumption stream. Under the usual assumption of price taking behavior, an individual agent cannot influence such consumption aggregates, hence the term "external". If, for example, all elements of $\boldsymbol{\chi}$ are the same, then each agent forms habits on some moving average of per capita consumption and we have externally formed habits. If, instead, the elements of $\boldsymbol{\chi}$ are non-zero but not always equal, then habits are formed through a blend of internal and external influences with the additional complication that the aggregate of consumption used in the external component may change over time and over states of uncertainty, potentially differing considerably from per capita consumption. For instance, the components of $\boldsymbol{\chi}$ could depend differently on different real macroeconomic factors, such as aggregate endowment or unemployment. It is hoped that this richer habit formation mechanism can provide new ways to link financial and real markets as well as yield asset price behavior not seen in the usual representative agent models of habit formation.

We now specify how individual agents derive utility from a consumption configuration $\mathbf{C} \in \boldsymbol{\mathcal{B}}$, which will involve the habit configuration $\mathbf{H}$ it generates. Agent $a$ is assumed to derive instantaneous utility from intermediate consumption rates according to the measurable function $u^a : [0,T] \times \Omega \times (0,\infty) \to \mathbb{R}$ which is assumed to satisfy the following properties:

U1: For each $(t, \omega)$ we have $u^a(t,\omega,\cdot) : (0,\infty) \to \mathbb{R}$ is $C^3$.

For each $\omega$ we have $u^a(\cdot,\omega,\cdot) : [0,T] \times (0,\infty) \to \mathbb{R}$ is $C^0$.

For all $\mathbb{F}$-progressive, continuous, $\mathbb{R}_{++}$-valued processes $X$, $u^a(t,\omega,X_t(\omega))$ is also $\mathbb{F}$-progressive and continuous.

U2: For each $(t,\omega)$ we have $u^{a'}(t,\omega,x) > 0$ where the prime denotes the derivative in $x$.

For each $(t,\omega)$ we have $u^{a''}(t,\omega,x) < 0$ where the double prime denotes the second derivative in $x$.

U3: For each $(t,\omega)$ we have $u^{a'}(t,\omega,0) := \lim_{x\downarrow 0} u^{a'}(t,\omega,x) = \infty$ and $u^{a'}(t,\omega,\infty) := \lim_{x\uparrow\infty} u^{a'}(t,\omega,x) = 0$

U4: For each $x$ there exist constants $k_u(x), K_u(x)$ such that for all $(t,\omega)$ and $a$ we have $-\infty < k_u(x) \leq u^a(t,\omega,x) \leq K_u(x) < \infty$.

U5: From U2, $u^{a'}$ as a function of $x$ has an inverse $I^a$: $I^a(t,\omega,u^{a'}(t,\omega,x)) = x$ and $u^{a'}(t,\omega,I^a(t,\omega,y)) = y$ for all $x,y \in (0,\infty)$ and $(t,\omega)$. We assume that for each $y$ there are constants $k_I(y), K_I(y)$ such that for all $(t,\omega)$ and $a$ we have $-\infty < k_I(y) \leq I^a(t,\omega,y) \leq K_I(y) < \infty$.

Property U1 is a smoothness and measurability assumption. Property U2 captures the notion of risk aversion. U3 are the Inada conditions to help guarantee the existence of a unique, interior optimum. U4 and U5 are bounds on $u^a$ and $I^a$ that are uniform in $(t,\omega)$ for each $x$ or $y$. An immediate consequences of these properties is that for any strictly positive, $\mathbb{F}$-progressive and continuous process $X$, $I^a(t,\omega,X_t(\omega))$ is also $\mathbb{F}$-progressive and continuous. We extend the domain of $u_t^a$ to $\mathbb{R}$ by setting $u^a(t,\omega,x) = -\infty$ for all $x \leq 0$, $(t,\omega)$. When suppressing references to $\omega \in \Omega$ we use the notation $u_t^a(x)$ in place of $u^a(t,\omega,x)$.

A prototypical example of an instantaneous utility function, one we will be considering in more detail in Chapter 3 on simulations, is the following:

**Example 2.4.6** *Suppose future instantaneous utility for agent $a$ is discounted at the subjective rate $\beta_t^a$ (patience) where $\beta^a$ is a $\mathbb{R}_+$-valued, $\mathbb{F}$-progressive, uniformly bounded process and the agent has power utility with an $\mathbb{F}$-progressive, uniformly bounded, $\mathbb{R}_{++}$-valued process $\alpha^a$ describing relative risk aversion. Using power utility, agent a's instantaneous utility would then be given by*

$$u_t^a(x) = \begin{cases} \mathcal{D}_{0t}^{\beta^a}\left(\frac{x^{1-\alpha_t^a}-1}{1-\alpha_t^a}\right) & \text{if } \alpha_t^a \in (0,1)\cup(1,\infty) \\ \mathcal{D}_{0t}^{\beta^a}\ln x & \text{if } \alpha_t^a = 1 \end{cases}$$

*Note that the instantaneous utility is defined so that as the process $\alpha_t^a$ passes through 1, the utility function changes continuously from one part of the piecewise definition to the other.*

For any consumption configuration $\mathbf{C} \in \boldsymbol{\mathcal{B}}$ there is the uniquely determined associated habit configuration $\mathbf{H}$. Agent $a$ is assumed to derive utility from this configuration $\mathbf{C}$ via the consumption in excess of his habit level, $C^a - H^a$, according to the utility functional $\mathcal{U}^a : \boldsymbol{\mathcal{B}} \to \overline{\mathbb{R}}$ defined by

$$\mathcal{U}^a(\mathbf{C}) = \begin{cases} \mathbf{E}\left[\int_0^T u_\nu^a(C_\nu^a - H_\nu^a)d\nu\right] & \text{if the expectation is defined and is in } (-\infty, \infty] \\ -\infty & \text{otherwise} \end{cases}$$

Agent $a$ would like to maximize his utility functional subject to the resource restrictions he faces in his economic environment. Note that although agent $a$ can only directly change his own consumption component $C^a$, the argument of agent $a$'s utility functional is necessarily the entire configuration $\mathbf{C}$ since all of $\mathbf{C}$ is required to determine his habits $H^a$; agent $a$'s utility depends on the consumption choices of all other agents, that is, there are consumption externalities. Agent $a$ seeks to determine an optimal consumption response to any given consumption configuration presented to him by all other agents.

It is conceivable that some set of model parameters and some set of configurations in $\boldsymbol{\mathcal{B}}$ result in consumption falling below habit for some agents. Thus, we must determine model parameters and a subset of $\boldsymbol{\mathcal{B}}$ for which this does not occur, which we do in the next section.

## 2.5 Subsistence & Utilizable Consumption Configurations

In the last section, we set $u_t^a(x) = -\infty$ for $x \leq 0$, and so we are attaching an infinite penalty to utility when consumption is at or below habit. Moreover, with the Inada condition $u^{a\prime}(0) = \infty$, we see that agent $a$ becomes increasingly averse to the risk, in absolute terms, of consumption dropping to or below habit and so will exert considerable effort to avoid this risk. This sort of habit formation is therefore often referred to as "addictive". [6] Thus, we need to determine which moving average processes $\boldsymbol{\chi}$ and which elements of $\boldsymbol{\mathcal{B}}$ will enable each agent to avoid $-\infty$ for utility.

[6] See Shrikhande (1997) and Detemple & Karatzas (2003) for recent treatments of non-addictive habit formation.

The economic assumption we use to formulate necessary restrictions on $\chi$ and $\mathcal{B}$ is that it should always be possible for all agents to avoid using the financial assets and to simply consume their endowment as it arrives. In other words, if $\mathbf{C} = \mathbf{E}$, we require that all agents should have utilities greater than $-\infty$.

In order to proceed, fix $\mathbf{C} \in \mathcal{B}$ and fix an agent $a$. Note that agent $a$ does not have direct influence on any component of $\mathbf{C}$ other than his own consumption component $C^a$. He must therefore take $C^b$, $\forall b \neq a$, as given. For this fixed $\mathbf{C}$, agent $a$'s habit level is given by (2.4.4):

$$H_t^a = \psi_t^a + \int_0^t \chi_{\nu t}^{aa} C_\nu^a d\nu$$

Agent $a$ must be able to choose a consumption component, given $C^b$, $\forall b \neq a$, such that consumption remains sufficiently greater than habit. The boundary case to be avoided is where agent $a$'s consumption equals his habit. We therefore define *agent a's subsistence consumption given* $\mathbf{C}$ by $C_t^{as} = H_t^{as}$. Substituting $C_t^{as} = H_t^{as}$ into (2.4.4) shows that $C^{as}$ ought to satisfy

$$C_t^{as} = \psi_t^a + \int_0^t \chi_{\nu t}^{aa} C_\nu^{as} d\nu$$

Using Theorem (2.1.4), we know that there is a unique solution, up to indistinguishability, and that it has the following representation:

$$C_t^{as} = \psi_t^a + \int_0^t \overleftarrow{\chi_{\nu t}^{aa}} \psi_\nu^a d\nu \tag{2.5.1}$$

Since $\chi^{aa}$ is non-negative, $\overleftarrow{\chi^{aa}}$ is also non-negative. Also, since $\psi^a$ is non-negative, we have $C^{as} \in \mathcal{C}$, boundedness also being clear. The series representation (2.5.1) shows clearly how the moving average of consumption of all other agents, as expressed in $\psi^a$, helps determine subsistence consumption $C^{as}$ for agent $a$.

Starting from $\mathbf{C} \in \mathcal{B}$, agent $a$ can adjust his consumption component $C^a$ in $\mathbf{C}$ to $C^{as}$, resulting in the configuration $\mathbf{C}^{as} = (C^1, \ldots, C^{a-1}, C^{as}, C^{a+1}, \ldots, C^A)$. So, given $\mathbf{C}$, the configuration $\mathbf{C}^{as}$ is the "worst" that agent $a$ can afford. In other words, we require that $C^{as} \in \mathcal{B}^a$, which places restrictions on $C^b$, $\forall b \neq a$. Even if $C^{as}$ is affordable, the configuration $\mathbf{C}^{as}$ itself will yield $\mathcal{U}^a(\mathbf{C}^{as}) = -\infty$ since $C^{as} = H^{as}$ so we require something stronger; agent $a$ must be able to afford *strictly* more than $C^{as}$.

So, from agent $a$'s point of view, the subset of $\mathcal{B}$ he is willing to consider is defined by

$$\mathcal{U}^a\mathcal{B} = \left\{ \mathbf{C} \in \mathcal{B} : \mathbf{E}\left[\int_0^T \xi_\nu C_\nu^{as} d\nu\right] < \mathbf{E}\left[\int_0^T \xi_\nu E_\nu^a d\nu\right] \right\}$$

which we refer to as the set of consumption configurations utilizable by agent $a$. Consumption configurations outside of $\mathcal{U}^a\mathcal{B}$ will be strongly resisted by agent $a$, due to increasing absolute risk aversion of consumption falling toward habit. The set of configurations utilizable by all agents is therefore given by

$$\mathcal{U}\mathcal{B} = \bigcap_{a=1}^{A} \mathcal{U}^a\mathcal{B}$$

$\mathcal{U}\mathcal{B}$ is the set of consumption configurations such that all agents can afford strictly more than the subsistence consumption level established by all other agents and will be the set of strategies which are acceptable by all economic players. More will be said on this point in Section 2.7.

Of course, we require that $\mathcal{U}\mathcal{B} \neq \emptyset$, otherwise our model is vacuous. In particular, as mentioned, we require that it is always economically possible for all agents to simply consume their endowment as it arrives: we require $\mathbf{E} \in \mathcal{U}\mathcal{B}$.

**Proposition 2.5.2** *Suppose the averaging process $\chi$ and endowments $\mathbf{E}$ are such that*

$$AK_\chi T e^{K_\chi T} < \frac{k_E}{K_E}$$

*Then, for all $a$ we have $\mathcal{U}^a(\mathbf{E}) \in (-\infty, \infty)$, and, $\mathbf{E} \in \mathcal{U}\mathcal{B}$.*

This condition ensures that $K_\chi$, the upper bound on habit formation intensity, is kept sufficiently low for given bounds on individual endowments.

**Proof:** First, we check that for all $a$ we have $\mathcal{U}^a(\mathbf{E}) \in (-\infty, \infty)$. From (2.4.4) with $\mathbf{C} = \mathbf{E}$, individual habits are seen to satisfy

$$H_t^a = \psi_t^a + \int_0^t \chi_{\nu t}^{aa} E_\nu^a d\nu = \int_0^t \sum_{b=1}^{A} \chi_{\nu t}^{ab} E_\nu^b d\nu \leq AK_E K_\chi T e^{K_\chi T} < k_E \leq E_t^a$$

So, for $\epsilon = k_E - AK_E K_\chi T e^{K_\chi T} > 0$ we have $E_t^a - H_t^a \geq \epsilon > 0$ for all $a$ and $t$. This implies

$$\begin{aligned} \mathcal{U}^a(\mathbf{E}) &= \mathbf{E}\left[\int_0^T u_\nu^a(E_\nu^a - H_\nu^a)d\nu\right] \geq \mathbf{E}\left[\int_0^T u_\nu^a(\epsilon)d\nu\right] \\ &\geq \mathbf{E}\left[\int_0^T k_u(\epsilon)d\nu\right] = Tk_u(\epsilon) > -\infty \end{aligned}$$

for all $a = 1, \ldots, A$. With $E^a \leq K_E$ for all $a$

$$\begin{aligned} \mathcal{U}^a(\mathbf{E}) &= \mathbf{E}\left[\int_0^T u_\nu^a(E_\nu^a - H_\nu^a)d\nu\right] \leq \mathbf{E}\left[\int_0^T u_\nu^a(K_E)d\nu\right] \\ &\leq \mathbf{E}\left[\int_0^T K_u(K_E)d\nu\right] = TK_u(K_E) < \infty \end{aligned}$$

Next, we check strict affordability of subsistence consumption $C^{as}$ for each agent $a$ given $\mathbf{E}$. Fix $a$. $C^{as}$ given $\mathbf{E}$ has the representation

$$C_t^{as} = \psi_t^a + \int_0^t \overleftarrow{\chi}_{\nu t}^{aa} \psi_\nu^a d\nu$$

As just seen, $\psi_t^a \leq H_t^a \leq AK_E K_\chi t$. Hence,

$$\begin{aligned} C_t^{as} &\leq AK_E K_\chi t\left(1 + \int_0^t \overleftarrow{\chi}_{\nu t}^{aa} d\nu\right) \\ &\leq AK_E K_\chi t e^{K_\chi t} \\ &\leq AK_E K_\chi T e^{K_\chi T} \\ &< k_E \leq E_{t_0}^a \end{aligned}$$

using the bound for the scalar process $\overleftarrow{\chi}^{aa}$ derived in Theorem (2.1.4). As a result, for every agent $a$, $\mathbf{E}\left[\int_0^T \xi_\nu C_\nu^{as} d\nu\right] < \mathbf{E}\left[\int_0^T \xi_\nu E_\nu^a d\nu\right]$ showing $\mathbf{E} \in \mathcal{UB}^a$ for all $a$ which in turn shows $\mathbf{E} \in \mathcal{UB}$. ■

Note that this proposition essentially says that each component $\chi^{ab}$ must scale as $1/A$. With $\mathcal{UB} \neq \emptyset$ established for averaging processes and endowments satisfying the condition in Proposition (2.5.2), all agents can now consider maximizing utility over the set of utilizable $\mathbf{C}$ for each agent.

## 2.6 Individually Optimal Consumption Configurations

In the absence of interactions, and acting as price takers, each agent can individually determine his optimal portfolio/consumption policy without considering the consumption choices of other agents in the economy. If each agent does so, this optimization process leads to a configuration of portfolio/consumption policies that is optimal for all agents *simultaneously*. In other words, uncoordinated individual optimizations, in this case, lead to a simultaneous optimization.

However, with consumption externalities, the agents' optimization problems become connected and so the degree of coordination required to reach a configuration that is simultaneously optimal will be greater. As a result, one cannot simply solve for each individual's optimal policy, as in the no interaction, pure price-taking case, and expect to end up with simultaneous optimality as a by product. Nevertheless, we can approach this more coordinated, simultaneous utility maximization in a similar manner to the no-interaction case by first considering an individual's optimization problem.

For a fixed agent $a$ and from any starting configuration $\hat{\mathbf{C}} = (\hat{C}^1, \ldots, \hat{C}^A) \in \mathcal{UB}$, we show that agent $a$ has a unique optimal adjustment of his consumption component from $\hat{C}^a$ to $\hat{C}^{a\diamond}$ which is given by a unique mathematical form. This new configuration, denoted by

$$\hat{\mathbf{C}}^{a\diamond} = (\hat{C}^1, \ldots, \hat{C}^{a-1}, \hat{C}^{a\diamond}, \hat{C}^{a+1}, \ldots, \hat{C}^A)$$

is optimal for $a$, but, because this individual adjustment was not coordinated with those of all other agents, $\hat{\mathbf{C}}^{a\diamond}$ may not be optimal for some or all of the other agents. In fact, agent $a$ choosing $\hat{C}^{a\diamond}$ may establish subsistence levels for other agents that are not strictly affordable, and, consequently, agent $a$'s optimal choice therefore may not be in $\mathcal{UB}$. Again, we leave this issue to Section 2.7.

Assume from here on that the condition in Proposition (2.5.2) is satisfied so that $\mathrm{E} \in \mathcal{UB}$. To proceed with the individual optimization problem, fix an agent $a$ and fix an arbitrary starting configuration $\hat{\mathbf{C}} \in \mathcal{UB}$, not necessarily $\mathbf{E}$. In maximizing his utility, agent $a$ can consider the configurations resulting from varying only his consumption component $\hat{C}^a$ of $\hat{\mathbf{C}}$. Thus, we define the set of utilizable variations from $\hat{\mathbf{C}}$ open to agent $a$ by

$$\mathcal{U}^a\mathcal{B}(\hat{\mathbf{C}}) := \left\{ \mathbf{C} \in \mathcal{U}^a\mathcal{B} : \forall b \neq a, \quad C^b = \hat{C}^b \right\}$$

Given $\hat{\mathbf{C}}$, agent $a$ attempts to solve the problem

$$\sup \left\{ \mathcal{U}^a(\mathbf{C}) : \mathbf{C} \in \mathcal{U}^a\mathcal{B}(\hat{\mathbf{C}}) \right\}$$

Since $\hat{\mathbf{C}} \in \mathcal{UB} \subset \mathcal{U}^a\mathcal{B}$ we have $\hat{\mathbf{C}} \in \mathcal{U}^a\mathcal{B}(\hat{\mathbf{C}})$ and so $\mathcal{U}^a\mathcal{B}(\hat{\mathbf{C}}) \neq \emptyset$. Thus, the optimization problem is not trivial. As we shall see, agent $a$ is always able to find a $\boldsymbol{\lambda} \otimes \mathbf{P}$-a.s. unique $\hat{\mathbf{C}}^{a\diamond} \in \mathcal{U}^a\mathcal{B}(\hat{\mathbf{C}})$ which solves his optimization problem. Note that $\hat{\mathbf{C}}^{a\diamond}$ will depend on $\hat{C}^b$ for all $b \neq a$, and, $\mathbf{C}^{a\diamond}$ may not be optimal for some or all agents $b \neq a$. Moreover, as mentioned, $\mathbf{C}^{a\diamond}$ may not even be in $\mathcal{UB}$. We allow this possibility for the moment, but, when we consider all agents simultaneously, we expect that if economic coordination is functioning properly, and equilibrium obtains, coordinated optimal choices ought to lie in $\mathcal{UB}$. See Section 2.7.

To determine the optimal individual consumption policy for agent $a$ given the consumption choices $\hat{\mathbf{C}}$ of all other agents, we proceed, heuristically, with the optimization of

$$\mathcal{U}^a(\mathbf{C}) = \mathbf{E}\left[\int_0^T u_\nu^a(C_\nu^a - H_\nu^a)d\nu\right]$$

over the set $\mathcal{U}^a\mathcal{B}(\hat{\mathbf{C}})$. The main constraint that agent $a$ has to satisfy is the budget constraint (2.3.8) in the definition of $\mathcal{B}^a$:

$$\mathbf{E}\left[\int_0^T \xi_\nu(C_\nu^a - E_\nu^a)d\nu\right] \leq 0$$

Because of market completeness, it is well known that this optimization can instead be done separately at each time $t$ and state $\omega$.[7] Thus, we consider optimizing the remaining utility to be derived over $[t, T]$, for any $t \in [0, T]$,

$$\mathcal{U}_t^a(\mathbf{C}) = \mathbf{E}_t\left[\int_t^T u_\nu^a(C_\nu^a - H_\nu^a)d\nu\right]$$

subject to the dynamic version of the budget constraint (2.3.8) given in (2.3.7):

$$\mathbf{E}_t\left[\int_t^T \xi_\nu(C_\nu^a - E_\nu^a)d\nu\right] \leq \xi_t W_t^a$$

[7]See the references in Section 1.1

After solving the heuristically derived first order conditions for optimality and identifying a candidate optimal policy, we then establish rigorously that our candidate is indeed optimal.

To this end, define agent $a$'s Lagrangian at time $t$ by

$$\mathcal{L}_t^a(\mathbf{C}; y^a) := \mathcal{U}_t^a(\mathbf{C}) + y^a \left\{ \xi_t W_t^a - \mathbf{E}_t \left[ \int_t^T \xi_\nu (C_\nu^a - E_\nu^a) d\nu \right] \right\}$$

where $y^a > 0$ is a constant multiplier. Although we are writing this Lagrangian as a function of the full configuration $\mathbf{C}$, agent $a$ can only vary his component $C^a$. Now, differentiating this Lagrangian, formally expressing derivatives inside integrals as Dirac delta functions[7],

$$\frac{\partial C_\nu^a}{\partial C_t^a} = \delta(\nu - t)$$

we obtain the first order conditions for optimality:

$$\begin{aligned} \frac{\partial}{\partial C_t^a} \mathcal{L}_t^a(\mathbf{C}; y^a) &= \frac{\partial}{\partial C_t^a} \mathcal{U}_t^a(\mathbf{C}) - y^a \mathbf{E} \left[ \int_t^T \xi_\nu \delta(\nu - t) d\nu \right] = \frac{\partial}{\partial C_t^a} \mathcal{U}_t^a(\mathbf{C}) - y^a \xi_t = 0 \\ \frac{\partial}{\partial y^a} \mathcal{L}_t^a(\mathbf{C}; y^a) &= \mathbf{E}_t \left[ \int_t^T \xi_\nu (C_\nu^a - E_\nu^a) d\nu \right] - \xi_t W_t^a = 0 \end{aligned} \qquad (2.6.1)$$

We now need to compute $\frac{\partial}{\partial C_t^a} \mathcal{U}_t^a(\mathbf{C})$ then solve for $C^a$ and $y^a$. We continue to differentiate formally:

$$\begin{aligned} \frac{\partial}{\partial C_t^a} \mathcal{U}_t^a(\mathbf{C}) &= \mathbf{E}_t \left[ \int_t^T u_\nu^{a\prime}(C_\nu^a - H_\nu^a) \left( \delta(\nu - t) - \frac{\partial H_\nu^a}{\partial C_t^a} \right) d\nu \right] \\ &= u_t^{a\prime}(C_t^a - H_t^a) - \mathbf{E}_t \left[ \int_t^T u_\nu^{a\prime}(C_\nu^a - H_\nu^a) \left( \frac{\partial H_\nu^a}{\partial C_t^a} \right) d\nu \right] \end{aligned} \qquad (2.6.2)$$

In order to compute $\frac{\partial H_\nu^a}{\partial C_t^a}$ we recall our component-wise solution (2.4.4) for $H^a$:

$$H_\nu^a = \psi_\nu^a + \int_0^\nu \chi_{w\nu}^{aa} C_w^a dw$$

Differentiating, noting that $\nu \geq t$ in the integral of (2.6.2),

$$\frac{\partial H_\nu^a}{\partial C_t^a} = \int_0^\nu \chi_{w\nu}^{aa} \delta(w - t) dw = \chi_{t\nu}^{aa}$$

[7] Use of the Dirac delta function is motivated by a discrete time version of the model considered here.

The derivative of $\mathcal{U}^a(\mathbf{C})$ then takes the form

$$\begin{aligned} \frac{\partial}{\partial C_t^a}\mathcal{U}_t^a(\mathbf{C}) &= \mathbf{E}_t\left[\int_t^T u_\nu^{a'}(C_\nu^a - H_\nu^a)\left(\delta(\nu - t) - \frac{\partial H_\nu^a}{\partial C_t^a}\right)d\nu\right] \\ &= u_t^{a'}(C_t^a - H_t^a) - \mathbf{E}_t\left[\int_t^T \chi_{t\nu}^{aa} u_\nu^{a'}(C_\nu^a - H_\nu^a)d\nu\right] \end{aligned}$$

The first order condition (2.6.1) is

$$\frac{\partial}{\partial C_t^a}\mathcal{U}_t^a(\mathbf{C}) = y^a \xi_t$$

and so defining

$$\gamma_t^a = \frac{1}{y^a} u_t^{a'}(C_t^a - H_t^a)$$

and substituting $\gamma^a$ into the first order condition we obtain

$$\gamma_t^a = \xi_t + \mathbf{E}_t\left[\int_t^T \chi_{t\nu}^{aa}\gamma_\nu^a d\nu\right] \tag{2.6.3}$$

If we can solve this equation for $\gamma^a$ in terms of $\xi$ then we have that optimal consumption for agent $a$ is related to his habits by

$$C_t^a = H_t^a + I_t^a(y^a \gamma_t^a) \tag{2.6.4}$$

where $I_t^a = (u_t^{a'})^{-1}$, the inverse of marginal utility $u_t^{a'}$:

$$I_t^a(u_t^{a'}(x)) = x \qquad \textit{and} \qquad u_t^{a'}(I_t^a(y)) = y \qquad \forall x, y \in (0, \infty)$$

Also, $y^a > 0$ is a multiplier to be determined. Note that $\gamma^a$ is potentially different for each agent, determined by the strength of his internal habit formation process $\chi^{aa}$. With the heterogeneity that results when the $\chi^{aa}$ are different, significant complications arise in the study of equilibrium and the representative agent, as we shall see later.

Applying Theorem (2.1.4) we obtain the indistinguishably unique solution $\gamma^a$ to (2.6.3) in terms of $\xi$:

$$\gamma_t^a = \xi_t + \mathbf{E}_t\left[\int_t^T \overrightarrow{\chi_{t\nu}^{a\hat{a}}}\xi_\nu d\nu\right]$$

From this series representation, we immediately have the lower bound

$$0 < k_\xi \leq \xi_t \leq \gamma_t^a$$

We can also derive some upper bounds: Recalling the bound in Theorem (2.1.4), we have

$$\begin{aligned}\gamma_t^a &= \xi_t + \mathbf{E}_t\left[\int_t^T \overrightarrow{\chi_{t\nu}^{a\hat{a}}}\xi_\nu d\nu\right] \\ &\leq K_\xi\left(1 + \mathbf{E}_t\left[\int_t^T \overrightarrow{\chi_{t\nu}^{a\hat{a}}}d\nu\right]\right) \\ &\leq K_\xi e^{K_\chi(T-t)}\end{aligned}$$

So, we have

$$0 \;<\; k_\xi \;\leq\; \xi_t \;\leq\; \gamma_t^a \;\leq\; K_\xi e^{K_\chi T} \;<\; \infty$$

It is important to keep in mind that in the candidate expression $C_t^a = H_t^a + I_t^a(y^a\gamma_t^a)$ for optimal consumption, $H^a$ depends not only on $C^a$ but also on $\hat{C}^b$ for $b \neq a$ so implicit in this relationship are consumption externalities.

We need to check that a value $y^a \in (0, \infty)$ can be chosen so that $C^a \in \mathcal{B}^a$. In fact, since agent $a$ is insatiable (strictly positive marginal utility) and wants to maximize utility, agent $a$ will want to choose an optimal $y^{a\diamond} \in (0, \infty)$ which uses all of his endowment and have the associated $\hat{C}^{a\diamond}$ binding the constraint in $\mathcal{B}^a$. We will then need to show that the new configuration $\hat{\mathbf{C}}^{a\diamond} = (\hat{C}^1, \ldots, \hat{C}^{a-1}, \hat{C}^{a\diamond}, \hat{C}^{a+1}, \ldots, \hat{C}^A)$ resulting from agent $a$'s adjustment from $\hat{C}^a$ to $\hat{C}^{a\diamond}$ is in $\mathcal{U}^a\boldsymbol{\mathcal{B}}(\hat{\mathbf{C}})$. And, as this candidate for optimal consumption for agent $a$ was only derived heuristically, we must show rigorously that this consumption configuration $\hat{\mathbf{C}}^{a\diamond}$ actually optimizes agent $a$'s utility and that it is the $\boldsymbol{\lambda} \otimes \mathbf{P}$-a.s. unique policy in $\mathcal{U}^a\boldsymbol{\mathcal{B}}(\hat{\mathbf{C}})$ to do so. These steps are collected in the following proposition:

**Proposition 2.6.5** *For each agent $a$ and configuration $\hat{\mathbf{C}} \in \mathcal{U}^a\boldsymbol{\mathcal{B}}$ there exists a $\boldsymbol{\lambda}\otimes\mathbf{P}$-a.s. unique consumption configuration $\hat{\mathbf{C}}^{a\diamond}$ in $\mathcal{U}^a\boldsymbol{\mathcal{B}}(\hat{\mathbf{C}})$ which maximizes agent $a$'s utility $\mathcal{U}^a$ over the set $\mathcal{U}^a\boldsymbol{\mathcal{B}}(\hat{\mathbf{C}})$ of variations from $\hat{\mathbf{C}}$ open to the agent. Agent $a$'s optimal consumption $\hat{C}^{a\diamond}$ and habit $\hat{H}^{a\diamond}$ processes are related by*

$$\hat{C}_t^{a\diamond} = \hat{H}_t^{a\diamond} + I_t^a(y^{a\diamond}\gamma_t^a)$$

*where $y^{a\diamond} \in (0,\infty)$ is the unique value such that*

$$\mathbf{E}\left[\int_0^T \xi_\nu \hat{C}_\nu^{a\diamond} d\nu\right] = \mathbf{E}\left[\int_0^T \xi_\nu E_\nu^a d\nu\right]$$

*and $\gamma^a$ is the indistinguishably unique process satisfying*

$$\gamma_t^a = \xi_t + \mathbf{E}_t\left[\int_t^T \chi_{t\nu}^{aa}\gamma_\nu^a d\nu\right]$$

*Moreover, since $\hat{C}^{a\diamond}$ binds the constraint in $\mathcal{B}^a$, agent $a$ has a $\boldsymbol{\lambda} \otimes \mathbf{P}$-a.s. unique portfolio $\pi^{a\diamond} \in \mathcal{P}$ such that $(\pi^{a\diamond}, \hat{C}^{a\diamond}) \in \mathcal{A}^a$ with which to implement $\hat{C}^{a\diamond}$. The form of the optimal portfolio and wealth processes are as in Proposition (2.3.9).*

**Proof:** Fix an agent $a$ and fix a consumption configuration $\hat{\mathbf{C}} \in \mathcal{U}^a\boldsymbol{\mathcal{B}}$. From the component-wise representation of habits (2.4.4) and the form (2.6.4) that we expect optional consumption to take, agent $a$ ought to choose

$$C_t^a \;=\; H_t^a + I_t^a(y^a\gamma_t^a) \;=\; \psi_t^a + I_t^a(y^a\gamma_t^a) + \int_0^t \chi_{\nu t}^{aa} C_\nu^a d\nu$$

for some $y^a \in (0,\infty)$; this is the heuristically derived "necessary" form that optimal consumption for agent $a$ must take. We have yet to prove rigorously that this form does yield the optimal consumption. Applying Theorem (2.1.4) to this form, using the assumed utility bounds $k_\xi \le \gamma_t^a \le K_\xi \exp[K_\chi T]$ and $k_I(y) \le I_t^a(y) \le K_I(y)$, as well as the assumption that $t \mapsto I_t^a(\gamma_t^a)$ is a continuous mapping, and the fact that $\psi^a$ is bounded and continuous, this equation has an indistinguishably unique, continuous, $\mathbb{F}$-progressive solution with representation

$$\begin{aligned} C_t^a &= \psi_t^a + I_t^a(y^a\gamma_t^a) + \int_0^t \overleftarrow{\chi_{\nu t}^{aa}}\,(\psi_\nu^a + I_\nu^a(y^a\gamma_\nu^a))\,d\nu \\ &= \hat{C}_t^{as} + I_t^a(y^a\gamma_t^a) + \int_0^t \overleftarrow{\chi_{\nu t}^{aa}} I_\nu^a(y^a\gamma_\nu^a)d\nu \end{aligned}$$

where we have used the previously derived representation (2.5.1) for subsistence consumption $\hat{C}^{as}$ given $\hat{\mathbf{C}}^{as}$ to obtain the second line. Clearly, as $y^a$ increases to $\infty$, then $I_t^a(y^a\gamma_t^a)$ decreases to 0 and so by the monotone convergence theorem, we have that $\lim_{y^a\uparrow\infty} C_t^a = \hat{C}_t^{as}$, which is strictly affordable since $\hat{\mathbf{C}} \in \mathcal{U}^a\boldsymbol{\mathcal{B}}$. Conversely, as $y^a$ decreases to 0, then $I_t^a(y^a\gamma_t^a)$ increases to $\infty$ so again by monotone convergence, we have that $\lim_{y^a\downarrow 0} C_t^a = \infty$, which is clearly unaffordable. Hence, by the continuity and strict monotonicity of the mapping $y^a \mapsto \mathbf{E}\left[\int_0^T \xi_\nu C_\nu^a d\nu\right]$, there is a unique multiplier $y^{a\diamond} \in (0,\infty)$ for which the resulting consumption choice

$$C_t^{a\diamond} = \hat{C}_t^{as} + I_t^a(y^{a\diamond}\gamma_t^a) + \int_0^t \overleftarrow{\chi_{\nu t}^{aa}} I_\nu^a(y^{a\diamond}\gamma_\nu^a)d\nu$$

binds the budget constraint in $\mathcal{B}^a$:

$$\mathbf{E}\left[\int_0^T \xi_\nu \hat{C}_\nu^{a\diamond} d\nu\right] = \mathbf{E}\left[\int_0^T \xi_\nu E_\nu^a d\nu\right]$$

Using Proposition (2.3.9), agent $a$ has a $\boldsymbol{\lambda}\otimes\mathbf{P}$-a.s. unique optimal portfolio $\pi^{a\diamond} \in \mathcal{P}$ with which to implement $\hat{C}^{a\diamond}$. Let $\hat{\mathbf{C}}^{a\diamond} = (\hat{C}^1, \ldots, \hat{C}^{a-1}, \hat{C}^{a\diamond}, \hat{C}^{a+1}, \ldots, \hat{C}^A)$ denote the new configuration resulting from agent $a$'s consumption choice $\hat{\mathbf{C}}^{a\diamond}$. We show now that $\hat{\mathbf{C}}^{a\diamond}$ is in the set $\mathcal{U}^a\boldsymbol{\mathcal{B}}(\hat{\mathbf{C}})$.

