Dynamic Stability of Plane Structures

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Ph.D.
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ABSTRACT

The finite element method has been used to determine the natural frequencies and modal shapes of plane structures such as beams, grids, trusses and frames. A computer program has been written for this purpose. The members of the structure may be of non-uniform cross-section and varying properties. Several examples are given to show comparison with results obtained experimentally and using other methods.

A numerical method of determining the regions of dynamic instability of a column subjected to periodic axial force and periodic support motion has been developed. The column is idealized as consisting of massless springs and lumped masses. Comparison with known results and with experimental work, which was carried out as a part of the investigation is presented.

An experimental investigation of the dynamic instability of a six storey portal frame is reported. The analytical solution of the problem was attempted by the continuum method and the results obtained are presented.

Stabilité Dynamique des Structures Planes

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et de Méchanique Appliquée

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Résumé

La méthode des éléments finis a été utilisée afin de déterminer les fréquences et les mouvements propres de structures planes telles que poutres, treilles et cadres rigides.

Un programme pour calcutrice électronique a été développé à cette fin. Les membrures de la structure peuvent être de sections et propriétés variables.

Plusieurs examples sont traités; leurs résultats sont comparés avec des valeurs expérimentales ou obtenues par d'autres méthodes.

Une méthode numérique pour la détermination des domaines d'instabilité dynamique d'une colonne soumise à une charge et à un déplacement d'appui axiaux et périodiques a été développée.

La colonne est idéalisée par une série de masses concentrées, et de ressorts de masse nulle.

Les résultats sont comparés avec des valeurs connues ou expérimentales, obtenues dans le cadre du présent travail.

L'étude expérimentale de l'instabilité dynamique d'un cadre rigide de six étages est incluse.

Une solution analytique de ce probleme fut tentée en remplacant le portique par un milieu continu équivalent, et les resultats obtenus sont présentés.

DYNAMIC STABILITY OF PLANE STRUCTURES

A Thesis

by

S.Z.H. Burney, B.Sc. (Hons.), M.Sc.

Submitted to the Faculty of Graduate Studies and Research in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

McGill University

March 1971

To my wife and to my parents

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SYMBOLS

Unless otherwise defined, the following symbols are used.

a	length of link
$\bar{a}$	coefficient of Fourier series
A	cross-sectional area
b	breadth
$\bar{b}$	coefficient of Fourier series
$\underset{\sim}{B}$	strain matrix
C	amplitude of support motion
C_x, C_y	direction cosines
$\underset{\sim}{d}$	displacement vector
$\underset{\sim}{D}$	elasticity matrix
e	strain
e_o	initial strain
E	Young's modulus of elasticity
F	force
G	shear modulus
H	height
I	moment of inertia
k	element of stiffness matrix
K	spring stiffness, overall stiffness matrix
L	length
$\underset{\sim}{m}$	element of mass matrix

m_o	mass per unit length
M	mass
$\underset{\sim}{M}$	mass matrix
n, N	number of beams
$\underset{\sim}{N}$	shape matrix
p	distributed load
P	load, force
P_e	approximate Euler buckling load
r_n	K_n/a_n
$\underset{\sim}{Q}$	force matrix
q	amplitude of displacement
q*	inertia force
$\underset{\sim}{R}$	external modal force vector
S_n	K_n/a_{n-1}
$\underset{\sim}{S}$	stiffness matrix
t	time
T	time period
T_e	Kinetic Energy
u, v	displacements
u_s	support displacement.
U, V	Force
U_e	Potential Energy
w	frequency

x, y	coordinates or displacements
X, Y	Body Force Components
α	stiffness coefficient for a continuum
β	αH,a measure of the relative stiffness of beams w.r.t. columns
β_o	natural frequency of the bar loaded by a longitudinal compressive force.
γ	shear strain
δ	parameter of Mathieu's equation
ϵ	parameter of Mathieu's equation
ξ	acceleration of the support motion
η	coefficients associated with V, m and K
θ	angular displacement, $\frac{dy}{dx}$
λ	coefficients associated with V_o, m, and K
μ	$V_t/2(P_k-V_c)$
ν	Poisson's ratio
ρ	mass per unit volume
ρ	mass per unit height
σ	stress
ϕ	modal shape function
Ω	frequency of external load, or exciting frequency of base motion

CHAPTER I

INTRODUCTION

1.1. General Review

Earthquakes occur in various parts of the world and occasionally they cause considerable damage to life and property. In order to build economic and attractive structures which are resistant to earthquake effects engineers and scientists have been studying their causes and behaviour. From records of ground motion it has been observed that during an earthquake the principal component is in the horizontal direction [1]. On this basis in the design of structures subjected to seismic loading the vertical acceleration is usually ignored or else its effect is combined with that of the horizontal acceleration in an arbitrary manner. Very little is known about the response of structures when subjected to fluctuating vertical base motion. Therefore, before one can attempt to study the response of the structure to the combined effects of horizontal and vertical motions it is necessary to first resolve the problem of vertical motion. The first step in attacking the problem of seismic vibrations, which are of a random nature, is to seek solutions for cases of periodic support motion. Even with this simplifying assumption it will be observed that the formulation and solution of the problem is difficult. It is hoped that this work will help in the study of the response of structures to random fluctuating base motion.

The differential equations of motion of structures subjected to periodic horizontal support motion have constant coefficients [(2)]. Consider the structure shown in Figure 1.1 and let the support motion be $y_s(t)$. The equations of motion write as:-

$$m_1\ddot{y}_1 + k_{11}(y_1-y_s) + k_{12}(y_2-y_s) + \dots k_{1n}(y_n-y_s) = 0$$

$$m_2\ddot{y}_2 + k_{21}(y_1-y_s) + k_{22}(y_2-y_s) + \dots k_{2n}(y_n-y_s) = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$m_n\ddot{y}_n + k_{n1}(y_1-y_s) + k_{n2}(y_2-y_s) + \dots k_{nn}(y_n-y_s) = 0$$

i.e. $$m_1\ddot{y}_1 + k_{11}y_1 + k_{12}y_2 + \dots\dots + k_{1n}y_n = f_1(k)y_s$$

$$m_2\ddot{y}_2 + k_{21}y_1 + k_{22}y_2 + \dots\dots + k_{n2}y_2 = f_2(k)y_s$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$m_n\ddot{y}_n + k_{n1}y_1 + k_{n2}y_n + \dots\dots + k_{nn}y_n = f_n(k)y_s$$

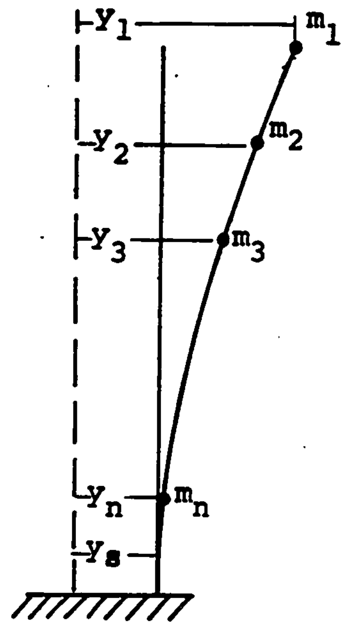


Figure 1.1: Column subjected to horizontal base motion.

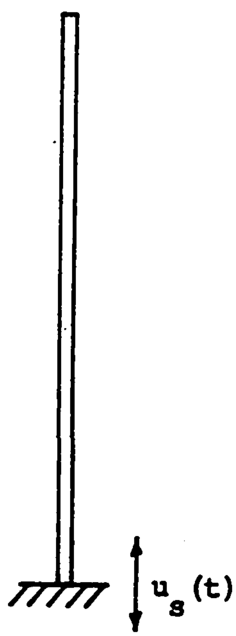


Figure 1.2: Column subjected to vertical support motion.

where k_{ij} are the stiffness coefficients and represent the force at station i due to a unit displacement at station j, the dots indicate differentiation w.r.t. time.

In matrix form the above equations may be written as:-

$$\left[\underset{\sim}{M}\right]\left\{\ddot{y}\right\} + \left[K\right]\left\{\underset{\sim}{y}\right\} = y_s\left\{f(k)\right\} \qquad (1.1)$$

Equations 1.1 are second order linear differential equations with constant coefficients.

By contrast, if the cantilever column of Figure 1.2 is subjected to fluctuating vertical support motion then vertical accelerations are present and this causes the axial force in the cantilever to vary with time; the effective column stiffnesses are no longer constant but become functions of time. The resulting differential equations have varying coefficients. If the vertical support motion is periodic, the differential equation will also have periodic coefficients. Such differential equations are termed Mathieu-Hill equations [3] and are encountered in various areas of physics, engineering and celestial mechanics. Therefore, the crucial difference between those structures subjected to horizontal support motion and those subjected to vertical support motion is that in the former case we get differential equations with constant coefficients whilst the latter gives rise to

Figure 1.3: Column subjected to periodic axial load.

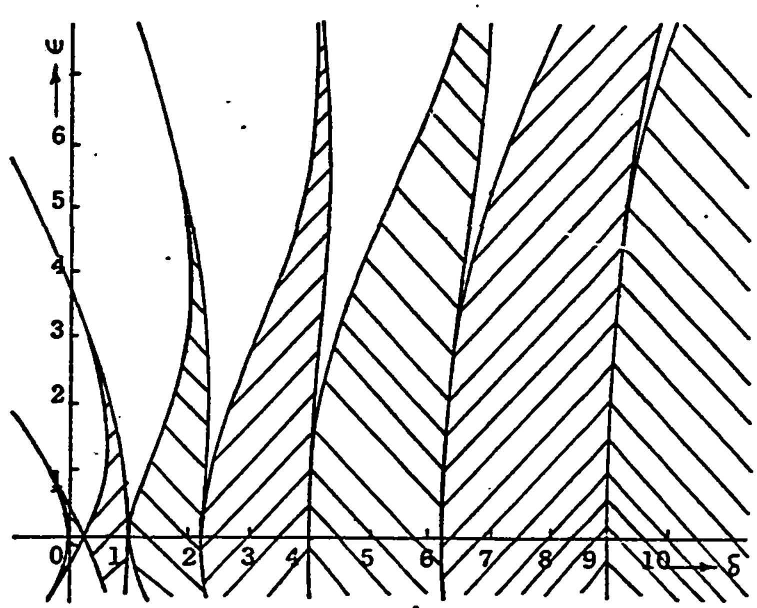


Figure 1.4: Stable and unstable regions for the Mathieu equation.
(Hatched areas represent stable regions)

differential equations with time-varying coefficients. As a result the characteristics of the response of structures to the two types of support motion are different and will be discussed later. In particular, when the governing equations are of the Mathieu-Hill type it is possible to have solutions which grow without limit as time progresses. Such behaviour is dynamic instability and is termed "parametric" to distinguish it from ordinary resonance phenomena.

The problem of the stability of a uniform bar subjected to time-varying axial forces was first studied by Beliaev [8] and has been reported by Timoshenko and Gere [4] and Bolotin [5]. Beliaev studied the case of a column with hinged ends (Figure 1.3) subjected to the action of an axial compressive force, $V_c+V_t \cos\Omega t$. Experience has shown that for certain cases slender bars can, without buckling, sustain instantaneously greater loads, (V_c+V_t) than the Euler buckling load for the column. Also, at certain values of the frequency Ω the lateral motion $y(t)$ of the column becomes unstable. The column then tends to oscillate with an amplitude which increases with time, being finally limited by system changes such as the nonlinearity of the restoring force at large amplitude. Such dynamically unstable behaviour occurs when the exciting frequency Ω is double the response frequency w of the system; this contrasts with the more familiar resonance behaviour of a forced oscillatory system, which occurs when the exciting frequency and response frequency are equal. Another distinguishing feature of parametric resonance lies in the possibility of producing resonance with exciting frequencies less than the natural frequency of the structure. Also parametric resonance differs from ordinary resonance by the fact that there exist continuous regions of dynamic instability. To illustrate the general nature of parametric resonance, consider the problem of the stability of a uniform bar with hinged ends and subjected to an axial pulsating load $V_c+V_t \cos \Omega t$ as shown in Figure 1.3. Assuming that the bar is initially straight and perfectly elastic, and ignoring the rotational inertia of the bar, the differential equation of motion is

$$EI \frac{\partial^4 y}{\partial x^4} + (V_c + V_t \cos \Omega t) \frac{\partial^2 y}{\partial x^2} + m_o \frac{\partial^2 y}{\partial t^2} = 0 \qquad (1.2)$$

where m_o is the mass per unit length of the bar. If we seek the solution of Equation 1.2 in the form

$$y(x,t) = f_k(t) \sin \frac{k \pi x}{L}, \quad (k = 1,2,3....) \qquad (1.3)$$

Equation 1.2 reduces to

$$\frac{d^2 f_k}{dt^2} + w_k^2 \left(1 - \frac{V_c + V_t \cos \Omega t}{P_k}\right) f_k = 0, \quad (k = 1,2,3..) \qquad (1.4)$$

where

$$w_k = \frac{k^2\pi^2}{L^2} \frac{EI}{m_o} \tag{1.5}$$

is the kth frequency of the free vibrations of an unloaded bar and

$$P_k = \frac{k^2\pi^2 EI}{L^2} \tag{1.6}$$

is the kth Euler buckling load.

Equation 1.4 may be further written as:-

$$\frac{d^2 f_k}{dt^2} + \beta_{ok}^2 (1 - 2\mu_k \cos \Omega t) f_k = 0, \quad (k = 1,2,3...) \tag{1.7}$$

where β_{ok} is the frequency of the free vibrations of the bar loaded by a constant longitudinal force V_c, i.e.

$$\beta_{ok}^2 = w_k^2 (1 - V_c/P_k) \tag{1.8}$$

and
$$2\mu_k = V_t/(P_k - V_c) \tag{1.9}$$

Since Equation 1.7 is identical for all forms of vibrations, the suffix k may be omitted without loss of generality and Equation 1.7 then writes as:-

$$\ddot{f} + \beta_o^2 (1 - 2\mu \cos \Omega t) f = 0 \tag{1.10}$$

or
$$\ddot{f} + (\delta + \epsilon \cos \Omega t) f = 0 \tag{1.10a}$$

where $\delta = \beta_o^2$ and $\epsilon = -2\mu\beta_o^2$

Equation 1.10 is the well-known Mathieu equation which has an extensive literature, for example the book by McLachlan[3].

To investigate the stability of the motion, the solutions of the Mathieu equation have to be studied. A given motion is stable if all the solutions of the Mathieu equation are bounded for all positive values of t, and unstable if the equation has an unbounded solution. The boundedness of the solutions depend upon the relationship between δ and ϵ. For certain values of δ and ϵ bounded solutions are obtained, whilst for other values unbounded solutions result. The regions of bounded and unbounded solutions in the δ-ϵ plane are shown in Figure 1.4.

Lubkin[6] and Lubkin and Stoker[7] studied the stability of columns and strings under periodically varying forces, sinusoidal in nature. Apparently, they were not aware of the work of Beliaev[8]. They showed that the problem of columns and strings subjected to periodically varying axial forces reduces to a Mathieu equation. They studied the stability of the Mathieu equation and determined the co-ordinates of the boundary points of the regions of instability. Utida and Sezawa[9] also independently studied the dynamic stability of columns under periodic longitudinal forces. Their study was both theoretical and experimental in

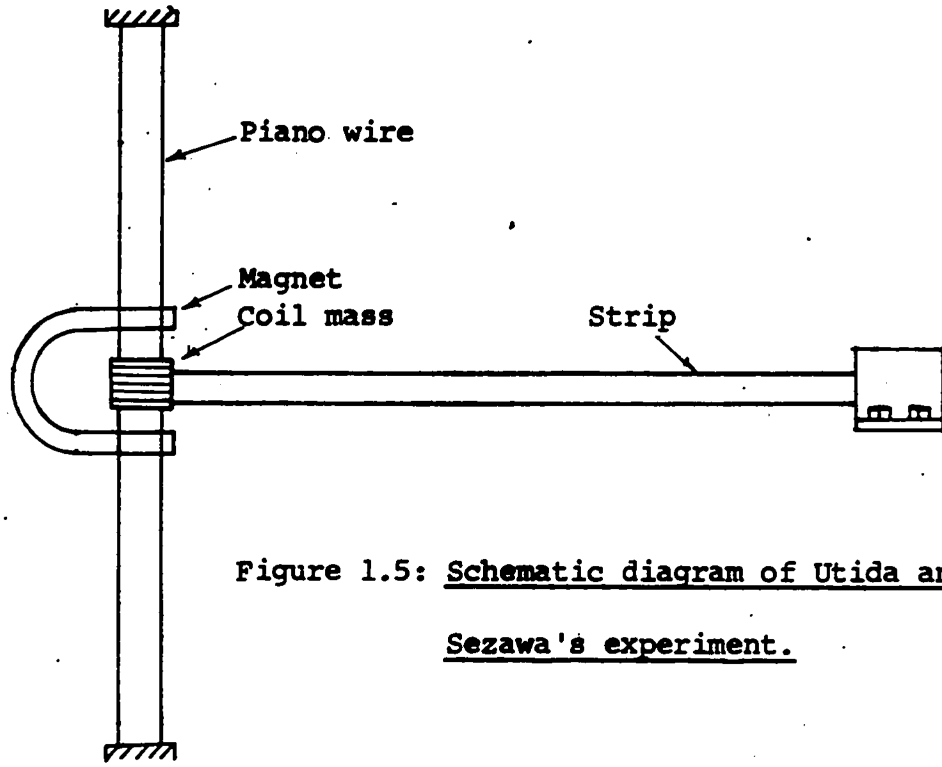


Figure 1.5: Schematic diagram of Utida and Sezawa's experiment.

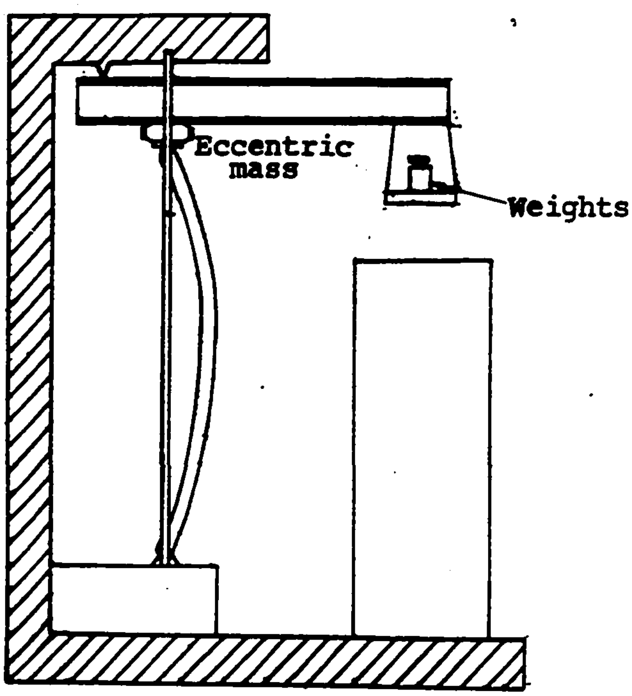


Figure 1.6: Schematic diagram of Bolotin's experiment.

character. They also showed that the problem of a column subjected to periodic longitudinal forces reduces to a Mathieu equation. Experimentally they investigated the parametric stability of brass strips 0.5 mm thick and 20.7 mm wide with one end being clamped and the other having a concentrated mass, which was supported by piano wires to permit axial movement as shown in Figure 1.5. They observed that the deflection and slope at the end carrying the concentrated mass were almost zero. The length of the strip measured from its clamped end to the centre of the concentrated mass was 353 mm. Concentrated mass weights of 1000 gms and 250 gms were used. The longitudinal force $V = V_t \cos \Omega t$ ($V_c = 0$) was generated electromagnetically. The concentrated mass consisted of a heavy coil around which a constant magnetic field was created by passing D.C. current. The passage of an alternating current gave rise to the axial load. Utida and Sezawa were the first to verify experimentally the existence of secondary regions of instability.

Bolotin[5] carried out an experimental investigation of the dynamic stability of columns. Unlike Utida and Sezawa the periodic axial force consisted of a constant part together with a harmonic portion, i.e. $V_c \neq 0$. A schematic diagram of his experimental set-up is shown in

Figure 1.6. The lower end of the column is pinned and fixed in position; the upper end is also pinned but allowed to move vertically. A set of weights placed at the end of a lever supplied the constant term V_c; the fluctuating component $V_t \cos \Omega t$ was generated by a rotating eccentric mass situated at the upper end of the column. From the tests carried out, Bolotin determined the principal region of dynamic instability for pinned columns and this was found to be in good agreement with theory. However, he failed to verify experimentally the existence of the higher (secondary) regions.

Weingarten[10] also conducted experiments to investigate the parametric instability of columns subjected to periodic axial loads of the form $V = V_t \cos \Omega t$. His results confirmed the existence of secondary regions of instability. He also investigated the effect of boundary conditions on the stability of the rod and studied the following two cases.

(a) Column simply-supported at both ends.

(b) Column clamped at both ends (for which he presented a theoretical formulation).

He found out that the instability regions for the two cases are similar, i.e. they are independent of the boundary conditions, and obtained an experimental verification.

Somerset and Evan-Iwanowski[11] also carried out an extensive experimental investigation of the principal region of dynamic instability of columns pinned at both ends subjected to periodic axial forces of the type $V = V_c + V_t \cos \Omega t$. The experiments were so designed as to vary independently the parameters V_c, V_t and Ω. They also studied the effect of damping and it was achieved by immersing the column in a fluid. Light oil and water were used as damping fluids.

Jaeger and Barr[2,12] investigated the problem of the stability of a cantilever column subjected to periodic vertical support motion. They showed that the governing equations of motion, initially a partial differential equation, may be reduced, as an approximation, to an ordinary differential equation of the Mathieu type by employing the Galerkin Method[13,14].

1.2. Purpose and Scope of Investigation

The purpose of the investigation is to study the dynamic stability of structures subjected to periodic axial forces and periodic support motion. A numerical method of determining the regions of dynamic stability of a uniform column subjected to a periodic longitudinal force and different boundary conditions has been developed. All the previous investigations had been confined to the testing of columns with periodic longitudinal forces only. The author carried out an experimental investigation of the parametric stability of columns subjected to periodic support motion. A numerical method formulation is also presented.

An experimental investigation of the dynamic stability of portal frames subjected to periodic support motion was undertaken. The study of the dynamic stability of frames requires a knowledge of their natural frequencies. A study was undertaken to find a suitable method for the determination of the natural frequencies of structures. It was found that the finite element method was an efficient and powerful method and was suitable for computer programming. Computer programs were written and the results obtained for different types of structures were compared with known experimental results.

CHAPTER II

NATURAL FREQUENCIES OF PLANE STRUCTURES BY A FINITE ELEMENT METHOD

2.1. Introduction

A number of methods are available in the literature for determining the natural frequencies of plane structures and some of them employ computer techniques. For simple structures like beams and columns various classical methods are available, a good account of which may be found in the texts of Timoshenko [15], Biggs [16], and Minhinnick [17]. A brief review of the major methods available for portal frames will now be presented.

Bishop [18,19,20] used the method of receptances to determine the natural frequencies. The forced vibrations of the system are expressed in terms of the receptances of its component parts, which have been tabulated for reference. At the natural frequencies the overall structure receptance vanishes and by a trial and error method the desired frequencies are obtained. Bishop's example involved only single-bay, single storey frames. Gladwell [21] extended Bishop's method of receptances to the analysis of multi-bay, multi-storey portal frames. Hurty [22] determined the natural modes and frequencies of structural systems by an energy method approach, which is a variation of the Rayleigh-Ritz method, using mode functions applicable to the complete system or subsystems.

Newmark (23) proposed a numerical integration technique for the computation of the dynamic response of structures. Chaudhury et al (24) also used a numerical integration method for the dynamic analysis of structural frameworks. However, numerical integration methods are not suitable for the determination of the natural frequency and the normal modes. The finite difference method has also been used for the determination of the natural frequencies, e.g. Livesley (25) employed it to obtain the natural frequencies of beams, Allman and Brotton (26) for plane structures, Cox and Denke (27) and Ellington and McCallion (28) for grillages. A good review of the other available methods for grillages is given by Hendry (29) and Rogers (30).

Ariaratnam (31) presented a method for analysing the dynamic behaviour of a plane structure which takes into account its distributed mass. The method is based on the stiffness approach of Livesley (32) for the elastic stability analysis of frameworks. Force-displacement equations of an individual member are obtained by combining the solutions of the differential equations governing the longitudinal and the transverse vibrations of the member. The natural frequencies are the roots of the resulting transcendental equation.

The matrix formulation of the general dynamic problem without damping leads to the equation

$$\left[\underset{\sim}{M}\right]\left\{\underset{\sim}{\ddot{u}}\right\} + \left[\underset{\sim}{K}\right]\left\{\underset{\sim}{u}\right\} = \left\{\underset{\sim}{Q}\right\} \tag{2.1}$$

where $\underset{\sim}{M}$ is the mass matrix

$\underset{\sim}{K}$ is the stiffness matrix

$\underset{\sim}{u}$ is the displacement matrix

$\underset{\sim}{Q}$ is the force matrix

In the early attempts to deal with dynamic problems the lumped mass procedure was used, and the mass matrix $\underset{\sim}{M}$, therefore, was constructed by lumping the masses at nodal or station points. The mass matrix $\underset{\sim}{M}$ obtained was a diagonal matrix. The natural frequencies may then be obtained by substituting $\underset{\sim}{Q} = 0$ in Equation (2.1) and this technique is well covered in the texts by Biggs [16] and Hurty and Rubinstein [33].

It is known that when the distributed mass of a body is replaced by an equal mass concentrated at the centre of gravity of the section or at node points, two primary inaccuracies are introduced into the analysis through the implied suppositions that

(i) The resultant of inertia actions of the elementary masses always passes through the centre of gravity of these masses.

(ii) A concentrated load produces the same deflection as a distributed load of which it is the resultant.

define the state of strain and stress within the element. By applying the principle of virtual work the work done by the internal forces may be equated to that done by the external forces, and thereby the solution for the unknown displacements may be obtained.

Different types of elements and various displacement fields may be chosen depending upon the type of the problem. The mathematical formulation is as follows:-

(a) Displacement field

Let the displacements at any point within the element be defined by $\underset{\sim}{f}$, a column vector.

$$\underset{\sim}{f}\,(x,y) = \left[\underset{\sim}{N}\right]\left\{\underset{\sim}{d}\right\} \tag{2.2}$$

where $\underset{\sim}{d}$ is the column vector of nodal displacements

$\underset{\sim}{N}$ is the 'shape' matrix and is a function of (x,y) and depends upon the assumed displacement field.

e.g. for a flat triangular element in plane stress (see fig. 2.1).

$$\underset{\sim}{f} \equiv \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} \equiv \begin{Bmatrix} \alpha_1 + \alpha_2 x + \alpha_3 y \\ \alpha_4 + \alpha_5 x + \alpha_6 y \end{Bmatrix}$$

$$\left\{\underset{\sim}{d}\right\} = \begin{Bmatrix} d_i \\ d_j \\ d_m \end{Bmatrix} \equiv \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix} .$$

$$\left[\underset{\sim}{N}\right] = \left[\underset{\sim}{N}_i \quad \underset{\sim}{N}_j \quad \underset{\sim}{N}_m\right] .$$

Archer [34] and Leckie and Lindberg [35] showed that an equivalent mass matrix, also called a consistent mass matrix, gives better results than the method of lumped masses. A good account of equivalent mass matrices is given by Przemieniecki [36]. The vibrations of beams using finite elements has been shown to yield good results [34,35,37,38,39].

The author has used the finite element method for determining the natural frequencies of non-uniform plane structures and the results obtained show good agreement with known experimental results [40].

2.2. The Finite Element Method

The theory of the finite element method is well covered in the texts by Zienkiewicz and Cheung [41] and Przemieniecki [36]. A brief account of its characteristics will be given.

In the finite element method the actual structure is divided into elements interconnected only at a finite number of points, called nodal points, at which some fictitious forces, representative of the distributed stresses actually acting on the element boundaries are supposed to act. The displacements of the nodal points, $\underset{\sim}{d}$, are the basic unknown parameters of the problem. A displacement function is assumed to define the displacement field within the element in terms of the nodal displacements. The displacement function serves to

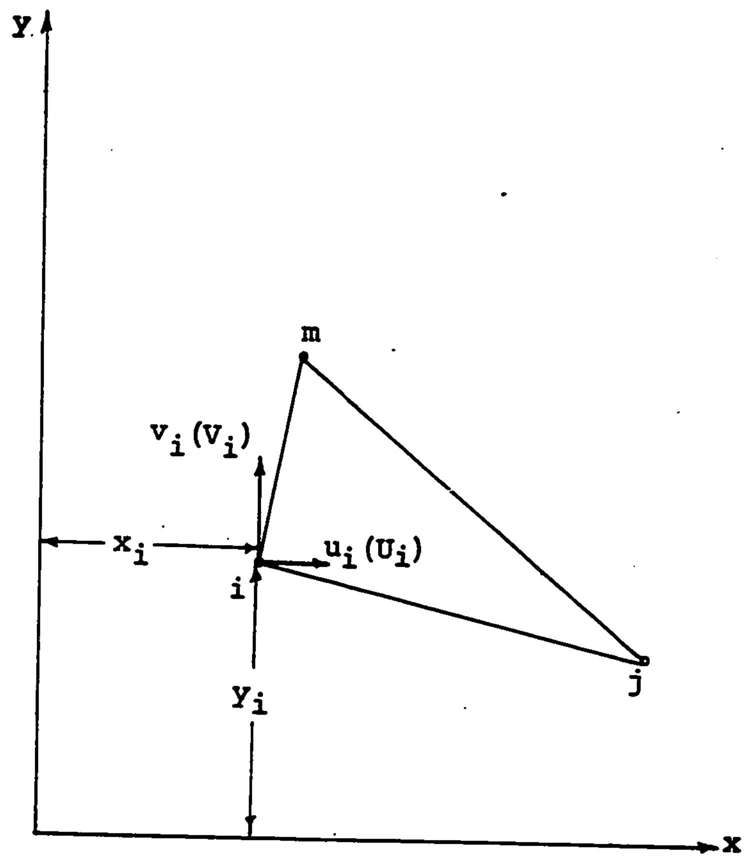


Figure 2.1:A triangular element in plane stress.

(b) Strains

The strain at any point within the element may be expressed as

$$\left\{ \underset{\sim}{e} \right\} = \left[\underset{\sim}{B} \right] \left\{ \underset{\sim}{d} \right\} \tag{2.3}$$

where $\underset{\sim}{e}$ is the strain vector

$\underset{\sim}{B}$ is the strain matrix and may be easily obtained from matrix $\left[\underset{\sim}{N} \right]$.

For the plane stress case

$$\left\{ \underset{\sim}{e} \right\} = \begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \tag{2.3a}$$

(c) Stresses

If the initial stresses within the element are denoted by $\left\{ \underset{\sim}{e}_o \right\}$, assuming general elastic behaviour, the relationship between stresses and strains may be expressed as

$$\left\{ \underset{\sim}{\sigma} \right\} = \left[\underset{\sim}{D} \right] \left(\left\{ \underset{\sim}{e} \right\} - \left\{ \underset{\sim}{e}_o \right\} \right) \tag{2.4}$$

where $\underset{\sim}{\sigma}$ is the stress vector

$\underset{\sim}{D}$ is the elasticity matrix containing the material properties.

Again, for the case of plane stress, we have

$$\left\{ \underset{\sim}{\sigma} \right\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} \tag{2.4a}$$

For an isotropic material

$$\underset{\sim}{D} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix} \tag{2.4b}$$

(d) <u>Equivalent Nodal Forces</u>

Let

$$\left\{\underset{\sim}{F}\right\} = \begin{Bmatrix} F_i \\ F_j \\ F_m \end{Bmatrix} \tag{2.5}$$

define the nodal forces which are statically equivalent to the free boundary stresses and distributed loads on the element. The distributed loads $\left\{\underset{\sim}{p}\right\}$ are defined as those acting on a unit volume of the material within the element.

For the case of plane stress

$$\left\{\underset{\sim}{F_i}\right\} = \begin{Bmatrix} U_i \\ V_i \end{Bmatrix} \tag{2.5a}$$

where U_i and V_i are the components of forces in the x and y direction respectively and the distributed load is

$$\left\{\underset{\sim}{p}\right\} = \begin{Bmatrix} X \\ Y \end{Bmatrix} \tag{2.5b}$$

in which X, Y are the 'body force' components in the x and y directions respectively.

The principle of virtual work is used to determine the fictitious nodal forces statically equivalent to the actual boundary stresses and distributed loads. By equating the work done by the external and internal forces it may be shown that

$$\{\underset{\sim}{F}\}^e = \left(\int [\underset{\sim}{B}]^T [\underset{\sim}{D}][\underset{\sim}{B}] \, d\,(Vol)\right) \{\underset{\sim}{d}\} - \int [\underset{\sim}{B}][\underset{\sim}{D}]\{\underset{\sim}{e}_o\} \, d\,(Vol)$$
$$- \int [\underset{\sim}{N}]^T \{\underset{\sim}{p}\} \, d\,(Vol) \qquad (2.6)$$

where the superscript e refers to the element and

$[\;]^T$ defines the transpose of a matrix.

Let $$[\underset{\sim}{k}]^e = \int [\underset{\sim}{B}]^T [\underset{\sim}{D}] [\underset{\sim}{B}] \, d\,(Vol) \qquad (2.7a)$$

and is called the stiffness matrix.

$$\{\underset{\sim}{F}\}^e_p = -\int [\underset{\sim}{N}]^T \{\underset{\sim}{p}\} \, d\,(Vol) \qquad (2.7b)$$

and is called the distributed nodal force vector.

$$\{\underset{\sim}{F}\}^e_o = -\int [\underset{\sim}{B}]^T [\underset{\sim}{D}] \{\underset{\sim}{e}_o\} \, d\,(Vol) \qquad (2.7c)$$

and is called the initial nodal force vector.

Having determined the element stiffness matrix, the structure stiffness matrix, $\underset{\sim}{S}$, may be obtained by the appropriate addition of the element stiffness matrices. Since the element stiffness matrix is generated in the local (member) axis system it must be transformed into the global axis system before the addition for the formation of the overall structure stiffness matrix, $\underset{\sim}{K}$.

For the equilibrium of the assembled structure, external nodal forces must be equal and opposite to the equivalent nodal forces

i.e. $\{\underset{\sim}{R}\} = \{\underset{\sim}{F}\}$

where $\underset{\sim}{R}$ is the external nodal force vector.

But $\{\underset{\sim}{F}\} = [\underset{\sim}{K}] \{\underset{\sim}{d}\} + \{\underset{\sim}{F}\}_p + \{\underset{\sim}{F}\}_o$

Hence $\left[\underset{\sim}{K}\right] \left\{\underset{\sim}{d}\right\} = \left\{\underset{\sim}{R}\right\} - \left\{\underset{\sim}{F}\right\}_p - \left\{\underset{\sim}{F}\right\}_o = \left\{\underset{\sim}{P}\right\}$ (2.8)

where $\underset{\sim}{P}$ gives the nodal forces.

Solving Equation (2.8) the unknown nodal displacements $\underset{\sim}{d}$ may be determined. Once the nodal displacements are known the strains and stresses may be obtained using relations (2.3) and (2.4) respectively.

If masses M_i are attached to the nodes of the structure, with no external forces acting there, we have

$$\left\{\underset{\sim}{R}_i\right\} = - M_i \frac{d^2}{dt^2} \left\{\underset{\sim}{d}_i\right\} = - M_i \left\{\ddot{\underset{\sim}{d}}_i\right\} \tag{2.9}$$

Let the inertia loading be represented by $\left\{\underset{\sim}{p}\right\}$. If the mass per unit volume is ρ, then again with no external forces acting we have

$$\left\{\underset{\sim}{p}\right\} = -\rho \frac{d^2 f}{dt^2} = -\rho \left\{\ddot{\underset{\sim}{f}}\right\} \tag{2.10}$$

Since $\left\{\underset{\sim}{f}\right\} = \left[\underset{\sim}{N}\right] \left\{\underset{\sim}{d}\right\}$, we obtain

$$\left\{\underset{\sim}{p}\right\} = -\rho\left[\underset{\sim}{N}\right] \left\{\ddot{\underset{\sim}{d}}\right\} \tag{2.11}$$

and since the equivalent nodal forces due to the inertia loading $\left\{\underset{\sim}{p}\right\}$ are

$$\left\{\underset{\sim}{F}\right\}_p^e = -\int \left[\underset{\sim}{N}\right]^T \left\{\underset{\sim}{p}\right\} d(\text{vol})$$

we have $\left\{\underset{\sim}{F}\right\}_p^e = \int \left[\underset{\sim}{N}\right]^T \rho \left[\underset{\sim}{N}\right] d(\text{vol}) \left\{\ddot{\underset{\sim}{d}}\right\}$

$$= \left[m\right]_e \left\{\ddot{\underset{\sim}{d}}\right\} \tag{2.12}$$

where $\left[m\right]_e$ is the element mass matrix.

A typical element m_{ij} of the mass matrix represents the mass inertia load at point i developed by a unit acceleration at point j with all other points stationary.

The equivalent nodal forces due to the inertia loading, $\{\underset{\sim}{F}\}_p^e$, may now be added to those already present at the nodes, thus

$$\{\underset{\sim}{P}\} = -\left([\underset{\sim}{M}^o] + [\underset{\sim}{M}_s]\right)\{\ddot{\underset{\sim}{d}}\} + \{\underset{\sim}{R}\}$$

$$\text{Where } [M^o] = \begin{bmatrix} M_1 & 0 & 0 \cdots 0 \\ 0 & M_2 & 0 \cdots 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0 \cdots M_n \end{bmatrix} \qquad (2.13)$$

is the matrix of external masses actually attached to the nodes. $\underset{\sim}{M}_s$ is the overall mass matrix obtained from the assembly of the element mass matrices, $[m]_e$.

The general formulation now writes as

$$[\underset{\sim}{K}]\{\underset{\sim}{d}\} = -\left([\underset{\sim}{M}^o] + [\underset{\sim}{M}_s]\right)\{\ddot{\underset{\sim}{d}}\} + \{\underset{\sim}{R}\} \qquad (2.14)$$

$\{\underset{\sim}{R}\}$ has been retained in case actual external forces are still active during the motion.

For the particular case of free vibrations, the above equation becomes

$$[\underset{\sim}{K}]\{\underset{\sim}{d}\} = -[\underset{\sim}{M}]\{\ddot{\underset{\sim}{d}}\} \qquad (2.15)$$

$$\text{where } [\underset{\sim}{M}] = \left([\underset{\sim}{M}^o] + [\underset{\sim}{M}_s]\right) \qquad (2.15a)$$

Assuming the free vibrations to be harmonic, the displacements $\underset{\sim}{d}$ may be written as

$$\{\underset{\sim}{d}\} = \underset{\sim}{q}e^{iwt} \qquad (2.16)$$

where $\underset{\sim}{q}$ is a column matrix of the amplitudes of the displacements $\underset{\sim}{d}$

w is the natural frequency of vibration

t is the time

Substituting Equation (2.16) in (2.15) it reduces to

$$(\underset{\sim}{K} - w^2 \underset{\sim}{M}) \underset{\sim}{q} = 0 \qquad (2.17)$$

This is now in the standard form of eigenvalue determination.

