AMALGAMATION OF G-METRIC SPACES AND HOMEOMORPHS OF $(2^\alpha)_\alpha$

The Amalgamation Property for G-metric Spaces
and Homeomorphs of the Space $(2^{\alpha})_{\alpha}$

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Summary

Two main results (I and II below) are proved in this thesis and some applications are considered.

I. The amalgamation property fails for all classes of G-metric spaces unless the ordered group G is $\mathbb{Z}$ or $\mathbb{R}$. In particular, the class of $\mathbb{Q}$-metric spaces does not have the amalgamation property.

II. The space $(2^{\alpha})_{\alpha}$, for all regular cardinals α, is homeomorphic to a P_{α}-space with no isolated points that has an α-subbasis $\mathcal{G} = \bigcup_{\xi<\alpha} \mathcal{G}_{\xi}$ such that

i) for each $\xi < \alpha$, $\mathcal{G}_{\xi}$ is a discrete open cover;

ii) either

(a) for some $\beta<\alpha$, $|\mathcal{G}_{\xi}| \leq 2^{\beta}$ for all $\xi < \alpha$; or

(b) $|\mathcal{G}_{\xi}| < \alpha$ for all $\xi < \alpha$; or still,

(c) the cardinal α is strongly inaccessible but not weakly compact and $|\mathcal{G}_{\xi}| \leq \alpha$ for all $\xi < \alpha$; and

iii) every subfamily of $\mathcal{G}$ with the finite intersection property has non-void intersection.

Applications include the identification (up to homeomorphism) of $(\Lambda(\alpha))_{\alpha}$, $(S_{\alpha})_{\alpha}$ and $(2^{\alpha})_{\alpha}$, for all infinite α, with $\alpha = \alpha^{\underline{\alpha}}$ and of $(U(\alpha))_{\alpha^+}$ (where $U(\alpha)$ is the space of uniform ultrafilters on α), for all infinite α, with $\alpha^{+} = 2^{\alpha}$. Characterizations of weakly compact cardinals and of cardinals for which $\alpha = \alpha^{\underline{\alpha}}$ are also given.

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by

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INTRODUCTION

This thesis consists of two parts. The purpose of the first part (Chapter II) is to prove that the amalgamation property fails for the class of all G-metric spaces, for every (totally) ordered (abelian) group G which is not equal to the additive group of the integers or that of the real numbers.

The amalgamation property, in its abstract form, was first formulated by Fraïssé [9] in connection with embedding problems. It has been studied by Jónsson [15], [16], [17] and [18], and Morley and Vaught [24] in connection with the general theory of homogeneous-universal structures in Jónsson classes.

Not too many examples of classes of relational systems are known for which the amalgamation property fails. Among them the following are included: (a) the class of semi-groups and hence of rings, (Kimura [22], Jónsson [15], [16], Howie [14]); (b) the class of modular lattices, (Jónsson [19]); (c) the class of ℓ-groups, (Pierce [31]). To these, because of our theorem, we can now add a whole family of such examples, among which is the class of all metric spaces on which the metric takes on only rational values.

Classes of metric spaces were studied by Urysohn [39], [40] and Sierpiński [33], [34] from the point of view of the existence of universal spaces; thus, Urysohn proved the existence of a universal separable, complete and ω-homogeneous metric space, unique up to isometries; while Sierpiński proved that for each cardinal α such that $2^{\omega} \leq \alpha = 2^{\underset{\smile}{\alpha}}$ $(= \Sigma\{2^{\beta}: \beta<\alpha\})$, there exists a universal metric space of cardinality α. It was implicitly proved by Sierpiński in [34], as pointed out by Morley and Vaught [24], that the class of all metric spaces has the amalgamation property, (and in fact forms a Jónsson class).

The notion of a metric has been generalized by replacing the additive group of real numbers by an arbitrary (totally) ordered (abelian) group G, (cf. e.g. Hausdorff [13], Cohen and Goffman [1], [2] and Sikorski [37]), and it is easy to see that the result of Sierpiński, and Morley-Vaught on the amalgamation property extends to G-metric spaces whenever G is order complete. The main result in Chapter II is the converse statement.

The second part (Chapter III) concerns the space $(2^{\alpha})_{\alpha}$, for infinite regular cardinals α. For any topological space X and any infinite cardinal α, we write X_{α} for the set X with the smallest P_{α}-topology (i.e. a topology closed under intersections of fewer than α elements) containing the topology of X. Thus by $(2^{\alpha})_{\alpha}$ we mean the cartesian product of α copies of the discrete space $\{0, 1\}$ with the smallest P_{α}-topology containing the usual product topology. We have some topological characterizations of the space $(2^{\alpha})_{\alpha}$ as follows.

For every infinite regular cardinal α, the space $(2^{\alpha})_{\alpha}$ is a P_{α}-space (i.e. a $T_{3\frac{1}{2}}$-space with some P_{α}-topology) with no isolated points which has an α-subbasis $\mathcal{G}$ of the form $\mathcal{G} = \bigcup_{\xi<\alpha} \mathcal{G}_{\xi}$ such that

i) for each $\xi<\alpha$, $\mathcal{G}_{\xi}$ is a discrete open cover of the space;

ii) either

a) for some $\beta<\alpha$, $|\mathcal{G}_{\xi}| \leq 2^{\beta}$ for all $\xi<\alpha$, or

b) $|\mathcal{G}_{\xi}|<\alpha$ for all $\xi<\alpha$, or still,

c) the cardinal α is strongly inaccessible but not weakly compact and $|\mathcal{G}_{\xi}| \leq \alpha$ for all $\xi<\alpha$; and

iii) every subfamily of $\mathcal{G}$ with the finite intersection property has non-void intersection.

This result is a generalization of a theorem of Comfort and Negrepontis in [3] which treats the case when α is the first uncountable cardinal.

Applications of this result are in two categories. First, we deduce from the main result the following statement (Theorem 3.13), which is also a generalization of a theorem of Comfort and Negrepontis in [3].

For a compact Hausdorff space X and an uncountable regular cardinal α, X_{α} is homeomorphic to the space $(2^{\alpha})_{\alpha}$, if

i) the cardinal number of continuous real-valued functions on X is α, and ii) no intersection of a family of fewer than α open sets of X is a singleton. The converse is true if α is such that $\alpha = \alpha^{\underline{\alpha}}$.

Denoting by $U(\alpha)$ the space of all uniform ultrafilters on α with the topology it inherits from that of the Stone-Čech compactification of α, we have the homeomorphism of the spaces $(U(\alpha))_{\alpha^+}$ and $(2^{\alpha^+})_{\alpha^+}$, for all infinite cardinals α such that $\alpha^+ = 2^\alpha$. Denoting by $\Lambda(\alpha)$ the space 2^α with the lexicographic order topology, we have the homeomorphism of the spaces $(\Lambda(\alpha))_\alpha$ and $(2^\alpha)_\alpha$, for all infinite cardinals α such that $\alpha = \alpha^{\underline{\alpha}}$. And, for all infinite cardinals α such that $\alpha = \alpha^{\underline{\alpha}}$, let us denote by S_α the Stone space of the α-homogeneous-universal Boolean algebra of cardinality α, whose existence is a consequence of results of Jónsson [15], [16], and Morley and Vaught [24]. Negrepontis mentioned in [25] without proof that the spaces $(\Lambda(\alpha))_\alpha$ and $(S_\alpha)_\alpha$ are homeomorphic. That result also follows from our Theorem 3.13.

Secondly we obtain a characterization of the weakly compact cardinals and a characterization of the cardinals α for which $\alpha = \alpha^{\underline{\alpha}}$. Thus, the space $(2^\alpha)_\alpha$ is homeomorphic to $(\alpha^\alpha)_\alpha$ if and only if α is not weakly compact. The space $(2^\alpha)_\alpha$ is homeomorphic to $(\gamma^\alpha)_\alpha$ for some $\gamma > \alpha$ if and only if α is such that $\alpha \neq \alpha^{\underline{\alpha}}$. The class of weakly compact cardinals has many characterizations which can be found in the work of Parovičenko [28], [29], [30], Erdös and Tarski [8], Keisler and Tarski [20], Hanf [12] and Monk and Scott [23], and are all collected together in a forthcoming book by Comfort and Negrepontis [6]. A number of conditions equivalent to the condition $\alpha^{\alpha} = \alpha$ can be found in a forthcoming work of Comfort and Negrepontis [5].

There is some connection between the two parts of the thesis in that the space $(2^{\alpha})_{\alpha}$ is a G_{α}-metric space, where G_{α} is the least ordered algebraic field containing the cardinal α, and has been studied from this point of view by Sikorski ([37]).

Chapter I contains all relevant preliminaries, with no new material. The main results in Chapters II and III are to the best of our knowledge original.

CHAPTER I: Preliminaries

In this chapter, we mention those notions and known results required for the exposition of our results. We also list the notations and conventions that we have adopted (and which may in some instances differ from those used by other authors).

This chapter is divided into parts A and B, relevant to Chapter II and Chapter III, respectively.

A. G-metric Spaces

1.1. Definition. A (totally) ordered (abelian) group is a triple $(G, +, <)$ such that $(G, +)$ is an (additive) abelian group, $(G, <)$ is a (totally) ordered set and such that if $a < b$, then $a+c < b+c$, for all $a, b, c \in G$.

We have the following conventions in this connection:

i) by $\leq$, we mean $<$ or $=$,

ii) sometimes, we write G for $(G, +, <)$ for convenience, even though strictly speaking G is only a set, and

iii) by G^+ we mean $\{x \in G: x \geq 0\}$.

1.2. Definitions. Two positive elements x, y of an ordered group are relatively archimedean if there are positive integers m, n such that $mx \geq y$ and $ny \geq x$.

If every two positive elements of an ordered group are relatively archimedean, then the ordered group is archimedean.

1.3. Notations. $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ denote the ordered group of integers, rational and real numbers respectively.

1.4. Proposition. Every archimedean ordered group is isomorphic to an ordered subgroup of $\mathbb{R}$.

(For a proof, see e.g. Theorem 8.12 of Rudin [32].)

1.5. Definition. A subgroup F of an ordered group G is convex, if whenever $x \in G$, $y \in F$ and $0 \leq x \leq y$, it follows $x \in F$.

1.6. Proposition. Let F be a convex subgroup of an ordered group G. The quotient group G/F can be made into an ordered group according to the following definition. The element of G/F that contains $a \in G$, $F(a)$, is ≥ 0 if there exists $x \in G^+$ such that $a \equiv x (\mathrm{mod}\, F)$.

1.7. Definition. A (totally) ordered set $(X, <)$ is order complete if and only if each non-void subset of X which has an upper bound has a least upper bound (i.e. a supremum).

1.8. Proposition. In an archimedean ordered group that is not order complete, 0 cannot be isolated. In fact, if $x > 0$ then there is $x' > 0$ such that $x \geq 2x'$.

Proof. If 0 is isolated, there is a smallest element to be called 1 and the group is isomorphic to $\mathbb{Z}$ which however is order complete. Therefore 0 cannot be isolated. If $x>0$, there is x_1 such that $x>x_1>0$ and one can let $x' = \min(x_1, x-x_1)$.

1.9. Definitions. Given an oredered group G, if there exist non-empty subsets X, Y of the set G, such that

i) $X \cup Y = G$,

ii) $x<y$ for all $x \in X$ and $y \in Y$,

iii) X has no last and Y no first element;

then there is said to be a Dedekind cut $X|Y$ in G.

A Dedekind cut $X|Y$ is positive if $0 \in X$.