Trivially, $\hat{\mathbf{C}} \in \mathcal{U}^a\boldsymbol{\mathcal{B}}(\hat{\mathbf{C}})$. Also, since $\hat{\mathbf{C}} \in \mathcal{U}^a\boldsymbol{\mathcal{B}}$, we know that $\hat{C}^{as}$ given $\hat{\mathbf{C}}$ is strictly affordable by agent $a$. In changing from $\hat{\mathbf{C}}$ to $\hat{\mathbf{C}}^{a\diamond}$, the consumption components $\hat{C}^b$, $\forall b \neq a$, do not change so $\psi^a$ does not change. Thus, inspecting (2.5.1), we see that $\hat{C}^{as}$ has not changed; it is the same given $\hat{\mathbf{C}}$ or given $\hat{\mathbf{C}}^{a\diamond}$. Hence, $\hat{\mathbf{C}}^{a\diamond}$ continues to be strictly affordable for agent $a$: $\hat{\mathbf{C}}^{a\diamond} \in \mathcal{U}^a\boldsymbol{\mathcal{B}}$. This, along with the fact that $\hat{C}^b$, $\forall b \neq a$, have not changed, we have that $\hat{\mathbf{C}}^{a\diamond} \in \mathcal{U}^a\boldsymbol{\mathcal{B}}(\hat{\mathbf{C}})$.

We show now that $\hat{\mathbf{C}}^{a\diamond}$ maximizes $\mathcal{U}^a$ over $\mathcal{U}^a\mathcal{B}(\hat{\mathbf{C}})$. The key property used here is concavity: if $f:(0,\infty)\to\mathbb{R}$ is twice differentiable, strictly increasing, and strictly concave, then for each fixed $y\in(0,\infty)$, we have for all $x\in(0,\infty)$ that $f(x)-xy\leq f(g(y))-yg(y)$ where $g$ is the inverse function of $f'$. Rearranged, we have $f(g(y))-f(x)\geq yg(y)-xy$. Moreover, if $x\neq g(y)$ then $f(g(y))-f(x)>yg(y)-xy$.

First, consider $\hat{\mathbf{C}}^{a\diamond}$ and let $\mathbf{C}$ be any other element of $\mathcal{U}^a\mathcal{B}(\hat{\mathbf{C}})$. Let $\mathbf{H}$ and $H^a$ be the habit processes determined by $\mathbf{C}$ and let $\hat{\mathbf{H}}^{a\diamond}$ and $\hat{H}^{a\diamond}$ be the habit processes determined by $\hat{\mathbf{C}}^{a\diamond}$. Recalling (2.4.4) we have

$$C_t^a=\psi_t^a+H_t^a \qquad \textit{and} \qquad \hat{C}_t^{a\diamond}=\hat{\psi}_t^{a\diamond}+\hat{H}_t^{a\diamond}$$

The only difference between $\hat{\mathbf{C}}^{a\diamond}$ and $\mathbf{C}$ is in component $a$: $\hat{C}^{a\diamond}$ and $C^a$. Thus, the terms that depend on all other consumption choices $\hat{C}^b$ for $b\neq a$ must be the same: $\psi^a=\hat{\psi}^{a\diamond}$. Now, using the stochastic integral equation relating $\xi_t$ and $\gamma_t^a$, changing the order of integration, and then using (2.4.4) for $H^a$ we have

$$\begin{aligned}
\mathbf{E}\left[\int_0^T \gamma_\nu^a C_\nu^a d\nu\right] &= \mathbf{E}\left[\int_0^T \left\{\xi_\nu+\mathbf{E}_\nu\left[\int_\nu^T \chi_{w\nu}^{aa}\gamma_w^a dw\right]\right\} C_\nu^a d\nu\right]\\
&= \mathbf{E}\left[\int_0^T \xi_\nu C_\nu^a d\nu\right]+\mathbf{E}\left[\int_0^T\left(\int_\nu^T \chi_{w\nu}^{aa}\gamma_w^a dw\right)C_\nu^a d\nu\right]\\
&= \mathbf{E}\left[\int_0^T \xi_\nu C_\nu^a d\nu\right]+\mathbf{E}\left[\int_0^T \gamma_w^a\left(\int_0^w \chi_{w\nu}^{aa}C_\nu^a d\nu\right)dw\right]\\
&= \mathbf{E}\left[\int_0^T \xi_\nu C_\nu^a d\nu\right]+\mathbf{E}\left[\int_0^T \gamma_w^a\left(H_w^a-\psi_w^a\right)dw\right] \qquad (2.6.6)
\end{aligned}$$

Similarly, with $\hat{\mathbf{C}}^{a\diamond}$ and the associated habit processes $\hat{H}^{a\diamond}$ and $\hat{\mathbf{H}}^{a\diamond}$, we have

$$\mathbf{E}\left[\int_0^T \gamma_\nu^a \hat{C}_\nu^{a\diamond} d\nu\right] = \mathbf{E}\left[\int_0^T \xi_\nu \hat{C}_\nu^{a\diamond} d\nu\right]+\mathbf{E}\left[\int_0^T \gamma_w^a\left(\hat{H}_w^{a\diamond}-\hat{\psi}_w^{a\diamond}\right)dw\right] \qquad (2.6.7)$$

Subtracting (2.6.6) from (2.6.7), using that $\psi^a=\hat{\psi}^{a\diamond}$, and rearranging, we obtain

$$\begin{aligned}\mathbf{E}\left[\int_0^T \gamma_\nu^a \left\{(\hat{C}_\nu^{a\diamond} - \hat{H}_\nu^{a\diamond}) - (C_\nu^a - H_\nu^a)\right\} d\nu\right] &= \mathbf{E}\left[\int_0^T \xi_\nu \left(\hat{C}_\nu^{a\diamond} - C_\nu^a\right) d\nu\right] \\ &= \mathbf{E}\left[\int_0^T \xi_\nu \left(E_\nu^a - C_\nu^a\right) d\nu\right] \\ &\geq 0 \qquad (2.6.8)\end{aligned}$$

Using $C_t^{a\diamond} - H_t^{a\diamond} = I_t^a(y^{a\diamond}\gamma_t^a)$ and that $u_t^a(I_t^a(y)) - u_t^a(x) \geq yI_t^a(y) - xy$ by concavity, we obtain

$$\begin{aligned}u_t^a(\hat{C}_\nu^{a\diamond} - \hat{H}_\nu^{a\diamond}) &- u_t^a(C_t^a - H_t^a) \\ &= u_t^a(I_t^a(y^{a\diamond}\gamma_t^a)) - u_t^a(C_t^a - H_t^a) \\ &\geq y^{a\diamond}\gamma_t^a I_t^a(y^{a\diamond}\gamma_t^a) - y^{a\diamond}\gamma_t^a(C_t^a - H_t^a) \\ &= y^{a\diamond}\left\{\gamma_t^a(\hat{C}_\nu^{a\diamond} - \hat{H}_\nu^{a\diamond}) - \gamma_t^a(C_t^a - H_t^a)\right\} \qquad (2.6.9)\end{aligned}$$

Now, with (2.6.8) and (2.6.9), we have

$$\begin{aligned}\mathcal{U}^a(\hat{\mathbf{C}}^{a\diamond}) - \mathcal{U}^a(\mathbf{C}) &= \mathbf{E}\left[\int_0^T \left\{u_\nu^a(\hat{C}_\nu^{a\diamond} - \hat{H}_\nu^{a\diamond}) - u_\nu^a(C_\nu^a - H_\nu^a)\right\} d\nu\right] \\ &\geq y^{a\diamond}\mathbf{E}\left[\int_0^T \gamma_\nu^a \left\{(\hat{C}_\nu^{a\diamond} - \hat{H}_\nu^{a\diamond}) - (C_\nu^a - H_\nu^a)\right\} d\nu\right] \\ &\geq 0 \qquad (2.6.10)\end{aligned}$$

establishing that $\hat{\mathbf{C}}^{a\diamond}$ is optimal. By construction, $\hat{C}^{a\diamond}$ binds the budget constraint in $\mathcal{B}^a$ so by Proposition (2.3.9) there is a $\boldsymbol{\lambda} \otimes \mathbf{P}$-a.s. unique portfolio $\pi^{a\diamond}$ such that $(\pi^{a\diamond}, \hat{C}^{a\diamond}) \in \mathcal{A}^a$ and hence to implement $\hat{C}^{a\diamond}$.

To prove uniqueness, we first need to establish that optimal utility $\mathcal{U}^a(\hat{\mathbf{C}}^{a\diamond})$ is a finite quantity. First, we have

$$\hat{C}_t^{a\diamond} - \hat{H}_t^{a\diamond} = I_t^a(y^{a\diamond}\gamma_t^a) \geq I_t^a(y^{a\diamond}K_\xi e^{K_x T}) \geq k_I(y^{a\diamond}K_\xi e^{K_x T}) > 0$$

and as a result $\mathcal{U}^a(\hat{\mathbf{C}}^{a\diamond}) > -\infty$. Similarly,

$$\hat{C}_t^{a\diamond} - \hat{H}_t^{a\diamond} = I_t^a(y^{a\diamond}\gamma_t^a) \leq I_t^a(y^{a\diamond}k_\xi) \leq K_I(y^{a\diamond}k_\xi) < \infty$$

and so $\mathcal{U}^a(\hat{\mathbf{C}}^{a\diamond}) < \infty$.

Now, suppose there is another configuration $\mathbf{C}$ in $\mathcal{U}^a\mathcal{B}(\hat{\mathbf{C}})$ yielding the optimal utility: $\mathcal{U}^a(\mathbf{C}) = \mathcal{U}^a(\hat{\mathbf{C}}^{a\diamond})$. If $\boldsymbol{\lambda} \otimes \mathbf{P}\{C^a - H^a \neq \hat{C}^{a\diamond} - \hat{H}^{a\diamond}\} > 0$ then, as a result of the finiteness of $\mathcal{U}^a(\hat{\mathbf{C}}^{a\diamond})$, the inequality (2.6.10) is strict so $\mathcal{U}^a(\mathbf{C}) < \mathcal{U}^a(\hat{\mathbf{C}}^{a\diamond})$, contradicting the assumption that $\mathcal{U}^a(\mathbf{C}) = \mathcal{U}^a(\hat{\mathbf{C}}^{a\diamond})$. Thus,

$$C^a - H^a = \hat{C}^{a\diamond} - \hat{H}^{a\diamond} \qquad \boldsymbol{\lambda} \otimes \mathbf{P} - a.s. \tag{2.6.11}$$

Recall that in shifting configurations from $\mathbf{C}$ to $\hat{\mathbf{C}}^{a\diamond}$, $\psi^a$ stays the same in the expressions

$$H_t^a = \psi_t^a + \int_0^t \chi_{\nu t}^{aa} C_\nu^a d\nu \qquad and \qquad \hat{H}_t^{a\diamond} = \psi_t^a + \int_0^t \chi_{\nu t}^{aa} \hat{C}_\nu^{a\diamond} d\nu$$

Subtracting these from consumption, we obtain:

$$C_t^a - H_t^a = C_t^a - \psi_t^a - \int_0^t \chi_{\nu t}^{aa} C_\nu^a d\nu \tag{2.6.12}$$

$$\hat{C}_t^{a\diamond} - \hat{H}_t^{a\diamond} = \hat{C}_t^{a\diamond} - \psi_t^a - \int_0^t \chi_{\nu t}^{aa} \hat{C}_\nu^{a\diamond} d\nu \tag{2.6.13}$$

Since (2.6.12) and (2.6.13) are $\boldsymbol{\lambda} \otimes \mathbf{P}$-a.s. equal, subtracting yields

$$\begin{aligned} 0 &= (\hat{C}_t^{a\diamond} - \hat{H}_t^{a\diamond}) - (C_t^a - H_t^a) \\ &= \hat{C}_t^{a\diamond} - C_t^a - \int_0^t \chi_{\nu t}^{aa} (\hat{C}_\nu^{a\diamond} - C_\nu^a) d\nu \qquad \boldsymbol{\lambda} \otimes \mathbf{P} - a.s. \end{aligned}$$

Hence,

$$\hat{C}_t^{a\diamond} - C_t^a = 0 + \int_0^t \chi_{\nu t}^{aa} (\hat{C}_\nu^{a\diamond} - C_\nu^a) d\nu \qquad \boldsymbol{\lambda} \otimes \mathbf{P} - a.s.$$

Applying Theorem (2.1.4), we see that $\hat{C}^{a\diamond} - C^a = 0$ is the indistinguishably unique solution. As $\hat{C}^a$ is the only component of the starting configuration $\hat{\mathbf{C}}$ that can be changed by agent $a$, we must have $\hat{\mathbf{C}}^{a\diamond}$ and $\mathbf{C}$ are indistinguishable. Thus, $\hat{\mathbf{C}}^{a\diamond}$ is the indistinguishably unique configuration in $\mathcal{U}^a\mathcal{B}(\hat{\mathbf{C}})$ which optimizes $\mathcal{U}^a$ for agent $a$.

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## 2.7 Mutually Optimal Consumption Configurations

The individually optimal consumption configuration $\hat{\mathbf{C}}^{a\diamond} \in \mathcal{U}^a\boldsymbol{\mathcal{B}}(\hat{\mathbf{C}})$ found in the last section is optimal for agent $a$, assuming that the original configuration is $\hat{\mathbf{C}}$ and that only agent $a$ makes an adjustment of his consumption component in moving from $\hat{\mathbf{C}}$ to $\hat{\mathbf{C}}^{a\diamond}$. In the absence of agent interactions, each agent could optimize separately, each adjusting his own consumption component from the starting configuration $\hat{\mathbf{C}}$ without regard for any other consumption component. The collective result is a configuration which is optimal for all agents, simultaneously. With agent interactions, the result of agent $a$ switching to $\hat{\mathbf{C}}^{a\diamond}$ is that agent $a$ changes the environment for all other agents and may even establish habit levels which are not strictly affordable by some or all of the other agents. Thus, simultaneous, or mutual, optimality does not automatically follow when agents must implement interdependent strategies as it does when agents are independent price-takers with dominant strategies.

We shall consider only those configurations in $\mathcal{U}\boldsymbol{\mathcal{B}}$ as candidates for mutual optimality. Mathematically, this ensures that all agents will have utility strictly larger than $-\infty$. Economically, this means that all agents can afford strictly more than the subsistence level established by all other agents' consumption choices. We do not provide a rigorous justification for confining mutually optimal configurations to $\mathcal{U}\boldsymbol{\mathcal{B}}$ but rather we give a heuristic economic argument to rule out configurations outside of $\mathcal{U}\boldsymbol{\mathcal{B}}$. First, note that as a result of market completeness, we have a complete set of Arrow-Debreu securities. It is therefore possible for all agents to negotiate and write consumption contracts for all points in time and for all states of uncertainty *in advance of markets opening.* Second, although our agents have a constant relative risk aversion at a given $(t, \omega)$, their absolute risk aversion to consumption fluctuation *increases* as consumption falls toward habit. In fact, marginal utility and hence absolute risk aversion is $+\infty$ at subsistence consumption. We interpret this to mean that, during pre-market negotiation, an agent whose consumption is close to his subsistence level will put more effort into acquiring (ie: will offer higher prices for) a consumption configuration that will increase the gap between his habit and consumption as compared to an agent that already has a sizable gap. As a group, agents approaching subsistence consumption in this negotiation will thus collectively bid up prices and steer the economy wide configuration away from an unfavorable one toward more favorable ones. In particular, in the pre-market negotiations, no rational agent would ever agree to sign a contract that yields $-\infty$ for utility and so no configuration outside of $\mathcal{UB}$ will ever be contracted.[8] Although this is only a heuristic argument, it is the most that can be provided with the level of model detail as given. In order to be rigorous, one would have to model the details of the consumption negotiation, probably as an economic game, and determine the core of this game. Our assumption that mutually optimal configurations are in $\mathcal{UB}$ is essentially assuming that the core of this unspecified game is in $\mathcal{UB}$. An alternative justification is that as the pre-market negotiation proceeds, any agent whose consumption drops to subsistence is thereafter excluded from the negotiation (either by dying off or being economically marginalized) so that when the negotiations are concluded and the contracts are executed, only those agents with utility larger than $-\infty$ for utility are left actively participating in the economy.

With the above considerations in mind, we now define a mutually optimal consumption configuration. We also provide conditions under which, for a given state price density $\xi$, mutual optimality is achieved. However, in the later discussion of equilibrium, $\xi$ will not be fixed but will instead be allowed to vary until optimal demand is brought into line with supply and all budget constraints are met. As a result of the variability of $\xi$, these sufficient conditions turn out not to be particularly useful in showing mutual optimality *before* studying equilibrium; the demonstration of mutual optimality must instead be incorporated into the proof of existence of equilibrium, in contrast to the approach taken in Karatzas et al. (1990), for instance. In any case, we include the sufficient conditions for simultaneous optimality here as they may prove useful in further studies of this model.

**Definition 2.7.1** *For any given state price density $\xi$, define an arbitrary consumption configuration $\mathbf{C}^\diamond = (C^{1\diamond}, \ldots, C^{A\diamond}) \in \boldsymbol{C}$ to be mutually optimal if*

*(1)* $\mathbf{C}^\diamond \in \mathcal{UB}$

*(2)* $\forall a = 1, \ldots, A, \qquad \mathcal{U}^a(\mathbf{C}^\diamond) = \sup\left\{\mathcal{U}^a(\mathbf{C}) : \mathbf{C} \in \mathcal{U}^a\mathcal{B}(\mathbf{C}^\diamond)\right\}$

[8] See Duffie & Huang (1985) for details or Duffie (1996) for a concise summary of the continuous time implementation of Arrow-Debreu complete market equilibrium.

We now characterize mutual optimality in a way that is useful here as well as in the discussion of equilibrium. Let $\Xi_{++}$ denote the subset of $\Xi$ consisting of those processes which are uniformly bounded above zero. For each $(\boldsymbol{\Upsilon}, \xi) \in (0,\infty)^A \times \Xi_{++}$, define, for $a = 1, \dots, A$, the maps

$$C^a(\boldsymbol{\Upsilon}, \xi)_t = I^a_t(y^a \gamma^a_t) + \int_0^t \sum_{b=1}^{A} \overleftarrow{\chi}^{ab}_{\nu t} I^b_\nu(y^b \gamma^b_\nu) d\nu \tag{2.7.2}$$

$$G^a(\boldsymbol{\Upsilon}, \xi) = \mathbf{E}\left[\int_0^T \xi_\nu \Big(C^a(\boldsymbol{\Upsilon}, \xi)_\nu - E^a_\nu\Big) d\nu\right] \tag{2.7.3}$$

where $\overleftarrow{\chi}^{ab}_{\nu t}$ denotes the $ab$ element of $\overleftarrow{\chi}$ constructed from $\chi$. Recall that

$$\gamma^a_t = \xi_t + \mathbf{E}_t\left[\int_t^T \overrightarrow{\chi^{aa}_{t\nu}} \xi_\nu\right] d\nu \tag{2.7.4}$$

where $\overrightarrow{\chi^{aa}}$ is the scalar process constructed from $\chi^{aa}$. Collecting the $C^a$ and $G^a$ functions, we have the maps

$$\mathbf{C}(\boldsymbol{\Upsilon}, \xi)_t = \mathbf{I}_t(\boldsymbol{\Upsilon}, \xi) + \int_0^t \overleftarrow{\chi}_{\nu t} \mathbf{I}_\nu(\boldsymbol{\Upsilon}, \xi) d\nu \tag{2.7.5}$$

$$\mathbf{G}(\boldsymbol{\Upsilon}, \xi) = (G^1(\boldsymbol{\Upsilon}, \xi), \dots, G^A(\boldsymbol{\Upsilon}, \xi))^\top \tag{2.7.6}$$

where we use the column vector notation

$$\mathbf{C}(\boldsymbol{\Upsilon}, \xi)_t = \begin{bmatrix} C^1(\boldsymbol{\Upsilon}, \xi)_t \\ \vdots \\ C^A(\boldsymbol{\Upsilon}, \xi)_t \end{bmatrix} \qquad \mathbf{I}_t(\boldsymbol{\Upsilon}, \xi) = \begin{bmatrix} I^1_t(y^1 \gamma^1_t) \\ \vdots \\ I^A_t(y^A \gamma^A_t) \end{bmatrix} \qquad \mathbf{G}(\boldsymbol{\Upsilon}, \xi) = \begin{bmatrix} G^1(\boldsymbol{\Upsilon}, \xi) \\ \vdots \\ G^A(\boldsymbol{\Upsilon}, \xi) \end{bmatrix}$$

We then have

**Proposition 2.7.7** $\mathbf{C}^\diamond \in \mathcal{C}$ *is mutually optimal if and only if there exists a pair* $(\boldsymbol{\Upsilon}, \xi) \in (0,\infty)^A \times \Xi_{++}$ *such that*

$$\mathbf{C}^\diamond = \mathbf{C}(\boldsymbol{\Upsilon}, \xi) \in \mathcal{UB} \qquad \textit{and} \qquad \mathbf{G}(\boldsymbol{\Upsilon}, \xi) = \mathbf{0}$$

**Proof:** From Definition (2.7.1), if $\mathbf{C}^\diamond = (C^{1\diamond}, \dots, C^{A\diamond})$ is a mutually optimal configuration for some given $\xi \in \Xi_{++}$ then for each $a$, $C^{a\diamond}$ must be individually optimal given $\mathbf{C}^\diamond$. As a result of Proposition (2.6.5), we must then have

$$C^{a\diamond}_t = H^{a\diamond}_t + I^a_t(y^{a\diamond} \gamma^a_t) \tag{2.7.8}$$

where $\mathbf{H}^\diamond$ is the habit configuration associated with $\mathbf{C}^\diamond$ and $y^{a\diamond} \in (0,\infty)$ are the unique values such that

$$\mathbf{E}\left[\int_0^T \xi_\nu\left(C_\nu^{a\diamond} - E_\nu^a\right)d\nu\right] = 0$$

In column vector form, consumption becomes

$$\mathbf{C}_t^\diamond = \mathbf{I}_t(\Upsilon,\xi) + \mathbf{H}_t^\diamond = \mathbf{I}_t(\Upsilon,\xi) + \int_0^t \chi_{\nu t}\mathbf{C}_\nu^\diamond d\nu \qquad (2.7.9)$$

From Theorem (2.1.4), the solution $\mathbf{C}^\diamond$ to (2.7.9) is given indistinguishably by

$$\mathbf{C}_t^\diamond = \mathbf{I}_t(\Upsilon,\xi) + \int_0^t \overleftarrow{\chi}_{\nu t}\mathbf{I}_\nu(\Upsilon,\xi)d\nu$$

Thus, $\mathbf{C}^\diamond = \mathbf{C}(\Upsilon,\xi)$ where $\xi \in \Xi_{++}$ is the given state price density for the model resulting in the mutual optimality of $\mathbf{C}^\diamond$ and $\Upsilon = (y^a,\ldots,y^A) \in (0,\infty)^A$ are the multipliers binding the budget constraints:

$$G^a(\Upsilon,\xi) = \mathbf{E}\left[\int_0^T \xi_\nu\left(C^a(\Upsilon,\xi)_\nu - E_\nu^a\right)d\nu\right] = \mathbf{E}\left[\int_0^T \xi_\nu\left(C_\nu^{a\diamond} - E_\nu^a\right)d\nu\right] = 0$$

Hence, $\mathbf{G}(\Upsilon,\xi) = \mathbf{0}$ also.

Now, suppose there is a $(\Upsilon,\xi) \in (0,\infty)^A \times \Xi_{++}$ such that $\mathbf{C}^\diamond = \mathbf{C}(\Upsilon,\xi) \in \mathcal{UB}$ and $\mathbf{G}(\Upsilon,\xi) = \mathbf{0}$. Hence, $C^{a\diamond}$ satisfies (2.7.8) and $G^a(\Upsilon,\xi) = 0$ for each $a$. By Proposition (2.6.5), $C^{a\diamond}$ is individually optimal given $\mathbf{C}^\diamond$ for each $a$. Since $\mathbf{C}^\diamond \in \mathcal{UB}$ we have that $\mathbf{C}^\diamond$ is mutually optimal. ■

For the mutual optimization problem, we emphasize that $\xi$ and $E^a$ are assumed given so that when various $\Upsilon$ are considered, only the consumption part of the budget constraints vary.[9] The existence and uniqueness of a mutually optimal consumption configuration for an exogenously given $\xi$ then becomes one of determining the existence and uniqueness of an appropriate vector of multipliers $\Upsilon^\diamond = (y^{1\diamond},\ldots,y^{A\diamond})^\top \in (0,\infty)^A$ so that all budget constraints are met simultaneously.

To explore this problem, write

$$G^a(\Upsilon) = g^a(y^a) + F^a(\Upsilon)$$

[9]Endowments $E^a$ will ***always*** be specified exogenously. However, $\xi$ will be varied in the search for equilibrium, and will thereby be endogenized.

for

$$\begin{aligned} g^a(y^a) &= \mathbf{E}\left[\int_0^T \xi_\nu (I^a_\nu(y^a\gamma^a_\nu) - E^a_\nu)d\nu\right] \\ F^a(\Upsilon) &= \mathbf{E}\left[\int_0^T \xi_\nu \sum_{b=1}^{A} \overleftarrow{\chi}^{ab}_{\nu T} I^b_\nu(y^b\gamma^b_\nu)d\nu\right] \end{aligned}$$

where, for the rest of this section, we suppress the argument $\xi$ for simplicity. We recall again that $\gamma^a$ is uniquely (up to indistinguishability) determined by $\xi$. The following properties are easily verifiable:

- For all $a$, $g^a(0) := \lim_{y^a \downarrow 0} g^a(y^a) = \infty$.
- For all $a$, $g^a(\infty) := \lim_{y^a \uparrow \infty} g^a(y^a) = -\mathbf{E}\left[\int_0^T \xi_\nu E^a_\nu d\nu\right] < 0$
- Defining $g^a_{min} = g^a(\infty)$ for all $a$, the maps $g^a : (0,\infty) \to (g^a_{min}, \infty)$ are $C^3$, strictly decreasing and onto.
- For all $a$, the map $g^a : (0,\infty) \to (g^a_{min}, \infty)$ is invertible with decreasing inverse $h^a : (g^a_{min}, \infty) \to (0,\infty)$ satisfying $h^a(g^a_{min}) = \infty$ and $h^a(\infty) = 0$.
- For all $a$, $F^a \equiv 0$ or $F^a : (0,\infty)^A \to (0,\infty)$ is onto and is strictly decreasing in at least one component.
- For all $a$, $\frac{\partial G^a}{\partial y^a} = \frac{dg^a}{dy^a} + \frac{\partial F^a}{\partial y^a} < 0$
- For all $a \neq b$, $\frac{\partial G^b}{\partial y^a} = \frac{\partial F^a}{dy^a} \leq 0$
- For all $a$, $F^a(\infty, \ldots, \infty) := \lim_{y^1,\ldots,y^A \uparrow \infty} F^b = 0$.

To show how this model is a generalization of a class of models already studied in the literature, we present two examples:

**Example 2.7.10** *If $\chi = 0$ then $F^a(\Upsilon) = 0$ and so $G^a(\Upsilon) = g^a(y^a)$. This case is that of time-additive utility; there is no habit formation or consumption externalities and is the case studied in Karatzas et al. (1990) and Karatzas et al. (1991) but with the addition of heterogeneous and dynamic risk aversion and patience. Since $0$ is in the range $(g^a_{min}, \infty)$ of $g^a$, and since $g^a$ is continuous and monotonically decreasing, there* *is a unique* $y^{a\diamondsuit} \in (0,\infty)$ *such that* $g^a(y^{a\diamondsuit}) = 0$ *by the intermediate value theorem for continuous functions. This can be done independently for each a, as each agent has a dominant strategy, and so one obtain a unique* $\mathbf{\Upsilon}^{\diamondsuit} \in (0,\infty)^A$ *such that* $\mathbf{G}(\mathbf{\Upsilon}^{\diamondsuit}) = \mathbf{0}$.

**Example 2.7.11** *If* $\chi$ *is diagonal, then* $F^a(\mathbf{\Upsilon}) = F^a(y^a)$ *so again we have that the budget constraint function* $G^a(\mathbf{\Upsilon}) = g^a(y^a) + F^a(y^a)$ *depends only on the* $y^a$ *multiplier. In this case, habits are purely internal. This is a multi-agent version of the representative agent model in Detemple & Zapatero (1991) and Detemple & Zapatero (1992) restricted to habit differences* $C^a - H^a$ *but with the addition of heterogeneous and dynamic risk aversion and patience. As in the prior example, since agents have dominant strategies, one can easily solve for a unique* $\mathbf{\Upsilon}^{\diamondsuit} \in (0,\infty)^A$ *such that* $\mathbf{G}(\mathbf{\Upsilon}^{\diamondsuit}) = \mathbf{0}$.

The above examples both show that mutual optimality follows simply from individual optimizations. A much more general class of $\chi$ we can consider, which includes cases for which agents do not have dominant strategies and for which mutual optimality is a subtler concept than that arising from implementing dominant strategies, are those $\chi$ which have identical row sums:

**Proposition 2.7.12** *If* $\chi$ *has the property that*

$$\forall a, \quad \sum_{b=1}^{A} \chi_{\nu t}^{ab} \;=\; \chi_{\nu t}^{r} \tag{2.7.13}$$

*for some row sum process* $\chi^r$ *then* $\mathbf{G}$ *is 1-1. Thus, if there exists a* $\mathbf{\Upsilon} \in (0,\infty)^A$ *such that* $\mathbf{G}(\mathbf{\Upsilon}^{\diamondsuit}) = 0$*, it is unique.*

**Proof:** Using the alternate series expansion for $\overleftarrow{\chi}$ discussed in Theorem (2.1.4), the terms in the expression for $F^a$ which depend on $a$ are

$$\sum_{b=1}^{A} \left\{ \prod_{i=0}^{k} \boldsymbol{\chi}_{t_{i+1}t_i} \right\}^{ab} \qquad k = 0, 1, \ldots$$

For the $k = 0$ case, this term is $\sum_{b=1}^{A} \left\{\boldsymbol{\chi}_{t_1 t_0}\right\}^{ab} = \sum_{b=1}^{A} \chi_{t_1 t_0}^{ab} = \chi_{t_1 t_0}^{r}$, which is free of $a$. For $k = 1$, this term is

$$\sum_{b=1}^{A} \left\{\boldsymbol{\chi}_{t_2 t_1}\boldsymbol{\chi}_{t_1 t_0}\right\}^{ab} = \sum_{b=1}^{A}\sum_{k=1}^{A} \chi_{t_2 t_1}^{ak}\chi_{t_1 t_0}^{kb} = \sum_{k=1}^{A} \chi_{t_2 t_1}^{ak} \sum_{b=1}^{A} \chi_{t_1 t_0}^{kb} = \sum_{k=1}^{A} \chi_{t_2 t_1}^{ak}\chi_{t_1 t_0}^{r} = \chi_{t_2 t_1}^{r}\chi_{t_1 t_0}^{r}$$

which is also free of $a$. Continuing by induction for $k > 1$ then shows that these terms are all free of $a$, hence, $F^a$ is free of $a$. Denote by $F$ these common functions $F^a$, $a = 1, \ldots, A$. Now suppose $\mathbf{G}(\Upsilon) = \mathbf{G}(\tilde{\Upsilon})$ for two distinct $\Upsilon, \tilde{\Upsilon} \in (0, \infty)^A$. Thus, $G^a(\Upsilon) = G^a(\tilde{\Upsilon})$ for all $a$ and so $g^a(y^a) + F(\Upsilon) = g^a(\tilde{y}^a) + F(\tilde{\Upsilon})$ for all $a$. If $F(\Upsilon) = F(\tilde{\Upsilon})$ then $g^a(y^a) = g^a(\tilde{y}^a)$ for all $a$ and since the $g^a$ are strictly decreasing, this implies $y^a = \tilde{y}^a$ for all $a$ and hence $\Upsilon = \tilde{\Upsilon}$, a contradiction. If $F(\Upsilon) < F(\tilde{\Upsilon})$ then $g^a(y^a) > g^a(\tilde{y}^a)$ for all $a$ and hence $y^a < \tilde{y}^a$ for all $a$. For all $a$, $G^a$ is strictly decreasing in $y^a$ so $G^a(\Upsilon) > G^a(\tilde{\Upsilon})$, a contradiction. A similar argument holds for the case $F(\Upsilon) > F(\tilde{\Upsilon})$. Thus, $\mathbf{G}$ is everywhere 1-1. In particular, $\mathbf{G}(\Upsilon) = \mathbf{0}$ has at most one solution. [10] ■

Under the condition (2.7.13) in Proposition (2.7.12), we have that $F^a = F$ is free of $a$ and $\mathbf{G}$ is 1-1. We also have the following proposition giving sufficient conditions for the existence of solution to $\mathbf{G}(\Upsilon) = \mathbf{0}$:

**Proposition 2.7.14** *Consider the function* $\mathbf{G} : (0, \infty)^A \to \mathbb{R}^A$. *Assume that* $\chi$ *has identical row sums indexes are relabeled so that* $g^1_{min} \le g^2_{min} \le \cdots \le g^A_{min}$. *Let* $h^a : (g^a_{min}, \infty) \to (0, \infty)$ *denote the inverse functions of the* $g^a : (0, \infty) \to (g^a_{min}, \infty)$. *Then, there exists a unique vector* $\Upsilon^\diamond \in (0, \infty)^A$ *for which* $\mathbf{G}(\Upsilon^\diamond) = \mathbf{0}$ *if and only if the ranges of the functions* $g^a$ *and the common function* $F$ *are such that*

$$g^A_{min} + F(h^1(g^A_{min}), \ldots, h^{A-1}(g^A_{min}), \infty) < 0$$

**Proof:** In order for $\Upsilon = (y^1, \ldots, y^A)$ to be a solution to $\mathbf{G}(\Upsilon) = \mathbf{0}$, we must have $g^a(y^a) = -F(y^1, \ldots, y^A)$ for all $a = 1, \ldots, A$. Because of the common term $-F(y^1, \ldots, y^A)$, we are immediately restricted to a set of candidates $(y^1, \ldots, y^A)$ satisfying:

$$g^1(y^1) = g^2(y^2) = \cdots = g^{A-1}(y^{A-1}) = g^A(y^A)$$

From $g^a(y^a) = g^A(y^A)$ we see that this family of candidates is a one parameter set:

$$y^a = h^a(g^A(y^A)) \qquad 1 \le a \le A-1$$

where the single parameter $y^A$ is restricted to a suitable domain. Note that for $y^a$ defined in this way we have $g^a(y^a) = g^a(h^a(g^A(y^A))) = g^A(y^A)$ for all $a \ne A$

[10] Thanks to Prof. Don Dawson for suggesting this simpler proof which separates uniqueness of a solution from its existence; uniqueness can also be seen as a by-product of Proposition (2.7.14).

