THEORY AND APPLICATION
OF
PHOTO-ELASTICITY

THESIS

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I INTRODUCTION

What is Photo-elasticity

Theory of Elasticity treats of the analysis of stresses set up in a body, subjected to external forces. It leads us to differential equations which have been solved only in a few simple cases.

For calculating stresses set up in structures the Theory of Strength of Materials has been developed from pure Elasticity, by making certain assumptions and approximations. It leads us to solutions, which, although not exact, are accurate enough for engineering purposes.

Cases arise, however, in which assumptions made by the Theory of Strength of Materials are not justified and solutions are apt to be of doubtful accuracy. In such cases it is desireable, if not necessary, to check calculations experimentally.

In addition of the usual methods of determining stresses by measuring strains with extensometers, a new, optical method has been developed within the last thirty five years. It is a result of the development of Optics in the XIX-th century and is based on a peculiar property of transparent materials to undergo a change in their optical behaviour when strained, as well as on general principles of Elasticity.

By this method - the Photo-elastic method, as it is called - it is possible to solve for internal stresses, by measuring the changes in optical behaviour of the material, utilising such optical phenomena as polarised light, double refraction, interference. In its present state of development, Photo-elasticity is limited to analysing models made out of a transparent material, and bound by plane parallel faces, with external forces uniformly distributed across the thickness. Its application to engineering problems is made practicable by the fact that stresses in two dimensions are independent of elastic constants of the material, so long as they are within its elastic limit.

If a plane plate is loaded in its own plane, there will be, at any point, two mutually perpendicular stresses, called principal stresses. Photo-elasticity completely determines the principal stresses and thus solves the problem of distribution of stresses; since, knowing the principal stresses at any point, it is simple enough to calculate direct stress or shear on any plane at that point.

Three methods

There are three distinct methods used in Photo-elasticity;

Coker's method - First used by Mesnager, but greatly developed and improved by Prof. Coker. It is partly optical and partly mechanical.

Filon's method - Insofar as the optical part is concerned it is the same as Coker's, but, instead of mechanical measurements, it utilises graphical integration along the lines of principal stress, beginning at the boundary.

Favre's method - It is the most recent one and is purely optical. It is this method that is followed in the Photo-elastic laboratory at McGill University.

The Filon method is complicated and not very accurate;, it is not much used at present, and will not be described here. The other two methods, however, are being used, and they will be, particularly the Favre method, described in detail. The latter one has been developed only within the last three or four years,and there are at present three laboratories in which ~~the~~ Favre's apparatus is installed. (Zurich, Buenos Ayres and McGill.)

II. UNDERLYING PRINCIPLES.

Forces in a plane are independent of elastic constants.

If a plane plate is subjected to external forces in its own plane, uniformly distributed across the thickness and is allowed to deform laterally, it can be shown by elementary Elasticity that the stresses set up in it are independent of all elastic constants, depending only on shape and loading.

Principal stresses in a co-planar system of forces.

Consider a block of material subjected to a system of co-planar forces, Fig. 1, there being no external force in the third direction (perpendicular to the plane of the paper). Any such system, holding the body in equilibrium, can be resolved into direct forces p_1 and p_2 and shears s, as shown, shears on all four faces being equal.

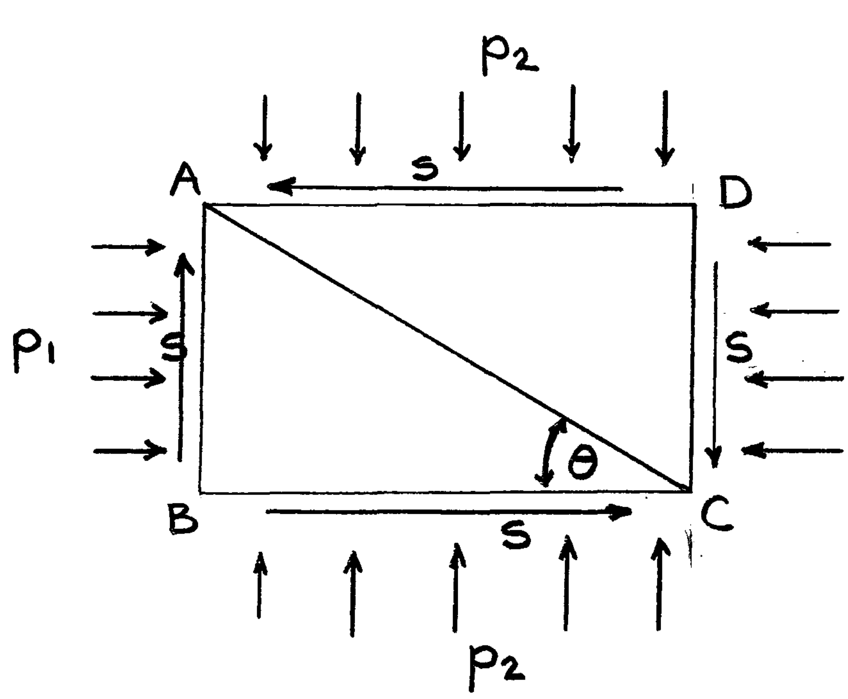


Fig 1

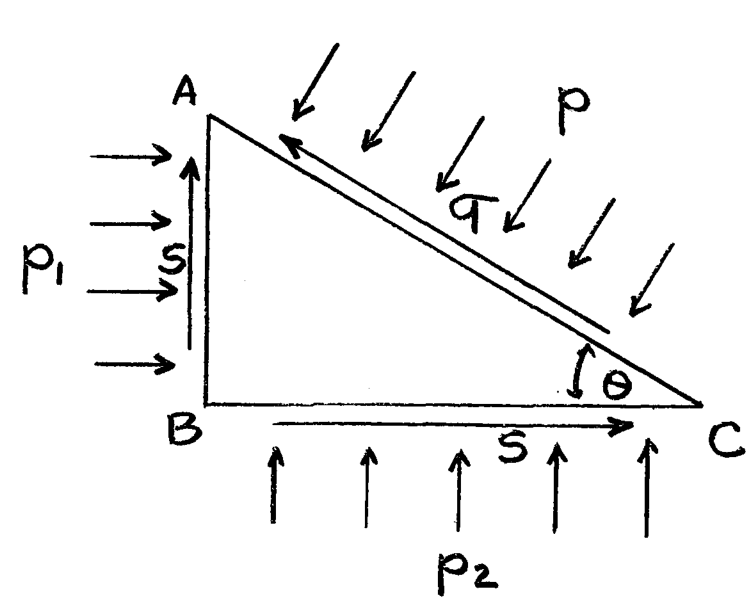


Fig. 2

Considering the equilibrium of the wedge ABC, Fig. 2, and resolving all forces parrallel to AC, we have:

$$p_1 . AB . \cos\theta + s . BC . \cos\theta - p_2 \, BC . \sin\theta - s . AB . \sin\theta - q \, AC = 0$$

where p and τ are the direct force and shear respectively, exerted by the other wedge.

$$\tau = (p_1 - p_2).\sin\theta.\cos\theta + q.(\cos^2\theta - \sin^2\theta)$$

For $\tau = 0$

$$\frac{p_1 - p_2}{2}\sin 2\theta = -q.\cos 2\theta$$

$$\tan 2\theta = -\frac{2q}{p_1 - p_2}$$

Two angles, 180° apart, satisfy the above equation. Therefore, there are two angles, two directions 90° apart, on which there is no shearing force and stresses are wholly normal.

Again, considering the equilibrium of the wedge ABC, but resolving perp. to AC, we have:

$$p.AC = p_1 AB.\sin\theta + p_2 BC.\cos\theta + q.AB.\cos\theta + q.BC.\sin\theta$$

$$p = p_1\sin^2\theta + p_2\cos^2\theta + q.\sin 2\theta$$

For maximum (or minimum) p, putting $\frac{dp}{d\theta} = 0$

$$2p_1.\sin\theta.\cos\theta - 2p_2.\sin\theta.\cos\theta + 2q.\cos 2\theta = 0$$

$$(p_1 - p_2).\sin 2\theta = -2q.\cos 2\theta$$

$$\tan 2\theta = -\frac{2q}{p_1 - p_2}$$

Thus, we see that the normal stress is maximum on planes on which shear is zero. At any point in a stressed body, then, there are two mutually perp. planes on which shear is zero and stresses, therefore, are wholly normal, and, moreover, one of these stresses is greater than (and the other one, according to the Ellipse of Stress, is smaller than) on any other plane at that point. These planes and stresses are called principal planes and principal stresses.

Incidentally, by putting $\frac{d\tau}{d\theta} = 0$, for planes of maximum shear, we find that $\tan 2\theta' = \frac{p_1 - p_2}{q}$

2θ and $2\theta'$ differ by 90°, and, therefore, θ and θ' by 45°. Maximum shear, then, acts on planes at 45° to the principal planes.

Optics.

1) The Wave Theory of Light.

The Wave theory of light, founded by Huygens, explains the propagation of light by means of vibrations of ether in a plane perpendicular to the direction of the ray, analogous to the propagation of waves on the surface of water. In the case of the water waves, the water particles oscillate about their equilibrium positions, at right angles to the direction in which the wave is travelling, but there is no movement of matter in the direction of the wave. It is found extremely helpful to think similarly of a ray of light, namely, that there is "something" which executes S.H.M. in a plane perp. to the ray. Since the vibrations are S.H.M., a ray of light can be represented by a sine curve. At any instant the illumination is proportional to the square of the ordinate.

Since light travels through "empty" inter-stellar space, and since a medium is necessary for the propagation of a wave of this kind, it is necessary to postulate the existence of a hypothetical medium, which was named ether, a fluid filling all space and penetrating all matter.

The speed of light in vacuum is 3.0×10^{10} cm. per sec. (186,000 miles per sec.) and $\frac{3.0 \times 10^{10}}{n}$ cms. per sec. in a medium of refractive index n.

Wave length, crest to crest, is denoted by λ, which is different for different colours. Below is a table of wave lengths of several colours.

Colour	Violet	Blue	Green	Gr.Yell.	Yellow	Orange	Red
λ	.41	.47	.52	.55	.575	.57	.65-.79
Comparative Effect on Eye	.01	.09	.7	1	.92	.63	.1-0

In the above table the wave lengths are in microns. (1 micron = 1.0 x 10^{-4} cm.). The last line shows the comparative sensitiveness of the human eye to the above colours, taking the sensitiveness to the greenish yellow, which is the greatest, as 1.

White light is a superposition of monochromatic vibrations.

2) Polarisation and Double Refraction.

a) Plane polarised light.

If a beam of parallel light OM falls on a plane mirror M, Fig.3, it will give rise to a reflected beam MF'. If the mirror M is turned about OX as axis, the reflected beam will turn with it and its intensity will not change.

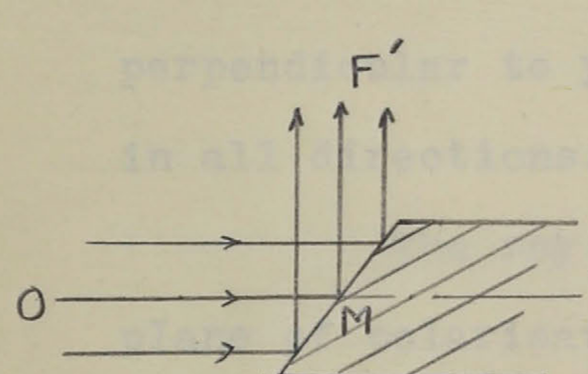


Fig. 3

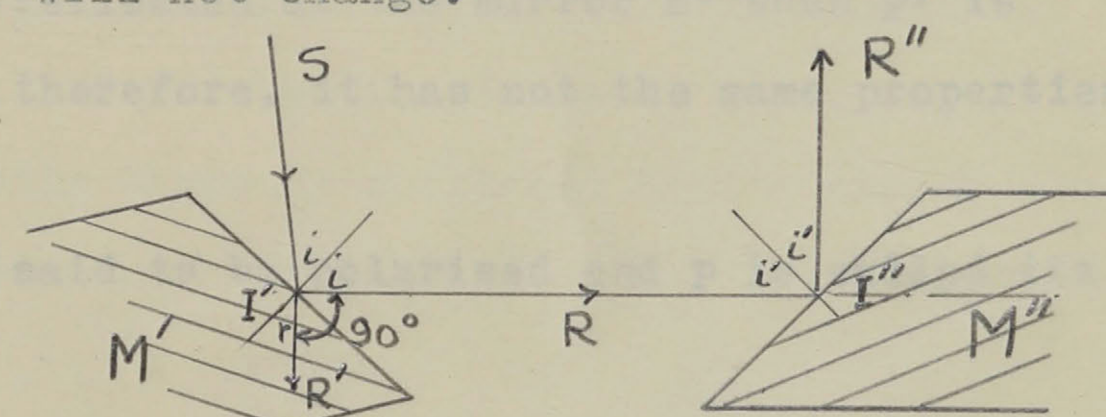


Fig. 4

If, however, the incident beam SI', Fig.4, is let fall on a plane mirror M' at a particular angle or incidence - discovered by Brewster and bearing his name, - such that the reflected and refrated rays are at right angles to each other, the reflected ray

I'R will exhibit a very peculiar property. This particular angle of incidence is different for glasses of different refractive index and is a simple function of it. Let n be the refractive index, i - the angle of incidence and r - the angle of refraction, then:

$$n = \frac{\sin i}{\sin r}\ ; \qquad \text{Also: } \sin r = \cos i$$

$$\therefore \tan i = n$$

For ordinary glass i is about 56°.

Let the reflected ray I'R fall on a second plane mirror M", Fig. 4, again at this peculiar angle of incidence. If the mirror M" is turned about I'I" as axis, the angle of incidence does not change, and the second plane of incidence RI"R" turns with the mirror. Let the plane of incidence at the first mirror be called p and that at the second mirror - p'. When plane p' coincides with plane p the intensity of reflected ray I"R" is maximum. As the second mirror is turned, intensity diminishes, until complete extinction is reached when p' is at right angles to p. Ray RI" is, therefore, unable to get reflected at the mirror M" when p' is perpendicular to p and, therefore, it has not the same properties in all directions.

The ray I'R is said to be polarised and p is called its plane of polarisation.

In a ray of ordinary light vibrations are assumed to take place in all directions around the ray, without any regularity whatever. In the case of polarised light, however, vibrations are supposed to take place only in one plane, at right angles to the plane of polarisation. Hence, its name - Plane polarised.

In the experiment described above, the ray I'I" is vibrating perpendicularly to the plane P, i.e. in and out of the plane of the paper. When P' coincides with P, these vibrations are still at right angles to the direction of the ray, and there is no decrease in intensity. When mirror M" is turned through 90^{o}, these vibrations are along the ray and, consequently, there is no amplitude of vibration, and no light. At any intermediate position the intensity of light is $\cos^2\phi$ where ϕ is the angle between P and P' (since illumination is proportional to the square of the ordinate, representing the transverse vibration).

b) Double Refraction in crystals.

A few natural crystals exhibit the property of double refraction, to be described presently. Crystals of Calcite, $CaCO_3$, found in Iceland possess particularly pronounced double refracting properties and are usually employed in this connection.

When a ray of light falls on such a crystal, two refracted rays are formed. Of the two rays, one obeys the laws of refraction and is called the ordinary ray, and the other does not, - and is called the extraordinary ray. The ordinary and extraordinary rays are both polarised, but in planes at right angles to each other, as can be easily shown by analysing them - one at a time - by a mirror on which they should fall at the Brewster angle of incidence.

In every crystal of Calcite, there is a line of optical symmetry, called the optical axis. A section parallel to the optical axis and normal to the face of entry is called the principal section.

The plane of polarisation of ordinary ray is parallel to the principal section.

c) Nicol Prisms.

The Nicol prism, Plate I, is made by cutting an Iceland spar crystal along a diagonal plane AC. The crystal is cut to such a length that this plane starts at one blunt corner A and finishes at the opposite on - C. The cut faces are polished and re-united with a thin film of Canada Balsam.

A ray of light entering the prism through one of the lozenge-shaped end faces, gives rise to the ordinary ray NP and the extraordinary ray NS. The angle of incidence of the extraordinary ray NP at the film of Canada Balsam is such (for a range of about 60° of the angle of incidence i) that it does not get refracted, but goes through emerging at the other end face BC. The ordinary ray, on the other hand, strikes the film at an angle sufficient to cause complete reflection, is reflected and is absorbed at T. Since the extraordinary ray (as well as the other one) is plane polarised, the net effect of the Nicol is to produce plane polarised light.

The Nicol prism is one of the most convenient means of producing polarised light and is used extensively for this purpose.

d) Eliptically polarised light.

Let us consider a thin plate, cut out of a double refracting crystal, Fig. 5, and let Ox be its optical axis, Oy at right angles to Ox.