1.10. Proposition. Every ordered group that is not order complete has a positive Dedekind cut.

1.11. Proposition. If $X|Y$ is a positive Dedekind cut in an archimedean ordered group, then for all $z>0$, there exist $x \in X$ and $y \in Y$ such that $y-x \le z$.

Proof. Let n be the smallest integer such that $nz \in Y$. One can let $x=(n-1)z$ and $y=nz$.

1.12. Definitions. Given an ordered group G, a G-metric on a set X is a mapping $\rho : X^2 \to G$ such that, for all $x, y, z \in X$,

i) $\rho(x, y) = \rho(y, x)$,

ii) $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$,

iii) $\rho(x, y) \geq 0$, $\rho(x, x) = 0$, and

iv) if $\rho(x, y) = 0$, then $x = y$.

A mapping ρ satisfying i), ii) and iii) is called a G-pseudo metric on X.

A G-(pseudo) metric space is a pair (X, ρ), where ρ is a G-(pseudo) metric on X. Sometimes we write X for (X, ρ).

Let (X, ρ) and (Y, σ) be G-(pseudo) metric spaces for some ordered group G. We say Y is a subspace of X if $Y \subset X$ and if $\sigma = \rho | Y^2$, the restriction of ρ to Y^2. If there is a mapping f from Y into X such that for all $x, y \in Y$, $\sigma(x, y) = \rho(f(x), f(y))$, then we say f is a G-isometry.

An $\mathbb{R}$-metric space is a metric space, an $\mathbb{R}$-isometry an isometry.

G-metric spaces have been studied by Hausdorff [13], Cohen and Goffman [1], [2], and Sikorski [37], among others.

1.13. Proposition. Let (X, ρ) be a G-pseudometric space for some ordered group G, let $\tilde{x} = \{y : \rho(y, x) = 0\}$ for all $x \in X$, let $\tilde{X}$ be $\{\tilde{x} : x \in X\}$, and let $\tilde{\rho}$ be a G-metric for $\tilde{X}$ such that for members A and B of $\tilde{X}$, $\tilde{\rho}(A, B) = \rho(a, b)$ for some $a \in A$ and $b \in B$. Then the quotient map π of X onto $\tilde{X}$ is a G-isometry.

1.14. Definitions. Given an arbitrary set I, let t be a function from I into ω the set of positive integers. A system, $\langle A, R_i \rangle_{i \in I}$, formed by a non-empty set A and t(i)-ary relations R_i over A is a relational system, having similarity type t, index set I. Relational systems are similar if they have the same similarity type.

Let $\mathfrak{A} = \langle A, R_i \rangle_{i \in I}$ and $\mathfrak{B} = \langle B, S_i \rangle_{i \in I}$ be similar relational systems of similarity type t. We say that $\mathfrak{B}$ is a subsystem of $\mathfrak{A}$, denoted by $\mathfrak{B} \subset \mathfrak{A}$, if $B \subset A$ and $S_i = R_i \cap B^{t(i)}$ for all $i \in I$. If there is a one-to-one mapping $f: B \to A$, and we let $f_{t(i)} : B^{t(i)} \to A^{t(i)}$ be given by $f_{t(i)}(b_0, \ldots, b_{t(i)-1}) = (f(b_0), \ldots, f(b_{t(i)-1}))$ for all $i \in I$; we say that f is an embedding of $\mathfrak{B}$ into $\mathfrak{A}$, denoted by $f: \mathfrak{B} \to \mathfrak{A}$, provided $\langle f[B], f_{t(i)}[S_i] \rangle_{i \in I}$ is a subsystem of $\mathfrak{A}$.

A class $\mathbb{K}$ of similar relational systems is said to have the amalgamation property, if given $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \mathbb{K}$ and given embeddings f, g of $\mathfrak{C}$ into $\mathfrak{A}, \mathfrak{B}$ respectively, there are $\mathfrak{D} \in \mathbb{K}$ and embeddings f', g' of $\mathfrak{A}, \mathfrak{B}$ respectively into $\mathfrak{D}$ such that $f' \circ f = g' \circ g$.

(Cf. Jónsson [15], [16], [18] and Morley and Vaught [24].)

1.15. Definitions and Conventions. Given an ordered group G and a non-empty set X, a G-metric ρ on X, like any function with range in G^+, can be written as $\bigcup\{\rho^{-1}(g) \times \{g\} : g \in G^+\}$, and therefore has the following equivalent definition. A G-metric on X is $\bigcup\{R_g \times \{g\} : g \in G^+\}$, where, for each $g \in G^+$, R_g is a binary relation on X, such that

i) $\bigcup\{R_g : g \in G^+\} = X^2$, and for all $x, y, z \in X$ and $f, g, h \in G^+$,

ii) if $R_f xy$, then $R_f yx$,

iii) if $R_f xy$, $R_g yz$ and $R_h xz$, then f, g, h as elements of G are such that $f+g \geq h$, and

iv) $R_0 xy$ if and only if $x=y$.

Observe that, from iii) and iv), it follows that $R_f \cap R_g = \phi$ if $f \neq g$.

It is then clear that a G-metric on X can be identified with a family $\{R_g : g \in G^+\}$ of binary relations such that i), ii), iii) and iv) are true; and a G-metric space X can be identified with a relational system $\langle X, R_g \rangle_{g \in G^+}$, having similarity type $t: G^+ \to \omega$, with $t(g) = 2$ for all $g \in G^+$, such that i), ii), iii) and iv) are true for the family of binary relations R_g indexed by G^+.

Let (X, ρ) and (Y, σ) be G-metric spaces for some ordered group G. (Y, σ) is a subspace of (X, ρ) if and only if $Y \subset X$ and $\sigma^{-1}(g) = \rho^{-1}(g) \cap Y^2$ for every $g \in G^+$. A mapping f from Y into X is a G-isometry of Y into X if and only if $f[Y] \subset X$ and $(f(x), f(y)) \in \rho^{-1}(g)$ for all $(x, y) \in \sigma^{-1}(g)$, i.e., if we let $f_2 : Y^2 \to X^2$ be given by $f_2(x, y) = (f(x), f(y))$; $(f[Y], \cup\{f_2[\sigma^{-1}(g)] \times \{g\} : g \in G^+\})$, is a subspace of X. Therefore, a subspace Y of X is a subsystem of the relational system X, G-isometries are embeddings of relational systems, and the amalgamation property for the class of all G-metric spaces for a given G has the following equivalent formulation, which is the one to be used in Chapter II.

For a given ordered group G, the class $\mathbb{K}_G$ of all G-metric spaces has the amalgamation property if given $A, B, C \in \mathbb{K}_G$ and given G-isometries f, g of C into A, B respectively there are $D \in \mathbb{K}_G$ and G-isometries f', g' of A, B respectively into D such that $f' \circ f = g' \circ g$.

(In the above, a G-metric space is identified with a relational system with G^+ as the index set. A relational system with an index set that is a dense subset of G^+ (with respect to the order of G) can also be used, which however necessitates a more complicated set of axioms that is to be satisfied by the relational systems identified with G-metric spaces, when G is not order complete. Such a relational system is in fact given in Morley and Vaught [24] for $G = \mathbb{R}$, and the positive rationals as the dense subset of $\mathbb{R}^+$.)

B. Spaces of Ultrafilters

1.16. Definitions and Notations. The axiom of choice is assumed. Each ordinal is the set of all smaller ordinals. Thus, the condition $\xi < \zeta$ is equivalent to the condition $\xi \in \zeta$. Nevertheless, we shall make the notational distinction between the first ordinal 0 and the empty set ϕ. Ordinal numbers are denoted by $\xi, \eta, \zeta, \mu, \nu$ and λ. Ordinal sums and products are assumed to be known and are denoted by $\xi+\eta$, $\xi \cdot \eta$ respectively. For any $\zeta \leq \xi$, we write $\xi - \zeta$ for the unique ordinal η such that $\xi = \zeta + \eta$.

A cardinal number is an initial ordinal. Cardinals are denoted by α, β, γ and $\varkappa$. The first infinite cardinal is ω. The least cardinal greater than α is denoted by α^+. A cardinal α is a limit cardinal if $\alpha \neq \beta^+$ for any β, and, α is regular if it is not equal to the sum of fewer than α cardinals each smaller than α. We denote by α^{β} the set of all mappings from β to α and, sometimes, the cardinal number of that set. A cardinal α is strongly inaccessible if it is regular and if $2^{\beta} < \alpha$ whenever $\beta < \alpha$. We denote by $\alpha^{\underline{\beta}}$ the cardinal $\Sigma\{\alpha^{\gamma} : \gamma < \beta\}$. If α is strongly inaccessible then $\alpha^{\underline{\alpha}} = 2^{\underline{\alpha}} = \alpha$; and, $\alpha^{\underline{\alpha}} = \alpha$ if and only if $2^{\underline{\alpha}} = \alpha$ and α is regular. The cardinality of a set A is denoted by $|A|$.

By the Generalized Continuum Hypothesis, we mean the statement that $\alpha^{+} = 2^{\alpha}$ for all infinite cardinals α.

(With some variations, material in this section can be found in Sierpiński [36].)

1.17. Definitions. A filter $\mathfrak{F}$ on a non-empty set X is a family of subsets of X which has the following properties:

i) the empty set is not in $\mathfrak{F}$,

ii) every subset of X which contains a member of $\mathfrak{F}$ belongs to $\mathfrak{F}$,

and

iii) every finite intersection of members of $\mathfrak{F}$ belongs to $\mathfrak{F}$.

A filter on a non-empty set not properly contained in any other filter on the same set is an ultrafilter. Ultrafilters are usually denoted by p, q.

A principal ultrafilter p on a non-empty set is an ultrafilter containing a singleton, or equivalently, is one such that $\bigcap p \neq \emptyset$.

A filter $\mathfrak{F}$ is α-complete if $\cap\mathcal{G}\in\mathfrak{F}$ whenever $\mathcal{G}\subset\mathfrak{F}$ and $|\mathcal{G}|<\alpha$.

A filter $\mathfrak{F}$ on X is uniform if $|F| = |X|$ for all $F\in\mathfrak{F}$.

1.18. Definitions. An infinite cardinal α is measurable if there is an α-complete, non-principal ultrafilter on α.

An infinite cardinal α is strongly measurable if α is regular and every α-complete filter on α can be extended to an α-complete ultrafilter on α.

Obviously a strongly measurable cardinal is measurable.

(The class of all non-measurable cardinals is denoted C_1 and that of all strongly non-measurable cardinals denoted C_1^* in Keisler and Tarski [20]. See also definitions of C_1 and C_1^* in Comfort and Negrepontis [4].)

1.19. Definitions. A topological space X is a T_1-space if for every $x\in X$, the singleton $\{x\}$ is closed.

A T_1-space X is a $T_{3\frac{1}{2}}$-space (or a completely regular space), if for every $x\in X$ and every open set A containing x, there exists a continuous real-valued function f on X such that $f(x) = 0$ and $f(y) = 1$ for all $y\in X\sim A$.

1.20. Definition. Given a topological space X, a family $\mathcal{A}$ of subsets of X is discrete if every point of X has a neighborhood that intersects at most one member of $\mathcal{A}$.

(Cf. e.g. Kelley [21] in connection with metrization, p.127.)

1.21. Notations. Given a function f on a set X into a set Y, a subset A of X and an element $y \in Y$, $f[A]$ denotes $\{f(x): x \in A\}$, $f^{-1}(y)$ denotes $\{x \in X, f(x) = y\}$, $f|A$ denotes the restriction of f to A and $cl_X A$ denotes the closure of A in X. Sometimes we write f_x for f(x).