so all $g^a$ functions have the same value. To check the domain, note that the $h^a$ are strictly decreasing with $h^a(g^a_{min}) = \infty$ and $h^a(\infty) = 0$. Now, $y^A \in (0, \infty)$ so $g^A(y^A) \in (g^A_{min}, \infty) \subset (g^a_{min}, \infty) = Dom(h^a)$. Hence, all $y^a$ for $1 \leq a \leq A-1$ can be defined in terms of $y^A$ as above for every $y^A \in (0, \infty)$. When $y^A \downarrow 0$ we have $y^a \downarrow h^a(g^A(0)) = h^a(\infty) = 0$ and when $y^A \uparrow \infty$ we have

$$y^a \uparrow h^a(g^A(\infty)) = h^a(g^A_{min}) \leq h^a(g^a_{min}) = \infty$$

Now, consider varying the parameter $y^A$ in the function

$$f(y^A) = g^A(y^A) + F(h^1(g^A(y^A)), \ldots, h^{A-1}(g^A(y^A)), y^A)$$

If we can find a value $y^{A\diamond} \in (0, \infty)$ such that $f(y^{A\diamond}) = 0$ then, by setting the other values as $y^{a\diamond} = h^a(g^A(y^{A\diamond})) \in (0, h^a(g^A_{min})) \subset (0, \infty)$ for all $a \neq A$ yields

$$0 = f(y^{A\diamond}) = g^A(y^{A\diamond}) + F(y^{1\diamond}, \ldots, y^{A-1\diamond}, y^{A\diamond}) = g^a(y^{a\diamond}) + F(y^{1\diamond}, \ldots, y^{A-1\diamond}, y^{A\diamond})$$

for all $a \neq A$ so we are done. Considering $f$ further, we have

$$\frac{df}{dy^A} = \frac{dg^A}{dy^A} + \frac{\partial F}{\partial h^1}\frac{dh^1}{dg^A}\frac{dg^A}{dy^A} + \cdots + \frac{\partial F}{\partial h^{A-1}}\frac{dh^{A-1}}{dg^A}\frac{dg^A}{dy^A} + \frac{\partial F}{\partial y^A} < 0$$

since $g^A$ is decreasing, since $h^a$ are decreasing, and since $F$ is decreasing in each coordinate. Thus, $f$ is a strictly decreasing function. Now, consider the range of $f$:

$$f(0) = g^A(0) + F(0, \ldots, 0) = \infty$$

so $f$ can be made to go above zero for some $y^A \in (0, \infty)$. Now, let $y^A \uparrow \infty$.

$$\begin{aligned} f(\infty) &= g^A(\infty) + F(h^1(g^A_{min}), \ldots, h^{A-1}(g^A_{min}), \infty) \\ &= g^A_{min} + F(h^1(g^A_{min}), \ldots, h^{A-1}(g^A_{min}), \infty) \end{aligned}$$

which is strictly negative by assumption. So, $f$ can be made to go below zero for some $y^A \in (0, \infty)$. Therefore, there is a $y^{A\diamond} \in (0, \infty)$ such that $f(y^{A\diamond}) = 0$ by the intermediate value theorem for single variable continuous functions. In fact, we see from the strictly decreasing behavior of $f$ that this $y^{A\diamond}$ is unique which then uniquely determines $y^{1\diamond}, \ldots, y^{A-1\diamond}$ via $y^{a\diamond} = h^a(y^{A\diamond})$, for $a = 1, \ldots, A-1$. ■

Note that in the cases for which $G^a(y^a) = g^a(y^a) + F^a(y^a)$ is a function of $y^a$ only, and $F$ is not necessarily common, the condition on the ranges in Proposition (2.7.14) becomes

$$G^a(\infty) = g^a_{min} + F^a(\infty) \;=\; g^a_{min} + 0 \;<\; 0$$

which is always satisfied, for any choice of $\xi$, explaining the ease of solution when $\chi$ is zero or diagonal. Unfortunately, the argument in Proposition (2.7.14) is not enough if the row sums of $\chi$ are not the same. Also, note that, in general, the condition on the functions' ranges depends on $\xi$ so this result is difficult to use when discussing equilibrium as $\xi$ will be allowed to vary. However, we show below that we always have that $\mathbf{G}$ is locally 1-1 for any $\chi$ and any $\xi$:

**Proposition 2.7.15** *For general $\chi$ and any $\xi \in \Xi_{++}$, the function $\mathbf{G}$ is locally 1-1.*

**Proof:** To show $\mathbf{G}$ is locally 1-1 for any $\xi$, we compute the Jacobian of $\mathbf{G}$. Note that for all $a$

$$\frac{\partial G^a}{\partial y^a} = \frac{dg^a}{dy^a} + \frac{\partial F^a}{\partial y^a} < 0$$

and for all $a \neq b$ we have

$$\frac{\partial G^a}{\partial y^b} = \frac{\partial F}{\partial y^b} \leq 0$$

so the Jacobian matrix of $\mathbf{G}$ has the form

$$J_{\mathbf{G}} = \begin{bmatrix} X_1 + Y_1 & Y_2 & Y_3 & \dots & Y_A \\ Y_1 & X_2 + Y_2 & Y_3 & \dots & Y_A \\ Y_1 & Y_2 & X_3 + Y_3 & \dots & Y_A \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Y_1 & Y_2 & Y_3 & \dots & X_A + Y_A \end{bmatrix}$$

where $X_a < 0$ and $Y_a \leq 0$ for all $a = 1, \dots, A$. Subtracting the bottom row from all others and omitting zero entries gives

$$\begin{bmatrix} X_1 & & & & & -X_A \\ & X_2 & & & & -X_A \\ & & X_3 & & & -X_A \\ & & & \ddots & & \vdots \\ & & & & X_{A-1} & -X_A \\ Y_1 & Y_2 & Y_3 & \dots & Y_{A-1} & X_A + Y_A \end{bmatrix}$$

Now, for $1 \leq a \leq A-1$, multiply row $a$ by $\frac{-Y_a}{X_a}$ and add it to row $A$. The result is

$$\begin{bmatrix} X_1 & & & & & -X_A \\ & X_2 & & & & -X_A \\ & & X_3 & & & -X_A \\ & & & \ddots & & \vdots \\ & & & & X_{A-1} & -X_A \\ & & & & & X_A + Y_A + X_A \sum_{a=1}^{A-1} \frac{Y_a}{X_a} \end{bmatrix}$$

Thus,

$$\det J_{\mathbf{G}} = X_1 \cdots X_{A-1} \left[ X_A + Y_A + X_A \sum_{a=1}^{A-1} \frac{Y_a}{X_a} \right] = X_1 \cdots X_A \left[ 1 + \sum_{a=1}^{A} \frac{Y_a}{X_a} \right] \neq 0$$

so $\mathbf{G}$ is locally $1-1$. Thus, if there exists a solution to $\mathbf{G}(\mathbf{\Upsilon}) = 0$ it is locally unique.

■

In light of these results, as well as the fact that the maps involved in constructing $\mathbf{G}$ are bijections, I make the following conjecture:

**Conjecture 2.7.16** *Let $\xi \in \Xi_{++}$ be given. (1) For general $\chi$ the map $\mathbf{G}(\cdot, \xi)$ is globally 1-1. (2) There exist conditions on the ranges of the functions $g^a$ and $F^a$ which depend on $\xi$ guaranteeing the existence of a solution to $\mathbf{G}(\mathbf{\Upsilon}, \xi) = \mathbf{0}$.*

## 2.8 Equilibrium: Definition, Existence & Uniqueness

Now that we have considered mutual optimality, we move on to equilibrium. The new ingredient for equilibrium is that in addition to mutual optimality, we also require that markets clear. The approach to finding equilibrium is to first impose market clearing so that, no matter what state prices $\xi$ agents are faced with, the form of their optimal consumption choices will clear the commodity markets. However, not all such consumption choices are optimal or affordable and so the final step is to vary $\xi$ until optimality and affordability are achieved. Rather than a direct variation of $\xi$, one establishes an implicit relationship between $\xi$ and the multipliers $\mathbf{\Upsilon}$, thereby allowing one, under certain conditions, to use market clearing to eliminate $\xi = \xi(\mathbf{\Upsilon})$ and express the budget constraints solely in terms of the multipliers $\mathbf{\Upsilon}$. The problem is then one of varying $\boldsymbol{\Upsilon}$, and hence $\xi(\boldsymbol{\Upsilon})$, to find the solution to $\mathbf{G}(\boldsymbol{\Upsilon},\xi(\boldsymbol{\Upsilon})) = \mathbf{0}$ as in the last section, but incorporating the added structure that imposing market clearing generates.

We now define equilibrium for this model, which is a rational expectations equilibrium, in the sense of Lucas (1978) and Radner (1979):

**Definition 2.8.1** *The model is in equilibrium with consumption configuration* $\mathbf{C}^* \in \mathcal{C}$ *if and only if*

*(1)* $\mathbf{C}^*$ *is mutually optimal: (no agent will deviate from* $\mathbf{C}^*$*):*

$$\forall a = 1, \ldots, A, \qquad \mathcal{U}^a(\mathbf{C}^*) = \sup\left\{\mathcal{U}^a(\mathbf{C}) : \mathbf{C} \in \mathcal{U}^a\mathcal{B}(\mathbf{C}^*)\right\}$$

*(2) The commodity market clears (aggregate consumption rate equals aggregate endowment rate):*

$$\sum_{a=1}^{A} C^{*a} = \sum_{a=1}^{A} E^a \qquad \boldsymbol{\lambda} \otimes \mathbf{P} - a.s.$$

*(3) The risky asset market clears (zero net supply):*

$$\sum_{a=1}^{A} \pi^{*a} = \mathbf{0} \qquad \boldsymbol{\lambda} \otimes \mathbf{P} - \text{a.s.}$$

*where* $\boldsymbol{\pi}^*$ *is the* $\boldsymbol{\lambda} \otimes \mathbf{P}$*-a.s. unique portfolio configuration implementing* $\mathbf{C}^*$.

*(4) The risk free asset market clears (zero net supply):*

$$\sum_{a=1}^{A} \left( W^{*a} - \sum_{k=1}^{N} \pi^{*ak} \right) = 0 \qquad \boldsymbol{\lambda} \otimes \mathbf{P} - a.s.$$

*where* $W^{*a}$ *is the indistinguishably unique wealth process generated by* $(\pi^{*a}, C^{*a})$.

In general, such an equilibrium resembles a Nash equilibrium for a continuous time, stochastic game, as the optimal strategy of each agent depends on those of all other agents. When $\boldsymbol{\chi} = 0$ or $\boldsymbol{\chi}$ is diagonal, agents have dominant strategies, corresponding to the usual assumption that agents act as independent price-takers, and are the cases dealt with in Karatzas et al. (1990), Karatzas et al. (1991), and Detemple & Zapatero

(1991). When the off-diagonal elements of $\chi$ are non-zero, we have agent interactions in the form of consumption externalities, involving interdependent strategies.[12]

In Definition (2.8.1) of equilibrium, two of the four conditions are actually redundant:

**Proposition 2.8.2** *The model is in equilibrium with consumption configuration* $\mathbf{C}^* \in \boldsymbol{C}$ *if and only if (1) and (2) in Definition (2.8.1) hold.*

**Proof:** If the model is in equilibrium with $\mathbf{C}^* \in \boldsymbol{C}$ then (1) and (2) hold by definition. Suppose (1) and (2) hold and prove that (3) and (4) follow.

Since $\mathbf{C}^*$ is mutually optimal, for each agent $a$, $C^{*a}$ is individually optimal. By Proposition (2.6.5), the $\lambda \otimes \mathbf{P}$-a.s. unique portfolio $\pi^{*a}$, the indistinguishably unique wealth process $W^{*a}$, and $\lambda \otimes \mathbf{P}$-a.s. unique integrand $\phi^{*a}$ in the martingale representation are given by:

$$\xi_t W_t^{*a} = \mathbf{E}_t \left[ \int_t^T \xi_\nu^* (C_\nu^{*a} - E_\nu^a) d\nu \right] \tag{2.8.3}$$

$$\int_0^t \phi_\nu^{*a\top} d\nu = \mathbf{E}_t \left[ \int_0^T \xi_\nu^* (C_\nu^{*a} - E_\nu^a) d\nu \right] \tag{2.8.4}$$

$$\pi_t^{*a} = (\sigma_t^\top)^{-1} (\xi_t^{-1} \phi_t^{*a} + W_t^{*a} \theta_t^*) \tag{2.8.5}$$

The commodity markets clear so $\sum_{a=1}^A C_t^{*a} = \sum_{a=1}^A E_t^a$ $\lambda \otimes \mathbf{P}$-a.s. Now, summing both (2.8.3) and (2.8.4), we then obtain $\xi_t \sum_{a=1}^A W_t^{*a} = 0$ indistinguishably and $\int_0^t \sum_{a=1}^A \phi_\nu^{*a\top} d\nu = 0$ $\lambda \otimes \mathbf{P}$-a.s. Since $\xi > 0$ we obtain $\sum_{a=1}^A W_t^{*a} = 0$, indistinguishably. Since the zero martingale has the $\lambda \otimes \mathbf{P}$-a.s. unique representation 0 we have that $\sum_{a=1}^A \phi_t^{*a\top} = 0$, $\lambda \otimes \mathbf{P}$-a.s. Using these and summing (2.8.5) gives $\sum_{a=1}^A \pi_t^{*a} = \mathbf{0}$, $\lambda \otimes \mathbf{P}$-a.s., showing that the risky asset market clears. And, the bond market clears, $\sum_{a=1}^A (W_t^{*a} - \sum_{k=1}^N \pi_t^{*ak}) = \sum_{a=1}^A W_t^{*a} - \sum_{k=1}^N (\sum_{a=1}^A \pi_t^{*ak}) = 0 - 0 = 0$ , $\lambda \otimes \mathbf{P}$-a.s. Thus, the model is in equilibrium. ■

To search for an equilibrium, we therefore need to focus on (1) and (2) of Proposition (2.8.2). As we saw in the prior section, mutual optimality requires that consumption have the form $\mathbf{C}(\boldsymbol{\Upsilon}, \xi)$ for some $(\boldsymbol{\Upsilon}, \xi) \in (0, \infty)^A \times \Xi_{++}$. For the commodity

[12] See Fudenberg & Tirole (1991). We only casually note here these game theoretic similarities, postponing a careful analysis for future work.

market to clear, we must also have

$$\begin{aligned} E_t &= \sum_{a=1}^{A} E_t^a = \sum_{a=1}^{A} C^a(\boldsymbol{\Upsilon}, \xi)_t \\ &= \sum_{a=1}^{A} I_t^a(y^a \gamma_t^a) + \int_0^t \sum_{a=1}^{A} \sum_{b=1}^{A} \overleftarrow{\chi}_{\nu t}^{ab} I_\nu^b(y^b \gamma_\nu^b) d\nu \end{aligned} \tag{2.8.6}$$

where, we recall, $\overleftarrow{\chi}^{ab}$ is the $ab$ entry in $\overleftarrow{\chi}$ constructed from $\chi$ as in Theorem (2.1.4), and

$$\gamma_t^a = \xi_t + \mathbf{E}_{t_0}\left[\int_t^T \overrightarrow{\chi_{t\nu}^{aa}} \xi_\nu d\nu\right] \tag{2.8.7}$$

where $\overrightarrow{\chi^{aa}}$ is constructed from $\chi^{aa}$ also as in Theorem (2.1.4). To simplify notation, we write the market clearing equation (2.8.6) as

$$E_t = \Theta(\boldsymbol{\Upsilon}, \xi)_t \tag{2.8.8}$$

where $E$ is aggregate endowment and $\Theta$ is the complicated functional of $\xi$, parameterized by $\boldsymbol{\Upsilon}$, and given by composing (2.8.6) and (2.8.7). As $E$ is specified exogenously, the equation (2.8.8) implicitly relates $\xi$ and $\boldsymbol{\Upsilon}$. Any pair $(\boldsymbol{\Upsilon}, \xi)$ satisfying (2.8.8) will result in cleared markets. In order to achieve mutual optimality, we must also have that the pair $(\boldsymbol{\Upsilon}, \xi)$ results in all budget constraints binding:

$$\mathbf{G}(\boldsymbol{\Upsilon}, \xi) = \mathbf{0} \tag{2.8.9}$$

**Proposition 2.8.10** $\mathbf{C}^* \in \boldsymbol{C}$ *is an equilibrium if and only if there exists a pair* $(\boldsymbol{\Upsilon}^*, \xi^*) \in (0,\infty)^A \times \Xi_{++}$ *such that*

$$\mathbf{C}^* = \mathbf{C}(\boldsymbol{\Upsilon}^*, \xi^*) \quad \textit{and} \quad E = \Theta(\boldsymbol{\Upsilon}^*, \xi^*) \quad \textit{and} \quad \mathbf{G}(\boldsymbol{\Upsilon}^*, \xi^*) = \mathbf{0}$$

**Proof:** Necessity has just been demonstrated: if $\mathbf{C}^*$ is mutually optimal, each individual consumption satisfies (2.7.2) which then leads to (2.8.8). Also, mutual optimality implies individual optimality which, by Proposition (2.6.5), implies (2.8.9) must hold. Sufficiency: since $\mathbf{C}^*$ is of the correct form for mutual optimality and $\mathbf{G}(\boldsymbol{\Upsilon}^*, \xi^*) = \mathbf{0}$, $\mathbf{C}^*$ is mutually optimal by Proposition (2.7.7). Moreover, $E = \Theta(\boldsymbol{\Upsilon}^*, \xi^*)$, so markets clear. By Proposition (2.8.2), the model is therefore in equilibrium. ■

Loosely speaking, locating an equilibrium requires that we solve the two equations $E = \Theta(\Upsilon^*, \xi^*)$ and $\mathbf{G}(\Upsilon^*, \xi^*) = \mathbf{0}$ for the two unknowns $(\Upsilon^*, \xi^*)$. After examining the structure of $\mathbf{G}$ in the last section, it is clear that one will not be able to analytically solve $\mathbf{G}(\Upsilon, \xi) = \mathbf{0}$ for $\xi$ in terms of $\Upsilon$, or, $\Upsilon$ in terms of $\xi$. However, as seen in the special cases in Karatzas et al. (1990), Karatzas et al. (1991), and Detemple & Zapatero (1991), the equation $E = \Theta(\Upsilon, \xi)$ can be quite easy to solve for $\xi = \xi(\Upsilon)$ uniquely, under certain conditions. Using this solution, one can "eliminate" $\xi$ from $\mathbf{G}(\Upsilon, \xi) = \mathbf{0}$ to obtain $\mathbf{G}(\Upsilon, \xi(\Upsilon)) = \mathbf{0}$. A variant of Brouwer's fixed point theorem, the Knaster-Kuratowski-Mazurkiewicz Lemma, can then be used to show the existence of a solution to $\mathbf{G}(\Upsilon, \xi(\Upsilon)) = \mathbf{0}$. Under certain conditions, one can show also that the solution is essentially unique.

In order to proceed, we must first consider the invertibility of an extension of $\Theta$. Continuously extend the inverse marginal utility functions $I^a(t, \omega, \cdot) : (0, \infty) \to (0, \infty)$ by taking limits, for each $(t, \omega) \in [0, T] \times \Omega$, to obtain $\tilde{I}^a(t, \omega, \cdot) : (0, \infty] \to [0, \infty)$:

$$\tilde{I}^a(t, \omega, y) = \begin{cases} I^a(t, \omega, y) & \text{if } y \in (0, \infty) \\ \lim_{\tilde{y} \uparrow \infty} I^a(t, \omega, \tilde{y}) \;=\; 0 & \text{if } y = \infty \end{cases}$$

This leads to the following continuous extensions of $C^a$ and $\Theta$: For each $(\Upsilon, \xi) \in (0, \infty]^A \times \Xi_{++}$ and $(t, \omega) \in [0, T] \times \Omega$, define

$$\begin{aligned} \tilde{C}^a(\Upsilon, \xi)_t &= \tilde{I}^a_t(y^a \gamma^a_t) + \int_0^t \sum_{b=1}^A \overleftarrow{\chi}^{ab}_{\nu t} \tilde{I}^b_\nu(y^b \gamma^b_\nu) d\nu \qquad (2.8.11) \\ \tilde{\Theta}(\Upsilon, \xi)_t &= \sum_{a=1}^A \tilde{C}^a(\Upsilon, \xi)_t = \sum_{a=1}^A \tilde{I}^a_t(y^a \gamma^a_t) + \int_0^t \sum_{a,b=1}^A \overleftarrow{\chi}^{ab}_{\nu t} \tilde{I}^b_\nu(y^b \gamma^b_\nu) d\nu \end{aligned}$$

Of course, for $\Upsilon \in (0, \infty)^A$, we have $\tilde{C}^a(\Upsilon, \xi) = C^a(\Upsilon, \xi)$ for all $a$ and $\tilde{\Theta}(\Upsilon, \xi) = \Theta(\Upsilon, \xi)$. For any choice of $\Upsilon \in (0, \infty]^A \setminus (0, \infty)^A$, $\tilde{C}^a$ and $\tilde{\Theta}$ are, by monotone convergence, the appropriate limits of $C^a$ and $\Theta$. One such limit is $\tilde{C}^a(\vec{\infty}, \xi) \equiv 0$ and $\tilde{\Theta}(\vec{\infty}, \xi) \equiv 0$ for any $\xi$. Since our aim is to solve $E = \Theta(\Upsilon, \xi)$ where $E$ is strictly positive, $\Upsilon = \vec{\infty}$ must be ruled out. Therefore, we confine ourselves to $\Upsilon \in (0, \infty]^A \setminus \{\vec{\infty}\}$.

Another limit to consider is $\Upsilon = (\infty, \ldots, \infty, y^b, \infty, \ldots, \infty)$ for $y^b \in (0, \infty)$. Assume that $\chi$ has identical column sums $\chi^c$; these are processes. The second part of

Lemma (2.1.16) then allows the simplification:

$$\begin{aligned}
\tilde{\Theta}(\xi, \Upsilon)_{t_0} &= \sum_{a=1}^{A} \tilde{I}^a_{t_0}(y^a\gamma^a_{t_0}) + \int_0^{t_0} \sum_{a,b=1}^{A} \overleftarrow{\chi}^{ab}_{\nu t_0} \tilde{I}^b_\nu(y^b\gamma^b_\nu)d\nu \\
&= I^b_{t_0}(y^b\gamma^b_{t_0}) + \int_0^{t_0} \sum_{a=1}^{A} \overleftarrow{\chi}^{ab}_{\nu t_0} I^b_\nu(y^b\gamma^b_\nu)d\nu \\
&= I^b_{t_0}(y^a\gamma^a_{t_0}) + \sum_{k=0}^{\infty} \int_0^{t_0} \cdots \int_0^{t_k} \sum_{a=1}^{A} \left\{ \prod_{i=0}^{k} \chi_{t_{i+1}t_i} \right\}^{ab} I^b_{t_{k+1}}(y^b\gamma^b_{t_{k+1}})dt_{k+1}\cdots dt_1 \\
&= I^b_{t_0}(y^a\gamma^a_{t_0}) + \sum_{k=0}^{\infty} \int_0^{t_0} \cdots \int_0^{t_k} \left\{ \prod_{i=0}^{k} \chi^c_{t_{i+1}t_i} \right\} I^b_{t_{k+1}}(y^b\gamma^b_{t_{k+1}})dt_{k+1}\cdots dt_1
\end{aligned}$$

See Theorem (2.1.4) for the derivation of this last series representation. Now, $E_{t_0} = \tilde{\Theta}(\xi, \Upsilon)_{t_0}$ can be solved indistinguishably for $\xi$: first apply Theorem (2.1.4) to solve for $I^b_t(y^b\gamma^b_t)$:

$$I^b_t(y^b\gamma^b_t) = E_t - \int_0^t \chi^c_{\nu t}E_\nu d\nu \tag{2.8.12}$$

As a result of the assumed condition $AK_\chi Te^{K_\chi T}K_E < k_E$ we have

$$\begin{aligned}
E_t - \int_0^t \chi^c_{\nu t}E_\nu d\nu &\geq Ak_E - \int_0^t K_\chi AK_E d\nu \\
&\geq k_E - K_\chi AK_E Te^{K_\chi T} > 0
\end{aligned} \tag{2.8.13}$$

Because of (2.8.13) we can invert (2.8.12) to obtain $\gamma^b$:

$$\gamma^b_t = \frac{1}{y^b}u^{b\prime}_t\left(E_t - \int_0^t \chi^c_{\nu t}E_\nu d\nu\right)$$

Finally, using the relationship between $\xi$ and $\gamma^b$ we obtain

$$\begin{aligned}
\xi_t &= \gamma^b_t - \mathbf{E}_t\left[\int_t^T \chi^{bb}_{t\nu}\gamma^b_\nu d\nu\right] \\
&= \frac{1}{y^b}u^{b\prime}_t\left(E_t - \int_0^t \chi^c_{\nu t}E_\nu d\nu\right) - \frac{1}{y^b}\mathbf{E}_t\left[\int_t^T \chi^{bb}_{t\nu}u^{b\prime}_\nu\left(E_\nu - \int_0^\nu \chi^c_{w\nu}E_w dw\right)d\nu\right]
\end{aligned}$$

Hence, in this instance, as long as $\|\chi\|$ is sufficiently small, $\xi > 0$ and so the map $\tilde{\Theta}(\cdot, \Upsilon) : \Xi_{++} \to \Xi_{++}$ is invertible. We postpone a discussion of the precise conditions under which $\tilde{\Theta}$ is invertible. For now, we make the following working assumption:

**Assumption 2.8.14** *For every* $\mathbf{\Upsilon} \in (0,\infty]^A \setminus \{\vec{\infty}\}$*, the map* $\tilde{\Theta}(\cdot,\mathbf{\Upsilon}) : \Xi_{++} \mapsto \Xi_{++}$ *is a bijection.*

We shall also make another working assumption but first it is convenient to introduce a reparametrization of the multipliers by defining $\iota : (0,\infty)^A \to (0,\infty)^A$ by

$$\iota(\mathbf{\Lambda}) = \left(\frac{1}{\lambda^1}, \ldots, \frac{1}{\lambda^A}\right) \quad \textit{for all } \mathbf{\Lambda} = (\lambda^1, \ldots, \lambda^A) \in (0,\infty)^A$$

**Assumption 2.8.15** *For every* $a = 1, \ldots, A$ *and* $(t,\omega) \in [0,T] \times \Omega$*, the process*

$$\Theta^{-1}\Big(\iota(\mathbf{\Lambda}), E\Big)\ C^a\Big(\iota(\mathbf{\Lambda}), \Theta^{-1}\Big(\iota(\mathbf{\Lambda}), E\Big)\Big)$$

*is non-increasing in each coordinate of* $\mathbf{\Lambda} = (\lambda^1, \ldots, \lambda^A) \in (0,\infty)^A$

Along with these working assumptions, we state the Knaster-Kuratowski-Mazurkiewicz Lemma here for reference:

**Lemma 2.8.16 (KKM)** *Let* $\{\vec{v}_1, \ldots, \vec{v}_M\} \subset \mathbb{R}^A$*. For any non-empty subset* $V \subset \{1, \ldots, A\}$ *of vertices define the closed simplices by*

$$\mathcal{S}(V) = \left\{ \sum_{a \in V} \lambda^a \vec{v}_a : \lambda^a \geq 0 \textit{ and } \sum_{a \in V} \lambda^a = 1 \right\}$$

*If* $F_1, \ldots, F_A$ *is a collection of closed subsets of* $\mathbb{R}^A$ *such that for every non-empty* $V \subset \{1, \ldots, A\}$ *we have* $\mathcal{S}(V) \subset \cup_{a \in V} F_a$ *then* $\cap_{a=1}^A F_a \neq \emptyset$*. We use the shorthand* $\mathcal{S}$ *to denote the full simplex* $\mathcal{S}(\{1, \ldots, A\})$*.*

**Proof:** See Border (1985), Chapter 5. ■

Under Assumption (2.8.14) we can demonstrate the existence of an equilibrium. Under Assumption (2.8.15) we can demonstrate the uniqueness of an equilibrium.

**Theorem 2.8.17** *If Assumption (2.8.14) holds, then there exists an equilibrium consumption configuration. If Assumption (2.8.15) holds as well, the equilibrium consumption configuration that exists is* $\boldsymbol{\lambda} \otimes \mathbf{P}$*-a.s. unique.*

**Proof:** First, extend $\iota$ continuously to $\iota : [0,\infty)^A \setminus \{\mathbf{0}\} \to (0,\infty]^A \setminus \{\vec{\infty}\}$ by taking limits. For each $\mathbf{\Lambda} \in [0,\infty)^A \setminus \{\mathbf{0}\}$, we have $\iota(\mathbf{\Lambda}) \in (0,\infty]^A \setminus \{\vec{\infty}\}$, so we can, by Assumption (2.8.14), define a state price density by

$$\xi(\iota(\mathbf{\Lambda})) = \tilde{\Theta}^{-1}(E, \iota(\mathbf{\Lambda}))$$

Hence,

$$E = \tilde{\Theta}(\iota(\mathbf{\Lambda}), \xi(\iota(\mathbf{\Lambda}))) = \sum_{a=1}^{A} \tilde{C}^a(\iota(\mathbf{\Lambda}), \xi(\iota(\mathbf{\Lambda}))) \tag{2.8.18}$$

So, we see that $\{\xi(\iota(\mathbf{\Lambda})) : \mathbf{\Lambda} \in [0,\infty)^A \setminus \{\mathbf{0}\}\}$ is a parametrized family of state price densities such that the continuously extended consumptions clear markets. Consider the set of standard unit vectors $\{\hat{e}_1, \ldots, \hat{e}^A\}$ of $\mathbb{R}^A$ and use them to construct $\mathcal{S}$, as in the KKM Lemma (2.8.16). Now, noting that $\mathcal{S} \subset [0,\infty)^A \setminus \{\mathbf{0}\}$, define $f_a : \mathcal{S} \to \mathbb{R}$ by

$$f_a(\mathbf{\Lambda}) = \mathbf{E}\left[\int_0^T \tilde{\Theta}^{-1}(E, \iota(\mathbf{\Lambda}))_\nu \left(\tilde{C}^a\Big(\iota(\mathbf{\Lambda}), \tilde{\Theta}^{-1}(E, \iota(\mathbf{\Lambda}))\Big)_\nu - E^a_\nu\right) d\nu\right]$$

Motivating this definition is the observation that when $\mathbf{\Lambda} \in \mathcal{S} \cap (0,\infty)^A$, we obtain $\iota(\mathbf{\Lambda}) \in (0,\infty)^A$ and hence

$$\begin{aligned} f_a(\mathbf{\Lambda}) &= \mathbf{E}\left[\int_0^T \tilde{\Theta}^{-1}(E, \iota(\mathbf{\Lambda}))_\nu \left(\tilde{C}^a\Big(\iota(\mathbf{\Lambda}), \tilde{\Theta}^{-1}(E, \iota(\mathbf{\Lambda}))\Big)_\nu - E^a_\nu\right) d\nu\right] \\ &= \mathbf{E}\left[\int_0^T \Theta^{-1}(E, \iota(\mathbf{\Lambda}))_\nu \left(C^a\Big(\iota(\mathbf{\Lambda}), \Theta^{-1}(E, \iota(\mathbf{\Lambda}))\Big)_\nu - E^a_\nu\right) d\nu\right] \\ &= G^a\Big(\iota(\mathbf{\Lambda}), \Theta^{-1}(E, \iota(\mathbf{\Lambda}))\Big) \\ &= G^a\Big(\iota(\mathbf{\Lambda}), \xi(\iota(\mathbf{\Lambda}))\Big) \end{aligned}$$

which is just a reparametrized version of agent $a$'s budget constraint function. What we are after is therefore a $\mathbf{\Lambda}^* \in \mathcal{S} \cap (0,\infty)^A$ such that $f_a(\mathbf{\Lambda}^*) = G^a\Big(\iota(\mathbf{\Lambda}^*), \xi(\mathbf{\Lambda}^*)\Big) = 0$ for all $a = 1, \ldots, A$ for then the vector $\mathbf{\Upsilon}^* = \iota(\mathbf{\Lambda}^*) \in (0,\infty)^A$ contains the equilibrium multipliers we are looking for; consumption is of the correct form for mutual optimality, all budget constraints bind, and markets clear. We start by solving $f_a(\mathbf{\Lambda}^*) = 0$ for $\mathbf{\Lambda}^* \in \mathcal{S}$ and then show that $\mathbf{\Lambda}^* \in \mathcal{S} \cap (0,\infty)^A$. Note that restricting ourselves to the simplex $\mathcal{S}$ in solving $f_a(\mathbf{\Lambda}) = 0$, rather than considering all of $[0,\infty)^A \setminus \{\mathbf{0}\}$, is not really a restriction as $f_a$ is homogeneous of degree 1 in $\mathbf{\Lambda}$. To see this, consider $\mathbf{\Lambda} \in \mathcal{S}$ and $k \neq 0$. First, $\iota(k\mathbf{\Lambda}) = \frac{1}{k}\iota(\mathbf{\Lambda})$. Next, $\xi$ and $\gamma^a$ are related by (2.8.7), and, by linearity, $k\xi$ and $k\gamma^a$ also satisfy (2.8.7). Considering (2.8.11), we then see that

$$\tilde{C}^a(\iota(k\mathbf{\Lambda}), k\xi) = \tilde{C}^a\left(\frac{1}{k}\iota(\mathbf{\Lambda}), k\xi\right) = \tilde{C}^a(\iota(\mathbf{\Lambda}), \xi)$$

and, from (2.8.12),

$$\tilde{\Theta}(\iota(k\mathbf{\Lambda}), k\xi) = \tilde{\Theta}(\iota(\mathbf{\Lambda}), \xi)$$

Solving $E = \tilde{\Theta}(\iota(k\mathbf{\Lambda}), k\xi)$ and $E = \tilde{\Theta}(\iota(\mathbf{\Lambda}), \xi)$ yields $k\xi = \tilde{\Theta}^{-1}(\iota(k\mathbf{\Lambda}), E)$ and $\xi = \tilde{\Theta}^{-1}(\iota(\mathbf{\Lambda}), E)$, respectively, and hence

$$\tilde{\Theta}^{-1}(\iota(k\mathbf{\Lambda}), E) = k\tilde{\Theta}^{-1}(\iota(\mathbf{\Lambda}), E)$$

Finally, we see that

$$\begin{aligned} f_a(k\mathbf{\Lambda}) &= \mathbf{E}\left[\int_0^T \tilde{\Theta}^{-1}(E, \iota(k\mathbf{\Lambda}))_\nu \left(\tilde{C}^a\left(\iota(k\mathbf{\Lambda}), \tilde{\Theta}^{-1}(E, \iota(k\mathbf{\Lambda}))\right)_\nu - E^a_\nu\right) d\nu\right] \\ &= \mathbf{E}\left[\int_0^T k\tilde{\Theta}^{-1}(E, \iota(\mathbf{\Lambda}))_\nu \left(\tilde{C}^a\left(\iota(k\mathbf{\Lambda}), k\tilde{\Theta}^{-1}(E, \iota(\mathbf{\Lambda}))\right)_\nu - E^a_\nu\right) d\nu\right] \\ &= \mathbf{E}\left[\int_0^T k\tilde{\Theta}^{-1}(E, \iota(\mathbf{\Lambda}))_\nu \left(\tilde{C}^a\left(\iota(\mathbf{\Lambda}), \tilde{\Theta}^{-1}(E, \iota(\mathbf{\Lambda}))\right)_\nu - E^a_\nu\right) d\nu\right] \\ &= k f_a(\mathbf{\Lambda}) \end{aligned}$$

Thus, $f_a(\mathbf{\Lambda}) = 0$ if and only if $f_a(k\mathbf{\Lambda}) = 0$; if $f_a(\mathbf{\Lambda}) = 0$ then $f_a = 0$ along an entire ray.[12]

Summing the $f_a$ for each $\mathbf{\Lambda} \in \mathcal{S}$, and using (2.8.18), we obtain

$$\sum_{a=1}^{A} f_a(\mathbf{\Lambda}) = \mathbf{E}\left[\int_0^T \tilde{\Theta}^{-1}(E, \iota(\mathbf{\Lambda}))_\nu \left(\sum_{a=1}^{A} \tilde{C}^a\left(\iota(\mathbf{\Lambda}), \tilde{\Theta}^{-1}(E, \iota(\mathbf{\Lambda}))\right)_\nu - E_\nu\right) d\nu\right] = 0$$

To get at $f_a(\mathbf{\Lambda}^*) = 0$ for each $a$, consider the sets defined by $F_a = \{\mathbf{\Lambda} \in \mathcal{S} : f_a(\mathbf{\Lambda}) \geq 0\}$. Since the $f_a$ are continuous, the $F_a$ are closed sets. For each $\mathbf{\Lambda} \in \mathcal{S}$ let $V \subset \{1, \ldots, A\}$ be the subset of indeces such that $\lambda^a \in (0, \infty)$ for $a \in V$ and $\lambda^a = 0$ for $a \notin V$. Note that $V \neq \emptyset$ as $\mathbf{0}$ is excluded from $\mathcal{S}$. Note also that, using the results on subsistence consumption in Sections 2.5 and 2.6, $\lim_{y^a \uparrow \infty} \tilde{C}^a(\mathbf{\Upsilon}, \xi) = C^{as}$. Thus, for $a \notin V$ we have $f_a(\mathbf{\Lambda}) = \mathbf{E}\left[\int_0^T \xi_\nu (C^{as}_\nu - E^a_\nu) d\nu\right] < 0$. We shall show that for all non-empty $V \subset \{1, \ldots, A\}$ we have $\mathcal{S}(V) \subset \cup_{a \in V} F_a$ so that we can apply Lemma

[12] As discussed in Chapter 3, this property is important in optimizing our simulation procedure.