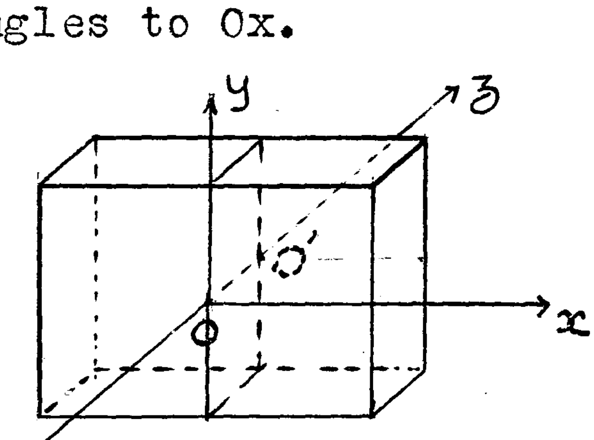


Fig. 5

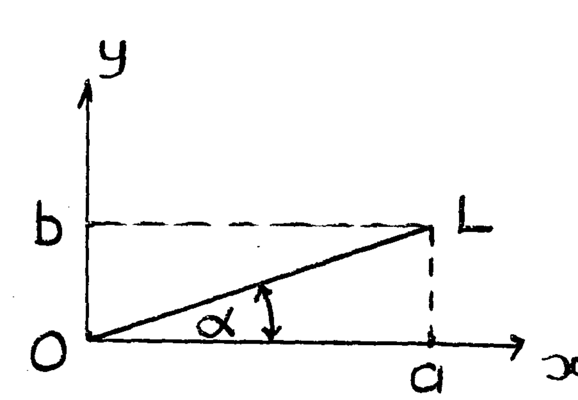
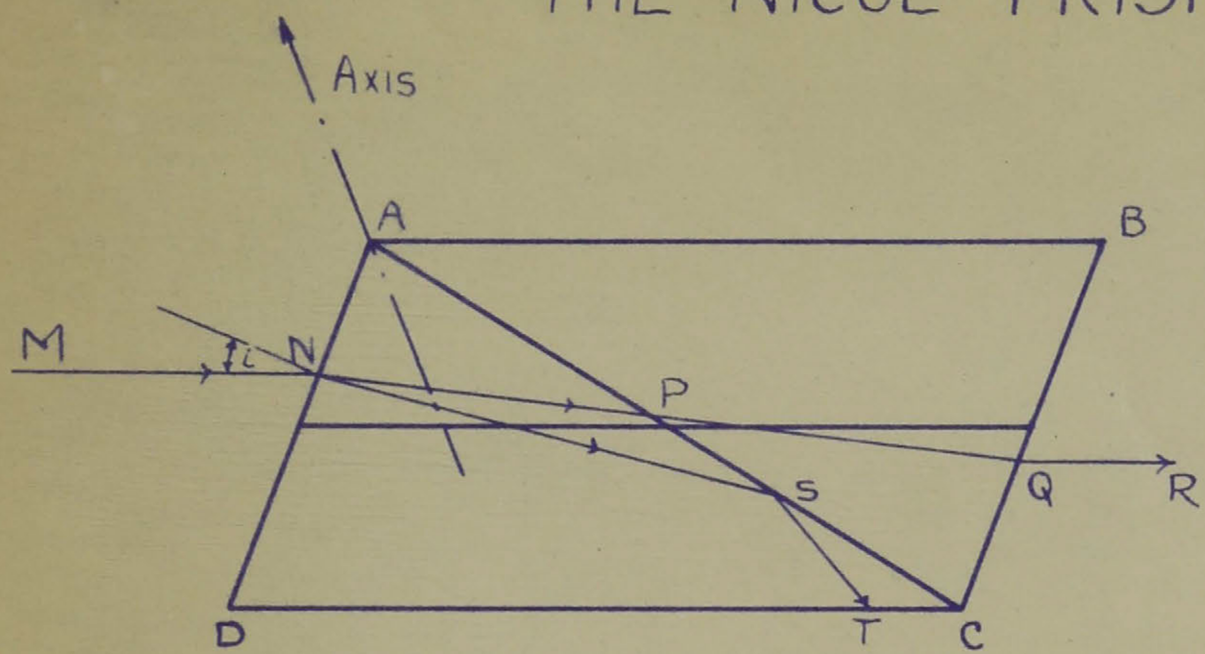


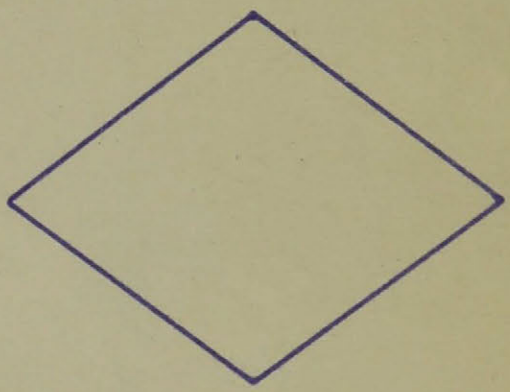
Fig. 6

PLATE I

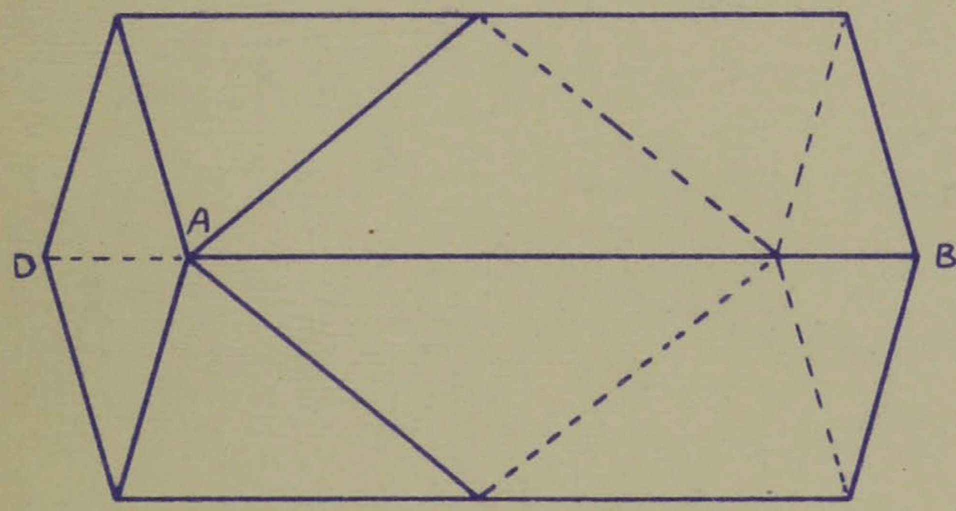
THE NICOL PRISM



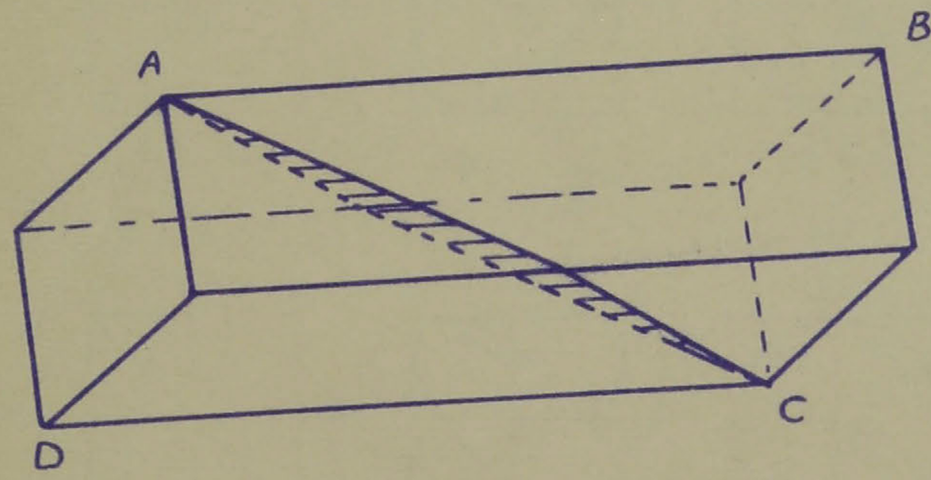
SIDE ELEV.



END VIEW



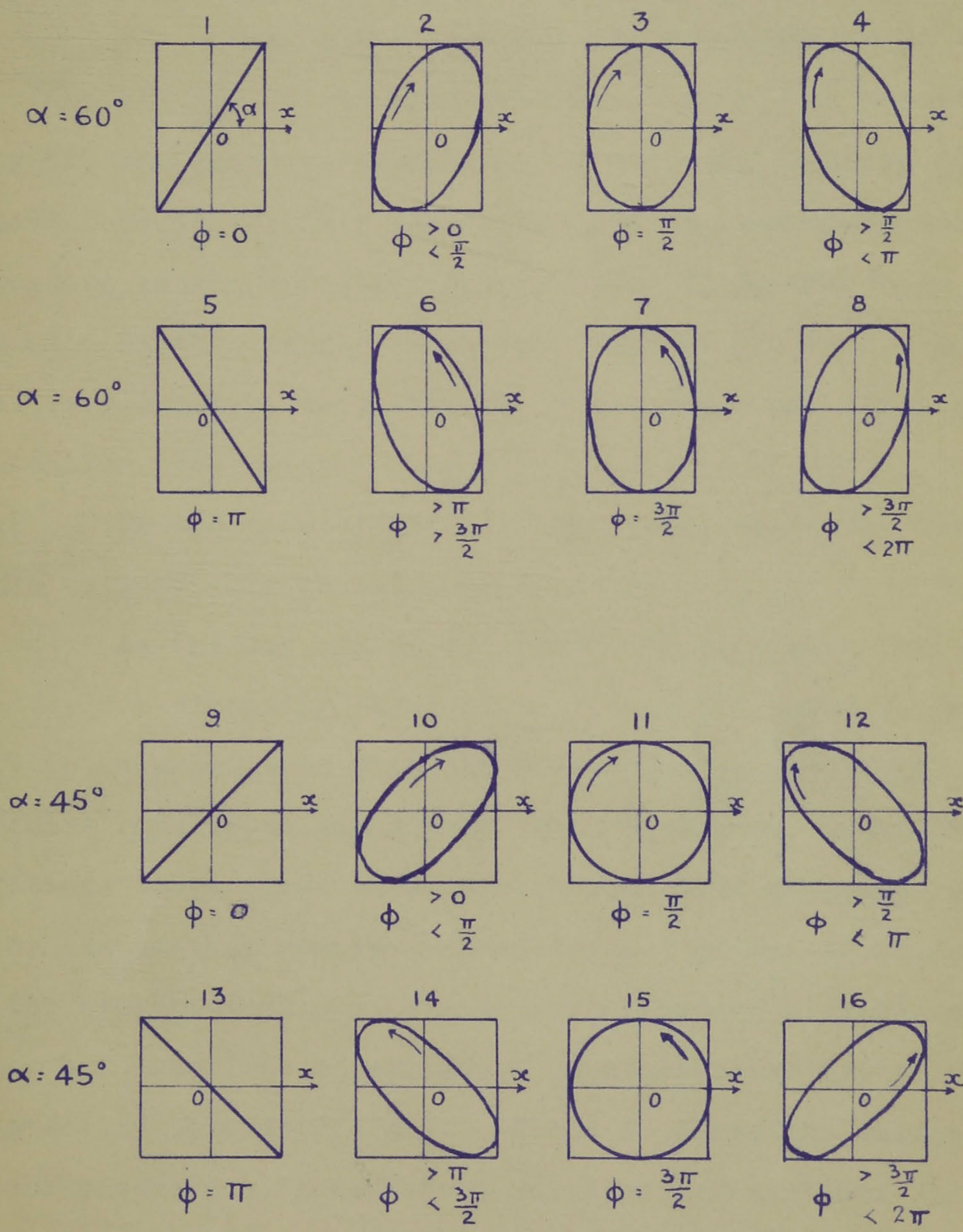
PLAN



PERSPECTIVE VIEW

PLATE II

VARIOUS POLARISATIONS OBTAINED BY VARYING α AND ϕ.



Then, a plane polarised ray, vibrating parallel to Ox, will go through the plate without any change, save a slight retardation while going through the plate, due to a different index, and will, therefore, emerge at the other face as still a plane polarised light vibrating parallel to Ox; similarly a ray vibrating parallel to Oy will undergo no change.

If, however, the incident ray is vibrating in some direction OL, Fig. 6, it will break up into two components, one along Ox, the other along Oy. Now, velocity of light in the direction Ox is Vx, to which correspond λ_x and n_x. Similarly, corresponding to the direction Oy, we have V_y, λ_y and n_y. We shall call axis Ox the one for which the index is greater, and, therefore, Vx - smaller, i.e. Axis Ox is the one along which greater retardation occurs.

In the case of uniaxial crystals, Ox is the projection of the optical axis on the face of entry.

As the ray enters the plate, it is broken into two components, one vibrating parallel to Ox and the other to Oy, and the first is retarded more than the second. The deviation of the two component rays is so samll that they recombine. Each of the two vibrations takes place as before, except that they are now out of phase, and as they combine, an elliptical motion is produced, that is, the hypothetical particle does not execute vibrations in a plane, but describes an ellipse. Such emergent light is called elliptically polarised light. Plate II shows the various polarisations produced with different relative retardations and different angles between Ox and the plane of vibrations of the incident ray (angle α).

It is readily seen that when the incident ray makes an angle of 45^{o} with the axes, and the relative retardation is $\frac{\pi}{2}$ the ellipse becomes a circle, and the emergent light is called circularly polarised.

3) <u>Full-, Half-, and Quarter- wave plates.</u>

Relative retardation between Ox and Oy components is proportional to thickness of the plate and to $(n_x - n_y)$. Plates of double refracting crystals may be cut to such a thickness as to produce any required retardation. Full-wave, half-wave and quarter-wave plates are commonly used in Optics. They are made of very thin mica laminae. The axis of greater retardation is usually indicated.

Full-wave plates produce a full wave length retardation, and, at the exit, the two components combine into vibrations identical to that of the incident ray.

Half-wave plates, produce a half wave retardation, that is, Ox component is exactly π radians behind Oy. This is the case shown at 5 and 13, Plate II. It will be seen that the effect of the half-wave plate is to turn the plane of polarisation symmetrically about the axis. Thus, if before the vibrations were at an angle α° with Ox, now they will be at the angle $-\alpha^{\circ}$.

Quarter-wave plates produce a quarter wave retardation. This is the case shown at 3 and 11, Plate II. When α is 45^{o}, the components along Ox and Oy are equal and the ellipse becomes a circle.

Depending on the relative retardation of components the elliptical motion is either clockwise or anti-clockwise.

When light received on any one of these plates (or any other double refracting plate, for that matter) is not plane polarised, but is already elliptically polarised, the effect of the plates is unchanged: - they still produce the same relative retardation, which may change the shape of the ellipse or may change it into plane polarised light, depending on values of α, phase difference produced by the plate and the phase difference already existing in the incident ray.

4) Interference.

Light, like other vibrations with S.H.M., is capable of interfering, Thus, certain conditions being fulfilled, dark bands could be seen on a screen lit by two sources.

Condition of main importance for interference is that the two sources should be identical: since it is, of course, impossible to realize this experimentally, two images of the same source, or a source and its image are used.

Representing schematically an arrangement for producing interference, Fig. 7, let S and S' be two identical sources sending out beams of light, as shown. In a region illuminated by both beams, alternate dark and light bands would be observed on a screen placed in it.

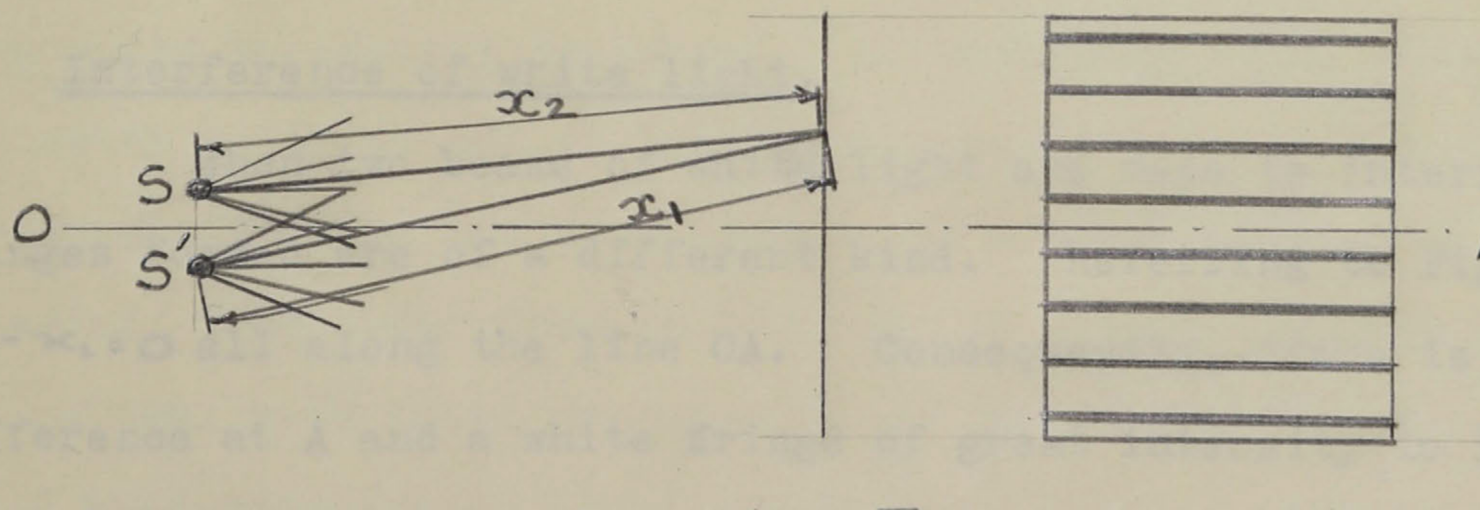


Fig. 7.

Interference is explained as follows: let two waves, out of phase with each other be represented by two sine curves, Fig. 8, and let one of them be δ behind the other. The angular difference will be $\frac{\delta}{\lambda} \cdot 2\pi$.

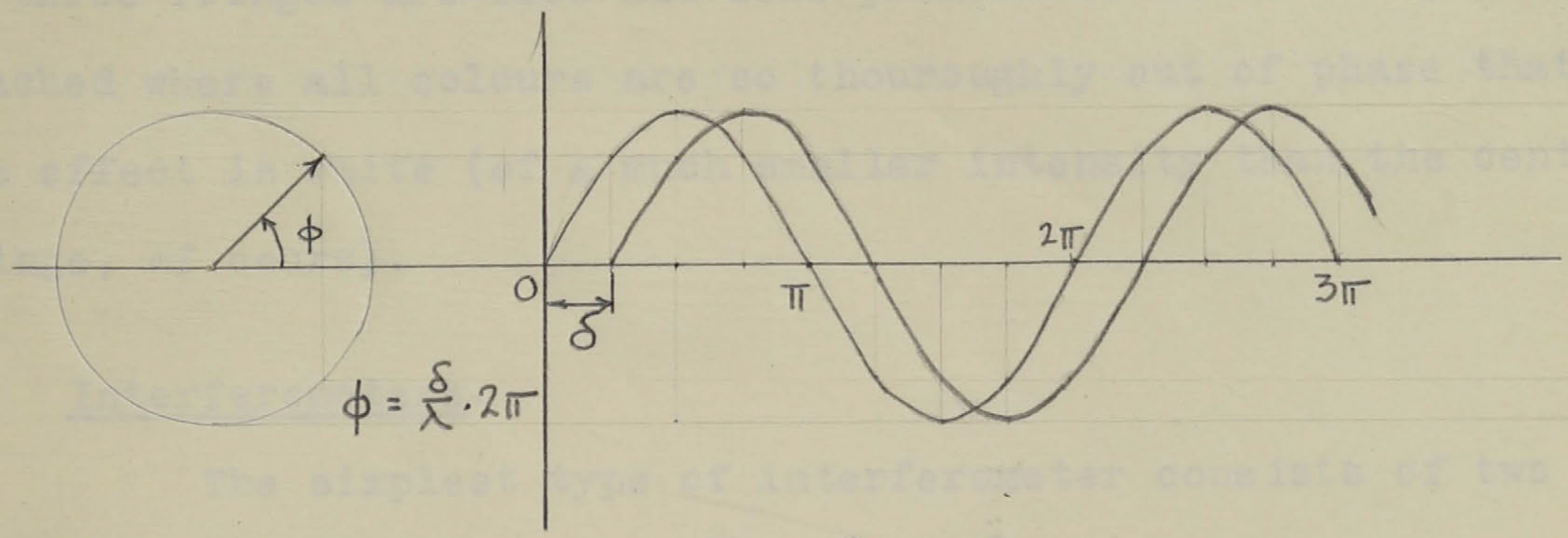


Fig. 8.

