DYNAMICS AND STABILITY OF CURVED PIPES CONVEYING FLUID

by

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ABSTRACT

The dynamics and stability of curved pipes conveying fluid are studied in this Thesis. The pipes are usually supported at both ends, with clamped or pinned supports, but some cases of cantilevered pipes are also considered.

The governing equations of small motion of the system are derived by the Newtonian approach, in a general form that applies to all three variants of the theory:

(i) the conventional inextensible theory;

(ii) the extensible theory;

(iii) the modified inextensible theory.

In all cases, the analysis is conducted by the finite element method.

According to the conventional inextensible theory, the centerline remains inextensible and the steady-state stress resultant of flow/pressure-induced forces on the pipe does no work. As the flow is increased, buckling is predicted for pipes supported at both ends. The extensible theory does not make these assumptions; as the flow is increased, no buckling occurs according to this theory. The new theory developed in this Thesis, the modified inextensible theory, retains the assumption of zero deformation of the centerline, but does take into account the work done by the flow/pressure-induced forces on the pipe. Its predictions are close to those of the extensible theory, while being considerably simpler and much more economical.

The case of cantilevered pipes and pipes with one end free to slide axially are also studied in this Thesis. Instabilities are predicted by all three variants of the theory, but the threshold flow velocities are different, depending on which theory is being used.

SOMMAIRE

La dynamique et la stabilité des tubes courbés, transportant des fluides sont étudiées dans cette thèse. Ces tubes sont souvent supportés aux extrémités avec des supports encastrés ou appuyés, mais des cas de tubes encastrés-libres sont aussi considérés.

Les équations gouvernant les petits mouvements du système sont derivées en utilisant l'approche Newtonienne, dans une forme générale applicable aux trois variantes de la théorie:

i) la théorie conventionnelle inextensible;

ii) la théorie extensible;

iii) la théorie inextensible modifiée.

Dans tous les cas, l'analyse est basée sur la méthode des éléments finis.

D'après la théorie conventionnelle inextensible, la ligne centrale du tube reste inextensible et les contraintes stationnaires resultant des forces induites par l'écoulement et la pression interne ne font aucun travail. Quand le flux augmente, selon cette théorie les tubes supportés aux extremités deviennent instables par flambage. Ces hypothèses ne sont pas utilisées dans la théorie extensible, selon laquelle le flambage des tubes aux etrémités supportées n'a pas lieu.

La nouvelle théorie developpée dans cette thèse, la théorie inextensible modifiée, retient l'hypothèse d'une déformation nulle de

la ligne centrale, mais elle tient compte du travail fait par les forces d'écoulement et de la pression interne. Les prédictions de cette théorie sont proches à celles de la théorie extensible, mais les calculs sont considérablement plus simples et encore plus économiques.

Le cas des tubes encastrés-libres et des tubes a une extrémité supportée mais libre à glisser dans la direction axiale est aussi étudié dans cette thèse. Des instabilités sont prédites par les trois variantes de la théorie, mais les vitesses critiques du fluide sont différentes selon le cas.

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NOMENCLATURE

Ordinary Roman Symbols

A_t	Cross-sectional area of the pipe $(= \pi(D_o^2 - D_i^2)/4)$.
A_i	Internal cross-sectional area of the pipe.
A_o	External cross-sectional area of the pipe.
$\vec{a}_f$	Fluid acceleration vector.
$\vec{a}_t$	Tube acceleration vector.
a_{fx_o}, a_{fy_o}, a_{fz_o}	Components of the fluid acceleration vector referred in the coordinate system (x_o, y_o, z_o).
a_{tx}, a_{ty}, a_{tz}	Components of the tube acceleration vector referred in the coordinate system (x, y, z).
c, c'	Coefficient of viscous damping due to the surrounding fluid, associated with the transverse and axial motions, respectively.
D_i	Internal diameter of the pipe.
D_o	External diameter of the pipe.
EI	Flexural rigidity of the pipe.
$\vec{e}_{x_o}$, $\vec{e}_{y_o}$, $\vec{e}_{z_o}$	Unit vectors associated with the coordinate system (x_o, y_o, z_o).
$\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$	Unit vectors associated with the coordinate system (x, y, z).
GJ	Shear rigidity of the pipe.
G_{x_o}, G_{y_o}, G_{z_o}	Components of the effective gravity force of the pipe per unit length.

G_{fx_o}, G_{fy_o}, G_{fz_o}	Components of the gravity force of the fluid per unit length.
$G^*_{x_o}$, $G^*_{y_o}$, $G^*_{z_o}$	Components of the gravity force for the combined pipe-fluid system.
g	Acceleration due to gravity.
I_z	Mass-moment of inertia of the pipe cross-section in the z-direction
M_x, M_y, M_z	Stress couples around the axes (x,y,z)
M_{x_o}, M_{y_o}, M_{z_o}	Components around the axes (x_o, y_o, z_o) of the resultant of the stress couples.
M_a, M'_a	Added mass per unit length, in transverse and axial directions, respectively.
M_t	Mass of empty pipe per unit length.
M_f	Mass of fluid conveyed per unit length (= $\rho_f A_f$).
$[N_2]$, $[N_3]$, $[N_4]$	Shape functions of η_2^*, η_3^* and ψ^*.
n	Number of finite elements.
P	Combined force (= $A_i P_i - A_o P_e - Q_z$).
P_o	Steady state value of P.
P_e	External fluid pressure.
P_i	Internal fluid pressure.
Q_x, Q_y, Q_z	Shear forces (Q_x, Q_y) and axial force (Q_z).
Q_{x_o}, Q_{y_o}, Q_{z_o}	Components of shear forces along the axes (x_o, y_o, z_o).
R_{x_o}, R_{y_o}, R_{z_o}	Components of the reaction force per unit length referred in the system (x_o, y_o, z_o).
R_o	Radius of curvature of the pipe centerline.

$\vec{r}$	Displacement vector of the pipe centerline whose components are u, v, w.
r_o	Dimensionless parameter defined in equations (2.74), or the total angle of the pipe for the case of constant radius of curvature R_o (= L/R_o).
s	Curvilinear coordinate directed along centerline of pipe.
t	time.
U	Flow velocity of the internal fluid in the pipe.
$\bar{u}$	Dimensionless form of U associated with L (= $(\frac{M_f}{EI})^{\frac{1}{2}}$ LU).
$\bar{u}_c$	Dimensionless form of U according to Chen, associated with R_o (= $(\frac{M_f}{EI})^{\frac{1}{2}} R_oU$).
u, u_o, u*	Total, static and perturbation displacements of the pipe in the x_o-direction.
$\vec{V}_f$	Vector of fluid velocity.
$\vec{V}_t$	Vector of tube velocity.
v, v_o, v*	Total, static and perturbation displacements of the pipe in the y_o-direction.
w, w_o, w*	Total, static and perturbation displacements of the pipe in the z_o-direction.
X_o, Y_o, Z_o	Inertial Cartesian coordinates defined in section (2.1).
X, Y, Z	Inertial Cartesian coordinates defined in section (2.1).
x_o, y_o, z_o	Frenet-Serret coordinates defined in Section (2.1).
x, y, z	Torsion-flexure coordinates defined in Section (2.1).

Greek Symbols

$\alpha_{x_o}, \alpha_{y_o}, \alpha_{z_o}$	Direction cosines of gravity acceleration vector.
α_i	Generalized coordinates for in-plane displacement model.
α_i^*	Generalized coordinates for out-of-plane displacement model.
β, β_a, β_a'	Dimensionless mass parameters defined in equations (2.74).
γ	Dimensionless gravitational parameter.
δ	Variational operator.
ε	Centerline strain.
$\{\eta^*_i\}^e, \{\eta_o^*\}^e$	In-plane and out-of-plane nodal displacement vectors.
$\{\eta_i^o\}^e$	Nodal static displacement vectors.
η_i	Dimensionless total displacements of (u,v,w) defined in equations (2.74).
η_i^o	Dimensionless static displacement of (u_o, v_o, w_o) defined in equations (2.74).
η_i^*	Dimensionless perturbation displacement of (u^*, v^*, w^*).
$\bar{\theta}$	Angle between the x_o- and z_o-axes defined in Fig. 1(b).
κ, κ'	Components of curvature about the x- and y-axes of the torsion-flexure coordinate system.
κ_o, κ_o'	Components of curvature about the x_o- and y_o-axes of the Frenet-Serret system.

λ	Eigenvalue (= $i\omega$).
λ^*	Dimensionless resistance coefficient.
ξ	Dimensionless form of s (= s/L).
π	= 3.14159.
Π, Π_o	Dimensionless combined-force and its static value, respectively.
ρ_{fe}, ρ_{fi}	Density of surrounding and internal fluid.
σ	Dimensionless parameter defined in equations (2.74).
τ	Dimensionless t, defined in equations (2.74).
τ^*	Twist of pipe around the strained centerline.
τ_o	Twist of pipe around the initial centerline.
$\bar{\Phi}$	Rotation around the z-axis defined in Fig. 1(b).
ψ, ψ_o, ψ^*	Total, static and perturbation of rotation angle of the pipe cross-section.
$\bar{\psi}$	Rotation around the z_o-axis.
Ω	Radian frequency of oscillation.
ω	Dimensionless frequency associated with L $\left(= \left(\frac{M_t+M_f}{EI}\right)^{\frac{1}{2}} \Omega L^2\right)$.
ω_c	Dimensionless frequency according to Chen $\left(\left(\frac{M_t+M_f}{EI}\right)^{\frac{1}{2}} \Omega R_o^2\right)$.
ω_{ci}	Dimensionless frequency of in-plane motion.
ω_{co}	Dimensionless frequency of out-of-plane motion.
	Slenderness parameter, defined in equations (2.74).
$\mathcal{H}$, $\mathcal{H}'$	Dimensionless coefficient of viscous damping due to the surrounding fluid on the tube, defined in equations (2.74).

CHAPTER 1

INTRODUCTION

1.1 LITERATURE REVIEW

The dynamics of pipes conveying fluid has received considerable attention over the past thirty years, and continues to do so, partly because it has applications to design of pipelines, reactor system components, pump discharge lines, propellant lines, and so forth, but mainly because of its inherent interest as a fundamental problem in applied mechanics. Most studies to-date have been concerned with straight tubes; relatively little effort was directed to curved tubes.

The study of the dynamics of straight tubes conveying fluid began with an attempt by Ashley & Haviland (1950) to describe the vibrations observed in the Trans-Arabian pipeline. However, their formulation of the problem was erroneous, as shown by Feodos'yev (1951), who derived the correct equation of motion for a tube conveying fluid and analysed the case of a tube with both ends simply supported. The same problem was studied independently by Housner (1952) using a different appraoch. Both Feodos'yev and Housner found that for sufficiently high flow velocities the tube may buckle, essentially like a column subjected to axial loading. The critical flow velocities for this buckling instability were shown to be directly related to the Euler buckling load for columns. A subsequent, elegant and more general study by Niordson (1953) led to the same equation of motion and essentially the same conclusions regarding stability for tubes with simply supported ends.

The case of tubes with one end free, i.e. tubular cantilevers, containing flowing fluid was first studied theoretically in an outstanding paper by Benjamin (1961), as a limiting case of a system of articulated pipes conveying fluid, as the number of degrees of freedom tends to infinity, Benjamin was the first author to show that the dynamical problem is independent of the effect of fluid friction. The problem was further studied theoretically and experimentally by Gregory & Paidoussis (1966a,b) who considered the case of a continuously flexible cantilever. It was found that for sufficiently high flow velocities the system is subjected to flexural oscillatory instability (flutter).

The stability of tubular cantilevers conveying fluid was further discussed by Nemat-Nasser, Prasad & Herrmann (1966), with emphasis on the effect on stability of velocity-dependent forces, such as dissipative and Coriolis forces; they showed that such forces may destabilize the system, an effect also reported previously by Gregory & Paidoussis (1966a,b). Subsequent papers by Herrmann (1967) and Herrmann & Nemat-Nasser (1967) stressed the connection between the problem of a cantilever conveying fluid and the more general problem of instability of a cantilever subjected to a "follower"-type force at the free end; i.e. a force retaining the same angular disposition relative to the free end in the course of small motions of the cantilever.

An interesting variation was studied by Paidoussis (1970), namely that of a vertical tubular cantilever conveying fluid, with the free end being either below the clamped one ('hanging' cantilever) or above it ('standing' cantilever). Gravity forces in that study are not considered to be negligible, as was the case in most of the foregoing. It is shown that, when the velocity of the fluid exceeds a certain value, the cantilever in all cases becomes subject to oscillatory instability (flutter). In the case of hanging cantilevers, buckling instability does not occur at all. Standing cantilevers, on the other hand, may buckle under their own weight; it was shown that in some cases, flow (within a certain range of flow velocities) may render stable a system which would have buckled in the absence of flow.

Thurman & Mote (1969) presented a non-linear analysis for a pipe with simply-supported ends conveying fluid, using a perturbation technique. They found that, in determining the natural frequencies of the system, the importance of non-linear terms increases with flow velocity, and hence that the range of applicability of linear theory becomes more restricted. The non-linear aspects of the problem were further studied, later, by Holmes (1977, 1978) and Rousselet & Herrmann (1981).

Chen (1971a) studied the stability of a pipe conveying fluid with the upstream end clamped and the downstream end constrained by a linear spring, so that the boundary conditions are intermediate between clamped-free and clamped-pinned; accordingly, both buckling and oscillatory instabilities are possible in general, depending on the spring constant.

Paidoussis & Denise (1971, 1972) studied the dynamics of very thin elastic pipes conveying fluid, by using thin-shell theory to describe the pipe motions and potential flow theory to obtain the fluid forces. They analyzed both cantilevered pipes and pipes with clamped ends. They found that, in addition to instabilities in the beam modes of the system (corresponding to those found previously by beam theory for thicker pipes), instabilities in the shell modes are also possible, as verified by their experiments. Of particular interest was the finding that thin pipes with clamped ends are not only subject to buckling (divergence) but also to coupled-mode flutter. Similar theoretical results were obtained later by a different analytical method by Weaver & Unny (1973), in the case of simply-supported shells, and by other investigators.

Most of the investigations were concerned with pipes conveying fluid at a constant flow velocity. Chen (1971b) seems to be the first one to have studied the case of pulsating flow. He investigated the stability of simply supported pipes conveying fluid, whose flow velocity has a harmonic fluctuation about a mean value. However, an error in his equation of motion was found (Paidoussis and Issid 1974). The same problem was also studied by Ginsberg (1973), and by Paidoussis & Issid (1974), the latter study not being confined to simply supported pipes, but also dealing with pipes either clamped at both ends or cantilevered. Although the foregoing systems can generally have both parametric and combination resonances (instabilities), all the aforementioned studies were restricted to the examination of parametric resonance only. However, Bohn & Herrmann (1974) considered a two degree-of-freedom articulated pipe, and studied both parameteric and combination resonances. Moreover, the equivalent problem for a continuously flexible pipe was studied by Paidoussis & Sundararajan (1975). Two methods of analysis were presented: Bolotin's method, which can only give the boundaries of regions of parametric resonance, and a numerical Floquet analysis, which gives also the boundaries of combination resonance. A number of calculations for cantilevered pipes showed that, generally, combination resonance is less important than parametric resonance, except for flow velocities near the critical (where the system loses stability in steady flow); parametric resonances are selectively associated with only some of the modes of the system, and combination resonances involve only the difference (not sum) of the eigenfrequencies. For pipes clamped at both ends, the behaviour of the system is similar to that of a column subjected to a pulsating load; combination resonances in this case involve the sum of the eigenfrequencies. Good agreement with theory was found in the only experiments available in this area, by Paidoussis & Issid (1976).

Recently, Hannoyer & Paidoussis (1978) studied the dynamics and stability of cylindrical tubular beams conveying fluid and simultaneously subjected to axial external flow. In deriving the equation of small motions, inviscid hydrodynamic forces were obtained by slender-body theory, modified to account for the boundary-layer thickness of the external flow; internal dissipation and gravity effects were also taken into account. Solutions were obtained by means of a method similar to Galerkin's method, with eigenfunctions approximated by Fourier series. They found that, in the case of tubular beams supported at both ends, the system eventually loses stability by divergence (buckling), when either the internal or external flow velocity, or both, become sufficiently large. In the case of cantilevered tubular beams, the behaviour of the system is much more complex, depending on the shape of the downstream end. Cantilevered beams with a blunt free end can only lose stability by flutter, in a process dominated by the internal flow; if the free end is streamlined, however, a complex sequence of buckling and flutter instabilities may result, where both internal and external flow effects come into play.

More recently, Luu & Paidoussis (1983) examined the dynamics and stability of a long, vertically disposed, cantilevered pipe, submerged in and aspirating fluid from the free lower end, and conveying it upwards to the supported upper end; the pipe had a large mass attached to its free end. The Euler-Bernoulli beam theory and a Galerkin-like method of solution were used in the analysis. It was found that the system is inherently unstable, in the sense that, if dissipation were totally absent, it would lose stability at infinitesimally small internal flow velocities.

The foregoing is a literature survey of some of the important papers dealing with the dynamics of straight tubes conveying fluid. More complete and extensive literature surveys are given by Chen (1977), Paidoussis (1980), and Paidoussis & Issid (1974) for beam-like (thick) pipes conveying fluid; in the case of thin shell-like pipes and coaxial shells conveying fluid, a complete literature survey is found in a recent paper by Paidoussis, Chan & Misra (1984).

Let us now turn our attention to previous work in the area of concern of this Thesis, namely on the dynamics and stability of curved pipes conveying fluid.

Geometrical considerations make the analysis of curved tubes conveying fluid somewhat more difficult. The natural frequencies of a curved rod were studied by Den Hartog (1928), Archer (1960), Nelson (1962), and Ojalvo & Newman (1968). Among the first to study hydroelastic vibrations of curved tubes was Svetlitskii (1966). He investigated the out-of-plane motion of a fluid-conveying perfectly flexible hose with fixed ends, whose initial shape was a catenary. Unny et al. (1970), considered the in-plane buckling of initially-circular tubes with fixed ends. The equations of motion were derived using Hamilton's principle, and critical flow velocities for instability were obtained for pinned and clamped ends, but the equations obtained were subsequently found to be incomplete (Chen 1972a).

The same problem for tubes shaped in the form of circular arcs was further studied by Chen (1972a, 1972b, 1973). He derived equations governing the in-plane motion via a Newtonian formulation (Chen 1972a) and via a Hamiltonian formulation (Chen 1972b) and equations governing the out-of-plane motion form the Hamiltonian viewpoint (Chen 1972b, 1973). In both cases, Chen made the assumptions of inextensibility of the axis of symmetry, to simplify the equations of motion. It was shown that in the case of in-plane motion the tube becomes subject to buckling-type instability for clamped-clamped, pinned-pinned, and clamped-pinned end conditions, when the flow velocity or the fluid pressure exceeds a certain value (Chen 1972a, 1972b); in the case of out-of-plane motion, the tube also becomes subject to buckling-type instability for the clamped-clamped and pinned-pinned end conditions, but generally at lower flow velocities than those necessary to cause instability for in-plane motions.

Chen also studied the stability of cantilevered curved tubes. He found that for in-plane motions such tubes are generally subject to both buckling and flutter instabilities, with buckling occurring at lower flow velocities; except in cases where the subtended angle is very small, when only flutter was found to arise (Chen 1972b). In the case of out-of-plane motions, only flutter was found to occur, with stability characteristics similar to those for a straight tube (Chen 1973). Earlier, Springfield (1970) had also presented an analysis of the in-plane and out-of-plane hydroelastic vibration of unpressurized circular arcs with clamped-clamped and clamped-free ends.

More recently Hill & Davis (1974) studied the dynamics and stability of curved and clamped-clamped tubes conveying fluid, shaped as circular arcs, S- and L-shaped tubes and spiral configurations. They obtained the equations of motion by the Newtonian force-balance approach, but with the following significant difference to Unny's et al. and Chen's work: they included the effect of initial forces (due to pressure and centrifugal forces) in the equations of motion. The importance of these terms was first brought to light by Svetliskii (otherwise transcribed as Svetlisky), in 1969 in Russian and later in English (Svetlisky 1977). Hill & Davis used the finite element technique to solve the problem, whereas Svetliskii used a Galerkin-type solution. The interesting result was obtained by both sets of investigators that, if these initial forces are taken into account, then tubes with both ends supported will not lose stability by buckling, no matter how high the flow velocity may be. On the other hand, Svetliskii (1977) finds that cantilevered tubes will lose stability by flutter at sufficiently high flow velocities.

Doll & Mote (1974, 1976) also studied the same problem. They obtained the equations of motion for an even more general case, i.e. that where the tube is both curved and twisted, via Hamilton's principle, and obtained solutions by the finite element method. Doll & Mote considered two forms of the equations of motion: (i) an "initial curvature formulation", which corresponds to Unny's et al. and Chen's inextensibility assumption, and (ii) an extensional formulation, based on initial axial stresses induced by a continually varying equilibrium curvature - in this case arising again from pressure and centrifugal forces. One of the fundamental differences in approach between Hill & Davis' and Doll & Mote's work is that the former calculate the equilibrium configuration (and forces) via a linearized set of equations, on the assumption that the initial form and the flow-deformed equilibrium form are close; the latter, on the other hand, utilizes a cumulative application of a linearization for small flow velocity increments, which is more general.

Doll & Mote (1974, 1976) although using equations somewhat different from those of Hill & Davis' or Svetliskii's nevertheless came to the same basic conclusions. If the initial stresses induced by the flow are taken into account, tubes supported at both ends are not subject to instability, but cantilevers may lose stability by buckling and flutter. For tubes supported at both ends, their results are quite close to those of Hill & Davis', showing that the eigenfrequencies are not very sensitive to flow velocity. On the other hand, if the initial stresses induced by the flow are neglected, then they obtain results which are very similar, qualitatively and quantitatively, to Chen's.

Doll & Mote also compare their results to experimental data by Liu & Mote (1974); these experiments were conducted with very slightly curved tubes, supported at both ends. Surprisingly, either the straight-tube theoretical results or, better, the inextensible theory gave better agreement with experiments than the extensible theory which Doll & Mote consider to be the most correct!

Finally, some very recent, important work on the dynamics of curved pipes conveying fluid by Dupuis & Rousselet (1985) has come to the author's attention at the time of writing this Thesis. This study deals with cantilevered curved pipes by using the transfer matrix method, in preference to either analytical or finite element techniques. Once more, flutter instabilities are predicted for cantilevered curves tubes.

The above brief and selective review of literature is intended to give the reader some idea of the developments in the subject of dynamics of cylindrical pipes conveying fluid; no attempt has been made to give an exhaustive list of all papers in this area. Moreover, vigorous research is being pursued on many other aspects of the problem, which has not been mentioned and is out of the limited scope of the present work.

1.2 OBJECTIVE AND ORGANIZATION OF THE THESIS

The objective of this thesis is to study the dynamics and stability of curved pipes, conveying fluid (and submerged in fluid). One of the motivations of this work is related to the fact that the previous studies of this topic (i.e. by Unny et al. (1970), Chen (1972a, b, 1973), Hill & Davis (1974) and Doll & Mote (1974, 1976) obtained significantly different results by different approaches; as mentioned in the foregoing literature survey. It is hoped to throw some light on the reasons for the drastically different dynamical behaviour of the system predicted by the aforementioned investigators, and hopefully come up with a theoretical model which is more correct than any of the foregoing, or at least one that is as correct as one of them - in which case, pin-pointing which one may be considered to be truly correct!

In this thesis the pipes are considered to be initially planar with arbitrary center-line shape. However, the radius of curvature of the centerline and the overall length of the pipe are assumed to be sufficiently large in comparison with the radius of pipe; thus, according to Timoshenko & von Kármán (1961), the effects of shear deformation and rotatory inertia can be neglected and plane sections may be considered to remain plane after deformation. In addition, the fluid flow may be assumed to be approximately a plug flow, the fluid being essentially an infinitely flexible "rod" travelling through the pipe

(Paidoussis, Luu & Laithier 1986). Hence the effects of secondarly flow may be neglected.

In line with the basic aims of this thesis, three sets of equations are generated, and calculations are conducted with all of them, selectively. The first two are based on the assumption of inextensibility of the centerline; in the first variant, the effect of the initial forces is neglected, thus being close to Chen's theory; in the second, the effect of the initial pressure - centrifugal forces is taken into account. The third formulation is a truly extensible formulation, where the assumption that the length of the centerline remains constant is no longer made, and initial forces are taken fully into account; thus this formulation is based on the same basic postulate as Hill & Davis' and Doll & Mote's.

The structure of this thesis is as follows:

In Chapter II, the kinematic relations of deformation of the pipe are derived in terms of a three-dimensional rod theory. The curved and straight pipe configurations result from specific simplifications in the general development. Subsequently, the pipe geometry is completely described by the curvature-torsion relationships in terms of three parameters (two curvatures and the twist). Then the equations of motion of the system are derived using the Newtonian approach. Finally, the equations of static equilibrium are obtained by deleting the time-derivative terms.

In Chapter III are presented (i) the analysis of the system for the case where the centerline of the pipe is assumed to be inextensible, (ii) the discretization of the system using the finite element technique, and (iii) the calculation of the so-called "combined force", which is the name given in this thesis to the initial forces induced by the fluid flow and the gravity field.

In Chapter IV are discussed the finite-element results obtained for the inextensible case. Extensive calculations for in-plane and out-of-plane motions are presented for various configurations, with the two variants of the theory, namely (a) ignoring the combined force (initial forces) and (b) taking the combined force into account. Extensive testing of the theory against results obtained by other researchers is given in this Chapter.

In Chapter V is presented the analysis of the system for the extensible case, followed by the discretization scheme, once more utilizing the finite-element technique.

In Chapter VI are discussed the finite element results obtained for the extensible case. Extensive calculations for the in-plane and out-of-plane case are again presented for various configurations. In this Chapter the results obtained by the different forms of the theory and the different sets of equations of Chapter III and V are compared among themselves, as well as to those obtained by the extensional theories of Doll & Mote's and Hill & Davis'.

Finally, in Chapter VII, general conclusions and some suggestions for future work are presented.

CHAPTER II

FORMULATION OF THE PROBLEM

2.1 KINEMATICAL FORMULAE

Consider a pipe, curved in one plane only, conveying fluid, as shown in Fig. 1. The kinematics of the curved pipe may be developed by the same approach as that used by Love (1944) for the curved "rod". This can be accomplished easily, provided that the external diameter of the pipe is small compared to the radius of curvature of the pipe centerline and to the overall length of the pipe (i.e., provided that the effect of shell-type and shear deformation can be neglected, and plane sections may be considered to remain plane after deformation). In order to describe the kinematics of the problem, we shall use the following reference frames:

(x_o, y_o, z_o) – In this reference frame, the origin, P_o, is on the initial (unstrained) centerline, the z_o-axis is directed along the tangent of the initial centerline, and the axes x_o and y_o are directed along the principal normal and binormal axes of this line, respectively, as shown in Fig. 1(a). When the origin of this frame moves along the centerline with unit velocity, the triad of axes will rotate with an angular velocity, the components of which, referred to the instantaneous position of the axes, will be denoted by κ_o, κ_o', and τ_o. Then, κ_o and κ_o' are the components of the initial curvature, and τ_o is the initial twist. Moreover, since in this work the pipe is considered to be initially plane and untwisted, one can set $\kappa_o = 0$, $\kappa'_o = 1/R_o$ and $\tau_o = 0$. This system is called the Frenet-Serret system, and has unit vectors denoted by $\bar{e}_{x_o}$, $\bar{e}_{y_o}$ and $\bar{e}_{z_o}$.

(x,y,z) This reference system has its origin at P_1, which is the displaced position of P_o. Because the pipe is not initially straight, its deformation will generally introduce twisting and out-of-plane deformation. The z-axis is tangential to the deformed (strained) centerline at P_1, and the plane (x,z) is the tangential plane of the surface made up of the aggregate of particles which, in the initial state, lie in the plane of (x_o, z_o); the y-axis is determined by the condition that the (x,y,z)-axes form a right-handed system, as shown in Fig. 1(a). Therefore, the origin P_1 of this reference system moves along the deformed centerline, and the triad of axes will rotate with an angular velocity, the components of which, referred in the instantaneous position of the axes, will be denoted by κ, κ' and τ^*. This system is called the torsion-flexure system, and is associated with unit vectors denoted by $\bar{e}_x$, $\bar{e}_y$, $\bar{e}_z$.

(X_o, Y_o, Z_o) This is an inertial system, the origin of which is located at P_o, and which is sometimes set to coincide with (x_o, y_o, z_o).

(X,Y,Z) This is another inertial system, the origin of which is located at P_1, and sometimes it is set to coincide with (x,y,z).

When the pipe is slightly deformed, any particle P of the centerline undergoes a small displacement, the components of which, referred to the system (x_o,y_o,z_o) with origin at P_o, will be denoted by u,v,w. Furthermore, the orientation of the system (x,y,z) may be completely determined by the three Eulerian angles $\bar{\psi}$, $\bar{\theta}$ and $\bar{\phi}$ shown in Fig. 1(b). $\bar{\psi}$ is the rotation around the z_o-axis, $\bar{\theta}$ is the angle between the z_o- and z-axes, while $\bar{\phi}$ is the rotation around the z-axis. Because the angle $\bar{\theta}$ is assumed small in this work (small deformation assumption), the angle between axes x_o and x may be written as

$$\psi = \bar{\psi} + \bar{\phi} \,, \tag{2.1}$$

where ψ may be considered as the angle through which a plane section of the pipe is rotated around the centerline. Hence, the stressed state of the pipe is determined by the generalized coordinates u,v,w and ψ.

The relative orientation of the two systems (x_o,y_o,z_o) and (x,y,z) may be determined by the following orthogonal transformation:

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 1 & \psi & -(\frac{\partial u}{\partial s} + \frac{w}{R_o}) \\ -\psi & 1 & -\frac{\partial v}{\partial s} \\ (\frac{\partial u}{\partial s} + \frac{w}{R_o}) & \frac{\partial v}{\partial s} & 1 \end{bmatrix} \begin{Bmatrix} x_o \\ y_o \\ z_o \end{Bmatrix} \,, \tag{2.2}$$

as shown in Appendices A and B. Relation (2.2) implies the assumption that the deformations are small.

The components of curvature and twist in the deformed state may also be expressed in terms of the displacements, as derived in Appendix C, as follows:

$$\kappa = \left(\frac{\psi}{R_o} - \frac{\partial^2 v}{\partial s^2}\right) , \tag{2.3}$$

$$\kappa' = \left(\frac{1}{R_o} + \frac{\partial^2 u}{\partial s^2} + \frac{1}{R_o}\frac{\partial w}{\partial s}\right), \tag{2.4}$$

$$\tau^* = \left(\frac{\partial \psi}{\partial s} + \frac{1}{R_o}\frac{\partial v}{\partial s}\right). \tag{2.5}$$

Finally, to complete the kinematic development, the velocity and acceleration of the internal fluid will be derived. The fluid flow may be assumed to be approximately a plug flow, the fluid being essentially an infinitely flexible "rod" travelling through the pipe. As it has been assumed that the radius of curvature is very large compared with the pipe radius, the effects of secondary flow are neglected.

The displacement vector of the deformed centerline, expressed in the inertial reference system that coincides with the system (x_o, y_o, z_o), is given by

$$\vec{r} = u\,\vec{e}_{x_o} + v\,\vec{e}_{y_o} + w\,\vec{e}_{z_o} . \tag{2.6}$$

By differentiating equation (2.6), the velocity and acceleration of the pipe may be written as

$$\vec{V}_t = \frac{\partial u}{\partial t}\,\vec{e}_{x_o} + \frac{\partial v}{\partial t}\,\vec{e}_{y_o} + \frac{\partial w}{\partial t}\,\vec{e}_{z_o} , \tag{2.7}$$

$$\vec{a}_t = \frac{\partial^2 u}{\partial t^2}\,\vec{e}_{x_o} + \frac{\partial^2 v}{\partial t^2}\,\vec{e}_{y_o} + \frac{\partial^2 w}{\partial t^2}\,\vec{e}_{z_o} . \tag{2.8}$$

Therefore, the absolute velocity of the internal fluid is

$$\vec{V}_f = \vec{V}_t + U\,\vec{e}_z, \tag{2.9}$$

where $\vec{e}_z$ is the unit vector along the z-axis, which is tangential to the strained centerline. From equation (2.2), one can obtain

$$\vec{e}_z = (\frac{\partial u}{\partial s} + \frac{w}{R_o}) \vec{e}_{x_o} + \frac{\partial v}{\partial s} \vec{e}_{y_o} + \vec{e}_{z_o} \quad . \qquad (2.10)$$

Combining equations (2.7), (2.9), and (2.10) yields the velocity

$$\vec{V}_f = [\frac{\partial u}{\partial t} + U(\frac{\partial u}{\partial s} + \frac{w}{R_o})]\vec{e}_{x_o} + [\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial s}]\vec{e}_{y_o} + [\frac{\partial w}{\partial t} + U]\vec{e}_{z_o} . \qquad (2.11)$$

To obtain the acceleration of the fluid, we differentiate the fluid velocity $\vec{V}_f$, yielding

$$\vec{a}_f = \frac{\partial \vec{V}_f}{\partial t} + (\vec{V}_f \cdot \vec{\nabla})\vec{V}_f , \qquad (2.12)$$

where $\vec{\nabla}$ is the gradient operator.

Substituting equations (2.9) into equation (2.12), the acceleration of the fluid can be rewritten as follows:

$$\vec{a}_f = \frac{\partial \vec{V}_f}{\partial t} + U(\vec{e}_z \cdot \vec{\nabla})\vec{V}_f + (\vec{V}_t \cdot \vec{\nabla})\vec{V}_f . \qquad (2.13)$$

Now examining the last term on the right-hand side of equation (2.13), we can rewrite it as

$$(\vec{V}_t \cdot \vec{\nabla})\vec{V}_f = (\frac{\partial u}{\partial t} \frac{\partial}{\partial x_o} + \frac{\partial v}{\partial t} \frac{\partial}{\partial y_o} + \frac{\partial w}{\partial t} \frac{\partial}{\partial z_o})\vec{V}_f ; \qquad (2.14)$$

then combining equations (2.11) and (2.14), we can see that this term is of higher order, and can be neglected. In addition we note that

$$\vec{e}_z \cdot \vec{\nabla} = \frac{\partial}{\partial z} , \qquad \frac{\partial}{\partial z} = \frac{\partial}{\partial s} , \qquad (2.15)$$

where s is the coordinate measured along the pipe centerline.

Hence, we obtain

$$\vec{a}_f = \frac{\partial \vec{V}_f}{\partial t} + U\frac{\partial \vec{V}_f}{\partial s} . \qquad (2.16)$$

Substituting equation (2.11) into equation (2.16), we can write the fluid acceleration in the x_o-, y_o- and z_o-directions, as follows:

$$a_{fx_o} = \frac{\partial^2 u}{\partial t^2} + 2U(\frac{\partial^2 u}{\partial t \partial s} + \frac{1}{R_o}\frac{\partial w}{\partial t}) + U^2(\frac{\partial^2 u}{\partial s^2} + \frac{1}{R_o}\frac{\partial w}{\partial s} + \frac{1}{R_o}), \qquad (2.17)$$

$$a_{fy_o} = \frac{\partial^2 v}{\partial t^2} + 2U\frac{\partial^2 v}{\partial t \partial s} + U^2\frac{\partial^2 v}{\partial x^2}, \qquad (2.18)$$

$$a_{fz_o} = \frac{\partial^2 w}{\partial t^2} + U(\frac{\partial^2 w}{\partial t \partial s} - \frac{1}{R_o}\frac{\partial u}{\partial t}) - \frac{U^2}{R_o}(\frac{\partial u}{\partial s} + \frac{w}{R_o}). \qquad (2.19)$$

Details of the derivation of equations (2.17)-(2.19) may be found in Appendix D.

2.2 THE EQUATIONS OF MOTION OF THE PIPE

The system under consideration is shown in Fig. 1(a). It consists of a uniform curved pipe of length L, cross sectional area A_t, mass per unit length M_t, flexural rigidity EI and shear modulus G; the pipe is initially plane, with an arbitrary centerline shape (with a radius of curvature not necessarily constant along its length); it conveys a stream of fluid of mass M_t per unit length and velocity U; furthermore, the pipe is considered to be fully submerged in a quiescent fluid.

Consider now an infinitesimal element of the pipe, contained between two cross-sections normal to the deformed centerline, and the forces and moments acting on it, as shown in Fig. 2. Balance of forces and moments along the directions of x_o, y_o, z_o yields

$$\frac{\partial}{\partial s} Q_{x_o} - \tau_o Q_{y_o} + \kappa_o' Q_{z_o} - c\frac{\partial u}{\partial t} + R_{x_o} + G_{x_o} - (M_t+M_a)a_{tx_o} = 0, \qquad (2.20)$$

$$\frac{\partial}{\partial s} Q_{y_o} - \kappa_o Q_{z_o} + \tau_o Q_{x_o} - c\frac{\partial v}{\partial t} + R_{y_o} + G_{y_o} - (M_t+M_a)a_{ty_o} = 0, \qquad (2.21)$$

$$\frac{\partial}{\partial s} Q_{z_o} - \kappa_o' Q_{x_o} + \kappa_o Q_{y_o} - c'\frac{\partial w}{\partial t} + R_{z_o} + G_{z_o} - (M_t+M_a')a_{tz_o} = 0, \qquad (2.22)$$

$$\frac{\partial}{\partial s} M_{x_o} - \tau_o M_{y_o} + \kappa_o' M_{z_o} + \frac{\partial v}{\partial s} Q_{z_o} - Q_{y_o} = 0, \qquad (2.23)$$

$$\frac{\partial}{\partial s} M_{y_o} - \kappa_o M_{z_o} + \tau_o M_{x_o} + Q_{x_o} - (\frac{\partial u}{\partial s} + \frac{w}{R_o}) Q_{z_o} = 0, \qquad (2.24)$$

$$\frac{\partial}{\partial s} M_{z_o} - \kappa_o' M_{x_o} + \kappa_o M_{y_o} + (\frac{\partial u}{\partial s} + \frac{w}{R_o})Q_{y_o} - \frac{\partial v}{\partial s} Q_{x_o} - I_z \frac{\partial^2 \psi}{\partial t^2} = 0. \qquad (2.25)$$

Here Q_{x_o}, Q_{y_o}, Q_{z_o} are components, along the axes (x_o,y_o,z_o) of the resultant of the transverse shear forces Q_x, Q_y, and of the combined force Q_z^* arising from the axial force Q_z and the external pressure force A_oP_e; M_{x_o}, M_{y_o}, M_{z_o} are components around the axes (x_o,y_o,z_o), of the resultant of the bending moments M_x, M_y and the twisting couple M_z; M_a is the added mass per unit length, and c the coefficient of viscous damping due to the surrounding fluid, associated with the transverse motion; M_a' and c' play similar roles in longitudinal motion; R_{x_o}, R_{y_o}, R_{z_o} are the components of the reaction force per unit length arising from the internal fluid, and G_{x_o}, G_{y_o}, G_{z_o} are the components of the effective gravity force. Details of the derivation of equations (2.20)-(2.25) may be found in Appendix E.

From the relation between the systems (x_o,y_o,z_o) and (x,y,z) given by equation (2.2), one can obtain

$$Q_{x_o} = Q_x - \psi Q_y + (\frac{\partial u}{\partial s} + \frac{w}{R_o})Q_z^* , \qquad (2.26)$$

$$Q_{y_o} = Q_y + \psi Q_x + \frac{\partial v}{\partial s} Q_z^*, \tag{2.27}$$

$$Q_{z_o} = Q_z^* - (\frac{\partial u}{\partial s} + \frac{w}{R_o}) Q_x - \frac{\partial v}{\partial s} Q_y, \tag{2.28}$$

$$M_{x_o} = M_x - \psi M_y + (\frac{\partial u}{\partial s} + \frac{w}{R_o}) M_z, \tag{2.29}$$

$$M_{y_o} = M_y + \psi M_x + \frac{\partial v}{\partial s} M_z, \tag{2.30}$$

$$M_{z_o} = M_z - (\frac{\partial u}{\partial s} + \frac{w}{R_o}) M_x - \frac{\partial v}{\partial s} M_y, \tag{2.31}$$

where

$$Q_z^* = Q_z + A_o P_e, \tag{2.32}$$

Substituting equations (2.26)-(2.31) and the values of κ_o, κ_o' and τ_o into equations (2.20)-(2.25), the equations of motion for the pipe may be written in the form

$$\frac{\partial}{\partial s}[Q_x - \psi Q_y + (\frac{\partial u}{\partial s} + \frac{w}{R_o}) Q_z^*] + \frac{1}{R_o}[Q_z^* - (\frac{\partial u}{\partial s} + \frac{w}{R_o}) Q_x - \frac{\partial v}{\partial s} Q_y] - c\frac{\partial u}{\partial t}$$
$$+ R_{x_o} + G_{x_o} - (M_t + M_a) a_{tx_o} = 0, \tag{2.33}$$

$$\frac{\partial}{\partial s}[Q_y + \psi Q_x + \frac{\partial v}{\partial s} Q_z^*] - c\frac{\partial v}{\partial t} + R_{y_o} + G_{y_o} - (M_t + M_a) a_{ty_o} = 0, \tag{2.34}$$

$$\frac{\partial}{\partial s}[Q_z^* - (\frac{\partial u}{\partial s} + \frac{w}{R_o}) Q_x - \frac{\partial v}{\partial s} Q_y] - \frac{1}{R_o}[Q_x - \psi Q_y + (\frac{\partial u}{\partial s} + \frac{w}{R_o}) Q_z^*] - c\frac{\partial w}{\partial t}$$
$$+ R_{z_o} + G_{z_o} - (M_t + M_a') a_{tz_o} = 0, \tag{2.35}$$

$$\frac{\partial}{\partial s}[M_x - \psi M_y + (\frac{\partial u}{\partial s} + \frac{w}{R_o}) M_z] + \frac{1}{R_o}[M_z - (\frac{\partial u}{\partial s} + \frac{w}{R_o}) M_x - \frac{\partial v}{\partial s} M_y] - Q_y - \psi Q_x = 0, \tag{2.36}$$

$$\frac{\partial}{\partial s}[M_y + \psi M_x + \frac{\partial v}{\partial s} M_z] + Q_x - \psi Q_y = 0, \tag{2.37}$$

$$\frac{\partial}{\partial s}[M_z - (\frac{\partial u}{\partial s} + \frac{w}{R_o}) M_x - \frac{\partial v}{\partial s} M_y] - \frac{1}{R_o}[M_x - \psi M_y + (\frac{\partial u}{\partial s} + \frac{w}{R_o}) M_z]$$
$$+ (\frac{\partial u}{\partial s} + \frac{w}{R_o}) Q_y - \frac{\partial v}{\partial s} Q_x - I_z \frac{\partial^2 \psi}{\partial t^2} \cdot \tag{2.38}$$

2.3 THE EQUATIONS OF MOTION OF THE FLUID

Consider now an infinitesimal element of the fluid and forces acting on it, as shown in Fig. 3. Let the internal cross-sectional area be A_i, and the internal pressure $P_i(s)$. Components of the pressure force $-A_iP_i$, in the directions (x_o, y_o, z_o) can be written in terms of the direction cosines (L_{31}, L_{32}, L_{33}) of the unit vector $\vec{e}_z$, referred in the reference frame (x_o, y_o, z_o), as follows:

$$\left.\begin{aligned} P_{x_o} &= -A_iP_i\left(\frac{\partial u}{\partial s} + \frac{w}{R_o}\right), \\ P_{y_o} &= -A_iP_i\frac{\partial v}{\partial s}, \\ P_{z_o} &= -A_iP_i. \end{aligned}\right\} \quad (2.39)$$

By comparing Fig. 2(b) with Fig. 3, and the forces $(P_{x_o}, P_{y_o}, P_{z_o})$ with $(Q_{x_o}, Q_{y_o}, Q_{z_o})$, and then using equations (2.20)-(2.22), the equations of motion of the fluid can be written in the form

$$\left.\begin{aligned} \frac{\partial P_{x_o}}{\partial s} - \tau_oP_{y_o} + \kappa_o'P_{z_o} - R_{x_o} + G_{fx_o} - M_fa_{fx_o} &= 0, \\ \frac{\partial P_{y_o}}{\partial s} - \kappa_oP_{z_o} + \tau_oP_{x_o} - R_{y_o} + G_{fy_o} - M_fa_{fy_o} &= 0, \\ \frac{\partial P_{z_o}}{\partial s} - \kappa_o'P_{x_o} + \kappa_oP_{y_o} - R_{z_o} + G_{fy_o} - M_fa_{fz_o} &= 0, \end{aligned}\right\} \quad (2.40)$$

where G_{fx_o}, G_{fy_o}, G_{fz_o} are components of the gravity force per unit length on the fluid.

Combination of equations (2.39)-(2.40) yields

$$\frac{\partial}{\partial s}\left[-A_iP_i\left(\frac{\partial u}{\partial s} + \frac{w}{R_o}\right)\right] - \frac{1}{R_o}A_iP_i - R_{x_o} + G_{fx_o} - M_fa_{fx_o} = 0, \quad (2.41)$$

$$\frac{\partial}{\partial s}\left[-A_iP_i\frac{\partial v}{\partial s}\right] - R_{y_o} + G_{fy_o} - M_f a_{fy_o} = 0, \tag{2.42}$$

$$\frac{\partial}{\partial s}\left[-A_iP_i\right] + \frac{1}{R_o}A_iP_i\left(\frac{\partial u}{\partial s} + \frac{w}{R_o}\right) - \ddot{R}_{z_o} + G_{fz_o} - M_f a_{fz_o} = 0. \tag{2.43}$$

2.4 THE RELATION BETWEEN THE TRANSVERSE SHEAR FORCES AND THE DISPLACEMENTS

According to the generalization of the Euler-Bernoulli beam theory, the stress couples M_x, M_y, M_z in the beam when bent and twisted from the state expressed by $\kappa_o, \kappa_o', \tau_o$ to that expressed by κ, κ', τ^* would be given by the formulae

$$\left.\begin{aligned} M_x &= EI(\kappa-\kappa_o)\ , \\ M_y &= EI(\kappa'-\kappa_o')\ , \\ M_z &= GJ(\tau^*-\tau_o). \end{aligned}\right\} \tag{2.44}$$

From the discretization to be introduced later, the pipe is divided into a series of constant curvature elements, i.e., within a given beam element

$$\frac{\partial R_o}{\partial s} = 0, \tag{2.45}$$

where the number of elements required will depend on the shape of the pipe centerline and the accuracy desired.

Combining equations (2.3), (2.5) and (2.44) yields

$$M_x = EI\left(\frac{\psi}{R_o} - \frac{\partial^2 v}{\partial s^2}\right), \tag{2.46}$$

$$M_y = EI\left(\frac{\partial^2 u}{\partial s^2} + \frac{1}{R_o}\frac{\partial w}{\partial s}\right), \tag{2.47}$$

$$M_z = GJ\left(\frac{\partial \psi}{\partial s} + \frac{1}{R_o}\frac{\partial v}{\partial s}\right). \tag{2.48}$$

Then, substituting equations (2.46)-(2.48) into (2.36)-(2.37), and neglecting the higher order terms, one may obtain

$$Q_x = \psi Q_y - EI\left(\frac{\partial^3 u}{\partial s^3} + \frac{1}{R_o}\frac{\partial^2 w}{\partial s^2}\right) , \tag{2.49}$$

$$Q_y = -\psi Q_x + EI\left(\frac{1}{R_o}\frac{\partial \psi}{\partial s} - \frac{\partial^3 v}{\partial s^3}\right) + \frac{GJ}{R_o}\left(\frac{\partial \psi}{\partial s} + \frac{1}{R_o}\frac{\partial v}{\partial s}\right) . \tag{2.50}$$

We can see that, if we again combine equations (2.49) and (2.50) and neglect higher order terms, the transverse shear forces may be written as follows:

$$Q_x = - EI\left(\frac{\partial^3 u}{\partial s^3} + \frac{1}{R_o}\frac{\partial^2 w}{\partial s^2}\right) , \tag{2.51}$$

$$Q_y = EI\left(\frac{1}{R_o}\frac{\partial \psi}{\partial s} - \frac{\partial^3 v}{\partial s^3}\right) + \frac{GJ}{R_o}\left(\frac{\partial \psi}{\partial s} + \frac{1}{R_o}\frac{\partial v}{\partial s}\right) . \tag{2.52}$$

2.5 THE EQUATIONS OF MOTION OF THE FLUID-PIPE SYSTEM

By adding equations (2.33), (2.34) and (2.35) to (2.41), (2.42) and (2.43) respectively, one may obtain the equations of motion of the system, which no longer depend on the reaction forces R_{x_o}, R_{y_o} and R_{z_o} between the pipe and the fluid, as follows:

$$\frac{\partial}{\partial s}\left[Q_x - \psi Q_y + \left(\frac{\partial u}{\partial s} + \frac{w}{R_o}\right)(Q_z + A_o P_e - A_i P_i)\right] + \frac{1}{R_o}\left[(Q_z + A_o P_e - A_i P_i)\right.$$
$$\left. - \left(\frac{\partial u}{\partial s} + \frac{w}{R_o}\right)Q_x - \frac{\partial v}{\partial s} Q_y\right] + G^*_{x_o}$$
$$= (M_t + M_a)a_{tx_o} + M_f a_{fx_o} + c\,\frac{\partial u}{\partial t} , \tag{2.53}$$

$$\frac{\partial}{\partial s}\left[Q_y + \psi Q_x + \frac{\partial v}{\partial s}(Q_z + A_o P_e - A_i P_i)\right] + G^*_{y_o} = (M_t + M_a)a_{ty_o} + M_f a_{fy_o} + c\,\frac{\partial v}{\partial t} , \tag{2.54}$$

$$\frac{\partial}{\partial s}\,[(Q_z+A_oP_e-A_iP_i)-(\frac{\partial u}{\partial s}+\frac{w}{R_o})Q_x-\frac{\partial v}{\partial s}\,Q_y]-\frac{1}{R_o}[Q_x-\psi Q_y+(\frac{\partial u}{\partial s}+\frac{w}{R_o})$$

$$x\ (Q_z+A_oP_e-A_iP_i)]+G^*_{z_o}=(M_t+M_a')a_{tz_o}+M_fa_{fz_o}+c'\frac{\partial w}{\partial t}\,, \tag{2.55}$$

$$\frac{\partial}{\partial s}\,[M_z-(\frac{\partial u}{\partial s}+\frac{w}{R_o})M_x-\frac{\partial v}{\partial s}\,M_y]-\frac{1}{R_o}[M_x-\psi M_y+(\frac{\partial u}{\partial s}+\frac{w}{R_o})M_z]$$

$$+(\frac{\partial u}{\partial s}+\frac{w}{R_o})Q_y-\frac{\partial v}{\partial s}\,Q_x=I_z\,\frac{\partial^2\psi}{\partial t^2}\,, \tag{2.56}$$

where

$$\left.\begin{aligned} G^*_{x_o} &= G_{x_o}+G_{fx_o}\,,\\ G^*_{y_o} &= G_{y_o}+G_{fy_o}\,,\\ G^*_{z_o} &= G_{z_o}+G_{fz_o}\,,\end{aligned}\right\} \tag{2.57}$$

are the components of the gravity force for the combined pipe-fluid system.

Finally, utilizing equations (2.8), (2.17)-(2.19), (2.46)-(2.48) (2.51)-(2.52) and (2.53)-(2.56) and neglecting higher order terms, the governing equations of motion for the dynamic system may be obtained, namely:

$$EI\,(\frac{\partial^4 u}{\partial s^4}+\frac{1}{R_o}\,\frac{\partial^3 w}{\partial s^3})+\frac{\partial}{\partial s}[(A_iP_i-A_oP_e-Q_z)\,(\frac{\partial u}{\partial s}+\frac{w}{R_o})]+\frac{1}{R_o}(A_iP_i-A_oP_e-Q_z)-G^*_{x_o}$$

$$+M_fU^2\,(\frac{\partial^2 u}{\partial s^2}+\frac{1}{R_o}\,\frac{\partial w}{\partial s}+\frac{1}{R_o})+2M_fU\,(\frac{\partial^2 u}{\partial t\partial s}+\frac{1}{R_o}\,\frac{\partial w}{\partial t})+c\,\frac{\partial u}{\partial t}$$

$$+(M_t+M_f+M_a)\,\frac{\partial^2 u}{\partial t^2}=0, \tag{2.58}$$

$$EI\left(\frac{\partial^4 v}{\partial s^4} - \frac{1}{R_o}\frac{\partial^2 \psi}{\partial s^2}\right) - \frac{GJ}{R_o}\left(\frac{\partial^2 \psi}{\partial s^2} + \frac{1}{R_o}\frac{\partial^2 v}{\partial s^2}\right) + \frac{\partial}{\partial s}\left[(A_iP_i - A_oP_e - Q_z)\frac{\partial v}{\partial s}\right] - G^*_{y_o}$$

$$+ M_fU^2\frac{\partial^2 v}{\partial s^2} + 2M_fU\frac{\partial^2 v}{\partial t\partial s} + c\frac{\partial v}{\partial t} + (M_t + M_f + M_a)\frac{\partial^2 v}{\partial t^2} = 0, \qquad (2.59)$$

$$-\frac{\partial}{\partial s}(A_iP_i - A_oP_e - Q_z) + \frac{EI}{R_o}\left(\frac{\partial^3 u}{\partial s^3} + \frac{1}{R_o}\frac{\partial^2 w}{\partial s^2}\right) + \frac{1}{R_o}(A_iP_i - A_oP_e - Q_z)\left(\frac{\partial u}{\partial s} + \frac{w}{R_o}\right)$$

$$+ G^*_{z_o} + \frac{M_fU^2}{R_o}\left(\frac{\partial u}{\partial s} + \frac{w}{R_o}\right) - M_fU\left(\frac{\partial^2 w}{\partial t\partial s} - \frac{1}{R_o}\frac{\partial u}{\partial t}\right) - c'\frac{\partial w}{\partial t} - (M_t + M_f + M_a')\frac{\partial^2 w}{\partial t^2} = 0, \qquad (2.60)$$

$$-GJ\left(\frac{\partial^2 \psi}{\partial s^2} + \frac{1}{R_o}\frac{\partial^2 v}{\partial s^2}\right) + \frac{EI}{R_o}\left(\frac{\psi}{R_o} - \frac{\partial^2 v}{\partial s^2}\right) + I_z\frac{\partial^2 \psi}{\partial t^2} = 0. \qquad (2.61)$$

These equations are, of course coupled, but similarly to the shell equations, each may be identified as being principally related to motions in one particular direction; thus, the first is related to in-plane deformations, the second to out-of-plane deformations, the third to deformations along the pipe, and the last equation is related to twist of the pipe. Hence, in-plane motions are governed by equations (2.58) and (2.60), while out-of-plane motions by equations (2.59) and (2.61). Note that, if the radius of curvature R_o is made equal to infinity, the curved pipe becomes a straight pipe. Moreover, if the pipe is vertical and the axial motion is ignored, equations (2.58)-(2.60) can be reduced to

$$EI\frac{\partial^4 u}{\partial s^4} + M_f\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial s}\right)^2 u + (M_t + M_a)\frac{\partial^2 u}{\partial t^2} + c\frac{\partial u}{\partial t} + \frac{\partial}{\partial s}\left[(A_iP_i - A_oP_e - Q_z)\frac{\partial u}{\partial s}\right] = 0, \qquad (2.62)$$

$$EI\frac{\partial^4 v}{\partial s^4} + M_f\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial s}\right)^2 v + (M_t + M_a)\frac{\partial^2 v}{\partial t^2} + c\frac{\partial v}{\partial t} + \frac{\partial}{\partial s}\left[(A_iP_i - A_oP_e - Q_z)\frac{\partial v}{\partial s}\right] = 0, \qquad (2.63)$$

$$\frac{\partial}{\partial s}(Q_z - A_i P_i) + (M_t + M_f)g = 0 , \qquad (2.64)$$

which, it may be verified, are identical to Luu's (1983) equations of motion for a straight pipe conveying fluid and fully submerged in a quiescent fluid. It may be noted that equation (2.62) is identical to equation (2.63), because for a straight pipe motions in the x_o- and y_o-directions are uncoupled and identical. Finally, if the surrounding fluid has negligible effect on the dynamics of the system, setting $M_a = 0$, $c = 0$, and $P_e = 0$ vis-à-vis the atmospheric pressure, then these equations reduce to those of Paidoussis' (1970).

2.6 THE BOUNDARY CONDITIONS

The boundary conditions associated with the governing equations of motion for the system can be obtained, as follows:

(i) If an end is clamped, the deflection and the slope of the deflection curve must be zero, Therefore, one obtains

$$\left.\begin{aligned} u &= 0 , \\ v &= 0 , \\ w &= 0 , \\ \psi &= 0 , \\ \frac{\partial u}{\partial s} &= 0 , \\ \frac{\partial v}{\partial s} &= 0 . \end{aligned}\right\} \qquad (2.65)$$

(ii) If an end is pinned, the deflections, the bending moments and the twist couple are all zero. Therefore, one obtains

$$\begin{aligned} u &= 0 , \\ v &= 0 , \\ w &= 0 , \end{aligned}$$

$$\left.\begin{aligned}&(\psi/R_o - \frac{\partial^2 v}{\partial s^2}) = 0\ ,\\&(\frac{\partial^2 u}{\partial s^2} + \frac{1}{R_o}\frac{\partial w}{\partial s}) = 0\ ,\\&(\frac{\partial \psi}{\partial s} + \frac{1}{R_o}\frac{\partial v}{\partial s}) = 0\ .\end{aligned}\right\} \quad (2.66)$$

(iii) If an end is free, the transverse shear forces, the axial force, the bending moments and the twisting couple are zero. Therefore, one obtains

$$\left.\begin{aligned}&(\frac{\partial^3 u}{\partial s^3} + \frac{1}{R_o}\frac{\partial^2 w}{\partial s^2}) = 0\ ,\\&(\frac{1}{R_o}\frac{\partial \psi}{\partial s} - \frac{\partial^3 v}{\partial s^3}) + \frac{GJ}{EI}\frac{1}{R_o}(\frac{\partial \psi}{\partial s} + \frac{1}{R_o}\frac{\partial v}{\partial s}) = 0\ ,\\&Q_z = 0\ ,\\&(\frac{\psi}{R_o} - \frac{\partial^2 v}{\partial s^2}) = 0\ ,\\&(\frac{\partial^2 u}{\partial s^2} + \frac{1}{R_o}\frac{\partial w}{\partial s}) = 0\ ,\\&(\frac{\partial \psi}{\partial s} + \frac{1}{R_o}\frac{\partial v}{\partial s}) = 0\ .\end{aligned}\right\} \quad (2.67)$$

2.7 THE EQUATIONS GOVERNING STATIC EQUILIBRIUM

Let u_o, v_o, w_o and ψ_o be the displacements and twist of the pipe at equilibrium. By deleting the time-dependent terms in equations (2.58)-(2.61), the equations governing static equilibrium can be obtained as follows :

$$EI(\frac{\partial^4 u_o}{\partial s^4} + \frac{1}{R_o}\frac{\partial^3 w_o}{\partial s^3}) + \frac{\partial}{\partial s}[P_o(\frac{\partial u_o}{\partial s} + \frac{w}{R_o})] + P_o/R_o + M_f U^2(\frac{\partial^2 u_o}{\partial s^2} + \frac{1}{R_o}\frac{\partial w_o}{\partial s}$$

$$+ \frac{1}{R_o}) - G^*_{x_o} = 0, \quad (2.68)$$

$$EI\left(\frac{\partial^4 v_o}{\partial s^4} - \frac{1}{R_o}\frac{\partial^2 \psi_o}{\partial s^2}\right) - \frac{GJ}{R_o}\left(\frac{\partial^2 \psi_o}{\partial s^2} + \frac{1}{R_o}\frac{\partial^2 v_o}{\partial s^2}\right) + \frac{\partial}{\partial s}\left[P_o \frac{\partial v_o}{\partial s}\right]$$

$$+ M_f U^2 \frac{\partial^2 v_o}{\partial s^2} - G^*_{y_o} = 0, \tag{2.69}$$

$$-\frac{\partial P_o}{\partial s} + \frac{EI}{R_o}\left(\frac{\partial^3 u_o}{\partial s^3} + \frac{1}{R_o}\frac{\partial^2 w_o}{\partial s^2}\right) + \frac{P_o}{R_o}\left(\frac{\partial u_o}{\partial s} + \frac{w_o}{R_o}\right) + \frac{M_f U^2}{R_o}\left(\frac{\partial u_o}{\partial s} + \frac{w_o}{R_o}\right) + G^*_{z_o} = 0, \tag{2.70}$$

$$GJ\left(\frac{\partial^2 \psi_o}{\partial s^2} + \frac{1}{R_o}\frac{\partial^2 v_o}{\partial s^2}\right) - \frac{EI}{R_o}\left(\frac{\psi_o}{R_o} - \frac{\partial^2 v_o}{\partial s^2}\right) = 0, \tag{2.71}$$

where P_o is the steady (static) form of the combined effect of the internal and external pressure and axial forces, i.e. the steady form of

$$P = (A_i P_i - A_o P_e - Q_z). \tag{2.72}$$

Here it is implicitly assumed that the departures from the original equilibrium state are small, so that these linearized equations hold true.

Moreover, if we eliminate u_o and w_o from equations (2.68) and (2.70), we obtain the differential equation governing the static value P_o of the combined force,

$$\frac{\partial^2 P_o}{\partial s^2} + \frac{P_o}{R_o^2} = \frac{G^*_{x_o}}{R_o} + \frac{\partial G^*_{z_o}}{\partial s} - \frac{M_f U^2}{R_o^2}. \tag{2.73}$$

2.8 THE DIMENSIONLESS EQUATIONS OF MOTION, STATIC EQUILIBRIUM EQUATIONS AND THE CORRESPONDING BOUNDARY CONDITIONS

The system may be expressed in dimensionless terms by defining the following quantities:

$$\left.\begin{aligned}
&\xi = s/L,\ \eta_1 = u/L,\ \eta_2 = v/L,\ \eta_3 = w/L,\ \eta_1^o = u_o/L,\ \eta_2^o = v_o/L,\ \eta_3^o = w_o/L,\\
&\tau = \left(\frac{EI}{M_f+M_t}\right)^{1/2}\frac{t}{L^2},\ \bar{u} = \left(\frac{M_f}{EI}\right)^{1/2} LU,\ \mathcal{K} = cL^2/[EI(M_f+M_t)\ ]^{1/2},\\
&\beta_a = \frac{M_a}{M_f+M_t},\ \beta_a' = \frac{M_a'}{M_f+M_t},\ \Lambda = \frac{GJ}{EI},\ \sigma = \frac{I_z}{(M_f+M_t)L^2},\ r_o = L/R_o,\\
&\beta = \frac{M_f}{M_f+M_t},\ \gamma = (M_f+M_t-A_o\rho_{fe})gL^3/EI,\quad \mathcal{K}' = c'L^2/[EI(M_f+M_t)]^{1/2},\\
&\mathcal{A} = A_tL^2/I,\ \Pi_p = (A_iP_i-A_oP_e)L^2/EI \text{ and } \Pi = (A_iP_i-A_oP_e-Q_z)L^2/EI\ .
\end{aligned}\right\} \quad (2.74)$$

Substitution of these terms into equations (2.58)-(2.61), (2.65)-(2.67) and (2.68)-(2.73) yields the dimensionless governing equations of motion

$$\left(\frac{\partial^4\eta_1}{\partial\xi^4} + r_o\frac{\partial^3\eta_3}{\partial\xi^3}\right) + \frac{\partial}{\partial\xi}\left[\Pi\left(\frac{\partial\eta_1}{\partial\xi} + r_o\eta_3\right)\right] + r_o\Pi - \gamma\alpha_{x_o} + \bar{u}^2\left(\frac{\partial^2\eta_1}{\partial\xi^2} + r_o\frac{\partial\eta_3}{\partial\xi} + r_o\right)$$

$$+ 2\beta^{1/2}\bar{u}\left(\frac{\partial^2\eta_1}{\partial\tau\partial\xi} + r_o\frac{\partial\eta_3}{\partial\tau}\right) + \mathcal{K}\frac{\partial\eta_1}{\partial\tau} + (1+\beta_a)\frac{\partial^2\eta_1}{\partial\tau^2} = 0, \quad (2.75)$$

$$\left(\frac{\partial^4\eta_2}{\partial\xi^4} - r_o\frac{\partial^2\psi}{\partial\xi^2}\right) - \Lambda r_o\left(\frac{\partial^2\psi}{\partial\xi^2} + r_o\frac{\partial^2\eta_2}{\partial\xi^2}\right) + \frac{\partial}{\partial\xi}\left[\Pi\frac{\partial\eta_2}{\partial\xi}\right] - \gamma\alpha_{y_o} + \bar{u}^2\frac{\partial^2\eta_2}{\partial\xi^2}$$

$$+ 2\beta^{1/2}\bar{u}\frac{\partial^2\eta_2}{\partial\tau\partial\xi} + \mathcal{K}\frac{\partial\eta_2}{\partial\tau} + (1+\beta_a)\frac{\partial^2\eta_2}{\partial\tau^2} = 0, \quad (2.76)$$

$$-\frac{\partial\Pi}{\partial\xi} + r_o\left(\frac{\partial^3\eta_1}{\partial\xi^3} + r_o\frac{\partial^2\eta_3}{\partial\xi^2}\right) + r_o\left(\frac{\partial\eta_1}{\partial\xi} + r_o\eta_3\right)\Pi + \gamma\alpha_{z_o} + r_o\bar{u}^2\left(\frac{\partial\eta_1}{\partial\xi} + r_o\eta_3\right)$$

$$- \beta^{1/2}\bar{u}\left(\frac{\partial^2\eta_3}{\partial\xi\partial\tau} - r_o\frac{\partial\eta_1}{\partial\tau}\right) - \mathcal{K}'\frac{\partial\eta_3}{\partial\tau} - (1+\beta_a')\frac{\partial^2\eta_3}{\partial\tau^2} = 0, \quad (2.77)$$

$$r_o\left(r_o\psi - \frac{\partial^2\eta_2}{\partial\xi^2}\right) - \Lambda\left(\frac{\partial^2\psi}{\partial\xi^2} + r_o\frac{\partial^2\eta_2}{\partial\xi^2}\right) + \sigma\frac{\partial^2\psi}{\partial\tau^2} = 0, \quad (2.78)$$

where α_{x_o}, α_{y_o} and α_{z_o} are the direction cosines of the gravity acceleration vector.

The corresponding dimensionless boundary conditions are as follows:

(i) at a clamped end

$$\left.\begin{aligned} \eta_1 &= 0, \\ \eta_2 &= 0, \\ \eta_3 &= 0, \\ \psi &= 0 \\ \frac{\partial \eta_1}{\partial \xi} &= 0, \\ \frac{\partial \eta_2}{\partial \xi} &= 0, \end{aligned}\right\} \qquad (2.79)$$

(ii) at a pinned end

$$\left.\begin{aligned} &\eta_1 = 0, \\ &\eta_2 = 0, \\ &\eta_3 = 0, \\ &(r_o\psi - \frac{\partial^2 \eta_2}{\partial \xi^2}) = 0, \\ &(\frac{\partial^2 \eta_1}{\partial \xi^2} + r_o \frac{\partial \eta_3}{\partial \xi}) = 0, \\ &(\frac{\partial \psi}{\partial \xi} + r_o \frac{\partial \eta_2}{\partial \xi}) = 0, \end{aligned}\right\} \qquad (2.80)$$

(iii) at a free end

$$\left.\begin{aligned} &(\frac{\partial^3 \eta_1}{\partial \xi^3} + r_o \frac{\partial^2 \eta_3}{\partial \xi^2}) = 0, \\ &(r_o \frac{\partial \psi}{\partial \xi} - \frac{\partial^3 \eta_2}{\partial \xi^3}) + r_o\Lambda (\frac{\partial \psi}{\partial \xi} + r_o \frac{\partial \eta_2}{\partial \xi}) = 0, \\ &(\Pi - \Pi_p) = 0, \\ &(r_o\psi - \frac{\partial^2 \eta_2}{\partial \xi^2}) = 0, \end{aligned}\right\} \qquad (2.81)$$

$$\left.\begin{aligned}&\left(\frac{\partial^2 \eta_1}{\partial \xi^2} + r_o \frac{\partial \eta_3}{\partial \xi}\right) = 0 \ , \\ &\left(\frac{\partial \psi}{\partial \xi} + r_o \frac{\partial \eta_2}{\partial \xi}\right) = 0 \cdot\end{aligned}\right|$$

The dimensionless equations of static equilibrium are given by

$$\left(\frac{\partial^4 \eta_1^o}{\partial \xi^4} + r_o \frac{\partial^3 \eta_3^o}{\partial \xi^3}\right) + \frac{\partial}{\partial \xi}\left[\Pi_o\left(\frac{\partial \eta_1^o}{\partial \xi} + r_o \eta_3^o\right)\right] + r_o \Pi_o + \bar{u}^2 \left(\frac{\partial^2 \eta_1^o}{\partial \xi^2} + r_o \frac{\partial \eta_3^o}{\partial \xi} + r_o\right)$$

$$- \gamma \alpha_{x_o} = 0, \tag{2.82}$$

$$\left(\frac{\partial^4 \eta_2^o}{\partial \xi^4} - r_o \frac{\partial^2 \psi_o}{\partial \xi^2}\right) - \Lambda r_o\left(\frac{\partial^2 \psi_o}{\partial \xi^2} + r_o \frac{\partial^2 \eta_2^o}{\partial \xi^2}\right) + \frac{\partial}{\partial \xi}\left[\Pi_o \frac{\partial \eta_2^o}{\partial \xi}\right] + \bar{u}^2 \frac{\partial^2 \eta_2^o}{\partial \xi^2} - \gamma \alpha_{y_o} = 0$$

$$\tag{2.83}$$

$$-\frac{\partial \Pi_o}{\partial \xi} + r_o\left(\frac{\partial^3 \eta_1^o}{\partial \xi^3} + r_o \frac{\partial^2 \eta_3^o}{\partial \xi^2}\right) + r_o \Pi_o\left(\frac{\partial \eta_1^o}{\partial \xi} + r_o \eta_3^o\right) + r_o \bar{u}^2\left(\frac{\partial \eta_1^o}{\partial \xi} + r_o \eta_3^o\right) + \gamma \alpha_{z_o} = 0, \tag{2.84}$$

$$\Lambda\left(\frac{\partial^2 \psi_o}{\partial \xi^2} + r_o \frac{\partial^2 \eta_2^o}{\partial \xi^2}\right) - r_o\left(r_o \psi_o - \frac{\partial^2 \eta_2^o}{\partial \xi^2}\right) = 0, \tag{2.85}$$

and the corresponding boundary conditions are:

(i) at a clamped end

$$\left.\begin{aligned}&\eta_1^o = 0 \ , \\ &\eta_2^o = 0 \ , \\ &\eta_3^o = 0 \ ; \\ &\psi_o = 0 \ , \\ &\frac{\partial \eta_1^o}{\partial \xi} = 0 \ , \\ &\frac{\partial \eta_2^o}{\partial \xi} = 0 ;\end{aligned}\right\} \tag{2.86}$$

(ii) at a pinned end

$$\left.\begin{aligned} &\eta_1^o = 0 , \\ &\eta_2^o = 0 , \\ &\eta_3^o = 0, \\ &(r_o\psi_o - \frac{\partial^2\eta_2^o}{\partial\xi^2}) = 0, \\ &(\frac{\partial^2\eta_1^o}{\partial\xi^2} + r_o \frac{\partial\eta_3^o}{\partial\xi}) = 0 , \\ &(\frac{\partial\psi_o}{\partial\xi} + r_o \frac{\partial\eta_2^o}{\partial\xi}) = 0 ; \end{aligned}\right\} \tag{2.87}$$

(iii) at a free end

$$\left.\begin{aligned} &(\frac{\partial^3\eta_1^o}{\partial\xi^3} + r_o \frac{\partial^2\eta_3^o}{\partial\xi^2}) = 0, \\ &(r_o \frac{\partial\psi_o}{\partial\xi} - \frac{\partial^3\eta_2^o}{\partial\xi^3}) + r_o\Lambda(\frac{\partial\psi_o}{\partial\xi} + r_o \frac{\partial\eta_2^o}{\partial\xi}) = 0 , \\ &(\Pi_o - \Pi_p) = 0 , \\ &(r_o\psi_o - \frac{\partial^2\eta_2^o}{\partial\xi^2}) = 0, \\ &(\frac{\partial^2\eta_1^o}{\partial\xi^2} + r_o \frac{\partial\eta_3^o}{\partial\xi}) = 0 , \\ &(\frac{\partial\psi_o}{\partial\xi} + r_o \frac{\partial\eta_2^o}{\partial\xi}) = 0 . \end{aligned}\right\} \tag{2.88}$$

Finally, the dimensionless differential equation governing the steady value of the combined effect (in dimensionless notation Π_o) is

$$\frac{\partial^2\Pi_o}{\partial\xi^2} + r_o^2\Pi_o = r_o\gamma\alpha_{x_o} + \gamma\frac{\partial\alpha_{z_o}}{\partial\xi} - r_o^2\,\bar{u}^2. \tag{2.89}$$

CHAPTER III

ANALYSIS FOR THE INEXTENSIBLE CASE

In Chapter II, the governing equations of motion of the dynamic system have been derived in terms of the dimensionless displacements and twist $(\eta_1, \eta_2, \eta_3, \psi)$, and the "combined force" Π. It is recalled that this combined force, defined in equation (2.74), is associated with pressure of the internal and external fluid and the axial tension. Solution of these equations may be very difficult, because the combined force is generally unknown and hard to evaluate. In this chapter, analysis of this problem will be performed, using the following assumptions:

(i) the centerline of the pipe is inextensible (when the pipe is subjected to small displacements and twist);

(ii) the perturbation to the combined force Π, due to departure from the original equilibrium state, is small (i.e., $\bar{\Pi}$ is considered to remain constant and equal to Π_0).

Based on the above assumptions, the combined force Π may be found from equation (2.89), and then the governing equations of motion about the equilibrium position may also be obtained. Moreover, we shall see later on that these governing equations will separate into in-plane and out-of-plane motions, which allows indepedent investigation of the dynamics of the system in these two planar directions.

3.1 THE EQUATIONS OF MOTION ABOUT THE EQUILIBRIUM POSITION

Let η_1^*, η_2^*, η_3^* and ψ^* be the dimensionless perturbation displacements and twist from the equilibrium position. They may be expressed in terms of the steady (static) and total displacements and twist, as follows:

$$\left.\begin{aligned} \eta_1 &= \eta_1^o + \eta_1^*, \\ \eta_2 &= \eta_2^o + \eta_2^*, \\ \eta_3 &= \eta_3^o + \eta_3^*, \\ \psi &= \psi_o + \psi^* . \end{aligned}\right\} \qquad (3.1)$$

Substituting equation set (3.1) into equations (2.72)-(2.75), neglecting higher order terms, and combining equations (2.79)-(2.82) one obtains the dimensionless equations governing the motion of the system about the equilibrium position, namely,

$$\left(\frac{\partial^4\eta_1^*}{\partial\xi^4} + r_o\frac{\partial^3\eta_3^*}{\partial\xi^3}\right) + \frac{\partial}{\partial\xi}\left[\Pi_o\left(\frac{\partial\eta_1^*}{\partial\xi} + r_o\eta_3^*\right)\right] + (1+\beta_a)\frac{\partial^2\eta_1^*}{\partial\tau^2} + 2\beta^{1/2}\bar{u}\left(\frac{\partial^2\eta_1^*}{\partial\tau\partial\xi} + r_o\frac{\partial\eta_3^*}{\partial\tau}\right)$$
$$+ \bar{u}^2\left(\frac{\partial^2\eta_1^*}{\partial\xi^2} + r_o\frac{\partial\eta_3^*}{\partial\xi}\right) + \mathcal{K}\frac{\partial\eta_1^*}{\partial\tau} = 0, \qquad (3.2)$$

$$\left(\frac{\partial^4\eta_2^*}{\partial\xi^4} - r_o\frac{\partial^2\psi^*}{\partial\xi^2}\right) - \Lambda r_o\left(\frac{\partial^2\psi^*}{\partial\xi^2} + r_o\frac{\partial^2\eta_2^*}{\partial\xi^2}\right) + \frac{\partial}{\partial\xi}\left[\Pi_o\frac{\partial\eta_2^*}{\partial\xi}\right] + (1+\beta_a)\frac{\partial^2\eta_2^*}{\partial\tau^2}$$
$$+ 2\beta^{1/2}\bar{u}\frac{\partial^2\eta_2^*}{\partial\tau\partial\xi} + \bar{u}^2\frac{\partial^2\eta_2^*}{\partial\xi^2} + \mathcal{K}\frac{\partial\eta_2^*}{\partial\tau} = 0, \qquad (3.3)$$

$$r_o\left(\frac{\partial^3\eta_1^*}{\partial\xi^3} + r_o\frac{\partial^2\eta_3^*}{\partial\xi^2}\right) + r_o\left(\frac{\partial\eta_1^*}{\partial\xi} + r_o\eta_3^*\right)\Pi_o - (1+\beta_a')\frac{\partial^2\eta_3^*}{\partial\tau^2} - \beta^{1/2}\bar{u}\left(\frac{\partial^2\eta_3^*}{\partial\tau\partial\xi}\right.$$
$$\left. - r_o\frac{\partial\eta_1^*}{\partial\tau}\right) + r_o\bar{u}^2\left(\frac{\partial\eta_1^*}{\partial\xi} + r_o\eta_3^*\right) - \mathcal{K}'\frac{\partial\eta_3^*}{\partial\tau} = 0, \qquad (3.4)$$

$$\Lambda\left(\frac{\partial^2\psi^*}{\partial\xi^2} + r_o\frac{\partial^2\eta_2^*}{\partial\xi^2}\right) - r_o\left(r_o\psi^* - \frac{\partial^2\eta_2^*}{\partial\xi^2}\right) - \sigma\frac{\partial^2\psi^*}{\partial\tau^2} = 0. \qquad (3.5)$$

It is seen that equations (3.2) and (3.4) pertain to in-plane displacements, while equations (3.3) and (3.5) are associated with out-of-plane displacement and twist. There is no coupling between in-plane and out-of-plane motions, as suggested earlier; therefore, they can be studied independently.

Furthermore, according to the assumption of inextensibility (i.e., the centerline strain $\varepsilon = (\partial w/\partial s - u/R_o)$ should vanish), the dimensionless displacements η_1 and η_3 are related by

$$\frac{\partial \eta_3}{\partial \xi} - r_o \eta_1 = 0 , \tag{3.6}$$

and hence

$$\frac{\partial \eta_3^o}{\partial \xi} - r_o \eta_1^o = 0, \tag{3.7}$$

$$\frac{\partial \eta_3^*}{\partial \xi} - r_o \eta_1^* = 0. \tag{3.8}$$

Then, utilizing equations (3.2), (3.4) and (3.8), the equations of in-plane motion may be reduced to a single sixth order partial differential equation for η_3^*; this equation is

$$\left(\frac{\partial^6 \eta_3^*}{\partial \xi^6} + 2r_o^2 \frac{\partial^4 \eta_3^*}{\partial \xi^4} + r_o^4 \frac{\partial^2 \eta_3^*}{\partial \xi^2}\right) + \bar{u}^2\left(\frac{\partial^4 \eta_3^*}{\partial \xi^4} + 2r_o^2 \frac{\partial^2 \eta_3^*}{\partial \xi^2} + r_o^4 \eta_3^*\right) + (1+\beta_a)$$
$$\times \frac{\partial^4 \eta_3^*}{\partial \tau^2 \partial \xi^2} - r_o^2 (1+\beta_a') \frac{\partial^2 \eta_3^*}{\partial \tau^2} + 2\beta^{1/2} \bar{u} \left(\frac{\partial^4 \eta_3^*}{\partial \tau \partial \xi^3} + r_o^2 \frac{\partial^2 \eta_3^*}{\partial \tau \partial \xi}\right)$$
$$+ \mathcal{K} \frac{\partial^3 \eta_3^*}{\partial \xi^2 \partial \tau} - r_o^2 \mathcal{K}' \frac{\partial \eta_3^*}{\partial \tau} + \frac{\partial^2}{\partial \xi^2} \left[\Pi_o \left(\frac{\partial^2 \eta_3^*}{\partial \xi^2} + r_o^2 \eta_3^*\right)\right] + r_o^2 \Pi_o \left(\frac{\partial^2 \eta_3^*}{\partial \xi^2} + r_o^2 \eta_3^*\right) = 0. \tag{3.9}$$

It is of interest to note that when the combined force Π is neglected and external fluid effects are absent ($\beta_a = \beta_a' = \mathcal{K} = \mathcal{K}' = 0$), this equation reduces to that obtained by Chen (1972) by Hamilton's principle.

The associated boundary conditions are also reduced, as follows:

(i) at a clamped end

$$\left.\begin{aligned} \eta_3^* &= 0 , \\ \frac{\partial \eta_3^*}{\partial \xi} &= 0 , \\ \frac{\partial^2 \eta_3^*}{\partial \xi^2} &= 0 ; \end{aligned}\right\} \tag{3.10}$$

(ii) at a pinned end

$$\left.\begin{aligned} \eta_3^* &= 0 , \\ \frac{\partial \eta_3^*}{\partial \xi} &= 0 , \\ \left(\frac{\partial^3 \eta_3^*}{\partial \xi^3} + r_o^2 \frac{\partial \eta_3^*}{\partial \xi}\right) &= 0; \end{aligned}\right\} \tag{3.11}$$

(iii) at a free end

$$\left(\frac{\partial^4 \eta_3^*}{\partial \xi^4} + r_o^2 \frac{\partial^2 \eta_3^*}{\partial \xi^2}\right) = 0 , \tag{3.12}$$

$$\left(\frac{\partial^3 \eta_3^*}{\partial \xi^3} + r_o^2 \frac{\partial \eta_3^*}{\partial \xi}\right) = 0 , \tag{3.13}$$

$$\left(\frac{\partial^5 \eta_3^*}{\partial \xi^5} + r_o^2 \frac{\partial^3 \eta_3^*}{\partial \xi^3}\right) + 2\beta^{1/2} \bar{u} \left(\frac{\partial^3 \eta_3^*}{\partial \tau \partial \xi^2} + r_o^2 \frac{\partial \eta_3^*}{\partial \tau}\right) + (1+\beta_a) \frac{\partial^3 \eta_3^*}{\partial \tau^2 \partial \xi}$$

$$+ \mathcal{K} \frac{\partial \eta_3^*}{\partial \tau \partial \xi} \pm \frac{\partial}{\partial \xi} \left[\Pi_o \left(\frac{\partial^2 \eta_3^*}{\partial \xi^2} + r_o^2 \eta_3^*\right)\right] = 0. \tag{3.14}$$

Details of the derivation of equation (3.14) may be found in Appendix F. Once more, these boundary conditions are identical to Chen's (1972).

Similarly, the equations of out-of-plane motion are given by

$$\left(\frac{\partial^4 \eta_2^*}{\partial \xi^4} - r_o \frac{\partial^2 \psi^*}{\partial \xi^2}\right) - \Lambda r_o \left(\frac{\partial^2 \psi^*}{\partial \xi^2} + r_o \frac{\partial^2 \eta_2^*}{\partial \xi^2}\right) + \frac{\partial}{\partial \xi} \left[\Pi_o \frac{\partial \eta_2^*}{\partial \xi}\right] + \bar{u}^2 \frac{\partial^2 \eta_2^*}{\partial \xi^2}$$

$$+(1+\beta_a) \frac{\partial^2 \eta_2^*}{\partial \tau^2} + 2\beta^{1/2} \bar{u} \frac{\partial^2 \eta_2^*}{\partial \tau \partial \xi} + \mathcal{K} \frac{\partial \eta_2^*}{\partial \tau} = 0, \tag{3.15}$$

$$r_o\left(r_o\psi^* - \frac{\partial^2\eta_2^*}{\partial\xi^2}\right) - \Lambda\left(\frac{\partial^2\psi^*}{\partial\xi^2} + r_o\frac{\partial^2\eta_2^*}{\partial\xi^2}\right) + \sigma\frac{\partial^2\psi^*}{\partial\tau^2} = 0, \qquad (3.16)$$

and the boundary conditions are as follows:

(i) at a clamped end

$$\left.\begin{aligned} &\eta_2^* = 0\,, \\ &\psi^* = 0\,, \\ &\frac{\partial\eta_2^*}{\partial\xi} = 0\,; \end{aligned}\right\} \qquad (3.17)$$

(ii) at a pinned end

$$\left.\begin{aligned} &\eta_2^* = 0\,, \\ &\left(r_o\psi^* - \frac{\partial^2\eta_2^*}{\partial\xi^2}\right) = 0\,, \\ &\left(\frac{\partial\psi^*}{\partial\xi} + r_o\frac{\partial\eta_2^*}{\partial\xi}\right) = 0\,; \end{aligned}\right\} \qquad (3.18)$$

(iii) at a free end

$$\left.\begin{aligned} &\left(r_o\frac{\partial\psi^*}{\partial\xi} - \frac{\partial^3\eta_2^*}{\partial\xi^3}\right) + r_o\Lambda\left(\frac{\partial\psi^*}{\partial\xi} + r_o\frac{\partial\eta_2^*}{\partial\xi}\right) = 0\,, \\ &\left(r_o\psi^* - \frac{\partial^2\eta_2^*}{\partial\xi^2}\right) = 0\,, \\ &\left(\frac{\partial\psi^*}{\partial\xi} + r_o\frac{\partial\eta_2^*}{\partial\xi}\right) = 0. \end{aligned}\right\} \qquad (3.19)$$

Once again, it may be verified that if the combined force Π_o is suppressed and external fluid effects are eliminated ($\beta_a = \mathcal{K} = 0$), then the equations of motion and boundary conditions above are identical to Chen's (1973), obtained by the use of Hamilton's principle.

3.2 ANALYSIS OF IN-PLANE MOTION IN THE INEXTENSIBLE CASE

In order to obtain the solution for in-plane motions, the finite-element method is applied. Accordingly, the variational statement used for the finite-element technique is utilized, i.e.,

$$\sum_{i=0}^{n} \int_{o}^{\xi_i} \delta\eta_3^* A_i^*(\eta_3^*)\, d\xi = 0, \tag{3.20}$$

where $\delta\eta_3^*$ is an arbitrary variational displacement, $A_i^*(\eta_3^*)$ represents the left-hand side of equation (3.9), n is the number of elements, and ξ_i is the length of the i^{th} element.

By performing integrations by parts and using the boundary conditions, one obtains

$$\sum_{i=0}^{n} \int_{o}^{\xi_i} \{\delta(\frac{\partial^3\eta_3^*}{\partial\xi^3} + r_o^2 \frac{\partial\eta_3^*}{\partial\xi})(\frac{\partial^3\eta_3^*}{\partial\xi^3} + r_o^2 \frac{\partial\eta_3^*}{\partial\xi}) + \bar{u}^2[\frac{\partial\delta\eta_3^*}{\partial\xi}(\frac{\partial^3\eta_3^*}{\partial\xi^3} + r_o^2\frac{\partial\eta_3^*}{\partial\xi})$$

$$-r_o^2\delta\eta_3^*(\frac{\partial^2\eta_3^*}{\partial\xi^2} + r_o^2\eta_3^*)] + \frac{\partial\delta\eta_3^*}{\partial\xi}\cdot\frac{\partial}{\partial\xi}[\Pi_o(\frac{\partial^2\eta_3^*}{\partial\xi^2} + r_o^2\eta_3^*)]$$

$$-r_o^2\Pi_o\delta\eta_3^*(\frac{\partial^2\eta_3^*}{\partial\xi^2} + r^2\eta_3^*) + 2\beta^{1/2}\bar{u}\frac{\partial\delta\eta_3^*}{\partial\xi}(\frac{\partial^3\eta_3^*}{\partial\tau\partial\xi^2} + r_o^2\frac{\partial\eta_3^*}{\partial\tau})$$

$$+\mathcal{K}\frac{\partial\delta\eta_3^*}{\partial\xi}\cdot\frac{\partial^2\eta_3^*}{\partial\tau\partial\xi} + r_o^2\mathcal{K}'\delta\eta_3^*\frac{\partial\eta_3^*}{\partial\tau}) + (1+\beta_a)\frac{\partial\delta\eta_3^*}{\partial\xi}\cdot\frac{\partial^3\eta_3^*}{\partial\tau^2\partial\xi}$$

$$+ r_o^2(1+\beta_a')\,\delta\eta_3^*\frac{\partial^2\eta_3^*}{\partial\tau^2}\}\, d\xi = 0. \tag{3.21}$$

Details of the derivation of equation (3.21) may be found in Appendix G.

According to the finite element process, solutions are being sought in the approximate form

$$\eta_3^* = [N_3]\{\eta_i^*\}^e, \tag{3.22}$$

where $[N_3]$ is a matrix of shape functions prescribed in terms of the space coordinate ξ, and $\{\eta_i^*\}^e$ is the element-displacement vector which depends only on time. Thus, one may write

$$\left.\begin{aligned}
\delta\eta_3^* &= [N_3]\,\delta\{\eta_i^*\}^e ,\\
\frac{\partial^m \eta_3^*}{\partial\xi^m} &= \frac{d^m[N_3]}{d\xi^m}\cdot\{\eta_i^*\}^e ,\\
\frac{\partial^m \delta\eta_3^*}{\partial\xi^m} &= \frac{d^m[N_3]}{d\xi^m}\cdot\delta\{\eta_i^*\}^e ,\\
\frac{\partial^m \eta_3^*}{\partial\tau^m} &= [N_3]\,\frac{d^m\{\eta_i^*\}^e}{d\tau^m} .
\end{aligned}\right\} \qquad (3.23)$$

Using equations (3.21)–(3.23), the discretized equation of in-plane motion may be written as

$$[M_i^*]\{\ddot{\eta}_i^*\}^e + [D_i^*]\cdot\{\dot{\eta}_i^*\}^e + [K_i^*]^e\{\eta_i^*\}^e = 0, \qquad (3.24)$$

where

$$[M_i^*]^e = \int_0^{\xi_e} [(1+\beta_a)[N_3]'^T[N_3]' + r_o^2\,(1+\beta_a')[N_3]^T[N_3]]\,d\xi, \qquad (3.25)$$

$$[D_i^*]^e = 2\beta^{1/2}\,\bar{u}\int_0^{\xi_e}[N_3]'^T([N_3]'' + r_o^2[N_3])\,d\xi + \int_0^{\xi_e}[\mathcal{K}[N_3]'^T[N_3]'$$

$$+ r_o^2\,\mathcal{K}'[N_3]^T[N_3]]\,d\xi, \qquad (3.26)$$

$$[K_i^*]^e = \int_0^{\xi_e}([N_3]''' + r_o^2[N_3]')^T([N_3]''' + r_o^2[N_3]')\,d\xi$$

$$+ \bar{u}^2\int_0^{\xi_e}\{[N_3]'^T([N_3]''' + r_o^2[N_3]') - r_o^2[N_3]^T([N_3]'' + r_o^2[N_3])\}\,d\xi$$

$$+ \int_0^{\xi_e}[N_3]'^T\frac{\partial}{\partial\xi}[\Pi_o([N_3]'' + r_o^2[N_3])]\,d\xi$$

$$- \int_0^{\xi_e} r_o^2\,\Pi_o[N_3]^T([N_3]'' + r_o^2[N_3])\,d\xi . \qquad (3.27)$$

Here, it is noted that the prime denotes differentiation with respect to ξ, and the dot denotes differentiation with respect to τ; ξ_e is the element length.

3.3 NUMERICAL ANALYSIS OF IN-PLANE MOTION IN THE INEXTENSIBLE CASE

The highest order of derivative of the shape function $[N_3]$, which exists in the integrands of equation (3.25)-(3.27), is the third; hence, it is necessary to ensure that η_3^*, $\eta_3^{*\prime}$ and $\eta_3^{*\prime\prime}$ be continuous between elements. This is easily accomplished if the nodal displacements at each node are taken as the values of η_3^*, $\eta_3^{*\prime}$ and $\eta_3^{*\prime\prime}$.

Thus for the j^{th} node

$$\{\eta_i^*\}_j = \begin{Bmatrix} \eta_{3j}^* \\ \eta_{3j}^{*\prime} \\ \eta_{3j}^{*\prime\prime} \end{Bmatrix}, \qquad (3.28)$$

as shown in Fig. 4.

The shape function $[N_3]$ will be derived next. If one accepts that in an element, two nodes (i.e., six degrees of freedom for each element) define the deflected shape, one may assume this deflected shape to be given by a fifth-order polynomial

$$\eta_3^* = \alpha_1 + \alpha_2\xi + \alpha_3\xi^2 + \alpha_4\xi^3 + \alpha_5\xi^4 + \alpha_6\xi^5, \qquad (3.29)$$

where α_i are the generalized coordinates, and the element displacement vector is

$$\{\eta_i^*\}^e = \begin{Bmatrix} \{\eta_i^*\}_j \\ \hline \{\eta_i^*\}_{j+1} \end{Bmatrix}. \qquad (3.30)$$

Here ξ is the local coordinate along the pipe element.

