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Interpolation of Operators using Peetre's K-method.

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Abstract

This thesis investigates properties of spaces generated by Peetre's K -method and uses the K -method to interpolate linear, quasilinear and Lipschitz operators. Variants of classical results such as the Riesz convexity theorem and the Marcinkiewicz Interpolation Theorem will be proved using the K -method.

Résumé

Cette thèse examine la méthode des espaces de moyenne à deux paramètres et l'utilise pour interpoler les opérations linéaires, quasilineaires et de type Lipschitz et pour prouver variations des résultats classiques comme le théorème de Riesz et le théorème de Marcinkiewicz.

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Chapter 0

Introduction

0.1 Notation

The following notation is frequently used in what follows.

$\mathbb{N}$ natural numbers: $\{1, 2, 3, \dots\}$

$\mathbb{Z}$ integers: $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

$\chi_A(x)$ characteristic function: $\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$

$m(A)$ Lebesgue measure of set A

$$\|f\|_{L^p(X,\mu)} = \begin{cases} \left(\int_X |f(x)|^p d\mu \right)^{\frac{1}{p}}, & 0 < p < \infty \\ \inf\{C \geq 0: \mu(\{x \in X: |f(x)| > C\}) = 0\}, & p = \infty \end{cases}$$

$$L^p(X, \mu) = \{f: \|f\|_{L^p(X, \mu)} < \infty\}$$

$$\|f\|_{L^p(\frac{dt}{t})} = \begin{cases} \left(\int_0^\infty |f(x)|^p \frac{dt}{t} \right)^{\frac{1}{p}}, & 0 < p < \infty \\ \inf\{C \geq 0: m(\{t > 0: |f(t)| > C\}) = 0\}, & p = \infty \end{cases}$$

$$[x]_+ = \max(x, 0)$$

Definition 0.1.1 *Let f be a function from $\mathbb{R} \rightarrow \mathbb{R}$ then f is said to be an increasing function if $x > y$ implies $f(x) \geq f(y)$ and is said to be a decreasing function if $x > y$ implies $f(x) \leq f(y)$.*

The point of the above definition is that we use the term increasing or decreasing to also include functions which may be constant in places.

0.2 Background

Suppose T is a bounded linear operator from $L_p \rightarrow L_p$ and from $L_r \rightarrow L_r$ where $p < r$. A natural question to ask is whether this guarantees that T is a bounded linear operator from $L_q \rightarrow L_q$ when $p < q < r$. This question was answered affirmatively in 1926, by Marcel Riesz. He proved the following theorem:

Theorem 0.2.1 (Riesz Convexity Theorem) *Let (X, μ) and (Y, λ) be arbitrary σ -finite measure spaces and let T be a bounded linear operator from $L^{p_i}(X) \rightarrow L^{q_i}(Y)$*

with bound M_i where $1 \leq p_i \leq q_i \leq \infty$ for $i = 0, 1$. Let p and q be such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ where $0 < \theta < 1$. Then T is a bounded linear operator from $L^p(X) \rightarrow L^q(Y)$ with bound M_θ where $M_\theta \leq M_0^{1-\theta} M_1^\theta$.

One can reformulate this result in geometric terms. Let $M(s, t)$ be a function from $[0, 1] \times [0, 1] \rightarrow [0, \infty]$ defined as $M(s, t) = \sup_{\|f\|_{L^p(X)}} \|Tf\|_{L^q(Y)}$ where $p = 1/s$ and $q = 1/t$. The above theorem states that $\log M(s, t)$ is a convex function on the triangle with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$ (which is why the above result is called the convexity theorem).

Theorem 0.2.2 (Hausdorff-Young Inequality) Let $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$. If $f \in L^p(\mathbb{T})$, then $\hat{f} \in l^{p'}(\mathbb{Z})$ and $\|\hat{f}\|_{l^{p'}} \leq \|f\|_{L^p}$.

This result remains true when we replace $\mathbb{T}$ with an arbitrary locally compact abelian group. A special case of this result (when p' is an even integer) was proved by Young in 1912 (see [15]). The general case was considerably more difficult and was solved by Hausdorff [17] in 1923. However, thanks to the Riesz convexity theorem, we no longer need to prove the result for $\forall p \in [1, 2]$ but only for the endpoint cases $p = 1$ and $p = 2$ which turn out to be simpler to derive than for general p . The proof is as follows:

Proof: When $(p, p') = (2, 2)$, the desired result follows from Parseval's Theorem. Now we examine the $(p, p') = (1, \infty)$ case. If $f \in L^1(\mathbb{T})$, then $|\hat{f}(n)| = |\int_{\mathbb{T}} f(\theta) e^{-in\theta} \frac{d\theta}{2\pi}| \leq \int_{\mathbb{T}} |f(\theta)| \frac{d\theta}{2\pi} = \|f\|_{L^1(\mathbb{T})}$ and taking the supremum over all $n \in \mathbb{Z}$, we get $\|\hat{f}\|_{l^\infty(\mathbb{Z})} \leq$

$\|f\|_{L^1(\mathbb{T})}$. Since the Fourier transform is a linear operator, the Hausdorff-Young Inequality now follows from the Riesz Convexity Theorem. ■

The Riesz convexity theorem was the genesis of the study of the interpolation of operators. The obvious usefulness of the Riesz convexity theorem in proving results such as Theorem 0.2.2 spurred further research in this field. In 1939, two major new results were discovered. Using methods from complex variable theory, Thorin was able to remove the restriction $p_i \leq q_i$ from Theorem 0.2.1. This new result came to be called the Riesz-Thorin Convexity theorem. Also in that same year, J. Marcinkiewicz discovered an interpolation theorem which could be used in situations in where the Riesz Convexity theorem no longer applies. These two results served as models for later results in the interpolation of operators. In the early 1960's, Calderón and Lions extended Thorin's methods from L^p spaces to general Banach spaces. These results formed the basis of the complex method of interpolation. Similarly, results inspired by the Marcinkiewicz Interpolation Theorem have come to be known as the real method of interpolation. One of the most widely used of the real methods of interpolation is the K -method developed by Jaak Peetre in 1963.

In this thesis we will study the K -method and its application to the Interpolation of Operators in quasi-normed vector spaces. This thesis aims to be as accessible and self-contained as possible. Chapter 1 will review any terminology or results used in the thesis which would not usually be covered in an introductory graduate analysis course. We formally define interpolation spaces in Chapter 2, and show how the K -

method can be used to find interpolation spaces. In Chapter 3, we use the K -method to study the interpolation of L^p spaces. In particular, we introduce the Lorentz spaces as the interpolation spaces generated by using the K -method on a pair of L^p spaces and we use the K -method to derive the Marcinkiewicz interpolation theorem and a variant of the Riesz convexity theorem. In Chapter 4, we examine Peetre's J -method, its applications, and its relation to the K -method. In our final chapter, we use the K -method to interpolate Lipschitz operators.

Chapter 1

Preliminaries

1.1 Quasi-Normed Vector Spaces

Definition 1.1.1 *Let V be a vector space over a scalar field K . Then $\|\cdot\|_V$, a function from $V \rightarrow [0, \infty)$, is called a quasi-seminorm if*

$$(1) \quad \|kv\|_V = |k|\|v\|_V \quad \forall k \in K, v \in V$$

$$(2) \quad \exists c_V \geq 1 \text{ s.t. } \|u + v\|_V \leq c_V(\|u\|_V + \|v\|_V) \quad \forall u, v \in V$$

Condition (1) is called homogeneity and Condition (2) is called the c -triangle inequality. If $c_V = 1$ in (2), then $\|\cdot\|_V$ is called a seminorm and (2) is simply called the triangle inequality.

In this thesis the scalar field K will always be either the real or complex numbers.

Definition 1.1.2 $\|\cdot\|_V$ is a quasi-norm (resp. norm) if $\|\cdot\|_V$ is a quasi-seminorm (resp. seminorm) with the added property that if $v \in V$ and $\|v\|_V = 0$ then $v = 0$.

Definition 1.1.3 A Banach space is a complete normed vector space.

Definition 1.1.4 Let V be a vector space having $\| \cdot \|$ as a quasi-norm. Let $\| \cdot \|'$ be a functional on V with the property that $\|kv\|' = |k|\|v\|' \forall k \in K, v \in V$. Then $\| \cdot \|$ and $\| \cdot \|'$ are said to be equivalent quasi-norms iff $\exists c, C > 0$ s.t. $c\|v\| \leq \|v\|' \leq C\|v\|$ for $\forall v \in V$.

The conditions on $\| \cdot \|'$ given in the definition are enough to guarantee that it is indeed a quasi-norm. Since $0 \leq c\|v\| \leq \|v\|' \leq C\|v\| < \infty$, we have $\|v\|'$ finite and nonnegative. Also $\|v\|' = 0$ implies $\|v\| = 0$ which means $v = 0$. Now we need only check the c-triangle inequality. Let $u, v \in V$ then $\|u + v\|' \leq C\|u + v\| \leq c_v C(\|u\| + \|v\|) \leq \frac{c_v C}{c}(\|u\|' + \|v\|')$.

1.2 L^p spaces

It is assumed the reader is familiar with L^p spaces when the exponent $p \in [1, \infty]$. We now show that the L^p spaces are quasi-normed vector spaces for $p \in (0, \infty]$. We begin with the following inequalities which will then be used to prove the c-triangle inequality for L^p spaces.

Lemma 1.2.1 Let $x, y \geq 0$. Then $(x+y)^p \leq (x^p + y^p) \leq 2^{1-p}(x+y)^p$ when $0 < p \leq 1$ and $2^{1-p}(x+y)^p \leq (x^p + y^p) \leq (x+y)^p$ when $1 \leq p < \infty$.

Proof: If either x or y is zero, the result is immediate. Now fix $y > 0$ and let $F(x) = [(x+y)^p - (x^p + y^p)]$. Differentiating, we get $F'(x) = p[(x+y)^{p-1} - x^{p-1}]$. For

$p \leq 1$, $F'(x) \leq 0$. Hence $[(x+y)^p - (x^p + y^p)] = F(x) \leq F(0) = 0$. The L.H.S. of the first inequality follows. For $p \geq 1$, $F'(x) \geq 0$. Hence $[(x+y)^p - (x^p + y^p)] = F(x) \geq F(0) = 0$. The R.H.S. of the second inequality follows. x^p is a concave function on $[0, \infty)$ for $p \leq 1$ and therefore we have $\frac{x^p + y^p}{2} \leq (\frac{x+y}{2})^p$. Multiplying both sides by 2, gives us $(x^p + y^p) \leq 2^{1-p}(x+y)^p$ which is the R.H.S. of the first inequality. For $p \geq 1$, x^p is a convex function on $[0, \infty)$ and as a result we have $(\frac{x+y}{2})^p \leq \frac{x^p + y^p}{2}$. Again multiplying both sides by two gives us $2^{1-p}(x+y)^p \leq (x^p + y^p)$ which is the L.H.S. of the second inequality. ■

Theorem 1.2.2 L^p is a Banach space for $1 \leq p \leq \infty$.

This result is well known so the proof is omitted. The following is a corresponding result for $p \in (0, \infty]$.

Theorem 1.2.3 L^p is a quasi-normed space for $0 < p \leq \infty$.

Proof: For $p \geq 1$ the result follows from above, so let $0 < p < 1$. We first prove that L^p is a quasi-seminorm. Homogeneity is obvious. To establish the c-triangle inequality requires repeated uses of the previous lemma.

$$\begin{aligned} \|f + g\|_{L^p} &= (\int |f + g|^p d\mu)^{1/p} \leq (\int (|f| + |g|)^p d\mu)^{1/p} \leq (\int |f|^p d\mu + \int |g|^p d\mu)^{1/p} \\ &\leq 2^{(1/p)-1} [(\int |f|^p d\mu)^{1/p} + (\int |g|^p d\mu)^{1/p}]. \end{aligned}$$

Hence $\|\cdot\|_{L^p}$ is a quasi-seminorm. Now let $f \in L^p$ such that $\|f\|_{L^p} = 0$. Then $f^p \in L^1$ with $\|f^p\|_{L^1} = 0$. Since L^1 is a Banach space, $f = 0$ a.e. and therefore $\|\cdot\|_{L^p}$ is a quasi-norm. ■

Lemma 1.2.1 can also be used to prove the following useful inequality for nonnegative functions in an L^p space.

Lemma 1.2.4 (Reverse Triangle Inequality) *Let $f, g \in L^p$ with $f, g \geq 0$. Then $(\|f\|_{L^p} + \|g\|_{L^p}) \leq 2^{1-1/p} \|f + g\|_{L^p}$.*

Proof: Let f, g be as above. Now let $p \geq 1$, then $(\int f^p d\mu)^{1/p} + (\int g^p d\mu)^{1/p} \leq 2^{1-1/p} (\int f^p + g^p d\mu)^{1/p} \leq 2^{1-1/p} (\int (f + g)^p d\mu)^{1/p}$.

Now let $p \leq 1$, then $(\int f^p d\mu)^{1/p} + (\int g^p d\mu)^{1/p} \leq (\int f^p + g^p d\mu)^{1/p} \leq (2^{1-p} \int (f + g)^p d\mu)^{1/p} = 2^{1/p-1} \|f + g\|_{L^p}$. ■

1.3 The Lattice Property

Let (X, μ) be an arbitrary measure space. We now consider quasi-normed spaces whose elements are real or complex valued functions on X . It is often useful to have a quasi-norm which is related to the size of a function. This idea motivates the following definition:

Definition 1.3.1 *Let V be a quasi-normed vector space whose elements are scalar valued functions on an arbitrary measure space (X, μ) . Then V is said to have the lattice property if $f, g : X \rightarrow K$, $g \in V$, $|f(x)| \leq |g(x)|$ μ -a.a. $x \in X$ then $f \in V$ and $\|f\|_V \leq \|g\|_V$.*

In references such as [39], normed spaces with the lattice property are called pre-ideal spaces and Banach spaces with the lattice property are called ideal spaces.

Similarly, we will define a quasi-ideal space as a quasi-normed vector space with the lattice property.

The following is a useful elementary property of quasi-ideal spaces.

Lemma 1.3.2 *Let V be a quasi-ideal space whose elements are functions from $X \rightarrow K$. Let $g \in V$ and $f : X \rightarrow K$ with $|f(x)| = |g(x)|$ μ -a.e. on X . Then $f \in V$ and $\|f\|_V = \|g\|_V$.*

Proof: We have $|f(x)| \leq |g(x)|$ a.e. and $g \in V$; therefore the lattice property implies that $f \in V$ and $\|f\|_V \leq \|g\|_V$. However, we also have $|g(x)| \leq |f(x)|$ a.e. which implies $\|g\|_V \leq \|f\|_V$. Hence we conclude that $\|f\|_V = \|g\|_V$. ■

For L^p spaces we have the following result.

Proposition 1.3.3 *L^p spaces for $p > 0$ have the lattice property.*

Proof: If $p = \infty$, the result is trivial. For $0 < p < \infty$, the result follows from the fact that x^p and $x^{1/p}$ are increasing on $(0, \infty)$. ■

One can extend the reverse triangle inequality from L^p spaces to arbitrary quasi-ideal spaces.

Lemma 1.3.4 *Let V be a quasi-ideal space Then $\exists c'_V \leq 2$ such that for $f, g \in V$ with $f, g \geq 0$ a.e. $\|f\|_V + \|g\|_V \leq c'_V \|f + g\|_V$.*

Proof: Since $f, g \geq 0$ a.e, $f \leq f + g$ a.e. and $g \leq f + g$ a.e. Hence $\|f\|_V + \|g\|_V \leq 2\|f + g\|_V$. ■

1.4 The Sum and the Intersection of Quasi-normed Vector Spaces.

The results in this chapter are central to the theory of interpolation. Most are from [1] or [3].

Definition 1.4.1 *Let V, W be Hausdorff topological vector spaces with $V \subseteq W$, V and W not necessarily having the same topologies. Then V is said to be continuously embedded in W if given $\{x_n\}_{n \in \mathbb{N}}$ with $x_n \in V$, $x_n \rightarrow 0$ in V , implies $x_n \rightarrow 0$ in W as well.*

A quasi-normed space V is a topological vector space whose neighborhood basis at 0 consists of the open balls $\{v : \|v\|_V < \varepsilon\}$ (see [24] for more details). Hence when V and W are quasi-normed spaces, we have the following:

Theorem 1.4.2 *Let V and W be quasi-normed spaces with $V \subseteq W$. Then V is continuously embedded in W iff $\exists M$ such that $\|f\|_W \leq M\|f\|_V$ for $\forall f \in V$.*

Proof If $\exists M$ such that $\|f\|_W \leq M\|f\|_V$ for $\forall f \in V$, then it is clear that V is continuously embedded in W . Now suppose no such M exists; then we can find a sequence $\{x_n\}_{n \in \mathbb{N}}$ such that $\|x_n\|_V = 1$ but $\|x_n\|_W \geq n$. Then $\frac{x_n}{n} \rightarrow 0$ in V but $\|\frac{x_n}{n}\|_W \geq 1$ and hence V is not continuously embedded in W . ■

Definition 1.4.3 Let V and W be quasi-normed vector space and M be a Hausdorff topological vector space. Then (V, W) is a compatible couple of quasi-normed spaces (in M) if V and W are continuously embedded subspaces of M .

Definition 1.4.4 Let (V, W) be a compatible couple (in M) of quasi-normed vector spaces. Then $V + W = \{f \in M \text{ such that } \exists v \in V \text{ and } w \in W \text{ such that } f = v + w\}$. Also, let $V \cap W$ denote the usual set-theoretic intersection (i.e. $V \cap W = \{f \text{ such that } f \in V \text{ and } f \in W\}$).

Lemma 1.4.5 Let (V, W) be a compatible couple (in M) of quasi-normed (resp. normed) vector spaces. Then $V \cap W$ and $V + W$ are also quasi-normed (resp. normed) vector spaces.