Referring to Figs. 7 and 8, $x_1 - x_2 = \delta$ for any point on the screen. When $x_1 - x_2 = k.\lambda$, where k is 0 or a whole number, the crests of the two waves superimpose, and the troughs superimpose; the luminous vactor is thus doubled, and the intensity quadrupled - this results in a bright fringe.

At points for which $x_1 - x_2 = (k + \frac{1}{2})\lambda$, the crests of one wave correspond to the troughs of the other, the two waves thus neutralising each other - this results in a dark fringe.

The interference fringes, although appearing straight, are really slightly curved, since they are traces of large hyperboloids on the plane of the screen.

5) Interference of white light.

When two beams of white light are made to interfere, the fringes formed are of a different kind. Referring to Fig. 7 $x_1 - x_2 = 0$ all along the line OA. Consequently, there is no phase difference at A and a white fringe of great intensity is formed. On each side of it a dark band appears and is not black but a succession of dark colours due to different colours having their dark zones not at the same poiht because of different λ's. For the same reason the next light fringe is not white but coloured successively with blue, green, yellow, orange and red. The next two or three fringes are less and less pronounced until a region is reached where all colours are so thouroughly out of phase that the effect is white (of a much smaller intensity than the centre fringe, of cours).

6) Interferometers.

The simplest type of interferometer consists of two parallel plates of glass, silvered on faces f, Fig. 9.

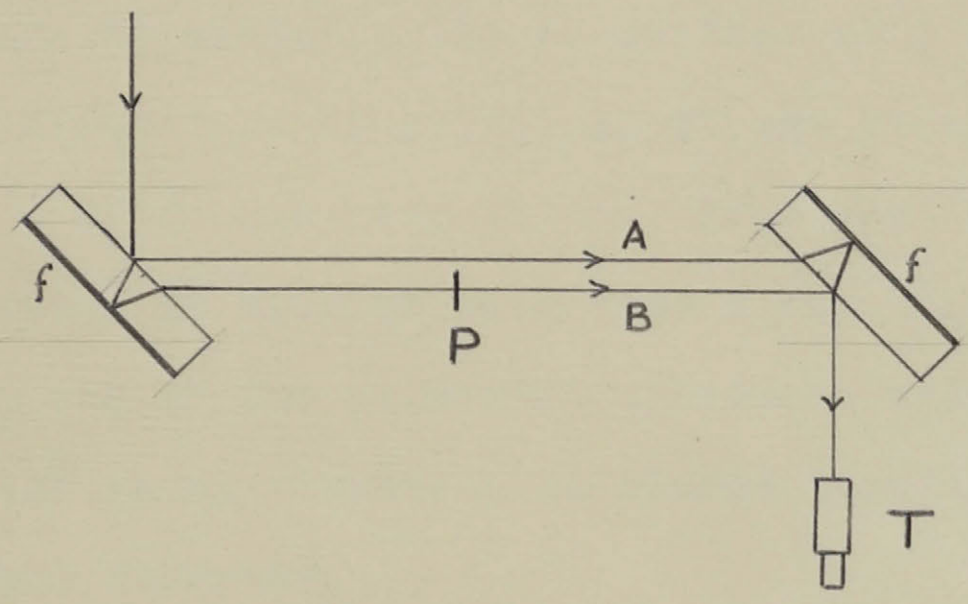


Fig. 9

If the two plates were of exactly the same thickness and were absolutely parallel there would be no interference, as the two rays would be exactly in phase. Actually, however, these conditio are never realized and on looking through telescope T vertical interference fringes would be seen. If a thin plate of transparent material, P be interposed on the path of one of the rays as shown, its optical path would be changed, and the whole interference field would move sideways. It can be brought back by turning slightly one of the mirrors. The movement is measured with a micrometer and is a measure of retardation of ray B. Incidentally, thickness of the Plate P or its index can be computed from this.

The Mach-Zehnder Interferometer which is used in connection with Favre's method, will be described in detail later.

III. PRINCIPLES OF PHOTO-ELASTICITY

Double refracting properties of stressed materials; Law of Wortheim.

As has been shown formerly, in a plane body, under the action of forces in that plane, there are two directions at right angles to each other, on which the stress is wholly normal. Therefore, at any point there are three unknowns: magnitude of one, magnitude of the other and direction of one of them.

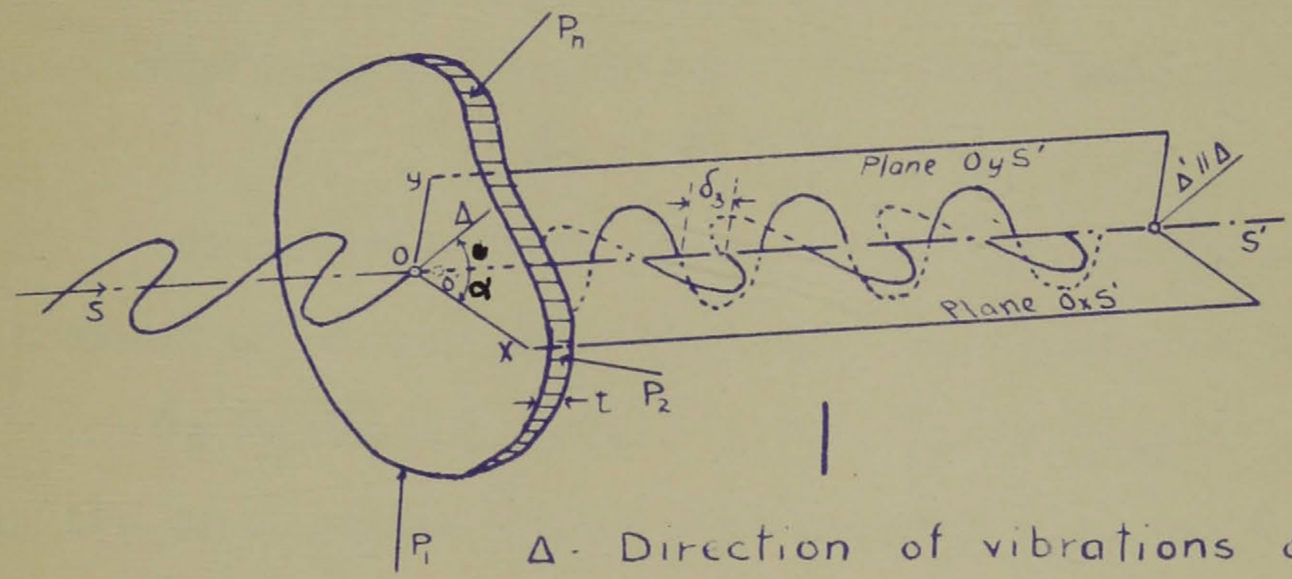
A plate of glass when unstrained is isotropic, that is, it has the same optical properties in all directions (provided it is homogenous and has no initial stress). When, however, it is loaded in its own plane it exhibits the properties of a double refracting material, the directions of principal stress at any point corresponding to the axes of double refraction. From what has been said about double refraction, it will be seen that a ray at an angle to the direction of any one of the principal stresses, will be decomposed into two component rays, vibrating in planes parallel to the directions of the principal stresses.

Referring to Plate III, SO is the incident ray vibrating in plane OΔ at an angle α with Ox; O_H and O_V are the components in the direction Ox and Oy. Assuming the stresses to be unequal, retardations suffered by the two rays will be unequal and on leaving the plates they will be out of phase. We shall call δ_3 the relative retardation suffered by the two rays.

Double refraction is due to destruction of properties of isotropy, - the index in one of the directions of principal stresses is now different from that in the other direction. The greater be the difference of principal stresses at a point, the greater will be the relative retardation δ_3. Therefore, δ_3 is proportional to p-q.

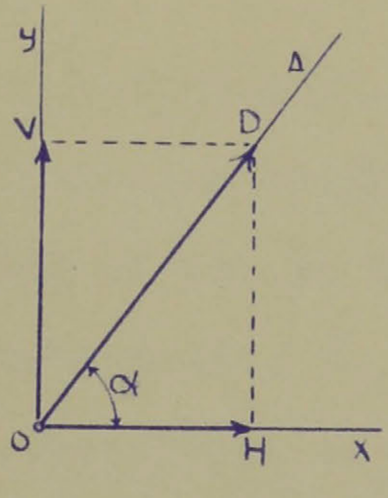
PLATE III

ILLUSTRATING DOUBLE REFRACTING PROPERTY OF A TRANSPARENT PLATE UNDER STRESS.



1

Δ - Direction of vibrations of incident ray SO
Ox, Oy - " " principal stresses at O.



2

OD - Amplitude of vibration of incident ray SO
OH, OV - Amplitudes of vibrations of new component rays.

δ_3 is also proportional to thickness t

$$\delta_3 = c \cdot t.(p-q)$$

where c is a constant, the value of which depends on kind of glass and wave length of light used. Wortheim was the first to formulate and prove by experiments the above law, which is known by his name.

When the plate is not loaded the emergent ray is identical to the incident ray. When the plate is loaded, each of the components undergoes a retardation. We shall call them δ_1 and δ_2, so that

$$\delta_3 = \delta_1 - \delta_2$$

Since it is only Favre's method tnat is concerned with measurement of δ_1 and δ_2, they will be discussed later

δ_3 is usually measured by compensating it with a calibrating piece or a compensator. Knowing δ_3, p - q can be calculated from Wortheim's relation.

p and q can be determined either separately or as p + q; in the latter case p and q are obtained by solving two simultaneous equations.

Determination of directions of the principal stress depends mainly on the following principle. When the incident ray vibrates in the direction of one of them, it will undergo no decomposition but simply a retardation δ_1, and will still vibrate in the same plane, and will, therefore, be completely extinguished by the second, analysing, nicol. Thus, extinction results at all points where the direction of stress coincides with planes of polarisation of light, which is known. The direction of the principal stress is usually referred to the vertical, and the angle the vertical makes with it is usually designated by α.

Historical

Seebeck and Sir David Brewster were the first to discover the double refracting properties of strained glass.

Later, Fresnel experimented with it and confirmed their theories.

Wortheim was the next to make the important discovery, vis. that $\delta_3 = ct(p-q)$

Neumann did considerable amount of work on double refraction and formulated a theory of behaviour of glass loaded in three dimensions.

But it was not until Mesnager that all that knowledge was applied to engineering. In 1900 Mesnager constructed a glass model of a concrete arch bridge and analysed it by means of polarised light and measurements of lateral deformation.

Soon after, Prof Coker developed the subject still further, introduced many changes and improvements and did a good many tests.

A few years later, Filon contributed another development - the graphical integration along the lines of stress.

And, finally, Prof. Favre, of Zurich, developed a purely optical method by introducing interferometer measurements.

IV MESNAGER-COKER OPTICO-MECHANICAL METHOD.

The following description mainly deals with Coker's apparatus, but methods used by Mesnager (when they are different from Coker's) are also outlined.

Mesnager used glass models; Coker uses models made of celluloid or xylonite.

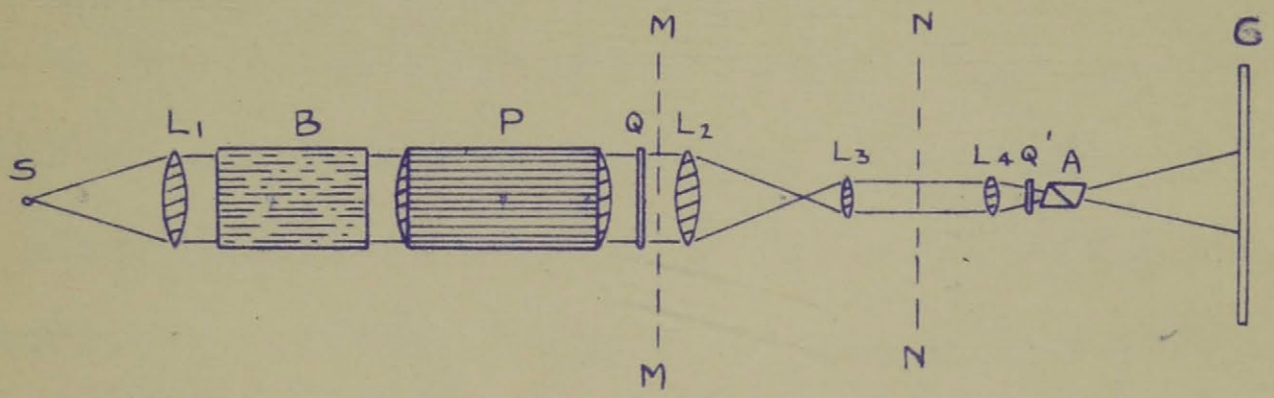
1) General arrangement of Coker's apparatus.

A schematic~~ally~~ lay-out of the apparatus is shown on Plate IV.

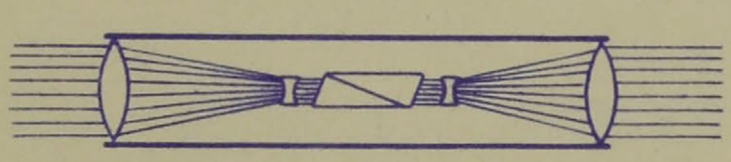
White light from a source S passes through a condensing lens L and water screen B (to reduce the heat rays), and is then passed through a polariser P. This unit consists of two larger condensing lenses and two small concave lenses, arranged as shown at 2, Plate IV, in combination with a Nicol. The latter is of less than one inch circular opeture, but in combination with lenses gives a beam of polarised light about 4" in diameter. The combination thus accomplishes what a nicol of 4" circular aperture would do, were there any of that size obtainable. The plane polarised light is then passed through a quarter wave plate Q, which alters it to circularly polarised light, which, in turn, is passed through the transparent specimen under load to be examined, which is in plane MM. The light then passes through lenses E, F and G, and through the quarter-wave plate K,which is similar to J except that its axis is at 90° to that of J, and, therefore, counteracts the effect of J. The resulting light is analysed by another nicol A, set at 90° with the polarising nicol. The light is finally

PLATE IV

LAY-OUT OF COKER'S APPARATUS



1. SCHEMATIC DIAGRAM OF OPTICAL SET.



2. POLARISER P.

projected upon the screen C so that points in plane MM (or NN, - another position for the specimen) are brought to a focus on this screen. The colours produced depend directly on the stress distribution in the specimen.

2) Determination of direction of stresses; iso-clinic, iso-chromic and iso-static lines.

For determination of direction the quarter-wave plates are removed and light remains plane polarised. In general, such light after passing through a loaded model, would become elliptically polarised, and some light would get past the analysing nicol. At points where one of the principal stresses is parallel to the plane of vibrations, the emergent light would still vibrate in that plane and would be completely extinguished by the analyser. If the whole specimen is examined simultaneously, a dark band would be seen, passing through all points at which the direction of one of the principal stresses is parallel to the principal section of the nicol, the direction of which is known. If the nicols are turned through an angle, the dark band would move, to pass through all points at which one of the principal stresses is parallel to the new direction of the principal section of the nicol.

These dark bands connect points at which the principal stresses are equally inclined to a fixed direction, and are called iso-clinic bands. The inclinations are usually in reference to the vertical; thus, the 30° iso-clinic passes through points at which the principal stresses are at 30° and 120° to the vertical.

Iso-clinic lines, as such, are rather useless in conveying the picture of the distribution of stresses, and are plotted only as a means of obtaining the lines of principal stress or iso-static lines, which at any point are tangent to one of the principal stresses, and thus, at any point show the direction of the principal stresses. When a number of iso-static lines are drawn for a specimen, they form a net-work of orthogonal lines.

The iso-static lines can be obtained from the iso-clinic lines in one of two ways:

(i) As practiced by Mesnager. If tangents are drawn at wo infinitesimally near points on a curve, then the lengths of these tangents (from the point of tangency to the point of their intersection) are regarded as equal.

Applying this to iso-clinic lines, the lines of principal stress can be obtained from them in this way; suppose we have two iso-clinic lines as shown in Fig.10, and suppose we want to derive iso-static lines from them.

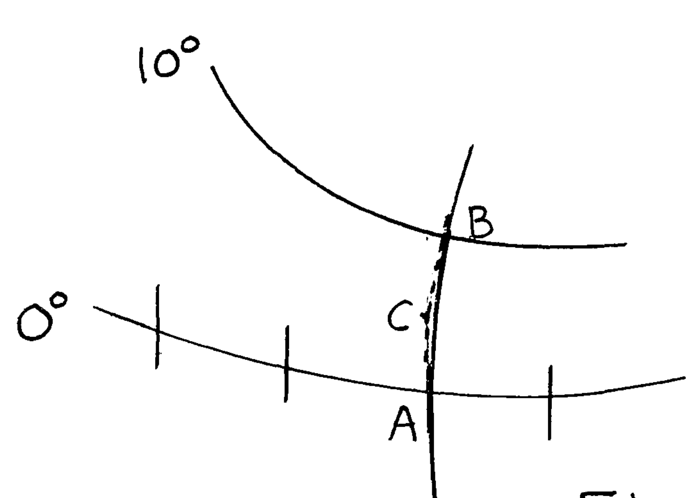


Fig. 10

On one of them, say the 0° line several equally spaced short lines are drawn in the direction of the principal stresses. Now, to find where each of them cuts the next iso-clinic, say the 10° one, a ruler is placed at 10° with line CA and is made to slide up and down until the intercept CB is equal to CA. When this position is found, the point B is marked and joined by an easy curve with A since it is on the same iso-static.