1.22. Definition. Given an infinite cardinal α, a $T_{3\frac{1}{2}}$-space X is a P_α-space if any intersection of fewer than α open sets of X is open; a topology is a P_α-space topology if it is closed under intersection of less than α members.

(Thus P_ω-spaces are ordinary $T_{3\frac{1}{2}}$-spaces. P-spaces as defined by Gillman and Henriksen in [10] are P_{ω^+}-spaces. P_α-spaces are called α-additive spaces in Sikorski [37], $T^{\mu}_{3\frac{1}{2}}$-spaces, if α is the μ-th infinite cardinal, in Parovičenko [26] and α-complete in Monk and Scott [23].)

1.23. Proposition. For all uncountable cardinals α, P_α-spaces are totally disconnected.

(This is given in (iv) of Sikorski [37].)

1.24. Definition. On a topological space X, for an infinite cardinal α, a family $\mathcal{G}$ of open sets is an α-subbasis for its topology if the family of all intersections of fewer than α members of $\mathcal{G}$ is a basis for the topology of X. A $T_{3\frac{1}{2}}$-space that has an α-subbasis for its topology is a P_α-space.

1.25. Definition. Given a topological space X and an infinite cardinal α, X_α denotes the set X with that topology for which the topology of the space X is an α-subbasis.

As sets, X and X_α are identical.

1.26. Definition. Given an infinite cardinal α, $\Lambda(\alpha)$ denotes the set 2^α, topologized with the lexicographic order topology.

(Cf. Sierpiński [35] for the definition of $\Lambda(\alpha)$. $\Lambda(\omega^+)$ and $(\Lambda(\omega^+))_{\omega^+}$ are denoted by Λ and Λ_π respectively in Comfort and Negrepontis [3].)

1.27. Definition. Given an infinite cardinal α, a Hausdorff space X is α-compact if each of its open covers admits a subcover by fewer than α elements..

(This definition can be found in Sikorski [37], Parovičenko [27] and Monk and Scott [23].)

1.28. Definition. An infinite cardinal α is weakly compact if $(2^\alpha)_\alpha$ is α-compact.

(This is equivalent to the usual definition in terms of the Boolean algebra representation problem, cf. Parovičenko [28], [29], [30], Erdös and Tarski [8], Keisler and Tarski [20], Hanf [12], Monk and Scott [23] and Comfort and Negrepontis [6].)

1.29. Proposition. If α is weakly compact, then α is strongly inaccessible.

(Cf. Monk and Scott [23].)

1.30 . Definition and Simple Facts. Given a $T_{3\frac{1}{2}}$-space X, βX denotes the Stone-Čech compactification of X, characterized by the following properties:

a) βX is a compact Hausdorff space

b) X is (homeomorphic with) a dense subspace of βX, and

c) X is C^*-embedded in βX, i.e. every bounded continuous real-valued function on X extends continuously to βX.

(The Stone-Čech compactification is defined and discussed in Chapter 6 of Gillman and Jerison [11].)

For any non-empty discrete space D, the Stone-Čech compactification βD of D can be regarded as the set of all ultrafilters on the set D, such that every element of D is identified with the principal ultrafilter consisting of all subsets of D containing itself, with the smallest topology generated by sets of the form $\{p \in \beta D : A \in p\}$ for some $A \subset D$.

1.31. Notations and Simple Facts. We sometimes use α to denote the discrete space of cardinality α.

Given a cardinal α, $U(\alpha)$ denotes the space of all uniform ultrafilters on α and $\Omega(\alpha)$ denotes the space of all α-complete non-principal ultrafilters on α, (both considered as subspaces of $\beta\alpha$). Clearly $\Omega(\alpha) \subset U(\alpha)$ for all α.

If $\omega \leq \alpha$, then $U(\alpha) \neq \phi$. Indeed, the family of complements of all subsets of α of cardinality less than α forms a filter which produces an element of

$U(\alpha)$. However, it is not known whether any uncountable cardinal exists such that $\Omega(\alpha) \neq \phi$, i.e. whether there exists any measurable cardinals besides ω. But $\Omega(\omega) = U(\omega) = \beta\omega \sim \omega$.

(The spaces $U(\alpha)$ and $\Omega(\alpha)$ are used in Comfort and Negrepontis [4] for the characterization of strongly measurable cardinals.)

1.32. Proposition. For all infinite cardinals α,

i) the space $\Omega(\alpha)$ is a P_α-space,

ii) if α is (strongly) measurable, then α is weakly compact and in particular strongly inaccessible,

iii) if α is strongly measurable, then $\Omega(\alpha)$ is α-compact.

(Items i) and iii) are derived from or contained in Lemmas 2.5 and 2.6 respectively of Comfort and Negrepontis [4]. Item ii) follows essentially from the results of Monk and Scott [23]. That (strongly) measurable cardinals are strongly inaccessible is essentially the classical result of Ulam [38] and Tarski, (cf.e.g. [8]).)

CHAPTER II: The Amalgamation Property for G-metric Spaces

As remarked in the Introduction, the class of all metric spaces (with isometries) satisfies the amalgamation property as a consequence of results by Sierpiński as pointed out by Morley and Vaught. The proof makes use of order completeness of the ordered group of real numbers. First of all, we notice that the same proof carries over to the classes of all G-metric spaces (with G-isometries), where G is not order complete. (However, the only such groups are $\mathbb{R}$ and $\mathbb{Z}$).

This part of the thesis is devoted to establishing the fact that these are the only instances of classes of G-metric spaces for which the amalgamation holds.

We remark on our proof. For any group G that is not order complete, we first define a subset C of G, making use of a Dedekind cut of G with a special property which is defined in 2.3 and whose existence is proved in Lemmas 2.4-2.6; and define on C a G-metric which is a modification of its natural G-metric structure as a subset of G. We then define two spaces A and B by adjoining to C in each case one single point, in such a way that the triangle inequality always fails in the triangle formed by these two points and a certain third point, when a G-metric is attempted on the set $A \cup B$.

We begin with some definitions and remarks on notations.

2.1. <u>Definition</u>. A positive Dedekind cut $X|Y$ will be called <u>archimedean</u> if there exist $x_1, x_2 \in X$ such that $x_1+x_2 \in Y$.

2.2. <u>Notations</u>. We shall write, for all subsets A of an ordered group G, $[A]$ for $\{x \in G: a<x$ for some $a \in A\}$ and $2A$ for $\{2a: a \in A\}$.

2.3. <u>Definition</u>. A positive Dedekind cut $X|Y$ will be called <u>quotient</u> if $G \sim [2Y] | [2Y]$ is also a Dedekind cut.

<u>Remark</u>. To show that $X|Y$ is quotient it suffices to show that $G \sim [2Y]$ has no last element.

The next three lemmas establish the existence of quotient positive Dedekind cuts in all ordered groups that are not order complete, and lead up to the main theorem (Theorem 2.7).

2.4. <u>Lemma</u>. Every archimedean ordered group that is not order complete has a quotient positive Dedekind cut.

Proof. Let G be an archimedean ordered group that is not order complete. Let $X|Y$ be a positive Dedekind cut. Let $M = \{z \in G: 2z \in X\}$, $N = \{z \in G: 2z \in Y\}$. We show that M has no last element. Suppose on the contrary that M has a last element m and $2m \in X$. Since X has no last element, there exists $x \in X$, $2m<x$, which can be assumed to be such that $2(x-2m)+2m \in X$. This however implies that $2((x-2m)+m) \in X$ and

m is not the last element of M. Similarly N has no first element. Thus $M|N$ is clearly a positive Dedekind cut.

Now $[2N] = Y$ from which it follows that $M|N$ is quotient. For, if not, there exist $y_1, y_2 \in Y \sim [2N]$, $y_1 < y_2$, and there exist $m \in M$ and $n \in N$ such that $2(n-m) \leq (y_2 - y_1)$, which is a contradiction since $2n > y_2 > y_1 > 2m$.

2.5. Lemma. Every non-archimedean ordered group has a non-archimedean positive Dedekind cut.

Proof. Let G be a non-archimedean ordered group. Let x, y, $0 < x < y$, be relatively non-archimedean. Let

$$X = \{z \in G\colon nz < y, \text{ for all integers } n\},$$

$$Y = \{z \in G\colon nz \geq y, \text{ for some integer } n\}.$$

Clearly X and Y are non-empty such that (i) $X \cup Y = G$, (ii) $a < b$ for all $a \in X$, $b \in Y$, and $0 \in X$. Further, X has no last element. For, if $x_0 \in X$ is the last element, then since $2x_0 > x_0 > 0$, it follows that $2x_0 \in Y$ and $x_0 \notin X$ by the definition of X. Also, Y has no first element. For, if $y_0 \in Y$ is the first element, $y_0 - x$, being smaller than y_0, is in X. But $(y_0 - x) + x \in Y$ which implies either $y_0 - x$ or x is in Y contrary to our assumption. Therefore $X|Y$ is a positive Dedekind cut. Its being non-archimedean is evident from its definition.

2.6. Lemma. Every non-archimedean ordered group has a quotient positive Dedekind cut.

Proof. Let G be a non-archimedean ordered group. Let $X|Y$ be a non-archimedean positive Dedekind cut. We shall prove it to be quotient. Let $F = \{g \in X: -g \in X\}$, F is a subgroup. For, if $f, g \in F$, then $\pm f, \pm g \in X$, $\pm f \pm g \in X$ ($X|Y$ being non-archimedean) and $\pm f \pm g \in F$. F is clearly convex. Consider $\tilde{G} = G/F$ which is an ordered group with the order induced in the usual way, (cf. 1.6). If the zero element in $\tilde{G}$ is isolated, then there exists a smallest positive element, denoted $\tilde{1}$, in $\tilde{G}$. Clearly $G \sim [2Y] = X \cup \tilde{1}$ which has no last element. If the zero element of $\tilde{G}$ is not isolated, then $[2Y] = Y$. For, otherwise there exist $y_0, y_1, y_2 \in Y \sim [2Y]$ such that $F(y_1) < F(y_2)$ and $F(y_0) < \min\{F(y_2) - F(y_1), F(y_1)\}$. But then $2y_0 \in Y \sim [2Y]$ which is a contradiction to the fact that $2y_0 \in [2Y]$. In either case $G \sim [2Y]$ has no last element, from which it follows that $X|Y$ is quotient.

We are now ready to state and prove the main result of this chapter.

2.7. Theorem. Let G be a (totally) ordered (abelian) group. The class $\mathbb{K}_G$ of all G-metric spaces satisfies the amalgamation property, if and only if G is either the ordered group of the integers or that of the real numbers.

Proof. As is well known (and is easy to prove), an ordered group G is order complete if and only if G is either the ordered group of the integers, $\mathbb{Z}$, or that of the real numbers, $\mathbb{R}$. As it was mentioned in the Introduction, it is known that the amalgamation property holds for $\mathbb{K}_{\mathbb{R}}$; the proof, which uses the order completeness of $\mathbb{R}$, carries over to every order complete group. We outline a proof of the statement that $\mathbb{K}_G$ satisfies the amalgamation property for every order complete group G.

Let A, B and C be any G-metric spaces with G-metrics α, β and γ respectively. Let there be G-isometries $f:C\to A$, $g:C\to B$. Let E be the disjoint union of the sets A and B. We define $\epsilon: E\times E\to G$ as follows. For $a, b\in E$,

$$\epsilon(a,b) = \begin{cases} \alpha(a,b) & \text{if } a,b\in A, \\ \beta(a,b) & \text{if } a,b\in B, \\ \inf_{c\in C}[\alpha(a,f(c))+\beta(b,g(c))] & \text{if } a\in A, b\in B. \end{cases}$$

It can be easily verified that ϵ is G-pseudometric by checking the triangle inequalities as follows.