Proof: Let $\|f\|_{V \cap W} = \max\{\|f\|_V, \|f\|_W\}$ where f is an arbitrary element of $V \cap W$. It is easy to verify that $\|\cdot\|_{V \cap W}$ is a quasi-norm (resp. norm) for $V \cap W$. Now let $\|f\|_{V+W} = \inf\{\|g\|_V + \|h\|_W \text{ where } g \in V \text{ and } h \in W \text{ such that } f = g + h\}$. It can easily be shown that $\|\cdot\|_{V+W}$ is a quasi-seminorm (resp. seminorm) for $V + W$. To show that $\|\cdot\|_{V+W}$ is also a quasi-norm (resp. norm), let $f \in V + W$ with $\|f\|_{V+W} = 0$. Then $\forall n, \exists v_n \in V$ and $w_n \in W$ such that $v_n + w_n = f$ and $\|v_n\|_V + \|w_n\|_W \leq n^{-1}$. Now v_n tends to 0 in V and therefore also tends to 0 in M . Similarly, w_n tends to 0 in M and therefore so does $v_n + w_n$. Hence $f = 0$ and $\|\cdot\|_{V+W}$ is a quasi-norm (resp. norm). ■

Theorem 1.4.6 Let (V, W) be a compatible couple of Banach spaces. Then $V \cap W$ and $V + W$ are also Banach spaces.

Proof: By the previous lemma, $V \cap W$ and $V + W$ are normed vector spaces and so we only need to show that they are also complete. Let f_n be a Cauchy sequence in $V \cap W$, then it is also a Cauchy sequence in V . By completeness of V , $\exists v \in V$ such that f_n tends to v in V and therefore also in M . By completeness of W , $\exists w \in W$ such that f_n tends to w in W and hence also in M . Since limits are unique in Hausdorff spaces, $v = w$. Hence $v \in V \cap W$ and f_n tends to v in $V \cap W$. So $V \cap W$ is complete and hence is a Banach space.

Now we turn our attention to $V + W$. Let f_n be an absolutely convergent sequence in $V + W$ (i.e. $\sum_{n=1}^{\infty} \|f_n\|_{V+W} < \infty$). $\forall n$, $\exists v_n \in V$, $w_n \in W$ such that $v_n + w_n = f_n$ and $\|v_n\|_V + \|w_n\|_W \leq \|f_n\|_{V+W} + n^{-2}$. Therefore, $\sum_{n=1}^{\infty} v_n$ converges absolutely in the V -norm. Since V is complete, $\exists v \in V$ such that $\sum_{n=1}^{\infty} v_n$ converges to v . Similarly $\exists w \in W$ such that $\sum_{n=1}^{\infty} w_n$ converges to w . Now let $f = v + w$. Then $f \in V + W$ and $\|f - \sum_{n=1}^N f_n\|_{V+W} \leq \|v - \sum_{n=1}^N v_n\|_V + \|w - \sum_{n=1}^N w_n\|_W$. Since R.H.S. tends to 0 as $N \rightarrow \infty$, so does the L.H.S. Hence every absolutely convergent sequence in $V + W$ converges in $V + W$ and hence $V + W$ is complete. ■

Theorem 1.4.7 *Let (A_0, A_1) and (B_0, B_1) be compatible couples of quasi-normed spaces. Let T be a linear operator from $A_i \rightarrow B_i$ where $i = 0, 1$. Then T can be extended uniquely to a linear operator from $A_0 + A_1 \rightarrow B_0 + B_1$.*

Proof: Let $f \in A_0 + A_1$, then $\exists g \in A_0, h \in A_1$ such that $f = g + h$. Now define $Tf = Tg + Th$. It is clear that the extension is linear; now we must show it is unique. Let $g' \in A_0$ and $h' \in A_1$ be such that $f = g' + h'$. Then $g - g' = h' - h$ and since

both sides are $\in A_0 \cap A_1$, we have $Tg - Tg' = T(g - g') = T(h' - h) = Th' - Th$. Rearranging, we get $Tg + Th = Tg' + Th'$ and hence Tf is independent of the choice of decomposition of f . ■

The above result motivates the following definition.

Definition 1.4.8 *Let A be quasi-normed vector space and let (V, W) be a compatible couple of quasi-normed vector spaces. Then A is said to be an intermediate space with respect to (V, W) if $V \cap W \subseteq A \subseteq V + W$ with $V \cap W$ continuously embedded in A and A continuously embedded in $V + W$.*

If a linear operator is defined on V and W , it will be defined on $V + W$ and hence on every intermediate space of (V, W) .

Example 1.4.9 *Let V and W be quasi-normed vector spaces. Then V , W , $V \cup W$ and $V + W$ are intermediate spaces for (V, W) .*

Turning our attention back to L^p spaces, we get the following example of intermediate spaces.

Example 1.4.10 *Let $0 < p \leq q \leq r \leq \infty$, then $L^q(X, \mu)$ is an intermediate space of the pair $(L^p(X, \mu), L^r(X, \mu))$.*

Proof: First we assume that $0 < p < q < r \leq \infty$. (If not, the result follows from previous example.) Let $f \in L^p \cap L^r$. Since $p < q < r$, $\exists \theta$ such that $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{r}$. Then using Holder's Inequality and $1 = \frac{q(1-\theta)}{p} + \frac{q\theta}{r}$, we get $\|f\|_{L^p \cap L^r} = \max(\|f\|_{L^p}, \|f\|_{L^r}) \geq$

$(\|f\|_{L^p})^{1-\theta}(\|f\|_{L^r})^\theta \geq \|f\|_{L^q}$. Hence $L^p \cap L^r$ is continuously embedded in L^q . Now let $f \in L^q$, we now prove $\|f\|_{L^p+L^r} \leq 2\|f\|_{L^q}$. Without loss of generality, let $\|f\|_{L^q} = 1$. Define $E = \{x \in X : |f(x)| > 1\}$. Since $p < q$, $\exists s > 0$ such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{s}$ and using Holder's inequality and $\mu(E) \leq 1$, we have $\|f\chi_E\|_{L^p} \leq \|f\|_{L^q}\|\chi_E\|_{L^s} \leq \|f\|_{L^q}$. Also $\|f\chi_{X \setminus E}\|_{L^r} = (\int_{X \setminus E} f^r d\mu)^{1/r} \leq (\int_{X \setminus E} f^q d\mu)^{1/r} \leq \|f\|_{L^q}^{q/r} = \|f\|_{L^q}$ where the first inequality follows from f being ≤ 1 on $X \setminus E$. Therefore we have $\|f\|_{L^p+L^r} \leq \|f\chi_E\|_{L^p} + \|f\chi_{X \setminus E}\|_{L^r} \leq 2\|f\|_{L^q}$. ■

Thus our intuitive notion of what is intermediate for L^p spaces corresponds to our definition of intermediate spaces.

1.5 Rearrangements

In this section we introduce the theory of rearrangements which was first systematically studied by Hardy, Littlewood and Pólya. This theory is intimately connected to the real method of interpolation. For instance, distribution functions were used in the first proof of the Marcinkiewicz Interpolation Theorem [40]. Our main results, Propositions 1.5.3 and 1.5.8 are a combination of results from [1],[14],[20],[30] and [38].

In the following section f and g are real or complex-valued functions on an arbitrary measure space, (X, μ) .

Definition 1.5.1 $S_f(s) = \{x \in X : |f(x)| > s\}$.

Definition 1.5.2 The distribution function of f is denoted as $\mu_f(s)$ and is defined as $\mu_f(s) = \mu(S_f(s))$ for $\forall s > 0$.

It is clear that $\mu_f(s)$ is a nonnegative decreasing function from $(0, \infty) \rightarrow [0, \infty]$

The following are more properties of the distribution function.

Proposition 1.5.3 The distribution function has the following properties:

- a) $\mu_f(s)$ is right-continuous on $(0, \infty)$.
- b) If $|f(x)| \leq |g(x)|$ a.a.x, then $\mu_f(s) \leq \mu_g(s)$ for $\forall s > 0$.
- c) If $|f_n| \uparrow |f|$ μ -a.e., then $\mu_{f_n} \uparrow \mu_f$.
- d) If $c \neq 0$, then $\mu_{cf}(s) = \mu_f(s/|c|)$.
- e) $\mu_{f+g}(s+t) \leq \mu_f(s) + \mu_g(t)$ for $\forall s, t > 0$.

Proof: a) Let s_n be a decreasing sequence of non-negative real numbers with $s_n \downarrow s$.

Then $S_f(s_1) \subseteq S_f(s_2) \subseteq \dots \subseteq \bigcup_{n=1}^{\infty} S_f(s_n) = S_f(s)$ and hence $\lim_{n \rightarrow \infty} \mu_f(s_n) = \mu_f(s)$.

b) Since $|f(x)| \leq |g(x)|$, $S_f(s) \subseteq S_g(s)$ and therefore $\mu_f(s) \leq \mu_g(s)$ for $\forall s > 0$.

c) For any $s > 0$, $S_{f_1}(s) \subseteq S_{f_2}(s) \subseteq \dots \subseteq \bigcup_{n=1}^{\infty} S_{f_n}(s) = S_f(s)$ and therefore $\mu_{f_1}(s) \leq \mu_{f_2}(s) \leq \dots \leq \lim_{n \rightarrow \infty} \mu_{f_n}(s) = \mu_f(s)$.

d) $S_{cf}(s) = \{x : |cf(x)| > s\} = \{x : |f(x)| > s/|c|\} = S_f(s/|c|)$. Hence $\mu_{cf}(s) = \mu_f(s/|c|)$.

e) Now if $|f(x) + g(x)| > s+t$ and hence we have $|f(x)| + |g(x)| \geq |f(x) + g(x)| > s+t$ so either $|f(x)| > s$ or $|g(x)| > t$. This gives us $S_{f+g}(s+t) \subseteq S_f(s) \cup S_g(t)$ and thus $\mu_{f+g}(s+t) = \mu(S_{f+g}(s+t)) \leq \mu(S_f(s) \cup S_g(t)) \leq \mu(S_f(s)) + \mu(S_g(t)) = \mu_f(s) + \mu_g(t)$.

■

The following is a useful inequality which bounds $\mu_f(s)$ in terms of the L^p norm.

Theorem 1.5.4 (Chebyshev's Inequality) *Let $s > 0$ and $f \in L^p$. Then $\mu_f(s) \leq (\frac{\|f\|_{L^p}}{s})^p$*

Proof: R.H.S. = $\frac{1}{s^p} \int_X |f|^p d\mu \geq \frac{1}{s^p} \int_{S_f(s)} |f|^p d\mu \geq \frac{1}{s^p} \int_{S_f(s)} s^p d\mu = \mu_f(s)$. ■

One can rewrite this inequality in the following form: $\sup_{s>0} s(\mu_f(s))^{1/p} \leq \|f\|_{L^p}$ and we can use this to define the following spaces.

Definition 1.5.5 *Weak L^p consists of all functions f such that $\sup_{s>0} s(\mu_f(s))^{1/p} < \infty$.*

Thus we can see that all L^p functions are in weak L^p but the converse fails as $x^{-1/p}$ is a weak L^p function which is not in L^p . More about this space later.

Definition 1.5.6 *Two functions f and g whose domains are respectively (X, μ) and (Y, λ) are called equimeasurable if $\mu_f(s) = \lambda_g(s)$ for $\forall s > 0$.*

Since μ_f is a function from $(0, \infty) \rightarrow [0, \infty)$ and $(0, \infty)$ is a measure space under Lebesgue measure, we can talk about the distribution function of a distribution function. So we have $m_{\mu_f} = m(\{s : \mu_f(s) > t\}) = \sup\{s : \mu_f(s) > t\} = \inf\{s : \mu_f(s) \leq t\}$. This function occurs often enough to be given a special name.

Definition 1.5.7 *The decreasing rearrangement of f is denoted as f^* and is defined as follows: $f^*(t) = \inf\{s : \mu_f(s) \leq t\}$.*

Proposition 1.5.8 *The following are properties of the decreasing rearrangement.*

a) $f^*(s)$ is nonnegative decreasing right-continuous on $(0, \infty)$.

b) If $|f(x)| \leq |g(x)|$ a.a.x, then $f^*(s) \leq g^*(s)$ for $\forall s > 0$.

c) If $|f_n| \uparrow |f|$ μ -a.e., then $f_n^* \uparrow f^*$.

d) $(cf)^*(s) = |c|f^*(s)$ where c is an arbitrary constant.

e) Let $s, t > 0$. Then $\mu_f(s) \leq t$ iff $f^*(t) \leq s$.

f) f and f^* are equimeasurable functions.

g) $(f + g)^*(s + t) \leq f^*(s) + g^*(t)$ for $\forall s, t > 0$.

Proof: a) follows immediately from Proposition 1.5.3.a and the fact that f^* is a distribution function. b), c) and d) also follow from their counterparts in Prop. 1.5.3.

e) Suppose $f^*(t) \leq s$ and let $\{s_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers such that $s_n \downarrow s$. Then $f^*(t) = \inf\{x > 0 : \mu_f(x) \leq t\} < s_n$ which since μ_f is decreasing implies that $\mu_f(s_n) \leq t$. Since the distribution function is right continuous (Prop. 1.5.3.a), we have $\mu_f(s) \leq t$. Now suppose $\mu_f(s) \leq t$. Then $f^*(t) = \inf\{x > 0 : \mu_f(x) \leq t\} \leq s$.

f) $m_{f^*}(s) = m\{x > 0 : f^*(x) > s\} = m\{x > 0 : \mu_f(s) > x\} = m((0, \mu_f(s))) = \mu_f(s)$. Here the second equality follows from part e above.

g) Let $\alpha = f^*(s)$ and $\beta = g^*(t)$. Our result in part e now gives us $\mu_f(\alpha) \leq s$ and $\mu_g(\beta) \leq t$. Using Prop. 1.5.3.e, $\mu_{f+g}(\alpha + \beta) \leq \mu_f(\alpha) + \mu_g(\beta) \leq s + t$. Now we use part e again to obtain $(f + g)^*(s + t) \leq f^*(s) + g^*(t)$. ■

It is two of the above properties of f^* (i.e. that f^* is a decreasing function equimeasurable with f) from which f^* derives its name. In fact, one can prove that f^* is the

only decreasing right-continuous functions on $(0, \infty)$ which is equimeasurable to f . We have the following result due to Burkil [4].

Lemma 1.5.9 *Let f be a measurable function on (X, μ) and let g be a decreasing right-continuous function on $(0, \infty)$ with $m_g(t) = \mu_f(t)$ for $\forall t > 0$ then $g = f^*$.*

Proof: Since f and g are equimeasurable, $f^* = g^*$. Now we only need show that g and g^* are equal everywhere. Assume $\exists a > 0$ such that $g(a) \neq g^*(a)$. Suppose $g(a) > g^*(a)$ and choose b such that $g(a) > b > g^*(a)$ and therefore $a \notin S_{g^*}(b)$ and since g^* is decreasing, we have $S_{g^*}(b) \subseteq (0, a)$. Now since g is right continuous $\exists \varepsilon > 0$ such that $g(a + \varepsilon) > b$ and hence $(0, a + \varepsilon) \subseteq S_g(b)$ and $\mu_g(b) \geq a + \varepsilon > a \geq \mu_{g^*}(b)$ which contradicts Prop. 1.5.8.f. If $g^*(a) > g(a)$, the symmetric argument applies.

1.6 Quasi-Symmetric Spaces

Definition 1.6.1 *Let V be a quasi-ideal space of functions from $(X, \mu) \rightarrow K$ and let f, g be μ -measurable functions from $X \rightarrow K$. Then V is a quasi-symmetric space if $f \in V$ and f equimeasurable to g implies $g \in V$ and $\|g\|_V = \|f\|_V$.*

Corollary 1.6.2 *Let V be a quasi-normed space. Then V is a quasi-symmetric space iff $f \in V$, g μ -measurable function from $X \rightarrow K$ and $g^*(t) \leq f^*(t)$ for $\forall t > 0$ implies that $g \in V$ and $\|g\|_V \leq \|f\|_V$.*

Proof: This easily follows from part b of Proposition 1.5.8.

It seems intuitively obvious that L^p spaces are quasi-symmetric spaces. We make this rigorous in the following theorem:

Theorem 1.6.3 [1] [30] *For $0 < p < \infty$, $(\|f\|_{L^p})^p = p \int_0^\infty s^{p-1} \mu_f(s) ds = \int_0^\infty (f^*)^p(t) dt$ and $\|f\|_{L^\infty} = \inf\{s : \mu_f(s) = 0\} = \|f^*\|_{L^\infty}$. Therefore for $0 < p \leq \infty$, L^p spaces are quasi-symmetric spaces.*

Proof: First let f be a nonnegative simple function with $f(x) = \sum_{i=1}^n a_i \chi_{E_i}(x)$. Without loss of generality we can assume the a_i to be in descending order.

Then $\mu_f(s) = \sum_{j=1}^n (\sum_{i=1}^j \mu(E_i)) \chi_{[a_{j+1}, a_j)}(s)$

$$\begin{aligned} \text{Therefore } (\|f\|_{L^p})^p &= \sum_{i=1}^n a_i^p \mu(E_i) \\ &= \sum_{j=1}^n (a_j^p - a_{j+1}^p) (\sum_{i=1}^j \mu(E_i)) \\ &= p \sum_{j=1}^n \int_{a_{j+1}}^{a_j} (\sum_{i=1}^j \mu(E_i)) s^{p-1} ds \\ &= p \int_0^\infty s^{p-1} \sum_{j=1}^n (\sum_{i=1}^j \mu(E_i)) \chi_{[a_{j+1}, a_j)}(s) ds \\ &= p \int_0^\infty s^{p-1} \mu_f(s) ds. \end{aligned}$$

Now, let f be an arbitrary function in L^p then let f_n be nonnegative simple functions such that $f_n \uparrow |f|$ then by part c of Proposition 1.5.3, $\mu_{f_n} \uparrow \mu_f$. Using the monotone convergence theorem, we get $(\|f\|_{L^p})^p = p \int_0^\infty s^{p-1} \mu_f(s) ds$. Since f and f^* are equimeasurable, $(\|f\|_{L^p})^p = p \int_0^\infty s^{p-1} \mu_f(s) ds = \int_0^\infty (f^*)^p(t) dt$. The L^∞ case is simple to verify. ■

The theory of rearrangements was systematically studied by Hardy, Littlewood and Pólya in their work [15] where this machinery is used to prove inequalities (see also [27]). For the sake of illustration, we give one example (Theorem 1.6.6) of this

type. As far as we know, the proof is new. We begin with the following two lemmas.