2.3 Development of Element Characteristics

a) Plane Frames

For the case of plane frameworks a beam element with three degrees of freedom (two translations and one rotation*) at each node, as shown in Figure 2.2, has been used. The mass matrix for a beam element, referred to member coordinate axes and ignoring its rotary inertia is

$$\underset{\sim}{m}' = \frac{\rho AL}{420} \begin{bmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ 0 & 156 & 22L & 0 & 54 & -13L \\ 0 & 22L & 4L^2 & 0 & 13L & -3L^2 \\ 70 & 0 & 0 & 140 & 0 & 0 \\ 0 & 54 & 13L & 0 & 156 & -22L \\ 0 & -13L & -3L^2 & 0 & -22L & 4L^2 \end{bmatrix} \qquad (2.18)$$

where ρ is the density of the material of the element

A is the cross-sectional area of the element

L is the length of the element.

If the element is oriented as shown in Figure 2.3 the element mass matrix referred to the global coordinate axes is given by Equation (2.19) where $C_x = \cos\theta$ is the direction cosine of

* Rotation about an axis is represented by a double headed arrow.

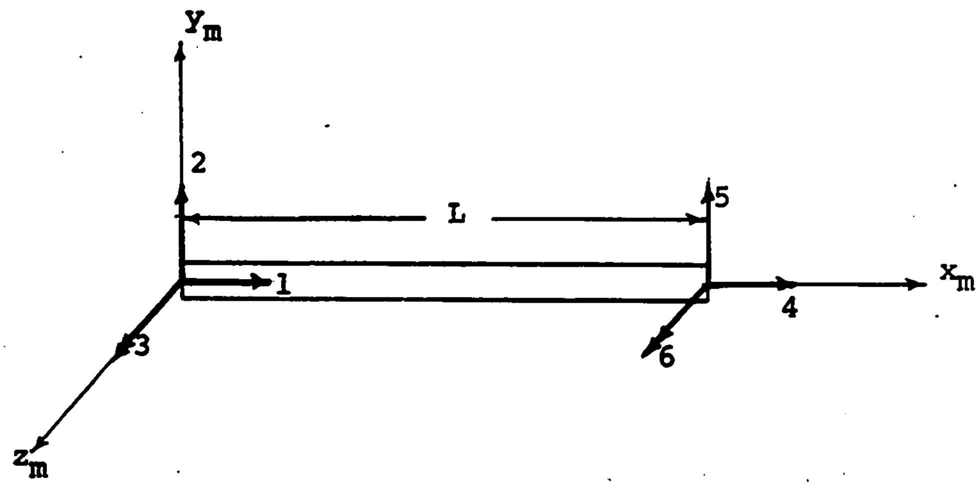


Figure 2.2: Beam element in member coordinate axes.

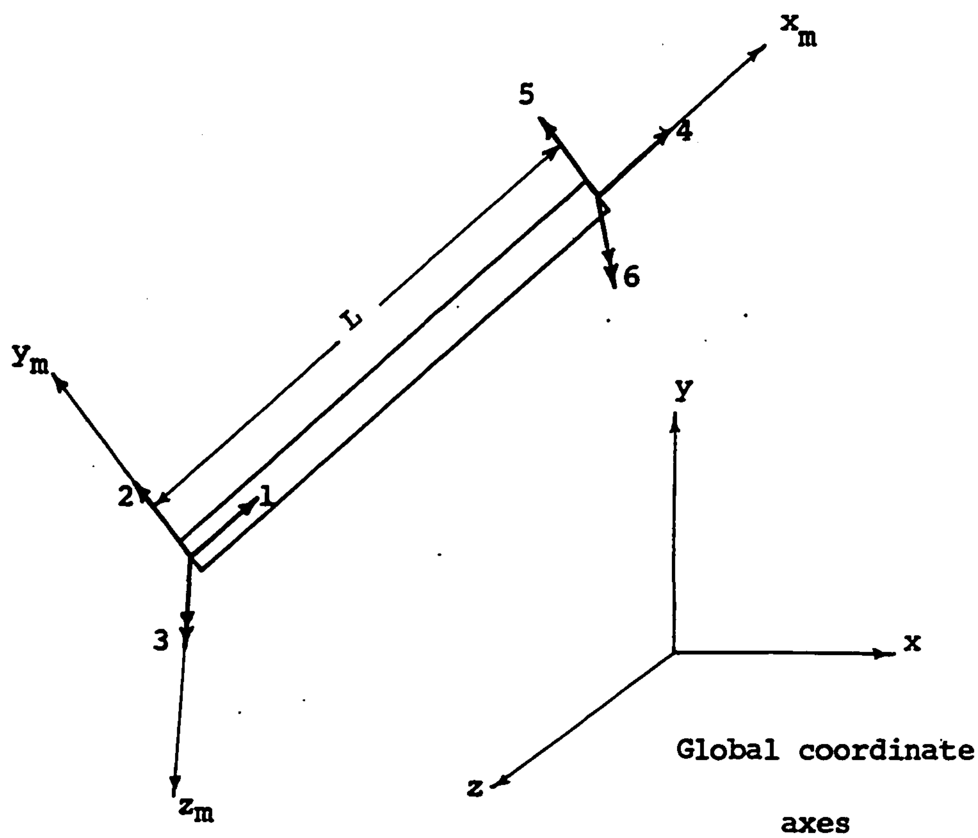


Figure 2.3: Beam element in global coordinate axes.

the member with respect to the X axis, $C_y = \sin\theta$ is the direction cosine of the member with respect to the Y axis.

In a similar way the element stiffness matrix referred to global coordinate axes is given by Equation (2.20) where I_z = Moment of inertia about the Z axis.

b) Grids

In the case of grillages, a beam element with three degrees of freedom (two rotations and one translation) at each node, as shown in Figure 2.4a has been used. The mass matrix for a beam element with respect to member coordinate axes and ignoring its rotary inertia is

$$\underset{\sim}{m}' = \frac{\rho AL}{420}\begin{bmatrix} 140\ J_x/A & 0 & 0 & 70\ J_x/A & 0 & 0 \\ 0 & 4L^2 & -22L & 0 & -3L^2 & -13L \\ 0 & -22L & 156 & 0 & 13L & 63 \\ 70\ J_x/A & 0 & 0 & 140\ J_x/A & 0 & 0 \\ 0 & -3L^2 & 13L & 0 & 4L^2 & 22L \\ 0 & -13L & 63 & 0 & 22L & 156 \end{bmatrix} \qquad (2.21)$$

where J_x is the Torsion constant.

If the element is oriented as shown in Figure 2.4b the element mass matrix in global coordinate axes is given by Equation (2.22); and the element stiffness matrix by Equation (2.23).

Based on the above formulation a computer program was written to obtain the natural frequencies and nodal shapes of plane structures such as beams, grids, frames and trusses. The members of the structure may be of non-uniform cross-section and varying properties. A listing of the program is given in

$$
m = \frac{\rho AL}{420}\begin{bmatrix}
140C_x^2+156C_y^2 & -16C_xC_y & -22LC_y & 70C_x^2+54C_y^2 & 16C_xC_y & 13LC_y \\
-16C_xC_y & 140C_y^2+156C_x^2 & 22LC_x & 16C_xC_y & 70C_y^2+54C_x^2 & -13LC_x \\
-22LC_y & 22LC_x & 4L^2 & -13LC_y & 13LC_x & -3L^2 \\
70C_x^2+54C_y^2 & 16C_xC_y & -13LC_y & 140C_x^2+156C_y^2 & -16C_xC_y & 22LC_y \\
16C_xC_y & 70C_y^2+54C_x^2 & 13LC_x & -16C_xC_y & 140C_y^2+156C_x^2 & -22LC_x \\
13LC_y & -13LC_x & -3L^2 & 22LC_y & -22LC_x & 4L^2
\end{bmatrix} \tag{2.19}
$$

$$
k = \frac{EI_z}{L^3}\begin{bmatrix}
\left(\frac{AL^2}{I_z}C_x^2+12C_y^2\right) & \left(\frac{AL^2}{I_z}-12\right)C_xC_y & -6LC_y & -\left(\frac{AL^2}{I_z}C_x^2+12C_y^2\right) & -\left(\frac{AL^2}{I_z}-12\right)C_xC_y & -6LC_y \\
\left(\frac{AL^2}{I_z}-12\right)C_xC_y & \left(\frac{AL^2}{I_z}C_y^2+12C_x^2\right) & 6LC_x & -\left(\frac{AL^2}{I_z}-12\right)C_xC_y & -\left(\frac{AL^2}{I_z}C_y^2+12C_x^2\right) & 6LC_x \\
-6LC_y & 6LC_x & 4L^2 & 6LC_y & -6LC_x & 2L^2 \\
-\left(\frac{AL^2}{I_z}C_x^2+12C_y^2\right) & -\left(\frac{AL^2}{I_z}-12\right)C_xC_y & 6LC_y & \left(\frac{AL^2}{I_z}C_x^2+12C_y^2\right) & \left(\frac{AL^2}{I_z}-12\right)C_xC_y & 6LC_y \\
-\left(\frac{AL^2}{I_z}-12\right)C_xC_y & -\left(\frac{AL^2}{I_z}C_y^2+12C_x^2\right) & -6LC_x & \left(\frac{AL^2}{I_z}-12\right)C_xC_y & \left(\frac{AL^2}{I_z}C_y^2+12C_x^2\right) & -6LC_x \\
-6LC_y & 6LC_x & 2L^2 & 6LC_y & -6LC_x & 4L^2
\end{bmatrix} \tag{2.20}
$$

$$m=\frac{\rho AL}{420}\begin{bmatrix} 140\frac{J_x}{A}C_x^2+4L^2C_y^2 & 140\frac{J_x}{A}C_xC_y-4L^2C_xC_y & 22LC_y & 70\frac{J_x}{A}C_x^2-3L^2C_y^2 & 70\frac{J_x}{A}C_xC_y+3L^2C_xC_y & 13LC_y \\ 140\frac{J_x}{A}C_xC_y-4L^2C_xC_y & 140\frac{J_x}{A}C_y^2+4L^2C_x^2 & -22LC_x & 70\frac{J_x}{A}C_xC_y+3L^2C_xC_y & 70\frac{J_x}{A}C_y^2-3L^2C_x^2 & -13LC_x \\ 22LC_y & -22LC_x & 156 & -13LC_y & 13LC_x & 63 \\ 70\frac{J_x}{A}C_x^2-3L^2C_y^2 & 70\frac{J_x}{A}C_xC_y+3L^2C_xC_y & -13LC_y & 140\frac{J_x}{A}C_x^2+4L^2C_y^2 & 140\frac{J_x}{A}C_xC_y-4L^2C_xC_y & -22LC_y \\ 70\frac{J_x}{A}C_xC_y+3L^2C_xC_y & 70\frac{J_x}{A}C_y^2-3L^2C_x^2 & 13LC_x & 140\frac{J_x}{A}C_xC_y-4L^2C_xC_y & 140\frac{J_x}{A}C_y^2+4L^2C_x^2 & 22LC_x \\ 13LC_y & -13LC_x & 63 & -22LC_y & 22LC_x & 156 \end{bmatrix} \quad (2.22)$$

$$k=\begin{bmatrix} \frac{GI_x}{L}C_x^2+\frac{4EI_y}{L}C_y^2 & \left(\frac{GI_x}{L}-\frac{4EI_y}{L}\right)C_xC_y & \frac{6EI_y}{L^2}C_y & -\frac{GI_x}{L}C_x^2+\frac{2EI_y}{L}C_y^2 & -\left(\frac{GI_x}{L}+\frac{2EI_y}{L}\right)C_xC_y & -\frac{6EI_y}{L^2}C_y \\ \left(\frac{GI_x}{L}-\frac{4EI_y}{L}\right)C_xC_y & \frac{GI_x}{L}C_y^2+\frac{4EI_y}{L}C_x^2 & -\frac{6EI_y}{L^2}C_x & -\left(\frac{GI_x}{L}+\frac{2EI_y}{L}\right)C_xC_y & -\frac{GI_x}{L}C_y^2+\frac{2EI_y}{L}C_x^2 & \frac{6EI_y}{L^2}C_x \\ \frac{6EI_y}{L^2}C_y & -\frac{6EI_y}{L^2}C_x & \frac{12EI_y}{L^3} & \frac{6EI_y}{L^2}C_y & -\frac{6EI_y}{L^2}C_x & -\frac{12EI_y}{L^3} \\ -\frac{GI_x}{L}C_x^2+\frac{2EI_y}{L}C_y^2 & -\left(\frac{GI_x}{L}+\frac{2EI_y}{L}\right)C_xC_y & \frac{6EI_y}{L^2}C_y & \frac{GI_x}{L}C_x^2+\frac{4EI_y}{L}C_y^2 & \left(\frac{GI_x}{L}-\frac{4EI_y}{L}\right)C_xC_y & -\frac{6EI_y}{L^2}C_y \\ -\left(\frac{GI_x}{L}+\frac{2EI_y}{L}\right)C_xC_y & -\frac{GI_x}{L}C_y^2+\frac{2EI_y}{L}C_x^2 & -\frac{6EI_y}{L^2}C_x & \left(\frac{GI_x}{L}-\frac{4EI_y}{L}\right)C_xC_y & \frac{GI_x}{L}C_y^2+\frac{4EI_y}{L}C_x^2 & \frac{6EI_y}{L^2}C_x \\ -\frac{6EI_y}{L^2}C_y & \frac{6EI_y}{L^2}C_x & -\frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2}C_y & \frac{6EI_y}{L^2}C_x & \frac{12EI_y}{L^3} \end{bmatrix} \quad (2.23)$$

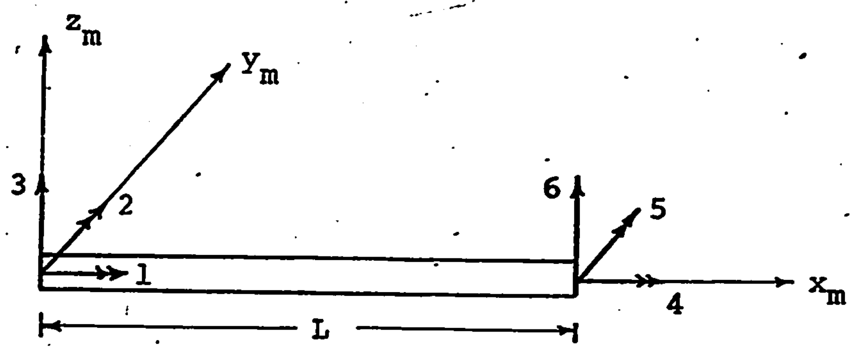


Figure 2.4a: Grid element in member coordinate axes.

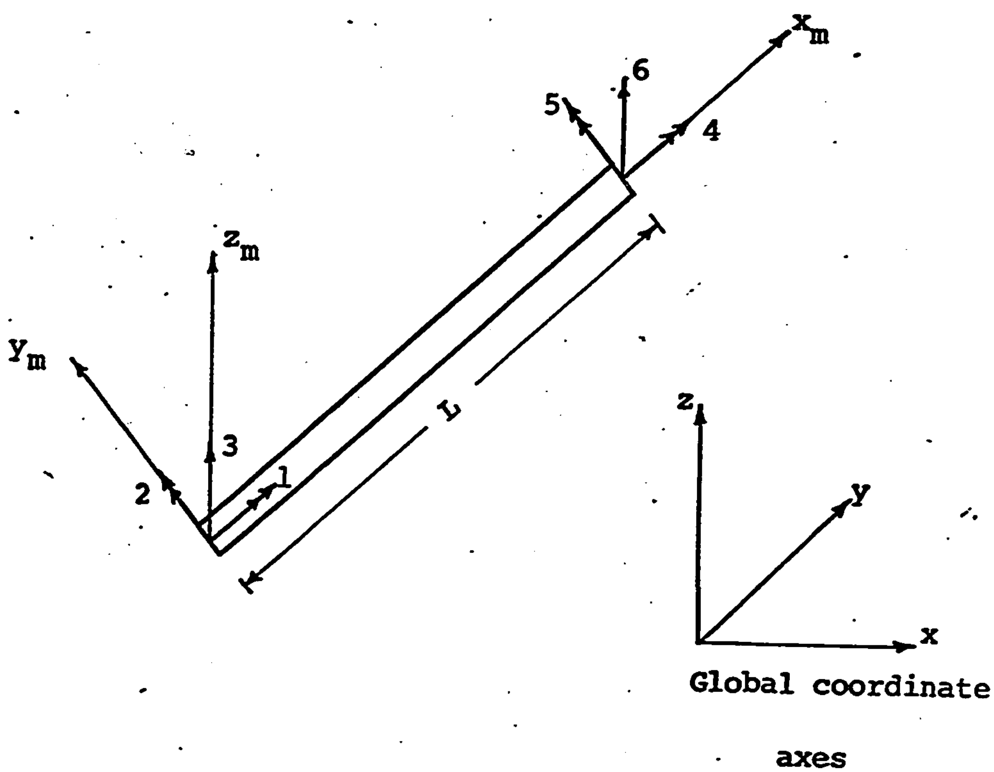


Figure 2.4b: Grid element in global coordinate axes.

Appendix C .

The program was used to determine the natural frequencies of the structures shown in Figures 2.5 to 2.8 and the results obtained have been reported in [39]. The results of the program were compared with those obtained experimentally and analytically by Bishop and Johnson [20], Rieger and McCallion [42], Cheng [43], and Zeidan [44].

2.4. DISCUSSION OF RESULTS

Bishop tested the frames shown in Fig. 2.5. The theoretical natural frequencies were obtained by the method of receptances. Rieger and McCallion employed a finite differences approach in obtaining the theoretical natural frequencies. The frames tested by them are shown in Fig. 2.6. Cheng determined the theoretical natural frequencies of the frame of Fig. 2.7 by a stiffness approach but employed the lumped mass technique in generating the mass matrix. Zeidan tested a 3-girder and a 4-girder grid as shown in Fig. 2.8.

For the analysis by the finite element method the frames of Fig. 5 were divided into elements as shown in Fig. 2.9. In Fig. 2.9(i) and 2.9(iv) each member of the frame is considered as one element, whilst in Fig. 2.9(ii) and 2.9(v) each member is subdivided into two elements, and in Fig. 2.9(iii) into 3 elements. Fig. 2.9(vi) shows the frame divided in such a way that each element is 2" long. The comparison of results is illustrated in Table 2.1.

Figure 2.10 shows two different subdivisions for the structure of Fig. 2.6. In the first one each member is considered as one element, whilst in the second it is divided into two elements. Since Rieger and McCallion did not quote material density for the frames tested by them, the author has used a value of $\rho A = .000725$ lbs. sec^2/in^4 based on a specific weight of 480 lbs./ft^3 for steel. The results for Fig. 2.6(a) to 6(d) and 6(h) are shown in Table 2.2.

Three cases of subdivision as shown in Fig. 2.11 are analysed for the frame of Fig. 2.7 and the results are shown in Table 2.3.

For the 3-girder grid of Fig. 2.8(a) two subdivisions are considered, with each member being taken as one element or as two elements as shown in Fig. 2.12(i) and 2.12(ii) respectively. For the 4-girder grid of Fig. 2.8(b) only one case, that of each member as one element, is considered as is shown in Fig. 2.12(iii). The results are shown in Table 2.4.

From the results illustrated in Tables 2.1 to 2.4 it can be observed that very good results may be obtained by the finite element method. If only the first few frequencies are required a coarse subdivision can yield sufficiently accurate results. Finer subdivision results in a rapid convergence towards the "exact" answer. From Table 2.4 it will be observed that there are no experimental values for the second harmonic both for the 3-girder and the 4-girder grids. This is due to the fact that Zeidan's experimental work was concerned with those nodes that are symmetric with respect to the central longitudinal axis of the grid. Thus, the second frequency reported by him is actually the third frequency of the complete set.

The program also gives the eigenvectors together with the eigenvalues, from which the modal shapes may be determined. The modal shapes corresponding to Fig. 2.9 (iii) are illustrated in Fig. 2.13.

Due to the coarse idealization of the structure the points of contraflexure cannot be determined unless a finer subdivision is used. Such a fine idealization was not undertaken since the frequencies compared favourably with the experimental values.

2.5. CONCLUSIONS

From the examples presented in this Chapter it may be concluded that a simple beam element may be used to obtain the natural frequencies and modal shapes of structures of non-uniform cross-section and varying properties. The presence of different boundary conditions presents no basic difficulty and may be easily incorporated. Though the effects of rotary inertia and shear deformations have not been included they can be easily incorporated. For most structures in which the ratio of the length to the depth is large, ignoring the rotary inertia does not appreciably affect the results (40). This was confirmed for a number of cases and the program for portal frames is presented in Appendix C and it includes the effect of rotary inertia.

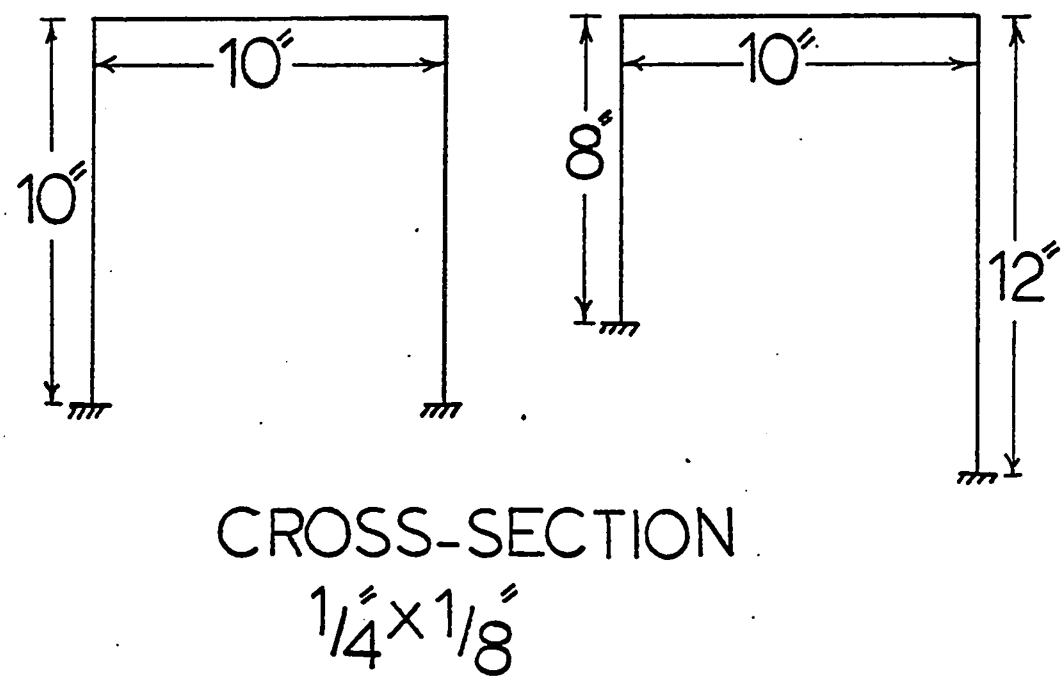


Figure 2.5: Bishop and Johnson's frames.

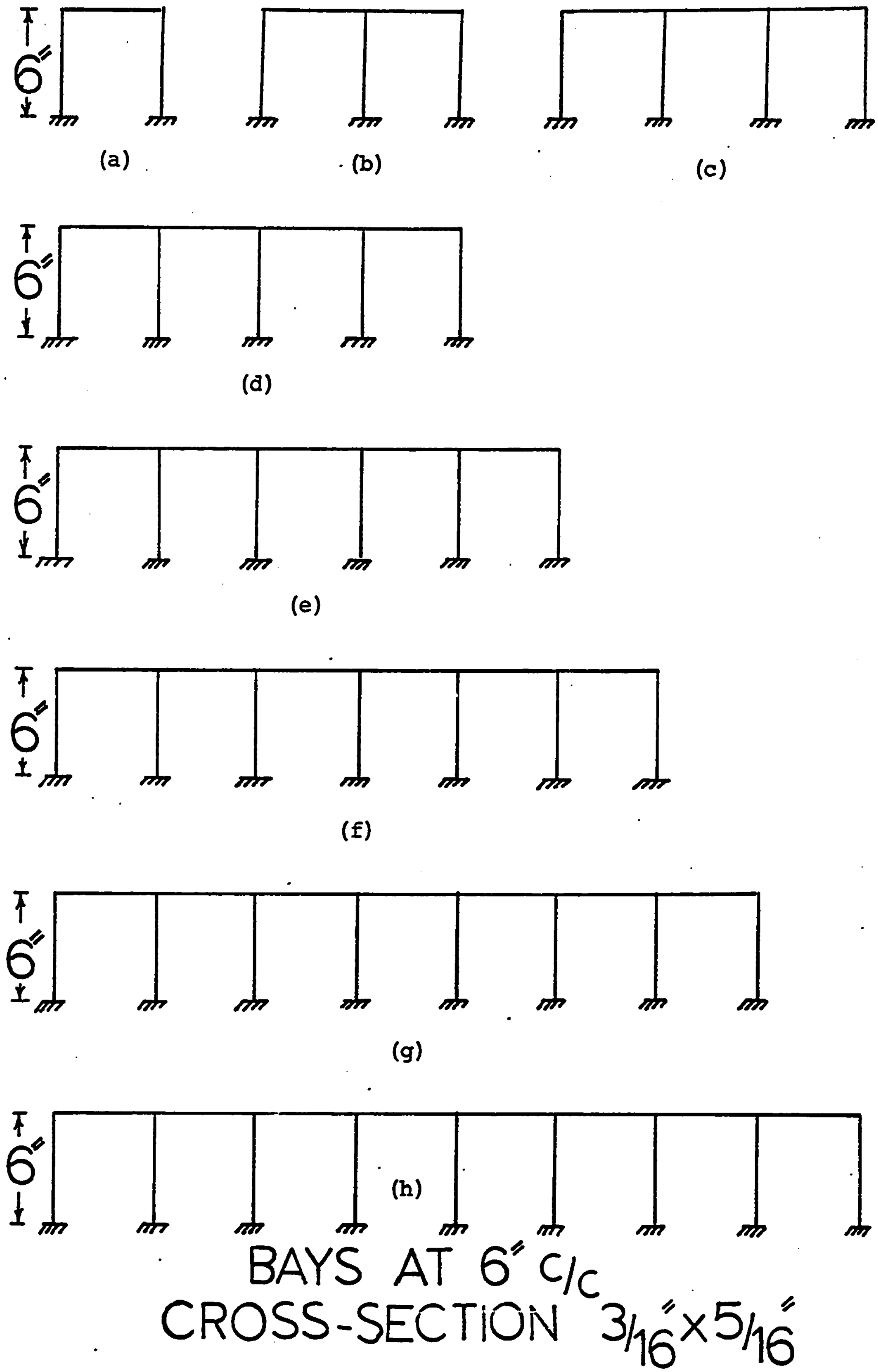


Figure 2.6: Rieger and McCallion's frames.

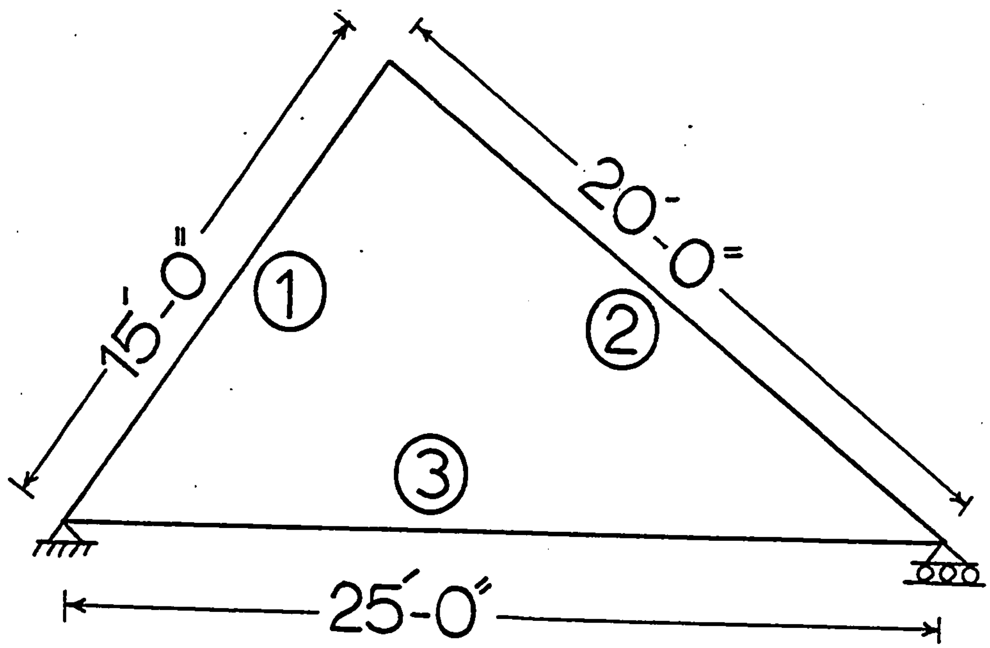


Figure 2.7: Cheng's plane truss.

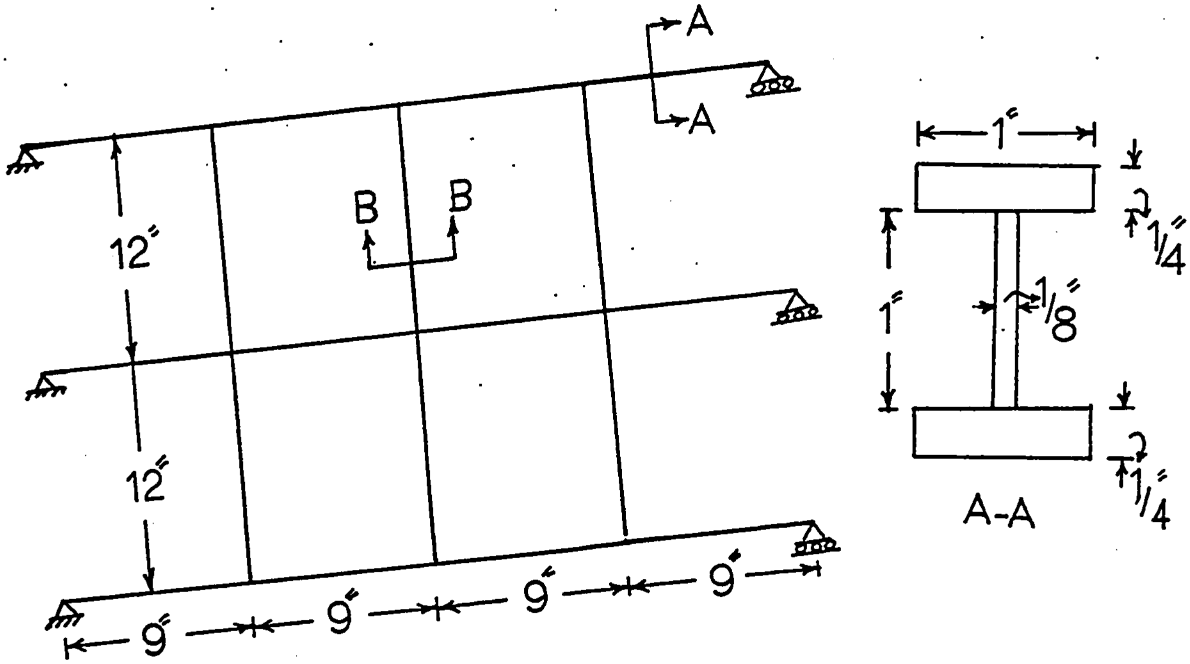


Figure 2.8(a)

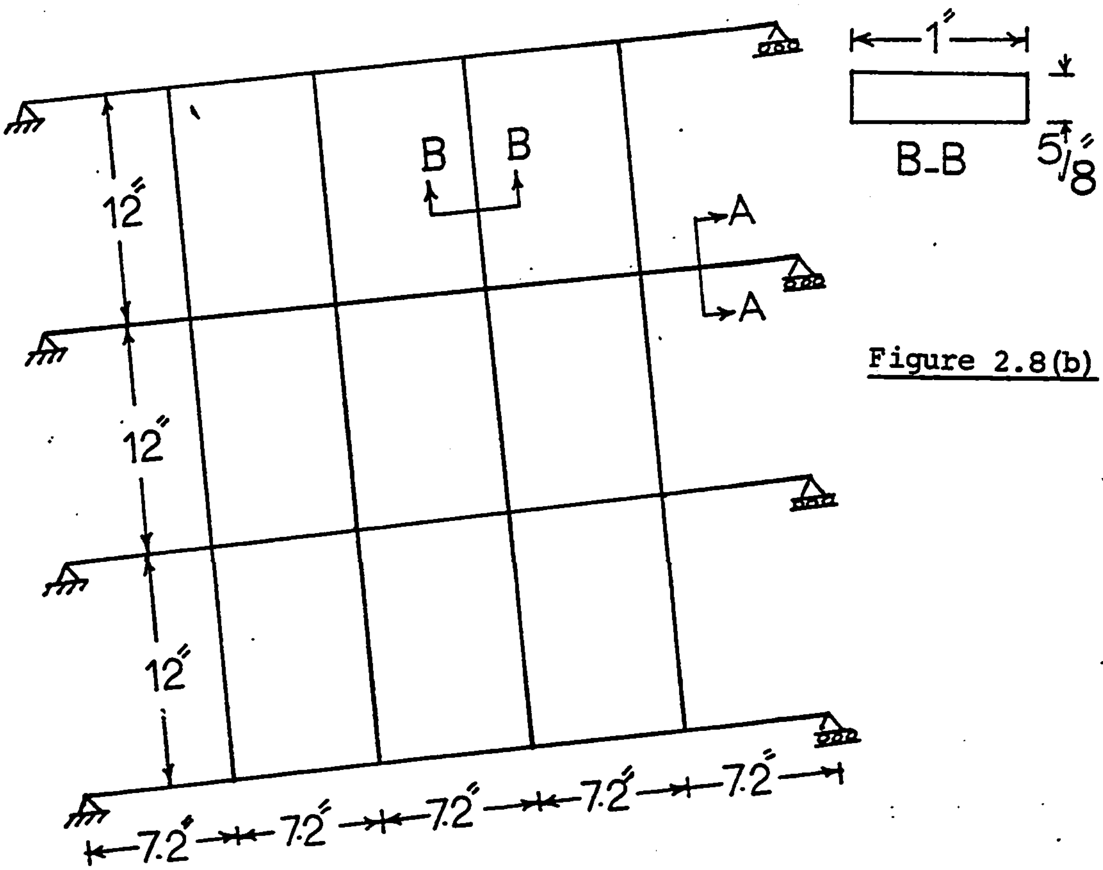


Figure 2.8(b)

Figure 2.8: Ziedan's 3-girder and 4 girder grids.

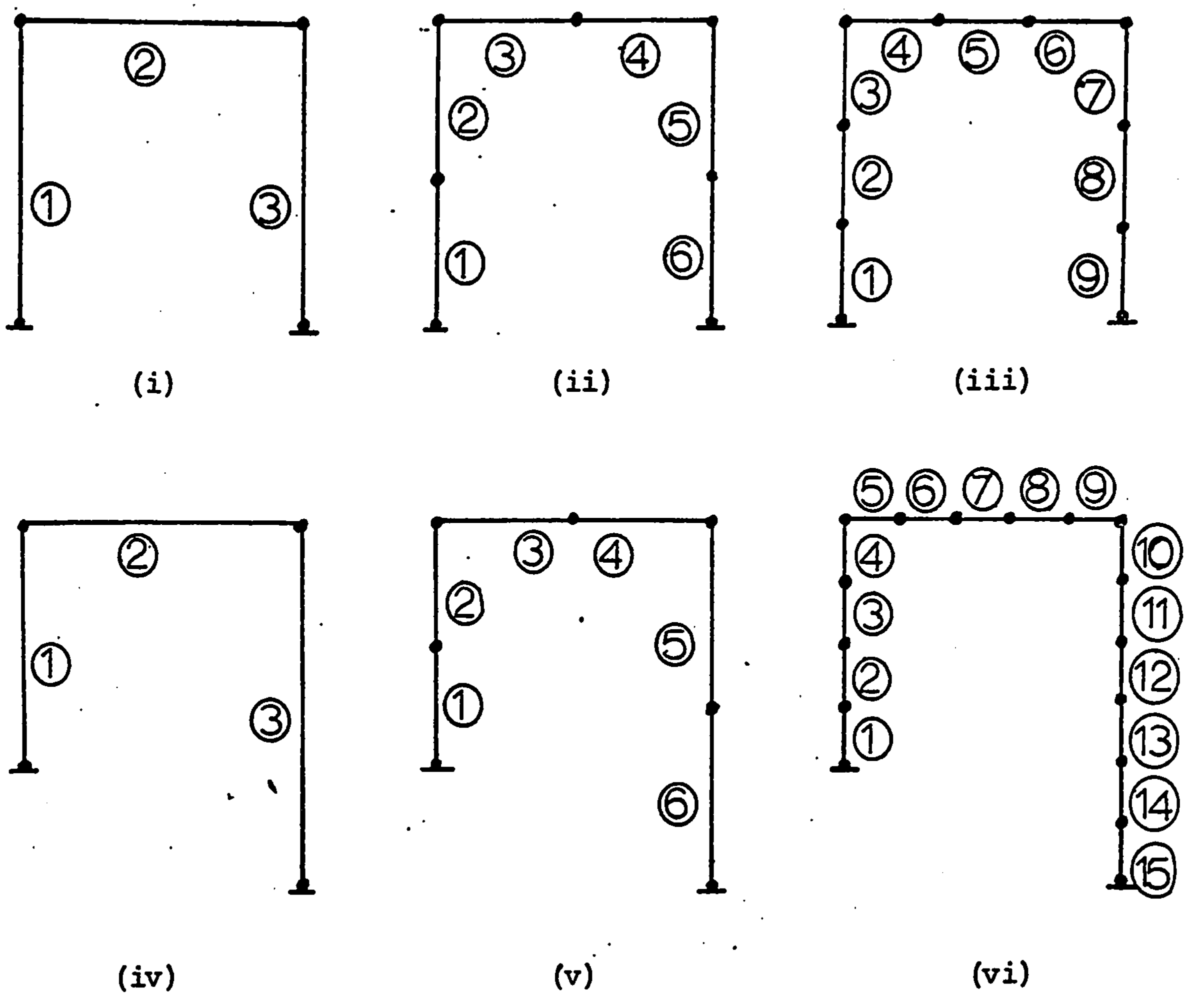


Figure 2.9: <u>Finite element idealizations of Bishop and Johnson's frames.</u>

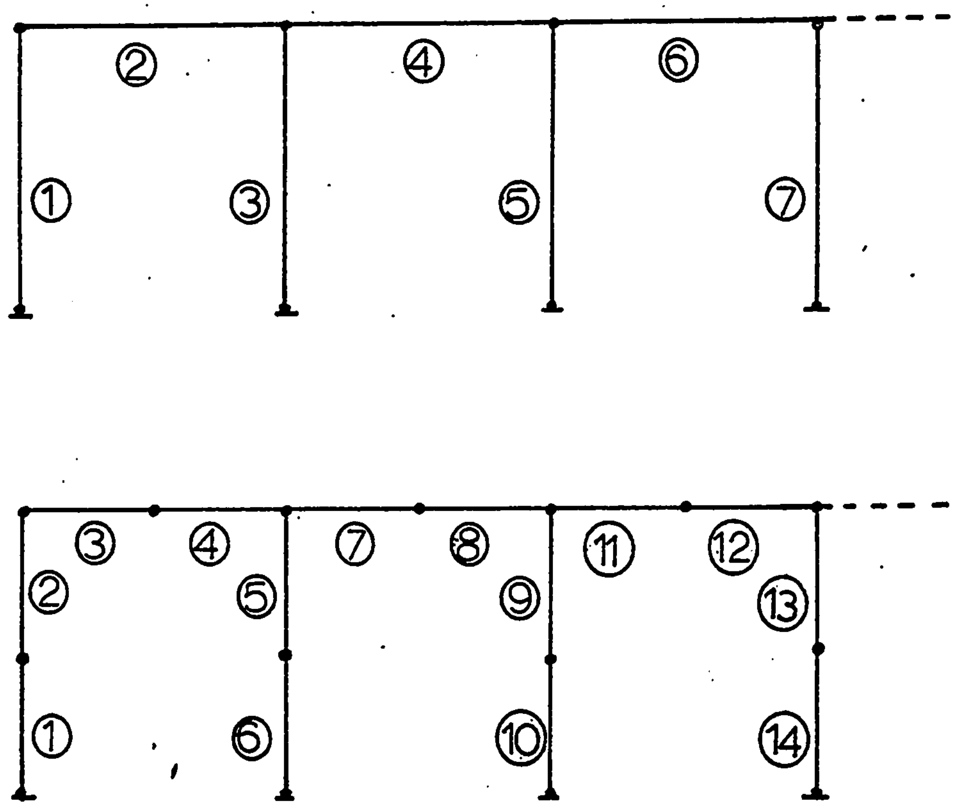
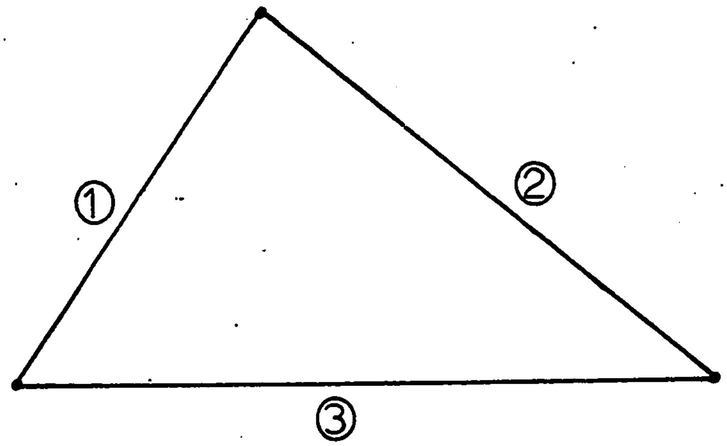
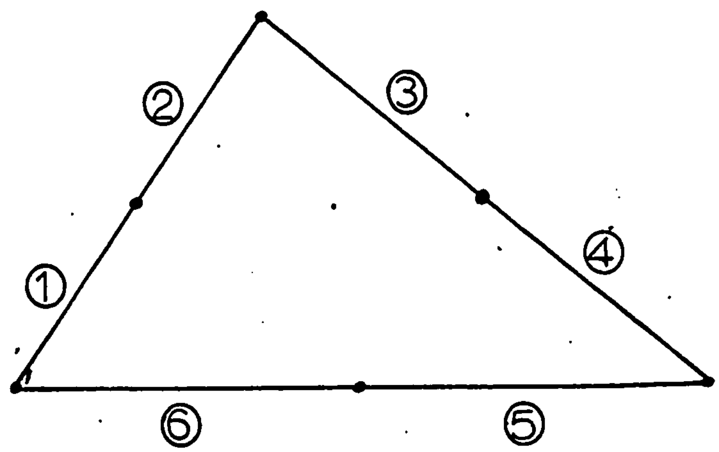


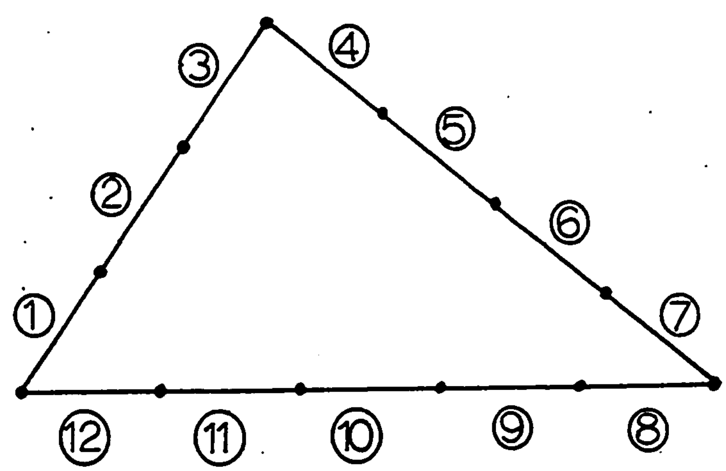
Figure 2.10: Finite element idealizations of Rieger and McCallion's frames.