(ii) Coker uses a faster method, if somewhat not quite as accurate. He draws little crosses all along the iso-clinic lines, such that their arms are in the direction of the principal stresses. Then the lines of principal stress are drawn in by eye. In drawing the iso-static lines it is important to keep in mind that they cut each other at right angles.

3) <u>Measurement of p - q.</u>

When a specimen is analysed in white, plane polarised light, in addition to dark iso-clinic bands there will be coloured bands, called iso-chromes, which are due to interference of white light at points where the plane of vibrations is not parallel to either principal stress. At any point on an iso-chrome the interference effect is the same, hence δ_3 or the difference of phase between the two rays is the same, hence (p - q) is the same. Points for which (p - q) is zero would lie on a dark band, which must not be confused with an iso-clinic line. Going from the dark band for which (p - q) is zero the succession of colours would be as follows: - Grey, white, yellow, orange, red, purple, blue, yellow, red, purple. Thus, for higher differences of stresses some of the colours repeat.

When analysing a model for p - q to eliminate confusion of the black iso-chromes and iso-clinics, two methods are available.

(i) To rotate the nicols at a high speed. For the field of iso-chrome is independent of the orientation of the nicols and will remain unchanged, while a different iso-clinic corresponds to every position of the nicols, and with nicols rotating fast, the iso-clinic lines will not be seen because of persistence of vision. The whole field will be somewhat dimmed.

(ii) A better method and one which is now universally used with Coker's apparatus is to use circularly polarised light, i.e. to rotate the plane of polarisation instead of the nicols, which, nevertheless gives the same effect. This is accomplished by inserting two quarter-wave plates, as shown on Plate IV, their axes at right angles to each other. It will be seen from what has been said of Optics that plane polarised light when falling on a quarter-wave plate and vibrating at 45° to its axis will result in circularly polarised light; this light is passed through the model, and then through the other quarter-wave plate which being at 90° to the first will neutralise its effect. The resulting light will be elliptically polarised with the same difference of phase at any point as if there were no quarter-wave plates; thus, the field of iso-chromes will not be changed, with iso-clinics eliminated.

(p - q) can be determined in connection with this method in one of three ways:

(i) A prismatic calibration piece of the same material and thickness as the model is placed near the model and loaded in direct tension or compression until it developes the colour of the iso-chrome under consideration, taking into account the repetition of some colours. When that is reached, (p - q) at any point on that iso-chrome is equal to the intensity of the direct stress in the calibration piece. By looking at the colour alone it is impossible to tell whether (p - q) is tensile or compressive, but it can always be determined by inspection. This method is fast but rather inaccurate and is not very much used.

(ii) Law of Wortheim states that $\delta_3 = c.t.(p-q)$. Mesnager measured δ_3 by means of a Babinet compensator, to be described presently. In the above equation t is known, δ_3 can be measured by the Babinet compensator; hence (p - q) can be calculated, c can be determined by measuring δ_3 for a known (p - q).

The Babinet compensator consists of two blocks of double refracting material, shaped as shown in Fig. 11, and placed with the slanting faces together so as to form a rectangular block. The axis of greater retardation of one of them, say block M, is horizontal and the similar axis of the other is vertical. A ray of light going through the centre of the block, that is in the plane CAOB will be unchanged when it emerges out at the other side, because the greater retardation of the horizontal component by the block M will be neutralisedby an equal greater retardation of the vertical component by the block N.

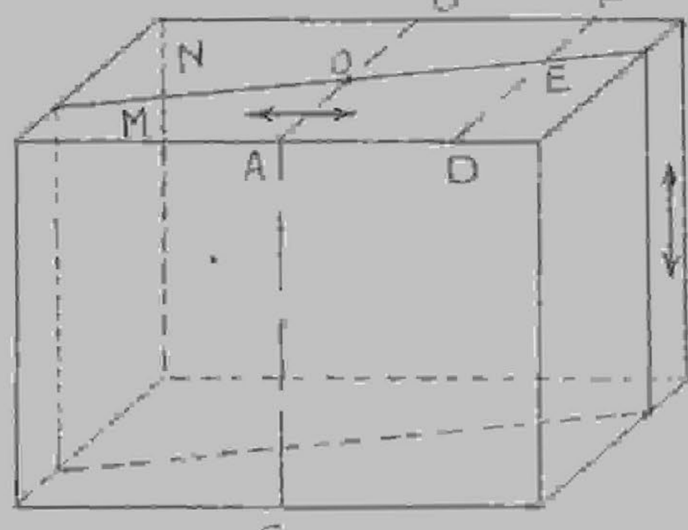


Fig. 11

If, however, the ray penetrates the block nearer to the side as at DEF, then the retardation of the horizontal component by the block M is proportional to DE and the retardation of the vertical component with respect to the horizontal one by the block N is proportional to EF, and the net effect of the compensator is retardation of the horizontal component with respect to the vertical, which is proportional to (DE - EF). Similarly to the left of centre AOB the compensator is equivalent to a single block with the vertical axis and thickness equal to the difference of thicknesses of blocks M and N at that point. Thus by letting the incident ray go through the block at different points various phase differences can be produced. And, having calibrated the compensator δ_3 can be measured by compensating it by producing $-\delta_3$.

In fact, a compensator like this is usually calibrated for measuring (p - q) directly. It is placed between the calibrating prism and the analysing nicol. When the prism is not stressed a dark line appears at the centre of the compensator. By putting a known direct stress on the calibrating piece, a certain δ_3 is produced which is compensated when the ray is passed through the compensator somewhere off the centre. The ray being fixed, it is the compensator that is made to slide, its movement being measured by a micrometer. It is obvious that for half the stress the compensator would be moved half the amount for extinction. Movement in the opposite direction corresponds to (p - q) of a different sign.

Having thus calibrated it, the compensator is placed behind the specimen to be tested and (p -q) is determined with the proper sign at any point by turning the micrometer screw until extinction is obtained, reading it and dividing it by the calibration constant.

(iii) Coker's method of determining (p - q) is as follows:

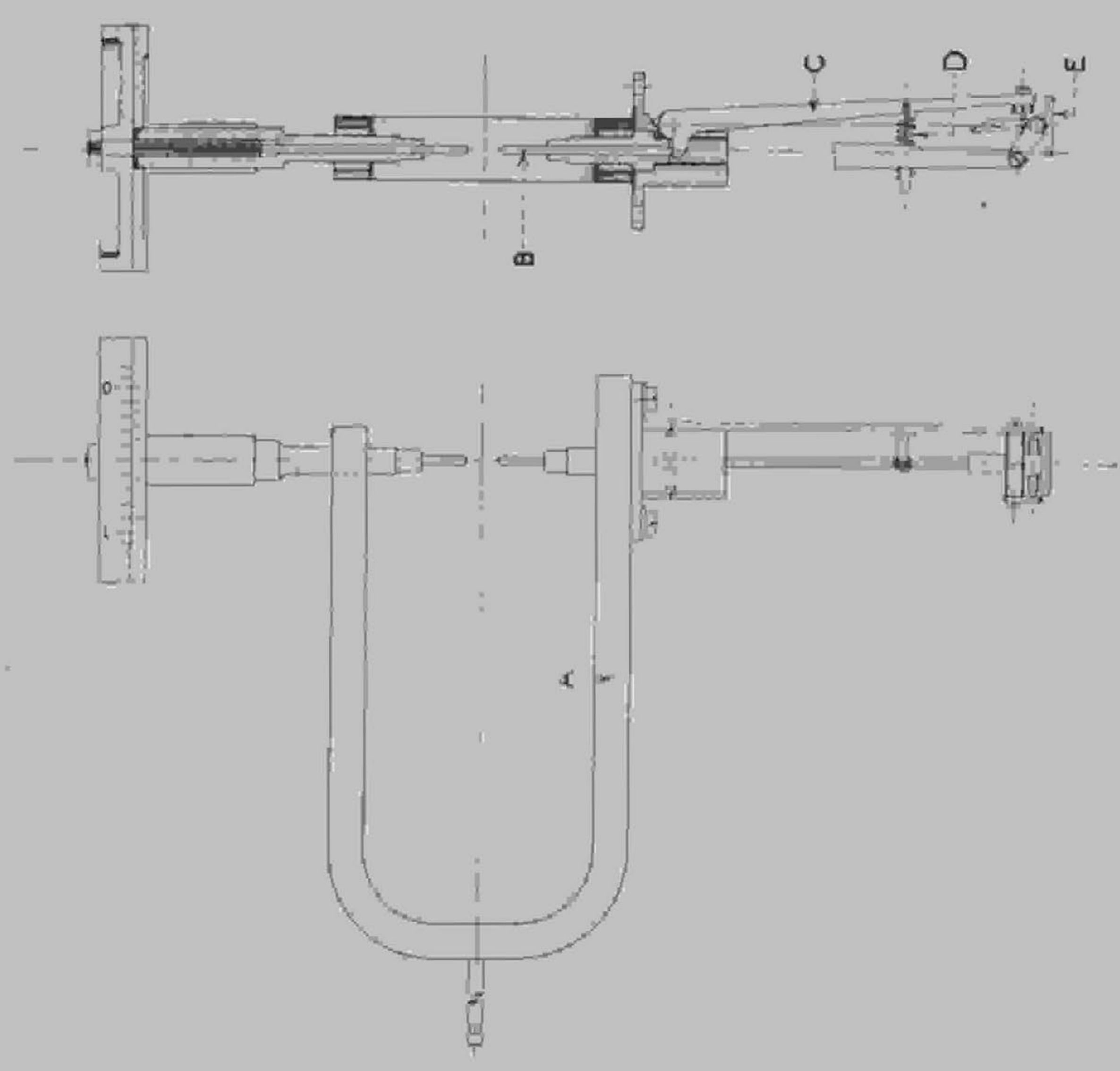
To neutralise the effect of the loaded model, a simple tension member is used. It is placed so that its axis coincides with the direction of one of the principal stresses. If they both are compressive then the axis of the tension member is placed along the greater; if they are both tensile, then it is placed along the smaller; if one is tensile and the other compressive, then it is placed along the compressive one. Since the tension member is to neutralise the effect of the model, the reasons for this are obvious. At points where it is impossible to say the kind of stresses, the calibrating piece would be placed along either stress and if it is found to augment rather than to compensate the action of the model, it would be placed along the other direction.

To find (p - q) the calibrating piece is first-orientated and then loaded until extinction results. At that moment the intensity of tension is equal to the intensity of (p - q).

4) <u>Determination of (p + q)</u>

Having obtained (p - q), stresses p and q can be calculated if (p + q) can be measured. In 1900 Mesnager suggested that (p + q) could be obtained by measuring the lateral deformation of the model. He suggested that, since this lateral expansion or contraction is so small, an extensometer employing two parallel mirrors producing interference might be used. One of the mirrors being fixed and the other one rotating with the lateral strain of the model, a minute strain would result in considerable shifting of the interference fringes. Due to extreme sensitiveness of interference, the above method of measuring lateral deformation has not been developed.

PLATE V



COKER'S LATERAL EXTENSOMETER

Coker's lateral extensometer.

Prof. Coker invented a lateral extensometer of great accuracy, Plate V, of which the description follows:

On one limb of a U-frame the measuring needle B is mounted, the outer end being kept in contact with the lever C by a very fine spring D. At the op osite end this lever operates a mirror E. A beam of light is passed through a lens and a reticule, projected onto the mirror E, is reflected by it, and is focussed on a scale. Thus, for a slight movement of needle B the image of the cross hair moves across the scale.

The extensometer is first calibrated by means of the micrometer mounted at the other limb of the U-frame. This micrometer is also utilised for measuring the thickness of a model.

By means of the Coker extensometer, lateral deformations of the order of $\frac{1}{1,000,000}$ * of an inch can be measured.

Considering an elemental cube under the action of external forces perpendicular to its faces, Fig.12 we have: -

$$E \cdot \frac{\Delta t}{t} = p_x - \frac{1}{m}(p_y + p_z)$$

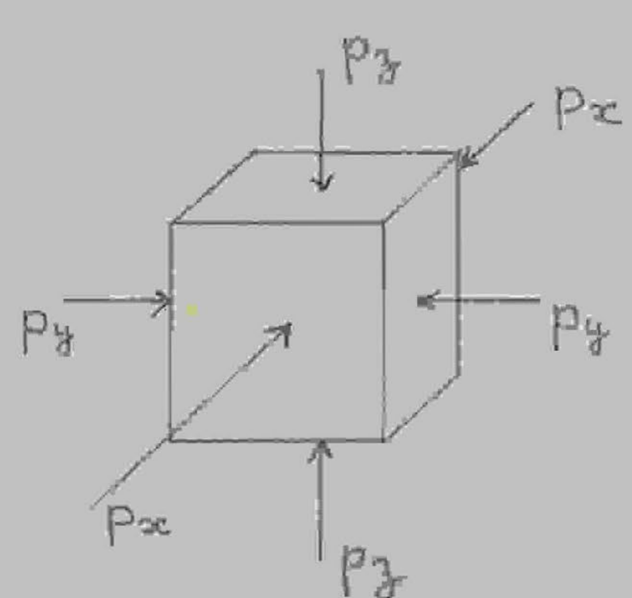


Fig. 12

If $p_x = 0$, as is the case with models tested by the method being described,
$$E \cdot \frac{\Delta t}{t} = \cancel{p_x} - \frac{1}{m}(p_y + p_z)$$
$$\therefore \; p_y + p_z = - \frac{mE}{e} \cdot \Delta e$$
Instead of determining E and m separately $\frac{mE}{e} = k$ is determined for each model by measuring lateral deformation under known force p_z, p_y being zero. Having determined k, the sum of the stresses at any point is obtained by multiplying it by the lateral deformation at that point.

5) Calculations and graphs.

As has been already mentioned a network of iso-static lines is drawn directly after iso-clinic lines have been determined.

Having (p - q) and (p + q) we can solve the two simultaneous equations for p and q with their proper sign, and obtain their direction from the iso-static lines.

Curves showing the variation of stress along any line can then be plotted.

V. THEORY OF FAVRE'S PURELY OPTICAL METHOD

This method which utilises no mechanical meaurements, but only optical, has been developed during the last four years by Prof. Favre, at Ecole Polytechnique de Zurich, Switzerland.

The theory underlying it as well as the description and adjustments of the machine will now be taken up in detail.

Three equations.

This method utilises green monochromatic light.

It will be recalled that a ray of light while passing through a stressed transparent plate undergoes double refraction and is decomposed into two rays vibrating at right angles to each other and parallel to the principal stresses.

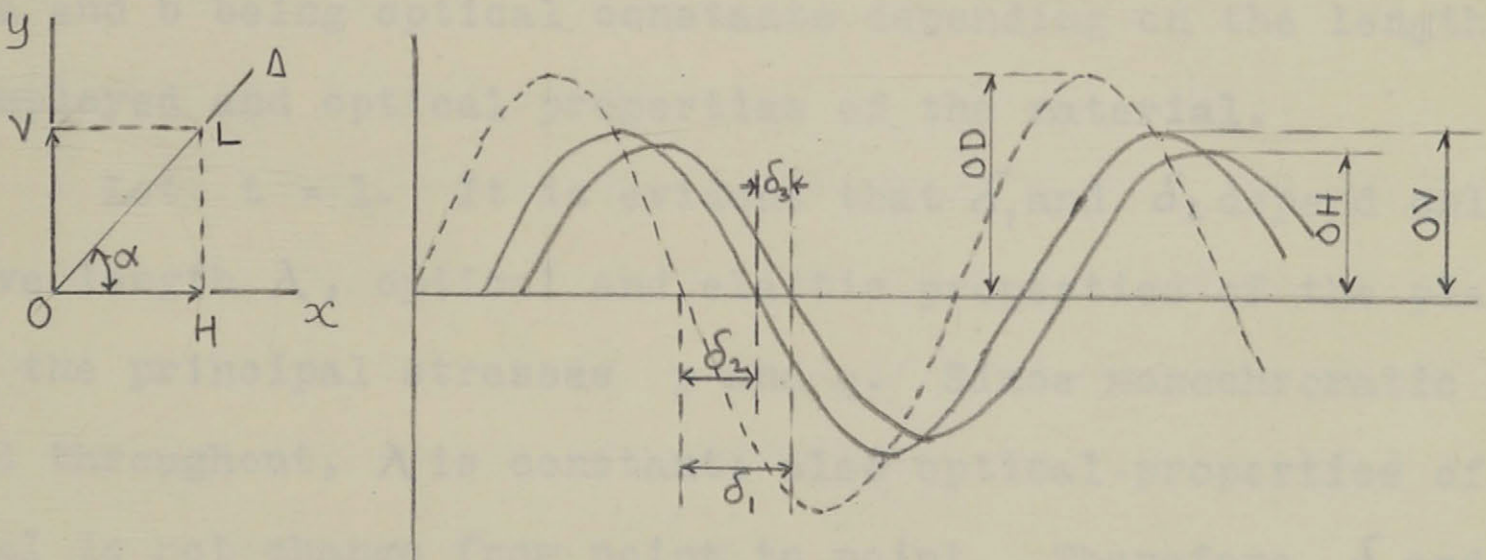


Fig. 13.

Referring to Fig. 13, δ_3 is the relative retardation of the two rays; δ_1 is the retardation of one of the rays as compared to the ray passing through the unstrained plate; δ_2 is the retardation of the other ray.

$$\therefore \quad \delta_3 = \delta_1 - \delta_2$$

There are three unknowns at each point: α , p and q (using the usual notation).

α is determined using the same principle as in Coker's method, namely, that complete extinction results when the principal section of the nicol is parallel to one of the principal stresses.

(p - q) is again determined by compensating the effect produced by loading the model, only using another kind of compensator.

But, instead of measuring (p + q) mechanically, p and q are obtained independently, by measuring δ_1 and δ_2.