Given any triangle in E, if all three vertices are in A (or in B), the inequality is clearly satisfied. The only other possibility is that two vertices a, b are in say A and c is in B. Then

$$\begin{cases} \epsilon(a, b) = \alpha(a, b), \\ \epsilon(b, c) = \inf_{d \in C}[\alpha(b, f(d)) + \beta(c, g(d))], \\ \epsilon(a, c) = \inf_{d \in C}[\alpha(a, f(d)) + \beta(c, g(d))]. \end{cases}$$

But

$$\begin{aligned} \alpha(a, b) &\leq \alpha(a, f(d)) + \alpha(f(d), f(d')) + \alpha(b, f(d')) \\ &= \alpha(a, f(d)) + \beta(g(d), g(d')) + \alpha(b, f(d')) \\ &\leq \alpha(a, f(d)) + \beta(c, g(d)) + \beta(c, g(d')) + \alpha(b, f(d')), \end{aligned}$$

for all $d, d' \in C$; and hence

$$\begin{aligned} \alpha(a, b) \leq\ & \inf_{d \in C}[\alpha(a, f(d)) + \beta(c, g(d))] \\ & + \inf_{d' \in C}[\alpha(b, f(d')) + \beta(c, g(d'))]. \end{aligned}$$

Therefore

$$\epsilon(a, b) \leq \epsilon(a, c) + \epsilon(b, c).$$

Also

$$\begin{aligned} \epsilon(b, c) &= \inf_{d \in C}[\alpha(b, f(d)) + \beta(c, g(d))] \\ &\leq \alpha(b, f(e)) + \beta(c, g(e)) \\ &\leq \alpha(a, b) + \alpha(a, f(e)) + \beta(c, g(e)), \end{aligned}$$

for all $e \in C$; and therefore

$$\epsilon(b, c) \leq \alpha(a, b) + \inf_{d \in C}[\alpha(a, f(d)) + \beta(c, g(d))]$$

or

$$\epsilon(b, c) \leq \epsilon(a, b) + \epsilon(a, c).$$

Similarly

$$\epsilon(a, c) \leq \epsilon(a, b) + \epsilon(b, c).$$

Now that (E, ϵ) is a G-pseudometric space, we can get a G-metric space (D, δ) out of it in the usual way (by identifying points of zero ϵ-distance as in 1.13). The G-metric space (D, δ), together with the G-isometries resulting from the natural embeddings of A, B into E satisfies the amalgamation property.

For the converse, let G be an ordered group that is not order complete. Let $X|Y$ be a quotient positive Dedekind cut which exists according to Lemmas 2.4 and 2.6. We are to construct three G-metric spaces (A, α), (B, β) and (C, γ) together with G-isometries $f: C \to A$, $g: C \to B$ such that there does not exist a G-metric space (D, δ) with G-isometries f', g' from A and B respectively into it such that $f' \circ f \doteq g' \circ g$.

Choose an arbitrary $y_0 \in Y$. Let $Z = \{-2y_0 - z : z \in G \sim [2Y], z \geq 0\}$. Let $C = Y \dot{\cup} Z$ (the disjoint union of Y and Z), and define the G-metric γ on C as follows:

$$\gamma(a, b) = \begin{cases} |a-b|, & \text{if } a, b \in Y \text{ or } a, b \in Z, \\ 2y_0 + a, & \text{if } a \in Y, b \in Z. \end{cases}$$

Let $A = \{0_A\} \dot{\cup} C$, and define the G-metric α on A as follows:

$$\alpha(a, b) = \begin{cases} \gamma(a, b), & \text{if } a, b \in C \\ b, & \text{if } a = 0_A, b \in Y \\ 2y_0 & \text{if } a = 0_A, b \in Z. \end{cases}$$

Let $B = \{0_B\} \dot{\cup} C$, and define the G-metric β on B as follows:

$$\beta(a, b) = \begin{cases} \gamma(a, b), & \text{if } a, b \in C \\ b, & \text{if } a = 0_B,\ b \in Y \\ -b, & \text{if } a = 0_B,\ b \in Z. \end{cases}$$

Let f, g be the natural embeddings of C into A, B respectively. We can verify that A, B, C are indeed G-metric spaces by checking the triangle inequality in detail as follows:

1) γ is a G-metric on C. For, given any triangle abc on C, there are only 4 cases that are fundamentally different.

i) $a, b, c \in Y$. The triangle inequality is evidently satisfied as γ restricted to Y is the only natural G-metric on Y.

ii) $a, b, c \in Z$. Same conclusion for the same reason.

iii) $a, b \in Y$, $c \in Z$, $a > b$. We have

$$\begin{cases} \gamma(a, b) = (a-b), \\ \gamma(b, c) = 2y_0 + b, \\ \gamma(c, a) = 2y_0 + a, \text{ thus} \end{cases}$$

$$\begin{cases} \gamma(c, a) + \gamma(b, c) = 4y_0 + a + b > a - b = \gamma(a, b), \\ \gamma(c, a) + \gamma(a, b) > \gamma(c, a) > \gamma(b, c), \\ \gamma(a, b) + \gamma(b, c) = \gamma(c, a). \end{cases}$$

iv) $a \in Y$, $b, c \in Z$, $b < c$. We have

$$\begin{cases} \gamma(a, b) = 2y_0 + a, \\ \gamma(b, c) = (c-b), \\ \gamma(c, a) = 2y_0 + a, \text{ and thus} \end{cases}$$

$$\begin{cases} \gamma(c,a)+\gamma(b,c) > \gamma(c,a) = \gamma(a,b), \\ \gamma(c,a)+\gamma(a,b) > 2a > (c-b) = \gamma(b,c), \text{ as } 2a \in 2Y, \\ \gamma(a,b)+\gamma(b,c) > \gamma(a,b) = \gamma(c,a). \end{cases}$$

2) α is a G-metric on A. For, given any triangle abc on A, there are only 4 cases that are fundamentally different.

i) $a, b, c \in C$. The triangle inequality is evidently satisfied as $\alpha(a,b) = \gamma(a,b)$, $\alpha(b,c) = \gamma(b,c)$ and $\alpha(c,a) = \gamma(c,a)$.

ii) $a = 0_A$, $b, c \in Y$, $b > c$. We have

$$\begin{cases} \alpha(a,b) = b, \\ \alpha(b,c) = b-c, \\ \alpha(c,a) = c, \text{ and thus} \end{cases}$$

$$\begin{cases} \alpha(c,a)+\alpha(b,c) = b = \alpha(a,b), \\ \alpha(c,a)+\alpha(a,b) > b > b-c = \alpha(b,c), \\ \alpha(a,b)+\alpha(b,c) > b > c = \alpha(c,a). \end{cases}$$

iii) $a = 0_A$, $b, c \in Z$, $b > c$. We have

$$\begin{cases} \alpha(a,b) = 2y_0, \\ \alpha(b,c) = b-c, \\ \alpha(c,a) = 2y_0, \text{ and thus} \end{cases}$$

$$\begin{cases} \alpha(c,a)+\alpha(b,c) > 2y_0 = \alpha(a,b), \\ \alpha(c,a)+\alpha(a,b) > 2y_0 > b-c = \alpha(b,c) \text{ as } 2y_0 \in 2Y, \\ \alpha(a,b)+\alpha(b,c) > 2y_0 = \alpha(c,a). \end{cases}$$

iv) $a = 0_A$, $b \in Y, c \in Z$. We have

$$\begin{cases} \alpha(a, b) = b, \\ \alpha(b, c) = 2y_0 + b, \\ \alpha(c, a) = 2y_0, \text{ and thus} \end{cases}$$

$$\begin{cases} \alpha(c, a)+\alpha(b, c) > b = \alpha(a, b), \\ \alpha(c, a)+\alpha(a, b) = 2y_0+b = \alpha(b, c), \\ \alpha(a, b)+\alpha(b, c) > \alpha(b, c) > \alpha(c, a). \end{cases}$$

3) β is a G-metric on B. For, given any triangle abc on B, there are only 4 cases that are fundamentally different.

i) $a, b, c \in C$. The triangle inequality is evidently satisfied as

$\beta(a, b) = \gamma(a, b)$, $\beta(b, c) = \gamma(b, c)$ and $\beta(c, a) = \gamma(c, a)$.

ii) $a = 0_B$, $b, c \in Y$, $b > c$. We have

$$\begin{cases} \beta(a, b) = b, \\ \beta(b, c) = b-c, \\ \beta(c, a) = c, \text{ and thus} \end{cases}$$

$$\begin{cases} \beta(c, a)+\beta(b, c) = b = \beta(a, b), \\ \beta(c, a)+\beta(a, b) > b > b-c = \beta(b, c), \\ \beta(a, b)+\beta(b, c) > b > c = \beta(c, a). \end{cases}$$

iii) $a = 0_A$, $b, c \in Z$, $b > c$. We have

$$\begin{cases} \beta(a, b) = -b, \\ \beta(b, c) = b-c, \\ \beta(c, a) = -c, \text{ and thus} \end{cases}$$

$$\begin{cases} \beta(c,a)+\beta(b,c) > -c > -b = \beta(a,b), \\ \beta(c,a)+\beta(a,b) > -c > b-c = \beta(b,c), \\ \beta(a,b)+\beta(b,c) = -c = \beta(c,a). \end{cases}$$

iv) $a = 0_A$, $b \in Y$, $c \in Z$. We have

$$\begin{cases} \beta(a,b) = b, \\ \beta(b,c) = 2y_0+b, \\ \beta(c,a) = -c, \text{ and thus} \end{cases}$$

$$\begin{cases} \beta(c,a)+\beta(b,c) > \beta(b,c) > \beta(a,b), \\ \beta(c,a)+\beta(a,b) = b-c > 2y_0+b = \beta(b,c), \text{ as } -c > 2y_0, \\ \beta(a,b)+\beta(b,c) = 2y_0+2b > -c = \beta(c,a). \end{cases}$$

The natural embeddings f, g are G-isometries by definition.

Now we show that there does not exist a G-metric space (D, δ) with G-isometries f' and g' from A and B respectively into it such that $f' \circ f = g' \circ g$.

Suppose the contrary. Let $\delta(f'(0_A), g'(0_B)) = \Delta$. We can prove $\Delta \in G \sim [2Y]$. For, $\Delta \le \delta(f'(0_A), f'(y)) + \delta(g'(0_B), g'(y)) = 2y$, for all $y \in Y$, and therefore by definition of $[2Y]$, $\Delta \in G \sim [2Y]$. There then exists $z_0 \in G \sim [2Y]$ such that $z_0 > \Delta$.

If we consider the triangle $f'(0_A)\, g'(0_B)\, f'(-2y_0-z_0)$ on D, we will find a contradiction to the triangle inequality as follows.

$$\begin{cases} \delta(f'(0_A), f'(-2y_0 - z_0)) = 2y_0, \\ \delta(g'(0_B), f'(-2y_0 - z_0)) = 2y_0 + z_0, \\ \delta(f'(0_A),\ g'(0_B)) = \Delta, \end{cases}$$

and $2y_0 + \Delta \not\geq 2y_0 + z_0$. This completes the proof of the theorem.