Lemma 1.6.4 (Hardy's lemma) *Let f, g, h be measurable functions from $(0, \infty) \rightarrow [0, \infty)$ and let h be decreasing everywhere. If $\int_0^t g(x)dx \leq \int_0^t f(x)dx$ for $\forall t > 0$, then $\int_0^\infty g(x)h(x)dx \leq \int_0^\infty f(x)h(x)dx$.*

Proof: First we assume that in addition to the above $h(x)$ is also a simple function.

Hence $\exists a_i, t_i > 0$ for $1 \leq i \leq n$ such that $h(x) = \sum_{i=1}^n a_i \chi_{(0, t_i)}(x)$.

$$\begin{aligned} \text{Therefore } \int_0^\infty g(x)h(x)dx &= \int_0^\infty g(x) \sum_{i=1}^n a_i \chi_{(0, t_i)}(x) \\ &= \sum_{i=1}^n a_i \int_0^{t_i} g(x)dx \leq \sum_{i=1}^n a_i \int_0^{t_i} f(x)dx \\ &= \int_0^\infty f(x) \sum_{i=1}^n a_i \chi_{(0, t_i)}(x) = \int_0^\infty f(x)h(x)dx. \end{aligned}$$

Now if h is not a simple function, we can find a sequence of decreasing simple functions

$h_n \uparrow h$ and using the monotone convergence theorem, we obtain our result. ■

Lemma 1.6.5 *Let f be a measurable function on (X, μ) . For $t > 0$, let $f_t(x) = f(x)$ if $|f(x)| \leq t$ and $f_t(x) = t \frac{f(x)}{|f(x)|}$ when $|f(x)| > t$. Then $\mu_{f_t}(s) = \mu_f(s)$ if $s < t$ and $\mu_{f_t}(s) = 0$ when $s \geq t$. ■*

Proof: If $s < t$, then $|f_t(x)| > s$ iff $|f(x)| > s$ which implies $S_{f_t}(s) = S_f(s)$ and $\mu_{f_t}(s) = \mu_f(s)$. Since $|f_t(x)| \leq t$, $S_{f_t}(s) = \emptyset$ for $s \geq t$ which gives us $\mu_{f_t}(s) = 0$. ■

Now we are ready to present the following result.

Theorem 1.6.6 *Let $0 < p < \infty$ and let f, g be measurable functions on arbitrary measure spaces. If $\|g_t\|_{L^p} \leq \|f_t\|_{L^p}$ for $\forall t > 0$ and if $f \in L^q$ for some q such that $0 < q \leq p$ then $g \in L^q$ and $\|g\|_{L^q} \leq \|f\|_{L^q}$.*

Proof: Now $(\|g_t\|_{L^p})^p = p \int_0^\infty s^{p-1} \mu_{g_t}(s) ds$ by Theorem 1.6.3,

$$= p \int_0^t s^{p-1} \mu_g(s) ds \text{ by Lemma 1.6.5.}$$

$$\text{Similarly, } (\|f_t\|_{L^p})^p = p \int_0^t s^{p-1} \mu_f(s) ds.$$

Now use Hardy's lemma with $h(x) = x^{q-p}$. We obtain $\int_0^\infty s^{q-1} \mu_g(s) ds \leq \int_0^\infty s^{q-1} \mu_f(s) ds$.

From our result in Theorem 1.6.3, we get $\|g\|_{L^q} \leq \|f\|_{L^q}$. ■

There is a degenerate case of quasi-symmetric spaces which contains only functions which are zero a.e. We will exclude these cases from the theory that follows.

Theorem 1.6.7 *Let V be a non-degenerate quasi-symmetric space of measurable functions on X , where (X, μ) is a non-atomic σ -finite measure space. If $E \subseteq X$ and $\mu(E) < \infty$, then $\chi_E \in V$.*

Proof: Since V is non-degenerate, $\exists f \in V$ with f not μ -a.e. zero on X . Hence $\exists P \subseteq X$ and $\alpha > 0$ with $\mu(P) > 0$ and $|f(x)| \geq \alpha > 0$ for $\forall x \in P$. Since $f \geq \alpha \chi_P$, $\chi_P \in V$. Since $\mu(E) < \infty$, $\exists \{S_i\}_{i=1}^n$ a finite family of pairwise disjoint subsets of E with $\mu(S_i) \leq \mu(P)$ for $1 \leq i \leq n$ and $\bigcup_{i=1}^n S_i = E$. Therefore $\chi_{S_i}^* \leq \chi_P^*$ and hence $\chi_{S_i} \in V$. Since $\chi_E = \sum_{i=1}^n \chi_{S_i}$, $\chi_E \in V$. ■

Definition 1.6.8 *Let (X, μ) be a measure space. Then $f : X \rightarrow \mathbb{C}$ is said to be a strictly simple function if $\exists \{E_i\}_{i=1}^n$ such that $E_i \subseteq X$ with $\mu(E_i) < \infty$ such that $f(t) = \sum_{i=1}^n a_i \chi_{E_i}(t)$ where the $a_i \in \mathbb{C}$.*

The following result now follows easily from Theorem 1.6.7 and the linearity of V .

Corollary 1.6.9 *Let V be a quasi-symmetric space whose elements are functions*

from $X \rightarrow K$, then the set of strictly simple functions from $X \rightarrow K$ is a subspace of V .

Theorem 1.6.7 also allows us to define the following:

Definition 1.6.10 *Let V be a non-degenerate quasi-symmetric space of measurable functions on (X, μ) , then the fundamental function of V , $\phi_V(t)$ is defined as $\phi_V(t) = \|\chi_E\|_V$ for $t \geq 0$ where $E \subseteq X$ with $\mu(E) = t$.*

Since χ_E and χ_F are equimeasurable whenever $\mu(E) = \mu(F)$, $\phi_V(t)$ is independent of the choice of E and therefore the above definition is well-defined. Theorem 1.6.7 shows us that the fundamental function is defined for $\forall t$ for quasi-symmetric spaces of functions on a non-atomic measure space. The lattice property of V implies that $\phi_V(t)$ is an increasing function. Finally, $\phi_V(t) > 0$ for $t > 0$ and $\phi_V(0) = 0$ which both follow from the properties of the quasi-norms. One can easily see that $\phi_{L^p} = t^{1/p}$. The fundamental function was first defined by Semenov [37].

Since many spaces of great interest such as $L^p(X, \mu)$ spaces are quasi-symmetric spaces, we will now show that two symmetric spaces of functions defined on the same σ -finite measure space (X, μ) will be a compatible couple. First we recall the concept of convergence in measure and one of its consequences.

Definition 1.6.11 *A sequence f_n of complex measurable functions on (X, μ) is said to converge in measure to f if for $\forall \varepsilon > 0$, $\exists N$ such that $\mu(\{x : |f_n(x) - f(x)| > \varepsilon\}) < \varepsilon$ for $\forall n > N$.*

Lemma 1.6.12 *Let f_n be a sequence of complex measurable functions on (X, μ) which converges in measure to f and let $E \subseteq X$ where $\mu(E) < \infty$. Then $\int_E \frac{|f_n - f|}{1 + |f_n - f|} d\mu \rightarrow 0$ as $n \rightarrow \infty$.*

Proof: Let N be as in previous definition. Fix $n > N$. Let $S = \{x : |f_n(x) - f(x)| > \varepsilon\}$; then $\mu(S) < \varepsilon$. We have $\int_E \frac{|f_n - f|}{1 + |f_n - f|} d\mu = \int_S \frac{|f_n - f|}{1 + |f_n - f|} d\mu + \int_{E \setminus S} \frac{|f_n - f|}{1 + |f_n - f|} d\mu$. The first integral on the R.H.S. is less than ε since $\mu(S) < \varepsilon$ and the integrand is non-negative and ≤ 1 . The second integral on the R.H.S. is $\leq \varepsilon \mu(E)$ since the integrand is $\leq \varepsilon$. The result follows. ■

Theorem 1.6.13 *Let (X, μ) be a non-atomic σ -finite measure space and V, W be quasi-symmetric spaces whose elements are measurable μ -a.e. finite real or complex valued functions on X , then (V, W) is a compatible couple.*

Proof: Let M be the set of all measurable μ -a.e. finite real or complex valued functions on X . Clearly, M is a linear space and V and W are subspaces of M . Now we construct a metric d on M . Since X is a σ -finite measure space, X is a countable union of subsets $\{E_n\}_{n=1}^{\infty}$ where $0 < \mu(E_n) < \infty$. Now let $d(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n \mu(E_n)} \int_{E_n} \frac{|f - g|}{1 + |f - g|} d\mu$ and this is a metric on M . Since M is metrizable, it is a Hausdorff topological space. Now suppose f_n is a sequence of functions in V with $\|f_n\|_V \rightarrow 0$ as $n \rightarrow \infty$. We now show that f_n converges to 0 in measure on the sets E_n . The proof is by contradiction. Suppose $\exists E \subseteq X$ with $0 < \mu(E) < \infty$ such that f_n does not converge to 0 in measure on E . Then $\exists f_{n_k}$ a subsequence of f_n

and $\varepsilon > 0$ such that $\mu(\{x \in E : |f_{n_k}| > \varepsilon\}) > \varepsilon$. Thus $f_{n_k}^* \geq \varepsilon \chi_{(0,\varepsilon)} \geq \chi_P^*$ where P is any measurable subset of E with $0 < \mu(P) < \varepsilon$. Since V is a quasi-symmetric space $\chi_P \in V$ and $0 < \varepsilon \|\chi_P\|_V \leq \|f_{n_k}\|_V$ which contradicts our assumption that $\|f_n\|_V \rightarrow 0$. Hence f_n converges in measure to 0 on all sets of finite measure which implies that $d(f_n, 0) \rightarrow 0$ and therefore V is continuously embedded in M . Similarly for W . Hence (V, W) is a compatible couple.

1.7 Quasilinear Operators

Much of analysis deals with linear operators. Interpolation theory is no exception. However there are cases where the theory can be extended to larger classes of operators. For this purpose we introduce the concept of a quasilinear operator.

Definition 1.7.1 *Let T be an operator whose domain and range are topological vector spaces whose elements are measurable functions on (S, λ) and (X, μ) respectively. Then if $\exists C \geq 1$ such that for $\forall f, g$ in the domain of T , $|T(f+g)(x)| \leq C(|Tf(x)| + |Tg(x)|)$ for $\mu - a.a. x \in X$, T is said to be a quasilinear operator. If in addition we can take $C = 1$, T is said to be sublinear.*

All linear operators are sublinear. We can define the bound (or norm) of a quasilinear operator in exactly the same way as we do for linear operators.

Definition 1.7.2 *Let T be a quasilinear operator and let V and W be quasi-normed spaces contained in the domain and range of T respectively. Then T is said to be a*

quasilinear operator from $V \rightarrow W$ with bound M (or norm M) if M is the smallest positive real number such that $\|Tf\|_W \leq M\|f\|_V$ for $\forall f \in V$.

The use of the word norm here comes from the theory of linear operators on Banach spaces where the norm actually is a norm (i.e. satisfies the axioms in Definition 1.1.2). In the general case, the word norm is often still used (as in [18]) but in this case the norm is not necessarily a norm. Hence, in this thesis, the word bound will be used.

We end this section by noting that if V and W are two quasi-normed spaces in the domain of T , $V + W$ and hence every intermediate space (Definition 1.4.8) of (V, W) is also in the domain of T .

Chapter 2

Interpolation and the K -method

2.1 Interpolation Spaces

Definition 2.1.1 *Let (A_0, A_1) and (B_0, B_1) both be compatible couples (as defined in Definition 1.4.3) of quasi-normed spaces. Then T is an admissible operator from (A_0, A_1) to (B_0, B_1) if T is an operator from $A_0 + A_1 \rightarrow B_0 + B_1$ with the added property that T maps $A_i \rightarrow B_i$ boundedly for both $i = 0$ and $i = 1$.*

Definition 2.1.2 *Let A and B be quasi-normed spaces and (A_0, A_1) and (B_0, B_1) be as above. Then (A, B) is said to be an interpolation pair for (A_0, A_1) and (B_0, B_1) if A is an intermediate space for (A_0, A_1) , B is an intermediate space for (B_0, B_1) and if every admissible linear operator from (A_0, A_1) to (B_0, B_1) maps $A \rightarrow B$ boundedly.*

Definition 2.1.3 *A is said to be an interpolation space for (A_0, A_1) if (A, A) is an interpolation pair for (A_0, A_1) and (A_0, A_1) .*

Many mathematicians have studied methods of generating interpolation pairs. In the remainder of this thesis we explore one of the more widely used of these methods; the K -method developed by J. Peetre in 1963.

2.2 The K -functional

Definition 2.2.1 [34] *Let (V, W) be a compatible couple of quasi-normed vector spaces and $f \in V + W$. Then $K(f, t, V, W) = \inf_{g+h=f} (\|g\|_V + t\|h\|_W)$ where the infimum is taken over $\forall g \in V, h \in W$ with $f = g + h$. $K(f, t, V, W)$ is called the K -functional.*

We sometimes write $K(f, t, V, W)$ as $K(f, t)$ if the identity of the spaces V and W are clear from the context.

Lemma 2.2.2 *Let (V, W) be a compatible couple of quasi-normed vector spaces. Then K -functional satisfies the inequality: $K(f, s, V, W) \leq \max(1, s/t)K(f, t, V, W)$ for $\forall f \in V + W$ and $\forall s, t > 0$. In particular $K(f, t)$ is an increasing function of t and $K(f, t)/t$ is a decreasing function of t .*

Proof: Now let $g \in V, h \in W$ with $f = g + h$. Suppose $s < t$, then $K(f, s, V, W) \leq \|g\|_V + s\|h\|_W \leq \|g\|_V + t\|h\|_W$. Taking the infimum over the R.H.S gives us $K(f, s) \leq K(f, t)$. Hence $K(f, t)$ is increasing. Now suppose $s > t$, then $K(f, s, V, W) \leq \|g\|_V + s\|h\|_W \leq (s/t)(\|g\|_V + t\|h\|_W)$. Taking the infimum over the R.H.S gives us

$K(f, s) \leq (s/t)K(f, t)$. Dividing both sides by s we see that $K(f, t)/t$ is decreasing.

■

Lemma 2.2.3 [1] *Let (V, W) be a compatible couple of quasi-normed (resp. normed) vector spaces and let $t > 0$. Then $K(\cdot, t, V, W)$ is a quasi-norm (resp. norm) of $V + W$ equivalent to the usual norm $\|\cdot\|_{V+W}$.*

Proof: It is easy to verify that $K(\cdot, t, V, W)$ is a quasi-seminorm (resp. seminorm).

From Lemma 2.2.2, we obtain $\min(1, t)\|f\|_{V+W} \leq K(f, t, V, W) \leq \max(1, t)\|f\|_{V+W}$

and thus $K(f, t) = 0$ implies $\|f\|_{V+W} = 0$ which by Lemma 1.4.5 implies $f = 0$ and

hence $K(\cdot, t, V, W)$ is a quasi-norm (resp. norm) equivalent to $\|\cdot\|_{V+W}$. ■

$K(f, t, V, W)$ is defined for $\forall f \in V + W$. If we restrict f to be in V or W we get the following bounds for the K -functional.

Proposition 2.2.4 $K(f, t, V, W) \leq \|f\|_V$ for $\forall f \in V$ and $K(f, t, V, W) \leq t\|f\|_W$ for $\forall f \in W$

Proof: This follows immediately from Definition 2.2.1. Use the decomposition $f = f + 0$.

Lemma 2.2.5 *Let (V, W) be a compatible couple of quasi-ideal spaces, $f, g \in V + W$ and $|g| \leq |f|$ a.e. then $K(g, t, V, W) \leq K(f, t, V, W)$.*

Proof: Let f, g be as above. Let $\eta(x) = \frac{g(x)}{f(x)}$ if $f(x) \neq 0$ and $\eta(x) = 0$ if $f(x) = 0$.

Then $|\eta(x)| \leq 1$ for a.a. x and $g(x) = \eta(x)f(x)$. Fix t, ε . Now $\exists h \in V, k \in W$ such that

$f = h + k$ and $\|h\|_V + t\|k\|_W \leq K(f, t, V, W) + \varepsilon$. Then $g(x) = \eta(x)f(x) = \eta(x)h(x) + \eta(x)k(x)$. Since $|\eta(x)h(x)| \leq |h(x)|$ and V is a quasi-ideal space, $\|\eta h\|_V \leq \|h\|_V$. Similarly, $\|\eta k\|_W \leq \|k\|_W$. Therefore $K(g, t, V, W) \leq \|\eta h\|_V + t\|\eta k\|_W \leq \|h\|_V + t\|k\|_W \leq K(f, t, V, W) + \varepsilon$. Since ε is arbitrary, we get $K(g, t, V, W) \leq K(f, t, V, W)$.

■

Corollary 2.2.6 *Let (V, W) be a compatible couple of quasi-ideal spaces, $f, g \in V + W$ and $|g| = |f|$ a.e. then $K(g, t, V, W) = K(f, t, V, W)$. More specifically $K(|f|, t, V, W) = K(f, t, V, W)$.*

Proof: Use previous lemma twice.

2.3 K -spaces and Interpolation

In this chapter, we examine the properties of K -spaces. All results except where otherwise noted can be found in [1] or [3].

Definition 2.3.1 *Let (V, W) be a compatible couple of quasi-normed spaces and let $0 < \theta < 1$ and $0 < q \leq \infty$. We define the space $(V, W)_{\theta, q} = \{f : f \in V + W \text{ such that } \|t^{-\theta}K(f, t, V, W)\|_{L^q(\frac{dt}{t})} < \infty\}$ (The definition of $\|f\|_{L^q(\frac{dt}{t})}$ was given in section 0.1).*

□

Theorem 2.3.2 *Let (V, W) be a compatible couple of quasi-normed spaces, $0 < \theta < 1$ and $0 < q \leq \infty$. Then $(V, W)_{\theta, q}$ is a quasi-normed space with the quasi-norm*

$\|f\|_{\theta,q} = \|t^{-\theta}K(f,t,V,W)\|_{L^q(\frac{dt}{t})}$. Furthermore, if V and W are normed spaces and $1 \leq q \leq \infty$ then $(V,W)_{\theta,q}$ is a normed space.