Let

$$[\Phi_3] = [1,\xi,\xi^2,\xi^3,\xi^4,\xi^5] , \tag{3.31}$$

$$\{\alpha_i\}^T = \{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6\}. \tag{3.32}$$

Then, equation (3.29) may be rewritten in the matrix form

$$\eta_3^* = [\Phi_3]\{\alpha_i\}, \tag{3.33}$$

and also

$$\eta_3^{*\prime} = [\Phi_3]^{\prime}\{\alpha_i\}, \tag{3.34}$$

$$\eta_3^{*\prime\prime} = [\Phi_3]^{\prime\prime}\{\alpha_i\}. \tag{3.35}$$

Substituting equations (3.33)-(3.35) into equation (3.30) yields

$$\{\eta_i^*\}^e = \begin{Bmatrix} [\Phi_3]_j \\ [\Phi_3]_j^{\prime} \\ [\Phi_3]_j^{\prime\prime} \\ [\Phi_3]_{j+1} \\ [\Phi_3]_{j+1}^{\prime} \\ [\Phi_3]_{j+1}^{\prime\prime} \end{Bmatrix} \{\alpha_i\}. \tag{3.36}$$

Using (3.31) and (3.36), one may express the nodal displacements in terms of the generalized coordinates $\{\alpha_i\}$ as

$$\{\eta_i^*\}^e = [A]\{\alpha_i\}, \tag{3.37}$$

where

$$[A] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & \xi_e & \xi_e^2 & \xi_e^3 & \xi_e^4 & \xi_e^5 \\ 0 & 1 & 2\xi_e & 3\xi_e^2 & 4\xi_e^3 & 5\xi_e^4 \\ 0 & 0 & 2 & 6\xi_e & 12\xi_e^2 & 20\xi_e^3 \end{bmatrix}. \tag{3.38}$$

Now inverting equation (3.37), one obtains

$$\{\alpha\}_i = [A]^{-1}\{\eta_i^*\}^e , \tag{3.39}$$

where

$$[A]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 \\ -10\xi_e^{-3} & -6\xi_e^{-2} & -\frac{3}{2}\xi_e^{-1} & 10\xi_e^{-3} & -4\xi_e^{-2} & \frac{1}{2}\xi_e^{-1} \\ 15\xi_e^{-4} & 8\xi_e^{-3} & \frac{3}{2}\xi_e^{-2} & -15\xi_e^{-4} & 7\xi_e^{-3} & -\xi_e^{-2} \\ -6\xi_e^{-5} & -3\xi_e^{-4} & -\frac{1}{2}\xi_e^{-3} & 6\xi_e^{-5} & -3\xi_e^{-4} & \frac{1}{2}\xi_e^{-3} \end{bmatrix} . \tag{3.40}$$

Details of the calculation to obtain matrix $[A]^{-1}$ can be found in Appendix H.

Utilizing equations (3.22), (3.33) and (3.39), one obtains the shape function $[N_3]$

$$[N_3] = [\Phi_3][A]^{-1} . \tag{3.41}$$

Then, substituting equation (3.41) into equations (3.25)-(3.27) yields

$$[M_i^*]^e = [A]^{-1T}\{ \int_0^{\xi e} ((1+\beta_a)[\Phi_3]'^T[\Phi_3]' + r_o^2(1+\beta_a')[\Phi_3]^T[\Phi_3])\, d\xi\} [A]^{-1}, \tag{3.42}$$

$$[D_i^*]^e = 2\beta^{1/2}\bar{u}\, [A]^{-1T}\{ \int_0^{\xi_e} [\Phi_3]'^T([\Phi_3]'' + r_o^2[\Phi_3])\, d\xi\} [A]^{-1}$$
$$+ [A]^{-1T}\{ \int_0^{\xi_e} (\mathcal{H}[\Phi_3]'^T[\Phi_3]' + r_o^2\mathcal{H}'[\Phi_3]^T[\Phi_3])d\xi\} [A]^{-1}, \tag{3.43}$$

$$[K_i^*]^e = [A]^{-1T}\int_0^{\xi_e} ([\Phi_3]''' + r_o^2[\Phi_3]')^T([\Phi_3]''' + r_o^2[\Phi_3]')\, d\xi\, [A]^{-1}$$
$$+\bar{u}^2[A]^{-1T}\int_0^{\xi_e}\{[\Phi_3]'^T([\Phi_3]''' + r_o^2[\Phi_3]') - r_o^2[\Phi_3]^T([\Phi_3]'' + r_o^2[\Phi_3])\}d\xi\, [A]^{-1}$$
$$+[A]^{-1T}\{\int_0^{\xi_e} [\Phi_3]'^T \frac{\partial}{\partial\xi}[\Pi_o([\Phi_3]'' + r_o^2[\Phi_3])]d\xi - \int_0^{\xi_e} r_o^2\Pi_o[\Phi_3]^T([\Phi_3]''$$
$$+ r_o^2[\Phi_3])d\xi\} [A]^{-1}. \tag{3.44}$$

To evaluate the last two integrals in $[K_i^*]^e$, the combined force Π_o, and its derivative $\partial\Pi_o/\partial\xi$ in each element are approximated by linear functions, for convenience, as shown in Fig. 5. Therefore, we may write

$$\left.\begin{aligned} \Pi_o &= a_1 + a_2\,\xi, \\ \frac{\partial\Pi_o}{\partial\xi} &= b_1 + b_2\,\xi, \end{aligned}\right\} \qquad (3.45)$$

where

$$\left.\begin{aligned} a_1 &= \Pi_o|_j\,, \\ a_2 &= (\Pi_o|_{j+1} - \Pi_o|_j)/\,\xi_e\,, \\ b_1 &= \frac{\partial\Pi_o}{\partial\xi}\Big|_j\,, \\ b_2 &= \left(\frac{\partial\Pi_o}{\partial\xi}\Big|_{j+1} - \frac{\partial\Pi_o}{\partial\xi}\Big|_j\right)/\xi_e\,. \end{aligned}\right\} \qquad (3.46)$$

Substituting equation set (3.45) into equation (3.44), and then evaluating the integrals in equations (3.42)-(3.44), one obtains the element matrices

$$[M_i^*]^e = [A]^{-1T}((1+\beta_a)[J_2] + r_o^2(1+\beta_a')[J_1])\ [A]^{-1}, \qquad (3.47)$$

$$[D_i^*]^e = 2\beta^{1/2}\bar{u}\ [A]^{-1T}([J_5]+r_o^2[J_4])[A]^{-1}+[A]^{-1T}(\mathcal{K}[J_2]$$
$$+ r_o^2\mathcal{K}'[J_1])\ [A]^{-1} \qquad (3.48)$$

$$[K_i^*]^e = [A]^{-1T}\{[J_3]+r_o^2([J_6]+[J_6]^T)+r_o^4[J_2]+\bar{u}^2([J_6]+r_o^2([J_2]-[J_7])$$
$$-r_o^4[J_1])+a_1([J_6]+r_o^2([J_2]-[J_7])-r_o^4[J_1])+b_1([J_5]+r_o^2[J_4])$$
$$+a_2([J_8]+r_o^2([J_{10}]-[J_{11}])-r_o^4[J_9])+b_2([J_{13}]+r_o^2[J_{12}])\}[A]^{-1},$$
$$(3.49)$$

where

$$\left.\begin{aligned}
[J_1] &= \int_0^{\xi_e} [\Phi_3]^T [\Phi_3] \, d\xi \,, \\
[J_2] &= \int_0^{\xi_e} [\Phi_3]'^T [\Phi_3]' \, d\xi \,, \\
[J_3] &= \int_0^{\xi_e} [\Phi_3]'''^T [\Phi_3]''' \, d\xi \,, \\
[J_4] &= \int_0^{\xi_e} [\Phi_3]'^T [\Phi_3] \, d\xi \,, \\
[J_5] &= \int_0^{\xi_e} [\Phi_3]'^T [\Phi_3]'' \, d\xi \,, \\
[J_6] &= \int_0^{\xi_e} [\Phi_3]'^T [\Phi_3]''' \, d\xi \,, \\
[J_7] &= \int_0^{\xi_e} [\Phi_3]^T [\Phi_3]'' \, d\xi \,, \\
[J_8] &= \int_0^{\xi_e} \xi [\Phi_3]'^T [\Phi_3]''' \, d\xi \\
[J_9] &= \int_0^{\xi_e} \xi [\Phi_3]^T [\Phi_3] \, d\xi \,, \\
[J_{10}] &= \int_0^{\xi_e} \xi [\Phi_3]'^T [\Phi_3]' \, d\xi \,, \\
[J_{11}] &= \int_0^{\xi_e} \xi [\Phi_3]^T [\Phi_3]'' \, d\xi \,, \\
[J_{12}] &= \int_0^{\xi_e} \xi [\Phi_3]'^T [\Phi_3] \, d\xi \,, \\
[J_{13}] &= \int_0^{\xi_e} \xi [\Phi_3]'^T [\Phi_3]'' \, d\xi \,,
\end{aligned}\right\} \qquad (3.50)$$

which are evaluated in Appendix I.

3.4 ANALYSIS OF OUT-OF-PLANE MOTION IN THE INEXTENSIBLE CASE

Similarly to the case of in-plane motion, the variational statement used for the finite-element model of out-of-plane motion is

$$\sum_{i=1}^{n} \int_{0}^{\xi_i} \{\delta\eta_2^* A_{01}^*(\eta_2^*,\psi^*) + \delta\psi^* A_{02}^*(\eta_2^*,\psi^*)\}\, d\xi = 0 \ , \quad (3.51)$$

where $\delta\eta_2^*$ and $\delta\psi^*$ are the variations in the displacement and twist,

$A_{01}^*(\eta_2^*,\psi^*)$ and $A_{02}^*(\eta_2^*,\psi^*)$ represent the left-hand sides of equations (3.15) and (3.16) respectively,

n is the number of elements, and

ξ_i is the length of the i^{th} element.

Performing integrations by parts and using the boundary conditions, one obtains

$$\sum_{i=1}^{n} \int_{0}^{\xi_i} \{\frac{\partial^2\delta\eta_2^*}{\partial\xi^2}(\frac{\partial^2\eta_2^*}{\partial\xi^2} - r_0\psi^*) + r_0\delta\psi^*(r_0\psi^* - \frac{\partial^2\eta_2^*}{\partial\xi^2}) + \Lambda[r_0 \frac{\partial\delta\eta_2^*}{\partial\xi}(\frac{\partial\psi^*}{\partial\xi}$$

$$+ r_0\frac{\partial\eta_2^*}{\partial\xi}) + \frac{\partial\delta\psi^*}{\partial\xi}(\frac{\partial\psi^*}{\partial\xi} + r_0\frac{\partial\eta_2^*}{\partial\xi})] + \bar{u}^2\ \delta\eta_2^* \frac{\partial^2\eta_2^*}{\partial\xi^2} + \delta\eta_2^*(\frac{\partial\Pi_0}{\partial\xi}\cdot \frac{\partial\eta_2^*}{\partial\xi} + \Pi_0 \frac{\partial^2\eta_2^*}{\partial\xi^2})$$

$$+ 2\beta^{1/2}\bar{u}\ \delta\eta_2^* \frac{\partial^2\eta_2^*}{\partial\xi\partial\tau} + \mathcal{K}\delta\eta_2^* \frac{\partial\eta_2^*}{\partial\tau} + (1+\beta_a)\delta\eta_2^* \frac{\partial^2\eta_2^*}{\partial\tau^2} + \sigma\delta\psi^* \frac{\partial^2\psi^*}{\partial\tau^2}\}\, d\xi = 0 .$$

$$(3.52)$$

Details of the derivation of equation (3.52) may be found in Appendix J.

Accordingly, applying the finite-element process, solutions will be sought in the approximate form

$$\left.\begin{aligned} \eta_2^* &= [N_2]\{\eta_0^*\}^e , \\ \psi^* &= [N_4]\{\eta_0^*\}^e , \end{aligned}\right\} \quad (3.53)$$

where $[N_2]$ and $[N_4]$ are the shape functions prescribed in terms of the space coordinate ξ, and $\{\eta_o^*\}^e$ is the element displacement vector which depends only on time.

Substituting equation set (3.53) into equation (3.52) yields the discretized equation of out-of-plane moetion, as follows:

$$[M_o^*]^e \{\ddot{\eta}_o^*\} + [D_o^*]^e\{\dot{\eta}_o^*\}^e + [K_o^*]^e\{\eta_o^*\}^e = 0, \qquad (3.54)$$

where

$$[M_o^*]^e = (1+\beta_a)\int_o^{\xi_i} [N_2]^T[N_2]\, d\xi + \sigma \int_o^{\xi_i} [N_4]^T[N_4]\, d\xi, \qquad (3.55)$$

$$[D_o^*]^e = 2\beta^{1/2}\bar{u}\int_o^{\xi_i} [N_2]^T[N_2]'\, d\xi + \mathcal{K}\int_o^{\xi_i} [N_2]^T[N_2] d\xi, \qquad (3.56)$$

$$[K_o^*]^e = \int_o^{\xi_i} \{[N_2]''^T([N_2]''-r_o[N_4]) + r_o[N_4]^T(r_o[N_4]-[N_2]'')$$

$$+ \Lambda[r_o[N_2]'^T([N_4]'+ r_o[N_2]')+[N_4]'^T([N_4]'+r_o [N_2]')]$$

$$+ \bar{u}^2 [N_2]^T[N_2]''+[N_2]^T(\frac{\partial \Pi_o}{\partial \xi}\cdot [N_2]'+\Pi_o[N_2]'')\}\, d\xi . \qquad (3.57)$$

3.5 NUMERICAL ANALYSIS OF OUT-OF-PLANE MOTION IN THE INEXTENSIBLE CASE

The highest order of derivatives of the shape functions $[N_2]$ and $[N_4]$, which exist in the integrands of equation (3.55)-(3.57), are second and first respectively, and it is necessary to ensure that $\eta_2^*, \eta_2^{*\prime}$ and ψ^* are continuous between elements. Thus, one choses the nodal displacements as

$$\{\eta_o^*\}_j = \begin{Bmatrix} \eta_{2j}^* \\ \eta_{2j}^{*\prime} \\ \psi_j^* \end{Bmatrix}, \qquad (3.58)$$

as shown in Fig. 6.

The shape functions $[N_2]$ and $[N_4]$ are now derived. If one also accepts that in an element two nodes define the deflected shape one may assume that

$$\eta_2^* = \alpha_1^* + \alpha_2^*\xi + \alpha_3^*\xi^2 + \alpha_4^*\xi^3 , \tag{3.59}$$

$$\psi^* = \alpha_5^* + \alpha_6^*\xi, \tag{3.60}$$

where α_i^* are the generalized coordinates, and ξ is the local coordinate. The element displacement vector is

$$\{\eta_o^*\}^e = \left\{\begin{array}{c} \{\eta_o^*\}_j \\ \hline \{\eta_o^*\}_{j+1} \end{array}\right\} . \tag{3.61}$$

Let

$$[\Phi_2] = [1,\xi,\xi^2,\xi^3, 0, 0] , \tag{3.62}$$

$$[\Phi_4] = [0,0,0,0,1,\xi] , \tag{3.63}$$

$$\{\alpha_i^*\}^T = \{\alpha_1^*,\alpha_2^*,\alpha_3^*,\alpha_4^*,\alpha_5^*,\alpha_6^*\} . \tag{3.64}$$

Then, equations (3.59) and (3.60) may be rewritten in the matrix form,

$$\eta_2^* = [\Phi_2]\{\alpha_i^*\}, \tag{3.65}$$

$$\psi^* = [\Phi_4]\{\alpha_i^*\}. \tag{3.66}$$

Substituting equations (3.65) and (3.66) into equation (3.61) yields

$$\{\eta_o^*\}^e = \left\{\begin{array}{c} [\Phi_2]_j \\ [\Phi_2]'_j \\ [\Phi_4]_j \\ [\Phi_2]_{j+1} \\ [\Phi_2]'_{j+1} \\ [\Phi_4]_{j+1} \end{array}\right\} \{\alpha_i^*\} . \tag{3.67}$$

Hence, one may express the nodal displacements in terms of the generalized coordinates $\{\alpha_i^*\}$ as

$$\{\eta_o^*\}^e = [A]_o\{\alpha_i^*\}, \tag{3.68}$$

where

$$[A]_o = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & \xi_e & \xi_e^2 & \xi_e^3 & 0 & 0 \\ 0 & 1 & 2\xi_e & 3\xi_e^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \xi_e \end{bmatrix}. \tag{3.69}$$

Now inverting equation (3.68), one obtains

$$\{\alpha_i^*\} = [A]_o^{-1}\{\eta_o^*\}^e , \tag{3.70}$$

where

$$[A]_o^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -3\xi_e^{-2} & -2\xi_e^{-1} & 0 & 3\xi_e^{-2} & -\xi_e^{-1} & 0 \\ 2\xi_e^{-3} & \xi_e^{-2} & 0 & -2\xi_e^{-3} & \xi_e^{-2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\xi_e^{-1} & 0 & 0 & \xi_e^{-1} \end{bmatrix}. \tag{3.71}$$

Details of the calculation of this matrix may be found in Appendix K.

Using equations (3.53), (3.65)-(3.66) and (3.70), one may write the shape functions as

$$[N_2] = [\Phi_2][A]_o^{-1}, \qquad (3.72)$$

$$[N_4] = [\Phi_4][A]_o^{-1}. \qquad (3.73)$$

Then, substituting equations (3.72) and (3.73) into equations (3.55)-(3.57) yields

$$[M_o^*]^e = [A]_o^{-1T}[(1+\beta_a)\int_o^{\xi_e}[\Phi_2]^T[\Phi_2]d\xi + \sigma\int_o^{\xi_e}[\Phi_4]^T[\Phi_4]d\xi]\;[A]_o^{-1}, \qquad (3.74)$$

$$[D_o^*]^e = [A]_o^{-1T}\{2\beta^{1/2}\bar{u}\int_o^{\xi_e}[\Phi_2]^T[\Phi_2]'d\xi + \mathcal{K}\int_o^{\xi_e}[\Phi_2]^T[\Phi_2]d\xi\}\;[A]^{-1}, \qquad (3.75)$$

$$\begin{aligned}[K_o^*]^e = [A]_o^{-1T}\{&\int_o^{\xi_e}[[\Phi_2]''^T[\Phi_2]'' - r_o([\Phi_2]''^T[\Phi_4]+[\Phi_4]^T[\Phi_2]'')+r_o^2[\Phi_4]^T[\Phi_4]]\;d\xi \\ &+\Lambda\int_o^{\xi_e}[r_o^2[\Phi_2]'^T[\Phi_2]' + r_o([\Phi_2]'^T[\Phi_4]' + [\Phi_4]'^T[\Phi_2]')+[\Phi_4]'^T[\Phi_4]']d\xi \\ &+\bar{u}^2\int_o^{\xi_e}[\Phi_2]^T[\Phi_2]''d\xi \\ &+\int_o^{\xi_e}[\Phi_2]^T(\frac{\partial\Pi_o}{\partial\xi}[\Phi_2]' + \Pi_o[\Phi_2]'')d\xi\;\}\;[A]_o^{-1}. \end{aligned} \qquad (3.76)$$

Here, the combined force Π_o and its derivative $\partial\Pi_o/\partial\xi$ in each element are again approximated by linear functions, as given in equation set (3.45). By substituting equation set (3.45) into equations (3.74)-(3.76), and evaluating the integrals in equations (3.74)-(3.76), one obtains the element matrices, as follows:

$$[M_o^*]^e = [A]_o^{-1T}\;[(1+\beta_a)[J_1^*] + \sigma\;[J_4^*]]\,[A]_o^{-1}, \qquad (3.77)$$

$$[D_o^*]^e = [A]_o^{-1T}\;[2\beta^{1/2}\bar{u}\;[J_5^*] + \mathcal{K}[J_1^*]]\;[A]_o^{-1}\;, \qquad (3.78)$$

$$\begin{aligned}[K_o^*]^e = [A]_o^{-1T}\;\{&[J_3^*]-r_o([J_6^*]+[J_6^*]^T)+r_o^2[J_4^*]+\Lambda[r_o^2[J_2^*]+r_o([J_7^*]+[J_7^*]^T)+[J_8^*]] \\ &+\bar{u}^2[J_9^*]+a_1[J_9^*]+b_1[J_5^*]+a_2[J_{10}^*]+b_2[J_{11}^*]\}\;[A]_o^{-1}\;, \end{aligned} \qquad (3.79)$$

where

$$
\left.
\begin{aligned}
[J_1^*] &= \int_0^{\xi_e} [\Phi_2]^T[\Phi_2]\, d\xi, \\
[J_2^*] &= \int_0^{\xi_e} [\Phi_2]'^T[\Phi_2]'\, d\xi \\
[J_3^*] &= \int_0^{\xi_e} [\Phi_2]''^T[\Phi_2]''\, d\xi, \\
[J_4^*] &= \int_0^{\xi_e} [\Phi_4]^T[\Phi_4]\, d\xi\ , \\
[J_5^*] &= \int_0^{\xi_e} [\Phi_2]^T[\Phi_2]'\, d\xi\ , \\
[J_6^*] &= \int_0^{\xi_e} [\Phi_2]''^T[\Phi_4]\, d\xi, \\
[J_7^*] &= \int_0^{\xi_e} [\Phi_2]'^T[\Phi_4]'\, d\xi\ , \\
[J_8^*] &= \int_0^{\xi_e} [\Phi_4]'^T[\Phi_4]'\, d\xi \\
[J_9^*] &= \int_0^{\xi_e} [\Phi_2]^T[\Phi_2]''\, d\xi\ , \\
[J_{10}^*] &= \int_0^{\xi_e} \xi[\Phi_2]^T[\Phi_2]''\, d\xi\ , \\
[J_{11}^*] &= \int_0^{\xi_e} \xi[\Phi_2]^T[\Phi_2]'\, d\xi\ ,
\end{aligned}
\right\} \qquad (3.80)
$$

which are evaluated in Appendix L.

3.6 CALCULATION OF THE DIMENSIONLESS COMBINED FORCE FOR THE INEXTENSIBLE CASE

Before calculating the dimensionless combined force in a pipe with arbitrary shape, we first perform the calculation of this force in an incomplete circular pipe, subtending an angle r_o, as shown in Fig. 7(a); then, the result of this simple case may be applied to obtain this force in each finite element for the case of arbitrary shape. Since the discretization has been introduced in the previous sections, the pipe is divided into a series of constant curvature elements, each of which may be treated as an incomplete circular pipe.

The differential equation of the dimensionless combined force derived in Chapter II, equation (2.89), will be re-written as

$$\frac{d^2\Pi_o}{d\xi^2} + r_o^2\Pi_o = -r_o^2\bar{u}^2 + \gamma(r_o\alpha_{x_o} + \frac{\partial\alpha_{z_o}}{\partial\xi}) , \qquad (3.81)$$

where α_{x_o} and α_{z_o} are the direction cosines of the gravity acceleration g along the x_o- and z_o-axes, and which are only functions of ξ.

From Fig. 7(a), if we assume that the pipe initially lies in a vertical plane, the angle of the vectors $\overrightarrow{P_ox_o}$ and $\vec{g}$ can be written as

$$(\overrightarrow{P_ox_o},\vec{g}) = (r_o\xi + \alpha) + K2\pi, \qquad (3.82)$$

where K is a positive or negative integer, and α is the principal value of the angle $(\overrightarrow{AO},\vec{g})$ which can be positive or negative depending on the position of the vector $\vec{g}$, as shown in Fig. 7(b,c). Therefore, one can write the direction cosines as follows:

$$\left.\begin{aligned} \alpha_{x_o} &= \text{Cos } (\overrightarrow{P_ox_o},\vec{g}) = \text{Cos } (r_o\xi + \alpha), \\ \alpha_{z_o} &= \text{Cos } (\overrightarrow{P_oz_o},\vec{g}) = \text{Sin } (r_o\xi + \alpha). \end{aligned}\right\} \qquad (3.83)$$

Substitution of equation set (3.83) into equation (3.81) yields

$$\frac{d^2\Pi_o}{d\xi^2} + r_o^2\Pi_o = - r_o^2\bar{u}^2 + 2r_o\gamma \ \mathrm{Cos}(r_o\xi_o + \alpha) , \quad (3.84)$$

which is a second order linear differential equation. Hence, one obtains the general solution in the form

$$\Pi_o = C_1 \ \mathrm{Sin} \ (r_o\xi + C_2) - \bar{u}^2 + \gamma\xi \ \mathrm{Sin} \ (r_o\xi + \alpha) , \quad (3.85)$$

where C_1 and C_2 are two integral constants which can be determined from the boundary conditions.

It is noted that when r_o is set equal to zero (i.e., in the case of an inclined straight pipe), the combined force can be reduced to

$$\Pi_o = (\gamma \ \mathrm{Sin} \ \alpha)\xi + C_1' . \quad (3.86)$$

If the gravity effect is assumed to be negligible, the boundary conditions associated with equation (3.84) are as follows:

(i) for a clamped-free incomplete circular pipe

$$\Pi_o|_0 = \Pi_p|_1 \ \mathrm{Cos} \ r_o - (1 - \mathrm{Cos} \ r_o)\bar{u}^2 ,$$

$$\Pi_o|_1 = \Pi_p|_1 , \quad (3.87)$$

$$\frac{d\Pi_o}{d\xi}|_1 = - \frac{1}{r_o} (r_o - \mathrm{Sin} \ r_o)(\bar{u}^2 + \Pi_p|_1)^2$$

where, it is recalled that $\Pi_p = (A_iP_i - A_eP_e)L^2/EI$, which is the pressure-related component of Π_o. Details of the derivation of equation set (3.87), may be found in Appendices M, N and O. In these Appendices it is assumed that (a) the pressure distribution of the internal fluid is a linear function along the length of the pipe, and (b) the initial centerline may be used as an approximate shape of the strained centerline after static deformation.

(ii) for clamped-clamped, clamped-pinned and pinned-pinned incomplete circular pipes

$$\left.\begin{aligned} \Pi_o\big|_1 &= -\bar{u}^2 , \\ \frac{d\Pi_o}{d\xi}\bigg|_1 &= 0 . \end{aligned}\right\} \qquad (3.88)$$

Details of the derivation of equation set (3.88) based on the same assumptions as above may be found in Appendices P,Q and R.

Combination of equations (3.85), (3.87) and (3.88) yields the combined force Π_o. Hence,

(i) for a clamped-free incomplete circular pipe

$$\Pi_o = C_1 \text{ Sin } (r_o\xi + C_2) - \bar{u}^2 , \qquad (3.89)$$

where

$$\left.\begin{aligned} C_1 &= (\Pi_p|_1 + \bar{u}^2)/\text{Sin } (r_o + C_2) , \\ C_2 &= -\text{ Tan}^{-1} [r_o^2/(r_o - \sin r_o)(\bar{u}^2 + \Pi_p|_1)] - r_o ; \end{aligned}\right. \qquad (3.90)$$

(ii) for clamped-clamped, clamped-pinned and pinned-pinned incomplete circular pipes

$$\Pi_o = -\bar{u}^2. \qquad (3.91)$$

Finally, to obtain the dimensionless combined force in a pipe with arbitrary shape, we proceed in a manner similar to what was done in the case of an incomplete circular pipe. First, the boundary conditions on one end are obtained. Second, the combined force in the finite-element at that boundary, where the conditions are specified, are obtained based on discretization, whereby a pipe element may be treated as an incomplete circular pipe, as mentioned earlier. Next, this force in the adjacent finite element is obtained through the continuity condition at the nodes.

3.7 THE VISCOUS DAMPING COEFFICIENT c

The only undetermined quantity in equations (2.58)-(2.60) is the viscous damping coefficient c, associated with viscous resistance due to the surrounding fluid. Assuming Stokes-flow conditions to hold, Batchelor (1967) and Paidoussis & Luu (1985) estimated the viscous damping coefficient for a circular cylinder in a quiescent fluid to be

$$c = 2\pi \sqrt{2}\, \Omega\, \rho_{fe}\, r_{ou}^{2}\, (\nu/\Omega r_{ou}^{2})^{1/2}, \qquad (3.92)$$

where ρ_{fe} is the fluid density, ν the kinematic viscosity, r_{ou} the cylinder outer radius and Ω the radian frequency of oscillation. This equation is identical in form to the experimental result obtained by Ramberg and Griffin (1977) from extensive tests on stranded cables.

3.8 EIGENVALUE ANALYSIS OF THE PROBLEM FOR BOTH THE IN-PLANE AND OUT-OF-PLANE CASES

In both cases of in-plane and out-of-plane motions the global equation of motion may be written as follows:

$$[M]\{\ddot{\eta}\} + [D]\{\dot{\eta}\} + [K]\{\eta\} = \{0\}, \qquad (3.93)$$

where [M], [D] and [K] are the global mass, damping and stiffness matrices, which are assembled from the corresponding matrices for an element of the dynamic system (as given in equations (3.47)-(3.49) or (3.77)-(3.79), respectively); $\{\eta\}$ is the global displacement vector.

By setting

$$\{Y\} = \begin{Bmatrix} \{\eta\} \\ \hline \{\dot{\eta}\} \end{Bmatrix}, \qquad (3.94)$$

one obtains

$$[[I], [O]]\{\dot{Y}\} = [[O], [I]] \{Y\}, \tag{3.95}$$

and then equation (3.93) may be rewritten as

$$[[O], [M]]\{\dot{Y}\} = - [[K],[D]]\{Y\}. \tag{3.96}$$

Finally, combination of equations (3.95) and (3.96) yields the equation of motion in the form of the first order differential equation

$$\begin{bmatrix} [I] & [O] \\ & \\ [O] & [M] \end{bmatrix} \{\dot{Y}\} = \begin{bmatrix} [O] & [I] \\ & \\ -[K] & -[D] \end{bmatrix} \{Y\}. \tag{3.97}$$

For solution of the eigenvalue problem, equation (3.97) may be cast into the general form

$$\lambda \ [B]\{X\} = [A]\{X\}. \tag{3.98}$$

To implement the procedure, let

$$\{Y\} = \{X\}e^{i\omega t}, \tag{3.99}$$

where ω is a dimensionless frequency related to the circular frequency of motion, Ω, by

$$\omega = (\frac{M_t+M_f}{EI})^{1/2} \ \Omega \ L^2 \ . \tag{3.100}$$

In general, ω will be complex and the dynamic system will be stable or unstable accordingly as the imaginary part of ω is positive or negative.

Substitution of equation (3.99) into equation (3.97) leads to an asymmetrical eigenvalue problem or order 2N (N being the total number of degrees of freedom of the system) as follows:

$$\lambda \begin{bmatrix} [I] & [O] \\ [O] & [M] \end{bmatrix} \{X\} = \begin{bmatrix} [O] & [I] \\ -[K] & -[D] \end{bmatrix} \{X\} , \tag{3.101}$$

where

$$\lambda = i\omega. \tag{3.102}$$

Implied in the above equation is the subsequent reduction of the system to order 2K < 2N from inclusion of the essential boundary conditions and condensation of the inertialess degrees of freedom.

For any given system the dimensionless parameters β, $\beta_a, \beta_a', \Lambda, \sigma$ and r_o are known. The complex frequencies of the system will be calculated for any value of the dimensionless flow velocity $\bar{u}$, and the critical velocity of the system will also be obtained. The method of computation using the IMSL subroutine, EIGZF, from the McGill computer library is described in the IMSL manual.

CHAPTER IV

RESULTS AND DISCUSSIONS FOR THE INEXTENSIBLE CASE

This chapter presents the results of an investigation of dynamical in-plane and out-of-plane behaviour of curved pipes conveying fluid under the effect of the internal flow, for the inextensible case.

These results were obtained by using the finite-element method discussed in the previous Chapter III for both the in-plane and out-of-plane motions. Two computer codes were developed to solve the eigenvalue problems, and a listing of these programs for in-plane and out-of-plane motions may be found in Appendix S.

The computer programs were used to obtain the complex frequencies of some typical dynamic systems. The calculations in the programs were conducted in double precision.

Two variants of the theory are used in the calculations. In the first, the steady-state, "initial" forces due to pressure and centrifugal forces are entirely suppressed; this is done by setting $Q_z = 0$. Hence, this form of the theory is the traditional inextensible theory, making the same assumptions as those made previously, for example by Chen (1971c, 1972a,b, 1973). The second variant of the theory may be referred to as the "modified inextensible theory", where, in effect, the combined force Π (and hence the centrifugal and pressure-induced forces therein) are taken into account - yet, retaining the assumptions of inextensibility of the centerline. Thus, this latter theory is intermediate between the true extensible theory (to be presented in Chapter V) and the traditional inextensible theory. Since there is no question that $\Pi \neq 0$ in any real physical system this second variant of the theory is considered to be the more realistic of the two.

Before calculating the new results, the finite element programs are verified by considering several configurations of the tube and comparing the results obtained from the present programs with existing results calculated by analytical or other methods, as described in Sections 4.1 to 4.3.

4.1 VERIFICATION OF THE FINITE-ELEMENT PROGRAM FOR THE CASE OF STRAIGHT PIPES

In this section the finite-element programs are verified, in the first instance, by solving the problem of straight pipes, i.e. the case of $r_o = L/R_o$ set equal to zero, and comparing the complex frequencies thereby obtained with existing results.

Of course straight pipes have the same eigenfrequencies for in-plane and out-of-plane motions; hence, calculations may be conducted with either the in-plane or out-of-plane program.

4.1.1 Straight Tubular Beams with No Internal Flow

Tables 1-4 show the natural frequencies of oscillation of a simple tubular beam (no flow and neglecting the effect of gravity force) obtained by the finite element methods for in-plane and out-of-plane motions. It is seen that the displacement models used in the finite-element methods of in-plane and out-of-plane motions (quartic and cubic, respectively) are satisfactory: the finite element solutions obtained by these models converge to the exact solutions, the convergence being very fast; the in-plane calculations are generally more successful than the out-of-plane ones for the same number of finite elements, reflecting perhaps the higher-order polynomial displacement functions utilized in the former case. Only six or less elements are needed to get a high accuracy of results, as seen in the tables, namely to within 2.5% of the analytical results, and usually better, for the first to fifth modes of these systems.

4.1.2 Horizontal Cantilevered Pipe Conveying Fluid

Figures 9 and 10 deal with horizontal cantilevers with $\gamma = 0$ and $\beta = 0.200$ and 0.295 respectively, which were studied by Gregory and Paidoussis (1966a). We note that, at $\bar{u} = 0$ (no flow), the system behaves as a simple cantilever beam. The effect of flow, for small values of $\bar{u}$, is to damp free motions of the system in all modes (i.e., the imaginary part of the eigenfrequencies, $Im(\omega)$, which is associated when damping is positive). However, the system will lose stability by flutter in the second mode, when the flow velocity exceeds a certain value, referred to as the critical flow velocity.

We can see that the finite-element results, obtained by using a six-element discretization scheme in the program for out-of-plane motions, are very close to the analytical results obtained by Gregory and Paidoussis (1966a). As expected, the discrepancies become more pronounced for the higher modes and at the higher flow velocities. As shown by Gregory and Paidoussis (1966a), each mode contains higher beam-mode components, with the importance of these components increasing with increasing values of $\bar{u}$. Thus, the fourth mode shape at say $\bar{u} = 9$, would contain appreciable components of the fifth and sixth beam modes, which could evidently not be truly well represented by a six-element discretization scheme - at least in the out-of-plane solution scheme involving cubic displacement functions.

4.1.3 Vertical Tubular Cantilevers Conveying Fluid

Figures 11 and 12 deal with hanging cantilevers with $\gamma = 100$ and $\beta = 0.4$ and 0.65. We can see that, in all cases, low velocities damp the system in all modes ($\mathrm{Im}(\omega) > 0$) and the effect of the flowing fluid is to reduce the frequencies of oscillation, $\mathrm{Re}(\omega)$. As the flow velocity increases, the locus of at least one mode crosses the $\mathrm{Re}(\omega)$-axis, and the system loses stability by flutter.

In this case also, the finite-element results obtained by using a six-element discretization scheme for the in-plane motion, are found to agree very well with the analytical results obtained by Paidoussis (1970). Indeed, comparatively to the results obtained in Figs. 9 and 10 with the out-of-plane solution, the agreement in this case is much better, and very good even for the fourth mode, up to $\bar{u} = 14$; the reason is very likely related to the use of quintic displacement functions in the in-plane solution, as compared to cubic ones in the out-of-plane solution.

Figure 13 deals with a standard cantilever (the free end above the fixed one) with $\gamma = -10$ and $\beta = 0.2$. We note that with no flow both the analytical and finite-element methods yield $\omega = \pm\, 1.8524i$ for the first mode; this implies that the cantilever is unstable due to its own weight and that of the enclosed fluid. The instability is of the buckling type, similarly to the case of a real column. If the flow velocity increases, the negative branch of the first mode locus eventually becomes positive and the system is stabilized. At still higher flow, however, the system loses stability by flutter in the second mode. For this problem also, the finite-element results (obtained by using the program for in-plane motion) with six-element discretization agree well with the analytical results of Paidoussis (1970).

4.1.4 A Long, Vertical Cantilevered Pipe Conveying Fluid and Submerged in a Quiescent Fluid

Figure 14 shows the Argand diagram for the lowest three modes of the long vertical tubular cantilever studied by Luu and Paidoussis (1985). An identical system with the following associated parameters has been used for the calculation to be presented here:

(i) a steel tube with Young's modulus $E = 200*10^6$ kN/m^2 and density $\rho_t = 7.83*10^3$ kg/m^3, conveying fluid and submerged in a quiescent fluid;

(ii) the length of the pipe $L = 1{,}000$ m; internal and external diameters $D_i = 0.45$ m and $D_o = 0.50$ m, respectively;

(iii) a constant fluid density for both internal and external fluid, equal to 998 kg/m^3.

We note that, in the case of downward flow (here $\bar{u} > 0$), the system remains stable when the flow velocity increases, at least up to the maximum values of $\bar{u}$ shown in Fig. 14. However, the system loses stability by flutter in the case of upward flow ($\bar{u} < 0$), when the flow velocity exceeds a relatively small critical value. The shape of the curves in the case of upward internal flow is very flat, as compared to the previous cases, since in this case the tension on the pipe due to its own weight (the pipe here being very long) is very large. Therefore, the loci of all modes are essentially straight lines parallel to the Im(ω)-axes.

For this case, the programs for in-plane and out-of-plane motion produce the same result by using six- and seven-element discretization schemes, respectively. The finite-element results are similar to those obtained by Luu & Paidoussis (1985), but not as close to them as in previous cases (Figs. 9-13); one reason could be the effect of damping due to the surrounding fluid, which was not calculated in exactly the same way as by Luu & Paidoussis. It is believed that the present results are more self-consistent and accurate.

4.1.5 General Comments on the Straight-Pipe Results

From the comparisons conducted between the eigenfrequencies calculated by the finite element programs of this Thesis and those calculated by other analytical, semi-analytical or numerical schemes by other investigators (Figs. 9-14), it may be concluded that, the agreement being generally very good, the finite element formulation and computer programs of this Thesis have been verified - at least insofar as the dynamics of straight pipes conveying fluid is concerned.

We next turn our attention to the dynamics of curved pipes conveying fluid, which is the topic of principal interest of this Thesis.

4.2 VERIFICATION OF THE FINITE ELEMENT PROGRAM FOR THE CASE OF IN-PLANE BEHAVIOUR OF CURVED PIPES, WHERE THE AXIAL FORCE Q_z IS NEGLECTED

In this section, calculations by means of the finite-element method for in-plane motion are continued, to verify previously obtained results for curved pipes conveying fluid. Here, the axial force Q_z is neglected, and the internal pressure is constant. This is done, so that the equations of motion (3.2)-(3.5) then become identical to those obtained by Chen (1972a,b) without implying that these equations are correct, from the physical point of view. Therefore, the finite-element results of this case may be compared with the analytical results obtained by Chen (1972a,b), thereby testing the finite-element formulation developed in this Thesis, for curved pipes (with the limiting condition of $Q_z = 0$).

4.2.1 Semi-Circular Pipe Conveying Fluid

Figures 15-20 show the in-plane natural frequencies of clamped-clamped, clamped-pinned and pinned-pinned semi-circular tubes conveying fluid, as functions of flow velocity. At $\bar{u} = 0$, the tube behaves as a semi-circular ring. When the flow velocity increases, the natural frequencies become smaller according to this formulation of the problem ($Q_z = 0$) and if the flow velocity exceeds a certain value, the tube becomes unstable in the first mode, the instability being of the buckling type. With further increase in the flow velocity, instability may occur in the higher modes also.

It can be seen in Figs. 15-20 that the finite-element results using an eight-element discretization scheme are very close to the analytical results of Chen (1972a).

4.2.2 Natural Frequencies of a Semi-Circular Pipe Conveying Fluid as Functions of the Internal Fluid Pressure

Figures 21-22 show the natural frequencies of a clamped-clamped semi-circular pipe conveying fluid at a dimensionless flow velocity $\bar{u} = \pi$, as a function of the internal fluid pressure. The effects of the internal fluid pressure are quite similar to those of the flow velocity; i.e., the natural frequencies monotonically decrease with increasing fluid pressure, until buckling instability occurs at the critical pressure.

For the total number of elements N > 7 the finite-element results converge, and as can be seen in Figs. 21-22 the eigenfrequencies then agree quite well with Chen's (1972a) analytical results.

4.2.3 General Comments on the Inextensible In-Plane Dynamics of Semi-Circular Pipes Conveying Fluid

It may be concluded that the in-plane dynamical behaviour of semi-circular pipes conveying fluid, when the 'initial' axial force due to pressure and centrifugal forces represented by Q_z is neglected, is as presented by Chen (1972a,b).

This may be considered to be a verification of the basic finite element solution formulated in this Thesis, as the equations obtained in Chapter III for this special case were shown to be identical to Chen's (1972a,b).

4.3 VERIFICATION OF THE FINITE-ELEMENT PROGRAM FOR THE CASE OF OUT-OF-PLANE MOTION OF UNIFORMLY CURVED PIPES, WHEN THE COMBINED FORCE Π IS NEGLECTED

In this section, the finite-element program for out-of-plane motion is verified by comparing with the analytical results obtained by Chen (1973), once again neglecting the axial force Q_z as well as the internal pressure force (i.e. $\Pi = 0$), as was done by Chen (1973).

4.3.1 Clamped-Clamped Semi-Circular Pipe Conveying Fluid

Similarly to the case of in-plane motion, Figs. 23-25 show the natural frequencies of clamped-clamped semi-circular pipes conveying fluid, as functions of the flow velocity. At $\bar{u} = 0$, the pipe behaves as a semi-circular ring. When the flow velocity increases, the natural frequencies become smaller according to this formulation of the problem ($\Pi = 0$), and if the flow velocity exceeds the critical value, the pipe becomes unstable in the first mode, the instability being of the buckling type. With further increase in the flow velocity, instability may occur in higher modes also. We can see that the out-of-plane frequencies are lower than the in-plane ones, and that the critical flow velocity is also lower.

The finite-element results with an eleven-element (or more than eleven elements) discretization scheme are very close to the analytical results obtained by Chen (1973), except for the second mode of Figs. 24-25, where Chen's results were probably plotted incorrectly.

4.3.2 Clamped-Free Semi-Circular Pipe Conveying Fluid

Figure 26 shows the Argand diagram of the lowest four complex frequencies of a cantilevered semi-circular pipe. The effect of low flow velocities is to damp the system in all modes. However, the system will lose stability by flutter in the second mode, when the flow velocity exceeds the critical value.

The finite-element results (using a ten-element discretization scheme) are similar, but numerically not very close to Chen's (1973) results, except for the second mode throughout and for the first mode when $\bar{u} > 1.5$, where the results were found to be very different. Even though the number of elements in the discretization scheme was increased to fourteen elements, this discrepancy remains. It is possible that Chen's six-term solution routine is insufficiently refined (note the discrepancies at $\bar{u} = 0$ for the third and fourth modes), or that an error may have crept in, in his computer program; however, it is impossible to pin-point the source of the discrepancy at this point.

4.3.3 General Comments on the Inextensible Out-of-Plane Dynamics of Semi-Circular Pipes Conveying Fluid

It may be concluded that the out-of-plane dynamical behaviour of semi-circular pipes conveying fluid, when the combined force Π is set to zero, is as presented by Chen (1973) - at least for pipes supported at both ends. This verifies the finite element formulation of this Thesis for out-of-plane motions of curved pipes.

The behaviour for cantilevered pipes, as predicted here is considerably different than that given by Chen (1973), despite the fact that the equations of motion and boundary conditions when $\Pi = 0$ are identical. The discrepancy is substantial in quantitative terms, and its source cannot be identified; however, the qualitative behaviour of the system as obtained by two solutions is the same.

4.4 RESULTS FOR IN-PLANE MOTION OF A CURVED PIPE FOR THE INEXTENSIBLE CASE AND INCLUDING THE COMBINED FORCE Π

The work presented so far verified the finite-element model for in-plane motion of curved pipes, when the combined force Π is neglected. Calculations will now be conducted to obtain the in-plane eigenfrequencies of curved pipes conveying fluid in the inextensible case, when the combined axial force Π is properly taken into account. Thus, the calculations presented here are with the second variant of the inextensible theory referred to at the beginning of this Chapter : the modified inextensible theory.

4.4.1 Clamped-Clamped, Clamped-Pinned and Pinned-Pinned Semi-Circular Pipes Conveying Fluid

Figures 27-29 show the in-plane natural frequencies of clamped-clamped, clamped-pinned and pinned-pinned semi-circular pipes conveying fluid as functions of the flow velocity.

When the axial force Q_z is neglected, the natural frequencies monotonically decrease with increasing flow velocity or internal pressure, until buckling instability occurs at the critical flow velocity or the critical pressure, as discussed in Sections 4.2.1 and 4.2.2. On the other hand, if the axial force Q_z is taken into account, the effect of the fluid flow is less pronounced. It tends to reduce the first-mode natural frequency, but does not cause buckling in the flow range investigated. It is also interesting to observe that the eigenfrequencies of some of the higher modes actually increase with flow velocity. Thus, whether Q_z is taken into account or not is very important: in the first case, where Q_z is neglected, the effect of internal flow on the natural frequencies consists of the centrifugal and Coriolis forces, whereas in the second case (where Q_z is not neglected), the internal flow exerts only a Coriolis force. This is because the terms associated with the dimensionless combined force Π given in equation (3.88), (i.e. the terms in the equation of in-plane motion (3.44) involving this force Π) cancel out those arising from the centrifugal force; it is recalled that it is the centrifugal forces that are responsible for the buckling instability obtained before, in the case of pipes with both ends supported.

4.4.2 Uniformly Curved Clamped-Free Pipe Conveying Fluid

Figure 30 shows the Argand diagram for the eigenfrequencies of in-plane vibrations of a clamped-free semi-circular pipe in the two cases of not including and including the combined axial force Π. For both the cases, we can see that, small flow velocities damp the system and result in a reduction of the frequencies of oscillation, $\mathrm{Re}(\omega)$. In the case of $\Pi = 0$, when the flow velocity is increased, the frequency of oscillation in the first mode becomes zero and the system loses stability by buckling. If the flow velocity is further increased, the system becomes unstable by flutter in the third mode. In the case of the same system, but including the combined axial force, Π, we can see that low velocities damp the system in this case also. However, when the flow velocity is increased, the system loses stability by buckling, not only in the first mode (at approximately the same flow rate as for $\Pi = 0$), but also in the second mode, at higher flow velocity. If the flow velocity is further increased, the system becomes unstable by flutter in the third mode, but at much lower flow velocity than for $\Pi = 0$.

Figure 31 deals with the same system but with $\beta = 0.25$, instead of $\beta = 0.75$. Here, it is seen that this system loses stability by buckling and flutter in the same modes as the previous system ($\beta = 0.75$), at similar critical flow velocities.

Figure 32 shows the Argand diagram for the first two modes of a uniformly curved pipe conveying fluid with $\beta = 0.25$ and a total angle $r_o = \pi/2$. By comparing these loci with the ones shown in Figs. 30 and 31, it becomes obvious that there exist significant differences: in the previous case ($r_o = \pi$), the system buckles in the first and second modes, whereas here the system loses stability by buckling in the first mode and by flutter in the second mode; there is no buckling associated with the second mode. Moreover, the critical flow velocities are higher in this case, as compared to the case of $r_o = \pi$. The reason for this is that the system in this case is stiffer than the previous case. Thus, the frequencies of this system are higher, and the locus of the second mode seems to be shifted far away from the Im(ω)-axis.

4.4.3 Effects of Total Angle r_o and Radius of Curvature R_o of a Curved Pipe on Its Frequencies

It is of interest to study the effects of total angle and radius of curvature of a pipe on its frequencies, partly because it has application to design.

Consider a clamped-clamped uniformly curved pipe, the radius of curvature R_o of which remains constant. Figure 33 shows the variation of the in-plane frequencies Re(ω) with the total angle r_o of the pipe. It is seen that when the total angle r_o decreases, the pipe becomes stiffer (in other words less flexible), and hence the frequencies become higher. Here, the finite-element results can be found to agree with those of Chen's (1972a) for the case of $\bar{u} = 0$, when the combined force Π has been neglected.

Now consider another case, where the total length L of the pipe remains fixed. Figure 34 shows the variation of the in-plane frequencies with the radius of curvature R_o. We can see that at small radius of curvature R_o, the frequencies are sensitive to R_o, while at large radius of curvature the frequencies become insensitive to R_o, and the

n^{th} frequency converges to the $(n+1)^{th}$ frequency of a straight pipe. Here, there is a difference in the mode number of straight and curved pipes because the boundary conditions are different. In the present case the axial displacement w ($\partial w/\partial s = \bar{u}/R_o$) is set equal to zero at the boundaries, whereas in the case of straight pipes these conditions are not specified.

If one does not specify these conditions in the case of large radius of curvature the results in both cases are the same.

Perhaps an easier way of understanding this, at first sight strange finding, is by considering the modal shapes involved. For the inextensible theory, the total length of the pipe remains constant during vibration, and it is easy to see that the equivalent of the first mode of a straight beam, involving no nodal points other than the two support points, is in fact extensional; hence, in this theory it does not exist. The lowest inextensional mode of the circular pipe corresponds to the asymmetric shape which, as the radius of curvature tends to infinity, converges to the second mode of a straight beam.

4.5 RESULTS FOR OUT-OF-PLANE MOTION FOR A CURVED PIPE CONVEYING FLUID FOR THE INEXTENSIBLE CASE, INCLUDING THE COMBINED AXIAL FORCE Π

In Sections 4.1 and 4.3 it was shown that the finite-element method for out-of-plane motion gave results in very good agreement with those obtained previously by other methods in the cases of straight pipes and curved pipes, when the axial force Π is neglected. The task of this section is to compute the out-of-plane eigenfrequencies of curved pipes conveying fluid, when this force is taken into account.

4.5.1 Clamped-Clamped Semi-Circular Pipe Conveying Fluid

Figure 35 shows the natural frequencies of a clamped-clamped semi-circular pipe conveying fluid, as functions of the flow velocity. Similarly to the in-plane motion, in the absence of the axial force Π the out-of-plane frequencies monotonically decrease with increasing flow velocity, until buckling instability occurs at the critical velocity, as previously discussed at greater length in Section 4.3. In the case where the axial force Π is taken into account, on the other hand, the effect of internal flow on the frequencies is small, and the system is predicted to remain stable.

4.5.2 Clamped-Free Semi-Circular Pipe Conveying Fluid

Figure 36 shows the Argand diagram of the four lowest out-of-plane eigenfrequencies of a clamped-free semi-circular pipe conveying fluid. Similarly to the case when the combined axial force Π, is neglected, the effect of the internal flow, for low velocities, is to damp free motion of the system in all modes. However, the system loses stability by flutter in the second mode at much lower flow velocities.

Figure 37 shows the Argand diagram of the two lowest out-of-plane frequencies of another clamped-free uniformly curved pipe conveying fluid with $\beta = 0.25$ and a total subtended angle $r_o = \pi/2$. It is seen that in the case $\bar{u} > 0$, this system is stable in the first mode and loses stability by flutter in the second mode. On the other hand, in the case $\bar{u} < 0$ (i.e., when the pipe sucks fluid at the free end) the system loses stability right away. However, the system is restablized in the second mode if the flow velocity is increased enough.

4.6 GENERAL DISCUSSION ON THE TWO VARIANTS OF THE INEXTENSIBLE THEORY

The two variants of the inextensible theory, it is recalled are (i) the traditional inextensible theory, in which the steady-state initial force associated with centrifugal forces and pressure are neglected ($Q_z = 0$); and (ii) the modified inextensible theory, where this force is taken into account ($\Pi \neq 0$), while still retaining the assumption of inextensibility of the centerline of the pipe.

As was seen in the foregoing (Sections 4.4 and 4.5), the effect of including Π is profound, insofar as the dynamics of the system is concerned. Thus, for pipes supported at both ends, the system is no longer subject to flow-induced instabilities; indeed, the natural frequencies of the system are rather insensitive to increasing flow velocity, increasing or decreasing somewhat, but remaining sensibly constant. This dynamical behaviour is similar to that predicted by the extensible theories of Hill & Davis (1974) and Doll & Mote (1974, 1977), which tends to suggest that the main differences between these theories and the traditional inextensible theory is not the extensibility of the centerline at all, but rather whether the combined steady-state axial forces (Π) are taken into account or not. This will be further discussed, once the results of the extensible theory have been presented in Chapter VI.

The effect of including Π is also profound in the case of cantilevered pipes (Figs. 30 and 36), albeit in this case the effect being mainly quantitative, rather than altering the fundamental dynamical behaviour of the system. Buckling and flutter instabilities occur in both cases, whether $\Pi = 0$ or $\Pi \neq 0$. However, inclusion of the combined axial force in this case has a destabilizing effect on the system, and the critical flow velocities for buckling and especially for flutter are substantially diminished.

As previously discussed by Svetlisky (1977) and by Doll & Mote (1974, 1977) and Hill & Davis (1974) the steady state force Π does exist and, of course, it must do work in the course of vibration of the system - just as pre-stress would do in any structure. However, unlike pre-stress, which may be negligible, Π increases in magnitude with increasing flow velocity ($\bar{u}$) at the same rate as the destabilizing (centrifugal) forces do. Hence, in the case of pipes with both ends supported, where the instability would occur when these centrifugal forces exceed the structural restoring forces, the inclusion of the steady state forces continually counterbalances the destabilizing forces, thus precluding the onset of instability.

In the case of cantilevered pipes, not discussed by Hill & Davis and Doll & Mote, but briefly discussed by Svetlisky, the situation is more complex, since the system is nonconservative and the Coriolis force also does work. As was shown by Paidoussis (1966), in connection with a similar dynamical system, tension in this case may indeed be destabilizing - and, indeed this is what has been observed here.

CHAPTER V

ANALYSIS OF THE EXTENSIBLE CASE

In Chapter III, the centerline of the pipe has been assumed to be inextensible for the analysis of the problem. Then, elaborate calculations were conducted to give results for various configurations, which were found to agree very well with those obtained in previous work. However, some researchers, e.g. Hill & Davis (1974) and Doll & Mote (1976), did not adopt this assumption; they proposed that because the geometrical configuration of the pipe and the forces acting on it are very complex, they may result in non-zero centerline strain. By this reasoning they proposed the following:

(i) the centerline of the pipe should not be considered to be inextensible, i.e., the centerline strain in the pipe,

$$\varepsilon = (\frac{\partial w}{\partial s} - \frac{u}{R_o}) \qquad (5.1)$$

does not vanish:

(ii) the axial force Q_z is linearly dependent on centerline strain, i.e.,

$$Q_z = EA_t \, \varepsilon, \qquad (5.2)$$

or

$$Q_z = EA_t(\frac{\partial w}{\partial s} - \frac{u}{R_o}) . \qquad (5.3)$$

The objective of the present chapter is to present the analysis of the problem based on the above assumptions. One of the motivations for this analysis is to study the difference between results obtained by the inextensible and extensible assumptions.

Based on the above assumptions, the dimensionless combined force Π depends only on the dimensionless displacements η_1 and η_3.

If the pipe initially lies in a vertical plane, or the gravity effect is neglected, we shall see later that again the governing equations of motion about the equilibrium position will separate into in-plane and out-of-plane motions, which allows independent investigation of the dynamics of the system in these two planar directions.

5.1 THE DIMENSIONLESS EQUATIONS OF STATIC EQUILIBRIUM AND THE ASSOCIATED BOUNDARY CONDITIONS IN THE EXTENSIBLE CASE

It should be noted that the dimensionless equations of static equilibrium, equations (2.82)-(2.85) derived in Chapter II, are valid for both cases of an inextensible and an extensible centerline.

From equation (5.3), the dimensionless combined force may be written as

$$\Pi = \Pi_p - \mathcal{A}\left(\frac{\partial \eta_3}{\partial \xi} - r_o \eta_1\right), \qquad (5.4)$$

where Π, Π_p and $\mathcal{A}$ are given in equation set (2.74).

Equation (5.4) implies that Π_o, which is the steady (static) form of the dimensionless combined force Π, does not depend on the static dimensionless displacement η_2^o, and the twist ψ_o. Therefore, the equations of in-plane static equilibrium (2.82) and (2.84) are decoupled from the equations of out-of-plane static equilibrium (2.83) and (2.85).

Moreover, if the gravity effect is neglected, or the pipe initially lies in a vertical plane, we can see that the forces acting on the pipe at the static equilibrium configuration are planar and lie in this same plane. Then, static out-of-plane deformation cannot happen (i.e., the dimensionless displacement η_2^o and the twist ψ_o

should vanish), and the equations of out-of-plane static equilibrium are satisfied automatically, or do not exist.

The equations of in-plane static equilibrium are very important for the investigation of static deformation of the system and also for the calculation of the dimensionless combined force. Now let us rewrite them, as follows:

$$\left(\frac{\partial^4 \eta_1^o}{\partial \xi^4} + r_o \frac{\partial^3 \eta_3^o}{\partial \xi^3}\right) + \frac{\partial}{\partial \xi}\left[\Pi_o\left(\frac{\partial \eta_1^o}{\partial \xi} + r_o \eta_3^o\right)\right] + r_o \Pi_o + \bar{u}^2\left(\frac{\partial^2 \eta_1^o}{\partial \xi^2} + r_o \frac{\partial \eta_3^o}{\partial \xi} + r_o\right) - \gamma \alpha_{x_o} = 0, \tag{5.5}$$

$$-\frac{\partial \Pi_o}{\partial \xi} + r_o\left(\frac{\partial^3 \eta_1^o}{\partial \xi^3} + r_o \frac{\partial^2 \eta_3^o}{\partial \xi^2}\right) + r_o \Pi_o\left(\frac{\partial \eta_1^o}{\partial \xi} + r_o \eta_3^o\right) + r_o \bar{u}^2\left(\frac{\partial \eta_1^o}{\partial \xi} + r_o \eta_3^o\right) + \gamma \alpha_{z_o} = 0. \tag{5.6}$$

Substituting equation (5.4) into equations (5.5)-(5.6), and neglecting the higher order terms, we may write the linearized equations of in-plane static equilibrium in the form

$$\left(\frac{\partial^4 \eta_1^o}{\partial \xi^4} + r_o \frac{\partial^3 \eta_3^o}{\partial \xi^3}\right) + \frac{\partial}{\partial \xi}\left[\Pi_p\left(\frac{\partial \eta_1^o}{\partial \xi} + r_o \eta_3^o\right)\right] - \mathcal{A} r_o\left(\frac{\partial \eta_3^o}{\partial \xi} - r_o \eta_1^o\right) + \bar{u}^2\left(\frac{\partial^2 \eta_1^o}{\partial \xi^2} + r_o \frac{\partial \eta_3^o}{\partial \xi}\right) + r_o(\Pi_p + \bar{u}^2) - \gamma \alpha_{x_o} = 0, \tag{5.7}$$

$$-\mathcal{A}\left(\frac{\partial^2 \eta_3^o}{\partial \xi^2} - r_o \frac{\partial \eta_1^o}{\partial \xi}\right) - r_o\left(\frac{\partial^3 \eta_1^o}{\partial \xi^3} + r_o \frac{\partial^2 \eta_3^o}{\partial \xi^2}\right) - r_o \Pi_p\left(\frac{\partial \eta_1^o}{\partial \xi} + r_o \eta_3^o\right) - r_o^2 \bar{u}^2\left(\frac{\partial \eta_1^o}{\partial \xi} + r_o \eta_3^o\right) + \frac{\partial \Pi_p}{\partial \xi} - \gamma \alpha_{z_o} = 0; \tag{5.8}$$

also, from equations (2.86)-(2.88), the associated boundary conditions are written as follows:

(i) at a clamped end

$$\left.\begin{aligned} \eta_1^o &= 0, \\ \frac{\partial \eta_1^o}{\partial \xi} &= 0, \\ \eta_3^o &= 0; \end{aligned}\right\} \tag{5.9}$$

(ii) at a pinned end

$$\left.\begin{aligned} \eta_1^o &= 0 , \\ \eta_3^o &= 0, \\ \left(\frac{\partial^2 \eta_1^o}{\partial \xi^2} + r_o \frac{\partial \eta_3^o}{\partial \xi}\right) &= 0; \end{aligned}\right\} \tag{5.10}$$

(iii) at a free end

$$\left.\begin{aligned} \left(\frac{\partial^2 \eta_1^o}{\partial \xi^2} + r_o \frac{\partial \eta_3^o}{\partial \xi}\right) &= 0, \\ \left(\frac{\partial^3 \eta_1^o}{\partial \xi^3} + r_o \frac{\partial^2 \eta_3^o}{\partial \xi^2}\right) &= 0, \\ \left(\frac{\partial \eta_3^o}{\partial \xi} - r_o \eta_1^o\right) &= 0. \end{aligned}\right\} \tag{5.11}$$

5.2 THE GOVERNING EQUATIONS OF MOTION ABOUT THE EQUILIBRIUM POSITION AND THE ASSOCIATED BOUNDARY CONDITIONS IN THE EXTENSIBLE CASE

Here again, similarly to Chapter III, we are interested in the motion about the equilibrium position of the dynamic system. Accordingly, the dimensionless displacements and twist are also expressed in terms of steady (static) and perturbation dimensionless displacements and twist, as follows:

$$\left.\begin{aligned} \eta_1 &= \eta_1^o + \eta_1^* \\ \eta_2 &= \eta_2^o + \eta_2^*, \\ \eta_3 &= \eta_3^o + \eta_3^*, \\ \psi &= \psi_o + \psi^* . \end{aligned}\right\} \tag{5.12}$$

Noting again that the governing equations of motion (2.75)-(2.78), derived in Chapter II, are also valid for the extensible case, substituting equations (5.4) and (5.12) into equations (2.75)-(2.78), and neglecting only the higher order terms of perturbation, the equations of motion about the equilibrium position for the dynamic system in the extensible case may be obtained, namely

$$\left(\frac{\partial^4 \eta_1^*}{\partial \xi^4} + r_o \frac{\partial^3 \eta_3^*}{\partial \xi^3}\right) + \frac{\partial}{\partial \xi}\left[\Pi_o\left(\frac{\partial \eta_1^*}{\partial \xi} + r_o \eta_3^*\right)\right] - \mathcal{A}\frac{\partial}{\partial \xi}\left[\left(\frac{\partial \eta_1^o}{\partial \xi} + r_o \eta_3^o\right)\left(\frac{\partial \eta_3^*}{\partial \xi} - r_o \eta_1^*\right)\right]$$
$$- r_o \mathcal{A}\left(\frac{\partial \eta_3^*}{\partial \xi} - r_o \eta_1^*\right) + \bar{u}^2\left(\frac{\partial^2 \eta_1^*}{\partial \xi^2} + r_o \frac{\partial \eta_3^*}{\partial \xi}\right) + 2\beta^{1/2}\bar{u}\left(\frac{\partial^2 \eta_1^*}{\partial \tau \partial \xi} + r_o \frac{\partial \eta_3^*}{\partial \tau}\right)$$
$$+ \mathcal{K}\frac{\partial \eta_1^*}{\partial \tau} + (1 + \beta_a)\frac{\partial^2 \eta_1^*}{\partial \tau^2} = 0, \qquad (5.13)$$

$$\left(\frac{\partial^4 \eta_2^*}{\partial \xi^4} - r_o \frac{\partial^2 \psi^*}{\partial \xi^2}\right) - r_o \Lambda\left(\frac{\partial^2 \psi^*}{\partial \xi^2} + r_o \frac{\partial^2 \eta_2^*}{\partial \xi^2}\right) + \frac{\partial}{\partial \xi}\left[\Pi_o \frac{\partial \eta_2^*}{\partial \xi}\right] - \mathcal{A}\frac{\partial}{\partial \xi}\left[\frac{\partial \eta_2^o}{\partial \xi}\left(\frac{\partial \eta_3^*}{\partial \xi}\right.\right.$$
$$\left.\left. - r_o \eta_1^*\right)\right] + \bar{u}^2 \frac{\partial^2 \eta_2^*}{\partial \xi^2} + 2\beta^{1/2}\bar{u}\frac{\partial^2 \eta_2^*}{\partial \xi \partial \tau} + \mathcal{K}\frac{\partial \eta_2^*}{\partial \tau} + (1+\beta_a)\frac{\partial^2 \eta_2^*}{\partial \tau^2} = 0, \qquad (5.14)$$

$$r_o\left(\frac{\partial^3 \eta_1^*}{\partial \xi^3} + r_o \frac{\partial^2 \eta_3^*}{\partial \xi^2}\right) + r_o \Pi_o\left(\frac{\partial \eta_1^*}{\partial \xi} + r_o \eta_3^*\right) - r_o \mathcal{A}\left(\frac{\partial \eta_1^o}{\partial \xi} + r_o \eta_3^o\right)\left(\frac{\partial \eta_3^*}{\partial \xi} - r_o \eta_1^*\right)$$
$$+ \mathcal{A}\left(\frac{\partial^2 \eta_3^*}{\partial \xi^2} - r_o \frac{\partial \eta_1^*}{\partial \xi}\right) + r_o \bar{u}^2\left(\frac{\partial \eta_1^*}{\partial \xi} + r_o \eta_3^*\right) - \beta^{1/2}\bar{u}\left(\frac{\partial^2 \eta_3^*}{\partial \xi \partial \tau} - r_o \frac{\partial \eta_1^*}{\partial \tau}\right)$$
$$- \mathcal{K}\frac{\partial \eta_3^*}{\partial \tau} - (1+\beta_a')\frac{\partial^2 \eta_3^*}{\partial \tau^2} = 0, \qquad (5.15)$$

$$r_o\left(r_o \psi^* - \frac{\partial^2 \eta_2^*}{\partial \xi^2}\right) - \Lambda\left(\frac{\partial^2 \psi^*}{\partial \xi^2} + r_o \frac{\partial^2 \eta_2^*}{\partial \xi^2}\right) + \sigma \frac{\partial^2 \psi^*}{\partial \tau^2} = 0, \qquad (5.16)$$

where, zero-order terms have been absorbed in the static equilibrium equations discussed previously.

It can be seen that equations (5.13) and (5.15) pertain to in-plane displacements, while equations (5.14) and (5.16) are associated with in-plane and out-of-plane dispacements and twist. There is coupling between in-plane and out-of-plane motion. However, in-plane motion can still be studied independently. After investigating in-plane motion, the fourth term in equation (5.14) is completely determined (known), and then out-of-plane motion may be studied as a linear problem of forced vibration.

Furthermore, if the gravity effect is neglected or the pipe initially lies in a vertical plane, the dimensionless displacment η_2^o and twist ψ_o should vanish, and there is no coupling whatsoever between in-plane and out-of-plane motions. In addition, the combined force Π obtained in both cases of inextensibility and extensibility is very close, as will be seen later, in Chapter VI; hence, equations (5.14) and (5.16) can be verified to be identical to the equations of out-of-plane motion according to the modified inextensible theory (i.e., where Π is not omitted) and the results obtained in the two cases are the same. Hence, we are only interested in studying the in-plane motion for this case, and the equations of in-plane motion are given by

$$\left(\frac{\partial^4 \overset{*}{\eta}_1}{\partial\xi^4} + r_o \frac{\partial^3 \overset{*}{\eta}_3}{\partial\xi^3}\right) + \frac{\partial}{\partial\xi}\left[\Pi_o\left(\frac{\partial \overset{*}{\eta}_1}{\partial\xi} + r_o \overset{*}{\eta}_3\right)\right] - \mathcal{A}\frac{\partial}{\partial\xi}\left[\left(\frac{\partial \eta_1^o}{\partial\xi} + r_o\eta_3^o\right)\left(\frac{\partial \overset{*}{\eta}_3}{\partial\xi} - r_o\overset{*}{\eta}_1\right)\right]$$

$$- r_o\mathcal{A}\left(\frac{\partial \overset{*}{\eta}_3}{\partial\xi} - r_o\overset{*}{\eta}_1\right) + \bar{u}^2\left(\frac{\partial^2 \overset{*}{\eta}_1}{\partial\xi^2} + r_o\frac{\partial \overset{*}{\eta}_3}{\partial\xi}\right) + 2\beta^{1/2}\,\bar{u}\left(\frac{\partial^2 \overset{*}{\eta}_1}{\partial\tau\partial\xi} + r_o\frac{\partial \overset{*}{\eta}_3}{\partial\tau}\right)$$

$$+ \mathcal{K}\frac{\partial \overset{*}{\eta}_1}{\partial\tau} + (1+\beta_a)\frac{\partial^2 \overset{*}{\eta}_1}{\partial\tau^2} = 0, \qquad (5.17)$$

$$-r_o\left(\frac{\partial^3\eta_1^*}{\partial\xi^3} + r_o\frac{\partial^2\eta_3^*}{\partial\xi^2}\right) - r_o\Pi_o\left(\frac{\partial\eta_1^*}{\partial\xi} + r_o\eta_3^*\right) + r_o\mathcal{A}\left(\frac{\partial\eta_1^o}{\partial\xi} + r_o\eta_3^o\right)\left(\frac{\partial\eta_3^*}{\partial\xi} - r_o\eta_1^*\right)$$

$$-\mathcal{A}\left(\frac{\partial^2\eta_3^*}{\partial\xi^2} - r_o\frac{\partial\eta_1^*}{\partial\xi}\right) - r_o\bar{u}^2\left(\frac{\partial\eta_1^*}{\partial\xi} + r_o\eta_3^*\right) + \beta^{1/2}\bar{u}\left(\frac{\partial^2\eta_3^*}{\partial\tau\partial\xi} - r_o\frac{\partial\eta_1^*}{\partial\tau}\right)$$

$$+\mathcal{K}'\frac{\partial\eta_3^*}{\partial\tau} + (1+\beta_a')\frac{\partial^2\eta_3^*}{\partial\tau^2} = 0. \tag{5.18}$$

From equations (2.79)-(2.81) the associated boundary conditions are written as follows:

(i) at a clamped end

$$\left.\begin{aligned} \eta_1^* &= 0, \\ \eta_3^* &= 0, \\ \frac{\partial\eta_1^*}{\partial\xi} &= 0\ ; \end{aligned}\right\} \tag{5.19}$$

(ii) at a pinned end

$$\left.\begin{aligned} \eta_1^* &= 0, \\ \eta_3^* &= 0, \\ \left(\frac{\partial^2\eta_1^*}{\partial\xi^2} + r_o\frac{\partial\eta_3^*}{\partial\xi}\right) &= 0; \end{aligned}\right\} \tag{5.20}$$

(iii) at a free end

$$\left.\begin{aligned} \left(\frac{\partial^2\eta_1^*}{\partial\xi^2} + r_o\frac{\partial\eta_3^*}{\partial\xi}\right) &= 0, \\ \left(\frac{\partial^3\eta_1^*}{\partial\xi^3} + r_o\frac{\partial^2\eta_3^*}{\partial\xi^2}\right) &= 0, \\ \left(\frac{\partial\eta_3^*}{\partial\xi} - r_o\eta_1^*\right) &= 0. \end{aligned}\right\} \tag{5.21}$$

5.3 ANALYSIS OF THE STATIC EQUILIBRIUM IN THE EXTENSIBLE CASE

In order to obtain the solution for the in-plane static deformation, the finite-element method is applied once again. Accordingly, the variational statement used for the finite-element technique is utilized, i.e.:

$$\sum_{i=1}^{n} \int_{0}^{\xi_i} \{\delta\eta_1^o A_{i1}^o (\eta_1^o, \eta_3^o) + \delta\eta_3^o A_{i3}^o (\eta_1^o, \eta_3^o)\} \, d\xi = 0, \qquad (5.22)$$

where $\delta\eta_1^o$ and $\delta\eta_3^o$ are the variations in the dimensionless displacements η_1^o and η_3^o, $A_{i1}^o (\eta_1^o, \eta_3^o)$ and $A_{i3}^o (\eta_1^o, \eta_3^o)$ represent the left-hand sides of equations (5.7) and (5.8) respectively, n is the number of finite elements utilized in the discretization, and ξ_i is the length of the ith element.

By performing integrations by parts and using the boundary conditions, one obtains

$$\begin{aligned}\sum_{i=1}^{n} \int_{0}^{\xi_i} & \{\delta\eta_1^{o''} (\eta_1^{o''} + r_o \eta_3^{o'}) + \delta\eta_1^o [\frac{\partial}{\partial\xi} [\bar{\Pi}_p (\eta_1^{o'} + r_o \eta_3^o)] - r_o \mathcal{A} (\eta_3^{o'} \\ & - r_o \eta_1^o) + \bar{u}^2 (\eta_1^{o''} + r_o \eta_3^{o'})] + \delta\eta_1^o (r_o (\bar{\Pi}_p + \bar{u}^2) - \gamma\alpha_{x_o}) \\ & + \delta\eta_3^{o'} [\mathcal{A}(\eta_3^{o'} - r_o \eta_1^o) + r_o (\eta_1^{o''} + r_o \eta_3^{o'})] - r_o \bar{\Pi}_p \delta\eta_3^o (\eta_1^{o'} \\ & + r_o \eta_3^o) - r_o \bar{u}^2 \delta\eta_3^o (\eta_1^{o'} + r_o \eta_3^o) + \delta\eta_3^o (\frac{\partial\bar{\Pi}_p}{\partial\xi} - \gamma\alpha_{z_o})\} \, d\xi = 0. \end{aligned} \qquad (5.23)$$

Here, it is noted that the prime denotes differentiation with respect to ξ, and details of the derivation of equation (5.23) may be found in Appendix T.

According to the finite-element process, solutions will be sought in the form

$$\left.\begin{aligned} \eta_1^{o\prime} &= [N_{e1}^o]\ \{\eta_i^o\}^e, \\ \eta_3^o &= [N_{e3}^o]\ \{\eta_i^o\}^e, \end{aligned}\right\} \qquad (5.24)$$

where $[N_{e1}^o]$ and $[N_{e3}^o]$ are the shape functions prescribed in terms of the space coordinate ξ, and $\{\eta_i^o\}^e$ is the element displacement vector, which does not depend on either space or time variables.

Substituting equation set (5.24) into equation (5.23) yields the discretized equation of in-plane static equilibrium in the form of linear algebraic equations

$$[K_i^o]^e\ \{\eta_i^o\}^e = \{F\}^e , \qquad (5.25)$$

where

$$\begin{aligned} [K_i^o]^e = \int_0^{\xi_e} \{ & ([N_{e1}^o]''^T[N_{e1}^o]'' + r_o([N_{e1}^o]''^T[N_{e3}^o]' + [N_{e3}^o]'^T[N_{e1}^o]'') \\ & + r_o^2\ [N_{e3}^o]'^T\ [N_{e3}^o]' \\ & + \mathcal{A}[[N_{e3}^o]'^T[N_{e3}^o]' - r_o([N_{e1}^o]^T[N_{e3}^o]' + [N_{e3}^o]'^T[(N_{e1}^o]) \\ & + r_o^2\ [N_{e1}^o]^T[N_{e1}^o]] \\ & + \bar{u}^2\ [[N_{e1}^o]^T([N_{e1}^o]'' + r_o[N_{e3}^o]') - r_o[N_{e3}^o]^T([N_{e1}^o]' + r_o\ [N_{e3}^o])] \\ & + [N_{e1}]^T\ \frac{\partial}{\partial\xi}[\Pi_p([N_{e1}^o]' + r_o[N_{e3}^o])] - r_o\Pi_p\ [N_{e3}^o]^T([N_{e1}^o]' \\ & + r_o[N_{e3}^o])\}\ d\xi, \end{aligned} \qquad (5.26)$$

$$\{F\}^e = -\int_0^{\xi_e}\{(r_o(\Pi_p + \dot{\bar{u}}^2) - \gamma\alpha_{x_o})[N^o_{e1}]^T + (\frac{\partial\Pi_p}{\partial\xi} - \gamma\alpha_{z_o})[N^o_{e3}]^T\}\, d\xi. \quad (5.27)$$

If the variation of the external pressure p_e along the centerline is not very large and the internal pressure is assumed to be linear along the centerline, we may write the dimensionless pressure force as

$$\Pi_p = -\lambda^* \bar{u}^2 \xi + \Pi_p|_o \ , \quad (5.28)$$

where λ^* has been derived in Appendix U.

At this step, proceeding in the same manner as was done in the numerical analysis of out-of-plane motion in the inextensible case*, the matrices are obtained, as follows:

$$\begin{aligned}[K^o_1]^e = [A]_o^{-1T}\{&[[J^*_3] + r_o([J^*_{15}] + [J^*_{15}]^T) + r_o^2[J^*_8]] + \mathcal{A}[[J^*_8] \\ &- r_o([J^*_{12}] + [J^*_{12}]^T) + r_o^2[J^*_1]] \\ &+ \bar{u}^2[[J^*_9] + r_o([J^*_{12}] - [J^*_{13}]) - r_o^2[J^*_4]] \\ &+ \Pi_{p|o}[[J^*_9] + r_o([J^*_{12}] - [J^*_{13}]) - r_o^2[J^*_4]] \\ &-\lambda^*\bar{u}^2[[J^*_5] + [J^*_{10}] + r_o([J^*_{16}] + [J^*_{14}] - [J^*_{17}]) \\ &- r_o^2[J^*_{18}]]\}[A]_o^{-1},\end{aligned} \quad (5.29)$$

---

* Note that in the inextensible case, the in-plane motion involved sixth order derivatives after combining the two equations for η_1^* and η_3^*. Corresponding shape functions are inappropriate here. One can, however, use the shape functions used for the out-of-plane motion in the inextensible case for the in-plane motion in the extensible case, since the maximum order of partial derivatives with respect to ξ match.

$$\{F\}^e = -[A]_o^{-1T} \{r_o(\bar{u}^2 + \Pi_p|_o)\ \{F_1\} - \lambda^* \bar{u}^2 (r_o\{F_2\} + \{F_3\}) + \{F_G\}\},\tag{5.30}$$

where

$$\left.\begin{aligned}
\{F_G\} &= \int_o^{\xi_e} -\gamma(\alpha_{x_o} [\phi_2]^T + \alpha_{z_o} [\phi_4']^T) d\xi, \\
\{F_1\} &= \int_o^{\xi_e} [\phi_2]^T d\xi = \begin{Bmatrix} \xi_e \\ 1/2\ \xi_e^2 \\ 1/3\ \xi_e^3 \\ 1/4\ \xi_e^4 \\ 0 \\ 0 \end{Bmatrix}, \\
\{F_2\} &= \int_o^{\xi_e} \xi [\phi_2]^T d\xi = \begin{Bmatrix} 1/2\ \xi_e^2 \\ 1/3\ \xi_e^3 \\ 1/4\ \xi_e^4 \\ 1/5\ \xi_o^5 \\ 0 \\ 0 \end{Bmatrix}, \\
\{F_3\} &= \int_o^{\xi_e} [\phi_4]^T d\xi = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \xi_e \\ 1/2\ \xi_e^2 \end{Bmatrix},
\end{aligned}\right\}\tag{5.31}$$

$$\begin{aligned}
[J_1^*] &= \int_o^{\xi_e} [\phi_2]^T [\phi_2]\ d\xi, \\
[J_3^*] &= \int_o^{\xi_e} [\phi_2]''^T [\phi_2]''\ d\xi, \\
[J_4^*] &= \int_o^{\xi_e} [\phi_4]^T [\phi_4]\ d\xi, \\
[J_5^*] &= \int_o^{\xi_e} [\phi_2]^T [\phi_2]'\ d\xi, \\
[J_8^*] &= \int_o^{\xi_e} [\phi_4]'^T [\phi_4]'\ d\xi, \\
[J_9^*] &= \int_o^{\xi_e} [\phi_2]^T [\phi_2]''\ d\xi,
\end{aligned}$$

$$\left.\begin{aligned}
[J^*_{10}] &= \int_o^{\xi_e} [\phi_2]^T[\phi_2]''\xi \, d\xi \,, \\
[J^*_{12}] &= \int_o^{\xi_e} [\phi_2]^T[\phi_4]' \, d\xi \,, \\
[J^*_{13}] &= \int_o^{\xi_e} [\phi_4]^T [\phi_2]' \, d\xi, \\
[J^*_{14}] &= \int_o^{\xi_e} [\phi_2]^T [\phi_4] \, d\xi \,, \\
[J^*_{15}] &= \int_o^{\xi_e} [\phi_2]''^T[\phi_4]' \, d\xi, \\
[J^*_{16}] &= \int_o^{\xi_e} [\phi_2]^T[\phi_4]'\xi \, d\xi \,, \\
[J^*_{17}] &= \int_o^{\xi_e} [\phi_4]^T [\phi_2]'\xi \, d\xi \,, \\
[J^*_{18}] &= \int_o^{\xi_e} [\phi_4]^T[\phi_4]\xi \, d\xi \,.
\end{aligned}\right\} \qquad (5.32)$$

Details of the derivation of equations (5.29), (5.30) and (5.32) may be found in Appendices V and L.

5.5 ANALYSIS OF IN-PLANE MOTION IN THE EXTENSIBLE CASE

Again similarly to the analysis of out-of-plane motion in the inextensible case, the variational statement used for the finite-element model of in-plane motion in the present case is

$$\sum_{i=1}^{n} \int_o^{\xi_i} \{\delta\eta^*_1 \, A^*_{11} \, (\eta^*_1, \, \eta^*_3) + \delta\eta^*_3 \, A^*_{13} \, (\eta^*_1, \, \eta^*_3)\} \, d\xi = 0, \qquad (5.33)$$

where $\delta\eta^*_1$ and $\delta\eta^*_3$ are the variations in the dimensionless displacements η^*_1 and η^*_3, $A^*_{11}(\eta^*_1, \eta^*_3)$ and $A^*_{13}(\eta^*_1,\eta^*_3)$ represent the left-hand sides of equations (5.17) and (5.18) respectively, n is the number of elements and ξ_i is the length of the ith element.

By performing integrations by parts and using the boundary conditions, one obtains

$$\sum_{i=1}^{n} \int_0^{\xi_i} \{\delta\eta_1^{*''}(\eta_1^{*''} + r_o\eta_3^{*'}) + r_o\delta\eta_3^{*'}(\eta_1^{*''} + r_o\eta_3^{*'}) + \mathcal{A}[r_o\delta\eta_1^*(r_o\eta_1^* - \eta_3^{*'})$$

$$+ \delta\eta_3^{*'}(\eta_3^{*'} - r_o\eta_1^*)]$$

$$+ \bar{u}^2 [\delta\eta_1^*(\eta_1^{*''} + r_o\eta_3^{*'}) - r_o\delta\eta_3^*(\eta_1^{*'} + r_o\eta_3^*)] + \delta\eta_1^* \frac{\partial}{\partial\xi}[\Pi_o(\eta_1^{*'} + r_o\eta_3^*)]$$

$$- r_o\Pi_o\delta\eta_3^*(\eta_1^{*'} + r_o\eta_3^*)$$

$$+ \mathcal{A}(\eta_1^{o'} + r_o\eta_3^o)[\delta\eta_1^{*'}(\eta_3^{*'} - r_o\eta_1^*) + r_o\delta\eta_3^*(\eta_3^{*'} - r_o\eta_1^*)]$$

$$+ \beta^{1/2}\bar{u}[2\delta\eta_1^*(\dot{\eta}_1^{*'} + r_o\dot{\eta}_3^*) + \delta\eta_3^*(\dot{\eta}_3^{*'} - r_o\dot{\eta}_1^*)]$$

$$+ \mathcal{K}\delta\eta_1^*\dot{\eta}_1^* + \mathcal{K}'\delta\eta_3^*\dot{\eta}_3^* + (1+\beta_a)\delta\eta_1^*\ddot{\eta}_1^* + (1+\beta_a')\delta\eta_3^*\ddot{\eta}_3^*\}d\xi = 0. \tag{5.34}$$

Here again, it is noted that the prime denotes differentiation with respect to ξ, and the dot denotes differentiation with respect to τ. Details of the derivation of equation (5.34) may be found in Appendix W.

According to the finite-element process, solutions will be sought in the approximate form

$$\left.\begin{aligned} \eta_1^* &= [N_{e1}^*] \; \{\eta_i^*\}^e \;, \\ \eta_3^* &= [N_{e3}^*] \; \{\eta_i^*\}^e \;, \end{aligned}\right\} \tag{5.35}$$

where $[N_{e1}^*]$ and $[N_{e3}^*]$ are the shape functions prescribed in terms of the space coordinate ξ, and $\{\eta_i^*\}^e$ is the element-displacement vector which depends only on time.

Substituting equation (5.35) into equation (5.34), the discretized equation of in-plane motion in the extensible case may be written in the form

$$[M_i^*]^e \{\ddot{\eta}_i^*\}^e + [D_i^*]^e\{\dot{\eta}_i^*\}^e + [K_i^*]^e\{\eta_i^*\}^e = [0] \ , \qquad (5.36)$$

where

$$[M_i^*]^e = \int_0^{\xi_e} \{(1+\beta_a)[N_{e1}^*]^T[N_{e1}^*]+(1+\beta_a')[N_{e3}^*]^T[N_{e3}^*]\} \, d\xi, \qquad (5.37)$$

$$[D_i^*]^e = \beta^{1/2}\,\bar{u} \int_0^{\xi_e} \{2[N_{e1}^*]^T([N_{e1}^*]' + r_o[N_{e3}^*])+[N_{e3}^*]^T([N_{e3}^*]' - r_o[N_{e1}^*])\} \, d\xi$$

$$+ \int_0^{\xi_e} (\mathcal{K}[N_{e1}^*]^T[N_{e1}^*] + \mathcal{K}[N_{e3}^*]^T[N_{e3}^*]) \, d\xi, \qquad (5.38)$$

$$[K_i^*]^e = \int_0^{\xi_e} \cdot\{ [N_{e1}^*]''^T[N_{e1}^*]''+r_o([N_{e1}^*]''^T[N_{e3}^*]' + [N_{e3}^*]'^T[N_{e1}^*]'')+r_o^2[N_{e3}^*]'^T[N_{e3}^*]'$$

$$+ \mathcal{A}[[N_{e3}^*]'^T[N_{e3}^*]' - r_o([N_{e1}^*]^T[N_{e3}^*]' + [N_{e3}^*]'^T[N_{e1}^*])+r_o^2[N_{e1}^*]^T[N_{e1}^*]]$$

$$+ \mathcal{A}(\eta_1^{o\,\prime} + r_o\eta_3^o)[[N_{e1}^*]'^T([N_{e3}^*]' - r_o[N_{e1}^*])+r_o[N_{e3}^*]^T([N_{e3}^*]'$$

$$- r_o[N_{e1}^*])]$$

$$+ \bar{u}^2 [[N_{e1}^*]^T([N_{e1}^*]'' + r_o[N_{e3}^*]') - r_o[N_{e3}^*]^T([N_{e1}^*]' + r_o[N_{e3}^*])]$$

$$+ [N_{e1}^*]^T \frac{\partial}{\partial\xi}[\Pi_o([N_{e1}^*]' + r_o[N_{e3}^*])] - r_o\Pi_o[N_{e3}^*]^T([N_{e1}^*]'$$

$$+ r_o[N_{e3}^*])\} \, d\xi \ . \qquad (5.39)$$

To evaluate the integrals in the third and fifth group of terms in $[K_i^*]$, the steady (static) parameters $\Pi_o, \partial\Pi_o/\partial\xi$ and $(\eta_1^{o\,\prime} + r_o\eta_3^o)$ in each element are approximated by linear functions for convenience, as shown in Fig. 8, namely

$$\left.\begin{aligned} \Pi_o &= a_1 + a_2\xi, \\ \frac{\partial\Pi_o}{\partial\xi} &= b_1 + b_2\xi, \\ (\eta_1^{o'} + r_o\eta_3^o) &= c_1 + c_2\xi, \end{aligned}\right\} \tag{5.40}$$

where

$$\left.\begin{aligned} a_1 &= \Pi_o|_j, \\ a_2 &= (\Pi_o|_{j+1} - \Pi_o|_j)/\xi_e , \\ b_1 &= \partial\Pi_o/\partial\xi|_j \\ b_2 &= (\frac{\partial\Pi_o}{\partial\xi}|_{j+1} - \frac{\partial\Pi_o}{\partial\xi}|_j)/\xi_e , \\ c_1 &= (\eta_1^{o'} + r_o\eta_3^o)|_j , \\ c_2 &= ((\eta_1^{o'} + r_o\eta_3^o)|_{j+1} - (\eta_1^{o'} - r_o\eta_3^o)|_j)/\xi_e . \end{aligned}\right\} \tag{5.41}$$

At this point, proceeding in the same way as was done in the numerical analysis of out-of-plane motion in the inextensible case, the element matrices are obtained, as follows:

$$[M_1^*]^e = [A]_o^{-1T}\{(1+\beta_a)[J_1^*] + (1+\beta_a')[J_4^*]\}\ [A]_o^{-1} \tag{5.42}$$

$$[D_1^*]^e = [A]_o^{-1T}\ \{\beta^{1/2}\,\bar{u}[2([J_5^*]+r_o[J_{14}^*])+([J_{20}^*]-r_o[J_{14}^*]^T)] + \mathcal{K}[J_1^*] + \mathcal{K}'[J_4^*])\}\ [A]_o^{-1}, \tag{5.43}$$

$$[K_1^*]^e = [A]_o^{-1T}\{[[J_3^*] + r_o([J_{15}^*]+[J_{15}^*]^T)+r_o^2[J_8^*]] + \mathcal{A}[[J_8^*]$$

$$-r_o([J_{12}^*] + [J_{12}^*]^T) + r_o^2 [J_1^*]]$$

$$+ \mathcal{A}[c_1[[J_7^*]-r_o([J_5^*]^T-[J_{20}^*])-r_o^2[J_{14}^*]^T]+c_2[[J_{22}^*]-r_o([J_{11}^*]^T - [J_{21}^*])$$

$$-r_o^2[J_{23}]^T]]$$

$$+ \bar{u}^2 [[J_9^*] + r_o([J_{12}^*]-[J_{13}^*]) - r_o^2 [J_4^*]]$$

$$+ a_1[[J_9^*] + r_o([J_{12}^*]-[J_{13}^*])-r_o^2[J_4^*]]+ b_1[[J_5^*] + r_o[J_{14}^*]]$$

$$+ a_2[[J_{10}^*] + r_o([J_{16}^*]-[J_{17}^*])-r_o^2[J_{18}^*]] + b_2[[J_{11}^*]+r_o[J_{23}^*]]\}[A]_o^{-1},$$

(5.44)

where

$$[J_1^*] = \int_o^{\xi_e} [\phi_2]^T[\phi_2]\, d\xi,$$

$$[J_3^*] = \int_o^{\xi_e} [\phi_2]''^T[\phi_2]''d\xi ,$$

$$[J_4^*] = \int_o^{\xi_e} [\phi_4]^T[\phi_4]\, d\xi ,$$

$$[J_5^*] = \int_o^{\xi_e} [\phi_2]^T[\phi_2]'\, d\xi,$$

$$[J_7^*] = \int_o^{\xi_e} [\phi_2]'^T[\phi_4]'\, d\xi,$$

$$[J_8^*] = \int_o^{\xi_e} [\phi_4]'^T[\phi_4]'\, d\xi,$$

$$[J_9^*] = \int_o^{\xi_e} [\phi_2]^T[\phi_2]''\, d\xi,$$

$$[J_{10}^*] = \int_o^{\xi_e} \xi[\phi_2]^T[\phi_2]''\, d\xi,$$

$$[J_{11}^*] = \int_o^{\xi_e} \xi[\phi_2]^T[\phi_2]'\, d\xi,$$

$$[J_{12}^*] = \int_o^{\xi_e} [\phi_2]^T[\phi_4]'\, d\xi,$$

$$
\left.
\begin{aligned}
[J_{13}^{*}] &= \int_{o}^{\xi_e} [\phi_4]^T[\phi_2]'\, d\xi, \\
[J_{14}^{*}] &= \int_{o}^{\xi_e} [\phi_2]^T[\phi_4]\, d\xi, \\
[J_{15}^{*}] &= \int_{o}^{\xi_e} [\phi_2]''^T[\phi_4]'\, d\xi, \\
[J_{16}^{*}] &= \int_{o}^{\xi_e} \xi[\phi_2]^T[\phi_4]'\, d\xi, \\
[J_{17}^{*}] &= \int_{o}^{\xi_e} [\phi_4]^T[\phi_2]'\xi\, d\xi, \\
[J_{18}^{*}] &= \int_{o}^{\xi_e} [\phi_4]^T[\phi_4]\xi\, d\xi, \\
[J_{20}^{*}] &= \int_{o}^{\xi_e} [\phi_4]^T[\phi_4]'\, d\xi, \\
[J_{21}^{*}] &= \int_{o}^{\xi_e} \xi[\phi_4]^T[\phi_4]'\, d\xi, \\
[J_{22}^{*}] &= \int_{o}^{\xi_e} \xi[\phi_2]'^T[\phi_4]'\, d\xi, \\
[J_{23}^{*}] &= \int_{o}^{\xi_e} \xi[\phi_2]^T[\phi_4]\, d\xi,
\end{aligned}
\right\}
\qquad (5.45)
$$

and $[\phi_2]$ and $[\phi_4]$ are defined by equations (3.62) and (3.63). Details of the derivation of equations (5.42)-(5.45) may be found in Appendices X and L.