(2.8.16). To do this, let $V \subset \{1, \ldots, A\}$ be non-empty, let $\mathbf{\Lambda} \in \mathcal{S}(V)$ and show that $\mathbf{\Lambda} \in \cup_{a \in V} F_a$.

First case: $V = \{1, \ldots, A\}$. If $\mathbf{\Lambda} \notin \cup_{a \in V} F_a = \cup_{a=1}^{A} F_a$ then $\mathbf{\Lambda} \in F_a^c$ and hence $f_a(\mathbf{\Lambda}) < 0$ for $1 \leq a \leq A$. Thus, $\sum_{a=1}^{A} f_a(\mathbf{\Lambda}) < 0$, contradicting that $\sum_{a=1}^{A} f_a(\mathbf{\Lambda}) = 0$ for all $\mathbf{\Lambda} \in \mathcal{S}$.

Second case: $V$ is proper but not empty. By the definition of $\mathcal{S}(V)$, $\mathbf{\Lambda} \in \mathcal{S}(V)$ implies that for all $a \notin V$ we have $\lambda^a = 0$. Thus, for all $a \notin V$ we have $f_a(\mathbf{\Lambda}) < 0$ so $\sum_{a \notin V} f_a(\mathbf{\Lambda}) < 0$

However, $\mathbf{\Lambda} \notin \cup_{a \in V} F_a$ implies $\mathbf{\Lambda} \in F_a^c$ for all $a \in V$ which means that $f_a(\mathbf{\Lambda}) < 0$ for all $a \in V$ so that $\sum_{a \in V} f_a(\mathbf{\Lambda}) < 0$. Thus, $\sum_{a=1}^{A} f_a(\mathbf{\Lambda}) = \sum_{a \in V} f_a(\mathbf{\Lambda}) + \sum_{a \notin V} f_a(\mathbf{\Lambda}) < 0$, a contradiction. So, $\mathbf{\Lambda} \in \cup_{a \in V} F_a$.

Now, applying KKM we have $\cap_{a=1}^{A} F_a \neq \emptyset$ so there is a $\mathbf{\Lambda}^* \in \cap_{a=1}^{A} F_a$ which means that $\mathbf{\Lambda}^* \in F_a$ and hence $f_a(\mathbf{\Lambda}^*) \geq 0$ for all $1 \leq a \leq A$. If there exists $1 \leq a \leq A$ such that $f_a(\mathbf{\Lambda}^*) > 0$ then $\sum_{a=1}^{A} f_a(\mathbf{\Lambda}^*) > 0$, a contradiction of $\sum_{a=1}^{A} f_a(\mathbf{\Lambda}^*) = 0$. Thus, $f_a(\mathbf{\Lambda}^*) = 0$ for all $1 \leq a \leq A$. Moreover, if for some $1 \leq a \leq A$ we have $\lambda^{a*} = 0$ then $f_a(\mathbf{\Lambda}^*) < 0$, contradicting that $f_a(\mathbf{\Lambda}^*) = 0$ for all $1 \leq a \leq A$. So, all $\lambda^{a*} > 0$ which means there is a $\mathbf{\Lambda}^* \in (0, \infty)^A$ such that $f_a(\mathbf{\Lambda}^*) = 0$ for all $1 \leq a \leq A$. Now, $\mathbf{\Upsilon}^* = \iota(\mathbf{\Lambda}^*) \in (0, \infty)^A$ are the equilibrium multipliers from which we can construct all equilibrium processes:

$$
\begin{aligned}
\xi^* &= \Theta^{-1}(\iota(\mathbf{\Lambda}^*), E) \\
C^{a*} &= C^a(\iota(\mathbf{\Lambda}^*), \xi^*) = C^a(\iota(\mathbf{\Lambda}^*), \Theta^{-1}(\iota(\mathbf{\Lambda}^*), E))
\end{aligned}
$$

Equilibrium wealth processes $W^{a*}$ and portfolio processes $\pi^{a*}$ follow from Proposition (2.6.5). Note that if $k\mathbf{\Lambda}^*$ is used instead of $\mathbf{\Lambda}^*$ we have

$$
\Theta^{-1}(\iota(k\mathbf{\Lambda}^*), E) = k\Theta^{-1}(\iota(\mathbf{\Lambda}^*), E) = k\xi^*
$$

and

$$
\begin{aligned}
C^a(\iota(k\mathbf{\Lambda}^*), \Theta^{-1}(\iota(k\mathbf{\Lambda}^*), E)) &= C^a\Big(\frac{1}{k}\iota(\mathbf{\Lambda}^*), k\Theta^{-1}(\iota(\mathbf{\Lambda}^*), E)\Big) \\
&= C^a(\iota(\mathbf{\Lambda}^*), \Theta^{-1}(\iota(\mathbf{\Lambda}^*), E)) \\
&= C^{a*}
\end{aligned}
$$

so that any rescaling affects only state prices. In Chapter 3, we will need to find the value of $k$ such that state prices at $t = 0$ are 1, as this is how we defined $\xi$ and is needed in the simulation of equilibrium interest rates $r^*$ and prices of risk $\theta^*$.

In order to show $k\mathbf{\Lambda}^*$ is a unique positive ray of equilibrating multipliers, under Assumption (2.8.15), consider the partial ordering on $(0,\infty)^A$ defined by

$$\mathbf{\Lambda} \leq \hat{\mathbf{\Lambda}} \iff \lambda^a \leq \hat{\lambda}^a \qquad \forall 1 \leq a \leq A$$

Write $\mathbf{\Lambda} < \hat{\mathbf{\Lambda}}$ if and only if $\mathbf{\Lambda} \leq \hat{\mathbf{\Lambda}}$ and for at least one $a$ we have $\lambda^a < \hat{\lambda}^a$. Now, consider the maps

$$\begin{aligned} C^a(\iota(\mathbf{\Lambda}),\xi)_t &= I^a_t(\gamma^a_t/\lambda^a) + \int_0^t \sum_{b=1}^A \overleftarrow{\chi}^{ab}_{\nu t} I^b_\nu(\gamma^b_\nu/\lambda^b)d\nu \\ \Theta(\iota(\mathbf{\Lambda}),\xi)_t &= \sum_{a=1}^A C^a(\iota(\mathbf{\Lambda}),\xi)_t = \sum_{a=1}^A I^a_t(\gamma^a_t/\lambda^a) + \int_0^t \sum_{a,b=1}^A \overleftarrow{\chi}^{ab}_{\nu t} I^b_\nu(\gamma^b_\nu/\lambda^b)d\nu \end{aligned}$$

For any given $\xi \in \Xi_{++}$, we have

$$\mathbf{\Lambda} \leq \hat{\mathbf{\Lambda}} \implies \Theta(\iota(\mathbf{\Lambda}),\xi) \leq \Theta(\iota(\hat{\mathbf{\Lambda}}),\xi)) \implies \Theta^{-1}(\iota(\mathbf{\Lambda}),E) \leq \Theta^{-1}(\iota(\hat{\mathbf{\Lambda}}),E))$$

and

$$\mathbf{\Lambda} < \hat{\mathbf{\Lambda}} \implies \Theta(\iota(\mathbf{\Lambda}),\xi) < \Theta(\iota(\hat{\mathbf{\Lambda}}),\xi)) \implies \Theta^{-1}(\iota(\mathbf{\Lambda}),E) < \Theta^{-1}(\iota(\hat{\mathbf{\Lambda}}),E))$$

Let both $\mathbf{\Lambda}$ and $\hat{\mathbf{\Lambda}}$ be solutions of the equilibrium condition: $f_a(\mathbf{\Lambda}) = f_a(\hat{\mathbf{\Lambda}}) = 0$ for all $a = 1,\ldots,A$. However, suppose that $\mathbf{\Lambda}$ and $\hat{\mathbf{\Lambda}}$ are not on the same ray. Define

$$k = \max_{1\leq a\leq A}\left(\frac{\lambda^a}{\hat{\lambda}^a}\right) > 0$$

Now $f_a(\mathbf{\Lambda}) = f_a(k\hat{\mathbf{\Lambda}}) = 0$ for all $a$ and $\mathbf{\Lambda} \leq k\hat{\mathbf{\Lambda}}$. However, $\mathbf{\Lambda}$ and $\hat{\mathbf{\Lambda}}$ are not on the same ray so the ratio $\frac{\lambda^{a'}}{\hat{\lambda}^{a'}}$ is strictly less than $k$ for at least one $a'$. Thus, $\mathbf{\Lambda} < k\hat{\mathbf{\Lambda}}$ so $\Theta^{-1}(\iota(\mathbf{\Lambda}),E) < \Theta^{-1}(\iota(k\hat{\mathbf{\Lambda}}),E))$ and hence

$$-\mathbf{E}\left[\int_0^T \Theta^{-1}(\iota(\mathbf{\Lambda}),E)_\nu E^a_\nu d\nu\right] > -\mathbf{E}\left[\int_0^T \Theta^{-1}(\iota(k\hat{\mathbf{\Lambda}}),E)_\nu E^a_\nu d\nu\right] \quad (2.8.19)$$

Using the assumption that $\Theta^{-1}\Big(\iota(\mathbf{\Lambda}),E\Big)C^a\Big(\iota(\mathbf{\Lambda}),\Theta^{-1}(\iota(\mathbf{\Lambda}),E)\Big)$ is non-increasing in $\mathbf{\Lambda}$, we also obtain

$$\begin{aligned} &\mathbf{E}\left[\int_0^T \Theta^{-1}\Big(\iota(\mathbf{\Lambda}),E\Big)_\nu C^a\Big(\iota(\mathbf{\Lambda}),\Theta^{-1}(\iota(\mathbf{\Lambda}),E)\Big)_\nu d\nu\right] \quad (2.8.20) \\ &\geq \mathbf{E}\left[\int_0^T \Theta^{-1}\Big(\iota(k\hat{\mathbf{\Lambda}}),E\Big)_\nu C^a\Big(\iota(k\hat{\mathbf{\Lambda}}),\Theta^{-1}(\iota(k\hat{\mathbf{\Lambda}}),E)\Big)_\nu d\nu\right] \end{aligned}$$

Adding (2.8.19) and (2.8.20) gives

$$0 \;=\; f_a(\mathbf{\Lambda}) \;>\; f_a(k\hat{\mathbf{\Lambda}}) \;=\; 0$$

which is a contradiction. Thus, $\mathbf{\Lambda}$ and $\hat{\mathbf{\Lambda}}$ must be on the same ray.

Finally, from the boundedness of $E$, $\xi = \Theta^{-1}(E;\mathbf{\Upsilon})$ is bounded. Since $r$ and $\theta$ are in the definition of $\xi$:

$$d\xi_t = -\xi_t r_t dt - \xi_t \theta_t^\top dZ_t$$

it must be that case that $r$ and $\theta$ are bounded, as required. We already have that $\sigma$ is bounded and non-degenerate. As a result, $\mu_t = r_t \mathbf{1} + \sigma_t \theta_t$ is bounded. Finally, by the continuity of all processes involved, consumption $\mathbf{C}^*(\xi^*, \mathbf{\Upsilon}^*)$ is also strictly positive, continuous, and bounded above, as required. ■

We now present some partial results on the invertibility of $\tilde{\Theta}$.

**Proposition 2.8.21** *Suppose that the habit kernel process $\boldsymbol{\chi}$ satisfies the following two properties:*

1. *$\boldsymbol{\chi}$ is of the form $\boldsymbol{\chi}_{\nu t} = \chi_{\nu t}\mathbf{I} + \mathbf{D}_{\nu t}$ where $\chi$ is a scalar process, $\mathbf{I}$ is the identity matrix, and $\mathbf{D}$ is a matrix valued process with zeros on its diagonal*
2. *$\boldsymbol{\chi}$ has identical column sums $\chi^c$*

*then for each $\mathbf{\Upsilon} \in (0,\infty]^A \setminus \{\vec{\infty}\}$, the map $\tilde{\Theta}(\mathbf{\Upsilon},\cdot) : \Xi_{++} \to \Xi_{++}$ is invertible.*

**Proof:** Consider the more explicit series representations for $\tilde{\Theta}$ and $\gamma^a$:

$$\tilde{\Theta}(\mathbf{\Upsilon},\xi)_{t_0} = \sum_{a=1}^{A} \tilde{I}^a_{t_0}(y^a\gamma^a_{t_0}) + \sum_{k=0}^{\infty}\int_0^{t_0}\cdots\int_0^{t_k}\sum_{a,b=1}^{A}\left\{\prod_{i=0}^{k}\boldsymbol{\chi}_{t_{i+1}t_i}\right\}^{ab}\tilde{I}^b_{t_{k+1}}(y^b\gamma^b_{t_{k+1}})dt_{k+1}\cdots dt_1$$

and

$$\gamma^a_{t_0} = \xi_{t_0} + \mathbf{E}_{t_0}\left[\sum_{k=0}^{\infty}\int_{t_0}^{T}\cdots\int_{t_k}^{T}\left\{\prod_{i=0}^{k}\chi^{aa}_{t_{i+1}t_i}\right\}\xi_{t_{k+1}}dt_{k+1}\cdots dt_1\right]$$

As already observed in this section, under the assumption that $\boldsymbol{\chi}$ has identical column sum processes $\chi^c$ we have

$$\tilde{\Theta}(\mathbf{\Upsilon},\xi)_{t_0} \;=\; \sum_{a=1}^{A} \tilde{I}^a_{t_0}(y^a\gamma^a_{t_0}) + \sum_{k=0}^{\infty}\int_0^{t_0}\cdots\int_0^{t_k}\left\{\prod_{i=0}^{k}\chi^c_{t_{i+1}t_i}\right\}\sum_{a=1}^{A}\tilde{I}^a_{t_{k+1}}(y^a\gamma^a_{t_{k+1}})dt_{k+1}\cdots dt_1$$

Since $\boldsymbol{\chi} = \chi\mathbf{I} + \mathbf{D}$ we have $\chi^{aa} = \chi$ is the same for each agent and hence $\gamma^a$ is the same for each agent. Let $\gamma$ denote this common $\gamma^a$. Hence,

$$\tilde{\Theta}(\boldsymbol{\Upsilon}, \xi)_{t_0} = \sum_{a=1}^{A} \tilde{I}^a_{t_0}(y^a\gamma_{t_0}) + \sum_{k=0}^{\infty} \int_0^{t_0} \cdots \int_0^{t_k} \left\{ \prod_{i=0}^{k} \chi^c_{t_{i+1}t_i} \right\} \sum_{a=1}^{A} \tilde{I}^a_{t_{k+1}}(y^a\gamma_{t_{k+1}}) dt_{k+1} \cdots dt_1$$

Now, applying Theorem (2.1.4), we can solve $E_{t_0} = \tilde{\Theta}(\boldsymbol{\Upsilon}, \xi)_{t_0}$ for $\sum_{a=1}^{A} \tilde{I}^a_{t_0}(y^a\gamma_{t_0})$:

$$\sum_{a=1}^{A} \tilde{I}^a_t(y^a\gamma_t) = E_t - \int_0^t \chi^c_{\nu t} E_\nu d\nu$$

for which, as already mentioned, the right hand side is strictly positive. Now, since the map $y \mapsto \sum_{a=1}^{A} \tilde{I}^a_t(y^a y)$ is invertible with inverse $\tilde{J}_t(x; \boldsymbol{\Upsilon})$, we can solve for $\gamma$:

$$\gamma_t = \tilde{J}_t\Big(E_t - \int_0^t \chi^c_{\nu t} E_\nu d\nu; \boldsymbol{\Upsilon}\Big)$$

Finally, from $\gamma$, we can obtain $\xi$ indistinguishably, again with Theorem (2.1.4)

$$\xi_t = \gamma_t - \mathbf{E}_t \left[ \int_t^T \chi_{t\nu}\gamma_\nu d\nu \right]$$

■

The conditions of Proposition (2.8.21) are restrictive but not too restrictive. When $A = 2$, we are restricted to habit kernels of the form

$$\boldsymbol{\chi} = \begin{bmatrix} \chi & a \\ a & \chi \end{bmatrix}$$

which leaves two processes to specify. It is this case which is simulated in Chapter 3 for some preliminary specifications of $\chi$ and $a$. When $A = 3$ we have

$$\boldsymbol{\chi} = \begin{bmatrix} \chi & a & b \\ b & \chi & a \\ a & b & \chi \end{bmatrix}$$

which leaves three processes to specify. When $A = 4$ we have

$$\boldsymbol{\chi} = \begin{bmatrix} \chi & a & b & c \\ a+b+c-e-f & \chi & e & f \\ e+f-j & c-e+j & \chi & a+b-f \\ j & b+e-j & a+c-e & \chi \end{bmatrix}$$

which leaves seven processes to specify. It appears impossible for the identical row sums to be different from the identical column sums, although we do not provide a general proof. In any case, there is a great deal more flexibility as the number of agents increases.

If we only insist on $\boldsymbol{\chi}$ having an identical column sum process $\chi^c$ then we have existence of a solution $\xi$ but unfortunately we have not been able to establish indistinguishability:

**Proposition 2.8.22** *Suppose that the habit kernel process* $\boldsymbol{\chi}$ *has identical column sums* $\chi^c$. *Then, for each* $\boldsymbol{\Upsilon} \in (0,\infty]^A \setminus \{\vec{\infty}\}$ *and for each* $E \in \Xi_{++}$ *there exists a solution* $\xi \in \Xi_{++}$ *to* $E = \tilde{\Theta}(\boldsymbol{\Upsilon}, \xi)$.

**Proof:** As in prior proposition, assuming $\boldsymbol{\chi}$ has an identical column sum process $\chi^c$ leads to

$$\begin{aligned} E_{t_0} &= \tilde{\Theta}(\boldsymbol{\Upsilon}, \xi)_{t_0} \\ &= \sum_{a=1}^{A} \tilde{I}^a_{t_0}(y^a \gamma^a_{t_0}) + \sum_{k=0}^{\infty} \int_0^{t_0} \cdots \int_0^{t_k} \left\{ \prod_{i=0}^{k} \chi^c_{t_{i+1}t_i} \right\} \sum_{a=1}^{A} \tilde{I}^a_{t_{k+1}}(y^a \gamma^a_{t_{k+1}}) dt_{k+1} \cdots dt_1 \end{aligned}$$

Applying Theorem (2.1.4), we can solve for $\sum_{a=1}^{A} \tilde{I}^a_t(y^a \gamma^a_t)$:

$$\sum_{a=1}^{A} \tilde{I}^a_t(y^a \gamma^a_t) = E_t - \int_0^t \chi^c_{\nu t} E_\nu d\nu$$

Now, the issue is to show that there exists a $\xi \in \Xi_{++}$ satisfying this relationship. The complication here is that $\gamma^a$ can be different for each agent as we are not assuming a common diagonal process in $\boldsymbol{\chi}$.

It is convenient to use the more compact series representation for $\gamma^a$:

$$\gamma^a_t = \xi_t + \mathbf{E}_t \left[ \int_t^T \chi^a_{t\nu} \xi_\nu d\nu \right]$$

where, for this proof, we are using $\chi^a$ to more simply denote $\overrightarrow{\chi^{aa}}$. Let $X_t = E_t - \int_0^t \chi^c_{\nu t} E_\nu d\nu$. Thus $X$ has the bounds $0 < k \le X \le K < \infty$ for constants $k, K$. To begin, note that at time $T$ we have $\gamma^a_T = \xi_T$ for all $a$ and so

$$X_T = \sum_{a=1}^{A} I^a_T(y^a \xi_T)$$

Since the map $y \mapsto \sum_{a=1}^{A} I_t(y^a y)$ is invertible for each $(t,\omega)$, we can uniquely solve for $\xi_T$.

To find $\xi$ at all other times, we approximate $\gamma^a$. For each $N \geq 1$, divide $[0,T]$ into $2^N$ intervals each of width $\frac{T}{2^N}$. And, for $0 \leq n \leq 2^N$ we approximate the integral by a Riemann sum, using left end points:

$$\int_{\frac{nT}{2^N}}^{T} \chi^a_{\frac{nT}{2^N}\nu} \xi_\nu d\nu \;\approx\; \frac{T}{2^N} \sum_{m=n}^{2^N-1} \chi^a_{\frac{nT}{2^N}\frac{mT}{2^N}} \xi_{\frac{mT}{2^N}}$$

Our approximation $\gamma^{a,N}$ for $\gamma^a$ at time points $\frac{nT}{2^N}$ is then defined for a given $\xi$ by

$$\gamma^{a,N}(\xi)_{\frac{nT}{2^N}} \;=\; \xi_{\frac{nT}{2^N}} + \frac{T}{2^N} \sum_{m=n}^{2^N-1} \mathbf{E}_{\frac{nT}{2^N}} \left[ \chi^a_{\frac{nT}{2^N}\frac{mT}{2^N}} \xi_{\frac{mT}{2^N}} \right]$$

For all other time points, $\gamma^{a,N}$ is defined by linear interpolation. We show that for each $N$ we can find a unique process $\xi^{(N)}$ such that

$$X_{\frac{nT}{2^N}} = \sum_{a=1}^{A} I^a_{\frac{nT}{2^N}} \left(y^a \gamma^{a,N}(\xi^{(N)})_{\frac{nT}{2^N}}\right) \tag{2.8.23}$$

for all $0 \leq n < 2^N$, the case $n = 2^N$ corresponding to time $T$ already solved above. As with $\gamma^a$, we have the terminal value at $t = T$ is

$$\gamma^{a,N}(\xi)_{\frac{2^N T}{2^N}} \;=\; \gamma^{a,N}(\xi)_T = \xi_T$$

Working backwards in time, we obtain

$$\begin{aligned}
\gamma^{a,N}(\xi)_{\frac{(2^N-1)T}{2^N}} &= \xi_{\frac{(2^N-1)T}{2^N}} + \frac{T}{2^N} \mathbf{E}_{\frac{(2^N-1)T}{2^N}} \left[ \chi^a_{\frac{(2^N-1)T}{2^N}\frac{(2^N-1)T}{2^N}} \xi_{\frac{(2^N-1)T}{2^N}} \right] \\
&= \xi_{\frac{(2^N-1)T}{2^N}} \left( 1 + \frac{T}{2^N} \chi^a_{\frac{(2^N-1)T}{2^N}\frac{(2^N-1)T}{2^N}} \right)
\end{aligned}$$

and another time step back, we have

$$\begin{aligned}
&\gamma^{a,N}(\xi)_{\frac{(2^N-2)T}{2^N}} \\
= \quad & \xi_{\frac{(2^N-2)T}{2^N}} + \frac{T}{2^N} \mathbf{E}_{\frac{(2^N-2)T}{2^N}} \left[ \chi^a_{\frac{(2^N-2)T}{2^N}\frac{(2^N-2)T}{2^N}} \xi_{\frac{(2^N-2)T}{2^N}} + \chi^a_{\frac{(2^N-2)T}{2^N}\frac{(2^N-1)T}{2^N}} \xi_{\frac{(2^N-1)T}{2^N}} \right] \\
= \quad & \xi_{\frac{(2^N-2)T}{2^N}} \left( 1 + \frac{T}{2^N} \chi^a_{\frac{(2^N-2)T}{2^N}\frac{(2^N-2)T}{2^N}} \right) + \frac{T}{2^N} \mathbf{E}_{\frac{(2^N-1)T}{2^N}} \left[ \chi^a_{\frac{(2^N-2)T}{2^N}\frac{(2^N-1)T}{2^N}} \xi_{\frac{(2^N-1)T}{2^N}} \right]
\end{aligned}$$

Repeating, we can recursively isolate $\xi_{\frac{nT}{2^N}}$ in terms of $\xi_{\frac{(n+1)T}{2^N}}, \ldots, \xi_{\frac{(2^N-1)T}{2^N}}$ for all $n = 0, \ldots, 2^N - 2$ in the expression of $\gamma^{a,N}$:

$$\begin{aligned}\gamma^{a,N}(\xi)_{\frac{nT}{2^N}} &= \xi_{\frac{nT}{2^N}}\left(1 + \frac{T}{2^N}\chi^a_{\frac{nT}{2^N}\frac{nT}{2^N}}\right) + \frac{T}{2^N}\mathbf{E}_{\frac{nT}{2^N}}\left[\sum_{m=n+1}^{2^N-1}\chi^a_{\frac{nT}{2^N}\frac{mT}{2^N}}\xi_{\frac{T}{2^N}}\right] \\ &= \xi_{\frac{nT}{2^N}}\mathcal{M}^a_{N,n} + \mathcal{N}^a_{N,n}\end{aligned}$$

where $\mathcal{M}^a_{N,n}$ and $\mathcal{N}^a_{N,n}$ are used to simplify notation. Note that $\mathcal{N}^a_{N,n}$ depends on $\xi$ only after time $\frac{nT}{2^N}$. Now, we use $\gamma^{a,N}(\xi)_{\frac{nT}{2^N}} = \xi_{\frac{nT}{2^N}}\mathcal{M}^a_{N,n} + \mathcal{N}^a_{N,n}$ to determine $\xi^{(N)}$ satisfying

$$X_{\frac{nT}{2^N}} = \sum_{a=1}^{A} I^a_{\frac{nT}{2^N}}(y^a\gamma^{a,N}(\xi^{(N)})_{\frac{nT}{2^N}}) = \sum_{a=1}^{A} I^a_{\frac{nT}{2^N}}\left(y^a\left(\xi^{(N)}_{\frac{nT}{2^N}}\mathcal{M}^a_{N,n} + \mathcal{N}^a_{N,n}\right)\right)$$

At time $\frac{(2^N-1)}{2^N}$ we have

$$X_{\frac{(2^N-1)T}{2^N}} = \sum_{a=1}^{A} I^a_{\frac{(2^N-1)T}{2^N}}\left(y^a\xi^{(N)}_{\frac{(2^N-1)T}{2^N}}\mathcal{M}^a_{N,2^N-1}\right)$$

As $\xi^{(N)}_{\frac{(2^N-1)T}{2^N}}$ is common to all $I^a$ and since $y \mapsto \sum_{a=1}^{A} I_t(y^a(y\mathcal{M}_{N,n}+\mathcal{N}^a_{N,n}))$ is invertible, we can solve for $\xi^{(N)}_{\frac{(2^N-1)T}{2^N}}$. Repeating at time $\frac{(2^N-2)T}{2^N}$ using the known value $\xi^{(N)}_{\frac{(2^N-1)T}{2^N}}$ we obtain

$$X_{\frac{(2^N-2)T}{2^N}} = \sum_{a=1}^{A} I^a_{\frac{(2^N-2)T}{2^N}}\left(y^a\left(\xi^{(N)}_{\frac{(2^N-2)T}{2^N}}\mathcal{M}^a_{N,2^N-2} + \mathcal{N}^a_{N,2^N-2}\right)\right)$$

which, again, gives $\xi^{(N)}_{\frac{(2^N-2)T}{2^N}}$ uniquely. In this way, we can inductively find a unique process $\{\xi^{(N)}_0, \ldots, \xi^{(N)}_{\frac{(2^N-1)T}{2^N}}, \xi_T\}$ which satisfies (2.8.23) at all time points $\frac{nT}{2^N}$.

Next, we show that $\{\xi^{(N)}\}_{N=1}^{\infty}$ is, pathwise, a Cauchy sequence on $\Xi$ relative to the sup-norm metric. Consider the time $\frac{nT}{2^N}$:

$$X_{\frac{nT}{2^N}} = \sum_{a=1}^{A} I^a_{\frac{nT}{2^N}}\left(y^a\left(\xi^{(N)}_{\frac{nT}{2^N}}\mathcal{M}^a_{N,n} + \mathcal{N}^a_{N,n}\right)\right)$$

which is equal to

$$\begin{aligned} X_{\frac{nT}{2^N}} &= X_{\frac{2nT}{2^{N+1}}} \\ &= \sum_{a=1}^{A} I^a_{\frac{2nT}{2^{N+1}}}\left(y^a\left(\xi^{(N+1)}_{\frac{2nT}{2^{N+1}}}\mathcal{M}^a_{N+1,2n}+\mathcal{N}^a_{N+1,2n}\right)\right) \\ &= \sum_{a=1}^{A} I^a_{\frac{nT}{2^N}}\left(y^a\left(\xi^{(N+1)}_{\frac{nT}{2^N}}\mathcal{M}^a_{N+1,2n}+\mathcal{N}^a_{N+1,2n}\right)\right) \end{aligned}$$

Since $y \mapsto I^a_t(y^a y)$ is invertible (strictly decreasing) for each $a$ and since the ordering of the arguments does not change in switching from $N$ to $N+1$, the arguments of the $N$ and $N+1$ cases must equal:

$$\begin{aligned} \xi^{(N)}_{\frac{nT}{2^N}}\mathcal{M}^a_{N,n}+\mathcal{N}^a_{N,n} &= \xi^{(N+1)}_{\frac{nT}{2^N}}\mathcal{M}^a_{N+1,2n}+\mathcal{N}^a_{N+1,2n} \\ \xi^{(N+1)}_{\frac{nT}{2^N}}\mathcal{M}^a_{N+1,2n}-\xi^{(N)}_{\frac{nT}{2^N}}\mathcal{M}^a_{N,n} &= \mathcal{N}^a_{N,n}-\mathcal{N}^a_{N+1,2n} \\ \xi^{(N+1)}_{\frac{nT}{2^N}}\left(1+\frac{T}{2^{N+1}}\chi^a_{\frac{nT}{2^N}\frac{nT}{2^N}}\right)-\xi^{(N)}_{\frac{nT}{2^N}}\left(1+\frac{T}{2^N}\chi^a_{\frac{nT}{2^N}\frac{nT}{2^N}}\right) &= \mathcal{N}^a_{N,n}-\mathcal{N}^a_{N+1,2n} \end{aligned}$$

Rearranging yields

$$\begin{aligned} \left(\xi^{(N+1)}_{\frac{nT}{2^N}}-\xi^{(N)}_{\frac{nT}{2^N}}\right)\left(1+\frac{T}{2^{N+1}}\chi^a_{\frac{nT}{2^N}\frac{nT}{2^N}}\right) &= \frac{T}{2^{N+1}}\xi^{(N)}_{\frac{nT}{2^N}}\chi^a_{\frac{nT}{2^N}\frac{nT}{2^N}}+\mathcal{N}^a_{N,n}-\mathcal{N}^a_{N+1,2n} \\ \left|\xi^{(N+1)}_{\frac{nT}{2^N}}-\xi^{(N)}_{\frac{nT}{2^N}}\right|\left(1+\frac{T}{2^{N+1}}\chi^a_{\frac{nT}{2^N}\frac{nT}{2^N}}\right) &\le \frac{T}{2^{N+1}}\xi^{(N)}_{\frac{nT}{2^N}}\chi^a_{\frac{nT}{2^N}\frac{nT}{2^N}}+\left|\mathcal{N}^a_{N,n}-\mathcal{N}^a_{N+1,2n}\right| \\ \left|\xi^{(N+1)}_{\frac{nT}{2^N}}-\xi^{(N)}_{\frac{nT}{2^N}}\right| &\le \frac{T}{2^{N+1}}\xi^{(N)}_{\frac{nT}{2^N}}\chi^a_{\frac{nT}{2^N}\frac{nT}{2^N}}+\left|\mathcal{N}^a_{N,n}-\mathcal{N}^a_{N+1,2n}\right| \end{aligned}$$

Now, consider taking the limit as $N \to \infty$ keeping $\frac{nT}{2^N} = t$ a constant. We need to show that $\xi^{(N)}$ is uniformly bounded and that $\left|\mathcal{N}^a_{N,n}-\mathcal{N}^a_{N+1,2n}\right|$ goes to zero as $N \to \infty$. Using the lower bound on $X$ we have

$$0 < k \le X_{\frac{nT}{2^N}} = \sum_{a=1}^{A} I^a_{\frac{nT}{2^N}}\left(y^a\xi^{(N)}_{\frac{nT}{2^N}}\mathcal{M}^a_{N,n}+\mathcal{N}^a_{N,n}\right) \le K < \infty$$

hence, for each $a$,

$$0 < k \le I^a_{\frac{nT}{2^N}}\left(y^a\left(\xi^{(N)}_{\frac{nT}{2^N}}\mathcal{M}^a_{N,n}+\mathcal{N}^a_{N,n}\right)\right) \le K < \infty$$

Thus,

$$0 < u^{a'}_{\frac{nT}{2^N}}(K) \le y^a\left(\xi^{(N)}_{\frac{nT}{2^N}}\mathcal{M}^a_{N,n}+\mathcal{N}^a_{N,n}\right) \le u^{a'}_{\frac{nT}{2^N}}(k) < \infty$$