It will be now shown that δ_1 and δ_2 are linear functions of p and q and are related in the following manner:

$$\delta_1 = t\,(ap + bq)$$
$$\delta_2 = t\,(bp + aq)$$

~~these~~

these equations holding good within the elastic limit of the material, a and b being optical constants depending on the length of wave employed and optical properties of the material.

Let t = 1. It is evident that δ_1 and δ_2 depend only on the wave length λ, optical and elastic properties of the plate and on the principal stresses p and q. Since monochromatic light is used throughout, λ is constant; also optical properties of the material do not change from point to point. Therefore, δ_1 and δ_2 depend only on p and q.

We thus have:

$$\delta_1 = f_1(p, q)$$
$$\delta_2 = f_2(p, q)$$

Developing $f_1(p,q)$ by means of McLaurin's series with two variables, we have:

$$\delta_1 = f_1(p,q) = f_1(0,0) + \frac{\left(\frac{\partial f_1(0,0)}{\partial p} \cdot p + \frac{\partial f_1(0,0)}{\partial q} \cdot q \right)}{1!} +$$

When p = 0 and q = 0, $\delta_1 = 0$ $\quad f_1(0,0) = 0$.

$$\delta_1 = \frac{\partial f_1(0,0)}{\partial p} \cdot p + \frac{\partial f_1(0,0)}{\partial q} \cdot q$$

Now, $\frac{\partial f_1(0,0)}{\partial p}$ and $\frac{\partial f_1(0,0)}{\partial q}$ are both constants.

Putting $\frac{\partial f_1(0,0)}{\partial p} = a$ and $\frac{\partial f_1(0,0)}{\partial q} = b$,

we have $\delta_1 = ap + bq$

and, by symmetry: $\delta_2 = bp + aq$.

If the thickness in not unity:

$$\delta_1 = atp + btq$$

$$\delta_2 = btp + atq$$

Also, by Wortheim: $\delta_3 = c.t.(p - q)$

In these three equations, only p and q are unknown, being measured by means of Bravais Compensator, and δ_1 and δ_2 by means of Mach-Zehnder Interferometer, yet to be descibed. a, b, and c are determined by measuring δ_1, δ_2 and δ_3 for known p and q, p usually being direct compression and q - 0.

We thus have three equations with only two unknowns, which not only provides a check, but also enables us to calculate the most probab_e values of p and q as well as their mean error.

Let the errors of measurements be v_1, v_2 and v_3. Then:

$$aep + beq = \delta_1 + v_1$$

$$bep + aeq = \delta_2 + v_2$$

$$c.t.(p - q) = \delta_3 + v_3$$

v_1, v_2 and v_3 may be regarded as functions of p and q related to them by the above equations. Let p_1, p_2 and p_3 be the "weights" of measurements of δ_1, δ_2 and δ_3, respectively. Therefore, $p_1 v_1^2 + p_2 v_2^2 + p_3 v_3^2 = \Sigma (pv^2)$ may be also regarded as a function of p and q. Then, accorking to Gauss's method of Least Squares, the most probable values of p and q are those which would make $\Sigma(pv^2)$ a minimum. That is:

$$\frac{\partial \Sigma (pv^2)}{\partial p} = 0 \quad \text{and} \quad \frac{\partial \Sigma (pv^2)}{\partial q} = 0$$

We can thus calculate the most probable values of p and q and their mean errors μ_p and μ_q.

We end up with the following expressions for p, q, μ_p and μ_q ::

$$p = R_{11}\delta_1 + R_{21}\delta_2 + R_{31}\delta_3$$

$$q = R_{12}\delta_1 + R_{22}\delta_2 + R_{32}\delta_3$$

$$\mu_p = \pm\theta_1 (\delta_1 - \delta_2 - \delta_3)$$

$$\mu_q = \pm\theta_2 (\delta_1 - \delta_2 \ \delta_3)$$

where R's and θ's are constants for each model, depending on a, b, c, t, p_1, p_2 and p_3, and stand for the following expressions:

$$R_{11} = \frac{a(\frac{1}{p_2} + \frac{1}{p_3}) - b(\frac{1}{p_2})}{t \ c \ (a+b).(\frac{1}{p})} \qquad R_{12} = \frac{a(\frac{1}{p_2}) - b(\frac{1}{p_2} + \frac{1}{p_3})}{t.c \ (a+b)(\frac{1}{p})}$$

$$R_{21} = \frac{a(\frac{1}{p_1}) - b(\frac{1}{p_1} + \frac{1}{p_3})}{t.c \ (a+b).(\frac{1}{p})} \qquad R_{22} = \frac{a(\frac{1}{p_1} + \frac{1}{p_3}) - b(\frac{1}{p_1})}{t \ c \ (a+b)(\frac{1}{p})}$$

$$R_{31} = \frac{a(\frac{1}{p_1}) + b(\frac{1}{p_2})}{t \ c.(a+b).(\frac{1}{p})} \qquad R_{32} = \frac{-a(\frac{1}{p_2}) - b(\frac{1}{p_1})}{t \ c \ (a+b)(\frac{1}{p})}$$

$$\theta_1 = \frac{\sqrt{p_1 b^2 + p_2 a^2 + p_3 c^2}}{t \ c \ (a+b)\left[\frac{1}{p}\right]\sqrt{p_1 p_2 p_3}} \qquad \theta_2 = \frac{\sqrt{p_1 a^2 + p_2 b^2 + p_3 c^2}}{t.c \ (a+b)(\frac{1}{p})\sqrt{p_1 p_2 p_3}}$$

$$\text{where} \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}$$

They can be calculated once for all for a given model.

VI. FAVRE'S APPARATUS, ITS ADJUSTMENTS AND CONSTANTS.

The general view of the apparatus is shown on Photo No.1. The frame Cm supporting the apparatus rests on three piers. Lm, at one end, is a mercury lamp, giving light rich in green, Mo is the monochromator which disperses the light into component colours. The colour of any desired wave length can be picked out by turning the drum (just behind Mo) until it reads the desired wave-length. All work is usually done with greenish yellow colour of $\lambda = .5461$ which is of greater intensity than colour of any other wave length.

Models are made of a fine quality glass, very carefully annealed and moulded. In this method the model is not analysed as a whole but point by point. A diaphragm made of a thin sheet of aluminium is pasted over the portion of the model to be analysed. This diaphragm has holes 0.5 mm. in dia. located at points at which the determination of p and q is deemed desirable. It will be seen that the apparatus consists of two distinct optical paths - one along the rail I, the other along the rail II. The former is used for determination of α and δ_3 and will be called Line I; the latter - for δ_1 and δ_2 and will be called Line II.

1) Line I

Referring to Plate VI, it will be seen that Line I consists of the following:

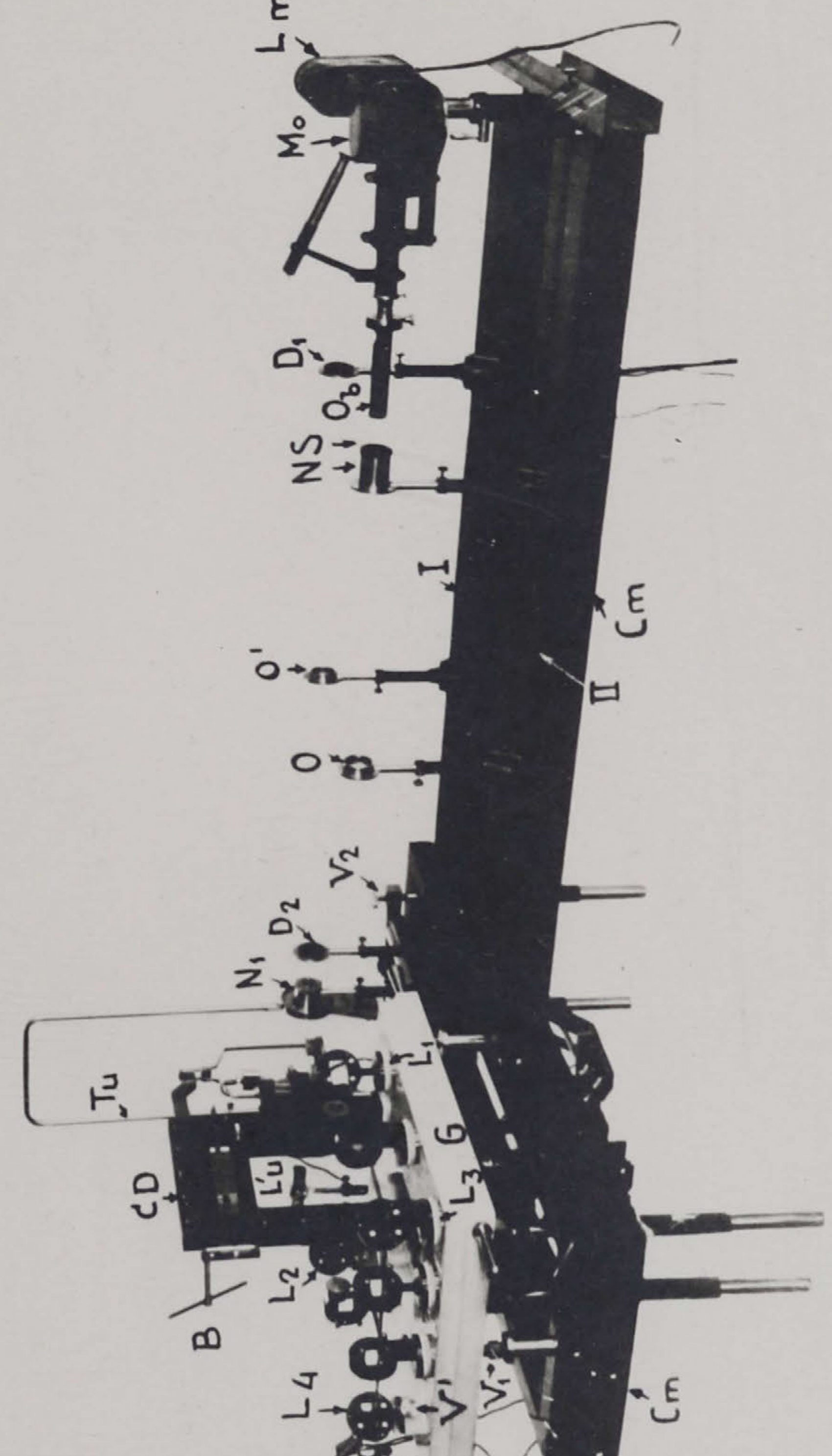
Mercury lamp Lm

Monochromator Mo

Adjustable shutter D_1

Converging lens O'

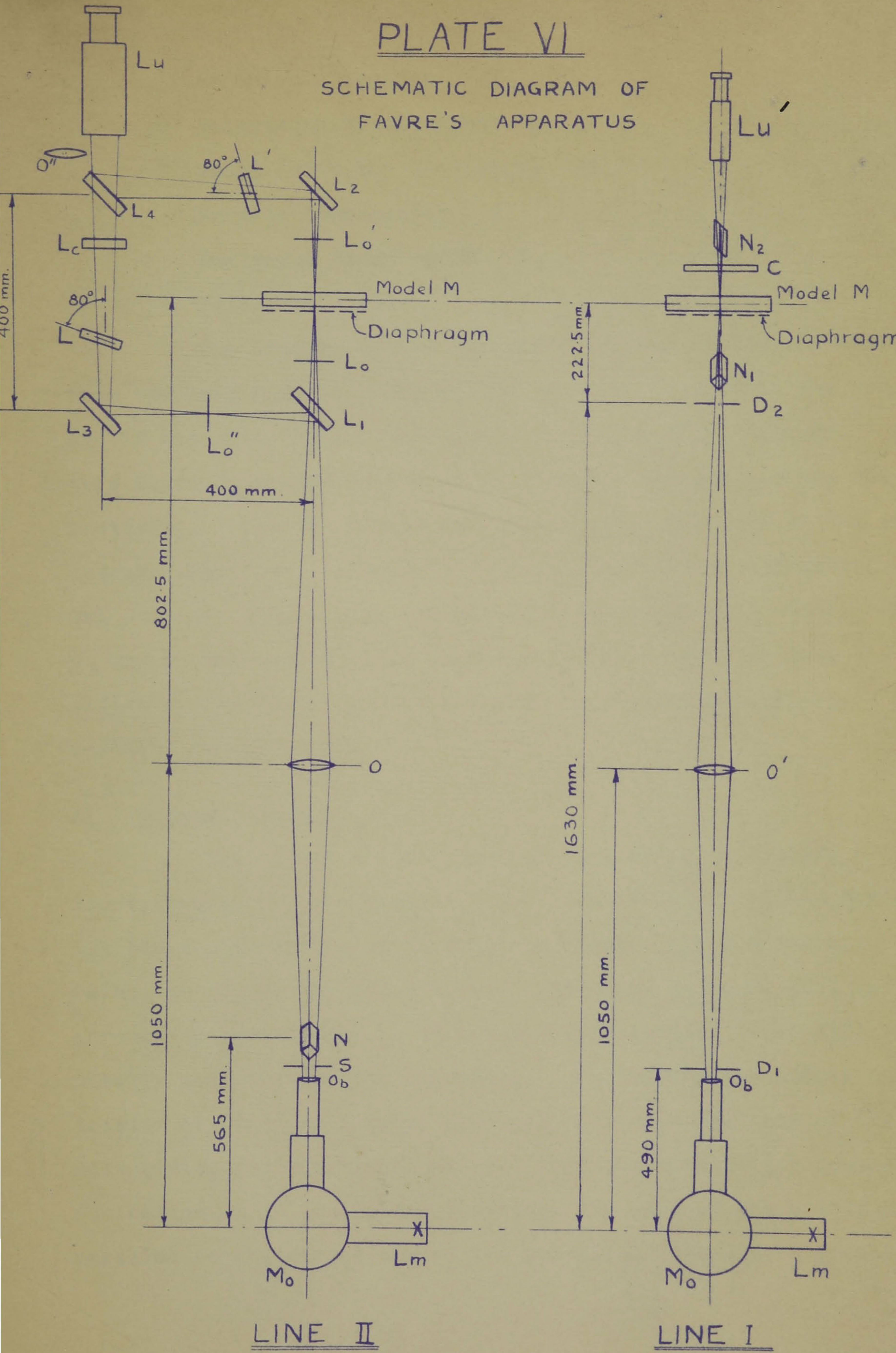
PHOTO. I



FAVRE'S APPARATUS

PLATE VI

SCHEMATIC DIAGRAM OF FAVRE'S APPARATUS



LINE II

LINE I

Adjustable shutter D_2

Polarising Nicol prism N_1

Model (with the diaphragm) M

Bravais Compensator C

Analysing Nicol prism N_2

Telescope Lu'

The dimensions are such that the ray of light emanating from the lamp has the diameter 0.5 mm. when it reaches the model and it can pass through the 0.5 mm. holes in the diaphragm. Shutters D_1 and D_2 should be opened to diameter 2.5 and 3.5 mm. respectively. The two Nicols can rotate about the axis of the ray and are kept fixed relatively to each other by the connecting arm Tu. Their optical axes are at right angles to each other. Nicol N_2 has an index travelling over a calibrated drum, so that the position of the Nicols can be referred to the vertical (in$^\circ$). The vertical is marked 0°.

a) Determination of α

The nicols are so adjusted that for the position of the index reading 0°, the axes of the Nicols are vertical and horizontal (it is immaterial as to which is which). The compensator being out of the way and the model unloaded, no light will go through the two nicols. If the model is loaded, light will be generally seen through the telescope. On turning the nicols the intensity of light will vary, andthere will be a position of the nicols giving maximum light, as well as a position giving complete extinction. In the latter position, the axes of the nicols are parallel to the directions of the principal stresses. Position of the nicols being indicated on the drum we can thus read off the inclination of one of the principal stresses to the vertical, the other one being at right angles to that. Directions of the principal stresses are obtained for each point in succession.

b) Determination of δ_3

δ_3 is measured by means of the Bravais compensator. The bravais compensator consists of three rectangular plates of quartz, possessing the property of double refraction, Fig.14.

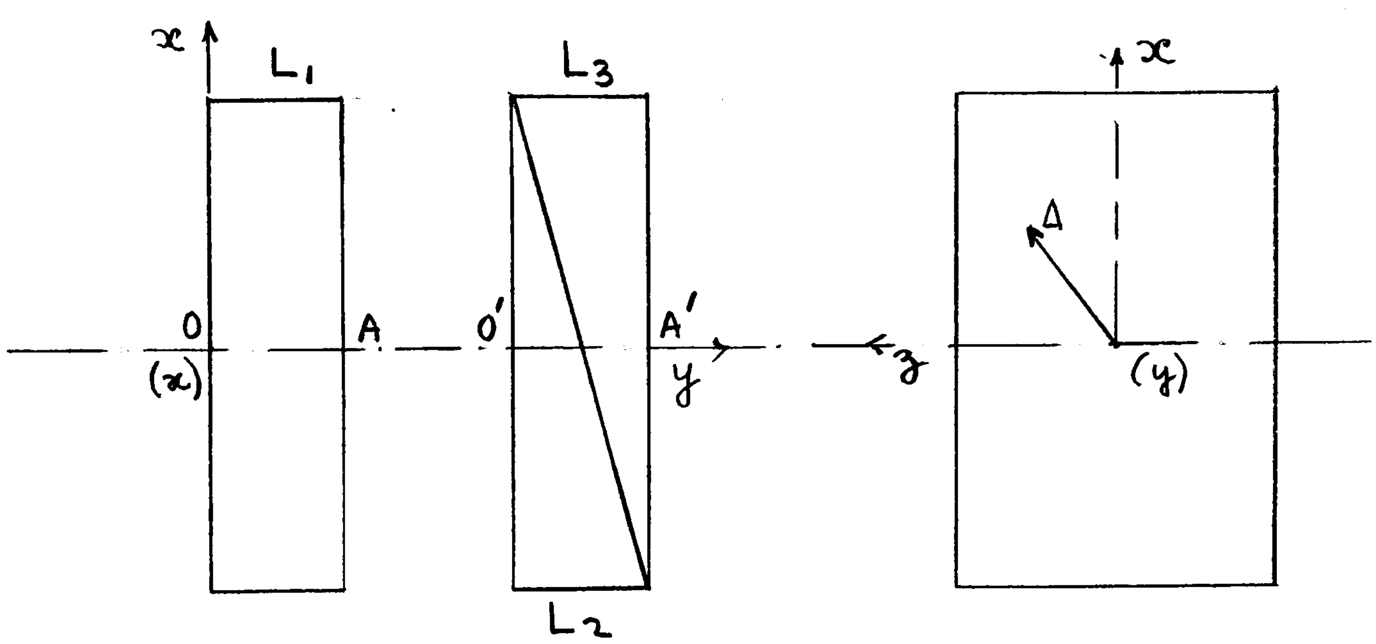


Fig. 14.