Proof: This follows from Lemma 2.2.3 and the fact $\|\cdot\|_{L^q(\frac{dt}{t})}$ is a quasi-norm for $0 < q \leq \infty$ and a norm for $1 \leq q \leq \infty$. ■

The members of these two parameter family of spaces are collectively called K -spaces. They satisfy the following important inclusion relations:

Lemma 2.3.3 *Let $0 < q < \infty$, then $(V,W)_{\theta,q} \subseteq (V,W)_{\theta,\infty}$ and $\|f\|_{\theta,\infty} \leq [q\theta(1-\theta)]^{1/q}\|f\|_{\theta,q}$ for $\forall f \in (V,W)_{\theta,q}$.*

Proof: Let $f \in (V,W)_{\theta,\infty}$.

Then $\|f\|_{\theta,q} = (\int_0^\infty [t^{-\theta}K(f,t)]^q \frac{dt}{t})^{1/q} = (\int_0^s [t^{-\theta}K(f,t)]^q \frac{dt}{t} + \int_s^\infty [t^{-\theta}K(f,t)]^q \frac{dt}{t})^{1/q}$

Using Lemma 2.2.2:

$$\begin{aligned} &\geq (\int_0^s [t^{-\theta}(t/s)K(f,s)]^q \frac{dt}{t} + \int_s^\infty [t^{-\theta}K(f,s)]^q \frac{dt}{t})^{1/q} \\ &= K(f,s)(s^{-q} \int_0^s t^{(1-\theta)q} \frac{dt}{t} + \int_s^\infty t^{-\theta q} \frac{dt}{t})^{1/q} = K(f,s)(\frac{s^{-\theta q}}{q(1-\theta)} + \frac{s^{-\theta q}}{q\theta})^{1/q} \\ &= K(f,s)s^{-\theta}[q\theta(1-\theta)]^{-1/q}. \end{aligned}$$

Take sup over $0 < s < \infty$ of the R.H.S. and we obtain our desired result. ■

Theorem 2.3.4 *Let $0 < q < r \leq \infty$, then $(V,W)_{\theta,q} \subseteq (V,W)_{\theta,r}$ and $\|f\|_{\theta,r} \leq [q\theta(1-\theta)]^{\frac{1}{q}-\frac{1}{r}}\|f\|_{\theta,q}$ for $\forall f \in (V,W)_{\theta,q}$.*

Proof: The case where $r = \infty$ was covered by the previous lemma. So let $r < \infty$,

then $\|f\|_{\theta,r} = (\int_0^\infty [t^{-\theta}K(f,t)]^r \frac{dt}{t})^{1/r} = (\int_0^\infty [t^{-\theta}K(f,t)]^{r-q} [t^{-\theta}K(f,t)]^q \frac{dt}{t})^{1/r}$.

$$\leq (\sup_{0 < s < \infty} K(f,s)s^{-\theta})^{1-q/r} (\|f\|_{\theta,q})^{q/r} = (\|f\|_{\theta,\infty})^{1-q/r} (\|f\|_{\theta,q})^{q/r}.$$

Now using Lemma 2.3.3,

$$\leq [q\theta(1-\theta)]^{\frac{1}{q}-\frac{1}{r}} \|f\|_{\theta,q}. \blacksquare$$

Theorem 2.3.5 *Let (V, W) be a compatible couple of quasi-normed spaces. Then $(V, W)_{\theta,q}$ is an intermediate space for (V, W) .*

Proof: First, let $f \in V \cap W$. Then from Lemma 2.2.4 we obtain the estimate $K(f, t) \leq \min(1, t) \|f\|_{V \cap W}$. Multiplying by $t^{-\theta}$ on both sides and applying the $L^q(\frac{dt}{t})$ norm on both sides, we get $\|f\|_{\theta,q} \leq \|t^{-\theta} \min(1, t)\|_{L^q(\frac{dt}{t})} \|f\|_{V \cap W}$. Since $\|t^{-\theta} \min(1, t)\|_{L^q(\frac{dt}{t})} < \infty$, $V \cap W$ is continuously embedded in $(V, W)_{\theta,q}$. Now let $f \in (V, W)_{\theta,q}$. If $q = \infty$, then $\|f\|_{\theta,\infty} = \sup_{t>0} t^{-\theta} K(f, t) \geq K(f, 1) = \|f\|_{V+W}$. If $q < \infty$, we use Lemma 2.3.3 to obtain $\|f\|_{V+W} \leq \|f\|_{\theta,\infty} \leq [q\theta(1-\theta)]^{1/q} \|f\|_{\theta,q}$. Hence $(V, W)_{\theta,q}$ is continuously embedded in $V + W$ and $(V, W)_{\theta,q}$ is an intermediate space for (V, W) . $\blacksquare$

Now we are ready to show that the K -method interpolates quasilinear operators. The following proof is original. This result is contained in the more general but less elementary results of Sagher [36] and Komatsu [23].

Theorem 2.3.6 *Let (A_0, A_1) and (B_0, B_1) be compatible couples of quasi-ideal spaces and let T be a quasi-linear operator from $A_0 + A_1 \rightarrow B_0 + B_1$ where T maps $A_i \rightarrow B_i$ with bound $M_i > 0$ when $i = 0$ or 1 . Then T maps $(A_0, A_1)_{\theta,q} \rightarrow (B_0, B_1)_{\theta,q}$ with bound $CM_0^{1-\theta} M_1^\theta$.*

Proof: In what follows C is a positive constant independent of t but not necessarily the same in all occurrences. Fix $t, \epsilon > 0$ and let $f \in A_0 + A_1$. Then $\exists g \in A_0, h \in A_1$

such that $f = g + h$ and $\|g\|_{A_0} + \frac{M_1}{M_0}t\|h\|_{A_1} \leq K(f, \frac{M_1}{M_0}t, A_0, A_1) + \epsilon$

Then $K(Tf, t, B_0, B_1) = K(|Tf|, t, B_0, B_1)$ using Corollary 2.2.6,

$\leq CK(|Tg| + |Th|, t, B_0, B_1)$ by the quasilinearity of T and Lemma 2.2.5,

$\leq C[K(|Tg|, t, B_0, B_1) + K(|Th|, t, B_0, B_1)]$

$= C[K(Tg, t, B_0, B_1) + K(Th, t, B_0, B_1)]$ using Corollary 2.2.6,

$\leq C[\|Tg\|_{B_0} + t\|Th\|_{B_1}]$ using Proposition 2.2.4,

$\leq C[M_0\|g\|_{A_0} + M_1t\|h\|_{A_1}]$

$\leq CM_0[K(f, \frac{M_1}{M_0}t, A_0, A_1) + \epsilon]$.

Since ϵ is arbitrary, we have $K(Tf, t, B_0, B_1) \leq CM_0K(f, \frac{M_1}{M_0}t, A_0, A_1)$.

Therefore $\|Tf\|_{(B_0, B_1)_{\theta, q}} = (\int_0^\infty [t^{-\theta}K(Tf, t, B_0, B_1)]^q \frac{dt}{t})^{1/q}$

$\leq CM_0(\int_0^\infty [t^{-\theta}K(f, \frac{M_1}{M_0}t, A_0, A_1)]^q \frac{dt}{t})^{1/q} = CM_0^{1-\theta}M_1^\theta\|f\|_{(A_0, A_1)_{\theta, q}}$. The last equal-

ity can be obtained by making a change of variables $s = \frac{M_1}{M_0}t$. ■

Since every linear operator is also quasi-linear, we have

Corollary 2.3.7 *Let (A_0, A_1) and (B_0, B_1) be compatible couples of quasi-ideal spaces.*

Then $((A_0, A_1)_{\theta, q}, (B_0, B_1)_{\theta, q})$ is an interpolation pair for (A_0, A_1) and (B_0, B_1) .

2.4 The Reiteration Theorem

All results in this section with the exceptions of Lemmas 2.4.2 and 2.4.3 are due to Tord Holmstedt and can be found in [18]. We have streamlined certain proofs. Let (V, W) be a compatible couple of quasi-normed spaces. Applying the K -method, we obtain a two parameter family of spaces $(V, W)_{\theta, q}$. Now choose two of these

spaces: $A_i = (V, W)_{\alpha_i, q_i}$ for $i = 0, 1$ where $0 < \alpha_0 < \alpha_1 < 1$ and $0 < q_0, q_1 \leq \infty$. Both A_0 and A_1 are continuously embedded in $V + W$ (proved in Theorem 2.3.5). Hence (A_0, A_1) is a compatible couple and we can apply the K -method on this new couple. The reiteration (or stability) theorem states that $(A_0, A_1)_{\theta, q} = (V, W)_{\eta, q}$ where $\eta = (1 - \theta)\alpha_0 + \theta\alpha_1$.

For ease of notation let $K(f, t) = K(f, t, V, W)$ and let $\bar{K}(f, t) = K(f, t, A_0, A_1)$. It is often very difficult to find the exact value of $\bar{K}(f, t)$. The K -method can still be used provided we can find $H(f, t) : (V + W) \times (0, \infty) \rightarrow \mathbb{R}$ with the property that $\forall f \in V + W$, $H(cf, t) = |c|H(f, t)$ and $C_1 H(f, t) \leq \bar{K}(f, t) \leq C_2 H(f, t)$, $\forall f \in V + W$, $\forall t > 0$ where C_1 and C_2 are positive constants. We write $K(f, t) \sim H(f, t)$ when $H(f, t)$ satisfies the above conditions. Then $\|t^{-\theta} H(f, t)\|_{L^q(\frac{ds}{s})}$ is a quasi-norm for the space $(A_0, A_1)_{\theta, q}$ and is equivalent to the usual quasi-norm (see Definition 1.1.4) for $(A_0, A_1)_{\theta, q}$. The following is a result of this type.

Theorem 2.4.1 (Holmstedt's Formulas) *Let V, W, A_0, A_1 be as above.*

Then $\bar{K}(f, t^{\alpha_1 - \alpha_0}) \sim (\int_0^t [s^{-\alpha_0} K(f, s)]^{q_0 \frac{ds}{s}})^{1/q_0} + t^{\alpha_1 - \alpha_0} (\int_t^\infty [s^{-\alpha_1} K(f, s)]^{q_1 \frac{ds}{s}})^{1/q_1}$

and $K(f, t^{\alpha_1}, V, A_1) \sim t^{\alpha_1} (\int_t^\infty [s^{-\alpha_1} K(f, s)]^{q_1 \frac{ds}{s}})^{1/q_1}$

and $K(f, t^{1 - \alpha_0}, A_0, W) \sim (\int_0^t [s^{-\alpha_0} K(f, s)]^{q_0 \frac{ds}{s}})^{1/q_0}$

All three of the above formulas are closely related. We will prove the first one and give indications in the text where the argument needs to be modified to prove the others.

Proof: In what follows C is a positive constant independent of t but not necessarily

the same in all its occurrences. Fix $t > 0$ and let $f \in A_0 + A_1$.

First we show that the L.H.S. $\leq C \cdot$ R.H.S. Since $A_0, A_1 \subseteq V + W$, $\exists g \in V$ and $h \in W$ such that $f = g + h$ and $\|g\|_V + t\|h\|_W \leq 2K(f, t)$.

Then by Lemma 2.2.4, $K(g, s) \leq \|g\|_V \leq 2K(f, t)$ and $K(h, s) \leq s\|h\|_W \leq 2\frac{s}{t}K(f, t)$.

Using Lemma 2.2.3, we obtain $K(g, s) = K(f - h, s) \leq C[K(f, s) + K(h, s)]$
 $\leq C[K(f, s) + \frac{s}{t}K(f, t)] = Cs[\frac{K(f, s)}{s} + \frac{K(f, t)}{t}]$. Now since $K(f, t)/t$ is a decreasing function of t , $K(g, s) \leq Cs[K(f, \tau)/\tau]$ where $\tau = \min(s, t)$. Hence we have $\|g\|_{A_0} = (\int_0^\infty [s^{-\alpha_0} K(g, s)]^{q_0} \frac{ds}{s})^{1/q_0} = C(\int_0^t [s^{-\alpha_0} K(f, s)]^{q_0} \frac{ds}{s} + \int_t^\infty [s^{-\alpha_0} \frac{s}{t} K(f, t)]^{q_0} \frac{ds}{s})^{1/q_0}$.

Now we consider the second term on the R.H.S. of the above inequality.

$$\begin{aligned} \int_t^\infty [s^{-\alpha_0} \frac{s}{t} K(f, t)]^{q_0} &= (\frac{K(f, t)}{t})^{q_0} \int_t^\infty [s^{1-\alpha_0}]^{q_0} \frac{ds}{s} \leq C(\frac{K(f, t)}{t} t^{1-\alpha_0})^{q_0} \\ &= C(\frac{K(f, t)}{t})^{q_0} \int_0^t (t^{1-\alpha_0})^{q_0} \leq C(\int_0^t [s^{-\alpha_0} K(f, s)]^{q_0} \frac{ds}{s}). \end{aligned}$$

Therefore we have $\|g\|_{A_0} \leq C(\int_0^t [s^{-\alpha_0} K(f, s)]^{q_0} \frac{ds}{s})^{1/q_0}$.

(In the case where A_0 is replaced by V we use $\|g\|_V \leq 2K(f, t)$)

$$\leq Ct^{\alpha_1} (\int_t^\infty [s^{-\alpha_1} K(f, t)]^{q_1} \frac{ds}{s})^{1/q_1} \leq Ct^{\alpha_1} (\int_t^\infty [s^{-\alpha_1} K(f, s)]^{q_1} \frac{ds}{s})^{1/q_1}.$$

Using Lemma 2.2.3, we obtain $K(h, s) = K(f - g, s) \leq C[K(f, s) + K(g, s)]$
 $\leq C[K(f, s) + K(f, t)] = C[K(f, \tau)]$ where $\tau = \max(s, t)$ since $K(f, t)$ is increasing.

$$\begin{aligned} \text{Hence we have } \|h\|_{A_1} &= (\int_0^\infty [s^{-\alpha_1} K(h, s)]^{q_1} \frac{ds}{s})^{1/q_1} \\ &= C(\int_0^t [s^{-\alpha_1} K(f, t)]^{q_1} \frac{ds}{s} + \int_t^\infty [s^{-\alpha_1} K(f, s)]^{q_1} \frac{ds}{s})^{1/q_1}. \end{aligned}$$

Now we consider the first term on the R.H.S. of the above inequality.

$$\begin{aligned} \int_0^t [s^{-\alpha_1} K(f, t)]^{q_1} \frac{ds}{s} &= [K(f, t)]^{q_1} \int_0^t [s^{-\alpha_1}]^{q_1} \frac{ds}{s} \leq C[K(f, t)]^{q_1} [s^{-\alpha_1}]^{q_1} \\ &\leq C(\int_t^\infty [s^{-\alpha_1} K(f, s)]^{q_1}). \end{aligned}$$

Hence we have $\|h\|_{A_1} \leq C(\int_t^\infty [s^{-\alpha_1} K(f, s)]^{q_1} \frac{ds}{s})^{1/q_1}$.

(In the case where A_1 is replaced by W we use $\|h\|_W \leq 2K(f, t)/t$
 $\leq Ct^{\alpha_0-1}(\int_0^t[s^{1-\alpha_0}K(f, t)/t]^{q_0}\frac{dt}{t})^{1/q_0} \leq Ct^{\alpha_0-1}(\int_0^t[s^{-\alpha_0}K(f, s)]^{q_0}\frac{ds}{s})^{1/q_0}$).

Therefore, we have $K(f, t^{\alpha_1-\alpha_0}, A_0, A_1) \leq \|g\|_{A_0} + t^{\alpha_1-\alpha_0}\|h\|_{A_1}$

$$\leq C[(\int_0^t[s^{-\alpha_0}K(f, s)]^{q_0}\frac{ds}{s})^{1/q_0} + t^{\alpha_1-\alpha_0}(\int_t^\infty[s^{-\alpha_1}K(f, s)]^{q_1}\frac{ds}{s})^{1/q_1}]$$

Therefore, L.H.S. $\leq C$ *R.H.S.

Now, we prove the reverse inequality. Fix $t, \epsilon > 0$. Since $f \in A_0 + A_1$, $\exists g \in A_0$, $h \in A_1$ such that $f = g + h$ and $\|g\|_{A_0} + t^{\alpha_1-\alpha_0}\|h\|_{A_1} \leq K(f, t^{\alpha_1-\alpha_0}, A_0, A_1) + \epsilon$. Using Lemma 2.2.3, $K(f, t) \leq C[K(g, t) + K(h, t)]$. Since $g \in A_0 = (V, W)_{\alpha_0, q_0} \subseteq (V, W)_{\alpha_0, \infty}$ (Last inclusion follows from Lemma 2.3.3), $\sup_{t>0} t^{-\alpha_0} K(g, t) \leq \infty$ and thus $K(g, t) \leq Ct^{\alpha_0}\|g\|_{A_0}$. Similarly $K(h, t) \leq Ct^{\alpha_1}\|h\|_{A_1}$. (For the second and third results we use instead $K(g, t) \leq \|g\|_V$ or $K(h, t) \leq t\|h\|_W$ respectively. The remainder of the argument is unchanged).

$$\begin{aligned} & \text{Therefore } (\int_0^t[s^{-\alpha_0}K(f, s)]^{q_0}\frac{ds}{s})^{1/q_0} \\ & \leq C[(\int_0^t[s^{-\alpha_0}K(g, s)]^{q_0}\frac{ds}{s})^{1/q_0} + (\int_0^t[s^{-\alpha_0}K(h, s)]^{q_0}\frac{ds}{s})^{1/q_0}] \\ & \leq C[(\int_0^\infty[s^{-\alpha_0}K(g, s)]^{q_0}\frac{ds}{s})^{1/q_0} + \|h\|_{A_1}(\int_0^t[s^{\alpha_1-\alpha_0}]^{q_0}\frac{ds}{s})^{1/q_0}] \\ & \leq C[\|g\|_{A_0} + t^{\alpha_1-\alpha_0}\|h\|_{A_1}] \leq C[K(f, t^{\alpha_1-\alpha_0}, A_0, A_1)] + \epsilon. \end{aligned}$$

$$\begin{aligned} & \text{Similarly } (\int_t^\infty[s^{-\alpha_1}K(f, s)]^{q_1}\frac{ds}{s})^{1/q_1} \\ & \leq C[(\int_t^\infty[s^{-\alpha_1}K(g, s)]^{q_1}\frac{ds}{s})^{1/q_1} + (\int_t^\infty[s^{-\alpha_1}K(h, s)]^{q_1}\frac{ds}{s})^{1/q_1}] \\ & \leq C[\|g\|_{A_0}(\int_t^\infty[s^{\alpha_0-\alpha_1}]^{q_1}\frac{ds}{s})^{1/q_1} + (\int_0^\infty[s^{-\alpha_1}K(h, t)]^{q_1}\frac{ds}{s})^{1/q_1}] \\ & \leq C[t^{\alpha_0-\alpha_1}\|g\|_{A_0} + \|h\|_{A_1}] \\ & \leq Ct^{\alpha_0-\alpha_1}[K(f, t^{\alpha_1-\alpha_0}, A_0, A_1)] + \epsilon. \end{aligned}$$

Since ε is arbitrary, $R.H.S. \leq C * L.H.S.$ ■

Before proving the reiteration theorem, we state the following well-known result.