5.6 EIGENVALUE ANALYSIS OF THE PROBLEM FOR THE IN-PLANE EXTENSIBLE CASE

Here again, similarly to the inextensible case, the global equation of motion may be written as follows:

$$[M]\ \{\ddot{\eta}\} + [D]\ \{\dot{\eta}\} + [K]\ \{\eta\} = \{0\}, \qquad (5.46)$$

where [M], [D] and [K] are respectively, the global mass, damping and stiffness matrices, which are assembled from the corresponding matrices for an element of the dynamic system (as given in equations (5.42)–(5.45) respectively); and $\{\eta\}$ is the global displacement vector.

We proceed by setting

$$\{Y\} = \begin{Bmatrix} \{\eta\} \\ \{\dot{\eta}\} \end{Bmatrix}, \tag{5.47}$$

and

$$\{Y\} = \{X\}\, e^{i\omega\tau}, \tag{5.48}$$

where ω is a dimensionless frequency related to the circular frequency of motion, Ω, by

$$\omega = \left(\frac{M_t + M_f}{EI}\right)^{1/2} \Omega\, L^2 . \tag{5.49}$$

In general, ω will be complex and the dynamic system will be stable or unstable accordingly as the imaginary part of ω is positive or negative.

Now proceeding in a similar manner as was done in the eigenvalue analysis for the inextensible case, the standard eigenvalue equation may be obtained as

$$\lambda \begin{bmatrix} [I] & [0] \\ [0] & [M] \end{bmatrix} \{X\} = \begin{bmatrix} [0] & [I] \\ -[K] & -[D] \end{bmatrix} \{X\}, \tag{5.50}$$

where

$$\lambda = i\omega. \tag{5.51}$$

Thus, the problem is reduced to one of solving the eigenvalue equation which can easily be done with the IMSL subroutine EIGZF from the McGill computer library.

CHAPTER VI

RESULTS AND DISCUSSION FOR THE EXTENSIBLE CASE

In Chapter IV, the results for the inextensible case have been discussed. This Chapter presents the results of an investigation of static and dynamic in-plane behaviour of extensible curved pipes conveying fluid under the effect of the internal flow.

These results were obtained by using the finite-element method discussed in Chapter V. Computer programs were developed to obtain both the static displacement field and complex frequencies of the dynamic system. The calculations in the programs were conducted in double precision, and a listing of these programs may be found in Appendix Y.

6.1 STATIC RESULTS

The task in this section is to compute the static equilibrium displacements by solving the discretized equation of static equilibrium (5.24). Then the static axial force Q_z is obtained, and this force is used to reformulate the stiffness matrix $[K_i^*]^e$ for use in the solution of the dynamic eigenvalue problem. Moreover, in the process of this calculation, we can see whether the system buckles or not depends on whether the global stiffness matrix, which is assembled from the element stiffness matrices $[K_i^O]^e$, is singular or not.

6.1.1 Stressed Configuration of a Clamped-Clamped Semi-Circular Pipe Conveying Fluid

Figures 38-40 show the stressed configurations (i.e. the static equilibrium configurations) of a clamped-clamped semi-circular pipe conveying fluid for various internal flow velocities for $\mathcal{A} = 10^4$ and $\gamma = 0$. These results are obtained by using the finite element method discussed in Section 5.3.

We can see that in the case of inviscid flow (Figs. 38-39), the stressed shape is symmetric because the fluid force acting on the pipe is only the centrifugal force and symmetric if the gravity force is neglected. When the flow velocity increases, the pipe is deformed out of the unstressed (initial) shape by the centrifugal force, and then the stressed shape changes according to the flow velocity. On the other hand, in the case of viscous flow (Fig. 40) the stressed shape is no longer symmetric since the pressurization effects of the friction force are not symmetric.

It is noted that for all cases the system does not buckle since the determinant of the global stiffness matrix in the static case does not vanish, although the stressed configuration changes to various different shapes. It will be verified later on, in the dynamic analysis of the problem, that the real parts of the complex frequencies do not vanish in the range of the flow velocity used in the calculation, implying that the system does not buckle.

6.1.2 Distribution of Static Combined Force Along the Pipe

Figures 41-42 show the distribution of the static combined force Π_o along the clamped-clamped semi-circular pipe for various internal flow velocities. It is seen that for the inviscid flow, the results of the extensible and inextensible* cases are very close. In the case of viscous

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* Recall that for the inextensible case, $\Pi_o = -\bar{u}^2$ for both inviscid and viscous flow and is shown by center or dashed lines in Figs. 41 and 42.

flow, the results of inextensibility and extensibility can be compared and the maximum difference between the two results can be seen to be less than 10% (except for the flow velocity at 3π, when the difference can be as high as 39% close to the midpoint of the tube).

6.2 DYNAMIC RESULTS

Using the results obtained in Section 6.1, the dynamic stiffness matrices $[K_i^*]^e$ are reformulated. In this section, calculations are conducted to compute the frequencies of the dynamic system and to check for buckling.

6.2.1 Convergence Study

Figures 43-45 show the convergence, with increasing number of finite elements, of the lowest three natural frequencies associated with $\bar{u} = 0$, for various values of $\mathcal{A}$. It can be seen that the convergence is very slow, and that it is affected by the slenderness parameter $\mathcal{A}$ (i.e. $A_t L^2/I$). For a small number of elements, the results associated with various values of this parameter are very different. However, with a large number of elements the results are comparable. It is noted that a high value of the parameter $\mathcal{A}$ is required to satisfy one of the assumptions involved in this theory*, but it results in high computational cost. Therefore, a value of the parameter $\mathcal{A}$ which provides a good trade off between cost and accuracy, is used in this calculation ($\mathcal{A} = 10^4$).

* In this theory, the radius of curvature R_o of the centerline and the total length of the pipe L are assumed to be large in comparison with the radius of the pipe. Therefore, this means that $A_t L^2$ must be much larger than I.

6.2.2 The In-Plane Behaviour of an Extensible Clamped-Clamped Semi-Circular Pipe Conveying Fluid (Neglecting the Combined Force Π_o)

The calculations in this Section are conducted with $\Pi_o = 0$, strictly to compare with the results of the first (conventional) form of the inviscid theory, in which $\Pi_o = 0$ is assumed. The calculations were conducted with thirty four finite elements.

Figures 46-47 show the in-plane natural frequencies of clamped-clamped pipes conveying fluid with $\mathcal{A} = 10^4$, as functions of the flow velocity. Similarly to the inextensible case, at $\bar{u} = 0$, the pipe behaves as a semi-circular ring. As the flow velocity is increased, the natural frequencies become smaller, and if the flow velocity exceeds a certain value, the pipe is predicted to buckle in the first mode. With further increase in the flow velocity, instability may occur in the higher modes also. Comparing with Fig. 16, it can be seen in Fig. 46 that the finite element results for the extensible and inextensible cases are close. Moreover, in Fig. 47 the present results can be seen to agree with the result of Doll and Mote (1976). It must be emphasized, however, that these results are for $\Pi_o = 0$, which is not realizable from physical considerations.

6.2.3 The In-Plane Behaviour of an Extensible Clamped-Clamped Semi-Circular Pipe Conveying Fluid (Including the Combined Force Π_o)

In this Section are presented results obtained with the proper form of the extensible theory, in which the combined force Π_o is taken into account. However, several variants of this theory are investigated; in one in order to assess the effect of steady-state (initial) deformation on the dynamics of the system, calculations were conducted in which the terms involving $\mathcal{A}(\eta_1^{o'} + r_o\eta_3^o)$ in equations (5.17) and (5.18) and hence in equation (5.44) are neglected; in another variant of the theory the fluid is supposed to be inviscid rather than viscous.

The calculations here have been conducted for a system with $\beta = 0.5$, $\gamma = 0$, $\mathcal{A} = 10^4$, and for viscous flow ($\lambda^* \neq 0$).

Figures 48-50 show the variation of the in-plane frequencies of the system with increasing dimensionless flow velocity $\bar{u}$, and how they compare with some previously obtained results, in this thesis or by other investigators.

Figure 48 shows the results obtained by this theory with the terms involving $\mathcal{A}(\eta_1^{o'} + r_o\eta_3^o)$ either taken into account or neglected, in the case of inviscid flow ($\lambda^* = 0$). It is seen that for $\bar{u} \leq 3$ the effect of the $\mathcal{A}(\eta_1^{o'} + r_o\eta_3^o)$ terms is not very important; this agrees with the results shown in Fig. 38, where it is seen that the static deformation of the pipe is small. However, for $\bar{u} > 3$, these effects become more pronounced, in the second and third modes particularly, which again reflects the greater departure from the unstressed state of the pipe shown in Fig. 39, as compared to $\bar{u} < 3$.

The most important feature of Fig. 48, however, is the fact that extensible theory, properly taking into account the combined force Π, predicts that no instability should occur, similarly to the modified inextensible theory (also taking Π into account), the results of which have been presented in Chapter IV. The frequencies of the system change very little with increasing $\bar{u}$, and none reach a zero value, unlike the case when $\Pi = 0$ (also shown in Fig. 48 for comparison).

It is recalled that Hill & Davis (1974) and Doll & Mote (1974, 1977) also presented extensible theories and reached the same general conclusion, namely that pipes with clamped (or otherwise supported) ends will not buckle. Fig. 49 shows the results obtained by these two investigators, compared to those obtained by this theory – when making the assumption that the fluid is inviscid ($\lambda^* = 0$). It is seen that the general character of the solutions is similar in all three cases, albeit that the results are not identical.

Hill & Davis' equations of motion are perhaps closest to those utilized here and the results from these two theories should therefore be closest; some parameters are different nevertheless, namely $\beta = 0.43$ and $\mathcal{A} = 1.4 \times 10^5$, as compared to 0.5 and 10^4, respectively, used in the calculations with the present theory. Hill & Davis, similarly to the present theory, considered motions about the deformed initial state calculated in a linearized fashion. On the other hand, Doll & Mote (1974, 1977) calculated the deformed state by a more sophisticated algorithm, involving a cumulative application of the linearized equations; their β was the same as in these calculations ($\beta = 0.5$*) and $\mathcal{A} = 1.6\pi^2 \times 10^3$.

It should be noted that Doll & Mote's and Hill & Davis' work was for effectively inviscid flow. The present theory, on the other hand, takes slightly different forms, depending on whether the flow is taken to be inviscid or viscous.

* Note that this is so, despite what appears in their published work ($\beta = 1$), due to a typographical error (Paidoussis 1983).

The results presented so far have been for inviscid theory. Some results, for the same system as that of Figs. 48 and 49, for a pipe conveying viscous flow are presented in Fig. 50. It is seen that frictional effects are not very pronounced, so far as first-mode frequencies are concerned, but they are important insofar as the second- and third-mode frequenices are concerned. The reason why this is so is not understood at present. The important point, however, is that even with this, the most complete form of the extensible theory, it is predicted that instability of pipes with both ends supported will not occur.

6.2.4 The Case Where the Downstream End is Free to Slide Axially

Here, it is of interest to present a result concerning the dynamical behaviour of a semi-circular pipe conveying fluid, with one end completely clamped and the other end clamped but free to slide axially. In many practical situations, in order to allow for thermal stress relief, it is not desirable to use totally clamped or pinned supports at two ends of the pipe, using instead a sliding support.

The calculations here have been conducted for a system with $\beta = 0.5$, $\gamma = 0$, $\mathcal{A} = 10^4$, and for viscous flow.

Figures 51 and 51a show the variation of the in-plane frequencies of this system with increasing dimensionless flow velocity $\bar{u}$. We can see that the natural frequencies $\mathrm{Re}(\omega_{ci})$ monotonically decrease with increasing $\bar{u}_c$(Figure 51); however, as seen in Figure 51a the system loses stability not by divergence, but rather by flutter, in the first mode, when the flow velocity exceeds the critical velocity ($\bar{u}_c \simeq 0.96$). Moreover, if the flow velocity is increased further, the system may lose stability in the higher modes also; here it should be mentioned that, these calculations being very expensive, they were not pursued to larger values of $\bar{u}$.

The important thing to note is that, unlike the case of true clamped-clamped and pinned-pinned pipes, the presence of a sliding end is sufficient to permit an instability to occur.

6.2.5 The Out-of-Plane Behaviour of an Extensible Clamped-Clamped Pipe Conveying Fluid (Including the Combined Force Π_o)

This Section is here for the sake of completeness. As discussed in Chapter V the equations of motion in this case are identical to those of the modified inextensible theory (i.e., including the combined force Π_o). Hence, the extensible results are the same as those presented in Fig. 35 with $\Pi \neq 0$.

In Fig. 52 are presented these same results compared with those of Hill & Davis and Doll & Mote, as obtained from the latter's work (Doll & Mote 1974) in the case of the first mode. It is seen that, although not identical, the results are quite similar in these three sets of calculations. The important thing is that, if the combined force is properly taken into account, no instability occurs.

CHAPTER VII

CONCLUSION

The dynamics and stability of curved pipes conveying fluid were studied in this Thesis. The pipe was assumed to have uniform physical properties and its initial shape was restricted to lie in one plane. The motions of the pipe in this plane, the so-called in-plane motions, and those normal to this plane, the out-of-plane motions, were considered separately. The pipes were usually supported at both ends, with clamped or pinned supports, but some cases of cantilevered pipes were also considered. In most cases, the curved pipes were semi-circular, i.e. the subtended angle between one end and the other was 180°, but some cases with this angle being 90° were also investigated.

The dynamics and stability of this system have been studied by three theoretical models, as follows:

(i) the conventional inextensible theory;

(ii) the modified inextensible theory;

(iii) the extensible theory;

in the above, inextensible or extensible refers to whether the assumption has been made that the centerline of the pipe from one end to the other is inextensible or not.

The conventional inextensible theory is similar to that produced by Chen (1972a,b, 1973), in which the pipe at any flow velocity is presumed to be unstressed, excluding the stresses introduced by the oscillation. Thus, the initial or steady-state stresses introduced by the flow even without oscillation, i.e. the stress introduced by steady-state centrifugal forces and pressure forces are entirely neglected, which, as commented upon by several investigators, is unrealistic.

The modified inextensible theory is entirely new and has not been considered heretofore by other researchers. In this theory, the assumption of inextensibility of the centerline is retained, but the steady-state initial stresses are taken into account. These forces are represented by the dimensionless parameter Π, so that in this theory $\Pi \neq 0$, whereas in the conventional inextensible theory $\Omega_z = 0$.

The extensible theory does not make the assumption of inextensibility of the centerline, and hence the shape of the pipe under the action of the forces represented by Π changes with flow velocity. Moreover, it should be noted that, in both forms of the inextensible theory, it makes no difference whether the fluid is considered to be inviscid or viscous - in the latter case friction-induced tension in the pipe being exactly counterbalanced by pressure-drop-induced tension, giving a zero net effect, as was the case for straight pipes conveying fluid (Benjamin 1961). In the extensible theory, however, it does make a difference, and in the general form of this theory the fluid is considered to be viscous. Other forms of the extensible theory were generated previously by Hill & Davis (1974) and Doll & Mote (1974, 1977).

The general linearized equations of motion are obtained in Chapter II, in a form that applies to all three variants of the theory. These are then modified and simplified, as the case may be, according to the particular assumptions pertinent to each. Thus, in all cases the theory is linear. For the two inextensible theories this is perfectly legitimate, since the pipe at any flow velocity has the same initial shape, or approximately so, and departures from this initial state are associated with vibration, which can be assumed to be small. On the other hand, in the extensible theory, the deformation at any given value of the dimensionless flow velocity $\bar{u}$ away from the initial state at $\bar{u} = 0$, under the action of the forces associated with Π, can be large. Accordingly, the state of a system at $\bar{u} \neq 0$ should generally have been obtained by means of nonlinear theory, but this was not done here, and is therefore a limitation of the present theory.

In all three cases the analysis was conducted by the finite element method, which has the great advantage of being easily adaptable to studying curved pipes of variable radius of curvature along their length, e.g. S-shaped, U-shaped pipes and other practically important situations.

The major findings of this Thesis will be presented according to which theory has been used to obtain the eigenfrequencies of the system, as will be discussed in the Sections that follow. In each case, the results obtained will be compared with those of other investigators, and a general discussion of the relative merits and correctness of these theories will be presented in Section 7.4. Finally, in Section 7.5 will be presented some suggestions for future work.

7.1 THE CONVENTIONAL INEXTENSIBLE THEORY

The equations of both in-plane and out-of-plane motions obtained in this Thesis according to this theory, in Chapter III, were found to be identical to Chen's (1972a,b, 1973). Also, if the radius of curvature of the pipe is taken to be infinite, it was confirmed that these equations become identical to those for a straight pipe conveying fluid, previously obtained by Gregory & Paidoussis (1966) and Paidoussis (1970) or those of Paidoussis & Luu (1985) for pipes immersed in a dense fluid medium. Hence, the results obtained by this theory should be identical to those obtained by these earlier investigators. On the other hand, since in all these earlier studies solutions were obtained by analytical methods, it was possible to compare with them the results obtained by the methods of this Thesis, in order to verify the finite-element formulation of the problem and the computer programs based thereon. Before commenting on this, however, a few words on the general character of the dynamical behaviour of the system, as predicted by this theory would be useful.

For curved pipes with supported ends, the effect of increasing the flow velocity or the internal pressure is to reduce the natural frequencies of both in-plane and out-of-plane motions, in all modes of the system. This effect is associated with the centrifugal force exerted on the curved pipe, associated with flow or pressure, which acts in a similar way to a compressive load. For a sufficiently large $\bar{u}$, this force overcomes the flexural restoring force and the pipe buckles in the first mode - corresponding to the point at which the first-mode natural frequency vanishes; the same effect is produced by increasing internal pressure. Identical behaviour is obtained in the in-plane and out-of-plane motions, but the critical flow velocities (or pressures) for out-of-plane motions are lower. For higher flow velocities, buckling instability may also occur in higher modes of the system, as well as coupled-mode flutter.

For cantilevered curved pipes, increasing flow induces a lowering of the natural frequencies, but is also responsible for the appearance of flow-induced damping, since the Coriolis forces in this case actually do work, hence, the eigenfrequencies in this case are generally complex. For in-plane motions it was found that both buckling and flutter instabilities are possible - buckling occurring at lower flow velocities than flutter. For out-of-plane motions, only flutter was found to occur, the behaviour of the system in this case being similar to that of a cantilevered straight pipe.

Indeed, apart from the dynamical behaviour of cantilevered pipes in in-plane motions, the dynamics of the curved pipes according to this theory are qualitatively the same as those of straight pipes: pipes with both ends supported are subject to buckling and coupled-mode flutter for sufficiently high flow velocities; cantilevered pipes, on the other hand, lose stability only by flutter - the exception being for in-plane motions of curved pipes, where buckling also occurs.

The results obtained by this theory exactly reproduced the results previously obtained by Gregory & Paidoussis (1966) and Paidoussis (1970) and were nearly the same as those of Paidoussis & Luu (1985) in the case of cantilevered straight pipes. The eigenfrequencies were identical, or almost identical.

The results obtained for curved pipes, either semi-circular (with subtended angle, r_o, equal to 180^o) or with $r_o = 90^o$, were found to be identical to those of Chen's (1972a, 1973) for pipes with both ends supported, but rather different in the case of cantilevered pipes. In view of the fact that the results obtained in this Thesis were all obtained with the same computer program, and that the equations of motion and boundary conditions were both the same as Chen's, whereas he utilized different analytical methods for cantilevered pipes, it is suspected that there is an error in Chen's analysis or computer program for obtaining the eigenfrequencies of cantilevered curved pipes.

Therefore, in the overwhelming majority of cases investigated, the earlier results of other investigators were reproduced with excellent accuracy and, hence, it may be concluded, that the finite element formulation and computer programs developed in this Thesis have been verified. Moreover, the rate of convergence, with increasing number of finite elements, was found to be very fast and the computer time necessary for the calculations very low. Thus, it was demonstrated that the finite element analysis is a useful tool for obtaining the dynamical behaviour of both straight and curved pipes conveying fluid.

A final set of calculations was undertaken to study the effect of the subtended angle r_o and the radius of curvature R_o of the pipe on its natural frequencies at $\bar{u} = 0$. It was found that, for a constant R_o, the natural frequencies decrease with r_o (and hence with length), for pipes with both ends supported. However, for a pipe of fixed total length, at small R_o the frequencies are very sensitive to R_o, while at large R_o the frequencies change little with varying radius of curvature. As $R_o \to \infty$, it was found that the n^{th} frequency converges to the $(n+1)^{th}$ frequency of a straight pipe, this being associated with the different boundary conditions in the two cases (in the curved pipe, axial motion at the support is specified to be zero, whereas for a straight pipe this condition does not exist and hence the pipe is not truly inextensible in that sense).

7.2 THE MODIFIED INEXTENSIBLE THEORY

As mentioned previously, the initial, steady-state forces associated with centrifugal and pressure forces are taken into account in this case, i.e. the parameter $\Pi \neq 0$. These forces do work in the course of free motions of the system, in the same way as initial stresses do in any such problem - although extension of the centerline is still presumed to be null, or at least negligible. Thus, this theory is intermediate between the conventional inextensible theory and the extensible theory. The motivation for undertaking the formulation of this theory was within the general aim of the work of this Thesis, namely to identify the source of the differences in behaviour predicted by the conventional inextensible theory (Chen 1972a, 1973) and the extensible theory (Doll & Mote 1974, 1976; Hill & Davis 1974); these differences are very profound, and it is recalled that extensible theory, for instance, predicts that curved pipes conveying fluid do not lose stability, no matter how high the flow velocity (when both ends of the pipe are supported).

The behaviour of curved pipes with supported ends according to this theory is similar to that predicted by the aforementioned extensible theories of Hill & Davis (1974) and Doll & Mote's (1974, 1976). With increasing flow velocity, the natural frequencies are found to either decrease or increase slightly, depending on the mode number and system parameters, but these changes from the zero-flow values are small. Thus, the natural frequency does not vanish for any flow velocity, in any of the modes of the system, and, moreover, the mode loci never coalesce (at least for the cases investigated); accordingly, the system cannot lose stability for either in-plane or out-of-plane motions.

The physical interpretation of this dramatic change in predicted behaviour when the forces associated with Π are taken into account is simple. The destabilizing forces that caused instability according to the conventional inextensible theory are the centrifugal-type forces proportional to $\bar{u}^2$ in equations (2.75)-(2.78), which may generally be denoted as $\mathcal{L}(\eta_1,\eta_3)\bar{u}^2$, where $\mathcal{L}(\eta_1,\eta_3)$ is a linear differential operator involving the dimensionless displacements η_1 and η_3. Now, it has been shown in the Appendices related to Chapter III, that for pipes supported at both ends, the combined force Π is simply $\Pi = -\bar{u}^2$. By referring to equations (2.75)-(2.78) again, it may easily be verified that, in all cases and for each $\mathcal{L}(\eta_1,\eta_3)\bar{u}^2$ term, there appears also a $\mathcal{L}(\eta_1,\eta_3)\Pi$ term; hence, since $\Pi \doteq -\bar{u}^2$, the destabilizing centrifugal terms in the equations of motion are exactly cancelled out by tensile-centripetal terms associated with the increased tension in the pipe due to these centrifugal forces. The net effect is zero, and the flow velocity (or pressure) can neither stabilize nor destabilize the system.

For cantilevered pipes, the effect of Π is not so radical. Considering in-plane motions, the system is still predicted to be subject to both buckling and flutter instabilities, whilst for out-of-plane motions only to flutter (similarly to a straight pipe). Quantitatively, however, there are differences. When the Π-related forces are taken into account, the critical flow velocities for instability are diminished; i.e., in this case these forces are destabilizing.

The physical reasons for this are more difficult to assess in this case, firstly because the cantilevered system is nonconservative, and secondly because Π in this case is not as simple as $\Pi = -\bar{u}^2$, but rather is as given by equations (3.89) and (3.90). In any event, the destabilizing centrifugal forces are not entirely cancelled out, and the presence of Π is also felt in one of the boundary conditions.

It should nevertheless be noted that the assumption of the pipe curvature not changing with $\bar{u}$, which is implicit in all inextensible theories, although justifiable rigorously or approximately for pipes supported at both ends, is rather weak for cantilevered ones. Hence, the results obtained by this theory for cantilevered pipes must be viewed with caution.

Finally, the dependence of natural frequencies on r_o and R_o discussed at the end of Section 7.1 for $\bar{u} = 0$, is the same in this case also, except that, since the frequency is but a weak function of $\bar{u}$, what was concluded there applies in this case for all values of $\bar{u}$.

The comparison of this theory to the extensible theory is deferred to Section 7.4.

7.3 THE EXTENSIBLE THEORY

In the extensible theory the initial shape of the pipe (at $\bar{u} = 0$) changes under the action of the centrifugal forces acting on the system at any given $\bar{u}$, and, if the flow is realistically considered to be viscous, by the action of frictional forces also. In this case the equations of motion thus generally depend on those of static equilibrium for any given flow velocity, which therefore must be solved first and ~~then used~~ for the solution of the equations of motion. This is true for the in-plane equations of motion, whereas, at least with the approximations introduced in formulating this theory, out-of-plane motions are unaffected (Chapter V). Indeed, the interesting finding was made that the out-of-plane motions are identical to those of the modified inextensible theory. Hence, attention in Chapter VI, in which the calculations with the extensible theory are presented, is confined to in-plane motions.

Calculations with this theory were made with several variants of the theory depending on (i) whether the fluid was considered to be inviscid or viscous, and (ii) whether the deformation due to initial, steady-state forces is taken into account or not. It is of interest that if this latter deformation is neglected, then the equations of motion of the extensible theory may be obtained from those of the modified inextensible one, simply by taking $\Pi = \Pi_p - \mathcal{A}(\partial\eta_3/\partial\xi - r_o\eta_1)$, as given by equation (5.4).

It was shown that under the effect of internal flow (i.e., the effect of centrifugal force) a pipe supported at both ends is deflected out of the initial, unstressed configuration, and this stressed shape changes continually with increasing flow (Figs. 38-40). For inviscid flow, the stressed shape changes symmetrically vis-à-vis the unstressed one, but this change is unsymmetrical if the flowing fluid is viscous. The combined force Π calculated in these two cases (inviscid and viscous fluid) was found to be almost the same, and to be similar to that obtained by the modified inextensible theory, but this does not apply generally for all values of $\bar{u}$.

As shown in Chapter VI, no matter which variant of the extensible theory is employed, the results are qualitatively the same: the natural frequencies of a pipe with supported ends do not change greatly with increasing flow velocity, and the system thus never loses stability.

The results were compared with those obtained previously by Hill & Davis (1974) and Doll & Mote (1974,1976). Bearing in mind the differences in some of the parameters in the calculations of the other investigators and differences in the method of solution (e.g., in calculating the deformed steady-state shape of the pipe), the agreement between all three theories, qualitative and quantitative, is remarkably good.

Thus, the results obtained by the extensible theory are qualitatively similar to those obtained by the modified inextensible theory, in the case of in-plane motions, and identical in the case of out-of-plane motions. Quantitative differences as well as a more general discussion of these theories is undertaken in Section 7.4.

No calculations were conducted with the extensible theory for cantilevered pipes conveying fluid.

7.4 THE MODIFIED INEXTENSIBLE AND EXTENSIBLE THEORIES COMPARED

It should first be said that the conventional inextensible theory is left out of this discussion, because it is now considered to be totally unrealistic. This is the opinion of not only the author, but all researchers who have subsequently worked on this subject (Svetlisky, Hill & Davis and Doll & Mote).

As mentioned in the previous Sections, the dynamical behaviour of the system is similar or the same, whether calculated by the modified inextensible theory or the extensible one, at least for pipes supported at both ends. In the case of out-of-plane motions the natural frequencies at least as obtained in this Thesis, are identical in the two cases. The differences for the in-plane motions are also quite small, as may be seen in Fig. 51 or Table 5).

This shows that the main effect of "extensibility", which produces the profound differences in the dynamical behaviour as predicted by Doll & Mote (1974) and Hill & Davis (1974), on the one hand, and Chen (1972a,b, 1973) on the other, is not the extensibility of the centerline of the pipe at all! Rather, it is whether the steady-state initial stress is taken into account or not; i.e., whether $\Pi \neq 0$, as is the case of Doll & Mote and Hill & Davis, and Svetlisky (1977), or $\Pi = 0$, as in the case of Chen.

Hence, although small differences in the frequencies are obtained in the results obtained by the modified inextensible and extensible theories it is shown in this Thesis that the former may be used instead of the latter for any practical purposes. One reason for doing so is that obtaining solutions by the extensible theory is rather expensive. As shown in Chapter VI (Figs. 43-45), to achieve convergent results by the extensible theory, many finite elements are necessary - easily 20 to 30 - this number increasing with increasing $\mathcal{A} = A_t L^2/I$. This contrasts with similar convergence being achieved by means of the (modified) inextensible theory, with 5 to 10 elements.

Of course, the foregoing discussion applies provided that the stressed centerline remains close to the unstressed one. If it does not, then the extensible theory must be used. However, generally speaking, the deformed shape of the centerline in such cases must be obtained by means of nonlinear theory, in preference to the approximate linear approach adopted in this Thesis.

7.5 ON THE EXISTENCE OF INSTABILITIES IN CURVED PIPES CONVEYING FLUID

The work presented in this Thesis makes it clear that, provided the combined force is properly taken into account - be it by means of the modified inextensible theory or the extensible theory - no instabilities arise with increasing flow, provided that the ends of the pipe are supported.

However, as shown in Section 6.2.4, this is true only if the ends of the pipe are completely supported, i.e., so long as u,v,w are all zero at the support. If sliding is permitted, however, then instability (flutter) is possible, as shown for in-plane motions in Section 6.2.4. This is an entirely new result, not previously reported by any of the earlier investigators.

Of course, if one end of the pipe is entirely free, then instabilities have been shown to be possible - both buckling and flutter.

7.6 SUGGESTIONS FOR FUTURE WORK

The physical system analyzed in this Thesis has been idealized in many respects. Therefore, there are several possible directions in which work can be extended.

Let us consider the fluid mechanics first. Throughout the work, it was assumed that the internal flow is a plug flow and that the flow velocity is constant. However, for the flow in a curved pipe, secondary flow and even separation may exist and the flow velocity may also change along the pipe and in terms of its distribution in a cross section of the pipe. These effects may affect the dynamics of the system, yet the work presented in this Thesis cannot account for them.

Now, let us consider large displacements and an arbitrary shape of the centerline of the pipe, which lead to a nonlinear problem. In this work, the displacements were assumed to be small; then, the nonlinear terms which couple in-plane and out-of-plane motions, as well as the perturbations of the bending moments and twist couple, are neglected. All these simplifications may cause the theory to lose accuracy.

Finally, because of the lack of experimental results the results of this work are difficult to confirm. Hence, some experiments involving curved pipes would be useful.

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TABLE 1 Dimensionless Frequencies of a Clamped-Free Tubular Beam

MODE	ANALYTICAL RESULTS	FINITE-ELEMENT RESULTS* (in-plane)	FINITE-ELEMENT RESULTS* (out-of-plane)
1	3.5156	3.5160	3.5160
2	22.0340	22.0368	22.0455
3	61.7010	61.8389	61.9188
4	120.9120	121.3581	122.3196
5	199.8500	204.5024	203.0202
6	298.4602	310.7852	337.2727

* Note that four-element discretization scheme was used in the in-plane program, and five-element discretization scheme was used in the out-of-plane program.

TABLE 2 Dimensionless Frequencies of a Clamped-Clamped Tubular Beam

MODE	ANALYTICAL RESULTS	FINITE-ELEMENT RESULTS* (in-plane)	FINITE-ELEMENT RESULTS* (out-of-plane)
1	22.3729	22.3733	22.3792
2	61.6696	61.6739	61.7939
3	120.9120	120.9214	121.7697
4	199.8545	200.0054	203.3525
5	298.5638	299.5346	305.1028

* Note that six-element discretization scheme was used in both the in-plane and out-of-plane programs.

TABLE 3 Dimensionless Frequencies of a Clamped-Pinned Tubular Beam

MODE	ANALYTICAL RESULTS	FINITE-ELEMENT RESULTS* (in-plane)	FINITE-ELEMENT RESULTS (out-of-plane)
1	15.4313	15.4182	15.4201
2	49.9310	49.9665	50.0295
3	104.2440	104.2796	104.8099
4	178.2759	178.5612	180.8799
5	272.0190	272.4762	279.5706

*Note that five-element discretization scheme was used in the in-plane program, and six-element discretization scheme was used in the out-of-plane program.

TABLE 4 Dimensionless Frequencies of a Pinned-Pinned Tubular Beam

MODE	ANALYTICAL RESULTS	FINITE-ELEMENT RESULTS* (in-plane)	FINITE-ELEMENT RESULTS* (out-of-plane)
1	$\pi^2 = 9.8696$	9.8696	9.8706
2	$4\pi^2 = 39.4784$	39.4812	39.5438
3	$9\pi^2 = 88.8264$	88.8921	89.5318
4	$16\pi^2 = 157.9136$	157.9604	161.5514
5	$25\pi^2 = 246.7401$	249.8259	273.8613
6	$36\pi^2 = 355.3057$	366.5469	395.3176

* Note that five-element discretization scheme was used in the in-plane and out-of-plane programs.

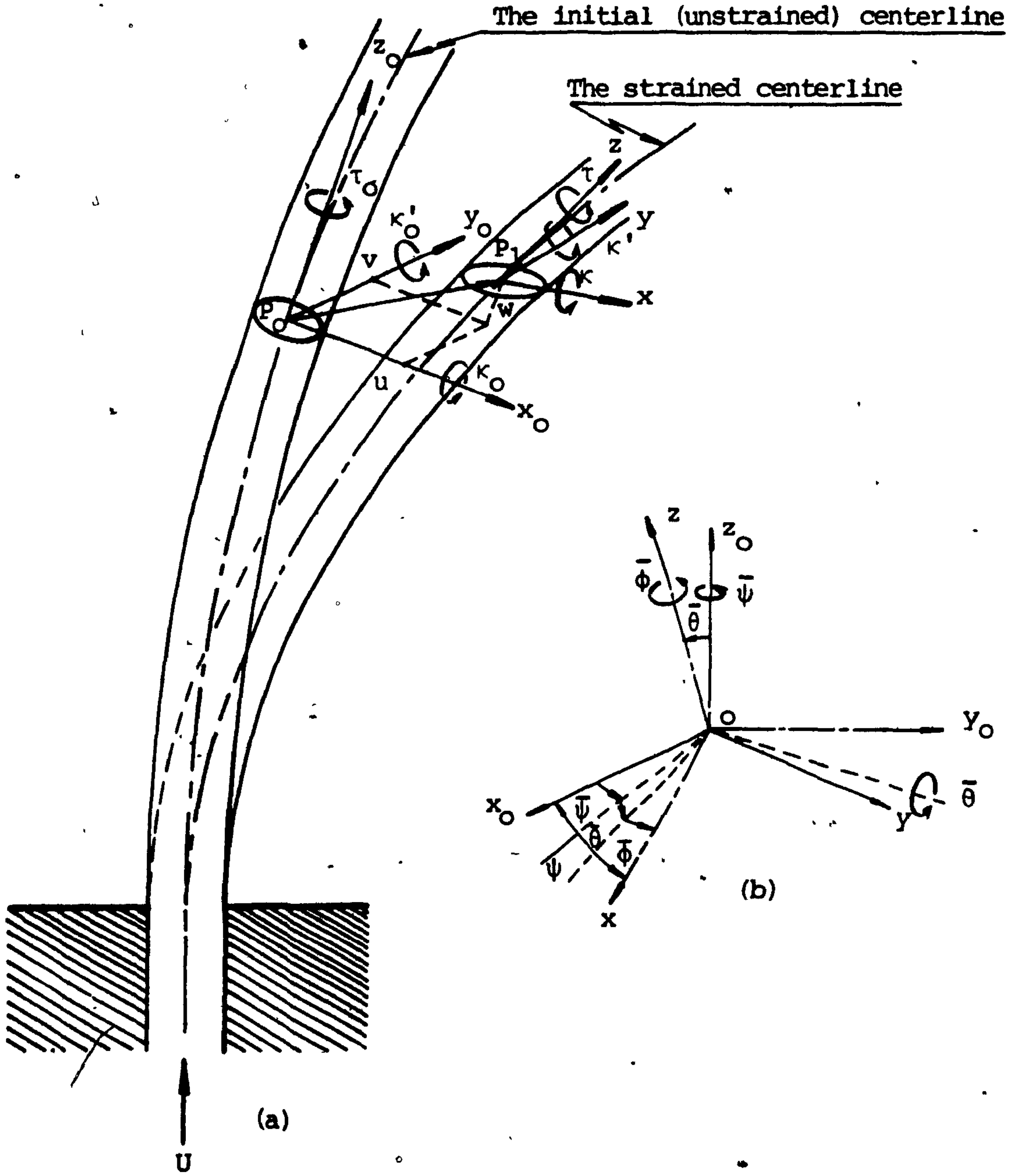
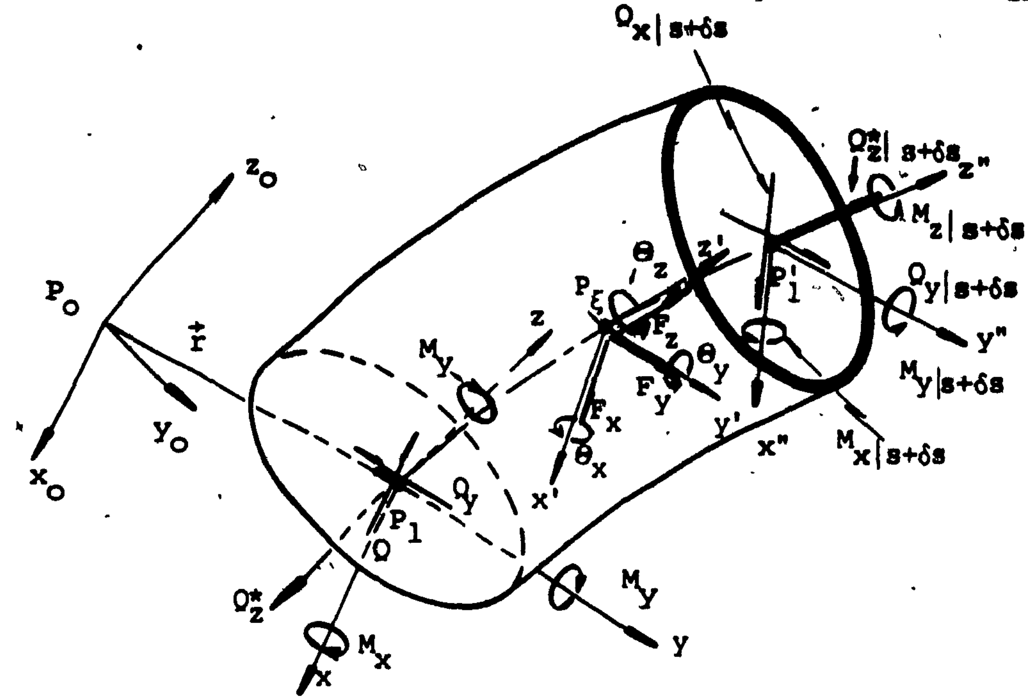
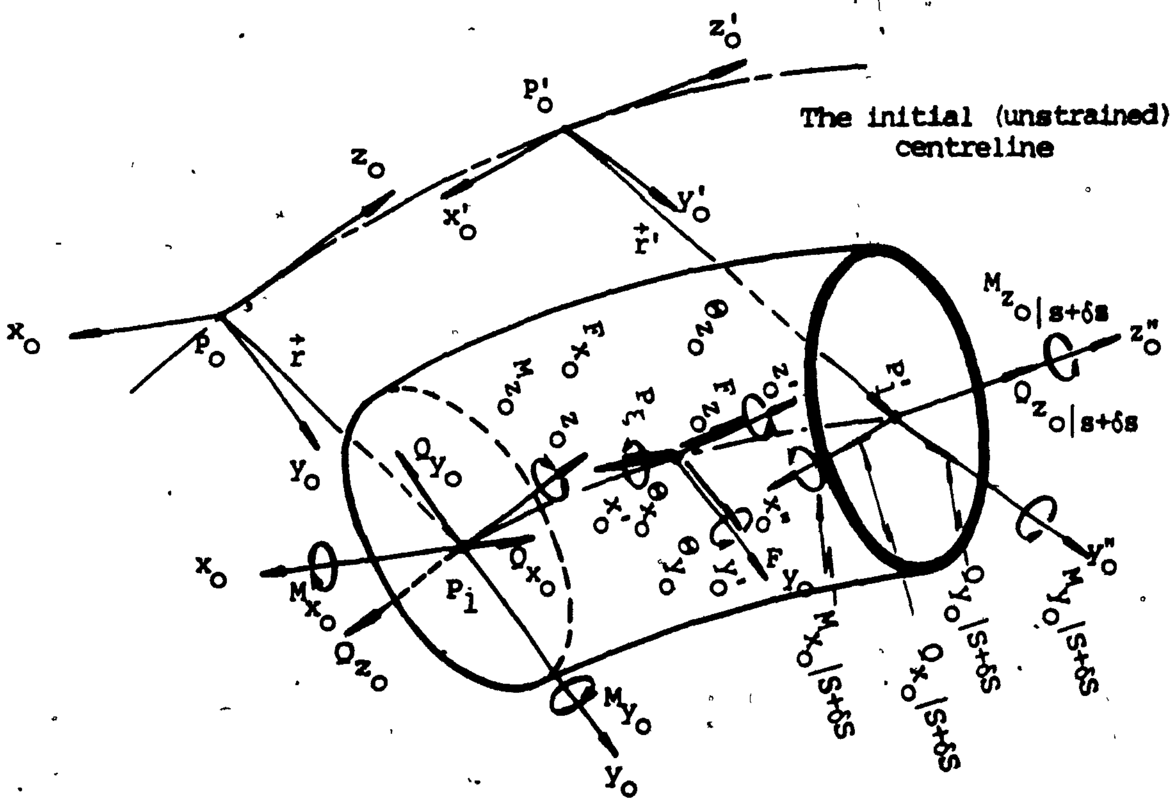


Fig. 1 (a) The system under consideration, showing a curved pipe conveying fluid and submerged in a quiescent fluid

(b) The two systems of axes (x_o, y_o, z_o) and (x, y, z) and the corresponding angles of rotation.



(a)



(b)

Fig. 2 Forces and Moments Acting on the Pipe Element Expressed

(a) in the reference system (x,y,z)

(b) in the reference system (x_o,y_o,z_o)

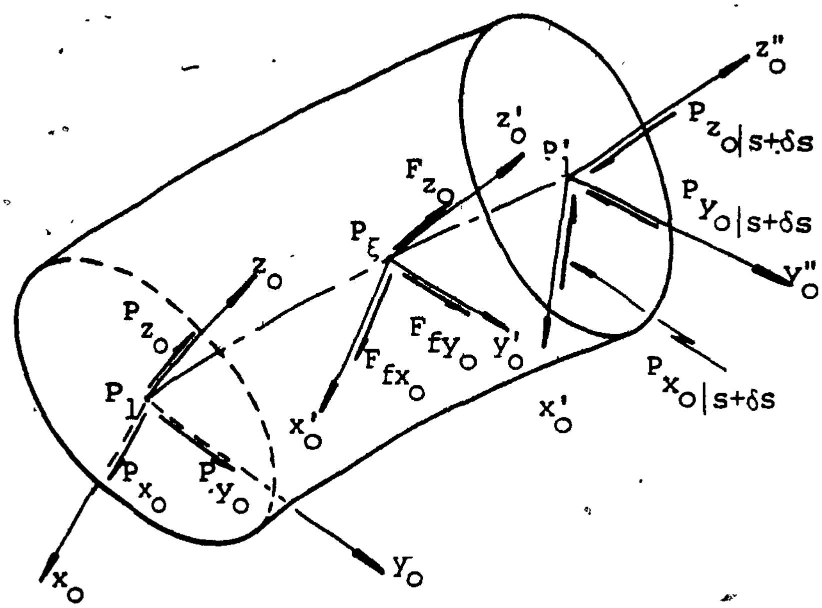


Fig. 3 Forces Acting on the Fluid Element, Expressed in the Reference System (x_o, y_o, z_o).

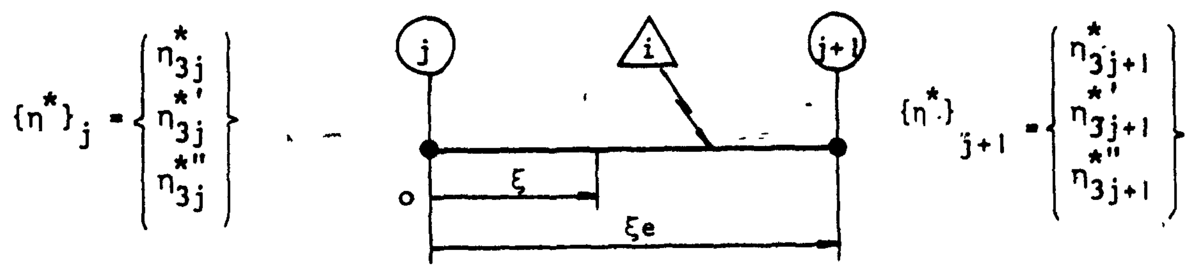


Fig. 4 Two-Node Pipe Element for the Case of In-Plane Motion (inextensible case)

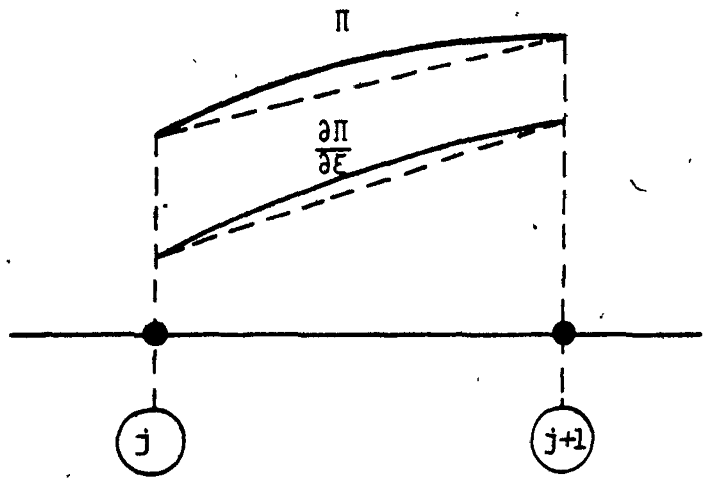


Fig. 5 Linear Polynomial Approximation Used in the Finite-Element Formulation (inextensible case)

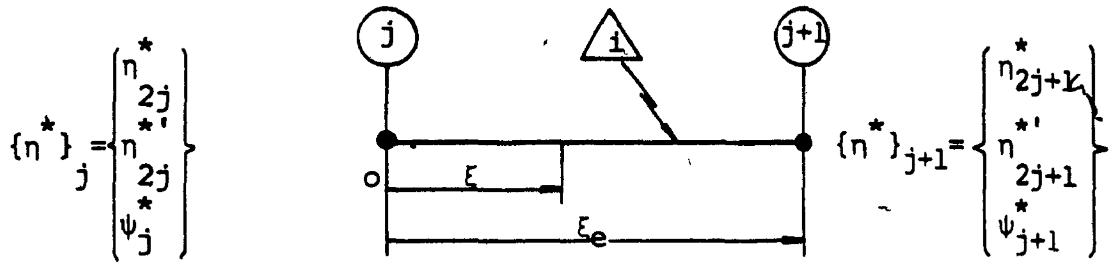


Fig. 6 Two-Node Element for the Case of Out-of-Plane Motion (inextensible case)

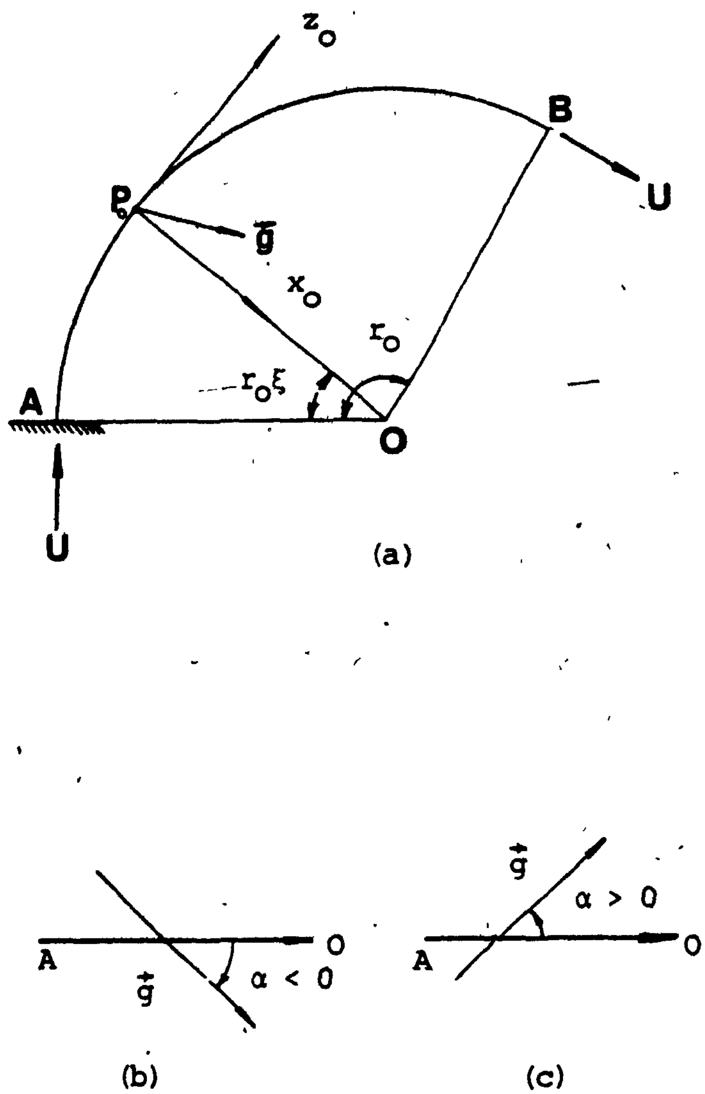


Fig. 7 (a) An incomplete circular pipe conveying fluid

(b) The position of $\vec{g}$ associated with $\alpha < 0$

(c) The position of $\vec{g}$ associated with $\alpha > 0$

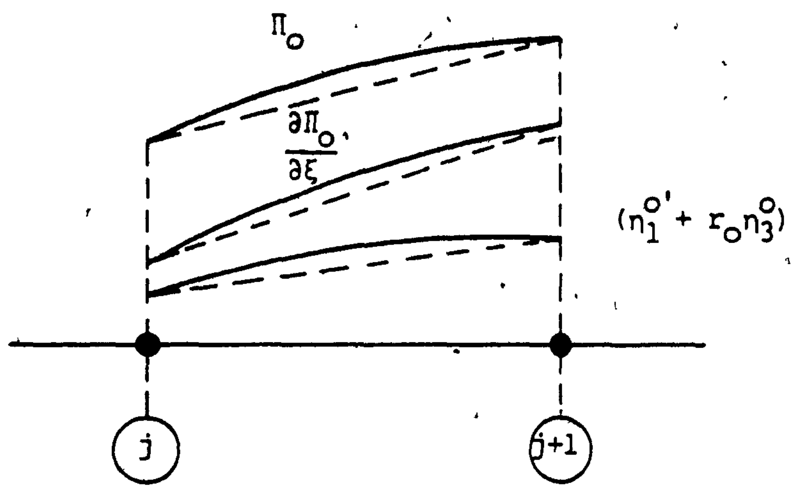


Fig. 8 Linear Polynomial Approximation Used in the Finite-Element Formulations (extensible case)

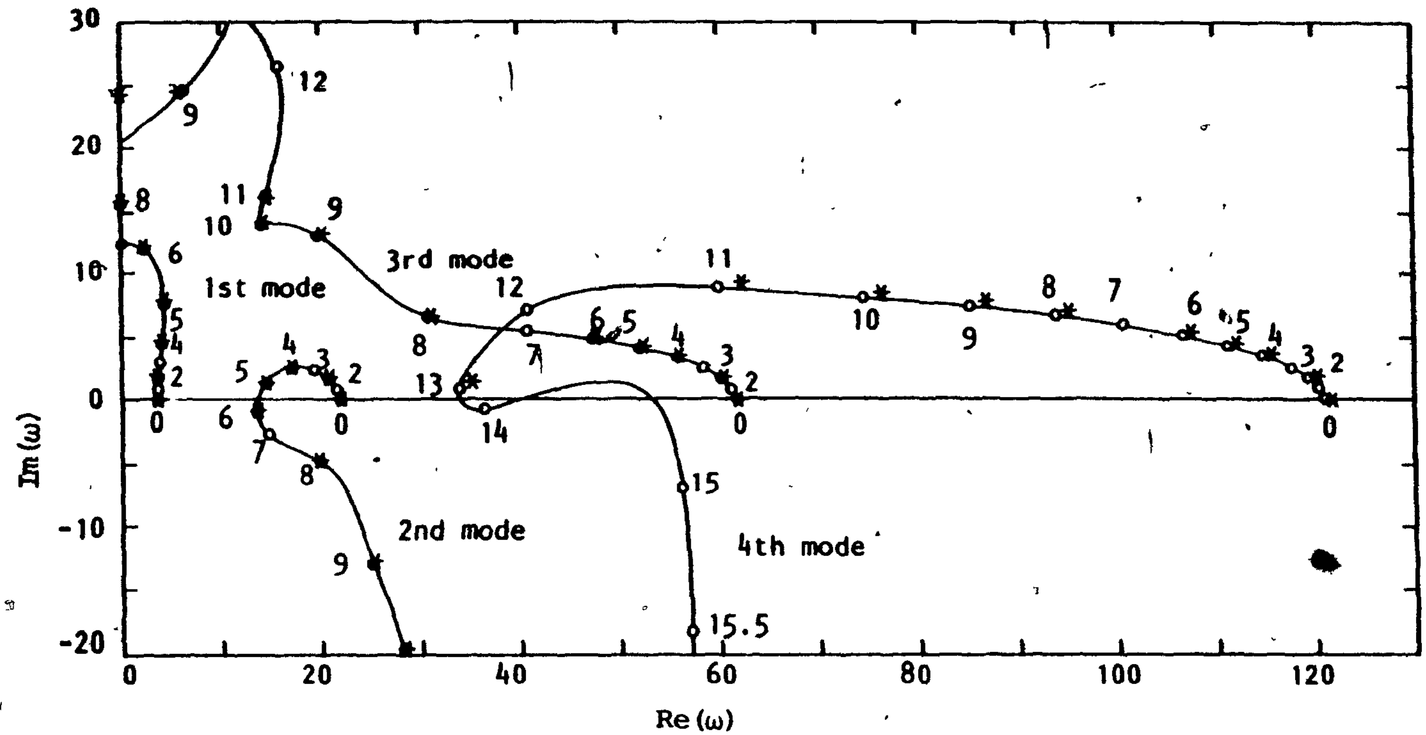


Fig. 9 The Dimensionless Complex Frequency of the Four Lowest Modes of a Horizontal Straight Tubular Cantilever Conveying Fluid as a Function of the Dimensionless Flow Velocity for $\beta = 0.200$, $\gamma = 0$

o Gregory & Paidoussis, (1966a)

* Finite-Element Method

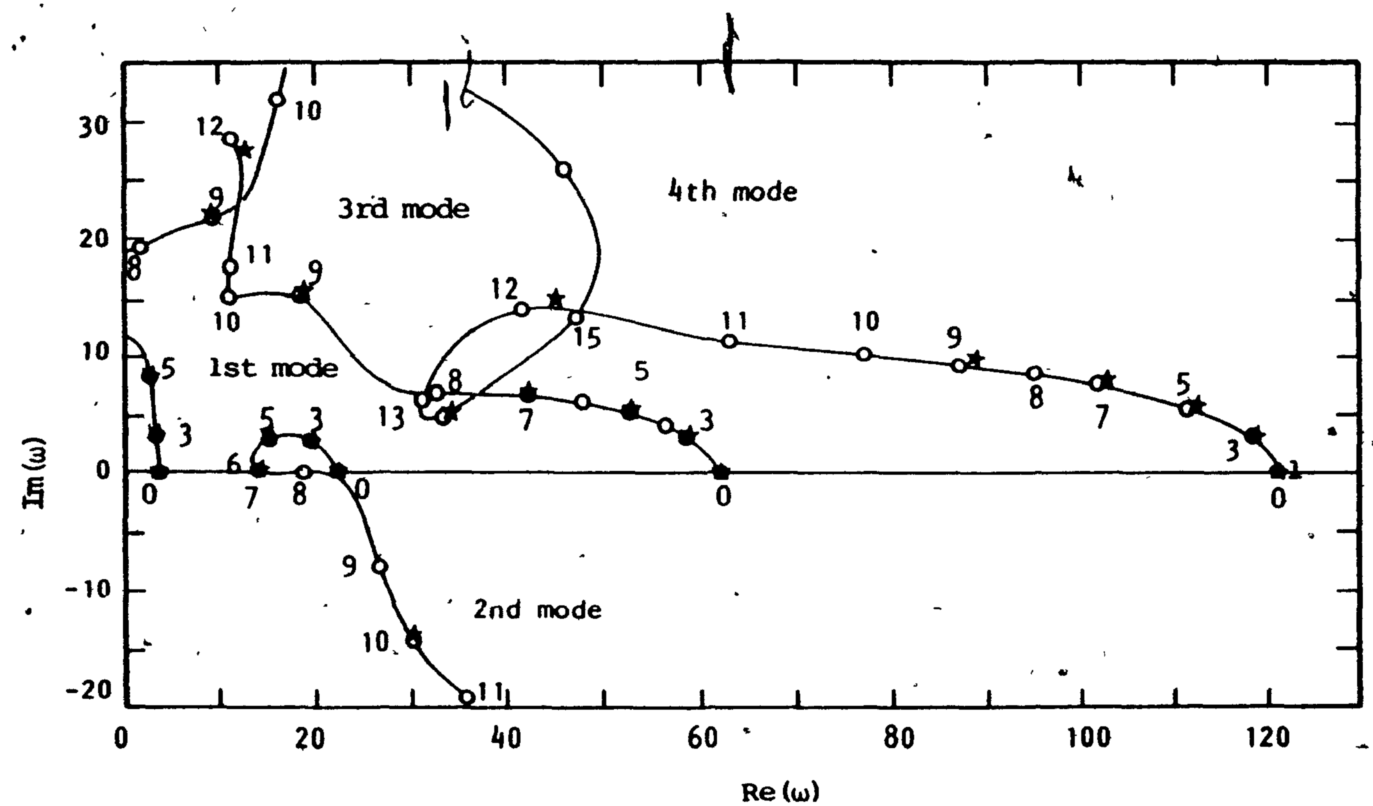


Fig. 10 The Dimensionless Complex Frequency of the Four Lowest Modes of a Horizontal Straight Tubular Cantilever Conveying Fluid as a Function of the Dimensionless Flow Velocity for $\beta = 0.295$, $\gamma = 0$.

o Gregory & Paidoussis (1966a)

★ Finite-Element Method

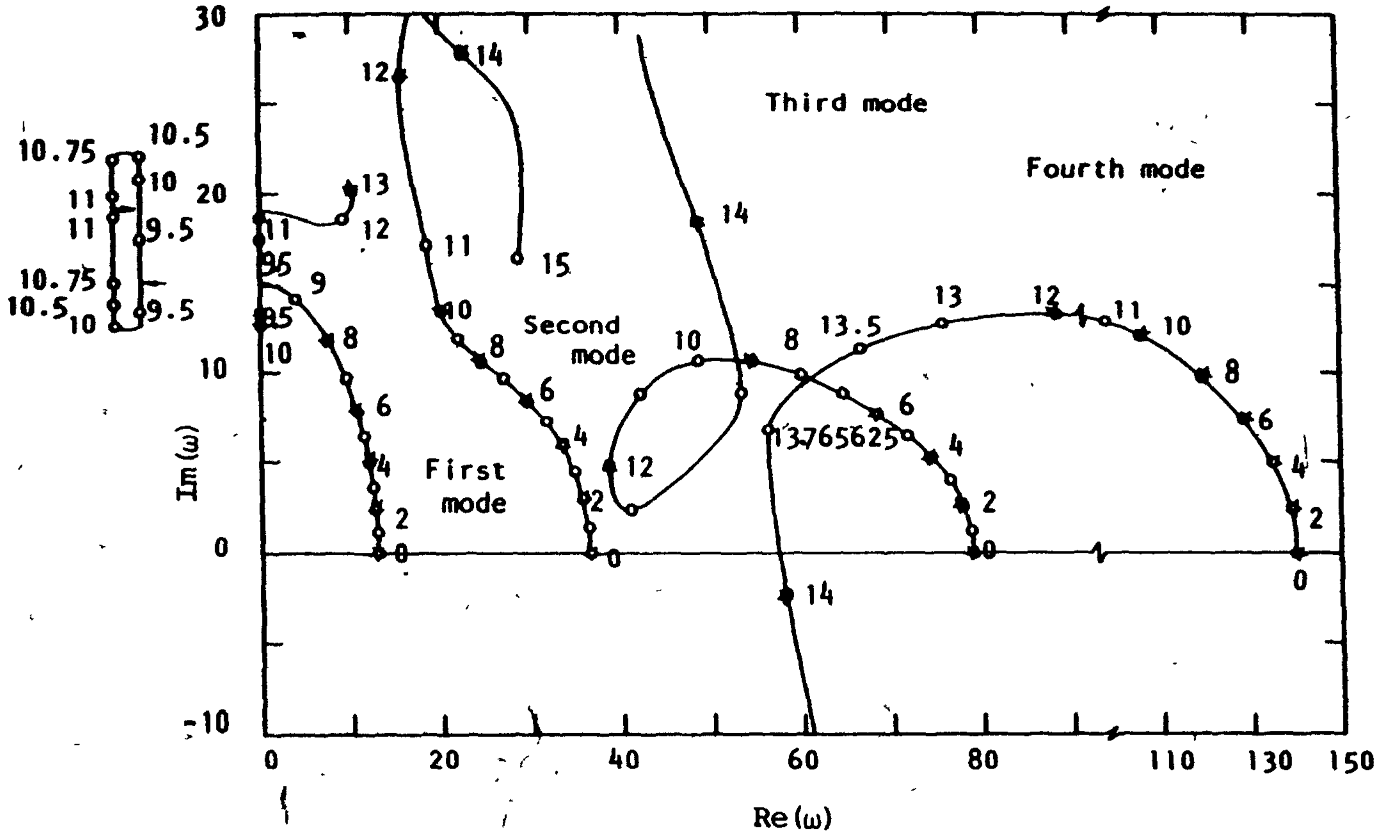


Fig. 11 The Dimensionless Complex Frequency of the Four Lowest Modes of a Vertical Straight Tubular Cantilever Conveying Fluid as a Function of the Dimensionless Flow Velocity for $\gamma = 100$ and $\beta = 0.4$.

o Paidoussis (1970)

☆ Finite-Element Method

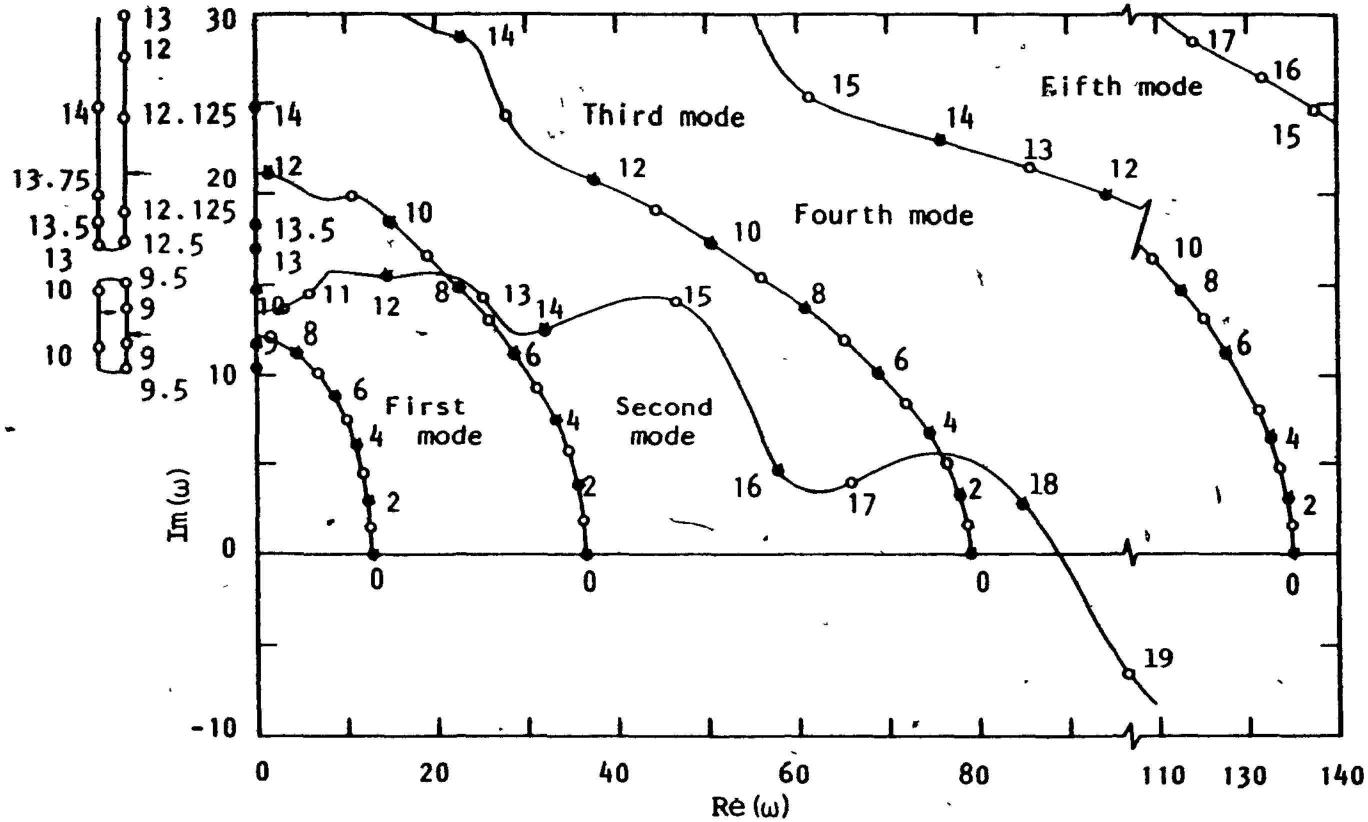


Fig. 12 The Dimensionless Complex Frequency of the Four Lowest Modes of a Vertical Straight Tubular Cantilever Conveying Fluid as a Function of the Dimensionless Flow Velocity for $\gamma = 100$ and $\beta = 0.65$.

o Paidoussis (1970)

★ Finite-Element Method

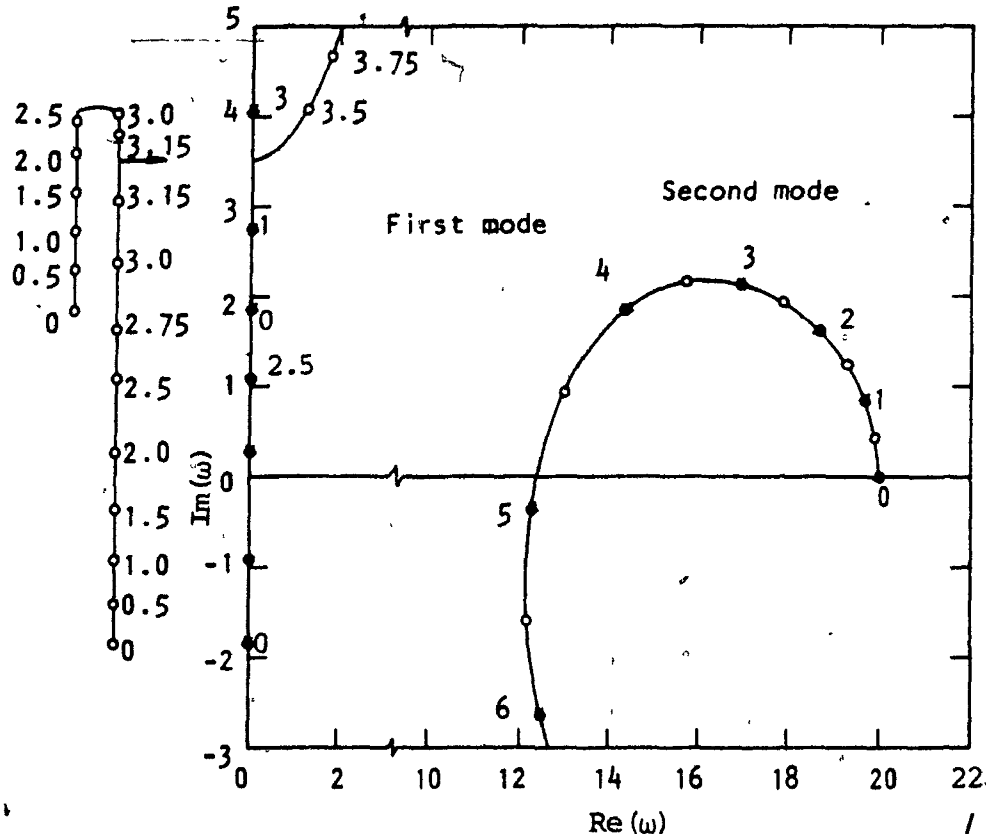


Fig. 13 The Dimensionless Complex Frequency of the Two Lowest Modes of a Standing Straight Cantilever Conveying Fluid as a Function of the Dimensionless Flow Velocity $\gamma = -10$ and $\beta = 0.20$

o Paidoussis (1970)

★ Finite-Element Method

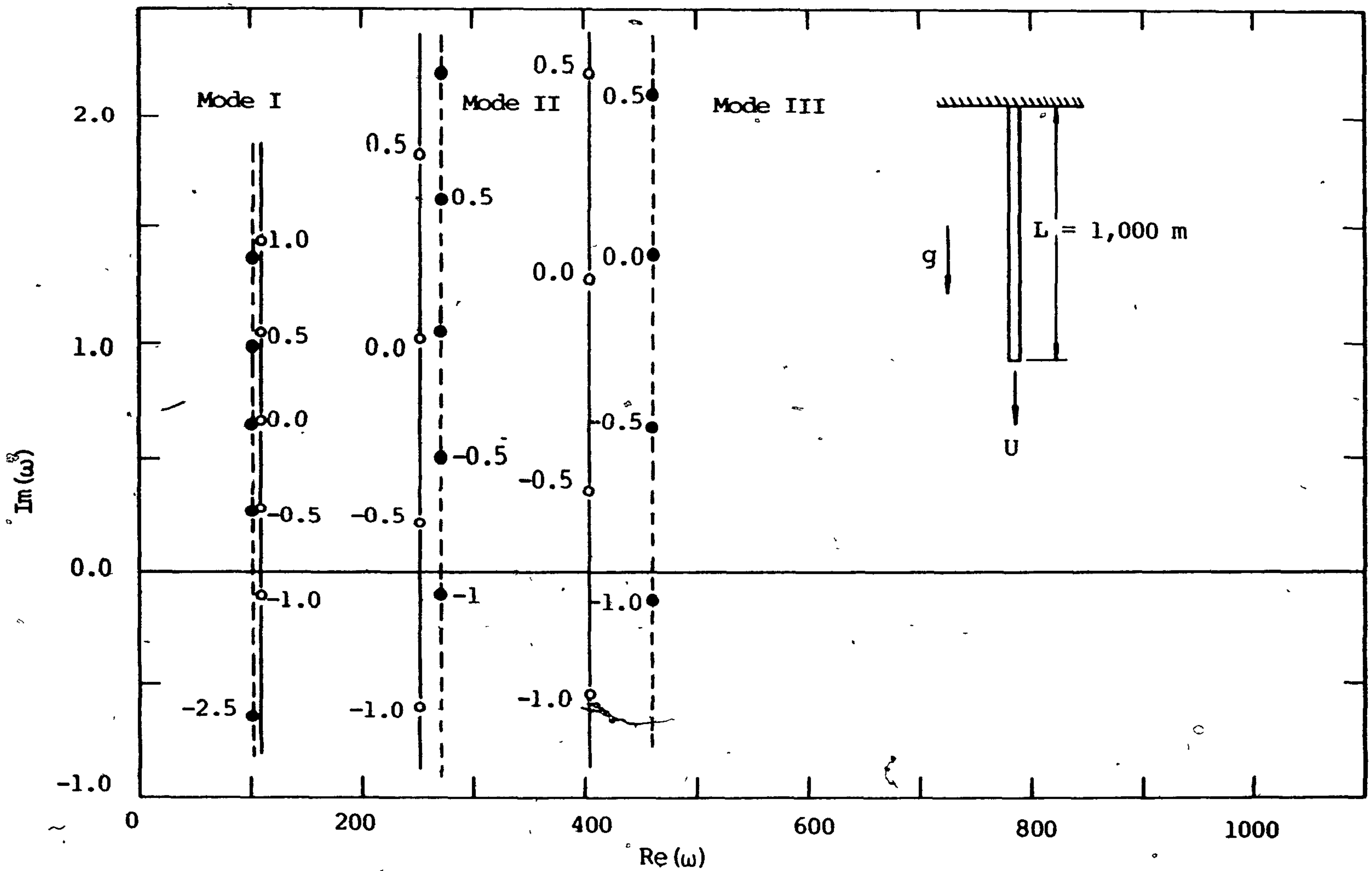


Fig. 14 Dimensionless Complex-Frequency Diagram of a Long, Vertical, No Cap Cantilevered Pipe Conveying Fluid and Submerged in a Quiescent Fluid.

- o Luu and Paidoussis (1985)
- ● Finite-Element Method of in-plane motion and of out-of-plane motion

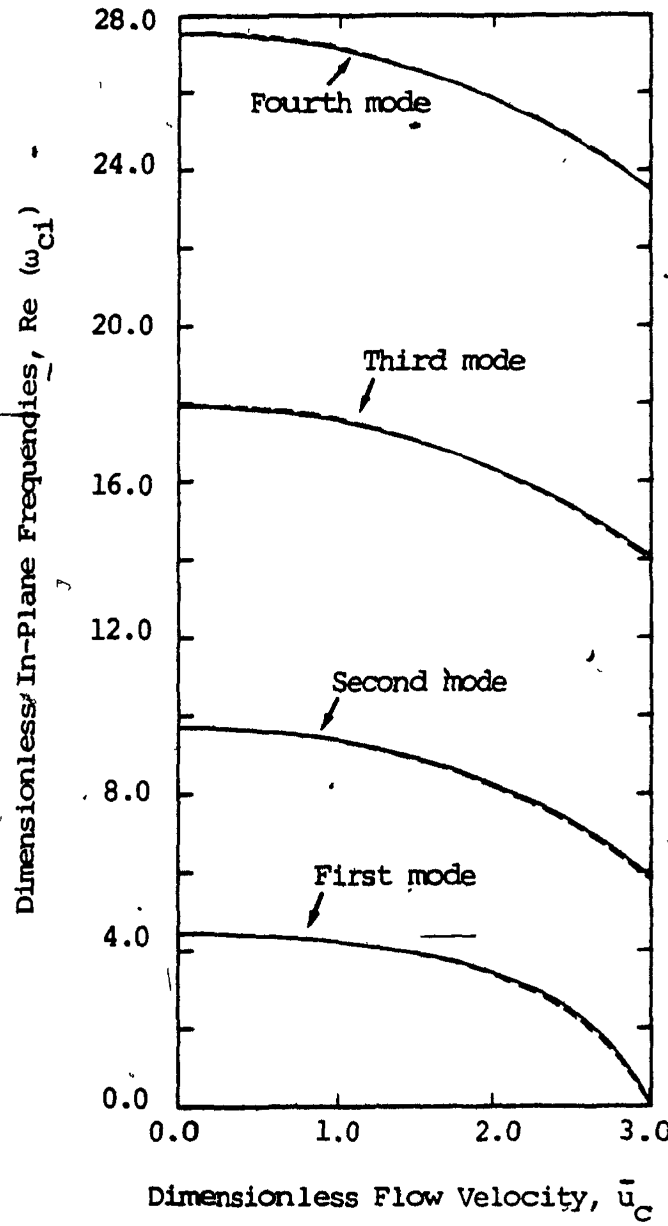


Fig. 15 Dimensionless In-Plane Frequencies of a Clamped-Clamped Semi-Circular Pipe Conveying Fluid as a Function of the Dimensionless Flow Velocity for $\beta = 0$ and $\Pi_o = 0$

- - - - - Chen (1972a)

——— Finite-Element

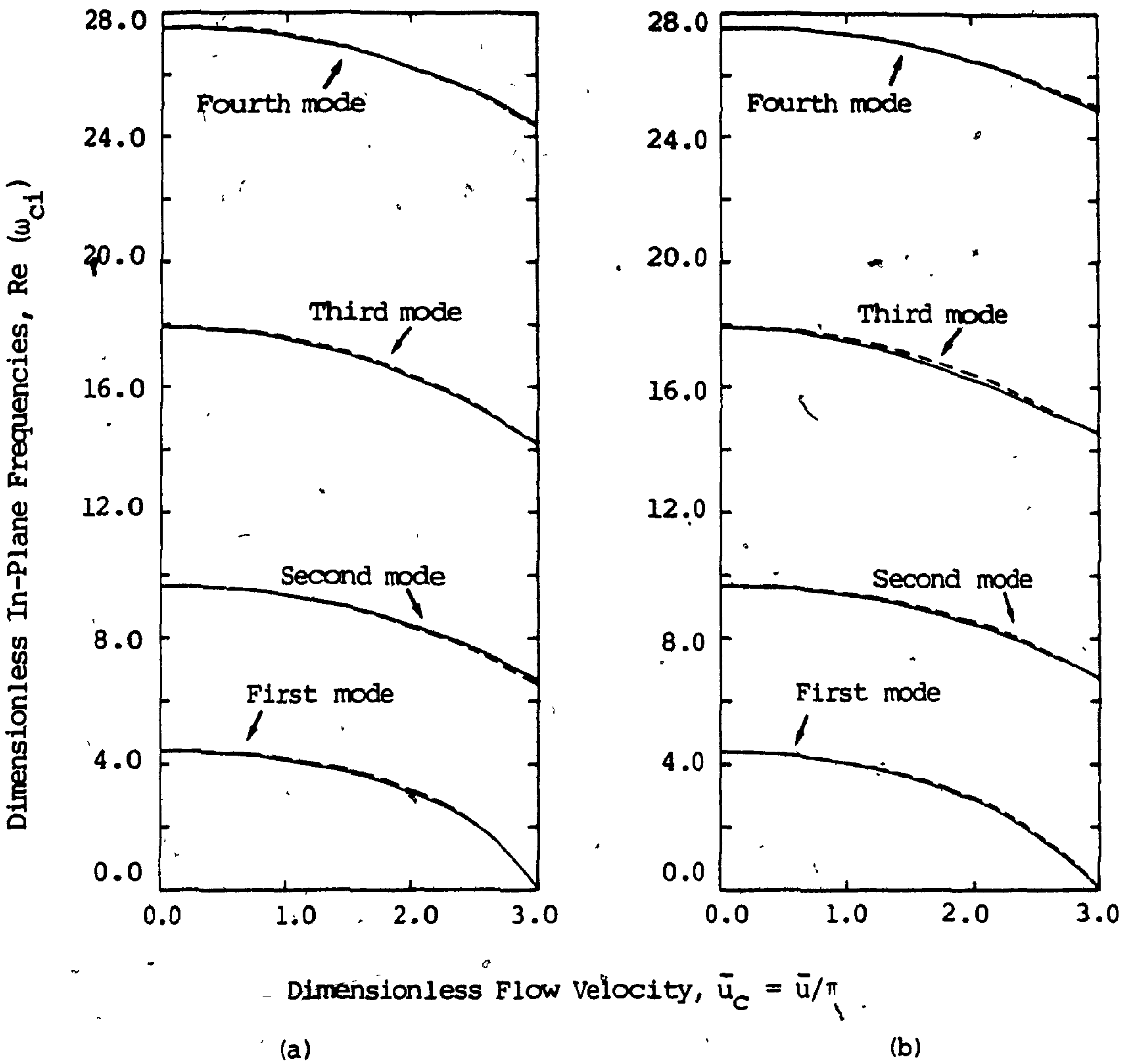


Fig. 16 Dimensionless In-Plane Frequencies of a Clamped-Clamped Semi-Circular Pipe Conveying Fluid as a Fucntion of the Dimensionless Flow Velocity for

(a) $\beta = 0.5$ and $\Pi_o = 0$

(b) $\beta = 1.0$ and $\Pi_o = 0$

------ Chen (1972a)

——— Finite-Element

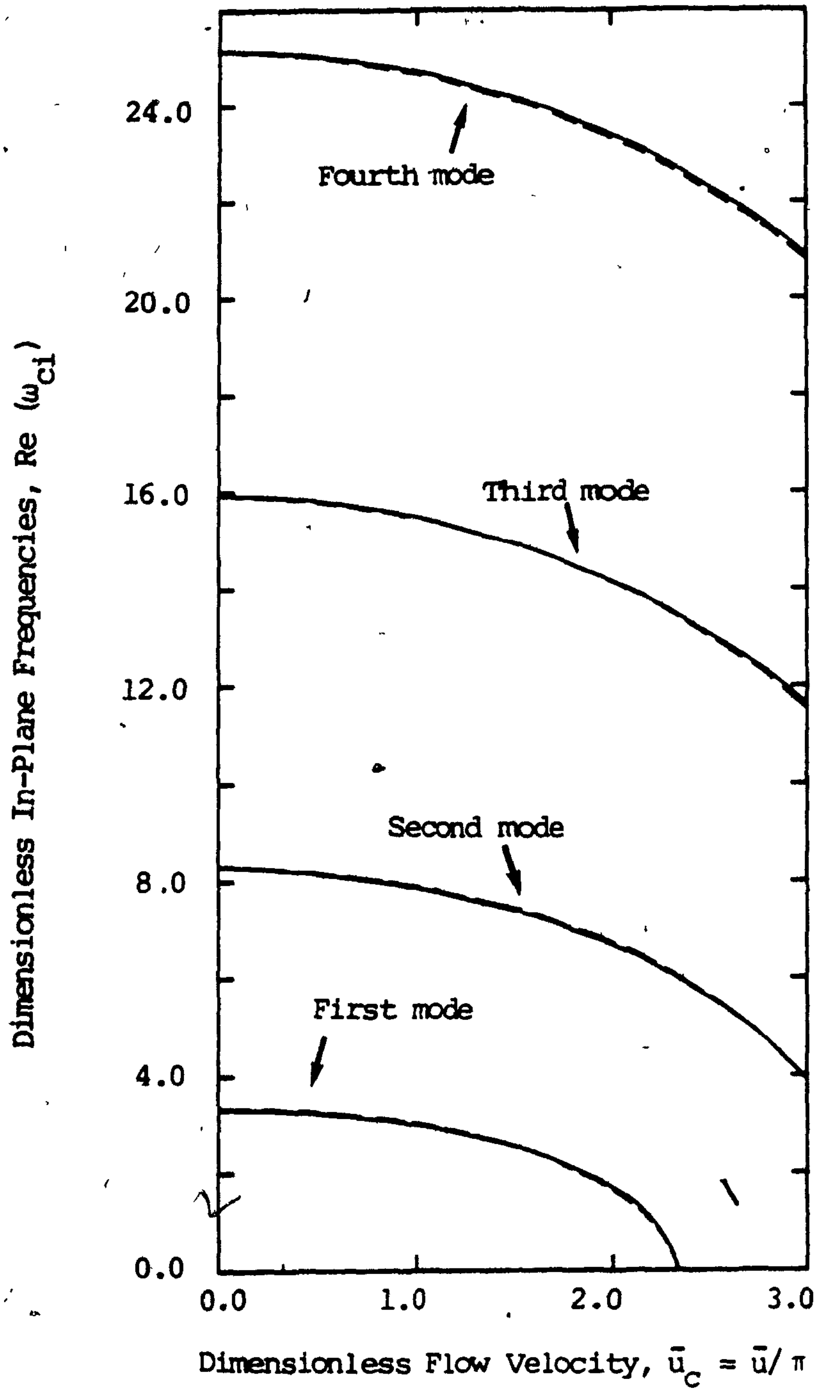


Fig. 17 Dimensionless In-Plane Frequencies of a Clamped-Pinned Semi-Circular Pipe Conveying Fluid as a Function of the Dimensionless Flow Velocity for $\beta = 0$ and $\Pi_o = 0$.

------ Chen (1972a)

——— Finite-Element

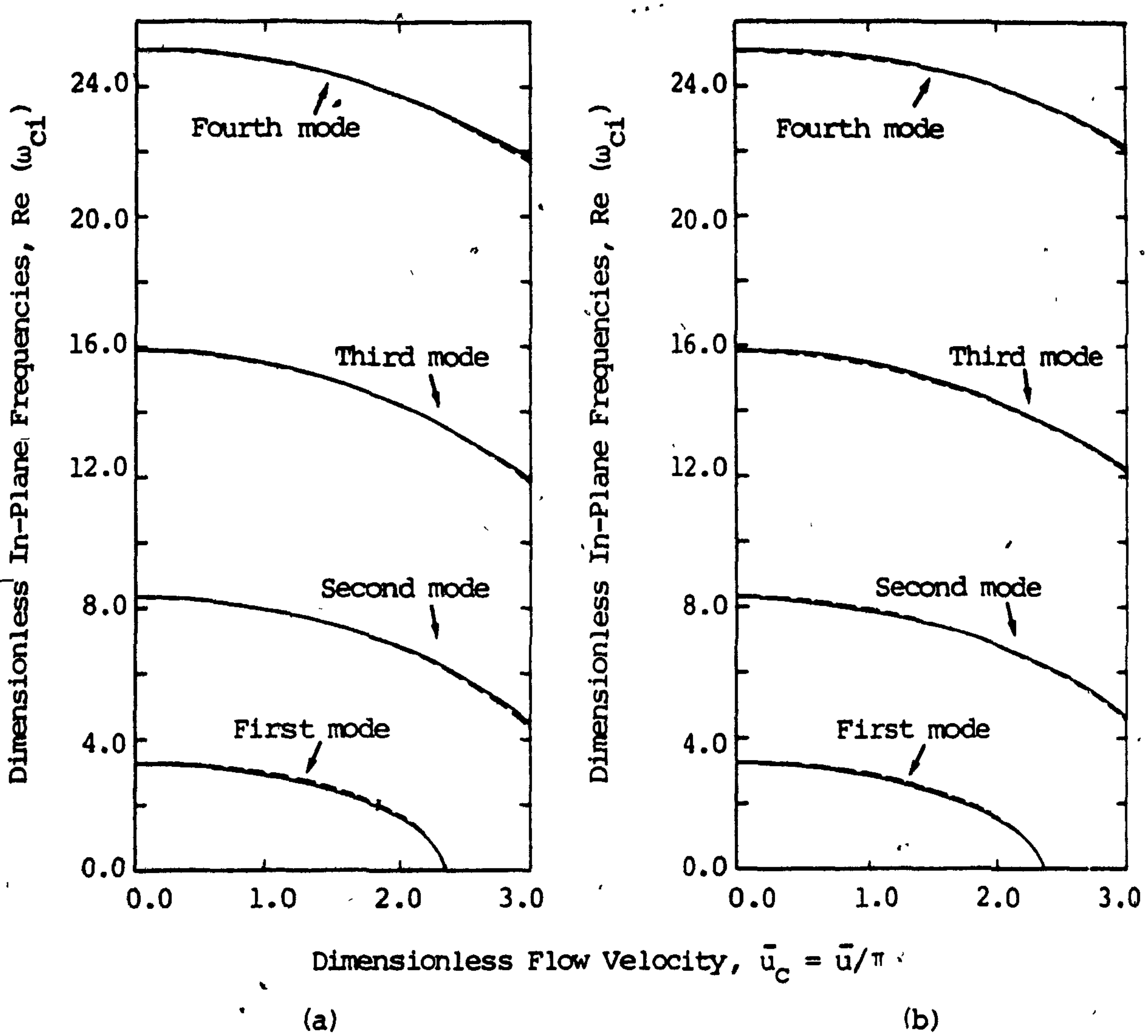


Fig. 18 Dimensionless In-Plane Frequencies of a Clamped-Pinned Semi-Circular Pipe Conveying Fluid as a Function of the Dimensionless Flow Velocity for

(a) $\beta = 0.5$ and $\Pi_o = 0$

(b) $\beta = 1.0$ and $\Pi_o = 0$

------ Chen (1972a)

——— Finite-Element

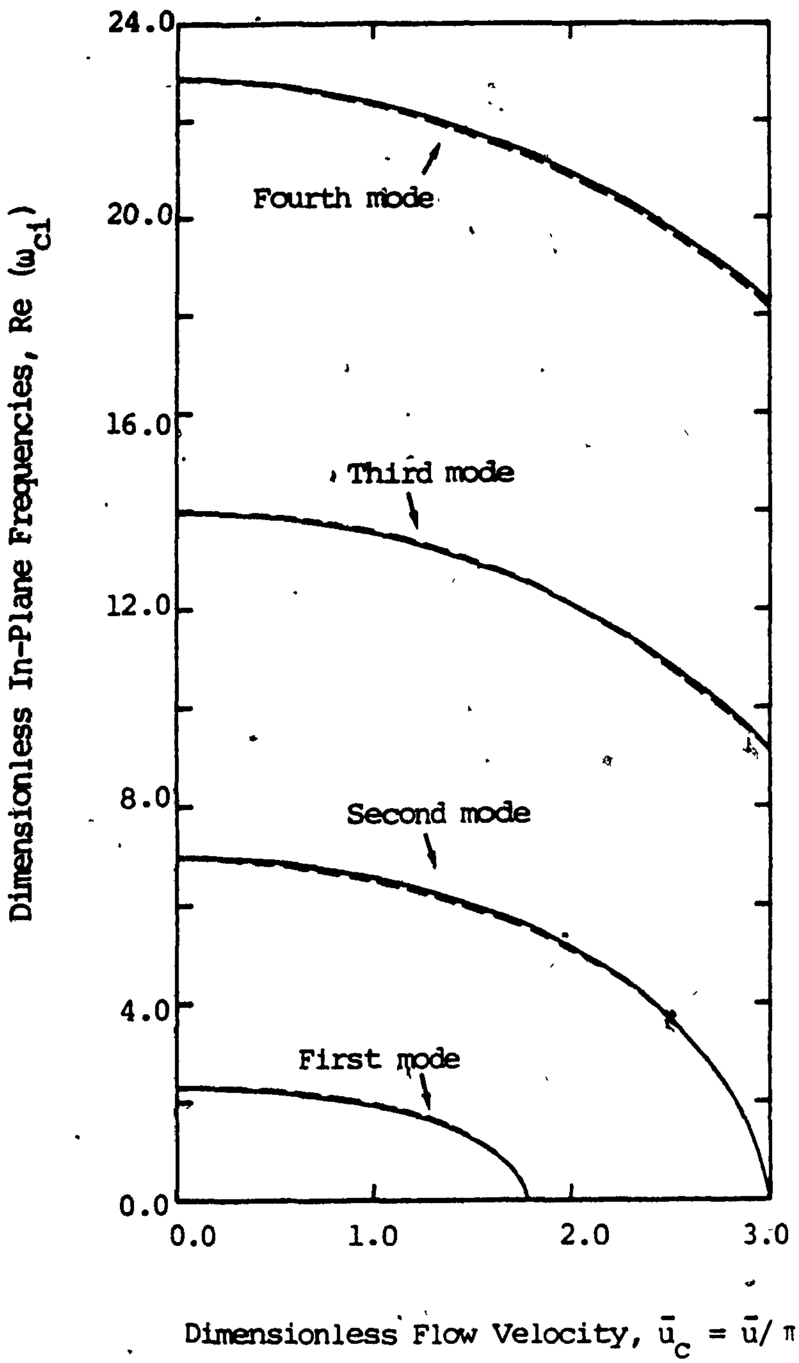


Fig. 19 Dimensionless In-Plane Frequencies of a Pinned-Pinned Semi-Circular Pipe Conveying Fluid as a Function of the Dimensionless Flow Velocity for $\beta = 0$ and $\Pi_o = 0$

----- Chen (1972a)

——— Finite-Element

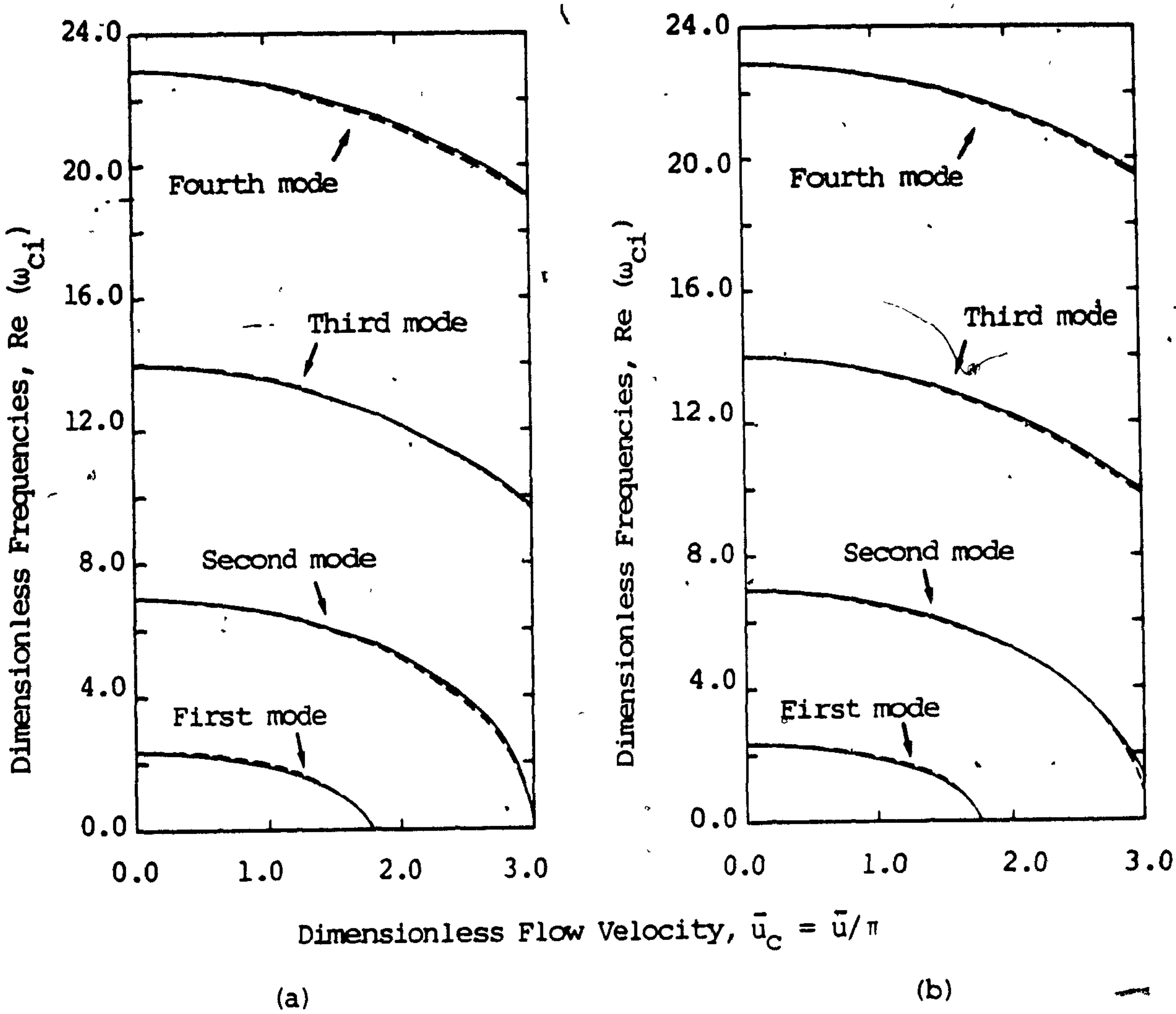


Fig. 20 Dimensionless In-Plane Frequencies of a Pinned-Pinned Semi-Circular Pipe Conveying Fluid as a Function of the Dimensionless Flow Velocity for

(a) $\beta = 0.5$ and $\Pi_o = 0$

(b) $\beta = 1.0$ and $\Pi_o = 0$

----- Chen (1972a)

——— Finite-Element

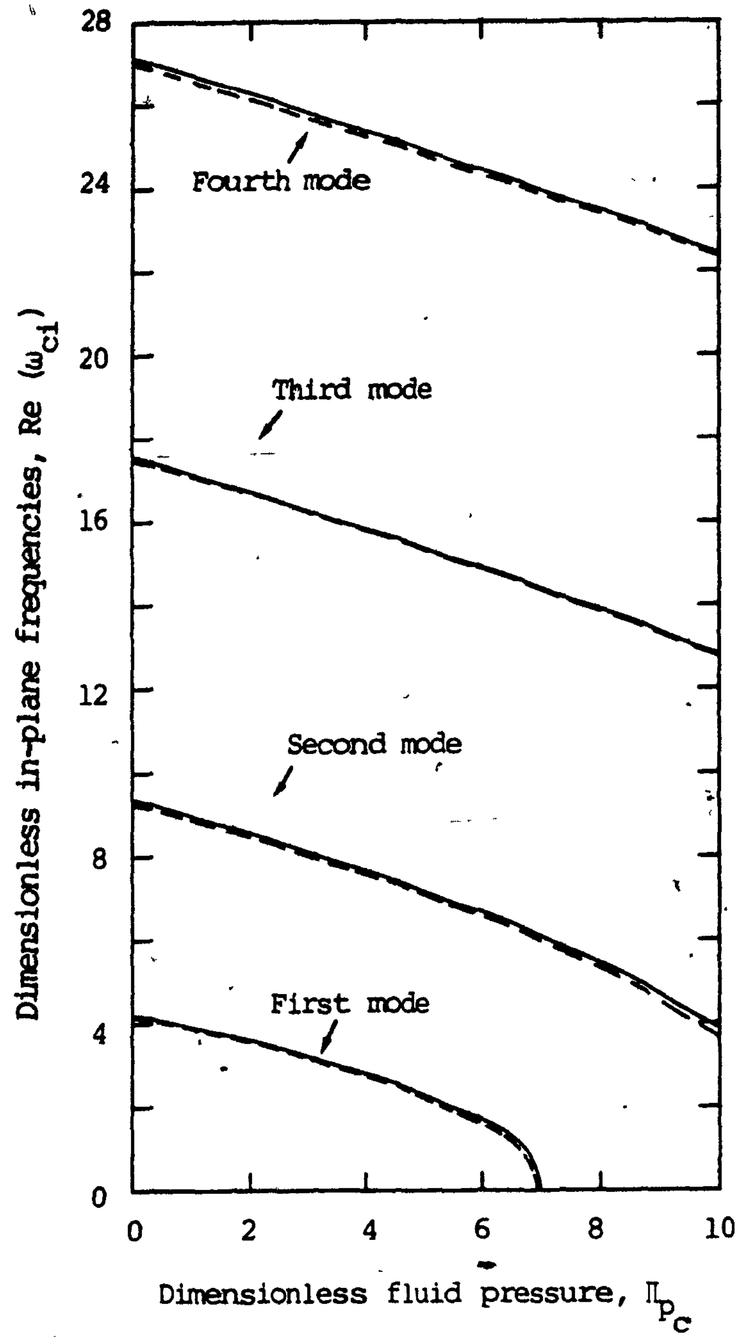


Fig. 21 Dimensionless in-plane frequencies of a clamped-Clamped semi-circular pipe conveying fluid as a function of the dimensionless fluid pressure for β = 0.0 and $\bar{u} = \pi(\bar{u}_c=1)$.