The plate L_1 is OA thick and has its optical axis (the axis of greater retardation) along Ox. The plates L_2 and L_3 have their optical axes along Oz. L_2 and L_3 are triangular in cross-section and can be made to slide along each other, thus varying their combined thickness. When thickness O'A' is equal to OA, L_2 and L_3 are neutralising L_1 and a ray going through the compensator is unchanged. If, however, the wedges are displaced, increasing thickness O'A', then the compensator acts as a double refracting plate of thickness O'A'-OA whose axis is parallel to Oz. If O'A' is smaller than OA then the optical axis is along Ox. The wedges are made to slide by turning a drum, one revolution of the drum corresponding to a certain displacement of the wedges. Revolutions of the drum are indicated on a scale travelling with the moving wedge and hundredths are read off the graduations on the drum.

Thus, by turning the drum any required difference of phase may be created.

Any δ_3 present in the ray before passing through the compensator is obtained as follows:

Before loading the model, the compensator is set at extinction and the reading taken; when the model is loaded, light will generally be seen through the telescope. The drum of the compensator is turned until an extinction is reached and the corresponding reading taken. The difference of the two readings divided by the Constant of the compensator gives δ_3 in terms of wavelength. (To determine the direction in which the drum is to be turned only a part of the load is applied - the required direction is the one for which the intensity decreases, as the drum is turned. Only a part of the load is applied, because, if δ_3 is over half wave-length when full load is applied, the intensity will increase as the drum is turned, reaching maximum, and then fade away until extinction is accomplished).

It is important that the axis of the compensator should be in the direction of one of the principal stresses.

2) <u>Line II.</u>

It consists of the following: (See Plate VI)

Mercury lamp Lm

Monochromator Mo

Adjustable shutter S

Nicol prism N

Converging lens O

Interferometer Mach-Zehnder $L_1 - L_4$

Model and diaphragm M

Telescope Lu.

The mercury lamp and the model can be moved until they are in Line II.

The dimensions are again such that the ray has diameter 0.5 mm. when it goes through the model. Before description of the interferometer, as used, its principle of operation will be outlined.

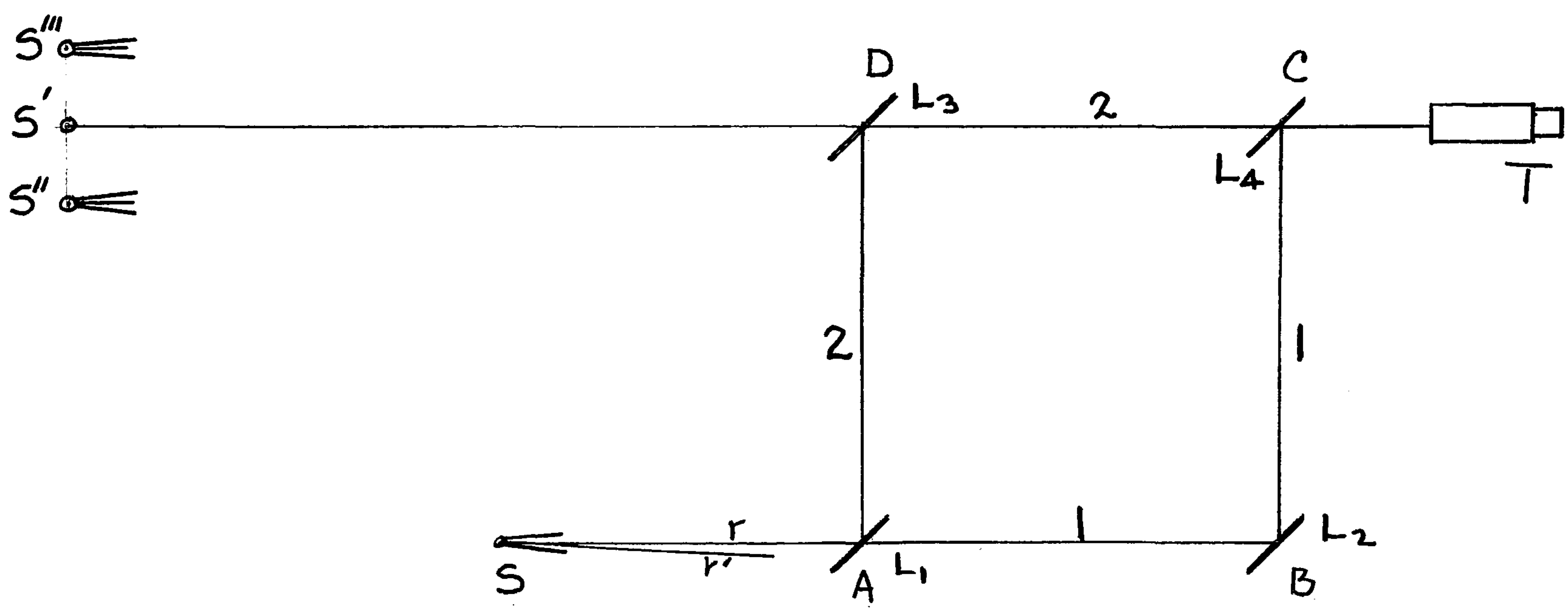


Fig. 15

Four equal glass plates are placed parallel to each other at the four corners of a square ABCD, Fig. 15. They are all inclined at 45° to the ray, emanating from the source S. Plates A and C are covered with a thin layer of platinum and are partly reflecting, partly transparent. Plates B and D are covered with a film of platinum, of sufficient thickness to reflect practically all the light falling upon them, letting none through. The ray SA will then be decomposed into two rays ABC and ADC. The two rays, which we shall call Ray 1 and 2, are of the same intensity. On looking through the telescope T, a virtual image of S will be seen at S'; any other ray Sr' will also appear to be coming from S'. Assuming all mirrors to be of the same thickness, the optical paths, traversed by the two rays will be exactly equal. If one of the mirrors be turned very little out of parallelism, the two rays 1 and 2 will have no longer the same virtual image S', but each will have its own virtual image at S" and S''' The two images are images of the same source S and rays coming from them are in a condition to interfere. On looking through the telescope, interference fringes will be seen. They will appear as vertical, equally spaced, black bands on uniform green background; by changing optical path of one of the rays, the fringes will move bodily to one side or the other. If the difference of optical paths is exactly one wave length, then each fringe will take place of its neighbour.

The interferometer as used on the machine at McGill consists of the following: (See Plate VI)

L_2 & L_3 full mirrors

L_1 & L_4 partial mirrors

L Compensator

L' A plate compensating the thickness of L

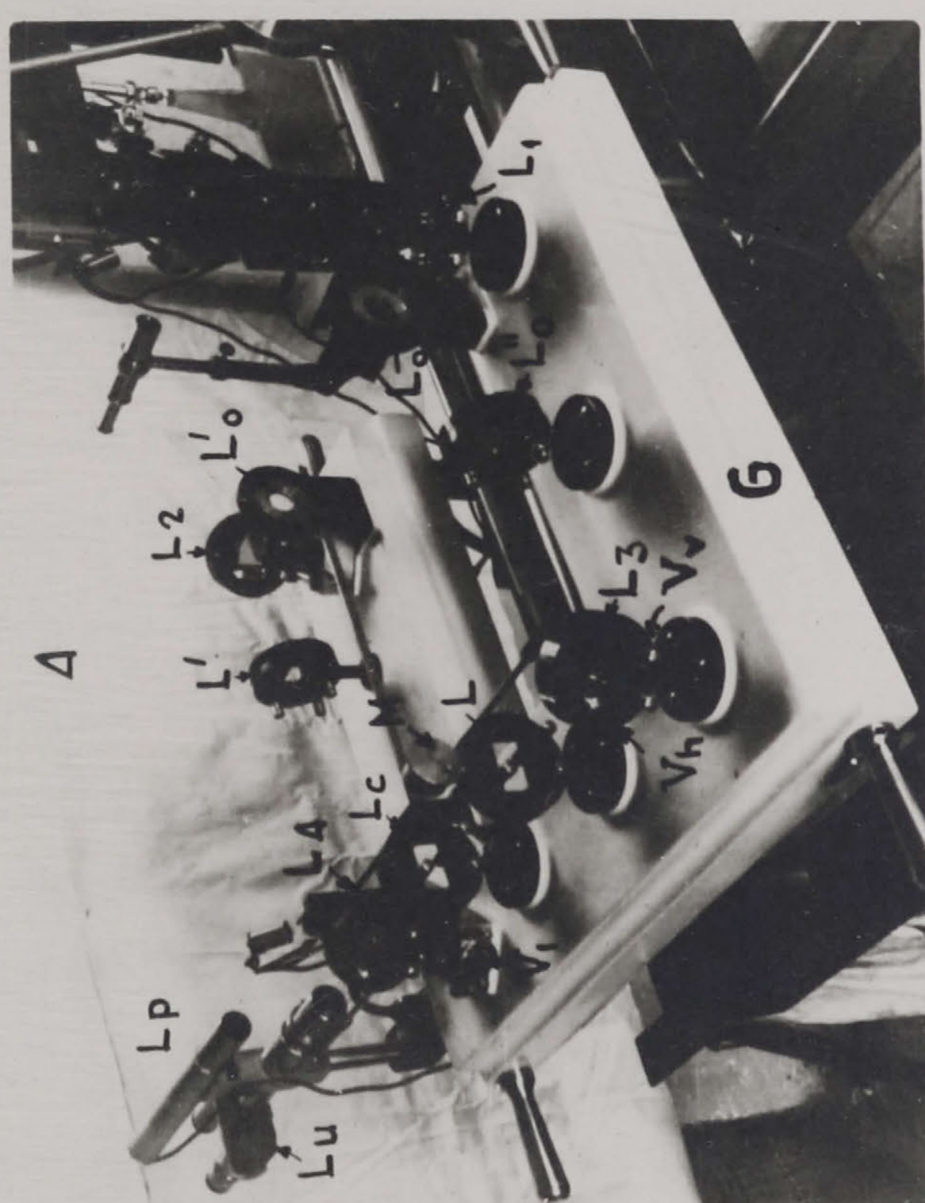
Lc A plate compensating the thickness of the model.

Lo and Lo' Half-wave plates

Lo" Full wave plate to compensate thickness of Lo and Lo'

Lu Telescope, with a scale in it to gauge the movement of fringes as will be described.

PHOTO. II



INTERFEROMETER

The frames of plates L_1 and L_3 are fixed to the frame of the interferometer, while L_2 and L_4 can slide along rails keeping parallel to themselves. All four plates can be adjusted in their frames by screws Vv and Vh.

Compensator L is a plate of glass, mounted in a frame capable of turning about the vertical axis. It is turned by means of a graduated drum operating a slow motion screw, thus changing the optical path of ray $L_1 L_3 L_4$. This plate is vertical and inclined at about 80^o to the ray.

L' is a plate identical to L and is set at the same angle as L, but it has no slow motion arangement to turn it. Its purpose is to compensate for thickness of L.

Lc is of the same thickness and material as the model and is placed on the path of ray $L_1 L_3 L_4$ to compensate for the thickness of the model.

Lo and Lo' are two half-wave plates. They can be turned about the axis of the ray by means of two long toothed rods, engaging their frames at one end and a common shaft at the other. This shaft can be turned and carries a graduated circular plate situated near Lu. The plates are so placed that when the circle is reading zero their axes are vertical. Now, nicol N produces polarised light vibrating in the vertical plane. To obtain retardation suffered by the ray in one of the directions of the principal stresses, the vibrations must be parallel to it. As has been explained, a half-wave plate has the property of changing the plane of polarisation of light going through it, symmetrically about its axis. If one of the principal stresses makes an angle

α° with the vertical then the axes of both half-wave plates are to be set at $\frac{\alpha}{2}^{\circ}$ to the vertical. The second plate renders the plane of vibrations again vertical. The drum is so graduated that it has to be turned through the number of divisions equal to the number of degrees in angle α.

Lo" is a full wave plate and is there to compensate for combined thickness of half-wave plates.

A section of the telescope is seen on Plate VIII. X is a source of light, B a lens and A a scale with very fine divisions (about 200 to the inch) etched on it. M_1 and M_2 are two mirrors. From Photo 2, it will be seen that the compensator L carries a mirror M. This morror is set in such a way that light from source X travels to it, is reflected and comes back into the telescope through the slot D; it is then reflected by mirrors M_1 and M_2 and goes out through the eye-piece. By moving Lp the image of the scale can be focussed at the reticule. When this is done the image of a portion of the scale will be seen, greatly magnified, through the eye-piece. The reticule consists of one horizontal and three vertical hairs arranged as shown on Fig. 16.

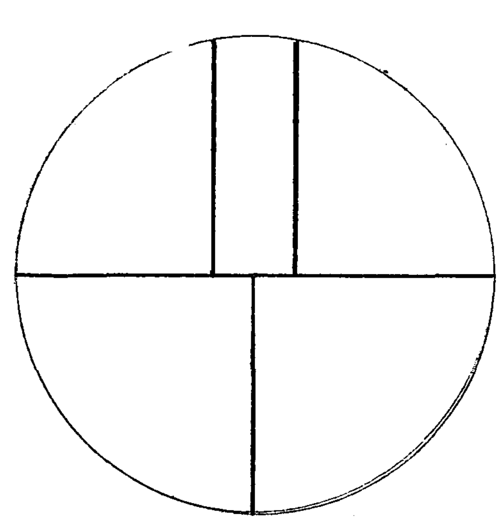
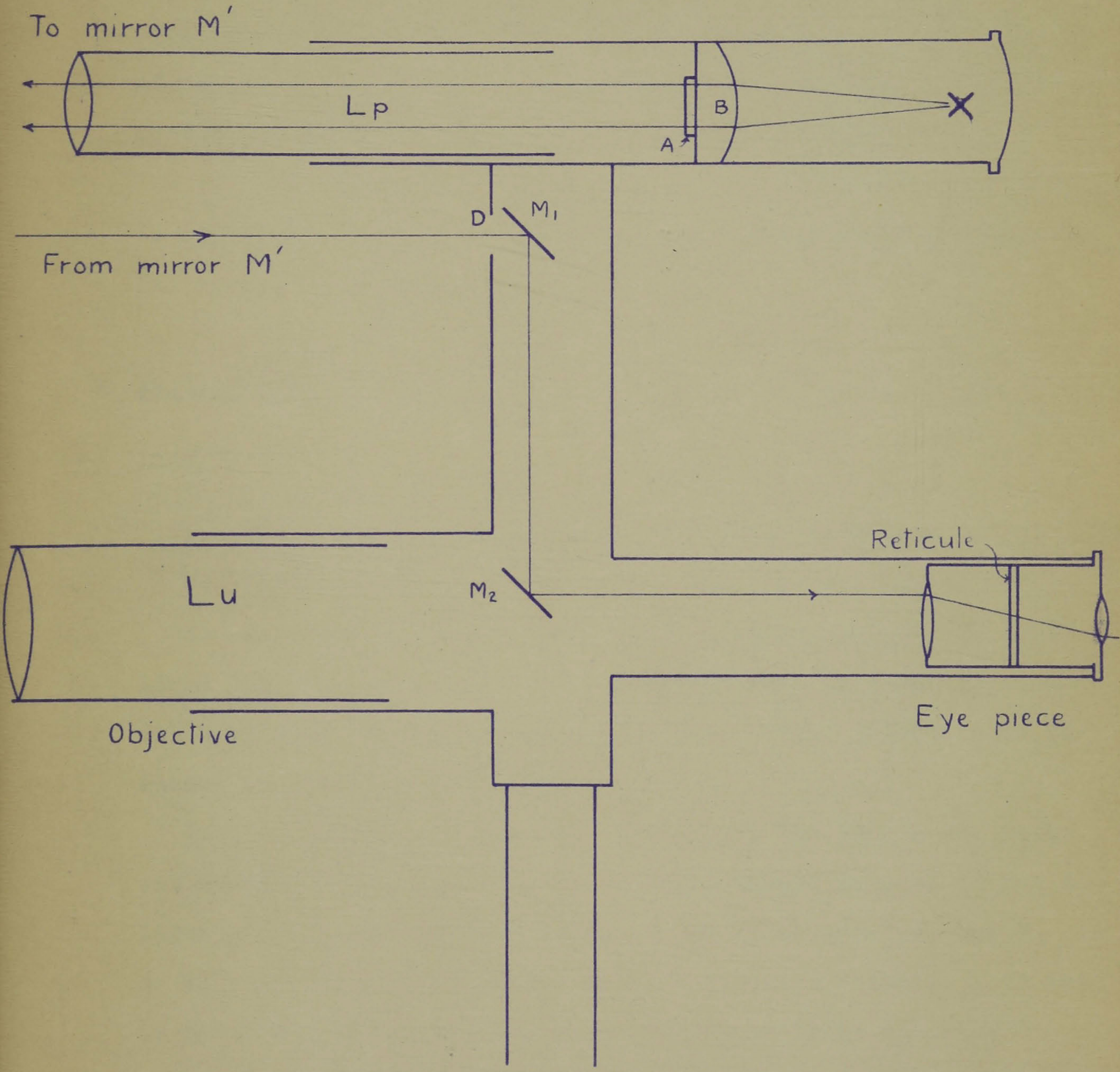


Fig. 16

The single hair is in the field of the image of the scale, while the two top hairs are in the interference field.

PLATE VII



SECTION OF TELESCOPE Lu

Any retardation introduced in one of the rays displaces the interference fringes sideways and is measured in this way: - the drum operating the compensator L is turned, thus turning it and mirror M and, consequently, moving the image of the scale across the reticule, until the interference field is restored to its original position. The scale is read before and after; the difference of readings divided by the compensator constant (see adjustments) gives us the values of δ_1 or δ_2 in terms of wave-length.

