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**CONTRIBUTIONS TO AUTOMATED-THEOREM-PROVING AND
FORMAL METHODS WITH APPLICATIONS TO CONTROL
SYSTEMS**

Tom Mackling

Departement of Electrical Engineering
McGill University, Montréal

October 1997

A Thesis submitted to the Faculty of Graduate Studies and Research
in partial fulfilment of the requirements for the degree of
Doctor of Philosophy

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For

Ian Kelly

Abstract

Part 1. We show that the standard notions of *tautology* and *subsumption* can be naturally generalized, (so that refutation completeness is preserved with respect to the associated deletion), within the context of (a specified set of) modified deduction rules for binary clausal resolution-refutation which *build-in* the *reflexivity*, *symmetry*, *transitivity* and *predicate substitutivity axioms* for equality. To this end, first a resolution system, $PC_{<}$, in which equality has no special status, is introduced and its refutation completeness under subsumption and tautology deletion shown. The generalized notions of subsumption and tautology, Σ_0 -subsumption and Σ_0 -tautology, respectively, are then formulated and analyzed. $EPC_{<}$, a system which provides an adequate deductive context for refutation completeness under deletion with respect to these generalized notions, is then introduced. To prove this completeness, some technical notions (principally, to allow an induction) are introduced, and a number of technical results are obtained. Additionally, some clause “normalization” techniques including generalized forms of *replacement factoring* and a form of *demodulation*, are suitably formulated, and the preservation of refutation completeness with the further introduction of these normalization procedures is shown. Some refinements for the rules are offered, as well as a possible further generalization for subsumption, and a generalized form of a special case of *replacement resolution*, *folding*.

Part 2. We present and analyze some primitives for the analysis and control specification of non-deterministic finite state machines (FSMs), analogous to the *Reachability* and *Current State Estimate Set* primitives, introduced as basic predicates, in *COCOLOG* [8, 9, 7, 38]. In terms of these predicates, an aggregation method for the hierarchical decomposition of a FSM is given, in terms of *centered covers* of the state space, whereby a hierarchical control may be synthesized and applied. A special class of covers is presented which possesses lattice theoretic properties generalizing the results of [13, 42, 41, 43, 40, 3]. A broader class of (so called) *regular* covers is introduced which, though lacking closure under any similar lattice operations, is sufficient to guarantee, in an obvious sense,

the completeness of the abstraction, (which guarantees meta-level *controllability* under appropriate circumstances). Other cover classes, suitable for controllable FSMs are then introduced, and the lattice theoretic properties are therein recovered.

Résumé

1^{ère} Partie: On montre que les notions normales de tautologie et de subsumption peuvent être généralisées (tel que le l'état complet, pour la réfutation, par rapport à l'effacement associé est préservée) dans le contexte (d'un ensemble spécifié) de règles de déduction modifiées pour des propositions résolution-réfutation binaires. Ces propositions sont intégrées dans le contexte des axiomes de réflexivité, de symétrie, de transitivité et de substitutivité des prédicats pour l'égalité. On introduit initialement un système de résolution $PC_{<}$ dans lequel l'égalité n'a pas un statut particulier. On démontre le l'état complet (pour la réfutation) de ce système dans le cas de subsumption et d'effacement de tautologie. Les notions générales de subsumption et de tautologie sont alors respectivement formulées et analysées. Ensuite on introduit un système $EPC_{<}$ qui fournit un contexte adéquat de déduction par l'état complet sous effacement par rapport à ces notions généralisées. Pour prouver l'état complet, des outils techniques sont introduits (principalement pour permettre une induction) et des résultats techniques sont obtenus. En plus, des techniques de normalisation des propositions incluent des formes généralisées d'affecturage de remplacement ainsi qu'un genre de démodulation sont formulées. On montre aussi la préservation de l'état complet avec l'introduction de ces procédures de normalisation. Des raffinements pour les règles ainsi qu'une généralisation pour subsumption et une forme généralisée du cas spécial de repliement sont présentées.