Hence, using the boundedness of $\mathcal{M}^a$,

$$\xi^{(N)}_{\frac{nT}{2^N}} \leq \hat{K} < \infty$$

where $\hat{K}$ is a constant free of $N$, establishing the uniform boundedness of $\xi^{(N)}$. Next, look at $\mathcal{N}^a_{N,n} - \mathcal{N}^a_{N+1,2n}$:

$$\begin{aligned}
\mathcal{N}^a_{N,n} & - \mathcal{N}^a_{N+1,2n} \\
&= \frac{T}{2^N}\mathbf{E}_{\frac{nT}{2^N}}\left[\sum_{m=n+1}^{2^N-1}\chi^a_{\frac{nT}{2^N}\frac{mT}{2^N}}\xi_{\frac{mT}{2^N}}\right] - \frac{T}{2^{N+1}}\mathbf{E}_{\frac{2nT}{2^{N+1}}}\left[\sum_{m=2n+1}^{2^{N+1}-1}\chi^a_{\frac{2nT}{2^{N+1}}\frac{mT}{2^{N+1}}}\xi_{\frac{mT}{2^{N+1}}}\right] \\
&= \frac{T}{2^N}\mathbf{E}_{\frac{nT}{2^N}}\left[\sum_{m=n+1}^{2^N-1}\chi^a_{\frac{nT}{2^N}\frac{mT}{2^N}}\xi_{\frac{mT}{2^N}}\right] - \frac{T}{2^{N+1}}\mathbf{E}_{\frac{nT}{2^N}}\left[\sum_{m=2n+1}^{2^{N+1}-1}\chi^a_{\frac{nT}{2^N}\frac{mT}{2^{N+1}}}\xi_{\frac{mT}{2^{N+1}}}\right] \\
&= \frac{T}{2^N}\mathbf{E}_{\frac{nT}{2^N}}\left[\sum_{m=n+1}^{2^N-1}\chi^a_{\frac{nT}{2^N}\frac{mT}{2^N}}\xi^{(N)}_{\frac{mT}{2^N}}\right] - \frac{T}{2^{N+1}}\mathbf{E}_{\frac{nT}{2^N}}\left[\chi^a_{\frac{nT}{2^N}\frac{(2^{N+1}-1)T}{2^{N+1}}}\xi^{(N+1)}_{\frac{(2^{N+1}-1)T}{2^{N+1}}}\right] \\
&\quad -\frac{T}{2^N}\mathbf{E}_{\frac{nT}{2^N}}\left[\sum_{m=n+1}^{2^N-1}\left(\frac{\chi^a_{\frac{nT}{2^N}\frac{(2m-1)T}{2^{N+1}}}\xi^{(N+1)}_{\frac{(2m-1)T}{2^{N+1}}} + \chi^a_{\frac{nT}{2^N}\frac{2mT}{2^{N+1}}}\xi^{(N+1)}_{\frac{2mT}{2^{N+1}}}}{2}\right)\right]
\end{aligned}$$

By the continuity of $\xi^{(N+1)}$ and $\chi^a$, we can write

$$\chi^a_{\frac{nT}{2^N}\frac{(2m-1)T}{2^{N+1}}}\xi^{(N+1)}_{\frac{(2m-1)T}{2^{N+1}}} = \chi^a_{\frac{nT}{2^N}\frac{2mT}{2^{N+1}}}\xi^{(N+1)}_{\frac{2mT}{2^{N+1}}} + \epsilon^{(N+1)}$$

where $\epsilon^{(N+1)} \to 0$ as $N \to \infty$. Hence, recalling that $\frac{nT}{2^N} = t$, a constant,

$$\begin{aligned}
\mathcal{N}^a_{N,n} & - \mathcal{N}^a_{N+1,2n} \\
&= \frac{T}{2^N}\mathbf{E}_t\left[\sum_{m=n+1}^{2^N-1}\chi^a_{t\frac{mT}{2^N}}\xi^{(N)}_{\frac{mT}{2^N}}\right] - \frac{T}{2^{N+1}}\mathbf{E}_t\left[\chi^a_{t\frac{(2^{N+1}-1)T}{2^{N+1}}}\xi^{(N+1)}_{\frac{(2^{N+1}-1)T}{2^{N+1}}}\right] \\
&\quad -\frac{T}{2^N}\mathbf{E}_t\left[\sum_{m=n+1}^{2^N-1}\chi^a_{t\frac{2mT}{2^{N+1}}}\xi^{(N+1)}_{\frac{2mT}{2^{N+1}}}\right] - \frac{T}{2^{N+1}}\mathbf{E}_t\left[\sum_{m=n+1}^{2^N-1}\epsilon^{(N+1)}\right] \\
&= \frac{T}{2^N}\mathbf{E}_t\left[\sum_{m=n+1}^{2^N-1}\chi^a_{t\frac{mT}{2^N}}\left(\xi^{(N)}_{\frac{mT}{2^N}} - \xi^{(N+1)}_{\frac{mT}{2^N}}\right)\right] \\
&\quad -\frac{T}{2^{N+1}}\mathbf{E}_t\left[\chi^a_{t\frac{(2^{N+1}-1)T}{2^{N+1}}}\xi^{(N+1)}_{\frac{(2^{N+1}-1)T}{2^{N+1}}}\right] - \frac{T}{2^{N+1}}\mathbf{E}_t\left[\epsilon^{(N+1)}\right](2^N - 1 - (t/T)2^N)
\end{aligned}$$

Now, using $\chi^a \leq K_{\vec{\chi}} := K_\chi e^{K_\chi T}$ and $\xi^{(N)} \leq \hat{K}$, we have

$$\left|\mathcal{N}^a_{N,n} - \mathcal{N}^a_{N+1,2n}\right| \leq \frac{T}{2^N} K_{\vec{\chi}} \mathbf{E}_t \left[ \sum_{m=n+1}^{2^N-1} \left| \xi^{(N)}_{\frac{mT}{2^N}} - \xi^{(N+1)}_{\frac{mT}{2^N}} \right| \right] + \frac{T}{2^{N+1}} K_{\vec{\chi}} \hat{K} + \frac{T}{2} \mathbf{E}_t \left[ \left| \epsilon^{(N+1)} \right| \right]$$

Now, we have

$$\begin{aligned}
\left| \xi^{(N+1)}_t - \xi^{(N)}_t \right| &\leq \frac{T}{2^{N+1}} \xi^{(N)}_t \chi^a_t + \left|\mathcal{N}^a_{N,n} - \mathcal{N}^a_{N+1,2n}\right| \\
&\leq \frac{T}{2^{N+1}} \xi^{(N)}_{\frac{nT}{2^N}} \chi^a_{\frac{nT}{2^N}} + \frac{T}{2^N} K_{\vec{\chi}} \mathbf{E}_t \left[ \sum_{m=n+1}^{2^N-1} \left| \xi^{(N)}_{\frac{mT}{2^N}} - \xi^{(N+1)}_{\frac{mT}{2^N}} \right| \right] \\
&\qquad + \frac{T}{2^{N+1}} K_{\vec{\chi}} \hat{K} + \frac{T}{2} \mathbf{E}_t \left[ \left| \epsilon^{(N+1)} \right| \right] \\
&\leq \frac{T}{2^N} K_{\vec{\chi}} \mathbf{E}_t \left[ \sum_{m=n+1}^{2^N-1} \left| \xi^{(N)}_{\frac{mT}{2^N}} - \xi^{(N+1)}_{\frac{mT}{2^N}} \right| \right] + \frac{T}{2^N} K_{\vec{\chi}} \hat{K} + \frac{T}{2} \mathbf{E}_t \left[ \left| \epsilon^{(N+1)} \right| \right]
\end{aligned}$$

which we write as

$$\left| \xi^{(N+1)}_{\frac{nT}{2^N}} - \xi^{(N)}_{\frac{nT}{2^N}} \right| \leq \frac{KT}{2^N} \mathbf{E}_{\frac{nT}{2^N}} \left[ \sum_{m=n+1}^{2^N-1} \left| \xi^{(N)}_{\frac{mT}{2^N}} - \xi^{(N+1)}_{\frac{mT}{2^N}} \right| \right] + K_N$$

where $K_N$ converges to zero. This inequality can be iterated. To make iteration easier, let

$$Y^N_m = \mathbf{E}_{\frac{nT}{2^N}} \left[ \left| \xi^{(N)}_{\frac{mT}{2^N}} - \xi^{(N+1)}_{\frac{mT}{2^N}} \right| \right]$$

It is easy to show by induction that for $1 \leq k \leq 2^N - n - 1$

$$\sum_{m_1=n+1}^{2^N-1} \cdots \sum_{m_k=m_{(k-1)}+1}^{2^N-1} 1 = \frac{(2^N-n-1)\cdots(2^N-n-k)}{k!} \leq \frac{2^{kN}}{k!}$$

where we take $m_0 = n$. In particular, when $k = 2^N - n - 1$ we obtain

$$\sum_{m_1=n+1}^{2^N-1} \cdots \sum_{m_{(2^N-n-1)}=m_{(2^N-n-2)}+1}^{2^N-1} 1 = \frac{(2^N-n-1)!}{(2^N-n-1)!} = 1$$

Hence,

$$
\begin{aligned}
Y_n^N &\le \frac{KT}{2^N}\sum_{m_1=n+1}^{2^N-1} Y_{m_1}^N + K_N \\
&\le \left(\frac{KT}{2^N}\right)^2 \sum_{m_1=n+1}^{2^N-1}\sum_{m_2=m_1+1}^{2^N-1} Y_{m_2}^N + K_N\frac{KT}{2^N}\sum_{m_1=n+1}^{2^N-1} 1 + K_N \\
&= \left(\frac{KT}{2^N}\right)^2 \sum_{m_1=n+1}^{2^N-1}\sum_{m_2=m_1+1}^{2^N-1} Y_{m_2}^N + K_N\frac{KT}{2^N}\frac{2^N}{1!} + K_N \\
&= \left(\frac{KT}{2^N}\right)^2 \sum_{m_1=n+1}^{2^N-1}\sum_{m_2=m_1+1}^{2^N-1} Y_{m_2}^N + K_N(KT+1) \\
&\le \left(\frac{KT}{2^N}\right)^3 \sum_{m_1=n+1}^{2^N-1}\sum_{m_2=m_1+1}^{2^N-1}\sum_{m_3=m_2+1}^{2^N-1} Y_{m_3}^N \\
&\qquad + K_N\left(\frac{KT}{2^N}\right)^2 \sum_{m_1=n+1}^{2^N-1}\sum_{m_2=m_1+1}^{2^N-1} 1 + K_N(KT+1) \\
&= \left(\frac{KT}{2^N}\right)^3 \sum_{m_1=n+1}^{2^N-1}\sum_{m_2=m_1+1}^{2^N-1}\sum_{m_3=m_2+1}^{2^N-1} Y_{m_3}^N + K_N\left(\frac{KT}{2^N}\right)^2\frac{(2^N)^2}{2!} + K_N(KT+1) \\
&= \left(\frac{KT}{2^N}\right)^3 \sum_{m_1=n+1}^{2^N-1}\sum_{m_2=m_1+1}^{2^N-1}\sum_{m_3=m_2+1}^{2^N-1} Y_{m_3}^N + K_N\left(\frac{(KT)^2}{2!} + KT + 1\right) \\
&\;\vdots \\
&\le \left(\frac{KT}{2^N}\right)^{2^N-n-1} \sum_{m_1=n+1}^{2^N-1}\cdots\sum_{m_{(2^N-n-1)}=m_{(2^N-n-2)}+1}^{2^N-1} Y_{m_{2^N-n-1}}^N \\
&\qquad + K_N\left(\frac{(KT)^{2^N-n-2}}{(2^N-n-2)!} + \cdots + \frac{(KT)^2}{2!} + KT + 1\right) \\
&\le \left(\frac{KT}{2^N}\right)^{2^N-n-1} \hat{K} + K_N e^{KT} \to 0
\end{aligned}
$$

Thus, for all dyadic time points, $\xi_{\frac{nT}{2^N}}^{(N)}$ is uniformly Cauchy. By linear interpolation, $\xi_t^{(N)}$ is uniformly Cauchy at all time points. Hence, $\xi^{(N)}$ is a Cauchy sequence in supremum norm. As we are working in the complete space of continuous paths, $\xi^{(N)}$

converges to some continuous $\xi$. For any $N$ and dyadic time $\frac{nT}{2^N}$ we have

$$X_{\frac{nT}{2^N}} = \sum_{a=1}^{A} I^a_{\frac{nT}{2^N}}(y^a \gamma^a (\xi^{(N)})_{\frac{nT}{2^N}})$$

Taking limits, holding $\frac{nT}{2^N} = t$, a constant, and using dominated convergence, we obtain

$$X_t = \sum_{a=1}^{A} I^a_t(y^a \gamma^a (\xi)_t)$$

at all dyadic time points and hence all times $t \in [0, T]$ by continuity.

As $\xi^{(N)} \leq \hat{K}$ we then have $\xi \leq \hat{K}$ so $\xi \in \Xi_{++}$. Thus, we have existence. ■

Unfortunately, uniqueness has not yet been established. We are currently exploring the use Laplace, Fourier and other linear integral transforms and exploiting the convexity of the $I^a$ in the definition of $\tilde{\Theta}$; it is hoped that we will be able to establish both existence and uniqueness without having to assume that column sums of $\boldsymbol{\chi}$ are identical.

# Chapter 3

# Model Simulation

## 3.1 Introduction

In this chapter we present the methodology and results of a preliminary Monte Carlo simulation study of the interdependent habits model analyzed in Chapter 2. Simulation is required as the model's equilibrium processes involve both a history dependence as well as a conditional expectation over future trajectories, making their mathematical forms too complex to interpret directly, except in a limited set of special cases.

We refer to these simulations as preliminary for two reasons. Firstly, our theoretical results are limited to (i) habit kernels of the form $\boldsymbol{\chi} = \chi\mathbf{I} + \mathbf{D}$ with identical column sums so as to ensure the invertibility of $\Theta$ and (ii) utility functions such that $\Theta^{-1}(\iota(\boldsymbol{\Lambda}), E)C^a(\iota(\boldsymbol{\Lambda}), \Theta^{-1}(\iota(\boldsymbol{\Lambda}), E))$ are all non-increasing in $\boldsymbol{\Lambda}$ so as to guarantee the uniqueness of equilibrium. We conjecture that the invertibility of $\Theta$ can be established for any $\boldsymbol{\chi}$ with bounded entries, and, we are currently working on a constructive proof of invertibility which will provide an algorithm for its computation. However, without this more general invertibility result, our simulations are necessarily limited to habit kernels of the form in (i). Further, in Karatzas et al. (1990) and Karatzas et al. (1991) it is shown that if relative risk aversion is at most 1 then, in their special class of models for which $\boldsymbol{\chi} = \mathbf{0}$, the map $\Theta^{-1}(\iota(\boldsymbol{\Lambda}), E)C^a(\iota(\boldsymbol{\Lambda}), \Theta^{-1}(\iota(\boldsymbol{\Lambda}), E))$ is non-increasing in $\boldsymbol{\Lambda}$ and hence the equilibrium is unique. They provide examples to show that even though this condition on relative risk aversion is sufficient, it is far from necessary. We conjecture that uniqueness of equilibrium holds quite generally in our model as well and we proceed with the simulation as if this is the case. Our simulation results do provide some numerical evidence of the uniqueness of equilibrium but it should be kept in mind that uniqueness is only theoretically assured for risk aversion at most 1. Secondly, our model allows relative risk aversion and subjective discounting to be stochastic processes, it allows any number $N$ underlying Brownian motions for the sources of uncertainty, and it allows for any number $A$ of agents. But, this first round of simulations has been conducted with constant risk aversion and subjective discounting, with $N = 1$ and with $A = 2, 3$. While not as general as one would like, these preliminary simulations allow us to determine whether or not our model has the potential for generating interesting equilibrium asset price behavior, which is their primary purpose. Our conclusion from this preliminary simulation study is that this model of interdependent habit formation definitely warrants further investigation. With the simulation program now in place, we plan to calibrate our simulation model to real data and extensively investigate the more general cases of heterogeneous, stochastic risk aversion and patience for their economic significance and implications. We also plan to explore more general habit kernels and the more complicated interactions between a larger number of agents. In general, we wish to further explore the potential of agent interaction and state dependent preferences within a complete market model as a route to explain empirical anomalies currently discussed in the consumption-based asset pricing literature; the simulation study here should therefore be viewed as only a first step in this program.

## 3.2 Methodology

Before describing the special cases simulated here and presenting the results of our diagnostic simulations as well as the results of some preliminary choices of general $\boldsymbol{\chi}$, we provide more detail on the overall simulation methodology.

We follow a two-step procedure in our approach to this Monte Carlo simulation. First, for a given set of agent parameters, we find the associated equilibrium multipliers $\Upsilon^*$. Second, with these equilibrium multipliers in hand, we re-simulate the equilibrium processes and extract information of economic interest: the means and standard deviations of (i) the equilibrium state price density $\xi^*$, (ii) the equilibrium real interest rate $r^*$, and (iii) the equilibrium market price of risk $\theta^*$.

The separation of the simulation into two main steps is not strictly required; in the computation of the equilibrium multipliers we have, as a by-product, equilibrium state prices $\xi^*$ and consumptions $\mathbf{C}^*$. However, separating the computation into two steps has several advantages. From an implementation point of view, it is always preferable to break a large simulation into as many smaller simulations as possible so one can monitor progress through intermediate stages of the overall computation. The re-simulation of $\xi^*$ and $\mathbf{C}^*$ in the second step, with a slightly different algorithm, also provides a valuable diagnostic consistency check of the correctness of the code that finds the equilibrium multipliers $\Upsilon^*$ and the code that computes the equilibrium processes $\xi^*$ and $\mathbf{C}^*$. The extra computational work in resimulating $\xi^*$ and $\mathbf{C}^*$ is relatively small compared to that of simulating the conditional expectations needed to extract the first and second moments of $r^*$ and $\theta^*$ from $\xi^*$. Unfortunately, since the second task primarily involves the simulation of conditional expectations, the only way to split this second task up is by writing and reading very large data files, a very slow and disk-space intensive procedure. We therefore chose to run this second task as a complete job. The first task of locating the equilibrium multipliers is fast relative to the second task and so we ran it as a complete job also.

All component subprograms comprising the overall simulation program were written in *C* and were individually and thoroughly tested on a local workstation. In particular, careful attention was paid to each component's performance (numerical precision and accuracy, cpu time, memory use, etc.) and the parameters that control the components' simulations were adjusted optimally. The full program was then assembled out of these components, converted to a parallel program using MPICH (a Message Passing Interface library for *C* which implements interprocessor communication), and run on a Beowulf style distributed memory computing cluster called "Helix", housed at Massey University in Auckland, New Zealand (see `http://helix.massey.ac.nz`). The Helix cluster consists of a server and 65 nodes, each with dual AMD 2.1GHz Athlon processors, 1 Gig RAM, gigabit ethernet cards and a gigabit ethernet switch to manage the interprocessor communications. The full program on Helix was then tested against special cases of the model having explicit solutions, or near explicit solutions that are easy to compute numerically with Mathematica; we demonstrate the accuracy of our full simulation procedure by a direct comparison with these special cases. To obtain reasonable variances in our Monte Carlo estimators, we found it necessary to incorporate some variance reduction; we opted for the use of antithetic variables applied to the underlying Brownian motions. After some additional fine tuning of the full collection of parameters governing the simulation, we then began a preliminary numerical exploration of the model for cases with unknown behavior.

For the preliminary simulation study we present here, we work with a single Brownian motion and we focus on a small number of agents that are identical in all ways (except in their habit formation) for several reasons. Firstly, although the simulation program is designed to run with any number of agents, we initially confine ourselves to two and three agents so as to determine execution time, memory requirements, and to optimize numerical accuracy. Secondly, a model with identical agents enables fairly explicit calculations which we then use to diagnose the accuracy of our simulation procedure. Thirdly, and most importantly, by first considering only identical agents, we can isolate the effect that interactions have on equilibrium asset prices. Identical agents with identical habit formation will share aggregate endowment in precisely the same way with or without interactions; each will have half of aggregate endowment. However, interactions may influence the intensity and dynamics of demand which in turn may manifest themselves in different equilibrium asset prices.

### 3.2.1 Equilibrium Lagrange Multipliers

The first task consists of computing the equilibrium multipliers $\boldsymbol{\Upsilon}^*$. To do this, we define a procedure for computing the value of a function $L : (0,\infty)^A \to \mathbb{R}_+$, for any given $\boldsymbol{\Upsilon}$, such that $L = 0$ only at the unique value $\boldsymbol{\Upsilon}^*$. With this procedure to compute $L$, we then apply another routine to find the location of the global minimum of $L$, which yields $\boldsymbol{\Upsilon}^*$. Here are the details:

Let $\boldsymbol{\Upsilon} \in (0,\infty)^A$ be given. The general class of utility functions for which we have written the simulation program is that presented in Example (2.4.6): agent $a$'s patience $\beta^a$ and relative risk aversion $\alpha^a$ are $\mathbb{F}$-progressive, and uniformly bounded processes in $\mathbb{R}$ and $\mathbb{R}_{++}$, respectively. Agent $a$'s instantaneous utility is

$$u_t^a(x) = \begin{cases} \mathcal{D}_{0t}^{\beta^a} \left( \frac{x^{1-\alpha_t^a}-1}{1-\alpha_t^a} \right) & if\ \alpha_t^a \in (0,1) \cup (1,\infty) \\ \mathcal{D}_{0t}^{\beta^a} \ln x & if\ \alpha_t^a = 1 \end{cases}$$

Marginal utility and inverse marginal utility for agent $a$ are then

$$u_t^{a'}(x) = \mathcal{D}_{0t}^{\beta^a} x^{-\alpha_t^a} \qquad and \qquad I_t^a(y) = \mathcal{D}_{0t}^{\beta^a/\alpha_t^a} y^{-1/\alpha_t^a}$$

Aggregate inverse marginal utility is

$$I_t(y; \Upsilon) = \sum_{a=1}^{A} I_t^a(y^a y)$$

As seen in Chapter 2, the mutually optimal consumption configuration takes the form

$$\mathbf{C}_t = \mathbf{I}_t + \mathbf{H}_t = \mathbf{I}_t + \int_0^t \boldsymbol{\chi}_{\nu t} \mathbf{C}_\nu d\nu$$

which has the indistinguishably unique solution

$$\mathbf{C}_t = \mathbf{I}_t + \int_0^t \overleftarrow{\boldsymbol{\chi}}_{\nu t} \mathbf{I}_\nu d\nu$$

Now, imposing market clearing, we obtain

$$E_t = \sum_{a=1}^{A} C_t^a = \sum_{a=1}^{A} I_t^a(y^a \gamma_t^a) + \int_0^t \sum_{a,b=1}^{A} \overleftarrow{\chi}_{\nu t}^{ab} I_\nu^b(y^b \gamma_\nu^b) d\nu$$

where

$$\gamma_t^a = \xi_t + \mathbf{E}_t \left[ \int_t^T \overrightarrow{\chi_{t\nu}^{ab}} \xi_\nu d\nu \right]$$

For kernels of the form $\boldsymbol{\chi} = \chi \mathbf{I} + \mathbf{D}$ having identical column sums, we have that the processes $\gamma^a$ are the same for all agents:

$$\gamma_t^a = \gamma_t = \xi_t + \mathbf{E}_t \left[ \int_t^T \overrightarrow{\chi_{t\nu}} \xi_\nu d\nu \right]$$

Market clearing then takes the form

$$\begin{aligned} E_t &= \sum_{a=1}^{A} I_t^a(y^a \gamma_t) + \int_0^t \sum_{a,b=1}^{A} \overleftarrow{\chi}_{\nu t}^{ab} I_\nu^b(y^b \gamma_\nu) d\nu \\ &= \sum_{a=1}^{A} I_t^a(y^a \gamma_t) + \int_0^t \overleftarrow{\chi}_{\nu t}^{c} \sum_{a=1}^{A} I_\nu^b(y^b \gamma_\nu) d\nu \\ &= I_t(\gamma_t; \Upsilon) + \int_0^t \overleftarrow{\chi}_{\nu t}^{c} I_\nu(\gamma_\nu; \Upsilon) d\nu \end{aligned}$$

which inverts to yield the "adjusted" aggregate endowment

$$I_t(\gamma_t; \mathbf{\Upsilon}) = E_t - \int_0^t \chi^c_{\nu t} E_\nu d\nu$$

$I$ has inverse $J$ for each $\mathbf{\Upsilon}$ and so

$$\gamma_t = J_t \left( E_t - \int_0^t \chi^c_{\nu t} E_\nu d\nu; \mathbf{\Upsilon} \right) \tag{3.2.1}$$

Now, with $E$, $J$, and $\chi$ known or computable, we use (3.2.1) to compute $\gamma$. For the integration in (3.2.1), we use a trapezoidal Riemann sum approximation. As shown in Chapter 2, the bounds on $\chi$ and $E$ ensure that adjusted aggregate endowment is strictly positive. For the inversion of $I$ we use a Newton-Raphson algorithm that is supplemented with a bisection method; Newton-Raphson on its own is not globally convergent and if this algorithm steps outside of the domain, or is converging too slowly, the bisection algorithm takes over temporarily to restart the Newton-Raphson procedure at an improved starting point; this combines the speed of Newton-Raphson with the global convergence of the bisection method. See Press, Teukolsky, Vetterling & Flannery (1992), Chapters 4 & 9, for details.

Now, with $\gamma$ in hand, we can solve

$$\mathbf{C}_t = \mathbf{I}_t + \mathbf{H}_t = \mathbf{I}_t + \int_0^t \boldsymbol{\chi}_{\nu t} \mathbf{C}_\nu d\nu$$

We could numerically simulate the known analytical solution but this would involve a computation that approximates an infinite sum of multiple integrals of increasing order, as is present in $\overleftarrow{\boldsymbol{\chi}}$, which is overly burdensome. It is more efficient to directly solve the linear Volterra equation for $\mathbf{C}$ using a standard technique. The method entails a discretization of the time interval, a trapezoidal Riemann sum approximation, and then, at each successive time point starting from $t = 0$, one recursively solves a linear system of equations. The linear equations are solved using an efficient LU decomposition (Crout's algorithm). See Press et al. (1992), Chapters 2 & 10, for implementation details and see Delves & Walsh (1974) for a general convergence proof for this approximation scheme for Volterra equations.

From

$$\xi_t = \gamma_t - \mathbf{E}_t \left[ \int_t^T \chi_{t\nu} \gamma_\nu d\nu \right]$$

we could compute $\xi$, but note that this involves a conditional expectation. We will need to simulate this conditional expectation in the second step of the overall simulation, but, for the purposes of finding the equilibrium multipliers, we can avoid this since the expectation of a conditional expectation eliminates the conditional expectation in $G^a$:

$$\begin{aligned}
G^a(\Upsilon) &= \mathbf{E}\left[\int_0^T \xi_\nu (C_\nu^a - E_\nu^a) d\nu\right] \\
&= \mathbf{E}\left[\int_0^T \left(\gamma_\nu - \mathbf{E}_\nu\left[\int_\nu^T \chi_{\nu w}\gamma_w dw\right]\right)(C_\nu^a - E_\nu^a) d\nu\right] \\
&= \mathbf{E}\left[\int_0^T \mathbf{E}_\nu\left[\left(\gamma_\nu - \int_\nu^T \chi_{\nu w}\gamma_w dw\right)(C_\nu^a - E_\nu^a)\right] d\nu\right] \\
&= \int_0^T \mathbf{E}\left[\mathbf{E}_\nu\left[\left(\gamma_\nu - \int_\nu^T \chi_{\nu w}\gamma_w dw\right)(C_\nu^a - E_\nu^a)\right]\right] d\nu \\
&= \mathbf{E}\left[\int_0^T \left(\gamma_\nu - \int_\nu^T \chi_{\nu w}\gamma_w dw\right)(C_\nu^a - E_\nu^a) d\nu\right]
\end{aligned}$$

Thus, for a given $\Upsilon$, we have $\gamma$, $C^a$, $E^a$ and $\chi$ and so, using the above, we can compute $G^a(\Upsilon)$, for all $a$: we use a trapezoidal Riemann sum approximation for the pathwise integrations, we generate a sample of such integrals, and then we estimate the mean. Finally, we assemble these $A$ values into a function $L$ defined by

$$L(\Upsilon) = \sum_{a=1}^{A} \left[G^a(\Upsilon)\right]^2$$

The above procedure enables us to compute $L(\Upsilon)$ for any $\Upsilon \in (0, \infty)^A$. In order to find the equilibrium multipliers we find the global minimum of $L$. We approach the minimization of this multivariable function through a well known procedure developed by Nelder & Mead (1965). Surprisingly, no general convergence results are available for the Nelder-Mead simplex algorithm aside from a result in dimension 1 and partial results in dimension 2 (see Lagarias, Reeds, Wright & Wright (1998)). However, we continue to use this algorithm for two important reasons: it works extremely well in practice (including our model). So much so that it is the default optimization method used in the Mathematica V optimization package; it is also implemented in Matlab 13 and Maple 9.5. Also, a quick scan of the literature shows countless successful applications of the algorithm in various fields of science and engineering. The second important reason is that the simplex algorithm falls into the class of "direct search" optimization methods for which many convergence results do exist (see for example Pardalos & Resende (2002)) and it appears that convergence results on the Nelder-Mead simplex algorithm for a restricted set of functions is forthcoming (see references in Lagarias et al. (1998)). In any case, our numerical evaluation of $L$ is fully supported by convergence results and since $L$ was shown, theoretically, to have a minimum of zero along a ray we take the pragmatic view that if the simplex algorithm locates a zero of $L$ then the lack of a convergence proof for the minimization of $L$ is of secondary importance.

We demonstrate the regularity of the function $L$ in the Figures (3.1) and (3.2) on the following two pages. The first three plots of $L$ are with $\boldsymbol{\chi} = \mathbf{0}$ and the following three are with

$$\boldsymbol{\chi} = \left\{(0.19)\mathbf{1}\Delta^{+}\right\} \wedge 0.19 = \left\{(0.19)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\Delta^{+}\right\} \wedge 0.19$$

a case of habit kernel which is discussed later in the chapter. These examples represent the two "extremes" in the parameter specifications we consider here for two agents. Note that in Figures (3.1) and (3.2), we progressively magnify the vertical scale so as to examine the fine structure of $L$ near the ray of global minimums emanating from the origin. Also note that if $L$ exceeds the maximum on the vertical axis, $L$ is plotted as this maximum value.

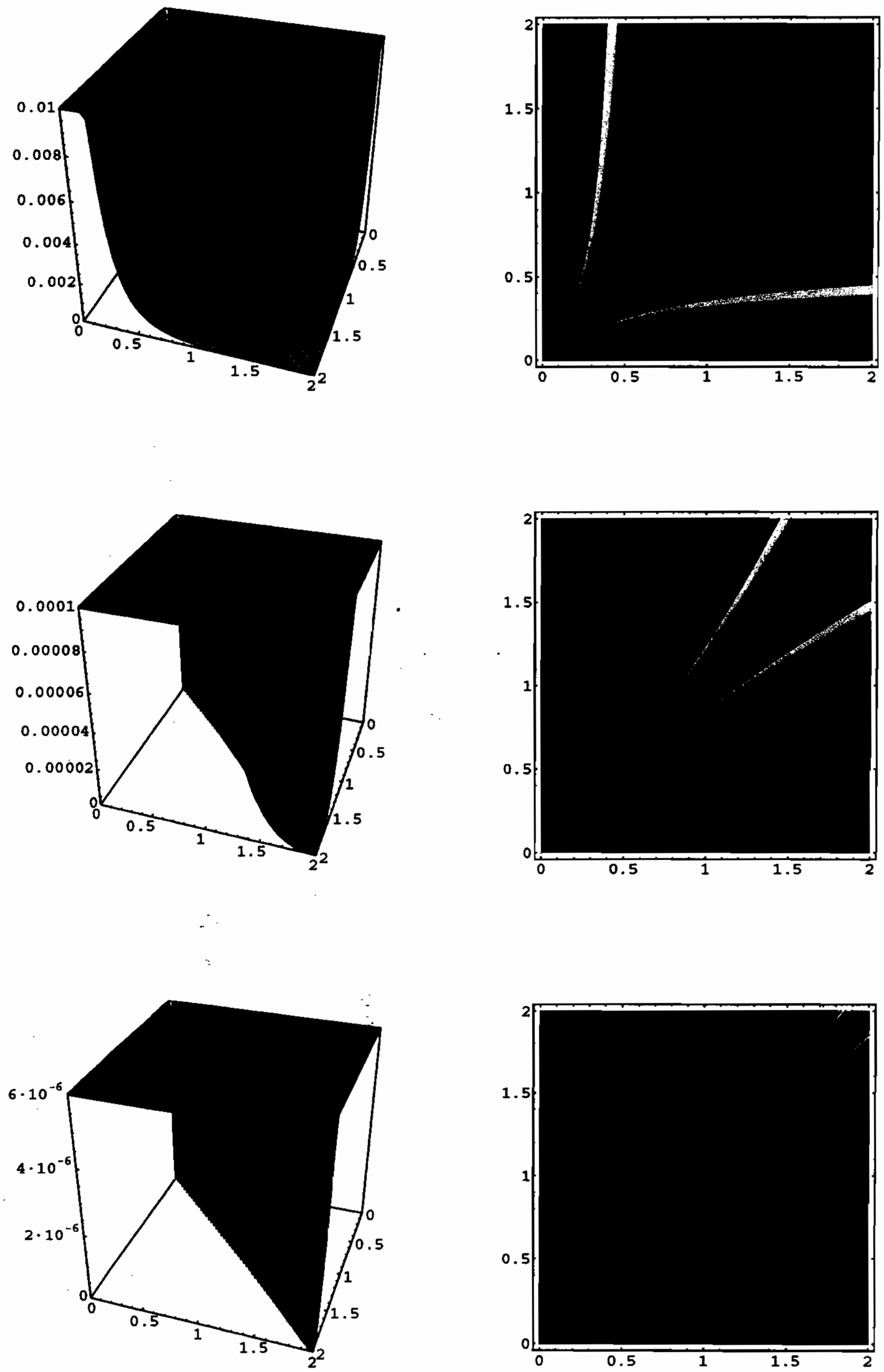


Figure 3.1: Surface and contour plot of $L(y^1, y^2)$ for $(y^1, y^2) \in (0, 2)^2$ with $\boldsymbol{\chi} = \mathbf{0}$ and constant parameters $\alpha = 1.0$ and $\beta = 1.0$. Endowments are constant at $E^1 = E^2 = 10$.

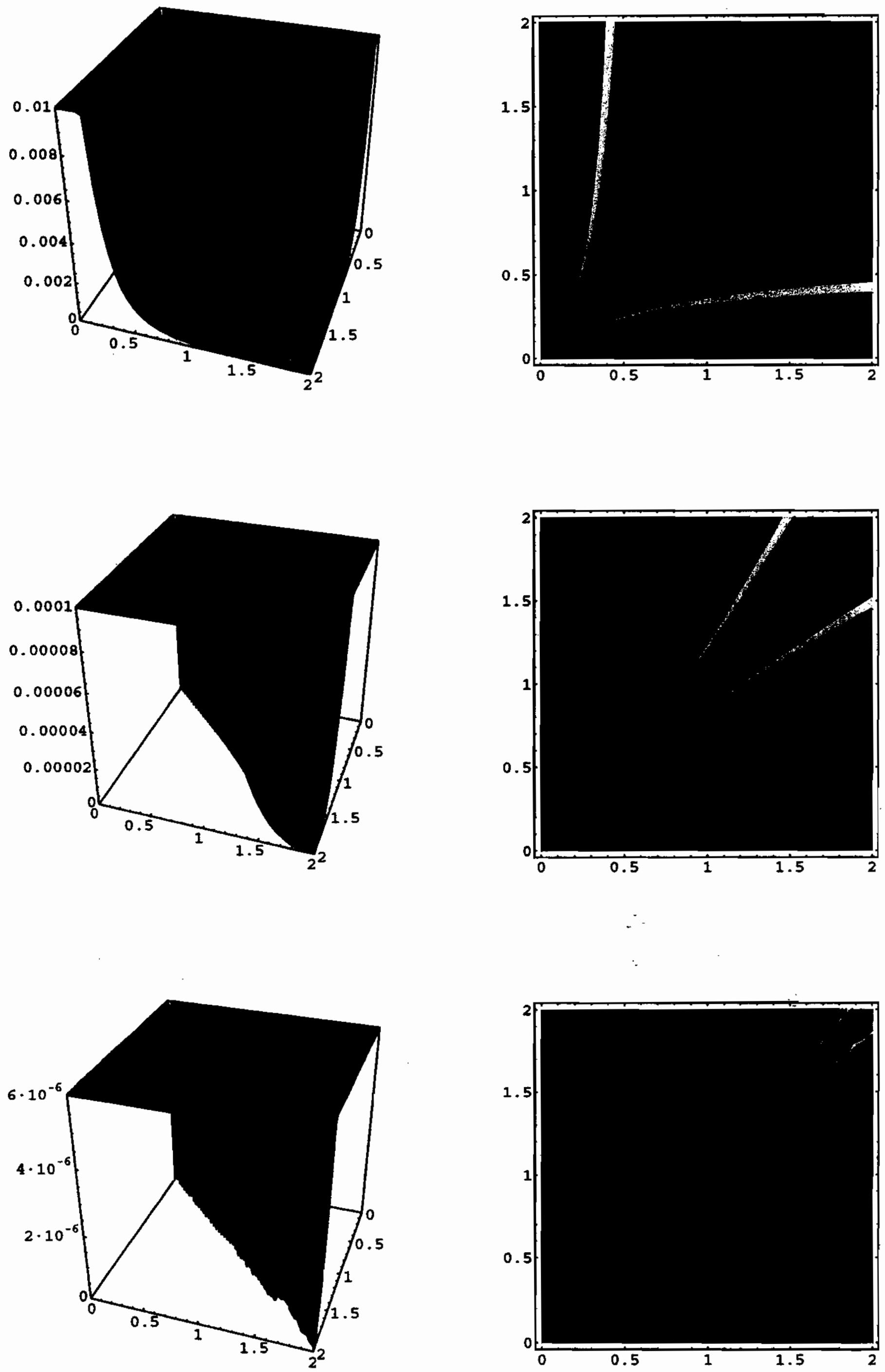


Figure 3.2: Surface and contour plot of $L(y^1, y^2)$ for $(y^1, y^2) \in (0,2)^2$ with $\chi = \{(0.19)\mathbf{1}\Delta^+\} \wedge 0.19$ and constant parameters $\alpha = 2.0$, $\beta = 0.55$. Geometric Brownian motion for endowments with $E_0^1 = E_0^2 = 10$, constant growth $g = 0.0122$ and constant standard deviation $\varrho = 0.0189$.