Lemma 2.4.2 (Hardy's Inequalities) *Let $f(t)$ be a function from $(0, \infty) \rightarrow [0, \infty)$*

Let $\beta, t > 0$ and $1 < r \leq \infty$

$$\|t^{-\beta} \int_0^t f(s) \frac{ds}{s}\|_{L^r(\frac{dt}{t})} \leq \beta^{-1} \|t^{-\beta} f(t)\|_{L^r(\frac{dt}{t})}$$

$$\|t^{\beta} \int_t^{\infty} f(s) \frac{ds}{s}\|_{L^r(\frac{dt}{t})} \leq \beta^{-1} \|t^{\beta} f(t)\|_{L^r(\frac{dt}{t})}$$

If $0 < r < 1$ then the above inequalities are reversed. Furthermore, if $r = 1$ the inequalities become equalities.

The proof of Hardy's inequalities may be found in [12],[1],[30] or [33]. We also need the following inequalities.

Lemma 2.4.3 *Let $s > 0$, $0 < \theta < 1$ and $0 < p < q \leq \infty$, then the following inequalities hold*

$$\|\chi_{(0,s)} t^{-\theta} K(f, t)\|_{L^q(\frac{dt}{t})} \leq C \|\chi_{(0,s)} t^{-\theta} K(f, t)\|_{L^p(\frac{dt}{t})}$$

and

$$\|\chi_{(s,\infty)} t^{-\theta} K(f, t)\|_{L^q(\frac{dt}{t})} \leq C \|\chi_{(s,\infty)} t^{-\theta} K(f, t)\|_{L^p(\frac{dt}{t})}$$

Here C may depend on s, θ, p or q but is independent of f .

Proof: First we prove these inequalities when $q = \infty$. Now $\|\chi_{(0,s)} t^{-\theta} K(f, t)\|_{L^p(\frac{dt}{t})} \geq \frac{K(f,s)}{s} \|\chi_{(0,s)} t^{1-\theta}\|_{L^p(\frac{dt}{t})} = CK(f,s)s^{-\theta}$ where our first inequality follows from $\frac{K(f,t)}{t}$

being a decreasing function (Lemma 2.2.2). Now $\|\chi_{(0,s)} t^{-\theta} K(f, t)\|_{L^p(\frac{dt}{t})}$ is an increasing function of s , so for $t < s$ we have $\|\chi_{(0,s)} t^{-\theta} K(f, t)\|_{L^p(\frac{dt}{t})} \geq CK(f, t) t^{-\theta}$. If we take sup of R.H.S. over $\forall t < s$ we obtain our first inequality for $q = \infty$.

For the second inequality, we have $\|\chi_{(s,\infty)} t^{-\theta} K(f, t)\|_{L^p(\frac{dt}{t})} \geq K(f, s) \|\chi_{(s,\infty)} t^{-\theta}\|_{L^p(\frac{dt}{t})} = CK(f, s) s^{-\theta}$ where our first inequality follows from $K(f, t)$ being an increasing function (Lemma 2.2.2). Now $\|\chi_{(s,\infty)} t^{-\theta} K(f, t)\|_{L^p(\frac{dt}{t})}$ is an decreasing function of s , so for $t > s$ we have $\|\chi_{(0,s)} t^{-\theta} K(f, t)\|_{L^p(\frac{dt}{t})} \geq CK(f, t) t^{-\theta}$. If we take sup of R.H.S. over $\forall t > s$ we obtain our second inequality for $q = \infty$.

Now using an argument identical to Theorem 2.3.4, we can use the $q = \infty$ case to prove the inequalities for general q . ■

Now we ready to prove the reiteration theorem. This result is also due to Holmstedt.

Theorem 2.4.4 *Let V, W, A_0, A_1 be as above, then $(A_0, A_1)_{\theta, p} = (V, W)_{\eta, p}$ where $\eta = (1 - \theta)\alpha_0 + \theta\alpha_1$. Furthermore $C_1 \theta^{-\min(1/p, 1/q_0)} (1 - \theta)^{-\min(1/p, 1/q_1)} \|f\|_{(V, W)_{\eta, p}} \leq \|f\|_{(A_0, A_1)_{\theta, p}} \leq C_2 \theta^{-\max(1/p, 1/q_0)} (1 - \theta)^{-\max(1/p, 1/q_1)} \|f\|_{(V, W)_{\eta, p}}$ where C_1 and C_2 are positive constants which do not depend on θ .*

Proof: In the following C is a constant which is independent of θ but is not necessarily the same in all its occurrences. $\|f\|_{(A_0, A_1)_{\theta, p}} = (\int_0^\infty [s^{-\theta} K(f, s, A_0, A_1)]^p \frac{ds}{s})^{1/p} = C (\int_0^\infty [t^{-\theta(\alpha_1 - \alpha_0)} K(f, t^{\alpha_1 - \alpha_0}, A_0, A_1)]^p \frac{dt}{t})^{1/p}$ where we have made the substitution $t^{\alpha_1 - \alpha_0} = s$. The same change of variables can be made for the supremum in the

$p = \infty$ case. Now using Holmstedt's formula, we get:

$$\|f\|_{(A_0, A_1)_{\theta, p}} \sim \|t^{-\theta(\alpha_1 - \alpha_0)} [(\int_0^t [s^{-\alpha_0} K(f, s)]^{q_0} \frac{ds}{s})^{1/q_0} + t^{\alpha_1 - \alpha_0} (\int_t^\infty [s^{-\alpha_1} K(f, s)]^{q_1} \frac{ds}{s})^{1/q_1}]\|_{L^p(\frac{dt}{t})}$$

Using the triangle and reverse triangle inequalities for $L^p(\frac{dt}{t})$, we now obtain:

$$\|f\|_{(A_0, A_1)_{\theta, p}} \sim \|t^{-\theta(\alpha_1 - \alpha_0)} (\int_0^t [s^{-\alpha_0} K(f, s)]^{q_0} \frac{ds}{s})^{1/q_0}\|_{L^p(\frac{dt}{t})} + \|t^{(1-\theta)(\alpha_1 - \alpha_0)} (\int_t^\infty [s^{-\alpha_1} K(f, s)]^{q_1} \frac{ds}{s})^{1/q_1}\|_{L^p(\frac{dt}{t})}.$$

Let I_0 be the first term on the R.H.S. and let I_1 be the second term on the R.H.S.

$$\text{Then } I_0 = (\int_0^\infty [t^{-\theta(\alpha_1 - \alpha_0)} \int_0^t [s^{-\alpha_0} K(f, s)]^{q_0} \frac{ds}{s}]^{p/q_0} \frac{dt}{t})^{1/p}.$$

If $p \geq q_0$, we use the first of Hardy's Inequalities with $\beta = \theta(\alpha_1 - \alpha_0)$, $r = p/q_0$ and $f(s) = [s^{-\alpha_0} K(f, s)]^{q_0}$ to obtain $I_0 \leq ([\theta(\alpha_1 - \alpha_0)]^{-1} \int_0^\infty [t^{\theta(\alpha_1 - \alpha_0) - \alpha_0} K(f, t)]^p \frac{dt}{t})^{1/p}$.

Remembering that $\eta = (1 - \theta)\alpha_0 + \theta\alpha_1 = \theta(\alpha_1 - \alpha_0) - \alpha_0$, we see that the $I_0 \leq C\theta^{1/q_0} \|f\|_{(V, W)_{\eta, p}}$.

If $p < q_0$, then we first use Lemma 2.4.3 to get $I_0 \leq (\int_0^\infty [t^{-\theta(\alpha_1 - \alpha_0)} \int_0^t [s^{-\alpha_0} K(f, s)]^p \frac{ds}{s}] \frac{dt}{t})^{1/p}$.

Now we can use Hardy's Inequalities with β as in previous case, $r = 1$ and $f(s) = [s^{-\alpha_0} K(f, s)]^p$, we get $I_0 \leq C\theta^{1/p} \|f\|_{(V, W)_{\eta, p}}$.

Combining our two cases, we get $I_0 \leq C\theta^{-\max(1/q_0, 1/p)} \|f\|_{(V, W)_{\eta, p}}$.

We can also show that $I_0 \geq C\theta^{-\min(1/q_0, 1/p)} \|f\|_{(V, W)_{\eta, p}}$ using the same methods.

If $p \leq q_0$, we use Hardy's Inequality with $\beta = \theta(\alpha_1 - \alpha_0)$, $r = p/q_0$ and $f(s) = [s^{-\alpha_0} K(f, s)]^{q_0}$ to obtain $I_0 \geq ([\theta(\alpha_1 - \alpha_0)]^{-1} \int_0^\infty [t^{\theta(\alpha_1 - \alpha_0) - \alpha_0} K(f, t)]^p \frac{dt}{t})^{1/p}$.

Remembering that $\eta = (1 - \theta)\alpha_0 + \theta\alpha_1 = \theta(\alpha_1 - \alpha_0) - \alpha_0$, we see that the $I_0 \geq C\theta^{1/q_0} \|f\|_{(V, W)_{\eta, p}}$.

If $p > q_0$, we use Lemma 2.4.3 to get $I_0 \geq (\int_0^\infty [t^{-\theta(\alpha_1 - \alpha_0)} \int_0^t [s^{-\alpha_0} K(f, s)]^p \frac{ds}{s}] \frac{dt}{t})^{1/p}$.

Now we can use Hardy's Inequalities with β as in previous case, $r = 1$ and $f(s) = [s^{-\alpha_0} K(f, s)]^p$, we get $I_0 \geq C\theta^{1/p} \|f\|_{(V,W)_{\eta,p}}$.

Combining our two cases, we get $I_0 \geq C\theta^{-\min(1/q_0, 1/p)} \|f\|_{(V,W)_{\eta,p}}$.

Now we also have $I_1 = (\int_0^\infty [t^{(1-\theta)(\alpha_1-\alpha_0)} \int_t^\infty [s^{-\alpha_1} K(f, s)]^{q_1} \frac{ds}{s}]^{p/q_1} \frac{dt}{t})^{1/p}$.

Using the same methods as with I_0 , we can show that $I_1 \leq C\theta^{-\max(1/q_1, 1/p)} \|f\|_{(V,W)_{\eta,p}}$ and $I_1 \geq C\theta^{-\min(1/q_1, 1/p)} \|f\|_{(V,W)_{\eta,p}}$. The only differences here are that we use the second of Hardy's Inequalities instead of the first and that we take $\beta = (1-\theta)(\alpha_1 - \alpha_0)$. Adding our inequalities for I_0 and I_1 we obtain the required result. ■

Note that in all of the previous results, we have had $A_i = (V, W)_{\alpha_i, q_i}$ with $\alpha_0 < \alpha_1$. The reiteration theorem also works when $\alpha_1 < \alpha_0$; all we have to do is interchange the roles of A_0 and A_1 . One can easily see that $K(f, t, A_0, A_1) = tK(f, 1/t, A_1, A_0)$ and therefore multiplying by $t^{-\theta}$ and taking $L^q(\frac{dt}{t})$ of both sides we get $(A_0, A_1)_{\theta, q} = (A_1, A_0)_{1-\theta, q}$. Now let $\eta = (1-\theta)\alpha_0 + \theta\alpha_1$. Since $\alpha_1 < \alpha_0$ we can use Theorem 2.4.4 on the reversed couple (A_1, A_0) to obtain $(A_0, A_1)_{\theta, q} = (A_1, A_0)_{1-\theta, q} = (V, W)_{\eta, p}$ with equivalent norms where η is as before.

We used the first of Holmstedt's formulas in the proof of the reiteration theorem. Using the second and third of the formulas and using an argument identical to the usual reiteration theorem we get the following:

Theorem 2.4.5 *Let V, A_1 be as above, then $(V, A_1)_{\theta, p} = (V, W)_{\eta, p}$ where $\eta = \theta\alpha_1$ with equivalent norms and $C_1(1-\theta)^{-\min(1/p, 1/q_1)} \|f\|_{(V,W)_{\eta,p}} \leq \|f\|_{(V,A_1)_{\theta,p}} \leq C_2(1-\theta)^{-\max(1/p, 1/q_1)} \|f\|_{(V,W)_{\eta,p}}$ where C_1 and C_2 are positive constants which do not depend*

on θ .

Theorem 2.4.6 *Let A_0, W be as above, then $(A_0, W)_{\theta, p} = (V, W)_{\eta, p}$ where $\eta = (1 - \theta)\alpha_0 + \theta$ with equivalent norms and $C_1 \theta^{-\min(1/p, 1/q_0)} \|f\|_{(V, W)_{\eta, p}} \leq \|f\|_{(A_0, W)_{\theta, p}} \leq C_2 \theta^{-\max(1/p, 1/q_0)} \|f\|_{(V, W)_{\eta, p}}$ where C_1 and C_2 are positive constants which do not depend on θ .*

Chapter 3

Interpolation of L^p spaces

3.1 The Lorentz spaces

One of the goals of this chapter is to describe the interpolation spaces generated by the L^p spaces and then use the results to prove classical L^p interpolation theorems. Some of the K -spaces generated by a pair of L^p spaces are not themselves L^p spaces but lie in a larger family of spaces called the Lorentz spaces. The results in this section are from [30].

Definition 3.1.1 *Let (X, μ) be a σ -finite measure space and let $0 < p, q \leq \infty$; then $\|f\|_{L^{p,q}} = (\frac{q}{p} \int_0^\infty [t^{1/p} f^*(t)]^q \frac{dt}{t})^{1/q}$ if $q < \infty$ and $\|f\|_{L^{p,q}} = \sup_{t>0} t^{1/p} f^*(t)$ when $q = \infty$. Then the set $L^{p,q}$ consists of all the μ -measurable functions f on X with $\|f\|_{L^{p,q}} < \infty$.*

Note that when $p = q$, we have $\|f\|_{L^{p,p}} = (\int_0^\infty [f^*(t)]^p dt)^{1/p} = \|f\|_{L^p}$ (second equality comes from Theorem 1.6.3). Therefore L^p spaces are all realizable as Lorentz spaces.

The following proposition shows that $L^{p,\infty}$ is essentially weak L^p (Definition 1.5.5).

Proposition 3.1.2 $\sup_{t>0} t^{1/p} f^*(t) = \sup_{s>0} s(\mu_f(s))^{1/p}$.

This follows from Proposition 1.5.8.e. Let $t = \mu_f(s)$, then $f^*(t) \leq s$. Hence $t^{1/p} f^*(t) \leq s(\mu_f(s))^{1/p}$. Taking supremums of both sides, we get L.H.S. $\leq$ R.H.S. Now let $s = f^*(t)$, then $\mu_f(s) \leq t$. Hence L.H.S. $\geq$ R.H.S. ■

Many of the properties of Lorentz spaces will follow from those of K -spaces once we prove that $(L^{p_0,q_0}, L^{p_1,q_1})_{\theta,q} = L^{p,q}$ where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. In order to apply this theory, we must show that $L^{p,q}$ are quasi-normed spaces.

Theorem 3.1.3 *The $L^{p,q}$ spaces are quasi-normed spaces.*

Homogeneity follows from Proposition 1.5.8.d. Now we prove the triangle inequality.

For what follows let $s = 2t$.

First let $q = \infty$. Then using Proposition 1.5.8.g, $\|f + g\|_{L^{p,\infty}} = \sup_{t>0} t^{1/p} (f + g)^*(t) \leq \sup_{t>0} t^{1/p} (f^*(t/2) + g^*(t/2)) \leq \sup_{t>0} (2s)^{1/p} (f^*(s) + g^*(s)) = 2^{1/p} (\|f\|_{L^{p,\infty}} + \|g\|_{L^{p,\infty}})$.

Secondly let $q < \infty$, Then using Proposition 1.5.8.g, $\|f + g\|_{L^{p,q}} = (\int_0^\infty [t^{1/p} (f + g)^*(t)]^q \frac{dt}{t})^{1/q} \leq (\int_0^\infty [t^{1/p} f^*(t/2) + t^{1/p} g^*(t/2)]^q \frac{dt}{t})^{1/q} = (\int_0^\infty [(2s)^{1/p} f^*(s) + (2s)^{1/p} g^*(s)]^q \frac{ds}{s})^{1/q} \leq 2^{1/p + [1/q - 1]} ((\int_0^\infty [s^{1/p} f^*(s)]^q \frac{ds}{s})^{1/q} + (\int_0^\infty [s^{1/p} g^*(s)]^q \frac{ds}{s})^{1/q}) = 2^{1/p + [1/q - 1]} (\|f\|_{L^{p,q}} + \|g\|_{L^{p,q}})$.