(i)

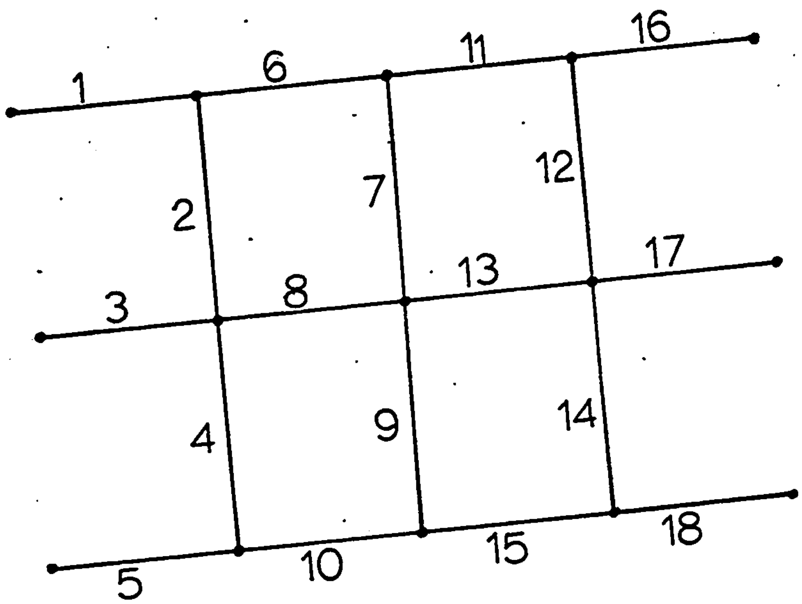


(ii)

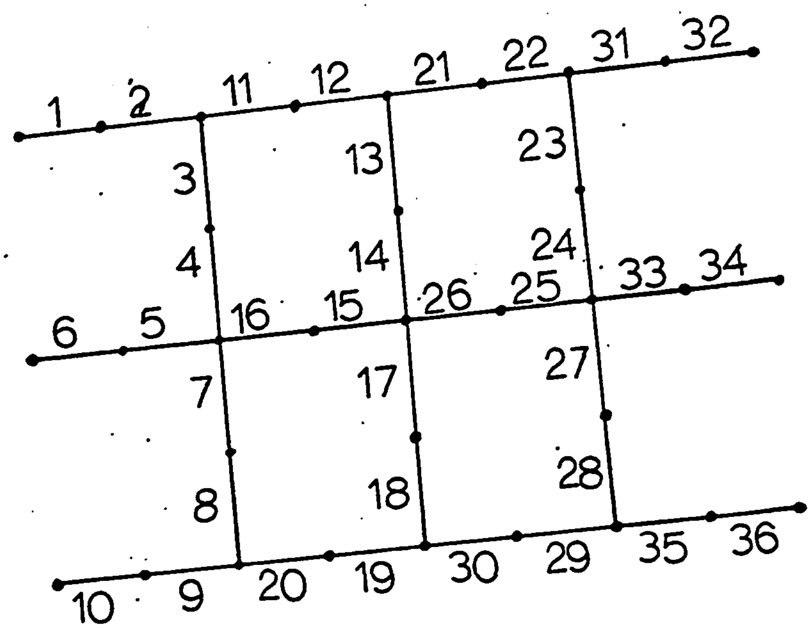


(iii)

Figure 2.11: <u>Finite element idealizations of Cheng's truss.</u>

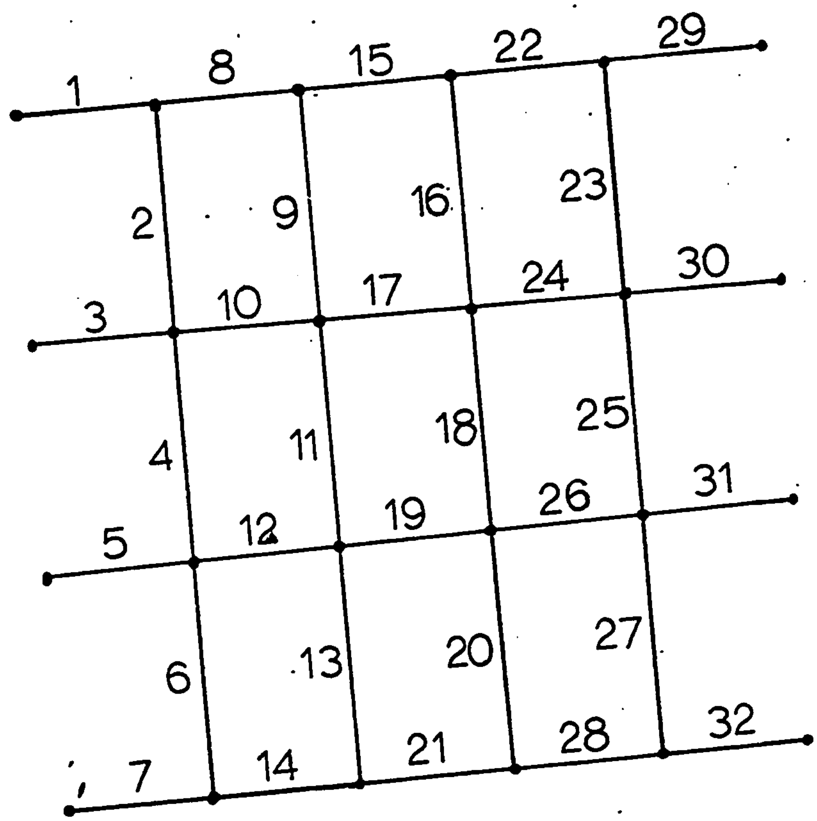


(i)



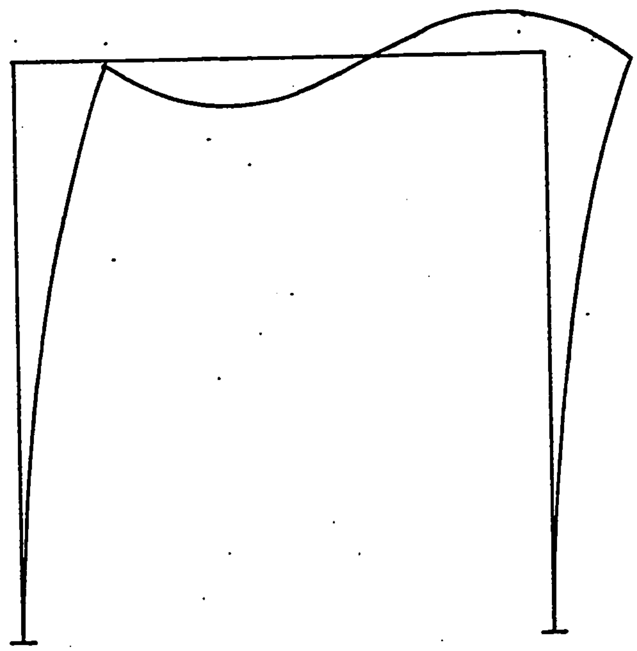
(ii)

Figure 2.12: Finite element idealizations of Ziedan's grids.

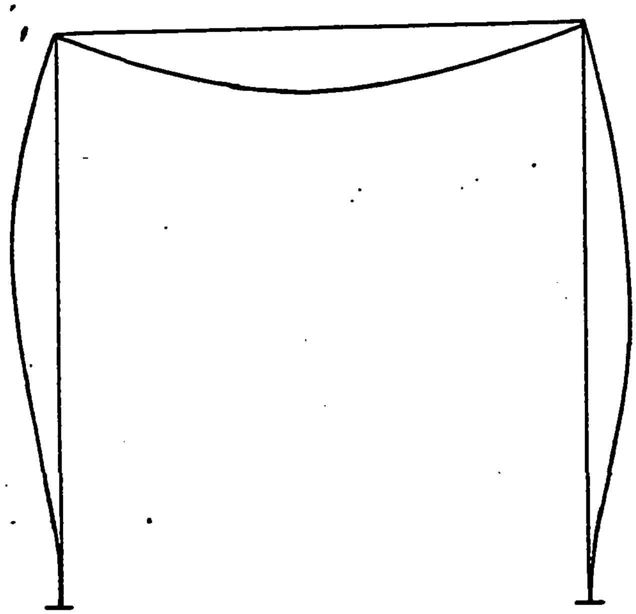


(iii)

Figure 2.12: Finite element idealization's of Ziedan's grids.

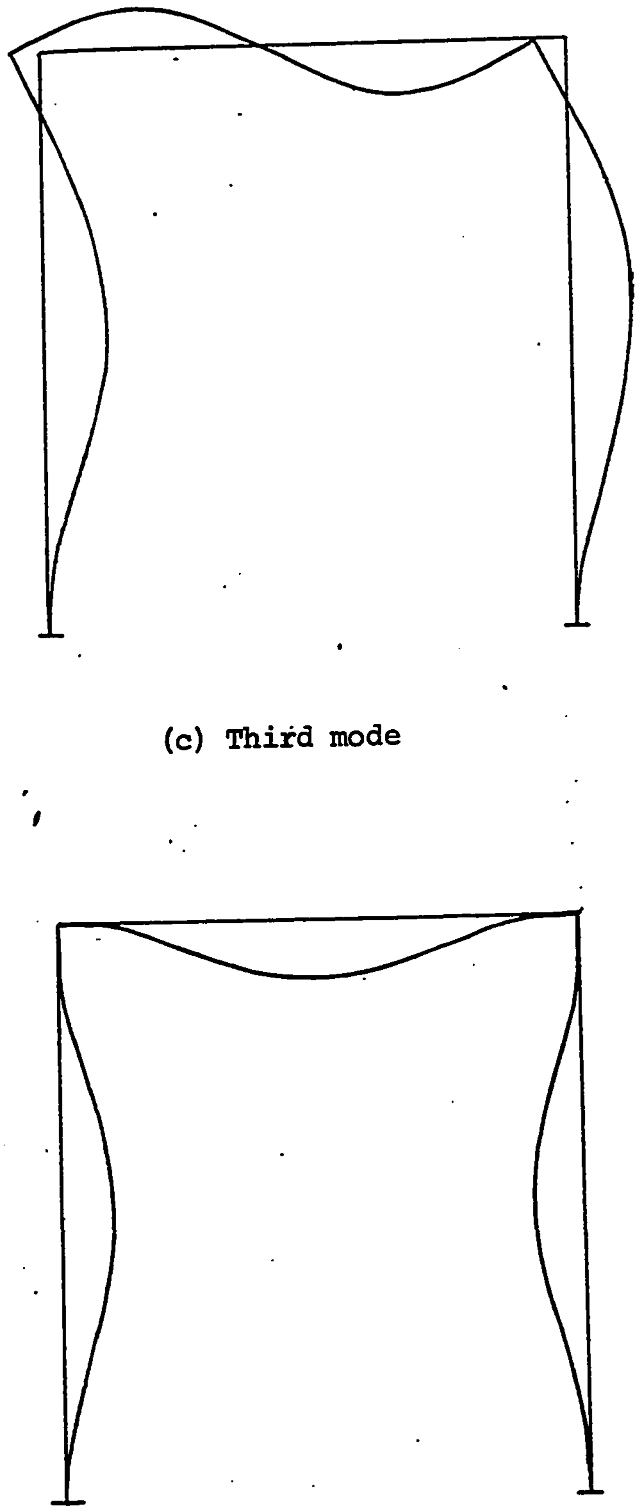


(a) First mode.



(b) Second mode.

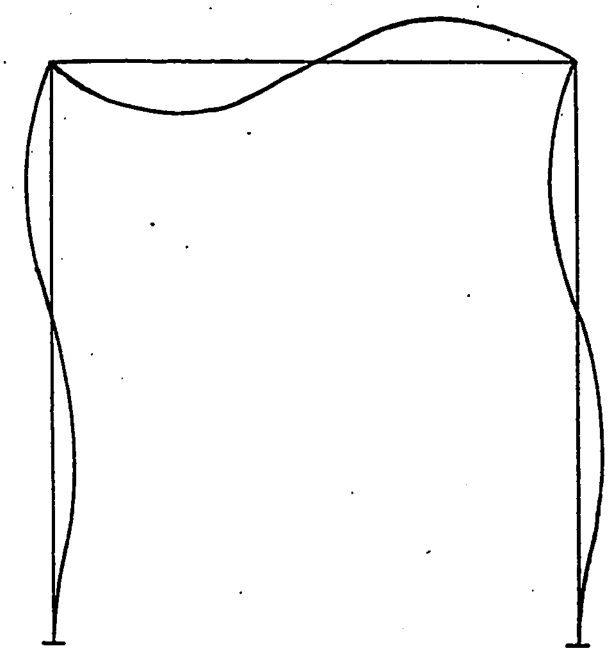
Figure 2.13: Modal shapes of portal frame.



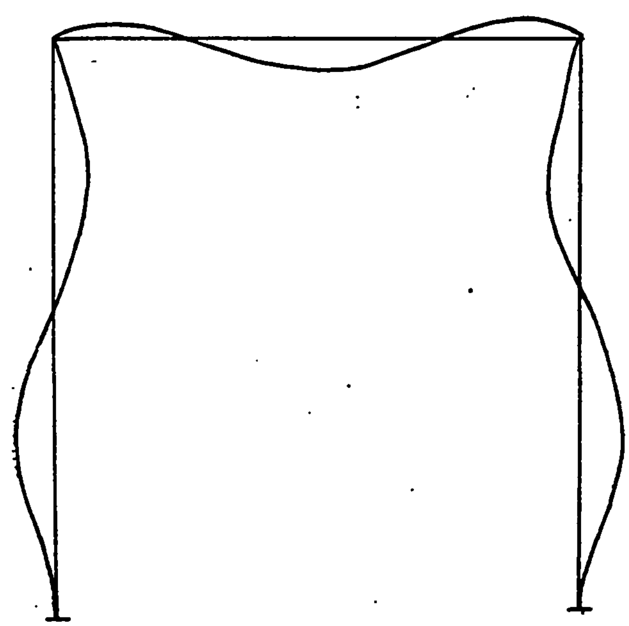
(c) Third mode

(d) Fourth mode

Figure 2.13: Modal shapes of portal frame.



(e) Fifth mode.



(f) Sixth mode.

Figure 2.13: Modal shapes of portal frame.

TABLE 2.1

SYMMETRICAL FRAME

Bishop and Johnson		Finite Element Solution		
Calculated	Experimental	Case (i)	Case (ii)	Case (iii)
cps.	cps.	cps.	cps.	cps.
36	36	36	36	36
145	145	171	143	143
233	233	368	235	233
256	254	4021	256	253
513	517	4318	579	517
628	626	7349	757	634

UNSYMMETRICAL FRAME

Calculated	Experimental	Case (iv)	Case (v)	Case (vi)
cps.	cps.	cps.	cps.	cps.
38	39	39	39	39
134	135	167	135	134
194	197	378	196	194
346	346	3532	351	345
439	436	5151	512	432
606	598	7426	717	585

TABLE 2.2

No. of Bays	Rieger and McCallion		Finite Element Solution	
	Calc.	Exper.	Case (a)	Case (b)
	cps.	cps.	cps.	cps.
1	152	152.7	152	152
	602.8	602.8	714	601
		980	1544	986
		1069	6756	1071
2	142.3	142.3	140.5	140.1
	583	583	682	581
	734	736	963	733
	990	986	1600	994
	1072	1067	5181	1067
3	138.3	138.6	137	137
	575	574	669	573
	663	663	818	660
	812	808.5	1148	810
	994	986	1626	997
	1072	1064	4527	1064
4	136	137.3	135	135
	571	571.5	663	569
	626	614.5	754	615
	736	739.5	965	734
	850	849	1258	850
	995	989	1640	999
	1072	1064	3651	1062
8	132.5	132	132	132
	565	565	653	562
	584	580	682	581
	620	619	742	617
	673	672	835	669
	736	742	964	733
	801	792	1118	799
	862	827	1290	860
	904	871	1342	889
	998	998	1663	1002
	1072	1069	2037	1057

TABLE 2.3

Cheng	Finite Element Solution		
Calculated	Case (i)	Case (ii)	Case (iii)
rad/sec.	rad/sec.	rad/sec.	rad/sec.
737.5	760.9	743.6	739.7
1116.5	1164.5	1123.5	1120.0
1519.4	1712.1	1573.3	1534.0
	2129.2	1780.0	1758.7
	3616.6	2713.3	2636.6
	7940.6	3017.8	2840.1

TABLE 2.4

3-Girder Grid

Experimental	Finite Element Solution	
	Case (i)	Case (ii)
cps.	cps.	cps.
30	29	29
-	35.5	35
49	51	50

4-Girder Grid

Experimental	Finite Element Solution
	Case (iii)
cps.	cps.
29.8	26.4
-	31
40.0	40.5

CHAPTER III

PARAMETRIC INSTABILITIES OF COLUMNS

3.1 Introduction

When a column is subjected to a periodic longitudinal force, or when its base is given a periodic vertical motion the differential equation of motion reduces to a Mathieu equation. In Section 1.1 it was shown that for the case of a simply supported column carrying a periodic axial load ($V_c + V_t \cos \Omega t$), the equation of motion, Equation (1.2), may be reduced to the following form of the Mathieu equation

$$\ddot{f} + \beta_o^2 (1-2\mu \cos \Omega t) f = o \qquad (1.10)$$

To investigate the stability of the motion the solutions of the Mathieu equation have to be studied. Since Equation (1.10) is a second order linear differential equation, it will have two independent non-zero solutions. (A brief account of some of the properties of the Mathieu equation is given in Appendix A.) If all the solutions of Equation (1.10) are bounded for all positive values of t, the corresponding motion is regarded as stable; however, for an unbounded solution, the resulting motion will continue to grow with time and is termed as unstable.

One may seek the periodic solutions of the Mathieu equations in the form of a Fourier series. For a periodic solution with period 2T, where T is defined as $\frac{2\pi}{\Omega}$, the solution may be obtained in the form

$$f(t) = \sum_{k=1,3,5,\ldots} \left(\bar{a}_k \sin \frac{k\Omega t}{2} + \bar{b}_k \cos \frac{k\Omega t}{2}\right) \qquad (3.1)$$

Substituting the series (3.1) into Equation (1.10) and equating the coefficients of $\sin \frac{k\Omega t}{2}$ and $\cos \frac{k\Omega t}{2}$, the following system of linear homogeneous algebraic equations is obtained

$$\left(1 + \mu - \frac{\Omega^2}{4\beta_o{}^2}\right) \bar{a}_1 - \mu \, \bar{a}_3 = o$$

$$\left(1 - \frac{k^2\Omega^2}{4\beta_o{}^2}\right) \bar{a}_k - \mu \left(\bar{a}_{k-2} + \bar{a}_{k+2}\right) = o \quad (k = 3,5,7,\ldots)$$

$$\left(1 - \mu - \frac{\Omega^2}{4\beta_o{}^2}\right) \bar{b}_1 - \mu \, \bar{b}_3 = o$$

$$\left(1 - k^2\frac{\Omega^2}{4\beta_o{}^2}\right) \bar{b}_k - \mu \left(\bar{b}_{k-2} + \bar{b}_{k+2}\right) = o \quad (k = 3,5,7,\ldots)$$

The non-zero values of $\bar{a}_k$ and $\bar{b}_k$ require that

$$\begin{vmatrix} 1 \pm \mu - \frac{\Omega^2}{4\beta_o{}^2} & -\mu & o & \cdots \\ -\mu & 1 - \frac{9\Omega^2}{4\beta_o{}^2} & -\mu & \cdots \\ o & -\mu & 1-25\frac{\Omega^2}{4\beta_o{}^2} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = o \qquad (3.2)$$

For periodic solutions of period T, one may seek the solution in the form

$$f(t) = \bar{b}_o + \sum_{k=2,4,6,\ldots} (\bar{a}_k \sin \frac{k\Omega t}{2}) + \bar{b}_k \cos \frac{k\Omega t}{2}) \qquad (3.3)$$

which again, after equating coefficients, results in the following system of equations

$$\left(1 - \frac{\Omega^2}{\beta_o^2}\right) \bar{a}_2 - \mu a_4 = o$$

$$\left(1 - \frac{k^2\Omega^2}{4\beta_o^2}\right) \bar{a}_k - \mu \left(\bar{a}_{k-2} + \bar{a}_{k+2}\right) = o$$

$$\bar{b}_o - \mu \bar{b}_2 = o$$

$$\left(1 - \frac{\Omega^2}{\beta_o^2}\right) \bar{b}_2 - \mu \left(2 \bar{b}_o + \bar{b}_4\right) = o$$

$$\left(1 - \frac{k^2\Omega^2}{\beta_o^2}\right) \bar{b}_k - \mu \left(\bar{b}_{k-2} + \bar{b}_{k+2}\right) = o$$

For non-trivial solutions

$$\begin{vmatrix} 1 - \frac{\Omega^2}{\beta_o^2} & -\mu & o & \cdots \\ -\mu & 1 - \frac{4\Omega^2}{\beta_o^2} & -\mu & \cdots \\ o & -\mu & 1-\frac{16\Omega^2}{\beta_o^2} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = o \qquad (3.4a)$$

and

$$\begin{vmatrix} 1 & -\mu & o & o & \cdots \\ -2\mu & 1-\frac{\Omega^2}{\beta_o^2} & -\mu & o & \cdots \\ o & -\mu & 1-\frac{4\Omega^2}{\beta_o^2} & -\mu & \cdots \\ o & o & -\mu & 1-\frac{16\Omega^2}{\beta_o^2} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{vmatrix} = o \qquad (3.4b)$$

Equations (3.2) and (3.4) are called the equations of boundary frequencies. The boundary frequencies are defined as the frequencies of the fluctuation of the external loading corresponding to the boundaries of the region of instability.

As a first approximation, we may determine the boundaries of the principal region of instability, by retaining the upper diagonal element only, and equating it to zero.

$$1 \pm \mu - \frac{\Omega^2}{4\beta_o^2} = o$$

which gives

$$\Omega = 2\beta_o \sqrt{1 \pm \mu} \qquad (3.5)$$

The accuracy of Equation (3.5) may be improved by considering the terms contained in the first two columns and rows of Equation (3.2)

$$\begin{vmatrix} 1 \pm \mu - \frac{\Omega^2}{4\beta_o^2} & -\mu \\ -\mu & 1-9\frac{\Omega^2}{4\beta_o^2} \end{vmatrix} = o$$

which gives

$$\Omega = 2\beta_o \sqrt{1 \pm \mu + \frac{\mu^2}{8 \pm 9\mu}} \tag{3.6}$$

The last term under the square root takes into account the correction for the second approximation. This correction increases with μ but even for a value of $\mu = 0.3$, the error is less than 1%. Hence, for all practical purposes, Equation (3.5) is sufficiently accurate.

Similarly, for the second region of instability, we get the following approximate formulas for the boundary frequencies [6].

$$\Omega = \beta_o \sqrt{1 + \tfrac{1}{3}\mu^2} \qquad \Omega = \beta_o \sqrt{1 - 2\mu^2} \tag{3.7}$$

and for the third regions of instability we have

$$\Omega = \frac{2}{3}\ \beta_o \sqrt{1 - \frac{9\mu^2}{8 \pm 9\mu}} \tag{3.8}$$

The typical diagram for the regions of instability is shown in Figure 3.1.

3.2 The Proposed Method

The author has developed a numerical method approach to obtain the regions of dynamic stability [45]. A uniform column is idealized by a system of lumped masses and massless springs to represent the stiffness of the column.

It is assumed that the mass and stiffness of each segment is concentrated at its end. For instance, if the length

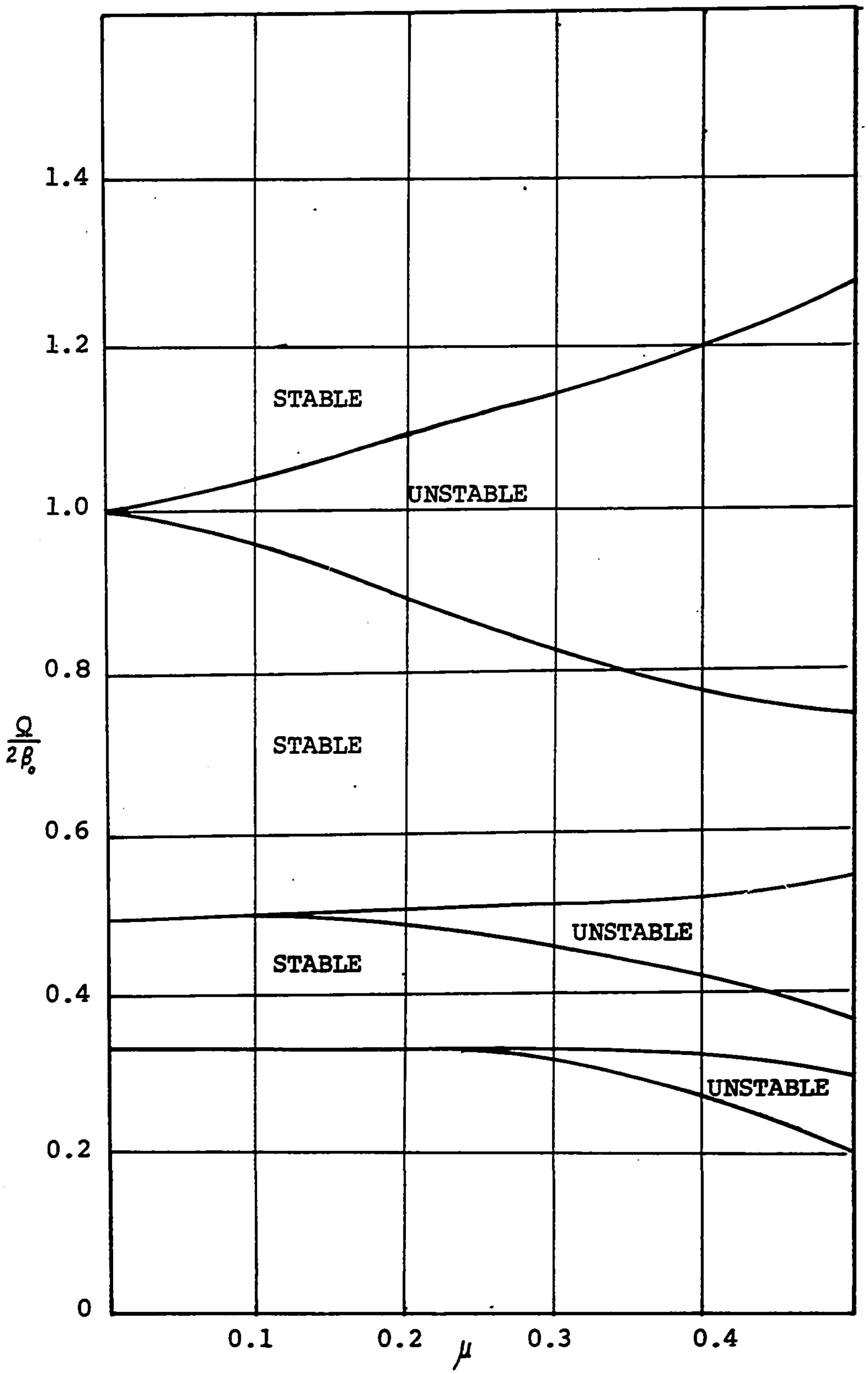


Figure 3.1: Regions of instability and stability.

of the segment is a, and its mass per unit length is $A\rho$, then the mass of the segment is replaced by two equal concentrated masses $A\rho a/2$ at the ends. It is shown that the equations of motion are of the Mathieu type and from their solutions one may establish the regions of dynamic stability. From the equation of motion it is also possible to obtain the natural frequencies and buckling loads; these when compared with known values serve as a measure for the validity of the idealization. The following three cases have been studied.

Case A : Column hinged at both ends and subjected to a periodically fluctuating axial compressive force. (Figure 1.3)

Case B : Column fixed at the base and free at the top and subjected to a periodically fluctuating compressive force. (Figure 3.2)

Case C : Column fixed at the base and free at the top and subjected to a periodic support motion (Figure 3.3)

3.2A. : Case A: Simply Supported Column Subjected to Periodically Varying Axial Force

The mathematical development is based on the following assumptions:

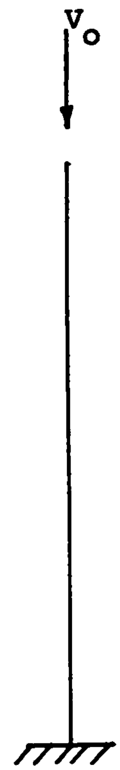


Figure 3.2: Column subjected to periodically fluctuating compressive force.

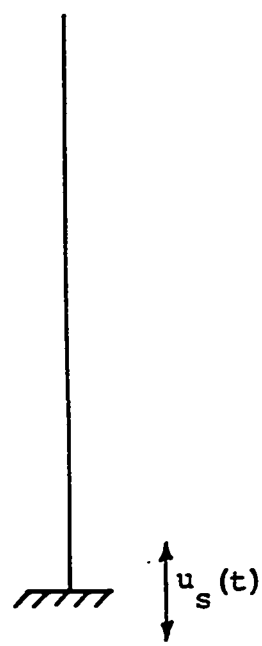


Figure 3.3: Column subjected to periodically varying vertical support motion.

1. The column consists of links of equal length, a, except the first and last links which are of length a/2. It may be noted that the total mass and stiffness of the first and last links is concentrated at node 1 and n-1 respectively.

2. Vertical accelerations are so small as to be neglected. Therefore, at any instant we have the same vertical force at each end of . link.

The idealization is shown in Figure 3.4. The spring stiffness is designated by K. The forces acting on the ith link are shown in Figure 3.5. Clearly

$$x_i = a\left(\frac{1}{2} \cos \theta_1 + \cos \theta_2 + . \quad . \quad . \quad . + \cos \theta_i\right)$$

$$y_i = a\left(\frac{1}{2} \sin \theta_1 + \sin \theta_2 + . \quad . \quad . \quad . + \sin \theta_i\right) \qquad (3.9)$$

$$H_{i-1} - H_i = M_{i-1} \ddot{y}_{i-1}$$

where H represents the horizontal shear forces.

If θ's are small, the equation of motion for the i-th member writes as

$$-V_o a \theta_i + H_{i-1} a - M_{i-1} a \ddot{y}_{i-1} = K_{i-1}\left(\theta_{i-1} - \theta_i\right) - K_i\left(\theta_i - \theta_{i+1}\right) \qquad (3.10)$$

Similarly, for the (i + 1) th member, we get

$$-V_o a\,\theta_{i+1} + H_i a - M_i a\ddot{y}_i = K_i\left(\theta_i - \theta_{i+1}\right) - K_{i+1}\left(\theta_{i+1} - \theta_{i+2}\right) \qquad (3.11)$$

Subtracting Equation (3.11) from (3.10) and simplifying gives

$$-V_o a\left(\theta_i - \theta_{i+1}\right) + M_i a^2\left(\ddot{\theta}_{1/2} + \ddot{\theta}_2 + \ddot{\theta}_3 + \ldots + \ddot{\theta}_i\right) = K_{i-1}\left(\theta_{i-1} - \theta_i\right)$$

$$-2K_i\left(\theta_i - \theta_{i+1}\right) + K_{i+1}\left(\theta_{i+1} - \theta_{i+2}\right) \qquad (3.12)$$

If all masses and springs are equal, Equation (3.12) reduces to

$$-V_o a\left(\theta_i - \theta_{i+1}\right) + Ma^2\left(\frac{\ddot{\theta}_1}{2} + \ddot{\theta}_2 + \ldots + \ddot{\theta}_i\right)$$

$$= K\left(\theta_{i-1} - 3\theta_i + 3\,\theta_{i+1} - \theta_{i+2}\right) \quad (i=2,3,\ldots n-2), \qquad (3.13)$$

where n is the number of members.

Similarly, the first and second members yield

$$-V_o a\left(\theta_1 - \theta_2\right) + M_1 \frac{a^2}{2}\,\ddot{\theta}_1 = K\left(-3\,\theta_1 + 4\theta_2 - \theta_3\right) \qquad (3.13a)$$

and the (n - 1) and n-th members yield

$$-V_o a\left(\theta_{n-1} - \theta_n\right) + M_{n-1} a^2\left(\frac{\ddot{\theta}_1}{2} + \ddot{\theta}_2 + \ldots + \ddot{\theta}_{n-1}\right) = K\left(\theta_{n-2} - 4\,\theta_{n-1} + 3\theta_n\right) \quad (3.13b)$$

Since there are n coordinates and since there is an equation of constraint, arising from the geometry,

$$\frac{a}{2}\sin\theta_1 + a\sin\theta_2 + \ldots\ldots + \frac{a}{2}\sin\theta_n = 0 \qquad (3.14)$$

there are only (n - 1) degrees of freedom and thus (n - 1) equations of type (3.12). It may be noted that the above equations may be also obtained by the Lagrangian formulation.

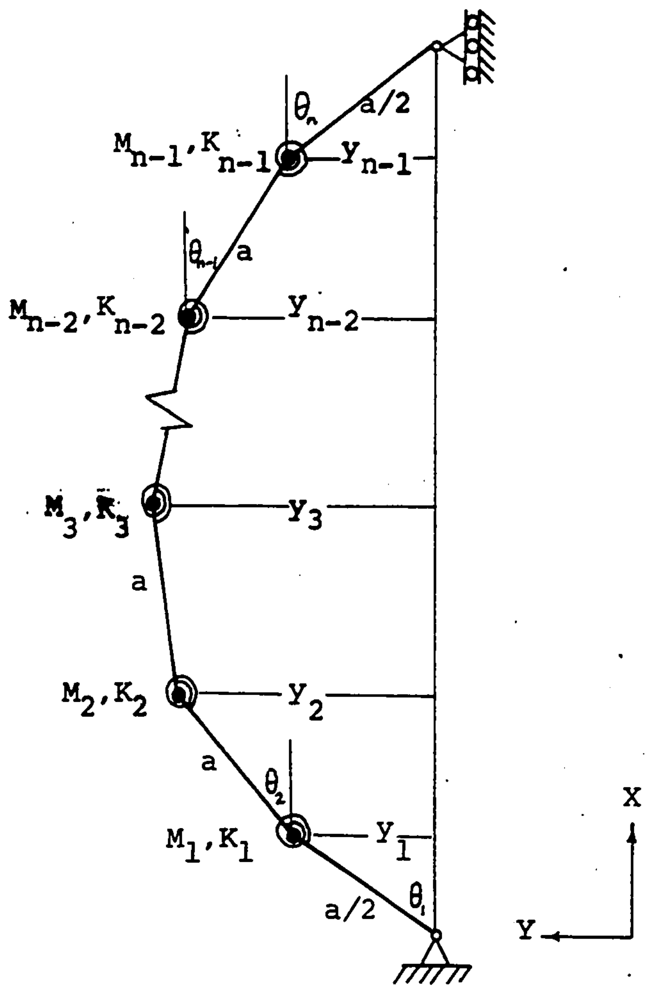


Figure 3.4: Idealization for column simply supported at both ends.