CHAPTER III: Homeomorphs of the Space $(2^{\alpha})_{\alpha}$

The main theorem in this chapter, Theorem 3.12, establishes some characterizations of the topological space $(2^{\alpha})_{\alpha}$, where α is an infinite regular cardinal. In all, we have two characterizations of $(2^{\alpha})_{\alpha}$ for an infinite regular α and a third only for cardinals α that are strongly inaccessible but not weakly compact. The method of proof for all three is essentially the same. The proofs, owing to their tedious complexity, are broken up into lemmas. These lemmas assert that bases with some more desirable properties can be constructed from the given ones (Lemmas 3.2 and 3.3) and that homeomorphisms between spaces can be deduced from isomorphisms (with respect to set inclusions) between bases with these properties (Lemma 3.4 and Corollary 3.5), and provide the steps of transfinite induction towards the construction of such isomorphisms (Lemmas 3.7-3.10). They are inevitably long to state, but they provide all the intermediate results and hopefully make the proof of the main theorem clearer. We remark that Lemma 3.4 gives a condition for homeomorphisms between spaces in terms of some very general relation between some bases on those spaces. This Lemma is more general than necessary, though not more difficult to prove. Its Corollary 3.5, a particular case, is all that is necessary. Lemma 3.11 shows that the image of the isomorphism (constructed with the help of Lemmas 3.7-3.10) is a basis of $(2^{\alpha})_{\alpha}$.

These lemmas together prove our main theorem. The remainder of the chapter is devoted to applications of this theorem. The more important ones are Theorem 3.13, Corollaries 3.14, 3.15 and 3.16 and Theorem 3.19, as outlined in the Introduction. We begin with some definitions.

3.1. Definitions. Given a set X, a family $\mathfrak{A}$ of non-void subsets of X is a partition of X if $\cup\mathfrak{A} = X$ and if $A \cap B = \phi$ whenever $A, B \in \mathfrak{A}$ and $A \neq B$.

An (ordered) family of partitions $\{\mathfrak{A}_\xi : \xi < \alpha\}$ of a set X is said to be i) refining, (respectively strictly refining) if $\mathfrak{A}_{\xi_1}$ refines (respectively strictly refines) $\mathfrak{A}_{\xi_0}$, i.e. for any $A_1 \in \mathfrak{A}_{\xi_1}$ there is $A_0 \in \mathfrak{A}_{\xi_0}$ such that $A_1 \subset$ (respectively $\subsetneq$) A_0, whenever $\xi_0 < \xi_1 < \alpha$; ii) continuously refining if it is refining and if for all limit ordinals λ, $0 < \lambda < \alpha$, $\mathfrak{A}_\lambda = \{A : A = \cap\{f_\xi : \xi < \lambda\} \neq \phi$, for some $f \in \Pi_{\xi<\lambda}\mathfrak{A}_\xi\}$.

Given a topological space X and a family $\mathfrak{A}$ of closed subsets of X, we say X is $\mathfrak{A}$-complete if every subfamily of $\mathfrak{A}$ with the finite intersection property has non-void intersection.

3.2. Lemma. Let α be an infinite regular cardinal. Let X be a P_α-space with no isolated points. If X has an α-subbasis $\mathfrak{A}$ of the form $\mathfrak{A} = \cup_{\xi<\alpha}\mathfrak{A}_\xi$ such that for each $\xi < \alpha$, $\mathfrak{A}_\xi$ is a discrete open cover of X, then there is a continuously refining family $\{\mathfrak{B}_\xi : \xi < \alpha\}$ of partitions of X such that

i) the family $\mathcal{B} = \bigcup_{\xi<\alpha} \mathcal{B}_\xi$ is a basis for the topology of X,

ii) if X is $\mathcal{A}$-complete, X is $\mathcal{B}$-complete, and

iii) for every $\xi<\alpha$ and every $B \in \mathcal{B}_\xi$, $1 < |\{C \in \mathcal{B}_{\xi+1} : C \subset B\}| \le |\mathcal{A}_\zeta|$ for some $\zeta<\alpha$ dependent on ξ and B.

Proof. Let $\mathcal{C}_0 = \{X\}$. For all $0<\xi<\alpha$, let $\mathcal{C}_\xi = \{C : C = \bigcap\{f_\zeta : \zeta<\xi\} \neq \emptyset$, for some $f \in \Pi_{\zeta<\xi} \mathcal{A}_\zeta\}$. Clearly, $\{\mathcal{C}_\xi : \xi<\alpha\}$ is a continuously refining family of partitions of X such that

a) the family $\mathcal{C} = \bigcup_{\xi<\alpha} \mathcal{C}_\xi$ is a basis for the topology of X.

b) if X is $\mathcal{A}$-complete, X is $\mathcal{C}$-complete, and

c) for every $\xi<\alpha$ and every $C \in \mathcal{C}_\xi$, there exists ζ, $\xi<\zeta<\alpha$ dependent on ξ and C, such that

$$1 < |\{D \in \mathcal{C}_\zeta : D \subset C\}| \le |\mathcal{A}_\zeta|.$$

Item c) is true because there exists some η, $\xi<\eta<\alpha$, such that $|\{A : A \in \mathcal{A}_\eta,\ A \cap C \neq \emptyset\}| > 1$ (since X has no isolated points) and ζ can be taken to be the least of such.

However $\{\mathcal{C}_\xi : \xi<\alpha\}$ need not be strictly refining, i.e. $\{\mathcal{C}_\xi : \xi<\alpha\}$ need not be disjoint, and we shall rearrange $\mathcal{C}$ into a family $\{\mathcal{B}_\xi : \xi<\eta\}$ that is strictly refining (or disjoint).

First, for every $C \in \mathcal{C}$, we define $\eta(C)$ to be the unique ordinal which is order-isomorphic to the set of all members of $\mathcal{C}$ properly containing C inversely ordered by set inclusion. Thus, $0 \le \eta(C) < \alpha$ for all $C \in \mathcal{C}$. For every $0 \le \xi < \alpha$, we let $\mathcal{B}_\xi = \{C \in \mathcal{C} : \eta(C) = \xi\}$.

We shall show that $\mathcal{B} = \bigcup_{\xi<\alpha} \mathcal{B}_\xi$ has all the required properties. Note first $\mathcal{B} = \mathcal{C}$, and therefore i) and ii) are satisfied; iii) is satisfied because of c). Because of c), $\mathcal{B}_\xi$, for every $\xi < \alpha$, is a cover of X and therefore a partition of X. Furthermore, $\{\mathcal{B}_\xi : \xi < \alpha\}$ is continuously refining because $\mathcal{B} = \mathcal{C}$.

This completes the proof of the lemma.

3.3. Lemma. Let α be an uncountable regular cardinal and let $\omega \leq \beta < \alpha$. If ona topological space X there is a continuously refining family $\{\mathcal{A}_\xi : \xi < \alpha\}$ of partitions such that the family $\mathcal{A} = \bigcup_{\xi < \alpha} \mathcal{A}_\xi$ is basis for the topology of X, and, for every $\xi < \alpha$ and every $A \in \mathcal{A}_\xi$ we have $1 < |\{B \in \mathcal{A}_{\xi+1} : B \subset A\}| \leq 2^\beta$; then there exists on X a continuously refining family $\{\mathcal{B}_\xi : \xi < \alpha\}$ of partitions such that

i) the family $\mathcal{B} = \bigcup_{\xi<\alpha} \mathcal{B}_\xi$ is a basis for the topology of X,

ii) if X is $\mathcal{A}$-complete, X is $\mathcal{B}$-complete, and

iii) for every $\xi < \alpha$ and every $B \in \mathcal{B}_\xi$, $|\{C \in \mathcal{B}_{\xi+1} : C \subset B\}| = 2^\beta$.

Proof: We can let $\mathcal{B}_\xi = \mathcal{A}_{\beta \cdot \xi}$ for all $\xi < \alpha$. Clearly the family $\{\mathcal{B}_\xi : \xi < \alpha\}$ has all the required properties. (Property iii) is clear if we note that $(2^\beta)^\beta = 2^\beta$.)

3.4. Lemma. If X and Y are T_1-spaces, with subbases $\mathcal{A}$ and $\mathcal{B}$ respectively for their topologies and if there is a one-to-one function φ from $\mathcal{A}$ onto $\mathcal{B}$, such that void intersections on X of members of $\mathcal{A}$, and only such, imply void intersections of their images under φ; then X and Y are homeomorphic.

Proof: We construct a function h from X to Y as follows. We show that for each $x \in X$, $\cap\{\varphi(A): A \in \mathcal{A}, x \in A\}$ is a singleton. First, we note that it cannot by hypothesis be void. Secondly, if it contains two distinct points y_1, y_2, there exists $B \in \mathcal{B}$ such that $y_1 \in B$, $y_2 \notin B$ and therefore $B \notin \{\varphi(A): A \in \mathcal{A}, x \in A\}$. It follows that $\varphi^{-1}(B) \notin \{A: A \in \mathcal{A}, x \in A\}$, $\varphi^{-1}(B) \cap \cap\{A: A \in \mathcal{A}, x \in A\} = \phi$, $B \cap \cap\{\varphi(A): A \in \mathcal{A}, x \in A\} = \phi$ and y_1 cannot exist. Therefore, for all $x \in X$ we can let h(x) be the only point in $\cap\{\varphi(A): A \in \mathcal{A}, x \in A\}$.

We show that this h is one-to-one and onto as follows. There clearly exists a function g from Y to X defined in a symmetrical way. For every $y \in Y$, $\{h \circ g(y)\} = \cap\{\varphi(A): A \in \mathcal{A}, g(y) \in A\} \subset \cap\{\varphi(A): A = \varphi^{-1}(B), B \in \mathcal{B}, y \in B\}$ $= \cap\{B: B \in \mathcal{B}, y \in B\} = \{y\}$ and therefore $h \circ g(y) = y$. It follows that h is onto and by symmetry h is one-to-one.

To prove h is a homeomorphism it suffices to prove that for all $A \in \mathcal{A}$, $\varphi(A) = h[A]$, because both h and φ are one-to-one. That $\varphi(A) \supset h[A]$ is clear from the definition of h. By symmetry $A \supset h^{-1}[\varphi(A)]$, i.e. $h[A] \supset \varphi(A)$. This completes the proof of the lemma.

3.5. <u>Corollary</u>. Let α be an infinite regular cardinal. Let X and Y be two topological spaces, on which are respectively refining families

$\{\mathcal{A}_\xi : \xi < \alpha\}$ and $\{\mathcal{B}_\xi : \xi < \alpha\}$ of partitions, such that the families $\mathcal{A} = \bigcup_{\xi<\alpha} \mathcal{A}_\xi$ and $\mathcal{B} = \bigcup_{\xi<\alpha} \mathcal{B}_\xi$ are bases for the topologies of X and Y respectively and such that X is $\mathcal{A}$-complete and Y is $\mathcal{B}$-complete. If there is an order-isomorphism φ from $\mathcal{A}$ onto $\mathcal{B}$ ($\mathcal{A}$ and $\mathcal{B}$ considered as partially ordered sets inversely ordered by set inclusion); then X and Y are homeomorphic.

Proof. (Straightforward.)

To use Corollary 3.5 to prove homeomorphism of a space with $(2^\alpha)_\alpha$, we need to have some specific basis for the topology of $(2^\alpha)_\alpha$. We describe a particularly simple one in the next section which we denote by $\mathcal{E}_\alpha$ and which has the important property that $(2^\alpha)_\alpha$ is $\mathcal{E}_\alpha$-complete.