$\|g\|_{L^{p,q}}$. Now $\|f\|_{L^{p,q}} = 0$ implies $f^* = 0$ and hence $f = 0$. Hence $L^{p,q}$ are quasi-normed spaces. ■

It is not too hard to see that, in fact, the $L^{p,q}$ spaces are quasi-symmetric spaces. One can compute the fundamental function of these spaces (Definition 1.6.10). For $p < \infty$ or $p = q = \infty$ we obtain $\phi_{L^{p,q}}(t) = t^{1/p}$. However, when we consider the case where $p = \infty$ and $q < \infty$ we find something quite different. Let $E \subseteq X$ with $\mu(E) = t < \infty$, then $\|\chi_E\|_{L^{\infty,q}} = (\int_0^\infty \chi_{(0,t)}(s) \frac{ds}{s})^{1/q} = \infty$. Hence $\chi_E \notin L^{p,q}$ and by Theorem 1.6.7, $L^{\infty,q}$ contains only functions which are zero a.e. We exclude these degenerate cases in what follows.

3.2 $K(f, t, A, L^\infty)$

Lemma 3.2.1 *Let (X, μ) be a non-atomic σ -finite measure space. Let A be a quasi-symmetric space whose elements consist of μ -a.e. finite functions on X . We take L^∞ to be $L^\infty(X, \mu)$. Let $f \in A + L^\infty$ and $E \subseteq X$ with $\mu(E) < \infty$ then $f\chi_E \in A$.*

Proof: If $f \in A + L^\infty$, then $\exists g \in A, h \in L^\infty$ such that $f = g + h$. First we note that A is a quasi-ideal space and therefore $g\chi_E \in A$, since $g \in A$ and $|g\chi_E(t)| \leq |g(t)|$. Furthermore, $|h\chi_E(t)| \leq \|h\|_{L^\infty} |\chi_E(t)|$ and $\chi_E \in A$ (Theorem 1.6.7) which gives us $h\chi_E \in A$. Since $f\chi_E = g\chi_E + h\chi_E$, $f\chi_E \in A$. ■

Before continuing, we recall the content of Definitions 1.5.1 and 1.5.2:

$$S_f(s) = \{x \in X : |f(x)| > s\} \text{ and } \mu_f(s) = \mu(S_f(s)).$$

$$\text{Also } S_f(s^-) = \lim_{u \rightarrow s^-} S_f(u) = \{x \in X : |f(x)| \geq s\}.$$

Definition 3.2.2 We define $M_f(t)$ as follows. Suppose $\exists s > 0$ such that $\mu_f(s) = t$ then let $M_f(t) = S_f(s)$. If no such s exists, take $s = f^*(t) = \inf\{\mu_f(s) \leq t\}$. Then $\mu(S_f(s)) < t$ and $\mu(S_f(s^-)) \geq t$. Then let $M_f(t)$ be a set of μ -measure t with $S_f(s) \subset M_f(t) \subseteq S_f(s^-)$.

In the second case above, our choice of $M_f(t)$ may not be unique. However, the $f\chi_{M_f(t)}$ are equimeasurable over all choices of $M_f(t)$ and hence the quantity $\|f\chi_{M_f(t)}\|_A$ which is needed for the following results is well defined. The following is due to Krée [25] (see also [2]).

Theorem 3.2.3 Let A, L^∞ and f be as in the previous lemma, let $\phi_A(t)$ be the fundamental function (Definition 1.6.10) of A and let c_A, c'_A be the constants in the c -triangle inequality and the reverse triangle inequality for A respectively. Then $(c_A)^{-1}\|f\chi_{M_f(t)}\|_A \leq K(f, \phi_A(t), A, L^\infty) \leq c'_A\|f\chi_{M_f(t)}\|_A$ for $\forall f \in A + L^\infty$.

Proof: Fix $t > 0$. Let $f \in A + L^\infty$, then $\exists g \in A, h \in L^\infty$ with $f = g + h$ and $\|g\|_A + \phi_A(t)\|h\|_{L^\infty} \leq K(f, \phi_A(t), A, L^\infty) + \varepsilon$.

Now using the c -triangle inequality, $(c_A)^{-1}\|f\chi_{M_f(t)}\| \leq \|g\chi_{M_f(t)}\|_A + \|h\chi_{M_f(t)}\|_A \leq \|g\|_A + \|h\|_{L^\infty}\|\chi_{M_f(t)}\|_A$ since $|g\chi_{M_f(t)}| \leq |g|, |h\chi_{M_f(t)}| \leq \|h\|_{L^\infty}|\chi_{M_f(t)}|$.

Hence $(c_A)^{-1}\|f\chi_{M_f(t)}\| \leq \|g\|_A + \phi_A(t)\|h\|_{L^\infty} \leq K(f, \phi_A(t), A, L^\infty) + \varepsilon$. Since ε is arbitrary we have the L.H.S. of the required inequalities.

Now we prove the R.H.S. Using Corollary 2.2.6, we may assume without loss of generality that $f \geq 0$. Now let $g(x) = [f(x) - f^*(t)]\chi_{M_f(t)}(x)$ and let $h(x) =$

$f(x) - g(x)$. Then $h(x) = \min(f(x), f^*(t))$ and $\|h\|_{L^\infty} = f^*(t)$. Furthermore since $f(x) \in A + L^\infty$ and $f^*(t) \in L^\infty \subseteq A + L^\infty$, $f(x) - f^*(t) \in A + L^\infty$ and by Lemma 3.2.1, $g(x) = [f(x) - f^*]\chi_{M_f(t)} \in A$.

$$\begin{aligned} \text{Thus } K(f, \phi_A(t), A, L^\infty) &\leq \|g\|_{A+\phi_A(t)} \|h\|_{L^\infty} \leq \|g\chi_{M_f(t)}\|_A + \|\chi_{M_f(t)}\|_A \|h\|_{L^\infty} \leq \\ &\|g\chi_{M_f(t)}\|_A + f^*(t) \|\chi_{M_f(t)}\|_A \leq \|[f(x) - f^*(t)]\chi_{M_f(t)}\|_A + \|f^*(t)\chi_{M_f(t)}\|_A \leq c'_A \|f\chi_{M_f(t)}\|_A. \end{aligned}$$

The last inequality follows from the reverse triangle inequality (Lemmas 1.2.4 and 1.3.4).

$$\text{Hence we have } (c_A)^{-1} \|f\chi_{M_f(t)}\|_A \leq K(f, \phi_A(t), A, L^\infty) \leq c'_A \|f\chi_{M_f(t)}\|_A. \blacksquare$$

Now we can substitute $\|f\chi_{M_f(t)}\|_A$ for $K(f, t, A, L^\infty)$ and obtain an equivalent norm for $(A, L^\infty)_{\theta, q}$. The following result gives us the decreasing rearrangement of $f\chi_{M_f(t)}$.

Proposition 3.2.4 *For fixed $t > 0$, $(f\chi_{M_f(t)})^* = f^*\chi_{(0,t)}$.*

Proof: Let $g = f\chi_{M_f(t)}$ and $h = f^*\chi_{(0,t)}$. Then h is clearly decreasing and right-continuous. When $S_f(s) < t$, $S_g(s) = S_f(s) = S_h(s)$ and when $S_f(s) \geq t$, $S_g(s) = t = S_h(s)$. Hence g and h are equicontinuous and by Lemma 1.5.9, $g^* = h$. $\blacksquare$

For $A = L^1$, we obtain an especially nice simplification of Theorem 3.2.3. Here $c_{L^1} = c'_{L^1} = 1$ giving us the following result due to Peetre [34]:

Corollary 3.2.5 *For $\forall f \in L^1 + L^\infty, t > 0$; $K(f, t, L^1, L^\infty) = \int_0^t f^*(s) ds$.*

In the theory above, we restricted X to be a non-atomic σ -finite measure space. In some applications, we may want X to be $\mathbb{Z}$ with counting measure. We sketch

what can be done in this case. Let $f(n)$ be a function on $\mathbb{Z}$, then let $f'(t) = \sum_{i \in \mathbb{N}} f(i) \chi_{[i, i+1)}(t)$. For Lorentz spaces, we now have $\|f'\|_{L^{p,q}(\mathbb{R})} = \|f\|_{L^{p,q}(\mathbb{Z})}$. Hence we have $K(f, t^{1/p}, l^{p,q}, l^\infty) \sim \|f' \chi_{M_{f'}(t)}\|_{L^{p,q}(\mathbb{R})}$.

3.3 Interpolation of Lorentz spaces

If A is an $L^{p,q}$ space we can say more. Since $(f \chi_{M_f(t)})^* = f^* \chi_{(0,t)}$, we have $\|f \chi_{M_f(t)}\|_{L^{p,q}} = (\int_0^t [s^{1/p} f^*(s)]^q \frac{ds}{s})^{1/q_0}$. Using this we can obtain the following result which will allow us to interpolate Lorentz spaces.

Theorem 3.3.1 *Let $0 < \theta < 1$ and let L^{p_0, q_0} and L^{p_1, q_1} be two nonempty Lorentz spaces with $p_0 \neq p_1$. Then $(L^{p_0, q_0}, L^{p_1, q_1})_{\theta, q} = L^{p, q}$ where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. Furthermore $\exists c, C > 0$ such that $c\theta^{-\min(1/q, 1/q_0)}(1-\theta)^{-\min(1/q, 1/q_1)}\|f\|_{L^{p,q}} \leq \|f\|_{(L^{p_0, q_0}, L^{p_1, q_1})_{\theta, q}} \leq C\theta^{-\max(1/q, 1/q_0)}(1-\theta)^{-\max(1/q, 1/q_1)}\|f\|_{L^{p,q}}$ for $\forall f \in L^{p,q}$.*

Proof: Now let us suppose that $p_i < \infty$ for $i = 0, 1$. Choose r such that $0 < r < p_i, q_i$ for $i = 0, 1$. Theorem 3.2.3 gives us $K(f, t^{1/r}, L^r, L^\infty) \sim (\int_0^t (f^*(s))^r ds)^{1/r}$. Making the substitution $\tau = t^{1/r}$, we obtain $K(f, \tau, L^r, L^\infty) \sim (\int_0^{\tau^r} (f^*(s))^r ds)^{1/r}$. Therefore $\|f\|_{(L^r, L^\infty)_{\theta, q}} \sim (\int_0^\infty (\tau^{-\theta r} \int_0^{\tau^r} (f^*(s))^r ds)^{q/r} \frac{d\tau}{\tau})^{1/q}$. Now let $t = \tau^r$. This gives us $\|f\|_{(L^r, L^\infty)_{\theta, q}} \sim (\int_0^\infty (t^{-\theta} \int_0^t s (f^*(s))^r \frac{ds}{s})^{q/r} \frac{dt}{t})^{1/q}$.

Now use Hardy's Inequality (Lemma 2.4.2). Let $p = \frac{r}{1-\theta}$. For $q > r$, we get $R.H.S. \leq C(\int_0^\infty [t^{\frac{1-\theta}{r}} f^*(s)]^q \frac{ds}{s})^{1/q} = C\|f\|_{L^{p,q}}$. Since f^* is a decreasing function, we have $\int_0^t [f^*(s)]^r ds \geq t[f^*(t)]^r$ and therefore $R.H.S \geq (\int_0^\infty [t^{\frac{1-\theta}{r}} f^*(s)]^q \frac{ds}{s})^{1/q} =$

$c\|f\|_{L^{p,q}}$. Therefore $c\|f\|_{L^{p,q}} \leq \|f\|_{(L^r, L^\infty)_{\theta,q}} \leq C\|f\|_{L^{p,q}}$ and $(L^r, L^\infty)_{\theta,q} = L^{p,q}$ when $q > r$ and $p = \frac{r}{1-\theta}$. Now since $p_i > r$, $\exists \theta_i \in (0, 1)$ such that $p_i = \frac{r}{1-\theta_i}$ and $q_i > r$ gives us $L^{p_i, q_i} = (L^r, L^\infty)_{\theta_i, q_i}$ for $i = 0, 1$. Now we can use the reiteration theorem (Theorem 2.4.4) to obtain our result.

Now let one of the p_i be infinite. Without loss of generality let $p_0 < \infty$ and let $p_1 = \infty$. Hence the pair we are interpolating is (L^{p_0, q_0}, L^∞) . We follow the same procedure as above except that we use Theorem 2.4.6 instead of Theorem 2.4.4. ■

Proposition 3.3.2 $L^{p,q} \subseteq L^{p,r}$, $0 < p < \infty$ and $0 < q \leq r \leq \infty$

Proof: Using Theorems 3.3.1 and 2.3.4, we obtain $L^{p,q} = (L^\alpha, L^\beta)_{\theta,q} \subseteq (L^\alpha, L^\beta)_{\theta,r} = L^{p,r}$ for properly chosen α, β and θ . ■

Now we can prove the following interpolation theorem for Lorentz spaces. This proof follows that of a similar result in [18].

Theorem 3.3.3 Let L^{p_i, r_i} and L^{q_i, s_i} where $i = 0$ or 1 be non-empty Lorentz spaces. Let T be a quasilinear operator from $L^{p_i, r_i} \rightarrow L^{q_i, s_i}$ with bound M_i for $i = 0$ or 1 and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ and $r \leq s$. Then T maps $L^{p,r} \rightarrow L^{q,s}$ with bound M where $M \leq CM_0^{1-\theta} M_1^\theta \theta^{-([1/s-1/s_0]_+ + [1/r_0-1/r]_+)} (1-\theta)^{-([1/s-1/s_1]_+ + [1/r_1-1/r]_+)}$ and C is independent of θ .

Proof: Using our main interpolation result (Theorem 2.3.6), we see that M is an operator from $(L^{p_0, r_0}, L^{p_1, r_1})_{\theta, r} \rightarrow (L^{q_0, s_0}, L^{q_1, s_1})_{\theta, r}$ with bound $\leq CM_0^{1-\theta} M_1^\theta$ or to put it in the form of an inequality: $\|Tf\|_{(L^{q_0, s_0}, L^{q_1, s_1})_{\theta, r}} \leq CM_0^{1-\theta} M_1^\theta \|f\|_{(L^{p_0, r_0}, L^{p_1, r_1})_{\theta, r}}$. Now

by Theorem 2.3.4, we have $\|Tf\|_{(L^{q_0, s_0}, L^{q_1, s_1})_{\theta, s}} \leq [\theta(1 - \theta)]^{1/r - 1/s} \|Tf\|_{(L^{q_0, s_0}, L^{q_1, s_1})_{\theta, r}} \leq CM_0^{1-\theta} M_1^\theta [\theta(1 - \theta)]^{1/r - 1/s} \|f\|_{(L^{p_0, r_0}, L^{p_1, r_1})_{\theta, r}}.$

Now by Theorem 3.3.1, $L^{q, s} = (L^{q_0, s_0}, L^{q_1, s_1})_{\theta, s}$ and $\|Tf\|_{L^{q, s}} \leq C\theta^{\min(1/s, 1/s_0)}(1 - \theta)^{\min(1/s, 1/s_1)} \|Tf\|_{(L^{q_0, s_0}, L^{q_1, s_1})_{\theta, s}}.$ Similarly $L^{p, r} = (L^{p_0, r_0}, L^{p_1, r_1})_{\theta, r}$ and $\|f\|_{(L^{p_0, r_0}, L^{p_1, r_1})_{\theta, r}} \leq C\theta^{-\max(1/r, 1/r_0)}(1 - \theta)^{-\max(1/r, 1/r_1)} \|f\|_{L^{p, r}}.$ Combining the last three inequalities, we have $\|f\|_{L^{q, s}} \leq CM_0^{1-\theta} M_1^\theta \theta^{-([1/s - 1/s_0]_+ + [1/r_0 - 1/r]_+)} (1 - \theta)^{-([1/s - 1/s_1]_+ + [1/r_1 - 1/r]_+)} \|f\|_{L^{p, r}}$ and hence our result is proved. ■

We can use the preceding theorem to get a sharper version of the Hausdorff-Young Inequality which is due to Payley [32].

Theorem 3.3.4 *Let $f \in L^p(\mathbb{T})$ for $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then $\hat{f} \in l^{p', p}(\mathbb{Z})$ and $\exists c_p > 0$ such that $\|\hat{f}\|_{l^{p', p}(\mathbb{Z})} \leq c_p \|f\|_{L^p(\mathbb{T})}.$*

Proof: Let $\mathcal{F}$ be the linear operator taking f to $\hat{f}$. Then $\mathcal{F} : L^1 \rightarrow l^\infty$ and $\mathcal{F} : L^2 \rightarrow l^2$ (see the proof of Theorem 0.2.2). Now using previous theorem with $p_0 = r_0 = 1, q_0 = s_0 = \infty, p_1 = q_1 = r_1 = s_1 = 2$ and $r = s = p$ we obtain the above result. ■

Since for $1 \leq p \leq 2, p \leq p', L^{p', p} \subseteq L^{p'}$ (Proposition 3.3.2). Hence Payley's result is indeed stronger than the Hausdorff-Young Inequality. It is interesting to note that Payley published this result almost two decades before Lorentz formally introduced the Lorentz spaces.

Theorem 3.3.3 can be used to derive many of the classical results of interpolation. We start by deriving the Marcinkiewicz Interpolation Theorem.

Theorem 3.3.5 (Marcinkiewicz Interpolation Theorem) Let $0 < p_i \leq q_i \leq \infty$. Let T be a quasilinear operator from $L^{p_i} \rightarrow L^{q_i, \infty}$ with bound M_i for $i = 0, 1$; and let $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ where $0 < \theta < 1$. Then T is a quasilinear operator from $L^p \rightarrow L^q$ with bound M where $M \leq CM_0^{1-\theta} M_1^\theta [\theta(1-\theta)]^{-1/q}$ and C is independent of θ .