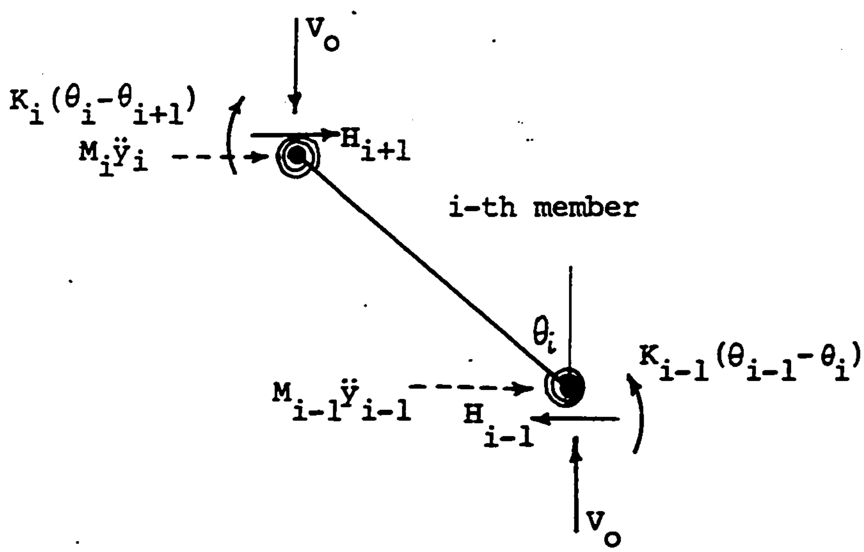


Figure 3.5: Free body diagram of the i-th link.

In matrix form the equations become

$$
-V_o a\begin{Bmatrix} \theta_1-\theta_2 \\ \theta_2-\theta_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \theta_{n-3}-\theta_{n-2} \\ \theta_{n-2}-\theta_{n-1} \\ \theta_{n-1}-\theta_n \end{Bmatrix} + M_a^2 \begin{bmatrix} 0.5 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0.5 & 1.0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & & & & & & & & & & \cdot \\ \cdot & & & & & & & & & & \cdot \\ \cdot & & & & & & & & & & \cdot \\ \cdot & & & & & & & & & & \cdot \\ \cdot & & & & & & & & & & \cdot \\ 0.5 & 1.0 & \cdot & \cdot & \cdot & 1.0 & & 0 & & & 0 \\ 0.5 & 1.0 & \cdot & \cdot & \cdot & 1.0 & & 1.0 & & & 0 \\ 0.5 & 1.0 & \cdot & \cdot & \cdot & 1.0 & & 1.0 & & & 1.0 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \ddot{\theta}_{n-3} \\ \ddot{\theta}_{n-2} \\ \ddot{\theta}_{n-1} \end{Bmatrix}
$$

$$
= K \begin{bmatrix} -3 & 4 & -1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 1 & -3 & 3 & -1 & 0 & & & & & & 0 \\ 0 & 1 & -3 & 3 & -1 & & & & & & 0 \\ \cdot & & & & & & & & & & \cdot \\ \cdot & & & & & & & & & & \cdot \\ \cdot & & & & & & & & & & \cdot \\ \cdot & & & & & & & & & & \cdot \\ \cdot & & & & & & & & & & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & 1 & -3 & 3 & -1 \\ 1 & 2 & 2 & 2 & 2 & \cdot & \cdot & 2 & 3 & -1 & 5 \\ -3 & -6 & -6 & -6 & -6 & \cdot & \cdot & -6 & -6 & -5 & -10 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \theta_{n-3} \\ \theta_{n-2} \\ \theta_{n-1} \end{Bmatrix} \qquad (3.15)
$$

In the matrix which appears above on the right hand side of Equation (3.15) the last two rows have a non-typical form which arises from substitution for θ_n in terms of θ_1, θ_2, . . . θ_{n-1} utilizing Equation (3.14).

By applying Galerkin's Method as applicable to discrete systems (see Appendix B) equations (3.15) may be reduced to a single equation for each mode of the type

$$\lambda_1 V_o a\,\theta + \lambda_2\, Ma^2 \ddot{\theta} + \lambda_3\, K\theta = o \qquad (3.16)$$

where the λ's are the coefficients associated with V_o, m and k.

From Appendix B and in particular from Equation (B.5) it may be seen that the application of the Galerkin Method requires the knowledge of the modal shape vector, $\underset{\sim}{\theta}$. To determine $\underset{\sim}{\theta}$ one sets V_o = o, then Equations (3.15) represent the free vibration case. If $\underset{\sim}{\theta} = \underset{\sim}{q}e^{i\,wt}$, where $\underset{\sim}{q}$ is a column matrix of the amplitudes of the displacement $\underset{\sim}{\theta}$, w is the natural frequency, there results

$$(\underset{\sim}{S} - w^2\, \underset{\sim}{M}) = o \qquad (3.17)$$

Equation (3.17) is in the standard form for eigenvalue problems, and its solution yields the required modal vector .

The natural frequency may now be determined from Equation (3.16) as

$$w^2 = \frac{\lambda_3}{\lambda_2}\;\frac{K}{Ma^2} \qquad (3.18)$$

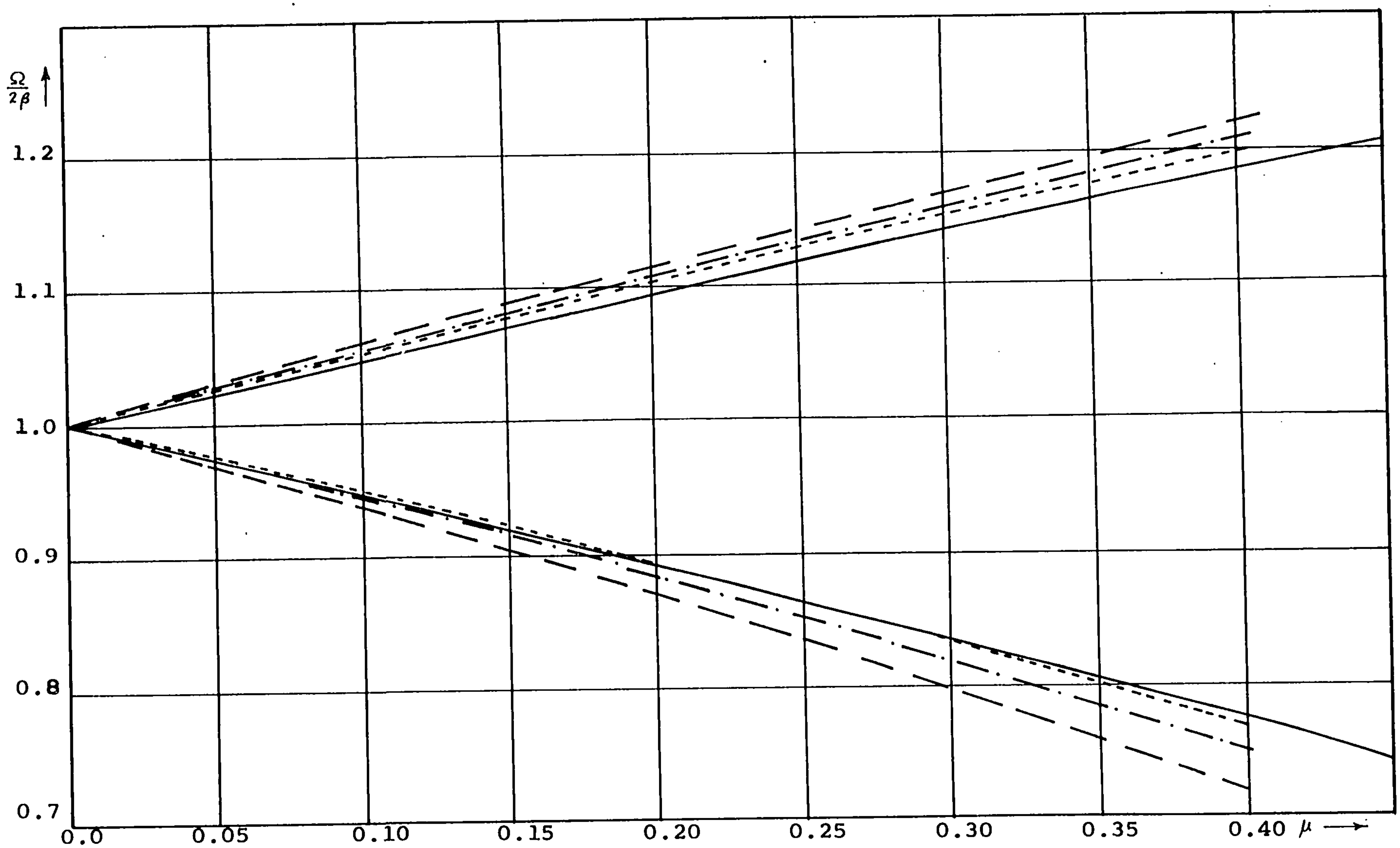


Figure 3.6: Instability diagram for column pinned at both ends.— —, n=3; —·—, n=4; -----, n=6; ———, n=∞.

The Euler buckling load may also be obtained from Equation (3.16) by putting $\ddot{\theta} = o$,

$$P_e = -\frac{\lambda_3}{\lambda_1} \quad \frac{K}{a} \tag{3.19}$$

Substituting Equations (3.18) and (3.19) in Equation (3.16) we get

$$\ddot{\theta} + w^2 \left(1 - \frac{V_o}{P_e}\right) \theta = o \tag{3.20}$$

Since $V_o = V_c + V_t \cos \Omega t$, Equation (3.20) gives

$$\ddot{\theta} + \beta_o^2 \; (1-2\mu \; \cos\Omega \; t) \; \theta \; = o \tag{3.21}$$

Equation (3.21) is the differential equation of motion for the particular mode and is of the Mathieu type.

A uniform column of length L was analyzed by this procedure and the results obtained are shown in Table 3.1. From this it may be seen that as the column is divided into more parts, the values for the natural frequency and the Euler buckling load converge to their true values. As regards the mapping of the principal zone of instability each subdivision of the column into elements yields its associated Mathieu equation. Figure 3.6 illustrates the rapid convergence to the true configuration for the principal region of instability. Even for the case of n = 6, though the results obtained for the natural frequency and the Euler buckling load are not very accurate, the error for the region of dynamic instability is not very great.

3.2B: Case B: Cantilever Column Subjected to Periodic Axial Force.

The mathematical formulation for Case B is based on the following assumptions:

1. On account of the base fixity the first link does not rotate and remains vertical.
2. Vertical accelerations may be neglected as for Case A.

Consider the column idealization of figure 3.7.

Clearly

$$y_i = a_1 \sin \theta_1 + a_2 \sin \theta_2 + \ldots + a_i \sin \theta_i \quad (i = 1,2,\ldots,n) \tag{3.22}$$

$$\begin{aligned} T_e &= \frac{1}{2} M_1 \dot{y}_1^2 + \frac{1}{2} M_2 \dot{y}_2^2 + \ldots + \frac{1}{2} M_n \dot{y}_n^2 \\ &= \frac{1}{2} M_1 \left(a_1 \dot{\theta}_1\right)^2 + \frac{1}{2} M_2 \left(a_1 \dot{\theta}_1 + a_2 \dot{\theta}_2\right)^2 + \ldots \\ &\quad + \frac{1}{2} M_n \left(a_1 \dot{\theta}_1 + a_2 \dot{\theta}_2 + \ldots + a_n \dot{\theta}_n\right)^2 \end{aligned} \tag{3.23}$$

$$U_e = \frac{1}{2} K_1 \theta_1^2 + \frac{1}{2} K_2 \left(\theta_1 - \theta_2\right)^2 + \ldots + \frac{1}{2} K_n \left(\theta_{n-1} - \theta_n\right)^2 + \int V_o dh \tag{3.24}$$

where T_e is the kinetic energy of the system

U_e is the potential energy of the system

$h = a_1 \cos \theta_1 + a_2 \cos \theta_2 + \ldots + a_n \cos \theta_n$

$$\begin{aligned} \frac{\partial T_e}{\partial \dot{\theta} i} &= M_i a_i \left(a_1 \dot{\theta}_1 + a_2 \dot{\theta}_2 + \ldots + a_i \dot{\theta}_i\right) + M_{i+1} a_i \left(a_1 \dot{\theta}_1 + a_2 \dot{\theta}_2 + \ldots + a_{i+1} \dot{\theta}_{i+1}\right) \\ &\quad + \ldots + M_n a_i \left(a_1 \dot{\theta}_1 + a_2 \dot{\theta}_2 + \ldots + a_n \dot{\theta}_n\right) \quad (i = 1,2,\ldots,n) \end{aligned} \tag{3.23a}$$

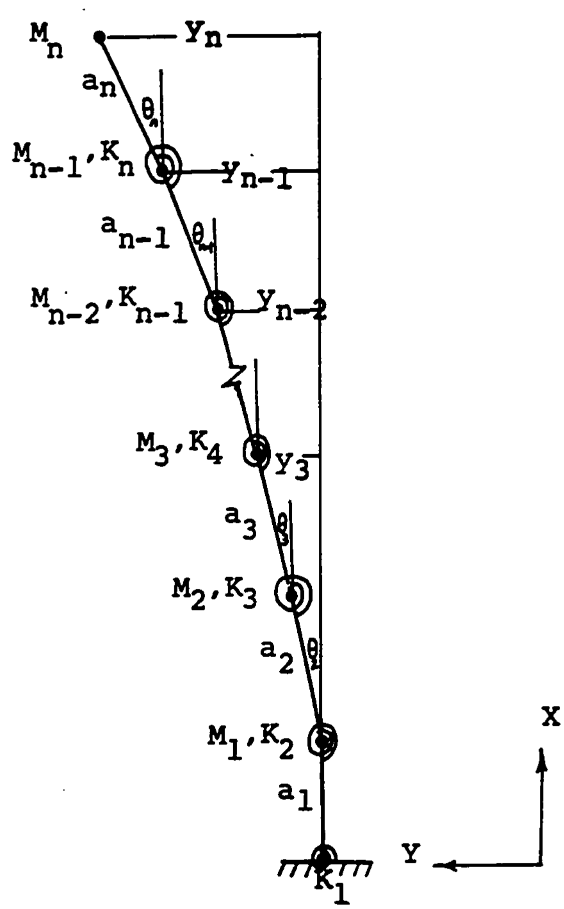


Figure 3.7: Column idealization for cases B and C

(θ_1 assumed to be zero).

$$\frac{\partial T_e}{\partial \theta_i} = o \quad (i = 1,2, \ldots,n) \tag{3.23b}$$

$$\frac{\partial U_e}{\partial \theta_i} = -K_i\theta_{i-1} + (K_i + K_{i+1})\ \theta_i - K_i\ \theta_{i+1} - V_o a_i \theta_i \quad (i = 2,3,\ldots n-1) \tag{3.24a}$$

$$\frac{\partial U_e}{\partial \theta_1} = (K_1+K_2)\theta_1 - K_2\theta_2 - V_o a_1 \theta_1 \tag{3.24b}$$

$$\frac{\partial U_e}{\partial \theta_n} = -K_n\theta_{n-1} + K_n\theta_n - V_o a_n \theta_n \tag{3.24c}$$

Substituting in Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial T_e}{\partial \theta_i}\right) - \frac{\partial T_e}{\partial \theta_i} + \frac{\partial U_e}{\partial \theta_i} = o, \quad (i = 1,2,\ldots,n), \text{ we obtain}$$

$$\begin{bmatrix} a_1^2 M_1^* & a_1 a_2 M_2^* & a_1 a_3 M_3^* \ldots\ldots a_1 a_n M_n^* \\ a_2 a_1 M_2^* & a_2^2\ M_2^* & a_2 a_3 M_3^* \ldots\ldots a_2 a_n M_n^* \\ a_3 a_1 M_3^* & a_3 a_2 M_3^* & a_3^2 M_3^* \ldots\ldots a_3 a_n M_n^* \\ \vdots & & \vdots \\ a_n a_1 M_n^* & a_n a_2 M_n^* & a_n a_3 M_n^* \ldots\ldots a_n^2\ M_n^* \end{bmatrix} \begin{Bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \\ \ddot\theta_3 \\ \vdots \\ \ddot\theta_n \end{Bmatrix} - V_o \begin{Bmatrix} a_1\theta_1 \\ a_2\theta_2 \\ a_3\theta_3 \\ \vdots \\ a_n\theta_n \end{Bmatrix}$$

$$+ \begin{bmatrix} (K_1+K_2) & -K_2 & o & o & \ldots\ldots\ldots\ldots\ldots\ldots o \\ -K_2 & (K_2+K_3) & -K_3 & o & \ldots\ldots\ldots\ldots\ldots\ldots o \\ o & -K_3 & (K_3+K_4) & -K_4 & \ldots\ldots\ldots\ldots\ldots\ldots o \\ \vdots & & & & \vdots \\ o & o & o & o \ldots\ldots\ldots -K_{n-1} & (K_{n-1}+K_n) \quad -K_n \\ o & o & o & o \ldots\ldots\ldots\ o & -K_n \quad K_n \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \vdots \\ \theta_{n-1} \\ \theta_n \end{Bmatrix} = o \tag{3.25}$$

where $M_i^* = M_i + M_{i+1} + \ldots + M_n$ $(i = 1,2, \ldots, n)$

In abbreviated form we may write

$$\underset{\sim}{M}\,\ddot{\underset{\sim}{\theta}} - V_o\,\underset{\sim}{a}\,\underset{\sim}{\theta} + \underset{\sim}{S}\,\underset{\sim}{\theta} = o \tag{3.25a}$$

The results for the idealization of Figure 3.7 are shown in Table 2. It may be noted that if one were to follow the procedure for Case A for the determination of the Euler buckling load the results obtained would not be very good. This is because the modal shapes for buckling under a tip load and for free vibration of a cantilever column are significantly different. The error in the buckling loads determined on the basis of the free vibration modal shapes is quite appreciable even after applying the Galerkin Method.

A superior method is due to the fact that the buckling loads are the latent roots of the matrix $\underset{\sim}{S}$ and this procedure has been used in obtaining Table 3.2.This is a more general method; however, there is no advantage in using it for Cases A and C because the free vibration modal shape vector itself yields good results.

Figure 3.8 shows the region of principal instability for a uniform column of length L free at the top and fixed at the base. It may again be observed that as the number of segments increases the curves converge rapidly to the true one.

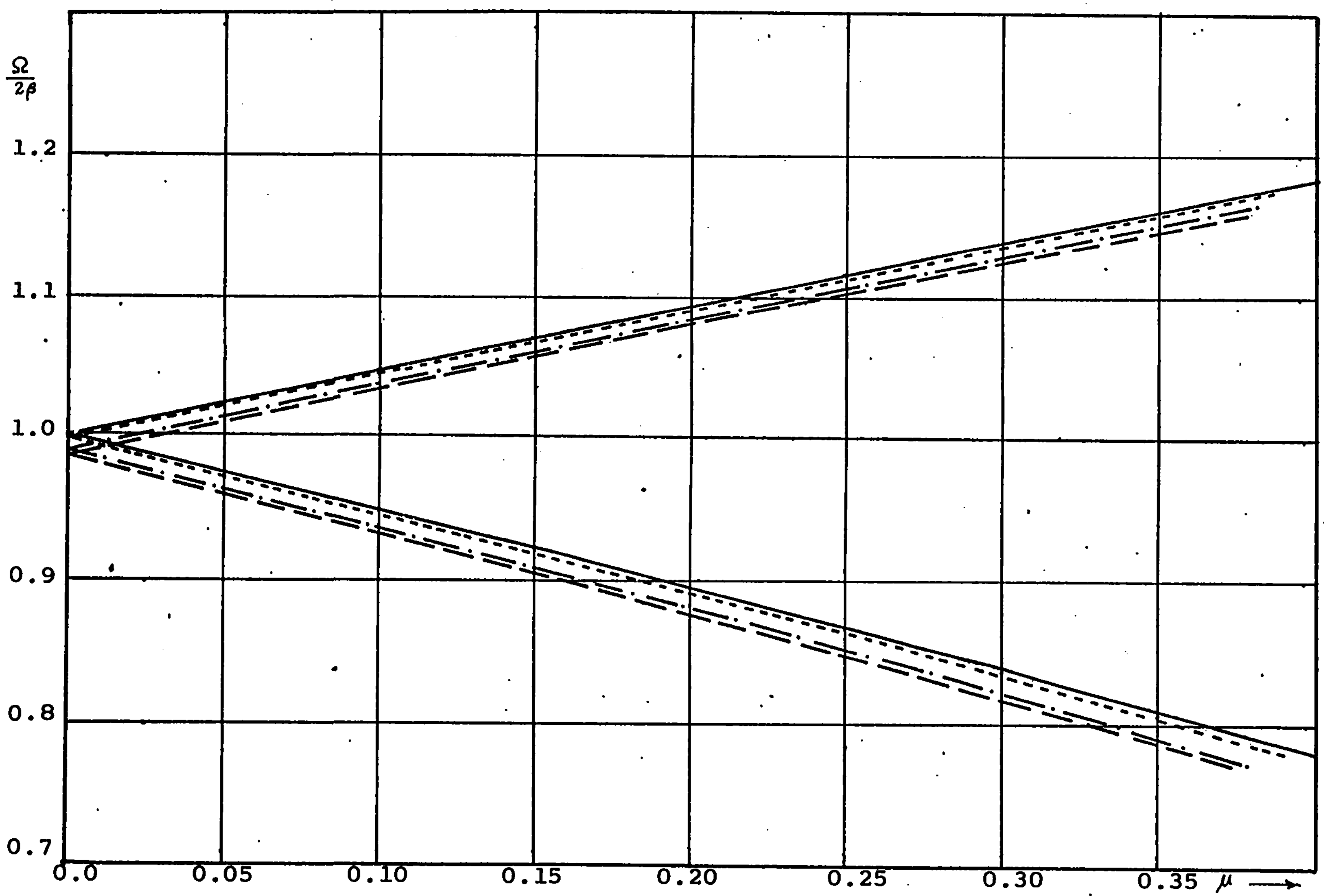


Figure 3.8: <u>Instability diagram for cantilever column</u> (case B) — —, n=5; —·—, n=6; ------, n=11; ——, n=∞.

3.2C. : Case C: Cantilever Column Subjected to Periodically Varying Vertical Support Motion.

In addition to the assumption of base fixity (as in Case B) it is now assumed that the vertical acceleration throughout the length of the column remains constant. A step-wise variation of the axial force in the column, resulting from this latter assumption, is shown in Figure 3.9. The mathematical development is based on the idealization shown in Figure 3.7.

Consider the top link of the column shown in Figure 3.7. The forces acting on it are shown in Figure 3.10. Taking moments about mass M_{n-1} gives

$$M_n \ddot{x}_n a_n \theta_n - M_n \ddot{y}_n a_n = K_n \left(\theta_n - \theta_{n-1}\right) \tag{3.26}$$

The vertical equilibrium of free-body of mass M_n gives

$$M_n \ddot{x}_n = V_n \tag{3.27}$$

From (3.26) and (3.27) we get

$$V_n a_n \theta_n - M_n \ddot{y}_n a_n = K_n\left(\theta_n - \theta_{n-1}\right) \tag{3.28}$$

Similarly, for the next link, taking moments about mass M_{n-2} (Figure 3.11) gives

$$M_n \ddot{x}_n\left(a_n \theta_n + a_{n-1}\theta_{n-1}\right) + M_{n-1} \ddot{x}_{n-1} \left(a_{n-1} \theta_{n-1}\right) - M_n \ddot{y}_n \left(a_n + a_{n-1}\right)$$

$$-M_{n-1}\ddot{y}_{n-1} a_{n-1} + K_{n-1} \left(\theta_{n-2} - \theta_{n-1}\right) = o \tag{3.29}$$

From the vertical equilibrium of the free-body of mass M_{n-1} we get

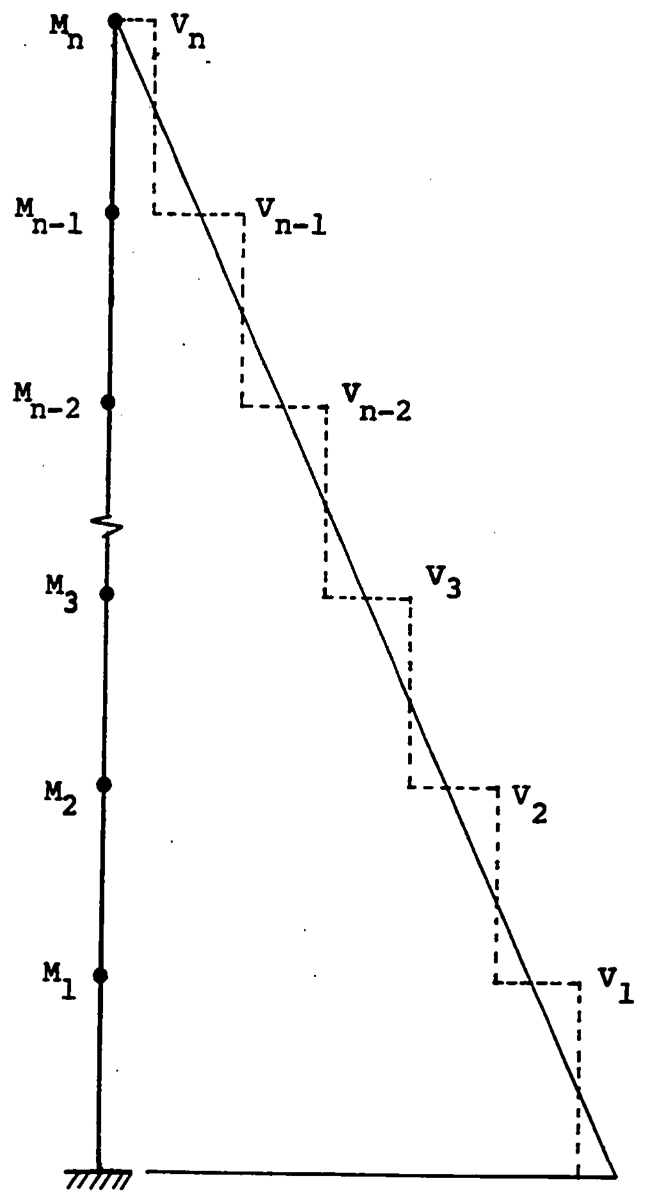


Figure 3.9: Stepwise variation of axial force.

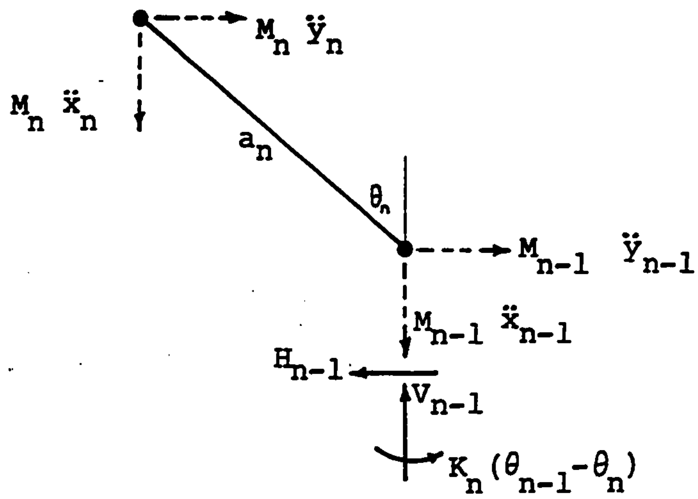


Figure 3.10: Free body diagram of n-th link.

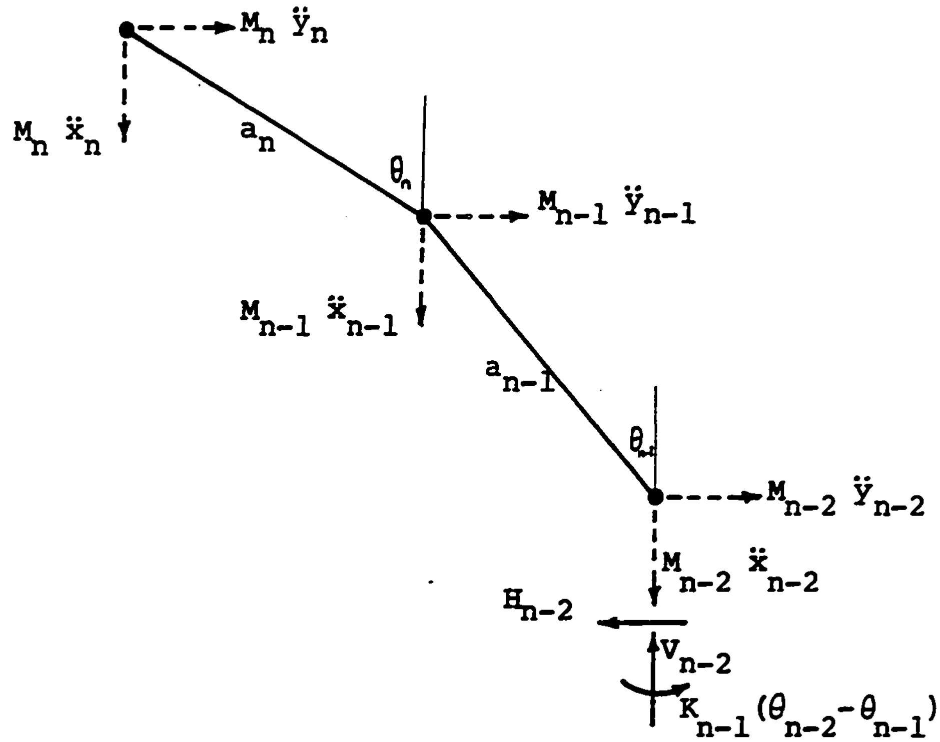


Figure 3.11: Free body diagram of n-th and (n-1)th links.

$$V_{n-1} - V_n = M_{n-1} \ddot{x}_{n-1} \tag{3.30}$$

Substituting Equations (3.27) and(3.30) in Equation (3.29) and subtracting Equation (3.26) from it results in

$$V_{n-1} a_{n-1} \theta_{n-1} - \left(M_n \ddot{y}_n + M_{n-1} \ddot{y}_{n-1}\right) a_{n-1} = K_{n-1}\left(\theta_{n-1} - \theta_{n-2}\right) - K_n \left(\theta_n - \theta_{n-1}\right) \tag{3.31}$$

Recasting Equation (3.26) and Equation (3.31) in terms of deflections gives

$$V_n (y_n - y_{n-1}) - M_n \ddot{y}_n a_n = r_n y_n - \left(r_n + S_n\right) y_{n-1} + S_n y_{n-2} \tag{3.26a}$$

and

$$V_{n-1}\left(y_{n-1} - y_{n-2}\right) - \left(M_n \ddot{y}_n + M_{n-1} \ddot{y}_{n-1}\right) a_{n-1} = S_{n-1} y_{n-3} - \left(r_{n-1} + S_{n-1} + S_n\right) y_{n-2}$$

$$+ \left(r_{n-1} + r_n + S_n\right) y_{n-1} - r_n y_n \tag{3.31a}$$

where

$$r_n = \frac{K_n}{a_n}$$

$$S_n = \frac{K_n}{a_{n-1}}$$

$$r_{n-1} = \frac{K_{n-1}}{a_{n-1}}$$

$$S_{n-1} = \frac{K_{n-1}}{a_{n-2}}, \text{ etc.}$$

In this way, the equations of motion in matrix form are written as

$$
\begin{Bmatrix}
V_1 y_1 \\
V_2(y_2-y_1) \\
\cdot \\
\cdot \\
\cdot \\
V_{n-1}(y_{n-1}-y_{n-2}) \\
V_n(y_n-y_{n-1})
\end{Bmatrix}
-
\begin{bmatrix}
M_1a_1 & M_2a_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & M_na_1 \\
o & M_2a_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & M_na_2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
o & o & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & M_{n-1} & a_{n-1} & M_na_{n-1} \\
o & o & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & o & & M_na_n
\end{bmatrix}
\begin{Bmatrix}
\ddot{y}_1 \\
\ddot{y}_2 \\
\cdot \\
\cdot \\
\cdot \\
\ddot{y}_{n-1} \\
\ddot{y}_n
\end{Bmatrix}
=
$$

$$
\begin{bmatrix}
(r_1+r_2+s_2) & -r_2 & o & o & \cdot & \cdot & \cdot & \cdot & o \\
-(r_2+s_2+s_3) & (r_2+r_3+s_3) & -r_3 & o & \cdot & \cdot & \cdot & \cdot & o \\
s_3 & -(r_3+s_3+s_4) & (r_3+r_4+s_4) & -r_4 & \cdot & \cdot & \cdot & \cdot & o \\
o & s_4 & -(r_4+s_4+s_5) & (r_4+r_5+s_5) & -r_5 & \cdot & \cdot & \cdot & o \\
\cdot & & & & & & & & \cdot \\
\cdot & & & & & & & & \cdot \\
\cdot & & & & & & & & \cdot \\
o & \cdot & \cdot & \cdot & \cdot & s_{n-1} & -(r_{n-1}+s_{n-1}+s_n) & (r_{n-1}+r_n+s_n) & -r_n \\
o & \cdot & \cdot & \cdot & \cdot & o & s_n & -(r_n+s_n) & r_n
\end{bmatrix}
\begin{Bmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4 \\
\cdot \\
\cdot \\
\cdot \\
y_{n-1} \\
y_n
\end{Bmatrix}
\qquad (3.32)
$$

Since a step-wise variation has been assumed for the axial forces, they may be expressed in terms of the base axial force, V . As before, applying the Galerkin Method, Equations (3.32) may be reduced to a single equation

$$\eta_1 V y + \eta_2 \, Ma \, \ddot{y} + \eta_3 \frac{K}{a} y = o \tag{3.33}$$

where η's are the coefficients associated with V , m, and K.

By putting V = o, the natural frequency is obtained as

$$w^2 = \frac{\eta_3}{\eta_2} \frac{K}{Ma^2} \tag{3.34}$$

and if y = o, we get

$$V^* = -\frac{\eta_3}{\eta_1} \frac{K}{a} \tag{3.35}$$

where V* denotes the inertia force which if applied at the base would cause the column to buckle (cf. buckling of the column under its own weight). If the base acceleration varies, the axial force in the column will also vary. If the base acceleration varies in a periodic manner the resulting equations of motion (Equation (3.33) will be of the Mathieu type).

Substituting Equations (3.34) and (3.35) in Equation (3.33) yields

$$\ddot{y} + w^2 \left(1 - \frac{V}{V^*}\right) y = o \tag{3.36}$$

Now $\quad V = \rho L A \xi$

where A = cross-sectional area

ξ = acceleration

Let the support motion be $u_s = C \cos \Omega t$ where C is the amplitude of the base motion, then $\ddot{u}_s = -\Omega^2 C \cos \Omega t$, whence substituting in Equation (3.36) we get

$$\ddot{y} + w^2 \left(1 + \frac{\rho LA}{V^*} \Omega^2 C \cos \Omega t\right) y = o$$

or $$\ddot{y} + w^2 \left(1 + b\Omega^2 C \cos \Omega t\right) y = o \qquad (3.37)$$

where $$b = \frac{\rho LA}{V^*} \qquad (3.38)$$

Let $$\epsilon = b\Omega^2 C \qquad (3.39)$$

Then we get $\ddot{y} + w^2 (1 + \epsilon \cos \Omega t) y = o$ (3.40)

Equation (3.40) is the familiar Mathieu equation.

The principal region of instability corresponding to the first mode for a cantilever was determined by the proposed method and is shown by full lines in Figure 3.12. The column was divided into 11 parts as shown in Figure 3.13.

An experimental investigation was also carried out to determine the principal region of instability of such a cantilever column. An aluminum specimen 23.5" long, 0.75" wide and 0.040" thick was used. The circled points in Figure 3.12 are experimental observations and it may be seen that these results are in good agreement with the theory. A comprehensive account of the experimental set-up and results is given in Chapter IV.

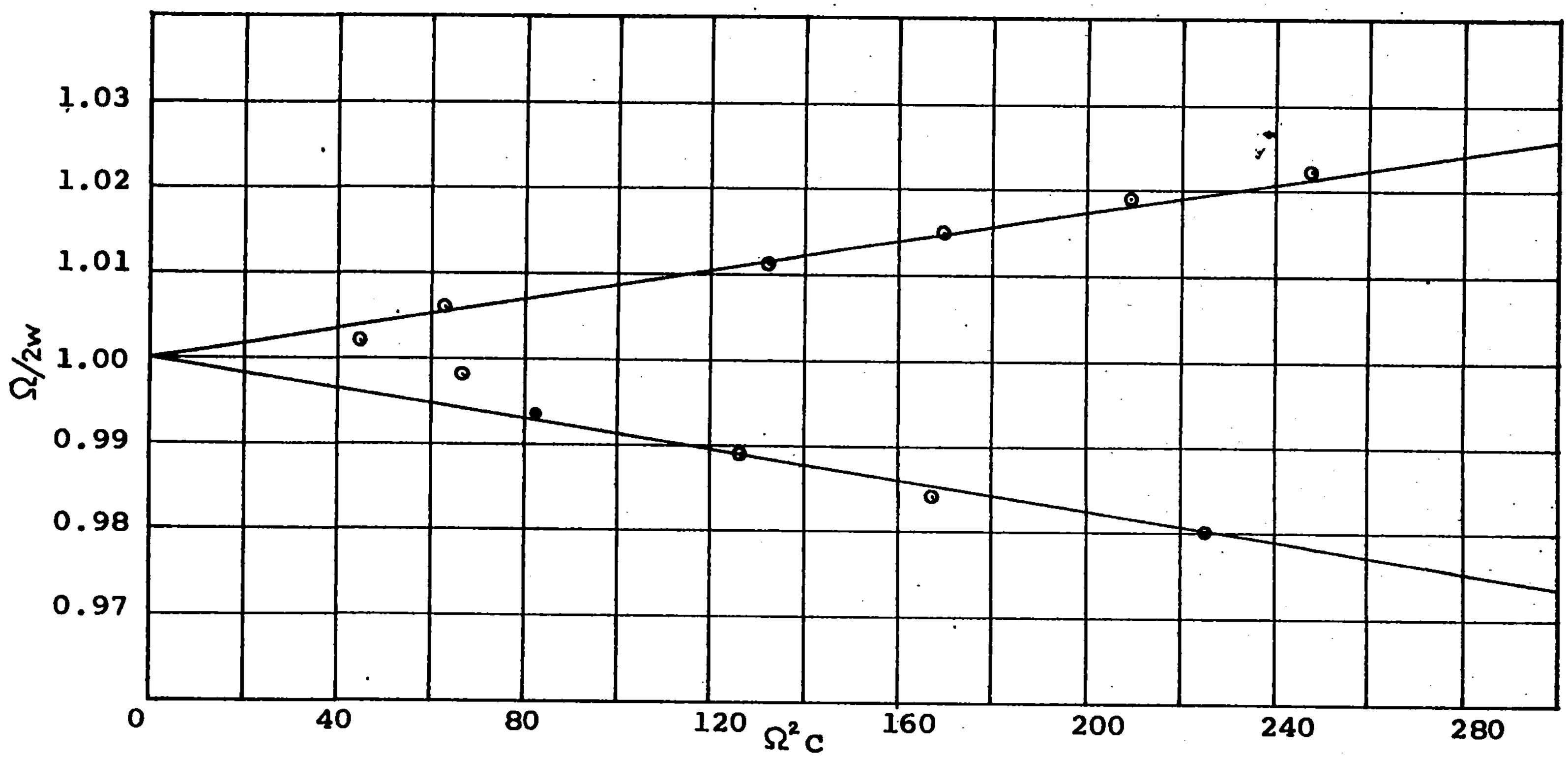


Figure 3.12: <u>Region of instability for a cantilever column subjected to periodic base motion (first mode).</u>

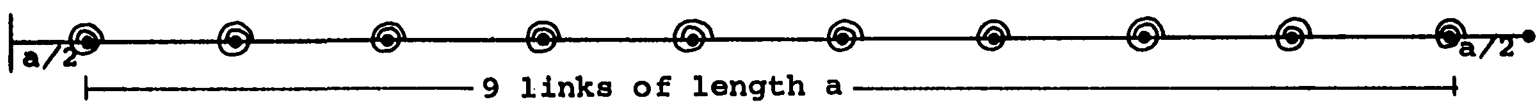


Figure 3.13: Column subdivision for case C.