From Figures (3.1) and (3.2) we see quite clearly that the global structure of $L$ is reassuringly robust to variations in the specification of $\chi$, utility parameters $\alpha$ and $\beta$, as well as endowments. We view this robustness as initial evidence supporting our conjectures that $\Theta$ is invertible and equilibrium is unique quite generally.

The fact that $L = 0$ along an entire ray is an issue, raised in Chapter 2: if $\Upsilon$ solves $L(\Upsilon) = 0$ then so does $k\Upsilon$ for any $k > 0$, which also has the effect of scaling the associated state price density. So, we need to locate the unique $\Upsilon$ such that $\xi_0(\Upsilon) = 1$. However, our theoretical results do not always guarantee this uniqueness. This presents a problem and an opportunity: if we add a penalty to $L$ for values of $\xi_0$ that deviate from 1, we can numerically test for uniqueness. Thus, consider a penalized version of $L$:

$$L(\Upsilon) = \sum_{a=1}^{A} \left[G^a(\Upsilon)\right]^2 + M|\xi_0(\Upsilon) - 1|$$

for some $M > 0$. The value of $\xi_0(\Upsilon)$ is available as a by-product in the computation of $L$ and so the penalty introduces very little extra effort to obtain. We observe that using a penalty leads us to a unique set of multipliers in both instances studied, further supporting our conjecture that equilibrium is unique. However, while unique multipliers are located when a penalty is used, computation is extremely slow compared with the $M = 0$ case (no penalty), and then simply rescaling the multipliers until $\xi_0(\Upsilon) = 1$, as discussed in Chapter 2.

Our explanation for the slower computation is that the penalty usually results in significantly more function evaluations. To confirm this intuition as well as to explore the global behavior of $L$ with the penalty for various $M$, in particular, the uniqueness of the multipliers, we plot the behavior of $L$ over a fairly large range. As one can see from Figures (3.1) and (3.2), when there is no penalty, the ray $k\Upsilon^*$ along which $L = 0$ is at the bottom of a very steep valley. The simplex algorithm, while not exactly a method of steepest descent, does direct itself toward larger changes in $L$, and hence, will quickly move from its initial starting point toward the ray $k\Upsilon^*$, essentially along a normal to the ray. Comparing this with Figure (3.3) on the following page, we see that as the penalty is increased, the steepness of the function lessens and the function has a unique minimum rather than a ray of minimums. Depending on the initial starting point, the penalty forces the algorithm to move toward the unique minimum more directly, which may be along a direction that is not normal to the underlying ray and of greater overall distance. Therefore, the penalty usually increases the number of function evaluations and hence requires longer run times to locate the minimum. Figure (3.4) shows that even for our "extreme" parameter specification, the minimum appears to be unique. In light of this, we chose to locate the ray and then rescale so that the initial state price density is 1, rather than work with a penalty. We also view the numerical uniqueness in Figure (3.4) as additional evidence supporting our conjectures about uniqueness.

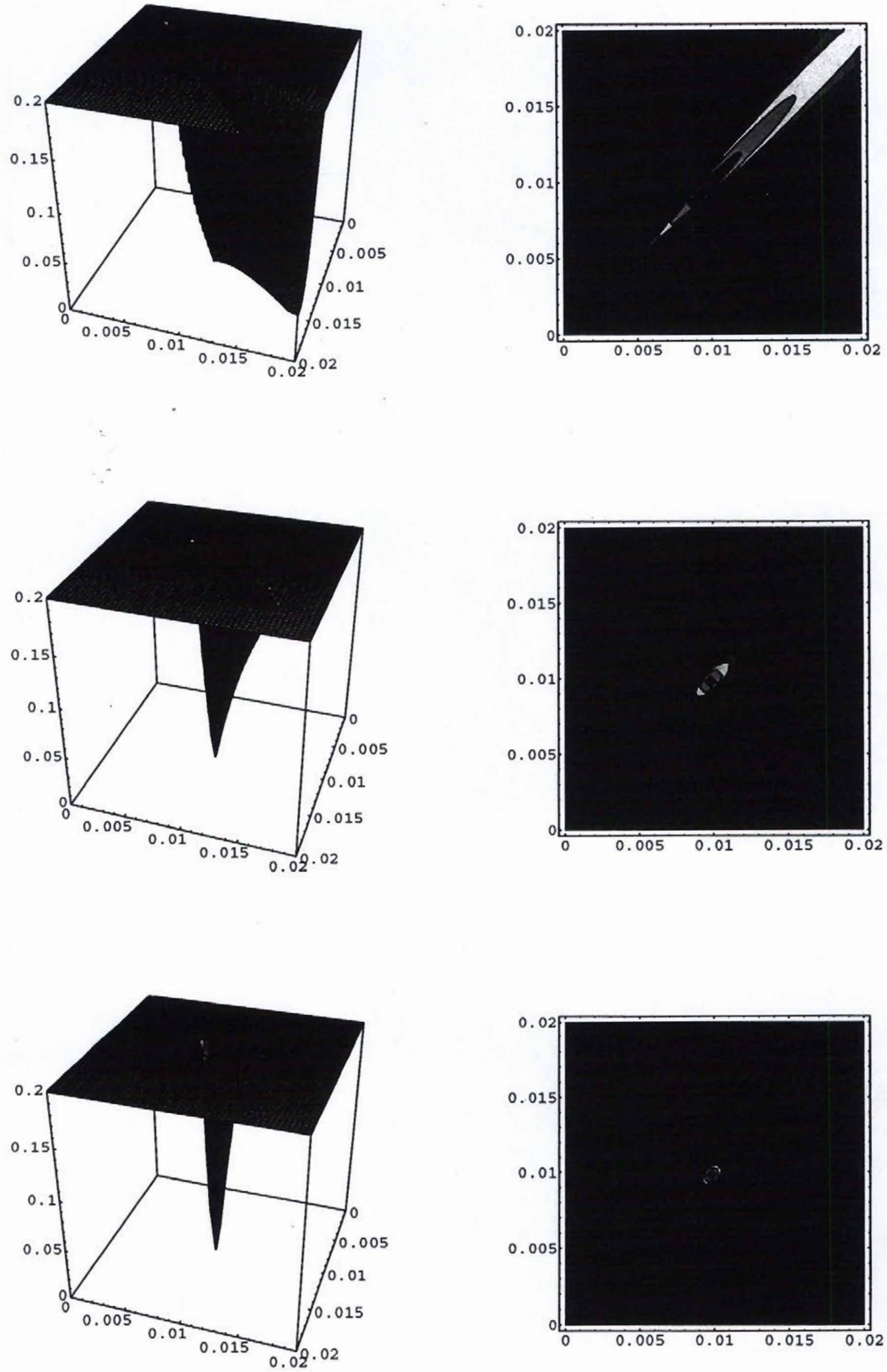


Figure 3.3: Surface and contour plot of $L(y^1, y^2)$ for $(y^1, y^2) \in (0, 0.02)^2$ with $\boldsymbol{\chi} = \mathbf{0}$ and penalties $M = 0.1, 0.5, 1.0$, from top to bottom. Constant parameters $\alpha = 1.0$ and $\beta = 1.0$. Geometric Brownian motion for endowments with $E_0^1 = E_0^2 = 10$, constant growth $g = 0.012$ and constant standard deviation $\varrho = 0.0189$.

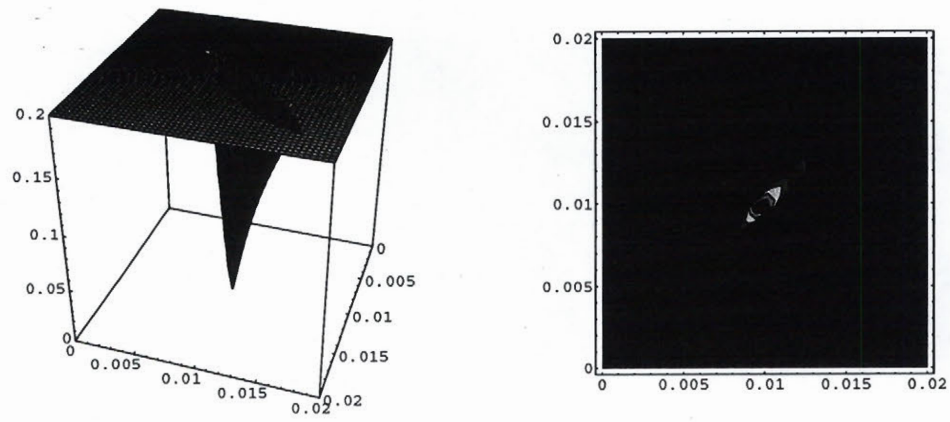


Figure 3.4: Surface and contour plot of $L(y^1, y^2)$ for $(y^1, y^2) \in (0, 0.02)^2$ with $\chi = \{(0.19)\mathbf{1}\Delta^+\} \wedge 0.19$, penalty $M = 0.5$, and constant parameters $\alpha = 2.0$, $\beta = 0.55$. Geometric Brownian motion for endowments with $E_0^1 = E_0^2 = 10$, constant growth $g = 0.0122$ and standard constant deviation $\varrho = 0.0189$.

After extensive testing, experimenting with (i) the time discretization size, (ii) the error tolerance in the optimization algorithm, (iii) the error tolerance in the inversion algorithm, and (iv) the number of sample paths, and after incorporating antithetic variance reduction by matching every simulated Brownian path by its reflection, we have seen that it is possible to obtain the equilibrium multipliers with a high degree of accuracy in a reasonable amount of computation time.

### 3.2.2 Moments of Equilibrium Processes

With the equilibrium multipliers $\boldsymbol{\Upsilon}^*$ in hand, we compute

$$\gamma_t^* = J_t\left(E_t - \int_0^t \chi_{\nu t}^c E_\nu d\nu; \boldsymbol{\Upsilon}^*\right)$$

We then compute equilibrium consumption $\mathbf{C}^*$ by solving the Volterra equation

$$\mathbf{C}_t^* = I_t^* + \int_0^t \boldsymbol{\chi}_{\nu t} \mathbf{C}_\nu^* d\nu$$

again using the numerical routine described above. This is done so as to check that aggregate equilibrium consumption does equal aggregate endowment. All runs have so far passed this test. For these preliminary simulations, we are focusing on identical agents but when we consider heterogeneous agents, information about how the aggregate endowment is shared between agents will be of considerable interest (see Chan & Kogan (2002)).

The next task is to set up a program to simulate equilibrium state prices $\xi_t^*$. Except in very special cases, such as that outlined in Chapter 1, extracting $r^*$ and $\theta^*$ from $\xi^*$ and expressing them in terms of aggregate endowment and a representative agent is analytically difficult. We must therefore simulate $r^*$ and $\theta^*$ and their first two moments. From the definition of $\xi^*$ we have

$$d\xi_t^* = -\xi_t r_t^* dt - \xi^* \theta_t^* dZ_t$$

and so

$$-\frac{d\xi_t^*}{\xi_t^*} = r_t^* dt + \theta_t^* dZ_t$$

Consider extracting $r^*$ and $\theta^*$ from $\xi^*$ from its conditional moments:

$$\mathbf{E}_t\left[\frac{\xi_{t+h}^* - \xi_t^*}{\xi_t^*}\right] \approx -r_t^* h \qquad \textit{and} \qquad \mathbf{V}_t\left[\frac{\xi_{t+h}^* - \xi_t^*}{\xi_t^*}\right] \approx (\theta_t^*)^2 h$$

We then have that

$$\begin{aligned} r_t^* &\approx \frac{1}{h}\left(1 - \frac{1}{\xi_t^*}\mathbf{E}_t\left[\xi_{t+h}^*\right]\right) \\ \theta_t^* &\approx \sqrt{\frac{1}{h}\mathbf{V}_t\left[\frac{\xi_{t+h}^* - \xi_t^*}{\xi_t^*}\right]} = \frac{1}{\xi_t^*}\sqrt{\frac{1}{h}\mathbf{V}_t\left[\xi_{t+h}^*\right]} = \frac{1}{\xi_t^*}\sqrt{\frac{1}{h}\left(\mathbf{E}_t\left[(\xi_{t+h}^*)^2\right] - \mathbf{E}_t\left[\xi_{t+h}^*\right]^2\right)} \end{aligned}$$

Hence, we need to simulate $\xi_t^*$ and $\xi_{t+h}^*$. Then we can simulate $\mathbf{E}_t\left[\xi_{t+h}^*\right]$ and $\mathbf{E}_t\left[(\xi_{t+h}^*)^2\right]$, which, in turn, enables us to simulate the unconditional moments of $\xi^*$, $r^*$ and $\theta^*$:

$$\mathbf{E}[\xi_t^*] \qquad \mathbf{V}[\xi_t^*] \qquad \mathbf{E}[r_t^*] \qquad \mathbf{V}[r_t^*] \qquad \mathbf{E}[\theta_t^*] \qquad \mathbf{V}[\theta_t^*]$$

We briefly explain the algorithm behind simulating $\xi_t^*$, $\xi_{t+h}^*$ and the moments mentioned above. The structure of the algorithm indicates the difficulty involved in simulating conditional expectations and how we incorporated antithetic variables to reduce sampling error as well as the recycling of pseudo random numbers to reduce computation time.

First, fix $t$ and $t+h$, where $h$ is the size of the time discretization. Recall that

$$\xi_t^* = \gamma_t^* - \mathbf{E}_t\left[\int_t^T \chi_{t\nu}\gamma_\nu^* d\nu\right]$$

To get one realization of $\xi_t^*$, we need one trajectory of $\gamma^*$ over $[0,t]$ and then, branching at time $t$, we need a sample of trajectories over $[t,T]$ to estimate the conditional expectation. We achieve this by generating one Brownian path over $[0,T]$. This path is divided into a primary branch over $[0,t]$ and a secondary branch over $[t,T]$. Then at time $t$, we reflect the secondary branch about $Z_t$ to obtain the antithetic version of the secondary branch. This is illustrated diagrammatically in Figure (3.5):

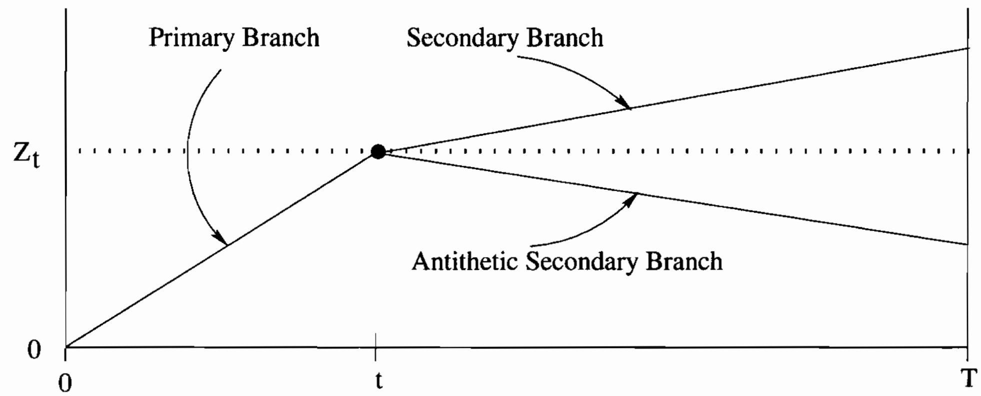


Figure 3.5: Primary and secondary Brownian branches.

Further, we reflect these three branches about 0 to obtain a mirror image, as seen in Figure (3.6).

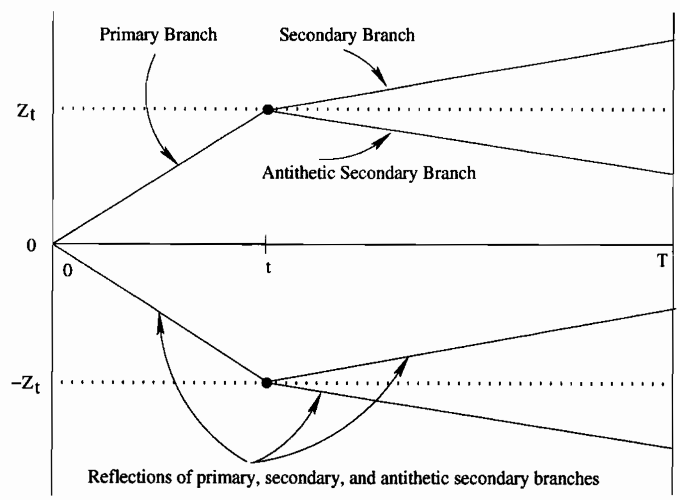


Figure 3.6: Primary and secondary Brownian branches and their reflections about 0.

As a result of this construction, we obtain four Brownian paths for the computational cost of only one. On this Brownian lattice, one can compute four $\gamma^*$ trajectories, using the expression mentioned at the start of this section, and recycling information that is used several times (ie: via simple reflections). Integrating $\gamma^*$ along the secondary and antithetic secondary branches then yields a sample of size two with which to estimate $\xi_t^*$ associated with the primary branch. Repeating with the reflection, we obtain another sample of size two for estimating $\xi_t^*$ associated with the reflected primary branch. This procedure yields a sample of size two for $\xi_t^*$ with which to estimate $\mathbf{E}[\xi_t^*]$ and $\mathbf{V}[\xi_t^*]$. Note that this estimation of the mean and variance of $\xi_t^*$ incorporates antithetic variance reduction.

In order to obtain the moments of $r_t^*$ and $\theta_t^*$, we also need the time $t$ conditional expectations $\mathbf{E}_t[\xi_{t+h}^*]$ and $\mathbf{E}_t[(\xi_{t+h}^*)^2]$. To do this, we similarly introduce an additional tertiary branching into our Brownian lattice, as shown in Figure (3.7):

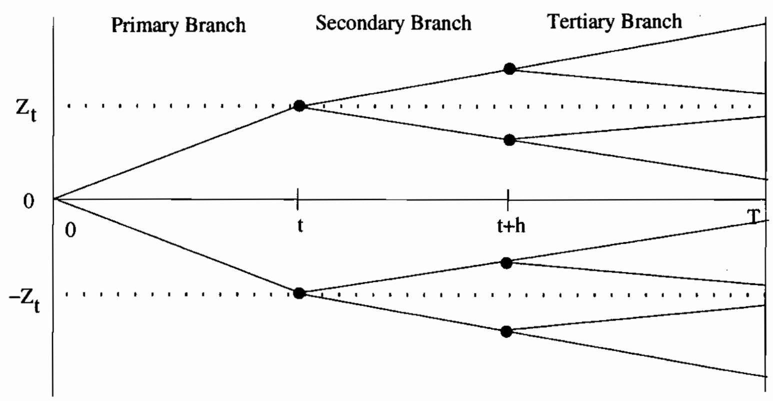


Figure 3.7: Full Brownian lattice.

At each of the four time $t+h$ nodes, the tertiary branches provide a 2-sample for estimating $\xi_{t+h}^*$, as before, which includes variance reduction. Then, the two time $t+h$ nodes emanating from the primary branch gives a 2-sample with which to estimate $\mathbf{E}_t[\xi_{t+h}^*]$ and $\mathbf{E}_t[(\xi_{t+h}^*)^2]$ associated with the primary branch, again with variance reduction. Repeating with the reflected primary branch then gives us a 2-sample for estimating $\mathbf{E}_t[\xi_{t+h}^*]$ and $\mathbf{E}_t[(\xi_{t+h}^*)^2]$ associated with the reflected primary branch. With this 2-sample of $\mathbf{E}_t[\xi_{t+h}^*]$ and $\mathbf{E}_t[(\xi_{t+h}^*)^2]$, we can then estimate the first and second unconditional moments of $r^*$ and $\theta^*$, again with variance reduction. This calculation entails significant recycling of variables and antithetic variance reduction as most Brownian paths are reflections.

In our figures we have, for simplicity, illustrated only a binary branching so that for each primary branch we have two secondary and four tertiary branches. However, we found it necessary to insert additional secondary and tertiary paths (and their antithetic and reflected counterparts) to obtain good estimators. In the simulation program we have separate parameters controlling the number of primary, secondary and tertiary branches. A large amount of experimentation was required to choose the three branch parameters so that the estimators were well behaved and the simulation did not take too much time. We demonstrate the accuracy of this approach in the next section.

## 3.3 Special Cases: Diagnostics

In this section we present the results of the special cases $\chi = 0$ for $A = 2$ and 3 which serve as diagnostics, primarily for the second task of simulating the conditional expectations. It is this case only for which explicit results for $\xi^*$, $r^*$ and $\theta^*$ are known and for which we can directly compare our simulation results (see Karatzas et al. (1991)). For cases with $\chi \neq 0$, the only additional concern was with the procedure that solves the Volterra equation for optimal consumption and various integrations involving $\chi$. Before the program was converted to a parallel program, the routines for all integrations and, particularly, the numerical solution to the Volterra equation, were thoroughly tested by duplicating these isolated computations in Mathematica under various assumptions about the underlying parameters.

For diagnostics, we considered $u_t^a(x) = e^{-\beta^a t}\left(\frac{x^{1-\alpha}-1}{1-\alpha}\right)$ where $\beta^a$ and $\alpha$ are all constants. Note that $\alpha$ is the same for all agents whereas $\beta^a$ is allowed to be different. In the case of common $\alpha$, the inversion can be done explicitly, enabling a certain degree of diagnostic testing. Now, $u_t^{a'}(x) = e^{-\beta^a t}x^{-\alpha}$ and hence $I_t^a(y) = e^{-(\beta^a/\alpha)t}y^{-1/\alpha}$. This leads to the representative inverse marginal utility

$$\begin{aligned} I_t(y;\Upsilon) &= \sum_{a=1}^{A} I_t^a(y^a y) \\ &= \sum_{a=1}^{A} e^{-(\beta^a/\alpha)t}(y^a y)^{-1/\alpha} \\ &= \left(\sum_{a=1}^{A} e^{-(\beta^a/\alpha)t}(y^a)^{-1/\alpha}\right) y^{-1/\alpha} \\ &= \left(\sum_{a=1}^{A} f(t,a;\Upsilon)\right) y^{-1/\alpha} \end{aligned}$$

where we set $f(t,a;\Upsilon) = e^{-(\beta^a/\alpha)t}(y^a)^{-1/\alpha}$. Now, $I_t(y;\Upsilon) = z$ is easily invertible:

$$J_t(z;\Upsilon) = \left(\sum_{a=1}^{A} f(t,a;\Upsilon)\right)^{\alpha} z^{-\alpha}$$

Market clearing occurs if

$$\xi_t = J_t(E_t;\Upsilon) = \left(\sum_{b=1}^{A} f(t,b;\Upsilon)\right)^{\alpha} E_t^{-\alpha}$$

We can then compute, for any agent $a$, the form of optimal consumption

$$\begin{aligned} C_t^a &= I_t^a(y^a \xi_t) \\ &= e^{-(\beta^a/\alpha)t}(y^a)^{-1/\alpha}\xi_t^{-1/\alpha} \\ &= f(t,a;\Upsilon)\xi_t^{-1/\alpha} \\ &= \left(\frac{f(t,a;\Upsilon)}{\sum_{b=1}^{A} f(t,b;\Upsilon)}\right) E_t \end{aligned}$$

At the equilibrium multipliers $\Upsilon^*$, the fraction

$$\frac{f(t,a;\Upsilon^*)}{\sum_{b=1}^{A} f(t,b;\Upsilon^*)}$$

represents the Pareto optimal sharing of aggregate endowment. Next, we have

$$\begin{aligned} G^a(\Upsilon) &= \mathbf{E}\left[\int_0^T \xi_\nu (C_\nu^a - E_\nu^a) d\nu\right] \\ &= \mathbf{E}\left[\int_0^T \left(\sum_{b=1}^{A} f(t,b;\Upsilon)\right)^{\alpha} E_\nu^{-\alpha}\left(\left(\frac{f(t,a;\Upsilon)}{\sum_{b=1}^{A} f(t,b;\Upsilon)}\right) E_\nu - E_\nu^a\right) d\nu\right] \\ &= \mathbf{E}\left[\int_0^T \left\{\left(\frac{f(t,a;\Upsilon)}{(\sum_{b=1}^{A} f(t,b;\Upsilon))^{1-\alpha}}\right) E_\nu^{1-\alpha} - \left(\sum_{b=1}^{A} f(t,b;\Upsilon)\right)^{\alpha} E_\nu^a E_\nu^{-\alpha}\right\} d\nu\right] \end{aligned}$$

which is easy to simulate directly for various choices of endowments. We then compute

$$F(\Upsilon) = \sum_{a=1}^{A} [G^a(\Upsilon)]^2$$

and find a global minimum of 0 at $\Upsilon^*$. If $\xi_0^* = J_0(E_0; \Upsilon^*) \neq 1$ then we rescale $\Upsilon^*$ until $\xi_0^* = 1$.

To specify individual endowments, define the process $\mathcal{E}$ by

$$d\mathcal{E}_t = g\mathcal{E}_t dt + \varrho \mathcal{E}_t dZ_t$$

where $\mathcal{E}_0$, $g$, and $\varrho$ are constants. Thus,

$$\mathcal{E}_t = \mathcal{E}_0 \exp\left[\left(g - \frac{1}{2}\varrho^2\right) t + \varrho Z_t\right]$$

Then, for each agent, we define $E^a$ by

$$E_t^a = (\mathcal{E}_t \vee k_E) \wedge K_E$$

which ensures that $k_E \leq E_t^a \leq K_E$. Aggregate endowment is then $E_t = \sum_{a=1}^{A} E_t^a$. If $k_E = 0$ and $K_E = \infty$ then $E^a$ satisfies the same SDE as $\mathcal{E}$ with $E_0^a = \mathcal{E}_0$. In the simulations, we chose $k_E$ and $K_E$ so that the probability of $\mathcal{E}$ leaving the interval $[k_E, K_E]$ was less that 0.005.

In our case of common and constant $\alpha$ and $\beta$, $u_t^{a'}(x) = e^{-\beta t} x^{-\alpha}$ and one can explicitly solve for $\Upsilon^*$, $\xi^*$, optimal consumption $C^{a*}$ as well as $r^*$ and $\theta^*$:

$$\begin{aligned}
y^{a*} &= \frac{1}{E_0^\alpha} \left( \frac{\mathbf{E}\left[\int_0^T e^{-\beta\nu} E_\nu^{1-\alpha} d\nu\right]}{\mathbf{E}\left[\int_0^T e^{-\beta\nu} E_\nu^a E_\nu^{-\alpha} d\nu\right]} \right)^\alpha \\
C_t^{a*} &= \left(\frac{1}{y^{a*}}\right)^{\frac{1}{\alpha}} \left(\frac{E_t}{E_0}\right) \\
\xi_t^* &= \left(\sum_{a=1}^{A} \frac{1}{y^{a*}}\right)^{\frac{1}{\alpha}} e^{-\beta t} E_t^{-\alpha} \\
r_t^* &= \beta + \alpha \left[g - \left(\frac{1+\alpha}{2}\right) \varrho^2\right] \\
\theta_t^* &= \alpha \varrho
\end{aligned}$$

We now present the results of our simulations for the cases just discussed above. All plots that follow have the same format. They consist of surface plots and their corresponding contour plots. For both types of plots, the horizontal axis is the time axis; time increases from $t = 0$ on the left to $t = 1$ on the right. The axis going into the page for the surface plots and the vertical axis for the contour plots is the relative risk aversion axis; our common $\alpha$ takes the values $0.5, 1.0, 1.5, 2.0, 2.5, 3.0$ from front to back for the surface plots and from bottom to top for the contour plots. The vertical axis on the surface plots is a mean or standard deviation of $\xi^*$, $r^*$ or $\theta^*$. The plotting ranges are kept the same for easy comparison with later simulations and are listed here:

$$\mathbf{E}[\xi^*]: \quad [0.0, 1.5] \qquad \mathbf{S}[\xi^*]: \quad [0.0, 0.1]$$

$$\mathbf{E}[r^*]: \quad [-2.0, 1.0] \qquad \mathbf{S}[r^*]: \quad [0.0, 0.1]$$

$$\mathbf{E}[\theta^*]: \quad [0.0, 0.1] \qquad \mathbf{S}[\theta^*]: \quad [0.0, 0.02]$$

The contour plots give lines of constant mean or standard deviation and are included to aid in visualizing the dependence on the parameters. There are 3 columns of plots corresponding to three values of patience used: $\beta = 0.11, 0.55, 0.89$. We work with geometric Brownian motion for endowments with growth $g = 0.0122$ and standard deviation $\varrho = 0.0189$; these parameter values are taken from Campbell & Cochrane (1999). $E_0^1 = E_0^2 = 10.0$.

The next two plots (Figures (3.8) and (3.9)) are the two diagnostic cases $\boldsymbol{\chi} = \mathbf{0}$, $A = 2$ and $\boldsymbol{\chi} = \mathbf{0}$, $A = 3$. All parameters governing the simulation were adjusted so that simulated results match the theoretical values to within 0.1%, as confirmed visually by the graphs. With $k_E = 7.0$ and $K_E = 15$, the probability of $E^a$ hitting the boundaries was simulated to be around 0.003 on average and never exceeding 0.005.

Many cases of different $\alpha^a$ and $\beta^a$ were also tested by comparing the simulation results with those produced from a more direct numerical simulation in Mathematica. Agreement was found to be excellent.

$$\chi = \begin{bmatrix} 0.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix}$$

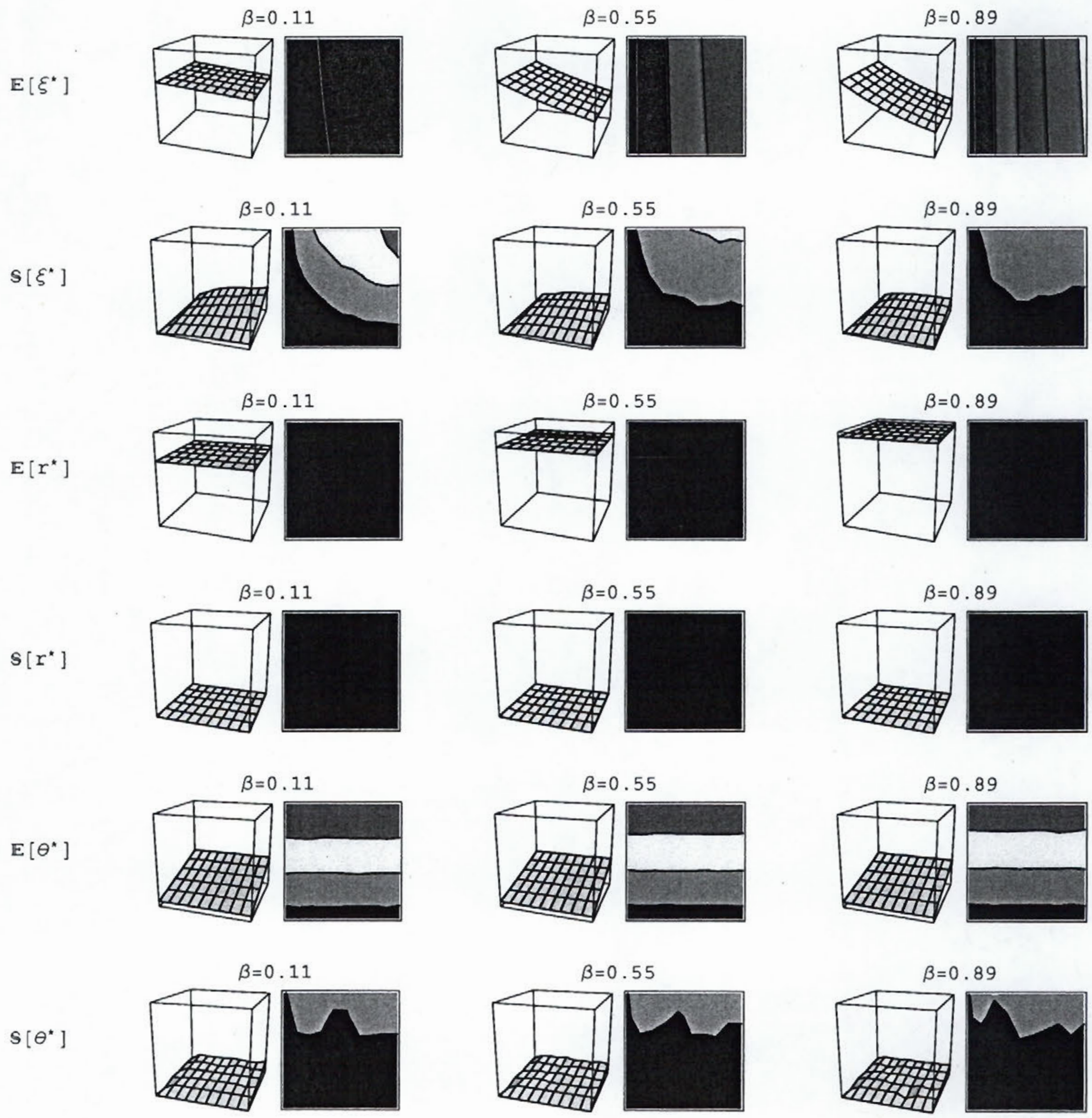


Figure 3.8: Plots of the unconditional moments of state prices, real interest rates, and market prices of risk

$$\chi = \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix}$$

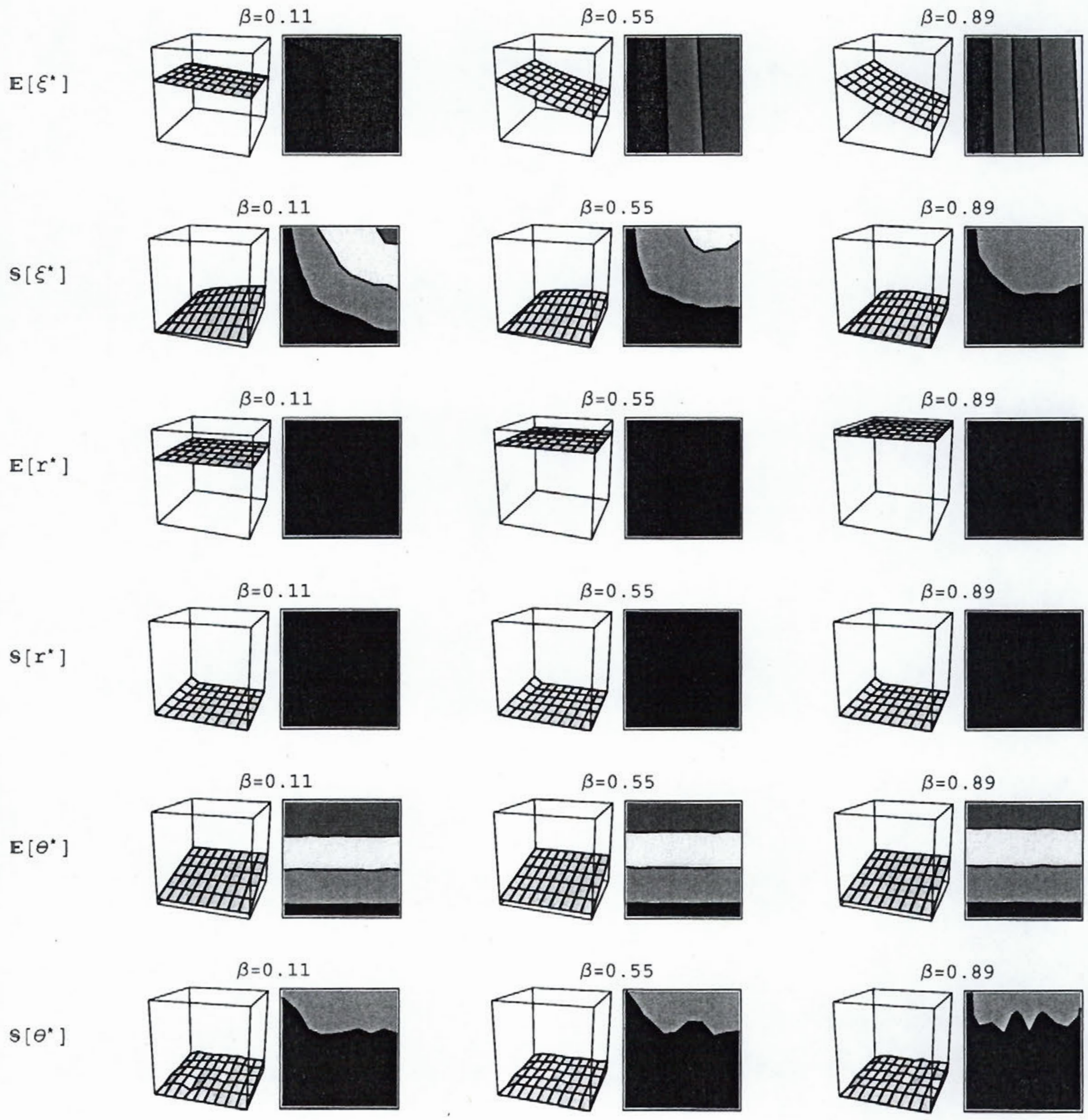


Figure 3.9: Plots of the unconditional moments of state prices, real interest rates, and market prices of risk

## 3.4 General Cases: Results

In this section we leave the restrictive case of $\boldsymbol{\chi} = \mathbf{0}$ considered for diagnostics. All the parameters specified in the diagnostics are retained except for $\boldsymbol{\chi}$. With $\boldsymbol{\chi} \neq \mathbf{0}$, we must ensure that the bound $AK_\chi Te^{K_\chi T} < \frac{k_E}{K_E}$ is satisfied so that consumption configurations are utilizable. As we are working with $T = 1$, $k_E = 7$ and $K_E = 15$. Our condition is then

$$K_\chi e^{K_\chi} < \frac{7}{15A}$$

and for $A = 2$ this means $K_\chi = 0.19$ and for $A = 3$ this means $K_\chi = 0.13$.