2^{ième} Partie: On présente et on analyse quelques primitives pour les spécifications de contrôle et d'analyse des machines à état fini non-déterministe (FSMs). Ces primitives sont similaires introduites comme prédicats de base à COCOLOG (aux primitives d'accessibilité et d'estimé de l'état), [8, 9, 7, 38]. En terme de ces prédicats, une méthode d'agrégation pour une décomposition hiérarchique d'une FSM est présentée en fonction des couvertures centrées de l'espace d'état par lesquelles une commande hiérarchique peut être développer et implémenter. Une classe spéciale de couverture est présentée. Cette classe possède les propriétés théoriques du "treillis" généralisant ainsi les résultats de [13, 42, 41, 43, 40, 3]. Une classe plus générale de couvertures est introduite et même si la

RÉSUMÉ

fermeture manque pour des opérations similaires du treillis, cette classe garantie, d'une façon évidente, l'état complet de l'abstraction, (qui assure le contrôlabilité du méta-niveau sous des circonstances appropriées). D'autres classes couvertures adéquats pour FSM contrôlable sont introduits et les propriétés théorique du treillis sont ainsi retrouvés.

Acknowledgements

First and foremost, I would like to thank my thesis supervisor, Professor Peter E. Caines, for his encouragement, inspiration and sponsorship, without which this dissertation would certainly never have been written, and for his unabashed intellectual curiosity, which for a wonderful example, made *Proofs and Types* into a respectable subject for engineering study. I would like to thank the late Professor George Zames, for teaching me *linear systems*, and for his candour and unassuming style, and Professor Michael Makkai, for taking time for me and for setting an inspiring example.

Kaouthar Benameur translated my abstract for me. Carlos Martínez-Mascarúa help me take the leap from *vi* to *GNU Emacs*, helped me with *LaTeX*, (and let me use his manuals), introduced me to drawing packages such as *ipe*, helped me with my windows set up, (letting me copy his!), and with innumerable other software related points. Thank you also to the resident computer system staff at CIM, in particular, Jan Binder and Mike Parker, who pretty much unfailingly helped me out whenever I had computer related problems. Peter Whaite also provided me speedy aid with my serious *LaTeX*, window and *Emacs* related, problems. Tal Arbel was the beauty I could admire and adore from afar. Thank you also to Jennifer Quinn, for looking after all those forms, to Yuan-Yun Wei, Carlos Martínez-Mascarúa, Paul Hubbard, Gang Shen, Ekatarina Lemch, Fazal Ur Rehman, and Kaouthar Benameur for many fine conversations and for listening to me when I felt “inspired”, and to all the other fine people who spent time with me at CIM.

I thank my beloved wife, Jennifer, to whom I probably owe my life, and our son, Ian, who is a heart warming joy, at the center of my world. I thank my mother, who should feel no shame nor any pangs of disappointment over her son, my father, for showing me love as a child, and my late Uncle John, for never abandoning me. I want to thank all those whom I’ve loved, all those whom I never loved, but wanted to, and all those whom I yet may love. I would like to thank all those who have given me the chance to benefit from them, and all those who will yet do so. I need to thank the

ACKNOWLEDGEMENTS

Lord Jesus Christ, of whose essential presence I can bear witness, and our heavenly father, keeper of the Truth, for His mercy and His love, praise be to God.

This dissertation was produced using *GNU Emacs* text editing software, the document preparation system *LaTeX_{2 ϵ}* , and a *sun microsystems* workstation computer, at CIM, McGill University. The research was supported by NSERC Grant A1329, and the NCE-IRIS-B5 and NCE-IRIS-II-IS5 grants.

Ode to Truth

Sometimes invisible. but always present.
beneath and behind all things, in each thing manifest.
Sometimes hard to face, often hard to know.
independent of belief and opinion, usually beyond comprehension.
Yet to shun thee is to be confounded.
to embrace thee is to receive reward.
Indestructible and incorruptible, eternal, though dynamic.
inexorable, and, when made manifest and known, undeniable.
Ultimate redeemer, make me ready.