----- Chen (1972a)

——— Finite-Element

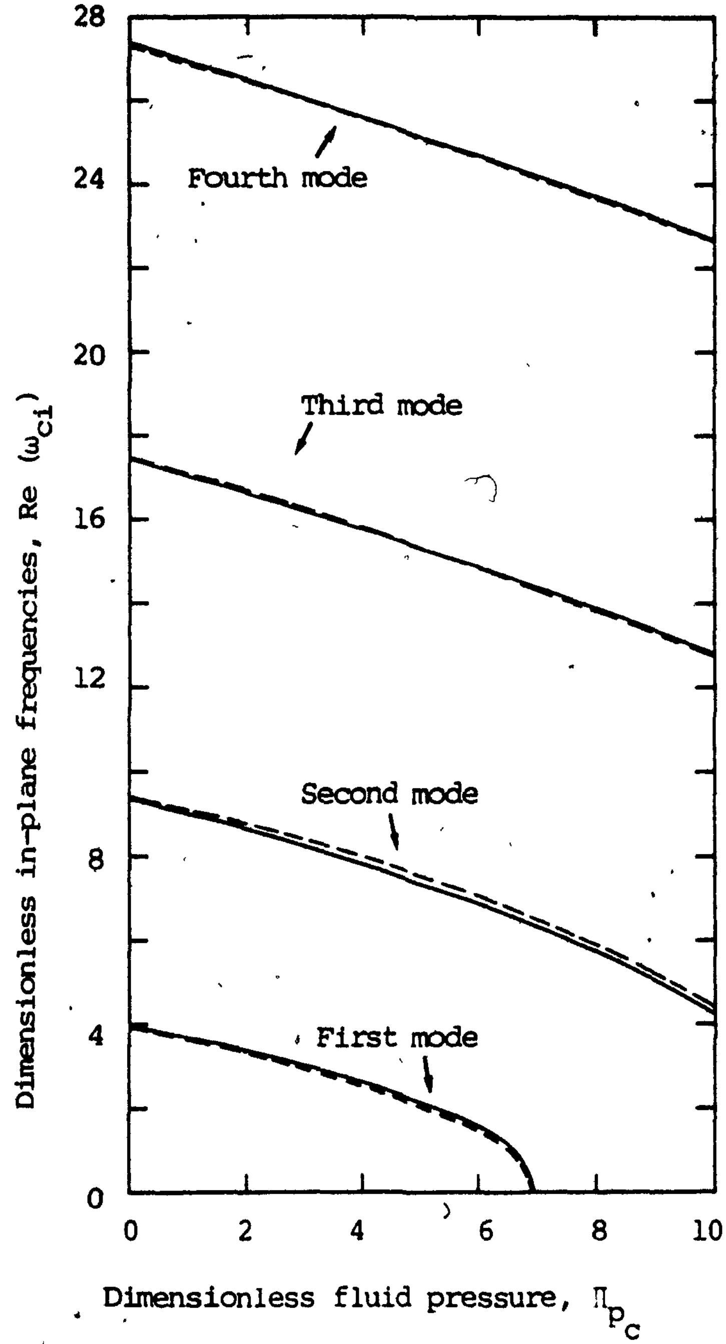


Fig. 22 Dimensionless frequencies of a clamped-clamped semi-circular pipe conveying fluid as a function of the dimensionless fluid pressure for $\beta = 1.0$ and $\bar{u} = \pi$(or $\bar{u}_c = 1$)

------ Chen (1972a)

——— Finite-Element

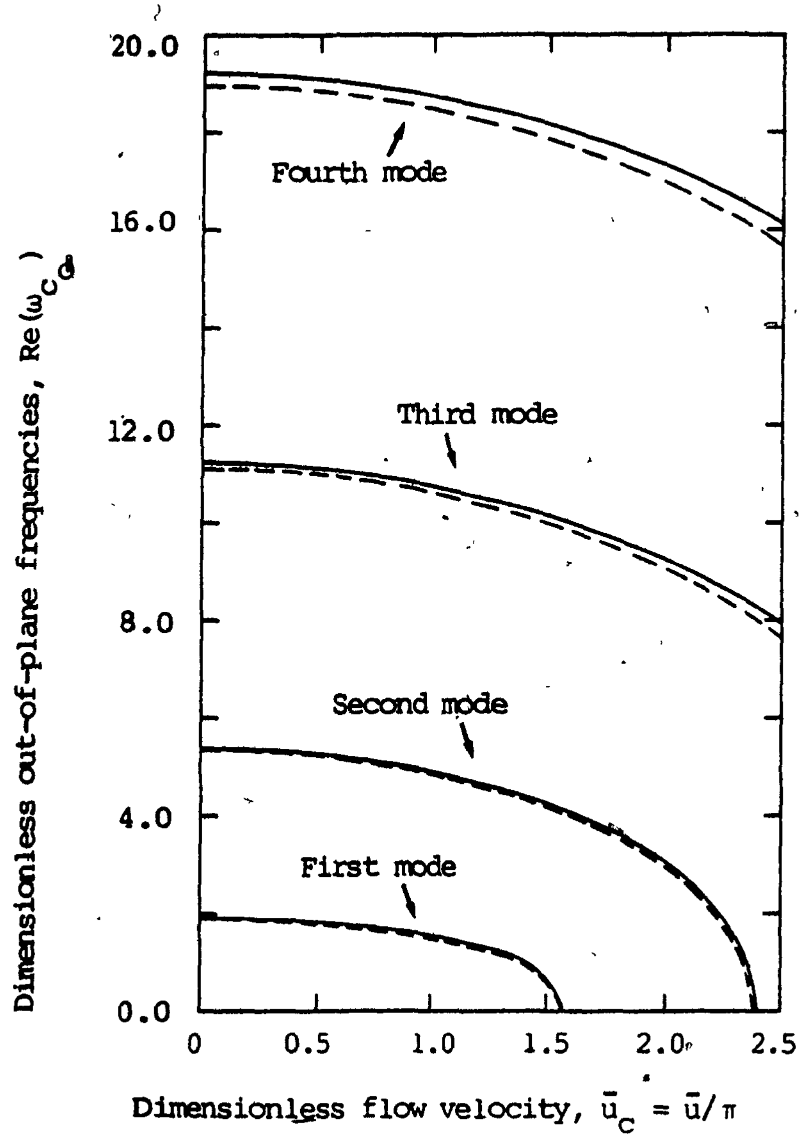


Fig. 23 Dimensionless out-of-plane frequencies of a clamped-clamped semi-circular pipe conveying fluid as a function of the dimensionless flow velocity for $\beta = 0$, $\Lambda = 0.769$, and $\Pi = 0$.

---- Chen (1973)

——— Finite-Element

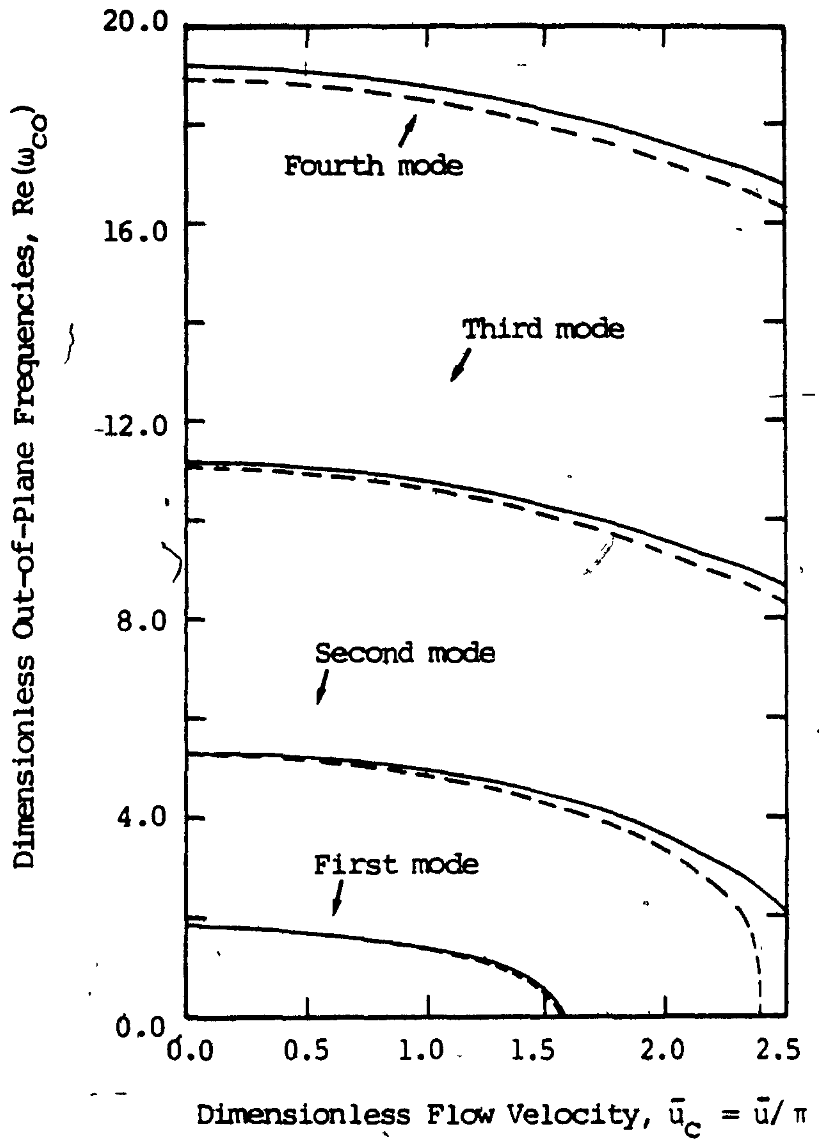


Fig. 24 Dimensionless Out-of-Plane Frequencies of a Clamped-Clamped Semi-Circular Pipe Conveying Fluid as a Function of the Dimensionless Flow Velocity for $\beta = 0.5$, $\Lambda = 0.769$ and $\Pi = 0$

----- Chen (1973)

——— Finite-Element

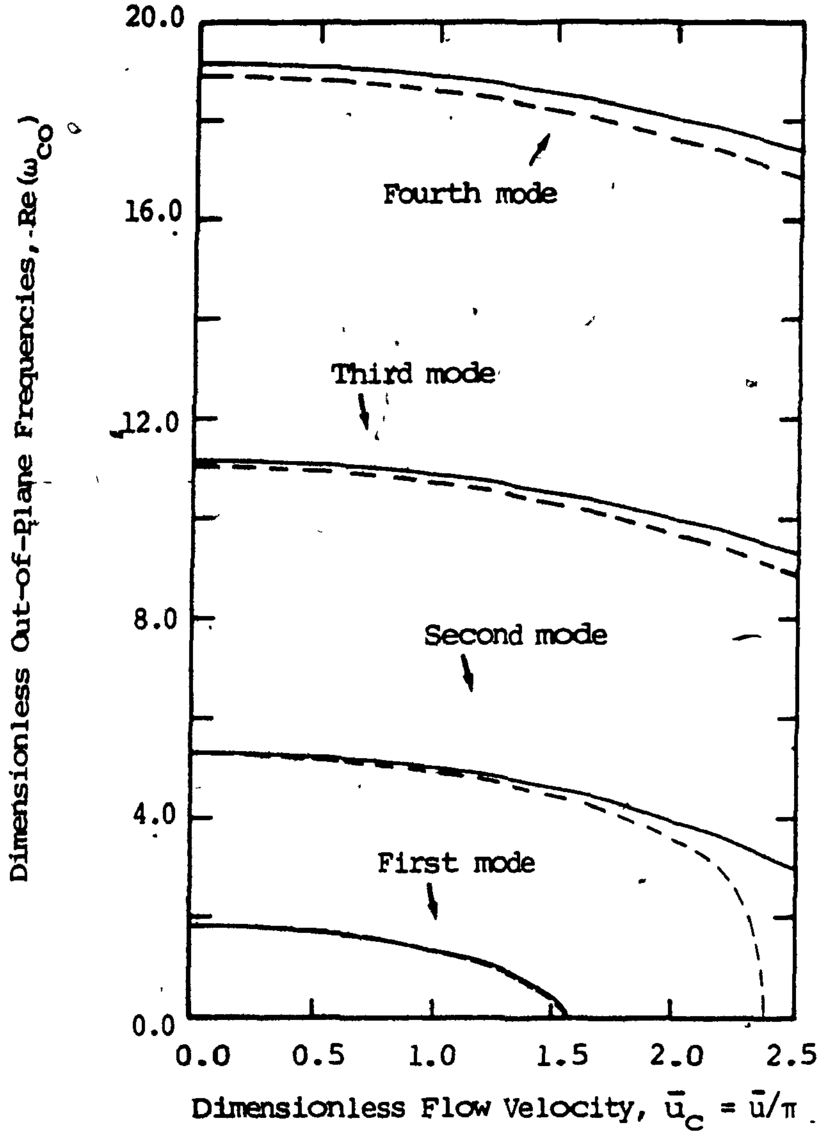


Fig. 25 Dimensionless Out-of-Plane Frequencies of a Clamped-Clamped Semi-Circular Pipe Conveying Fluid as a Function of the Dimensionless Flow Velocity for $\beta = 1.0$, $\Lambda = 0.769$ and $\Pi = 0$

----- Chen (1973)

——— Finite-Element

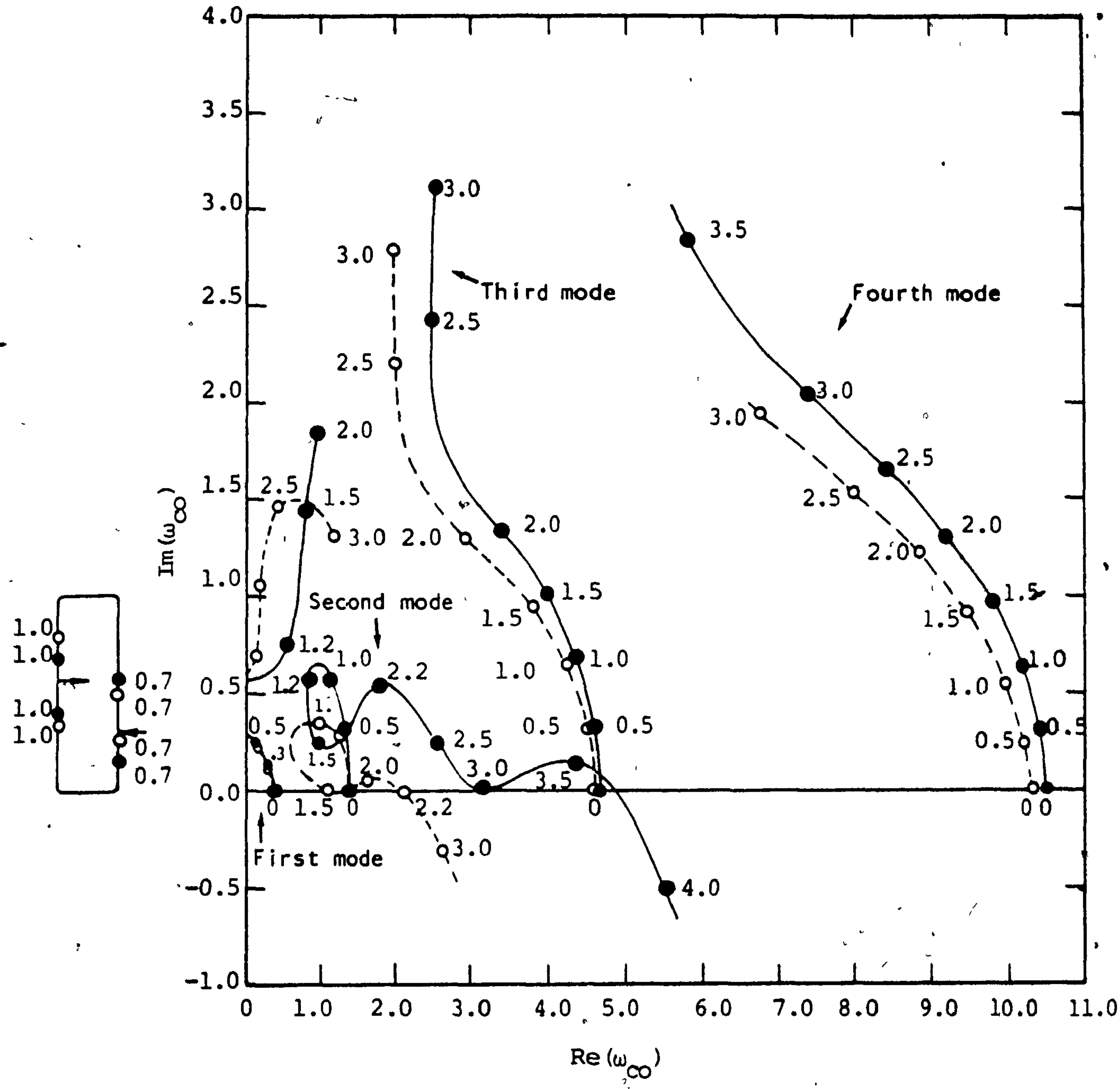


Fig. 26 Dimensionless Complex-Frequency Diagram of Out-of-Plane Motion of a Clamped-Free Semi-Circular Pipe Conveying Fluid for $\beta = 0.75$, $\Lambda = 0.769$, $r_o = \pi$ and $\Pi = 0$.

○ Chen (1973)

● Finite-Element

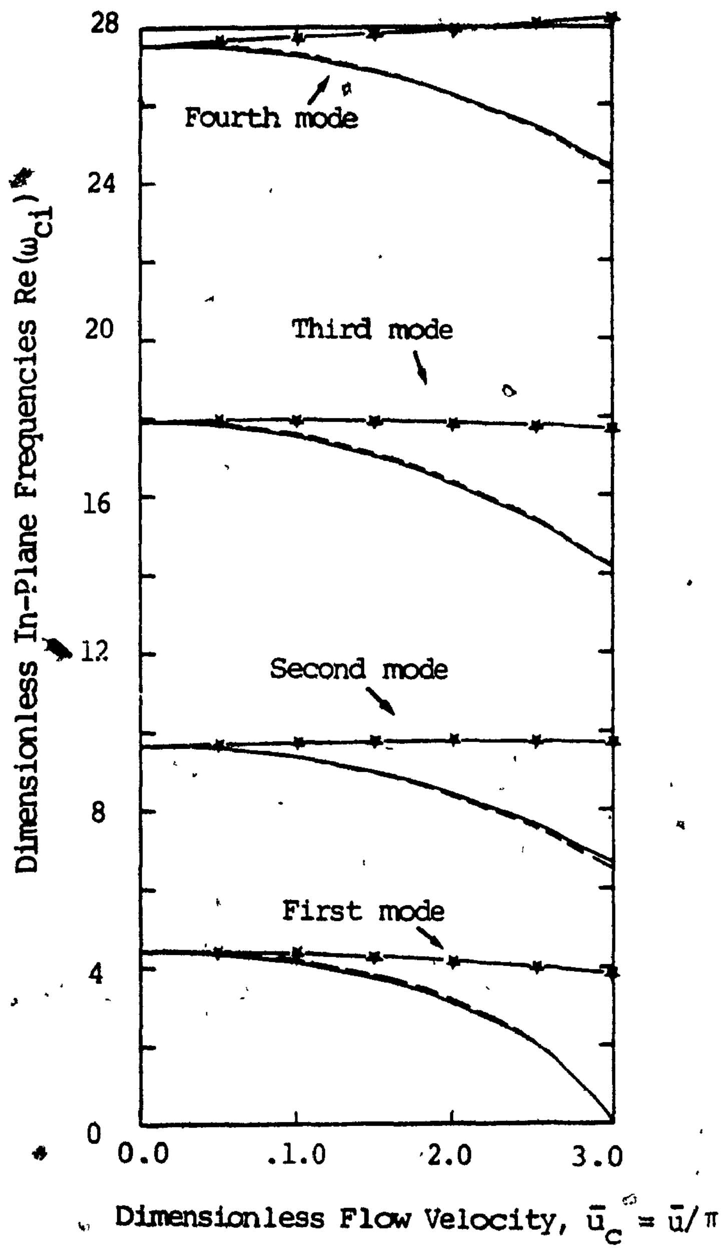


Fig. 27 Dimensionless In-Plane Frequencies of a Clamped-Clamped Semi-Circular Pipe Conveying Fluid as a Function of the Dimensionless Flow Velocity for $\beta = 0.5$

----- Chen (1972a) ($\Pi = 0$)

——— Finite-Element ($\Pi = 0$)

—★— Finite-Element (Including Π)

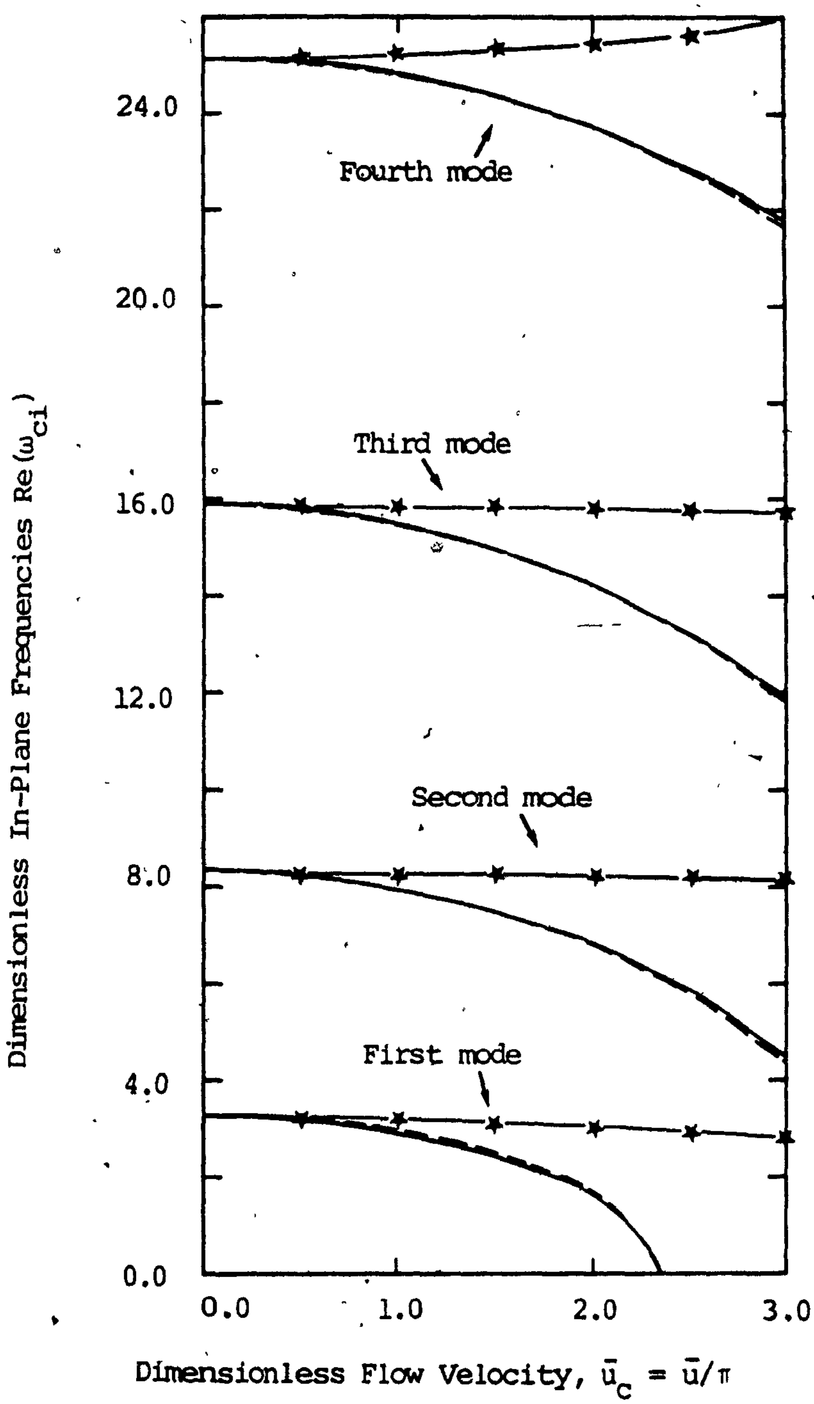


Fig. 28 Dimensionless In-Plane Frequencies of a Clamped-Pinned Semi-Circular Pipe Conveying Fluid as a Function of the Dimensionless Flow Velocity for $\beta = 0.5$

------ Chen (1972a) ($\Pi = 0$)

——— Finite-Element ($\Pi = 0$)

—★— Finite-Element (Including Π)

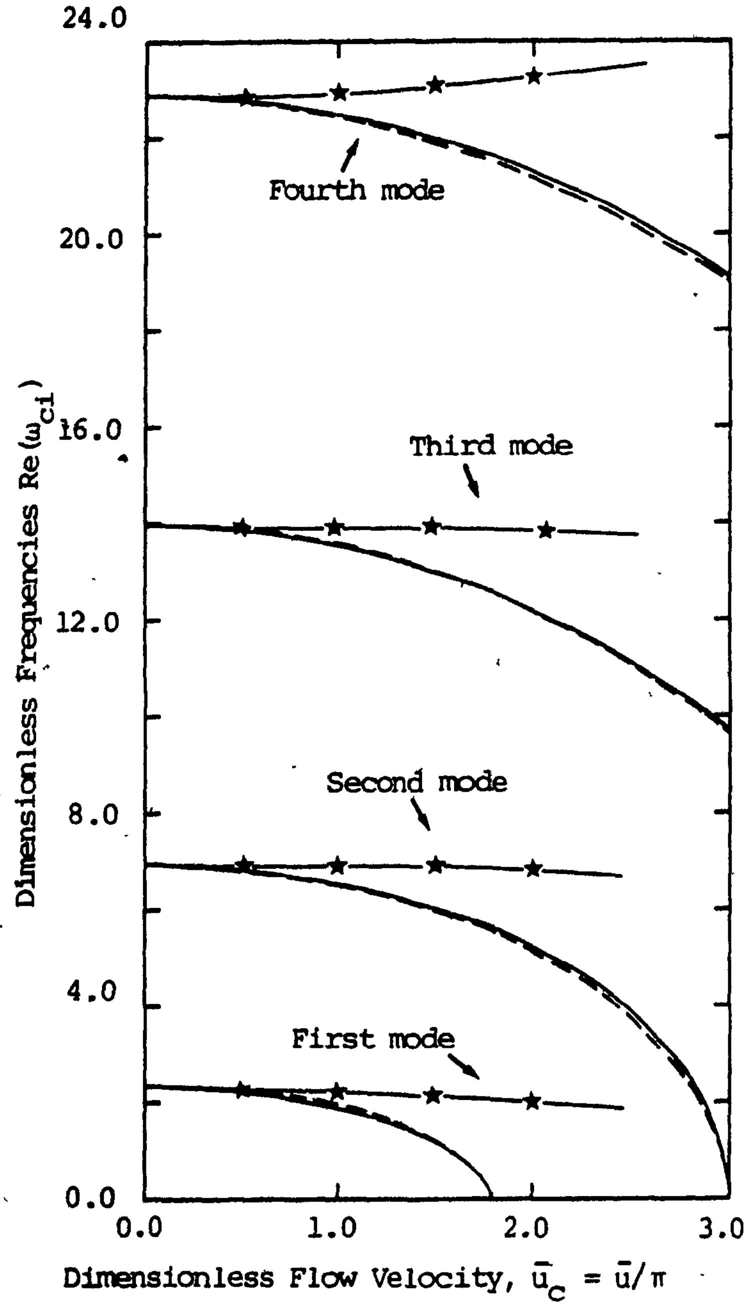


Fig. 29 Dimensionless In-Plane Frequencies of a Pinned-Pinned Semi-Circular Pipe Conveying Fluid as a Function of the Dimensionless Flow Velocity for $\beta = 0.5$

----- Chen (1972a) ($\Pi = 0$)

——— Finite-Element ($\Pi = 0$)

—★— Finite-Element (Including Π)

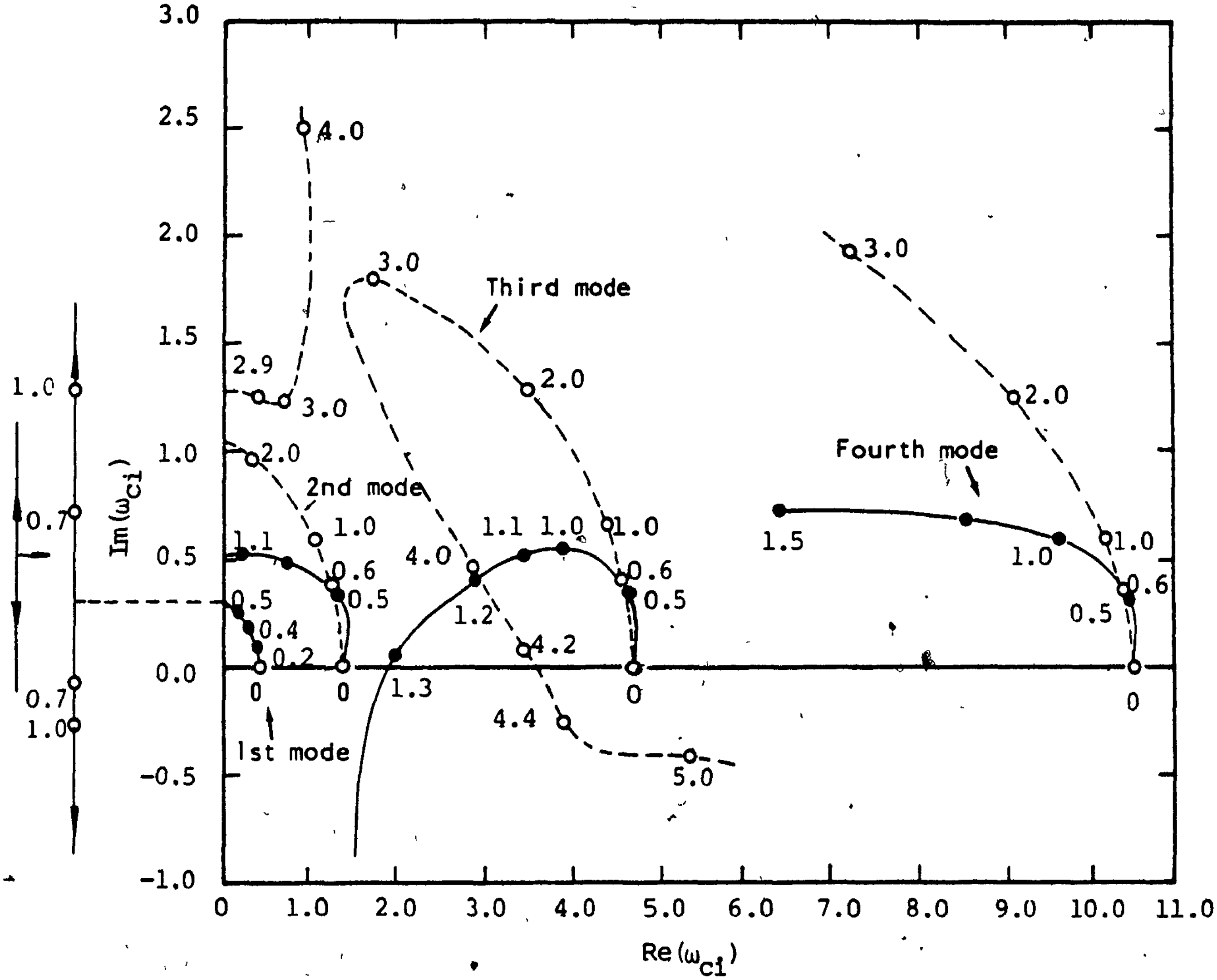


Fig. 30 Dimensionless Complex-Frequency Diagram of In-Plane Motion of a Clamped-Free Semi-Circular Pipe Conveying Fluid for $\beta = 0.75$, $\beta_a = 0$ and $\mathcal{K} = 0$.

○ Finite-Element ($\Pi = 0$)

● Finite-Element (Including Π)

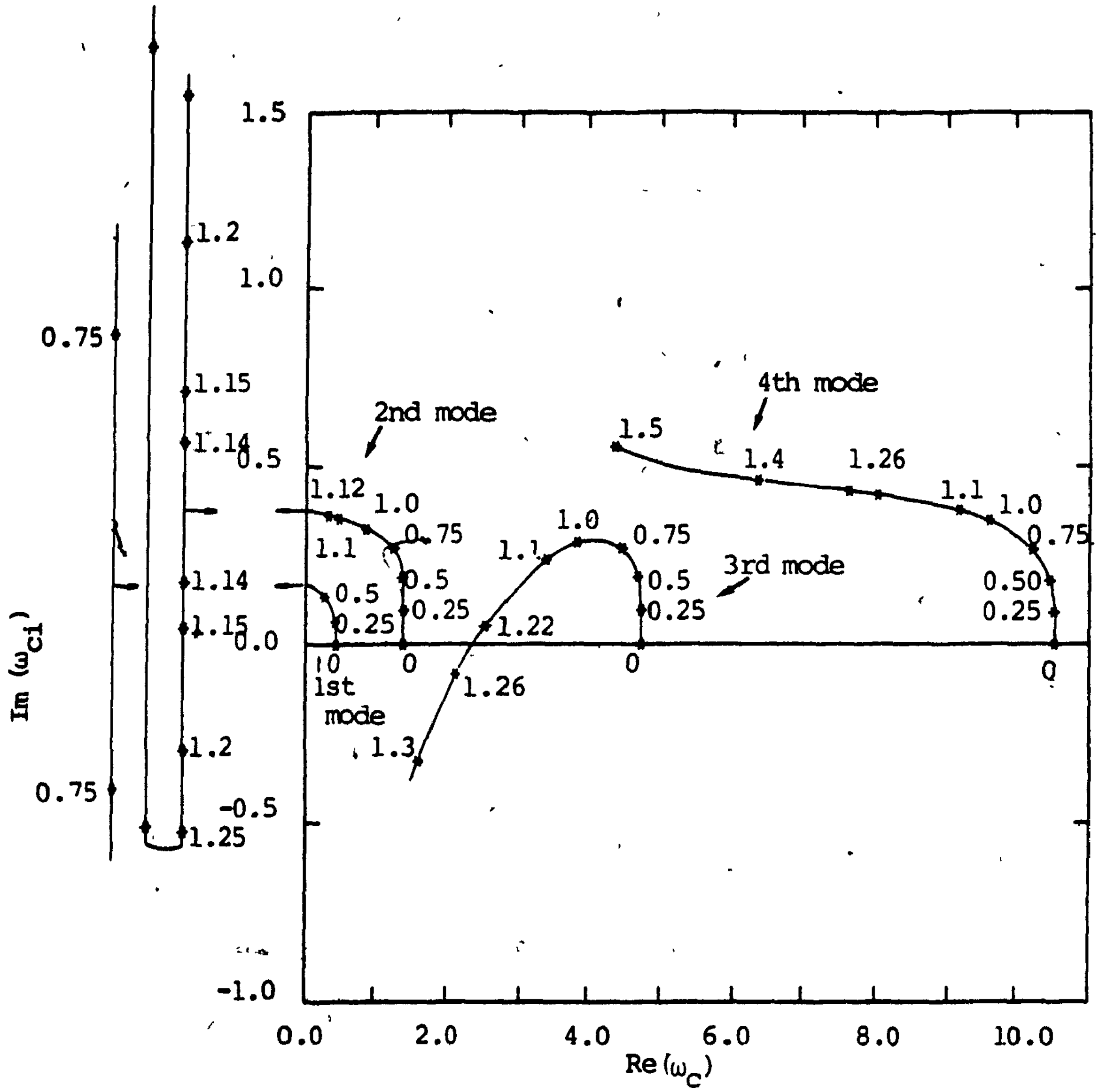


Fig. 31 Dimensionless Complex-Frequency Diagram of In-Plane Motion of a Clamped-Free Semi-Circular Pipe Conveying Fluid for $\beta = 0.25$, $\beta_a = 0$ and $\mathcal{H} = 0$.

✶ Finite-Element (Including Π)

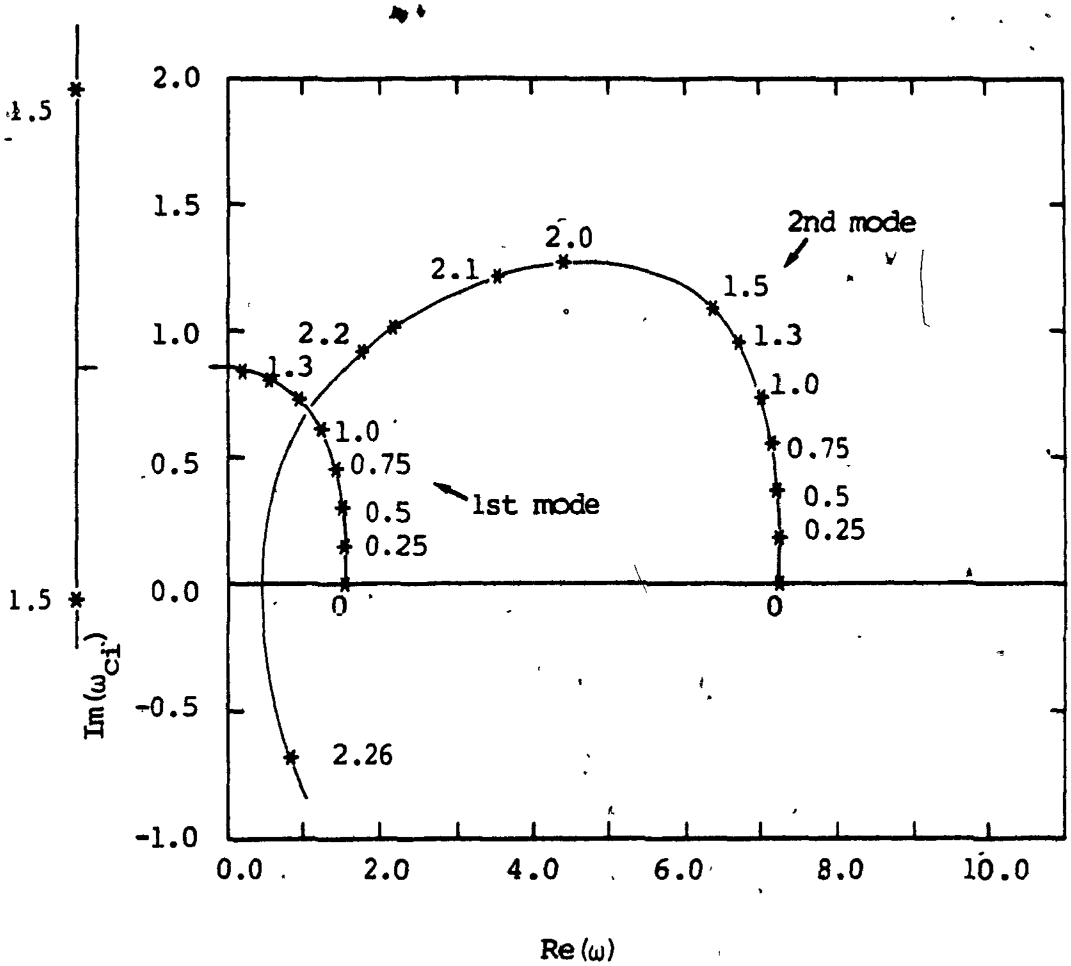


Fig. 32 Dimensionless Complex-Frequency Diagram of In-Plane Motion of a Clamped-Free Uniformly Curved Pipe Conveying Fluid for $\beta = 0.25$, $r_o = \pi/2$, $\beta_a = 0$ and $\mathcal{K} = 0$.

Finite-Element (Including Π)

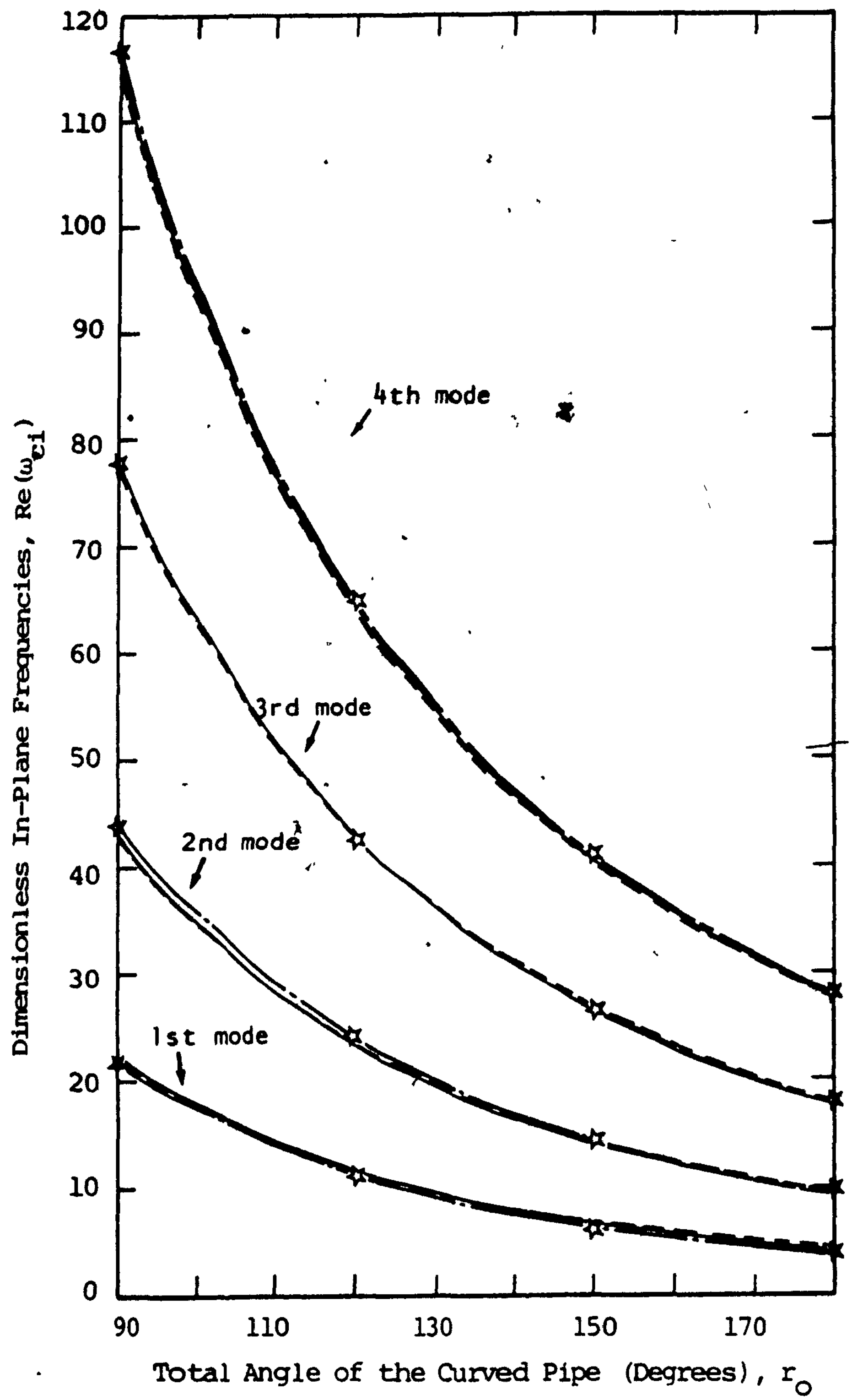


Fig. 33 Dependence of the Dimensionless In-Plane Frequencies on the Total Angle of the Curved Pipe for $\beta = 0.5$

---- Chen (1972a) ($\bar{u} = 0$)

—— Finite-Element ($\bar{u} = 0$)

—☆— Finite-Element ($\bar{u} = 2\pi$ and including Π)

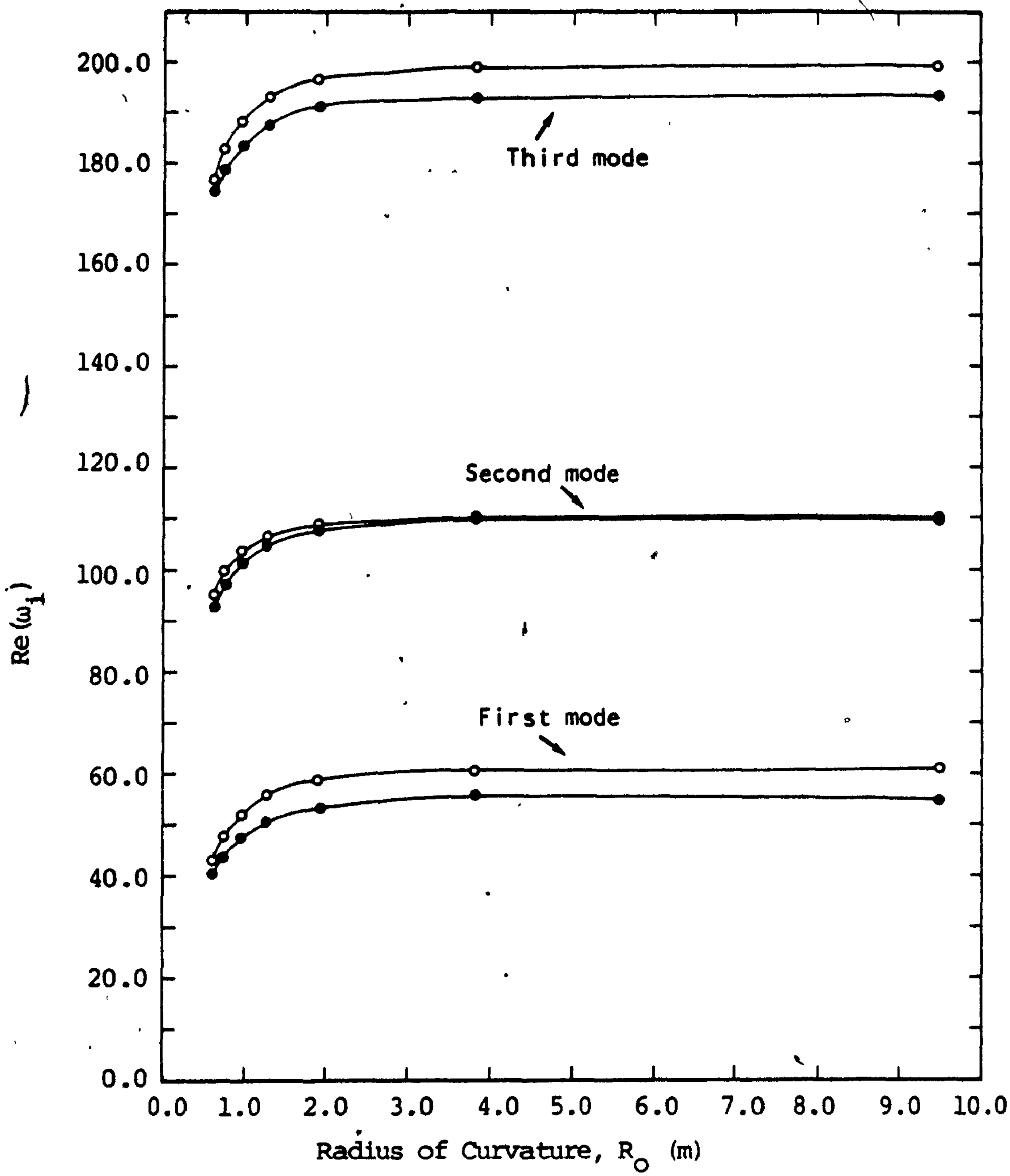


Fig. 34 Natural Frequencies of In-Plane Motion of a Clamped-Clamped Semi-Circular Pipe as a Function of the Radius of Curvature for $\beta = 0.5$, $L = 2$ m

o $\bar{u} = 0$

● $\bar{u} = \pi$

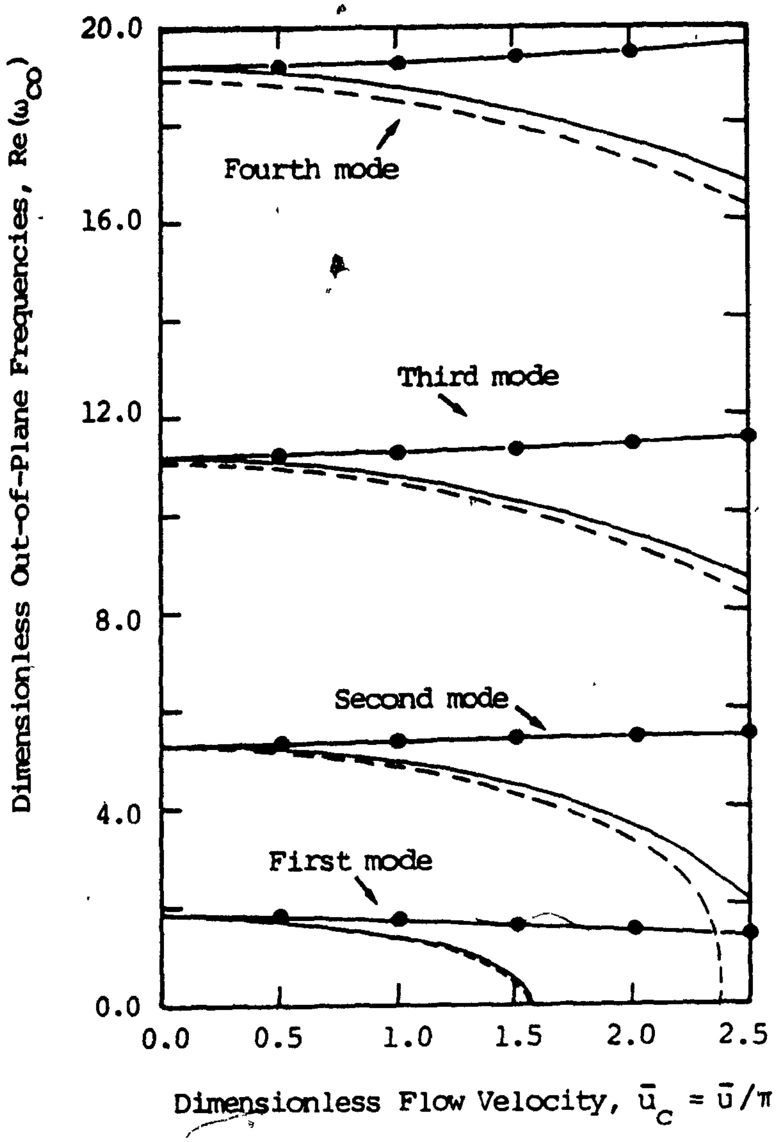


Fig. 35 Dimensionless Out-of-Plane Frequencies of a Clamped-Clamped Semi-Circular Pipe Conveying Fluid as a Function of the Dimensionless Flow Velocity for $\beta = 0.5$

----- Chen (1973) ($\Pi = 0$)

——— Finite-Element ($\Pi = 0$)

—●— Finite-Element (Including Π)

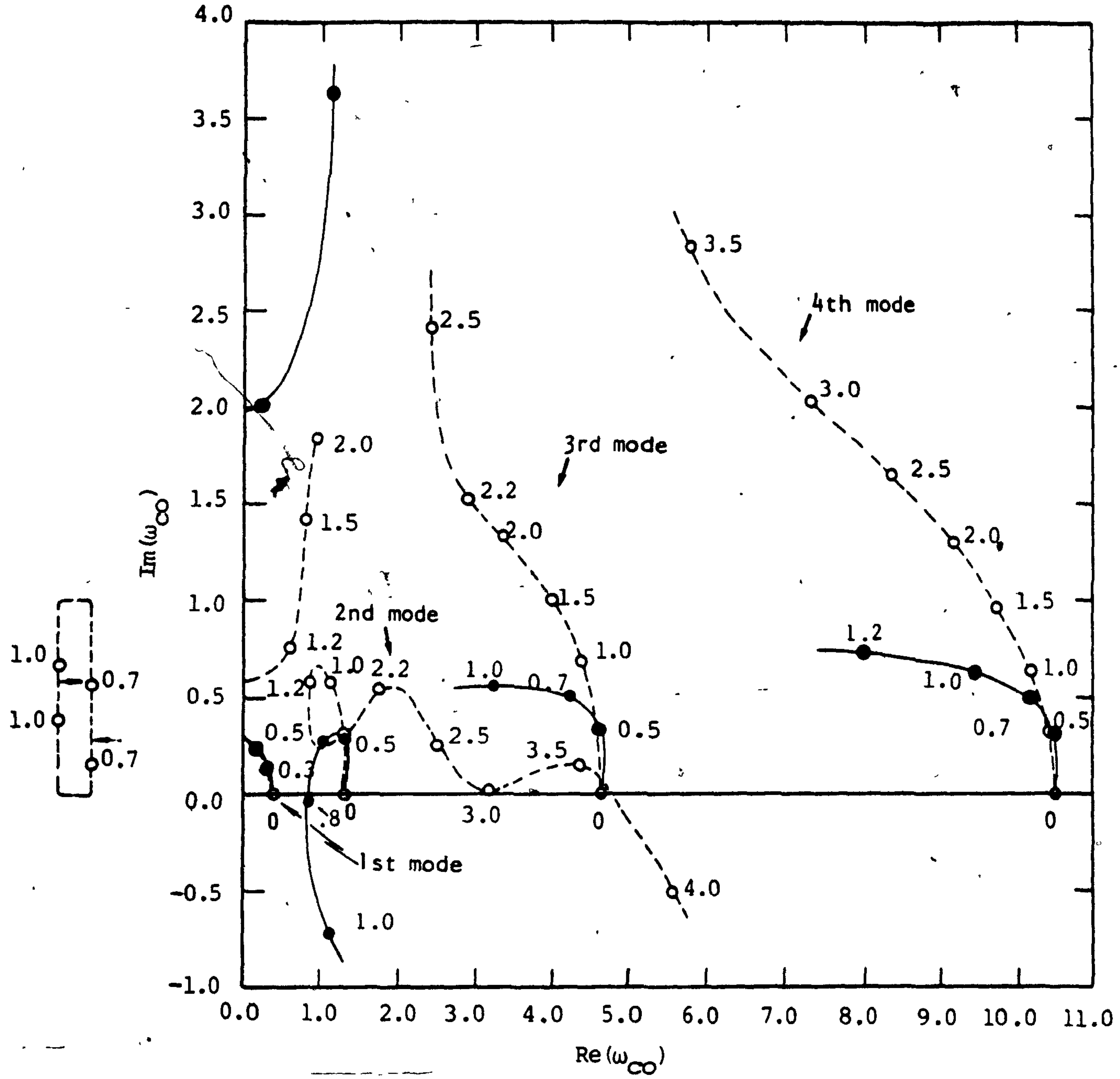


Fig. 36 Dimensionless Complex-Frequency Diagram of Out-of-Plane Motion of a Clamped-Free Semi-Circular Pipe Conveying Fluid for $\beta = 0.75$, $\Lambda = 0.769$ and $r_o = \pi$

○ Finite-Element ($\Pi = 0$)

● Finite-Element (Including Π)

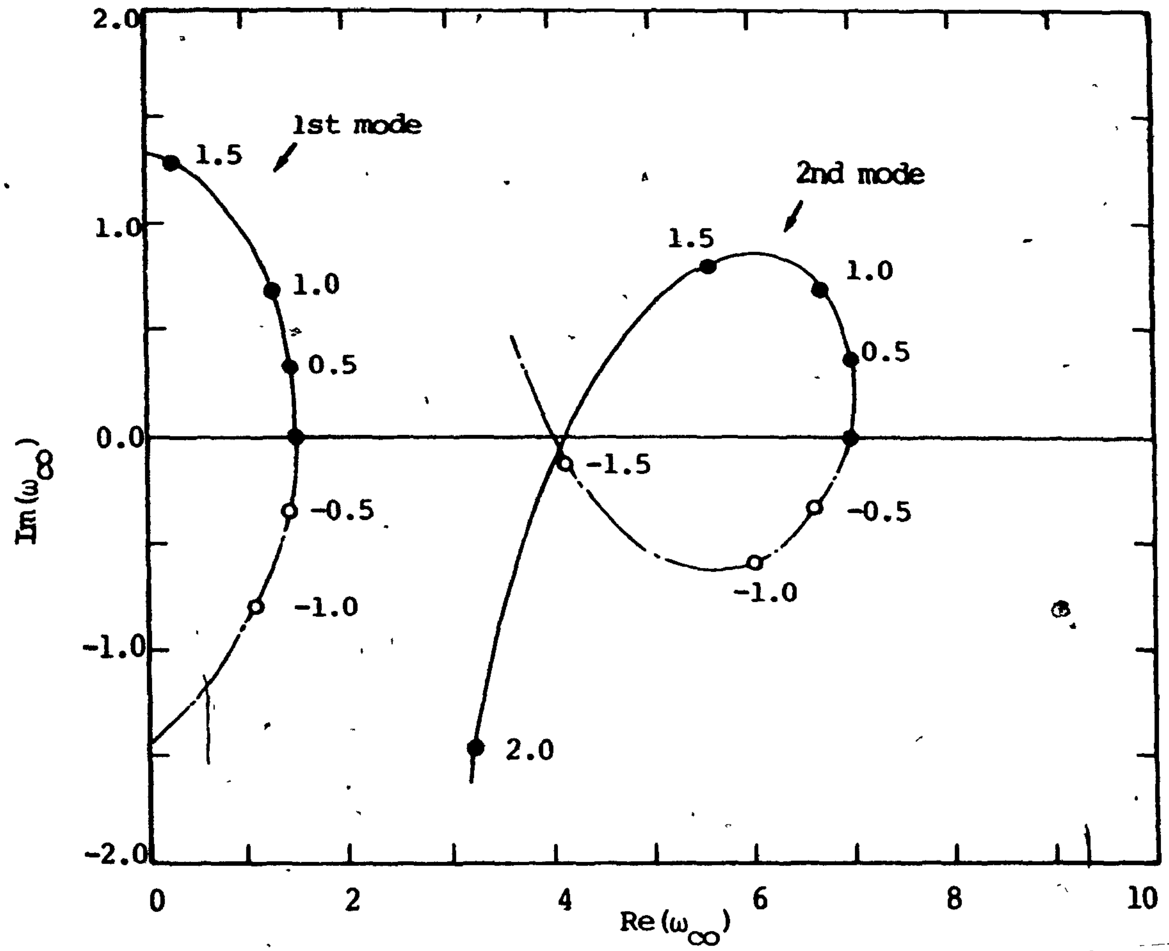


Fig. 37 Dimensionless Complex-Frequency Diagram of Out-of-Plane Motion of a Clamped-Free Uniformly Curved Pipe Conveying fluid for $\beta = 0.25$, $\Lambda = 0.769$ and $r_o = \pi/2$ (including π)

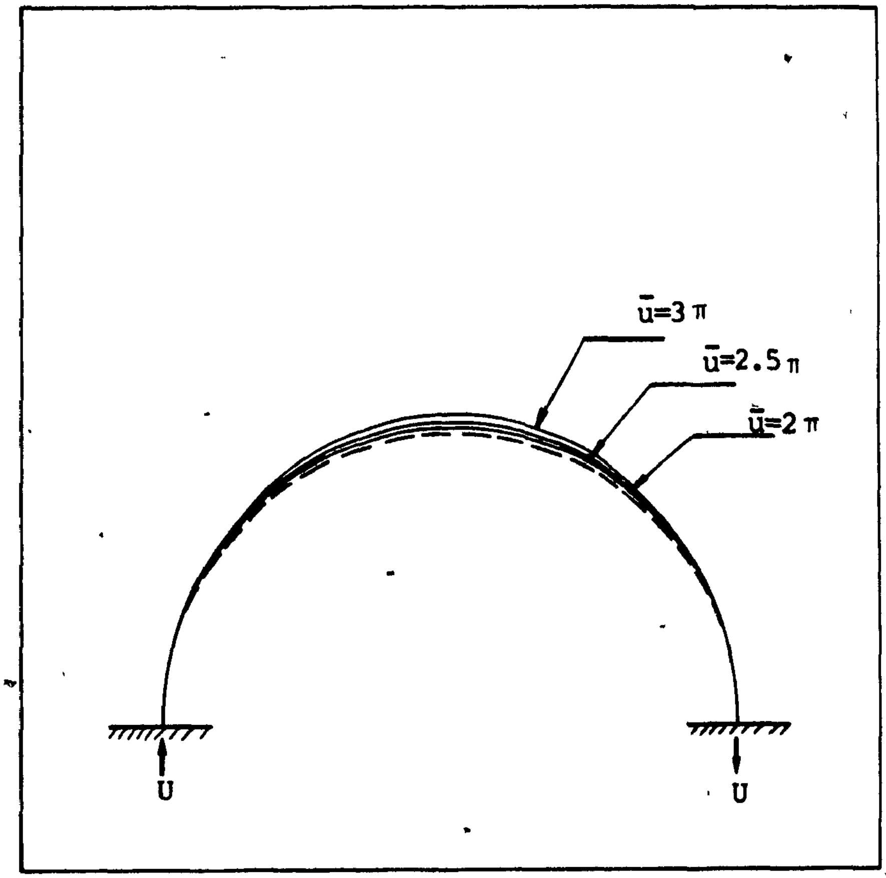


Fig. 38 The Static Equilibrium Configuration of a Clamped-Clamped Extensible Semi-Circular Pipe Conveying Inviscid Fluid: $\bar{u} = 2\pi,\ 2.5\pi,\ 3\pi;\ \mathcal{A} = 10^4,\ \gamma = 0.$ The dashed line represents the undeformed pipe. (Deformation is magnified by a factor of 28 approximately)

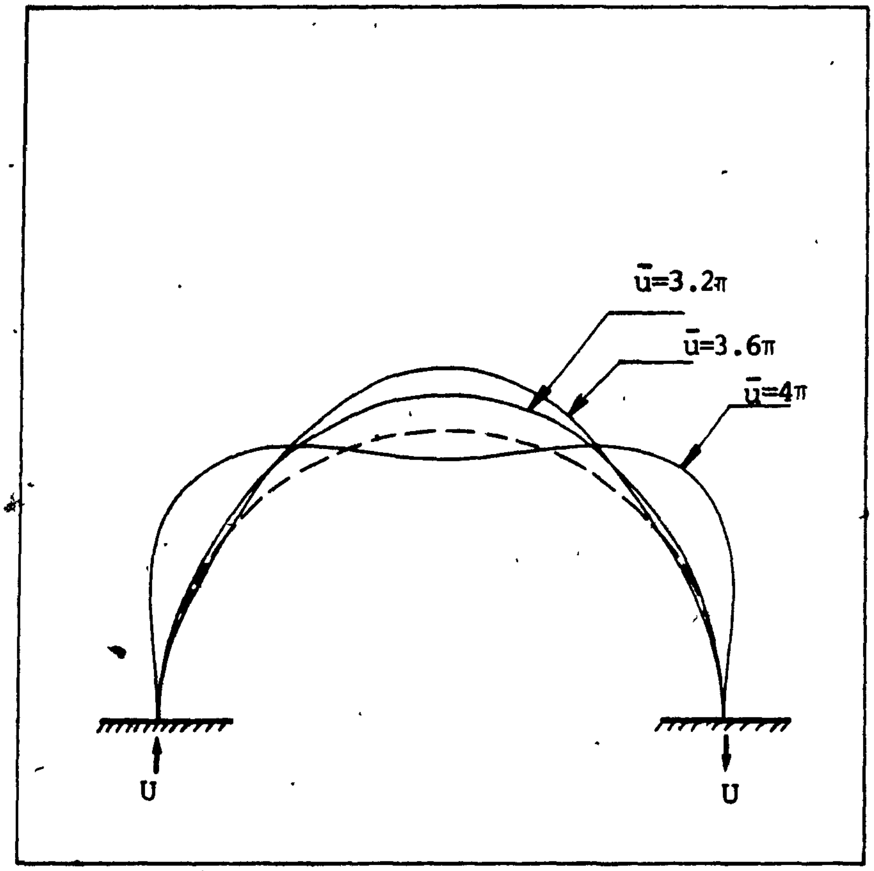


Fig. 39 The Static Equilibrium Configuration of a Clamped-Clamped Extensible Semi-Circular Pipe Conveying Inviscid Fluid: $\bar{u} = 3.2\pi$, 3.6π, and 4π; $\mathcal{A} = 10^4$, $\gamma = 0$. The dashed line represents the undeformed pipe. (Deformation is magnified by a factor of 30 Approximately)

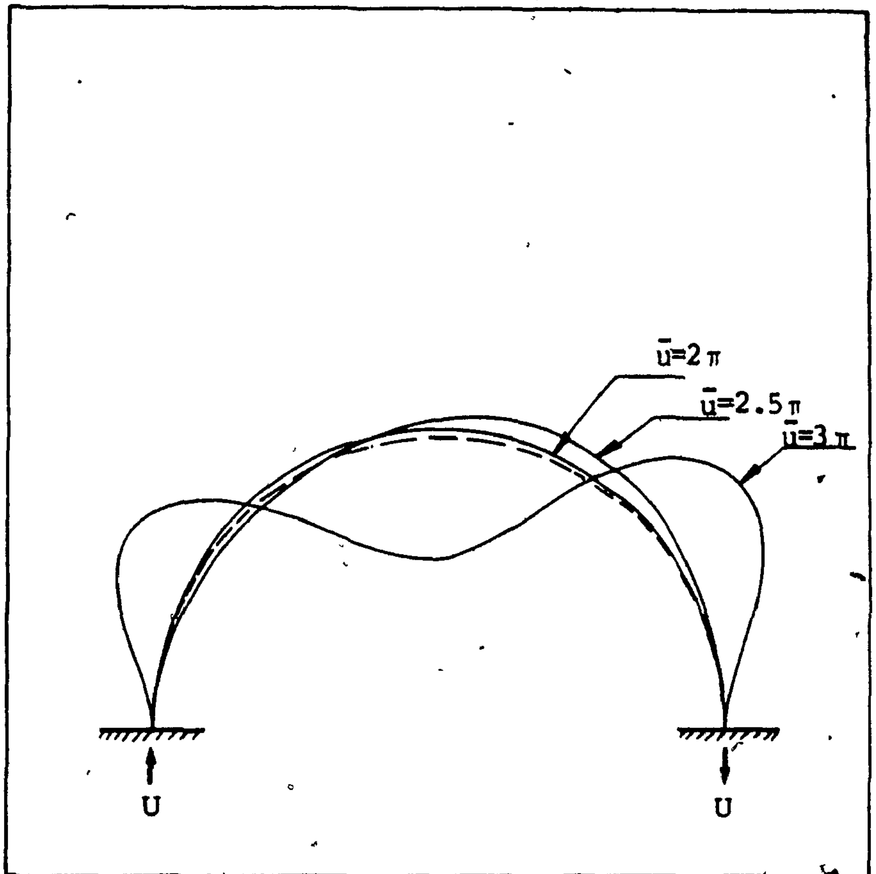


Fig. 40 The Static Equilibrium Configuration of a Clamped-Clamped Extensible Semi-Circular Pipe Conveying Viscous Fluid: $\bar{u} = 2\pi$, 2.5π and 3π; $\mathcal{A} = 10^4$, $\gamma = 0$. The dashed line represents the undeformed pipe. (Deformation has been magnified by a factor of 25 approximately)

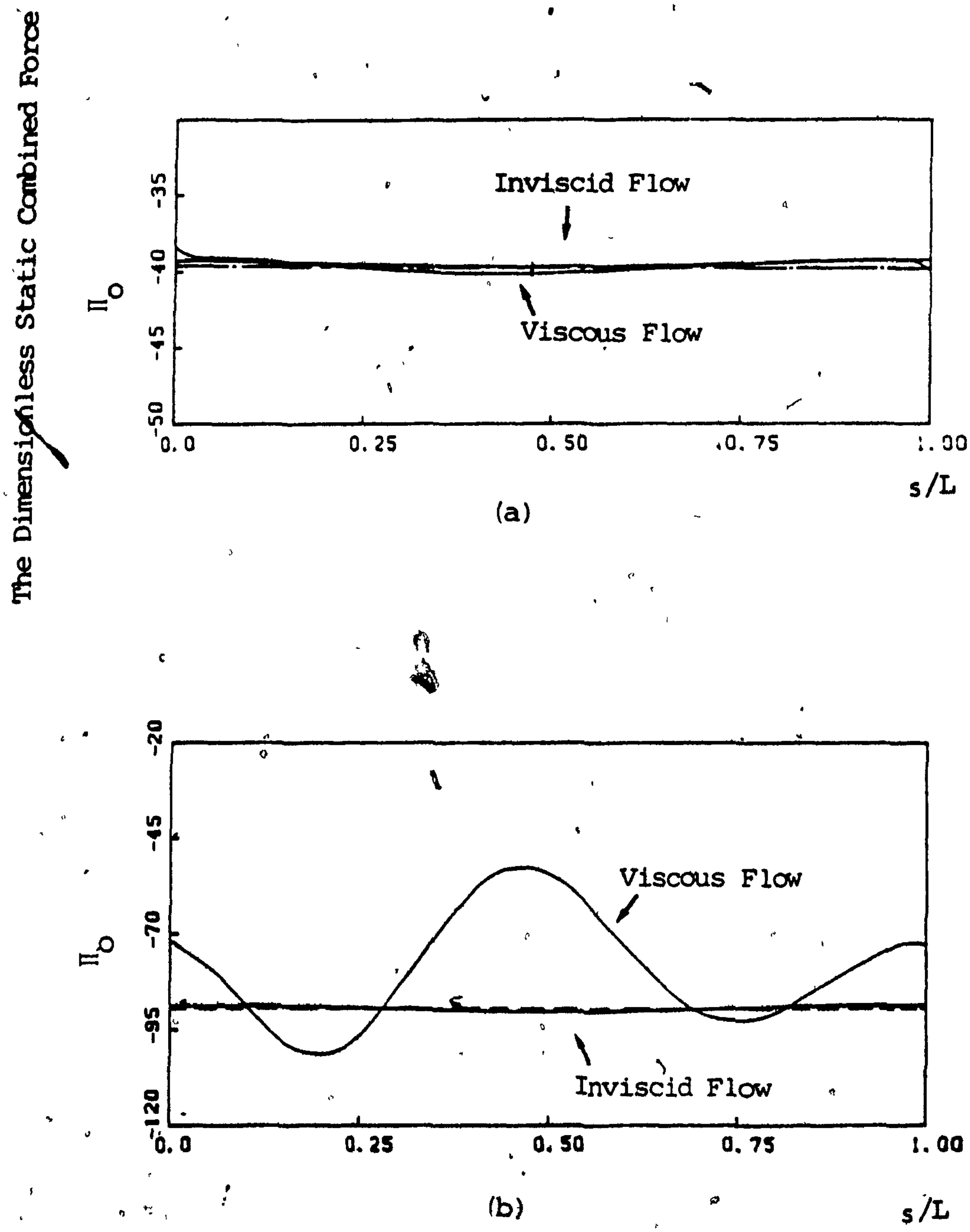


Fig. 41 The Distribution of Static Combined Force Along a Clamped-Clamped Semi-Circular Pipe Conveying Fluid for $\mathcal{A} = 10^4$, $\gamma = 0$

(a) $\bar{u} = 2\pi$ ($\bar{u}_c = 2$)

(b) $\bar{u} = 3\pi$ ($\bar{u}_c = 3$)

---- Inextensible Case

—— Extensible Case

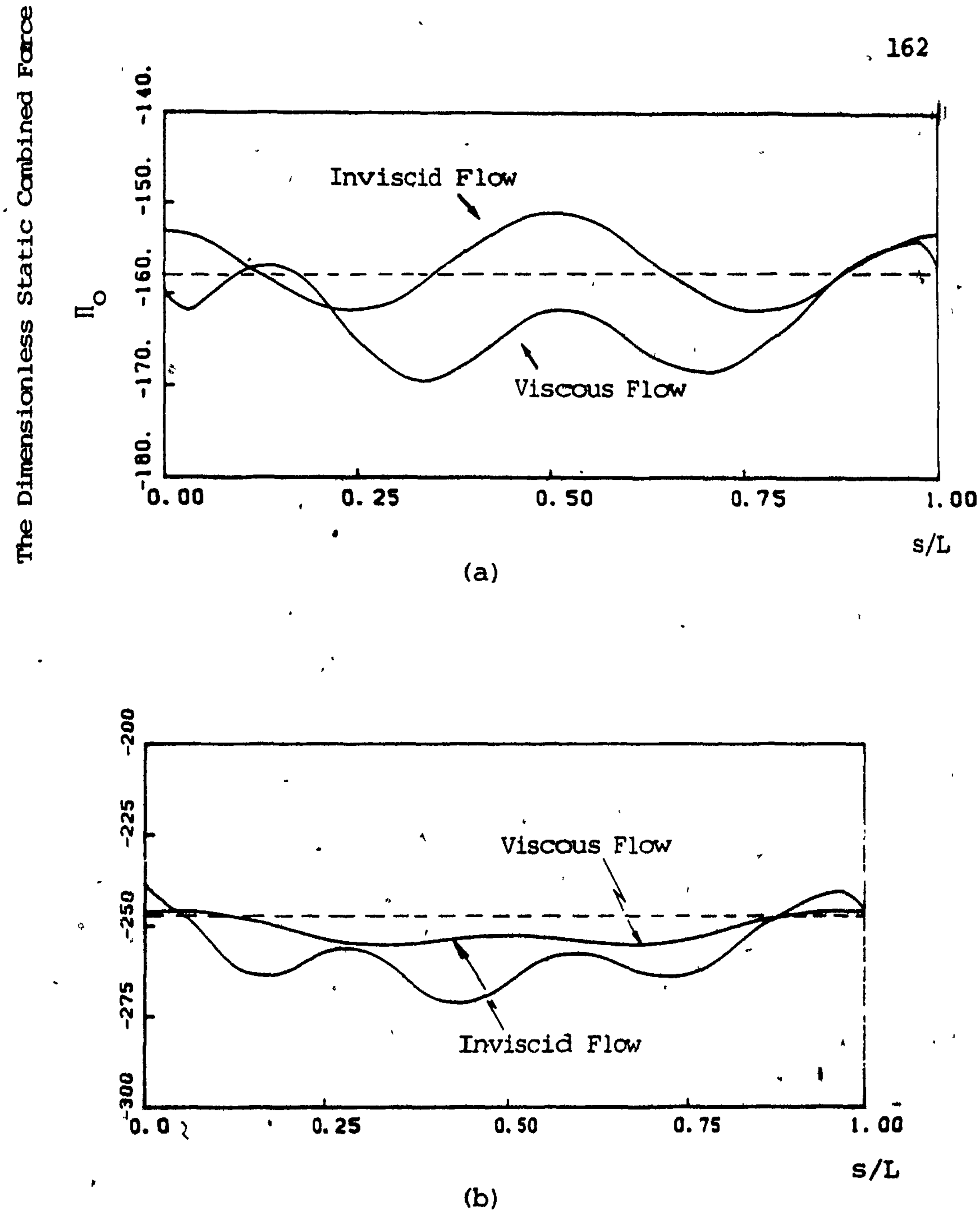


Fig. 42 The Distribution of Static Combined Force Along a Clamped-Clamped Semi-Circular Pipe Conveying Fluid for $\mathcal{A} = 10^4$, $\gamma = 0$

(a) $\bar{u} = 4\pi$ ($\bar{u}_c = 4$)

(b) $\bar{u} = 5\pi$ ($\bar{u}_c = 5$)

------- Inextensible Case

——— Extensible Case

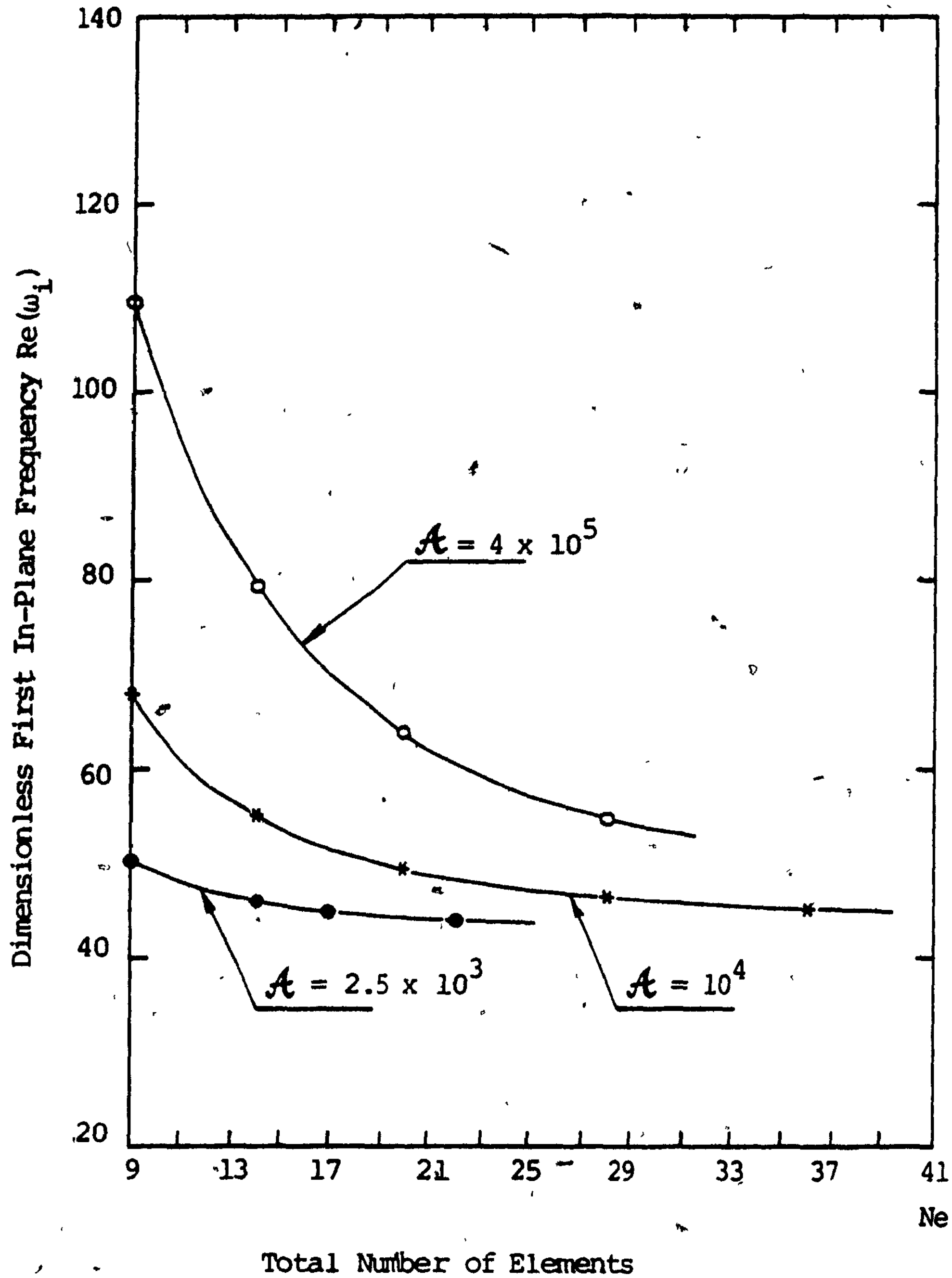


Fig. 43 The Convergence of the First In-Plane Natural Frequency of a Clamped-Clamped Extensible Semi-Circular Pipe for $\gamma = 0$ and $\bar{u} = 0$ Neglecting the Combined Force.

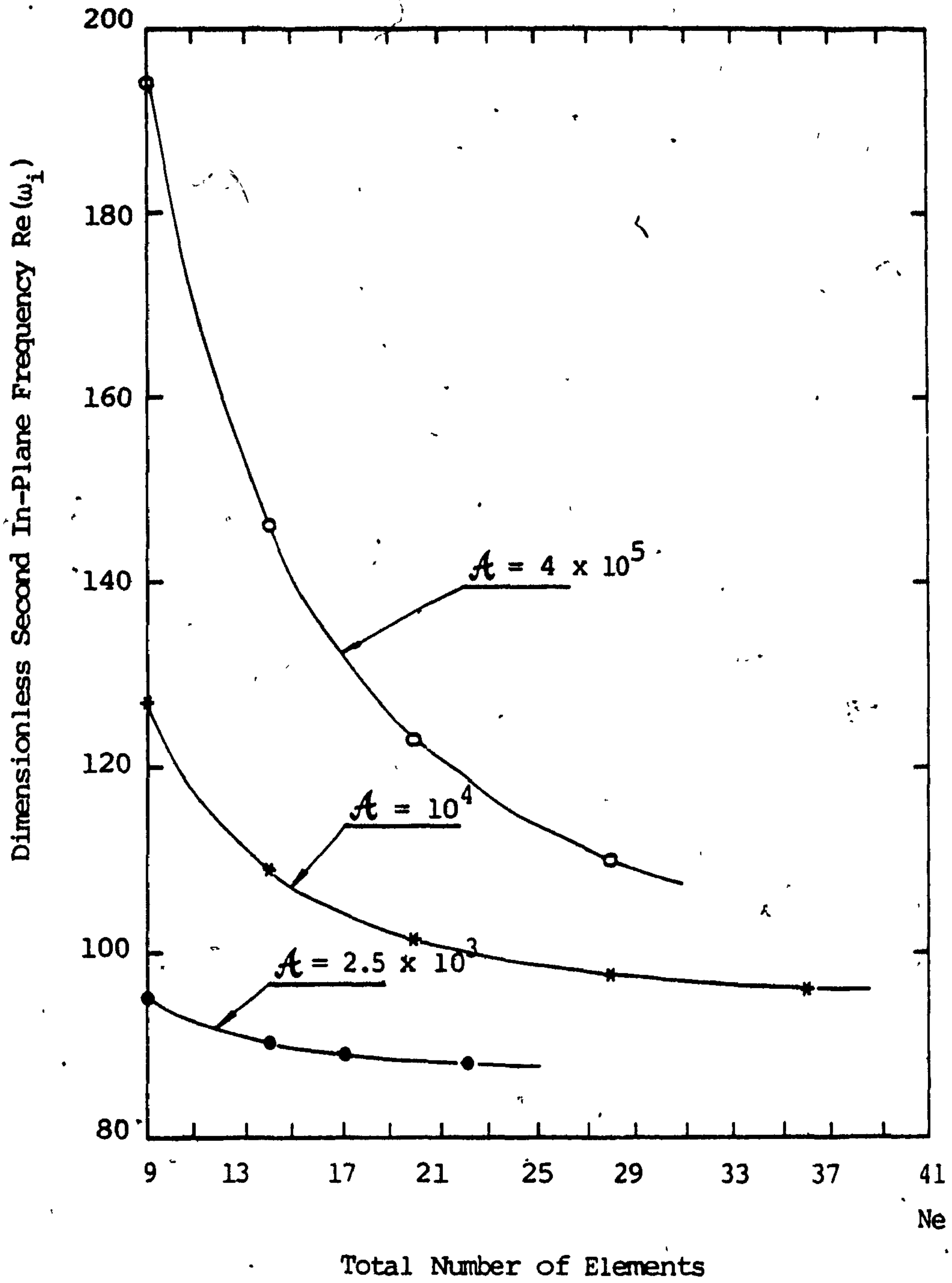


Fig. 44 The Convergence of the Second In-Plane Frequency of a Clamped-Clamped Extensible Semi-Circular Pipe for $\gamma = 0$ and $\bar{u} = 0$ Neglecting the Combined Force

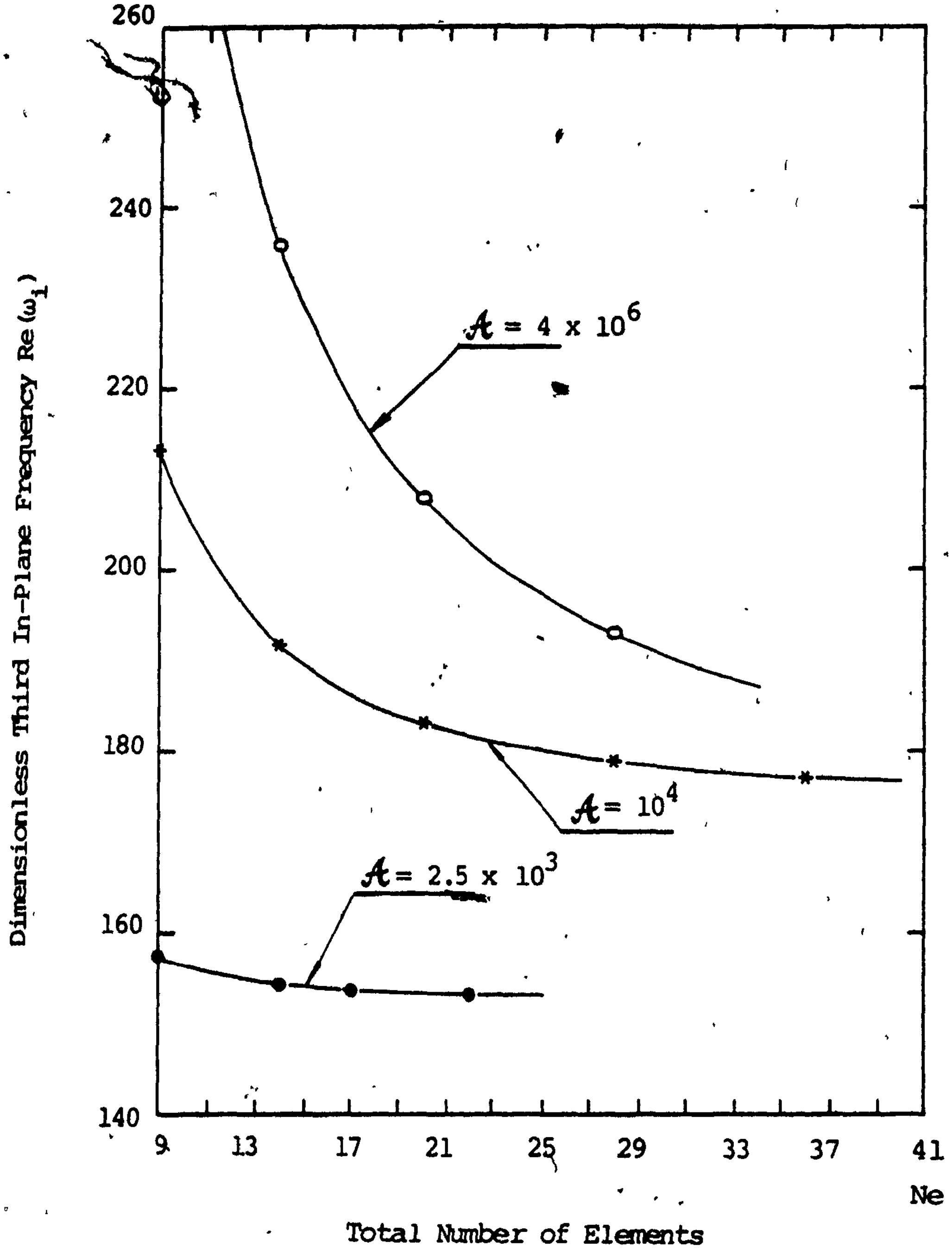


Fig. 45 The Convergence of the Third In-Plane Natural Frequency of a Clamped-Clamped Extensible Semi-Circular Pipe for $\gamma = 0$ and $\bar{u} = 0$ Neglecting the Combined Force.