The principal region of instability corresponding to the second mode for the cantilever of Figure 3.13 is shown in Figure 3.14. As before the full lines refer to the proposed method whilst the experimental points are represented by circles. It may be observed that the agreement between theoretical and experimental results is good.

3.3 CONCLUSIONS

The analysis presented shows that the idealization of columns by massless springs and lumped masses may be used for the construction of the regions of dynamic stability. The equation of motion for each mode may be transformed into its associated Mathieu equation whose solution yields the stability zones. It is shown that as the column is divided into more parts, the region of instability converges to the true one. The method is general and it may be applied to columns with different end conditions. A non-uniform column may be treated easily by an appropriate subdivision.

The above analysis was carried out by computer programs. The programs for Cases A and C are presented in Appendix D. The program for Case B is similar with only slight modifications. For a particular subdivision the values of the natural frequencies and the buckling load are given for Cases A and B. For Case C, the program gives the value of the natural frequency and the inertia force, V^*, which if applied at the base would cause the column to buckle. From these values the stability regions may be obtained as discussed above.

TABLE 3.1

No. of Parts	3		4		6		10		15		EXACT	
Mode	Nat Freq [1]	Buck Load [2]	Nat Freq	Buck Load	Nat Freq	Buck Load	Nat Freq	Buck Load	Nat Freq	Buck Load	Nat Freq	Buck Load
1	64.0	8.0	81.0	9.0	91.19	9.55	95.48	9.77	97.28	9.89	97.41	9.87
2	256.0	16.0	729.0	27.0	1193.65	34.55	1436.47	37.90	1507.23	38.83	1558.56	39.48
3			1296.0	36.0	4283.81	65.45	6561.18	81.00	7313.67	85.52	7890.21	88.83
4					8181.39	90.45	17920.85	133.87	21781.91	147.58	24936.96	157.91
5					10000.0	100.0	36149.39	199.13	49242.28	221.90	60881.25	246.74

1 Nat Freq = Natural Frequency (Coefficient x $\sqrt{\frac{EI}{mL^4}}$)

2 Buck Load = Buckling Load (Coefficient x $\frac{EI}{L^2}$)

TABLE 3.2

No. of Parts	3		4		6		11		16		EXACT	
Mode	Nat Freq	Buck Load	Nat Freq	Buck Load	Nat Freq	Buck Load	Nat Freq	Buck Load	Nat Freq	Buck Load	Nat Freq	Buck Load
1	3.455	2.343	3.479	2.411	3.499	2.447	3.511	2.462	3.514	2.465	3.516	2.467
2	18.39	13.657	19.47	18.00	20.86	20.61	21.68	21.80	21.87	22.02	22.03	22.20
3			49.70	33.59	52.57	50.00	59.03	58.58	60.44	60.29	61.70	61.68
4					84.91	79.39	110.70	109.20	116.09	115.58	120.64	120.90
5					138.42	97.55	172.23	168.71	186.77	185.50	199.81	199.83

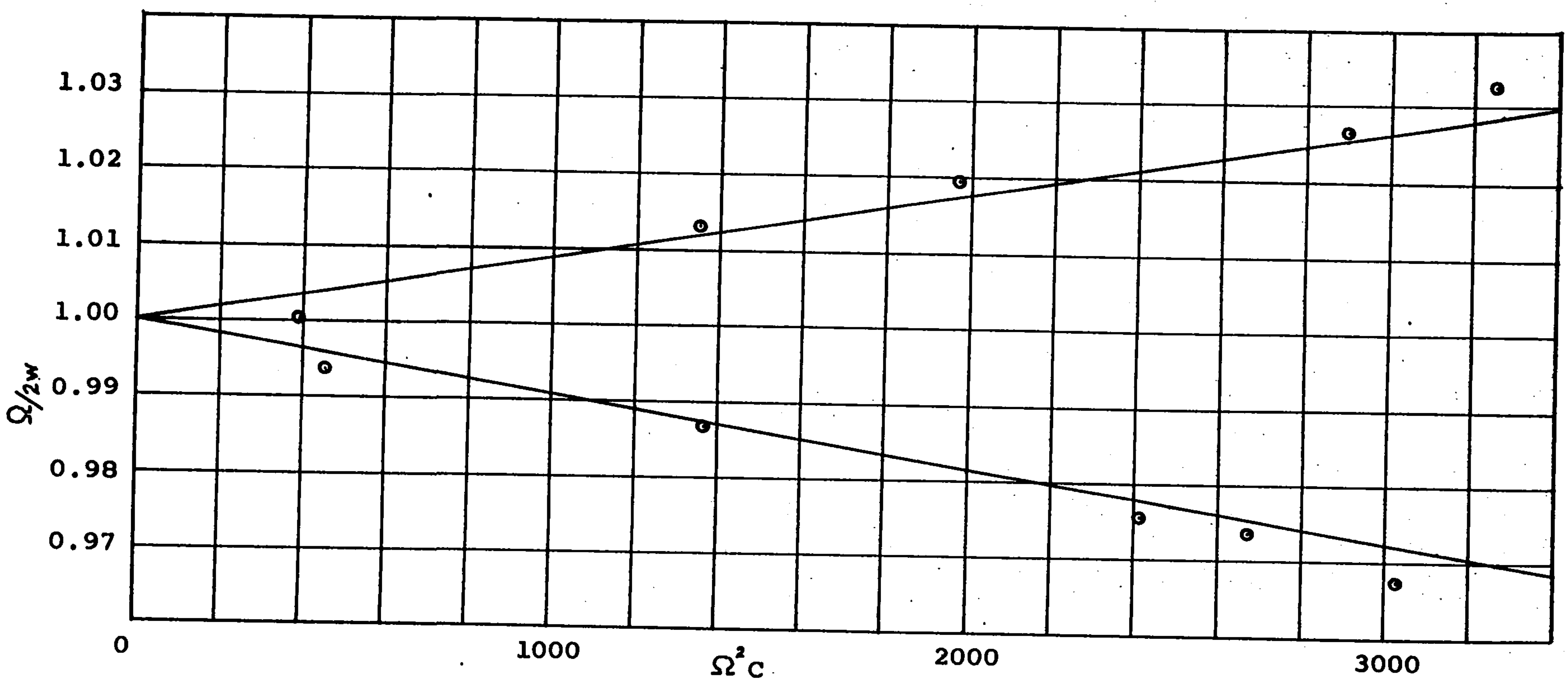


Figure 3.14 : Region of instability for a cantilever column subjected to periodic base motion (second mode).

CHAPTER IV

EXPERIMENTAL INVESTIGATION OF PARAMETRIC INSTABILITY OF COLUMNS

4.1 Introduction

For the most part previous experimental investigations have been directed towards studying the parametric instability of columns subjected to periodically varying axial forces. The author performed tests on a cantilever column to determine experimentally the regions of dynamic instability when its base is subjected to periodically varying vertical motion. The test specimen used was an aluminum alloy, 24S-T3, 3/4" wide and 0.040" thick. The specimen was clamped by means of screws and nuts as shown in Figure 4.1. The length of the column from its free tip to the centre of the 'clamps' was 23.5". Since an aluminum alloy was used the "deformations" at instability were fairly large and could be observed visually as evident from Figures 4.3 to 4.5. As the investigation was of a qualitative nature, i.e. it was intended only to establish whether the motion was stable or unstable, no elaborate instrumentation was necessary. The motion was termed as unstable when the column departed from its initial configuration. This was quite easy

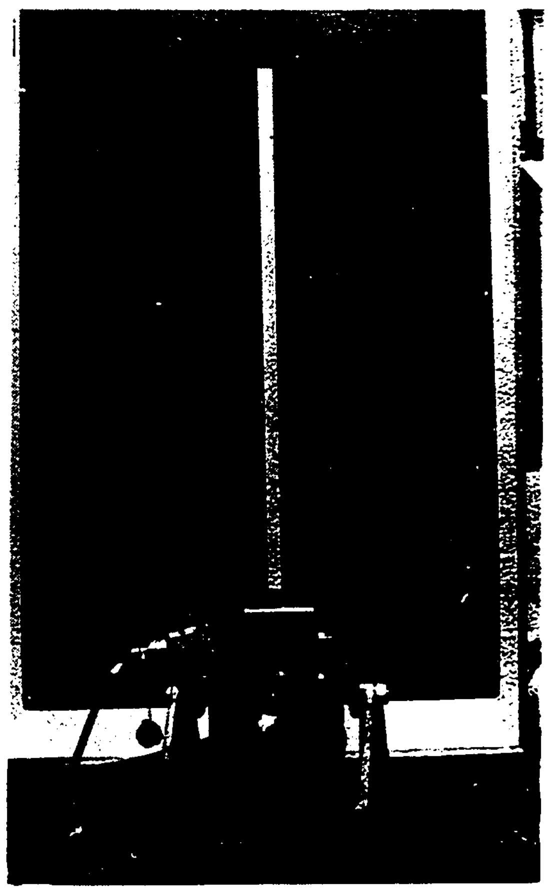


Figure 4.1: Cantilever column mounted on shaker

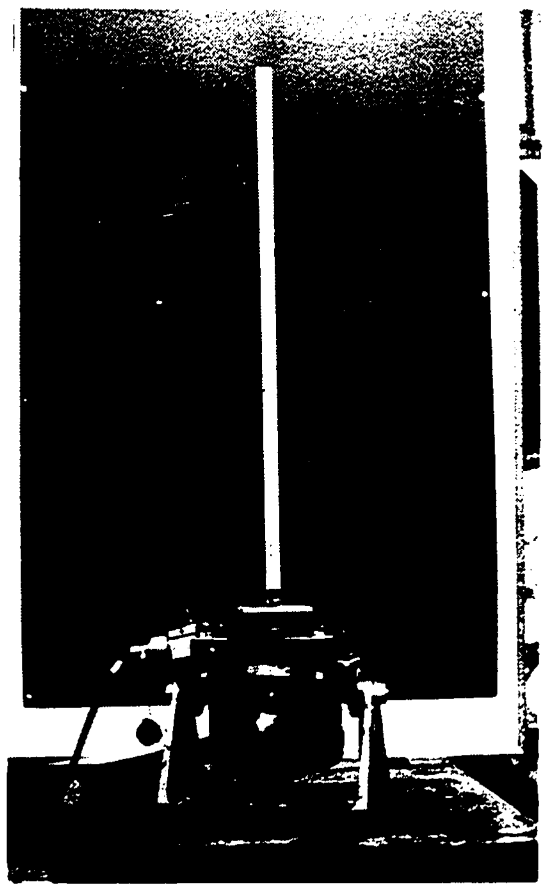


Figure 4.1: Cantilever column mounted on shaker

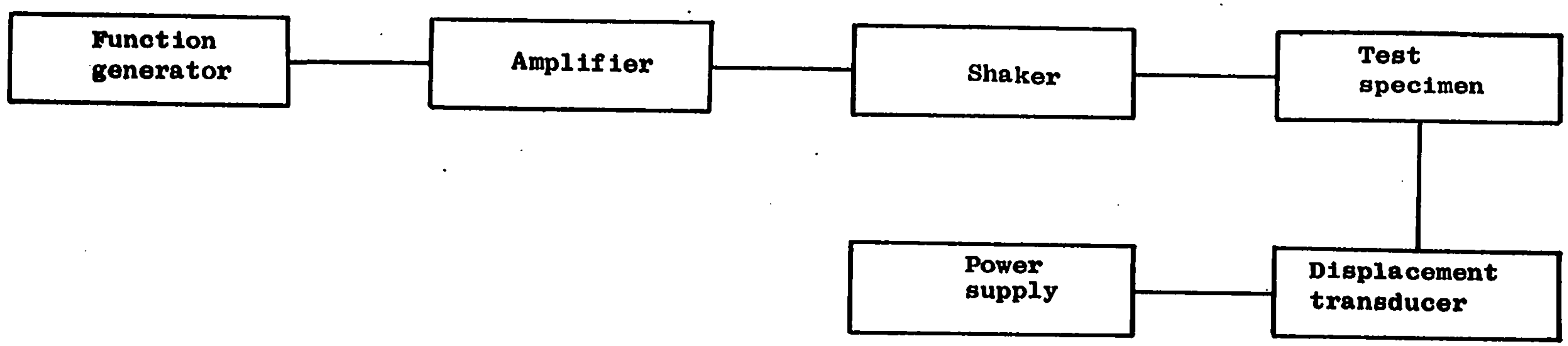


Figure 4.2: Schematic diagram of the experimental set up.

to detect.

To measure the amplitude of oscillation of the base a Sanborn displacement transducer (D.C. excited) was used. A schematic diagram of the experimental set-up is shown in Figure 4.2.

4.2 Equipment

The equipment used in the experiments is described below:-

1. Function Generator

The function generator used was the Hewlett-Packard Model 3300A which is capable of generating three types of waveforms, namely sine, square and triangular at frequencies ranging from 0.01 Hz to 100 kHz. For frequencies below 10 kHz the sine wave distortion is less than 1%.

2. Power Amplifiers

A Ling Model 300-2 power amplifier was used to drive the shaker manufactured by the same firm. The power amplifier was matched to the shaker used.

3. Shaker

A Goodman's V50 shaker manufactured by Pye-Ling was employed. The output force is generated when a current flows through a moving coil placed in a magnetic field produced by a permanent magnet. The coil is driven by the amplifiers and the motion of the coil is determined by the frequency and mode of the generator. The output force is imparted to the structure by means of a flexible diaphragm which moves with a maximum displacement of 0.7 in. between stops. The shaker can generate a maximum force of 48 lbs. A major drawback of this type of shaker is that the amplitude is inversely proportional to the applied frequency and hence at high frequencies the amplitude is very small.

4. Amplitude Measurement Device

A Sanborn displacement transducer, denomination 7DCDT-1000, was used to measure the amplitude of the movement of the base of the cantilever column. The moving core was connected to the base. A 6 volts D.C. power supply was used to excite the transducer. The device is capable of measuring displacements of up to ± 1 inch with an output of 4.8 volts for full scale displacement. Linearity of the output is guaranteed to 0.5% of full scale under the conditions used in the test.

5. Recording Unit

The output from the displacement transducer was recorded on photographic paper by an ultra-violet recorder, type 1050 manufactured by New Electronics Products Ltd., London. This recorder is of a galvanometric type; therefore, the input impedance is low (typically around 300 ohms). To retain the linearity of the transducer output with displacement an amplifier, Budd-Brown type 1631, was used so as to provide the required high input impedance, i.e. to match the galvo-impedance to the transducer.

6. Strobotac

A Type 1538A strobotac manufactured by the General Radio Company was used. The Strobotac consists of a power supply, an oscillator for controlling the rate at which the lamp is flashed and a Strobotac or flashing lamp. The frequency of the oscillator and hence the flashing speed of the lamp can be adjusted to any desired value between 100 c.p.m. to 400,000 c.p.m. The frequency controller is graduated directly in c.p.m. and was used to measure the frequency of the various instability modes.

4.3 Test Procedure

A sinusoidal waveform at the required frequency was generated by the function generator. This signal was amplified by the power amplifier and fed into the shaker thereby imparting a vertical sinusoidal motion to the base of the cantilever column. The base was directly mounted on to the shaker spindle.

The determination of the regions of dynamic instability entailed the measurement of the frequency and the amplitude of the base motion. For each frequency increment the amplitude was increased slowly until the column became unstable. If no instability occurred at that frequency, the frequency was increased and the whole procedure repeated. When instability was encountered at a certain frequency the determination of the 'critical' amplitude which caused instability was repeated at least five times to get an average reading. It was observed that the variation from the average was small. The record of the output from the displacement transducer was used to obtain the frequency and amplitude of the motion. The frequency of the instability mode was measured by the strobotac.

The modulus of elasticity was obtained experimentally from four tension test specimens made from the same strip as the column. The average value obtained was $E = 9.82 \times 10^6$ psi.

The sample was weighed and its density was found to be $.00025273$ lbs-sec^2/in^4.

4.4 Test Results

The average values of the amplitude of the support motion corresponding to the first mode are given in Table 4.1, and those corresponding to the second mode are given in Table 4.2.

Figures 4.3, 4.4, 4.5 respectively show the first mode, second mode and the third mode instabilities of the column corresponding to the first region of instability.

The amplitude of the base motion corresponding to the third mode was very small and could not be measured. Hence no readings are reported.

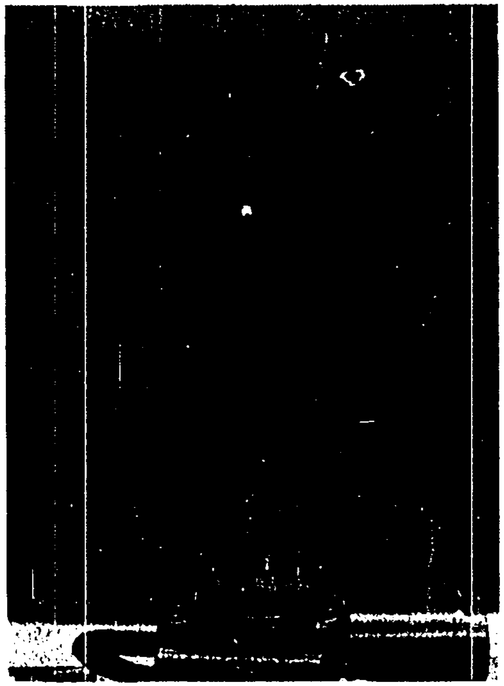


Figure 4.3: First Mode Instability

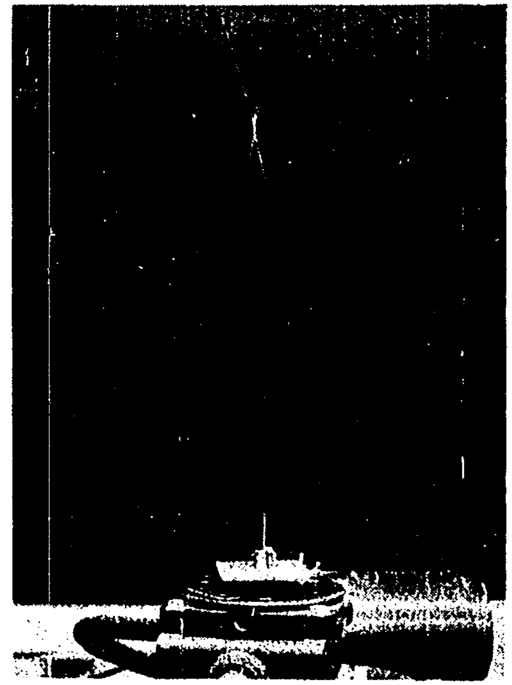


Figure 4.4: Second Mode Instability

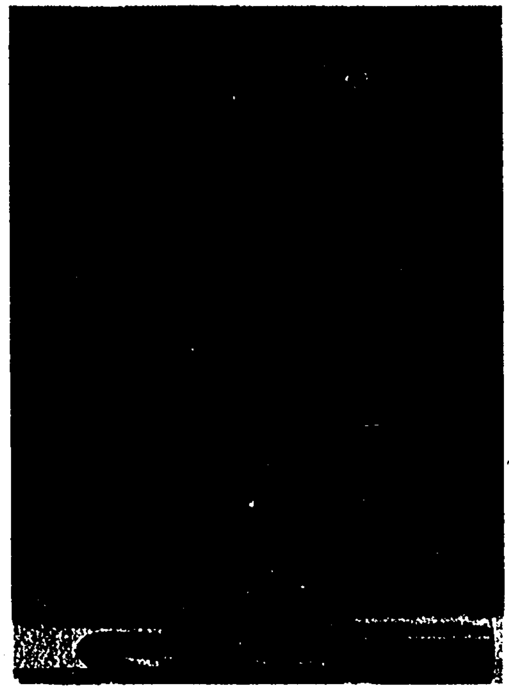


Figure 4.3: First Mode Instability

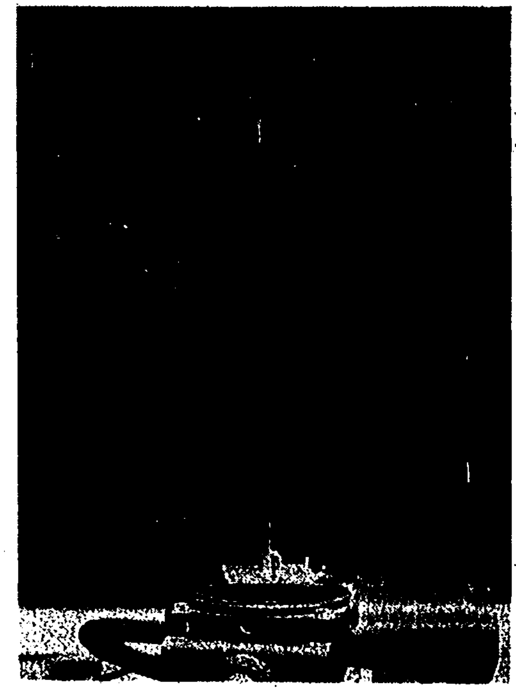


Figure 4.4: Second Mode Instability

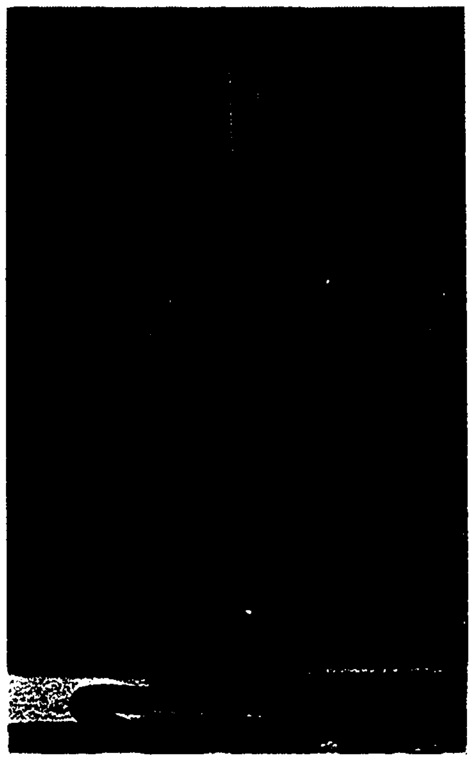


Figure 4.5: Third mode instability

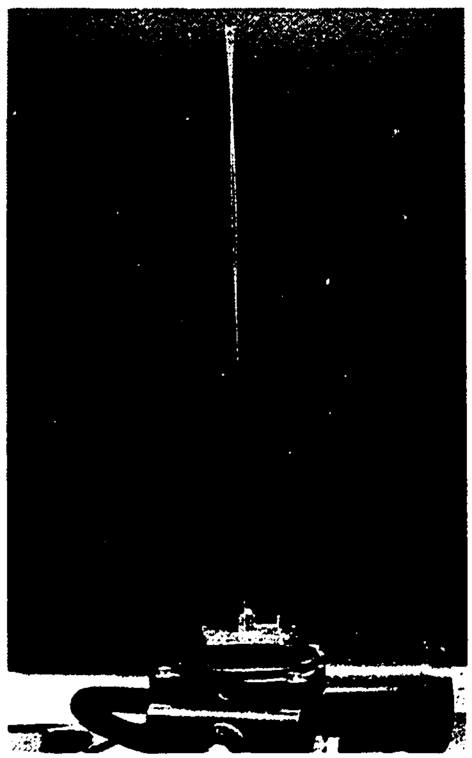


Figure 4.5: Third mode instability

TABLE 4.1

AVERAGE READINGS OF THE AMPLITUDE OF THE SUPPORT MOTION CORRESPONDING TO THE FIRST MODE

FREQUENCY (c.p.s.)	AMPLITUDE (inches)
4.52	0.2786
4.54	0.2053
4.56	0.1542
4.58	0.1120
4.60	0.0794
4.62	0.0538
4.64	0.0740
4.66	0.1538
4.68	0.1955
4.70	0.2411
4.71	0.2728

TABLE 4.2

AVERAGE VALUES OF THE AMPLITUDE OF THE SUPPORT MOTION CORRESPONDING TO THE SECOND MODE

FREQUENCY (c.p.s.)	AMPLITUDE (inches)
29.4	0.08860
29.6	0.07735
29.8	0.06327
30.0	0.03836
30.2	0.01240
30.4	0.01072
30.6	0.02001
30.8	0.03600
31.0	0.05170
31.2	0.07541
31.4	0.08326

CHAPTER V

EXPERIMENTAL INVESTIGATION OF PARAMETRIC INSTABILITY OF PORTAL FRAMES

5.1 Introduction

An experimental investigation of the parametric instability of portal frames was also undertaken. The same criterion of instability as for the columns was adopted in this case. This necessitated that the structure be such that its instability modes could be observed visually. To achieve this, different types of frames of various materials were constructed and tested. As remarked earlier the characteristics of the shaker are such that at high frequencies the available output amplitude is very small. Therefore, the frame had to be designed to permit the possibility of occurrence of as many of the modes as possible. The finite element method was used in selecting an appropriate shape and size. A six storey, single bay portal frame was considered to be a suitable shape.

The first frame was made from spring steel and was of the type shown in Figure 5.1. It was observed that this frame was very stiff and only the first mode could be observed visually.

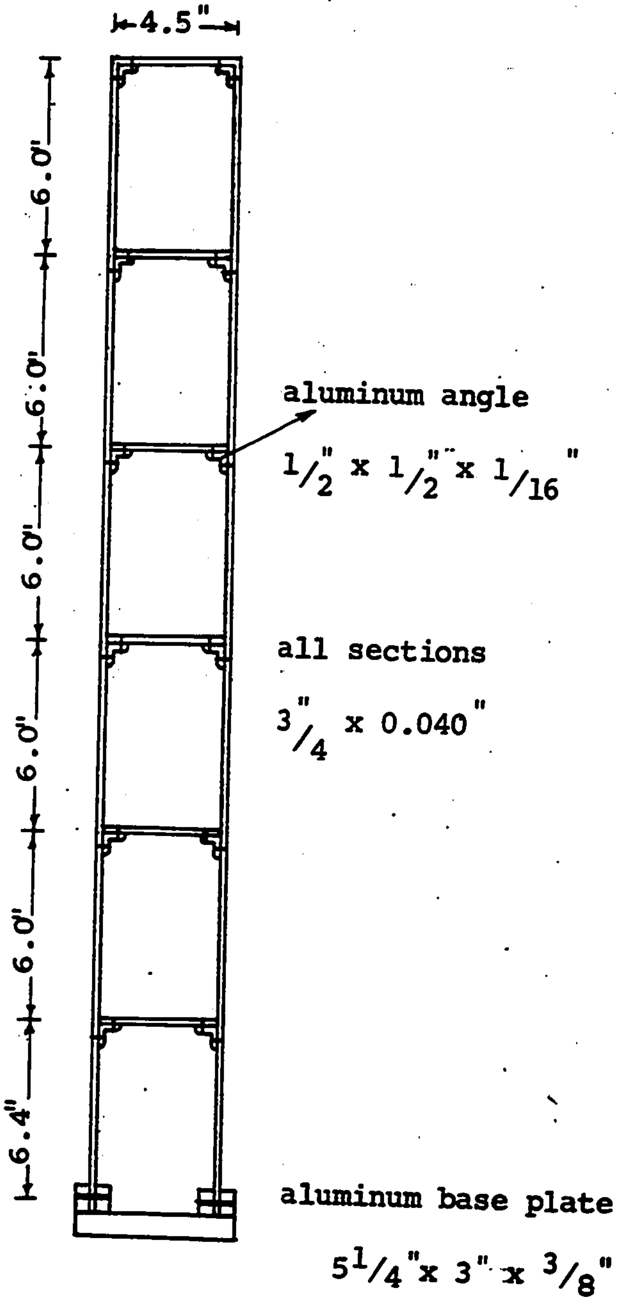


Figure 5.1: Six storey portal frame.

The second frame, similar to the previous one was made from aluminum, denomination 24S-TO. This frame was found to be too soft and it suffered inelastic changes of configuration when disturbed violently. Consequently, the third frame was made from aluminum of a slightly stiffer variety, denomination 24S-T3. This frame was used for the determination of the regions of dynamic instability. From the tension test on six specimens the modulus of elasticity was found to be 9.5×10^6 p.s.i.

All the above frames were constructed from strips 3/4" wide and 0.040" thick. The same section was used for the columns and beams. The beams were spaced at 6" centre to centre and were joined by screws and nuts to the columns by 1/2" x 1/2" x 1/16" aluminum brackets as shown in Figure 5.1. International locking washers were used to ensure that the nuts did not become loose during vibration and were found to perform satisfactorily. The frame was clamped to an aluminum plate 5-1/4" x 3" x 3/8" which was directly mounted on the shaker.

The testing procedure adopted was the same as for the cantilever column. For a particular frequency the amplitude was increased till the frame became unstable; the corresponding frequency and the amplitude were measured by means of the Sanborn displacement transducer. The experimental set-up is shown in Figure 5.2.

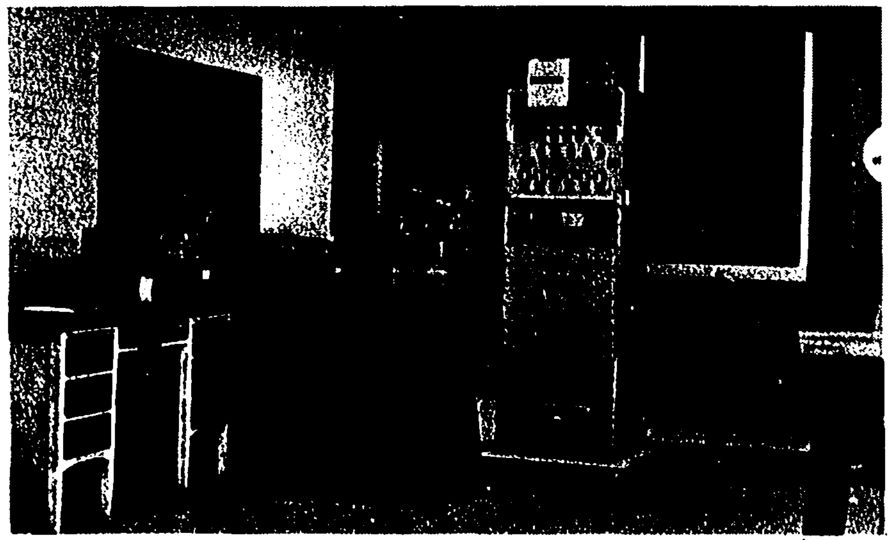


Figure 5.2(a): Experimental set-up

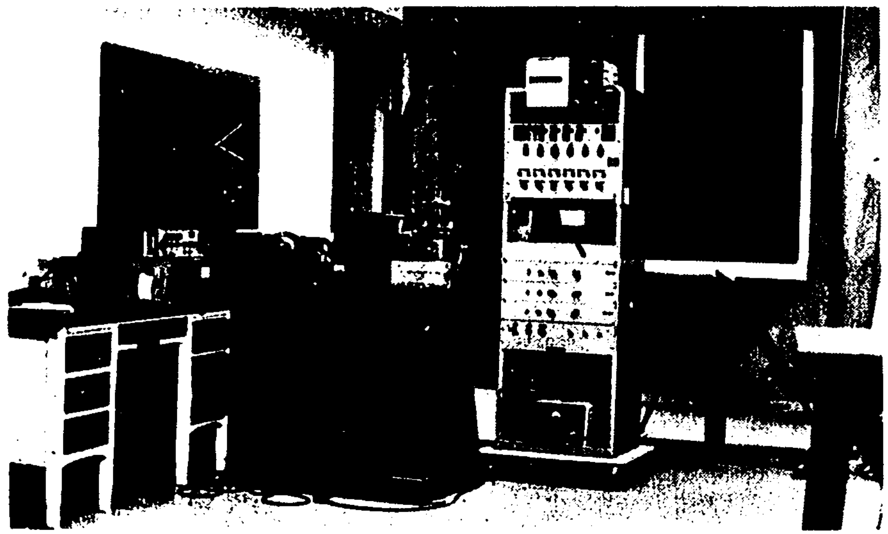


Figure 5.2(a): <u>Experimental set-up</u>

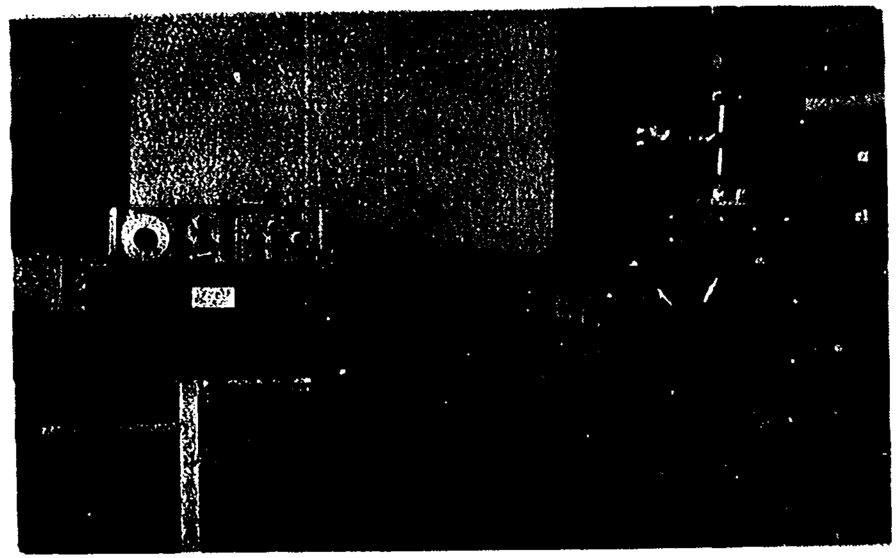


Figure 5.2(b): Experimental set-up

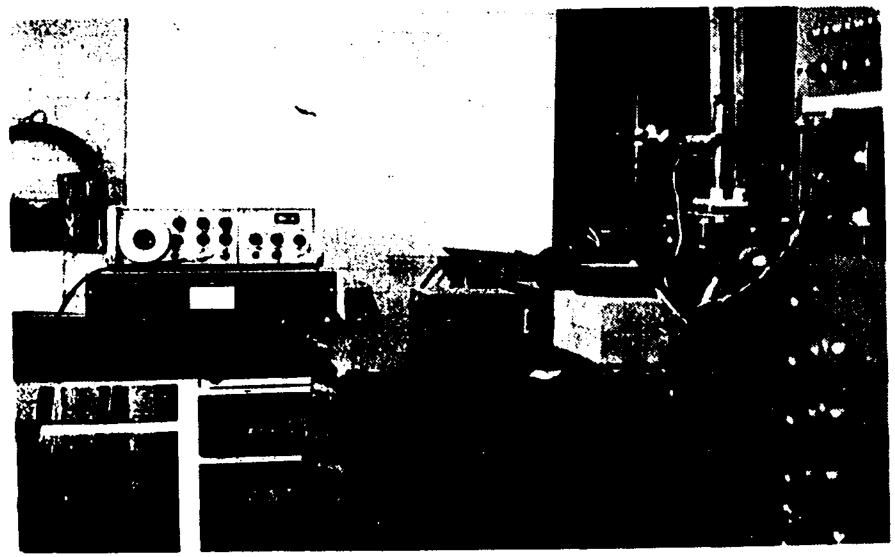


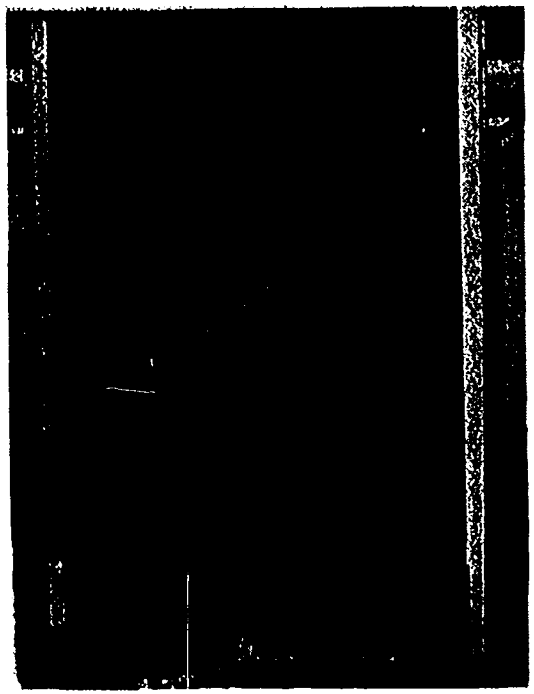
Figure 5.2(b): Experimental set-up

The first four modes were observed visually and are shown in Figures 5.3. The above photographs are for the natural modes corresponding to the principal region of instability. The instabilities corresponding to the first secondary region (i.e. the second region of instability) were also detected visually for the above four natural modes, but the amplitude of the instabilities was smaller, as was to be expected. The frequency of the instability mode was measured by the strobotac. A movie of the development of the instability modes was also taken; for which the author is greatly indebted to Professor B. Gersovitz and Mr. W. Hillgartner for their assistance and advice.

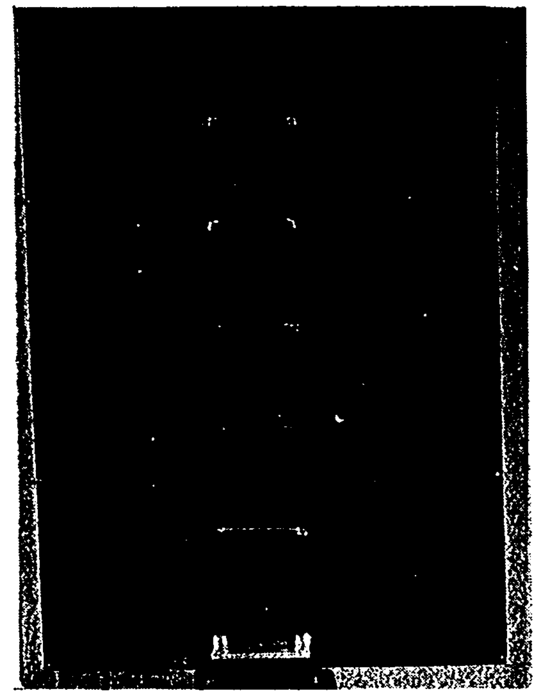
The amplitudes could only be measured for the first three modes. For the fourth mode the amplitudes were so small that they could not be detected by the deflection transducer. The experimental readings are given in Tables 5.1 to 5.3 and are plotted in Figures 5.4, 5.5 and 5.6.