Claim of Originality

Unless otherwise indicated, nearly everything in this dissertation is, as far as I know, original scholarship. Some notions are too obvious to be original. Otherwise, if something has been borrowed from elsewhere, the author has made an effort to provide a reference. The more specific indication of originality below is not intended to be inclusive, but rather, a highlighting of certain specific contributions. All the work, and of course, any mistakes herein, are my own.

In Part 1, in particular the introduced first-order clausal resolution-refutation systems $PC_{<}$ and $EPC_{<}$ are original, though variants exist for both; in particular, $PC_{<}$ may be considered to be a form of *Ordered Resolution* with *selection*. The notions of Σ_0 -*tautology*, Σ_0 -*subsumption* and *reduct* Σ_0 -*subsumption* are also deemed original, though the idea of Σ_0 -tautology was more or less obvious. The notions of *folding*, Σ_0 -*repf* and Σ_0 -*folding*, for clause reduction, are also new, though they may be regarded as specific instances of clause *simplification* techniques.

In Part 2, the introduced notion of a *centered cover* for a non-deterministic finite state machine (FSM) is considered original. The notions of *set collapsibility* and *collapsibility*, pertaining to observability in the (non-deterministic) FSM framework, and the formulation of *coherent reachability* are also deemed original. The notion of a *coherently centered cover*, and notions of the *(FOC) meta-machine* and the *(CC) meta-machine* associated with given centered and coherently centered covers of an FSM, are deemed original. The various properties and classes of centered covers, *weak-quasi-convexity*, *quasi-convexity*, *center separation*, *regularity*, *center controllability*, and *center completeness*, together with their associated analysis and lattice theoretical properties presented, are all considered original.

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Introduction

This dissertation really consists of research contributions to two distinct subject areas: 1) resolution-refutation based first-order theorem-proving, and 2) state aggregation methods for systems control. First-order resolution-refutation has been a central area of research in mechanized deduction ever since its inception, widely accredited to J. A. Robinson's article [34] of 1965. Though more than 30 years have now passed since this inception, theoretical interest in resolution-based automatic-theorem-proving has remained ongoing. With the availability of ever better methods, ever more powerful computing machinery, and ever greater potential application areas, largely consisting of ever more abundant requirements for system or software verification¹, practical interest in resolution-based theorem-proving², albeit sometimes wavering, seems to be correspondingly increasing. As for the second subject area, as engineering projects grow more complex, (systems responsible for controlling robotic manipulators on space platforms, for example), problems of computational intractability, and a concomitant need for methodological techniques for avoiding unmanageable complexity in requisite verification tasks, have been commonly acknowledged. As part of the effort underway to devise methods for the effective verification and synthesis of complex systems, state aggregation approaches have been proposed as methods for system model abstraction [32, 11, 12, 14, 40, 23, 13, 42, 41, 43, 3, 5, 4]. Apart from these basic remarks, there is probably no need here to say anything more about either of these two rather separate research areas in general, except for the assertion that they do have some point of contact and mutual relevance in the logic-based control project of COCOLOG [8, 9, 7, 38].

COCOLOG [8, 9, 7, 38] is essentially a dynamic first-order logic specification language for a plant (P) taken together with its controller (C). In this formulation, the plant, regarded as a (deterministic) finite state machine (FSM), is systematically specified by a first order theory (in "canonical" syntax), $\mathcal{T}$, and the controller is specified by a set $\mathcal{C}$, of *Conditional-Control Rules* (CCRs), where

¹principally, the verification that some requisite properties must follow from a given formal specification