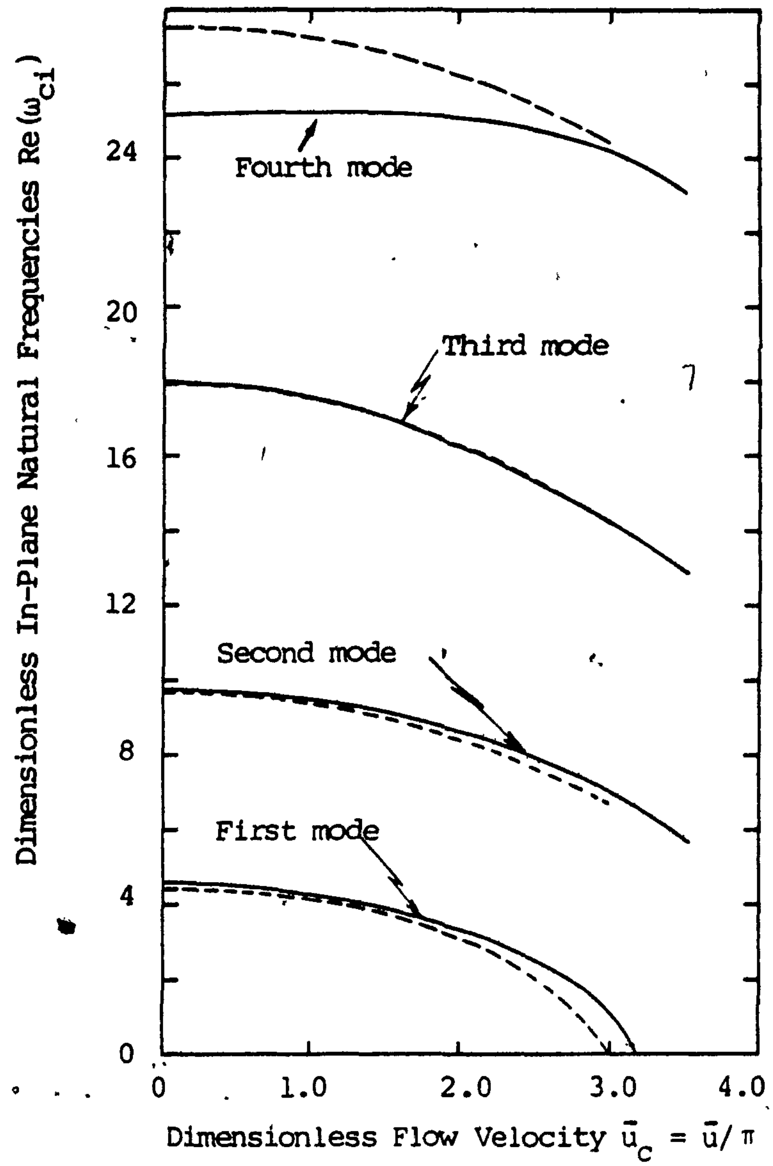


Fig. 46 Dimensionless In-Plane Frequencies of a Clamped-Clamped Semi-Circular Pipe Conveying Fluid as a Function of the Flow Velocity for $\beta = 0.5$, $\gamma = 0$, $\mathcal{A} = 10^4$ and $\Pi = 0$.

——— Extensible Case

------ Inextensible Case

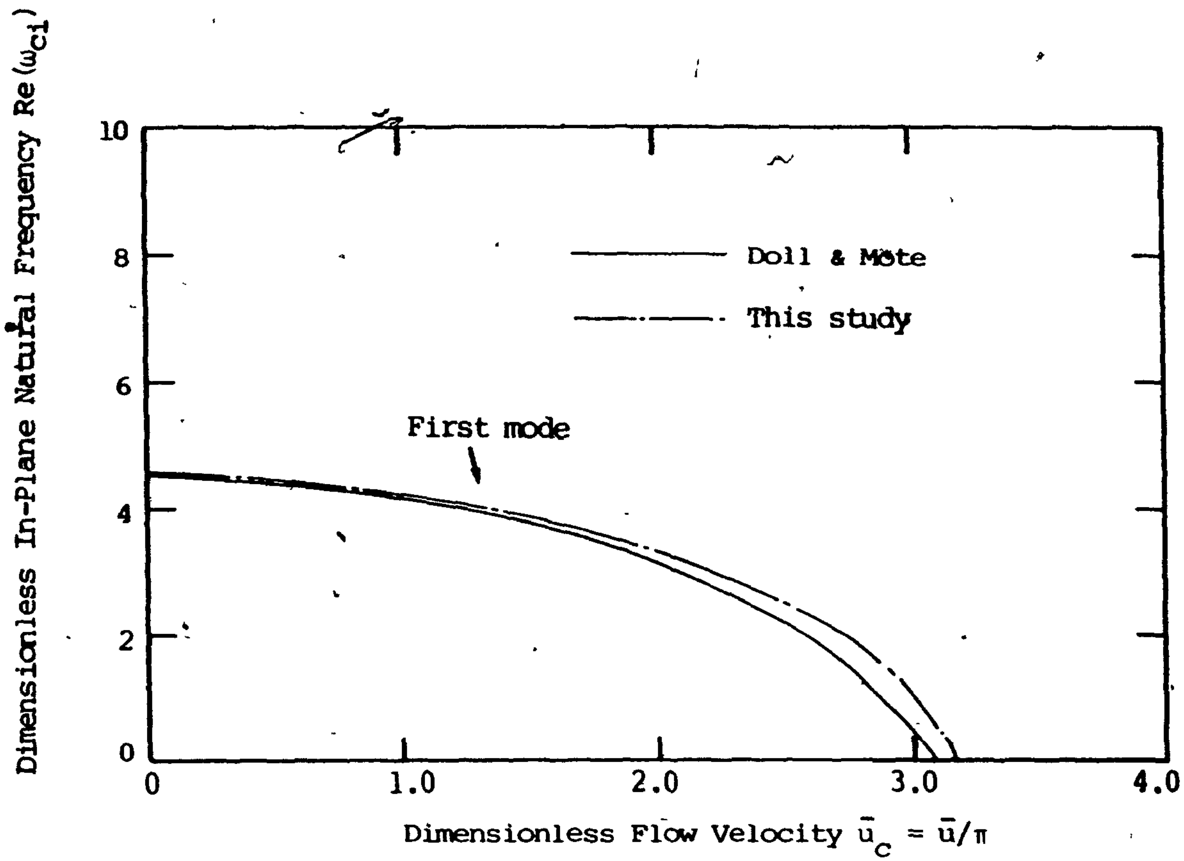
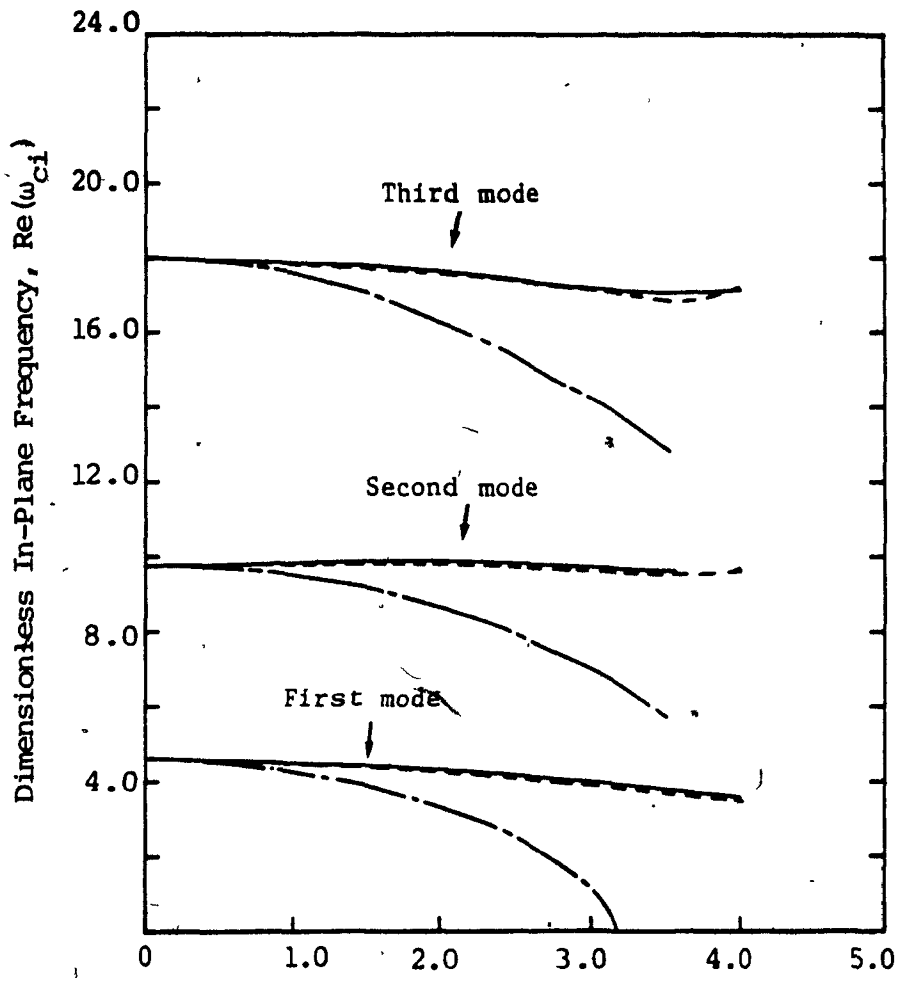


Fig. 47 Dimensionless In-Plane Fundamental Frequencies of a Clamped-Clamped Semi-Circular Pipe Conveying Fluid as a Function of the Flow Velocity for $\beta = 0.5$ and $\Pi = 0$.



Dimensionless Flow Velocity $\bar{u}_c = \bar{u}/\pi$

Fig. 48 Dimensionless In-Plane Frequencies of a Clamped-Clamped Extensible Semi-Circular Pipe Conveying Inviscid Fluid as Fucntions of the Dimensionless Flow Velocity for the Extensible Case; $\beta = 0.5$, $\gamma = 0$

——— Including the terms involving $\mathcal{A}(\eta_1^{o'} + r_o\eta_3^o)$

- - - - Neglecting the terms involving $\mathcal{A}(\eta_1^{o'} + r_o\eta_3^o)$

—·—·— Neglecting Π and $\mathcal{A}(\eta_1^{o'} + r_o\eta_3^o)$

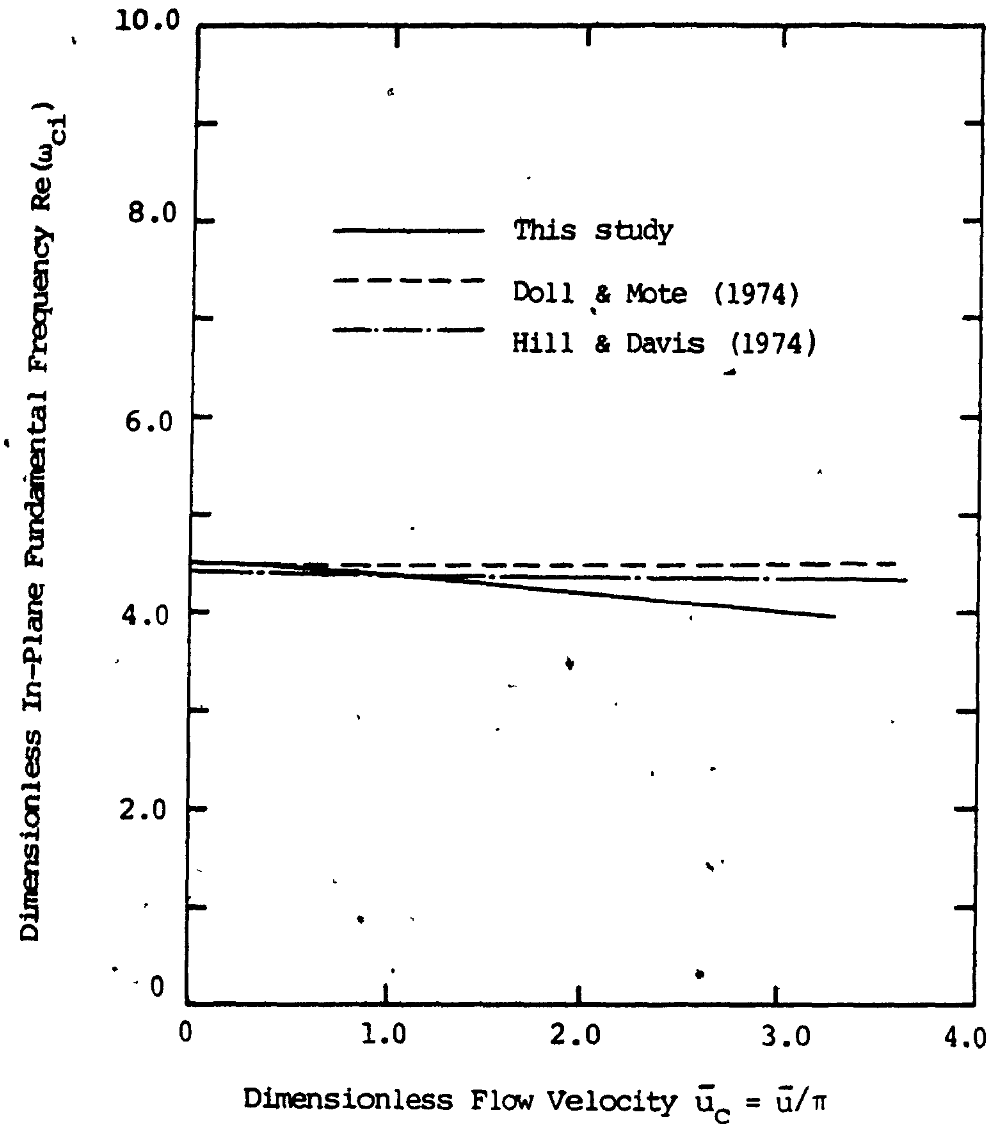


Fig. 49 Dimensionless In-Plane Fundamental Frequency of a Clamped-Clamped Extensible Semi-Circular Pipe Conveying Inviscid Fluid as a Function of the Flow Velocity for $\beta = 0.5$, $\gamma = 0$, $\mathcal{A} = 10^4$ and $\Pi \neq 0$

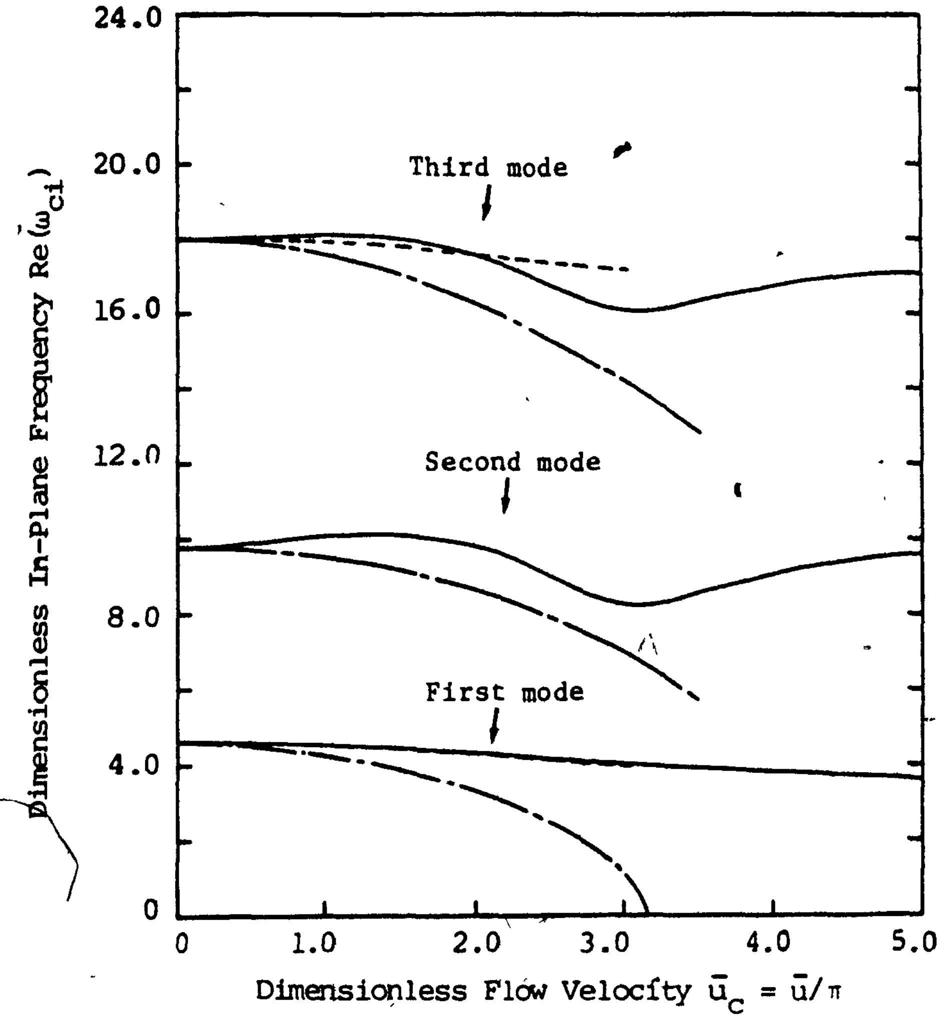


Fig. 50 Dimensionless In-Plane Frequencies of a Clamped-Clamped Extensible Semi-Circular Pipe Conveying Viscous Fluid as Functions of the Dimensionless Flow Velocity for the Extensible Case;

——— Including the terms involving $\mathcal{A}(\eta_1^{o'} + r_o\eta_3^o)$

– – – – For the case of the inviscid flow, $\mathcal{A}(\eta_1^{o'} + r_o\eta_3^o) \neq 0$

—·— For $\Pi = 0$ and $\mathcal{A}(\eta_1^{o'} + r_o\eta_3^o) = 0$

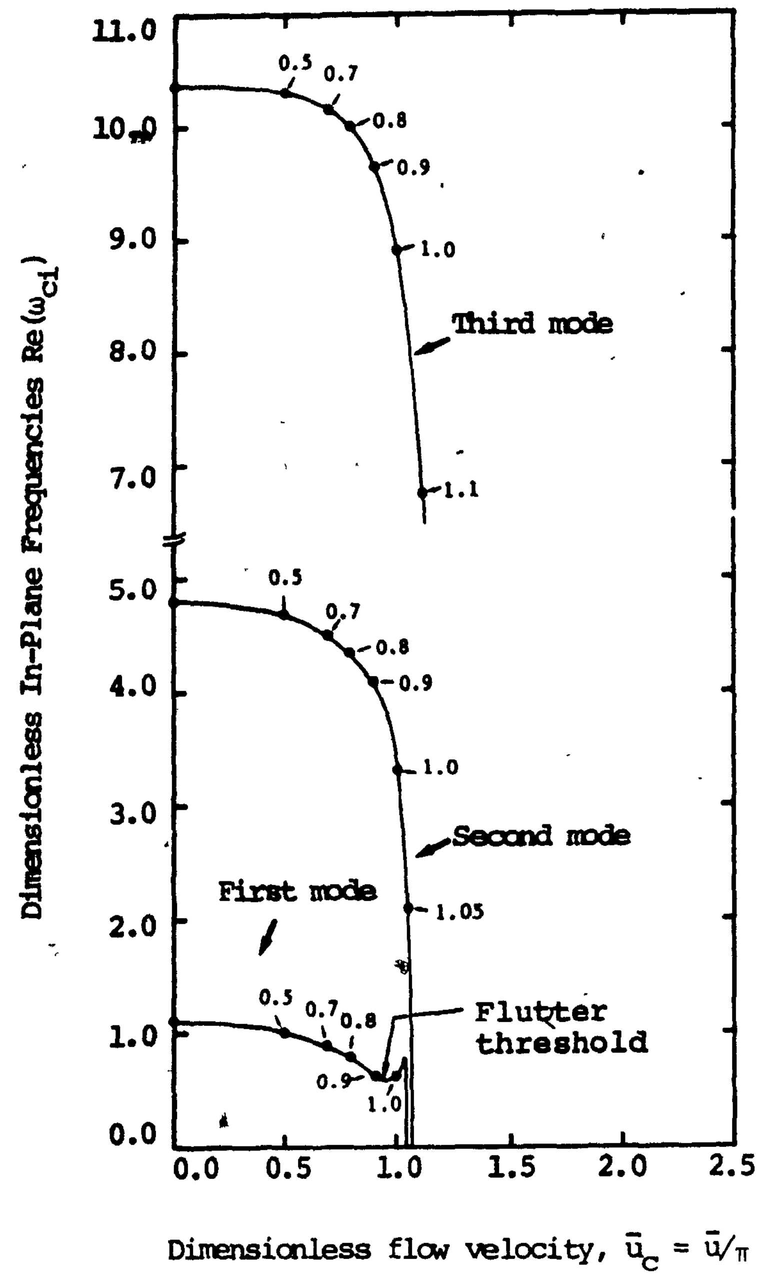


Fig. 51 Dimensionless In-Plane Frequencies of a Clamped-Sliding Extensible Semi-Circular Pipe Conveying Viscous Fluid as Functions of the Flow Velocity for $\beta = 0.5$, $\gamma = 0$, $\mathcal{A} = 10^4$ and $\mathcal{A}(\eta^{o\prime} + r_o \eta_3^o) \neq 0$

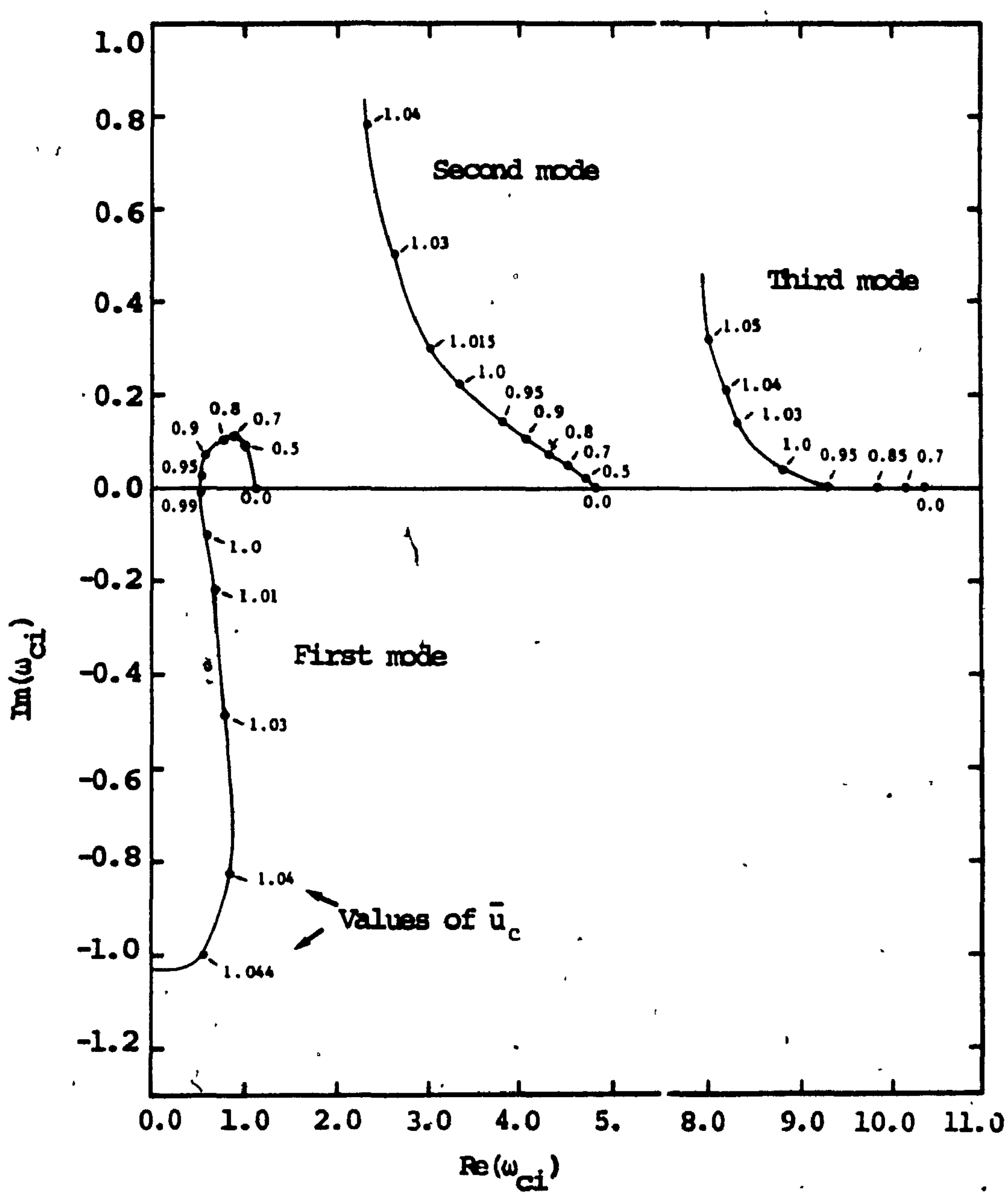


Fig. 51a Dimensionless Complex-Frequency Diagram of In-Plane Motion of a Clamped-Sliding Extensible Semi-Circular Pipe Conveying Viscous Fluid for $\beta = 0.5$, $\gamma = 0$, $\mathcal{A} = 10^4$ and $\mathcal{A}(\eta_1^{o'} + r_o\eta_3^o) \neq 0$.

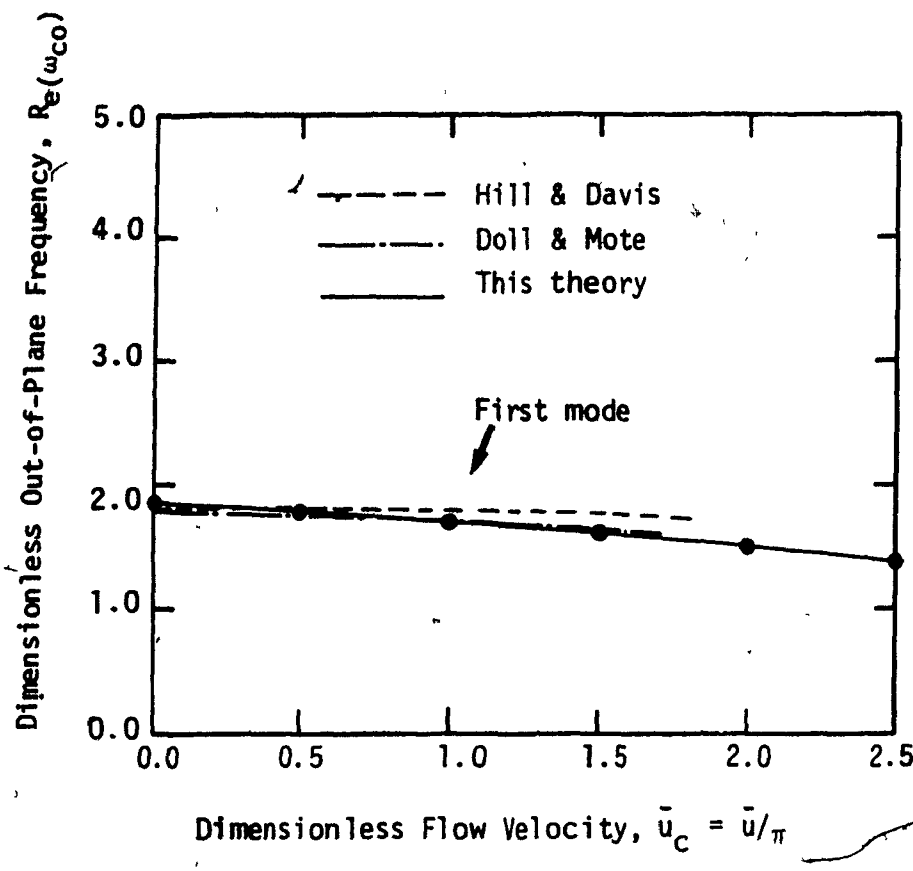


Fig. 52 Dimensionless Out-of-Plane Fundamental Frequency of a Clamped-Clamped Extensible Semi-Circular Pipe Conveying Fluid as a Function of the Flow Velocity for $\beta = 0.5$, $\Lambda = 0.769$ and including Π.

APPENDIX A

RELATIONSHIP BETWEEN THE SYSTEMS (x_o,y_o,z_o) and (x,y,z) USED IN CHAPTER II

Let u, v and w be the displacements of a point P_o on the initial centerline, referred to in the system (x_o,y_o,z_o) centered at P_o (Fig. A.1) also, let ψ be the angle between axes x_o and x, P_o' be a point on this centerline in the neighbourhood of P_o, and δs the arc P_oP_o', so that $(\delta x_o,\delta y_o,\delta z_o)$ are the coordinates of P_o' in the system (x_o,y_o,z_o). Furthermore, we define (ξ,η,ζ) to be the coordinates of P_1', the displaced position of P_o' referred to the system (x_o,y_o,z_o); also, (U',V',W') are defined as the displacements of P_o' referred in (x_o,y_o,z_o) as shown in Fig. A.1. Each of the system (x_o,y_o,z_o) and (x,y,z) is orthogonal. Therefore, we can write

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [L_{ij}] \begin{Bmatrix} x_o \\ y_o \\ z_o \end{Bmatrix} , \tag{A.1}$$

where L_{ij} are the direction cosines of the axes x,y,z referred to in the system (x_o,y_o,z_o).

Consider now the vector P_oP_o'. Where P_o' approaches P_o, P_1' will approach P_1, and the straight line P_1P_1' will become the tangent of the strained centerline at P_1. Hence, one obtains

$$\lim_{\delta s\to 0} \frac{\vec{P_oP_1}}{\partial s} = \vec{e}_z , \tag{A.2}$$

where $\vec{e}_z$ is the unit vector along the z-axis.

From Fig. A.1, the components of the unit vector $\vec{e}_z$ can be written as

$$\lim_{\delta s \to 0} \frac{\xi - u}{\delta s} = L_{31}, \tag{A.3}$$

$$\lim_{\delta s \to 0} \frac{\eta - v}{\partial s} = L_{32}, \tag{A.4}$$

$$\lim_{\partial s \to 0} \frac{\zeta - w}{\partial s} = L_{33} ; \tag{A.5}$$

On the other hand, we have

$$\xi = \delta x_o + U', \tag{A.6}$$

$$\eta = \delta y_o + V', \tag{A.7}$$

$$\zeta = \delta z_o + W', \tag{A.8}$$

$$\frac{\delta x_o}{\delta s} = 0, \ \frac{\delta y_o}{\delta s} = 0, \text{ and } \frac{\delta z_o}{\delta s} = 1 \quad \text{as} \quad \delta s \to 0. \tag{A.9}$$

Combination of equations (A.3) - (A.5) and (A.6) - (A.9) yields

$$L_{31} = \lim_{\delta s \to 0} \frac{U' - u}{\delta s}, \tag{A.10}$$

$$L_{32} = \lim_{\delta s \to 0} \frac{V' - v}{\delta s}, \tag{A.11}$$

$$L_{33} = \lim_{\delta s \to 0} \frac{W' - w}{\delta s} + 1. \tag{A.12}$$

From Appendix B and assuming that u, v, w and ψ are small, one can rewrite equations (A.10)-(A.12), as follows:

$$L_{31} = \frac{\partial u}{\partial s} - \tau_o v + \kappa_o' w, \tag{A.13}$$

$$L_{32} = \frac{\partial v}{\partial s} - \kappa_o w + \tau_o u, \tag{A.14}$$

$$L_{33} = \frac{\partial w}{\partial s} - \kappa_o' u + \kappa_o v + 1. \tag{A.15}$$

Similarly, the centerline strain is given by

$$\varepsilon = \lim_{\delta s \to 0} \frac{W'-w}{\delta s} = \frac{\partial w}{\partial s} - \kappa_o' u + \kappa_o v$$

Since L_{31}, L_{32}, L_{33} are the direction cosines of ~~the unit~~ vector $\vec{e}_z$, referred in the system (x_o, y_o, z_o), we obtain

$$L_{31}^2 + L_{32}^2 + L_{33}^2 = 1. \tag{A.16}$$

By neglecting squares and products of u, v and w, equation (A.16) leads to

$$\frac{\partial w}{\partial s} - \kappa_o' u + \kappa_o v = 0. \tag{A.17}$$

~~Here,~~ it is noted that if the quantities u, v, w and ψ are small, and if the initial centerline lies in the plane (x_o, z_o) (i.e., $\kappa_o = 0$, $\kappa_o' = \frac{1}{R_o}$ and $\tau_o = 0$), the centerline strain can be written as

$$\varepsilon = (\frac{\partial w}{\partial s} - \kappa_o' u) \ . \tag{A.18}$$

Hence, equation (A.17) implies that the centerline is inextensible.

Consequently, in view of equation (A.17), we have

$$L_{33} = 1. \tag{A.19}$$

Furthermore, since ψ is small we can set

$$L_{12} = \psi \ , \tag{A.20}$$

and since the scheme of the transformation (A.1) is orthogonal, one obtains

$$[L_{ij}]^{-1} = [L_{ij}]^T \ , \tag{A.21}$$

$$\det [L_{ij}] = 1. \tag{A.22}$$

Finally, utilizing equations (A.13) - (A.15) and (A.18) - (A.21), and noting that $\kappa_o = 0$, $\kappa_o' = 1/R_o$ and $\tau_o = 0$, one obtains

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 1 & \psi & -(\frac{\partial u}{\partial s} + \frac{w}{R_o}) \\ -\psi & 1 & -\frac{\partial v}{\partial s} \\ (\frac{\partial u}{\partial s} + \frac{w}{R_o}) & \frac{\partial v}{\partial s} & 1 \end{bmatrix} \begin{Bmatrix} x_o \\ y_o \\ z_o \end{Bmatrix} \qquad (A.22)$$

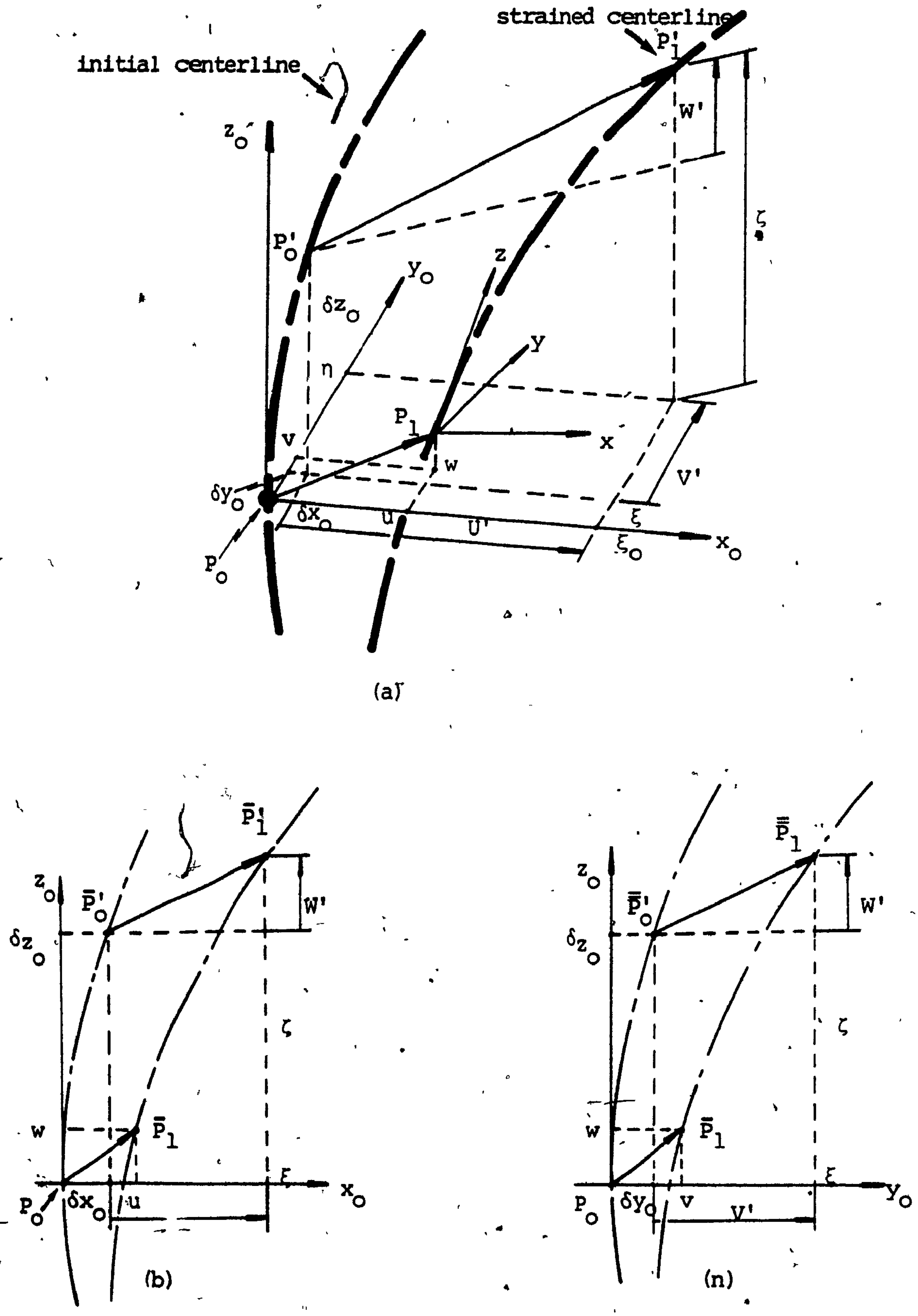


Fig. A.1

APPENDIX B

RELATIONSHIP BETWEEN THE DERIVATIVES WITH RESPECT TO s AS MEASURED IN (X,Y,Z) and (x,y,z) SYSTEMS

Consider a vector quantity $\vec{R}$ with components U*, V* and W* in the torsion-flexure reference frame (x,y,z) at P_1. When the origin of this frame moves along the deformed centerline, it will rotate with the angular velocity $\vec{\Omega}$ (having compoents (κ,κ',τ^*), and the vector $\vec{R}$ will change its direction as well as magnitude.

The derivative of $\vec{R}$ as measured from the inertial frame (X,Y,Z) may be written as

$$\left.\frac{\partial\vec{R}}{\partial s}\right|_{(X,Y,Z)} = \left.\frac{\partial\vec{R}}{\partial s}\right|_{(x,y,z)} + \vec{\Omega} \times \vec{R} \tag{B.1}$$

If the inertial frame (X,Y,Z) coincides with the system (x,y,z) at P_1, one may obtain the following relations:

$$\frac{\delta U^*}{\delta s} = \frac{\partial U^*}{\partial s} - \tau^* V^* + \kappa' W^* , \tag{B.2}$$

$$\frac{\delta V^*}{\delta s} = \frac{\partial V^*}{\partial s} - \kappa\, W^* + \tau^* U^* , \tag{B.2}$$

$$\frac{\delta W^*}{\delta s} = \frac{\partial W^*}{\partial s} - \kappa' U^* + \kappa V^* , \tag{B.4}$$

where $\delta/\delta s$ denotes the partial derivative as measured from the inertial frame (X,Y,Z), and $\partial/\partial s$ denotes the partial derivative as measured from the frame (x,y,z).

APPENDIX C

DERIVATION OF THE FORMULAE OF CURVATURE AND TWIST

Let (x_o, y_o, z_o) be a Frenet-Serret system at P_o on the initial center-line, (x,y,z) be a torsion-flexure reference system at P_1 on the deformed center-line, and (X_o, Y_o, Z_o) be the inertial system coincident with (x_o, y_o, z_o); as shown in Fig. C.1.

These systems are orthogonal. Therefore, one can write the following interrelationships:

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{Bmatrix} x_o \\ y_o \\ z_o \end{Bmatrix}, \qquad \text{(C.1)}$$

and

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \begin{Bmatrix} X_o \\ Y_o \\ Z_o \end{Bmatrix}, \qquad \text{(C.2)}$$

where L_{ij} are the direction cosines of the axes x_i, referred to the system (x_o, y_o, z_o) and derived in Appendix A; and ℓ_{ij} are the direction cosines of the axes x_i, referred to the coincident inertial system (X_o, Y_o, Z_o).

From equation (C.2), the unit vectors in the system (x,y,z) can be written as

$$\vec{e}_x = \ell_{11}\vec{i} + \ell_{12}\vec{j} + \ell_{13}\vec{k}, \qquad \text{(C.3)}$$

$$\vec{e}_y = \ell_{21}\vec{i} + \ell_{22}\vec{j} + \ell_{23}\vec{k}, \qquad \text{(C.4)}$$

$$\vec{e}_z = \ell_{31}\vec{i} + \ell_{32}\vec{j} + \ell_{33}\vec{k}, \qquad \text{(C.5)}$$

where $\vec{i}$, $\vec{j}$ and $\vec{k}$ are the unit vectors in the system (X_o, Y_o, Z_o).

Differentiating equations (C.3)-(C.5) with respect to s at a given time t yields

$$\frac{\partial \vec{e}_x}{\partial s} = \frac{\partial \ell_{11}}{\partial s}\vec{i} + \frac{\partial \ell_{12}}{\partial s}\vec{j} + \frac{\partial \ell_{13}}{\partial s}\vec{k} \,, \tag{C.6}$$

$$\frac{\partial \vec{e}_y}{\partial s} = \frac{\partial \ell_{21}}{\partial s}\vec{i} + \frac{\partial \ell_{22}}{\partial s}\vec{j} + \frac{\partial \ell_{23}}{\partial s}\vec{k} \,, \tag{C.7}$$

$$\frac{\partial \vec{e}_z}{\partial s} = \frac{\partial \ell_{31}}{\partial s}\vec{i} + \frac{\partial \ell_{32}}{\partial s}\vec{j} + \frac{\partial \ell_{33}}{\partial s}\vec{k} \,. \tag{C.8}$$

On the other hand, since the system (x,y,z) rotates about (x_o, y_o, z_o) with the angular velocity

$$\vec{\Omega} = \kappa\ \vec{e}_x + \kappa'\ \vec{e}_y + \tau^*\ \vec{e}_z \,, \tag{C.9}$$

when its origin moves along the deformed center-line. then, we obtain

$$\frac{\partial \vec{e}_x}{\partial s} = \vec{\Omega} \times \vec{e}_x \,, \tag{C.10}$$

$$\frac{\partial \vec{e}_y}{\partial s} = \vec{\Omega} \times \vec{e}_y \,, \tag{C.11}$$

$$\frac{\partial \vec{e}_z}{\partial s} = \vec{\Omega} \times \vec{e}_z \,. \tag{C.12}$$

By noting that $\vec{e}_x \times \vec{e}_y = \vec{e}_z$, $\vec{e}_y \times \vec{e}_z = \vec{e}_x$, and $\vec{e}_z \times \vec{e}_x = \vec{e}_y$, one can rewrite equations (C.10)-(C.12), as follows:

$$\frac{\partial \vec{e}_x}{\partial s} = -\kappa'\vec{e}_z + \tau^*\vec{e}_y \,, \tag{C.13}$$

$$\frac{\partial \vec{e}_y}{\partial s} = \kappa\vec{e}_z - \tau^*\ \vec{e}_x \,, \tag{C.14}$$

$$\frac{\partial \vec{e}_z}{\partial s} = -\kappa\ \vec{e}_y + \kappa'\vec{e}_x \,. \tag{C.15}$$

Substituting equations (C.3)-(C.5) into (C.13)-(C.15) yields

$$\frac{\partial \vec{e}_x}{\partial s} = (\ell_{21}\tau^* - \ell_{31}\kappa')\vec{i} + (\ell_{22}\tau^* - \ell_{32}\kappa')\vec{j} + (\ell_{23}\tau^* - \ell_{33}\kappa')\vec{k}\ , \qquad \text{(C.16)}$$

$$\frac{\partial \vec{e}_y}{\partial s} = (\ell_{31}\kappa - \ell_{11}\tau^*)\vec{i} + (\ell_{32}\kappa - \ell_{12}\tau^*)\vec{j} + (\ell_{33}\kappa - \ell_{13}\tau^*)\vec{k}\ , \qquad \text{(C.17)}$$

$$\frac{\partial \vec{e}_z}{\partial s} = (\ell_{11}\kappa' - \ell_{21}\kappa)\vec{i} + (\ell_{12}\kappa' - \ell_{22}\kappa)\vec{j} + (\ell_{13}\kappa' - \ell_{23}\kappa)\vec{k}\ . \qquad \text{(C.18)}$$

Combining equations (C.6)-(C.8) and (C.16)-(C.18), one obtains

$$\left.\begin{aligned} \frac{\partial \ell_{11}}{\partial s} &= \ell_{21}\tau^* - \ell_{31}\kappa', \\ \frac{\partial \ell_{12}}{\partial s} &= \ell_{22}\tau^* - \ell_{32}\kappa', \\ \frac{\partial \ell_{13}}{\partial s} &= \ell_{23}\tau^* - \ell_{33}\kappa', \end{aligned}\right\} \qquad \text{(C.19)}$$

$$\left.\begin{aligned} \frac{\partial \ell_{21}}{\partial s} &= \ell_{31}\kappa - \ell_{11}\tau^*\ , \\ \frac{\partial \ell_{22}}{\partial s} &= \ell_{32}\kappa - \ell_{12}\tau^*\ , \\ \frac{\partial \ell_{23}}{\partial s} &= \ell_{33}\kappa - \ell_{13}\tau^*\ , \end{aligned}\right\} \qquad \text{(C.20)}$$

$$\left.\begin{aligned} \frac{\partial \ell_{31}}{\partial s} &= \ell_{11}\kappa' - \ell_{21}\kappa, \\ \frac{\partial \ell_{32}}{\partial s} &= \ell_{12}\kappa' - \ell_{22}\kappa, \\ \frac{\partial \ell_{33}}{\partial s} &= \ell_{13}\kappa' - \ell_{23}\kappa. \end{aligned}\right\} \qquad \text{(C.21)}$$

Multiplying the equations in equation set (C.19) by ℓ_{21}, ℓ_{22} and ℓ_{23} respectively, and then adding the three equations, one obtains

$$\ell_{21}\frac{\partial \ell_{11}}{\partial s} + \ell_{22}\frac{\partial \ell_{12}}{\partial s} + \ell_{23}\frac{\partial \ell_{13}}{\partial s} = (\ell_{21}^2 + \ell_{22}^2 + \ell_{23}^2)\tau^* - (\ell_{21}\ell_{31} + \ell_{22} + \ell_{32} + \ell_{23}\ell_{33})\kappa'. \qquad \text{(C.22)}$$

However, each of the systems (x,y,z) and (X_o,Y_o,Z_o) is orthogonal; hence, we have

$$\sum_{K=1}^{3} \ell_{ik}\ell_{jk} = \delta_{ij}, \tag{C.23}$$

where δ_{ij} is the Kronecker delta.

Combining equations (C.22) and (C.23) yields

$$\tau^* = \ell_{21}\frac{\partial \ell_{11}}{\partial s} + \ell_{22}\frac{\partial \ell_{12}}{\partial s} + \ell_{23}\frac{\partial \ell_{13}}{\partial s} . \tag{C.24}$$

We can proceed similarly for κ, and κ'. In summary the expressions for κ, κ' and τ* are

$$\kappa = \ell_{31}\frac{\partial \ell_{21}}{\partial s} + \ell_{32}\frac{\partial \ell_{22}}{\partial s} + \ell_{33}\frac{\partial \ell_{23}}{\partial s}, \tag{C.25}$$

$$\kappa' = \ell_{11}\frac{\partial \ell_{31}}{\partial s} + \ell_{12}\frac{\partial \ell_{32}}{\partial s} + \ell_{13}\frac{\partial \ell_{33}}{\partial s}, \tag{C.26}$$

$$\tau^* = \ell_{21}\frac{\partial \ell_{11}}{\partial s} + \ell_{22}\frac{\partial \ell_{12}}{\partial s} + \ell_{23}\frac{\partial \ell_{13}}{\partial s} . \tag{C.27}$$

Using Appendix B, one can write the derivative of components of the unit vectors $\bar{e}_x$, $\bar{e}_y$ and $\bar{e}_z$ measured in the system (X_o,Y_o,Z_o), as follows:

$$\left.\begin{aligned} \frac{\partial \ell_{11}}{\partial s} &= \frac{\partial L_{11}}{\partial s} - L_{12}\tau_o + L_{13}\kappa_o', \\ \frac{\partial \ell_{12}}{\partial s} &= \frac{\partial L_{12}}{\partial s} - L_{13}\kappa_o + L_{11}\tau_o , \\ \frac{\partial \ell_{13}}{\partial s} &= \frac{\partial L_{13}}{\partial s} - L_{11}\kappa_o' + L_{12}\kappa_o , \end{aligned}\right\} \tag{C.28}$$

$$\left.\begin{aligned}\frac{\partial \ell_{21}}{\partial s} &= \frac{\partial L_{21}}{\partial s} - L_{22}\tau_o + L_{23}\kappa_o',\\ \frac{\partial \ell_{22}}{\partial s} &= \frac{\partial L_{22}}{\partial s} - L_{23}\kappa_o + L_{21}\tau_o,\\ \frac{\partial \ell_{23}}{\partial s} &= \frac{\partial L_{23}}{\partial s} - L_{21}\kappa_o' + L_{22}\kappa_o,\end{aligned}\right\} \qquad (C.29)$$

$$\left.\begin{aligned}\frac{\partial \ell_{31}}{\partial s} &= \frac{\partial L_{31}}{\partial s} - L_{32}\tau_o + L_{33}\kappa_o',\\ \frac{\partial \ell_{32}}{\partial s} &= \frac{\partial L_{32}}{\partial s} - L_{33}\kappa_o + L_{31}\tau_o,\\ \frac{\partial \ell_{33}}{\partial s} &= \frac{\partial L_{33}}{\partial s} - L_{31}\kappa_o' + L_{32}\kappa_o .\end{aligned}\right\} \qquad (C.30)$$

Substituting the values of $\partial\ell_{ij}/\partial s$ from equation sets (C.28)-(C.30) into equations (C.25)-(C.27), and then setting $\ell_{ij} = L_{ij}$, one obtains

$$\kappa = L_{31}\left(\frac{\partial L_{21}}{\partial s} - L_{22}\tau_o + L_{23}\kappa_o'\right) + L_{32}\left(\frac{\partial L_{22}}{\partial s} - L_{23}\kappa_o + L_{21}\tau_o\right) + L_{33}\left(\frac{\partial L_{23}}{\partial s} - L_{21}\kappa_o' + L_{22}\kappa_o\right), \qquad (C.31)$$

$$\kappa' = L_{11}\left(\frac{\partial L_{31}}{\partial s} - L_{32}\tau_o + L_{33}\kappa_o'\right) + L_{12}\left(\frac{\partial L_{32}}{\partial s} - L_{33}\kappa_o + L_{31}\tau_o\right) + L_{13}\left(\frac{\partial L_{33}}{\partial s} - L_{31}\kappa_o' + L_{32}\kappa_o\right), \qquad (C.32)$$

$$\tau^* = L_{21}\left(\frac{\partial L_{11}}{\partial s} - L_{12}\tau_o + L_{13}\kappa_o'\right) + L_{22}\left(\frac{\partial L_{12}}{\partial s} - L_{13}\kappa_o + L_{11}\tau_o\right) + L_{23}\left(\frac{\partial L_{13}}{\partial s} - L_{11}\kappa_o' + L_{12}\kappa_o\right) \qquad (C.33)$$

Now substituting the values of L_{ij} given in Appnedix A, and the values of $\kappa_o, \kappa_o', \tau_o$ into equation (C.31)-(C.33), and then neglecting higher order terms, one may write the expressions for κ, κ' and τ^* in terms of u, v, w and ψ, as follows:

$$
\left.
\begin{aligned}
\kappa &= \left(\frac{\psi}{R_o} - \frac{\partial^2 v}{\partial s^2}\right) , \\
\kappa' &= \left(\frac{1}{R_o} + \frac{\partial^2 u}{\partial s^2} + \frac{1}{R_o} \frac{\partial w}{\partial s}\right) , \\
\tau^* &= \left(\frac{\partial \psi}{\partial s} + \frac{1}{R_o} \frac{\partial v}{\partial s}\right) .
\end{aligned}
\right\} \qquad (C.34)
$$

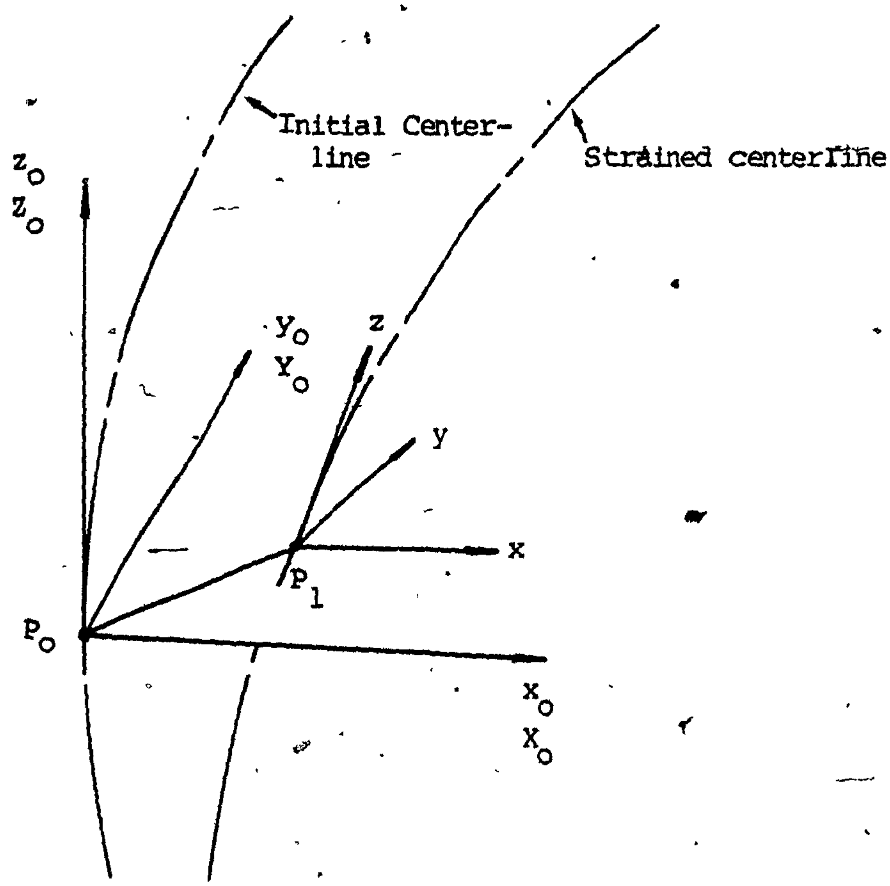

Initial Center-
line
Strained centerline
z_o
Z_o
y_o
Y_o
z
y
x
P_1
P_o
x_o
X_o

Fig. C.1

APPENDIX D

DERIVATION OF THE COMPONENTS OF FLUID-ACCELERATION VECTOR

Recall equation (2.16) in the main text

$$\vec{a}_f = \frac{\partial \vec{V}_f}{\partial t} + U \frac{\partial \vec{V}_f}{\partial s} , \qquad (D.1)$$

where

$$\vec{V}_f = [\frac{\partial u}{\partial t} + U(\frac{\partial u}{\partial s} + \frac{w}{R_o})]\vec{e}_{x_o} + [\frac{\partial v}{\partial t} + U \frac{\partial v}{\partial s}] \vec{e}_{y_o} + [\frac{\partial w}{\partial t} + U] \vec{e}_{z_o} . \qquad (D.2)$$

By differentiating $\vec{V}_f$ with respect to t and s yields

$$\frac{\partial \vec{V}_f}{\partial t} = [\frac{\partial^2 u}{\partial t^2} + U(\frac{\partial^2 u}{\partial t \partial s} + \frac{1}{R_o} \frac{\partial w}{\partial t})]\vec{e}_{x_o} + [\frac{\partial^2 v}{\partial t^2} + U \frac{\partial^2 v}{\partial t \partial s}]\vec{e}_{y_o} + \frac{\partial^2 w}{\partial t^2} \vec{e}_{z_o} , \qquad (D.3)$$

$$\frac{\partial \vec{V}_f}{\partial s} = [\frac{\partial^2 u}{\partial t \partial s} + U(\frac{\partial^2 u}{\partial s^2} + \frac{1}{R_o} \frac{\partial w}{\partial s})]\vec{e}_{x_o} + [\frac{\partial^2 v}{\partial t \partial s} + U \frac{\partial^2 v}{\partial s^2}]\vec{e}_{y_o} + \frac{\partial^2 w}{\partial t \partial s} \vec{e}_{z_o}$$

$$+ [\frac{\partial u}{\partial t} + U(\frac{\partial u}{\partial s} + \frac{w}{R_o})]\frac{\partial \vec{e}_{x_o}}{\partial s} + [\frac{\partial v}{\partial t} + U \frac{\partial v}{\partial s}] \frac{\partial \vec{e}_{y_o}}{\partial s} + [\frac{\partial w}{\partial t} + U] \frac{\partial \vec{e}_{z_o}}{\partial s} , \qquad (D.4)$$

and from Appendix C, one obtains

$$\frac{\partial \vec{e}_{x_o}}{\partial s} = - \frac{1}{R_o} \vec{e}_{z_o} ,$$
$$\frac{\partial \vec{e}_{y_o}}{\partial s} = \vec{0} , \qquad (D.5)$$
$$\frac{\partial \vec{e}_{z_o}}{\partial s} = \frac{1}{R_o} \vec{e}_{x_o}$$

Combining equations (D.1), (D.3), (D.4) and (D.5), the components of the vector of fluid acceleration can be written, as follows:

$$a_{fx_o} = \frac{\partial^2 u}{\partial t^2} + 2U\left(\frac{\partial^2 u}{\partial t \partial s} + \frac{1}{R_o}\frac{\partial w}{\partial t}\right) + U^2\left(\frac{\partial^2 u}{\partial s^2} + \frac{1}{R_o}\frac{\partial w}{\partial s} + \frac{1}{R_o}\right), \tag{D.6}$$

$$a_{fy_o} = \frac{\partial^2 v}{\partial t^2} + 2U\frac{\partial^2 v}{\partial t \partial s} + U^2\frac{\partial^2 v}{\partial s^2}, \tag{D.7}$$

$$a_{fz_o} = \frac{\partial^2 w}{\partial t^2} + U\left(\frac{\partial^2 w}{\partial t \partial s} - \frac{1}{R_o}\frac{\partial u}{\partial t}\right) - \frac{U^2}{R_o}\left(\frac{\partial u}{\partial s} + \frac{w}{R_o}\right), \tag{D.8}$$

which are equations (2.17)-(2.19) in the main text.

APPENDIX E

DERIVATION OF EQUATIONS OF MOTION FOR THE PIPE

Consider an infinitesimal element of the pipe contained between the cross section through P_1 and P_1' on the strained center-line, and the forces and moments acting on it, as shown in Fig. 2. By projecting the forces and moments in Fig. 2(b) on the inertial system (X_o, Y_o, Z_o), as shown in Fig. E.1, and balancing the forces and moments along axes of X_o, Y_o, Z_o, one obtains, the following:

$$\left.\begin{aligned}
&\delta(\ell^*_{11}Q_{x_o}+\ell^*_{21}Q_{y_o}+\ell^*_{31}Q_{z_o}) + \int_S^{S+\delta S}(\ell^{*\prime}_{11}F_{x_o}+\ell^{*\prime}_{21}F_{y_o}+\ell^{*\prime}_{31}F_{z_o})\,dS = 0\,,\\
&\delta(\ell^*_{12}Q_{x_o}+\ell^*_{22}Q_{y_o}+\ell^*_{32}Q_{z_o}) + \int_S^{S+\delta S}(\ell^{*\prime}_{12}F_{x_o}+\ell^{*\prime}_{22}F_{y_o}+\ell^{*\prime}_{32}F_{z_o})\,dS = 0\,,\\
&\delta(\ell^*_{13}Q_{x_o}+\ell^*_{23}Q_{y_o}+\ell^*_{33}Q_{z_o}) + \int_S^{S+\delta S}(\ell^{*\prime}_{13}F_{x_o}+\ell^{*\prime}_{23}F_{y_o}+\ell^{*\prime}_{33}F_{z_o})\,dS = 0\,,
\end{aligned}\right\}\quad (E.1)$$

$$\left.\begin{aligned}
&\delta(\ell^*_{11}M_{x_o}+\ell^*_{21}M_{y_o}+\ell^*_{31}M_{z_o}) + \delta Y_o\{(\ell^*_{13}Q_{x_o}+\ell^*_{23}Q_{y_o}+\ell^*_{33}Q_{z_o})+\delta(\ell^*_{13}Q_{x_o}+\ell^*_{23}Q_{y_o}+\ell^*_{33}Q_{z_o})\}\\
&\quad -\delta Z_o\{(\ell^*_{12}Q_{x_o}+\ell^*_{22}Q_{y_o}+\ell^*_{32}Q_{z_o})+\delta(\ell^*_{12}Q_{x_o}+\ell^*_{22}Q_{y_o}+\ell^*_{32}Q_{z_o})\} +\\
&\int_S^{S+\delta S}\{\delta Y_o'(\ell^{*\prime}_{13}F_{x_o}+\ell^{*\prime}_{23}F_{y_o}+\ell^{*\prime}_{33}F_{z_o})-\delta Z_o'(\ell^{*\prime}_{12}F_{x_o}+\ell^{*\prime}_{22}F_{y_o}+\ell^{*\prime}_{32}F_{z_o})\}\,dS + \int_S^{S+\delta S}(\ell^{*\prime}_{11}\bullet_{x_o}\\
&\quad +\ell^{*\prime}_{21}\bullet_{y_o}+\ell^{*\prime}_{31}\bullet_{z_o})\,dS = 0\,,\\
&\delta(\ell^*_{12}M_{x_o}+\ell^*_{22}M_{y_o}+\ell^*_{32}M_{z_o}) + \delta Z_o\{(\ell^*_{11}Q_{x_o}+\ell^*_{21}Q_{y_o}+\ell^*_{31}Q_{z_o})+\delta(\ell^*_{11}Q_{x_o}+\ell^*_{21}Q_{y_o}+\ell^*_{31}Q_{z_o})\}\\
&\quad -\delta X_o\{(\ell^*_{13}Q_{x_o}+\ell^*_{23}Q_{y_o}+\ell^*_{33}Q_{z_o})+\delta(\ell^*_{13}Q_{x_o}+\ell^*_{23}Q_{y_o}+\ell^*_{33}Q_{z_o})\} +\\
&\int_S^{S+\delta S}\{\delta Z_o'(\ell^{*\prime}_{11}F_{x_o}+\ell^{*\prime}_{21}F_{y_o}+\ell^{*\prime}_{31}F_{z_o})-\delta X_o'(\ell^{*\prime}_{13}F_{x_o}+\ell^{*\prime}_{23}F_{y_o}+\ell^{*\prime}_{33}F_{z_o})\}\,dS + \int_S^{S+\delta S}(\ell^{*\prime}_{12}\bullet_{x_o}\\
&\quad +\ell^{*\prime}_{22}\bullet_{y_o}+\ell^{*\prime}_{32}\bullet_{z_o})\,dS = 0\,,
\end{aligned}\right\}\quad (E.2$$

$$\begin{aligned}
&\delta(\ell^*_{13}M_{x_o}+\ell^*_{23}M_{y_o}+\ell^*_{33}M_{z_o})+\delta X_o\{(\ell^*_{12}Q_{x_o}+\ell^*_{22}Q_{y_o}+\ell^*_{32}Q_{z_o})+\delta(\ell^*_{12}Q_{x_o}+\ell^*_{22}Q_{y_o}+\ell^*_{32}Q_{z_o})\}\\
&\quad -\delta Y_o\{(\ell^*_{11}Q_{x_o}+\ell^*_{21}Q_{y_o}+\ell^*_{31}Q_{z_o})-\delta(\ell^*_{11}Q_{x_o}+\ell^*_{21}Q_{y_o}+\ell^*_{31}Q_{z_o})\} +\\
&\int_S^{S+\delta S}\{\delta X'_o(\ell^{*\prime}_{12}F_{x_o}+\ell^{*\prime}_{22}F_{y_o}+\ell^{*\prime}_{32}F_{z_o})-\delta Y'_o(\ell^{*\prime}_{11}F_{x_o}+\ell^{*\prime}_{21}F_{y_o}+\ell^{*\prime}_{31}F_{z_o})\}\,dS\\
&\quad +\int_S^{S+\delta S}(\ell^{*\prime}_{13}\Theta_{x_o}+\ell^{*\prime}_{23}\Theta_{y_z}+\ell^{*\prime}_{33}\Theta_{z_o})\,dS = 0\,,
\end{aligned} \qquad \text{(E.2) Cont'd}$$

where

ℓ^*_{ij} and $\ell^{*\prime}_{ij}$ are the direction cosines of the axes x_{oi} and x'_{oi}, referred to the inertial system (X_o,Y_o,Z_o);

$(Q_{x_o},Q_{y_o},Q_{z_o})$ are components, referred to the system (x_o,y_o,z_o), of the resultant of the transverse shear forces Q_x,Q_y and the axial force Q^*_z;

$(M_{x_o},M_{y_o},M_{z_o})$ are components of the resultant of the bending moments M_x, M_y and the twist couple M_z in the directions of X_o,Y_o,Z_o;

$(F_{x_o},F_{y_o},F_{z_o})$ are components, referred to the system (x'_o,y'_o,z'_o), of the force resultant at P_ξ per unit length of the centerline which includes the inertial forces, the viscous-damping force due to the surrounding fluid, the reaction force arising from the internal fluid, the gravity force, and the force due to the pressure distribution of the surrounding fluid;

$(\Theta_{x_o},\Theta_{y_o},\Theta_{z_o})$ are components, referred to the system (x'_o,y'_o,z'_o), of the moment resultant at P_ξ per unit length of the pipe centerline, which includes the moment of rotary inertia and external moments if they exist.

Now it is noted that when $\delta S \to 0$, the system (x_o', y_o', z_o') coincides with the system (x_o, y_o, z_o). Therefore, one obtains

$$\ell_{ij}^{*} = \ell_{ij}^{*\prime} , \tag{E.3}$$

$$\frac{1}{\delta S} \int_{S}^{S+\delta S} (\ell_{1i}^{*\prime} F_{x_o} + \ell_{2i}^{*\prime} F_{y_o} + \ell_{3i}^{*\prime} F_{z_o}) dS = \ell_{1i}^{*} F_{x_o} + \ell_{2i}^{*} F_{y_o} + \ell_{3i}^{*} F_{z_o} , \tag{E.4}$$

$$\frac{1}{\delta S} \int_{S}^{S+\delta S} (\ell_{1i}^{*\prime} \Theta_{x_o} + \ell_{2i}^{*\prime} \Theta_{y_o} + \ell_{3i}^{*\prime} \Theta_{z_o}) dS = \ell_{1i}^{*} \Theta_{x_o} + \ell_{2i}^{*} \Theta_{y_o} + \ell_{3i}^{*} \Theta_{z_o} , \tag{E.5}$$

and $\frac{\delta X_o}{\delta S}, \frac{\delta Y_o}{\delta S}, \frac{\delta Z_o}{\delta S}$ become components of the unit vector $\vec{e}_z$ at P_1, because $\delta X_o, \delta Y_o, \delta Z_o$ are components of the vector $\overline{P_1 P_1'}$, as shown in Fig. E.2; and they are denoted by

$$\left.\begin{aligned} \ell_1 &= \frac{\delta X_o}{\delta S} , \\ \ell_2 &= \frac{\delta Y_o}{\delta S} , \\ \ell_3 &= \frac{\delta Z_o}{\delta S} . \end{aligned}\right\} \tag{E.6}$$

Moreover, if the system X_{oi} coincides with the system x_{oi} at P_1, then

$$\left.\begin{aligned} \ell_1 &= L_{31} , \\ \ell_2 &= L_{32} , \\ \ell_3 &= L_{33} , \end{aligned}\right\} \tag{E.7}$$

where L_{3i} are the direction cosines of the z-axis, referred to the reference frame (x_o, y_o, z_o).

Dividing the equation sets (E.1) and (E.2) by δS, and taking the limit $\delta S \to 0$, and then combining equations (E.4)-(E.7) yields

$$\frac{\partial}{\partial s}\left[\ell^*_{11}Q_{x_o}+\ell^*_{21}Q_{y_o}+\ell^*_{31}Q_{z_o}\right] + \ell^*_{11}F_{x_o} + \ell^*_{21}F_{y_o} + \ell^*_{31}F_{z_o} = 0 \ ,$$

$$\frac{\partial}{\partial s}\left[\ell^*_{12}Q_{x_o}+\ell^*_{22}Q_{y_o}+\ell^*_{32}Q_{z_o}\right] + \ell^*_{12}F_{x_o} + \ell^*_{22}F_{y_o} + \ell^*_{32}F_{z_o} = 0 \ , \qquad \text{(E.8)}$$

$$\frac{\partial}{\partial s}\left[\ell^*_{13}Q_{x_o}+\ell^*_{23}Q_{y_o}+\ell^*_{33}Q_{z_o}\right] + \ell^*_{13}F_{x_o} + \ell^*_{23}F_{y_o} + \ell^*_{33}F_{z_o} = 0 \ ,$$

$$\frac{\partial}{\partial s}\left[\ell^*_{11}M_{x_o}+\ell^*_{21}M_{y_o}+\ell^*_{31}M_{z_o}\right]+\ell_2(\ell^*_{13}Q_{x_o}+\ell^*_{23}Q_{y_o}+\ell^*_{33}Q_{z_o})-\ell_3(\ell^*_{12}Q_{x_o}+\ell^*_{22}Q_{y_o}+\ell^*_{32}Q_{z_o})$$
$$+ (\ell^*_{11}\oplus_{x_o}+\ell^*_{21}\oplus_{y_o}+\ell^*_{31}\oplus_{z_o}) = 0 \ ,$$

$$\frac{\partial}{\partial s}\left[\ell^*_{12}M_{x_o}+\ell^*_{22}M_{y_o}+\ell^*_{32}M_{z_o}\right]+\ell_3(\ell^*_{11}Q_{x_o}+\ell^*_{21}Q_{y_o}+\ell^*_{31}Q_{z_o})-\ell_1(\ell^*_{13}Q_{x_o}+\ell^*_{23}Q_{y_o}+\ell^*_{33}Q_{z_o})$$
$$+ (\ell^*_{12}\oplus_{x_o}+\ell^*_{22}\oplus_{y_o}+\ell^*_{32}\oplus_{z_o}) = 0 \ ,$$

$$\frac{\partial}{\partial s}\left[\ell^*_{13}Q_{x_o}+\ell^*_{23}Q_{y_o}+\ell^*_{33}Q_{z_o}\right]+\ell_1(\ell^*_{12}Q_{x_o}+\ell^*_{22}Q_{y_o}+\ell^*_{32}Q_{z_o})-\ell_2(\ell^*_{11}Q_{x_o}+\ell^*_{21}Q_{y_o}+\ell^*_{31}Q_{z_o})$$
$$+ (\ell^*_{13}\oplus_{x_o}+\ell^*_{23}\oplus_{y_o}+\ell^*_{33}\oplus_{z_o}) = 0. \qquad \text{(E.9)}$$

From Equations (C.19)-(C.21) in Appendix C, one can write the derivatives of ℓ^*_{ij}, as follows:

$$\frac{\partial \ell^*_{11}}{\partial s} = \ell^*_{21}\tau_o - \ell^*_{31}\kappa'_o \ ,$$
$$\frac{\partial \ell^*_{12}}{\partial s} = \ell^*_{22}\tau_o - \ell^*_{32}\kappa'_o \ ,$$
$$\frac{\partial \ell^*_{13}}{\partial s} = \ell^*_{23}\tau_o - \ell^*_{33}\kappa'_o \ ,$$

$$
\left.
\begin{aligned}
\frac{\partial \ell^*_{21}}{\partial S} &= \ell^*_{31}\kappa_o - \ell^*_{11}\tau_o \,, \\
\frac{\partial \ell^*_{22}}{\partial S} &= \ell^*_{32}\kappa_o - \ell^*_{12}\tau_o \,, \\
\frac{\partial \ell^*_{23}}{\partial S} &= \ell^*_{33}\kappa_o - \ell^*_{13}\tau_o \\
\frac{\partial \ell^*_{31}}{\partial S} &= \ell^*_{11}\kappa'_o - \ell^*_{21}\kappa_o \,, \\
\frac{\partial \ell^*_{32}}{\partial S} &= \ell^*_{12}\kappa'_o - \ell^*_{22}\kappa_o \,, \\
\frac{\partial \ell^*_{33}}{\partial S} &= \ell^*_{13}\kappa'_o - \ell^*_{23}\kappa_o \,.
\end{aligned}
\right\} \tag{E.10}
$$

Substituting equation set (E.10) into equation sets (E.8)-(E.9), and then setting the system X_{oi} to coincide with the system x_{oi} at P_o, the equations of motion for the pipe along the axes of x_o, y_o, z_o may be written as

$$\frac{\partial Q_{x_o}}{\partial S} - \tau_o Q_{y_o} + \kappa'_o Q_{z_o} + F_{x_o} = 0 \,, \tag{E.11}$$

$$\frac{\partial Q_{y_o}}{\partial S} - \kappa_o Q_{z_o} + \tau_o Q_{x_o} + F_{y_o} = 0 \,, \tag{E.12}$$

$$\frac{\partial Q_{z_o}}{\partial S} - \kappa'_o Q_{x_o} + \kappa_o Q_{y_o} + F_{z_o} = 0 \,, \tag{E.13}$$

$$\frac{\partial M_{x_o}}{\partial S} - \tau_o M_{y_o} + \kappa'_o M_{z_o} + \bullet_{x_o} + L_{32} Q_{z_o} - L_{33} Q_{y_o} = 0 \,, \tag{E.14}$$

$$\frac{\partial M_{y_o}}{\partial S} - \kappa_o M_{z_o} + \tau_o M_{x_o} + \bullet_{y_o} + L_{33} Q_{x_o} - L_{31} Q_{z_o} = 0 \,, \tag{E.15}$$

$$\frac{\partial M_{z_o}}{\partial S} - \kappa_o' M_{x_o} + \kappa_o M_{y_o} + \Theta_{z_o} + L_{31} Q_{y_o} - L_{32} Q_{x_o} = 0 \ . \tag{E.16}$$

We now consider the force per unit length of the center-line due to the gravity force and the pressure distribution of the surrounding fluid. For convenience, the pressure distribution of the surrounding fluid acting on the external lateral surface per unit length of the pipe may be replaced by the buoyancy force $\vec{B}$ (i.e. $\vec{B} = A_o \rho_{fe} \vec{g}$) and the tensions $A_o P_e$ and $A_o P_e'$ applied on the top and bottom faces where P_e and P_e' are the pressures at levels P_1 and P_1', as shown in Fig. E.3. The buoyancy force $\vec{B}$ and the gravity force, can be combined into a single force, called the effective gravity force $\vec{G}$, and the pressure force $A_o P_e$ and the tension Q_z can also be combined into a single term Q_z^*. Let $(G_{x_o}, G_{y_o}, G_{z_o})$ denote components, referred to the system (x_o, y_o, z_o), of the effective gravity force $\vec{G}$; then, we can write

$$G_{x_o} = (M_t - A_o \rho_{fe}) g \, \alpha_{x_o} , \tag{E.17}$$

$$G_{y_o} = (M_t - A_o \rho_{fe}) g \, \alpha_{yo} , \tag{E.18}$$

$$G_{z_o} = (M_t - A_o \rho_{fe}) g \, \alpha_{z_o} , \tag{E.19}$$

$$Q_z^* = Q_z + A_o P_e , \tag{E.20}$$

where M_t is the mass per unit length of the pipe, A_o is the external cross sectional area of the pipe, ρ_{fe} is the density of the surrounding fluid, g is the acceleration due to gravity and α_{x_o}, α_{y_o}, α_{z_o} are the direction cosines, referred to the system (x_o, y_o, z_o) of the gravitational acceleration.

For the pipe vibrating in a quiescent fluid, fluid damping arises due to the fluid viscous effect and due to the energy carried away by acoustic waves. The damping force arising from these effects may be considered to be proportional to the pipe velocity. Let (f_{x_o}, f_{y_o}, f_{z_o}) denote the components of this force, referred to the system (x_o, y_o, z_o); then we can write

$$f_{x_o} = - c \frac{\partial u}{\partial t} , \tag{E.21}$$

$$f_{y_o} = - c \frac{\partial v}{\partial t} , \tag{E.22}$$

$$f_{z_o} = - c' \frac{\partial w}{\partial t} , \tag{E.23}$$

where c and c' are the coefficients of viscous damping due to the surrounding fluid, associated with lateral and axial motion of the pipe, respectively, and u,v,w are the displacements of the pipe along the x_o-, y_o- and z_o-axes.

Finally, components of the force resultant per unit length of the pipe center-line can be written as follows:

$$F_{x_o} = -(M_a + M_t) a_{tx_o} - c \frac{\partial u}{\partial t} + R_{x_o} + G_{x_o} , \tag{E.24}$$

$$F_{y_o} = -(M_a + M_t) a_{ty_o} - c \frac{\partial v}{\partial t} + R_{y_o} + G_{y_o} , \tag{E.25}$$

$$F_{z_o} = -(M'_a + M_t) a_{tz_o} - c' \frac{\partial w}{\partial t} + R_{z_o} + G_{z_o} , \tag{E.26}$$

where a_{tx_o}, a_{ty_o}, a_{tz_o} are components of the pipe acceleration; M_a or M'_a is the added mass per unit length, and R_{x_o}, R_{y_o}, R_{z_o} are components of the reaction force arising from the internal fluid.

Subject to the limitation that the dimensions of the cross section of the pipe are small as compared with the overall length of the pipe, the rotatory inertia about axes x and y can be neglected. Therefore, if external moments are absent one obtains

$$\bullet_{x_o} = 0 \; , \tag{E.27}$$

$$\bullet_{y_o} = 0 \; , \tag{E.28}$$

$$\bullet_{z_o} = I_z \frac{\partial^2 \psi}{\partial t^2} \; , \tag{E.29}$$

where I_z is the pipe moment of inertia about the z axis.

Substituting equations (E.24)-(E.29) and the values of L_{ij} obtained in Appendix A into equations (E.11)-(E.16), the equations of motion for the pipe may be obtained, i.e.,

$$\frac{\partial Q_{x_o}}{\partial S} - \tau_o Q_{y_o} + \kappa'_o Q_{z_o} - c\frac{\partial u}{\partial t} + R_{x_o} + G_{x_o} - (M_t + M_a)a_{tx_o} = 0 \; , \tag{E.30}$$

$$\frac{\partial Q_{y_o}}{\partial S} - \kappa_o Q_{z_o} + \tau_o Q_{x_o} - c\frac{\partial v}{\partial t} + R_{y_o} + G_{y_o} - (M_t + M_a)a_{ty_o} = 0, \tag{E.31}$$

$$\frac{\partial Q_{z_o}}{\partial S} - \kappa'_o Q_{x_o} + \kappa_o Q_{y_o} - c\frac{\partial w}{\partial t} + R_{z_o} + G_{z_o} - (M_t + M'_a)a_{tz_o} = 0 \; , \tag{E.32}$$

$$\frac{\partial M_{x_o}}{\partial S} - \tau_o M_{y_o} + \kappa'_o M_{z_o} + \frac{\partial v}{\partial S} Q_{z_o} - Q_{y_o} = 0 \; , \tag{E.33}$$

$$\frac{\partial M_{y_o}}{\partial S} - \kappa_o M_{z_o} + \tau_o M_{x_o} + Q_{x_o} - \left(\frac{\partial u}{\partial S} + \frac{w}{R_o}\right) Q_{z_o} = 0 \; , \tag{E.34}$$

$$\frac{\partial M_{z_o}}{\partial S} - \kappa'_o M_{x_o} + \kappa_o M_{y_o} + \left(\frac{\partial u}{\partial S} + \frac{w}{R_o}\right) Q_{y_o} - \frac{\partial v}{\partial S} Q_{x_o} - I_z \frac{\partial^2 \psi}{\partial t^2} = 0 \; . \tag{E.35}$$

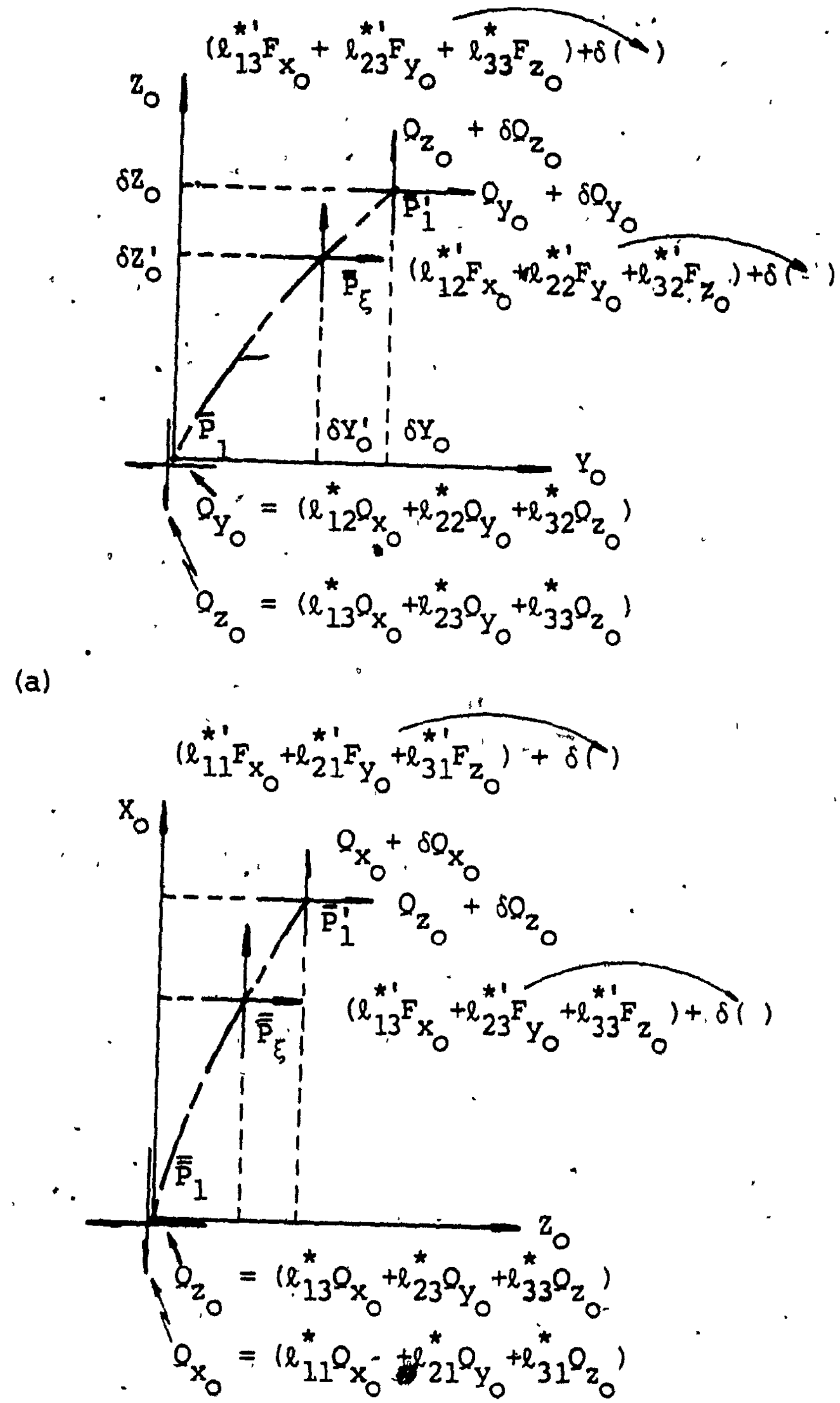


Fig. E.1 Forces in Fig. 2(b) Projected Into the (X_o, Y_o, Z_o)-Axes

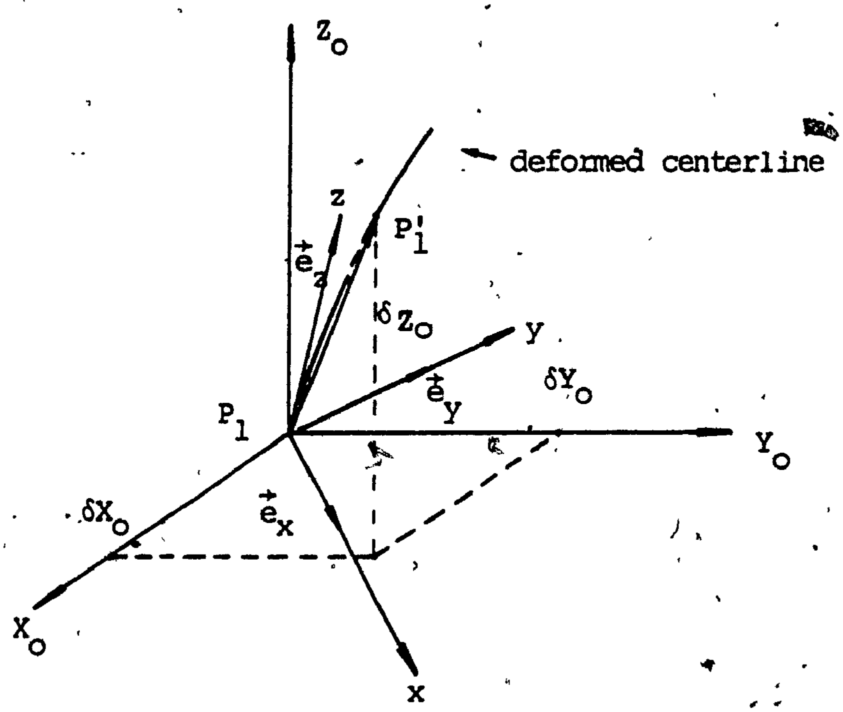


Fig. E.2 Components of $\overrightarrow{P_1P_1'}$ in the Reference Frame (X_o, Y_o, Z_o)

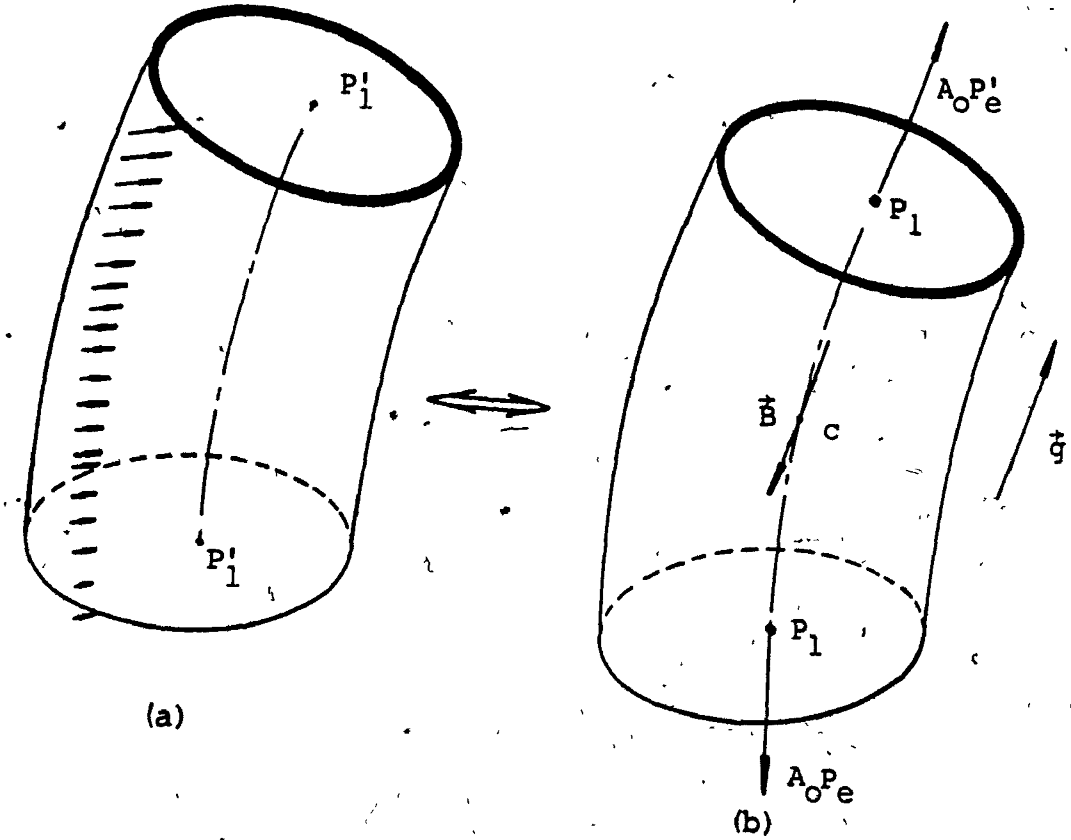


Fig. E.3 (a) Pressure Distribution of the Surrounding Fluid Acting on the External Lateral Surface of the Pipe.

(b) Pressure Distribution Replaced by the Buoyancy Force $\vec{B}$ and the Tensions $\overrightarrow{A_oP_e}$ and $\overrightarrow{A_oP_e'}$.

APPENDIX F

BOUNDARY CONDITION ASSOCIATED WITH A FREE END FOR IN-PLANE INEXTENSIBLE MOTION

From equation (3.2), the relationship of η_1^* and η_3^* at the free end may be written as

$$\left(\frac{\partial^4\eta_1^*}{\partial\xi^4} + r_o\frac{\partial^3\eta_3^*}{\partial\xi^3}\right) + \frac{\partial}{\partial\xi}\left[\Pi\left(\frac{\partial\eta_1^*}{\partial\xi} + r_o\eta_3^*\right)\right] + (1+\beta_a)\frac{\partial^2\eta_1^*}{\partial\tau^2} + 2\beta^{1/2}\bar{u}\left(\frac{\partial^2\eta_1^*}{\partial\tau\partial\xi} + r_o\frac{\partial\eta_3^*}{\partial\tau}\right) + \bar{u}^2\left(\frac{\partial^2\eta_1^*}{\partial\xi^2} + r_o\frac{\partial\eta_3^*}{\partial\xi}\right) + \mathcal{K}\frac{\partial\eta_1^*}{\partial\tau} = 0 \qquad \text{(F.1)}$$

On the other hand, at the free end one has obtained [equation (2.81)] that

$$\left(\frac{\partial^2\eta_1^*}{\partial\xi^2} + r_o\frac{\partial\eta_3^*}{\partial\xi}\right) = 0\ . \qquad \text{(F.2)}$$

Combining equations (F.1) and (F.2) yields

$$\left(\frac{\partial^4\eta_1^*}{\partial\xi^4} + r_o\frac{\partial^3\eta_3^*}{\partial\xi^3}\right) + \frac{\partial}{\partial\xi}\left[\Pi\left(\frac{\partial\eta_1^*}{\partial\xi} + r_o\eta_3^*\right)\right] + (1+\beta_a)\frac{\partial^2\eta_1^*}{\partial\tau^2} + 2\beta^{1/2}\bar{u}\left(\frac{\partial^2\eta_1^*}{\partial\tau\partial\xi} + r_o\frac{\partial\eta_3^*}{\partial\tau}\right) + \mathcal{K}\frac{\partial\eta_1^*}{\partial\tau} = 0\ . \qquad \text{(F.3)}$$

Hence, in the inextensible case, one has a modified boundary condition at a free end, namely

$$\left(\frac{\partial^5\eta_3^*}{\partial\xi^5} + r_o^2\frac{\partial^3\eta_3}{\partial\xi^3}\right) + \frac{\partial}{\partial\xi}\left[\Pi\left(\frac{\partial^2\eta_3^*}{\partial\xi^2} + r_o^2\eta_3^*\right)\right] + (1+\beta_a)\frac{\partial^3\eta_3^*}{\partial\xi\partial\tau^2} + 2\beta^{1/2}\bar{u}\left(\frac{\partial^3\eta_3^*}{\partial\xi^2\partial\tau} + r_o^2\frac{\partial\eta_3^*}{\partial\tau}\right) + \mathcal{K}\frac{\partial^2\eta_3^*}{\partial\xi\partial\tau} = 0 \qquad \text{(F.4)}$$

which in the main text is given as equation (3.14).