### 3.4.1 Constant $\boldsymbol{\chi}$

In this section, we present the results of various choices of $\boldsymbol{\chi}$ with constant entries having identical diagonals and identical column sums. Diagonal $\boldsymbol{\chi}$ correspond to purely internal habit formation. $\boldsymbol{\chi}$ with zeros on the diagonal and non-zero off diagonal elements correspond to pure interaction. $\boldsymbol{\chi}$ with all non-zero entries corresponds to a mixture of internal and external habits.

$$\chi = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

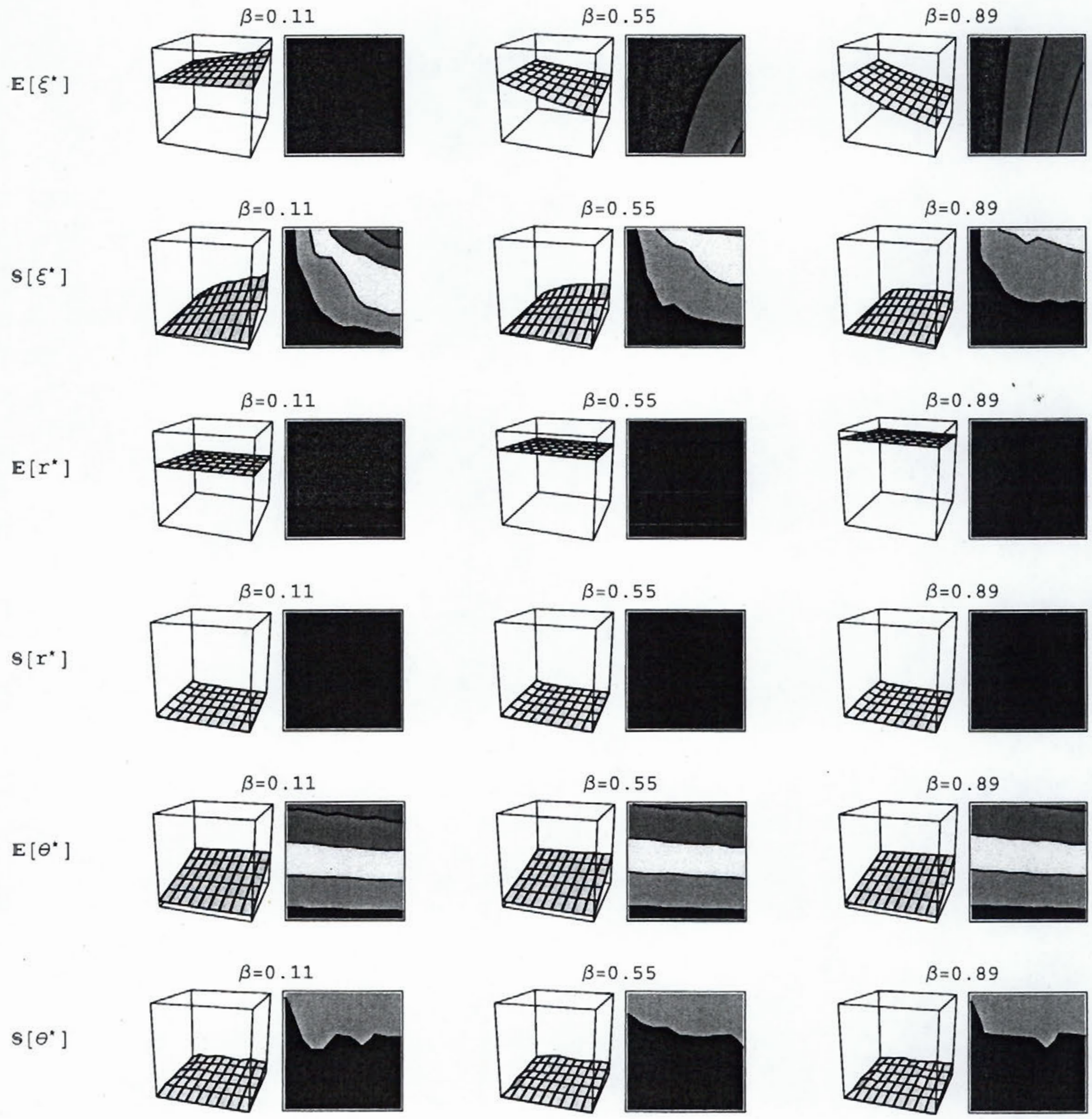


Figure 3.10: Plots of the unconditional moments of state prices, real interest rates, and market prices of risk

$$\chi = \begin{bmatrix} 0.0 & 0.1 \\ 0.1 & 0.0 \end{bmatrix}$$

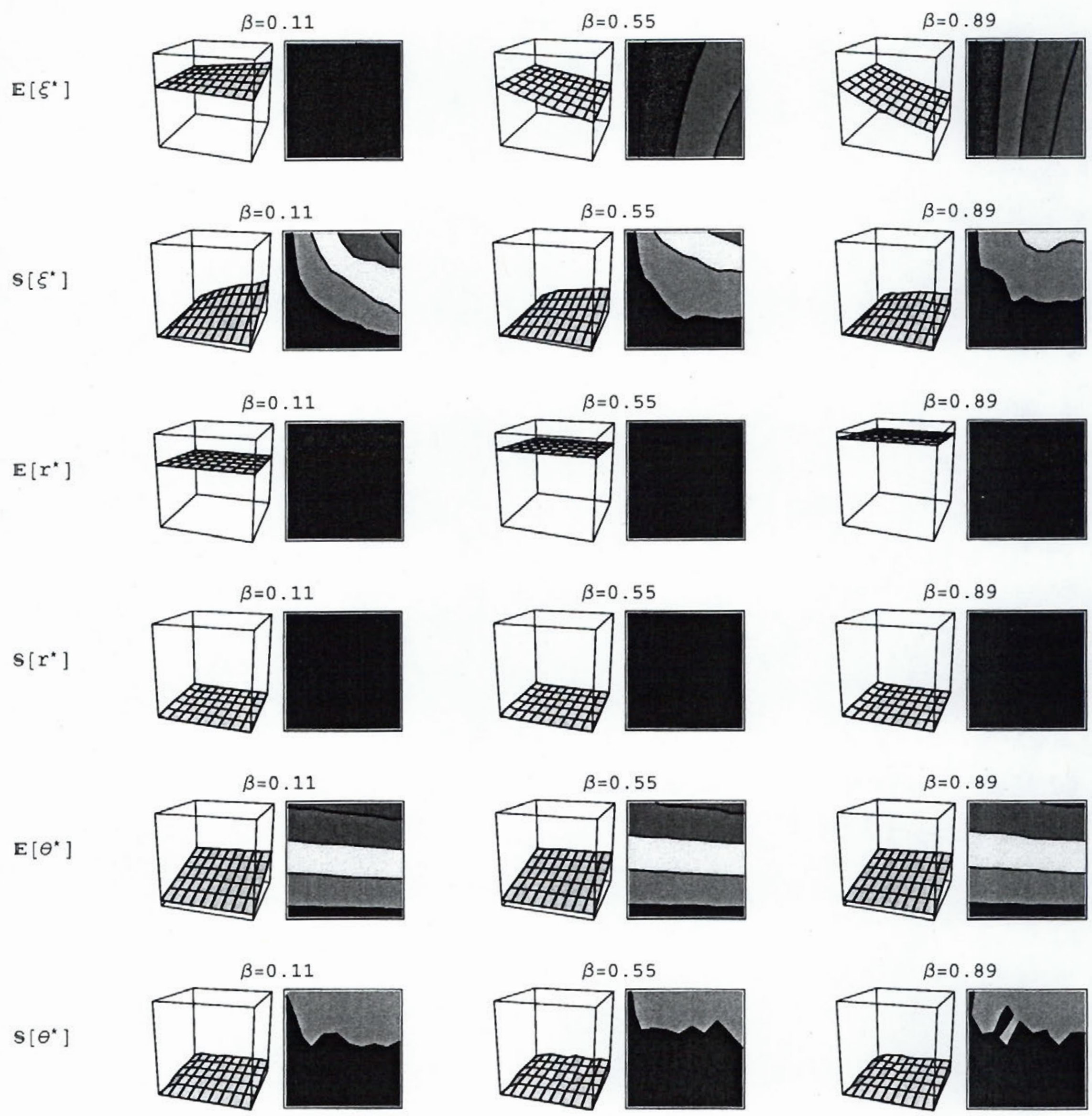


Figure 3.11: Plots of the unconditional moments of state prices, real interest rates, and market prices of risk

$$\chi = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}$$

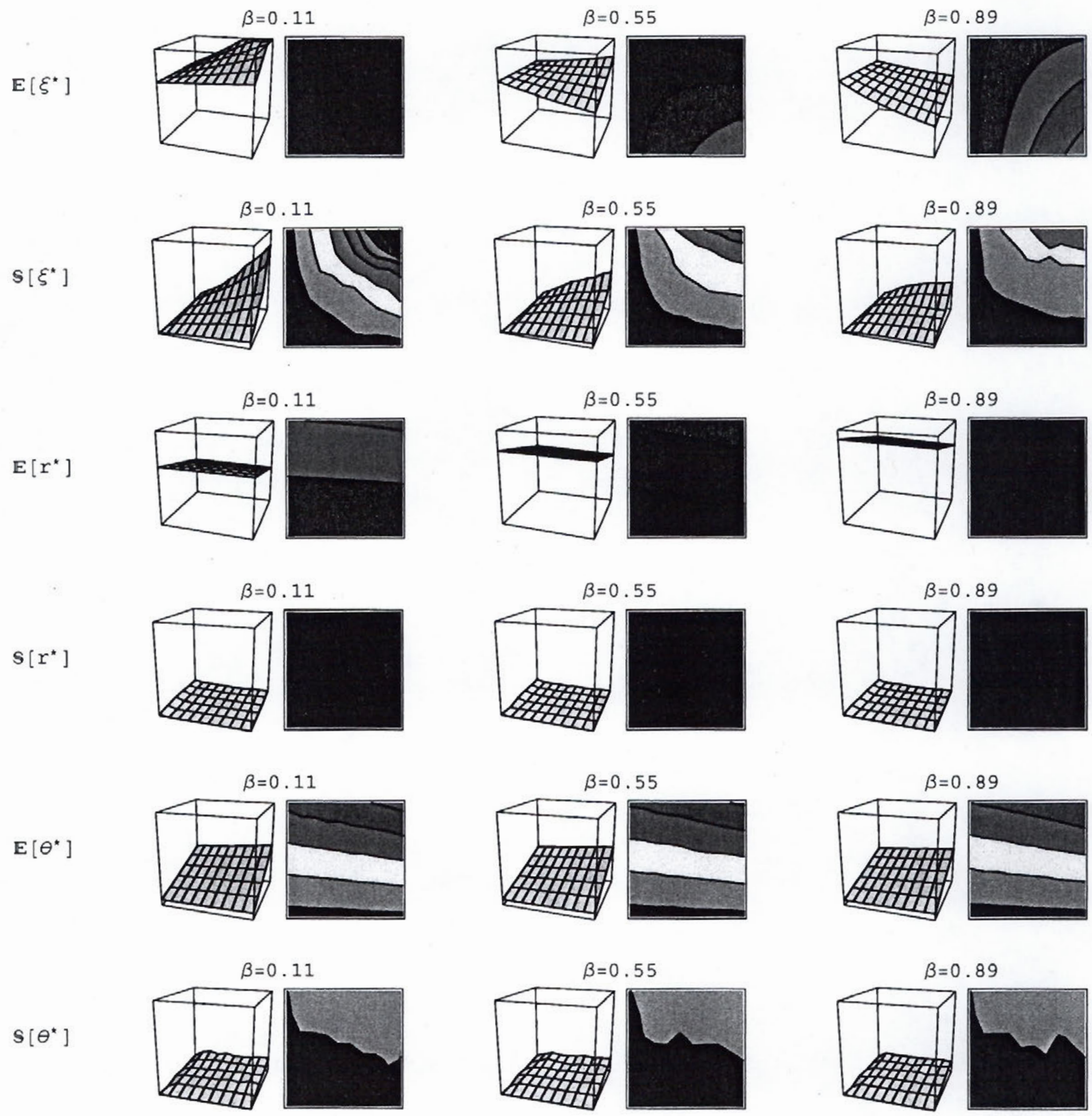


Figure 3.12: Plots of the unconditional moments of state prices, real interest rates, and market prices of risk

$$\chi = \begin{bmatrix} 0.19 & 0.0 \\ 0.0 & 0.19 \end{bmatrix}$$

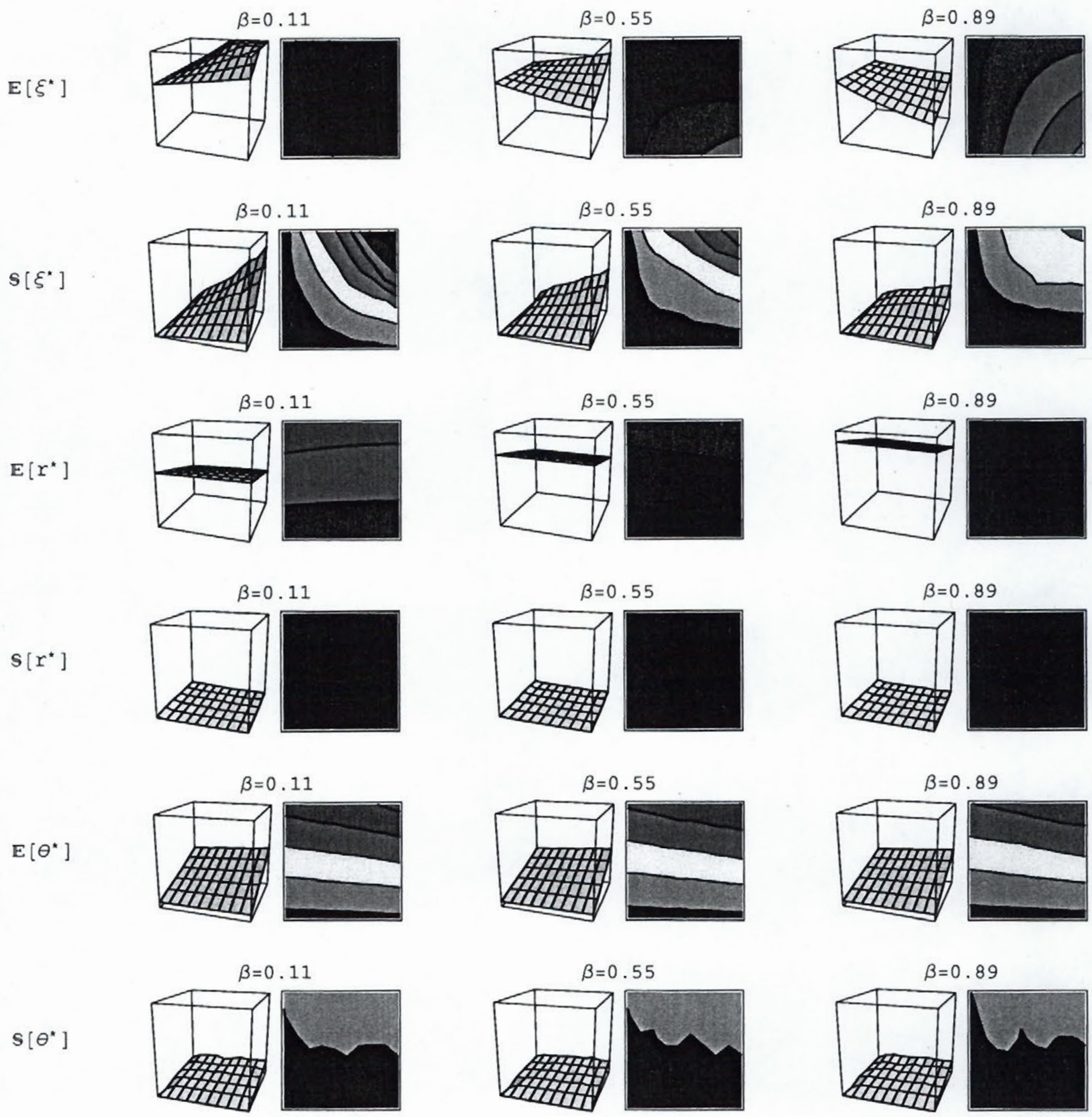


Figure 3.13: Plots of the unconditional moments of state prices, real interest rates, and market prices of risk

$$\chi = \begin{bmatrix} 0.0 & 0.19 \\ 0.19 & 0.0 \end{bmatrix}$$

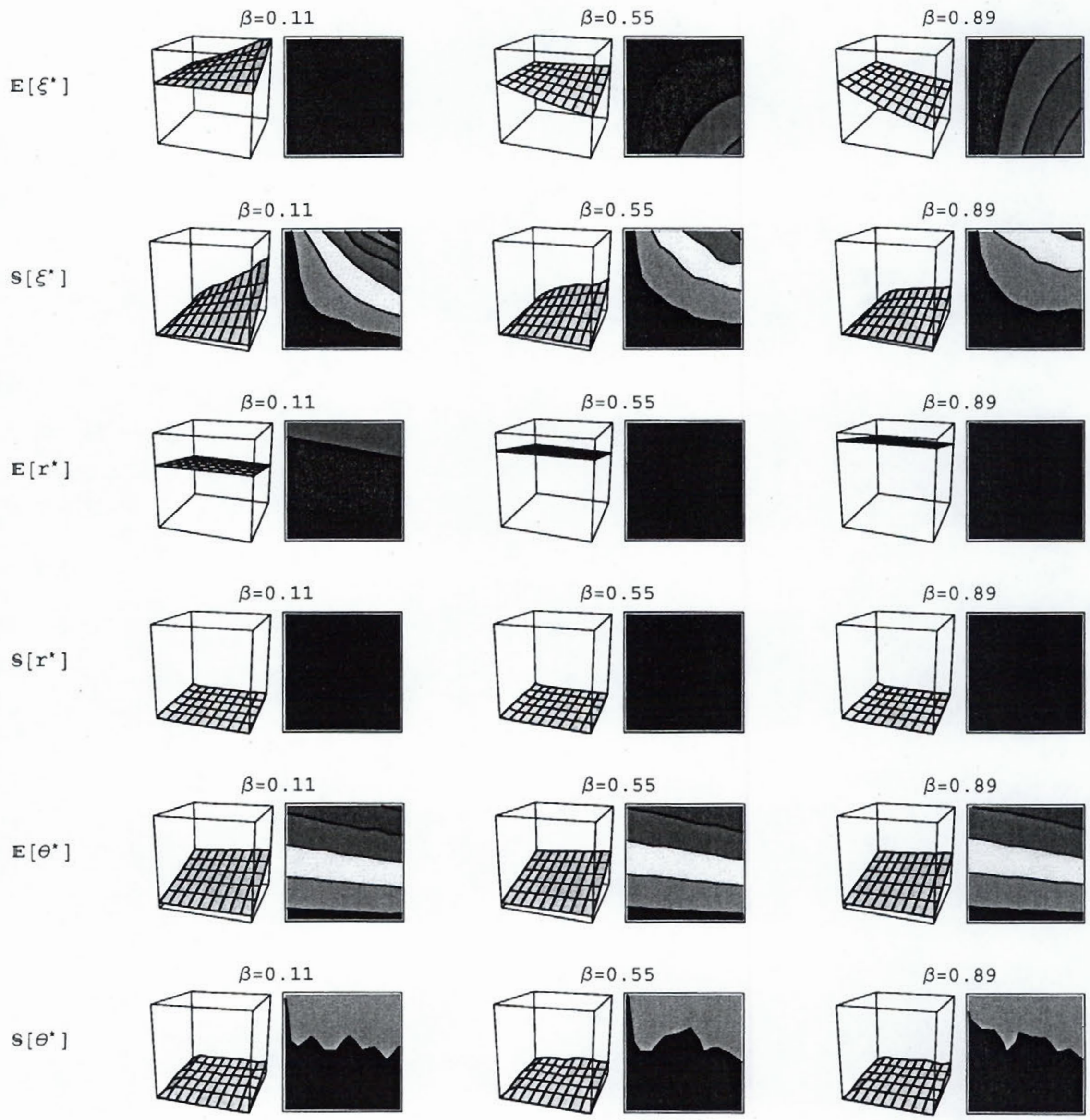


Figure 3.14: Plots of the unconditional moments of state prices, real interest rates, and market prices of risk

$$\chi = \begin{bmatrix} 0.19 & 0.19 \\ 0.19 & 0.19 \end{bmatrix}$$

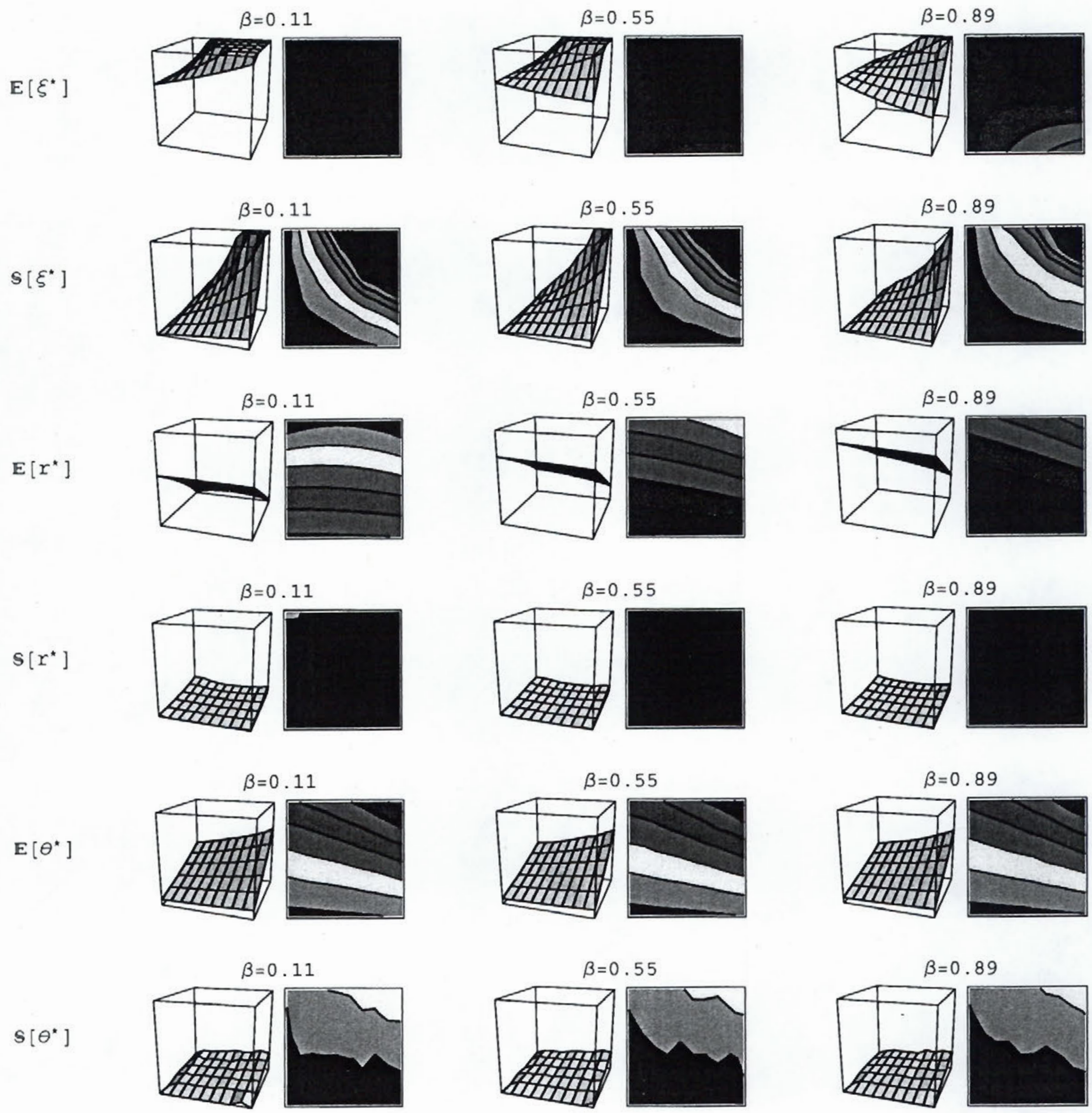


Figure 3.15: Plots of the unconditional moments of state prices, real interest rates, and market prices of risk

$$\chi = \begin{bmatrix} 1.3 & 1.3 & 0.0 \\ 0.0 & 1.3 & 1.3 \\ 1.3 & 0.0 & 1.3 \end{bmatrix}$$

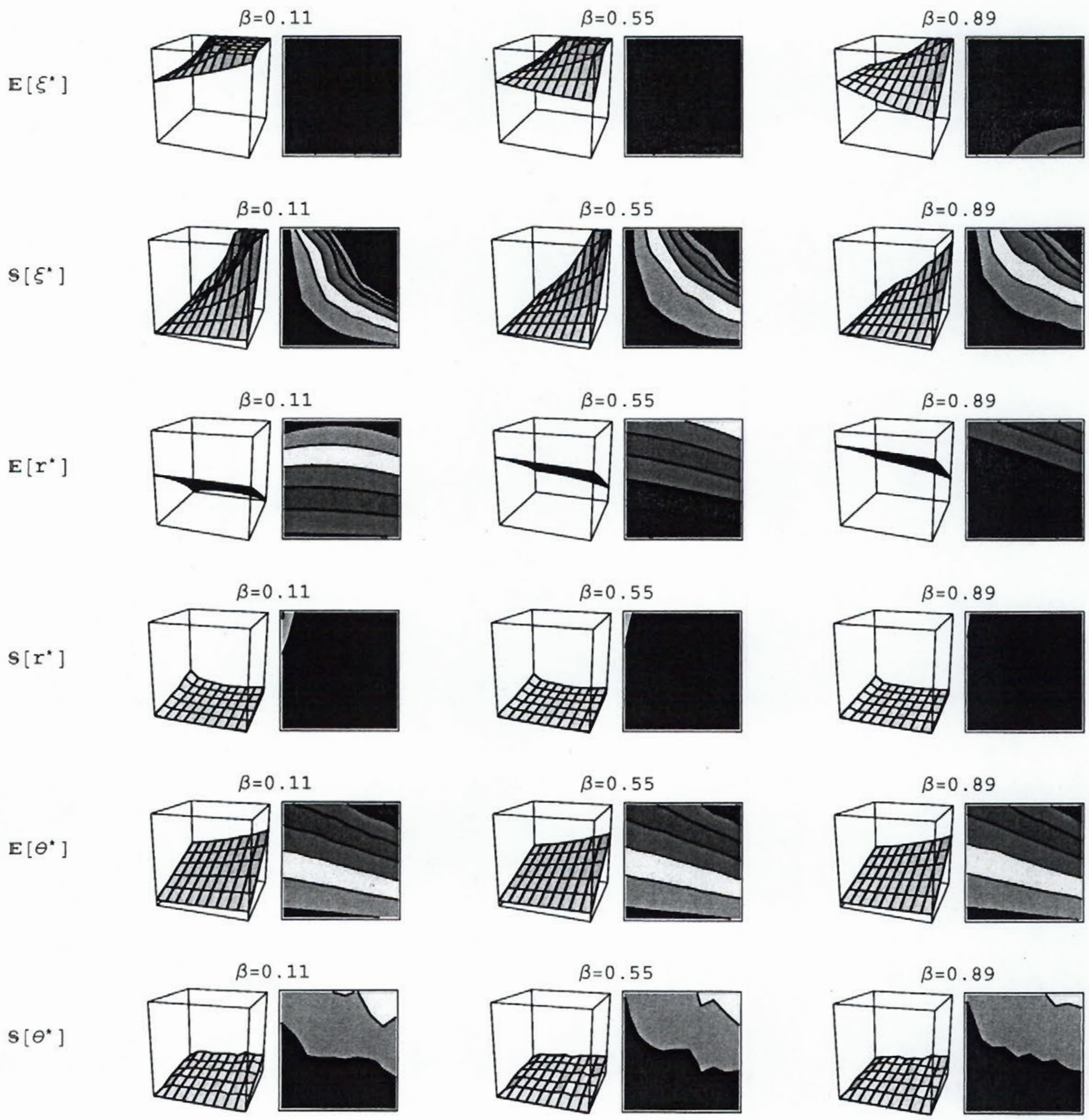


Figure 3.16: Plots of the unconditional moments of state prices, real interest rates, and market prices of risk

### 3.4.2 Variable $\chi$

As an initial exploration, constant $\chi$ aids the development of intuition about how parameter values influence equilibrium. We now proceed with some cases for $\chi$ in which the elements are stochastic.

There is a great deal of freedom in our choice of stochastic $\chi$. The examples here are merely experiments to test the behavior of the model. Recall that each of our identical agents has an endowment which satisfies

$$E_t^a = \left\{ E_0^a \exp\left[ \left( g - \frac{1}{2}\varrho^2 t \right) + \varrho Z_t \right] \vee k_E \right\} \wedge K_E$$

and hence

$$\mathbf{E}\left[E_t^a\right] \approx E_0^a \exp\left[gt\right]$$

since we have kept the probability of endowments being outside of $[k_E, K_E]$ very small. Define deviations of endowment from it's expected growth by

$$\Delta_t = E_t^a - E_0^a e^{gt}$$

The use of deviations from expected growth is motivated by the work of Kraus & Sagi (2004) in which this variable plays an important role in generating new model behavior. Now, we choose $\chi$ to be

$$\chi_{\nu t} = \left\{ \begin{bmatrix} X & a \\ a & X \end{bmatrix} \left( \Delta_\nu - \Delta_t \right)^+ \right\} \wedge .19$$

or, in the three agent case,

$$\chi_{\nu t} = \left\{ \begin{bmatrix} X & a & b \\ b & X & a \\ a & b & X \end{bmatrix} \left( \Delta_\nu - \Delta_t \right)^+ \right\} \wedge .13$$

where "$\wedge .19$" and "$\wedge .13$" is applied to each of the constant entries $X, a, b$. The motivation for these definitions is that for $\nu < t$ our agent is weighing time $\nu$ endowment rate against the time $t$ rate. If at time $\nu$, endowment is received faster than rate $g$ then $\Delta_\nu > 0$. If, later, endowment is received more slowly than rate $g$ at time $t$ then not only is our agent doing worse than expected (ie $g$), so that $\Delta_t < 0$, but there has also been an additional drop from a rate above $g$ at time $\nu$ down to the rate $g$. In this case $(\Delta_\nu - \Delta_t)^+$ is large. This will cause habits to quickly catch up to consumption and make agents more concerned about an additional worsening of their rate of endowment. Conversely, if one starts off receiving at a rate below $g$ and then later begins to receive above $g$, our agent feels that things are going extremely well; $(\Delta_\nu - \Delta_t)^+$ is small, habit is slow to catch up with consumption, and this agent enjoys this bettering of his situation for a longer period of time before habits develop. The parameters $X, a, b$ determine the intensity of the internal and external habit components in the formation of the agent's overall habit level.