²largely, as a verification tool

in each such CCR, a control input is associated to a first-order sentence, the *Conditional-Control Formula* (CCF), whose validity (or truth) is responsible for the issuance of the associated input. In principle the specification is dynamic, in that the theory $\mathcal{T}$, (and possibly, the set of CCRs, $\mathcal{C}$), is subject to systematic revision during the evolution of the plant under COCOLOG control. However, in implementations where the validity of CCFs is established (at least partially), through model verification techniques, the updated first-order theories need not actually be generated (as first-order syntax), but rather may be specified through a dynamically updated (partial) model. In fact, COCOLOG is formulated so that at every stage, each COCOLOG axiomatization is logically complete (in other words, truth is decidable), and so the CCFs can always be *verified* through a form of model verification, (instead of being logically deduced). Indeed, recent work includes the specification of a procedure whereby a (stationary, i.e. static CCR set, and Markovian [10, 6, 44, 45]), COCOLOG controller may be translated into, and realized as, a FSM [27, 24].

The basic syntax of COCOLOG, (see e.g. [8, 9, 7]), involves the specification and use of two basic predicates, *Rbl* (reachability), and *CSE* (current state estimate set). The specification and use of other predicate and function symbols not part of the basic COCOLOG syntax is addressed via the formulation of the *Macro-COCOLOG language*, (see [25, 26]).

In Hierarchical-COCOLOG [13, 42, 41, 43, 40], the specifications of plant (as a FSM), and (COCOLOG) controller are given hierarchically or modularly, together with a hierarchical control evaluation and “theory revision” strategy. More recently, the COCOLOG control of plants formulated as Hybrid-systems has been considered, in Hybrid-COCOLOG [11, 12, 14, 40, 23].

A program exists which realizes a finite state Markovian COCOLOG controller given a COCOLOG type specification. This software allows the user to specify a model, its input dependent dynamic updates, and the CCFs in a flexible high level language. This language supports the use of several general classes of user defined predicates and functions, and *macros* may be readily defined and used. The CCFs and model may be specified either directly, in a finite type bounded quantifier first-order logic formula format (F2), or indirectly by specifying the CCF's negation or model axioms in clausal format (F1), or in some combination of both.

If a given CCF is specified in “closed form”, (format F2), the software will attempt to verify the CCF using an effective model verification procedure to evaluate the truth of the given formula. This procedure may be regarded as essentially a language interpreter which includes several automatically

applied optimization techniques.

If (the negation of) a given CCF is specified in clausal form (format F1) the software will begin a theoretically complete binary resolution-refutation procedure in order to attempt to refute the clause set consisting of the given clause together with those clauses given as specifying the (current state of the) model, utilizing any model aspects specified non-clausally (in format F2) as *semantic-attachment* (see, eg. [29, 30]) specifications. The software which allows the user to supply (partial or complete) model specification in the high level language, may be regarded as, and in fact constitutes, a flexible semantic-attachment facility for the theorem-proving module of the software. The refutation attempt proceeds until either a new clause has been successfully added to the clausal data-base, or it has been established that no such addition is possible. The software then proceeds to the next CCF in a cyclic fashion, until a valid CCF is discovered, though in the event that no clausal addition was possible, the clause set corresponding to the given CCF must be satisfiable, and this CCF will be bypassed from further evaluation until some control has been successfully triggered by some other CCF and the model has been updated. The system returns to cycle through the CCFs once the model has been dynamically updated, (by another module of the software).

The resolution-refutation-based theorem-proving module is subsumption³ (or saturation) based, and incorporates a novel methodology for the special treatment of the equality predicate which uses correspondingly generalized tautology and subsumption deletion. This theorem-proving module is a partial implementation of the theory described in Part 1 of this thesis. This theory includes the specification of set of a binary resolution-style deduction rules, defining (in Chapter 4) a resolution procedure entitled $EPC_{<}$, which is refutation complete with respect to deletion associated with the equality generalized subsumption and tautology relations, Σ_0 -*subsumption* and Σ_0 -*tautology*, respectively, (introduced in Chapter 3). The relation of Σ_0 -*subsumption* is an extension of an earlier version implemented in the theorem-proving software, and the deduction rules, $EPC_{<}$ have been correspondingly modified from those actually implemented in the software. Thus Part 1 is essentially a theoretical offspring of some novel techniques actually introduced in order to allow theorem-proving sufficiently efficient for control applications. Earlier research efforts on attaining sufficiently efficient first-order mechanical theorem-proving for the COCOLOG project included *F.E.-resolution* [39].