Proof: Let T satisfy the above hypotheses. Using Theorem 3.3.3 with $r_i = p_i$, $r = p$, $s_i = \infty$ and $s = q$ for $i = 0, 1$, we find that T is a quasilinear operator from $L^p \rightarrow L^q$ with bound $\leq CM_0^{1-\theta} M_1^\theta \theta^{-(1/q + [1/p_0 - 1/p]_+)} (1-\theta)^{-(1/q + [1/p_1 - 1/p]_+)}$. Since p is between p_0 and p_1 , one of the terms of the form $[1/p_i - 1/p]_+$ is zero and will disappear. Now we consider the other term. $(1/p_i - 1/p) = \frac{1}{p_i} - \frac{(1-\theta)}{p_0} - \frac{\theta}{p_1}$. Hence if $i = 0$, $\theta^{1/p_0 - 1/p} = \theta^{(\theta(1/p_0 - 1/p_1))}$ and a simple calculus argument shows that this tends to 1 as $\theta \rightarrow 0$ and hence is bounded for $\theta \in (0, 1)$. Therefore, we can absorb this term into the C . Now similarly, if $i = 1$, then $(1-\theta)^{1/p_1 - 1/p} = (1-\theta)^{(1-\theta)(1/p_1 - 1/p_0)}$ and again this tends to 1 as $\theta \rightarrow 1$ and hence is also bounded for $\theta \in (0, 1)$. We also absorb this into the C . What we are left with is $M \leq CM_0^{1-\theta} M_1^\theta [\theta(1-\theta)]^{-1/q}$ exactly as required. ■

Often the only L^p spaces that are considered are the Banach space cases where $1 \leq p \leq \infty$. In this case $1/q \leq 1$ and we get the version of the Marcinkiewicz Interpolation theorem most often found in the literature.

Theorem 3.3.6 [40] Let $1 \leq p_i \leq q_i \leq \infty$. Let T be a quasilinear operator from $L^{p_i} \rightarrow L^{q_i, \infty}$ with bound M_i for $i = 0, 1$; and let $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ where

$0 < \theta < 1$. Then T is a quasilinear operator from $L^p \rightarrow L^q$ with bound M where $M \leq CM_0^{1-\theta} M_1^\theta [\theta(1-\theta)]^{-1}$ and C is independent of θ .

We may also state a version of the Riesz Convexity Theorem. While the bound is less sharp than in the original, our version also works when $p < 1$ and when T is quasi-linear and not just linear.

Theorem 3.3.7 Let $0 < p_i \leq q_i \leq \infty$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. Then if T is a quasilinear operator from $L^{p_i} \rightarrow L^{q_i}$ with bound M_i for $i = 0, 1$, T also is a quasilinear operator from $L^p \rightarrow L^q$ with the bound M where $M \leq CM_0^{1-\theta} M_1^\theta$ and C is independent of θ .

Proof: Let T satisfy the above hypotheses. Then Theorem 3.3.3 with $r_i = p_i$, $r = p$, $s_i = q_i$ and $s = q$ implies that T is a quasilinear operator from $L^p \rightarrow L^q$ with bound $M \leq CM_0^{1-\theta} M_1^\theta \theta^{-([1/q-1/q_0]_+ + [1/p_0-1/p]_+)} (1-\theta)^{-([1/q-1/q_1]_+ + [1/p_1-1/p]_+)}$. Using similar arguments as in the proof of Theorem 3.3.5, we can show that the powers of θ and $1-\theta$ are bounded for $\forall \theta \in (0, 1)$ and hence we absorb them into the constant C . The conclusion follows. ■

The proofs of Theorem 3.3.5 and 3.3.7 are due to Holmstedt [18]. His results were stated only for linear operators. Using Theorem 2.3.6, we have extended these results to quasilinear operators and are thus able to obtain the Marcinkiewicz interpolation theorem in full generality. It may be of interest to note that Calderón and Zygmund have also extended Riesz's result in these directions using complex-variable and sub-

harmonic function theory to prove results for the quasi-normed L^p spaces in [9] and for sublinear operators in [10].

3.4 The E -functional

In this section we explore the properties of the E -functional and use them to prove a result of Hardy, Littlewood and Pólya.

Definition 3.4.1 *Let (V, W) be a compatible couple of quasi-normed vector spaces. Then we define the E -functional as follows. For $\forall f \in V + W$, let $E(f, t, V, W) = \inf\{\|h\|_W : f = g + h, \|g\|_V \leq t, g \in V, h \in W\}$.*

In [26], Lorentz and Shimogaki write $s_f(t)$ instead of $E(f, t, V, W)$. We use the latter notation as it is much more commonly used. The main application of the E -functional is in approximation theory (see chapter 7 of [3]). We will not discuss the applications to approximation theory here; instead we will explore connections between the E -functional and the K -functional.

Theorem 3.4.2 *Let (V, W) be a compatible couple of quasi-normed spaces. Then $\forall t > 0$, $K(f, t, V, W) = \inf_{\alpha > 0}\{E(f, \alpha, W, V) + \alpha t\}$.*

Proof: Fix $t, \varepsilon > 0$. Now $\exists g \in V, h \in W$ such that $K(f, t, V, W) \geq \|g\|_V + t\|h\|_W + \varepsilon$. If we take s such that $s - \frac{\varepsilon}{t} \leq \|h\|_W \leq s$; then $\|g\|_V \geq E(f, s, W, V)$. Hence $K(f, t, V, W) \geq E(f, s, W, V) + st - 2\varepsilon \geq \inf_{\alpha > 0}\{E(f, \alpha, W, V) + \alpha t\} + 2\varepsilon$. Since ε is arbitrary, we have $K(f, t, V, W) \geq \inf_{\alpha > 0}\{E(f, \alpha, W, V) + \alpha t\}$.

Now we prove the converse inequality. Again fix $t, \varepsilon > 0$. $\exists s > 0$ such that $\inf_{\alpha > 0} \{E(f, \alpha, W, V) + \alpha t\} \geq E(f, s, W, V) + st - \varepsilon$. Now choose $g \in V$ and $h \in W$ such that $\|h\|_W \leq s$, $f = g + h$ and such that $\|g\|_V \leq E(f, s, W, V) + \varepsilon$. Then $E(f, s, W, V) + st \geq \|g\|_V + t\|h\|_W - \varepsilon \geq K(f, t, V, W) - \varepsilon$. Since ε is arbitrary we have $K(f, t, V, W) \leq \inf_{\alpha > 0} \{E(f, \alpha, W, V) + \alpha t\}$. ■

Therefore, we can obtain $K(f, t, V, W)$ from $E(f, t, W, V)$ using a variant of the Legendre transform. Now we attempt to solve for the E -functional for certain pairs of quasi-normed spaces. Just as in the case of the K -functional, it will turn out to be easier when one of our spaces is L^∞ . We have the following theorem.

Theorem 3.4.3 *Let A be a quasi-ideal space whose elements are measurable functions on the measure space (X, μ) . Let $L^\infty = L^\infty(X, \mu)$ and furthermore let (A, L^∞) be a compatible couple. Then $E(f, t, L^\infty, A) = \|(|f| - t)_+\|_A$ for $\forall t > 0$.*

Proof: Fix $t > 0$, then define f_t as $f_t(x) = f(x)$ if $|f(x)| \leq t$ and $f_t(x) = s \frac{f(x)}{|f(x)|}$. Note that $\|f_t\|_{L^\infty} \leq t$. Now suppose $\exists g \in L^\infty$ such that $\|g\|_{L^\infty} \leq t$. Then it is easy to see that $|f(x) - f_t(x)| \leq |f(x) - g(x)|$ and that $|f - f_t| = (|f| - t)_+$. Hence using the lattice property and Lemma 1.3.2 we obtain $E(f, t, L^\infty, A) = \inf_{\|g\|_{L^\infty} \leq t} \|f - g\|_A = \|(|f| - t)_+\|_A$. ■

Using our two previous results, we obtain the following characterization of the K -functional.

Corollary 3.4.4 *Let A and L^∞ be as above. Then $K(f, t, A, L^\infty) = \inf_{s > 0} \{ \|(|f| - s)_+\|_A + st \}$.*

One application of the above results is the following classical result of Hardy, Littlewood and Pólya given in [15] and [16].

Theorem 3.4.5 *Let f and g be integrable functions on the σ -finite measure spaces (X, μ) and (Y, λ) respectively. Then the following conditions are equivalent:*

$$(1) \int_0^t f^*(s)ds \leq \int_0^t g^*(s)ds \text{ for } \forall t > 0.$$

$$(2) \int_X \Phi(|f|)d\mu \leq \int_Y \Phi(|g|)d\lambda \text{ for every increasing convex function } \Phi \text{ from } [0, \infty) \rightarrow [0, \infty).$$

Proof: We will only prove that (2) $\implies$ (1) here. We first note that $\phi(x) = [|x| - t]_+$ is an increasing convex function from $[0, \infty) \rightarrow [0, \infty)$; hence $\int_X [|f| - t]_+ d\mu \leq \int_Y [|g| - t]_+ d\lambda$. Using Corollary 3.4.4, we get $K(f, t, L^1(X, \mu), L^\infty(X, \mu)) \leq K(g, t, L^1(Y, \lambda), L^\infty(Y, \lambda))$. By Corollary 3.2.5, this is the same as $\int_0^t f^*(s)ds \leq \int_0^t g^*(s)ds$. Hence we have (1). ■

(1) $\implies$ (2) can be established using similar machinery. More about the above theorem can be found in [15], [16], [28] and [33]. A nice proof of (1) $\implies$ (2) can be found in [13].

Chapter 4

The J -Method

4.1 The J -Functional

The results in this section can be found in [1] and [3]. Let (A, B) be a compatible couple of Banach spaces. The K -functional was constructed by a slight modification of the norm of the space $A + B$. Analogously, we construct the J -functional using a similar modification of the norm of the space $A \cap B$.

Definition 4.1.1 *Let $f \in A \cap B$ and $t > 0$. Then $J(f, t, A, B) = \max(\|f\|_A, t\|f\|_B)$.*

The J -functional can be related to the K -functional in the following way.

Lemma 4.1.2 *Let $f \in A \cap B$ and $t > 0$. Then $K(f, t, A, B) \leq J(f, s, A, B) \min(1, t/s)$.*

Proof: Using Proposition 2.2.4, we get $K(f, t) \leq \min(\|f\|_A, t\|f\|_B)$. Then $K(f, t) \leq \|f\|_A \leq \max(\|f\|_A, s\|f\|_B) = J(f, s)$. Also $K(f, t) \leq t\|f\|_B \leq \max(\frac{t}{s}\|f\|_A, t\|f\|_B) = (t/s)J(f, s)$. Hence $K(f, t) \leq \min(1, t/s)J(f, s)$ and our result follows. ■

This following lemma also gives another relation between the K - and J -functionals.

Lemma 4.1.3 (The fundamental lemma of Interpolation) *Let (A, B) be a compatible couple of Banach spaces. Let $f \in A + B$ such that $\lim_{t \rightarrow 0} K(f, t, A, B) = \lim_{t \rightarrow \infty} \frac{K(f, t, A, B)}{t} = 0$. Then for any $\epsilon > 0$, $\exists f_i \in A \cap B$ for $\forall i \in \mathbb{Z}$ such that $f = \sum_{i=-\infty}^{\infty} f_i$ where this sum converges absolutely in $A+B$. Furthermore, $J(f_i, 2^i, A, B) \leq \gamma(1 + \epsilon)K(f, 2^i, A, B)$ for $\forall i$ where γ is a universal constant ≤ 3 .*

Proof: Fix $\epsilon > 0$. For each i , $\exists g_i \in A$, $h_i \in B$ with $f = g_i + h_i$ and $\|g_i\|_A + 2^i\|h_i\|_B < (1 + \epsilon)K(f, 2^i)$. Thus we have $\|g_i\|_A < (1 + \epsilon)K(f, 2^i)$ and $\|h_i\|_B < (1 + \epsilon)2^{-i}K(f, 2^i)$.

Let $f_i = g_{i+1} - g_i = h_i - h_{i+1}$. Then $f_i \in A \cap B$.

Furthermore, $\|f - \sum_{i=M}^N f_i\|_{A+B} = \|f - \sum_{i=M}^N (g_{i+1} - g_i)\|_{A+B} = \|f - g_{N+1} + g_M\|_{A+B} = \|h_{N+1} + g_M\|_{A+B} \leq \|g_M\|_A + \|h_{N+1}\|_B \leq (1 + \epsilon)[2^{-N-1}K(f, 2^{N+1}) + K(f, 2^{-M})]$.

Since the R.H.S. tends to 0 when $M \rightarrow -\infty$ and $N \rightarrow \infty$, $\lim_{n \rightarrow \infty, m \rightarrow -\infty}$

$\|f - \sum_{i=M}^N f_i\|_{A+B} = 0$ and $f = \sum_{i=-\infty}^{\infty} f_i$ with absolute convergence in $A + B$.

Finally, $J(f_i, 2^i) = \max(\|f_i\|_A, 2^i\|f_i\|_B) = \max(\|g_{i+1} - g_i\|_A, 2^i\|h_i - h_{i+1}\|_B) \leq \max(\|g_i\|_A + \|g_{i+1}\|_A, 2^i\|h_i\|_B + 2^i\|h_{i+1}\|_B) \leq (\|g_i\|_A + 2^i\|h_i\|_B) + (\|g_{i+1}\|_A + 2^{i+1}\|h_{i+1}\|_B) \leq (1 + \epsilon)[K(f, 2^i) + K(f, 2^{i+1})] \leq 3(1 + \epsilon)K(f, 2^i)$ where the final inequality follows from Lemma 2.2.2. ■

The fundamental lemma has a corollary which shows that if f satisfies the hypotheses of the Fundamental lemma, then there exists a representation of f in the form $f(t) = \int_0^\infty u_s(t) \frac{ds}{s}$ where $u_s \in A \cap B$ for $\forall s > 0$.

Corollary 4.1.4 *Let f and $\{f_i\}_{i=-\infty}^{\infty}$ be as in the above lemma.*

Let $u_s(t) = \frac{1}{\log 2} \sum_{i=-\infty}^{\infty} f_i(t) \chi_{(2^i, 2^{i+1}]}(s)$. Then $f(t) = \int_0^{\infty} u_s \frac{ds}{s}$ where this integral converges absolutely in $A+B$ (i.e. $\int_0^{\infty} \|u_s\|_{A+B} \frac{ds}{s} < \infty$).

Let i be such that $2^i < s < 2^{i+1}$ then $u_s(t) = (\log 2)^{-1} f_i(t)$ and $\|u_s(t)\|_{A+B} = (\log 2)^{-1} \|f_i(t)\|_{A+B}$.

Thus $\int_0^{\infty} \|u_s\|_{A+B} \frac{ds}{s} = \sum_{i=-\infty}^{\infty} \int_{2^i}^{2^{i+1}} \|u_s\|_{A+B} \frac{ds}{s} = (\log 2)^{-1} \sum_{i=-\infty}^{\infty} \|f_i\|_{A+B} \int_{2^i}^{2^{i+1}} \frac{ds}{s} = \sum_{i=-\infty}^{\infty} \|f_i\|_{A+B} < \infty$.

Now $\int_0^{\infty} u_s \frac{ds}{s} = \sum_{i=-\infty}^{\infty} \int_{2^i}^{2^{i+1}} u_s \frac{ds}{s} = (\log 2)^{-1} \sum_{i=-\infty}^{\infty} f_i \int_{2^i}^{2^{i+1}} \frac{ds}{s} = \sum_{i=-\infty}^{\infty} f_i = f$. ■

Now we are ready to describe the J -method.

Definition 4.1.5 *Let (A, B) be a compatible couple of Banach spaces and let $0 < \theta < 1$ and $1 \leq q \leq \infty$. Then $(A, B)_{\theta, q; J}$ consist of all f , for which $\|f\|_{\theta, q; J} = \inf \|s^{-\theta} J(u_s, s)\|_{L^q(\frac{ds}{s})}$ is finite. Here the infimum is taken over all u_s such that $f(t) = \int_0^{\infty} u_s(t) \frac{ds}{s}$ where this integral converges absolutely in $A+B$ and $u_s \in A \cap B$ for $\forall s > 0$ (We follow the convention that the infimum of the empty set is $= \infty$).*

The following theorem will equate the spaces $(A, B)_{\theta, q; J}$ with the spaces $(A, B)_{\theta, q}$.

Theorem 4.1.6 (Equivalence Theorem) *Let (A, B) be a compatible couple of Banach spaces and $0 < \theta < 1$ and $1 \leq q \leq \infty$. Then $(A, B)_{\theta, q; J} = (A, B)_{\theta, q}$ with equivalent norms.*

Suppose $f \in (A, B)_{\theta, q; J}$ then $\exists u_s \in A \cap B$ such that $f = \int_0^{\infty} u_s \frac{ds}{s}$. Using the subadditivity of the K -functional and Lemma 4.1.2 we have:

$$K(f, t) \leq \int_0^\infty K(u_s, t) \frac{ds}{s} \leq \int_0^\infty J(u_s, t) \min(1, t/s) \frac{ds}{s} = \int_0^t J(u_s, s) \frac{ds}{s} + t \int_t^\infty \frac{J(u_s, s)}{s} \frac{ds}{s}.$$

Now multiplying each side by $t^{-\theta}$ and taking the $L^q(\frac{dt}{t})$ norm on each side, we get

$$\|f\|_{\theta, q} \leq \|t^{-\theta} \int_0^\infty J(u_s, t) \min(1, t/s) \frac{dt}{t}\| \leq \|t^{-\theta} \int_0^t J(u_s, t) \frac{dt}{t}\|_{L^q(\frac{dt}{t})} + \|t^{1-\theta} \int_t^\infty \frac{J(u_s, t)}{s} \frac{dt}{t}\|_{L^q(\frac{dt}{t})}.$$

Now using Hardy's Inequalities (Lemma 2.4.2) on the R.H.S. of the above inequality, we obtain $\|f\|_{\theta, q} \leq \frac{2}{\theta} \|t^{-\theta} J(u_s, t)\|_{L^q(\frac{dt}{t})}$.

Now taking the inf of the R.H.S. over all functions $u_s(t) \in A \cap B$ for $\forall s$ where $f(t) = \int_0^\infty u_s(t) \frac{ds}{s}$, we get $\|f\|_{\theta, q} \leq \frac{2}{\theta} \|f\|_{\theta, q; J}$. Hence $(A, B)_{\theta, q; J} \subseteq (A, B)_{\theta, q}$.

Now suppose $f \in (A, B)_{\theta, q}$. Then $\lim_{t \rightarrow 0} K(f, t) = \lim_{t \rightarrow \infty} \frac{K(f, t)}{t} = 0$. Then let u_s be as in Corollary 4.1.4. Now fix s and let i be the integer with the property that $2^i \leq s \leq 2^{i+1}$.