The natural frequencies for the first four modes in c.p.s. as estimated by the finite element method and as obtained experimentally were as follows:

Mode	1	2	3	4
Finite Element	3.83	12.75	22.77	34.33
Experimental	3.80	13.7	23.5	36.0

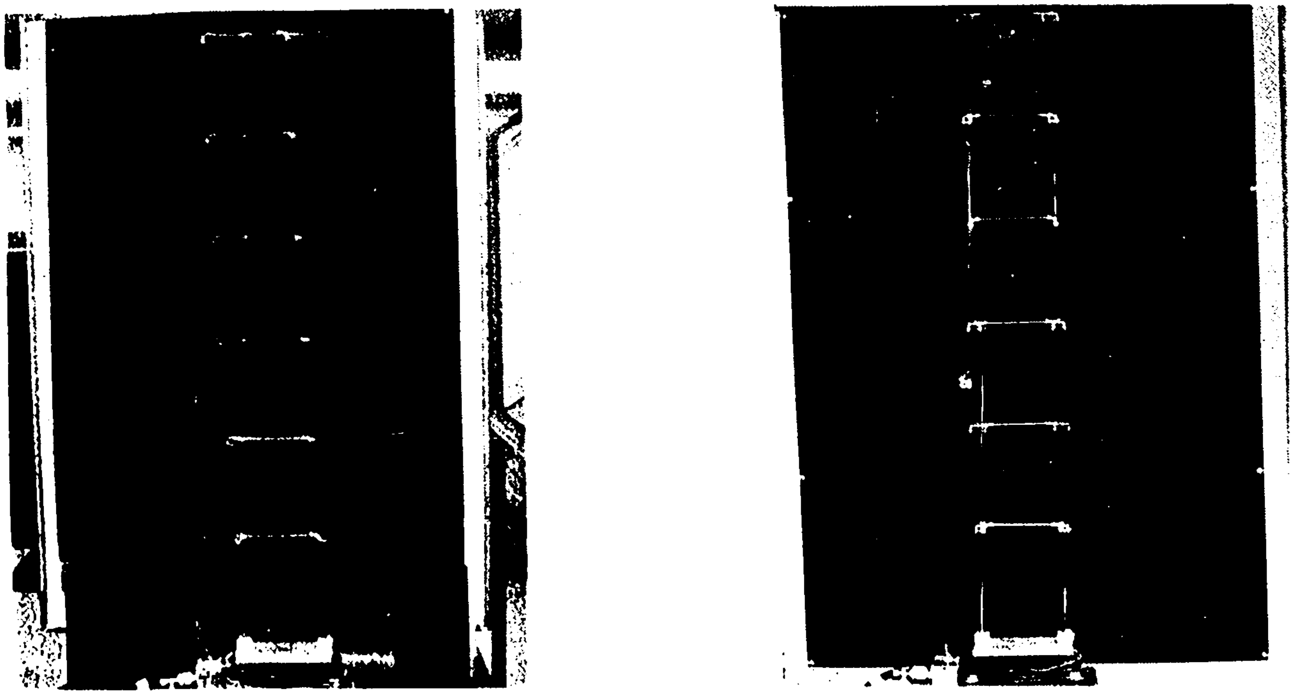


a) First Mode



b) Second Mode

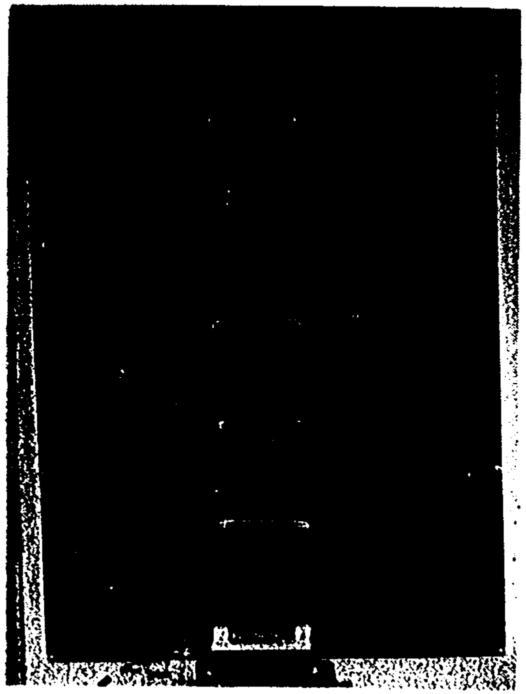
Figure 5.3: <u>Instability Modes of portal frame</u>



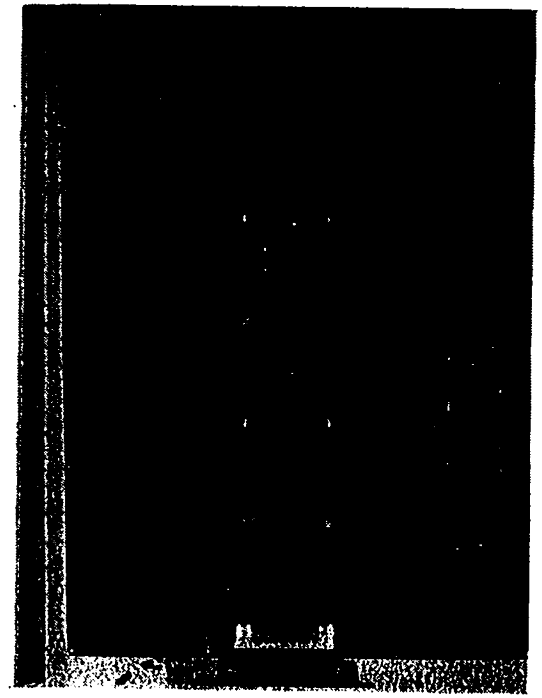
a) First Mode

b) Second Mode

Figure 5.3: Instability Modes of portal frame

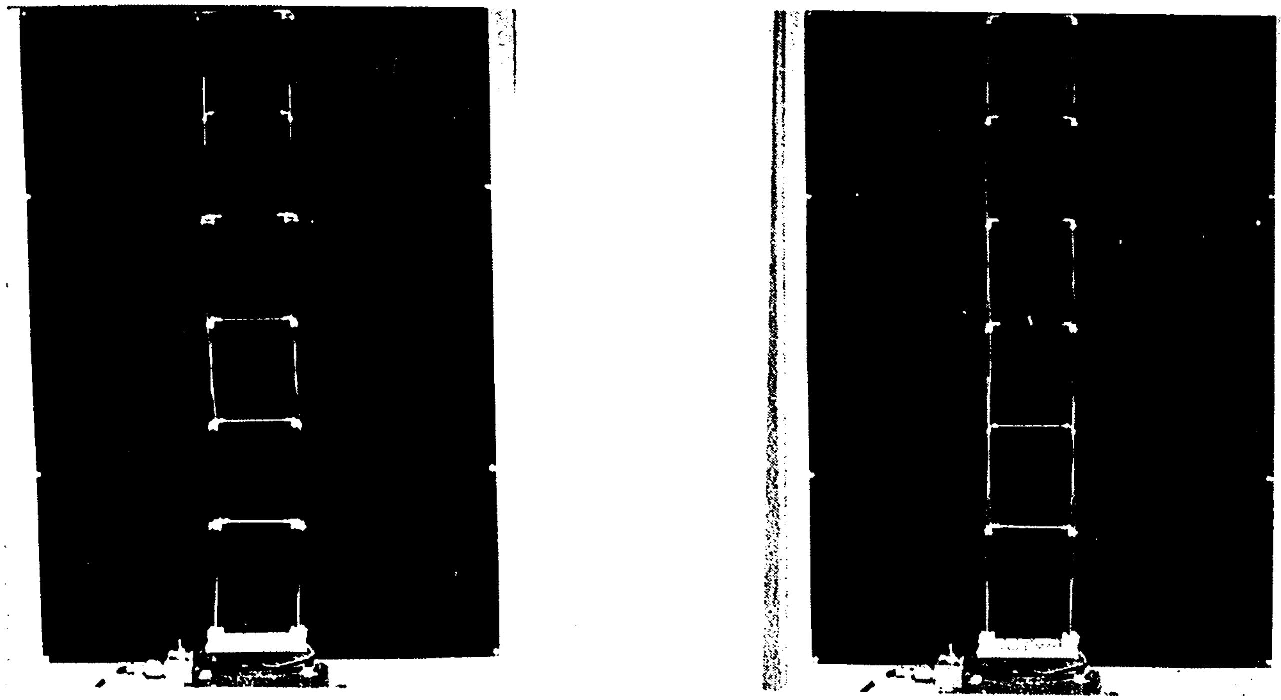


c) Third Mode



d) Fourth Mode

Figure 5.3: Instability Modes of portal frame



c) Third Mode

d) Fourth Mode

Figure 5.3: Instability Modes of portal frame

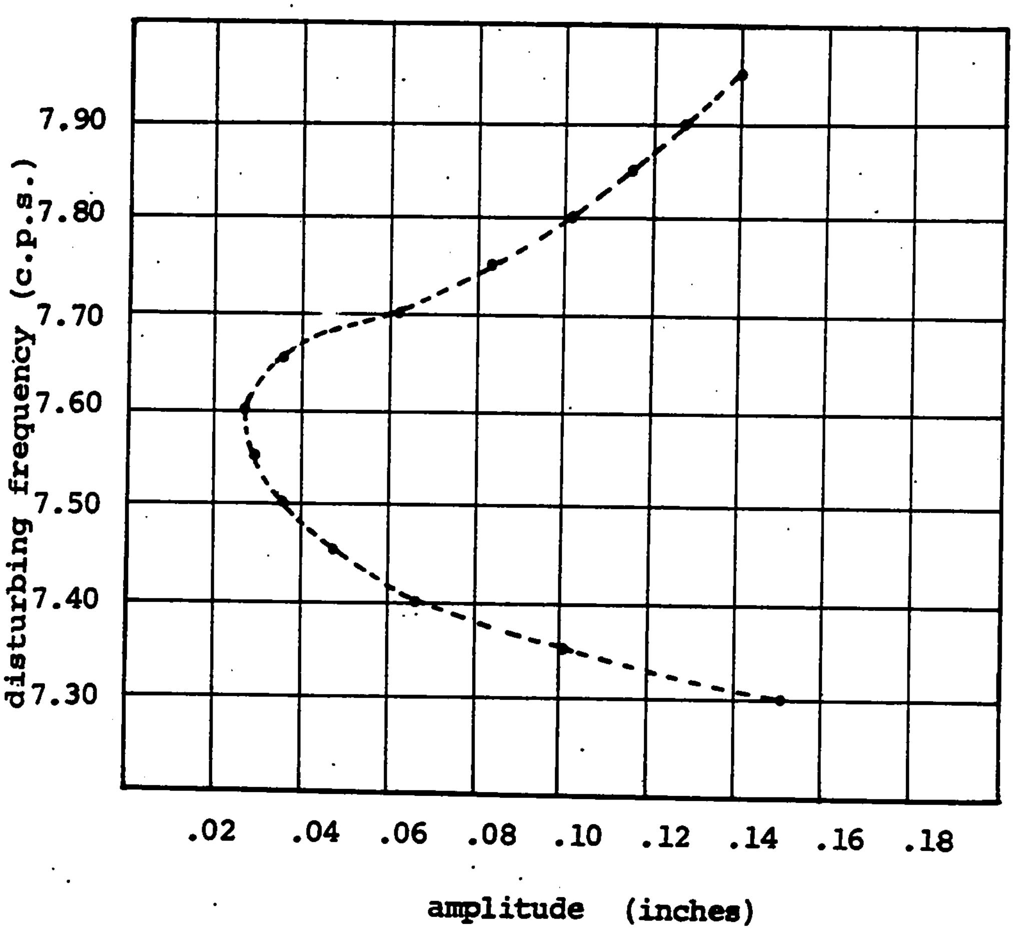


Figure 5.4: Principal instability region for first mode.

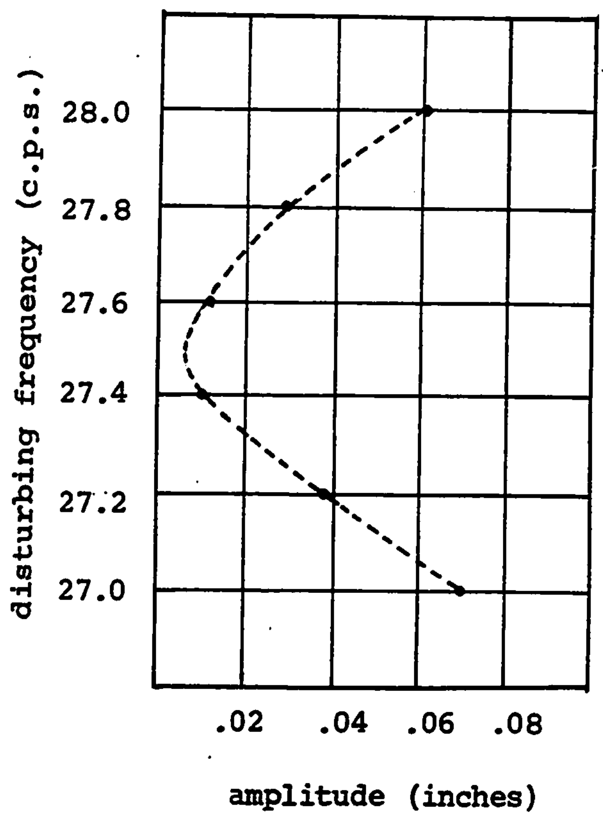


Figure 5.5: Principal instability region for second mode.

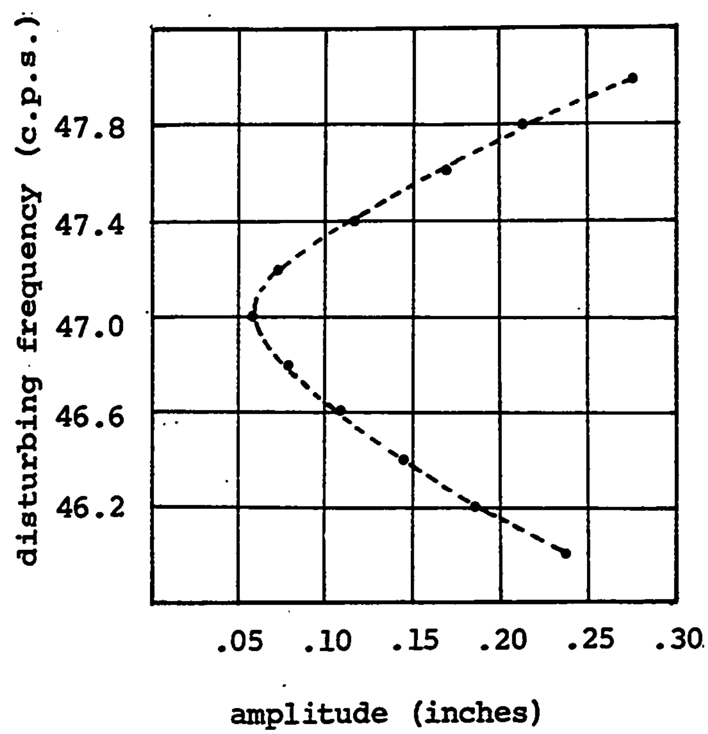


Figure 5.6: Principal instability region for third mode.

TABLE 5.1

AVERAGE VALUES OF THE AMPLITUDE OF THE SUPPORT MOTION CORRESPONDING TO THE FIRST MODE

FREQUENCY (c.p.s.)	AMPLITUDE (inches)
7.30	0.1499
7.35	0.1021
7.40	0.0629
7.45	0.0473
7.50	0.0348
7.55	0.0289
7.60	0.0261
7.65	0.0355
7.70	0.0622
7.75	0.0835
7.80	0.1028
7.85	0.1142
7.90	0.1254
7.95	0.1391

TABLE 5.2

AVERAGE VALUES OF THE AMPLITUDE OF THE SUPPORT MOTION

CORRESPONDING TO THE SECOND MODE

FREQUENCY (c.p.s.)	AMPLITUDE (inches)
27.0	0.0701
27.2	0.0365
27.4	0.0102
27.6	0.0113
27.8	0.0286
28.0	0.0610

TABLE 5.3

AVERAGE VALUES OF THE AMPLITUDE OF THE SUPPORT MOTION CORRESPONDING TO THE THIRD MODE

FREQUENCY (c.p.s.)	AMPLITUDE (inches)
46.0	0.0239
46.2	0.0185
46.4	0.0143
46.6	0.011
46.8	0.0082
47.0	0.0058
47.2	0.0071
47.4	0.0126
47.6	0.0170
47.8	0.0221
48.0	0.0274

CHAPTER VI

PARAMETRIC INSTABILITY OF PORTAL FRAMES

6.1 Introduction

The experimental investigation of the parametric instability of portal frames showed that the relationship between the frequency of the base motion and the natural frequencies of the frame is similar to that for the case of columns. When $\epsilon \to 0$ for instability to occur in the principal region the frequency of the base motion must be twice that of the natural frequency. Similarly, for the instability corresponding to the second region the frequency must be equal to the natural frequency of the structure. In other words, the following equation of boundary frequencies still holds good

$$\theta = \frac{2\Omega}{k}$$

To determine the equation of the principal region of instability a continuum approach was attempted and is discussed in detail in the next section.

6.2 The Continuum Approach

Consider the frame shown in Figure 6.1. Let the axial changes of length of the column be ignored and let the

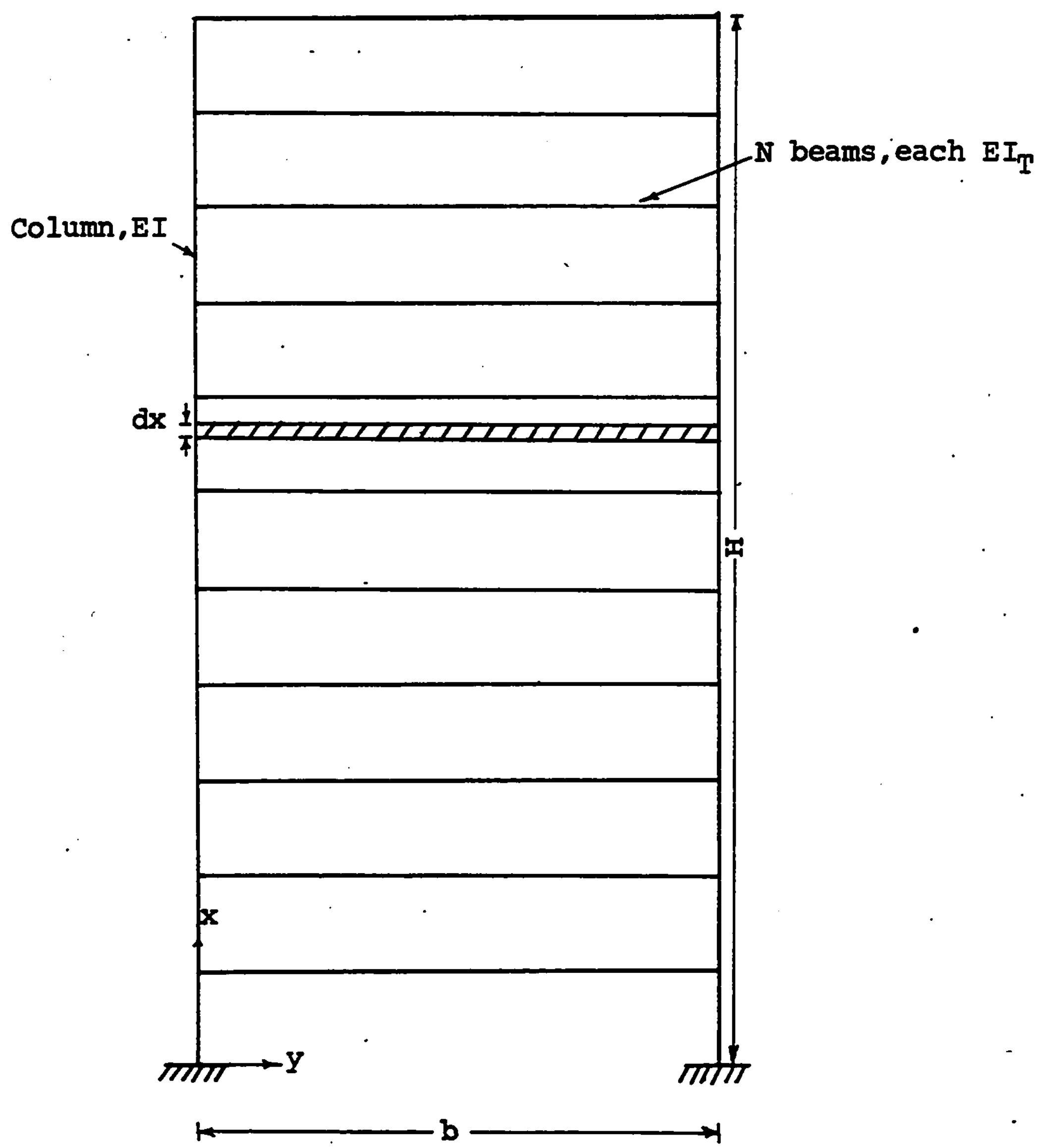


Figure 6.1: Multi-storey portal frame.

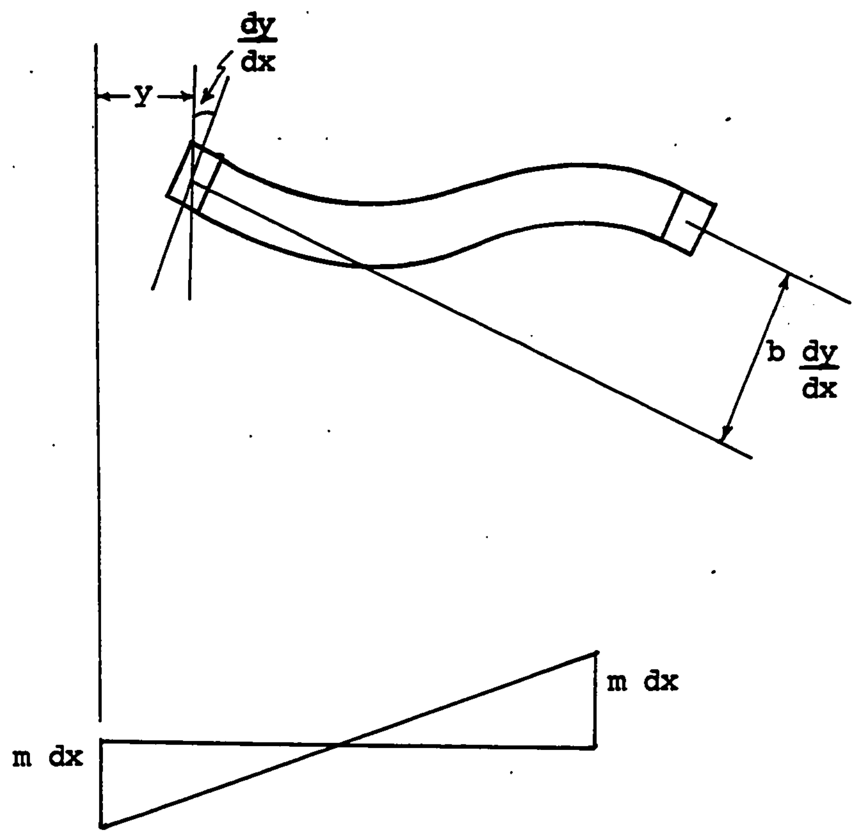


Figure 6.2: Bending moment diagram for strip of medium.

actual horizontal beams be replaced by a medium of bending stiffness $\frac{NEI_T}{H}$ per unit length. Consider a slice of thickness dx of the medium at height x above the base. Let end moment per unit height be called m. From Figure 6.2 and using the moment area method we have:

$$\frac{1}{2}(mdx)\,\frac{b}{2}\,\frac{2}{3}\,b = \frac{NEI_T}{H}\,dx\,b\,\frac{dy}{dx}$$

$$\text{so } m = 6\,\frac{NEI_T}{Hb}\,\frac{dy}{dx} \qquad (6.1)$$

Let q*(x) be a static external force acting per unit height on the left hand column. Consider the free-body of a differential element of the column as shown in Figure 6.3.

Horizontal equilibrium gives $\frac{dV}{dx} = -\frac{q^*}{2}$ (6.2)

Moment equilibrium gives $V\,dx = dM + mdx$

$$\text{or } V = \frac{dM}{dx} + m \qquad (6.3)$$

Differentiating Equation (6.3) w.r.t. x and eliminating V gives

$$\frac{d^2M}{dx^2} + \frac{dm}{dx} = -\frac{q^*}{2}$$

Now $M = -EI\,\frac{d^2y}{dx^2}$, $m = 6\,\frac{NEI_T}{Hb}\,\frac{dy}{dx}$

$$\text{so } -EI\,\frac{d^4y}{dx^4} + 6\,\frac{NEI_T}{Hb}\,\frac{d^2y}{dx^2} = -\frac{q^*(x)}{2} \qquad (6.4)$$

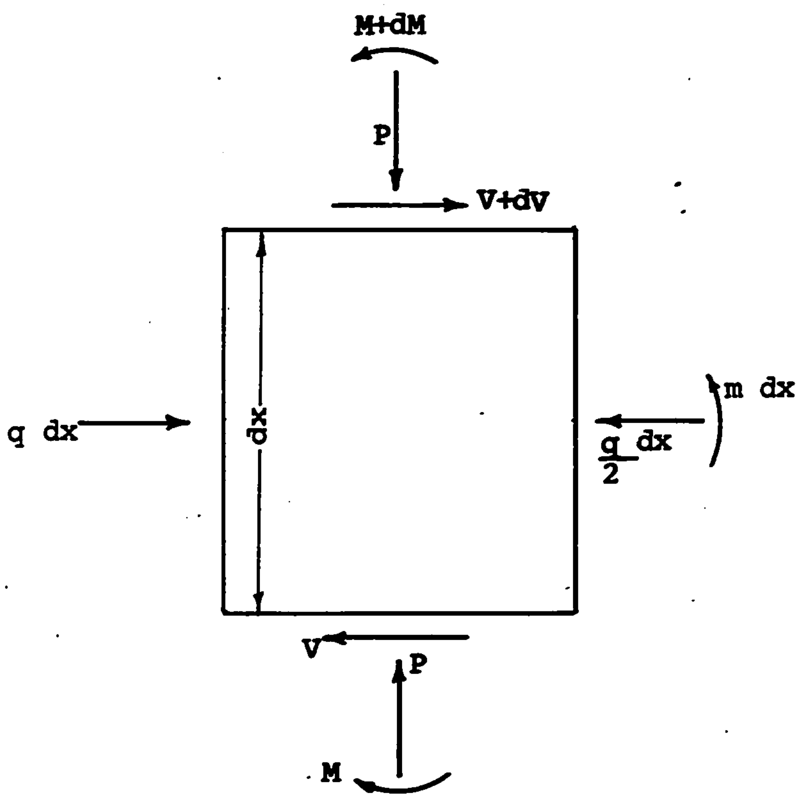


Figure 6.3: Forces on an element of the column.

or $$\frac{d^4y}{dx^4} - \frac{6NEI_T}{HbEI}\frac{d^2y}{dx^2} = \frac{q^*(x)}{2EI} \qquad (6.4a)$$

or $$\frac{d^2M}{dx^2} - \frac{6NEI_T}{HbEI} M = \frac{-q^*(x)}{2} \qquad (6.4b)$$

Put $\alpha^2 = \frac{6NEI_T}{HbEI}$

let $q^*(x) = \frac{q_o}{H} x$

Equation (6.4b) then becomes

$$\frac{d^2M}{dx^2} - \alpha^2 M = -\frac{q_o}{2H} x \qquad (6.4c)$$

The solution of Equation (6.4c) may be readily obtained as

$$M = A^* \sinh\alpha x + B^* \cosh\alpha x + \frac{q_o}{2H\alpha^2} x \qquad (6.5)$$

and since $M = -EI\frac{d^2y}{dx^2}$, one obtains

$$\frac{d^2y}{dx^2} = C^* \sinh\alpha x + D^* \cosh\alpha x - \frac{q_o}{2H\alpha^2 EI} x \qquad (6.6)$$

whence

$$\frac{dy}{dx} = \frac{C^*}{\alpha}\cosh\alpha x + \frac{D^*}{\alpha}\sinh\alpha x - \frac{q_o}{2H\alpha^2 EI}\frac{x^2}{2} + G^*$$

$$y = \frac{C^*}{\alpha^2}\sinh\alpha x + \frac{D^*}{\alpha^2}\cosh\alpha x - \frac{q_o}{2H\alpha^2 EI}\frac{x^3}{6} + G^*x + S^*$$

Conditions at the bottom end require $y = \frac{dy}{dx} = o$ at $x = o$, from which

$$S^* = -\frac{D^*}{\alpha^2}$$

$$G^* = -\frac{C^*}{\alpha}$$

and so $$y = \frac{C^*}{\alpha^2}(\sinh\alpha x - \alpha x) + \frac{D^*}{\alpha^2}(\cosh\alpha x - 1) - \frac{q_o x^3}{12H\alpha^2 EI} \qquad (6.7)$$

Conditions at the top end require $M = o$ i.e. $\frac{d^2y}{dx^2} = o$ (6.8a)

and also $V = o$, i.e. $-EI\frac{d^3y}{dx^3} + \frac{6NEI_T}{Hb}\frac{dy}{dx} = o$

or $$\frac{d^3y}{dx^3} - \alpha^2\frac{dy}{dx} = o \qquad (6.8b)$$

For brevity one puts $\alpha H = \beta$ and then Equation (6.8a) yields

$$\frac{q_o}{2\alpha^2 EI} = C^*\sinh\beta + D^*\cosh\beta \qquad (6.9)$$

whilst Equation (6.8b) gives $\alpha C^* - \frac{q_o}{2H\alpha^2 EI} + \frac{q_o\beta^2}{4H\alpha^2 EI} = o$

Hence $$C^* = \frac{q_o}{2H\alpha^3 EI}\left(1 - \frac{\beta^2}{2}\right) \qquad (6.10)$$

and $$D^* = \frac{q_o\{\beta - (1 - \beta^2/2)\sinh\beta\}}{2H\alpha^3 EI\cosh\beta} \qquad (6.11)$$

Substituting Equations (6.10) and (6.11) in Equation (6.7) gives

$$y = \frac{q_o}{2H\alpha^5 EI}\left\{\left(1-\frac{\beta^2}{2}\right)\left(\sinh\frac{\beta x}{H} - \frac{\beta x}{H}\right) + \frac{\left[\beta - \left(1-\frac{\beta^2}{2}\right)\sinh\beta\right]}{\cosh\beta}\left(\cosh\frac{\beta x}{H} - 1\right) - \frac{\left(\beta\frac{x}{H}\right)^3}{6}\right\}$$

(6.12)

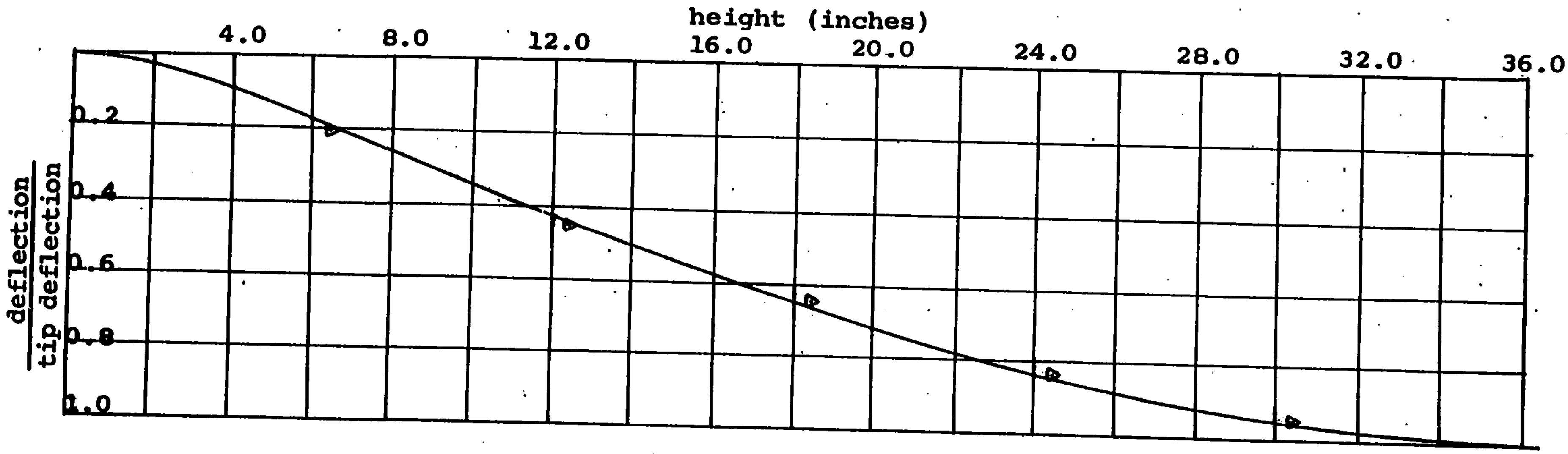


Figure 6.4: Comparison of deflection by continuum and finite element methods.

Equation (6.12) gives the deflected shape of the frame and is a function of β, a measure of the relative stiffness of the beams w.r.t. the columns. The case $\beta = 0$ represents a free cantilever column.

The modal shape for the frame shown in Figure (5.1) was determined using Equation (6.12) and is drawn in Figure 6.4 by the solid line. The modal shape was also obtained using the finite element method and is indicated by triangular markers in Figure 6.4, from which it may be concluded that the continuum approach gives a good prediction.

We are now in a position to study the dynamic instability of a portal frame. Let the vertical support motion be $u_s(t)$. Consider an element of the column shown in Figure 6.5.

Vertical mass-acceleration gives $dP = -\rho_o \ddot{u}_s \, dx$

$$P = -\int_x^H \rho_o \ddot{u}_s \, dx$$

$$= \rho_o (H-x) \ddot{u}_s \qquad (6.13)$$

where ρ_o = mass per unit height for half of the structure.

i.e. $\rho_o = \frac{1}{2} \rho^*$ where ρ^* is the mass per unit height for the whole structure.

Horizontal resolution gives $$\frac{dV}{dx} = -\frac{q^*}{2} \qquad (6.14)$$

Moment equilibrium gives $P\,dy + V dx = dM + m dx$

or

$$V = \frac{dM}{dx} - P(x)\frac{dy}{dx} + m \qquad (6.15)$$

Differentiating Equation (6.15) w.r.t. x we get

$$\frac{dV}{dx} = \frac{d^2M}{dx^2} - P\frac{d^2y}{dx^2} - \frac{dP}{dx}\frac{dy}{dx} + \frac{dm}{dx} \tag{6.16}$$

Substituting for V and m and simplifying gives

$$EI\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} + \frac{dP}{dx}\frac{dy}{dx} - \frac{6NEI_T}{Hb}\frac{d^2y}{dx^2} = \frac{q^*(x)}{2}$$

Since $\alpha^2 = \frac{6NEI_T}{HbEI}$, we have

$$\frac{d^4y}{dx^4} + \frac{\rho_o}{EI}(H-x)\,\ddot{u}_s\frac{d^2y}{dx^2} - \frac{\rho_o}{EI}\ddot{u}_s\frac{dy}{dx} - \alpha^2\frac{d^2y}{dx^2} = \frac{q^*(x)}{2EI}$$

Now $q^*(x) = -\rho^*\frac{\partial^2y}{\partial t^2}$ (6.16a)

Substituting in the above equation gives

$$\frac{\partial^4y}{\partial x^4} + \frac{\rho^*}{2EI}(H-x)\,\ddot{u}_s\frac{\partial^2y}{\partial x^2} - \frac{\rho^*}{2EI}\ddot{u}_s\frac{\partial y}{\partial x} - \alpha^2\frac{\partial^2y}{\partial x^2} + \frac{\rho^*}{2EI}\frac{\partial^2y}{\partial t^2} = 0 \tag{6.17}$$

From Equation (6.12) we get

$$\frac{\partial y}{\partial x} = \frac{q_o H^4}{2EI\beta^5}\left\{\frac{(1-\beta^2)}{2}\left(\frac{\beta}{H}\cosh\frac{\beta x}{H} - \frac{\beta}{H}\right) + \frac{[\beta-(1-\beta^2/2)\sinh\beta]}{\cosh\beta}\left(\frac{\beta}{H}\sinh\frac{\beta x}{H}\right) - \frac{\left(\frac{\beta}{H}\right)^3 x^2}{2}\right\} \tag{6.12a}$$

$$\frac{\partial^2y}{\partial x^2} = \frac{q_o H^4}{2EI\beta^5}\left\{\frac{(1-\beta^2)}{2}\left(\frac{\beta}{H}\right)^2\sinh\frac{\beta x}{H} + \frac{[\beta-(1-\beta^2/2)\sinh\beta]}{\cosh\beta}\left(\frac{\beta^2}{H^2}\cosh\frac{\beta x}{H}\right) - \left(\frac{\beta}{H}\right)^3 x\right\} \tag{6.12b}$$

$$\frac{\partial^4y}{\partial x^4} = \frac{q_o H^4}{2EI\beta^5}\left\{\frac{(1-\beta^2)}{2}\left(\frac{\beta}{H}\right)^4\sinh\frac{\beta x}{H} + \frac{[\beta-(1-\beta^2/2)\sinh\beta]}{\cosh\beta}\left(\frac{\beta}{H}\right)^4\cosh\frac{\beta x}{H}\right\} \tag{6.12c}$$

$$\frac{\partial^4y}{\partial x^4} - \alpha^2\frac{\partial^2y}{\partial x^2} = \frac{q_o H^4}{2EI\beta^5}\left(\frac{\beta}{H}\right)^5 x$$

$$= R^*\alpha^5 x \tag{6.18}$$

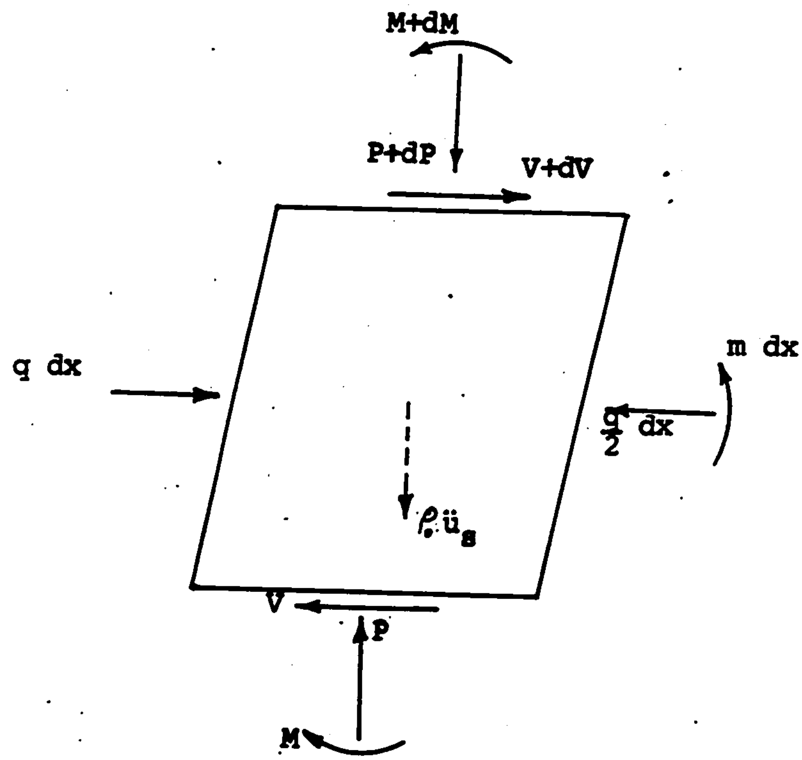


Figure 6.5: Forces on an element of the column.

where $$R^* = \frac{q_o H^4}{2EI\beta^5} \tag{6.19}$$

Substituting in Equation (6.17) yields

$$\frac{\rho^*}{2EI} (H-x)\, \ddot{u}_s \frac{\partial^2 y}{\partial x^2} - \frac{\rho^*}{2EI} \ddot{u}_s \frac{\partial y}{\partial x} + R^* \alpha^5 x + \frac{\rho^*}{2EI} \frac{\partial^2 y}{\partial t^2} = o \tag{6.20}$$

Let $y(x,t) = \sum_{i=1}^{\infty} R^* \phi_i (x) \; f_i(t)$

where $\phi_i(x)$ are a complete set of functions satisfying the boundary conditions, for instance the various modes of free vibration of the problem, and $f_i(t)$ are generalized coordinates. An approximate solution for y may be obtained by curtailing the series at some integer $i = n$. This approximation will not now satisfy Equation (6.20), but the error introduced by the substitution of the curtailed series may be minimized by applying the Galerkin Method.

$$\int R^* \left[\frac{\rho^*}{2EI} (H-x) \; \ddot{u}_s \sum_{i=1}^{n} f_i \phi_i'' - \frac{\rho^*}{2EI} \ddot{u}_s \sum_{i=1}^{n} f_i \phi_i' + \alpha^5 x f_i + \frac{\rho^*}{2EI} \cdot \sum_{i=1}^{n} \ddot{f}_i \phi_i \right] \phi_j \, dx = o \text{ for } j = 1,2,\ldots n \tag{6.22}$$

Consider a one term series approximation for y.

i.e. $y = R^* \phi_1(x) \; f_1(t)$

Equation (6.22) then writes as

$$\int \left[\left\{ \frac{\rho^*}{2EI} (H-x) \ddot{u}_s \frac{d^2\phi_1}{dx^2} - \frac{\rho^*}{2EI} \ddot{u}_s \frac{d\phi_1}{dx} + \alpha^5 x \right\} f_1(t) + \frac{\rho^*}{2EI} \frac{d^2 f_1}{dt^2} \phi_1 \right] \phi_1 \, dx = o$$

$$\text{or} \int \left[\left\{ \frac{\rho^*}{2EI} (H-x)\ddot{u}_s \, \phi_1 \frac{d^2 \phi_1}{dx^2} - \frac{\rho^*}{2EI} \ddot{u}_s \, \phi_1 \frac{d\phi_1}{dx} + \alpha^5 \, x\phi_1 \right\} f_1(t) + \frac{\rho^*}{2EI} \frac{d^2 f_1}{dt^2} \phi_1^2 \right] dx = o \tag{6.23}$$

The integrals $\int \phi_1 \frac{d^2\phi_1}{dx^2} dx$, $\int x\phi_1 \frac{d^2\phi_1}{dx^2} dx$, $\int \phi_1 \frac{d\phi_1}{dx} dx$,

$\int x \, \phi_1 dx$, $\int \phi_1^2 \, dx$

may be evaluated numerically using Equations (6.12).