³Though standard within the resolution-refutation theorem-proving literature, definitions for the terms *subsumption* and *tautology* are given in Chapter 1.

INTRODUCTION

The model verification based *semantic-attachment facility* module which interprets the high level user specification language is now computationally complete, in the sense that the use of lists and lists of lists, et cetera, over finite types, and of recursive functions over these domains, is supported.

A number of applications of COCOLOG, utilizing this program, have been realized. These include the synthesis, via the writing of extended COCOLOG style specifications, and modelling of at least two 7 state machine (one partially observed) steering exercises and several elevator controllers. The elevator system controllers began modestly with a *toy* elevator system consisting of a single car servicing 4 floors, and a single omnipresent control console consisting of four buttons, $b_1, \dots, b_4$, each button b_i , corresponding to the i^{th} floor, either *on* or *off*, $b_i = on$ indicating a request for the i^{th} floor. [31]. Another early effort included consideration of a passenger *frustration index*. [19].

With the improvement of the COCOLOG software, (the aforementioned program being the last of a several incarnations or generations, being preceded by weaker, earlier versions), the elevator systems became more complex. Later elevator systems included a single car, 9 floor, elevator system, with the regular 2 button *up*, *down* request console per floor, a 9 *on*, *off* button console, (one button per floor) for inside the car, with a typical elevator control policy, with and without a simulated *fire response*, and finally, a two car, 30 floor system, with the regular 2 button *up*, *down* request console per floor, and two 30 *on*, *off* button consoles, (one button per floor), one such for inside each car, with a typical elevator control policy effected.

Additionally, a hierarchical steering exercise was coded and tested. This application involved steering a hypothetical robot through a three story building, where the control tasks were hierarchically divided in three layers of control: in terms of floor to floor, room/hallway to room/hallway, and (inside room/hallway) position to position, control tasks. Though a COCOLOG controller was synthesized which accomplished the steering exercise in a hierarchical fashion, the COCOLOG controller itself was itself not executed as in Hierarchical-COCOLOG per se, but rather the hierarchical controller was simulated by a *single level* COCOLOG controller.

With the exception of the elevator controller described in [19], which work was performed when the software was still in a nascent form, (eventually) all of the other elevator system controllers, as well as the hierarchical steering robot controller, were used as demonstrations showing real-time control capability. Elevator button pushes (*requests*) were modelled as exogenous inputs, and the

COCOLOG software additionally allows the specification and modelling of pseudo-random “exogenous” inputs with user specified probabilities. Successful synthesis of the hierarchical control via the single level controller attested to the power of the COCOLOG software.

Finally, an implemented COCOLOG controller was developed and utilized for the control of a real mobile robot in the NCE-IRIS-II-IS5 mobile robots project at McGill University. The COCOLOG software, which was upgraded with communications interface software, was again used to realize the controller from COCOLOG style model and control specification input files. It was processed, on-line, on an off-board workstation computer, and interfaced with lower-level control software, also processed, on-line, on off-board workstation computers controlling the robot by radio link. The COCOLOG controller modelled the system in terms of rooms, doors and corridors, and the robots location up to a particular room or corridor. By discovery, the doors were either blocked, (obstructed), open or closed, and corridors similarly by discovery, were, between doors, either blocked or passable. Blockages within corridors were modeled as newly introduced constraints, which essentially indicated that: “traversal from a given door D_i to a given door D_j through a given corridor C_k is infeasible”. Using such a model, the COCOLOG controller steered the robot, in terms of the doors to be sequentially traversed, to given specified target rooms, dynamically updating its model with the discovery of untraversable (blocked or closed) doors and blocked corridors. This mobile robot project is the only implemented real-world application of COCOLOG to date, and demonstrated use of real-time logical specification based control.