3.6. <u>Notations</u>. Let α be an infinite regular cardinal. For any $0 < \nu < \alpha$ and any $s \in 2^\nu$, let $E_\alpha(s)$ denote the set $\{t \in 2^\alpha : t|\nu = s\}$. For any $0 < \nu < \alpha$, let $\mathcal{E}_{\alpha,\nu}$ denote the family $\{E_\alpha(s) : s \in 2^\nu\}$. We also denote the singleton $\{2^\alpha\}$ by $\mathcal{E}_{\alpha,0}$ and the family $\bigcup_{\xi<\alpha} \mathcal{E}_{\alpha,\xi}$ by $\mathcal{E}_\alpha$. Clearly $\mathcal{E}_\alpha$ is a basis of $(2^\alpha)_\alpha$, and $(2^\alpha)_\alpha$ is $\mathcal{E}_\alpha$-complete.

For any $0 < \mu, \nu$, and any $s \in 2^\mu$, $t \in 2^\nu$, we write s;t for the element $r \in 2^{\mu+\nu}$ such that $r(\xi) = s(\xi)$ for all $\xi < \mu$ and $r(\xi) = t(\xi - \mu)$ for all $\mu \le \xi < \mu+\nu$. For any $0 < \lambda, \mu, \nu$, and any $r \in 2^\lambda$, $s \in 2^\mu$, $t \in 2^\nu$, (r;s);t and r;(s;t) then denote the same element in $2^{\lambda+\mu+\nu}$ which can therefore be unambiguously written r;s;t. For any $0 < \xi$, we write $\underline{0}^{(\xi)}$ for the element in 2^ξ such that

$\underline{0}^{(\xi)}(\eta) = 0$ for all $\eta < \xi$ and write $\underline{1}^{(\xi)}$ for the element in 2^{ξ} such that $\underline{1}^{(\xi)}(\eta) = 1$ for all $\eta < \xi$. We also write $\underline{0}$ for $\underline{0}^{(1)}$ and $\underline{1}$ for $\underline{1}^{(1)}$.

3.7. Lemma. Let α be an infinite regular cardinal and let $\beta < \alpha$. Let $\mathfrak{a}_0$ and $\mathfrak{a}_1$ be two partitions of a set X such that

i) $\mathfrak{a}_1$ refines $\mathfrak{a}_0$, and

ii) for every $A \in \mathfrak{a}_0$, $|\{B \in \mathfrak{a}_1 : B \subset A\}| = 2^{\beta}$.

Suppose there is an order-isomorphism φ from $\mathfrak{a}_0$ into $\mathcal{E}_{\alpha}$ ($\mathfrak{a}_0$ and $\mathcal{E}_{\alpha}$ considered as partially ordered sets inversely ordered by set inclusion) such that $\varphi[\mathfrak{a}_0]$ is a partition of 2^{α}.

Then φ can be extended to an order isomorphism $\bar{\varphi}$ on $\mathfrak{a}_0 \cup \mathfrak{a}_1$ such that $\bar{\varphi}[\mathfrak{a}_1]$ is a partition of 2^{α}.

Proof: We first note that if $\mathfrak{a}_0 \neq \{X\}$ φ induces a one-to-one function f from $\mathfrak{a}_0$ into the set $\bigcup_{0<\xi<\alpha} 2^{\xi}$ such that, for all $A \in \mathfrak{a}_0$, $\varphi(A) = E_{\alpha}(f_A)$. We also note that for all $A \in \mathfrak{a}_0$, there exists a one-to-one function g_A from the set $\{B \in \mathfrak{a}_1 : B \subset A\}$ onto the set 2^{β}. In terms of f and g_A for all $A \in \mathfrak{a}_0$ we can define the required $\bar{\varphi}$ on $\mathfrak{a}_0 \cup \mathfrak{a}_1$ as follows. Let $\bar{\varphi}|\mathfrak{a}_0 = \varphi$. For each $A \in \mathfrak{a}_0$, and for each $B \subset A$, $(B \in \mathfrak{a}_1)$, let $\bar{\varphi}(B) = E_{\alpha}(f_A ; g_A(B))$ if $\mathfrak{a}_0 \neq \{X\}$, otherwise let $\bar{\varphi}(B) = E_{\alpha}(g_A(B))$. Clearly $\bar{\varphi}$ thus defined is an order-isomorphism on $\mathfrak{a}_0 \cup \mathfrak{a}_1$, extending φ such that $\bar{\varphi}[\mathfrak{a}_1]$ is a partition of 2^{α}.

3.8. Lemma. Let α be an infinite regular cardinal. Let $\mathfrak{a}_0$ and $\mathfrak{a}_1$ be two partitions of a set X such that

i) $\mathfrak{a}_1$ refines $\mathfrak{a}_0$, and

ii) for every $A \in \mathfrak{a}_0$, $1 < |\{B \in \mathfrak{a}_1 : B \subset A\}| < \alpha$.

Suppose there is an order-isomorphism φ from $\mathfrak{a}_0$ into $\mathcal{E}_\alpha$ ($\mathfrak{a}_0$ and $\mathcal{E}_\alpha$ considered as partially ordered sets inversely ordered by set inclusion) such that $\varphi[\mathfrak{a}_0]$ is a partition of 2^α.

Then φ can be extended to an order-isomorphism $\bar{\varphi}$ on $\mathfrak{a}_0 \cup \mathfrak{a}_1$ such that $\bar{\varphi}[\mathfrak{a}_1]$ is a partition of 2^α.

Proof: Again we note that if $\mathfrak{a}_0 \neq \{X\}$ φ induces a one-to-one function f from $\mathfrak{a}_0$ into the set $\cup_{0<\xi<\alpha} 2^\xi$ such that for all $A \in \mathfrak{a}_0$, $\varphi(A) = E_\alpha(f_A)$. For all $A \in \mathfrak{a}_0$, let $|\{B \in \mathfrak{a}_1 = B \subset A\}| = \varkappa(A)$ and note that there is a one-to-one function δ_A from the set $\{B \in \mathfrak{a}_1 : B \subset A\}$ onto $\varkappa(A)$. In terms of f and $\varkappa(A)$, δ_A for all $A \in \mathfrak{a}_0$, we can define the required $\bar{\varphi}$ on $\mathfrak{a}_0 \cup \mathfrak{a}_1$ as follows. Let $\bar{\varphi}|\mathfrak{a}_0 = \varphi$. For each $A \in \mathfrak{a}_0$, and for each $B \subset A$, $(B \in \mathfrak{a}_1)$; if $\mathfrak{a}_0 \neq \{X\}$, let

$$\bar{\varphi}(B) = \begin{cases} E_\alpha(f_A; \underline{0}), & \text{if } \delta_A(B) = 0, \\ E_\alpha(f_A; \underline{1}^{(\varkappa(A)-1)}), & \text{if } \delta_A(B) = 1, \\ E_\alpha(f_A; \underline{1}^{(\delta_A(B)-1)}; \underline{0}), & \text{if } 1 < \delta_A(B) < \varkappa(A); \end{cases}$$

otherwise let

$$\bar{\varphi}(B) = \begin{cases} E_\alpha(\underline{0}), & \text{if } \delta_A(B) = 0, \\ E_\alpha(\underline{1}^{(\varkappa(A)-1)}), & \text{if } \delta_A(B) = 1, \\ E_\alpha(\underline{1}^{(\delta_A(B)-1)};\underline{0}), & \text{if } 1 < \delta_A(B) < \varkappa(A). \end{cases}$$

Clearly $\bar{\varphi}$ thus defined is an order-isomorphism on $\mathfrak{a}_0 \cup \mathfrak{a}_1$, extending φ such that $\bar{\varphi}[\mathfrak{a}_1]$ is a partition of 2^α.

3.9. Lemma. Let α be a strongly inaccessible cardinal that is not weakly compact. Let $\mathfrak{a}_0$ and $\mathfrak{a}_1$ be two partitions of a set X such that

i) $\mathfrak{a}_1$ refines $\mathfrak{a}_0$, and

ii) for every $A \in \mathfrak{a}_0$, $1 < |\{B \in \mathfrak{a}_1 : B \subset A\}| \leq \alpha$.

Suppose there is an order-isomorphism φ from $\mathfrak{a}_0$ into $\mathcal{E}_\alpha$ ($\mathfrak{a}_0$ and $\mathcal{E}_\alpha$ considered as partially ordered sets inversely ordered by set inclusion) such that $\varphi[\mathfrak{a}_0]$ is a partition of 2^α.

Then φ can be extended to an order-isomorphism $\bar{\varphi}$ on $\mathfrak{a}_0 \cup \mathfrak{a}_1$ such that $\bar{\varphi}[\mathfrak{a}_1]$ is a partition of 2^α.

Proof. We note as before (in the proof of Lemma 3.8) that if $\mathfrak{a}_0 \neq \{X\}$ φ induces a one-to-one function f from $\mathfrak{a}_0$ into the set $\bigcup_{0<\xi<\alpha} 2^\xi$, and that for each $A \in \mathfrak{a}_0$, there is a one-to-one function δ_A from the set $\{B \in \mathfrak{a}_1 : B \subset A\}$ onto $\varkappa(A)$ ($= |\{B \in \mathfrak{a}_1 : B \subset A\}|$). We note in addition that $E_\alpha(s)$, for any $s \in \bigcup_{0<\xi<\alpha} 2^\xi$, as a subspace of $(2^\alpha)_\alpha$, is homeomorphic to $(2^{\alpha})_{\alpha}$ and therefore not α-compact. It follows then $E_{\alpha}(s)$, for any $s \in \bigcup_{0<\xi<\alpha} 2^{\xi}$, or 2^{α} itself (as a subspace of $(2^{\alpha})_{\alpha}$) has an open cover $\mathcal{G}$ of cardinality $\geq \alpha$ admitting no subcover of cardinality $<\alpha$. Furthermore because $\mathcal{E}_{\alpha}$ is a basis, and because α is strongly inaccessible and $|\mathcal{E}_{\alpha}| = 2^{\underline{\alpha}} = \alpha$, we can assume $\mathcal{G}$ to be a partition of $E_{\alpha}(s)$, consisting of exactly α elements from $\mathcal{E}_{\alpha}$. In particular for every $A \in \mathcal{A}_0$, there is a subfamily $\mathcal{G}_A \subset \mathcal{E}_{\alpha}$ such that $|\mathcal{G}_A| = \alpha$ and $\mathcal{G}_A$ is a partition of $\varphi(A)$. There is of course a one-to-one function g_A from α onto $\mathcal{G}_A$, for each $A \in \mathcal{A}_0$.

We can now define the required $\overline{\varphi}$ on $\mathcal{A}_0 \cup \mathcal{A}_1$ in terms of f, $\varkappa(A), \delta_A$ $\mathcal{G}_A$ and g_A for all $A \in \mathcal{A}_0$. Let $\overline{\varphi}|\mathcal{A}_0 = \varphi$. For each $A \in \mathcal{A}_0$ such that $\varkappa(A) < \alpha$ we define $\overline{\varphi}(B)$ for every $B \subset A$, $(B \in \mathcal{A}_1)$, as in 3.8. For each $A \in \mathcal{A}_0$ such that $\varkappa(A) = \alpha$, and for each $B \subset A$, $(B \in \mathcal{A}_1)$, we let $\varphi(B) = g_A(\delta_A(B))$. Clearly $\overline{\varphi}$ thus defined is an order-isomorphism on $\mathcal{A}_0 \cup \mathcal{A}_1$, extending φ such that $\overline{\varphi}[\mathcal{A}_1]$ is a partition of 2^{α}.