Equation (6.23) is a Mathieu equation and its solution would yield the region of dynamic instability.

The frame shown in Figure 5.1 was analysed by this procedure. Using Simpson's rule the above integrals were determined and the following values were obtained.

$$\int \phi_1'' \, \phi_1 \, dx = -\, 0.06772 \times 10^5$$

$$\int \phi_1^2 \, dx = 43.7656 \times 10^6$$

$$\int x \, \phi_1 \, dx = 1.4131 \times 10^6$$

$$\int \phi_1' \, \phi_1 \, dx = 0.8369 \times 10^6$$

$$\int x \, \phi_1'' \, \phi_1 \, dx = -\, 1.9173 \times 10^6$$

The frame was weighed and its mass was found to be 0.465 lbs. Substituting the value of the integrals in Equation (6.23) yields

$$21.8826 \frac{d^2 f}{dt^2} + (17600 - 0.9805 \, \ddot{u}_s) \, f(t) = o$$

Letting $u_s = C \cos \Omega t$, results in

$$21.8826\,\frac{d^2f}{dt^2} + (17600 + 0.9805\,\Omega^2 c \cos\Omega t)\,f = o$$

The solution of this Mathieu equation gives

$$\Omega^2 = \frac{3230}{1 \pm 0.09c} \qquad (6.24)$$

Equation (6.24) gives the value of the disturbing frequency required to cause the frame to go into instability in the first mode. $c = o$, refers to the hypothetical case when instability will occur even if the amplitude of the base motion is zero. From Equation (6.24) Ω = 57.0 rad./sec. when $c = o$. However, from the experimental tests, as reported in Chapter V, the value of the disturbing frequency, Ω, corresponding to the least amplitude of the base motion was 47.8 rad./sec. Eq. (6.24) does not give a good prediction of the width of the instability region. This is possibly due to the fact that the above analysis considered only a one term series approximation for y. If more terms are included a better estimate of the width of the instability region would be obtained.

6.3 CONCLUSIONS

The continuum approach has been used to determine the natural frequencies of structures e.g. grids (29,46). It was felt that this method could be used to obtain the region of dynamic instability of portal frames. As shown above, the method gives a good estimate of the deflected shape and the natural frequency of the first mode of the structure. However, the prediction of the width of the instability zone is not satisfactory. This may be attributed to several factors. In the above analysis a one-term series approximation for the deflected shape was considered. The inclusion of more terms would yield better results. The accuracy of the results depends upon the values of the integrals in Equation 6.23. Since the integrals were evaluated numerically, and even though the deflected shape was shown to be in good agreement with that given by the finite element solution, the relationship for the slope and curvature may not be good. This may account for the poor estimate of the region of instability.

Chapter VII

CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

This study deals with the parametric instability of plane structures. The dynamic stability of structures requires a knowledge of their natural frequencies. The finite element method was used for this purpose and has been shown to yield good results. One of the major advantages of the finite element method is that the governing differential equations need not be known and different boundary conditions may be easily handled. The method is applicable to structures of non-uniform cross-section and varying properties. Computer programs to obtain the natural frequencies and modal shapes of structures are also given.

A simple numerical method for determining the regions of dynamic instability of columns subjected to periodic axial forces or periodic support motion is presented. The validity of the results has been checked against experimental work and available classical solutions. A non-uniform column or one with varying properties can be easily handled.

Finally, the dynamic instability of portal frames due to periodic support motion was investigated. An extensive experimental program was carried out and the results have been reported in Chapter V. The continuum method gives a good estimate of the deflected shape and the natural frequency of the frame for the first mode, but the prediction of the width of the instability region was found to be unsatisfactory. By taking more terms in the series for the equation of the deflected shape a better relationship for the instability region could be obtained.

As the ground motion in earthquakes consists of horizontal and vertical movements the study of the response of structures subjected to continued horizontal and vertical support motions needs to be investigated and is proposed for future research. The present investigation was confined to the response of structures to periodic support motion. Since seismic vibrations are random in nature the study of the response of structures to random fluctuating base motion needs to be looked into.

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APPENDIX A

SOME PROPERTIES OF THE MATHIEU EQUATION

Consider the Mathieu equation

$$\frac{d^2y}{dz^2} + (\delta + \epsilon \cos 2z)\, y = o \qquad (A.1)$$

which is periodic in z.

The general solution of the Mathieu equation containing two arbitrary constants has not yet been found. However, periodic coefficients lead to periodic solutions and the above linear equation may be solved by Floquet's theory for the solution of linear differential equations with periodic coefficients. Since Equation (A.1) is a linear second-order differential equation, two independent non-zero solutions may be found as $y_1(z)$ and $y_2(z)$. These solutions may be combined linearly to give any other solution, say $y(z)$ so that

$$y(z) = C_1 y_1(z) + C_2 (y_2) z \qquad (A.2)$$

where C_1 and C_2 are constants.

For $y_1(z)$ and $y_2(z)$ to be independent, they must satisfy the determinant

$$W\,(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

The replacement of $y(z)$ by $y(-z)$ does not alter Equation (A.1). Therefore, if $y(z)$ is a solution of Equation (A.1) then $y(-z)$ is also a solution. If $y(z)$ has no symmetry about the point $z = o$, it can be split into components having even and odd symmetry by the usual process.

$$\text{Even:} \quad \frac{1}{2}\left[y(z) + y(-z)\right] = C_1 y_1(z)$$

$$\text{Odd :} \quad \frac{1}{2}\left[y(z) - y(-z)\right] = C_2 y_2(z)$$

Since $y(z)$ is periodic, $y(z + T) = y(z)$, and since $y_1(z)$ and $y_2(z)$ are solutions of Equation (A.1), then $y_1(z + T)$ and $y_2(z + T)$ also are solutions. It does not mean that $y_1(z + T)$ is the same as $y_1(z)$. Rather it means that $y_1(z + T)$ and $y_2(z + T)$ may be represented in the form of a linear combination of the primary functions.

$$\left.\begin{aligned} y_1(z + T) &= a_{11}y_1(z) + a_{12}y_2(z) \\ y_2(z + T) &= a_{21}y_1(z) + a_{22}y_2(z) \end{aligned}\right\} \qquad \text{(A.3)}$$

where a_{ik} are the constants called coefficients of transformations. Now $y(z + T) = C_1y_1(z + T) + C_2y_2(z + T)$ substituting from Equations (A.3) we get

$$y(z + T) = (C_1a_{11} + C_2\,a_{21})y_1(z) + (C_1a_{12} + C_2a_{22})y_2(z)$$

If $y(z)$ is suitably chosen, we can have

$$\left.\begin{aligned} C_1a_{11} + C_2\,a_{21} &= \rho\,C_1 \\ C_1a_{12} + C_2\,a_{22} &= \rho\,C_2 \end{aligned}\right\} \qquad \text{(A.4)}$$

and hence we get, $y(z + T) = \rho\, y(z)$ (A.5)

A solution that possesses the above property is called a normal solution e.g. if y denotes displacement, it means that the displacement at the end of the period is ρ times that at the beginning of the period.

If constants C_1 and C_2 are not to be zero in Equation (A.4) the determinant of their coefficients must be zero, i.e.

$$\begin{vmatrix} a_{11}-\rho & a_{21} \\ a_{12} & a_{22}-\rho \end{vmatrix} = 0 \qquad \text{(A6)}$$

This is called the characteristic equation.

We can select the two linearly independent solutions $y_1(z)$ and $y_2(z)$ such that they satisfy the initial conditions at $z = 0$.

$$\left.\begin{array}{ll} y_1(0) = 1 & y_1'(0) = 0 \\ y_2(0) = 0 & y_2'(0) = 1 \end{array}\right\} \qquad \text{(A.7)}$$

From these conditions we can determine the coefficients a_{ik} and the characteristic equation becomes

$$\rho^2 - 2A\rho + B = 0 \qquad \text{(A.8)}$$

where $A = \frac{1}{2}\left[y_1(T) + y_2'(T)\right]$

$$B = y_1(T)y_2'(T) - y_2(T)\, y_1'(T)$$

It can be readily shown that B is always equal to 1. Hence, Equation (A.8) becomes

$$\rho^2 - 2A\rho + 1 = 0$$

$$\text{or} \quad \rho = A \pm \sqrt{A^2-1} \tag{A.9}$$

and the roots are related by $\rho_1 \cdot \rho_2 = 1$ (A.10)

From Equation (A.5) we have

$$y_k\ (z + T) = \rho_k\ y_k(z) \qquad (k = 1,2)$$

These solutions may also be expressed as

$$y_k(z) = \chi_k(z)\ e^{z/T \ln \rho_k} \qquad (k = 1,2) \tag{A.11}$$

where $\chi_k(z)$ are periodic functions of period T.

From Equation (A.11) it may be observed that the behaviour of the solution as $t \to \infty$ depends upon the values of ρ_1, ρ_2.

Since $\log \rho = \log |\rho| + i \arg \rho$

Equation (A.11) may be written as

$$y_k(z) = \phi_k(z)\ e^{z/T \ln |\rho_k|} \quad (k = 1,2) \tag{A.12}$$

$$\text{where} \qquad \phi_k = \chi_k(t)\ e^{iz/T \arg \rho}$$

Equation (A.12) shows that if $\rho_k > 1$, the corresponding solution will have an unbounded exponential multiplier. If $\rho_k < 1$ the solution is damped as z increases; and if $\rho = 1$ then the solution is periodic (or almost periodic), i.e. it will be bounded in time. It may be readily shown that if $A > 1$, the roots of the characteristic equation will be real and one of them will be greater than unity and hence unbounded solutions will result. However, if $A < 1$, the characteristic equation has conjugate complex roots, and since their product must be equal to unity, their modulus will be equal to unity. The case of complex roots corresponds to the region of bounded solutions. On the boundaries separating the regions of bounded solutions from those of unbounded solutions we must have $A = 1$.

$$\text{i.e.} \quad y_1(T) + y_2'(T) = 2 \tag{A.13}$$

The regions of bounded solutions correspond to the regions of dynamic stability in the physical sense, and unbounded solutions to the regions of dynamic instability. Therefore, Equation (A.13) may be used to determine the regions of dynamic instability.

Since $\rho_1 \cdot \rho_2 = 1$, at the boundaries separating the regions of stability from that of instability we must have either $\rho_1 = \rho_2 = 1$ or $\rho_1 = \rho_2 = -1$. The former case corresponds to periodic solution of period T, while for the latter case the period is 2T.

Therefore, the regions of unbounded solutions are separated from those of bounded solutions by the periodic solutions with periods T and 2T. In fact, the regions of stability are bounded by solutions of different periods, while those of instability are bounded by solutions of identical period. Therefore, the determination of the boundaries of the regions of instability is reduced to finding the condition under which the given differential equation has periodic solutions with periods T and 2T.

APPENDIX B

1. GALERKIN'S METHOD

Let us represent the motion of a body by considering the motion of some discrete points. The equations of motion write as

$$\begin{matrix} \sum F_1 = \\ \sum F_2 = \\ \vdots \\ \sum F_n = \end{matrix} \begin{bmatrix} m_{11} & m_{12} \cdots m_{1n} \\ m_{21} & m_{22} \cdots m_{2n} \\ \vdots & \vdots \\ m_{n1} & m_{n2} \cdots m_{nn} \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \vdots \\ \ddot{\theta}_n \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \cdots k_{1n} \\ k_{21} & k_{22} \cdots k_{2n} \\ \vdots & \vdots \\ k_{n1} & k_{n2} \quad k_{nn} \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{Bmatrix} + \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix}_{ex} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \quad \text{(B.1)}$$

where $\{\theta\}$ is the displacement vector

$\{F\}_{ex}$ is the external force vector

Let us suppose that

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \theta_n \end{Bmatrix} = \begin{Bmatrix} \gamma_1 \\ \gamma_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \gamma_n \end{Bmatrix} \theta \quad \text{(B.2)}$$

where $\{\gamma\}$ is the modal shape or eigenvector. If the eigenvector,

$\{\gamma\}$, is only approximately known, then in general the forces will not be in equilibrium. Equation (B.1) then becomes

$$\begin{matrix} \sum F_1 = \\ \sum F_2 = \\ \cdot \\ \cdot \\ \cdot \\ \sum F_n = \end{matrix} \begin{bmatrix} m_{11} & m_{12} \cdots\cdots m_{1n} \\ m_{21} & m_{22} \cdots\cdots m_{2n} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ m_{n1} & m_{n2} \cdots\cdots m_{nn} \end{bmatrix} \begin{Bmatrix} \gamma_1 \\ \gamma_2 \\ \cdot \\ \cdot \\ \cdot \\ \gamma_n \end{Bmatrix} \ddot{\theta} + \begin{bmatrix} k_{11} & k_{12} \cdots\cdots k_{1n} \\ k_{21} & k_{22} \cdots\cdots k_{2n} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ k_{n1} & k_{n2} \qquad k_{nn} \end{bmatrix} \begin{Bmatrix} \gamma_1 \\ \gamma_2 \\ \cdot \\ \cdot \\ \cdot \\ \gamma_n \end{Bmatrix} \theta + \begin{Bmatrix} F_1 \\ F_2 \\ \cdot \\ \cdot \\ \cdot \\ F_n \end{Bmatrix} = \begin{Bmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ \cdot \\ e_n \end{Bmatrix} \qquad \text{(B.3)}$$

The i-th row of this equation represents the sum of the generalized forces $\sum F_i$ associated with the coordinate θ_i. The quantity e_i on the right-hand side of Equation (B.3) represents the error in equilibrium of the forces associated with θ_i.

Galerkin's method requires that the weighted sum of the errors $\{e\}$ be zero; each of the terms in $\{e\}$ is weighted by the corresponding term in the modal shape vector $\{\gamma\}$. Thus it requires that

$$\begin{bmatrix} \gamma_1 & \gamma_2 & \cdots\cdots & \gamma_n \end{bmatrix} \begin{Bmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ \cdot \\ n \end{Bmatrix} = o \qquad \text{(B.4)}$$

If we take $\{\underset{\sim}{Y}\}$ to be a displacement vector and $\{\underset{\sim}{e}\}$ as a force vector, we can say that Galerkin's Method requires the error in the forces to be orthogonal to the assumed shape of the displacements. Substitution of Equation (B.3) into Equation (B.4) gives

$$\lfloor \underset{\sim}{Y} \rfloor [\underset{\sim}{M}] \{\underset{\sim}{Y}\} \ddot{\theta} + \lfloor \underset{\sim}{Y} \rfloor [\underset{\sim}{S}] \{\underset{\sim}{Y}\} \theta + \lfloor \underset{\sim}{Y} \rfloor \{\underset{\sim}{F}\}_{ex} = 0 \tag{B.5}$$

APPENDIX C

PROGRAM TO FIND THE NATURAL FREQUENCIES OF PLANE FRAMES AND GRIDS

The program is coded in Fortran IV and can be run either on the RAX or O/S systems of IBM 360/65 or IBM 360/75. Nodal points and elements should be numbered in a consecutive order. The size of the problem to be solved can be adjusted by suitably changing the dimension statement.

The input to the program consists of the nodal co-ordinates, member properties and the boundary conditions. The program generates the stiffness and the mass matrices from which the natural frequencies and the modal amplitudes are computed. The computer print-out includes

a) Reprint of input data

b) Natural frequencies

c) Normalized modal amplitudes

1. Input data for plane frames.

Cols.	Notation	Description
A Structure Parameters and Material Properties (4I5, F10.0, F10.8) one card		
1-5	M	Number of elements or members
6-10	NJ	Number of nodes or joints
11-15	NR	Number of restraints
16-20	NRJ	Number of restrained joints
21-30	E	Modulus of Elasticity
31-40	R	Member density
B Joint Coordinates (I5, 2F10.2) NJ cards		
1-5	J	Counter
6-15	X	X coordinate
16-25	Y	Y coordinate
C Member Designations and Properties (3I5, F10.5, F10.8) M cards		
1-5	J	Counter
6-10	JJ	Designation of J end of member
11-15	JK	Designation of K end of member
16-25	AX	Area of member
26-35	IZ	Moment of inertia about Z-axis

2. Input data for grids

Cols.	Notation	Description
A		Structure Parameters and Material Properties (4I5, 2F10.0, F10.8) one card
1-5	M	Number of members
6-10	NJ	Number of joints
11-15	NR	Number of restraints
16-20	NRJ	Number of restrained joints
21-30	E	Modulus of Elasticity
31-40	G	Shear Modulus
41-50	RA	Member density
B		Joint Coordinates (I5, 2F10.2) NJ cards
1-5	J	Counter
6-15	X	X coordinate
16-25	Y	Y coordinate
C		Member Designations and Properties (3I5, 3F10.6) M cards
1-5	J	Counter
6-10	JJ	Designation of J end of member
11-15	JK	Designation of K end of member
16-25	IX	Torsion constant
26-35	IY	Moment of inertia about Y axis
36-45	AX	Area of member

PROGRAM TO FIND THE NATURAL FREQUENCIES OF PLANE FRAMES.

```
      DIMENSION L(20),X(20),Y(20),JJ(20),JK(20),AX(20),IZ(20),CX(20),CY(
     120),RL(50),CRL(50),SMD(6,6),S(50,50),MM(50,50),MD(6,6),
     1XL(15),XX(15,15),MSM(15,15),SM(15,15)
      REAL IZ,L,MD,MM,MSM
      INTEGFR CRL,RL
      WRITE(6,1)
C
C**********************************************************************
    1 FORMAT('1','NATURAL FREQUENCIES OF FRAME INCLUDING ROT. INERTIA')
C**********************************************************************
C
C                1. INPUT   AND   STRUCTURE   DATA
C
C                1.A STRUCTURE PARAMETERS,ELASTIC MODULUS AND DENSITY
C
      WRITE (6,2)
    2 FORMAT(//'0','STRUCTURE  DATA')
      WRITE (6,3)
    3 FORMAT ('0','    M     N    NJ    NR    NRJ          E            R')
      READ (5,4) M,NJ,NR,NRJ,E,R
    4 FORMAT(4I5,F10.0,F10.8)
      N=3*NJ-NR
      WRITE (6,5) M,N,NJ,NR,NRJ,E,R
    5 FORMAT(5I5,F15.0,F12.8)
C
C                1B.   JOINT   COORDINATES
C
      WRITE (6,6)
    6 FORMAT ('0','JOINT COORDINATES')
      WRITE (6,7)
    7 FORMAT ('0','JOINT     X COORD    Y COORD')
      DO 101 J=1,NJ
      READ (5,8) J,X(J),Y(J)
    8 FORMAT (I5,2F10.2)
      WRITE (6,9) J,X(J),Y(J)
    9 FORMAT (I5,2F10.2)
  101 CONTINUE
C
C                1C.  MEMBER DESIGNATIONS AND PROPERTIES
C
      WRITE (6,10)
   10 FORMAT ('0',' MEMBER  DESIGNATIONS  AND  PROPERTIES')
      WRITE (6,11)
   11 FORMAT (1H0,6HMEMBER,2X,2HJJ,3X,2HJK,5X,2HAX,9X,2HIZ,12X,1HL,9X,2H
     1CX,8X,2HCY)
      DO 102  I=1,M
      READ (5,12) J,JJ(I),JK(I),AX(I),IZ(I)
   12 FORMAT(3I5,F10.5,F10.8)
      JI=JJ(I)
      KI=JK(I)
      XCL=X(KI)-X(JI)
      YCL=Y(KI)-Y(JI)
      L(I)=SQRT(XCL**2&YCL**2)
      CX(I)=XCL/L(I)
      CY(I)=YCL/L(I)
      WRITE (6,13) I,JJ(I),JK(I),AX(I),IZ(I),L(I),CX(I),CY(I)
```

```
 13 FORMAT(3I5,F10.5,F14.8,3F10.2)
102 CONTINUE
C
C              10.  JOINT RESTRAINT  LIST, CUMULATIVE  RESTRAINT  LIST
C
C          INITIALISE  RL(I)  TO  ZERO
C
    NJ3=3*NJ
    DO 104 J3=1,NJ3
    RL(J3)=0
104 CONTINUE
    WRITE (6,14)
 14 FORMAT('0','JOINT RESTRAINTS')
    WRITE (6,15)
 15 FORMAT ('0','JOINT     X RSTRT  Y RSTRT  Z RSTRT')
    DO 103 J=1,NRJ
    READ (5,16) K,RL(3*K-2),RL(3*K-1),RL(3*K)
 16 FORMAT (4I10)
    WRITE(6,17) K,RL(3*K-2),RL(3*K-1),RL(3*K)
 17 FORMAT (I5,3I10)
103 CONTINUE
    CRL(1)=RL(1)
    DO 105 K=2,NJ3
    CRL(K)=CRL(K-1)&RL(K)
105 CONTINUE
C
C              2.A GENERATION OF OVERALL STIFNESS AND MASS MATRICES.
C
    NJ2=3*NJ
    DO 106 J3=1,NJ2
    DO 106 K3=1,NJ2
    S(J3,K3)=0.0
    MM(J3,K3)=0.0
106 CONTINUE
    DO 51 I=1,M
    J1=3*JJ(I)-2
    J2=3*JJ(I)-1
    J3=3*JJ(I)
    K1=3*JK(I)-2
    K2=3*JK(I)-1
    K3=3*JK(I)
    SCM1=(E*AX(I))/L(I)
    SCM2=(4.0*E*IZ(I))/L(I)
    SCM3=(1.5*SCM2)/L(I)
    SCM4=(2.0*SCM3)/L(I)
    Z=R*AX(I)*L(I)/420.0
    WRITE (6,73)
 73 FORMAT ('0','SCM')
    WRITE (6,72) SCM1,SCM2,SCM3,SCM4
 72 FORMAT (1H0,4F10.2)
    IF (RL(J1)) 18,19,18
 19 J1=J1-CRL(J1)
    GO TO 20
 18 J1=N&CRL(J1)
 20 IF (RL(J2)) 21,22,21
 22 J2=J2-CRL(J2)
    GO TO 23
 21 J2=N&CRL(J2)
 23 IF (RL(J3)) 24,25,24
 25 J3=J3-CRL(J3)
```

```
      GO TO 26
   24 J3=N&CRL(J3)
   26 IF (RL(K1))  27,28,27
   28 K1=K1-CRL(K1)
      GO TO 29
   27 K1=N&CRL(K1)
   29 IF (RL(K2)) 30,31,30
   31 K2=K2-CRL(K2)
      GO TO 32
   30 K2=N&CRL(K2)
   32 IF (RL(K3)) 33,34,33
   34 K3=K3-CRL(K3)
      GO TO 35
   33 K3=N&CRL(K3)
   35 WRITE(6,200) J1,J2,J3,K1,K2,K3,Z
  200 FORMAT(1H0,6I5,E15.8)
      CY2=CY(I)*CY(I)
      CX2=CX(I)*CX(I)
      CXY=CX(I)*CY(I)
      SMD(1,1)=SCM1*CX2&SCM4*CY2
      SMD(4,4)=SMD(1,1)
      SMD(1,4)=-SMD(1,1)
      SMD(4,1)=-SMD(1,1)
      SMD(1,2)=(SCM1-SCM4)*CXY
      SMD(2,1)=SMD(1,2)
      SMD(4,5)=SMD(1,2)
      SMD(5,4)=SMD(1,2)
      SMD(1,5)=-SMD(1,2)
      SMD(5,1)=-SMD(1,2)
      SMD(2,4)=-SMD(1,2)
      SMD(4,2)=-SMD(1,2)
      SMD(1,3)=-SCM3*CY(I)
      SMD(3,1)=SMD(1,3)
      SMD(1,6)=SMD(1,3)
      SMD(6,1)=SMD(1,3)
      SMD(3,4)=-SMD(1,3)
      SMD(4,3)=-SMD(1,3)
      SMD(4,6)=-SMD(1,3)
      SMD(6,4)=-SMD(1,3)
      SMD(2,2)=SCM1*CY2&SCM4*CX2
      SMD(5,5)=SMD(2,2)
      SMD(2,5)=-SMD(2,2)
      SMD(5,2)=-SMD(2,2)
      SMD(2,3)=SCM3*CX(I)
      SMD(3,2)=SMD(2,3)
      SMD(2,6)=SMD(2,3)
      SMD(6,2)=SMD(2,3)
      SMD(3,5)=-SMD(2,3)
      SMD(5,3)=-SMD(2,3)
      SMD(5,6)=-SMD(2,3)
      SMD(6,5)=-SMD(2,3)
      SMD(3,3)=SCM2
      SMD(6,6)=SMD(3,3)
      SMD(3,6)=SCM2/2.0
      SMD(6,3)=SMD(3,6)
C
      CX2Z=CX2*Z
      CY2Z=CY2*Z
      CXYZ=CXY*Z
      CXLZ=CX(I)*L(I)*Z
```

```
S.0151         CYLZ=CY(I)*L(I)*Z
S.0152         ZL2=L(I)*L(I)*Z
         C
S.0153         WRITE(6,800)
S.0154     800 FORMAT('0','CXY')
S.0155         WRITE(6,801) CX2,CY2,CXY,CX2Z,CY2Z,CXYZ,CXLZ,CYLZ,ZL2
S.0156     801 FORMAT (1H0,9E12.6)
S.0157         Z1=R/420.0
S.0158         ZL=IL*Z1
S.0159         CYIZ=CY(I)*IZ(I)*Z1
S.0160         CXIZ=CX(I)*IZ(I)*Z1
S.0161         ZIL=Z1*IZ(I)*L(I)
S.0162         CX2ZL=CX2*ZL
S.0163         CY2ZL=CY2*ZL
S.0164         CXYZL=CXY*ZL
S.0165         MD(1,1)=140.0*CX2Z&156.0*CY2Z&504.0*CY2ZL
S.0166         MD(4,4)=MD(1,1)
S.0167         MD(1,2)=-16.0*CXYZ-504.0*CXYZL
S.0168         MD(2,1)=MD(1,2)
S.0169         MD(1,3)=-22.0*CYLZ-42.0*CYIZ
S.0170         MD(3,1)=MD(1,3)
S.0171         MD(4,6)=-MD(1,3)
S.0172         MD(6,4)=-MD(1,3)
S.0173         MD(1,4)=70.0*CX2Z&54.0*CY2Z-504.0*CY2ZL
S.0174         MD(4,1)=MD(1,4)
S.0175         MD(1,5)=-MD(1,2)
S.0176         MD(5,1)=-MD(1,2)
S.0177         MD(2,4)=-MD(1,2)
S.0178         MD(4,2)=-MD(1,2)
S.0179         MD(1,6)=13.0*CYLZ-42.0*CYIZ
S.0180         MD(6,1)=MD(1,6)
S.0181         MD(2,2)=140.0*CY2Z&156.0*CX2Z&504.0*CX2ZL
S.0182         MD(5,5)=MD(2,2)
S.0183         MD(2,3)=22.0*CXLZ&42.0*CXIZ
S.0184         MD(3,2)=MD(2,3)
S.0185         MD(2,5)=70.0*CY2Z&54.0*CX2Z-504.0*CX2ZL
S.0186         MD(2,6)=-13.0*CXLZ&42.0*CXIZ
S.0187         MD(6,2)=MD(2,6)
S.0188         MD(3,3)=4.0*ZL2&56.0*ZIL
S.0189         MD(6,6)=MD(3,3)
S.0190         MD(3,4)= -MD(1,6)
S.0191         MD(4,3)= -MD(1,6)
S.0192         MD(3,5)=-MD(2,6)
S.0193         MD(5,3)=-MD(2,6)
S.0194         MD(3,6)=-3.0*ZL2-14.0*ZIL
S.0195         MD(6,3)=MD(3,6)
S.0196         MD(5,6)=-MD(2,3)
S.0197         MD(6,5)=-MD(2,3)
S.0198         MD(4,5)=MD(1,2)
S.0199         MD(5,4)=MD(1,2)
S.0200         WRITE(6,2001)
S.0201    2001 FORMAT('0','MD')
S.0202         DO 2003 IQ=1,6
S.0203    2003 WRITE(6,111) (MD(IQ,JQ),JQ=1,6)
         C
S.0204         JJ3=3*JJ(I)
         C
S.0205         IF (RL(JJ3-2)) 41,40,41
S.0206      40 S(J1,J1)=S(J1,J1)&SMD(1,1)
S.0207         S(J2,J1)=S(J2,J1)&SMD(2,1)
```

```
      S(J3,J1)=S(J3,J1)&SMD(3,1)
      S(K1,J1)=SMD(4,1)
      S(K2,J1)=SMD(5,1)
      S(K3,J1)=SMD(6,1)
      MM(J1,J1)=MM(J1,J1)&MD(1,1)
      MM(J2,J1)=MM(J2,J1)&MD(2,1)
      MM(J3,J1)=MM(J3,J1)&MD(3,1)
      MM(K1,J1)=MD(4,1)
      MM(K2,J1)=MD(5,1)
      MM(K3,J1)=MD(6,1)
   41 IF (RL(JJ3-1)) 43,42,43
   42 S(J1,J2)=S(J1,J2)&SMD(1,2)
      S(J2,J2)=S(J2,J2)&SMD(2,2)
      S(J3,J2)=S(J3,J2)&SMD(3,2)
      S(K1,J2)=SMD(4,2)
      S(K2,J2)=SMD(5,2)
      S(K3,J2)=SMD(6,2)
      MM(J1,J2)=MM(J1,J2)&MD(1,2)
      MM(J2,J2)=MM(J2,J2)&MD(2,2)
      MM(J3,J2)=MM(J3,J2)&MD(3,2)
      MM(K1,J2)=MD(4,2)
      MM(K2,J2)=MD(5,2)
      MM(K3,J2)=MD(6,2)
   43 IF (RL(JJ3)) 45,44,45
   44 S(J1,J3)=S(J1,J3)&SMD(1,3)
      S(J2,J3)=S(J2,J3)&SMD(2,3)
      S(J3,J3)=S(J3,J3)&SMD(3,3)
      S(K1,J3)=SMD(4,3)
      S(K2,J3)=SMD(5,3)
      S(K3,J3)=SMD(6,3)
      MM(J1,J3)=MM(J1,J3)&MD(1,3)
      MM(J2,J3)=MM(J2,J3)&MD(2,3)
      MM(J3,J3)=MM(J3,J3)&MD(3,3)
      MM(K1,J3)=MD(4,3)
      MM(K2,J3)=MD(5,3)
      MM(K3,J3)=MD(6,3)
C
   45 JK3=3*JK(I)
C
      IF (RL(JK3-2)) 47,46,47
   46 S(J1,K1)=SMD(1,4)
      S(J2,K1)=SMD(2,4)
      S(J3,K1)=SMD(3,4)
      S(K1,K1)=S(K1,K1)&SMD(4,4)
      S(K2,K1)=S(K2,K1)&SMD(5,4)
      S(K3,K1)=S(K3,K1)&SMD(6,4)
      MM(J1,K1)=MD(1,4)
      MM(J2,K1)=MD(2,4)
      MM(J3,K1)=MD(3,4)
      MM(K1,K1)=MM(K1,K1)&MD(4,4)
      MM(K2,K1)=MM(K2,K1)&MD(5,4)
      MM(K3,K1)=MM(K3,K1)&MD(6,4)
   47 IF (RL(JK3-1)) 49,48,49
   48 S(J1,K2)=SMD(1,5)
      S(J2,K2)=SMD(2,5)
      S(J3,K2)=SMD(3,5)
      S(K1,K2)=S(K1,K2)&SMD(4,5)
      S(K2,K2)=S(K2,K2)&SMD(5,5)
      S(K3,K2)=S(K3,K2)&SMD(6,5)
      MM(J1,K2)=MD(1,5)
```

```
      MM(J2,K2)=MD(2,5)
      MM(J3,K2)=MD(3,5)
      MM(K1,K2)=MM(K1,K2)&MD(4,5)
      MM(K2,K2)=MM(K2,K2)&MD(5,5)
      MM(K3,K2)=MM(K3,K2)&MD(6,5)
   49 IF (RL(JK3)) 51,50,51
   50 S(J1,K3)=SMD(1,6)
      S(J2,K3)=SMD(2,6)
      S(J3,K3)=SMD(3,6)
      S(K1,K3)=S(K1,K3)&SMD(4,6)
      S(K2,K3)=S(K2,K3)&SMD(5,6)
      S(K3,K3)=S(K3,K3)&SMD(6,6)
      MM(J1,K3)=MD(1,6)
      MM(J2,K3)=MD(2,6)
      MM(J3,K3)=MD(3,6)
      MM(K1,K3)=MM(K1,K3)&MD(4,6)
      MM(K2,K3)=MM(K2,K3)&MD(5,6)
      MM(K3,K3)=MM(K3,K3)&MD(6,6)
   51 CONTINUE
C
C              2.B RETAINING OF REDUCED STIFFNESS AND MASS MATRICES.
C
      DO 150 I=1,N
      DO 150 J=1,N
      SM(I,J)=S(I,J)
      MSM(I,J)=MM(I,J)
  150 CONTINUE
      WRITE(6,100)
  100 FORMAT (1H1,16HSTIFFNESS MATRIX)
      DO 110 I=1,N
      WRITE (6,111) (S(I,J),J=1,N)
  110 CONTINUE
      WRITE(6,250)
  250 FORMAT (1H0,11HMASS MATRIX)
      DO 310 I=1,N
      WRITE(6,111)(MSM(I,J),J=1,N)
  310 CONTINUE
  111 FORMAT (1H0,8E15.6)
C
C              3. CALCULATION OF EIGENVALUES AND EIGENVECTORS.
C
      CALL NROOT (N,SM,MSM,XL,XX)
C
C          PRINTING OF NATURAL FREQUENCIES.
C
      WRITE(6,500)
  500 FORMAT(//'1','NATURAL FREQUENCIES IN RADIANS PER SECOND ')
      DO 600 I=1,N
      EVALUE=SQRT(XL(I))
  600 WRITE(6,400) EVALUE
  400 FORMAT(1H0,F20.8)
      END
```