APPENDIX G

INTEGRATIONS BY PARTS ASSOCIATED WITH THE DERIVATION OF EQUATION (3.21)

Consider the equation

$$\sum_{i=1}^{n} \int_{o}^{\xi_i} \delta\eta_3^* A_i^*(\eta_3)\, d\xi = 0 , \tag{G.1}$$

where

$$A_i^*(\eta_3^*) = \left(\frac{\partial^6\eta_3^*}{\partial\xi^6} + 2r_o^2 \frac{\partial^4\eta_3^*}{\partial\xi^4} + r_o^4 \frac{\partial^2\eta_3^*}{\partial\xi^2}\right) + \bar{u}^2\left(\frac{\partial^4\eta_3^*}{\partial\xi^4} + 2r_o^2 \frac{\partial^2\eta_3^*}{\partial\xi^2} + r_o^4\eta_3^*\right) + \frac{\partial^2}{\partial\xi^2}\left[\Pi\left(\frac{\partial^2\eta_3^*}{\partial\xi^2} + r_o^2\eta_3^*\right)\right]$$

$$+ r_o^2\Pi\left(\frac{\partial^2\eta_3^*}{\partial\xi^2} + r_o^2\eta_3^*\right) + (1+\beta_a) \frac{\partial^4\eta_3^*}{\partial\tau^2\partial\xi^2} - r_o^2(1+\beta_a') \frac{\partial^2\eta_3^*}{\partial\xi^2} + 2\bar{u}\beta^{1/2}\left(\frac{\partial^4\eta_3^*}{\partial\tau\partial\xi^3} + r_o^2 \frac{\partial^2\eta_3^*}{\partial\tau\partial\xi}\right)$$

$$+ \mathcal{K} \frac{\partial^3\eta_3^*}{\partial\xi^2\partial\tau} - r_o^2\, \mathcal{K}' \frac{\partial\eta_3^*}{\partial\xi} . \tag{G.2}$$

Performing integrations by parts, one obtains the following:

$$\sum_{i=1}^{n} \int_{o}^{\xi_i} \delta\eta_3^*\left(\frac{\partial^6\eta_3^*}{\partial\xi^6} + 2r_o^2 \frac{\partial^4\eta_3^*}{\partial\xi^4} + r_o^4 \frac{\partial^2\eta_3}{\partial\xi^2}\right) d\xi = \sum_{i=1}^{n} \left[\int_{o}^{\xi_i} \delta\eta_3^*\left(\frac{\partial^6\eta_3^*}{\partial\xi^6} + r_o^2 \frac{\partial^4\eta_3^*}{\partial\xi^4}\right) d\xi\right.$$

$$\left. + \int_{o}^{\xi_i} \delta\eta_3^*\, r_o^2\left(\frac{\partial^4\eta_3^*}{\partial\xi^4} + r_o^2 \frac{\partial^2\eta_3^*}{\partial\xi^2}\right) d\xi\right],$$

$$= \sum_{i=1}^{n} \left\{\delta\eta_3^*\left(\frac{\partial^5\eta_3^*}{\partial\xi^5} + 2r_o^2 \frac{\partial^3\eta_3^*}{\partial\xi^3} + r_o^4 \frac{\partial\eta_3^*}{\partial\xi}\right) - \frac{\partial\delta\eta_3^*}{\partial\xi}\left(\frac{\partial^4\eta_3^*}{\partial\xi^4} + r_o^2 \frac{\partial^2\eta_3^*}{\partial\xi^2}\right) + \frac{\partial^2\delta\eta_3^*}{\partial\xi^2}\left(\frac{\partial^3\eta_3^*}{\partial\xi^3} + r_o^2 \frac{\partial\eta_3^*}{\partial\xi}\right)\right\}_{0}^{\xi_i}$$

$$- \sum_{i=1}^{n} \int_{o}^{\xi_i} \left\{\delta\left(\frac{\partial^3\eta_3}{\partial\xi^3} + r_o^2 \frac{\partial\eta_3^*}{\partial\xi}\right)\left(\frac{\partial^3\eta_3^*}{\partial\xi^3} + r_o^2 \frac{\partial\eta_3^*}{\partial\xi}\right)\right\} d\xi,$$

$$= -\sum_{i=1}^{n} \int_{o}^{\xi_i} \left\{\delta\left(\frac{\partial^3\eta_3^*}{\partial\xi^3} + r_o^2 \frac{\partial\eta_3^*}{\partial\xi}\right)\left(\frac{\partial^3\eta_3^*}{\partial\xi^3} + r_o^2 \frac{\partial\eta_3^*}{\partial\xi}\right)\right\} d\xi +$$

$$\left\{\delta\eta_3^*\left(\frac{\partial^5\eta_3^*}{\partial\xi^5} + 2r_o^2 \frac{\partial^3\eta_3^*}{\partial\xi^3} + r_o^4 \frac{\partial\eta_3^*}{\partial\xi}\right) - \frac{\partial\delta\eta_3^*}{\partial\xi}\left(\frac{\partial^4\eta_3^*}{\partial\xi^4} + r_o^2 \frac{\partial^2\eta_3^*}{\partial\xi^2}\right) + \frac{\partial^2\delta\eta_3^*}{\partial\xi^2}\left(\frac{\partial^3\eta_3^*}{\partial\xi^3}\right.\right.$$

$$\left.\left. + r_o^2 \frac{\partial\eta_3^*}{\partial\xi}\right)\right\}_{o}^{1} , \tag{G.3}$$

$$\sum_{i=1}^{n}\int_{0}^{\xi_i}\delta\eta_3^*\left(\frac{\partial^4\eta_3^*}{\partial\xi^4}+2r_o^2\frac{\partial^2\eta_3^*}{\partial\xi^2}+r_o^4\eta_3^*\right)d\xi=\sum_{i=1}^{n}\int_{0}^{\xi_i}\delta\eta_3^*\left(\frac{\partial^4\eta_3^*}{\partial\xi^4}+r_o^2\frac{\partial^2\eta_3^*}{\partial\xi^2}\right)d\xi$$

$$+\sum_{i=1}^{n}\int_{0}^{\xi_i}r_o^2\delta\eta_3^*\left(\frac{\partial^2\eta_3^*}{\partial\xi^2}+r_o^2\eta_3^*\right)d\xi\ ,$$

$$=\sum_{i=1}^{n}\left\{\delta\eta_3^*\left(\frac{\partial^3\eta_3^*}{\partial\xi^3}+r_o^2\frac{\partial\eta_3^*}{\partial\xi}\right)\right\}_0^{\xi_i}-\sum_{i=1}^{n}\int_{0}^{\xi_i}\left[\frac{\partial\delta\eta_3^*}{\partial\xi}\left(\frac{\partial^3\eta_3^*}{\partial\xi^3}+r_o^2\frac{\partial\eta_3^*}{\partial\xi}\right)\right.$$

$$\left.-\ r_o^2\delta\eta_3^*\left(\frac{\partial^2\eta_3^*}{\partial\xi^2}+r_o^2\eta_3^*\right)\right]d\xi\ ,$$

$$=-\sum_{i=1}^{n}\int_{0}^{\xi_i}\left[\frac{\partial\delta\eta_3^*}{\partial\xi}\left(\frac{\partial^3\eta^*}{\partial\xi^3}+r_o^2\frac{\partial\eta_3^*}{\partial\xi}\right)-r_o^2\delta\eta_3^*\left(\frac{\partial^2\eta_3^*}{\partial\xi^2}+r_o^2\eta_3^*\right)\right]d\xi$$

$$+\left\{\delta\eta_3^*\left(\frac{\partial^3\eta_3^*}{\partial\xi^3}+r_o^2\frac{\partial\eta_3^*}{\partial\xi}\right)\right\}_0^1, \tag{G.4}$$

$$\sum_{i=1}^{n}\int_{0}^{\xi_i}\delta\eta_3^*\left(\frac{\partial^4\eta_3^*}{\partial\xi^3\partial\tau}+r_o^2\frac{\partial\eta_3^*}{\partial\xi\partial\tau}\right)d\xi=-\sum_{i=1}^{n}\int_{0}^{\xi_i}\frac{\partial\delta\eta_3^*}{\partial\xi}\left(\frac{\partial^3\eta_3^*}{\partial\xi^2\partial\tau}+r_o^2\frac{\partial\eta_3^*}{\partial\tau}\right)d\xi$$

$$+\left\{\delta\eta_3^*\left(\frac{\partial^3\eta_3^*}{\partial\xi^2\partial\tau}+r_o^2\frac{\partial\eta_3^*}{\partial\tau}\right)\right\}_0^1, \tag{G.5}$$

$$\sum_{i=1}^{n}\int_{0}^{\xi_i}\delta\eta_3^*\left(\frac{\partial^4\eta_3^*}{\partial\xi^2\partial\tau^2}-r_o^2\frac{\partial^2\eta_3^*}{\partial\tau^2}\right)d\xi=-\sum_{i=1}^{n}\int_{0}^{\xi_i}\left(\frac{\partial\delta\eta_3^*}{\partial\xi}\cdot\frac{\partial^3\eta_3^*}{\partial\xi\partial\tau^2}+r_o^2\,\delta\eta_3^*\frac{\partial^2\eta_3^*}{\partial\tau^2}\right)d\xi$$

$$+\left\{\delta\eta_3^*\frac{\partial^3\eta_3^*}{\partial\xi\partial\tau^2}\right\}_0^1, \tag{G.6}$$

$$\sum_{i=1}^{n}\int_{0}^{\xi_i}\delta\eta_3^*\left(\frac{\partial^3\eta_3^*}{\partial\tau\partial\xi^2}-r_o^2\frac{\partial\eta_3^*}{\partial\tau}\right)d\xi=-\sum_{i=1}^{n}\int_{0}^{\xi_i}\left(\frac{\partial\delta\eta_3^*}{\partial\xi}\cdot\frac{\partial^2\eta_3^*}{\partial\xi\partial\tau}+r_o^2\delta\eta_3^*\frac{\partial\eta_3^*}{\partial\tau}\right)d\xi$$

$$+\left\{\delta\eta_3^*\frac{\partial^2\eta^*}{\partial\xi\partial\tau}\right\}_0^1, \tag{G.7}$$

$$\sum_{i=1}^{n}\int_{0}^{\xi_i}\delta\eta_3^*\frac{\partial^2}{\partial\xi^2}\left[\Pi\left(\frac{\partial^2\eta_3}{\partial\xi^2}+r_o^2\eta_3^*\right)\right]d\xi=-\sum_{i=1}^{n}\int_{0}^{\xi_i}\frac{\partial\delta\eta_3^*}{\partial\xi}\frac{\partial}{\partial\xi}\left[\Pi\left(\frac{\partial^2\eta_3^*}{\partial\xi^2}+r_o^2\eta_3^*\right)\right]d\xi$$

$$+\left\{\delta\eta_3^*\frac{\partial}{\partial\xi}\left[\Pi\frac{\partial^2\eta_3^*}{\partial\xi^2}+r_o^2\eta_3^*\right]\right\}_0^1. \tag{G.8}$$

It is noted that in equation (G.3)-(G.8), the following replacement was made

$$\sum_{i=1}^{n} \{\ \}_{o}^{\xi_i} = \{\ \}_{o}^{1}, \tag{G.9}$$

because of the continuity condition (i.e. the value of $\{\ \}^{\xi_{i-1}}$ at the i^{th} node in the $(i-1)^{th}$ element is equal to the value of $\{\ \}_o$ at the same node in the i^{th} element), as shown in Fig. G.1.

Substituting equations (G.3)-(G.8) into equation (G.1) yields

$$\sum_{i=1}^{n}\int_{o}^{\xi_i} [\delta(\frac{\partial^3\eta_3^*}{\partial\xi^3} + r_o^2 \frac{\partial\eta_3^*}{\partial\xi})(\frac{\partial^3\eta_3^*}{\partial\xi^3} + r_o^2 \frac{\partial\eta_3^*}{\partial\xi}) + \bar{u}^2 [\frac{\partial\delta\eta_3^*}{\partial\xi}(\frac{\partial^3\eta_3^*}{\partial\xi^3} + r_o^2 \frac{\partial\eta_3^*}{\partial\xi}) - r_o^2\delta\eta_3^*(\frac{\partial^2\eta_3^*}{\partial\xi^2}$$

$$+ r_o^2\eta_3^*)] + \frac{\partial\delta\eta_3^*}{\partial\xi} \cdot \frac{\partial}{\partial\xi}[\Pi(\frac{\partial^2\eta_3^*}{\partial\xi^2} + r_o^2\eta_3^*)] - r_o^2\Pi\delta\eta_3^*(\frac{\partial^2\eta_3^*}{\partial\xi^2} + r_o^2\eta_3^*) + 2\beta^{1/2}\bar{u}\frac{\partial\delta\eta_3^*}{\partial\xi}(\frac{\partial^3\eta_3^*}{\partial\tau\partial\xi^2}$$

$$+ r_o^2\frac{\partial\eta_3^*}{\partial\tau}) + \mathcal{H}\frac{\partial\delta\eta_3^*}{\partial\xi}\frac{\partial^2\eta_3^*}{\partial\tau\partial\xi} + r_o^2\mathcal{H}'\delta\eta_3^*\frac{\partial\eta_3^*}{\partial\tau} + (1+\beta_a)\frac{\partial\delta\eta_3^*}{\partial\xi}\frac{\partial^3\eta_3^*}{\partial\xi\partial\tau^2} + r_o^2(1+\beta_a')\delta\eta_3^*\frac{\partial^2\eta_3^*}{\partial\tau^2})]d\xi$$

$$+ \delta\eta_3^*\{(\frac{\partial^5\eta_3^*}{\partial\xi^3} + r_o^2\frac{\partial^3\eta_3^*}{\partial\xi^3}) + 2\beta^{1/2}\bar{u}(\frac{\partial^3\eta^*}{\partial\tau\partial\xi^2} + r_o^2\frac{\partial\eta_3^*}{\partial\tau}) + (1+\beta_a)\frac{\partial^3\eta_3^*}{\partial\xi\partial\tau^2} + \frac{\partial^2\eta_3^*}{\partial\xi\partial\tau} + \frac{\partial}{\partial\xi}[\Pi(\frac{\partial^2\eta_3^*}{\partial\xi^2}$$

$$+ r_o^2\eta_3^*)]\}_o^1 - \{\frac{\partial\delta\eta_3^*}{\partial\xi}(\frac{\partial^4\eta_3^*}{\partial\xi^4} + r_o^2\frac{\partial^2\eta_3^*}{\partial\xi^2})\}_o^1 + \{\frac{\partial^2\delta\eta_3^*}{\partial\xi^2}(\frac{\partial^3\eta_3^*}{\partial\xi^3} + r_o^2\frac{\partial\eta_3^*}{\partial\xi})\}_o^1 + \bar{u}^2\{\delta\eta_3^*(\frac{\partial^3\eta_3^*}{\partial\xi^3}$$

$$+ r_o^2\frac{\partial\eta_3^*}{\partial\xi})\}_o^1 + r_o^2\{\delta\eta_3^*(\frac{\partial^2\eta_3^*}{\partial\xi^3} + r_o^2\frac{\partial\eta_3^*}{\partial\xi})\}_o^1 = 0. \tag{G.10}$$

From the boundary conditions, equations (3.10)-(3.14), one can see that the integrated terms vanish. Therefore, one obtains

$$\sum_{i=1}^{n}\int_{o}^{\xi_1}\{[[\delta(\frac{\partial^3\eta_3^*}{\partial\xi^3}+r_o^2\frac{\partial\eta_3^*}{\partial\xi})]\,(\frac{\partial^3\eta_3^*}{\partial\xi^3}+r_o^2\frac{\partial\eta_3^*}{\partial\xi})+\bar{u}^2[\frac{\partial\delta\eta_3^*}{\partial\xi}(\frac{\partial^3\eta_3^*}{\partial\xi^3}+r_o^2\frac{\partial\eta_3^*}{\partial\xi})-r_o^2\delta\eta_3^*(\frac{\partial^2\eta_3^*}{\partial\xi^2}$$

$$+r_o^2\eta_3^*)]+\frac{\partial\delta\eta_3^*}{\partial\xi}\cdot\frac{\partial}{\partial\xi}[\Pi(\frac{\partial^2\eta_3^*}{\partial\xi^2}+r_o^2\eta_3^*)]-r_o^2\Pi\delta\eta_3^*(\frac{\partial^2\eta_3^*}{\partial\xi^2}+r_o^2\eta_3^*)+2\beta^{1/2}\bar{u}\,\frac{\partial\delta\eta_3^*}{\partial\xi}(\frac{\partial^3\eta_3^*}{\partial\tau\partial\xi^2}$$

$$+r_o^2\frac{\partial\eta_3^*}{\partial\tau})+\mathcal{K}\frac{\partial\delta\eta_3^*}{\partial\xi}\cdot\frac{\partial^2\eta_3^*}{\partial\tau\partial\xi}+r_o^2\mathcal{K}\delta\eta_3^*\frac{\partial\eta_3^*}{\partial\tau})+(1+\beta_a)\frac{\partial\delta\eta_3^*}{\partial\xi}\;\frac{\partial^3\eta_3^*}{\partial\xi\partial\tau^2}+r_o^2(1+\beta')\delta\eta_3^*\frac{\partial^2\eta_3^*}{\partial\tau^2}\}\,d\xi=0,\tag{G.11}$$

which is equation (3.21), appearing in the main text.

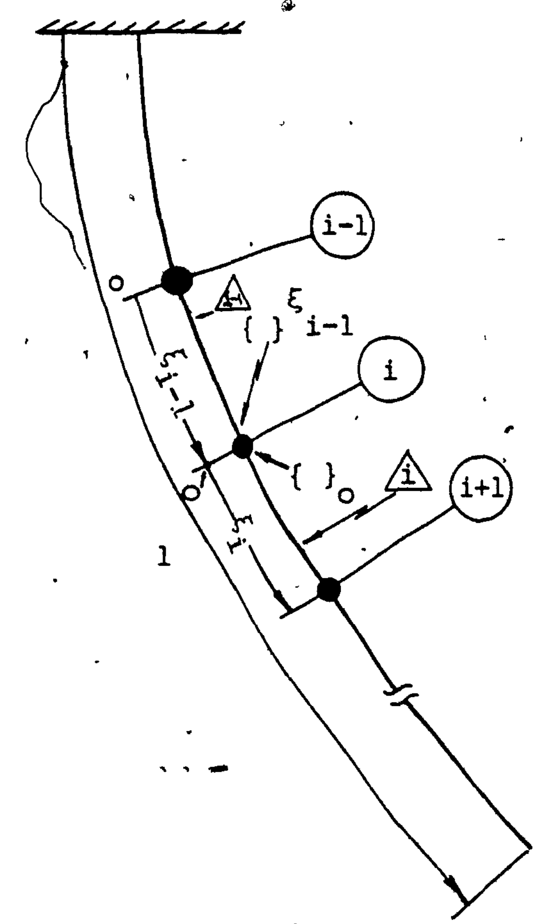


Fig. G.1 Two Finite Elements of the Pipe, Illustrating the Condition of Continuity Across a Node

APPENDIX H

THE INVERSE OF MATRIX [A]; EQUATIONS (3.38) AND (3.40)

Let the square matrix

$$[A] = [a_{ij}] \tag{H.1}$$

of order n and its inverse

$$[B] = [b_{ij}] \tag{H.2}$$

be partitioned into submatrices as indicated below:

$$\left[\begin{array}{c|c} \underset{(p \times p)}{A_{11}} & \underset{(p \times q)}{A_{12}} \\ \hline \underset{(q \times p)}{A_{21}} & \underset{(q \times q)}{A_{22}} \end{array}\right], \tag{H.3}$$

and

$$\left[\begin{array}{c|c} \underset{(p \times p)}{B_{11}} & \underset{(p \times q)}{B_{12}} \\ \hline \underset{(q \times p)}{B_{21}} & \underset{(q \times q)}{B_{22}} \end{array}\right] \tag{H.4}$$

where $$p + q = n \, . \tag{H.5}$$

Because

$$[A] \; [B] = [B] \; [A] = I_n \, , \tag{H.6}$$

where I_n is an identity matrix, one obtains

$$A_{11} \, B_{11} + A_{12} \, B_{21} = I_p, \tag{H.7}$$

$$A_{11} \, B_{12} + A_{12} \, B_{22} = 0 \, , \tag{H.8}$$

$$B_{21} \, A_{11} + B_{22} \, A_{21} = 0 \, , \tag{H.9}$$

$$B_{21} \, A_{12} + B_{22} \, A_{22} = I_q \, . \tag{H.10}$$

then, provided A_{11} is non-singular, one can rewrite equation (H.9) as

$$B_{21} = - B_{22} A_{21} A_{11}^{-1} . \qquad (H.11)$$

Substituting equation (H.11) into equation (H.10) yields

$$B_{22} = (A_{22} - A_{21}A_{11}^{-1} A_{12})^{-1} ; \qquad (H.12)$$

again substituting equation (H.12) into equation (H.8) and (H.11) yields

$$B_{12} = -A_{11}^{-1}A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}, \qquad (H.13)$$

$$B_{21} = -(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} A_{21}A_{11}^{-1}. \qquad (H.14)$$

Finally, combining equation (H.7) and (H.14), one may write

$$B_{11} = A_{11}^{-1} + A_{11}^{-1}A_{12}(A_{22}- A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1}. \qquad (H.15)$$

Hence, one obtains the following expressions for the submatrices of the inverse matrix B:

$$\begin{aligned} B_{11} &= A_{11}^{-1} + A_{11}^{-1}A_{12}\gamma^{-1}A_{21}A_{11}^{-1}, \\ B_{12} &= -A_{11}^{-1}A_{12}\gamma^{-1} , \\ B_{21} &= -\gamma^{-1} A_{21}A_{11}^{-1} , \\ B_{22} &= \gamma^{-1} , \end{aligned} \qquad (H.16)$$

where

$$\gamma = (A_{22} - A_{21}A_{11}^{-1}A_{12}). \qquad (H.17)$$

By applying formulae (H.16), one can invert the matrix

$$[A] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & \xi_e & \xi_e^2 & \xi_e^3 & \xi_e^4 & \xi_e^5 \\ 0 & 1 & 2\xi_e & 3\xi_e^2 & 4\xi_e^3 & 5\xi_e^4 \\ 0 & 0 & 2 & 6\xi_e & 12\xi_e^2 & 20\xi_e^3 \end{bmatrix}, \qquad (H.18)$$

where, as previously defined,

$$A_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \text{(H.19)}$$

$$A_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{(H.20)}$$

$$A_{21} = \begin{bmatrix} 1 & \xi_e & \xi_e^2 \\ 0 & 1 & 2\xi_e \\ 0 & 0 & 2 \end{bmatrix}, \quad \text{(H.21)}$$

$$A_{22} = \begin{bmatrix} \xi_e^3 & \xi_e^4 & \xi_e^5 \\ 3\xi_e^2 & 4\xi_e^3 & 5\xi_e^4 \\ 6\xi_e & 12\xi_e^2 & 20\xi_e^3 \end{bmatrix}, \quad \text{(H.22)}$$

From equations (H.19) and (H.22), one can write

$$A_{11}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}, \quad \text{(H.23)}$$

$$A_{22}^{-1} = \begin{bmatrix} 10\xi_e^{-3} & -4\xi_e^{-2} & \frac{1}{2}\xi_e^{-1} \\ -15\xi_e^{-4} & 7\xi_e^{-3} & -\xi_e^{-2} \\ 6\xi_e^{-5} & -3\xi_e^{-4} & \frac{1}{2}\xi_e^{-3} \end{bmatrix}. \quad \text{(H.24)}$$

Substituting equations (H.19)-(H.24) into equation set (H.16) yields

$$\gamma = A_{22}, \tag{H.25}$$

$$B_{22} = A_{22}^{-1}, \tag{H.26}$$

$$B_{21} = \begin{bmatrix} -10\xi_2^{-3} & -6\xi_e^{-2} & -\frac{3}{2}\xi_e^{-1} \\ 15\xi_e^{-4} & 8\xi_e^{-3} & \frac{3}{2}\xi_e^{-2} \\ -6\xi_e^{-5} & -3\xi_e^{-4} & -\frac{1}{2}\xi_e^{-3} \end{bmatrix}, \tag{H.27}$$

$$B_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{H.28}$$

$$B_{11} = A_{11}^{-1}. \tag{H.29}$$

Hence, the inverse matrix of A is given by

$$[A]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 \\ -10\xi_e^{-3} & -6\xi_e^{-2} & -\frac{3}{2}\xi_e^{-1} & 10\xi_e^{-3} & -4\xi_e^{-2} & \frac{1}{2}\xi_e^{-1} \\ 15\xi_e^{-4} & 8\xi_e^{-3} & \frac{3}{2}\xi_e^{-2} & -15\xi_e^{-4} & 7\xi_e^{-3} & -\xi_e^{-2} \\ -6\xi_e^{-5} & -3\xi_e^{-4} & -\frac{1}{2}\xi_e^{-3} & 6\xi_e^{-5} & -3\xi_e^{-4} & \frac{1}{2}\xi_e^{-3} \end{bmatrix}. \tag{H.30}$$

APPENDIX I

DERIVATION OF THE MATRICES $[J_1]$ - $[J_{13}]$

Consider the vector,

$$[\Phi_3] = [1,\ \xi,\ \xi^2,\ \xi^3,\ \xi^4,\ \xi^5]\ , \qquad \text{(I.1)}$$

clearly,

$$\begin{aligned} [\Phi_3]' &= [0,\ 1,\ 2\xi,\ 3\xi^2,\ 4\xi^3,\ 5\xi^4]\ , \\ [\Phi_3]'' &= [0,\ 0,\ 2,\ 6\xi,\ 12\xi^2,\ 20\xi^3]\ , \qquad \text{(I.2)} \\ [\Phi_3]''' &= [0,\ 0,\ 0,\ 6,\ 24\xi,\ 60\xi^2]\ , \end{aligned}$$

where the prime denotes differentiation with respect to ξ. Therefore, we obtain the following:

$$[J_1] = \int_0^{\xi_e} [\Phi_3]^T [\Phi_3]\ d\xi\ ,$$

$$= \begin{bmatrix} \xi_e & \frac{1}{2}\xi_e^2 & \frac{1}{3}\xi_e^3 & \frac{1}{4}\xi_e^4 & \frac{1}{5}\xi_e^5 & \frac{1}{6}\xi_e^6 \\ \frac{1}{2}\xi_e^2 & \frac{1}{3}\xi_e^3 & \frac{1}{4}\xi_e^4 & \frac{1}{5}\xi_e^5 & \frac{1}{6}\xi_e^6 & \frac{1}{7}\xi_e^7 \\ \frac{1}{3}\xi_e^3 & \frac{1}{4}\xi_e^4 & \frac{1}{5}\xi_e^5 & \frac{1}{6}\xi_e^6 & \frac{1}{7}\xi_e^7 & \frac{1}{8}\xi_e^8 \\ \frac{1}{4}\xi_e^4 & \frac{1}{5}\xi_e^5 & \frac{1}{6}\xi_e^6 & \frac{1}{7}\xi_e^7 & \frac{1}{8}\xi_e^8 & \frac{1}{9}\xi_e^9 \\ \frac{1}{5}\xi_e^5 & \frac{1}{6}\xi_e^6 & \frac{1}{7}\xi_e^7 & \frac{1}{8}\xi_e^8 & \frac{1}{9}\xi_e^9 & \frac{1}{10}\xi_e^{10} \\ \frac{1}{6}\xi_e^6 & \frac{1}{7}\xi_e^7 & \frac{1}{8}\xi_e^8 & \frac{1}{9}\xi_e^9 & \frac{1}{10}\xi_e^{10} & \frac{1}{11}\xi_e^{11} \end{bmatrix}\ , \qquad \text{(I.3)}$$

$$[J_2] = \int_0^{\xi_e} [\Phi_3]'^T [\Phi_3]'\ d\xi\ ,$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \xi_e & \xi_e^2 & \xi_e^3 & \xi_e^4 & \xi_e^5 \\ 0 & \xi_e^2 & \frac{4}{3}\xi_e^3 & \frac{3}{2}\xi_e^4 & \frac{8}{5}\xi_e^5 & \frac{5}{3}\xi_e^6 \\ 0 & \xi_e^3 & \frac{3}{2}\xi_e^4 & \frac{9}{5}\xi_e^5 & 2\xi_e^6 & \frac{15}{7}\xi_e^7 \\ 0 & \xi_e^4 & \frac{8}{5}\xi_e^5 & 2\xi_e^6 & \frac{16}{7}\xi_e^7 & \frac{5}{2}\xi_e^8 \\ 0 & \xi_e^5 & \frac{5}{3}\xi_e^6 & \frac{15}{7}\xi_e^7 & \frac{5}{2}\xi_e^8 & \frac{25}{9}\xi_e^9 \end{bmatrix}\ , \qquad \text{(I.4)}$$

$$[J_3] = \int_0^{\xi_e} [\Phi_3]'''^T[\Phi_3]''' d\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 36\xi_e & 72\xi_e^2 & 120\xi_e^3 \\ 0 & 0 & 0 & 72\xi_e^2 & 192\xi_e^3 & 360\xi_e^4 \\ 0 & 0 & 0 & 120\xi_e^3 & 360\xi_e^4 & 720\xi_e^5 \end{bmatrix}, \quad (I.5)$$

$$[J_4] = \int_0^{\xi_e} [\Phi_3]'^T[\Phi_3]\, d\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \xi_e & \frac{1}{2}\xi_e^2 & \frac{1}{3}\xi_e^3 & \frac{1}{4}\xi_e^4 & \frac{1}{5}\xi_e^5 & \frac{1}{6}\xi_e^6 \\ \xi_e^2 & \frac{2}{3}\xi_e^3 & \frac{1}{2}\xi_e^4 & \frac{2}{5}\xi_e^5 & \frac{1}{3}\xi_e^6 & \frac{2}{7}\xi_e^7 \\ \xi_e^3 & \frac{3}{4}\xi_e^4 & \frac{3}{5}\xi_e^5 & \frac{1}{2}\xi_e^6 & \frac{3}{7}\xi_e^7 & \frac{3}{8}\xi_e^8 \\ \xi_e^4 & \frac{4}{5}\xi_e^5 & \frac{2}{3}\xi_e^6 & \frac{4}{7}\xi_e^7 & \frac{1}{2}\xi_e^8 & \frac{4}{9}\xi_e^9 \\ \xi_e^5 & \frac{5}{6}\xi_e^6 & \frac{5}{7}\xi_e^7 & \frac{5}{8}\xi_e^8 & \frac{5}{0}\xi_e^9 & \frac{1}{2}\xi_e^{10} \end{bmatrix}, \quad (I.6)$$

$$[J_5] = \int_0^{\xi_e} [\Phi_3]'^T[\Phi_3]'' d\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\xi_e & 3\xi_e^2 & 4\xi_e^3 & 5\xi_e^4 \\ 0 & 0 & 2\xi_e^2 & 4\xi_e^3 & 6\xi_e^4 & 8\xi_e^5 \\ 0 & 0 & 2\xi_e^3 & \frac{9}{2}\xi_e^4 & \frac{36}{5}\xi_e^5 & 10\xi_e^6 \\ 0 & 0 & 2\xi_e^4 & \frac{24}{5}\xi_e^5 & 8\xi_e^6 & \frac{80}{7}\xi_e^7 \\ 0 & 0 & 2\xi_e^5 & 5\xi_e^6 & \frac{60}{7}\xi_e^7 & \frac{25}{2}\xi_e^8 \end{bmatrix}, \quad (I.7)$$

$$[J_6] = \int_0^{\xi_e} [\Phi_3]'^T[\Phi_3]'''d\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6\xi_e & 12\xi_e^2 & 20\xi_e^3 \\ 0 & 0 & 0 & 6\xi_e^2 & 16\xi_e^3 & 30\xi_e^4 \\ 0 & 0 & 0 & 6\xi_e^3 & 18\xi_e^4 & 36\xi_e^5 \\ 0 & 0 & 0 & 6\xi_e^4 & \frac{96}{5}\xi_e^5 & 40\xi_e^6 \\ 0 & 0 & 0 & 6\xi_e^5 & 20\xi_e^6 & \frac{300}{7}\xi_e^7 \end{bmatrix}, \tag{I.8}$$

$$[J_7] = \int_0^{\xi_e} [\Phi_3]^T[\Phi_3]''d\xi = \begin{bmatrix} 0 & 0 & 2\xi_e & 3\xi_e^2 & 4\xi_e^3 & 5\xi_e^4 \\ 0 & 0 & \xi_e^2 & 2\xi_e^3 & 3\xi_e^4 & 4\xi_e^5 \\ 0 & 0 & \frac{2}{3}\xi_e^3 & \frac{3}{2}\xi_e^4 & \frac{12}{5}\xi_e^5 & \frac{10}{3}\xi_e^6 \\ 0 & 0 & \frac{1}{2}\xi_e^4 & \frac{6}{5}\xi_e^5 & 2\xi_e^6 & \frac{20}{7}\xi_e^7 \\ 0 & 0 & \frac{2}{5}\xi_e^5 & \xi^6 & \frac{12}{7}\xi_e^7 & \frac{5}{2}\xi_e^8 \\ 0 & 0 & \frac{1}{3}\xi_e^6 & \frac{6}{7}\xi_e^7 & \frac{3}{2}\xi_e^8 & \frac{20}{9}\xi_e^9 \end{bmatrix}, \tag{I.9}$$

$$[J_8] = \int_0^{\xi_e} \xi[\Phi_3]'^T[\Phi_3]'''d\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3\xi_e^2 & 8\xi_e^3 & 15\xi_e^4 \\ 0 & 0 & 0 & 4\xi_e^3 & 12\xi_e^4 & 24\xi_e^5 \\ 0 & 0 & 0 & 4.5\xi_e^4 & \frac{72}{5}\xi_e^5 & 30\xi_e^6 \\ 0 & 0 & 0 & 4.8\xi_e^5 & 16\xi_e^6 & \frac{240}{7}\xi_e^7 \\ 0 & 0 & 0 & 5\xi_e^6 & \frac{120}{7}\xi_e^7 & 37.5\xi_e^8 \end{bmatrix}, \tag{I.10}$$

$$[J_9] = \int_0^{\xi_e} \xi[\Phi_3]^T[\Phi_3]\, d\xi = \begin{bmatrix} \frac{1}{2}\xi_e^2 & \frac{1}{3}\xi_e^3 & \frac{1}{4}\xi_e^4 & \frac{1}{5}\xi_e^5 & \frac{1}{6}\xi_e^6 & \frac{1}{7}\xi_e^7 \\ \frac{1}{3}\xi_e^3 & \frac{1}{4}\xi_e^4 & \frac{1}{5}\xi_e^5 & \frac{1}{6}\xi_e^6 & \frac{1}{7}\xi_e^7 & \frac{1}{8}\xi_e^8 \\ \frac{1}{4}\xi_e^4 & \frac{1}{5}\xi_e^5 & \frac{1}{6}\xi_e^6 & \frac{1}{7}\xi_e^7 & \frac{1}{8}\xi_e^8 & \frac{1}{9}\xi_e^9 \\ \frac{1}{5}\xi_e^5 & \frac{1}{6}\xi_e^6 & \frac{1}{7}\xi_e^7 & \frac{1}{8}\xi_e^8 & \frac{1}{9}\xi_e^9 & \frac{1}{10}\xi_e^{10} \\ \frac{1}{6}\xi_e^6 & \frac{1}{7}\xi_e^7 & \frac{1}{8}\xi_e^8 & \frac{1}{9}\xi_e^9 & \frac{1}{10}\xi_e^{10} & \frac{1}{11}\xi_e^{11} \\ \frac{1}{7}\xi_e^7 & \frac{1}{8}\xi_e^8 & \frac{1}{9}\xi_e^9 & \frac{1}{10}\xi_e^{10} & \frac{1}{11}\xi_e^{11} & \frac{1}{12}\xi_e^{12} \end{bmatrix}, \qquad (I.11)$$

$$[J_{10}] = \int_0^{\xi_e} \xi[\Phi_3]'^T[\Phi_3]'\, d\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\xi_e^2 & \frac{2}{3}\xi_e^3 & \frac{3}{4}\xi_e^4 & \frac{4}{8}\xi_e^5 & \frac{5}{6}\xi_e^6 \\ 0 & \frac{2}{3}\xi_e^3 & \xi_e^4 & \frac{6}{5}\xi_e^5 & \frac{8}{6}\xi_e^6 & \frac{10}{7}\xi_e^7 \\ 0 & \frac{3}{4}\xi_e^4 & \frac{6}{5}\xi_e^5 & \frac{9}{6}\xi_e^6 & \frac{12}{7}\xi_e^7 & \frac{15}{8}\xi_e^8 \\ 0 & \frac{4}{5}\xi_e^5 & \frac{8}{6}\xi_e^6 & \frac{12}{7}\xi_e^7 & \frac{16}{8}\xi_e^8 & \frac{20}{9}\xi_e^9 \\ 0 & \frac{5}{6}\xi_e^6 & \frac{10}{7}\xi_e^7 & \frac{15}{8}\xi_e^8 & \frac{20}{9}\xi_e^9 & \frac{25}{10}\xi_e^{10} \end{bmatrix}, \qquad (I.12)$$

$$[J_{11}] = \int_0^{\xi_e} \xi[\Phi_3]^T[\Phi_3]''\, d\xi = \begin{bmatrix} 0 & 0 & \xi_e^2 & \frac{6}{3}\xi_e^3 & \frac{12}{4}\xi_e^4 & \frac{20}{5}\xi_e^5 \\ 0 & 0 & \frac{2}{3}\xi_e^3 & \frac{6}{4}\xi_e^4 & \frac{12}{5}\xi_e^5 & \frac{20}{6}\xi_e^6 \\ 0 & 0 & \frac{2}{4}\xi_e^4 & \frac{6}{5}\xi_e^5 & \frac{12}{6}\xi_e^6 & \frac{20}{7}\xi_e^7 \\ 0 & 0 & \frac{2}{5}\xi_e^5 & \frac{6}{6}\xi_e^6 & \frac{12}{7}\xi_e^7 & \frac{20}{8}\xi_e^8 \\ 0 & 0 & \frac{2}{6}\xi_e^6 & \frac{6}{7}\xi_e^7 & \frac{12}{8}\xi_e^8 & \frac{20}{9}\xi_e^9 \\ 0 & 0 & \frac{2}{7}\xi_e^7 & \frac{6}{8}\xi_e^8 & \frac{12}{9}\xi_e^9 & \frac{20}{10}\xi_e^{10} \end{bmatrix}, \qquad (I.13)$$

$$[J_{12}] = \int_0^{\xi_e} \xi [\Phi_3]'^T [\Phi_3] \, d\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}\xi_e^2 & \frac{1}{3}\xi_e^3 & \frac{1}{4}\xi_e^4 & \frac{1}{5}\xi_e^5 & \frac{1}{6}\xi_e^6 & \frac{1}{7}\xi_e^7 \\ \frac{2}{3}\xi_e^3 & \frac{2}{4}\xi_e^4 & \frac{2}{5}\xi_e^5 & \frac{2}{6}\xi_e^6 & \frac{2}{7}\xi_e^7 & \frac{2}{8}\xi_e^8 \\ \frac{3}{4}\xi_e^4 & \frac{3}{5}\xi_e^5 & \frac{3}{6}\xi_e^6 & \frac{3}{7}\xi_e^7 & \frac{3}{8}\xi_e^8 & \frac{3}{9}\xi_e^9 \\ \frac{4}{5}\xi_e^5 & \frac{4}{6}\xi_e^6 & \frac{4}{7}\xi_e^7 & \frac{4}{8}\xi_e^8 & \frac{4}{9}\xi_e^9 & \frac{4}{10}\xi_e^{10} \\ \frac{5}{6}\xi_e^6 & \frac{5}{7}\xi_e^7 & \frac{5}{8}\xi_e^8 & \frac{5}{9}\xi_e^9 & \frac{5}{10}\xi_e^{10} & \frac{5}{11}\xi_e^{11} \end{bmatrix}, \tag{I.14}$$

$$[J_{13}] = \int_0^{\xi_e} \xi [\Phi_3]'^T [\Phi_3]'' \, d\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{2}\xi_e^2 & \frac{6}{3}\xi_e^3 & \frac{12}{4}\xi_e^4 & \frac{20}{5}\xi_e^5 \\ 0 & 0 & \frac{4}{3}\xi_e^3 & \frac{12}{4}\xi_e^4 & \frac{24}{5}\xi_e^5 & \frac{40}{6}\xi_e^6 \\ 0 & 0 & \frac{6}{4}\xi_e^4 & \frac{18}{5}\xi_e^5 & \frac{36}{6}\xi_e^6 & \frac{60}{7}\xi_e^7 \\ 0 & 0 & \frac{8}{5}\xi_e^5 & \frac{24}{6}\xi_e^6 & \frac{48}{7}\xi_e^7 & \frac{80}{8}\xi_e^8 \\ 0 & 0 & \frac{10}{6}\xi_e^6 & \frac{30}{7}\xi_e^7 & \frac{60}{8}\xi_e^8 & \frac{100}{9}\xi_e^9 \end{bmatrix} \tag{I.15}$$

APPENDIX J

INTEGRATIONS BY PARTS ASSOCIATED WITH THE DERIVATION OF EQUATION (3.52)

Consider the equation

$$\sum_{i=1}^{n} \int_{o}^{\xi_i} \{\delta\eta_2^* A_{o1}^*(\eta_2^*,\psi^*) + \delta\psi^* A_{o2}^*(\eta_2^*,\psi^*)\}\, d\xi = 0 \ , \qquad (J.1)$$

where

$$A_{o1}^*(\eta_2^*,\psi^*) = \left(\frac{\partial^4\eta_2^*}{\partial\xi^4} - r_o \frac{\partial^2\psi^*}{\partial\xi^2}\right) - \Lambda r_o\left(\frac{\partial^2\psi^*}{\partial\xi^2} + r_o \frac{\partial^2\eta_2^*}{\partial\xi^2}\right) + \frac{\partial}{\partial\xi}\left[\Pi \frac{\partial\eta_2^*}{\partial\xi}\right]$$

$$+ \bar{u}^{-2} \frac{\partial^2\eta_2^*}{\partial\xi^2} + (1+\beta_a) \frac{\partial^2\eta_2^*}{\partial\tau^2} + 2\beta^{1/2}\bar{u} \frac{\partial^2\eta_2^*}{\partial\tau\partial\xi} + \mathcal{K}\frac{\partial\eta_2^*}{\partial\tau} \ , \qquad (J.2)$$

$$A_{o2}^*(\eta_2^*,\psi^*) = r_o\left(r_o\psi^* - \frac{\partial^2\eta_2^*}{\partial\xi^2}\right) - \Lambda\left(\frac{\partial^2\psi^*}{\partial\xi^2} + r_o \frac{\partial^2\eta_2^*}{\partial\xi^2}\right) + \sigma \frac{\partial^2\psi^*}{\partial\tau^2}. \qquad (J.3)$$

Performing integrations by parts yields

$$\sum_{i=1}^{n} \int_{o}^{\xi_i} \delta\eta_2^*\left(\frac{\partial^4\eta_2^*}{\partial\xi^4} - r_o \frac{\partial^2\psi^*}{\partial\xi^2}\right) d\xi = \sum_{i=1}^{n} \left\{\delta\eta_2^*\left(\frac{\partial^3\eta_2^*}{\partial\xi^3} - r_o \frac{\partial\psi^*}{\partial\xi}\right) - \frac{\partial\delta\eta_2^*}{\partial\xi}\left(\frac{\partial^2\eta_2^*}{\partial\xi^2} - r_o\psi^*\right)\right\}_{o}^{\xi_i}$$

$$+ \sum_{i=1}^{n} \int_{o}^{\xi_i} \frac{\partial^2\delta\eta_2^*}{\partial\xi^2}\left(\frac{\partial^2\eta_2^*}{\partial\xi^2} - r_o\psi^*\right) d\xi,$$

$$= \sum_{i=1}^{n} \int_{o}^{\xi_i} \frac{\partial^2\delta\eta_2^*}{\partial\xi^2}\left(\frac{\partial^2\eta_2^*}{\partial\xi^2} - r_o\psi^*\right) d\xi + \left\{\delta\eta_2^*\left(\frac{\partial^3\eta_2^*}{\partial\xi^3} - r_o\frac{\partial\psi^*}{\partial\xi}\right)\right.$$

$$\left. - \frac{\partial\delta\eta_2^*}{\partial\xi}\left(\frac{\partial^2\eta_2^*}{\partial\xi^2} - r_o\psi^*\right)\right\}_{o}^{1} , \qquad (J.4)$$

$$\sum_{i=1}^{n} \int_{o}^{\xi_i} \delta\eta_2^*\left(\frac{\partial^2\psi^*}{\partial\xi^2} + r_o \frac{\partial^2\eta_2^*}{\partial\xi^2}\right) d\xi = - \sum_{i=1}^{n} \int_{o}^{\xi_i} \frac{\partial\delta\eta_2^*}{\partial\xi}\left(\frac{\partial\psi^*}{\partial\xi} + r_o \frac{\partial\eta_2^*}{\partial\xi}\right) d\xi + \sum_{i=1}^{n} \left\{\delta\eta_2^*\left(\frac{\partial\psi^*}{\partial\xi}\right.\right.$$

$$\left.\left. + r_o \frac{\partial\eta_2^*}{\partial\xi}\right)\right\}_{o}^{\xi_i} ,$$

$$= - \sum_{i=1}^{n} \int_{o}^{\xi_i} \frac{\partial\delta\eta_2^*}{\partial\xi}\left(\frac{\partial\psi^*}{\partial\xi} + r_o \frac{\partial\eta_2^*}{\partial\xi}\right) d\xi + \left\{\delta\eta_2^*\left(\frac{\partial\psi^*}{\partial\xi} + r_o \frac{\partial\eta_2^*}{\partial\xi}\right)\right\}_{o}^{1} , \qquad (J.5)$$

$$\sum_{i=1}^{n} \int_{o}^{\xi_i} \partial\psi^* \left(\frac{\partial^2\psi^*}{\partial\xi^2} + r_o \frac{\partial^2\eta_2^*}{\partial\xi^2}\right) d\xi = - \sum_{i=1}^{n} \int_{o}^{\xi_i} \frac{\partial\delta\psi^*}{\partial\xi} \left(\frac{\partial\psi^*}{\partial\xi} + r_o \frac{\partial\eta_2^*}{\partial\xi}\right) d\xi \; +$$

$$+ \left\{\partial\psi^* \left(\frac{\partial\psi^*}{\partial\xi} + r_o \frac{\partial\eta_2^*}{\partial\xi}\right) \right\}_o^1 . \tag{J.6}$$

It is noted that in equations (J.4)-(J.6), the following replacement was made

$$\sum_{i=1}^{n} \{ \quad \}_o^{\xi_i} = \{ \quad \}_o^1 , \tag{J.7}$$

because of the continuity condition: the value of $\{ \quad \}^{\xi_i - 1}$ at the i^{th} node in the $(i-1)^{th}$ element is equal to the value of $\{ \quad \}_o$ at the same node in the i^{th} element, as shown in Fig. G.1.

Substituting equations (J.4)-(J.6) into equation (J.1) yields

$$\sum_{i=1}^{n} \int_{o}^{\xi_i} \left\{ \frac{\partial^2\delta\eta_2^*}{\partial\xi^2} \left(\frac{\partial^2\eta_2^*}{\partial\xi^2} - r_o\psi^* \right) + \Lambda r_o \frac{\partial\delta\eta_2^*}{\partial\xi} \left(\frac{\partial\psi^*}{\partial\xi} + r_o \frac{\partial\eta_2^*}{\partial\xi} \right) + \delta\eta_2^* \left(\frac{\partial\Pi}{\partial\xi} \frac{\partial\eta_2^*}{\partial\xi} + \Pi \frac{\partial^2\eta_2^*}{\partial\xi^2} \right) \right.$$

$$\left. + \bar{u}^2 \delta\eta_2^* \frac{\partial^2\eta_2^*}{\partial\xi^2} + 2\beta^{1/2}\bar{u}\delta\eta_2^* \frac{\partial^2\eta_2^*}{\partial\xi\partial\tau} + \mathcal{K}\delta\eta_2^* \frac{\partial\eta_2^*}{\partial\tau^2} + (1+\beta_a)\delta\eta_2^* \frac{\partial^2\eta_2^*}{\partial\tau^2} \right\} d\xi$$

$$+ \sum_{i=1}^{n} \int_{o}^{\xi_i} \left\{ r_o\delta\psi^* \left(r_o\psi^* - \frac{\partial^2\eta_2^*}{\partial\xi^2} \right) + \Lambda \frac{\partial\delta\psi^*}{\partial\xi} \left(\frac{\partial\psi^*}{\partial\xi} + r_o \frac{\partial\eta_2^*}{\partial\xi} \right) + \sigma\delta\psi^* \frac{\partial^2\psi^*}{\partial\tau^2} \right\} d\xi$$

$$+ \left\{ \delta\eta_2^* \left(\frac{\partial^3\eta_2^*}{\partial\xi^3} - r_o \frac{\partial\psi^*}{\partial\xi} \right) - \frac{\partial\delta\eta_2^*}{\partial\xi} \left(\frac{\partial^2\eta_2^*}{\partial\xi^2} - r_o\psi^* \right) \right\}_o^1 - \left\{ \Lambda r_o \delta\eta_2^* \left(r_o \frac{\partial\eta_2^*}{\partial\xi} + \frac{\partial\psi^*}{\partial\xi} \right) \right\}_o^1$$

$$- \Lambda r_o \left\{ \delta\psi^* \left(\frac{\partial\psi^*}{\partial\xi} + r_o \frac{\partial\eta_2^*}{\partial\xi} \right) \right\}_o^1 = 0 . \tag{J.8}$$

From the boundary conditions (i.e. equations (3.17)-(3.19)), one can see that the integrated terms vanish. Therefore, equation (J.8) may be reduced to the final form

$$\sum_{i=1}^{n} \int_{0}^{\xi_i} \{ \frac{\partial^2 \delta\eta_2^*}{\partial\xi^2} (\frac{\partial^2 \eta_2^*}{\partial\xi^2} - r_o\psi^*) + r_o\delta\psi^* (r_o\psi^* - \frac{\partial^2 \eta_2^*}{\partial\xi^2}) + \Lambda [r_o \frac{\partial\delta\eta_2^*}{\partial\xi} (\frac{\partial\psi^*}{\partial\xi} + r_o \frac{\partial\eta_2^*}{\partial\xi})$$

$$+ \frac{\partial\delta\psi^*}{\partial\xi} (\frac{\partial\psi^*}{\partial\xi} + r_o \frac{\partial\eta_2^*}{\partial\xi})] + \bar{u}^2 \delta\eta_2^* \frac{\partial^2 \eta_2^*}{\partial\xi^2} + \overline{\delta\eta_2^*} (\frac{\partial\Pi}{\partial\xi} \frac{\partial\eta_2^*}{\partial\xi} + \Pi \frac{\partial^2 \eta_2^*}{\partial\xi^2}) + [2\beta^{1/2} \bar{u} \delta\eta_2^* \frac{\partial^2 \eta_2^*}{\partial\tau\partial\xi}$$

$$+ \mathcal{H} \delta\eta_2^* \frac{\partial\eta_2^*}{\partial\tau} + (1+\beta_a)\, \delta\eta_2^* \frac{\partial^2 \eta_2^*}{\partial\tau^2} + \sigma\delta\psi^* \frac{\partial^2 \psi^*}{\partial\tau^2} = 0, \qquad (J.9)$$

which is equation (3.52).

APPENDIX K

THE INVERSE OF MATRIX $[A]_o$; EQUATIONS (3.69) AND (3.71)

Consider the matrix

$$[A]_o = \left[\begin{array}{cc|cccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & \xi_e & \xi_e^2 & \xi_e^3 & 0 & 0 \\ 0 & 1 & 2\xi_e & 3\xi_e^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \xi_e \end{array}\right], \tag{K.1}$$

and partitioning it into submatrices as shown above, one obtains

$$A^*_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{K.2}$$

$$A^*_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{K.3}$$

$$A^*_{21} = \begin{bmatrix} 0 & 0 \\ 1 & \xi_e \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \tag{K.4}$$

$$A^*_{22} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ \xi_e^2 & \xi_e^3 & 0 & 0 \\ 2\xi_e & 3\xi_e^2 & 0 & 0 \\ 0 & 0 & 1 & \xi_e \end{bmatrix}. \tag{K.5}$$

From equations (K.2) and (K.5), one has

$$A^{*-1}_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{K.6}$$

$$A_{22}^{*-1} = \begin{bmatrix} 0 & 3\xi_e^{-2} & -\xi_e^{-1} & 0 \\ 0 & -2\xi_e^{-3} & \xi_e^{-2} & 0 \\ 1 & 0 & 0 & 0 \\ -\xi_e^{-1} & 0 & 0 & \xi_e^{-1} \end{bmatrix} \quad \text{(K.7)}$$

To obtain the inverse of the matrix $[A]_o$, equation set (H.16) derived in Appendix H is applied. Substitution of equations (K.2)-(K.7) into (H.16)-(H.17) yields

$$\gamma = A_{22}^{*} , \quad \text{(K.8)}$$

$$B_{22} = A_{22}^{*-1}, \quad \text{(K.9)}$$

$$B_{21} = -\gamma^{-1} A_{21}^{*} A_{11}^{*-1} \text{ , or} \quad \text{(K.10)}$$

$$B_{21} = - \begin{bmatrix} 0 & 3\xi_e^{-2} & -\xi_e^{-1} & 0 \\ 0 & -2\xi_e^{-3} & \xi_e^{-2} & 0 \\ 1 & 0 & 0 & 0 \\ -\xi_e^{-1} & 0 & 0 & \xi_e^{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & \xi_e \\ 0 & 1 \\ 0 & 0 \end{bmatrix} , \quad \text{(K.11)}$$

$$= \begin{bmatrix} -3\xi_e^{-2} & -2\xi_e^{-1} \\ 2\xi_e^{-3} & \xi_e^{-2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} , \quad \text{(K.12)}$$

$$B_{12} = -A_{11}^{*-1} A_{12}^{*} \gamma^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} , \quad \text{(K.13)}$$

$$B_{11} = A_{11}^{*-1} + A_{11}^{*-1} A_{12}^{*} \gamma^{-1} A_{21}^{*} A_{11}^{*-1} ,$$

$$= A_{11}^{*-1} . \quad \text{(K.14)}$$

Therefore, one obtains the inverse of the matrix $[A]_o$ as

$$[A_o]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -3\xi_e^{-2} & -2\xi_e^{-1} & 0 & 3\xi_e^{-2} & -\xi_e^{-1} & 0 \\ 2\xi_e^{-3} & \xi_e^{-2} & 0 & -2\xi_e^{-3} & \xi_e^{-2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\xi_e^{-1} & 0 & 0 & \xi_e^{-1} \end{bmatrix}, \qquad (K.15)$$

which is equation (3.71) of the main text.

APPENDIX L

DERIVATION OF MATRICES $[J_1^*]$ - $[J_{23}^*]$

Consider the vectors,

$$[\Phi_2] = [1,\ \xi,\ \xi^2,\ \xi^3,\ 0,\ 0]\ ,$$
$$[\Phi_4] = [0,\ 0,\ 0,\ 0,\ 1,\ \xi]\ ; \qquad \text{(L.1)}$$

Clearly,

$$[\Phi_2]' = [0,\ 1,\ 2\xi,\ 3\xi^2,\ 0,\ 0]\ ,$$
$$[\Phi_2]'' = [0,\ 0,\ 2,\ 6\xi,\ 0,\ 0], \qquad \text{(L.2)}$$
$$[\Phi_4]' = [0,\ 0,\ 0,\ 0,\ 0,\ 1]\ ,$$

where the prime denotes differentiation with respect to ξ. Therefore, one obtains the following:

$$[J_1^*] = \int_0^{\xi_e} [\Phi_2]^T[\Phi_2]\, d\xi = \begin{bmatrix} \xi_e & \frac{1}{2}\xi_e^2 & \frac{1}{3}\xi_e^3 & \frac{1}{4}\xi_e^4 & 0 & 0 \\ \frac{1}{2}\xi_e^2 & \frac{1}{3}\xi_e^3 & \frac{1}{4}\xi_e^4 & \frac{1}{5}\xi_e^5 & 0 & 0 \\ \frac{1}{3}\xi_e^3 & \frac{1}{4}\xi_e^4 & \frac{1}{5}\xi_e^5 & \frac{1}{6}\xi_e^6 & 0 & 0 \\ \frac{1}{4}\xi_e^4 & \frac{1}{5}\xi_e^5 & \frac{1}{6}\xi_e^6 & \frac{1}{7}\xi_e^7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(L.3)}$$

$$[J_2^*] = \int_0^{\xi_e} [\Phi_2]'^T[\Phi_2]'\, d\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \xi_e & \xi_e^2 & \xi_e^3 & 0 & 0 \\ 0 & \xi_e^2 & \frac{4}{3}\xi_e^3 & \frac{3}{2}\xi_e^4 & 0 & 0 \\ 0 & \xi_e^3 & \frac{3}{2}\xi_e^4 & \frac{9}{5}\xi_e^5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(L.4)}$$

$$[J_3^*] = \int_0^{\xi_e} [\Phi_2]''^T [\Phi_2]'' d\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4\xi_e & 6\xi_e^2 & 0 & 0 \\ 0 & 0 & 6\xi_e^2 & 12\xi_e^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(L.5)}$$

$$[J_4^*] = \int_0^{\xi_e} [\Phi_4]^T [\Phi_4] \, d\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi_e & \frac{1}{2}\xi_e^2 \\ 0 & 0 & 0 & 0 & \frac{1}{2}\xi_e^2 & \frac{1}{3}\xi_e^3 \end{bmatrix} \quad \text{(L.6)}$$

$$[J_5^*] = \int_0^{\xi_e} [\Phi_2]^T [\Phi_2]' d\xi = \begin{bmatrix} 0 & \xi_e & \xi_e^2 & \xi_e^3 & 0 & 0 \\ 0 & \frac{1}{2}\xi_e^2 & \frac{2}{3}\xi_e^3 & \frac{3}{4}\xi_e^4 & 0 & 0 \\ 0 & \frac{1}{3}\xi_e^3 & \frac{1}{2}\xi_e^4 & \frac{3}{5}\xi_e^5 & 0 & 0 \\ 0 & \frac{1}{4}\xi_e^4 & \frac{2}{5}\xi_e^5 & \frac{1}{2}\xi_e^6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(L.7)}$$

$$[J_6^*] = \int_0^{\xi_e} [\Phi_2]''^T [\Phi_4] \, d\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\xi_e & \xi_e^2 \\ 0 & 0 & 0 & 0 & 3\xi_e^2 & 2\xi_e^3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(L.8)}$$

$$[J_7^*] = \int_0^{\xi_e} [\Phi_2]'^T[\Phi_4]' d\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \xi_e \\ 0 & 0 & 0 & 0 & 0 & \xi_e^2 \\ 0 & 0 & 0 & 0 & 0 & \xi_e^3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (L.9)$$

$$[J_8^*] = \int_0^{\xi_e} [\Phi_4]'^T[\Phi_4]' d\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \xi_e \end{bmatrix} \quad (L.10)$$

$$[J_9^*] = \int_0^{\xi_e} [\Phi_2]^T[\Phi_2]'' d\xi = \begin{bmatrix} 0 & 0 & 2\xi_e & 3\xi_e^2 & 0 & 0 \\ 0 & 0 & \xi_e^2 & 2\xi_e^3 & 0 & 0 \\ 0 & 0 & \frac{2}{3}\xi_e^3 & \frac{3}{2}\xi_e^4 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\xi_e^4 & \frac{6}{5}\xi_e^5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (L.11)$$

$$[J_{10}^*] = \int_0^{\xi_e} \xi [\Phi_2]^T[\Phi_2]'' d\xi = \begin{bmatrix} 0 & 0 & \xi_e^2 & 2\xi_e^3 & 0 & 0 \\ 0 & 0 & \frac{2}{3}\xi_e^3 & \frac{3}{2}\xi_e^4 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\xi_e^4 & \frac{6}{5}\xi_e^5 & 0 & 0 \\ 0 & 0 & \frac{2}{5}\xi_e^5 & \xi_e^6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (L.12)$$

$$[J^*_{11}] = \int_0^{\xi_e} \xi[\Phi_2]^T[\Phi_2]'d\xi = \begin{bmatrix} 0 & \frac{1}{2}\xi_e^2 & \frac{2}{3}\xi_e^3 & \frac{3}{4}\xi_e^4 & 0 & 0 \\ 0 & \frac{1}{3}\xi_e^3 & \frac{1}{2}\xi_e^4 & \frac{3}{5}\xi_e^5 & 0 & 0 \\ 0 & \frac{1}{4}\xi_e^4 & \frac{2}{5}\xi_e^5 & \frac{1}{2}\xi_e^6 & 0 & 0 \\ 0 & \frac{1}{5}\xi_e^5 & \frac{1}{3}\xi_e^6 & \frac{3}{7}\xi_e^7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{L.13}$$

$$[J^*_{12}] = \int_0^{\xi_e} [\Phi_2]^T[\Phi_4]'d\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \xi_e \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\xi_e^2 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3}\xi_e^3 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4}\xi_e^4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{L.14}$$

$$[J^*_{13}] = \int_0^{\xi_e} [\Phi_4]^T[\Phi_2]'d\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \xi_e & \xi_e^2 & \xi_e^3 & 0 & 0 \\ 0 & \frac{1}{2}\xi_e^2 & \frac{2}{3}\xi_e^3 & \frac{3}{4}\xi_e^4 & 0 & 0 \end{bmatrix} \tag{L.15}$$

$$[J^*_{14}] = \int_0^{\xi_e} [\Phi_2]^T[\Phi_4]\, d\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & \xi_e & \frac{1}{2}\xi_e^2 \\ 0 & 0 & 0 & 0 & \frac{1}{2}\xi_e^2 & \frac{1}{3}\xi_e^3 \\ 0 & 0 & 0 & 0 & \frac{1}{3}\xi_e^3 & \frac{1}{4}\xi_e^4 \\ 0 & 0 & 0 & 0 & \frac{1}{4}\xi_e^4 & \frac{1}{5}\xi_e^5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{L.16}$$

$$[J^*_{15}] = \int_0^{\xi_e} [\Phi_2]''^T[\Phi_4]' d\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\xi_e \\ 0 & 0 & 0 & 0 & 0 & 3\xi_e^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(L.17)}$$

$$[J^*_{16}] = \int_0^{\xi_e} \xi[\Phi_2]^T[\Phi_4]' d\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\xi_e^2 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3}\xi_e^3 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4}\xi_e^4 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{5}\xi_e^5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(L.18)}$$

$$[J^*_{17}] = \int_0^{\xi_e} \xi[\Phi_4]^T[\Phi_2]' d\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\xi_e^2 & \frac{2}{3}\xi_e^3 & \frac{3}{4}\xi_e^4 & 0 & 0 \\ 0 & \frac{1}{3}\xi_e^3 & \frac{1}{2}\xi_e^4 & \frac{3}{5}\xi_e^5 & 0 & 0 \end{bmatrix} \quad \text{(L.19)}$$

$$[J^*_{18}] = \int_0^{\xi_e} \xi[\Phi_4]^T[\Phi_4] d\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}\xi_e^2 & \frac{1}{3}\xi_e^3 \\ 0 & 0 & 0 & 0 & \frac{1}{3}\xi_e^3 & \frac{1}{4}\xi_e^4 \end{bmatrix} \quad \text{(L.20)}$$

$$[J^*_{19}] = [J^*_{14}]^T \tag{L.21}$$

$$[J^*_{20}] = \int_o^{\xi_e} [\Phi_4]^T[\Phi_4]'d\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \xi_e \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\xi_e^2 \end{bmatrix} \tag{L.22}$$

$$[J^*_{21}] = \int_o^{\xi_e} \xi[\Phi_4]^T[\Phi_4]'d\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\xi_e^2 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3}\xi_e^3 \end{bmatrix} \tag{L.23}$$

$$[J^*_{22}] = \int_o^{\xi_e} \xi[\Phi_2]'^T[\Phi_4]'d\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\xi_e^2 \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3}\xi_e^3 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{4}\xi_e^4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{L.24}$$

$$[J_{23}^{*}] = \int_{0}^{\xi_e} \xi [\Phi_2]^T [\Phi_4] \, d\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{2}\xi_e^2 & \frac{1}{3}\xi_e^3 \\ 0 & 0 & 0 & 0 & \frac{1}{3}\xi_e^3 & \frac{1}{4}\xi_e^4 \\ 0 & 0 & 0 & 0 & \frac{1}{4}\xi_e^4 & \frac{1}{5}\xi_e^5 \\ 0 & 0 & 0 & 0 & \frac{1}{5}\xi_e^5 & \frac{1}{6}\xi_e^6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (L.25)$$

APPENDIX M

BOUNDARY CONDITIONS ASSOCIATED WITH EQUATION (3.84) FOR A CLAMPED-FREE INCOMPLETE CIRCULAR PIPE IN THE CASE OF INEXTENSIBILITY

Consider a clamped-free incomplete circular pipe conveying fluid, under static equilibrium, as shown in Fig. M.1. According to the dimensionless equation of static equilibrium in the z_o-direction (i.e. equation (2.84) derived in Chapter II), one can obtain the derivative of Π_o at the free end, as follows:

$$\frac{d\Pi_o}{d\xi}\Big|_1 = [r_o(\Pi_o + \bar{u}^2)(\frac{\partial \eta_1^o}{\partial \xi} + r_o\eta_3^o) + r_o(\frac{\partial^3 \eta_1^o}{\partial \xi^3} + r_o\frac{\partial^2 \eta_3^o}{\partial \xi^2}) + \gamma\alpha_{z_o}]\Big|_1 . \tag{M.1}$$

Substituting the boundary condition at the free end given in equation set (2.88), i.e.

$$(\frac{\partial^3 \eta_1^o}{\partial \xi^3} + r_o\frac{\partial^2 \eta_3^o}{\partial \xi^2})\Big|_1 = 0, \tag{M.2}$$

into equation (M.1) and neglecting the effect of gravity, one obtains

$$\frac{d\Pi_o}{d\xi}\Big|_1 = r_o(\Pi_o + \bar{u}^2)(\frac{\partial \eta_1^o}{\partial \xi} + r_o\eta_3^o)\Big|_1 . \tag{M.3}$$

It is noted that the rotation at the free end about the y_o-axis may be written as

$$\delta = (\frac{\partial \eta_1^o}{\partial \xi} + r_o\eta_3^o)\Big|_1 . \tag{M.4}$$

On the other hand, from equations (O.10) and (O.17) of Appendix O, one has

$$\delta = -\frac{1}{r_o^2}(r_o - \sin r_o)(\bar{u}^2 + \Pi_p|_1), \tag{M.5}$$

$$\Pi_o|_o = \Pi_p|_1 \cos r_o - (1 - \cos r_o)\bar{u}^2 . \tag{M.6}$$

Combining equations (M.3), (M.4) and (M.5) yields the value of $d\Pi_o/d\xi$ at the free end

$$\frac{d\Pi_o}{d\xi}\bigg|_1 = -\frac{1}{r_o}(r_o - \sin r_o)(\bar{u}^2 + \Pi_p|_1)^2. \qquad \text{(M.7)}$$

Here, use has been made of the fact that $\Pi_o|_1 = \Pi_p|_1$, since $Q_z = 0$ at the free end. Therefore, the boundary conditions associated with equation (3.84) are as follows:

$$\left.\begin{aligned} \Pi_o|_1 &= \Pi_p|_1, \\ \Pi_o|_o &= \Pi_p|_1 \cos r_o - (1 - \cos r_o)\bar{u}^2, \\ \frac{d\Pi_o}{d\xi}\bigg|_1 &= -\frac{1}{r_o}(r_o - \sin r_o)(\bar{u}^2 + \Pi_p|_1)^2, \end{aligned}\right\} \qquad \text{(M.8)}$$

which is equation set (3.87), appearing in the main text.

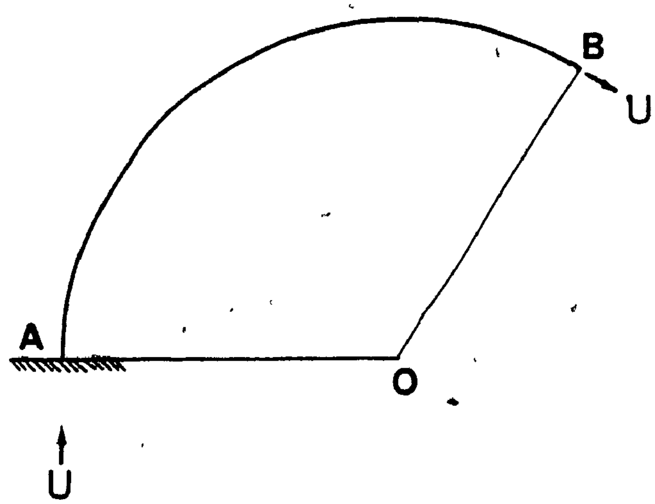


Fig. M.1 A Clamped-Free Incomplete Circular Pipe Conveying Fluid and Submerged in a Quiescent Fluid Under Static Equilibrium

APPENDIX N

DERIVATION OF COMPONENTS ALONG THE x_o AND z_o-AXES OF THE RESULTANT OF FORCES PER UNIT LENGTH ACTING ON AN INCOMPLETE CIRCULAR PIPE UNDER STATIC EQUILIBRIUM

By evaluating the reactions R_{x_o} and R_{z_o} from the internal fluid on the wall of an incomplete circular pipe under static equilibrium, the equations of motion of the internal fluid in this case can be written from equations (2.41) and (2.43) derived in Chapter II, as follows:

$$\left.\begin{aligned} &\frac{\partial}{\partial S}\left[A_iP_i\left(\frac{\partial u_o}{\partial S}+\frac{w_o}{R_o}\right)\right]+\frac{A_iP_i}{R_o}+R_{x_o}-G_{fx_o}+M_f a_{fx_o}=0 \\ &\frac{\partial}{\partial S}(A_iP_i)-\frac{1}{R_o}A_iP_i\left(\frac{\partial u_o}{\partial S}+\frac{w_o}{R_o}\right)+R_{z_o}-G_{fz_o}+M_f a_{z_o}=0 \end{aligned}\right\} \quad (N.1)$$

where

$$\left.\begin{aligned} a_{fx_o} &= U^2\left(\frac{1}{R_o}+\frac{\partial^2 u_o}{\partial S^2}+\frac{1}{R_o}\frac{\partial w_o}{\partial S}\right), \\ a_{fz_o} &= -\frac{U^2}{R_o}\left(\frac{\partial u_o}{\partial S}+\frac{w_o}{R_o}\right), \end{aligned}\right\} \quad (N.2)$$

are the time-independent components of equations (2.17) and (2.19).

In order to evaluate the approximate reactions at the ends of an incomplete circular pipe and according to the assumption of small deformations, one may use the undeformed centerline of the pipe as an approximate shape of the strained centerline of the pipe after static deformation. Therefore, equation set (N.1) is reduced to

$$\left.\begin{aligned} &A_iP_i/R_o+R_{x_o}-G_{fx_o}+M_fU^2/R_o=0, \\ &\frac{\partial(A_iP_i)}{\partial S}+R_{z_o}-G_{fz_o}=0. \end{aligned}\right\} \quad (N.3)$$

Hence, one obtains the reactions R_{x_o} and R_{z_o} per unit length on the wall of an incomplete circular pipe under static equilibrium due to the internal fluid, as follows:

$$\left.\begin{aligned} R_{x_o} &= -\frac{1}{R_o}(A_iP_i + M_fU^2) + G_{fx_o}, \\ R_{z_o} &= -\frac{d}{dS}(A_iP_i) + G_{fz_o}. \end{aligned}\right\} \qquad (N.4)$$

If it is assumed that the pressure distribution of the internal fluid is linear along the length of the pipe (i.e., assuming a constant pressure drop per unit length), one may write

$$A_iP_i = A_iP_i|_L + \left|\frac{d(A_iP_i)}{ds}\right| R_o\,\theta . \qquad (N.5)$$

Moreover, the pressure-distribution of the surrounding fluid acting on the external lateral surface of the pipe may be replaced by the buoyancy force and the tensions $A_oP_e|_o$ and $A_oP_e|_L$ applied on the faces at the ends of the pipe, as shown in Fig. N.1.

Finally, components along the x_o- and z_o-axes of the resultant for forces per unit length acting on an incomplete circular pipe under static equilibrium may be obtained as follows:

$$\left.\begin{aligned} q_{x_o} &= -\frac{1}{R_o}\left(A_iP_i|_L + M_fU^2 + \left|\frac{d(A_iP_i)}{dS}\right| R_o\theta\right) + G^*_{x_o}, \\ q_{z_o} &= -\frac{d(A_iP_i)}{dS} + G^*_{z_o}, \end{aligned}\right\} \qquad (N.6)$$

where $G^*_{x_o}$ and $G^*_{z_o}$ are defined by equation set (2.57) in Chapter II.

For convenience, we set

$$\left.\begin{aligned} q_c &= \frac{1}{R_o}(A_iP_i|_L + M_fU^2), \\ q_s &= \left|\frac{d(A_iP_i)}{dS}\right|, \end{aligned}\right\} \qquad (N.7)$$

which are constants. Therefore, we can rewrite components q_{x_o} and q_{z_o} in another form,

$$\left.\begin{aligned} q_{x_o} &= -(q_c + q_s\theta) + G^*_{x_o} , \\ q_{z_o} &= q_s + G^*_{z_o} . \end{aligned}\right\} \qquad (N.8)$$

Because the coordinate system has been chosen, then the resultant of forces per unit length acting on an incomplete circular pipe under static equilibrium condition is as shown in Fig. N.1.

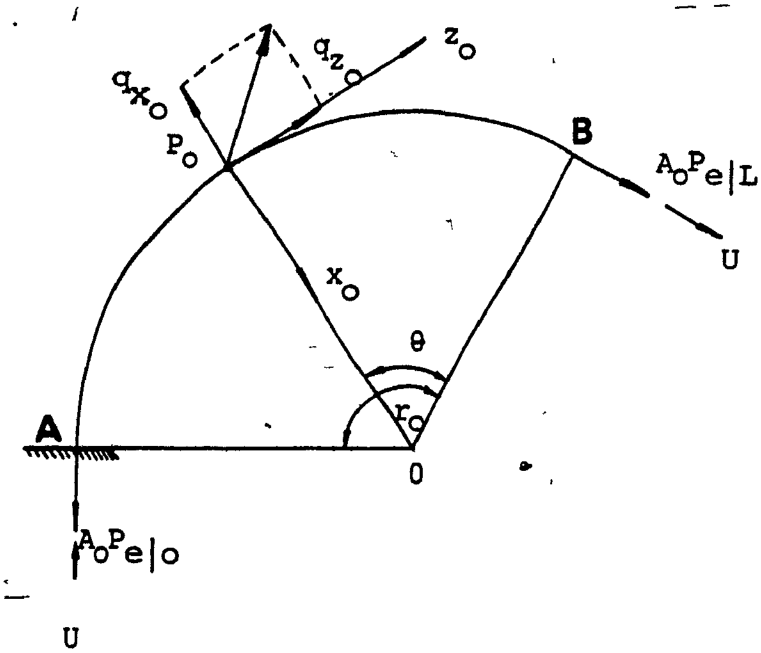


Fig. N.1 Forces Acting on the Incomplete Circular Pipe Under Static Equilibrium

APPENDIX O

CALCULATION OF THE ROTATION AT THE FREE END AND THE AXIAL FORCE AT THE CLAMPLED END FOR A CLAMPED-FREE INCOMPLETE CIRCULAR PIPE

Consider a clamped-free incomplete circular pipe under static equilibrium as shown in Fig. O.1. The resultant of forces per unit length acting on it has been derived in Appendix N. If the effect of gravity is neglected, the components along the x_o- and z_o-axes of this resultant of forces can be written as follows:

$$\left.\begin{aligned} q_{x_o} &= -(q_c + q_s(\theta - \theta^*)), \\ q_{z_o} &= q_s , \end{aligned}\right\} \qquad (O.1)$$

where

$$\left.\begin{aligned} q_c &= \frac{1}{R_o}(A_iP_i|_L + M_fU^2), \\ q_s &= \left|\frac{d(A_iP_i)}{ds}\right| , \end{aligned}\right\} \qquad (O.2)$$

and θ and θ^* are the angles defined in Fig. O.1.

To obtain the rotation at the free end using Castigliano's theorem, we place a moment M_o around the y_o-axis as shown in Fig. O.1, which is finally going to be set equal to zero. In this case the bending moment around the y_o-axis at any cross-section C is given by

$$M_{y_o}|_C = M_o + R_oA_oP_e|_L\,(1-\cos\theta) + \int_o^\theta q_{z_o}\,R_o^2(1-\cos\theta^*)d\theta^* + \int_o^\theta q_{x_o}\,R_o^2\sin\theta^*\,d\theta^* . \qquad (O.3)$$

Substituting equation set (O.1) into equation (O.3) yields

$$M_{y_o}\Big|_C = M_o + R_o A_o P_e\Big|_L (1-\cos\theta) + \int_o^\theta q_s R_o^2 (1-\cos\theta^*) d\theta^*$$

$$- \int_o^\theta R_o^2 [q_c + q_s (\theta-\theta^*)] \sin\theta^* \, d\theta^* \qquad (O.4)$$

Therefore, by evaluating the above integrals, one obtains the bending moment $M_{y_o}\Big|_C$ as

$$M_{y_o}\Big|_C = M_o + R_o (A_o P_e\Big|_L - q_c R_o)(1 - \cos\theta) \ . \qquad (O.5)$$

The total strain energy of the pipe is

$$\bar{U} = \frac{1}{2EI} \int_o^\theta M_{y_o}^2\Big|_C R_o \, d\theta \ , \qquad (O.6)$$

where r_o is the total angle subtended by the incomplete ring (Fig. O.1). According to the Castigliano theorem, the rotation of the cross-section of the pipe at the free end about the y_o-axis can be written as

$$\delta = \frac{\partial\bar{U}}{\partial M_o} \ . \qquad (O.7)$$

Combining equations (O.5), (O.6) and (O.7), and evaluating the integral, one obtains the rotation

$$\delta = \frac{R_o}{EI} [M_o r_o + R_o (A_o P_e\Big|_L - R_o q_c)(r_o - \sin r_o)] \ . \qquad (O.8)$$

Now, setting M_o equal to zero (since no moment is actually applied at the free end) and substituting equation set (O.2) into (O.8) yields

$$\delta = -\frac{R_o^2}{EI} (M_f U^2 + A_i P_i\Big|_L - A_o P_e\Big|_L)(r_o - \sin r_o) \qquad (O.9)$$

or

$$\bar{\delta} = -\frac{1}{r_o} (\bar{u}^2 + \Pi_p\Big|_1)(r_o - \sin r_o) \ . \qquad (O.10)$$

To obtain the axial force Q_{z_o} at the clamped end, a force balance along the Az'_o - axis yields

$$Q_{z_o}\Big|_o = -A_oP_e\big|_o + A_oP_e\big|_L \cos r_o + \int_0^{r_o} q_{z_o} \cos(\overrightarrow{Az'_o}, \overrightarrow{P_oz_o}) R_o d\bar{\theta}$$

$$+ \int_0^{r_o} - q_{x_o} \cos(\overrightarrow{Az'_o}\ \overrightarrow{OP_o}) R_o\, d\bar{\theta}\ , \qquad (0.11)$$

and from Fig. 0.1 one can write

$$\left.\begin{aligned} \cos(\overrightarrow{Az_o}, \overrightarrow{OP_o}) &= \sin(r_o - \bar{\theta})\ , \\ \cos(\overrightarrow{Az'_o}, \overrightarrow{P_oz_o}) &= \cos(r_o - \bar{\theta})\ , \\ \bar{\theta} &= \theta - \theta^*. \end{aligned}\right\} \qquad (0.12)$$

Combining equations (0.11), (0.12) and (0.1) and evaluating the integrals, one obtains the axial force

$$Q_{z_o}\Big|_o = -A_oP_e\big|_o + A_oP_e\big|_L \cos r_o + R_oq_c'(1-\cos r_o) + r_oR_oq_s \qquad (0.13)$$

On the other hand, we note that the value of A_iP_i at the clamped end can be written as

$$A_iP_i\big|_o = A_iP_i\big|_L + q_s\ (r_oR_o). \qquad (0.14)$$

Finally, combining equations (0.2), (0.13) and (0.14) yields

$$Q_{z_o}\Big|_o = (A_iP_i - A_oP_e)\big|_o - (A_iP_i - A_oP_e)\big|_L \cos r_o + M_fU^2\ (1-\cos r_o), \qquad (0.15)$$

and the combined force at this ends is given by

$$P\big|_o = +(A_iP_i - A_oP_e)\big|_L \cos r_o - M_fU^2\ (1 - \cos r_o). \qquad (0.16)$$

Therefore, the dimensionless combined force at $\xi = 0$ is

$$\Pi_o\big|_o = \Pi_p\big|_1 \cos r_o - \bar{u}^2\ (1 - \cos r_o). \qquad (0.17)$$

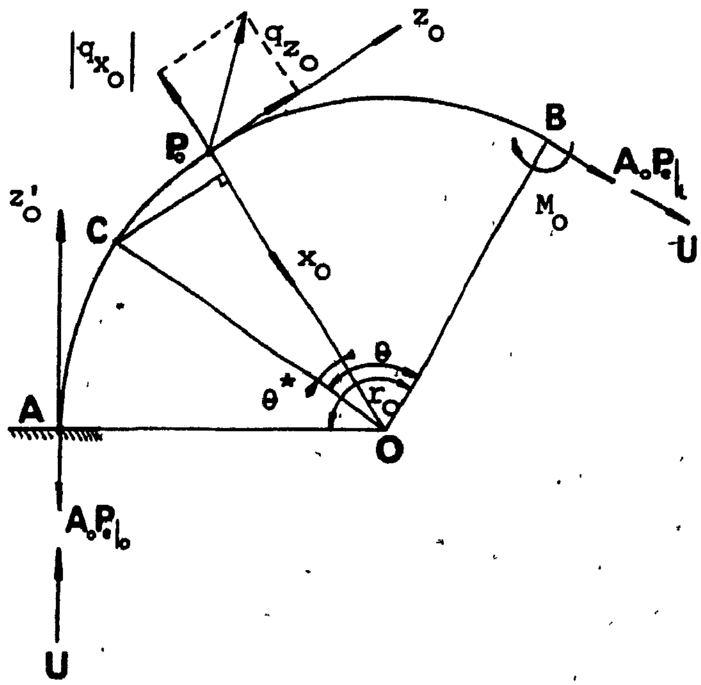


Fig. 0.1 Forces Acting on the Clamped-Free Incomplete Pipe
(Moment M_o is a virtual moment)

APPENDIX P

BOUNDARY CONDITIONS ASSOCIATED WITH EQUATION (3.84) FOR A CLAMPED-CLAMPED INCOMPLETE CIRCULAR PIPE UNDER STATIC EQUILIBRIUM CONDITION

Consider a clamped-clamped incomplete circular pipe conveying fluid under static equilibrium, as shown in Fig. P.1. According to Appendix M (Eq. M.1), the derivative of Π_o at the end ($\xi = 1$) can be written as

$$\frac{d\Pi_o}{d\xi}\Big|_1 = [r_o(\Pi_o + \bar{u}^2)(\frac{\partial \eta_1^o}{\partial \xi} + r_o\eta_3^o) + r_o(\frac{\partial^3 \eta_1^o}{\partial \xi^2} + r_o\frac{\partial^2 \eta_3^o}{\partial \xi^2}) + \gamma\alpha_{z_o}]\Big|_1 . \qquad \text{(P.1)}$$

Substituting the boundary condition at the clamped end ($\xi = 1$) given in equation set (2.86), i.e.,

$$(\frac{\partial \eta_1^o}{\partial \xi} + r_o\eta_3^o) = 0 , \qquad \text{(P.2)}$$

into equation (P.1), and neglecting the effect of gravity, one obtains

$$\frac{d\Pi_o}{d\xi}\Big|_1 = r_o(\frac{\partial^3 \eta_1^o}{\partial \xi^3} + r_o\frac{\partial^2 \eta_3^o}{\partial \xi^2}) . \qquad \text{(P.3)}$$

It is noted that equation (2.51) in Chapter II can be written in dimensionless form as

$$\frac{L^2 Q_x}{EI} = -(\frac{\partial^3 \eta_1^o}{\partial \xi^3} + r_o\frac{\partial^2 \eta_3^o}{\partial \xi^2}) . \qquad \text{(P.4)}$$

On the other hand, based on the small-deformation assumption, we may use the initial centerline of the pipe as an approximate shape of the deformed centerline after static deformation. Then, we obtain the forces Q_x and Q_z at the clamped end ($\xi = 1$), as follows:

$$\left.\begin{aligned} Q_z|_L &= M_f U^2 + (A_i P_i - A_o P_e)|_L \,, \\ Q_x|_L &= 0. \end{aligned}\right\} \qquad (P.5)$$

as shown in Appendix Q.

By combining equations (P.3), (P.4) and (P.5), one obtains the derivative of Π_o at the end $\xi = 1$

$$\frac{d\Pi_o}{d\xi}\Big|_1 = 0 \,. \qquad (P.6)$$

Hence, the boundary conditions associated with equation (3.84) for a clamped-clamped incomplete circular pipe are as follows:

$$\left.\begin{aligned} \Pi_o|_1 &= -\bar{u}^2 \,, \\ \frac{d\Pi_o}{d\xi}\Big|_1 &= 0 \,, \end{aligned}\right\} \qquad (P.7)$$

which is equation set (3.88), appearing in the main text, and which applies when gravity is neglected.

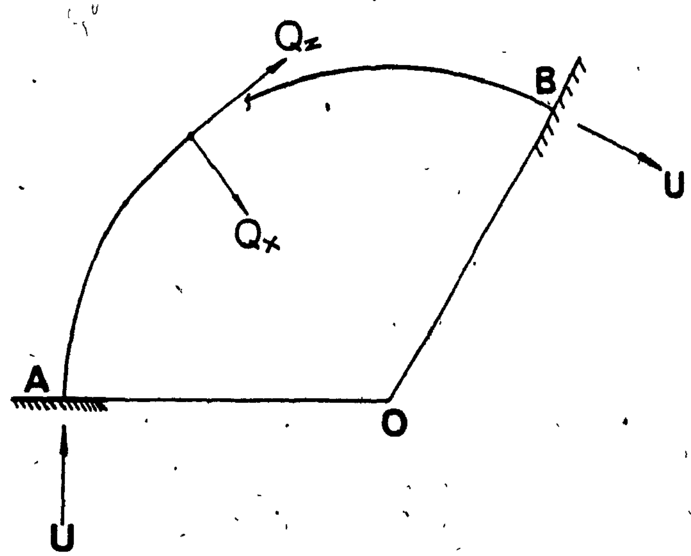


Fig. P.1 A Clamped-Clamped Incomplete Pipe Conveying Fluid Under Static Equilibrium

APPENDIX Q

CALCULATION OF THE FORCES Q_x AND Q_z AT ONE END OF A CLAMPED-CLAMPED INCOMPLETE CIRCULAR PIPE (INEXTENSIBLE CASE)

Consider the free-body diagram of a clamped-clamped incomplete circular pipe conveying fluid, as shown in Fig. Q.1. The forces and moments acting on it consist of the reactions applied at the ends and the resultant of forces per unit length acting on the pipe, which is derived in Appendix N.

If the effect of gravity is assumed to be negligible, one may write the components along the x_o- and z_o-axes of this resultant of forces as

$$\left.\begin{aligned} q_{x_o} &= -(q_c + q_s\,(\theta - \theta^*)) , \\ q_{z_o} &= q_s , \end{aligned}\right\} \qquad (Q.1)$$

where

$$\left.\begin{aligned} q_c &= \frac{1}{R_o}\,(A_i P_i|_L + M_f U^2) , \\ q_s &= \left|\frac{d(A_i P_i)}{dS}\right| , \end{aligned}\right\} \qquad (Q.2)$$

and θ and θ^* are defined in Fig. Q.1.

From Fig. Q.1 the bending moment around the y_o-axis at any cross-section c is given by

$$M_{y_o|c} = M_o + R_o(Q_z + A_o P_e)|_L (1-\cos\theta) + R_o Q_x|_L \sin\theta + \int_0^\theta q_{z_o} R_o^2 (1-\cos\theta^*)\,d\theta^* + \int_0^\theta q_{x_o} R_o^2 \sin\theta^*\,d\theta^* . \qquad (Q.3)$$

Substituting equation set (Q.1) into (Q.3) and evaluating the integrals yields the moment

$$M_{y_o}\Big|_c = M_o + R_o(Q_z + A_oP_e)\big|_L (1-\cos\theta) + R_oQ_x\big|_L \sin\theta - R_o^2\, q_c\,(1-\cos\theta). \qquad (Q.4)$$

Therefore, the total strain energy of the pipe can be written as

$$\bar{U} = \frac{1}{2EI}\int_0^{r_o} M^2_{y_o}\Big|_c R_o\, d\theta \qquad (Q.5)$$

According to the Castigliano theorem, and the conditions at the clamped end (i.e., the deflection and the slope of the deflection curve must be zero), one obtains the following:

$$\left.\begin{aligned} \frac{\partial\bar{U}}{\partial M_o} &= 0 , \\ \frac{\partial\bar{U}}{\partial Q_{z|L}} &= 0, \\ \frac{\partial\bar{U}}{\partial Q_{x|L}} &= 0. \end{aligned}\right\} \qquad (Q.6)$$

Combining equations (Q.4), (Q.5) and (Q.6) yields

$$\left.\begin{aligned} &\int_0^{r_o} [M_o + R_o(Q_z + A_oP_e)\big|_L (1-\cos\theta) + R_oQ_x\big|_L \sin\theta - q_cR_o^2(1-\cos\theta)]d\theta = 0, \\ &\int_0^{r_o} [M_o + R_o(Q_z + A_oP_e)\big|_L (1-\cos\theta) + R_oQ_x\big|_L \sin\theta - R_o^2q_c(1-\cos\theta)] \\ &\qquad (1-\cos\theta)d\theta = 0 , \\ &\int_0^{r_o} [M_o + R_o(Q_z + A_oP_e)\big|_L (1-\cos\theta) + R_oQ_x\big|_L \sin\theta - R_o^2q_c(1-\cos\theta)] \\ &\qquad \sin\theta\, d\theta = 0. \end{aligned}\right\} \qquad (Q.7)$$

Then, evaluating the integrals, one obtains

$$\left.\begin{aligned} r_oM_o + aR_o(Q_z+A_oP_e)\big|_L + bR_oQ_x\big|_L - aR_o^2\,q_c &= 0\ , \\ aM_o + cR_o(Q_z+A_oP_e)\big|_L + dR_oQ_x\big|_L - cR_o^2\,q_c &= 0\ , \\ bM_o + dR_o(Q_z+A_oP_e)\big|_L + eR_oQ_x\big|_L - dR_o^2\,q_c &= 0\ , \end{aligned}\right\} \qquad (Q.8)$$

where

$$\left.\begin{aligned} a &= (r_o - \sin r_o)\ , \\ b &= (1 - \cos r_o)\ , \\ c &= (\tfrac{3}{2}\, r_o - 2\sin r_o + \tfrac{1}{4}\sin\,(2r_o))\ , \\ d &= (1 - \cos r_o - \tfrac{1}{2}\sin^2 r_o)\ , \\ e &= \tfrac{1}{2}\,(r_o - \tfrac{1}{2}\sin\,(2\,r_o))\ . \end{aligned}\right\} \qquad (Q.9)$$

By solving equation set (Q.3), one obtains the reactions, as follows:

$$\left.\begin{aligned} Q_x\big|_L &= 0\ , \\ Q_z\big|_L &= -A_oP_e\big|_L + R_o\,q_c\ , \\ M_o &= 0\ , \end{aligned}\right\} \qquad (Q.10)$$

Therefore, combination of equations (Q.2) and (Q.10) yields

$$\left.\begin{aligned} Q_x\big|_L &= 0\ , \\ Q_z\big|_L &= M_fU^2 + (A_iP_i - A_oP_e)\big|_L\ , \\ M_o &= 0\ . \end{aligned}\right\} \qquad (Q.11)$$

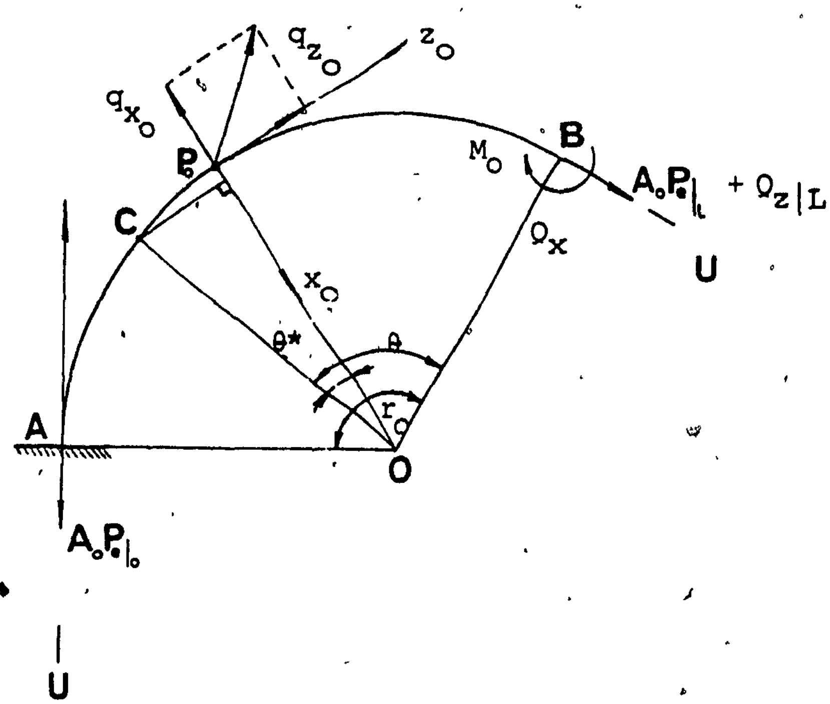


Fig. Q.1 Forces and Moments Acting on a Clamped-Clamped Incomplete Circular Pipe (here the pipe is cut at the end (ξ = 1) to show the reaction forces and moments acting on it)

APPENDIX R

BOUNDARY CONDITIONS ASSOCIATED WITH EQUATION (3.84) FOR CLAMPED-PINNED OR PINNED-PINNED INCOMPLETE CIRCULAR PIPES UNDER STATIC EQUILIBRIUM CONDITION

Consider a clamped-pinned incomplete pipe conveying fluid under static equilibrium, as shown in Fig. R.1. Using equations (M.1), (M.4) and (2.51), the derivative of Π at the pinned end, $\xi = 1$, can be written as

$$\frac{d\Pi_o}{d\xi}\Big|_1 = [r_o(\Pi_o + \bar{u}^2)\delta - r_o \frac{L^2 Q_x}{EI}\Big|_L] , \qquad \text{(R.1)}$$

where δ and $Q_x|_L$ are the rotation and the shearing force in the x-direction at the pinned end. The effect of gravity has been neglected.