3.10. <u>Lemma</u>. Let α be an infinite regular cardinal and let λ be any infinite limit ordinal $<\alpha$. Let $\{\mathcal{A}_{\xi} : \xi \leq \lambda\}$ be a continuously refining family of partitions of a set X such that any subfamily of $\bigcup_{\xi<\lambda} \mathcal{A}_{\xi}$ with the finite intersection property has non-void intersection.

Suppose there is an order-isomorphism φ from $\bigcup_{\xi<\lambda} \mathcal{A}_{\xi}$ into $\mathcal{E}_{\alpha}$ ($\bigcup_{\xi<\lambda} \mathcal{A}_{\xi}$ and $\mathcal{E}_{\alpha}$ considered as partially ordered sets inversely ordered by set inclusion) such that $\{\varphi[\mathcal{A}_\xi]:\xi<\lambda\}$ is a continuously refining family of partitions of 2^α.

Then φ can be extended uniquely to an order-isomorphism $\bar{\varphi}$ on $\bigcup_{\xi\leq\lambda}\mathcal{A}_\xi$ such that $\{\varphi[\mathcal{A}_\xi]:\xi\leq\lambda\}$ is a continuously refining family of partitions of 2^α.

Proof: We define the required $\bar{\varphi}$ on $\bigcup_{\xi\leq\lambda}\mathcal{A}_\xi$ as follows. Let $\bar{\varphi}|\bigcup_{\xi<\lambda}\mathcal{A}_\xi=\varphi$. For every $A\in\mathcal{A}_\lambda$, we note that $A=\bigcap\{f^A_\xi:\xi<\lambda\}$ for a unique $f^A\in\Pi_{\xi<\lambda}\mathcal{A}_\xi$, and let $\bar{\varphi}(A)=\bigcap\{\varphi(f^A_\xi):\xi<\lambda\}$, which is a member of $\mathcal{E}_\alpha$ because α is regular. Clearly $\bar{\varphi}$ thus defined is the unique order-isomorphism on $\bigcup_{\xi\leq\lambda}\mathcal{A}_\xi$, extending φ such that $\{\varphi[\mathcal{A}_\xi]:\xi\leq\lambda\}$ is a continuously refining family of partitions of 2^α.

3.11. Lemma. Let α be an infinite regular cardinal. If $\{\mathcal{A}_\xi:\xi<\alpha\}$ is a family of strictly refining partitions of 2^α such that $\mathcal{A}_\xi\subset\mathcal{E}_\alpha$ for all $\xi<\alpha$, then $\bigcup_{\xi<\alpha}\mathcal{A}_\xi$ is a basis for the topology of the space $(2^\alpha)_\alpha$.

Proof: It suffices to show that given any element $s\in 2^\alpha$ and any $0<\xi<\alpha$, there exists $A\in\bigcup_{\zeta<\alpha}\mathcal{A}_\zeta$ such that $s\in A\subset E_\alpha(s|\xi)$.

For every $\zeta<\alpha$, since $\mathcal{A}_\zeta$ is a partition of 2^α, there is a unique $A_\zeta\in\mathcal{A}_\zeta$ that contains s. There is evidently a unique function η from α into α such that $A_\zeta\in\mathcal{E}_{\alpha,\eta(\zeta)}$, for all $\zeta<\alpha$. Clearly the function η is strictly increasing and in particular $\eta(\zeta)\geq\zeta$. We can accordingly let $A=A_\xi$. This completes the proof of the lemma.

Now we are ready for the uniqueness theorem.

3.12. <u>Theorem</u>. Let α be an infinite regular cardinal. Let X be a P_α-space with no isolated points. If X has an α-subbasis $\mathfrak{G}$ of the form $\mathfrak{G} = \bigcup_{\xi<\alpha} \mathfrak{G}_\xi$ such that

i) for each $\xi<\alpha$, $\mathfrak{G}_\xi$ is a discrete open cover of X;

ii) either

a) for some $\beta<\alpha$, $|\mathfrak{G}_\xi| \leq 2^\beta$ for all $\xi<\alpha$; or

b) $|\mathfrak{G}_\xi|<\alpha$ for all $\xi<\alpha$; or still,

c) the cardinal α is strongly inaccessible but not weakly compact and $|\mathfrak{G}_\xi| \leq \alpha$ for all $\xi<\alpha$; and

iii) X is $\mathfrak{G}$-complete;

then X is homeomorphic to $(2^\alpha)_\alpha$.

Proof: a) If $\beta \geq \omega$, then Lemmas 3.2 and 3.3 apply and there is on X a continuously refining family $\{\mathfrak{B}_\xi : \xi<\alpha\}$ of partitions such that

i) the family $\mathfrak{B} = \bigcup_{\xi<\alpha} \mathfrak{B}_\xi$ is a basis for the topology of X,

ii) X is $\mathfrak{B}$-complete, and

iii) for every $\xi<\alpha$ and every $B \in \mathfrak{B}_\xi$, $|\{C \in \mathfrak{B}_{\xi+1} : C \subset B\}| = 2^\beta$.

Lemmas 3.7 and 3.10 imply the existence of an order-isomorphism φ from $\mathfrak{B}$ into $\mathcal{E}_\alpha$ ($\mathfrak{B}$ and $\mathcal{E}_\alpha$ considered as partially ordered sets inversely ordered by set inclusion) such that $\{\varphi[\mathfrak{B}_\xi] : \xi<\alpha\}$ is a strictly refining family of partitions of 2^α. By Lemma 3.11, $\varphi[\mathfrak{B}]$ is a basis of $(2^\alpha)_\alpha$.

Clearly, $(2^{\alpha})_{\alpha}$ is $\varphi[\mathcal{B}]$-complete and Corollary 3.5 is applicable. The spaces X and $(2^{\alpha})_{\alpha}$ are therefore homeomorphic.

If $\beta < \omega$, a) is included in b).

b) Lemma 3.2 applies and there is on X a continuously refining family $\{\mathcal{B}_{\xi} : \xi < \alpha\}$ of partitions such that

i) the family $\mathcal{B} = \bigcup_{\xi<\alpha} \mathcal{B}_{\xi}$ is a basis for the topology of X,

ii) X is $\mathcal{B}$-complete, and

iii) for every $\xi < \alpha$ and every $B \in \mathcal{B}_{\xi}$, $1 < |\{C \in \mathcal{B}_{\xi+1} : C \subset B\}| < \alpha$.

Lemmas 3.8, 3.10, 3.11 and Corollary 3.5 combine as before to produce the result that X is homeomorphic to $(2^{\alpha})_{\alpha}$, Lemma 3.8 taking the part of Lemma 3.7.

c) Lemma 3.2 applies and there is an X a continuously refining family $\{\mathcal{B}_{\xi} : \xi < \alpha\}$ of partitions such that

i) the family $\mathcal{B} = \bigcup_{\xi<\alpha} \mathcal{B}_{\xi}$ is a basis for the topology of X,

ii) X is $\mathcal{B}$-complete and,

iii) for every $\xi < \alpha$ and every $B \in \mathcal{B}_{\xi}$, $1 < |\{C \in \mathcal{B}_{\xi+1} : C \subset B\}| \leq \alpha$.

This time we need Lemma 3.9 in place of Lemma 3.8 for the final result that X is homeomorphic to $(2^{\alpha})_{\alpha}$.

This completes the proof of the theorem.

3.13. Theorem. Let α be an uncountable regular cardinal and let X be a compact Hausdorff space. The space X_α is homeomorphic to $(2^\alpha)_\alpha$ if

i) the set C(X) of all continuous real-valued functions on X has cardinality α, and

ii) intersection of fewer than α open sets on X is never a singleton.

The converse is true if $\alpha = \alpha^{\underline{\alpha}}$.

Proof. The first part follows from a straightforward application of Theorem 3.12.

Conversely, if X_α is homeomorphic to $(2^\alpha)_\alpha$, it is clear that ii) is true and it is true that $|C(X)| \geq \alpha$. If $\alpha^{\underline{\alpha}} = \alpha$ and therefore $2^{\underline{\alpha}} = \alpha$, the basis $\mathcal{E}_\alpha$ of $(2^\alpha)_\alpha$ has cardinality α. It follows that any other basis of $(2^\alpha)_\alpha$ has a subfamily of cardinality α which is also a basis, and, any basis of X_α, because of homeomorphism, has a subfamily of cardinality α which is also a basis. One basis of X_α is the family $\mathfrak{A}$ of all intersections of fewer than α open sets of X, and there is a basis $\mathfrak{B}$ of X_α, $\mathfrak{B} \subset \mathfrak{A}$, $|\mathfrak{B}| \leq \alpha$. In particular, this $\mathfrak{B}$ distinguishes points of X_α. There is therefore a family of α open sets of X which distinguishes points of X (as $\alpha^{\underline{\alpha}} = \alpha$). Because X is compact Hausdorff, it can be embedded into the cube $[0, 1]^\alpha$ according to the Embedding Lemma (cf. e.g. Chapter 4, §5 of Kelley [21]). In particular X has a basis of cardinality $\leq \alpha$ and using a result of Comfort and Hager in [7], we have $|C(X)| \leq \alpha^{\omega} = \alpha^{\underset{\smile}{\alpha}} = \alpha$. This together with the inequality in the opposite direction gives our result that $|C(X)| = \alpha$. The proof of our theorem is complete.

3.14. <u>Corollary</u>. Given an infinite cardinal α, suppose $\alpha^{+} = 2^{\alpha}$, then $(U(\alpha))_{\alpha^+}$ is homeomorphic to $(2^{\alpha^+})_{\alpha^+}$, and $(\Omega(\alpha))_{\alpha^+}$ is homeomorphic to a closed subset of $(2^{\alpha^+})_{\alpha^+}$.

Proof. That $(\Omega(\alpha))_{\alpha^+}$ is a closed subset of $(U(\alpha))_{\alpha^+}$ can be seen from the following consideration. Given any uniform ultrafilter p that is not α-complete, there is $\mathfrak{F} \subset p$, $|\mathfrak{F}| < \alpha$, $\cap\mathfrak{F} \notin p$. If we let $F^{*} = \{q \in U(\alpha) : F \in q\}$ $(= \mathrm{Cl}_{\beta\alpha} F \cap U(\alpha))$ for all $F \in \mathfrak{F}$, then $\cap\{F^{*} : F \in \mathfrak{F}\}$ is open in $(U(\alpha))_{\alpha^+}$, containing p and disjoint from $(\Omega(\alpha))_{\alpha^+}$.

The second conclusion then follows if the first is proved.

For the proof of the first statement, Theorem 3.13 applies, because $U(\alpha)$ is a compact Hausdorff space, α^{+} is uncountable and regular, the set of all real-valued continuous functions on $U(\alpha)$ has cardinality α^{+}, and no uniform ultrafilter on α can be generated by fewer than α^{+} elements.

3.15. <u>Corollary</u>. If α is an is an infinite cardinal such that $\alpha^{\underset{\smile}{\alpha}} = \alpha$, then $(\Lambda(\alpha))_{\alpha}$ is homeomorphic to $(2^{\alpha})_{\alpha}$.

Proof. We note that if $\alpha = \omega$, $(\Lambda(\alpha))_{\alpha}$ is homeomorphic to $(2^{\alpha})_{\alpha}$, (also to $\Lambda(\alpha)$ and to 2^{α}, the Cantor discontinuum). If $\alpha > \omega$, then

Theorem 3.13 applies, because $\Lambda(\alpha)$ is a compact Hausdorff space, α is regular, intersections of fewer than α open intervals are never singletons, and the set C(X) of all real-valued continuous functions on $\Lambda(\alpha)$ has cardinality α as shown below. $\Lambda(\alpha)$ has a basis of cardinality $2^{\underline{\alpha}}$ and $|C(X)| \le (2^{\underline{\alpha}})^{\omega} = \alpha$ by Comfort and Hager's estimate in [7]. It is also quite clear that $|C(X)| \ge \alpha$ and therefore $|C(X)| = \alpha$. This completes the proof.