By measuring interference when vibrations are parallel to one of the principal stresses, say p, δ_1 is obtained and when parallel to the other, q, δ_2 is obtained.

3) Adjustments.

a) Line I.

1. The reading on the drum at the monochromator is set at .5461.

2. The position of the objective glass is set at 1:1.

3. All optical pieces are spaced as shown on Plate VI, and lined up; the alignment must be horizontal. A green spot will then be seen through the telescope Lu'. The latter can be focussed by moving the eye-piece.

4. Compensator C being out of the way, the Nicols are to be set at 90°. In this position no light should be seen in the telescope. To accomplish this N_1 is not to be touched, but N_2 is to be turned inside its frame until extinction results. To get to nicol N_2, the protecting glass will have to be removed.

5. The prismatic calibrating piece, supplied with every model is next subjected to vertical compression, making p vertical and q - 0.

6. Both nicols are turned together until extinction is obtained. In that position the optical axis of one of the nicols is parallel to one of the stresses and the reading on the drum N_2 should be 0. If it is not, it will be necessary to keep the nicols fixed in space and to turn the arm Tu until the reading is zero. This adjustment is to be repeated until the reading is exactly zero, when extinction is reached.

7: Constant of the Bravais Compensator.

The model being out of the way, the axis of the compensator is rendered vertical by setting the vertical graduated disc at 0°. A bright spot will generally be seen. The nicols are placed at 45° to the vertical, for greater intensity. Turning the compensator drum a position of complete extinction is reached. The scale and the drum are read, as has been explained. Then the drum is turned in either direction - the spot will re-appear, reach maximum intensity and fade away again until another extinction is reached. The scale and the drum are again read. The difference of the two readings is called the constant of Bravais compensator and represents in terms of the divisions of the scale the relative movement of the wedges, necessary to produce a relative phase difference of one wave length Several determinations over the range of two or three extinctions are made and the mean is taken. Any other difference of readings divided by the constant will give the corresponding δ_3 in wave-lengths

b) Line II.

Different optical pieces are located according to Plate VI, and lined up, the alignment being horizontal and at the same height as that of Line II.

1. The four mirrors are placed parallel to each other. To do this adjustment, all pieces except L_1, L_2, L_3 and L_4 are taken off and the interferometer is taken to a place from where a point at least 1 km. away would be seen (e.g. a church spire). Carriages with L_2 and L_4 are also removed. The interferometer and a telescope (not Lu, but an ordinary telescope, such as a transit) are then placed as shown at 1, on Plate IX, and the images of the spire due to L_1 and L_3 are made to coincide by turning screws Vv and Vh. The two plates are now parallel within about 2'. The interferometer and the telescope are then placed in position 2, Plate IX. L_4 with its carriage is replaced and is brought parallel to L_1 and L_3 Similarly, L_2 is placed parallel to the other three plates.

2. The interferometer is placed back on the machine and put in line with the rest of the pieces in line. II. Uniform green field with vertical interference fringes across it will be seen in the upper part of the telescope field. Lens O" is to be out of the way; its purpose will be mentioned later.

3. All other pieces are now replaced on the interferometer, L and L' at 80° to the ray, passing through them.

4. Lens O ~~is~~ may be slightly moved to produce a sharp focus of the light on the diaphragm.

5. The fringes are made exactly vertical, spaced conveniently (5 or 6 in the field) and made as dark as possible by turning Vv, Vh and V_1 respectively

6. By turning slowly mirror M and focussing Lp, the image of the scale can be focussed on the reticule. It will be in the lower

half of the field. The source of light is fed by a battery through a resistance, which can be adjusted to produce the image of the scale of the most convenient intensity, permitting at the same time to see easily the interference fringes.

7. To ameliorate the fringes further, the mercury lamp is replaced with a carbon arc lamp, producing white light. On looking through the telescope there may or may not be fringes. If not, they are made to appear by again turning slightly the adjusting screws. They are again spaced so as to have about six in the field and the white fringe is placed at the centre. The carbon arc lamp is then removed and the mercury lamp replaced.

8. Constant of the compensator L. With the model placed in the path of ray ~~2~~ $L_1 L_2 L_4$, drum operating L and M is slowly turned until a fringe is exactly between the two vertical cross hairs. The scale in the lower half is read with reference to the vertical hair. The drum is then turned until four inter-fringe spaces go by, and the fourth fringe is brought between the cross hairs. The scale is again read. The difference of readings divided by four is called the constant of the compensator L and represents the amount which L has to be turned, necessary to produce a retardation of one wave length. This determination is repeated several times and the mean taken. It applies only within the range of the four fringes. Any other difference of readings divided by the constant gives δ_1 or δ_2 in terms of wave-length. [~~is measured by the above method when the light traversing the model is vibrating parallel to one of the stresses, say p, and when it is vibrating parallel to the other one~~.]

When the lens O" is inserted it is found that instead of uniform green field two pairs of bright spots are seen, the spots in each pair being superimposed. This lens may be useful when changing to another hole. When the light does not go through the hole the two bright spots, one from each pair, are somewhat dimmer than when the light does goes through a hole.

After all adjustments are completed the wooden cover is replaced over the interferometer, to screen it from draughts, etc.

c) Determination of constants a, b and c.

Coefficients a, b and c are determined by measuring δ_1 δ_2 and δ_3 when the principal stresses p and q are known. The most convenient way is to subject the test piece to a known direct compression. If the cross section of the prism is h by e and V is the vertical force, then at any point near the contre of the piece p will be $-\frac{V}{eh}$ and q - 0 (or nearly so). The three fundamental equations reduce to:

$$\delta_1 = aep \qquad \delta_2 = bep \qquad \delta_3 = cep$$

$$a = \frac{\delta_1}{ep} \qquad b = \frac{\delta_2}{ep} \qquad c = \frac{\delta_3}{ep}$$

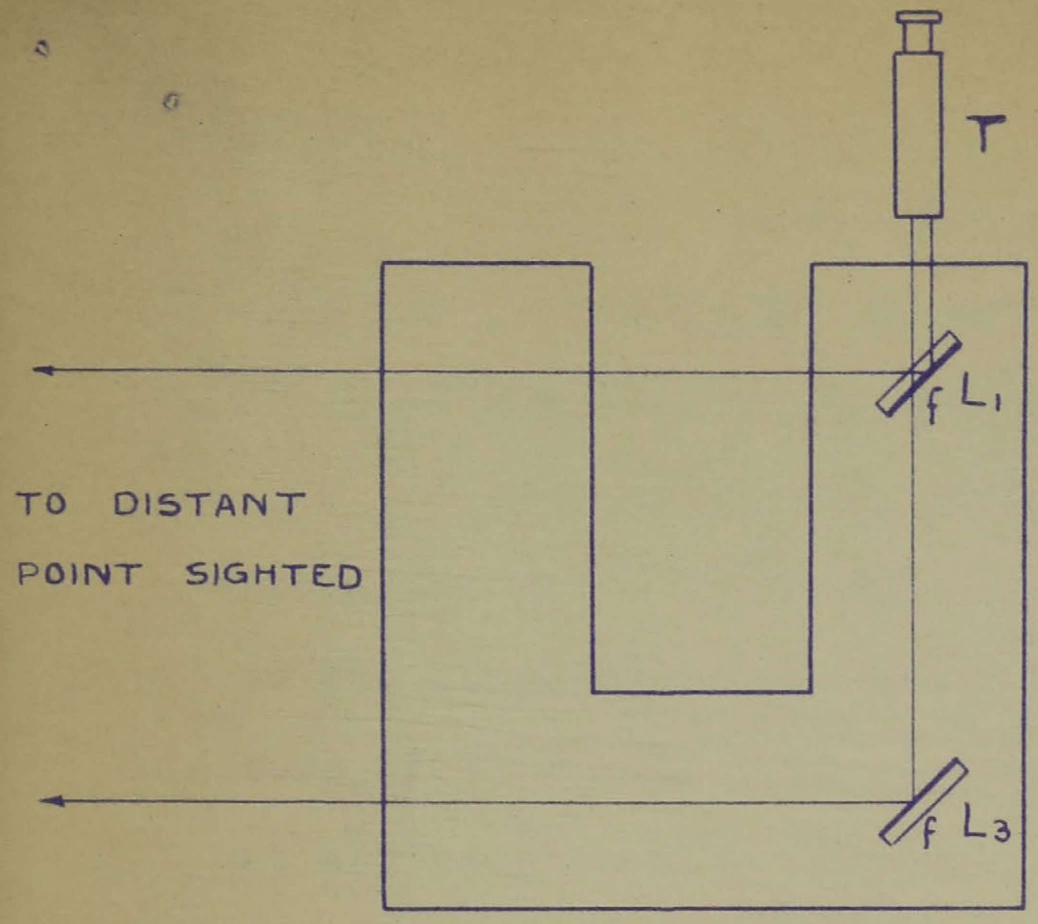
Since there is a certain "zero" correction due to the weight of the levers and the scale pan, δ's are determined for several loads. It is not the absolute δ's and not the absolute load that are taken but an increase in δ corresponding to the respective increase in the load. Several determinations ofeach constant are made and the mean calculated, as well as its mean error $\mu\delta$ Weights p are inversely proportional to squares of $\mu\delta$'s

Finally R's and θ's are calculated, from formulae given previously and the model may be now tested for p and q.

PLATE VIII

ADJUSTMENT OF INTERFEROMETER

PLATES L_1, L_2, L_3 & L_4 PLACED ||
TO EACH OTHER



1

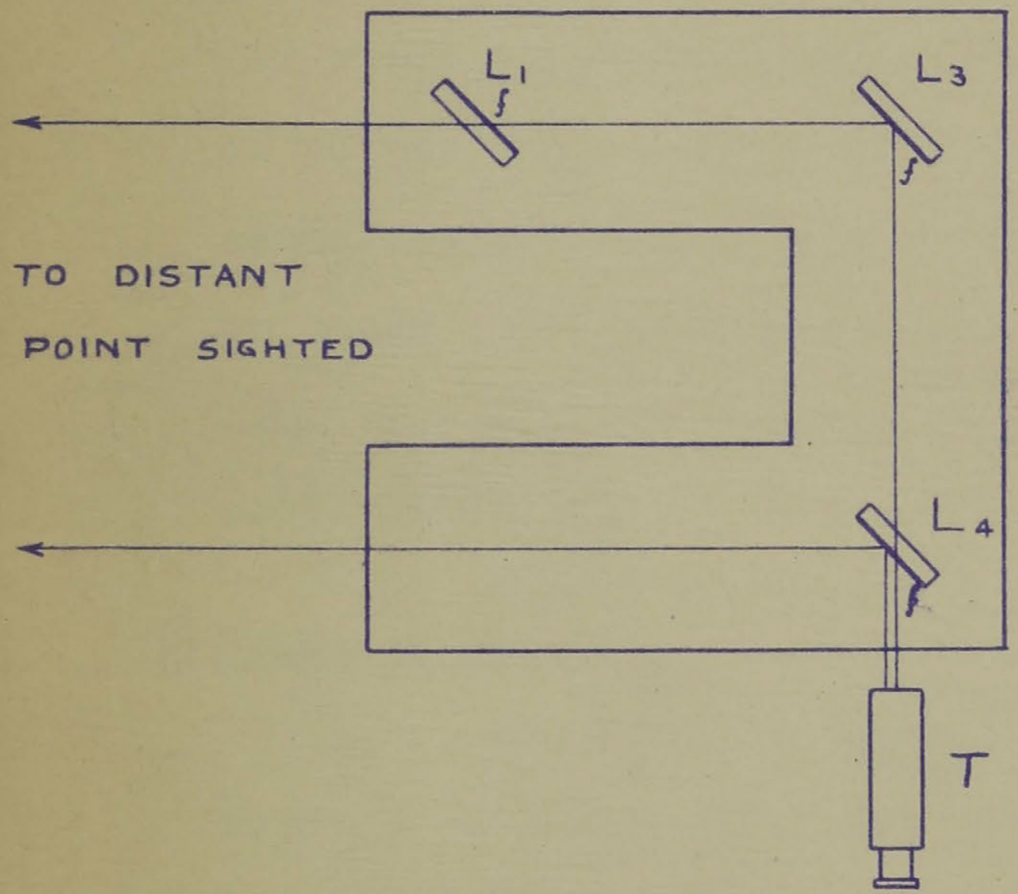
L_1 & L_3 PLACED
|| TO EACH OTHER

NOTES

L_1 & L_4 - PARTIALLY REFLECTING PL

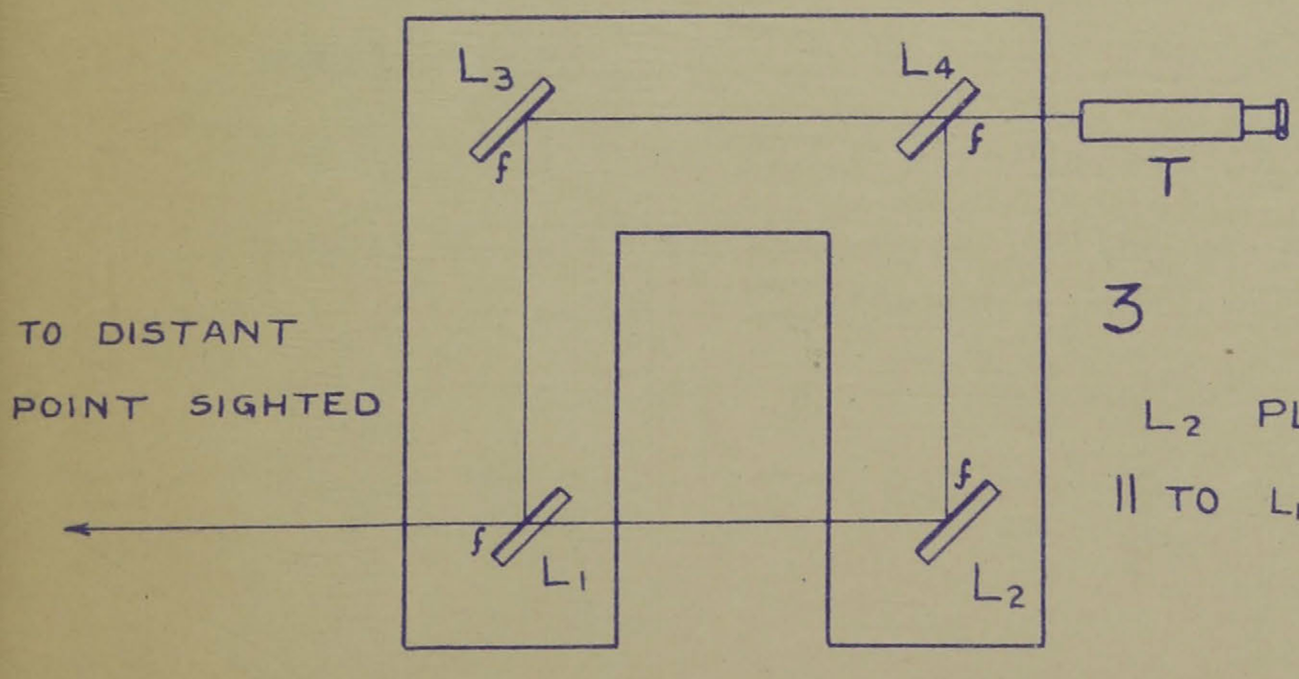
L_2 & L_3 - TOTALLY " "

f - FACE COVERED WITH PLATINUM



2

L_4 PLACED
|| TO L_1 & L_3



3

L_2 PLACED
|| TO L_1, L_3 & L_4.

d) Graphs.

Little crosses are marked at the position of each hole with their arms in the direction of the principal stresses. Lines of principal stress are then drawn in by eye.

Curves showing variation of stress along the boundary, along the ℄ of the model and along transversal lines corresponding to the lines of holes, would convey a sufficiently complete picture of stresses. If desired, vertical and horizontal stresses at any point can be calculated and plotted.

e) Accuracy of the method.

Dr. Favre made several tests on models choosing shapes for which a mathematical solution was available. And for majority of points the discrepancy between the mathematical solution and experimental results was within .02 kgms. per sq. mm. For points in zones where stresses vary greately his results are within .05 kgms. per sq. mm. Thus, in an ordinary test the error would be well within 5%.

f) Scale of loading.

Assuming the actual structural member and a model of it to be of the same material let their linear dimensions be in the ratio a : 1. If the ratio of external forces applied to the structure to the external forces applied to the model be b : 1. Then the ratio of intensities of stresses in the actual member to that in the model at the corresponding point will be $\frac{b}{a^2}$.

VII. EXPERIMENTAL WORK.

The experimental work done in connection with this thesis includes setting up and adjusting the apparatus and testing one of the models supplied with it.

The apparatus which is installed in the Testing Laboratory, Engineering Building, was presented to the University by Mr. C.M. Morssen, Honourary Research Fellow in the Department of Civil Engineering.

A sketch of the model which represents a built-in wall is shown below.