From Fig. R.1 and Appendix Q the moment around the y_o-axis at any cross-section c is given by

$$M_{y_o}\Big|_c = R_o(Q_z+A_oP_e)\Big|_L(1-\cos\theta) + R_oQ_x\Big|_L \sin\theta - R_o^2 q_c(1-\cos\theta), \qquad \text{(R.2)}$$

where in this case $M_o = 0$ because the end is pinned.

The total strain energy of the pipe is

$$\bar{U} = \frac{1}{2EI}\int_o^{r_o} M_{y_o}^2\Big|_c R_o \, d\theta . \qquad \text{(R.3)}$$

Using Castigliano's theorem and the conditions at the pinned end (i.e. the deflection must vanish) one obtains as follows:

$$\left.\begin{aligned} \frac{\partial\bar{U}}{\partial Q_z|_L} &= 0 , \\ \frac{\partial\bar{U}}{\partial Q_x|_L} &= 0 . \end{aligned}\right\} \qquad \text{(R.4)}$$

By combining equations (R.2), (R.3) and (R.4) and evaluating the integrals, one obtains

$$\left.\begin{aligned} &cR_o(Q_z+A_oP_e)|_L + dR_oQ_x|_L - cR_o^2\, q_c = 0 \ , \\ &dR_o(Q_z+A_oP_e)|_L + eR_oQ_x|_L - dR_o^2\, q_c = 0 \ , \end{aligned}\right\} \qquad \text{(R.5)}$$

where

$$\left.\begin{aligned} c &= (\tfrac{3}{2}\, r_o - 2 \sin r_o + \tfrac{1}{4} \sin (2r_o)) \ , \\ d &= (1 - \cos r_o - \tfrac{1}{2} \sin^2 r_o) \ , \\ e &= \tfrac{1}{2}\, (r_o - \tfrac{1}{2} \sin (2r_o)) \ . \end{aligned}\right\} \qquad \text{(R.6)}$$

From equation set (R.5), we obtain the forces Q_x and Q_z at the pinned end

$$\left.\begin{aligned} &Q_x|_L = 0 \ , \\ &Q_z|_L = M_fU^2 + (A_iP_i - A_oP_e)|_L \ . \end{aligned}\right\} \qquad \text{(R.7)}$$

To obtain the rotation at the pinned end ($\xi = 1$) we place a fictitious moment M_o around the y_o-axis at this end. In this case the moment around the y_o-axis at any cross-section c is given by

$$M_{y_o}\Big|_c = M_o + R_o(Q_z+A_oP_e)|_L(1-\cos\theta)+R_oQ_x|_L \sin\theta - R_o^2q_c(1-\cos\theta) \qquad \text{(R.8)}$$

The total strain energy of the pipe is again give by

$$\bar{U} = \frac{1}{2EI} \int_o^{r_o} M^2_{y_o|c}\, R_o\, d\theta \ , \qquad \text{(R.9)}$$

and the rotation at the pinned end

$$\delta = \frac{\partial \bar{U}}{\partial M_o} \ . \qquad \text{(R.10)}$$

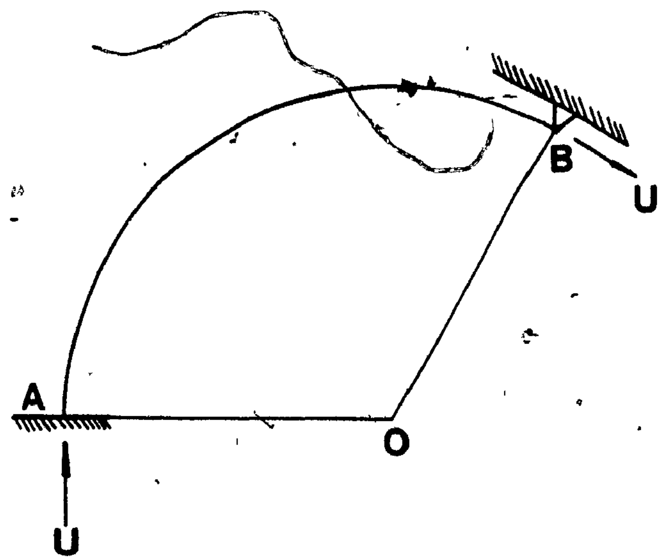
Finally, combining equations (R.7), (R.8), (R.9) and (R.10) yields the rotation at the pinned end

$$\delta = 0 \ . \qquad \text{(R.11)}$$

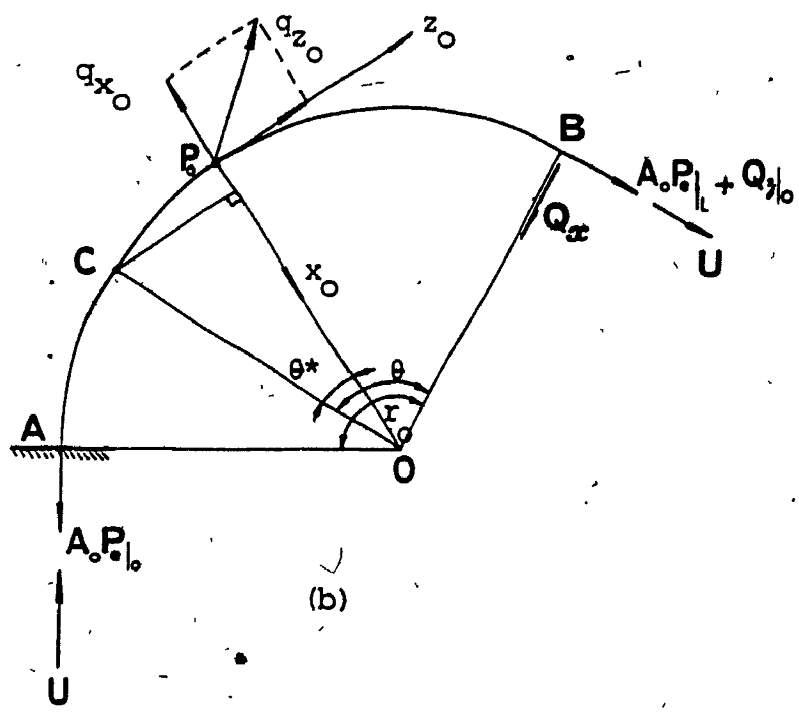
Hence, the boundary conditions associated with equation (3.88) are

$$\left.\begin{aligned} \left.\frac{d\Pi_o}{d\xi}\right|_1 &= 0, \\ \Pi_o\big|_1 &= -\bar{u}^2 . \end{aligned}\right\} \qquad (R.12)$$

It is noted that these boundary conditions also apply to a pinned-pinned incomplete circular pipe.



(a)



(b)

Fig. R.1 (a) A Clamped-Pinned Incomplete Pipe Conveying Fluid.
(b) Forces Acting on the Pipe in (a).

APPENDIX S

COMPUTER PROGRAMS FOR THE INEXTENSIBLE THEORY

```
LEVEL 1.4.0 (OCT 1984)          VS FORTRAN          DATE: AUG 25, 1986     TIME: 13:41:31

REQUESTED OPTIONS (EXECUTE): NODECK,NOLIST,OPT(2),NOFIPS,XREF,MAP,GOSTMT,GOSTMT,NOTEST,NOTF,NOSDUMP

OPTIONS IN EFFECT:  NOLIST   MAP   XREF   GOSTMT NODECK   SOURCE   TERM   OBJECT FIXED  NOTEST NOTRMFLG
                    NOSYM NORENT NOSDUMP AUTODBL(NONE)   NOSXM
                     OPT(2) LANGLVL(77) NOFIPS   FLAG(I)  NAME(MAIN  )  LINECOUNT(60)      CHARLEN(500

                    *....*...1.........2.........3.........4.........5.........6.........7.*.......8

                    C***********************************************************************
                    C                  FINITE ELEMENT PROGRAM OF INEXTENSIBLE THEORY
                    C                             FOR IN-PLANE MOTION
                    C                      ( CURVED PIPE CONVEYING FLUID )
                    C***********************************************************************
                    C
                    C     MAIN PROGRAM
                    C
 ISN      1               IMPLICIT REAL*8(A-H,O-Z)
 ISN      2               DIMENSION EM(6,6),ED(6,6),EK(6,6),NODE(6,10),RO(10),ELENG(10)
 ISN      3               DIMENSION GM(27,27),GD(27,27),GK(27,27),GMM(54,54),GKK(54,54)
 ISN      4               DIMENSION BETA(54),WK(7600)
 ISN      5               COMPLEX*16 EIVALU(54),EIVECT(54,54)
 ISN      6               COMMON /EMDK/EM,ED,EK
 ISN      7               COMMON /COEF/XPHIA,XPHI,XH
 ISN      8               DATA RO    /10*3.14159265D0/
 ISN      9               DO 9999 LLL=2,2
 ISN     10               NDT =27
 ISN     11               NET =8
 ISN     12               NDPE=6
 ISN     13               NDT2=2*NDT
                    C=========
                    C NODE DATA
                    C===========
 ISN     14               PRINT 100
 ISN     15         100   FORMAT('1',8X,'IN PLANE MOTION (CLAMPED-CLAMPED)',
                        *         //, 8X,'ELEMENT CONNECTIVITY:',
                        *         //,20X,'ELEMENT NUMBER', 20X,'NODE',//)
 ISN     16               DO 101 I=1,NET
                    C=======
                    C DATA FOR LENGHT OF ELEMENT
                    C==========================
 ISN     17               ELENG(I)=1.D0/DFLOAT(NET+2)
 ISN     18               DO 102 J=1,NDPE
 ISN     19         102   NODE(J,I)=3*(I-1)+J
 ISN     20               WRITE(6,103)I,(NODE(L,I),L=1,NDPE)
 ISN     21         101   CONTINUE
 ISN     22         103   FORMAT(29X,I2,10X,6(I3,3X)/)
                    C===================================
                    C  FORMING THE DIMENSIONLESS PARAMETERS
                    C=====================================
 ISN     23               VELU =RO(1)*DFLOAT(LLL)
 ISN     24               XPHIA=0.0D0
 ISN     25               XPHI =0.5D0
 ISN     26               XH   =0.0D0
 ISN     27               PI   =0.0D0
 ISN     28               WRITE(6,106)VELU,XPHIA,XPHI,XH,PI,RO(1)
 ISN     29         106   FORMAT(//,10X,'DIMENSIONLESS PARAMETERS:',///10X,'DIMENSIONLESS ',
                        *'VELOCITY=',D12.5,//,10X,'BETAA=',D12.5,',','BETA=',D12.5,',','H='
                        *,D12.5,//,10X,'PI=',D12.5,',','RO=',D12.5,/)
```

```
                  *....*...1.........2.........3.........4.........5.........6.........7.*.......8
                  C==============================================
                  C ESSAMBLY OF ELEMENT MATRIX YIELDS GLOBAL MATRIX
                  C===============================================
ISN     30              DO 107 II=1,NDT
ISN     31              DO 107 JJ=1,NDT
ISN     32              GM(II,JJ)=0.0D0
ISN     33              GD(II,JJ)=0.0D0
ISN     34              GK(II,JJ)=0.0D0
ISN     35        107   CONTINUE
                  C====
ISN     36              XELENG=ELENG(1)
ISN     37              ROO    =RO(1)
ISN     38              CALL ELMDKM(XELENG,VELU,ROO)
ISN     39              DO 333 JG=1,3
ISN     40              JE=3+JG
ISN     41              DO 333 KG=1,3
ISN     42              KE=3+KG
ISN     43              GM(JG,KG)=GM(JG,KG)+EM(JE,KE)
ISN     44              GD(JG,KG)=GD(JG,KG)+ED(JE,KE)
ISN     45              GK(JG,KG)=GK(JG,KG)+EK(JE,KE)
ISN     46        333   CONTINUE
                  C=====
ISN     47              DO 109 L=1,NET
ISN     48              XELENG=ELENG(L)
ISN     49              ROO    =RO(L)
ISN     50              CALL ELMDKM(XELENG,VELU,ROO)
ISN     51              DO 109 JL=1,NDPE
ISN     52              DO 109 KL=1,NDPE
ISN     53                     JG=NODE(JL,L)
ISN     54                     KG=NODE(KL,L)
ISN     55                     GM(JG,KG)=GM(JG,KG)+EM(JL,KL)
ISN     56                     GD(JG,KG)=GD(JG,KG)+ED(JL,KL)
ISN     57                     GK(JG,KG)=GK(JG,KG)+EK(JL,KL)
ISN     58        109   CONTINUE
                  C===
ISN     59              XELENG=ELENG(NET)
ISN     60              ROO    =RO(NET)
ISN     61              CALL ELMDKM(XELENG,VELU,ROO)
ISN     62              DO 334 JE=1,3
ISN     63              JG=NDT+JE-3
ISN     64              DO 334 KE=1,3
ISN     65              KG=NDT+KE-3
ISN     66              GM(JG,KG)=GM(JG,KG)+EM(JE,KE)
ISN     67              GD(JG,KG)=GD(JG,KG)+ED(JE,KE)
ISN     68              GK(JG,KG)=GK(JG,KG)+EK(JE,KE)
ISN     69        334   CONTINUE
                  C========
                  C FORMING THE AUGMENTED MATRIX OF M,D,K           C
                  C==============================================C
                  C
ISN     70              DO 120 I=1,NDT2
ISN     71              DO 120 J=1,NDT2
ISN     72              GMM(I,J)=0.0D0
ISN     73        120   GKK(I,J)=0.0D0
ISN     74              DO 121 I1=1,NDT
ISN     75              II=NDT+I1
```

```
              *....*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN    76           GMM(I1,I1)=1.00
ISN    77           GKK(II,II)=1.00
ISN    78           DO 121 J1=1,NDT
ISN    79           JJ=NDT+J1
ISN    80           GKK(II,J1)=-GK(I1,J1)
ISN    81           GMM(II,JJ)=GM(I1,J1)
ISN    82           GKK(II,JJ)=-GD(I1,J1)
ISN    83     121   CONTINUE
              C===================================
              C CALCULATING EIGENVALUES AND EIGENVECTORS
              C========================================
ISN    84           IA=NDT2
ISN    85           IB=NDT2
ISN    86           IZ=NDT2
ISN    87           N =NDT2
ISN    88           IJOB=2
ISN    89           CALL EIGZF(GKK,IA,GMM,IB,N,IJOB,EIVALU,BETA,EIVECT,IZ,WK,IER)
              C===========
              C  EIGENVALUES
              C======================
ISN    90           DO 145 J=1,NDT2
ISN    91           EIVALU(J)=EIVALU(J)/BETA(J)
ISN    92     145   CONTINUE
ISN    93           CALL ARRANG(NDT2,EIVALU,EIVECT)
ISN    94           CALL OUTEIG(NDT2,3,EIVALU,EIVECT)
ISN    95     9999  CONTINUE
ISN    96           STOP
ISN    97           END
```

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LEVEL 1.4.0 (OCT 1984)          VS FORTRAN          DATE: AUG 25, 1986     TIME: 13:41:32

OPTIONS IN EFFECT: NOLIST   MAP   XREF   GOSTMT NODECK   SOURCE   TERM   OBJECT FIXED  NOTEST NOTRMFLG
                   NOSYM NORENT NOSDUMP AUTODBL(NONE)    NOSXM
                      OPT(2) LANGLVL(77) NOFIPS    FLAG(I)  NAME(MAIN   )  LINECOUNT(60)      CHARLEN(500

                   *....*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN      1            SUBROUTINE ELMDKM(ELENG,VELU,RO)
                   C
                   C SUBROUTINE ESTIMATING -MASS MATRIX
                   C                       -DAMPING MATRIX
                   C                       -STIFFNESS MATRIX
                   C
ISN      2            IMPLICIT REAL*8(A-H,O-Z)
ISN      3            DIMENSION EM(6,6),ED(6,6),EK(6,6)
ISN      4            DIMENSION RJ1(6,6),RJ2(6,6),RJ3(6,6),RJ4(6,6),RJ5(6,6),
                     2          RJ6(6,6),RJ8(6,6),AINVE(6,6),AINVET(6,6)
ISN      5            COMMON /EMDK/EM,ED,EK
ISN      6            COMMON /COEF/PHIA,PHI,VISDAH
ISN      7            COMMON /MAJ/RJ1,RJ2,RJ3,RJ4,RJ5,RJ6,RJ8
ISN      8            COMMON /INV/AINVE,AINVET
ISN      9            RO2=RO**2
ISN     10            RO4=RO**4
ISN     11            PHI12=DSQRT(PHI)
ISN     12            VELU2=VELU**2
                   C
                   C CALL MATRICES J1-8
                   C
ISN     13            CALL GENMAJ(ELENG)
                   C
                   C CALL INVERSE OF THE MATRIX A
                   C
ISN     14            CALL INVERS(ELENG)
                   C======================
                   C FORMING ELEMENT MATRICES
                   C===========================
ISN     15            DO 80 I=1,6
ISN     16            DO 80 J=1,6
ISN     17            ED(I,J)=RJ2(I,J)+RO2*RJ1(I,J)
ISN     18            EM(I,J)=(1.D0+PHIA)*ED(I,J)
ISN     19            ED(I,J)=(2.D0*VELU*PHI12)*(RJ5(I,J)+RO2*RJ4(I,J))+
                     *          VISDAH*ED(I,J)
ISN     20            EK(I,J)=(RJ3(I,J)+RO2*(RJ6(I,J)+RJ6(J,I))+RO4*RJ2(I,J))+
                     *          VELU2*(RJ6(I,J)+RO2*(RJ2(I,J)-RJ8(I,J))-RO4*RJ1(I,J))
                     *         -VELU2*(RJ6(I,J)+RO2*(RJ2(I,J)-RJ8(I,J)-RO2*RJ1(I,J)))
ISN     21         80 CONTINUE
                   C
                   C ELEMENT MASS MATRIX
                   C
ISN     22            CALL PRODMA(6,6,6,AINVET,EM,RJ4)
ISN     23            CALL PRODMA(6,6,6,RJ4,AINVE,EM)
                   C
                   C ELEMENT DAMPING MATRIX
                   C
ISN     24            CALL PRODMA(6,6,6,AINVET,ED,RJ5)
ISN     25            CALL PRODMA(6,6,6,RJ5,AINVE,ED)
                   C
                   C ELEMENT STIFNESS MATRIX
                   C
```

```
                 *....*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN      26      CALL PRODMA(6,6,6,AINVET,EK,RJ4)                                        00199
ISN      27      CALL PRODMA(6,6,6,RJ4,AINVE,EK)                                         00200
ISN      28      RETURN                                                                  00201
ISN      29      END                                                                     00202
```

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LEVEL 1.4.0 (OCT 1984)           VS FORTRAN          DATE: AUG 25, 1986     TIME: 13:41:32

OPTIONS IN EFFECT:  NOLIST   MAP   XREF   GOSTMT NODECK   SOURCE   TERM   OBJECT FIXED   NOTEST NOTRMFLG
                    NOSYM NORENT NOSDUMP AUTODBL(NONE)   NOSXM
                      OPT(2) LANGLVL(77) NOFIPS   FLAG(I)  NAME(MAIN   )  LINECOUNT(60)      CHARLEN(500

                    *....*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN     1          SUBROUTINE  GENMAJ(ELENG)
              C
              C    SUBROUTINE GENERATING MATRICES J1-J13
              C
ISN     2          IMPLICIT REAL*8(A-H,O-Z)
ISN     3          DIMENSION RJ1(6,6),RJ2(6,6),RJ3(6,6),RJ4(6,6),RJ5(6,6),RJ6(6,6),
                *           RJ8(6,6),AJ2(6,6),AJ3(6,6),AJ5(6,6)
ISN     4          COMMON /MAJ/RJ1,RJ2,RJ3,RJ4,RJ5,RJ6,RJ8
ISN     5          DATA     AJ2/7*0.0D0,5*1.D0,0.0D0,1.D0,4.D0,1.5D0,1.6D0,5.D0,
                *            0.0D0,1.D0,1.5D0,1.8D0,2.D0,15.D0,0.0D0,1.D0,1.6D0,
                *            2.D0,16.D0,2.5D0,0.0D0,1.D0,5.D0,15.D0,2.5D0,25.D0/
ISN     6          DATA     AJ3/21*0.0D0,36.D0,72.D0,120.D0,3*0.0D0,72.D0,192.D0,
                *              360.D0,3*0.0D0,120.D0,360.D0,720.D0/
ISN     7          DATA     AJ5/13*0.0D0,5*2.D0,0.0D0,3.D0,4.D0,4.5D0,4.8D0,5.D0,
                *              0.0D0,4.D0,6.D0,7.2D0,8.D0,60.D0,0.0D0,5.D0,8.D0
                *             ,10.D0,80.D0,12.5D0/
ISN     8          DO 56 I=1,6
ISN     9          DO 56 J=1,6
ISN    10          RJ2(I,J)=AJ2(I,J)
ISN    11          RJ3(I,J)=AJ3(I,J)
ISN    12          RJ5(I,J)=AJ5(I,J)
ISN    13          RJ6(I,J)=0.0D0
ISN    14          RJ8(I,J)=0.0D0
ISN    15     56   CONTINUE
ISN    16          RJ2(3,3)=RJ2(3,3)/3.D0
ISN    17          RJ2(3,6)=RJ2(3,6)/3.D0
ISN    18          RJ2(6,3)=RJ2(3,6)
ISN    19          RJ2(4,6)=RJ2(4,6)/7.D0
ISN    20          RJ2(6,4)=RJ2(4,6)
ISN    21          RJ2(5,5)=RJ2(5,5)/7.D0
ISN    22          RJ2(6,6)=RJ2(6,6)/9.D0
              C
              C  GENERATING MATRICES J1,J2
              C
ISN    23     111   DO 1  I=1,6
ISN    24                 DO 2  J=1,6
ISN    25                 K=I+J-1
ISN    26     2           RJ1(I,J)=ELENG**(K)/DFLOAT(K)
ISN    27                 IF(I.EQ.1) GOTO 1
ISN    28                 DO 5  JJ=2,6
ISN    29                 KK=I+JJ-3
ISN    30     5           RJ2(I,JJ)=RJ2(I,JJ)*ELENG**(KK)
ISN    31     1     CONTINUE
              C
              C  GENERATING MATRICES J3
              C
ISN    32           DO   10  I=4,6
ISN    33           DO   10  J=4,6
ISN    34                    K=I+J-7
ISN    35     10             RJ3(I,J)=RJ3(I,J)*ELENG**(K)
              C
              C  GENERATING MATRICCES J4 AND J5
```

```
                      *....*...1.........2.........3.........4.........5.........6.........7.*.......8
                    C
ISN     36                RJ4(1,1)=0.0D0
ISN     37                RJ5(5,6)=RJ5(5,6)/7.D0
ISN     38                RJ5(6,5)=RJ5(6,5)/7.D0
ISN     39                DO  12  I1 = 2,6
ISN     40                        RJ4(1,I1)=0.0D0
ISN     41                        DO  14  J1 = 1,6
ISN     42                        K=I1-1
ISN     43                        KK=I1+J1-2
ISN     44          14            RJ4(I1,J1)=(DFLOAT(K)*ELENG**(KK))/DFLOAT(KK)
ISN     45                        DO  15  J2 =3,6
ISN     46                        K =I1+J2-4
ISN     47          15            RJ5(I1,J2)=RJ5(I1,J2)*ELENG**(K)
ISN     48          12    CONTINUE
                    C
                    C  GENERATING MATRICES J8 AND J6
                    C
ISN     49                DO  16  II=1,6
ISN     50                        K1 =II+1
ISN     51                        K2 =II+2
ISN     52                        K3= II+3
ISN     53                        RJ8(II,3)=2.D0*ELENG**(II)/DFLOAT(II)
ISN     54                        RJ8(II,4)=6.D0*ELENG**(K1)/DFLOAT(K1)
ISN     55                        RJ8(II,5)=12.D0*ELENG**(K2)/DFLOAT(K2)
ISN     56                        RJ8(II,6)=20.D0*ELENG**(K3)/DFLOAT(K3)
ISN     57                        IF(II.EQ.1) GOTO16
ISN     58                        K =II-1
ISN     59                        KK=II+1
ISN     60                        RJ6(II,4)=6.D0*ELENG**(K)
ISN     61                        RJ6(II,5)=(24.D0*DFLOAT(K)*ELENG**(II))/DFLOAT(II)
ISN     62                        RJ6(II,6)=(60.D0*DFLOAT(K)*ELENG**(KK))/DFLOAT(KK)
ISN     63          16    CONTINUE
ISN     64                RETURN
ISN     65                END
```

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LEVEL 1.4.0 (OCT 1984)            VS FORTRAN           DATE: AUG 25, 1986     TIME: 13:41:33

OPTIONS IN EFFECT:  NOLIST  MAP   XREF   GOSTMT NODECK   SOURCE   TERM   OBJECT FIXED  NOTEST NOTRMFLG
                    NOSYM NORENT NOSDUMP AUTODBL(NONE)    NOSXM
                      OPT(2) LANGLVL(77) NOFIPS    FLAG(I)  NAME(MAIN   )  LINECOUNT(60)      CHARLEN(500

                    *....*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN        1     .     SUBROUTINE  INVERS(ELENG)
                 C
                 C     SUBROUTINE INVERTING THE MATRIX A
                 C
ISN        2           IMPLICIT REAL*8(A-H,O-Z)
ISN        3           DIMENSION AINVE(6,6),AINVET(6,6),AI(6,6)
ISN        4           COMMON /INV/AINVE,AINVET
ISN        5           DATA  AI/1.D0,2*0.D00,-10.D0,15.D0,-6.D0,0.D00,1.D0,
                1             0.D00,-6.D0,8.D0,-3.D0,2*0.D00,.5D0,-1.5D0,1.5D0,
                2          -.5D0,3*0.D00,10.D0,-15.D0,6.D0,3*0.D00,-4.D0,7.D0,
                3          -3.D0,3*0.D00,.5D0,-1.D0,.5D0/
ISN        6           DO 54  I=1,6
ISN        7           DO 54  J=1,6
ISN        8     54    AINVE(I,J)=AI(I,J)
ISN        9           DO 50  I=4,6
ISN       10           DO 50  J=1,3
ISN       11                 JJ=J+3
ISN       12                 K =-(I-J)
ISN       13                 X =ELENG**(K)
ISN       14                 AINVE(I,J)=AINVE(I,J)*X
ISN       15                 AINVE(I,JJ)=AINVE(I,JJ)*X
ISN       16     50    CONTINUE
ISN       17           DO 51 I=1,6
ISN       18           DO 51 J=1,6
ISN       19           AINVET(I,J)=AINVE(J,I)
ISN       20     51    CONTINUE
ISN       21           RETURN
ISN       22           END
```

```
LEVEL 1.4.0 (OCT 1984)          VS FORTRAN          DATE: AUG 25, 1986     TIME: 13:41:33

OPTIONS IN EFFECT:  NOLIST   MAP   XREF   GOSTMT NODECK   SOURCE   TERM   OBJECT FIXED  NOTEST NOTRMFLG
                    NOSYM NORENT NOSDUMP AUTODBL(NONE)  , NOSXM
                      OPT(2) LANGLVL(77) NOFIPS   FLAG(I)  NAME(MAIN  )  LINECOUNT(60)       CHARLEN(500

              *....*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN      1          SUBROUTINE PRODMA(M,L,N,A,B,C)
              C
              C SUBROUTINE MULTIPLYING MATRICES
              C
ISN      2          IMPLICIT   REAL*8(A-H,O-Z)
ISN      3          DIMENSION  A(M,L),B(L,N),C(M,N)
ISN      4          DO 90  I=1,M
ISN      5          DO 90  J=1,N
ISN      6          C(I,J)=0.0D0
ISN      7          DO 90  K=1,L
ISN      8          C(I,J)=C(I,J)+A(I,K)*B(K,J)
ISN      9    90    CONTINUE
ISN     10          RETURN
ISN     11          END
```

```
LEVEL 1.4.0 (OCT 1984)          VS FORTRAN          DATE: AUG 25, 1986     TIME: 13:41:33

OPTIONS IN EFFECT:  NOLIST   MAP   XREF   GOSTMT NODECK   SOURCE   TERM   OBJECT FIXED   NOTEST NOTRMFLG
                    NOSYM NORENT NOSDUMP AUTODBL(NONE)   NOSXM
                     OPT(2) LANGLVL(77) NOFIPS   FLAG(I)  NAME(MAIN   )  LINECOUNT(60)      CHARLEN(500

                 *....*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN      1             SUBROUTINE ARRANG(N,X,PHI)
                 C
                 C  SUBROUTINE ARRANGING EIGENVALUES FROM SMALL TO BIGGER
                 C  AND ARRANGING EIGENVECTORS CORRESPONDING TO EIGENVALUES
                 C
ISN      2             IMPLICIT REAL*8(A-H,O-Z)
ISN      3             COMPLEX*16 X,PHI,PHIN,EIMIN,XEI
ISN      4             DIMENSION X(N),PHI(N,N)
ISN      5             LA =N-1
ISN      6             DO 2001 I=1,LA
ISN      7             EIMIN=X(I)
ISN      8             XMIN=DIMAG(EIMIN)
ISN      9             JMI =I
ISN     10             JF  =I+1
ISN     11             DO 501 M=JF,N
ISN     12             XEI =X(M)
ISN     13             IF(DABS(XMIN).LE.DABS(DIMAG(XEI))) GOTO 501
ISN     14             JMI=M
ISN     15             XMIN=DIMAG(XEI)
ISN     16             EIMIN=XEI
ISN     17       501   CONTINUE
ISN     18             X(JMI)=X(I)
ISN     19             X(I)  =EIMIN
ISN     20             DO 533 L=1,N
ISN     21             PHIN      =PHI(L,JMI)
ISN     22             PHI(L,JMI)=PHI(L,I)
ISN     23             PHI(L,I)  =PHIN
ISN     24       533   CONTINUE
ISN     25       2001  CONTINUE
ISN     26             RETURN
ISN     27             END
```

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LEVEL 1.4.0 (OCT 1984)              VS FORTRAN           DATE: AUG 25, 1986     TIME: 13:41:33

OPTIONS IN EFFECT:  NOLIST   MAP   XREF   GOSTMT NODECK   SOURCE   TERM   OBJECT FIXED   NOTEST NOTRMFLG
                    NOSYM NORENT NOSDUMP AUTODBL(NONE)   NOSXM
                     OPT(2) LANGLVL(77) NOFIPS   FLAG(I)  NAME(MAIN  )  LINECOUNT(60)     CHARLEN(500

                *....*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN      1            SUBROUTINE OUTEIG(NDTT,N,X,PHI)
                C
                C SUBROUTINE TO PRINT EIGENVALUES AND EIGENVECTORS
                C
ISN      2            IMPLICIT REAL*8(A-H,O-Z)
ISN      3            DIMENSION X(NDTT),PHI(NDTT,NDTT)
ISN      4            COMPLEX*16 X,PHI
ISN      5            PRINT 540
ISN      6      540   FORMAT(///,14X,'====FREQUENCIES===='//)
ISN      7            M=NDTT/3
ISN      8            DO 541 J1=1,M
ISN      9                  J2=M+J1
ISN     10                  J3=2*M+J1
ISN     11            WRITE(6,550)J1,X(J1),J2,X(J2),J3,X(J3)
ISN     12      550   FORMAT(/,4X,I2,'TH',2D18.8,I3,'TH',2D18.8,I3,'TH',2D18.8)
ISN     13      541   CONTINUE
                C==============
                C ∂=(∂,∂') ∂ IS VECTOR
                C======================
ISN     14            NN= NDTT/2
ISN     15            DO 551 JA=1,N,3
ISN     16                  JB=JA+1
ISN     17                  JC=JA+2
ISN     18            WRITE(6,560)JA,JB,JC
ISN     19      560   FORMAT(//,9X,I2,'TH EIGENVECTOR',30X,I2,'TH EIGENVECTOR',32X,
                     *      I2,'TH EINGVECTOR')
ISN     20            DO 561 I=1,NN
ISN     21            WRITE(6,562)PHI(I,JA),PHI(I,JB),PHI(I,JC)
ISN     22      562   FORMAT(/4X,3(2D18.8,2X))
ISN     23      561   CONTINUE
ISN     24      551   CONTINUE
ISN     25            RETURN
ISN     26            END
```

```
LEVEL 1.4.0 (OCT 1984)          VS FORTRAN          DATE: AUG 25, 1986     TIME: 13:41:34

OPTIONS IN EFFECT:  NOLIST    MAP   XREF   GOSTMT NODECK   SOURCE    TERM    OBJECT FIXED   NOTEST NOTRMFLG
                    NOSYM NORENT NOSDUMP AUTODBL(NONE)    NOSXM
                      OPT(2) LANGLVL(77) NOFIPS    FLAG(I)  NAME(MAIN  )  LINECOUNT(60)      CHARLEN(500

                 *....*...1.........2.........3.........4.........5.........6.........7.*.......8

                 C
                 C      SUBROUTINE TO PRINT MATRIX
                 C
ISN       1             SUBROUTINE  PRINT(N,A)
ISN       2             IMPLICIT REAL*8(A-H,O-Z)
ISN       3             DIMENSION A(6,6)
ISN       4             WRITE(6,22) N
ISN       5   22        FORMAT(///,10X,'MATRIX J',I2,//)
ISN       6             WRITE(6,23)((A(I,J),J=1,6),I=1,6)
ISN       7   23        FORMAT(/,6(4X,D12.5),/)
ISN       8             RETURN
ISN       9             END
```

IN PLANE MOTION (CLAMPED-CLAMPED)

ELEMENT CONNECTIVITY:

ELEMENT NUMBER	NODE					
1	1	2	3	4	5	6
2	4	5	6	7	8	9
3	7	8	9	10	11	12
4	10	11	12	13	14	15
5	13	14	15	16	17	18
6	16	17	18	19	20	21
7	19	20	21	22	23	24
8	22	23	24	25	26	27

DIMENSIONLESS PARAMETERS:

DIMENSIONLESS VELOCITY= 0.62832D+01

BETAA= 0.00000D+00,BETA= 0.50000D+00,H= 0.00000D+00

PI= 0.00000D+00,RO= 0.31416D+01

****FREQUENCIES****

1TH	-0.42124077D-06	0.40300008D+02	19TH	-0.11072724D-03	0.12914058D+04
2TH	-0.42124077D-06	-0.40300008D+02	20TH	-0.11072724D-03	-0.12914058D+04
3TH	-0.50586683D-07	0.95835131D+02	21TH	-0.20239003D-03	0.15356683D+04
4TH	-0.50586673D-07	-0.95835131D+02	22TH	-0.20239003D-03	-0.15356683D+04
5TH	0.118[illegible]209D-05	0.17555309D+03	23TH	0.15603152D-04	-0.18071989D+04
6TH	0.11800208D-05	-0.17555309D+03	24TH	0.15603152D-04	0.18071989D+04
7TH	0.10892956D-05	0.27522871D+03	25TH	0.24905698D-03	-0.21037520D+04
8TH	0.10892956D-05	-0.27522871D+03	26TH	0.24905698D-03	0.21037520D+04
9TH	0.20750260D-04	0.39312542D+03	27TH	0.614[illegible]-04	-0.24320157D+04
10TH	0.20750260D-04	-0.39312542D+03	28TH	0.61479809D-04	0.24320157D+04

11TH	0.131943370-04	-0.532951310+03	29TH	-0.684317000-04	-0.279028330+04
12TH	0.131943370-04	0.532951310+03	30TH	-0.684317000-04	0.279028330+04
13TH	-0.610111730-04	0.690573860+03	31TH	-0.382665180-04	0.318047160+04
14TH	-0.610111730-04	-0.690573860+03	32TH	-0.382665180-04	-0.318047160+04
15TH	-0.281598210-04	0.872021250+03	33TH	-0.242499400-04	-0.358668120+04
16TH	-0.281598210-04	-0.872021250+03	34TH	-0.242499400-04	0.358668120+04
17TH	0.172604640-03	0.106650290+04	35TH	0.122633050-05	-0.394608100+04
18TH	0.172604640-03	-0.106650290+04	36TH	0.122632690-05	0.394608100+04

1TH EIGENVECTOR		2TH EIGENVECTOR	
0.122477470-04	0.340860180-04	0.122477470-04	-0.340860180-04
0.313106040-03	0.945094160-03	0.313106040-03	-0.945094160-03
0.416309370-02	0.157913310-01	0.416309370-02	-0.157913310-01
0.582887650-04	0.210806040-03	0.582887650-04	-0.210806040-03
0.521941010-03	0.255057360-02	0.521941010-03	-0.255057360-02
-0.825096160-03	0.135577920-01	-0.825096160-03	-0.135577920-01
0.957716060-04	0.512347120-03	0.957716060-04	-0.512347120-03
0.134283440-03	0.322988490-02	0.134283440-03	-0.322988490-02
-0.631691510-02	-0.127096290-02	-0.631691510-02	0.127096290-02
0.752580240-04	0.799780640-03	0.752580240-04	-0.799780640-03
-0.538629770-03	0.224065460-02	-0.538629770-03	-0.224065460-02
-0.596755680-02	-0.177606320-01	-0.596755680-02	0.177606320-01
0.174458630-10	0.917760810-03	0.174458630-10	-0.917760810-03
-0.865329130-03	0.761542540-11	-0.865329130-03	-0.761542540-11
-0.431959110-09	-0.248138900-01	-0.431959110-09	0.248138900-01
-0.752579930-04	0.799780640-03	-0.752579930-04	-0.799780640-03
-0.538629850-03	-0.224065460-02	-0.538629850-03	0.224065460-02
0.596755620-02	-0.177606320-01	0.596755620-02	0.177606320-01
-0.957715850-04	0.512347120-03	-0.957715850-04	-0.512347120-03
0.134283330-03	-0.322988490-02	0.134283330-03	0.322988490-02

0.631691500-02	-0.127096290-02	0.631691500-02	0.127096290-02
-0.582887560-04	0.210806050-03	-0.582887560-04	-0.210806050-03
0.521940900-03	-0.255057360-02	0.521940900-03	0.255057360-02
0.825096500-03	0.135577920-01	0.825096500-03	-0.135577920-01
-0.122477450-04	0.340860190-04	-0.122477450-04	-0.340860190-04
0.313106000-03	-0.945094180-03	0.313106000-03	0.945094180-03
-0.416309280-02	0.157913310-01	-0.416309280-02	-0.157913310-01

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LEVEL 1.4.0 (OCT 1984)            VS FORTRAN              DATE: AUG 25, 1986      TIME: 14:24:46

REQUESTED OPTIONS (EXECUTE): NODECK,NOLIST,OPT(2),NOFIPS,XREF,MAP,GOSTMT,GOSTMT,NOTEST,NOTF,NOSDUMP

OPTIONS IN EFFECT:  NOLIST   MAP   XREF   GOSTMT NODECK   SOURCE   TERM   OBJECT FIXED   NOTEST NOTRMFLG
                    NOSYM NORENT NOSDUMP AUTODBL(NONE)    NOSXM
                      OPT(2) LANGLVL(77) NOFIPS    FLAG(I)  NAME(MAIN  )  LINECOUNT(60)      CHARLEN(500)

                  *....*...1.........2.........3.........4.........5.........6.........7.*.......8

                  C                                                                       00013
                  C***********************************************************************00014
                  C         FINITE ELEMENT PROGRAM OF INEXTENSIBLE THEORY                 00015
                  C                    FOR OUT-OFF-PLANE MOTION                           00016
                  C                 (CURVED PIPE CONVEYING FLUID)                         00016
                  C***********************************************************************00017
                  C                                                                       00019
                  C     MAIN PROGRAM                                                      00018
                  C                                                                       00019
ISN       1             IMPLICIT REAL*8(A-H,O-Z)                                          00020
ISN       2             DIMENSION EM(6,6),ED(6,6),EK(6,6),NODE(6,15),RO(15),ELENG(15)     00021
ISN       3             DIMENSION GM(45,45),GD(45,45),GK(45,45),GMM(90,90),GKK(90,90)     00022
ISN       4             DIMENSION BETA(90),WK(9900),PI(15)                                00023
ISN       5             COMPLEX*16 EIVALU(90),EIVECT(90,90)                               00024
ISN       6             COMMON /EMDK/EM,ED,EK                                             00025
ISN       7             COMMON /COEF/XPHIA,XPHI,XH,SIMA,CAPA                              00026
ISN       8             DATA   RO/15*3.14159265D0/                                        00027
ISN       9             DO 9999 LLL=2,2                                                   00028
ISN      10             NDT =45                                                           00029
ISN      11             NET =14                                                           00030
ISN      12             NDPE=6                                                            00031
ISN      13             NDT2=2*NDT                                                        00032
                  C=============                                                          00033
                  C NODE DATA                                                             00034
                  C============                                                           00035
ISN      14             PRINT 100                                                         00036
ISN      15       100   FORMAT('1',8X,'OUT-OF PLANE MOTION (CLAMPED-CLAMPED)',            00037
                       *         //,9X,'ELEMENT CONNECTIVITY:',                           00038
                       *         //20X,'ELEMENT NUMBER',20X,'NODE',//)                    00039
ISN      16             DO 101 I=1,NET                                                    00040
                  C========                                                               00041
                  C DATA FOR LENGHT OF ELEMENT                                            00042
                  C==========================                                             00043
ISN      17             ELENG(I)=1.D0/DFLOAT(NET+2)                                       00044
ISN      18             DO 102 J=1,NDPE                                                   00045
ISN      19       102   NODE(J,I)=3*(I-1)+J                                               00046
ISN      20             WRITE(6,103)I,(NODE(L,I),L=1,NDPE)                                00047
ISN      21       101   CONTINUE                                                          00048
ISN      22       103   FORMAT(29X,I2,10X,6(I3,3X)/)                                      00049
                  C====================================                                   00050
                  C  FORMING THE DIMENSIONLESS PARAMETERS                                 00051
                  C====================================                                   00052
ISN      23             VELU =RO(1)*DFLOAT(LLL)                                           00053
ISN      24             XPHIA=0.0D0                                                       00054
ISN      25             XPHI =.5D0                                                        00055
ISN      26             SIMA =0.0D0                                                       00056
ISN      27             CAPA =1.D0/(1.D0+.3D0)                                            00057
ISN      28             XH   =0.0D0                                                       00058
                  C=======                                                                00059
ISN      29             XMF  =162.6D0                                                     00060
```

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                 *....*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN    30             XMT   =299.3D0
ISN    31             XLEN  =1000.D0
ISN    32             GRAV  =9.81D0
ISN    33             EI    =221.3D+06
ISN    34             DENSIF=998.D0
ISN    35             XMASOI=3.1415D0*DENSIF*(.253**2)
ISN    36             GZEO  =(XMT+XMF-XMASOI)*GRAV
ISN    37             PI1   =(GZEO*XLEN**3)/EI
                 C  **** TEST FOR MR. CHEN'S CASE
ISN    38             PI1   =0.0D0
ISN    39             PIO=-PI1
ISN    40             DO 105 I=1,NET
ISN    41             PIO=PIO+PI1*ELENG(I)
ISN    42             PI(I)=PIO
ISN    43        105  CONTINUE
                 C
                 C=======
ISN    44             WRITE(6,106)VELU,XPHIA,XPHI,XH,SIMA,CAPA,RO(1)
ISN    45        106  FORMAT(//,10X,'DIMENSIONLESS PARAMETERS:',///10X,'DIMENSIONLESS ',
                     *'VELOCITY=',D12.5,//,10X,'BETAA=',D12.5,',','BETA=',D12.5,',','H='
                     *,D12.5,//,10X,'PI (INCLUDING)',',',' SIMA  =',D12.5,','//,10X,
                     *          'CAPA=',D12.5,',','RO=',D12.5,/)
                 C======================
                 C ESSAMBLY OF ELEMENT MATRIX YIELDS GLOBAL MATRIX
                 C==============================================
                 C
ISN    46             DO 107 II=1,NDT
ISN    47             DO 107 JJ=1,NDT
ISN    48             GM(II,JJ)=0.0D0
ISN    49             GD(II,JJ)=0.0D0
ISN    50             GK(II,JJ)=0.0D0
ISN    51        107  CONTINUE
                 C
ISN    52             PIO   =-PI1
ISN    53             XELENG=ELENG(1)
ISN    54             ROO=RO(1)
ISN    55             CALL ELMDKM(XELENG,VELU,PIO,PI1,ROO)
ISN    56             DO 115 JG=1,3
ISN    57             JE=3+JG
ISN    58             DO 115 KG=1,3
ISN    59             KE=3+KG
ISN    60             GM(JG,KG)=GM(JG,KG)+EM(JE,KE)
ISN    61             GD(JG,KG)=GD(JG,KG)+ED(JE,KE)
ISN    62             GK(JG,KG)=GK(JG,KG)+EK(JE,KE)
ISN    63        115  CONTINUE
                 C
ISN    64             DO 109 L=1,NET
ISN    65             XELENG=ELENG(L)
ISN    66             ROO  =RO(L)
ISN    67             PIX  =PI(L)
ISN    68             CALL ELMDKM(XELENG,VELU,PIX,PI1,ROO)
ISN    69             DO 109 JL=1,NOPE
ISN    70             DO 109 KL=1,NOPE
ISN    71                   JG=NODE(JL,L)
ISN    72                   KG=NODE(KL,L)
ISN    73                   GM(JG,KG)=GM(JG,KG)+EM(JL,KL)
```

```
                *....*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN    74                    GD(JG,KG)=GD(JG,KG)+ED(JL,KL)
ISN    75                    GK(JG,KG)=GK(JG,KG)+EK(JL,KL)
ISN    76       109    CONTINUE
ISN    77              XELENG=ELENG(NET)
ISN    78              ROO=RO(NET)
ISN    79              CALL ELMDKM(XELENG,VELU,PI0,PI1,ROO)
ISN    80              DO 199 JE=1,3
ISN    81              JG=NDT+JE-3
ISN    82              DO 199 KE=1,3
ISN    83              KG=NDT+KE-3
ISN    84              GM(JG,KG)=GM(JG,KG)+EM(JE,KE)
ISN    85              GD(JG,KG)=GD(JG,KG)+ED(JE,KE)
ISN    86              GK(JG,KG)=GK(JG,KG)+EK(JE,KE)
ISN    87       199    CONTINUE
                C
                C====================================
                C FORMING THE AUGMENTED MATRIX OF M,D,K
                C======================================
                C
ISN    88              DO 120 I=1,NDT2
ISN    89              DO 120 J=1,NDT2
ISN    90              GMM(I,J)=0.0D0
ISN    91       120    GKK(I,J)=0.0D0
ISN    92              DO 121 I1=1,NDT
ISN    93              II=NDT+I1
ISN    94              GMM(I1,I1)=1.D0
ISN    95              GKK(I1,II)=1.D0
ISN    96              DO 121 J1=1,NDT
ISN    97              JJ=NDT+J1
ISN    98              GKK(II,J1)=-GK(I1,J1)
ISN    99              GMM(II,JJ)=GM(I1,J1)
ISN   100              GKK(II,JJ)=-GD(I1,J1)
ISN   101       121    CONTINUE
                C==================================
                C CALCULATING EIGENVALUES AND EIGENVECTORS
                C========================================
ISN   102              IA=NDT2
ISN   103              IB=NDT2
ISN   104              IZ=NDT2
ISN   105              N =NDT2
ISN   106              IJOB=2
ISN   107              CALL EIGZF(GKK,IA,GMM,IB,N,IJOB,EIVALU,BETA,EIVECT,IZ,WK,IER)
                C===========
                C  EIGENVALUES
                C======================
ISN   108              DO 145 J=1,NDT2
ISN   109              IF(DABS(BETA(J)).LT.1.D-8) GOTO139
ISN   110              EIVALU(J)=EIVALU(J)/BETA(J)
ISN   111              GOTO 145
ISN   112       139    EIVALU(J)=EIVALU(J)*1.D07
ISN   113       145    CONTINUE
ISN   114              CALL ARRANG(NDT2,EIVALU,EIVECT)
ISN   115              CALL OUTEIG(NDT2,22,EIVALU,EIVECT)
ISN   116      9999    CONTINUE
ISN   117              STOP
ISN   118              END
```

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LEVEL 1.4.0 (OCT 1984)          VS FORTRAN          DATE: AUG 25, 1986     TIME: 14:24:47

OPTIONS IN EFFECT:  NOLIST   MAP   XREF   GOSTMT NODECK   SOURCE   TERM   OBJECT FIXED  NOTEST NOTRMFLG
                    NOSYM NORENT NOSDUMP AUTODBL(NONE)   NOSXM
                     OPT(2) LANGLVL(77) NOFIPS   FLAG(I)  NAME(MAIN  )  LINECOUNT(60)     CHARLEN(500

                  *....*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN       1             SUBROUTINE ELMDKM(ELENG,VELU,PIX,PI1,RO)
                  C=======================================================================
                  C SUBROUTINE ESTIMATING -MASS MATRIX
                  C                       -DAMPING MATRIX
                  C                       -STIFFNESS MATRIX
                  C=======================================================================
                  C
ISN       2             IMPLICIT REAL*8(A-H,O-Z)
ISN       3             DIMENSION EM(6,6),ED(6,6),EK(6,6)
ISN       4             DIMENSION RJS1(6,6),RJS2(6,6),RJS3(6,6),RJS4(6,6),RJS5(6,6),
                       2          RJS6(6,6),RJS8(6,6),RJS10(6,6),RJS11(6,6),RJS12(6,6),
                       3          AINVE(6,6),AINVET(6,6)
ISN       5             COMMON /EMDK/EM,ED,EK
ISN       6             COMMON /COEF/PHIA,PHI,VISDAH,SIMA,CAPA
ISN       7             COMMON /MAJS/RJS1,RJS2,RJS3,RJS4,RJS5,RJS6,RJS8,RJS10,RJS11,RJS12
ISN       8             COMMON /INV/AINVE,AINVET
ISN       9             RO2=RO**2
ISN      10             PHI12=DSQRT(PHI)
ISN      11             VELU2=VELU**2
                  C
                  C CALL MATRICES J*1-J*12
                  C
ISN      12             CALL GENMAJ(ELENG)
                  C
                  C CALL INVERSE OF THE MATRIX A
                  C
ISN      13             CALL INVERS(ELENG)
                  C=======================
                  C FORMING ELEMENT MATRICES
                  C=========================
ISN      14             DO 280 I=1,6
ISN      15             DO 280 J=1,6
ISN      16             EM(I,J)=(1.D0+PHIA)*RJS1(I,J)+SIMA*RJS4(I,J)
ISN      17             ED(I,J)=(2.D0*VELU*PHI12)*RJS5(I,J)+VISDAH*RJS1(I,J)
ISN      18             EK(I,J)=(RJS3(I,J)-RO*(RJS6(I,J)+RJS6(J,I))+RO2*RJS4(I,J))+
                       *   CAPA*(RO2*RJS2(I,J)+RO*(RJS8(I,J)+RJS8(J,I))+RJS10(I,J))
ISN      19       280    CONTINUE
                  C
                  C ELEMENT MASS MATRIX
                  C
ISN      20             CALL PRODMA(6,6,6,AINVET,EM,RJS4)
ISN      21             CALL PRODMA(6,6,6,RJS4,AINVE,EM)
                  C
                  C ELEMENT DAMPING MATRIX
                  C
ISN      22             CALL PRODMA(6,6,6,AINVET,ED,RJS5)
ISN      23             CALL PRODMA(6,6,6,RJS5,AINVE,ED)
                  C
                  C ELEMENT STIFNESS MATRIX
                  C
ISN      24             CALL PRODMA(6,6,6,AINVET,EK,RJS4)
ISN      25             CALL PRODMA(6,6,6,RJS4,AINVE,EK)


LEVEL 1.4.0 (OCT 1984)          VS FORTRAN          DATE: AUG 25, 1986     TIME: 14:24:47     NAME: ELM

                  *....*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN      26             RETURN
ISN      27             END
```

```
LEVEL 1.4.0 (OCT 1984)          VS FORTRAN          DATE: AUG 25, 1986    TIME: 14:24:47

OPTIONS IN EFFECT:  NOLIST   MAP   XREF   GOSTMT NODECK   SOURCE   TERM   OBJECT FIXED  NOTEST NOTRMFLG
                    NOSYM NORENT NOSDUMP AUTODBL(NONE)    NOSXM
                      OPT(2) LANGLVL(77) NOFIPS    FLAG(I)  NAME(MAIN   )  LINECOUNT(60)      CHARLEN(500)

                   *....*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN      1               SUBROUTINE  GENMAJ(ELENG)
                   C==================================================================
                   C  SUBROUTINE GENERATING MATRICES J*1-J*11
                   C==================================================================
                   C
ISN      2               IMPLICIT REAL*8(A-H,O-Z)
ISN      3               DIMENSION RJS1(6,6),RJS2(6,6),RJS3(6,6),RJS4(6,6),RJS5(6,6),
                        *          RJS6(6,6),RJS8(6,6),RJS10(6,6),RJS11(6,6),RJS12(6,6)
ISN      4               COMMON /MAJS/RJS1,RJS2,RJS3,RJS4,RJS5,RJS6,RJS8,RJS10,RJS11,RJS12
                   C=================
ISN      5               DO 400 I=1,6
ISN      6               DO 400 J=1,6
ISN      7               RJS1(I,J)=0.D00
ISN      8               RJS2(I,J)=0.D00
ISN      9               RJS3(I,J)=0.D00
ISN     10               RJS4(I,J)=0.D00
ISN     11               RJS5(I,J)=0.D00
ISN     12               RJS6(I,J)=0.D00
ISN     13               RJS8(I,J)=0.D00
ISN     14               RJS10(I,J)=0.D00
ISN     15               RJS11(I,J)=0.D00
ISN     16               RJS12(I,J)=0.D00
ISN     17         400   CONTINUE
                   C
                   C  GENERATING MATRIX J*1
                   C
ISN     18               DO 402 I=1,4
ISN     19               DO 402 J=1,4
ISN     20               K=I+J-1
ISN     21               RJS1(I,J)=ELENG**(K)/DFLOAT(K)
ISN     22         402   CONTINUE
                   C
                   C  GENERATING MATRIX J*2
                   C
ISN     23               DO 410 II=2,4
ISN     24               DO 410 JJ=2,4
ISN     25               KK=II+JJ-3
ISN     26               RJS2(II,JJ)=ELENG**(KK)
ISN     27         410   CONTINUE
ISN     28               RJS2(3,3)=4.D0*RJS2(3,3)/3.D0
ISN     29               RJS2(3,4)=1.5D0*RJS2(3,4)
ISN     30               RJS2(4,3)=RJS2(3,4)
ISN     31               RJS2(4,4)=1.8D0*RJS2(4,4)
                   C
                   C  GENERATING MATRICES J*3
                   C
ISN     32               RJS3(3,3)=4.D0*ELENG
ISN     33               RJS3(3,4)=6.D0*ELENG**(2)
ISN     34               RJS3(4,3)=RJS3(3,4)
ISN     35               RJS3(4,4)=12.D0*ELENG**(3)
                   C
                   C  GENERATING MATRIX  J*4
```

```
                 *....*...1.........2.........3.........4.........5.........6.........7.*.......8
                 C
ISN   36                RJS4(5,5)=ELENG
ISN   37                RJS4(5,6)=.5D0*ELENG**(2)
ISN   38                RJS4(6,5)=RJS4(5,6)
ISN   39                RJS4(6,6)=ELENG**(3)/3.D0
                 C
                 C   GENERATING MATRIX J*5
                 C
ISN   40                DO 420 I=1,4
ISN   41                DO 420 J=2,4
ISN   42                K2 =J-1
ISN   43                K3= I+J-2
ISN   44                RJS5(I,J)=DFLOAT(K2)*ELENG**(K3)/DFLOAT(K3)
ISN   45         420    CONTINUE
                 C
                 C GENERATING MATRIX RJS6
                 C
ISN   46                RJS6(3,5)=2.D0*ELENG
ISN   47                RJS6(3,6)=ELENG**(2)
ISN   48                RJS6(4,5)=3.D0*RJS6(3,6)
ISN   49                RJS6(4,6)=2.D0*ELENG**(3)
                 C
                 C   GENERATING MATRIX  J*8
                 C
ISN   50                DO 430 L=2,4
ISN   51                LL=L-1
ISN   52                RJS8(L,6)=ELENG**(LL)
ISN   53         430    CONTINUE
                 C
                 C   GENERATING MATRIX J*10
                 C
ISN   54                RJS10(6,6)=ELENG
                 C
                 C   GENERATING MATRICES  J*11 AND J*12
                 C
ISN   55                DO 440 JJ=1,4
ISN   56                RJS11(JJ,3)=2.D0*ELENG**(JJ)/DFLOAT(JJ)
ISN   57                J1=JJ+1
ISN   58                RJS11(JJ,4)=6.D0*ELENG**(J1)/DFLOAT(J1)
ISN   59                RJS12(JJ,3)=2.D0*ELENG**(J1)/DFLOAT(J1)
ISN   60                RJS12(JJ,4)=6.D0*ELENG**(JJ+2)/DFLOAT(JJ+2)
ISN   61         440    CONTINUE
ISN   62                RETURN
ISN   63                END
```

```
LEVEL 1.4.0 (OCT 1984)          VS FORTRAN          DATE: AUG 25, 1986     TIME: 14:24:48

OPTIONS IN EFFECT: NOLIST  MAP   XREF   GOSTMT NODECK   SOURCE   TERM   OBJECT FIXED  NOTEST NOTRMFLG
                   NOSYM NORENT NOSDUMP AUTODBL(NONE)   NOSXM
                     OPT(2) LANGLVL(77) NOFIPS    FLAG(I)  NAME(MAIN  )  LINECOUNT(60)     CHARLEN(500)

                   *....*..1.........2.........3.........4.........5.........6.........7.*.......8

ISN   1                  SUBROUTINE  INVERS(ELENG)
                   C=======================================================================
                   C  SUBROUTINE INVERTING THE MATRIX A
                   C=======================================================================
                   C
ISN   2                  IMPLICIT REAL*8(A-H,O-Z)
ISN   3                  DIMENSION AINVE(6,6),AINVET(6,6)
ISN   4                  COMMON /INV/AINVE,AINVET
ISN   5                  DO 500 I=1,6
ISN   6                  DO 500 J=1,6
ISN   7                  AINVE(I,J)=0.D00
ISN   8            500   CONTINUE
ISN   9                  AINVE(1,1)=1.D0
ISN  10                  AINVE(2,2)=1.D0
ISN  11                  AINVE(5,3)=1.D0
ISN  12                  AINVE(3,1)=-3.D0/ELENG**(2)
ISN  13                  AINVE(3,2)=-2.D0/ELENG
ISN  14                  AINVE(3,4)=-AINVE(3,1)
ISN  15                  AINVE(3,5)=-1.D0/ELENG
ISN  16                  AINVE(4,1)=2.D0/ELENG**(3)
ISN  17                  AINVE(4,2)=ELENG**(-2)
ISN  18                  AINVE(4,4)=-AINVE(4,1)
ISN  19                  AINVE(4,5)=AINVE(4,2)
ISN  20                  AINVE(6,6)=1.D0/ELENG
ISN  21                  AINVE(6,3)=-AINVE(6,6)
                   C=======
ISN  22                  DO 505 I=1,6
ISN  23                  DO 505 J=1,6
ISN  24                  AINVET(I,J)=AINVE(J,I)
ISN  25            505    CONTINUE
ISN  26                  RETURN
ISN  27                  END
```

```
LEVEL 1.4.0 (OCT 1984)          VS FORTRAN          DATE: AUG 25, 1986     TIME: 14:24:48

OPTIONS IN EFFECT: NOLIST   MAP   XREF   GOSTMT NODECK   SOURCE   TERM   OBJECT FIXED   NOTEST NOTRMFLG
                   NOSYM NORENT NOSDUMP AUTODBL(NONE)   NOSXM
                    OPT(2) LANGLVL(77) NOFIPS   FLAG(I)  NAME(MAIN   )  LINECOUNT(60)     CHARLEN(500

                     *....*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN      1                 SUBROUTINE PRODMA(M,L,N,A,B,C)
                     C
                     C SUBROUTINE MULTIPLYING MATRICES
                     C
ISN      2                 IMPLICIT   REAL*8(A-H,O-Z)
ISN      3                 DIMENSION  A(M,L),B(L,N),C(M,N)
ISN      4                 DO 590  I=1,M
ISN      5                 DO 590  J=1,N
ISN      6                 C(I,J)=0.0D0
ISN      7                 DO 590  K=1,L
ISN      8                 C(I,J)=C(I,J)+A(I,K)*B(K,J)
ISN      9           590    CONTINUE
ISN     10                 RETURN
ISN     11                 END
```

```
LEVEL 1.4.0 (OCT 1984)          VS FORTRAN          DATE: AUG 25, 1986    TIME: 14:24:48

OPTIONS IN EFFECT:  NOLIST   MAP   XREF   GOSTMT NODECK   SOURCE   TERM   OBJECT FIXED  NOTEST NOTRMFLG
                    NOSYM NORENT NOSDUMP AUTODBL(NONE)    NOSXM
                     OPT(2) LANGLVL(77) NOFIPS    FLAG(I)  NAME(MAIN  )  LINECOUNT(60)       CHARLEN(500)

                    *....*..1.........2.........3.........4.........5.........6.........7.*.......8

ISN    1          SUBROUTINE ARRANG(N,X,PHI)
                C
                C SUBROUTINE ARRANGING EIGENVALUES FROM SMALL TO BIGGER
                C AND ARRANGING EIGENVECTORS CORRESPONDING TO EIGENVALUES
                C
ISN    2          IMPLICIT REAL*8(A-H,O-Z)
ISN    3          COMPLEX*16 X,PHI,PHIN,EIMIN,XEI
ISN    4          DIMENSION X(N),PHI(N,N)
ISN    5          LA =N-1
ISN    6          DO 600 I=1,LA
ISN    7          EIMIN=X(I)
ISN    8          XMIN=DIMAG(EIMIN)
ISN    9          JMI =I
ISN   10          JF  =I+1
ISN   11          DO 601 M=JF,N
ISN   12          XEI =X(M)
ISN   13          IF(DABS(XMIN).LE.DABS(DIMAG(XEI))) GOTO 601
ISN   14          JMI=M
ISN   15          XMIN=DIMAG(XEI)
ISN   16          EIMIN=XEI
ISN   17      601 CONTINUE
ISN   18          X(JMI)=X(I)
ISN   19          X(I)  =EIMIN
ISN   20          DO 633 L=1,N
ISN   21          PHIN      =PHI(L,JMI)
ISN   22          PHI(L,JMI)=PHI(L,I)
ISN   23          PHI(L,I)  =PHIN
ISN   24      633 CONTINUE
ISN   25      600 CONTINUE
ISN   26          RETURN
ISN   27          END
```

```
LEVEL 1.4.0 (OCT 1984)            VS FORTRAN          DATE: AUG 25, 1986    TIME: 14:24:48

OPTIONS IN EFFECT:  NOLIST    MAP    XREF    GOSTMT NODECK    SOURCE    TERM    OBJECT FIXED    NOTEST NOTRMFLG
                    NOSYM NORENT NOSDUMP AUTODBL(NONE)    NOSXM
                      OPT(2) LANGLVL(77) NOFIPS    FLAG(I)  NAME(MAIN  )  LINECOUNT(60)       CHARLEN(500

                    *....*..1.........2.........3.........4.........5.........6.........7.*.......8

ISN       1           SUBROUTINE OUTEIG(NDTT,N,X,PHI)
                C
                C SUBROUTINE TO PRINT EIGENVALUES AND EIGENVECTORS
                C
ISN       2           IMPLICIT REAL*8(A-H,O-Z)
ISN       3           DIMENSION X(NDTT),PHI(NDTT,NDTT)
ISN       4           COMPLEX*16 X,PHI
ISN       5           PRINT 740
ISN       6     740   FORMAT(///,14X,'====FREQUENCIES===='//)
ISN       7           M=NDTT/3
ISN       8           DO 741 J1=1,M
ISN       9                  J2=M+J1
ISN      10                  J3=2*M+J1
ISN      11           WRITE(6,750)J1,X(J1),J2,X(J2),J3,X(J3)
ISN      12     750   FORMAT(/,4X,I2,'TH',2D18.8,I3,'TH',2D18.8,I3,'TH',2D18.8)
ISN      13     741   CONTINUE
                C=============
                C ∂=(∂,∂') ∂ IS VECTOR
                C======================
ISN      14           NN= NDTT/2
ISN      15           DO 751 JA=21,N,3
ISN      16                  JB=JA+1
ISN      17                  JC=JA+2
ISN      18           WRITE(6,760)JA,JB,JC
ISN      19     760   FORMAT(//,9X,I2,'TH EIGENVECTOR',30X,I2,'TH EIGENVECTOR',32X,
                     *       I2,'TH EINGVECTOR')
ISN      20           DO 761 I=1,NN
ISN      21           WRITE(6,762)PHI(I,JA),PHI(I,JB),PHI(I,JC)
ISN      22     762   FORMAT(/4X,3(2D18.8,2X))
ISN      23     761   CONTINUE
ISN      24     751   CONTINUE
ISN      25           RETURN
ISN      26           END
```

```
LEVEL 1.4.0 (OCT 1984)          VS FORTRAN          DATE: AUG 25, 1986     TIME: 14:24:48

OPTIONS IN EFFECT: NOLIST   MAP   XREF   GOSTMT NODECK   SOURCE   TERM   OBJECT FIXED  NOTEST NOTRMFLG
                   NOSYM NORENT NOSDUMP AUTODBL(NONE)   NOSXM
                    OPT(2) LANGLVL(77) NOFIPS   FLAG(I)  NAME(MAIN   )  LINECOUNT(60)      CHARLEN(500

                  *....*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN      1          SUBROUTINE PRINT(N,A)                                                     00433
                 C                                                                           00434
                 C SUBROUTINE TO PRINT MATRIX                                                00435
                 C                                                                           00436
ISN      2          IMPLICIT REAL*8(A-H,O-Z)                                                  00437
ISN      3          DIMENSION A(6,6)                                                          00438
ISN      4          WRITE(6,822) N                                                            00439
ISN      5   822    FORMAT(///,10X,'MATRIX J',I2,//)                                          00440
ISN      6          WRITE(6,823)((A(I,J),J=1,6),I=1,6)                                        00441
ISN      7   823    FORMAT(/,6(4X,D12.5),/)                                                   00442
ISN      8          RETURN                                                                    00443
ISN      9          END                                                                       00444
```

OUT-OF PLANE MOTION (CLAMPED-CLAMPED)

ELEMENT CONNECTIVITY:

ELEMENT NUMBER	NODE					
1	1	2	3	4	5	6
2	4	5	6	7	8	9
3	7	8	9	10	11	12
4	10	11	12	13	14	15
5	13	14	15	16	17	18
6	16	17	18	19	20	21
7	19	20	21	22	23	24
8	22	23	24	25	26	27
9	25	26	27	28	29	30
10	28	29	30	31	32	33
11	31	32	33	34	35	36
12	34	35	36	37	38	39
13	37	38	39	40	41	42
14	40	41	42	43	44	45

DIMENSIONLESS PARAMETERS:

DIMENSIONLESS VELOCITY= 0.62832D+01

BETAA= 0.00000D+00,BETA= 0.50000D+00,H= 0.00000D+00

PI (INCLUDING), SIMA = 0.00000D+00,

CAPA= 0.76923D+00,RO= 0.31416D+01

====FREQUENCIES====

1TH	-0.100000000D+08	0.000000000D+00	31TH	0.190757900D-10	-0.153404930D+02	61TH	0.258412070D-07	-0.298084910D+04
2TH	-0.100000000D+08	0.000000000D+00	32TH	0.190746650D-10	0.153404930D+02	62TH	0.258407050D-07	0.298084910D+04
3TH	0.100000000D+08	0.000000000D+00	33TH	-0.248815400D-09	-0.542101850D+02	63TH	0.227088620D-07	0.334670470D+04

4TH	0.10000000D+08	0.00000000D+00	34TH	-0.24881181D-09	0.54210185D+02	64TH	0.22708826D-07	-0.33467047D+04
5TH	-0.10000000D+08	0.00000000D+00	35TH	0.35607998D-09	-0.11243650D+03	65TH	0.48150245D-07	-0.37860959D+04
6TH	0.10000000D+08	0.00000000D+00	36TH	0.35607910D-09	0.11243650D+03	66TH	0.48150328D-07	0.37860959D+04
7TH	-0.10000000D+08	0.00000000D+00	37TH	-0.35775688D-09	0.19115490D+03	67TH	-0.53206811D-08	0.42800683D+04
8TH	-0.25744299D+10	0.00000000D+00	38TH	-0.35775957D-09	-0.19115490D+03	68TH	-0.53213596D-08	-0.42800683D+04
9TH	-0.10000000D+08	0.00000000D+00	39TH	0.14268629D-08	-0.29019594D+03	69TH	-0.12804345D-07	-0.48291112D+04
10TH	-0.17819123D+10	0.00000000D+00	40TH	0.14268797D-08	0.29019594D+03	70TH	-0.12804374D-07	0.48291112D+04
11TH	-0.10000000D+08	0.00000000D+00	41TH	0.88769541D-09	0.40950918D+03	71TH	-0.94094836D-07	0.54375210D+04
12TH	0.14879235D+10	0.00000000D+00	42TH	0.88768552D-09	-0.40950918D+03	72TH	-0.94093854D-07	-0.54375210D+04
13TH	-0.10000000D+08	0.00000000D+00	43TH	0.23442163D-08	0.54924673D+03	73TH	0.24119692D-07	0.61105667D+04
14TH	-0.13412771D+10	0.00000000D+00	44TH	0.23442160D-08	-0.54924673D+03	74TH	0.24119302D-07	-0.61105667D+04
15TH	-0.10000000D+08	0.00000000D+00	45TH	-0.43268914D-08	-0.70972156D+03	75TH	0.21597549D-06	-0.68530135D+04
16TH	-0.12533290D+10	0.00000000D+00	46TH	-0.43269276D-08	0.70972156D+03	76TH	0.21597545D-06	0.68530135D+04
17TH	-0.10000000D+08	0.00000000D+00	47TH	-0.27105892D-08	0.89140908D+03	77TH	-0.16423430D-07	0.76671678D+04
18TH	-0.11946996D+10	0.00000000D+00	48TH	-0.27105461D-08	-0.89140908D+03	78TH	-0.16423493D-07	-0.76671678D+04
19TH	-0.10000000D+08	0.00000000D+00	49TH	-0.12445012D-08	0.10949402D+04	79TH	0.15521690D-06	-0.85495856D+04
20TH	-0.11528283D+10	0.00000000D+00	50TH	-0.12444250D-08	-0.10949402D+04	80TH	0.15521710D-06	0.85495856D+04
21TH	-0.10000000D+08	0.00000000D+00	51TH	0.13298778D-08	0.13210547D+04	81TH	0.28631730D-06	-0.94856839D+04
22TH	-0.11214406D+10	0.00000000D+00	52TH	0.13298388D-08	-0.13210547D+04	82TH	0.28631843D-06	0.94856839D+04
23TH	-0.10000000D+08	0.00000000D+00	53TH	0.12765857D-07	0.15704164D+04	83TH	0.61052548D-06	-0.10441956D+05
24TH	-0.63067413D+09	0.00000000D+00	54TH	0.12766158D-07	-0.15704164D+04	84TH	0.61052569D-06	0.10441956D+05
25TH	0.19301314D+10	0.00000000D+00	55TH	-0.43397272D-08	-0.18429290D+04	85TH	-0.48668138D-06	0.11357318D+05
26TH	0.12873496D+10	0.00000000D+00	56TH	-0.43400801D-08	0.18429290D+04	86TH	-0.48668184D-06	-0.11357318D+05
27TH	-0.14979330D+10	0.00000000D+00	57TH	0.18342318D-07	0.21345781D+04	87TH	-0.18439138D-06	0.12139280D+05
28TH	0.15091857D+10	0.00000000D+00	58TH	0.18342267D-07	-0.21345781D+04	88TH	-0.18439184D-06	-0.12139280D+05
29TH	0.56865107D+09	0.00000000D+00	59TH	-0.53326565D-12	-0.24156293D+04	89TH	-0.26603283D-06	-0.12675409D+05
30TH	-0.93777651D+08	0.00000000D+00	60TH	-0.60947073D-12	0.24156293D+04	90TH	-0.26603293D-06	0.12675409D+05

APPENDIX T

INTEGRATIONS BY PARTS ASSOCIATED WITH THE DERIVATION OF EQUATION (5.23)

Consider the equation

$$\sum_{i=1}^{n} \int_{o}^{\xi_i} \{\delta\eta_1^o A_{i1}^o(\eta_1^o,\eta_3^o) + \delta\eta_3^o A_{i3}^o(\eta_1^o,\eta_3^o)\}\, d\xi = 0 \,, \tag{T.1}$$

where

$$A_{i1}^o(\eta_1^o,\eta_3^o) = \left(\frac{\partial^4\eta_1^o}{\partial\xi^4} + r_o \frac{\partial^3\eta_3^o}{\partial\xi^3}\right) + \frac{\partial}{\partial\xi}\left[\Pi_p\left(\frac{\partial\eta_1^o}{\partial\xi} + r_o\eta_3^o\right)\right] - \mathcal{A} r_o\left(\frac{\partial\eta_3^o}{\partial\xi} - r_o\eta_1^o\right) + \bar{u}^2\left(\frac{\partial^2\eta_1^o}{\partial\xi^2} + r_o\frac{\partial\eta_3^o}{\partial\xi}\right) + r_o(\Pi_p + \bar{u}^2) - \gamma\alpha_{x_o} \,, \tag{T.2}$$

$$A_{i3}^o(\eta_1^o,\eta_3^o) = -\mathcal{A}\left(\frac{\partial^2\eta_3^o}{\partial\xi^2} - r_o\frac{\partial\eta_1^o}{\partial\xi}\right) - r_o\left(\frac{\partial^3\eta_1^o}{\partial\xi^3} + r_o\frac{\partial^2\eta_3^o}{\partial\xi^2}\right) - r_o\Pi_p\left(\frac{\partial\eta_1^o}{\partial\xi} + r_o\eta_3^o\right) - r_o\bar{u}^2\left(\frac{\partial\eta_1^o}{\partial\xi} + r_o\eta_3^o\right) + \frac{\partial\Pi_p}{\partial\xi} - \gamma\alpha_{z_o} \,. \tag{T.3}$$

By performing integrations by parts, one obtains the following:

$$\sum_{i=1}^{n} \int_{o}^{\xi_i} \delta\eta_1^o\left(\frac{\partial^4\eta_1^o}{\partial\xi^4} + r_o\frac{\partial^3\eta_3^o}{\partial\xi^3}\right) d\xi = \sum_{i=1}^{n} \left\{\delta\eta_1^o\left(\frac{\partial^3\eta_1^o}{\partial\xi^3} + r_o\frac{\partial^2\eta_3^o}{\partial\xi^2}\right) - \delta\left(\frac{\partial\eta_1^o}{\partial\xi}\right)\left(\frac{\partial^2\eta_1^o}{\partial\xi^2} + r_o\frac{\partial\eta_3^o}{\partial\xi}\right)\right\}_o^{\xi_i} + \sum_{i=1}^{n}\int_o^{\xi_i} \delta\left(\frac{\partial^2\eta_1^o}{\partial\xi^2}\right)\left(\frac{\partial^2\eta_1^o}{\partial\xi^2} + r_o\frac{\partial\eta_3^o}{\partial\xi}\right) d\xi \tag{T.4}$$

$$= \left\{\delta\eta_1^o\left(\frac{\partial^3\eta_1^o}{\partial\xi^3} + r_o\frac{\partial^2\eta_3^o}{\partial\xi^2}\right) - \delta\left(\frac{\partial\eta_1^o}{\partial\xi}\right)\left(\frac{\partial^2\eta_1^o}{\partial\xi^2} + r_o\frac{\partial\eta_3^o}{\partial\xi}\right)\right\}_o^1 + \sum_{i=1}^{n}\int_o^{\xi_i} \delta\left(\frac{\partial^2\eta_1^o}{\partial\xi^2}\right)\left(\frac{\partial^2\eta_1^o}{\partial\xi^2} + r_o\frac{\partial\eta_3^o}{\partial\xi}\right) d\xi \,, \tag{T.5}$$

$$\sum_{i=1}^{n} \int_{o}^{\xi_i} \delta\eta_3^o\left(\frac{\partial^2\eta_3^o}{\partial\xi^2} - r_o\frac{\partial\eta_1^o}{\partial\xi}\right) d\xi = \left\{\delta\eta_3^o\left(\frac{\partial\eta_3^o}{\partial\xi} - r_o\eta_1^o\right)\right\}_o^1 - \sum_{i=1}^{n}\int_o^{\xi_i} \delta\left(\frac{\partial\eta_3^o}{\partial\xi}\right) \times \left(\frac{\partial\eta_3^o}{\partial\xi} - r_o\eta_1^o\right) d\xi \,, \tag{T.6}$$

$$\sum_{i=1}^{n} \int_{o}^{\xi_i} \delta\eta_3^o \left(\frac{\partial^3 \eta_1^o}{\partial \xi^3} + r_o \frac{\partial^2 \eta_3^o}{\partial \xi^2}\right) d\xi = \left\{\delta\eta_3^o \left(\frac{\partial^2 \eta_1^o}{\partial \xi^2} + r_o \frac{\partial \eta_3^o}{\partial \xi}\right)\right\}_o^1 - \sum_{i=1}^{n} \int_{o}^{\xi_i} \delta\left(\frac{\partial \eta_3^o}{\partial \xi}\right) \left(\frac{\partial^2 \eta_1^o}{\partial \xi^2} + r_o \frac{\partial \eta_3^o}{\partial \xi}\right) d\xi \ . \quad \text{(T.7)}$$

It is noted that in equations (T.5)-(T.7), the following replacement was made

$$\sum_{i=1}^{n} \{\ \}_o^{\xi_i} = \{\ \}_o^1 \ , \quad \text{(T.8)}$$

because of the continuity condition, as mentioned in Appendix J.

Substituting equations (T.5)-(T.7) into equation (T.1) yields

$$\sum_{i=1}^{n} \int_{o}^{\xi_i} \Big\{\delta\left(\frac{\partial^2 \eta_1^o}{\partial \xi^2}\right) \left(\frac{\partial^2 \eta_1^o}{\partial \xi^2} + r_o \frac{\partial \eta_3^o}{\partial \xi}\right) + \delta\eta_1^o \Big[\frac{\partial}{\partial \xi}\Big[\Pi_p \left(\frac{\partial \eta_1^o}{\partial \xi} + r_o \eta_3^o\right)\Big] + \bar{u}^2 \left(\frac{\partial^2 \eta_1^o}{\partial \xi^2} + r_o \frac{\partial \eta_3^o}{\partial \xi}\right) - \mathcal{A} r_o \left(\frac{\partial \eta_3^o}{\partial \xi} - r_o \eta_1^o\right) + r_o (\Pi_p + \bar{u}^2) - \gamma \alpha_{x_o}\Big] + \delta\left(\frac{\partial \eta_3^o}{\partial \xi}\right) \Big[\mathcal{A}\left(\frac{\partial \eta_3^o}{\partial \xi} - r_o \eta_1^o\right) + r_o \left(\frac{\partial^2 \eta_1^o}{\partial \xi^2} + r_o \frac{\partial \eta_3^o}{\partial \xi}\right)\Big] + \delta\eta_3^o \Big[-r_o (\Pi_p + \bar{u}^2) \left(\frac{\partial \eta_1^o}{\partial \xi} + r_o \eta_3^o\right) + \frac{\partial \Pi_p}{\partial \xi} - \gamma \alpha_{z_o}\Big]\Big\} d\xi$$

$$+ \left\{\delta \eta_1^o \left(\frac{\partial^3 \eta_1^o}{\partial \xi^3} + r_o \frac{\partial^2 \eta_3^o}{\partial \xi^2}\right) - \delta\left(\frac{\partial \eta_1^o}{\partial \xi}\right) \left(\frac{\partial^2 \eta_1^o}{\partial \xi^2} + r_o \frac{\partial \eta_3^o}{\partial \xi}\right)\right\}_o^1 - \left\{\delta\eta_3^o \Big[\mathcal{A}\left(\frac{\partial \eta_3^o}{\partial \xi} - r_o \eta_1^o\right) + r_o \left(\frac{\partial^2 \eta_1^o}{\partial \xi^2} + r_o \frac{\partial \eta_3^o}{\partial \xi}\right)\Big]\right\}_o^1 \ . \quad \text{(T.9)}$$

From the boundary conditions, equations (5.9)-(5.11), one can see that the integrated terms vanish. Therefore, one obtains

$$\sum_{i=1}^{n} \int_{o}^{\xi_i} \Big\{\delta\left(\frac{\partial^2 \eta_1^o}{\partial \xi^2}\right) \left(\frac{\partial^2 \xi_1^o}{\partial \xi^2} + r_o \frac{\partial \eta_3^o}{\partial \xi}\right) + \delta\eta_1^o \Big[\frac{\partial}{\partial \xi}\Big[\Pi_p \left(\frac{\partial \eta_1^o}{\partial \xi} + r_o \eta_3^o\right)\Big] - r_o \mathcal{A} \left(\frac{\partial \eta_3^o}{\partial \xi} - r_o \eta_1^o\right) + \bar{u}^2 \left(\frac{\partial^2 \eta_1^o}{\partial \xi^2} + r_o \frac{\partial \eta_3^o}{\partial \xi}\right)\Big] \ \delta\eta_1^o (r_o (\Pi_p + \bar{u}^2) - \gamma \alpha_{x_o}) + \delta\left(\frac{\partial \eta_3^o}{\partial \xi}\right) \Big[\mathcal{A}\left(\frac{\partial \eta_3^o}{\partial \xi} - r_o \eta_1^o\right) + r_o \left(\frac{\partial^2 \eta_1^o}{\partial \xi^2} + r_o \frac{\partial \eta_3^o}{\partial \xi}\right)\Big] - r_o (\Pi_p + \bar{u}^2) \delta\eta_3^o \left(\frac{\partial \eta_1^o}{\partial \xi} + r_o \eta_3^o\right) + \delta\eta_3^o \left(\frac{\partial \Pi_p}{\partial \xi} - \gamma \alpha_{z_o}\right)\Big\} d\xi = 0 \ , \quad \text{(T.10)}$$

which is equation (5.23), appearing in the main text.

APPENDIX U

THE RESISTANCE COEFFICIENT FOR TURBULENT FLOW IN A CURVED PIPE

Consider turbulent flow in a curved pipe. The influence of centrifugal force is to increase the frictional resistance coefficient for tubulent flow in a curved pipe vis-à-vis that of a straight pipe. According to the theory of White (1979), this resistance coefficient (for turbulent flow in a curved pipe) can be represented by the following equation

$$\lambda = \lambda_o \left(1 + 0.075\, Re_d^{0.25} \left(\frac{D_i}{2R_o}\right)^{0.5}\right), \qquad (U.1)$$

where λ_o is the resistance coefficient of a straight pipe, Re_d is the Reynolds number (i.e. UD_i/ν), D_i is the internal diameter of pipe and R_o is the radius of curvature of the pipe.

In general, λ_o is a function of Reynolds number and the dimensionless roughness of the pipe, and its value can be obtained from Moody's diagram. Therefore, the pressure gradient in the pipe can be written as

$$\frac{\partial P_i}{\partial s} = -\frac{\lambda}{D_i}\, \rho_f \frac{U^2}{2}\,. \qquad (U.2)$$

Finally, the dimensionless form of equation (T.2) is

$$\frac{\partial (P_i A_i L^2/EI)}{\partial \xi} = -\lambda^* \bar{u}^2, \qquad (U.3)$$

where

$$\lambda^* = \frac{\lambda L}{2D_i}, \qquad (U.4)$$

and, L is the total length of the pipe. Equation (U.3) is used in obtaining equation (5.28).

APPENDIX V

DERIVATION OF THE ELEMENT STIFFNESS MATRIX AND THE ELEMENT FORCE VECTOR IN THE CASE OF STATIC EQUILIBRIUM

Consider equations (5.26) and (5.27) in the main text. The highest order of derivatives of the shape functions $[N^o_{e1}]$ and $[N^o_{e3}]$ in the integrands of these equations are second and first, respectively, and it is necessary to ensure that η^o_1, $\eta^{o\prime}_1$ and η^o_3 are continuous at the element boundaries. Thus the nodal displacements chosen are

$$\{\eta^o_i\}_j = \begin{Bmatrix} \eta^o_{1j} \\ \eta^{o\prime}_{1j} \\ \eta^o_{3j} \end{Bmatrix}, \tag{V.1}$$

as shown in Fig. V.1, and the element displacement vector can be written as

$$\{\eta^o_i\}^e = \begin{Bmatrix} \{\eta^o_i\}_j \\ \text{-----} \\ \{\eta^o_i\}_{j+1} \end{Bmatrix}. \tag{V.2}$$

Accordingly, displacement functions associated with η^o_1 and η^o_3 can be chosen to be the same as for the out-of-plane motion in the inextensible case. Therefore, proceeding in the same way that was followed in the numerical analysis of the out-of-plane motion in the inextensible case, one obtains the following relations

$$\left.\begin{aligned} [N^o_{e1}] &= [\Phi_2][A]^{-1}_o, \\ [N^o_{e3}] &= [\Phi_4][A]^{-1}_o, \end{aligned}\right\} \tag{V.3}$$

where $[\Phi_2]$, $[\Phi_4]$ and $[A]_o^{-1}$ have been defined in equations (3.62), (3.63) and (3.71) in Chapter III.

By substituting equation (U.3) into equations (5.26) and (5.27) in the main text yields

$$\{K_1^o\}^e = [A]_o^{-1T}\{\int_0^{\xi_e}\{[\Phi_2]''^T[\Phi_2]'' + r_o([\Phi_2]''^T[\Phi_4]' + [\Phi_4]'^T[\Phi_2]'') + r_o^2[\Phi_4]'^T[\Phi_4]'$$

$$+ \mathcal{A}([\Phi_4]'^T[\Phi_4]' - r_o([\Phi_2]^T[\Phi_4]' + [\Phi_4]'^T[\Phi_2]) + r_o^2 [\Phi_2]^T[\Phi_2]$$

$$+ \bar{u}^2[[\Phi_2]^T([\Phi_2]'' + r_o[\Phi_4]') - r_o[\Phi_4]^T([\Phi_2]' + r_o[\Phi_4])]$$

$$+ [\Phi_2]^T \frac{\partial}{\partial\xi}[\Pi_p([\Phi_2]' + r_o[\Phi_4])] - r_o\Pi_p[\Phi_4]^T([\Phi_2]' + r_o[\Phi_4])\} d\xi \}[A]_o^{-1}, \tag{V.4}$$

$$\{F\}^e = -[A]_o^{-1T} \int_0^{\xi e} \{(r_o(\Pi_p + \bar{u}^2) - \gamma\alpha_{x_o})[\Phi_2]^T + (\frac{\partial\Pi_p}{\partial\xi} - \gamma\alpha_{z_o})[\Phi_4]^T\} d\xi.$$

Finally, substituting equation (5.28) into (V.4) and then evaluating the integrals, one obtains the element matrices, i.e.,

$$[K_1^o]^e = [A]_o^{-1T} \{[[J_3^*]+r_o([J_{15}^*]+[J_{15}^*]^T)+r_o^2[J_8^*]] + \mathcal{A}[[J_8^*]-r_o([J_{12}^*]+[J_{12}^*]^T) + r_o^2[J_1^*]]$$

$$+ \bar{u}^2 [[J_9^*]+r_o([J_{12}^*]-[J_{13}^*])-r_o^2[J_4^*]]$$

$$+ \Pi_p|_o [[J_9^*]+r_o([J_{12}^*]-[J_{13}^*])-r_o^2[J_4^*]] \tag{V.5}$$

$$-\lambda^*\bar{u}^2[[J_5^*]+[J_{10}^*]+r_o([J_{16}^*]+[J_{14}^*]-[J_{17}^*])-r_o^2[J_{18}^*]]\} [A]_o^{-1},$$

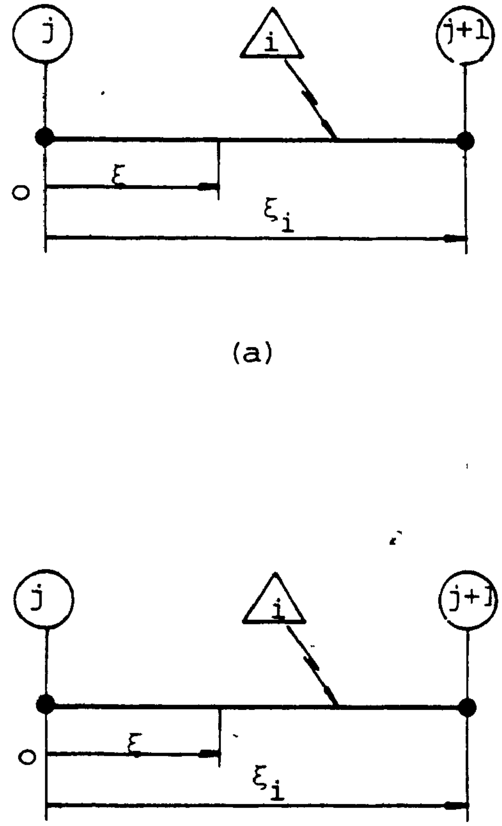
$$\{F\}^e = -[A]_o^{-1T} \{r_o(\bar{u}^2+\Pi_p|_o) \{F_1\} - \lambda^*\bar{u}^2(r_o\{F_2\} + \{F_3\}) + \{F_G\}\},$$

where

$$\left.\begin{aligned}
\{F_0\} &= \int_0^{\xi_e} -\gamma(\alpha_{x_0}[\Phi_2]^T + \alpha_{z_0}[\Phi_4]^T)\, d\xi \,, \\
\{F_1\} &= \int_0^{\xi_e} [\Phi_2]^T\, d\xi = \begin{Bmatrix} \xi_e \\ 1/2\ \xi_e^2 \\ 1/3\ \xi_e^3 \\ 1/4\ \xi_e^4 \\ 0 \\ 0 \end{Bmatrix}, \\
\{F_2\} &= \int_0^{\xi_e} \xi[\Phi_2]^T\, d\xi = \begin{Bmatrix} 1/2\ \xi_e^2 \\ 1/3\ \xi_e^3 \\ 1/4\ \xi_e^4 \\ 1/5\ \xi_e^5 \\ 0 \\ 0 \end{Bmatrix}, \\
\{F_3\} &= \int_0^{\xi_e} [\Phi_4]^T\, d\xi = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \xi_e \\ 1/2\ \xi_e^2 \end{Bmatrix}.
\end{aligned}\right\} \qquad (V.6)$$

$$
\left.
\begin{aligned}
[J_1^*] &= \int_0^{\xi_e} [\Phi_2]^T [\Phi_2] \, d\xi \, , \\
[J_3^*] &= \int_0^{\xi_e} [\Phi_2]''^T [\Phi_2]'' \, d\xi \, , \\
[J_4^*] &= \int_0^{\xi_e} [\Phi_4]^T [\Phi_4] \, d\xi \, , \\
[J_5^*] &= \int_0^{\xi_e} [\Phi_2]^T [\Phi_2]' \, d\xi \, , \\
[J_8^*] &= \int_0^{\xi_e} [\Phi_4]'^T [\Phi_4]' \, d\xi \, ; \\
[J_9^*] &= \int_0^{\xi_e} [\Phi_2]^T [\Phi_2]'' \, d\xi , \\
[J_{10}^*] &= \int_0^{\xi_e} [\Phi_2]^T [\Phi_2]'' \xi \, d\xi \, , \\
[J_{12}^*] &= \int_0^{\xi_e} [\Phi_2]^T [\Phi_4]' \, d\xi \, , \\
[J_{13}^*] &= \int_0^{\xi_e} [\Phi_4]^T [\Phi_2]' \, d\xi \, , \\
[J_{14}^*] &= \int_0^{\xi_e} [\Phi_2]^T [\Phi_4] \, d\xi \, , \\
[J_{15}^*] &= \int_0^{\xi_e} [\Phi_2]'' [\Phi_4]' \, d\xi \, , \\
[J_{16}^*] &= \int_0^{\xi_e} [\Phi_2]^T [\Phi_4]' \xi \, d\xi \, , \\
[J_{17}^*] &= \int_0^{\xi_e} [\Phi_4]^T [\Phi_2]' \xi \, d\xi \, , \\
[J_{18}^*] &= \int [\Phi_4]^T [\Phi_4] \xi \, d\xi \, ,
\end{aligned}
\right\} \qquad (V.7)
$$

which are evaluated in Appendix L. It is noted that equation set (V.5) corresponds to equations (5.29) and (5.30) in the main text.



$$\{\eta_i^o\}_j = \begin{Bmatrix} \eta^o_{1j} \\ \eta^{o\prime}_{1j} \\ \eta^o_{3j} \end{Bmatrix} \qquad \{\eta_i^o\}_{j+1} = \begin{Bmatrix} \eta^o_{1j+1} \\ \eta^{o\prime}_{1j+1} \\ \eta^o_{3j+1} \end{Bmatrix}$$

(a)

$$\{\eta_i^*\}_j = \begin{Bmatrix} \eta^*_{1j} \\ \eta^{*\prime}_{1j} \\ \eta^*_{3j} \end{Bmatrix} \qquad \{\eta_i^*\}_{j+1} \begin{Bmatrix} \eta^*_{1j+i} \\ \eta^{*\prime}_{1j+1} \\ \eta^*_{3j+1} \end{Bmatrix}$$

(b)

Fig. V.1 Two-Node Pipe Element for the Case of

(a) In-plane extensible static deformation

(b) In-plane extensible motion

APPENDIX W

INTEGRATIONS BY PARTS ASSOCIATED WITH THE DERIVATION OF EQUATION (5.34)

Consider the equation

$$\sum_{i=1}^{n} \int_{o}^{\xi_i} \{\delta\eta_1^* A_{i1}^*(\eta_1^*,\eta_3^*) + \delta\eta_3^* A_{i3}^*(\eta_1^*,\eta_3^*)\} d\xi = 0 , \qquad (W.1)$$

where, as given by equations (5.17) and (5.18),

$$A_{i1}^*(\eta_1^*,\eta_3^*) = (\frac{\partial^4\eta_1^*}{\partial\xi^4} + r_o \frac{\partial^3\eta_3^*}{\partial\xi^3}) + \frac{\partial}{\partial\xi}[\Pi_o(\frac{\partial\eta_1^*}{\partial\xi} + r_o\eta_3^*)] - \mathcal{A}\frac{\partial}{\partial\xi}[(\frac{\partial\eta_1^o}{\partial\xi} + r_o\eta_3^o)(\frac{\partial\eta_3^*}{\partial\xi} - r_o\eta_1^*)]$$
$$- r_o\mathcal{A}(\frac{\partial\eta_3^*}{\partial\xi} - r_o\eta_1^*) + \bar{u}^2(\frac{\partial^2\eta_1^*}{\partial\xi^2} + r_o\frac{\partial\eta_3^*}{\partial\xi}) + 2\beta^{1/1}\bar{u}(\frac{\partial^2\eta_1^*}{\partial\tau\partial\xi} + r_o\frac{\partial\eta_3^*}{\partial\tau})$$
$$+ \mathcal{K}\frac{\partial\eta_1^*}{\partial\tau} + (1+\beta_a)\frac{\partial^2\eta_1^*}{\partial\tau^2} , \qquad (W.2)$$

$$A_{i3}^*(\eta_1^*,\eta_3^*) = -r_o(\frac{\partial^3\eta_1^*}{\partial\xi^3} + r_o\frac{\partial^2\eta_3^*}{\partial\xi^2}) - r_o\Pi_o(\frac{\partial\eta_1^*}{\partial\xi} + r_o\eta_3^*) + r_o\mathcal{A}(\frac{\partial\eta_1^o}{\partial\xi} + r_o\eta_3^o)(\frac{\partial\eta_3^*}{\partial\xi} - r_o\eta_1^*)$$
$$- \mathcal{A}(\frac{\partial^2\eta_3^*}{\partial\xi^2} - r_o\frac{\partial\eta_1^*}{\partial\xi}) - r_o\bar{u}^2(\frac{\partial\eta_1^*}{\partial\xi} + r_o\eta_3^*) + \beta^{1/2}\bar{u}(\frac{\partial^2\eta_3^*}{\partial\tau\partial\xi} - r_o\frac{\partial\eta_1^*}{\partial\tau})$$
$$+ \mathcal{K}'\frac{\partial\eta_3^*}{\partial\tau} + (1+\beta_a')\frac{\partial^2\eta_3^*}{\partial\tau^2}, \qquad (W.3)$$

By performing intergrations by parts, one obtains the following:

$$\sum_{i=1}^{n}\int_o^{\xi_i} \delta\eta_1^*(\frac{\partial^4\eta_1^*}{\partial\xi^4} + r_o\frac{\partial^3\eta_3^*}{\partial\xi^3})d\xi = \sum_{i=1}^{n}\{\delta\eta_1^*(\frac{\partial^3\eta_1^*}{\partial\xi^3} + r_o\frac{\partial^2\eta_3^*}{\partial\xi^2}) - \delta(\frac{\partial\eta_1^*}{\partial\xi})(\frac{\partial^2\eta_1^*}{\partial\xi^2} + r_o\frac{\partial\eta_3^*}{\partial\xi})\}_o^{\xi_i}$$
$$+ \sum_{i=1}^{n}\int_o^{\xi_i}\delta(\frac{\partial^2\eta_1^*}{\partial\xi^2})(\frac{\partial^2\eta_1^*}{\partial\xi^2} + r_o\frac{\partial\eta_3^*}{\partial\xi})d\xi,$$
$$= \{\delta\eta_1^*(\frac{\partial^3\eta_1^*}{\partial\xi^3} + r_o\frac{\partial^2\eta_3}{\partial\xi^2}) - \delta(\frac{\partial\eta_1^*}{\partial\xi})(\frac{\partial^2\eta_1^*}{\partial\xi^2} + r_o\frac{\partial\eta_3^*}{\partial\xi})\}_o^1$$
$$+ \sum_{i=1}^{n}\int_o^{\xi_i}\delta(\frac{\partial^2\eta^*}{\partial\xi^2})(\frac{\partial^2\eta_1^*}{\partial\xi^2} + r_o\frac{\partial\eta_3^*}{\partial\xi})d\xi, \qquad (W.4)$$

$$\sum_{i=1}^{n}\int_{o}^{\xi_i} \delta\eta_1^{*}\frac{\partial}{\partial\xi}\left[\left(\frac{\partial\eta_1^{o}}{\partial\xi}+r_o\eta_3^{o}\right)\left(\frac{\partial\eta_3^{*}}{\partial\xi}-r_o\eta_1^{*}\right)\right]d\xi = \left\{\delta\eta_1^{*}\left(\frac{\partial\eta_1^{o}}{\partial\xi}+r_o\eta_3^{o}\right)\left(\frac{\partial\eta_3^{*}}{\partial\xi}-r_o\eta_1^{*}\right)\right\}_{o}^{1}$$

$$-\sum_{i=1}^{n}\int_{o}^{\xi_i}\delta\left(\frac{\partial\eta_1^{*}}{\partial\xi}\right)\left(\frac{\partial\eta_1^{o}}{\partial\xi}+r_o\eta_3^{o}\right)\left(\frac{\partial\eta_3^{*}}{\partial\xi}-r_o\eta_1^{*}\right)d\xi\ , \qquad \text{(W.5)}$$

$$\sum_{i=1}^{n}\int_{o}^{\xi_i}\delta\eta_3^{*}\left(\frac{\partial^3\eta_1^{*}}{\partial\xi^3}+r_o\frac{\partial^2\eta_3^{*}}{\partial\xi^2}\right)d\xi = \left\{\delta\eta_3^{*}\left(\frac{\partial^2\eta_1^{*}}{\partial\xi^2}+r_o\frac{\partial\eta_3^{*}}{\partial\xi}\right)\right\}_{o}^{1} - \sum_{i=1}^{n}\int_{o}^{\xi_i}\delta\left(\frac{\partial\eta_3^{*}}{\partial\xi}\right)$$

$$\times\left(\frac{\partial^2\eta_1^{*}}{\partial\xi^2}+r_o\frac{\partial\eta_3^{*}}{\partial\xi}\right)d\xi, \qquad \text{(W.6)}$$

$$\sum_{i=1}^{n}\int_{o}^{\xi_i}\delta\eta_3^{*}\left(\frac{\partial^2\eta_3^{*}}{\partial\xi^2}-r_o\frac{\partial\eta_1^{*}}{\partial\xi}\right)d\xi = \left\{\delta\eta_3^{*}\left(\frac{\partial\eta_3^{*}}{\partial\xi}-r_o\eta_1^{*}\right)\right\}_{o}^{1} - \sum_{i=1}^{n}\int_{o}^{\xi_i}\delta\left(\frac{\partial\eta_3^{*}}{\partial\xi}\right)\left(\frac{\partial\eta_3^{*}}{\partial\xi}-r_o\eta_1^{*}\right)d\xi\ . \qquad \text{(W.7)}$$

It is noted that in equations (W.4)-(W.7), the following substitution was made

$$\sum_{i=1}^{n}\{\ \}_{o}^{\xi_i} = \{\ \}_{o}^{1}, \qquad \text{(W.8)}$$

because of the continuity condition across the finite elements.

From the boundary conditions, equations (5.19)-(5.21), one can see that the integrated terms vanish. Therefore, one obtains

$$\sum_{i=1}^{n}\int_{o}^{\xi_i}\{\delta\eta_1^{*''}(\eta_1^{*''}+r_o\eta_3^{*'})+r_o\delta\eta_3^{*'}(\eta_1^{*''}+r_o\eta_3^{*'})+\mathcal{A}[r_o\delta\eta_1^{*}(r_o\eta_1^{*}-\eta_3^{*'})+\delta\eta_3^{*'}(\eta_3^{*'}-r_o\eta_1^{*})]$$

$$+\bar{u}^2[\delta\eta_1^{*}(\eta_1^{*''}+r_o\eta_3^{*'})-r_o\delta\eta_3^{*}(\eta_1^{*'}+r_o\eta_3^{*})]+\delta\eta_1^{*}\frac{\partial}{\partial\xi}[\Pi_o(\eta_1^{*'}+r_o\eta_3^{*})-r_o\Pi_o\delta\eta_3^{*}(\eta_1^{*'}+r_o\eta_3^{*})$$

$$+\mathcal{A}(\eta_1^{o'}+r_o\eta_3^{o})[\delta\eta_1^{*'}(\eta_3^{*'}-r_o\eta_1^{*})+r_o\delta\eta_3^{*}(\eta_3^{*'}-r_o\eta_1^{*})]+\beta^{1/2}\bar{u}[2\delta\eta_1^{*}(\dot{\eta}_1^{*'}+r_o\dot{\eta}_3^{*})$$

$$+\delta\eta_3^{*}(\dot{\eta}_3^{*'}-r_o\dot{\eta}_1^{*})]+\mathcal{K}\delta\eta_1^{*}\dot{\eta}_1^{*}+\mathcal{K}'\delta\eta_3^{*}\dot{\eta}_3^{*}$$

$$+(1+\beta_a)\delta\eta_1^{*}\ddot{\eta}_1^{*}+(1+\beta_a')\delta\eta_3^{*}\ddot{\eta}_3^{*}\}\ d\xi = 0, \qquad \text{(W.9)}$$

which is equation (5.34), appearing in the main text.

APPENDIX X

DERIVATION OF THE ELEMENT MASS, DAMPING AND STIFFNESS MATRICES

Consider equations (5.37)-(5.39) in the main text. The highest order of derivatives of the shape functions $[N^*_{e1}]$ and $[N^*_{e3}]$ which exist in the integrands of these equations are second and first respectively, and it is necessary to ensure that η^o_1, $\eta^{o'}_1$ and η^o_3 are continuous at the element boundaries. Thus nodal displacements chosen are

$$\{\eta^*_i\}_j = \begin{Bmatrix} \eta^o_{1j} \\ \eta^{o'}_{1j} \\ \eta^o_{3j} \end{Bmatrix}, \qquad \text{(X.1)}$$

as shown in Fig. U.1 and the element displacement vector can be written as

$$\{\eta^*\}^e_i = \begin{Bmatrix} \{\eta^*_i\}_j \\ \text{------} \\ \{\eta^*_i\}_{j+1} \end{Bmatrix}. \qquad \text{(X.2)}$$

Similarly to the case of static equilibrium, the displacement functions associated with η^*_{e1} and η^*_{e3} can be chosen the same as for the out-of-plane motion in the inextensible case. Therefore, from the numerical analysis of out-of-plane motion in the inextensible case one can write the following relations

$$\left.\begin{aligned} [N^*_{e1}] &= [\phi_2][A]^{-1}_o, \\ [N^*_{e3}] &= [\phi_4][A]^{-1}_o, \end{aligned}\right\} \qquad \text{(X.3)}$$

where $[\phi_2]$, $[\phi_4]$ and $[A]^{-1}_o$ have been defined in equations (3.62) (3.63), and (3.71) in Chapter III.

By substituting equation (W.3) into equations (5.36)-(5.39) yields

$$[M_i^*]^e = [A]_o^{-1T} \int_o^{\xi_e} \{(1+\beta_a)[\Phi_2]^T[\Phi_2]+(1+\beta_a')[\Phi_4]^T[\Phi_4]\}\, d\xi\, [A]_o^{-1},$$

$$[D_i^*]^e = [A]_o^{-1T} \int_o^{\xi_e} \{\beta^{1/2}\bar{u}[2[\Phi_2]^T([\Phi_2]'+r_o[\Phi_4])+[\Phi_4]^T([\Phi_4]'-r_o[\Phi_2])]$$
$$+ \mathcal{K}[\Phi_2]^T[\Phi_2] + \mathcal{K}'[\Phi_4]^T[\Phi_4]]\}\, d\xi\, [A]_o^{-1},$$

$$[K_i^*]^e = [A]_o^{-1T} \int_o^{\xi_e} \{[\Phi_2]''^T[\Phi_2]'' + r_o([\Phi_2]''^T[\Phi_4]' + [\Phi_4]'^T[\Phi_2]'')+r_o^2[\Phi_4]'^T[\Phi_4]'$$
$$+ \mathcal{A}[[\Phi_4]'^T[\Phi_4]' - r_o([\Phi_2]'^T[\Phi_4]' + [\Phi_4]'^T[\Phi_2]')+r_o^2[\Phi_2]^T[\Phi_2]]$$
$$+ \mathcal{A}(\eta_1^{o'}+r_o\eta_3^o)[[\Phi_2]'^T([\Phi_4]'-r_o[\Phi_2])+r_o[\Phi_4]^T([\Phi_4]'-r_o[\Phi_2])]$$
$$+ \bar{u}^2[[\Phi_2]^T([\Phi_2]''+r_o[\Phi_4]') - r_o[\Phi_4]^T([\Phi_2]'+r_o[\Phi_4])]$$
$$+ [\Phi_2]^T \frac{\partial}{\partial\xi}[\Pi_o([\Phi_2]'+r_o[\Phi_4])-r_o\Pi_o[\Phi_4]^T([\Phi_2]'+r_o[\Phi_4])\}\, d\xi\, [A]_o^{-1}. \quad \text{(X.4)}$$

Finally, substituting equation set (5.41) into equation set (W.4) and then evaluating the integrals, one obtains the element matrices

$$[M_i^*]^e = [A]_o^{-1T}\{(1+\beta_a)[J_1^*] + (1+\beta_a')[J_4^*]\}[A]_o^{-1}, \quad \text{(X.5)}$$

$$[D_i^*]^e = [A]_o^{-1T}\{\beta^{1/2}\bar{u}[2([J_5^*]+r_o[J_{14}^*])+([J_{20}^*]-r_o[J_{14}^*]^T)]+\mathcal{K}[J_1^*]+\mathcal{K}'[J_4^*]\}[A]_o^{-1}, \quad \text{(X.6)}$$

$$[K_i^*]^e = [A]_o^{-1T}\{[[J_3^*]+r_o([J_{15}^*]+[J_{15}^*]^T)+r_o^2[J_8^*]]+\mathcal{A}[[J_8^*]-r_o([J_{12}^*]+[J_{12}^*]^T)+r_o^2[J_1^*]]$$
$$+\mathcal{A}[C_1[[J_7^*]-r_o([J_5^*]^T-[J_{20}^*])-r_o^2[J_{14}^*]^T]+C_2[[J_{22}^*]-r_o([J_{11}^*]^T-[J_{21}^*])-r_o^2[J_{23}]^T]]$$
$$+\bar{u}^2[[J_9^*]+r_o([J_{12}^*]-[J_{13}^*])-r_o^2[J_4^*]]$$
$$+a_1[J_9^*]+\bar{r}_o([J_{12}^*]-[J_{13}^*])-r_o^2[J_4^*]]+b_1[[J_5^*]+r_o[J_{14}^*]]$$
$$+a_2[[J_{10}^*]+r_o([J_{16}^*]-[J_{17}^*])-r_o^2[J_{18}^*]]+b_2[[J_{11}^*]+r_o[J_{23}^*]]\}[A]_o^{-1}, \quad \text{(X.7)}$$

where

$$
\left.
\begin{aligned}
[J_1^*] &= \int_o^{\xi_e} [\Phi_2]^T[\Phi_2]\, d\,\xi, \\
[J_3^*] &= \int_o^{\xi_e} [\Phi_2]''^T[\Phi_2]''\, d\xi, \\
[J_4^*] &= \int_o^{\xi_e} [\Phi_4]^T[\Phi_4]\, d\,\xi, \\
[J_5^*] &= \int_o^{\xi_e} [\Phi_2]^T[\Phi_2]'\, d\xi, \\
[J_7^*] &= \int_o^{\xi_e} [\Phi_2]'^T[\Phi_4]'\, d\xi, \\
[J_8^*] &= \int_o^{\xi_e} [\Phi_4]'^T[\Phi_4]'\, d\xi, \\
[J_9^*] &= \int_o^{\xi_e} [\Phi_2]^T[\Phi_2]''\, d\xi, \\
[J_{10}^*] &= \int_o^{\xi_e} \xi[\Phi_2]^T[\Phi_2]''\, d\xi, \\
[J_{11}^*] &= \int_o^{\xi_e} \xi[\Phi_2]^T[\Phi_2]'\, d\xi, \\
[J_{12}^*] &= \int_o^{\xi_e} [\Phi_2]^T[\Phi_4]'\, d\xi, \\
[J_{13}^*] &= \int_o^{\xi_e} [\Phi_4]^T[\Phi_2]'\, d\xi, \\
[J_{14}^*] &= \int_o^{\xi_e} [\Phi_2]^T[\Phi_4]\, d\xi, \\
[J_{15}^*] &= \int_o^{\xi_e} [\Phi_2]''^T[\Phi_4]'\, d\xi, \\
[J_{16}^*] &= \int_o^{\xi_e} \xi[\Phi_2]^T[\Phi_4]'\, d\xi, \\
[J_{17}^*] &= \int_o^{\xi_e} [\Phi_4]^T[\Phi_2]'\,\xi d\xi, \\
[J_{18}^*] &= \int_o^{\xi_e} [\Phi_4]^T[\Phi_4]\xi\, d\xi, \\
[J_{19}^*] &= \int_o^{\xi_e} [\Phi_4]^T[\Phi_2]\, d\xi, \\
[J_{20}^*] &= \int_o^{\xi_e} [\Phi_4]^T[\Phi_4]'\, d\xi, \\
[J_{21}^*] &= \int_o^{\xi_e} \xi[\Phi_4]^T[\Phi_4]'\, d\xi, \\
[J_{22}^*] &= \int_o^{\xi_e} \xi[\Phi_2]'^T[\Phi_4]'\, d\xi, \\
[J_{23}^*] &= \int_o^{\xi_e} \xi[\Phi_2]^T[\Phi_4]\, d\xi .
\end{aligned}
\right\} \quad (X.8)
$$

Details of the manipulations leading to equation (X.6) may be found in

Appendix L. It should also be noted that equations (X.5)-(X.7) are equations (5.42)-(5.44) of the main text.

APPENDIX Y

COMPUTER PROGRAMS FOR THE EXTENSIBLE THEORY