3.16. Remark. It can be proved directly that $(\Lambda(\alpha))_{\alpha}$ is homeomorphic to $(2^{\alpha})_{\alpha}$ for all infinite regular α as follows. (If α is singular, both $(\Lambda(\alpha))_{\alpha}$ and $(2^{\alpha})_{\alpha}$ are homeomorphic to the discrete space of cardinality 2^{α}.)

1) We prove that the topology of $(\Lambda(\alpha))_{\alpha}$ contains that of $(2^{\alpha})_{\alpha}$. For any $s \in 2^{\nu}$, $0<\nu<\alpha$, we show that $E_{\alpha}(s)$ is open in $(\Lambda(\alpha))_{\alpha}$. Clearly $E_{\alpha}(s) = A \cap B$ where

$$A = \bigcap_{\eta<\zeta}\{t \in 2^{\alpha} : t < s|\eta;\ \underline{1}^{(\zeta-\eta)};\underline{0}^{(\alpha-\zeta)}\}$$

if there is a smallest $\zeta<\nu$ such that $s[\nu\sim\zeta] = \{1\}$, otherwise

$$A = \bigcap_{\eta<\nu}\{t \in 2^{\alpha} : t < s|\eta;\ \underline{1}^{(\nu-\eta)};\underline{0}^{(\alpha-\nu)}\};$$

and

$$B = \bigcap_{\eta<\xi}\{t \in 2^{\alpha} : t > s|\eta;\underline{0}^{(\xi-\eta)};\underline{1}^{(\alpha-\xi)}\};$$

if there is a smallest $\xi < \nu$ such that $s[\nu \sim \xi] = \{0\}$, otherwise

$$B = \cap_{\eta<\nu} \{t \in 2^{\alpha} : t > s|\eta;\ \underline{0}^{(\nu-\eta)};\ \underline{1}^{(\alpha-\nu)}\}.$$

Both A and B are open in $(\Lambda(\alpha))_{\alpha}$ and therefore $E_{\alpha}(s)$ is open in $(\Lambda(\alpha))_{\alpha}$.

ii) We prove that the topology of $(2^{\alpha})_{\alpha}$ contains that of $(\Lambda(\alpha))_{\alpha}$. For all, $r, s, t \in 2^{\alpha}$, $r < s < t$ lexicographically, we prove that there is some ν, $0 < \nu < \alpha$, such that $E_{\alpha}(s|\nu)$ (containing s) is contained in the open interval (r, t); from which the desired conclusion follows. Let ξ be the first ordinal such that $r_{\xi} \neq s_{\xi}$ and η the first ordinal such that $s_{\eta} \neq t_{\eta}$. We can clearly let $\nu = 1 + \max\{\xi, \eta\}$.

This completes the proof of our remark.

Theorem 3.13 also provides a proof of a result mentioned but not proved in Negrepontis [25]. There, it is pointed out that as a consequence of results of Jónsson [15], [16] and Morley and Vaught [24], for every infinite cardinal α such that $\alpha = \alpha^{\underline{\alpha}}$, there exists a unique (up to Boolean isomorphism) α-homogeneous-universal Boolean algebra of cardinality α, and that if S_{α} is the Stone space of this Boolean algebra, then $(S_{\alpha})_{\alpha}$ is homeomorphic to $(\Lambda(\alpha))_{\alpha}$. The latter of the two statements is a corollary to our Theorem 3.13, as we shall see.

3.17. Corollary. Let α be an infinite cardinal such that $\alpha = \alpha^{\underline{\alpha}}$. If S_α is the Stone space of the α-homogeneous-universal Boolean algebra of cardinality α, then $(S_\alpha)_\alpha$ is homeomorphic to $(2^\alpha)_\alpha$.

Proof: We note that if $\alpha = \omega$, $(S_\alpha)_\alpha$ is homeomorphic to the Cantor discontinuum and is therefore homeomorphic to $(2^\alpha)_\alpha$. If $\alpha > \omega$, Theorem 3.13 applies, according to Theorem 1.7 of Negrepontis [25] and Comfort and Hager's result in [7].

3.18. Theorem. For any strongly measurable cardinal α, $(2^\alpha)_\alpha$ is a continuous image of $\Omega(\alpha)$.

Proof. We construct a continuous function from $\Omega(\alpha)$ onto $(2^\alpha)_\alpha$. First, let R_α denote the subset

$$\{s \in (2^\alpha)_\alpha : s[\alpha \sim \xi] = \{0\} \text{ or } \{1\} \text{ for some } \xi < \alpha\}.$$

$|R_\alpha| = 2^{\underline{\alpha}} = \alpha$ (cf. 1.16 and Proposition 1.32 ii)). We can clearly define a continuous function f from α onto R_α such that $|f^{-1}(s)| = \alpha$ for every $s \in R_\alpha$. It is well known that f extends to a continuous function from $\beta\alpha$ into some compact Hausdorff space containing R_α (cf. e.g. §6.4 of Gillman and Jerison [11]). It is clearly also true that f extends to a continuous function $\bar{f}$ from $\Omega(\alpha) \cup \alpha$ into $(2^\alpha)_\alpha$, $(2^\alpha)_\alpha$ being α-compact (Proposition 1.32 ii)). The continuous function $\bar{f}$ can be explicitly defined as follows.

For every $p \in \Omega(\alpha) \cup \alpha$, that is, for every α-complete ultrafilter p on α, let

$$f^{\#}p = \{E \subset (2^{\alpha})_{\alpha} : E \text{ is closed, } f^{-1}[E] \in p\} ,$$

and $f^{\#}p$ being an α-complete prime filter of closed subsets of $(2^{\alpha})_{\alpha}$, we can let $\bar{f}(p)$ be the limit of $f^{\#}p$.

It remains only to prove that $\bar{f}[\Omega(\alpha)] = (2^{\alpha})_{\alpha}$. Given $s \in (2^{\alpha})_{\alpha} \sim R_{\alpha}$, the family $\{f^{-1}[E_{\alpha}(s|\xi)] : 0 < \xi < \alpha\}$ has void intersection and is contained in some non-principal α-complete ultrafilter p on α. Given $s \in R_{\alpha}$, $|\cap f^{-1}[E_{\alpha}(s|\xi)] : 0 < \xi < \alpha\}| = \alpha$ and the family $\{f^{-1}[E_{\alpha}(s|\xi)] : 0 < \xi < \alpha\}$ is again contained in some non-principal α-complete ultrafilter p on α. In either case, by definition $\bar{f}(p) = s$. The proof is therefore complete.

We provide an alternate proof using the idea of Theorem 3.12. This alternate proof does not make use of the fact that $(2^{\alpha})_{\alpha}$ is α-compact when α is strongly measurable and in fact can be taken to be a proof of that fact.

Alternate proof. First we shall establish that on any strongly measurable cardinal α, there exists a family $\{A_{\xi,i} \subset \alpha : \xi < \alpha, i=0,1\}$ such that

i) $|A_{\xi,i}| = \alpha$ for all $\xi < \alpha$, i=0, 1,

ii) $A_{\xi,0} \cap A_{\xi,1} = \phi$, $|\alpha \sim (A_{\xi,0} \cup A_{\xi,1})| < \alpha$, for all $\xi < \alpha$, and

iii) for any $s \in 2^{\alpha}$, and any $0 < \eta < \alpha$, $|\cap_{\xi<\eta} A_{\xi, s_{\xi}}| = \alpha$.

The set $S = \bigcup\{2^{\xi} : 0 < \xi < \alpha\}$ is of cardinality $2^{\underline{\alpha}} = \alpha$ (cf. 1.16 and Proposition 1.32 ii)) and can therefore be identified with α. For all $\xi < \alpha$, $i = 0, 1$, we can let $A_{\xi, i} = \{s \in S : s_{\xi}$ is defined and $= i\}$. It is clear that $A_{\xi, i}$, for all $\xi < \alpha$, $i=0, 1$, thus defined satisfy i)-iii) above.

For all $\xi < \alpha$, $i=0, 1$, we write $A^*_{\xi, i} = \{q \in \Omega(\alpha) : A_{\xi, i} \in q\} (= cl_{\beta\alpha} A_{\xi, i} \cap \Omega(\alpha))$. Clearly $\{\{A^*_{\xi, 0}, A^*_{\xi, 1}\} : \xi < \alpha\}$ is a family of partitions of $\Omega(\alpha)$ such that $\bigcap_{\xi<\eta} A^*_{\xi, s_\xi} \neq \phi$, for all $0 < \eta < \alpha$ and all $s \in 2^{\alpha}$, and hence, for all $s \in 2^{\alpha}$, $\bigcap_{\xi<\alpha} A^*_{\xi, s_\xi} \neq \phi$. Now we can define a function f from $\Omega(\alpha)$ onto $(2^{\alpha})_{\alpha}$ as follows. For every $s \in (2^{\alpha})_{\alpha}$, we let $f(t) = s$ for all $t \in \bigcap_{\xi<\alpha} A^*_{\xi, s_\xi}$. That f is continuous is clear because $A^*_{\xi, i}$ is open in $\Omega(\alpha)$ for every $\xi < \alpha$, $i=0, 1$. This completes the alternate proof.

3.19. Theorem. Let α be an infinite regular cardinal.

i) For all β such that $0 < \beta < \alpha$, the spaces $(2^{\alpha})_{\alpha}$ and $((2^{\beta})^{\alpha})_{\alpha}$ are homeomorphic.

ii) The space $(2^{\alpha})_{\alpha}$ is homeomorphic to $(\gamma^{\alpha})_{\alpha}$ for some $\gamma > \alpha$, if and only if α is such that $\alpha \neq 2^{\underline{\alpha}}$.

iii) The space $(2^{\alpha})_{\alpha}$ is homeomorphic to $(\alpha^{\alpha})_{\alpha}$, if and only if α is not weakly compact.

Proof. i) This is clear from part a) of Theorem 3.12.

ii) If $\alpha = 2^{\alpha}$, then $\mathcal{E}_{\alpha}$ has cardinality α and $(2^{\alpha})_{\alpha}$ must be γ-compact for all $\gamma > \alpha$, while $(\gamma^{\alpha})_{\alpha}$ is certainly not γ-compact. Therefore $(2^{\alpha})_{\alpha}$ cannot be homeomorphic to $(\gamma^{\alpha})_{\alpha}$ for any $\gamma > \alpha$. Conversely, there exists a $\beta < \alpha$ such that $\alpha < 2^{\beta}$ and by part a) of Theorem 3.12, $(2^{\alpha})_{\alpha}$ is homeomorphic to $((2^{\beta})^{\alpha})_{\alpha}$.

iii) If α is not strongly inaccessible, then $(2^{\alpha})_{\alpha}$ is homeomorphic to $(\alpha^{\alpha})_{\alpha}$ by part a) of Theorem 3.12. If α is strongly inaccessible but not weakly compact, the same is true by part c) of Theorem 3.12. Conversely, if α is weakly compact $(2^{\alpha})_{\alpha}$ is α-compact and cannot be homeomorphic to $(\alpha^{\alpha})_{\alpha}$ which is clearly not α-compact.

This completes the proof of this theorem.

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