Through all of its preceding incarnations, COCOLOG has been systematically developed for the deterministic case. In applications to Discrete Event System(s) (DES) problems, *uncontrollable events* have been largely treated as *exogenous inputs*, whose contribution towards the problem of effective COCOLOG controller *synthesis* was to be largely absorbed into the task of controller *verification*. However, for many DES problems, the formulation of a non-deterministic model seems more natural and straight-forward, or at least easier, than a deterministic counterpart. In particular, allowing non-determinism often allows significantly more compact FSM representations. Further, it may be argued that hybrid systems are essentially non-deterministic. Thus the definition of predicates suitable for control analysis, synthesis and specification, for non-deterministic FSMs, as well as aggregation methodologies for such, may be seen as ground work for future developments in Hybrid-COCOLOG. In Part 2 of this thesis, we introduce some analogues of the basic primitives of COCOLOG, suitable for the generalization to non-deterministic FSMs. In terms of one of these analogues, corresponding to the notion of *winning reachability*, we then propose and analyse some

state aggregation techniques for hierarchical control, suitable for a non-deterministic formulation of Hierarchical-COCOLOG.

Part 1.

We provide a positive partial answer to the following informally posed question:

Can the standard notions of tautology and subsumption be naturally generalized, (so that refutation completeness is preserved with respect to the associated deletion), within a context of reasonably modified deduction rules for binary clausal resolution-refutation which *builds-in* the equality axioms?

The answer is partial because, in this paper, we neglect the functional substitutivity axioms for equality, and provide appropriate generalizations for the notions of subsumption and tautology, together with a specification of a set of modified binary resolution-style deduction rules, (in which refutation completeness is preserved under deletion with respect to these generalized notions), for the theory consisting of only the reflexivity, symmetry, transitivity and predicate substitutivity axioms for equality. Nevertheless, we feel this paper is an instructive first step towards a system which provides the complete solution.

The equality axioms have been identified as a source of inefficiency or even incapacity for clause resolution/deduction oriented refutation provers (see e.g. [6, 17, 16, 22, 33, 21, 46]). A number of approaches to “building in equality”, have been offered in the literature of automated deduction. These include *paramodulation*, *hyper-paramodulation*, and *demodulation* [21, 46], *E-Resolution* [28], *RUE-Resolution* [17, 16], *Z-Resolution* [18], and the approach suggested by Sibert in [35]. More recently, restrictions of *ordered paramodulation* are described in, for example, [1, 2]. However, with the exception of [35], and the work described in [1, 2], very little has yet been written on how the notions of tautology and subsumption can be appropriately and feasibly generalized within the framework of such techniques.

We present $EPC_{<}$, (*Equality Predicate Colouring*), (which in the terminology of [37], is essentially an instance of binary partial narrow theory resolution), as one approach to “the first four” equality axioms, and describe how such a generalization is possible. Of course, generalizing these notions allows a strengthened deletion strategy, which potentially results in a significant pruning of the clause set search space.

Before the introducing $EPC_{<}$ and the notions of Σ_0 -subsumption and Σ_0 -tautology, we first introduce a resolution system $PC_{<}$, (*predicate colouring*), in which equality has no special status, and show its refutation complete under subsumption and tautology deletion. We do this for the following reasons: 1) simply to introduce $PC_{<}$, which we feel is, as a conceptual model at least, an important binary clausal resolution-refutation system in its own right, and 2) to provide a conceptual springboard for the system $EPC_{<}$, and in particular, for the proof-theoretical arguments we needed to justify those properties we claim it possesses, (such as its refutation completeness with the generalized subsumption and tautology deletion). Though “independently invented” (I was not aware of the development called *Ordered Resolution* during the preparation of this thesis), $PC_{<}$ is closely related to *ordered resolution* [36]⁴, and Ordered Resolution may be seen as a kind of corollary to Predicate Colouring. Our formulation and treatment of $PC_{<}$, (in Section 2), serves as a (partial) template for what we wish to do for $EPC_{<}$, and for most results obtained for $PC_{<}$, there are analogous ones later obtained for $EPC_{<}$.