PROGRAM TO FIND THE NATURAL FREQUENCIES OF GRIDS.

```
0001          DIMENSICN L(20),X(20),Y(20),JJ(20),JK(20),IX(20),IY(20),CX(20),CY(
             120),RL(50),CRL(50),SMD(6,6),S(50,50),MM(50,50),MD(6,6),AX(20),
             1XL(39),XX(39,39),MSM(39,39),SM(39,39)
0002          REAL IX,IY,L,MD,MM,MSM
0003          INTEGER CRL,RL
        C        RA MASS PER UNIT LENGTH
        C        IX TIRSION CCNSTANT
        C        IY M.O.I. ABOUT Y AXIS
        C        AX CROSS-SECTIONAL AREA
0004          WRITE(6,1)
        C
        C***********************************************************************
0005        1 FORMAT('1@,'NATURAL FREQUENCIES OF GRID@)
        C***********************************************************************
        C
        C               1. INPUT  AND  STRUCTURE  DATA
        C
        C               1.A STRUCTURE PARAMETERS,ELASTIC MODULUS AND DENSITY
        C
0006          WRITE (6,2)
0007        2 FORMAT(//'0@,'STRUCTURE  CATA@)
0008          WRITE (6,3)
0009        3 FORMAT ('0@,'    M    N    NJ    NR    NRJ         E           G
             1     RA@)
0010          READ (5,4) M,NJ,NR,NRJ,E,G,RA
0011        4 FCRMAT(4I5,2F10.0,F10.8)
0012          N=3*NJ-NR
0013          WRITE (6,5) M,N,NJ,NR,NRJ,E,G,RA
0014        5 FORMAT(5I5,F15.0,F10.0,F12.8)
        C
        C               1B.  JOINT  COORCINATES
        C
0015          WRITE (6,6)
0016        6 FORMAT ('0@,'JOINT COORDINATES@)
0017          WRITE (6,7)
0018        7 FORMAT ('0@,'JCINT    X COORD   Y COORD@)
0019          DO 101 J=1,NJ
0020          READ (5,8) J,X(J),Y(J)
0021        8 FORMAT (I5,2F10.2)
0022          WRITE (6,9) J,X(J),Y(J)
0023        9 FORMAT (I5,2F10.2)
0024      101 CCNTINUE
        C
        C               1C.  MEMBER DESIGNATIONS AND PROPERTIES
        C
0025          WRITE (6,10)
0026       10 FORMAT ('0@,' MEMBER  DESIGNATIONS  AND  PROPERTIES@)
0027          WRITE (6,11)
0028       11 FORMAT (1H0,6HMEMBER,2X,2HJJ,3X,2HJK,5X,2HIX,9X,2HIY,6X,2HAX,10X,1
             1HL,9X,2HCX,8X,2HCY)
0029          DO 102  I=1,M
0030          READ (5,12) I,JJ(I),JK(I),IX(I),IY(I),AX(I)
0031       12 FORMAT(3I5,3F10.6)
0032          JI=JJ(I)
```

```
      KI=JK(I)
      XCL=X(KI)-X(JI)
      YCL=Y(KI)-Y(JI)
      L(I)=SQRT(XCL**2+YCL**2)
      CX(I)=XCL/L(I)
      CY(I)=YCL/L(I)
      WRITE (6,13) I,JJ(I),JK(I),IX(I),IY(I),AX(I),L(I),CX(I),CY(I)
   13 FORMAT(3I5,3F10.5,3F10.2)
  102 CONTINUE
C
C             1D.  JOINT RESTRAINT  LIST, CUMULATIVE  RESTRAINT  LIST
C
C        INITIALISE  RL%I<  TO  ZERO
C
      NJ3=3*NJ
      DO 104 J3=1,NJ3
      RL(J3)=0
  104 CONTINUE
      WRITE (6,14)
   14 FORMAT('0@,'JOINT RESTRAINTS@)
      WRITE (6,15)
   15 FORMAT ('0@,'JOINT      X RSTRT  Y RSTRT  Z RSTRT@)
      DO 103 J=1,NRJ
      READ (5,16) K,RL(3*K-2),RL(3*K-1),RL(3*K)
   16 FORMAT (4I10)
      WRITE(6,17) K,RL(3*K-2),RL(3*K-1),RL(3*K)
   17 FORMAT (I5,3I10)
  103 CONTINUE
      CRL(1)=RL(1)
      DO 105 K=2,NJ3
      CRL(K)=CRL(K-1)+RL(K)
  105 CONTINUE
C
C             2.A GENERATION OF OVERALL STIFNESS AND MASS MATRICES.
C
      NJ2=3*NJ
      DO 106 J3=1,NJ2
      DO 106 K3=1,NJ2
      S(J3,K3)=0.0
      MM(J3,K3)=0.0
  106 CONTINUE
      DO 51 I=1,M
      J1=3*JJ(I)-2
      J2=3*JJ(I)-1
      J3=3*JJ(I)
      K1=3*JK(I)-2
      K2=3*JK(I)-1
      K3=3*JK(I)
      SCM1=G*IX(I)/L(I)
      SCM2=4.0*E*IY(I)/L(I)
      SCM3=1.5*SCM2/L(I)
      SCM4=2.0*SCM3/L(I)
      Z=RA*L(I)/420.0
      IF (RL(J1)) 18,19,18
```

```
   19 J1=J1-CRL(J1)
      GO TO 20
   18 J1=N+CRL(J1)
   20 IF (RL(J2)) 21,22,21
   22 J2=J2-CRL(J2)
      GO TO 23
   21 J2=N+CRL(J2)
   23 IF (RL(J3)) 24,25,24
   25 J3=J3-CRL(J3)
      GO TO 26
   24 J3=N+CRL(J3)
   26 IF (RL(K1))  27,28,27
   28 K1=K1-CRL(K1)
      GO TO 29
   27 K1=N+CRL(K1)
   29 IF (RL(K2)) 30,31,30
   31 K2=K2-CRL(K2)
      GO TO 32
   30 K2=N+CRL(K2)
   32 IF (RL(K3)) 33,34,33
   34 K3=K3-CRL(K3)
      GO TO 35
   33 K3=N+CRL(K3)
   35 WRITE(6,200) J1,J2,J3,K1,K2,K3,Z
  200 FORMAT(1H0,6I5,E15.8)
      CY2=CY(I)*CY(I)
      CX2=CX(I)*CX(I)
      CXY=CX(I)*CY(I)
      SMD(1,1)=SCM1*CX2+SCM2*CY2
      SMD(4,4)=SMD(1,1)
      SMD(1,2)=(SCM1-SCM2)*CXY
      SMD(2,1)=SMD(1,2)
      SMD(1,3)=SCM3*CY(I)
      SMD(3,1)=SMD(1,3)
      SMD(1,4)=-SCM1*CX2+SCM2*CY2/2.0
      SMD(4,1)=SMD(1,4)
      SMD(1,5)=-(SCM1+SCM2/2.0)*CXY
      SMD(5,1)=SMD(1,5)
      SMD(1,6)=-SMD(1,3)
      SMD(6,1)=-SMD(1,3)
      SMD(2,2)=SCM1*CY2+SCM2*CX2
      SMD(5,5)=SMD(2,2)
      SMD(2,3)=-SCM3*CX(I)
      SMD(3,2)=SMD(2,3)
      SMD(2,4)=SMD(1,5)
      SMD(4,2)=SMD(1,5)
      SMD(2,5)=-SCM1*CY2+SCM2*CX2/2.0
      SMD(5,2)=SMD(2,5)
      SMD(2,6)=-SMD(2,3)
      SMD(6,2)=-SMD(2,3)
      SMC(3,3)=SCM4
      SMD(6,6)=SMD(3,3)
      SMD(3,4)=SMD(1,3)
      SMD(4,3)=SMD(1,3)
```

```
      SMD(3,5)=SMD(2,3)
      SMD(5,3)=SMD(2,3)
      SMD(3,6)=-SCM4
      SMD(6,3)=SMD(3,6)
      SMD(4,5)=SMD(1,2)
      SMD(5,4)=SMD(1,2)
      SMD(4,6)=-SMD(1,3)
      SMD(6,4)=-SMD(1,3)
      SMD(5,6)=-SMD(2,3)
      SMD(6,5)=-SMD(2,3)
C
      AI=IX(I)/AX(I)
      CX2AI=AI*CX2
      CY2AI=AI*CY2
      CXYAI=AI*CXY
      CX2L=CX2*L(I)*L(I)
      CY2L=CY2*L(I)*L(I)
      CXYL=CXY*L(I)*L(I)
C
      MD(1,1)=(140.0*CX2AI+4.0*CY2L)*Z
      MD(4,4)=MD(1,1)
      MD(1,2)=(140.0*CXYAI-4.0*CXYL)*Z
      MD(2,1)=MD(1,2)
      MD(1,3)=22.0*L(I)*CY(I)*Z
      MD(3,1)=MD(1,3)
      MD(1,4)=(70.0*CX2AI-3.0*CY2L)*Z
      MD(4,1)=MD(1,4)
      MD(1,5)=(70.0*CXYAI+3.0*CXYL)*Z
      MD(5,1)=MD(1,5)
      MD(1,6)=13.0*L(I)*CY(I)*Z
      MD(6,1)=MD(1,6)
      MD(2,2)=(140.0*CY2AI+4.0*CX2L)*Z
      MD(5,5)=MD(2,2)
      MD(2,3)=-22.0*L(I)*CX(I)*Z
      MD(3,2)=MD(2,3)
      MD(2,4)=MD(1,5)
      MD(4,2)=MD(1,5)
      MD(2,5)=(70.0*CY2AI-3.0*CX2L)*Z
      MD(5,2)=MD(2,5)
      MD(2,6)=-13.0*L(I)*CX(I)*Z
      MD(6,2)=MD(2,6)
      MD(3,3)=156.0*Z
      MD(6,6)=MD(3,3)
      MD(3,4)= -MD(1,6)
      MD(4,3)= -MD(1,6)
      MD(3,5)=-MD(2,6)
      MD(5,3)=-MD(2,6)
      MD(3,6)=63.0*Z
      MD(6,3)=MD(3,6)
      MD(4,6)=-MD(1,3)
      MD(6,4)=-MD(1,3)
      MD(4,5)=MD(1,2)
      MD(5,4)=MD(1,2)
      MD(5,6)=-MD(2,3)
```

```
      MD(6,5)=-MD(2,3)
C
      JJ3=3*JJ(I)
C
      IF (RL(JJ3-2)) 41,40,41
   40 S(J1,J1)=S(J1,J1)+SMD(1,1)
      S(J2,J1)=S(J2,J1)+SMD(2,1)
      S(J3,J1)=S(J3,J1)+SMD(3,1)
      S(K1,J1)=SMD(4,1)
      S(K2,J1)=SMD(5,1)
      S(K3,J1)=SMD(6,1)
      MM(J1,J1)=MM(J1,J1)+MD(1,1)
      MM(J2,J1)=MM(J2,J1)+MD(2,1)
      MM(J3,J1)=MM(J3,J1)+MD(3,1)
      MM(K1,J1)=MD(4,1)
      MM(K2,J1)=MD(5,1)
      MM(K3,J1)=MD(6,1)
   41 IF (RL(JJ3-1)) 43,42,43
   42 S(J1,J2)=S(J1,J2)+SMD(1,2)
      S(J2,J2)=S(J2,J2)+SMD(2,2)
      S(J3,J2)=S(J3,J2)+SMD(3,2)
      S(K1,J2)=SMD(4,2)
      S(K2,J2)=SMD(5,2)
      S(K3,J2)=SMD(6,2)
      MM(J1,J2)=MM(J1,J2)+MD(1,2)
      MM(J2,J2)=MM(J2,J2)+MD(2,2)
      MM(J3,J2)=MM(J3,J2)+MD(3,2)
      MM(K1,J2)=MD(4,2)
      MM(K2,J2)=MD(5,2)
      MM(K3,J2)=MD(6,2)
   43 IF (RL(JJ3)) 45,44,45
   44 S(J1,J3)=S(J1,J3)+SMD(1,3)
      S(J2,J3)=S(J2,J3)+SMD(2,3)
      S(J3,J3)=S(J3,J3)+SMD(3,3)
      S(K1,J3)=SMD(4,3)
      S(K2,J3)=SMD(5,3)
      S(K3,J3)=SMD(6,3)
      MM(J1,J3)=MM(J1,J3)+MD(1,3)
      MM(J2,J3)=MM(J2,J3)+MD(2,3)
      MM(J3,J3)=MM(J3,J3)+MD(3,3)
      MM(K1,J3)=MD(4,3)
      MM(K2,J3)=MD(5,3)
      MM(K3,J3)=MD(6,3)
C
   45 JK3=3*JK(I)
C
      IF (RL(JK3-2)) 47,46,47
   46 S(J1,K1)=SMD(1,4)
      S(J2,K1)=SMD(2,4)
      S(J3,K1)=SMD(3,4)
      S(K1,K1)=S(K1,K1)+SMD(4,4)
      S(K2,K1)=S(K2,K1)+SMD(5,4)
      S(K3,K1)=S(K3,K1)+SMD(6,4)
      MM(J1,K1)=MD(1,4)
```

```
      MM(J2,K1)=MD(2,4)
      MM(J3,K1)=MD(3,4)
      MM(K1,K1)=MM(K1,K1)+MD(4,4)
      MM(K2,K1)=MM(K2,K1)+MD(5,4)
      MM(K3,K1)=MM(K3,K1)+MD(6,4)
   47 IF (RL(JK3-1)) 49,48,49
   48 S(J1,K2)=SMD(1,5)
      S(J2,K2)=SMD(2,5)
      S(J3,K2)=SMD(3,5)
      S(K1,K2)=S(K1,K2)+SMD(4,5)
      S(K2,K2)=S(K2,K2)+SMD(5,5)
      S(K3,K2)=S(K3,K2)+SMD(6,5)
      MM(J1,K2)=MD(1,5)
      MM(J2,K2)=MD(2,5)
      MM(J3,K2)=MD(3,5)
      MM(K1,K2)=MM(K1,K2)+MD(4,5)
      MM(K2,K2)=MM(K2,K2)+MD(5,5)
      MM(K3,K2)=MM(K3,K2)+MD(6,5)
   49 IF (RL(JK3)) 51,50,51
   50 S(J1,K3)=SMD(1,6)
      S(J2,K3)=SMD(2,6)
      S(J3,K3)=SMD(3,6)
      S(K1,K3)=S(K1,K3)+SMD(4,6)
      S(K2,K3)=S(K2,K3)+SMD(5,6)
      S(K3,K3)=S(K3,K3)+SMD(6,6)
      MM(J1,K3)=MD(1,6)
      MM(J2,K3)=MD(2,6)
      MM(J3,K3)=MD(3,6)
      MM(K1,K3)=MM(K1,K3)+MD(4,6)
      MM(K2,K3)=MM(K2,K3)+MD(5,6)
      MM(K3,K3)=MM(K3,K3)+MD(6,6)
   51 CONTINUE
C
C              2.B RETAINING OF REDUCED STIFFNESS AND MASS MATRICES.
C
      DO 150 I=1,N
      DO 150 J=1,N
      SM(I,J)=S(I,J)
      MSM(I,J)=MM(I,J)
  150 CONTINUE
      WRITE(6,250)
  250 FORMAT (1H0,11HMASS MATRIX)
      DO 310 I=1,N
      WRITE(6,111)(MSM(I,J),J=1,N)
  310 CONTINUE
  111 FORMAT (1H0,8E15.6)
C
C              3. CALCULATION OF EIGENVALUES AND EIGENVECTORS.
C
      CALL NROOT (N,SM,MSM,XL,XX)
C
C         PRINTING OF NATURAL FREQUENCIES.
C
      WRITE(6,500)
```

```
      500 FORMAT(//'1@,'NATURAL FREQUENCIES IN RADIANS PER SECOND @)
          DO 600 I=1,N
          EVALUE=SQRT(XL(I))/6.283184
      600 WRITE(6,400) EVALUE
      400 FORMAT(1H0,F20.8)
C             PRINTING OF NORMALIZED NODAL AMPLITUDES.
C
          WRITE(6,550)
      550 FORMAT(//'1@,'NORMALIZED NODAL AMPLITUDES@)
          DO 650 I=1,N
      650 WRITE(6,651) (XX(I,J),J=1,N)
      651 FORMAT(1H0,10F10.4)
          END
```

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APPENDIX D

PROGRAM FOR THE PARAMETRIC INSTABILITY OF COLUMNS

The programs in this section are also coded in Fortran IV and are used to obtain the natural frequencies and the buckling loads for a particular idealization of the column.

D1: Program to find the natural frequencies and buckling load of a pinned-ended column.

Input

Cols.	Notation	Description
A Number of Subdivisions (I5) one card		
1-5	NM	Number of parts in which the column is divided
B Stiffness Matrix (8F10.3) (NM-1) cards		
1-80	S	Elements of stiffness matrix
C Mass Matrix (8F10.3) (NM-1) cards		
1-80	M	Elements of mass matrix

Output

The output consists of the input data, the modal shape, the natural frequencies and the buckling loads for all the modes.

```
      C
      C     ****************************************************
      C        PROGRAM TO FIND THE EIGEN VALUES AND EIGEN VECTORS OF A PINNED
      C        ENDED COLUMN
      C     ****************************************************
            DOUBLE PRECISION A,B,C,U,X,Y,Y1,EIGV,DELT,DIV,EPS,R
            DIMENSION SS(8,8)
            DIMENSION A(8,8),U(8),DF(8),DV(8)
            DIMENSION B(8,8),C(8,8),X(8),Y1(8),Y(8),EIGV(8),
           1M(2,2),S(2,2),L1(2),M1(2)
            EQUIVALENCE (X(1),Y1(1))
            CALL PGMCHK
            REAL M
            READ(5,20) NM
      20    FORMAT(I5)
            WRITE(6,21) NM
         21 FORMAT(1H1,5X,3HNM=,I3)
            DO 2 I=1,NM
      2     READ(5,1) (S(I,J),J=1,NM)
          1 FORMAT(10F10.3)
            WRITE(6,3)
      3     FORMAT('0','  STIFFNESS MATRIX')
            DO 4 I=1,NM
      4     WRITE(6,1) (S(I,J),J=1,NM)
            DO 40 I=1,NM
         40 READ(5,1) (M(I,J),J=1,NM)
            WRITE(6,71)
      71    FORMAT('0','  MASS MATRIX')
            DO 8 I=1,NM
      8     WRITE(6,1) (M(I,J),J=1,NM)
            DO 72 I=1,NM
            DO 72 J=1,NM
         72 SS(I,J)=S(I,J)
            CALL MINV(S,NM,D,L1,M1)
            DO 31 I=1,NM
            DO 31 K=1,NM
            SUM=0.
            DO 30 J=1,NM
         30 SUM=SUM+S(I,J)*M(J,K)
         31 B(I,K)=SUM
            WRITE(6,33)
         33 FORMAT('0','  PRODUCT MATRIX ')
            DO 32 I=1,NM
         32 WRITE(6,1) (B(I,J),J=1,NM)
      C        MATRIX C=MATRIX B
        121 DO 98 I=1,NM
            DO 98 J=1,NM
         98 C(I,J)=B(I,J)
      C
      C   EPS  SMALL NUMBER TO TEST WHETHER ANY DIAGONAL IS ZERO OR NOT
      C
            EPS=.000000001
            WRITE(6,64) EPS
         64 FORMAT(1H0,5X,4HEPS=,F10.9)
```

```
            C
            C          TEST TO FIND IF LAST EIGENVALUE REACHED
0045                DO 110 II=1,NM
            C
0046                IF(II-NM) 116,117,116
0047            116 N=NM-II+1
0048                N1=N-1
            C
            C          SET Y1(I) AS A UNIT COLUMN MATRIX
            C
0049                DO 100 I=1,N
0050            100 Y1(I)=1.0
0051             99 KC=0
            C
            C          UP TO STATEMENT 115, SUCCESSIVE APPROXIMATION IN FINDING THE
            C          EIGENVALUES %AND THE FIRST EIGENVECTOR ONLY < IS CARRIED OUT
0052                DO 101 I=1,N
0053                Y(I)=0.
0054                DO 101 J=1,N
0055            101 Y(I)=Y(I)+C(I,J)*Y1(J)
0056                J=1
0057        92      IF (Y(J)) 89,91,89
0058        89      R=Y(J)
0059                GO TO 93
0060        91      J=J+1
0061                GO TO 92
            C
            C          EIGENVALUE IS THE FIRST ELEMENT OF THE EIGENVECTOR
0062        93      DO 90 I=1,N
0063                Y(I)=Y(I)/R
0064                IF (DABS(Y(I)-Y1(I))-EPS) 90,90,102
0065            102 KC=1
0066             90 CONTINUE
0067                IF(KC) 103,103,104
            C
            C          TRANSFER X(M+1)TH EIGENVECTOR TO X(M)TH EIGENVECTOR
0068            104 DO 115 I=1,N
0069            115 Y1(I)=Y(I)
0070                GO TO 99
            C
            C          TTHE EIGENVALUE IS THE FIRST ELEMENT IN THE EIGENVECTOR
0071            103 EIGV(II)=R
            C          UP TO STATEMENT 112, THE SIZE OF MATRIX C IS REDUCED BY ONE
0072                DO 109 I=1,N1
0073            109 X(I)=-Y(I+1)
0074                DO 112 I=2,N
0075                DO 112 J=2,N
0076            112 A(I-1,J-1)=C(I,J)+C(1,J)*X(I-1)
            C          REDUCTION OF SIZE OF MATRIX C BY ONE COMPLETED
0077                DO 113 I=1,N1
0078                DO 113 J=1,N1
0079            113 C(I,J)=A(I,J)
0080                GO TO 118
0081            117 EIGV(II)=C(1,1)
```

```
0082        118 WRITE(6,54) II,EIGV(II)
0083         54 FORMAT(I2,16HTH.  EIGENVALUE=,E12.5)
      C           UP TO STATEMENT 9015, THE EIGENVECTORS ARE SOLVED
0084            NN=NM-1
0085            NLAST=NM+1
0086            IF(II-1) 141,141,142
0087        142 DO 105 I=1,NN
0088            DO 107 J=1,NN
0089            A(I,J)=B(I,J)
0090            IF(I-J) 107,139,107
0091        139 A(I,J)=A(I,J)-EIGV(II)
0092        107 CONTINUE
0093        105 U(I)=-B(I,NM)
      C
      C   TEST TO SEE WHETHER DIAGONAL ELEMENT IS ZERO OR NOT
      C
0094            DO 9015 I=1,NN
0095            IF(I-NN) 9021,9007,9021
0096       9021 IF(A(I,I)-EPS) 9005,9006,9007
0097       9005 IF(-A(I,I)-EPS) 9006,9006,9007
0098       9006 U(I)=U(I)+U(I+1)
0099            DO 9023 J=1,NN
0100       9023 A(I,J)=A(I,J)+A(I+1,J)
0101            GO TO 9021
0102       9007 DIV=A(I,I)
0103            U(I)=U(I)/DIV
      C
      C   DIVIDE ALL ELEMENTS OF I-TH EQUATION BY A(I,I)
      C
0104            DO 9009 J=1,NN
0105       9009 A(I,J)=A(I,J)/DIV
      C
      C   REDUCE THE I-TH ELEMENT OF THE OTHER EQUATIONS TO ZERO
      C
0106            DO 9015 MM=1,NN
0107            DELT=A(MM,I)
0108       9016 IF(MM-I) 9010,9015,9010
0109       9010 U(MM)=U(MM)-U(I)*DELT
0110            DO 9011 J=1,NN
0111       9011 A(MM,J)=A(MM,J)-A(I,J)*DELT
0112       9015 CONTINUE
0113            Y(NM)=1.0
0114            DO 108 I=1,NN
0115        108 Y(I)=U(I)
      C
      C           CALCULATION OF ' LAST ANGLE '
      C
0116        141 Y(NLAST)=Y(1)
0117            DO 160 I=2,NM
0118        160 Y(NLAST)=Y(NLAST)+2.0*Y(I)
0119            Y(NLAST)=-Y(NLAST)
      C
      C           CALCUTATIUN UF DISPLACEMENT VECTOR
      C
```

```
      DV(1)=0.5*Y(1)                                          RHS  158
      DO 163 I=2,NM                                           RHS  159
  163 DV(I)=DV(I-1)+Y(I)                                      RHS  160
      WRITE(6,5) II                                           RHS  161
    5 FORMAT(I2,8X,16HTH. EIGEN VECTOR,5X,19HDISPLACEMENT VECTOR) RHS  162
      DO 10 K=1,NM                                            RHS  163
   10 WRITE(6,79) Y(K),DV(K)                                  RHS  164
      WRITE(6,161) Y(NLAST)                                   RHS  165
  161 FORMAT(1H ,6HYLAST=,F15.6)                              RHS  166
   79 FORMAT(F22.6,F15.6)                                     RHS  167
      DO 201 I=1,NM                                           RHS  168
      SUM=-EIGV(II)*Y(I)                                      RHS  169
      DO 200 J=1,NM                                           RHS  170
      SUM=SUM+B(I,J)*Y(J)                                     RHS  171
200   CONTINUE                                                RHS  172
204   FORMAT(1H ,2HI=,I2,4X,4HSUM=,F15.8)                     RHS  173
201   WRITE(6,204) I,SUM                                      RHS  174
      IF(II-1) 151,151,152                                    RHS  175
  151 DO 143 I=1,NM                                           RHS  176
  143 DF(I)=DV(I)                                             RHS  177
      GO TO 310
C                                                             RHS  179
C         TEST FOR ORTHOGONALITY OF EIGEN VECTORS             RHS  180
C                                                             RHS  181
  152 SUMM=0.0                                                RHS  182
      DO 153 I=1,NM                                           RHS  183
  153 SUMM=SUMM+DF(I)*DV(I)                                   RHS  184
      WRITE(6,154) II,SUMM                                    RHS  185
  154 FORMAT(1H0,3HII=,I2,4X,5HSUMM=,F15.8)                   RHS  186
C         ***************************************
C         APPLICATION OF GALERKIN'S METHOD.
C         *************************************
C         MULTIPLICATION OF MASS MATRIX
  310 DO 300 I=1,NM
      DV(I)=0.0
      DO 300 J=1,NM
      DV(I)=DV(I)+M(I,J)*Y(J)
  300 CONTINUE
      CMASS=0.0
      DO 301 I=1,NM
  301 CMASS=CMASS+DV(I)*Y(I)
      WRITE(6,302) CMASS
  302 FORMAT('0','COEFFICIENT OF MASS MATRIX',5X,6HCMASS=,E20.8)
C         MULTIPLICATION OF STIFFNESS MATRIX
      DO 303 I=1,NM
      DV(I)=0.0
      DO 303 J=1,NM
      DV(I)=DV(I)+SS(I,J)*Y(J)
  303 CONTINUE
      CSTIFF=0.0
      DO 304 I=1,NM
      CSTIFF=CSTIFF+DV(I)*Y(I)
  304 CONTINUE
      WRITE(6,305) CSTIFF
```

```
  305 FORMAT('0','COEFFICIENT OF STIFFNESS MATRIX',5X,7HCSTIFF=,E20.8)
C         MULTIPLICATION OF FCRCE VECTOR
      CFORCE=0.0
      DO 306 I=1,NM
  306 CFORCE=CFORCE+(Y(I)-Y(I+1))*Y(I)
      WRITE(6,307) CFORCE
  307 FORMAT('0','COEFFICIENT OF FORCE MATRIX',5X,7HCFORCE=,E20.8)       RHS
      GFREQ=CSTIFF/CMASS
      GLOAD=CSTIFF/CFORCE
      WRITE(6,308) GFREQ
  308 FORMAT('0','FREQUENCY BY GALERKINS  METHOD',5X,6HGFREQ=,E20.8)     RHS
      WRITE(6,309) GLOAD
  309 FORMAT('0','BUCKLING LOAD BY GALERKINS  METHOD',5X,6HGLOAD=,E20.8) RHS  187
  110 CONTINUE                                                           RHS  188
      END

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```

D2: Program to find the natural frequencies and the base inertia force for a column subjected to periodic support motion.

<u>Input</u>

Cols.	Notation	Description
A Number of Subdivisions (I5) one card		
1-5	NMM	Number of parts in which the column is divided
B Member Properties (I5, 3F10.3) NMM cards		
1-5	I	Counter
6-15	MS	Mass of link
16-25	KM	Stiffness of spring
26-35	L	Length of spring
C Axial Force Array (10F10.6) one card		
1-80	FM	Variation of the axial force along the length of the column

```
C
      DIMENSION MS(40),L(40),KM(40),RM(40),SM(40),RL(40),CRL(40)
      DIMENSION A(40,40),U(40),DF(40),DV(40),S(40,40)
      DIMENSION B(40,40),C(40,40),X(40),Y1(40),Y(40),EIGV(40),EIGG(40),
     1SS(10,10),M(10,10),L1(10),M1(10),FM(10)
      DOUBLE PRECISION L,KM,SM,RM,MS
      DOUBLE PRECISION A,B,C,U,X,Y,Y1,EIGV,DELT,DIV,EPS,R
      EQUIVALENCE (X(1),Y1(1))
      CALL PGMCHK
      REAL M
      INTEGER RL,CRL
      READ (5,20) NMM
   20 FORMAT(I5)
      WRITE(6,21) NMM
   21 FORMAT(1H1,5X,4HNMM=,I3)
      DO 252 I=1,NMM
  252 RL(I)=0
      CRL(1)=0
      DO 253 I=2,NMM
  253 CRL(I)=CRL(I-1)+RL(I)
      NM=NMM-CRL(NMM)
      WRITE(6,14) NMM,NM
   14 FORMAT(1H0,5X,4HNMM=,I3,5X,3HNM=,I3)
      WRITE (6,51)
   51 FORMAT('0','  JOINT   RL    CRL')
      DO 9 I=1,NMM
    9 WRITE(6,6) I,RL(I),CRL(I)
    6 FORMAT(3I6)
      DO 206 I=1,NMM
  206 READ(5,207) I,MS(I),KM(I),L(I)
  207 FORMAT(I5,3F10.3)
      WRITE(6,220)
  220 FORMAT(1H0,6HNUMBER,5X,4HMASS,6X,9HSTIFFNESS,6X,6HLENGTH)
      DO 208 I=1,NMM
  208 WRITE(6,209) I,MS(I),KM(I),L(I)
  209 FORMAT(1H0,I5,F15.11,2F10.5)
      READ (5,81) (FM(I),I=1,NM)
   81 FORMAT(10F8.5)
      WRITE(6,82)
   82 FORMAT('0','FORCE MATRIX')
      WRITE(6,83) (FM(I),I=1,NM)
   83 FORMAT(1H0,10F10.6)
      DO 210 I=1,NMM
      DO 210 J=1,NMM
      S(I,J)=0.0
  210 CONTINUE
      DO 1 I=1,NM
      DO 1 J=1,NM
      M(I,J)=0.0
    1 SS(I,J)=0.0
      ML=1
  250 DO 211 I=1,NMM
      IF (RL(I)) 240,241,240
  240 ML=ML+1
```

```
      GO TO 211
  241 ICRL=I-CRL(I)
      LL=ML-CRL(I)
      DO 212 J=1,NMM
      IF (LL-J) 244,244,243
  243 GO TO 212
  244 JRL=J+CRL(I)
      IF (NMM-JRL) 212,242,242
  242 JCRL=J-CRL(JRL)+CRL(I)
      IF (RL(JRL)) 230,231,230
  231 M(ICRL,JCRL)=MS(JRL)*L(I)
      GO TO 212
  230 M(ICRL,JCRL)=M(ICRL,JCRL)+MS(JRL)*L(I)
  212 CONTINUE
      ML=ML+1
  211 CONTINUE
      DO 215 I=1,NMM
  215 RM(I)=KM(I)/L(I)
      DO 216 I=2,NMM
  216 SM(I)=KM(I)/L(I-1)
      DO 18 I=1,NMM
   18 WRITE(6,11) I,RM(I),KM(I)
   11 FORMAT(I7,2F10.3)
      SS(1,1)=RM(1)+RM(2)+SM(2)
      SS(1,2)=-RM(2)
      SS(2,1)=-(RM(2)+SM(2)+SM(3))
      SS(2,2)=RM(2)+RM(3)+SM(3)
      SS(2,3)=-SM(3)
      S(1,1)=SS(1,1)
      S(1,2)=SS(1,2)
      S(2,1)=SS(2,1)
      S(2,2)=SS(2,2)
      S(2,3)=SS(2,3)
      S(NMM,NMM)=RM(NMM)
      S(NMM,NMM-1)=-(RM(NMM)+SM(NMM))
      S(NMM,NMM-2)=SM(NMM)
      MN=NMM-1
      LL=3
      DO  217 I=3,MN
      S(I,LL-2)=SM(I)
      S(I,LL-1)=-(RM(I)+SM(I)+SM(I+1))
      S(I,LL)=RM(I)+RM(I+1)+SM(I+1)
      S(I,LL+1)=-RM(I+1)
      LL=LL+1
  217 CONTINUE
      DO 2110 I=3,NMM
      IF(RL(I)) 2400,2410,2400
 2400 GO TO 2110
 2410 ICPL=I-CRL(I)
      DO 2120 J=1,NMM
      JRL=J+CPL(I)
      IF(NMM-JRL) 2120,2420,2420
 2420 JCRL=J-CRL(JRL)+CRL(I)
      IF (RL(JRL)) 2300,2310,2300
```

```
 2310 SS(ICRL,JCRL)=S(I,JRL)
      GO TO 2120
 2300 SS(ICRL,JCRL)=SS(ICRL,JCRL)+S(I,JRL)
 2120 CONTINUE
 2110 CONTINUE
      WRITE(6,3001)
 3001 FORMAT('0','STIFFNESS MATRIX')
      DO 3002 I=1,NM
 3002 WRITE(6,3003) (SS(I,J),J=1,NM)
      WRITE(6,3004)
 3004 FORMAT('0','MASS MATRIX')
      DO 3005 I=1,NM
 3005 WRITE(6,3003) (M(I,J),J=1,NM)
 3003 FORMAT(1H ,9F13.5)
      DO 3006 I=1,NMM
 3006 WRITE(6,3003) (S(I,J),J=1,NMM)
      CALL MINV(SS,NM,D,L1,M1)
      DO 31 I=1,NM
      DO 31 K=1,NM
      SUM=0.
      DO 30 J=1,NM
   30 SUM=SUM+SS(I,J)*M(J,K)
   31 B(I,K)=SUM
C           MATRIX C=MATRIX B
  121 DO 98 I=1,NM
      DO 98 J=1,NM
   98 C(I,J)=B(I,J)
C
C    EPS  SMALL NUMBER TO TEST WHETHER ANY DIAGONAL IS ZERO OR NOT
C
      EPS=.000000001
      WRITE(6,64) EPS
   64 FORMAT (1H0,5X,4HEPS=,E12.5)
C
C          TEST TO FIND IF LAST EIGENVALUE REACHED
C
      DO 110 II=1,5
      IF(II-NM) 116,117,116
  116 N=NM-II+1
      N1=N-1
C
C          SET Y1(I) AS A UNIT COLUMN MATRIX
C
      DO 100 I=1,N
  100 Y1(I)=1.0
   99 KC=0
C
C          UP TO STATEMENT 115, SUCCESSIVE APPROXIMATION IN FINDING THE
C          EIGENVALUES (AND THE FIRST EIGENVECTOR ONLY ) IS CARRIED OUT
      DO 101 I=1,N
      Y(I)=0.
      DO 101 J=1,N
  101 Y(I)=Y(I)+C(I,J)*Y1(J)
      J=N
```

```
   92   IF (Y(J)) 89,91,89
   89   R=Y(J)
        GO TO 93
     91 J=J-1
        GO TO 92
C
C          EIGENVALUE IS THE FIRST ELEMENT OF THE EIGENVECTOR
   93   DO 90 I=1,N
        Y(I)=Y(I)/R
        IF (DABS(Y(I)-Y1(I))-EPS) 90,90,102
    102 KC=1
     90 CONTINUE
        IF(KC) 103,103,104
C
C          TRANSFER X(M+1)TH EIGENVECTOR TO X(M)TH EIGENVECTOR
    104 DO 115 I=1,N
    115 Y1(I)=Y(I)
        GO TO 99
C
C          TTHE EIGENVALUE IS THE FIRST ELEMENT IN THE EIGENVECTOR
    103 EIGV(II)=R
C          UP TO STATEMENT 112, THE SIZE OF MATRIX C IS REDUCED BY ONE
        DO 109 I=1,N1
    109 X(I)=-Y(I)
        DO 112 I=1,N1
        DO 112 J=1,N1
    112 A(I,J)=C(I,J)+C(N,J)*X(I)
C          REDUCTION OF SIZE OF MATRIX C BY ONE COMPLETED
        DO 113 I=1,N1
        DO 113 J=1,N1
    113 C(I,J)=A(I,J)
        GO TO 119
    117 EIGV(II)=C(1,1)
    118 EIGG(II)=1.0/R
        WRITE(6,54) II,EIGV(II),EIGG(II)
        FREQ=SQRT(ABS(EIGG(II)))
        WRITE(6,53) FREQ
     53 FORMAT(1H0,5X,40HNATURAL FREQUENCY IN RADIANS PER SECOND=,F20.3)
     54 FORMAT(I4,18HTH. INVERSE VALUE=,E12.5,5X,16HTH. EIGEN VALUE=,E12.5
      1)
C          UP TO STATEMENT 9015, THE EIGENVECTORS ARE SOLVED
        NN=NM-1
        NLAST=NM+1
        IF(II-1) 141,141,142
    142 DO 105 I=1,NN
        DO 107 J=1,NN
        A(I,J)=B(I,J)
        IF(I-J) 107,139,107
    139 A(I,J)=A(I,J)-EIGV(II)
    107 CONTINUE
    105 U(I)=-B(I,NM)
C     TEST TO SEE WHETHER DIAGONAL ELEMENT IS ZERO OR NOT
C
        DO 9015 I=1,NN
```

```
      IF(I-NN) 9021,9007,9021
 9021 IF(A(I,I)-EPS) 9005,9006,9007
 9005 IF(-A(I,I)-EPS) 9006,9006,9007
 9006 U(I)=U(I)+U(I+1)
      DO 9023 J=1,NN
 9023 A(I,J)=A(I,J)+A(I+1,J)
      GO TO 9021
 9007 DIV=A(I,I)
      U(I)=U(I)/DIV
C
C   DIVIDE ALL ELEMENTS OF I-TH EQUATION BY A(I,I)
C
      DO 9009 J=1,NN
 9009 A(I,J)=A(I,J)/DIV
C
C   REDUCE THE I-TH ELEMENT OF THE OTHER EQUATIONS TO ZERO
C
      DO 9015 MM=1,NN
      DELT=A(MM,I)
 9016 IF(MM-I) 9010,9015,9010
 9010 U(MM)=U(MM)-U(I)*DELT
      DO 9011 J=1,NN
 9011 A(MM,J)=A(MM,J)-A(I,J)*DELT
 9015 CONTINUE
      Y(NM)=1.0
      DO 108 I=1,NN
  108 Y(I)=U(I)
C
C
C        CALCUTATION OF DISPLACEMENT VECTOR
C
  141 DV(1)=0.5*Y(1)
      DO 163 I=2,NM
  163 DV(I)=DV(I-1)+Y(I)
      DIV=DV(NM)
      DO 190 I=1,NM
  190 DV(I)=DV(I)/DIV
      WRITE(6,5) II
    5 FORMAT(I2,8X,16HTH. EIGEN VECTOR,5X,19HDISPLACEMENT VECTOR)
      DO 10 K=1,NM
   10 WRITE(6,79) Y(K),DV(K)
   79 FORMAT(F22.6,F15.6)
      DO 201 I=1,NM
      SUM=-EIGV(II)*Y(I)
      DO 200 J=1,NM
      SUM=SUM+B(I,J)*Y(J)
200   CONTINUE
204   FORMAT(1H ,2HI=,I2,4X,4HSUM=,F15.8)
201   WRITE(6,204) I,SUM
      IF(II-1) 151,151,152
  151 DO 143 I=1,NM
  143 DF(I)=DV(I)
      GO TO 310
C
```

```
C         TEST FO- O-T+OGONAL+TY OF E+GEN VFCTO-S                              LIN  271
C                                                                              LIN  272
  152 SUMM=0.0                                                                 LIN  273
      DO 153 I=1,NM                                                            LIN  274
  153 SUMM=SUMM+DF(I)*DV(I)*M(I,1)*2.0                                         LIN  275
      WRITE(6,154) II,SUMM                                                     LIN  276
  154 FORMAT(1H0,3HII=,I2,4X,5HSUMM=,F15.8)                                    LIN  277
C         ************************************                                 LIN  278
C         APPLICATION OF GALERKIN'S METHOD.                                    LIN  279
C         ************************************                                 LIN  280
C         MULTIPLICATION OF MASS MATRIX                                        LIN  281
  310 DO 300 I=1,NM                                                            LIN  282
      DV(I)=0.0                                                                LIN  283
      DO 300 J=1,NM                                                            LIN  284
      DV(I)=DV(I)+M(I,J)*Y(J)                                                  LIN  285
  300 CONTINUE                                                                 LIN  286
      CMASS=0.0                                                                LIN  287
      DO 301 I=1,NM                                                            LIN  288
  301 CMASS=CMASS+DV(I)*Y(I)                                                   LIN  289
      WRITE(6,302) CMASS                                                       LIN  290
  302 FORMAT('0','COEFFICIENT OF MASS MATRIX',5X,6HCMASS=,E20.8)               LIN  291
C         MULTIPLICATION OF STIFFNESS MATRIX                                   LIN  292
      DO 303 I=1,NM                                                            LIN  293
      DV(I)=0.0                                                                LIN  294
      DO 303 J=1,NM                                                            LIN  295
      DV(I)=DV(I)+S(I,J)*Y(J)                                                  LIN  296
  303 CONTINUE                                                                 LIN  297
      CSTIFF=0.0                                                               LIN  298
      DO 304 I=1,NM                                                            LIN  299
      CSTIFF=CSTIFF+DV(I)*Y(I)                                                 LIN  300
  304 CONTINUE                                                                 LIN  301
      WRITE(6,305) CSTIFF                                                      LIN  302
  305 FORMAT('0','COEFFICIENT OF STIFFNESS MATRIX',5X,7HCSTIFF=,E20.8)         LIN  303
C         MULTIPLICATION OF FORCE MATRIX                                       LIN  304
      CFORCE=0.0                                                               LIN  305
      CFORCE=Y(1)*Y(1)*FM(1)                                                   LIN  306
      DO 306 I=2,NM                                                            LIN  307
  306 CFORCE=CFORCE+(Y(I)-Y(I-1))*Y(I)*FM(I)                                   LIN  308
      WRITE(6,307) CFORCE                                                      LIN  309
  307 FORMAT('0','COEFFICIENT OF FORCE MATRIX',5X,7HCFORCE=,E20.8)             LIN  310
      GFREQ=CSTIFF/CMASS                                                       LIN  311
      GLOAD=CSTIFF/CFORCE                                                      LIN  312
      WRITE(6,308) GFREQ                                                       LIN  313
  308 FORMAT('0','FREQUENCY BY GALERKINS  METHOD',5X,6HGFREQ=,E20.8)           LIN  314
      WRITE(6,309) GLOAD                                                       LIN  315
  309 FORMAT('0','BUCKLING LOAD BY GALERKINS  METHOD',5X,6HGLOAD=,E20.8)LIN  316
  110 CONTINUE                                                                 LIN  317
      END                                                                      LIN  318
```