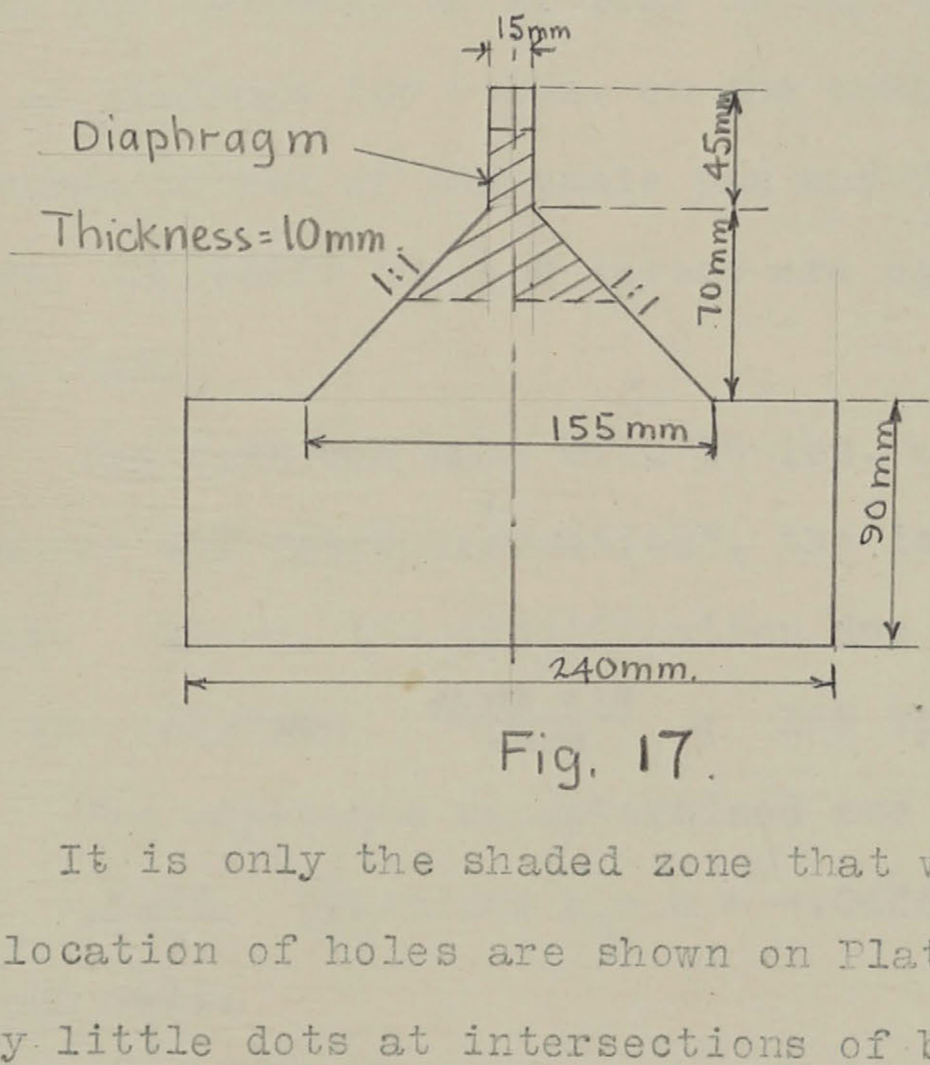


Fig. 17.

It is only the shaded zone that was tested. The diaphragm and the location of holes are shown on Plate X, holes being represented by little dots at intersections of broken lines.

Originally, the intention was to do a considerable amount of testing, but, due to a number of unexpected difficulties in settin up of the apparatus, and consequent lack of time only enough testing was done to check the validity of the method. Simple symmetrical loading was, therefore, chosen, the bearing plate extending over 4/5th of the width of the top of the model.

Constants of the Bravais Compensator and of Interferometer Compensator were first determined. Then, coefficients a, b, and c were determined by measuring δ_1, δ_2 and δ_3 for two different weights on the scale pan and taking the difference of readings corresponding to the difference of weights and calculating a, b, and c from equations given in the last chapter. This eliminates the zero correction due to weight of the levers and the scale pan which were afterwards determined as follows: - From readings obtained for determination of constant c, the difference of readings was -2.47 for increase of 11 lbs. on the scale pan. The difference of readings for 5 lbs. on the scale pan is -1.44. Therefore, the combined effect of the scale pan and 5 lbs. weight is 6.42 lbs. Therefore, the scale pan and levers are equivalent to 1.42 lbs. on the scale pan.

The test was made with 16 lbs. on the scale pan. Taking into account the "zero correction", the load on the scale pan was 17.42 lbs. Since, the amplification due to levers is 20, total load on the model was $\frac{17.42 \times 20}{2.205} = 158$ kgms.

The constants as determined are a = .0376, b = .0842 and c = -.0467. Therfore a - b = -.0466. Thus, the constants check very well.

The model was then analised for α, δ_1, δ_2 and δ_3. p and q were calculated from the two simultaneous equations

$$\delta_1 = atp + btq$$
$$\delta_2 = btp + atq$$

(p - q) as obtained from δ_3 serves as a check. In the circumstances it was deemed advisable not to go through laborious calculations of R's, θ's,

From the enclosed tables of δ's,p's and q's it will be seen that although a general check was obtained, quite a few points were considerably out, some of them as much as 20%. Such errors although not a reflection on the validity of the method are quite objectionable.

The main source of error, undoubtedly, is due to the fact that no readings were repeated. In the case of constants all readings were repeated and the constants checked very well.

Another source of error is a certain deffficiency of the Bravais Compensator;under certain circumstances it gives no complete extinction and the position of minimum light can then be only estimated.

A poor mounting of the telescope is, undoubtedly, responsible for some errors. It is held to the frame of the interferometer by only one bolt and is,consequently, rather unsteady.

VIII. CONCLUSION.

It is hard to compare the relative advantages of the two methods as the writer had no experience with the Coker method.

But it seems to him that for ordinary testing the Favre method is faster and little more accurate. It is generally the boundary conditions that are the worst and in which we are mostly interested. At the boundary one of the principal stresses is zero, and consequently, determination of (p - q) will give us directly the other stress. Determination of both, α and (p - q) is quite fast and accurate on Favre's machine.

Coker's method, however, provides a good illustration of the general distribution of stresses.

It seems, that a combination of the two, as suggested by Mr. Morssen, would be desirable. Coker's method would be used to obtain a general picture of the stresses and location of dengerous points, and Favre's - to determine the stresses quantitatively at such points.

As to the material, both glass and xylonite have certain advantages and disadvantages. Glass is very expensive to anneal and cut, but, when it is well annealled, it is almost perfectly isotropic. Celluloid and xylonite are easy to cut but they have the disadvantage of "flowing" under stress, and imperfect material near the edges. An advantage of celluloid, when applied to Coker's method is its high lateral deformation as compared to glass.

ACKNOWLEDGEMENTS.

I should like to express my gratitude to Professor Jamieson and Mr. Morssen for their guidance and valuable advice, as well as my appreciation to those responsible for awarding to me the J.B. Porter Scholarship for the last session, which made this work possible.

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RESULTS OF TEST.

Determination of constants.

C of Bravais Compensator.

Load off:	+.68	-7.32	+.70	-7.21
Load on:	-7.32	-15.30	-7.21	-15.12
Dofference:	8.00	7.98	7.91	7.91

Mean Cb = 7.95

C of Interferometer.

Load off:	10.68	10.17	10.62	
Load on:	12.68	12.15	12.62	
Difference:	2.00	1.98	2.00	(for 4 fringes)

Mean Ci = .50

Constant "a".

5 lbs. on scale pan;	Off:	10.53	11.05	11.54	12.02
	On:	10.61	11.11	11.60	12.09
	Diff:	.08	.06	.06	.07

Mean difference = .067

16 lbs. on scale pan;	Off:	10.58	11.09	11.59	12.09
	On:	10.78	11.28	11.78	12.28
	Diff:	.20	.19	.19	.19

Mean difference = .192

Formula: $a = \frac{\delta_1}{t \times p}$ $\delta_1 = \frac{.192-.067}{.50} = \frac{.125}{.50} = .25$ w.l.

$p = \frac{(16 - 5) \times 20}{2.205 \times 150} = .665$ kgms/sq.mm. $t = 10$ mm.

$$a = \frac{.25}{10 \times .665} = .0376$$

Constant "b"

5 lbs. on scale pan;	Off:	10.84	11.32	11.80	12.31
	On:	11.02	11.48	11.98	12.48
	Diff:	.18	.16	.18	.17

Mean difference = .173

16 lbs. on scale pan:	Off:	10.38	10.87	11.36	11.82
	On:	10.85	11.32	11.80	12.27
	Diff:	.47	.45	.44	.45

Mean difference = .453

Formula: $b = \frac{\delta_2}{t \times p}$ $\quad \delta_2 = \frac{.453 - .173}{.50} = \frac{.280}{.50} = .56$ wl.

p = .665 kgms/sq.mm. t = 10mm.

$$b = \frac{.56}{10 \times .665} = .0842$$

Constant "c"

5 lbs. on scale pan;	Off:	.23	.21
	On:	-1.24	-1.20
	Diff:	-1.47	-1.41

Mean difference = -1.44

16 lbs. on scale pan;	Off:	.23	.20
	On:	-3.68	-3.71
	Diff:	-3.91	-3.91

Mean difference: = -3.91

Formula: $c = \frac{\delta_3}{t \times p}$ $\quad \delta_3 = \frac{-3.91 - (-1.44)}{7.95} = \frac{-2.47}{7.95} = -.311$ wl.

p = .665 kgms/sq.mm. t = 10 mm.

$$c = -\frac{.311}{10 \times .665} = -.0467 \text{ wl.}$$

Adjusted values of a, b and c

a = .0376 b = .0843 c = - .0467

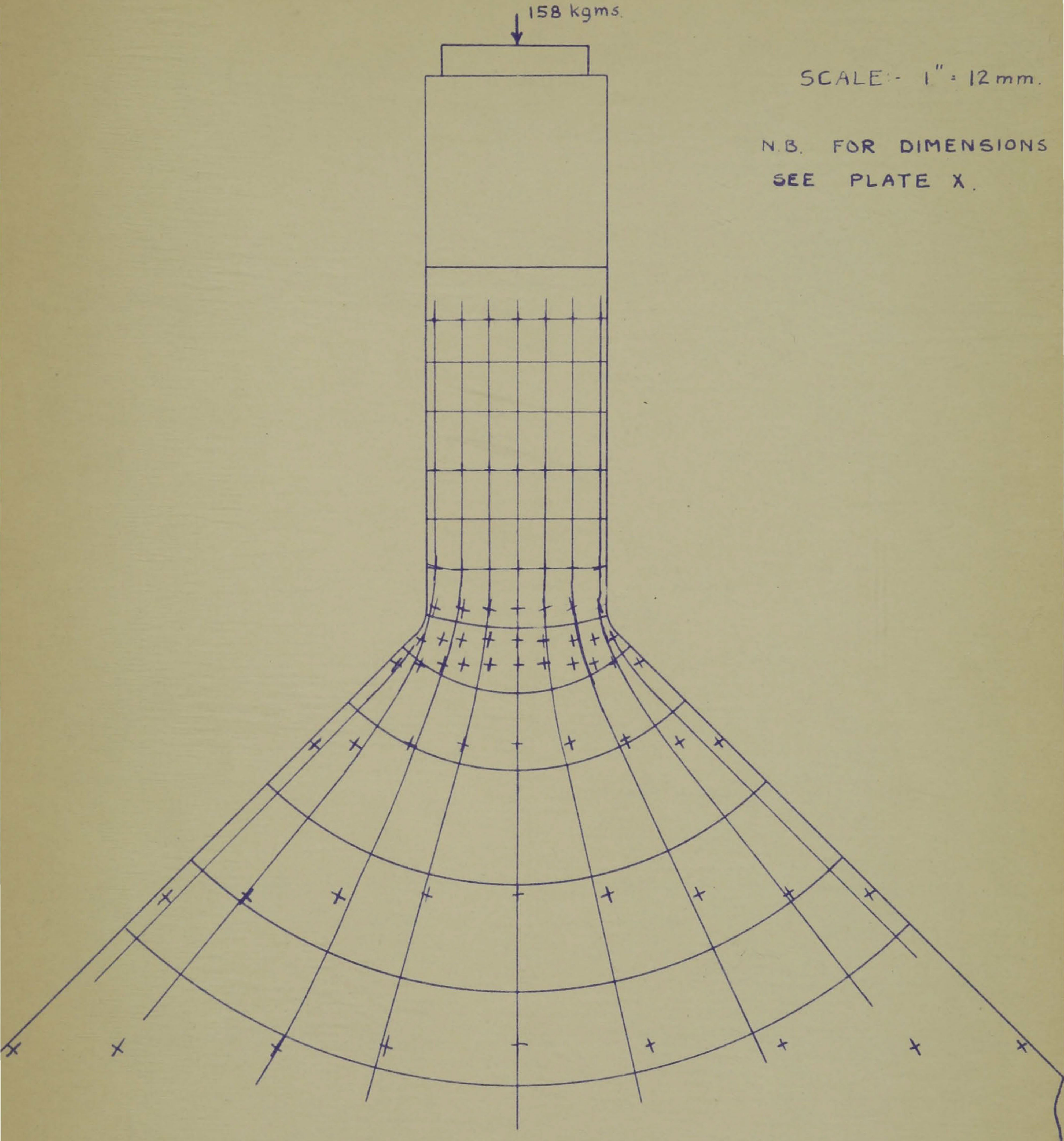
Row	Point	α°	δ_1	δ_2	p	q	p-q	δ_3 (neg.)	p-q from δ_3
A	-3	0	.40	.94	1.13	-.03	1.16	.503	1.08
	-2	0	.38	.88	1.05	-.02	1.07	.496	1.06
	-1	0	.40	.94	1.13	-.03	1.16	.492	1.05
	Centre	0	.40	.92	1.10	-.02	1.12	.498	1.07
	1	0	.42	.92	1.09	.01	1.08	.499	1.07
	2	0	.38	.86	1.09	-.01	1.10	.494	1.06
	3	0	.40	.84	.98	.04	.94	.489	1.05
B	-3	0	.40	.92	1.10	-.02	1.12	.465	.99
	-2	0	.40	.90	1.07	0	1.07	.483	1.03
	-1	0	.38	.92	1.11	-.05	1.16	.483	1.03
	Centre	0	.40	.92	1.10	-.02	1.12	.495	1.06
	1	0	.40	.90	1.07	0	1.07	.494	1.06
	2	0	.40	.92	1.10	-.02	1.12	.501	1.07
	3	0	.38	.92	1.11	-.05	1.16	.508	1.09
C	-3	1.5	.36	.94	1.15	-.09	1.24	.479	1.03
	-2	0	.40	.96	1.16	-.04	1.20	.468	1.00
	-1	0	-	-	-	-	-	.440	.94
	Centre	0	.42	.90	1.06	.03	1.03	.413	.88
	1	0	.42	.88	1.03	.04	.99	.439	.94
	2	0	.40	.90	1.07	0	1.07	.501	1.07
	3	-2.5	.40	.90	1.07	0	1.07	.546	1.17

Row	Point	$\alpha°$	δ_1	δ_2	p	q	p-q	δ_3 (neg.)	p-q from δ_3
D	-3	12	.52	1.04	1.20	.08	1.12	.600	1.28
	-2	12	.56	.84	.87	.13	.74	.391	.84
	-1	6	.54	.88	.95	.22	.73	.333	.71
	Centre	0	.32	.78	.94	.04	.90	.322	.69
	1	-6	.52	.80	.84	.24	.60	.342	.73
	2	-12	.58	.90	.95	.26	.69	.444	.95
	3	-11	.64	1.16	1.29	.18	1.11	.665	1.42
E	-4	28	.48	1.00	1.16	.05	1.11	.575	1.23
	-3	14	.58	.98	1.07	.21	.86	.386	.83
	-2	10	.54	.88	.95	.22	.73	.325	.69
	-1	4	.56	.76	.76	.33	.43	.303	.65
	Centre	0	.54	.78	.80	.28	.52	.288	.62
	1	-5	.48	.78	.78	.20	.58	.284	.61
	2	-9	.56	.82	.84	.29	.55	.326	.70
	3	-14	.60	.92	.97	.28	.69	.381	.82
	4	-29	.46	.94	1.09	.06	1.03	.503	1.08
F	-5	41.5	.22	.58	.71	-.06	.77	.340	.73
	-4	30	.32	.68	.79	-.03	.82	.391	.84
	-3	17.5	.40	.76	.86	.10	.76	.370	.79
	-2	9.5	.50	.72	.74	.29	.45	.322	.69
	-1	4	.42	.66	.70	.21	.49	.295	.63
	Centre	0	.40	.60	.62	.21	.41	.275	.59
	1	-4	.44	.72	.77	.19	.58	.285	.61
	2	-9	.46	.70	.77	.14	.63	.324	.69
	3	-16.5	.40	.64	.68	.19	.49	.358	.77
	4	-29	.28	.66	.79	-.02	.81	.372	.80
	5	-41	.22	.54	.65	-.03	.68	.304	.65

Row	Point	$\alpha°$	δ_1	δ_2	p	q	p-q	δ_3 (neg.)	p-q from δ_3
G	-4	43	.14	.28	.32	.02	.30	.184	.39
	-3	36.5	.16	.40	.49	-.03	.52	.218	.47
	-2	24	.22	.46	.54	.02	.52	.283	.60
	-1	12	.26	.56	.66	.02	.64	.277	.59
	Centre	0.5	.28	.54	.61	.06	.55	.289	.62
	1	-11	.26	.54	.63	.03	.60	.268	.57
	2	-22.5	.22	.42	.48	.05	.43	.244	.52
	3	-36	.14	.30	.35	.01	.34	.186	.40
	4	-43	.14	.22	.23	.06	.17	.142	.30
H	-4	44.5	.06	.20	.26	-.04	.30	.126	.27
	-3	37.5	.10	.22	.26	0	.26	.133	.28
	-2	27.5	.12	.30	.36	-.02	.38	.158	.34
	-1	15	.12	.32	.39	-.03	.42	.184	.39
	Centre	0.5	.14	.34	.43	-.02	.45	.178	.38
	1	-14	.10	.28	.35	-.04	.39	.174	.37
	2	-26.5	.12	.26	.31	.01	.30	.141	.30
	3	-37.5	.10	.20	.23	.02	.21	.118	.25
	4	-44.5	.08	.18	.21	0	.21	.104	.22
J	-4	45	.08	.14	.15	-.03	.18	.133	.28
	-3	41	.06	.18	.23	-.03	.26	.140	.30
	-2	32	.01	.14	.14	.06	.08	.140	.30
	-1	18	.12	.20	.22	.05	.17	.114	.24
	Centre	1	.12	.20	.22	.05	.17	.130	.28
	1	-19	.12	.22	.25	.03	.22	.121	.26
	2	-31	.10	.20	.23	.02	.21	.117	.25
	3	-42.5	.08	.18	.21	0	.21	.108	.23
	4	-45	.06	.14	.17	0	.17	.100	.21

PLATE IX

LINES OF PRINCIPAL STRESS



SCALE:- 1" = 12 mm.

N.B. FOR DIMENSIONS SEE PLATE X.

PLATE X

STRESS DIAGRAMS