```
LEVEL 1.4.0 (OCT 1984)          VS FORTRAN          DATE: AUG 25, 1986     TIME: 14:37:24

REQUESTED OPTIONS (EXECUTE): NODECK,NOLIST,OPT(2),NOFIPS,XREF,MAP,GOSTMT,GOSTMT,NOTEST,NOTF,NOSDUMP

OPTIONS IN EFFECT:  NOLIST  MAP  XREF  GOSTMT NODECK  SOURCE  TERM  OBJECT FIXED  NOTEST NOTRMFLG
                    NOSYM NORENT NOSDUMP AUTODBL(NONE)  NOSXM
                    OPT(2) LANGLVL(77) NOFIPS  FLAG(I)  NAME(MAIN   )  LINECOUNT(60)     CHARLEN(500

                    *....*...1.........2.........3.........4.........5.........6.........7.*.......8

                    C
                    C*****************************************************************************
                    C          FINITE-ELEMENT PROGRAM OF EXTENSIBLE THEORY
                    C              FOR THE CASE OF STATIC EQUILIBRIUM
                    C               ( CURVED PIPE CONVEYING FLUID )
                    C*****************************************************************************
                    C
                    C      MAIN PROGRAM
                    C
ISN   1                    IMPLICIT REAL*8(A-H,O-Z)
ISN   2                    DIMENSION EK(6,6),EFOR(6,1),NODE(6,38),RO(38),ELENG(38)
ISN   3                    DIMENSION GK(105,105),GF(105),GKN(105,105),WKAREA(18200)
ISN   4                    DIMENSION PINOD(37),COFPI(36),TOROT(37),COFTO(36),
                         *           DISN3(37),DEVN3(37)
ISN   5                    COMMON /EMDK/EK,EFOR
ISN   6                    COMMON /COEF/XPHIA,XPHI,XH,SIMA,CAPA
ISN   7                    DATA    RO/38*3.14159265D0/
                    C
ISN   8                    NDT   = 105
ISN   9                    NET   = 34
ISN  10                    NDPE  = 6
                    C=============
                    C NODAL DATA
                    C==============
ISN  11                    PRINT 100
ISN  12             100    FORMAT('1',8X,'IN-PLANE STATIC DEFORMATION (CLAMPED-CLAMPED)',
                         *        //,9X,'ELEMENT CONNECTIVITY:',
                         *        //20X,'ELEMENT NUMBER',20X,'NODE',//)
ISN  13                    DO 101 I=1,NET
                    C========
                    C DATA FOR LENGHT OF ELEMENT
                    C========================
ISN  14                    ELENG(I)=1.D0/DFLOAT(NET+2)
ISN  15                    DO 102 J=1,NDPE
ISN  16             102    NODE(J,I)=3*(I-1)+J
ISN  17                    WRITE(6,103)I,(NODE(L,I),L=1,NDPE)
ISN  18             101    CONTINUE
ISN  19             103    FORMAT(29X,I2,10X,6(I3,3X)/)
                    C==========================
                    C  FORMING THE DIMENSIONLESS PARAMETERS
                    C====================================
ISN  20                    VELU  =RO(1)*DFLOAT(2)
ISN  21                    XPHIA =0.D00
ISN  22                    XPHI  =.5D0
ISN  23                    SIMA  =0.D00
ISN  24                    CAPA  =1.D0/(1.D0+.3D0)
ISN  25                    XH    =0.D00
ISN  26                    AA    =1.D04
ISN  27                    XLAMDA=1.3D0
ISN  28                    PIO   =XLAMDA*VELU**2
```

```
                 *....*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN     29            WRITE(6,106)VELU,XPHIA,XPHI,XH,SIMA,CAPA,RO(1)
ISN     30     106    FORMAT(//,10X,'DIMENSIONLESS PARAMETERS:',///10X,'DIMENSIONLESS ',
                     *'VELOCITY=',D12.5,//,10X,'BETAA=',D12.5 ',',',','BETA=',D12.5,',',','H='
                     *,D12.5,//,10X,'PI (INCLUDING)',',',',' SIMA  =',D12.5,',','//,10X,
                     *           'CAPA=',D12.5,',',','RO=',D12.5,/)
               C========================
               C ESSAMBLY OF ELEMENT MATRIX YIELDS GLOBAL MATRIX
               C================================================
               C
ISN     31            DO 107 II=1,NDT
ISN     32            GF(II)=0.0D0
ISN     33            DO 107 JJ=1,NDT
ISN     34            GK(II,JJ)=0.0D0
ISN     35     107    CONTINUE
               C
ISN     36            XELENG=ELENG(1)
ISN     37            ROO=RO(1)
ISN     38            CALL ELMDKM(XELENG,VELU,ROO,AA,PIO,XLAMDA)
ISN     39            DO 115 JG=1,3
ISN     40            JE=3+JG
ISN     41            GF(JG)=GF(JG)+EFOR(JE,1)
ISN     42            DO 115 KG=1,3
ISN     43            KE=3+KG
ISN     44            GK(JG,KG)=GK(JG,KG)+EK(JE,KE)
ISN     45     115    CONTINUE
               C---
ISN     46            DO 109 L=1,NET
ISN     47            XELENG=ELENG(L)
ISN     48            PIO =PIO*(1.D0-L*XELENG)
ISN     49            ROO  =RO(L)
ISN     50            CALL ELMDKM(XELENG,VELU,ROO,AA,PIO,XLAMDA)
ISN     51            DO 109 JL=1,NDPE
ISN     52            DO 119 KL=1,NDPE
ISN     53                   JG=NODE(JL,L)
ISN     54                   KG=NODE(KL,L)
ISN     55                   GK(JG,KG)=GK(JG,KG)+EK(JL,KL)
ISN     56     119    CONTINUE
ISN     57            GF(JG)=GF(JG)+EFOR(JL,1)
ISN     58     109    CONTINUE
ISN     59            XELENG=ELENG(NET)
ISN     60            PIO =PIO*(1.0D0-DFLOAT(NET+1)*XELENG)
ISN     61            ROO=RO(NET)
ISN     62            CALL ELMDKM(XELENG,VELU,ROO,AA,PIO,XLAMDA)
ISN     63            DO 199 JE=1,3
ISN     64            JG=NDT+JE-3
ISN     65            GF(JG)=GF(JG)+EFOR(JE,1)
ISN     66            DO 199 KE=1,3
ISN     67            KG=NDT+KE-3
ISN     68            GK(JG,KG)=GK(JG,KG)+EK(JE,KE)
ISN     69     199    CONTINUE
               C============
               C  SOLVING EQUATION GK*U=F
               C======================
ISN     70            IDGT =8
ISN     71            CALL LEQT2F(GK,1,NDT,NDT,GF,IDGT,WKAREA,IR)
               C
```

```
                  *....*...1.........2.........3.........4.........5.........6.........7.*.......8

                  C CALCULATING THE COMBINED FORCE
                  C
ISN     72              WRITE(6,239)
ISN     73        239   FORMAT(///,T10,'NODE',T20,'DIM. LATERAL DISPL.  ',
                       *                  T45,'SLOP OF  ',
                       *      T62,'DIM. AXIAL DISPLACEMENT',//)
ISN     74              NE2=NET+2
ISN     75              NE3=NET+3
ISN     76              DISN3(1)=0.0D0
ISN     77              DISN3(NE3)=0.0D0
ISN     78              DO 230  JK=2,NE2
ISN     79              JK3 =(JK-1)*3
ISN     80              DISN3(JK)=GF(JK3)
ISN     81        230   CONTINUE
                  C--
ISN     82              DEVN3(1)=DISN3(2)*DFLOAT(NE2)
ISN     83              DEVN3(NE3)=-DISN3(NE2)*DFLOAT(NE2)
ISN     84              DO 231 JL=2,NE2
ISN     85              DEVN3(JL)=(DISN3(JL+1)-DISN3(JL-1))*DFLOAT(NE2)/2.D0
ISN     86        231   CONTINUE
                  C---
ISN     87              PINOD(1)=PIO-AA*DEVN3(1)
ISN     88              PINOD(NE3)=-AA*DEVN3(NE3)
ISN     89              DO 235 JH=2,NE2
ISN     90              JM=1+(JH-2)*3
ISN     91              PRENO=PIO*(1.D0-ELENG(1)*DFLOAT(JH-1))
ISN     92              ELONGA=DEVN3(JH)-RO(JH)*GF(JM)
ISN     93              PINOD(JH)=PRENO-AA*ELONGA
ISN     94        235   CONTINUE
ISN     95              TOROT(1)=0.D0D
ISN     96              TOROT(NE2)=GF(NDT-1)+RO(NET)*GF(NDT)
ISN     97              TOROT(NE3)=0.D0D
ISN     98              J9=0
ISN     99              ND3=NDT-3
ISN    100              DO 499 JJ=1,ND3,3
ISN    101              J9=J9+1
ISN    102              J91=J9+1
ISN    103              JJ1=JJ+1
ISN    104              JJ2=JJ1+1
ISN    105              JJ3=JJ2+3
ISN    106              TOROT(J91)=GF(JJ1)+RO(J91)*GF(JJ2)
                  C--
ISN    107              WRITE(6,241)J9,GF(JJ),GF(JJ1),GF(JJ2)
ISN    108        241   FORMAT(10X,I2,T22,D12.5,T42,D12.5,T66,D12.5,/)
ISN    109        499   CONTINUE
                  C---
ISN    110              WRITE(6,251)
ISN    111        251   FORMAT(///,T5,'NODE',T12,'DIM. COMB. FORCE',T32,
                       *      'SL. OF COMB. FORCE',T58,'TOTAL SLOP',T82,
                       *      'SL. OF TO. SLOP'//)
ISN    112              DO 255 II=1,NE2
ISN    113              COFPI(II)=(PINOD(II+1)-PINOD(II))*DFLOAT(NE2)
ISN    114              COFTO(II)=(TOROT(II+1)-TOROT(II))*DFLOAT(NE2)
ISN    115              WRITE(6,259)II,PINOD(II),COFPI(II),TOROT(II),COFTO(II)
ISN    116        259   FORMAT(T5,I2,T14,D12.5,T36,D12.5,T60,D12.5,T84,D12.5/)
ISN    117        255   CONTINUE
```

```
                  *....*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN     118              WRITE(2,899)PINOD,COFPI,TOROT,COFTO                                00175
ISN     119       899    FORMAT(D20.10)                                                     00176
ISN     120              STOP                                                               00177
ISN     121              END                                                                00178
```

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LEVEL 1.4.0 (OCT 1984)          VS FORTRAN          DATE: AUG 25, 1986     TIME: 14:37:25

OPTIONS IN EFFECT:  NOLIST   MAP   XREF   GOSTMT NODECK   SOURCE   TERM   OBJECT FIXED  NOTEST NOTRMFLG
                    NOSYM NORENT NOSDUMP AUTODBL(NONE)    NOSXM
                      OPT(2) LANGLVL(77) NOFIPS   FLAG(I)  NAME(MAIN   )  LINECOUNT(60)     CHARLEN(500

                    *....*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN      1          SUBROUTINE ELMDKM(ELENG,VELU,RO,AA,PIO,XLAMDA)
              C========================================C
              C SUBROUTINE ESTIMATING -STIFFNESS MATRIX  C
              C========================================C
              C                                        C
ISN      2          IMPLICIT REAL*8(A-H,O-Z)
ISN      3          DIMENSION EK(6,6),EFOR(6,1)
ISN      4          DIMENSION RJS1(6,6),RJS2(6,6),RJS3(6,6),RJS4(6,6),RJS5(6,6),
                   2          RJS6(6,6),RJS8(6,6),RJS10(6,6),RJS11(6,6),RJS12(6,6),
                   3        RJS15(6,6),RJS16(6,6),RJS17(6,6),RJS18(6,6),RJS19(6,6),
                   4                    RJS20(6,6),RJS21(6,6),EF1(6),EF2(6),EF3(6),
                   5                             EFI(6,1),AINVE(6,6),AINVET(6,6)
ISN      5          COMMON /EMDK/EK,EFOR
ISN      6          COMMON /COEF/PHIA,PHI,VISDAH,SIMA,CAPA
ISN      7          COMMON /MAJS/RJS1,RJS2,RJS3,RJS4,RJS5,RJS6,RJS8,RJS10,RJS11,
                   *        RJS12,RJS15,RJS16,RJS17,RJS18,RJS19,RJS20,RJS21
ISN      8          COMMON /MAFORC/EF1,EF2,EF3
ISN      9          COMMON /INV/AINVE,AINVET
ISN     10          RO2=RO**2
ISN     11          PHI12=DSQRT(PHI)
ISN     12          VELU2=VELU**2
              C
              C CALL MATRICES J*1-J*21
              C
ISN     13          CALL GENMAJ(ELENG)
              C
              C CALL INVERSE OF THE MATRIX A
              C
ISN     14          CALL INVERS(ELENG)
              C=======================
              C FORMING ELEMENT MATRICES
              C===========================
ISN     15          DO 280 I=1,6
ISN     16          DO 280 J=1,6
ISN     17          EK(I,J)=(RJS3(I,J)+RO*(RJS15(I,J)+RJS15(J,I))+RO2*RJS10(I,J))
                   *  +AA*(RJS10(I,J)-RO*(RJS16(I,J)+RJS16(J,I))+RO2*RJS1(I,J))
                   *  +PIO*(RJS11(I,J)+RO*(RJS16(I,J)-RJS17(I,J))-RO2*RJS4(I,J))
                   *  +VELU2*(RJS11(I,J)+RO*(RJS16(I,J)-RJS17(I,J))-RO2*RJS4(I,J))
                   *  -VELU2*XLAMDA*(RJS5(I,J)+RJS12(I,J)-RO2*RJS21(I,J)
                   *                 +RO*(RJS18(I,J)+RJS19(I,J)-RJS20(I,J)))
ISN     18    280   CONTINUE
              C
              C THE FORCE VECTOR OF ELEMENT
              C
ISN     19          DO 309 JJ=1,6
ISN     20          EFI(JJ,1)=-RO*(VELU2+PIO)*EF1(JJ)+
                   *        XLAMDA*VELU2*(RO*EF2(JJ)+EF3(JJ))
ISN     21    309   CONTINUE
ISN     22          CALL PRODMA(6,6,1,AINVET,EFI,EFOR)
              C
              C ELEMENT STIFNESS MATRIX
              C
```

```
                 *....*...1.........2.........3.........4.........5.........6.........7.*.......8
ISN    23       CALL PRODMA(6,6,6,AINVET,EK,RJS4)                                   00231
ISN    24       CALL PRODMA(6,6,6,RJS4,AINVE,EK)                                    00232
ISN    25       RETURN                                                              00233
ISN    26       END                                                                 00234
```

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LEVEL 1.6.0 (OCT 1984)          VS FORTRAN          DATE: AUG 25, 1986    TIME: 14:37:25

OPTIONS IN EFFECT:  NOLIST   MAP   XREF   GOSTMT NODECK   SOURCE   TERM   OBJECT FIXED  NOTEST NOTRMFLG
                    NOSYM NORENT NOSDUMP AUTODBL(NONE)   NOSXM
                      OPT(2) LANGLVL(77) NOFIPS   FLAG(I)  NAME(MAIN  )  LINECOUNT(60)     CHARLEN(500

                    *....*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN     1                SUBROUTINE  GENMAJ(ELENG)
                   C
                   C  SUBROUTINE GENERATING MATRICES J*1-J*23
                   C
                   C
ISN     2                IMPLICIT REAL*8(A-H,O-Z)
ISN     3                DIMENSION RJS1(6,6),RJS2(6,6),RJS3(6,6),RJS4(6,6),RJS5(6,6),
                        *      RJS6(6,6),RJS8(6,6),RJS10(6,6),RJS11(6,6),RJS12(6,6),
                        *    RJS15(6,6),RJS16(6,6),RJS17(6,6),RJS18(6,6),RJS19(6,6),
                        *    RJS20(6,6),RJS21(6,6),EF1(6),EF2(6),EF3(6)
ISN     4                COMMON /MAJS/RJS1,RJS2,RJS3,RJS4,RJS5,RJS6,RJS8,RJS10,RJS11,
                        *          RJS12,RJS15,RJS16,RJS17,RJS18,RJS19,RJS20,RJS21
ISN     5                COMMON /MAFORC/EF1,EF2,EF3
                   C=================
ISN     6                DO 400 I=1,6
ISN     7                DO 400 J=1,6
ISN     8                RJS1(I,J)=0.0D0
ISN     9                RJS2(I,J)=0.0D0
ISN    10                RJS3(I,J)=0.0D0
ISN    11                RJS4(I,J)=0.0D0
ISN    12                RJS5(I,J)=0.0D0
ISN    13                RJS6(I,J)=0.0D0
ISN    14                RJS8(I,J)=0.0D0
ISN    15                RJS10(I,J)=0.0D0
ISN    16                RJS11(I,J)=0.0D0
ISN    17                RJS12(I,J)=0.0D0
ISN    18                RJS15(I,J)=0.0D0
ISN    19                RJS16(I,J)=0.0D0
ISN    20                RJS17(I,J)=0.0D0
ISN    21                RJS18(I,J)=0.0D0
ISN    22                RJS19(I,J)=0.0D0
ISN    23                RJS20(I,J)=0.0D0
ISN    24                RJS21(I,J)=0.0D0
ISN    25          400   CONTINUE
                   C
                   C  GENERATING MATRIX J*1
                   C
ISN    26                DO 402 I=1,4
ISN    27                DO 402 J=1,4
ISN    28                K=I+J-1
ISN    29                RJS1(I,J)=ELENG**(K)/DFLOAT(K)
ISN    30          402   CONTINUE
                   C
                   C  GENERATING MATRIX J*2
                   C
ISN    31                DO 410 II=2,4
ISN    32                DO 410 JJ=2,4
ISN    33                KK=II+JJ-3
ISN    34                RJS2(II,JJ)=ELENG**(KK)
ISN    35          410   CONTINUE
ISN    36                RJS2(3,3)=4.D0*RJS2(3,3)/3.D0
ISN    37                RJS2(3,4)=1.5D0*RJS2(3,4)
```

```
                    *....*...1.........2.........3.........4.........5.........6.........7.*.......8
ISN      38               RJS2(4,3)=RJS2(3,4)
ISN      39               RJS2(4,4)=1.8D0*RJS2(4,4)
                    C
                    C  GENERATING MATRICES J*3
                    C
ISN      40               RJS3(3,3)=4.D0*ELENG
ISN      41               RJS3(3,4)=6.D0*ELENG**(2)
ISN      42               RJS3(4,3)=RJS3(3,4)
ISN      43               RJS3(4,4)=12.D0*ELENG**(3)
                    C
                    C  GENERATING MATRIX  J*4
                    C
ISN      44               RJS4(5,5)=ELENG
ISN      45               RJS4(5,6)=.5D0*ELENG**(2)
ISN      46               RJS4(6,5)=RJS4(5,6)
ISN      47               RJS4(6,6)=ELENG**(3)/3.D0
                    C
                    C  GENERATING MATRIX J*5
                    C
ISN      48               DO 420 I=1,4
ISN      49               DO 420 J=2,4
ISN      50               K2 =J-1
ISN      51               K3= I+J-2
ISN      52               RJS5(I,J)=DFLOAT(K2)*ELENG**(K3)/DFLOAT(K3)
ISN      53         420   CONTINUE
                    C
                    C GENERATING MATRIX RJS6
                    C
ISN      54               RJS6(3,5)=2.D0*ELENG
ISN      55               RJS6(3,6)=ELENG**(2)
ISN      56               RJS6(4,5)=3.D0*RJS6(3,6)
ISN      57               RJS6(4,6)=2.D0*ELENG**(3)
                    C
                    C  GENERATING MATRIX  J*8
                    C
ISN      58               DO 430 L=2,4
ISN      59               LL=L-1
ISN      60               RJS8(L,6)=ELENG**(LL)
ISN      61         430   CONTINUE
                    C
                    C  GENERATING MATRIX J*10
                    C
ISN      62               RJS10(6,6)=ELENG
                    C
                    C  GENERATING MATRICES  J*11 AND J*12
                    C
ISN      63               DO 440 JJ=1,4
ISN      64               RJS11(JJ,3)=2.D0*ELENG**(JJ)/DFLOAT(JJ)
ISN      65               J1=JJ+1
ISN      66               RJS11(JJ,4)=6.D0*ELENG**(J1)/DFLOAT(J1)
ISN      67               RJS12(JJ,3)=2.D0*ELENG**(J1)/DFLOAT(J1)
ISN      68               RJS12(JJ,4)=6.D0*ELENG**(JJ+2)/DFLOAT(JJ+2)
ISN      69         440   CONTINUE
                    C
                    C  GENERATING MATRIX R*15
                    C
```

```
                  *....*...1.........2.........3.........4.........5.........6.........7.*.......8
ISN     70              RJS15(3,6)=2.D0*ELENG
ISN     71              RJS15(4,6)=3.D0*ELENG**2
                  C
                  C  GENERATING MATRICES R*16 , R*18 AND R*19
                  C
ISN     72              DO 443 I5=1,4
ISN     73              RJS16(I5,6)=ELENG**(I5)/DFLOAT(I5)
                  C***
ISN     74              RJS18(I5,5)=ELENG**(I5)/DFLOAT(I5)
ISN     75              RJS18(I5,6)=ELENG**(I5+1)/DFLOAT(I5+1)
                  C***
ISN     76              RJS19(I5,6)=ELENG**(I5+1)/DFLOAT(I5+1)
ISN     77        443   CONTINUE
                  C
                  C  GENERATING MATRICES R*17 AND R*20
                  C
ISN     78              DO 446 J5=2,4
ISN     79              J51 =J5-1
ISN     80              RJS17(5,J5) =ELENG**(J51)
ISN     81              RJS17(6,J5) =DFLOAT(J51)*ELENG**(J5)/DFLOAT(J5)
                  C***
ISN     82              RJS20(5,J5)=DFLOAT(J51)*ELENG**(J5)/DFLOAT(J5)
ISN     83              RJS20(6,J5)=DFLOAT(J51)*ELENG**(J5+1)/DFLOAT(J5+1)
ISN     84        446   CONTINUE
                  C
                  C  GENERATING MATRIX R*21
                  C
ISN     85              DO 447 I=2,3
ISN     86              IJ          =3+I
ISN     87              RJS21(5,IJ)=ELENG**(I)/DFLOAT(I)
ISN     88              RJS21(6,IJ)=ELENG**(I+1)/DFLOAT(I+1)
ISN     89        447   CONTINUE
                  C
                  C  GENERATING FORCE VECTOR OF ELEMENT
                  C
ISN     90              DO 449 IF1=1,6
ISN     91              EF1(IF1)=0.D0
ISN     92              EF2(IF1)=0.D0
ISN     93              EF3(IF1)=0.D0
ISN     94        449   CONTINUE
ISN     95              DO 501 L=1,4
ISN     96              EF1(L)=ELENG**(L)/DFLOAT(L)
ISN     97              EF2(L)=ELENG**(L+1)/DFLOAT(L+1)
ISN     98        501   CONTINUE
ISN     99              EF3(5)=ELENG
ISN    100              EF3(6)=ELENG**(2)/2.D0
ISN    101              RETURN
ISN    102              END
```

```
LEVEL 1.4.0 (OCT 1984)           VS FORTRAN           DATE: AUG 25, 1986     TIME: 14:37:26

OPTIONS IN EFFECT:  NOLIST   MAP   XREF  GOSTMT NODECK   SOURCE   TERM   OBJECT FIXED  NOTEST NOTRMFLG
                    NOSYM NORENT NOSDUMP AUTODBL(NONE)    NOSXM
                      OPT(2) LANGLVL(77) NOFIPS    FLAG(I)  NAME(MAIN  )  LINECOUNT(60)      CHARLEN(500

                   *....*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN       1             SUBROUTINE  INVERS(ELENG)
                   C
                   C    SUBROUTINE INVERTING THE MATRIX A
                   C
                   C
ISN       2             IMPLICIT REAL*8(A-H,O-Z)
ISN       3             DIMENSION AINVE(6,6),AINVET(6,6)
ISN       4             COMMON /INV/AINVE,AINVET
ISN       5             DO 500 I=1,6
ISN       6             DO 500 J=1,6
ISN       7             AINVE(I,J)=0.0D0
ISN       8        500  CONTINUE
ISN       9             AINVE(1,1)=1.D0
ISN      10             AINVE(2,2)=1.D0
ISN      11             AINVE(5,3)=1.D0
ISN      12             AINVE(3,1)=-3.D0/ELENG**(2)
ISN      13             AINVE(3,2)=-2.D0/ELENG
ISN      14             AINVE(3,4)=-AINVE(3,1)
ISN      15             AINVE(3,5)=-1.D0/ELENG
ISN      16             AINVE(4,1)=2.D0/ELENG**(3)
ISN      17             AINVE(4,2)=ELENG**(-2)
ISN      18             AINVE(4,4)=-AINVE(4,1)
ISN      19             AINVE(4,5)=AINVE(4,2)
ISN      20             AINVE(6,6)=1.D0/ELENG
ISN      21             AINVE(6,3)=-AINVE(6,6)
                   C=======
ISN      22             DO 505 I=1,6
ISN      23             DO 505 J=1,6
ISN      24             AINVET(I,J)=AINVE(J,I)
ISN      25        505   CONTINUE
ISN      26             RETURN
ISN      27             END
```

```
LEVEL 1.4.0 (OCT 1984)            VS FORTRAN          DATE: AUG 25, 1986     TIME: 14:37:26

OPTIONS IN EFFECT:  NOLIST   MAP   XREF   GOSTMT NODECK   SOURCE   TERM   OBJECT FIXED  NOTEST NOTRMFLG
                    NOSYM NORENT NOSDUMP AUTODBL(NONE)   NOSXM
                      OPT(2) LANGLVL(77) NOFIPS   FLAG(I)  NAME(MAIN  )  LINECOUNT(60)     CHARLEN(500

                   *....*..1.........2.........3.........4.........5.........6.........7.*.......8

ISN      1           SUBROUTINE PRODMA(M,L,N,A,B,C)
               C
               C     SUBROUTINE MULTIPLYING MATRICES
               C
               C
ISN      2           IMPLICIT   REAL*8(A-H,O-Z)
ISN      3           DIMENSION  A(M,L),B(L,N),C(M,N)
ISN      4           DO 590   I=1,M
ISN      5           DO 590   J=1,N
ISN      6           C(I,J)=0.000
ISN      7           DO 590   K=1,L
ISN      8           C(I,J)=C(I,J)+A(I,K)*B(K,J)
ISN      9     590   CONTINUE
ISN     10           RETURN
ISN     11           END
```

```
LEVEL 1.4.0 (OCT 1984)          VS FORTRAN          DATE: AUG 25, 1986    TIME: 14:37:26

OPTIONS IN EFFECT:  NOLIST   MAP   XREF   GOSTMT NODECK   SOURCE   TERM   OBJECT FIXED   NOTEST NOTRMFLG
                    NOSYM NORENT NOSDUMP AUTODBL(NONE)    NOSXM
                     OPT(2) LANGLVL(77) NOFIPS   FLAG(I)  NAME(MAIN   )  LINECOUNT(60)      CHARLEN(500

                         *....*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN     1            SUBROUTINE PRINT(N,A)                                                    00438
                 C                                                                            00439
                 C SUBROUTINE TO PRINT MATRIX                                                 00440
                 C                                                                            00441
ISN     2            IMPLICIT REAL*8(A-H,O-Z)                                                 00442
ISN     3            DIMENSION A(6,6)                                                         00443
ISN     4            WRITE(6,822) N                                                           00444
ISN     5        822 FORMAT(///,10X,'MATRIX J',I2,//)                                         00445
ISN     6            WRITE(6,823)((A(I,J),J=1,6),I=1,6)                                       00446
ISN     7        823 FORMAT(/,6(4X,D12.5),/)                                                  00447
ISN     8            RETURN                                                                   00448
ISN     9            END                                                                      00449
```

IN-PLANE STATIC DEFORMATION (CLAMPED-CLAMPED)

ELEMENT CONNECTIVITY:

ELEMENT NUMBER	NODE					
1	1	2	3	4	5	6
2	4	5	6	7	8	9
3	7	8	9	10	11	12
4	10	11	12	13	14	15
5	13	14	15	16	17	18
6	16	17	18	19	20	21
7	19	20	21	22	23	24
8	22	23	24	25	26	27
9	25	26	27	28	29	30
10	28	29	30	31	32	33
11	31	32	33	34	35	36
12	34	35	36	37	38	39
13	37	38	39	40	41	42
14	40	41	42	43	44	45
15	43	44	45	46	47	48
16	46	47	48	49	50	51
17	49	50	51	52	53	54
18	52	53	54	55	56	57
19	55	56	57	58	59	60
20	58	59	60	61	62	63
21	61	62	63	64	65	66
22	64	65	66	67	68	69
23	67	68	69	70	71	72
24	70	71	72	73	74	75
25	73	74	75	76	77	78
26	76	77	78	79	80	81
27	79	80	81	82	83	84

28	82	83	84	85	86	87
29	85	86	87	88	89	90
30	88	89	90	91	92	93
31	91	92	93	94	95	96
32	94	95	96	97	98	99
33	97	98	99	100	101	102
34	100	101	102	103	104	105

DIMENSIONLESS PARAMETERS:

DIMENSIONLESS VELOCITY= 0.62832D+01

BETAA= 0.00000D+00,BETA= 0.50000D+00,H= 0.00000D+00

PI (INCLUDING), SIMA = 0.00000D+00,

CAPA= 0.76923D+00,RO= 0.31416D+01

NODE	DIM. LATERAL DISPL.	SLOP OF	DIM. AXIAL DISPLACEMENT
1	-0.39162D-04	-0.28338D-02	0.24901D-03
2	-0.15712D-03	-0.56530D-02	0.48726D-03
3	-0.35146D-03	-0.83134D-02	0.70794D-03
4	-0.61587D-03	-0.10681D-01	0.90465D-03
5	-0.94059D-03	-0.12641D-01	0.10717D-02
6	-0.13129D-02	-0.14100D-01	0.12043D-02
7	-0.17180D-02	-0.14993D-01	0.12989D-02
8	-0.21397D-02	-0.15287D-01	0.13535D-02
9	-0.25610D-02	-0.14977D-01	0.13673D-02
10	-0.29657D-02	-0.14087D-01	0.13410D-02
11	-0.33381D-02	-0.12669D-01	0.12768D-02
12	-0.36647D-02	-0.10794D-01	0.11781D-02
13	-0.39339D-02	-0.85535D-02	0.10494D-02

14	-0.413700-02	-0.604800-02	0.896160-03
15	-0.426820-02	-0.338540-02	0.724320-03
16	-0.432450-02	-0.674080-03	0.540350-03
17	-0.430610-02	0.198180-02	0.350790-03
18	-0.421580-02	0.448680-02	0.162040-03
19	-0.405910-02	0.675790-02	-0.198640-04
20	-0.384340-02	0.872680-02	-0.189430-03
21	-0.357780-02	0.103420-01	-0.341920-03
22	-0.327270-02	0.115690-01	-0.473430-03
23	-0.293920-02	0.123920-01	-0.580990-03
24	-0.258850-02	0.128100-01	-0.662620-03
25	-0.223160-02	0.128380-01	-0.717310-03
26	-0.187910-02	0.125040-01	-0.744990-03
27	-0.154040-02	0.118480-01	-0.746450-03
28	-0.122380-02	0.109170-01	-0.723290-03
29	-0.936310-03	0.976260-02	-0.677740-03
30	-0.683300-03	0.844180-02	-0.612600-03
31	-0.468610-03	0.701110-02	-0.531040-03
32	-0.294510-03	0.552580-02	-0.436520-03
33	-0.161790-03	0.403790-02	-0.332630-03
34	-0.698620-04	0.259460-02	-0.222960-03

NODE	DIM. COMB. FORCE	SL. OF COMB. FORCE	TOTAL SLOP	SL. OF TO. SLOP
1	-0.383230+02	-0.258350+02	0.000000+00	-0.738550-01
2	-0.390410+02	-0.114070+01	-0.205150-02	-0.745460-01
3	-0.390720+02	-0.192840+01	-0.412230-02	-0.708150-01
4	-0.391260+02	-0.263110+01	-0.608940-02	-0.629960-01
5	-0.391990+02	-0.321130+01	-0.783920-02	-0.516510-01
6	-0.392880+02	-0.364070+01	-0.927400-02	-0.375190-01

7	-0.39389D+02	-0.39014D+01	-0.10316D-01	-0.21464D-01
8	-0.39498D+02	-0.39856D+01	-0.10912D-01	-0.44071D-02
9	-0.39608D+02	-0.38958D+01	-0.11035D-01	0.12724D-01
10	-0.39717D+02	-0.36437D+01	-0.10681D-01	0.29054D-01
11	-0.39818D+02	-0.32486D+01	-0.98743D-02	0.43802D-01
12	-0.39908D+02	-0.27360D+01	-0.86576D-02	0.56320D-01
13	-0.39984D+02	-0.21351D+01	-0.70931D-02	0.66114D-01
14	-0.40043D+02	-0.14776D+01	-0.52566D-02	0.72864D-01
15	-0.40084D+02	-0.79539D+00	-0.32326D-02	0.76420D-01
16	-0.40106D+02	-0.11892D+00	-0.11098D-02	0.76800D-01
17	-0.40110D+02	0.52413D+00	0.10235D-02	0.74172D-01
18	-0.40095D+02	0.11100D+01	0.30838D-02	0.68834D-01
19	-0.40064D+02	0.16197D+01	0.49959D-02	0.61186D-01
20	-0.40019D+02	0.20394D+01	0.66955D-02	0.51701D-01
21	-0.39963D+02	0.23603D+01	0.81316D-02	0.40897D-01
22	-0.39897D+02	0.25791D+01	0.92677D-02	0.29306D-01
23	-0.39826D+02	0.26969D+01	0.10082D-01	0.17449D-01
24	-0.39751D+02	0.27191D+01	0.10566D-01	0.58126D-02
25	-0.39675D+02	0.26546D+01	0.10728D-01	-0.51703D-02
26	-0.39601D+02	0.25146D+01	0.10584D-01	-0.15133D-01
27	-0.[illegible]531D+02	0.23123D+01	0.10164D-01	-0.23785D-01
28	-0.39467D+02	0.20620D+01	0.95032D-02	-0.30916D-01
29	-0.39410D+02	0.17781D+01	0.86445D-02	-0.36398D-01
30	-0.39361D+02	0.14747D+01	0.76334D-02	-0.40180D-01
31	-0.39320D+02	0.11648D+01	0.65173D-02	-0.42280D-01
32	-0.39287D+02	0.86021D+00	0.53428D-02	-0.42781D-01
33	-0.39263D+02	0.57096D+00	0.41545D-02	-0.41815D-01
34	-0.39248D+02	0.30523D+00	0.29929D-02	-0.39556D-01
35	-0.39239D+02	0.69253D-01	0.18942D-02	-0.36206D-01
36	-0.39237D+02	-0.25858D+02	0.88845D-03	-0.31984D-01

```
LEVEL 1.4.0 (OCT 1984)          VS FORTRAN          DATE: AUG 25, 1986    TIME: 15:03:39

REQUESTED OPTIONS (EXECUTE): NODECK,NOLIST,OPT(2),NOFIPS,XREF,MAP,GOSTMT,GOSTMT,NOTEST,NOTF,NOSDUMP

OPTIONS IN EFFECT:  NOLIST   MAP   XREF   GOSTMT NODECK   SOURCE   TERM   OBJECT FIXED   NOTEST NOTRMFLG
                    NOSYM NORENT NOSDUMP AUTODBL(NONE)   NOSXM
                      OPT(2) LANGLVL(77) NOFIPS   FLAG(I)  NAME(MAIN  )  LINECOUNT(60)     CHARLEN(500

                 *....*...1.........2.........3.........4.........5.........6.........7.*.......8

                 C***********************************************************************
                 C                   FINITE ELEMENT PROGRAM OF EXTENSIBLE THEORY
                 C                            FOR IN-PLANE MOTION
                 C                       (CURVED PIPE CONVEYING FLUID)
                 C***********************************************************************
                 C
                 C      MAIN PROGRAM
                 C
ISN      1              IMPLICIT REAL*8(A-H,O-Z)
ISN      2              DIMENSION EM(6,6),ED(6,6),EK(6,6),NODE(6,38),RO(38),ELENG(38)
ISN      3              DIMENSION GM(105,105),GD(105,105),GK(105,105)
ISN      4              DIMENSION GMM(210,210),GKK(210,210)
ISN      5              DIMENSION BETA(210),WK(47000),PI(38)
ISN      6              DIMENSION PINOD(37),COFPI(36),TOROT(37),COFTO(36)
ISN      7              COMPLEX*16 EIVALU(210),EIVECT(210,210)
ISN      8              COMMON /EMDK/EM,ED,EK
ISN      9              COMMON /COEF/XPHIA,XPHI,XH,SIMA,CAPA,ALI
ISN     10              COMMON /COPI/PINOD,COFPI,TOROT,COFTO
ISN     11              DATA    RO/38*3.14159265D0/
                 C***** DATA    RO/38*1.570796325D0/
ISN     12              READ(02,1)PINOD,COFPI,TOROT,COFTO
ISN     13       1      FORMAT(D20.10)
ISN     14              DO 9999 LLL=1,1
ISN     15              NDT =105
ISN     16              NET =34
ISN     17              NDPE=6
ISN     18              NDT2=2*NDT
                 C************
                 C NODE DATA
                 C**************
ISN     19              PRINT 100
ISN     20       100    FORMAT('1',8X,'IN-PLANE MOTION (EXTENSIBLE & CLAMPED-CLAMPED)',
                      *          //,9X,'ELEMENT CONNECTIVITY:',
                      *          //20X,'ELEMENT NUMBER',20X,'NODE',//)
ISN     21              DO 101 I=1,NET
                 C********
                 C DATA FOR LENGHT OF ELEMENT
                 C************************
ISN     22              ELENG(I)=1.D0/DFLOAT(NET+2)
ISN     23              DO 102 J=1,NDPE
ISN     24       102    NODE(J,I)=3*(I-1)+J
ISN     25              WRITE(6,103)I,(NODE(L,I),L=1,NDPE)
ISN     26       101    CONTINUE
ISN     27       103    FORMAT(29X,I2,10X,6(I3,3X)/)
                 C*********************************
                 C  FORMING THE DIMENSIONLESS PARAMETERS
                 C*************************************
ISN     28              VELU =RO(1)*2.D0
ISN     29              XPHIA=0.0D0
ISN     30              XPHI =.5D0
```

```
                  *....*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN    31              SIMA =0.0D0
ISN    32              CAPA =1.D0/(1.D0+.3D0)
ISN    33              XH   =0.0D0
ISN    34              ALI  =1.0D0
                  C=======
ISN    35              XMF  =162.6D0
ISN    36              XMT  =299.3D0
ISN    37              XLEN =1000.D0
ISN    38              GRAV =9.81D0
ISN    39              EI   =221.3D+06
ISN    40              DENSIF=998.D0
ISN    41              XMASDI=3.1415D0*DENSIF*(.253**2)
ISN    42              GZEO =(XMT+XMF-XMASDI)*GRAV
ISN    43              PI1  =(GZEO*XLEN**3)/EI
                  C  **** TEST FOR MR. CHEN'S CASE
ISN    44              PI1  =0.0D0
ISN    45              PIO=-PI1
ISN    46              DO 105 I=1,NET
ISN    47              PIO=PIO+PI1*ELENG(I)
ISN    48              PI(I)=PIO
ISN    49         105  CONTINUE
                  C
                  C=======
ISN    50              WRITE(6,106)VELU,XPHIA,XPHI,XH,SIMA,CAPA,RO(1)
ISN    51         106  FORMAT(//,10X,'DIMENSIONLESS PARAMETERS:',///10X,'DIMENSIONLESS ',
                      *'VELOCITY=',D12.5,//,10X,'BETAA=',D12.5,',','BETA=',D12.5,',','H='
                      *,D12.5,//,10X,'AA = 10,000   ',',',' SIMA  =',D12.5,','//,10X,
                      *          'CAPA=',D12.5,',','RO=',D12.5,/)
                  C====================
                  C ESSAMBLY OF ELEMENT MATRIX YIELDS GLOBAL MATRIX
                  C=============================================
                  C
ISN    52              DO 107 II=1,NDT
ISN    53              DO 107 JJ=1,NDT
ISN    54              GM(II,JJ)=0.0D0
ISN    55              GD(II,JJ)=0.0D0
ISN    56              GK(II,JJ)=0.0D0
ISN    57         107  CONTINUE
                  C---
ISN    58              PIO   =-PI1
ISN    59              XELENG=ELENG(1)
ISN    60              IEL =1
ISN    61              ROO=RO(1)
ISN    62              CALL ELMDKM(XELENG,VELU,PIO,PI1,ROO,IEL)
ISN    63              DO 115 JG=1,3
ISN    64              JE=3+JG
ISN    65              DO 115 KG=1,3
ISN    66              KE=3+KG
ISN    67              GM(JG,KG)=GM(JG,KG)+EM(JE,KE)
ISN    68              GD(JG,KG)=GD(JG,KG)+ED(JE,KE)
ISN    69              GK(JG,KG)=GK(JG,KG)+EK(JE,KE)
ISN    70         115  CONTINUE
                  C
ISN    71              DO 109 L=1,NET
ISN    72              XELENG=ELENG(L)
ISN    73              IEL =L+1
```

```
                   *....*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN      74              ROO  =RO(L)
ISN      75              PIX  =PI(L)
ISN      76              CALL ELMDKM(XELENG,VELU,PIX,PI1,ROO,IEL)
ISN      77              DO 109 JL=1,NDPE
ISN      78              DO 109 KL=1,NDPE
ISN      79                     JG=NODE(JL,L)
ISN      80                     KG=NODE(KL,L)
ISN      81                     GM(JG,KG)=GM(JG,KG)+EM(JL,KL)
ISN      82                     GD(JG,KG)=GD(JG,KG)+ED(JL,KL)
ISN      83                     GK(JG,KG)=GK(JG,KG)+EK(JL,KL)
ISN      84        109   CONTINUE
ISN      85              XELENG=ELENG(NET)
ISN      86              ROO=RO(NET)
ISN      87              IEL =NET+2
ISN      88              CALL ELMDKM(XELENG,VELU,PIO,PI1,ROO,IEL)
ISN      89              DO 199 JE=1,3
ISN      90              JG=NDT+JE-3
ISN      91              DO 199 KE=1,3
ISN      92              KG=NDT+KE-3
ISN      93              GM(JG,KG)=GM(JG,KG)+EM(JE,KE)
ISN      94              GD(JG,KG)=GD(JG,KG)+ED(JE,KE)
ISN      95              GK(JG,KG)=GK(JG,KG)+EK(JE,KE)
ISN      96        199   CONTINUE
                   C
                   C=====================================
                   C FORMING THE AUGMENTED MATRIX OF M,D,K
                   C=======================================
                   C
ISN      97              DO 120 I=1,NDT2
ISN      98              DO 120 J=1,NDT2
ISN      99              GMM(I,J)=0.0D0
ISN     100        120   GKK(I,J)=0.0D0
ISN     101              DO 121 I1=1,NDT
ISN     102              II=NDT+I1
ISN     103              GMM(I1,I1)=1.D0
ISN     104              GKK(I1,II)=1.D0
ISN     105              DO 121 J1=1,NDT
ISN     106              JJ=NDT+J1
ISN     107              GKK(II,J1)=-GK(I1,J1)
ISN     108              GMM(II,JJ)=GM(I1,J1)
ISN     109              GKK(II,JJ)=-GD(I1,J1)
ISN     110        121   CONTINUE
                   C==================================
                   C CALCULATING EIGENVALUES AND EIGENVECTORS
                   C========================================
ISN     111              IA=NDT2
ISN     112              IB=NDT2
ISN     113              IZ=NDT2
ISN     114              N =NDT2
ISN     115              IJOB=2
ISN     116              CALL EIGZF(GKK,IA,GMM,IB,N,IJOB,EIVALU,BETA,EIVECT,IZ,WK,IER)
                   C===========
                   C  EIGENVALUES
                   C=====================
ISN     117              DO 145 J=1,NDT2
ISN     118              EIVALU(J)=EIVALU(J)/BETA(J)
```



```
                   *....*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN     119        145   CONTINUE
ISN     120              CALL ARRANG(NDT2,EIVALU,EIVECT)
ISN     121              CALL OUTEIG(NDT2,1,EIVALU,EIVECT)
ISN     122       9999   CONTINUE
ISN     123              STOP
ISN     124              END
```

```
LEVEL 1.4.0 (OCT 1984)          VS FORTRAN          DATE: AUG 25, 1986     TIME: 15:03:39

OPTIONS IN EFFECT:  NOLIST   MAP   XREF   GOSTMT NODECK   SOURCE   TERM   OBJECT FIXED  NOTEST NOTRMFLG
                    NOSYM NORENT NOSDUMP AUTODBL(NONE)   NOSXM
                      OPT(2) LANGLVL(77) NOFIPS   FLAG(I)  NAME(MAIN   )  LINECOUNT(60)     CHARLEN(500

                    *...+*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN    1          SUBROUTINE ELMDKM(ELENG,VELU,PIX,PI1,RO,IEL)
                C=============================================================C
                C SUBROUTINE ESTIMATING -MASS MATRIX                           C
                C                      -DAMPING MATRIX                         C
                C                      -STIFFNESS MATRIX                       C
                C=============================================================C
                C                                                              C
ISN    2          IMPLICIT REAL*8(A-H,O-Z)
ISN    3          DIMENSION EM(6,6),ED(6,6),EK(6,6)
ISN    4          DIMENSION RJS1(6,6),RJS2(6,6),RJS3(6,6),RJS4(6,6),RJS5(6,6),
                 2          RJS6(6,6),RJS8(6,6),RJS10(6,6),RJS11(6,6),
                 3          RJS12(6,6),RJS15(6,6),RJS16(6,6),RJS17(6,6),RJS18(6,6),
                 4          RJS19(6,6),RJS20(6,6),RJS21(6,6),RJS22(6,6),
                 5          RJS31(6,6),RJS41(6,6),RJS42(6,6),RJS43(6,6),
                 6          AINVE(6,6),AINVET(6,6),
                 7          PINOD(37),COFPI(36),TOROT(37),COFTO(36)
ISN    5          COMMON /EMDK/EM,ED,EK
ISN    6          COMMON /COEF/PHIA,PHI,VISDAH,SIMA,CAPA,ALI
ISN    7          COMMON /MAJS/RJS1,RJS2,RJS3,RJS4,RJS5,RJS6,RJS8,RJS10,
                 *        RJS11,RJS12,RJS15,RJS16,RJS17,RJS18,RJS19,RJS20,
                 *        RJS21,RJS22,RJS31,RJS41,RJS42,RJS43
ISN    8          COMMON /INV/AINVE,AINVET
ISN    9          COMMON /COPI/PINOD,COFPI,TOROT,COFTO
ISN   10          RO2=RO**2
ISN   11          PHI12=DSQRT(PHI)
ISN   12          VELU2=VELU**2
ISN   13          A1    =PINOD(IEL)
ISN   14          A2    =COFPI(IEL)
ISN   15          C1    =TOROT(IEL)
ISN   16          C2    =COFTO(IEL)
                C
                C CALL MATRICES J*1-J*12
                C
ISN   17          CALL GENMAJ(ELENG)
                C
                C CALL INVERSE OF THE MATRIX A
                C
ISN   18          CALL INVERS(ELENG)
                C
                C=======================
                C FORMING ELEMENT MATRICES
                C========================
ISN   19          DO 280 I=1,6
ISN   20          DO 280 J=1,6
ISN   21          EM(I,J)=(1.D0+PHIA)*(RJS1(I,J)+RJS4(I,J))
ISN   22          ED(I,J)=2.D0*VELU*PHI12*(RJS5(I,J)+RO*RJS18(I,J))+
                 *              VELU*PHI12*(RJS22(I,J)-RO*RJS18(J,I))+
                 *              VISDAH*(RJS1(I,J)+RJS4(I,J))
ISN   23          EK(I,J)=(RJS3(I,J)+RO*(RJS15(I,J)+RJS15(J,I))+RO2*RJS10(I,J))
                 *   +ALI*(RJS10(I,J)-RO*(RJS16(I,J)+RJS16(J,I))+RO2*RJS1(I,J))
                 *   +VELU2*(RJS11(I,J)+RO*(RJS16(I,J)-RJS17(I,J))-RO2*RJS4(I,J))
                 *      +A1*(RJS11(I,J)+RO*(RJS16(I,J)-RJS17(I,J))-RO2*RJS4(I,J))
```

```
                  *....*...1.........2.........3.........4.........5.........6.........7.*.......8
                       *     +A2*(RJS5(I,J)+RJS12(I,J)-RO2*RJS21(I,J)+
                       *      RO*(RJS18(I,J)+RJS19(I,J)-RJS20(I,J)))
                       *  +ALI*(C1*(RJS8(I,J)-RO*(RJS5(J,I)-RJS22(I,J))-RO2*RJS18(J,I))
                       *    +C2*(RJS42(I,J)-RO*(RJS31(J,I)-RJS41(I,J))-RO2*RJS43(I,J)))
ISN       24      200    CONTINUE
                  C
                  C ELEMENT MASS MATRIX
                  C
ISN       25            CALL PRODMA(6,6,6,AINVET,EM,RJS4)
ISN       26            CALL PRODMA(6,6,6,RJS4,AINVE,EM)
                  C
                  C ELEMENT DAMPING MATRIX
                  C
ISN       27            CALL PRODMA(6,6,6,AINVET,ED,RJS5)
ISN       28            CALL PRODMA(6,6,6,RJS5,AINVE,ED)
                  C
                  C ELEMENT STIFNESS MATRIX
                  C
ISN       29            CALL PRODMA(6,6,6,AINVET,EK,RJS4)
ISN       30            CALL PRODMA(6,6,6,RJS4,AINVE,EK)
ISN       31            RETURN
ISN       32            END
```

```
LEVEL 1.4.0 (OCT 1984)          VS FORTRAN          DATE: AUG 25, 1986    TIME: 15:03:39

OPTIONS IN EFFECT:  NOLIST   MAP   XREF   GOSTMT NODECK   SOURCE   TERM   OBJECT FIXED  NOTEST NOTRMFLG
                    NOSYM NORENT NOSDUMP AUTODBL(NONE)   NOSXM
                      OPT(2) LANGLVL(77) NOFIPS   FLAG(I)  NAME(MAIN  )  LINECOUNT(60)     CHARLEN(500

                    *....*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN      1          SUBROUTINE  GENMAJ(ELENG)
               C
               C    SUBROUTINE GENERATING MATRICES J*1-J*23 11
               C
ISN      2          IMPLICIT REAL*8(A-H,O-Z)
ISN      3          DIMENSION RJS1(6,6),RJS2(6,6),RJS3(6,6),RJS4(6,6),RJS5(6,6),
              *            RJS6(6,6),RJS8(6,6),RJS10(6,6),RJS11(6,6),
              *         RJS12(6,6),RJS15(6,6),RJS16(6,6),RJS17(6,6),RJS18(6,6),
              *         RJS19(6,6),RJS20(6,6),RJS21(6,6),RJS22(6,6),
              *         RJS31(6,6),RJS41(6,6),RJS42(6,6),RJS43(6,6)
ISN      4          COMMON /MAJS/RJS1,RJS2,RJS3,RJS4,RJS5,RJS6,RJS8,RJS10,
              *         RJS11,RJS12,RJS15,RJS16,RJS17,RJS18,RJS19,RJS20,
              *         RJS21,RJS22,RJS31,RJS41,RJS42,RJS43
               C=================
ISN      5          DO 400 I=1,6
ISN      6          DO 400 J=1,6
ISN      7          RJS1(I,J)=0.0D0
ISN      8          RJS2(I,J)=0.0D0
ISN      9          RJS3(I,J)=0.0D0
ISN     10          RJS4(I,J)=0.0D0
ISN     11          RJS5(I,J)=0.0D0
ISN     12          RJS6(I,J)=0.0D0
ISN     13          RJS8(I,J)=0.0D0
ISN     14          RJS10(I,J)=0.0D0
ISN     15          RJS11(I,J)=0.0D0
ISN     16          RJS12(I,J)=0.0D0
ISN     17          RJS15(I,J)=0.0D0
ISN     18          RJS16(I,J)=0.0D0
ISN     19          RJS17(I,J)=0.0D0
ISN     20          RJS18(I,J)=0.0D0
ISN     21          RJS19(I,J)=0.0D0
ISN     22          RJS20(I,J)=0.0D0
ISN     23          RJS21(I,J)=0.0D0
ISN     24          RJS22(I,J)=0.0D0
ISN     25          RJS31(I,J)=0.0D0
ISN     26          RJS41(I,J)=0.0D0
ISN     27          RJS42(I,J)=0.0D0
ISN     28          RJS43(I,J)=0.0D0
ISN     29     400  CONTINUE
               C
               C  GENERATING MATRIX J*1
               C
ISN     30          DO 402 I=1,4
ISN     31          DO 402 J=1,4
ISN     32          K=I+J-1
ISN     33          RJS1(I,J)=ELENG**(K)/DFLOAT(K)
ISN     34     402  CONTINUE
               C
               C  GENERATING MATRIX J*2
               C
ISN     35          DO 410 II=2,4
ISN     36          DO 410 JJ=2,4
```

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LEVEL 1.4.0 (OCT 1984)          VS FORTRAN          DATE: AUG 25, 1986     TIME: 15:03:39      NAME: GEN

                   *....*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN      37              KK=II+JJ-3
ISN      38              RJS2(II,JJ)=ELENG**(KK)
ISN      39        410   CONTINUE
ISN      40              RJS2(3,3)=4.D0*RJS2(3,3)/3.D0
ISN      41              RJS2(3,4)=1.5D0*RJS2(3,4)
ISN      42              RJS2(4,3)=RJS2(3,4)
ISN      43              RJS2(4,4)=1.8D0*RJS2(4,4)
                   C
                   C  GENERATING MATRICES J*3
                   C
ISN      44              RJS3(3,3)=4.D0*ELENG
ISN      45              RJS3(3,4)=6.D0*ELENG**(2)
ISN      46              RJS3(4,3)=RJS3(3,4)
ISN      47              RJS3(4,4)=12.D0*ELENG**(3)
                   C
                   C  GENERATING MATRIX  J*4
                   C
ISN      48              RJS4(5,5)=ELENG
ISN      49              RJS4(5,6)=.5D0*ELENG**(2)
ISN      50              RJS4(6,5)=RJS4(5,6)
ISN      51              RJS4(6,6)=ELENG**(3)/3.D0
                   C
                   C  GENERATING MATRIX J*5
                   C
ISN      52              DO 420 I=1,4
ISN      53              DO 420 J=2,4
ISN      54              K2 =J-1
ISN      55              K3= I+J-2
ISN      56              RJS5(I,J)=DFLOAT(K2)*ELENG**(K3)/DFLOAT(K3)
ISN      57        420   CONTINUE
                   C
                   C GENERATING MATRIX RJS6
                   C
ISN      58              RJS6(3,5)=2.D0*ELENG
ISN      59              RJS6(3,6)=ELENG**(2)
ISN      60              RJS6(4,5)=3.D0*RJS6(3,6)
ISN      61              RJS6(4,6)=2.D0*ELENG**(3)
                   C
                   C  GENERATING MATRIX  J*8
                   C
ISN      62              DO 430 L=2,4
ISN      63              LL=L-1
ISN      64              RJS8(L,6)=ELENG**(LL)
ISN      65        430   CONTINUE
                   C
                   C  GENERATING MATRIX J*10
                   C
ISN      66              RJS10(6,6)=ELENG
                   C
                   C  GENERATING MATRICES  J*11  AND J*12
                   C
ISN      67              DO 440 JJ=1,4
ISN      68              RJS11(JJ,3)=2.D0*ELENG**(JJ)/DFLOAT(JJ)
ISN      69              J1=JJ+1
ISN      70              RJS11(JJ,4)=6.D0*ELENG**(J1)/DFLOAT(J1)
ISN      71              RJS12(JJ,3)=2.D0*ELENG**(J1)/DFLOAT(J1)
```

```
                  *...,*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN      72             RJS12(JJ,4)=6.D0*ELENG**(JJ+2)/DFLOAT(JJ+2)
ISN      73       440   CONTINUE
                  C
                  C  GENERATING MATRIX R*15
                  C
ISN      74             RJS15(3,6)=2.D0*ELENG
ISN      75             RJS15(4,6)=3.D0*ELENG**2
                  C
                  C  GENERATING MATRICES R*16 , R*18 AND R*19
                  C
ISN      76             DO 443 I5=1,4
ISN      77             RJS16(I5,6)=ELENG**(I5)/DFLOAT(I5)
                  C===
ISN      78             RJS18(I5,5)=ELENG**(I5)/DFLOAT(I5)
ISN      79             RJS18(I5,6)=ELENG**(I5+1)/DFLOAT(I5+1)
                  C===
ISN      80             RJS19(I5,6)=ELENG**(I5+1)/DFLOAT(I5+1)
ISN      81       443   CONTINUE
                  C
                  C  GENERATING MATRICES R*17 AND R*20
                  C
ISN      82             DO 446 J5=2,4
ISN      83             J51 =J5-1
ISN      84             RJS17(5,J5) =ELENG**(J51)
ISN      85             RJS17(6,J5) =DFLOAT(J51)*ELENG**(J5)/DFLOAT(J5)
                  C===
ISN      86             RJS20(5,J5)=DFLOAT(J51)*ELENG**(J5)/DFLOAT(J5)
ISN      87             RJS20(6,J5)=DFLOAT(J51)*ELENG**(J5+1)/DFLOAT(J5+1)
ISN      88       446   CONTINUE
                  C
                  C  GENERATING MATRICES R*21 AND R*22
                  C
ISN      89             DO 447 I=2,3
ISN      90             IJ          =3+I
ISN      91             RJS21(5,IJ)=ELENG**(I)/DFLOAT(I)
ISN      92             RJS21(6,IJ)=ELENG**(I+1)/DFLOAT(I+1)
ISN      93       447   CONTINUE
ISN      94             RJS22(5,6)=ELENG
ISN      95             RJS22(6,6)      =.5D0*ELENG**2
                  C
                  C  GENERATING MATRICES R*31,R*41,R*42 AND R*43
                  C
ISN      96             RJS41(5,6)=0.5D0*ELENG**2
ISN      97             RJS41(6,6)=ELENG**3/DFLOAT(3)
ISN      98             RJS42(2,6)=ELENG**2/DFLOAT(2)
ISN      99             RJS42(3,6)=2.D0*RJS41(6,6)
ISN     100             RJS42(4,6)=3.D0*ELENG**4/DFLOAT(4)
ISN     101             DO 480 JI=1,4
ISN     102             JI1=JI+1
ISN     103             JI2=JI+2
ISN     104             JI3=JI+3
ISN     105             RJS43(JI,5)=ELENG**JI1/DFLOAT(JI1)
ISN     106             RJS31(JI,2)=RJS43(JI,5)
ISN     107             RJS43(JI,6)=ELENG**JI2/DFLOAT(JI2)
ISN     108             RJS31(JI,3)=2.D0*RJS43(JI,6)
ISN     109             RJS31(JI,4)=(3.D0*ELENG**JI3)/DFLOAT(JI3)
```



```
                  *...,*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN     110       480   CONTINUE
ISN     111             RETURN
ISN     112             END
```

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LEVEL 1.4.0 (OCT 1984)            VS FORTRAN           DATE: AUG 25, 1986     TIME: 15:03:40

OPTIONS IN EFFECT:  NOLIST    MAP   XREF   GOSTMT NODECK   SOURCE   TERM   OBJECT FIXED  NOTEST NOTRMFLG
                    NOSYM NORENT NOSDUMP AUTODBL(NONE)    NOSXM
                      OPT(2) LANGLVL(77) NOFIPS   FLAG(I)  NAME(MAIN  )  LINECOUNT(60)       CHARLEN(500)

                *....*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN     1               SUBROUTINE  INVERS(ELENG)
                C
                C       SUBROUTINE INVERTING THE MATRIX A
                C
                C
ISN     2               IMPLICIT REAL*8(A-H,O-Z)
ISN     3               DIMENSION AINVE(6,6),AINVET(6,6)
ISN     4               COMMON /INV/AINVE,AINVET
ISN     5               DO 500 I=1,6
ISN     6               DO 500 J=1,6
ISN     7               AINVE(I,J)=0.0D0
ISN     8       500     CONTINUE
ISN     9               AINVE(1,1)=1.D0
ISN    10               AINVE(2,2)=1.D0
ISN    11               AINVE(5,3)=1.D0
ISN    12               AINVE(3,1)=-3.D0/ELENG**(2)
ISN    13               AINVE(3,2)=-2.D0/ELENG
ISN    14               AINVE(3,4)=-AINVE(3,1)
ISN    15               AINVE(3,5)=-1.D0/ELENG
ISN    16               AINVE(4,1)=2.D0/ELENG**(3)
ISN    17               AINVE(4,2)=ELENG**(-2)
ISN    18               AINVE(4,4)=-AINVE(4,1)
ISN    19               AINVE(4,5)=AINVE(4,2)
ISN    20               AINVE(6,6)=1.D0/ELENG
ISN    21               AINVE(6,3)=-AINVE(6,6)
                C=======
ISN    22               DO 505 I=1,6
ISN    23               DO 505 J=1,6
ISN    24               AINVET(I,J)=AINVE(J,I)
ISN    25       505      CONTINUE
ISN    26               RETURN
ISN    27               END
```

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LEVEL 1.4.0 (OCT 1984)          VS FORTRAN          DATE: AUG 25, 1986   TIME: 15:03:40

OPTIONS IN EFFECT:  NOLIST    MAP    XREF    GOSTMT NODECK    SOURCE    TERM    OBJECT FIXED    NOTEST NOTRMFLG
                    NOSYM NORENT NOSDUMP AUTODBL(NONE)    NOSXM
                      OPT(2) LANGLVL(77) NOFIPS    FLAG(I)   NAME(MAIN    )  LINECOUNT(60)        CHARLEN(500

                *....*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN       1          SUBROUTINE PRODMA(M,L,N,A,B,C)
                C
                C SUBROUTINE MULTIPLYING MATRICES
                C
                C
ISN       2          IMPLICIT   REAL*8(A-H,O-Z)
ISN       3          DIMENSION  A(M,L),B(L,N),C(M,N)
ISN       4          DO 590  I=1,M
ISN       5          DO 590  J=1,N
ISN       6          C(I,J)=0.0D0
ISN       7          DO 590  K=1,L
ISN       8          C(I,J)=C(I,J)+A(I,K)*B(K,J)
ISN       9     590   CONTINUE
ISN      10          RETURN
ISN      11          END
```

```
LEVEL 1.4.0 (OCT 1984)          VS FORTRAN          DATE: AUG 25, 1986     TIME: 15:03:40

OPTIONS IN EFFECT:  NOLIST   MAP   XREF   GOSTMT NODECK   SOURCE   TERM   OBJECT FIXED  NOTEST NOTRMFLG
                    NOSYM NORENT NOSDUMP AUTODBL(NONE)   NOSXM
                      OPT(2) LANGLVL(77) NOFIPS    FLAG(I,  NAME(MAIN  )  LINECOUNT(60)     CHARLEN(500)

                   *....*...1.........2.........3.........4.........5.........6.........7.*.......8

ISN       1           SUBROUTINE ARRANG(N,X,PHI)
                C
                C     SUBROUTINE ARRANGING EIGENVALUES FROM SMALL TO BIGGER AND
                C     ARRANGING EIGENVECTORS CORRESPONDING TO EIGENVALUES
                C
                C
ISN       2           IMPLICIT REAL*8(A-H,O-Z)
ISN       3           COMPLEX*16 X,PHI,PHIN,EIMIN,XEI
ISN       4           DIMENSION X(N),PHI(N,N)
ISN       5           LA =N-1
ISN       6           DO 600 I=1,LA
ISN       7           EIMIN=X(I)
ISN       8           XMIN=DIMAG(EIMIN)
ISN       9           JMI =I
ISN      10           JF  =I+1
ISN      11           DO 601 M=JF,N
ISN      12           XEI =X(M)
ISN      13           IF(DABS(XMIN).LE.DABS(DIMAG(XEI))) GOTO 601
ISN      14           JMI=M
ISN      15           XMIN=DIMAG(XEI)
ISN      16           EIMIN=XEI
ISN      17     601   CONTINUE
ISN      18           X(JMI)=X(I)
ISN      19           X(I)  =EIMIN
ISN      20           DO 633 L=1,N
ISN      21           PHIN       =PHI(L,JMI)
ISN      22           PHI(L,JMI)=PHI(L,I)
ISN      23           PHI(L,I)   =PHIN
ISN      24     633   CONTINUE
ISN      25     600   CONTINUE
ISN      26           RETURN
ISN      27           END
```

```
LEVEL 1.4.0 (OCT 1984)            VS FORTRAN           DATE: AUG 25, 1986     TIME: 15:03:40

OPTIONS IN EFFECT:   NOLIST    MAP    XREF   GOSTMT NODECK    SOURCE    TERM    OBJECT FIXED  NOTEST NOTRMFLG
                     NOSYM NORENT NOSDUMP AUTODBL(NONE)    NOSXM
                       OPT(2) LANGLVL(77) NOFIPS    FLAG(I)  NAME(MAIN  )   LINECOUNT(60)      CHARLEN(500)

                 *....*..1.........2.........3.........4.........5.........6.........7.*.......8

ISN       1            SUBROUTINE OUTEIG(NDTT,N,X,PHI)
                 C
                 C     SUBROUTINE TO PRINT EIGENVALUES AND EIGENVECTORS
                 C
                 C
ISN       2            IMPLICIT REAL*8(A-H,O-Z)
ISN       3            DIMENSION X(NDTT),PHI(NDTT,NDTT)
ISN       4            COMPLEX*16 X,PHI
ISN       5            PRINT 740
ISN       6      740   FORMAT(///,14X,'====FREQUENCIES===='//)
ISN       7            M=NDTT/3
ISN       8            DO 741 J1=1,M
ISN       9                 J2=M+J1
ISN      10                 J3=2*M+J1
ISN      11            WRITE(6,750)J1,X(J1),J2,X(J2),J3,X(J3)
ISN      12      750   FORMAT(/,4X,I2,'TH',2D18.8,I3,'TH',2D18.8,I3,'TH',2D18.8)
ISN      13      741   CONTINUE
                 C============
                 C ∂=(∂,∂') ∂̀ IS VECTOR
                 C======================
ISN      14            NN= NDTT/2
ISN      15            DO 751 JA=1,1,1
ISN      16                 JB=JA+1
ISN      17                 JC=JA+2
ISN      18            WRITE(6,760)JA,JB,JC
ISN      19      760   FORMAT(//,9X,I2,'TH EIGENVECTOR',30X,I2,'TH EIGENVECTOR',32X,
                      *         I2,'TH EINGVECTOR')
ISN      20            DO 761 I=1,NN
ISN      21            WRITE(6,762)PHI(I,JA),PHI(I,JB),PHI(I,JC)
ISN      22      762   FORMAT(/4X,3(2D18.8,2X))
ISN      23      761   CONTINUE
ISN      24      751   CONTINUE
ISN      25            RETURN
ISN      26            END
```

```
LEVEL 1.4.0 (OCT 1984)            VS FORTRAN          DATE: AUG 25, 1986     TIME: 15:03:40

OPTIONS IN EFFECT:   NOLIST    MAP    XREF   GOSTMT NODECK    SOURCE   TERM    OBJECT FIXED   NOTEST NOTRMFLG
                     NOSYM NORENT NOSDUMP AUTODBL(NONE)      NOSXM
                       OPT(2) LANGLVL(77) NOFIPS    FLAG(I)  NAME(MAIN  )  LINECOUNT(60)      CHARLEN(500)

                *....*...1.........2.........3.........4.........5.........6.........7.*.......8
ISN      1            SUBROUTINE PRINT(N,A)                                                00535
               C                                                                           00536
               C      SUBROUTINE TO PRINT MATRIX                                           00537
               C                                                                           00538
               C                                                                           00539
ISN      2            IMPLICIT REAL*8(A-H,O-Z)                                             00540
ISN      3            DIMENSION A(6,6)                                                     00541
ISN      4            WRITE(6,822) N                                                       00542
ISN      5     822    FORMAT(///,10X,'MATRIX J',I2,//)                                     00543
ISN      6            WRITE(6,823)((A(I,J),J=1,6),I=1,6)                                   00544
ISN      7     823    FORMAT(/,6(4X,D12.5),/)                                              00545
ISN      8            RETURN                                                               00546
ISN      9            END                                                                  00547
```

IN-PLANE MOTION (EXTENSIBLE & CLAMPED-CLAMPED)

ELEMENT CONNECTIVITY:

ELEMENT NUMBER	NODE					
1	1	2	3	4	5	6
2	4	5	6	7	8	9
3	7	8	9	10	11	12
4	10	11	12	13	14	15
5	13	14	15	16	17	18
6	16	17	18	19	20	21
7	19	20	21	22	23	24
8	22	23	24	25	26	27
9	25	26	27	28	29	30
10	28	29	30	31	32	33
11	31	32	33	34	35	36
12	34	35	36	37	38	39
13	37	38	39	40	41	42
14	40	41	42	43	44	45
15	43	44	45	46	47	48
16	46	47	48	49	50	51
17	49	50	51	52	53	54
18	52	53	54	55	56	57
19	55	56	57	58	59	60
20	58	59	60	61	62	63
21	61	62	63	64	65	66
22	64	65	66	67	68	69
23	67	68	69	70	71	72
24	70	71	72	73	74	75
25	73	74	75	76	77	78
26	76	77	78	79	80	81
27	79	80	81	82	83	84

28	82	83	84	85	86	87
29	85	86	87	88	89	90
30	88	89	90	91	92	93
31	91	92	93	94	95	96
32	94	95	96	97	98	99
33	97	98	99	100	101	102
34	100	101	102	103	104	105

DIMENSIONLESS PARAMETERS:

DIMENSIONLESS VELOCITY= 0.62832D+01

BETAA= 0.00000D+00,BETA= 0.50000D+00,H= 0.00000D+00

AA = 10,000 , SIMA = 0.00000D+00,

CAPA= 0.76923D+00,RO= 0.31416D+01

****FREQUENCIES****

1TH	-0.15750822D+00	0.42172437D+02	71TH	-0.56402343D-01	0.50276740D+04
2TH	-0.15750822D+00	-0.42172437D+02	72TH	-0.56402343D-01	-0.50276740D+04
3TH	0.11358792D+00	0.96652755D+02	73TH	0.53295924D-01	0.50732216D+04
4TH	0.11358792D+00	-0.96652755D+02	74TH	0.53295924D-01	-0.50732216D+04
5TH	0.74200679D-01	0.17377118D+03	75TH	-0.36804906D-01	0.54488805D+04
6TH	0.74200679D-01	-0.17377118D+03	76TH	-0.36804906D-01	-0.54488805D+04
7TH	-0.90615231D-01	-0.25896990D+03	77TH	0.34412332D-01	0.55045268D+04
8TH	-0.90615231D-01	0.25896990D+03	78TH	0.34412332D-01	-0.55045268D+04
9TH	-0.11407508D+00	0.30719172D+03	79TH	-0.12898461D-01	0.58466242D+04
10TH	0.11407508D+00	-0.30719172D+03	80TH	-0.12898461D-01	-0.58466242D+04
11TH	-0.18652525D+00	0.39009733D+03	81TH	0.11941487D-01	-0.59907182D+04
12TH	-0.18652525D+00	-0.39009733D+03	82TH	0.11941487D-01	0.59907182D+04
13TH	0.13162190D+00	0.43962450D+03	83TH	-0.93718387D-02	0.62512579D+04

14TH	0.13162190D+00	-0.43962450D+03	84TH	-0.93718387D-02	-0.62512579D+04
15TH	-0.77595331D-01	0.55172153D+03	85TH	0.13156205D-01	-0.65018557D+04
16TH	-0.77595331D-01	-0.55172153D+03	86TH	0.13156205D-01	0.65018557D+04
17TH	0.11809616D+00	0.67353256D+03	87TH	-0.14601354D-01	-0.66645774D+04
18TH	0.11809616D+00	-0.67353256D+03	88TH	-0.14601354D-01	0.66645774D+04
19TH	-0.11325324D+00	0.72649379D+03	89TH	0.59285625D-01	0.70356267D+04
20TH	-0.11325324D+00	-0.72649379D+03	90TH	0.59285625D-01	-0.70356267D+04
21TH	0.50608929D-01	0.88107906D+03	91TH	-0.62652001D-01	-0.70873504D+04
22TH	0.50608929D-01	-0.88107906D+03	92TH	-0.62652001D-01	0.70873504D+04
23TH	-0.65242202D-01	-0.99167602D+03	93TH	0.59024449D-01	0.75142061D+04
24TH	-0.65242202D-01	0.99167602D+03	94TH	0.59024449D-01	-0.75142061D+04
25TH	0.43297875D-01	0.10835991D+04	95TH	-0.64344176D-01	0.75967927D+04
26TH	0.43297875D-01	-0.10835991D+04	96TH	-0.64344176D-01	-0.75967927D+04
27TH	-0.10550034D+00	0.12833078D+04	97TH	0.33721465D-01	0.79513535D+04
28TH	-0.10550034D+00	-0.12833078D+04	98TH	0.33721465D-01	-0.79513535D+04
29TH	0.98768203D-01	0.13149566D+04	99TH	-0.43800307D-01	0.81779118D+04
30TH	0.98768203D-01	-0.13149566D+04	100TH	-0.43800307D-01	-0.81779118D+04
31TH	-0.43834361D-01	0.15342376D+04	101TH	0.34678154D-01	0.83937219D+04
32TH	-0.43834361D-01	-0.15342376D+04	102TH	0.34678154D-01	-0.83937219D+04
33TH	0.41325479D-01	-0.16150926D+04	103TH	-0.83812230D-01	-0.87781674D+04
34TH	0.41325479D-01	0.16150926D+04	104TH	-0.83812230D-01	0.87781674D+04
35TH	-0.31510958D-01	-0.17933950D+04	105TH	0.79820057D-01	0.88440451D+04
36TH	-0.31510958D-01	0.17933950D+04	106TH	0.79820057D-01	-0.88440451D+04
37TH	0.30864532D-01	-0.19324265D+04	107TH	-0.37029358D-01	-0.92768764D+04
38TH	0.30864532D-01	0.19324265D+04	108TH	-0.37029358D-01	0.92768764D+04
39TH	-0.30038970D-01	-0.20715743D+04	109TH	0.43228081D-01	0.94189752D+04
40TH	-0.30038970D-01	0.20715743D+04	110TH	0.43228081D-01	-0.94189752D+04
41TH	0.29861193D-01	0.22558243D+04	111TH	-0.36338484D-01	-0.97195664D+04
42TH	0.29861193D-01	-0.22558243D+04	112TH	-0.36338484D-01	0.97195664D+04
43TH	-0.31164856D-01	-0.23695674D+04	113TH	0.15120281D+00	-0.10066352D+05