$J(u_s, s) \leq (\log 2)^{-1} J(f_i, s) \leq 2(\log 2)^{-1} J(f_i, 2^i)$ using the inequalities $u_s \leq (\log 2)^{-1} f_i$ and $\frac{s}{2^i} \leq 2$.

Using the fundamental lemma of interpolation with $\gamma(1+\epsilon) < 3.5$, we get $J(u_s, s) \leq 2(\log 2)^{-1} J(f_i, 2^i) \leq 7(\log 2)^{-1} K(f, 2^i) \leq 7(\log 2)^{-1} K(f, s)$.

Multiplying by $s^{-\theta}$ and taking $L^q(\frac{ds}{s})$ norms, we get $\|f\|_{\theta, q; J} \leq 7(\log 2)^{-1} \|f\|_{\theta, q}$ and therefore $(A, B)_{\theta, q} = (A, B)_{\theta, q; J}$ with equivalent norms. ■

4.2 The Density Theorem

Since the J -method generates the same spaces as the K -method, it may seem at first glance that the J -method is redundant. However the J -method can be used to examine the structure of the K -spaces. One of the most important applications of

the J -method is its use in proving the following theorem.

Theorem 4.2.1 (Density Theorem) [1] *Let (A, B) be a compatible couple of ideal spaces and suppose $0 < \theta < 1$, $1 \leq q < \infty$. Then $A \cap B$ is dense in $(A, B)_{\theta, q}$.*

Let $f \in (A, B)_{\theta, q}$. Then $\lim_{t \rightarrow 0} K(f, t) = \lim_{t \rightarrow \infty} t^{-1} K(f, t) = 0$. Hence let f_i be as in the fundamental lemma. For $\forall N > 0$, let $u_s^N = (\log 2)^{-1} \sum_{i \in \mathbb{Z} \setminus [-N, N]} f_i(t) \chi_{(2^i, 2^{i+1})}(s)$. Now clearly, $|u_s^N(t)| \leq |u_s(t)| \forall s, t$. Therefore $\int_0^\infty \|u_s^N\|_{A+B} \frac{ds}{s} \leq \int_0^\infty \|u_s\|_{A+B} \frac{ds}{s} < \infty$.

$$\text{Now } \int_0^\infty u_s^N \frac{ds}{s} = \sum_{i \in \mathbb{Z} \setminus [-N, N]} f_i(t) (\log 2)^{-1} \int_0^\infty \frac{\chi_{(2^i, 2^{i+1})}(s)}{s} ds = f - \sum_{i=-N}^N f_i$$

$$\text{Hence by the definition of the } J\text{-method } \|f - \sum_{i=-N}^N f_i\|_{\theta, q, J} \leq (\int_0^\infty [s^{-\theta} J(u_s^N, s)]^q \frac{ds}{s})^{1/q}.$$

We will now show that the R.H.S. (and hence also the L.H.S.) tends to 0 as $N \rightarrow \infty$. Then since $\sum_{i=-N}^N f_i \in A \cap B$, we are done.

Since $u_s^N(t) = 0$ for $2^{-N} < s < 2^N$, $[s^{-\theta} J(u_s^N, s)]^q = 0$ for $2^{-N} < s < 2^N$ and therefore $u_s^N(t) \rightarrow 0$ pointwise on $(0, \infty)$ when $N \rightarrow \infty$. Since $f \in (A, B)_{\theta, q}$, $[s^{-\theta} K(f, s)]^q/s \in L^1(0, \infty)$. Thus since $[s^{-\theta} J(u_s^N, s)]^q/s \leq [s^{-\theta} J(u_s, s)]^q/s$, Lebesgue's dominated convergence theorem implies that $(\int_0^\infty [s^{-\theta} J(u_s^N, s)]^q \frac{ds}{s})^{1/q}$ tends to 0 as $N \rightarrow \infty$. ■

The restriction that $q < \infty$ cannot in general be relaxed as seen in the following example.

Example 4.2.2 $L^1(0, \infty) \cap L^\infty(0, \infty)$ is not dense in $L^{p, \infty}(0, \infty)$ for $1 < p < \infty$.

Proof: Let $f(x) = x^{-1/p}$ and let $g(x) \in L^1 \cap L^\infty$. Let $h(x) = [f(x) - \|g\|_{L^\infty}]_+$ then $h(x) \leq |f(x) - g(x)|$. Since $L^{p, \infty}$ is an ideal space, $\|f - g\|_{L^{p, \infty}} \geq \|h\|_{L^{p, \infty}}$. Now

$\|h\|_{L^{p,\infty}} = \sup_{t>0} t^{1/p} h^*(t) = \sup_{t>0} (1 - t^{1/p} \|g\|_{L^\infty}) = 1$. Hence $L^1 \cap L^\infty$ is not dense in $L^{p,\infty}$. ■

One consequence of the density theorem is the following property of Lorentz spaces.

Theorem 4.2.3 *The set of strictly simple functions is dense in $L^{p,q}$ where $1 < p, q < \infty$.*

Let $f \geq 0$ and $f \in L^{p,q}(X, \mu)$ where X is an arbitrary σ -finite measure space and $\varepsilon > 0$; by the density theorem, $L^1 \cap L^\infty$ is dense in $L^{p,q}$ and therefore $\exists g \in L^1 \cap L^\infty$ such that $\|f - g\|_{L^{p,q}} < \frac{\varepsilon}{2}$. Since $|f(x) - |g(x)|| \leq |f(x) - g(x)|$, we make take g to be non-negative.

For $\forall x \in \mathbb{R}$, define $[x] =$ the greatest integer $\leq x$. For $\forall \sigma > 0$, let $g_\sigma(x) = \sigma [\frac{g(x)}{\sigma}]$. Now $g_\sigma(x)$ is only nonzero on $\{x : g(x) > \sigma\}$ which is a set of finite measure since $g \in L^1$. Furthermore g takes only values $\{i\sigma\}_{i \in \mathbb{N} \cup \{0\}}$ with $i\sigma \leq \|g\|_{L^\infty}$ and hence g_σ is a strictly simple function. Now $\|g - g_\sigma\|_{L^\infty} \leq \sigma$ and therefore $g(x) - g_\sigma(x) \rightarrow 0$ when $\sigma \rightarrow 0$. Since $|g(x) - g_\sigma(x)| \leq g(x)$, a simple dominated convergence argument shows that $\|g - g_\sigma\|_{L^1} \rightarrow 0$ as $\sigma \rightarrow 0$. Therefore $\|g - g_\sigma\|_{L^1 \cap L^\infty} \rightarrow 0$ as $\sigma \rightarrow 0$.

Since $L^{p,q}$ is a K -space for the couple (L^1, L^∞) , $L^{p,q}$ is an intermediate space for (L^1, L^∞) and $\|f\|_{L^{p,q}} \leq C\|f\|_{L^1 \cap L^\infty}$ for $\forall f \in L^1 \cap L^\infty$. Now pick σ such that $\|g - g_\sigma\|_{L^1 \cap L^\infty} \leq \frac{\varepsilon}{2C}$ and hence $\|g - g_\sigma\|_{L^{p,q}} \leq C\|g - g_\sigma\|_{L^1 \cap L^\infty} \leq \frac{\varepsilon}{2}$. Then $\|f - g_\sigma\|_{L^{p,q}} \leq \|f - g\|_{L^{p,q}} + \|g - g_\sigma\|_{L^{p,q}} \leq \varepsilon$. Now we can remove the restriction that $f \geq 0$ by remembering that $f = [\Re f]_+ - [-\Re f]_+ + i[\Im f]_+ - i[-\Im f]_+$. For each of these

four terms, there exists a strictly simple function that approximates it to within $\varepsilon/4$ and hence the sum of those simple functions approximates f to within ε . ■

This can be compared to [20] where Hunt gives a direct proof of the above result.

Chapter 5

Interpolation of Lipschitz Operators

5.1 The K -method and Lipschitz Operators

Definition 5.1.1 *Let A and B be Banach spaces and let $T : A \rightarrow B$: then T is said to be a Lipschitz operator with bound M if for $f, g \in A$, $\|Tf - Tg\|_B \leq M\|f - g\|_A$.*

The natural question to ask in the context of this thesis is whether the K -method can be used to interpolate Lipschitz operators. Peetre [35] and Maligranda [27] have shown that it can. We present the following theorem which is modeled after a result of Lorentz and Shimogaki [26].

Theorem 5.1.2 *Let (A_0, A_1) and (B_0, B_1) be two compatible couples of Banach spaces and let T be a Lipschitz operator from $A_i \rightarrow B_i$ with bound M_i .*

Then $K(Tf - Tg, t, B_0, B_1) \leq M_0 K(f - g, \frac{M_1}{M_0}t, A_0, A_1)$ for $\forall f, g \in A_0 \cap A_1$.

Proof: Let $f, g \in A_0 \cap A_1$ and fix $t, \varepsilon > 0$. Let $h_i \in A_i$ for $i = 0, 1$ be such that $f - g = h_0 + h_1$ and $\|h_0\|_{A_0} + \frac{M_1}{M_0}t\|h_1\|_{A_1} \leq K(f - g, \frac{M_1}{M_0}t, A_0, A_1) + \varepsilon$. Then $h_0, h_1 \in A_0 \cap A_1$ and $Th_0, Th_1 \in B_0 \cap B_1$. Now let $\alpha = T(g + h_0) - Tg$ and let $\beta = Tf - T(g + h_0)$. Therefore $\alpha, \beta \in B_0 \cap B_1$ and $\alpha + \beta = Tf - Tg$. Now $\|\alpha\|_{B_0} = \|T(g + h_0) - Tg\|_{B_0} \leq M_0\|h_0\|_{A_0}$ and $\|\beta\|_{B_1} = \|Tf - T(g + h_0)\|_{B_1} \leq M_1\|f - g - h_0\|_{A_1} = M_1\|h_1\|_{A_1}$. Then $K(Tf - Tg, t, B_0, B_1) \leq \|\alpha\|_{B_0} + t\|\beta\|_{B_1} \leq M_0(\|h_0\|_{A_0} + \frac{M_1}{M_0}t\|h_1\|_{A_1}) \leq M_0[K(f - g, \frac{M_1}{M_0}t, A_0, A_1) + \varepsilon]$. Since ε is arbitrary we have our result.

Corollary 5.1.3 *If T is as in the above theorem and $0 < \theta < 1$ and $1 \leq q < \infty$ then T is a Lipschitz operator from $(A_0, A_1)_{\theta, q} \rightarrow (B_0, B_1)_{\theta, q}$ with bound $\leq M_0^{1-\theta} M_1^\theta$*

Proof: From previous theorem we have $K(Tf - Tg, t, B_0, B_1) \leq M_0 K(f - g, \frac{M_1}{M_0}t, A_0, A_1)$ and thus $\|Tf - Tg\|_{(B_0, B_1)_{\theta, q}} = (\int_0^\infty [t^{-\theta} K(Tf - Tg, t, B_0, B_1)]^q \frac{dt}{t})^{1/q}$
 $\leq CM_0 (\int_0^\infty [t^{-\theta} K(f - g, \frac{M_1}{M_0}t, A_0, A_1)]^q \frac{dt}{t})^{1/q} = CM_0^{1-\theta} M_1^\theta \|f - g\|_{(A_0, A_1)_{\theta, q}}$. The last equality can be obtained by making a change of variables $s = \frac{M_1}{M_0}t$. Now we have $\|Tf - Tg\|_{(B_0, B_1)_{\theta, q}} \leq M_0^{1-\theta} M_1^\theta \|f - g\|_{(A_0, A_1)_{\theta, q}}$ for $\forall f, g \in A_0 \cap A_1$. Since the density theorem guarantees that $A_0 \cap A_1$ is dense in $(A_0, A_1)_{\theta, q}$ when $0 < \theta < 1$ and $1 \leq q < \infty$, we can use continuity to extend T uniquely to $(A_0, A_1)_{\theta, q}$. ■

Applications of the Interpolation of Lipschitz Operators to areas of applied mathematics are given in [6]. We present an application of the above results to the theory of rearrangements in the next section.

5.2 The Lorentz-Shimogaki Inequality

The following is an interesting example of a Lipschitz operator due to Lorentz and Shimogaki [26] and Maligranda [27].

Example 5.2.1 *Let (X, μ) be a nonatomic σ -finite measure space and let T be the operator that takes a function on (X, μ) to its nonincreasing rearrangement. (i.e. $Tf = f^*$). We call T the rearrangement operator. Then T is a Lipschitz operator with bound 1 from $L^1(X, \mu) \rightarrow L^1(R^+, m)$ and is also a Lipschitz operator with bound 1 from $L^\infty(X, \mu) \rightarrow L^\infty(R^+, m)$.*

Proof: It follows from Theorem 1.6.3 that $T : L^1(X, \mu) \rightarrow L^1(R^+, m)$ and similarly for L^∞ . Now let $f, g \in L^1$ and let $h(x) = \max(|f(x)|, |g(x)|)$.

$$\begin{aligned}
 \text{Then } \|f^* - g^*\|_{L^1} &= \int_0^\infty [f^*(t) - g^*(t)]_+ dt + \int_0^\infty [g^*(t) - f^*(t)]_+ dt \\
 &\leq \int_0^\infty [h^*(t) - g^*(t)]_+ dt + \int_0^\infty [h^*(t) - f^*(t)]_+ dt = \int_0^\infty h^*(t) - g^*(t) dt + \int_0^\infty h^*(t) - f^*(t) dt \\
 &\text{(since } |f| \leq |h| \text{ implies } f^* \leq h^* \text{ (Prop. 1.5.8.b) and similarly for } g\text{).} \\
 &= \int_0^\infty h^*(t) dt - \int_0^\infty g^*(t) dt + \int_0^\infty h^*(t) dt - \int_0^\infty f^*(t) dt \\
 &= \int_X h d\mu - \int_X |g| d\mu + \int_X h d\mu - \int_X |f| d\mu \text{ (Theorem 1.6.3 with } p = 1\text{)} \\
 &= \int_X h - |g| d\mu + \int_X h - |f| d\mu = \int_X [|f| - |g|]_+ d\mu + \int_X [|g| - |f|]_+ d\mu = \int_X ||f| - |g|| d\mu \leq \\
 &\int_X |f - g| d\mu = \|f - g\|_{L^1}.
 \end{aligned}$$

Since we have $|f(x)| \leq \|f - g\|_{L^\infty} + |g(x)|$, we can use proposition 1.5.8.b to obtain $f^*(t) \leq \|f - g\|_{L^\infty} + g^*(t)$ and therefore we have $f^*(t) - g^*(t) \leq \|f - g\|_{L^\infty}$. Interchanging the roles of f and g , we can also obtain $g^*(t) - f^*(t) \leq \|f - g\|_{L^\infty}$.

Therefore we have $|f^*(t) - g^*(t)| \leq \|f - g\|_{L^\infty}$ and taking sup of L.H.S. over all t gives us $\|f^* - g^*\|_{L^\infty} \leq \|f - g\|_{L^\infty}$. ■

Now using Corollary 5.1.3 we can show that the rearrangement operator is a Lipschitz operator from $L^{p,q}(X, \mu) \rightarrow L^{p,q}((0, \infty), m)$ when $1 < p < \infty$ and $1 \leq q < \infty$. We can also use Theorem 5.1.2 to prove the following inequality.

Theorem 5.2.2 (Lorentz-Shimogaki Inequality) *Let f, g be two functions in $L^1(X, \mu)$ and let $t > 0$. Then $\int_0^t [f^* - g^*]^*(s) ds \leq \int_0^t [f - g]^*(s) ds$.*

Proof: In what follows we denote $K(f, t, L^1, L^\infty)$ as $K(f, t)$. Using Theorem 5.1.2, we obtain $K(\phi^* - \psi^*, t) \leq K(\phi - \psi, t)$ for $\forall \phi, \psi \in L^1 \cap L^\infty$. Now let $f, g \in L^1$ and $s > 0$, then define f_s as $f_s(x) = f(x)$ if $|f(x)| \leq s$ and $f_s(x) = s \frac{f(x)}{|f(x)|}$ if $|f(x)| \geq s$. Define g_s analogously. Now $f_s, g_s \in L^1 \cap L^\infty$. Choose s such that $\int_X |f - f_s| d\mu < \varepsilon$ and $\int_X |g - g_s| d\mu < \varepsilon$.

Then since $|f_s| \leq |f|$, $f_s^* \leq f^*$ (Prop. 1.5.8.b) and $K(f^* - f_s^*, t) \leq \int_0^\infty (f^* - f_s^*)^*(t) dt$ (Corollary 3.2.5) $\leq \int_0^\infty |f^*(t) - f_s^*(t)| dt$ (Theorem 1.6.3 with $p = 1$) $= \int_0^\infty f^*(t) - f_s^*(t) dt$ (since integrand is positive) $= \int_X |f| d\mu - \int_X |f_s| d\mu$ (Theorem 1.6.3 with $p = 1$) $\leq \int_X |f - f_s| d\mu \leq \varepsilon$. Similarly $K(g^* - g_s^*, t) < \varepsilon$. Using Corollary 3.2.5, $K(f - f_s, t) = \int_0^t (f - f_s)^*(x) dx \leq \int_0^\infty (f - f_s)^*(x) dx = \int_X |f - f_s| d\mu \leq \varepsilon$. Similarly $K(g - g_s, t) \leq \varepsilon$.

Now $\int_0^t (f^* - g^*)^*(s) ds = K(f^* - g^*, t)$ (Corollary 3.2.5) $\leq K(f^* - f_s^*, t) + K(f_s^* - g_s^*, t) + K(g^* - g_s^*, t)$ (using the fact that $K(\cdot, t)$ is a norm.) $\leq K(f_s^* - g_s^*, t) + 2\varepsilon$ $\leq K(f_s - g_s, t) + 2\varepsilon \leq K(f - f_s, t) + K(f - g, t) + K(g - g_s, t) + 2\varepsilon \leq K(f - g, t) + 4\varepsilon$.

Since ε is arbitrary, $K(f^* - g^*, t) \leq K(f - g, t)$ and using Corollary 3.2.5 we get $\int_0^t (f^* - g^*)^*(s) ds \leq \int_0^t (f - g)^*(s) ds$.

The original proof of this result is quite different and can be found in [26]. The idea behind this version of the proof is due to Maligranda [27].

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