Another set of choices is to replace $\left(\Delta_\nu - \Delta_t\right)^+$ with $\Delta_\nu^+$. In this case, if the endowment rate exceeds $g$ most of the time, $\Delta_\nu^+$ will usually be large, and habits will quickly be formed at a higher level since it appears that the expected endowment rate is actually better than $g$. Conversely, if the endowment rate is less than $g$ most of the time then $\Delta_\nu^+$ will usually be zero, habits will form slowly, if at all, and agents are content with the lower rate of endowment; it is as if the agents resign themselves to the lower endowment rate and don't worry about what others are consuming. There is no point in keeping up with the Joneses' if there the funds to do so are simply not available. As we show, some interesting behavior occurs in this case.

$$\chi_{\nu t} = \left\{ \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix} \left(\Delta_\nu - \Delta_t\right)^+ \right\} \wedge .19$$

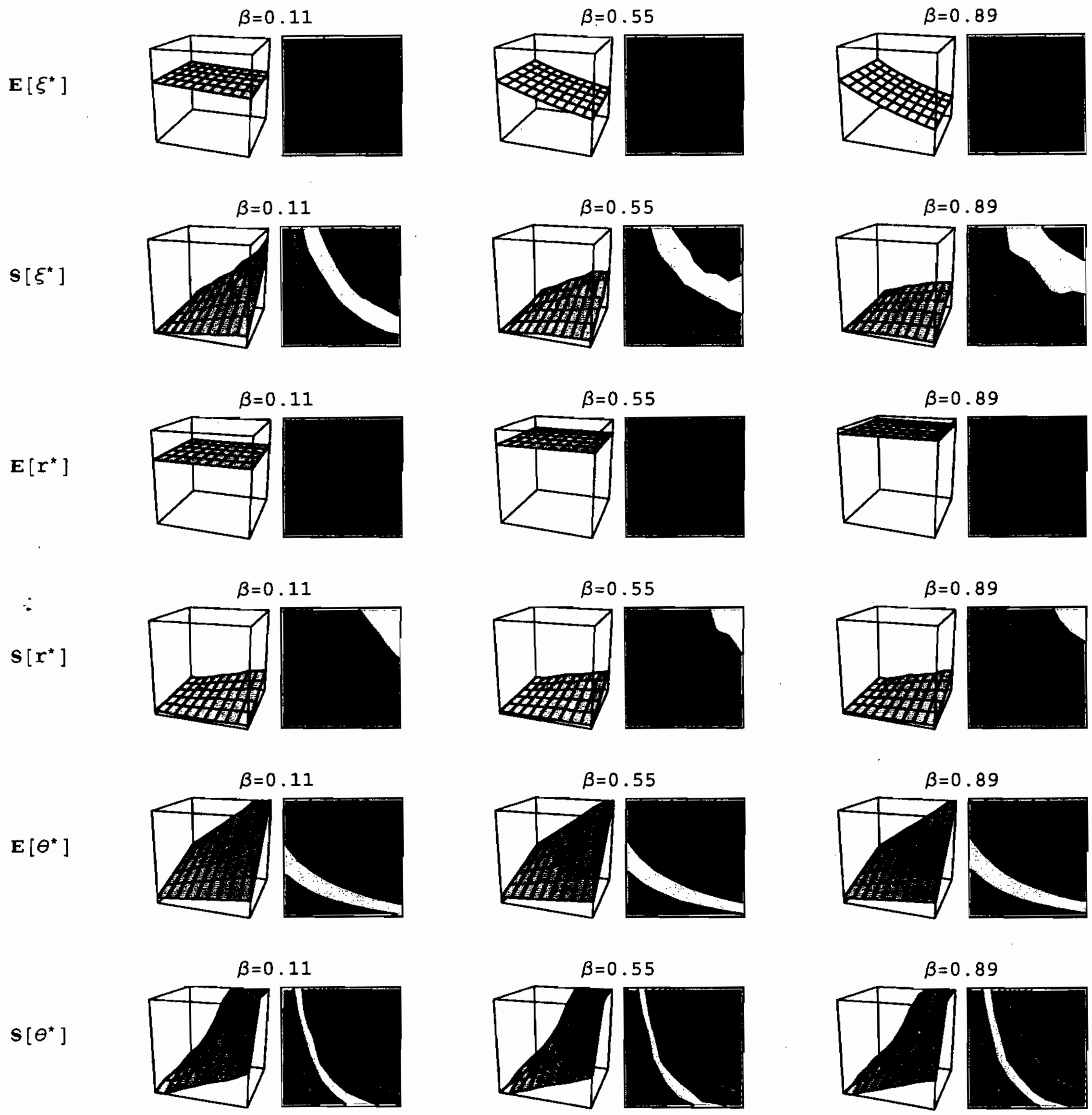


Figure 3.17: Plots of the unconditional moments of state prices, real interest rates, and market prices of risk

$$\chi_{\nu t} = \left\{ \begin{bmatrix} 0.19 & 0.19 \\ 0.19 & 0.19 \end{bmatrix} \left(\Delta_\nu - \Delta_t\right)^+ \right\} \wedge .19$$

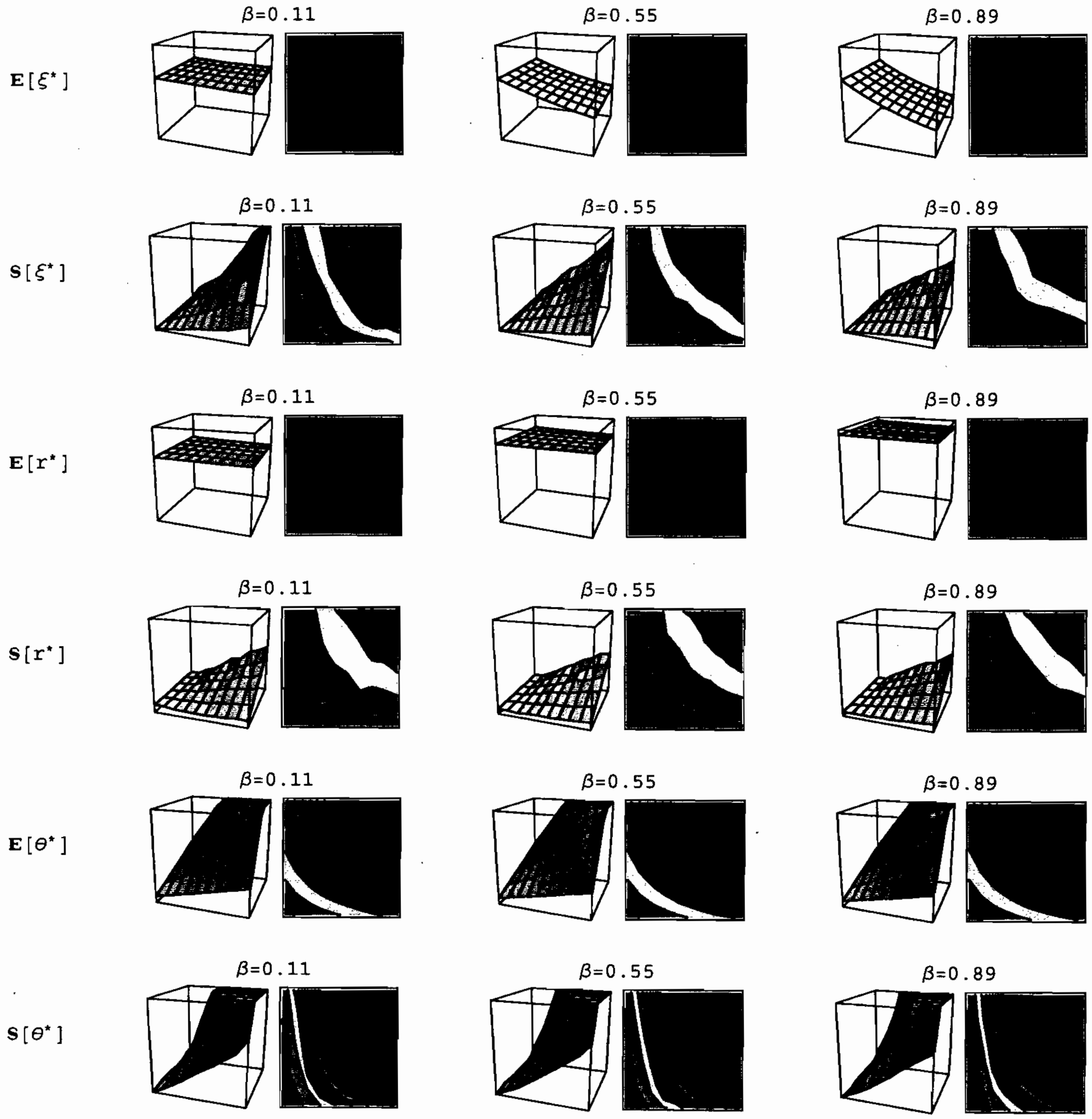


Figure 3.18: Plots of the unconditional moments of state prices, real interest rates, and market prices of risk

$$\chi_{\nu t} = \left\{ \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix} \Delta_\nu^+ \right\} \wedge .19$$

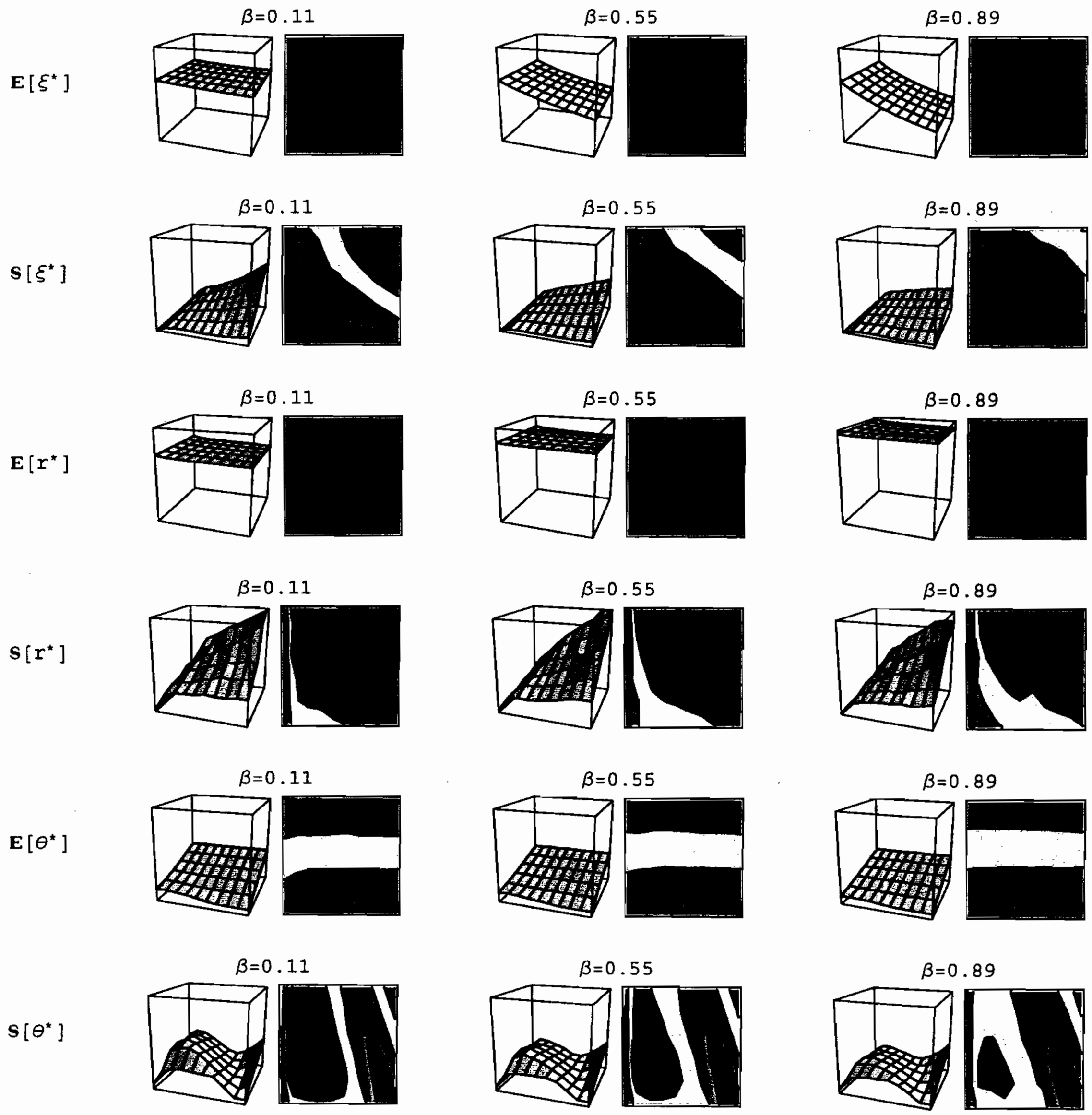


Figure 3.19: Plots of the unconditional moments of state prices, real interest rates, and market prices of risk

$$\chi_{\nu t} = \left\{ \begin{bmatrix} 0.19 & 0.19 \\ 0.19 & 0.19 \end{bmatrix} \Delta_\nu^+ \right\} \wedge .19$$

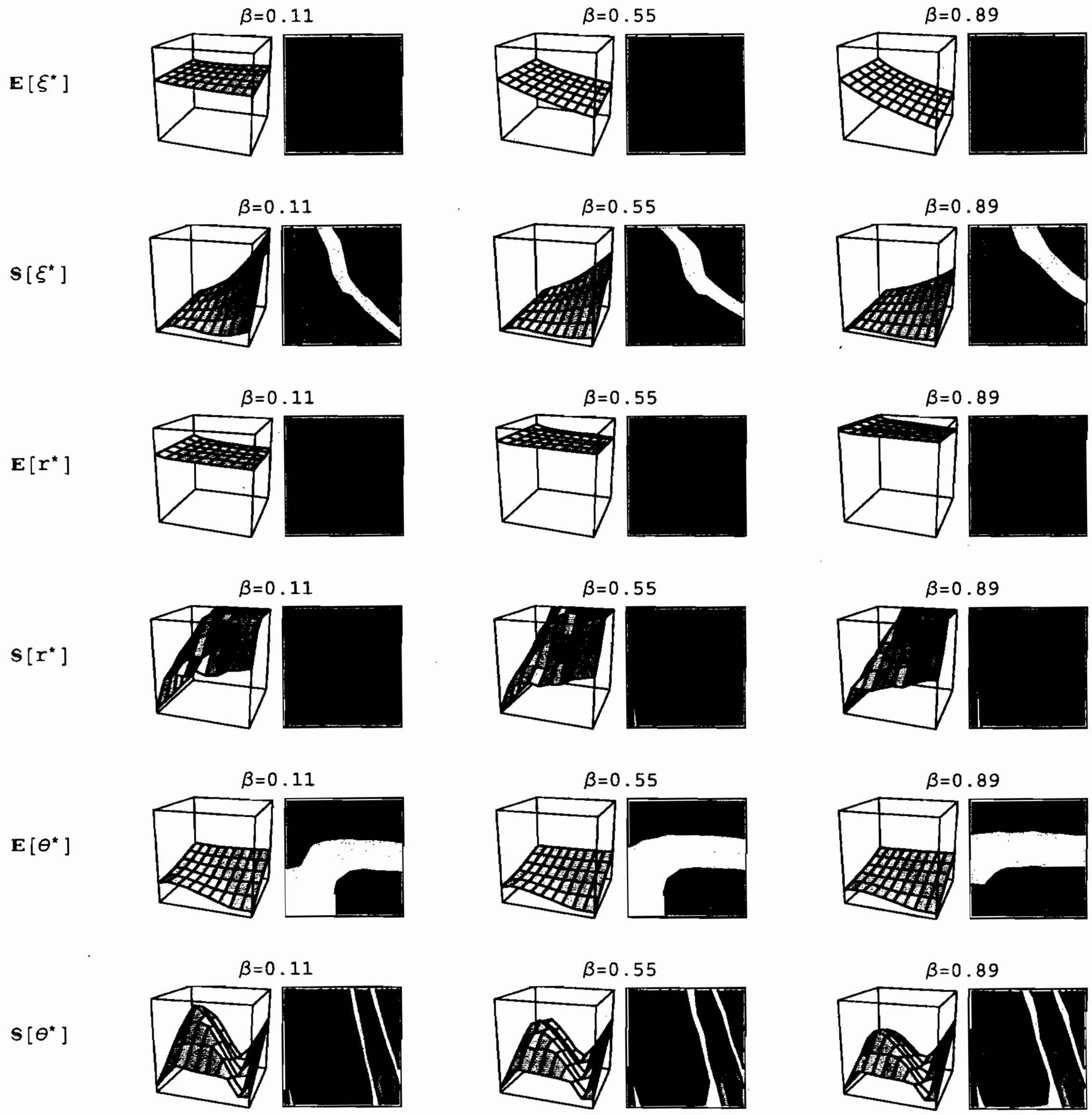


Figure 3.20: Plots of the unconditional moments of state prices, real interest rates, and market prices of risk

$$\chi = \left\{ \begin{bmatrix} 0.0 & 0.13 & 0.0 \\ 0.0 & 0.0 & 0.13 \\ 0.13 & 0.0 & 0.0 \end{bmatrix} \Delta_{\nu}^{+} \right\} \wedge 0.13$$

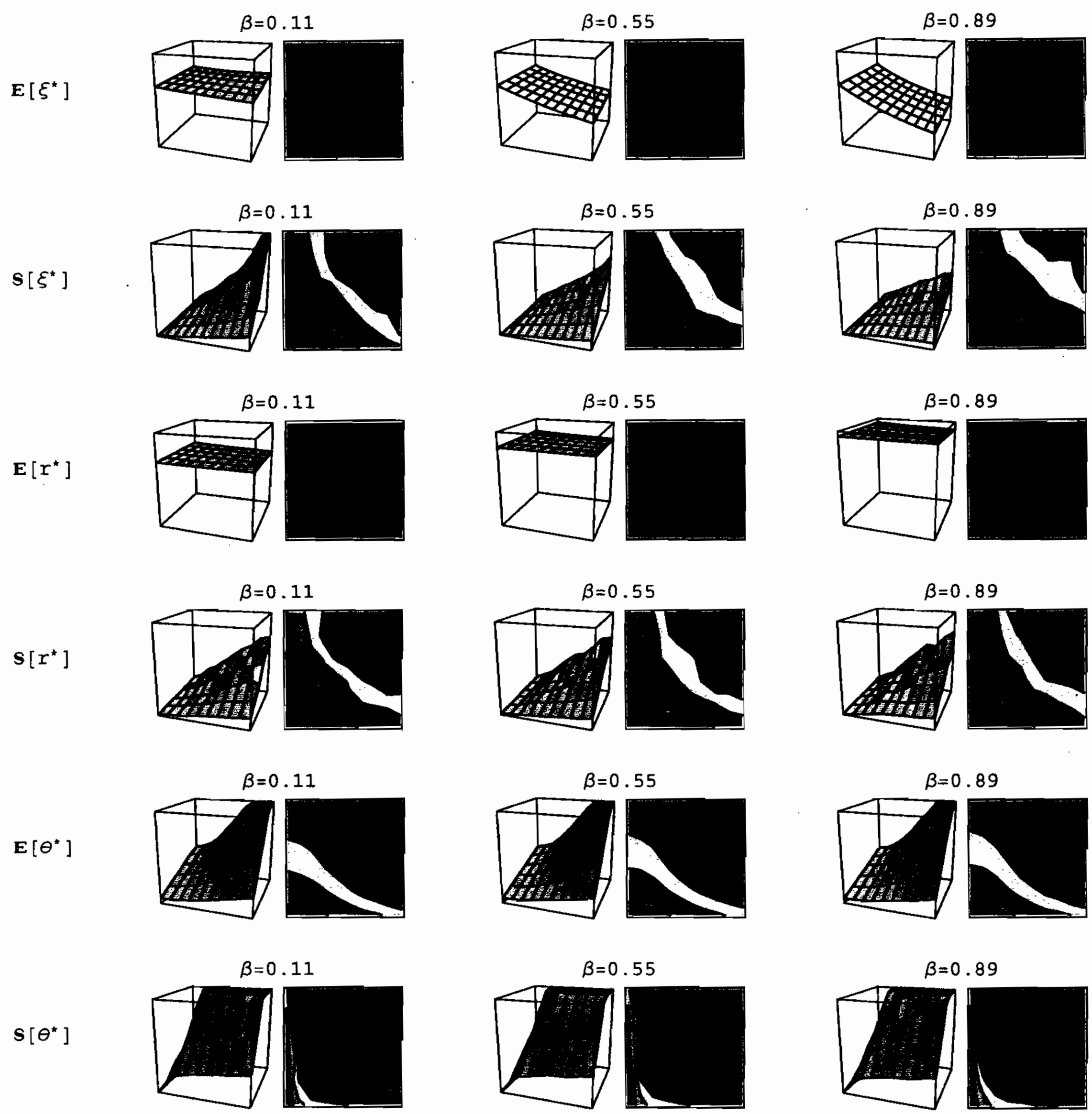


Figure 3.21: Plots of the unconditional moments of state prices, real interest rates, and market prices of risk

$$\chi = \left\{ \begin{bmatrix} 0.13 & 0.13 & 0.13 \\ 0.13 & 0.13 & 0.13 \\ 0.13 & 0.13 & 0.13 \end{bmatrix} \Delta_\nu^+ \right\} \wedge 0.13$$

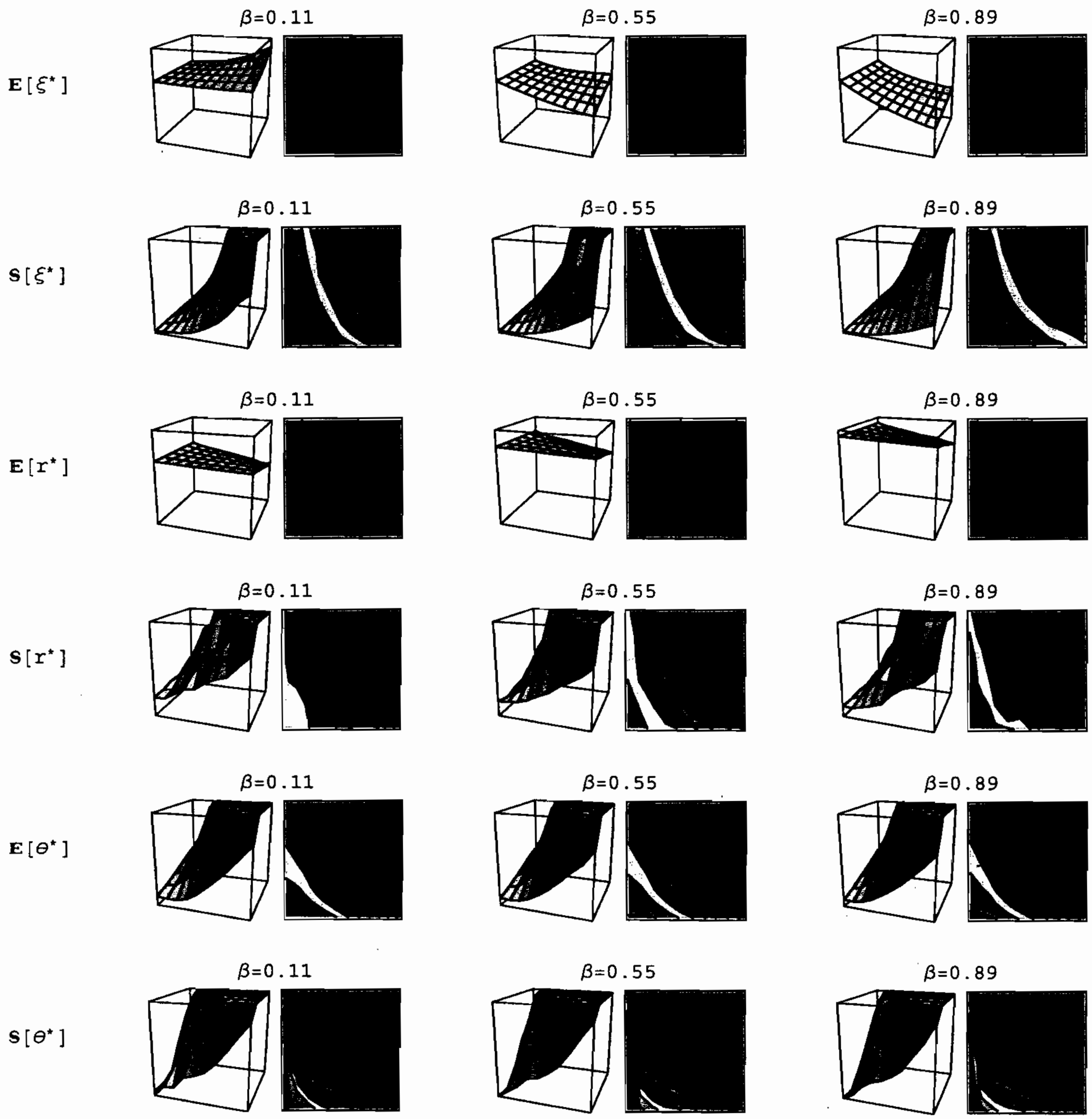


Figure 3.22: Plots of the unconditional moments of state prices, real interest rates, and market prices of risk

$$\chi = \left\{ \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 1.0 & 1.0 & 1.0 \\ 1.0 & 1.0 & 1.0 \end{bmatrix} \Delta_\nu^+ \right\} \wedge 0.13$$

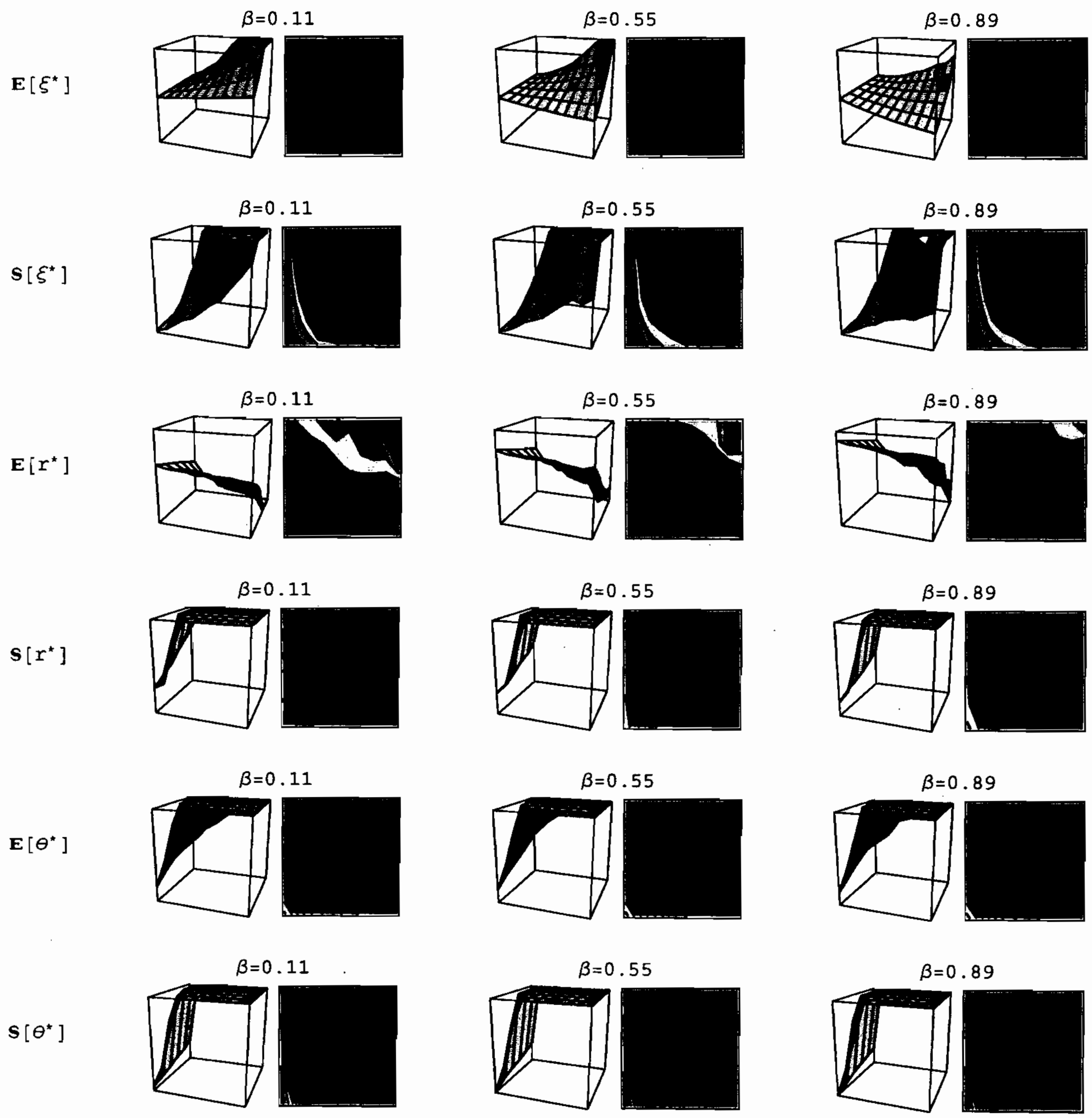


Figure 3.23: Plots of the unconditional moments of state prices, real interest rates, and market prices of risk

## 3.5 Conclusions

To summarize, we offer some interpretation of the plots just presented and indicate directions of future work that seem promising.

In Figures (3.8) and (3.9) we see clearly that our simulation procedure is reproducing known theoretical results extremely well. Note that $\mathbf{E}[\xi_t^*]$ is correctly decreasing in time at an exponential rate; if interest rates are positive then $\xi^*$ ought to be a supermartingale. We also see the correct power law behavior as $\alpha$ varies. $\mathbf{S}[\xi_t^*]$ varies as it should with $\alpha$ and time. Interest rates are constant, and at levels that vary correctly with the subjective utility discounting factors $\beta$. $\mathbf{E}[\theta_t^*]$ correctly increases linearly with $\alpha$ and is constant in time. $\mathbf{S}[\theta_t^*]$ is very close to zero, although there is some estimation error. Considering that the vertical axis for $\mathbf{S}[\theta_t^*]$ has range $[0.0, 0.02]$, this estimation noise represents a very small deviation from zero. We note here that the simulation of $\theta^*$ proved to be the most difficult in terms of accuracy and precision here and in later simulations.

In Figures (3.10) and (3.13), we have a constant and diagonal $\boldsymbol{\chi}$. This is a multi-agent version of the model studied in Sundaresan (1989), Constantinides (1990), Detemple & Zapatero (1991), and Detemple & Zapatero (1992) with only internal habit formation. First we note that if the subject discount rate $\beta$ is not large enough then $\xi^*$ is not a supermartingale and interest rates are negative. $\xi^*$ now has a small dependence on $\alpha$. Interest rates and prices of risk appear to have a very small dependence on time and on risk aversion. All other plots are very similar to those in the diagnostics.

Figures (3.11) and (3.14) are the cases with zero diagonals and constant, non-zero off-diagonals; agent 1 is influenced by agent 2 and agent 2 is influenced by agent 1. A feedback loop of only external habit formation. We see that the behavior is very similar to that in Figures (3.10) and (3.13), although the effect is weaker.

In Figures (3.12) and (3.15), we have the combined effect of the diagonals and off-diagonals. One can see that the behavior is similar to all prior plots but note that they exhibit much more sensitivity to the level of risk aversion. Figure (3.15) shows clearly that if we increase agents' impatience and set risk aversion around 1.5, then these off setting effects will leave interest rates at realistic interest levels with low volatility and supermartingale state prices density with high volatility. It is this property that account for the interest in habit formation models.

It also appears that the effect of combining the diagonal and off-diagonal effects is not just additive.

In Figure (3.16), we have a 3 agent case in which all have an internal component but the external component has the following form: agent 1 is influenced by agent 2, and agent 2 is influenced by agent 3, and, agent 3 is influenced by agent 1. The behavior is very similar to that of Figure (3.15).

In considering the cases of non-constant $\chi$, it is instructive to compare the three Figures (3.8), (3.12) and (3.17) to each other. Similarly for Figures (3.8), (3.15) and (3.18). Thus, we use Figures (3.8) as a benchmark to measure changes in behavior.

Comparing Figures (3.8) and (3.17), we see that the behavior of $\mathbf{E}[\xi^*]$ and $\mathbf{E}[r^*]$ are essentially the same. There is a small increase in $\mathbf{S}[\xi^*]$ and $\mathbf{S}[r^*]$ when the non-constant $\chi$ is introduced but nothing like the large increases observed in $\mathbf{E}[\theta^*]$ and $\mathbf{S}[\theta^*]$. However, in Figure (3.12), we see qualitatively different behavior to that in (3.17): $\mathbf{E}[\theta^*]$ and $\mathbf{S}[\theta^*]$ are much less affected whereas as there is a small increase in $\mathbf{S}[\xi^*]$ and dramatic changes in the behavior of $\mathbf{E}[\xi^*]$ and $\mathbf{E}[r^*]$. A similar comparison can be made with Figures (3.8), (3.15) and (3.18), although with generally enhanced effects.

In Figure (3.19) and, particularly in (3.20), we observe some intriguing behavior in the price of risk as well as in the standard deviation of $\xi^*$ and $r^*$. $\mathbf{S}[\theta^*]$ appears to be exhibiting a cyclical behavior in time. On a much smaller scale, so too does $\mathbf{E}[\theta^*]$, for risk aversion near 2 and patience 0.11. One sees a related behavior in $\mathbf{S}[\xi^*]$ and $\mathbf{S}[r^*]$. These graphs indicate that this model may have the potential to generate cycles in the price of risk even though the rate of endowment growth is a martingale. This behavior is similar to that observed in the simulations in Campbell & Cochrane (1999). In our simulations, the equity premium is $\mu^* - r^* = \sigma\theta^*$ where $\sigma$ is a constant so is proportional to $\theta^*$. These cyclical dynamics for the equity premium were observed in the empirical results of De Santis (2004) and definitely deserve further study since they arise in our model quite naturally from a specific form of habits. However, this cyclical effect on the equity premium appears to be washed out or dominated by some other effect when one looks at Figures (3.21), (3.22) and (3.23) which are similar models but with three agents and increasing habit intensities.

We conclude from these results that the effect of interaction is to enhance the dynamics observed with no interaction; in most cases, adding interaction does not result in significantly new behavior. However, Figures (3.19) through to (3.23) seem to indicate that there is a scale effect: some parameter regimes reveal an interesting cyclically varying equity premium and others do not. Whether or not the cyclical behavior is due in whole or in part to interactions is unclear at this stage and more experimentation is required. These preliminary results suggest that further experimentation may be very useful; we need to consider other forms of $\chi$, based on other economic assumptions about agent behavior, as well as exploring cases of dynamic and heterogeneous risk aversion and patience.

With these numerical results combined with our inability to find counter-examples to the unresolved theoretical problems of inversion and uniqueness of equilibrium, it also appears worthwhile to simultaneously pursue these theoretical issues. If solved, we could then explore more 2-agent cases in which the assumption that $\chi$ has identical diagonals and identical column sums has been dropped. Additional directions to pursue would be to increase the number of agents to see if these effects persist for larger groups of agents, first by simulation and then perhaps theoretically by returning to the originally considered approach via mean-field analysis and taking the mean-field limit as the number of agents increases to infinity. More broadly, many financial and economic questions can be addressed within the above model, with minor modifications. With the simulation program now in place, many of the models not yet studied numerically can be examined in greater detail with the potential of revealing dynamics that were not apparent from a purely analytical treatment. Finally, the Nash Equilibrium flavor of these results indicates that directly incorporating game-theoretic features into this standard model can be feasible under certain circumstances and, judging by the literature, is an avenue along which few have gone and should therefore be considered more carefully.

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