We then, (in Section 3), formulate and analyze Σ_0 -subsumption and Σ_0 -tautology, our generalized notions of subsumption and tautology. In particular, we give a kind of semantic characterization for these syntactic predicates. We follow this, (in Section 4), with the introduction of $EPC_{<}$, our system which provides an adequate deductive context for refutation completeness under deletion with respect to these generalized notions.

Unfortunately, our proof-theoretical argument of this completeness is not brief, and we require numerous technical results. Often we require a conclusion C from hypotheses H , were we have already shown that C' follows from H' , for C' and H' closely resembling C and H , respectively. However, possibly because of the very syntactic nature of our subject, we shied away from the problem of trying to establish, and prove our required results within, an appropriate and adequate abstract setting. Instead, we settled on avoiding redundant argumentation where possible through “modularization by lemma’s”, (using the analogy of ‘lemma is to proof’ as ‘subroutine is to program’), and, where it seemed expedient, by specifying proofs for results by indicating sufficient modification to the proofs given for similar results.⁵

⁴[36] contains a slick formulation, even though, it seems to me, that the proof given for completeness on pp. 162-163, is inadequate.

⁵We take advantage of the presentation of $PC_{<}$ here, for example, by sometimes referring to a proof of a result for $PC_{<}$, for the proof of a similar result for $EPC_{<}$.

There is an obvious need for something better here, presumably, either through adequate abstraction or (the use of) some sort of polymorphic proof-theory, but neither of these were felt to be immediately available and amenable to our task. In any case, in contrast to the situation typically assumed in automated-theorem-proving, the kind of proof structure found in proof-theoretical arguments we present, is, in a very transparent sense, (that of lemmas depending on other lemmas, et cetera), far from *flat*, and it is at least hoped that some of the proofs within this paper, can at some point be better (and more clearly) stratified and/or *encapsulated*, to the extent that, in particular, one may show that a property $\mathfrak{P}$, which we showed for $\text{EPC}_{<}$, Σ_0 -subsume, Σ_0 -tautology, QR_{full} , et cetera, must also hold for some other specific system and corresponding definitions, by similarly showing, all of the attendant “atomic lemmas” hold. Of course, we will not try to make exact the nature of such an undertaking here, but evidently, any such enterprise, if successful, should at least be suggestive as to the kind of abstraction and/or polymorphic proof-theory which would prove most useful.

Most of our proofs are by finitary induction, and virtually all of them are constructive. Though none of them utilize anything exotic, some are quite large. Incidentally, with respect to the “mathematization” of the proof-theory, (obtaining the results in a more general setting), almost every time it was discovered that some “nice mathematics” could be used, (for example, general properties of relations or simple algebraic structures), it was omitted, typically because a finer, more concrete characterization was ultimately required, and it could be swept aside by simply providing that concrete characterization in the first place.

Apologies aside, once we have established the refutation completeness of $\text{EPC}_{<}$ (in Sections 5), a battery of technical results, (in Sections 6 and 7), and thereby, a few big ones, (notably, Corollary 7.2.5), more or less as icing, we examine (in Section 8), the potential role of some clause “normalization” techniques including generalized forms of *replacement factoring* (see, eg. [20]), and a form of *demodulation* (see, eg. [15]), within the context of $\text{EPC}_{<}$, and offer some refinements for the rules, possible further generalization for subsumption, and a generalized form of a special case of *replacement resolution* (see, eg. [20]), which we refer to as *folding*. A few words on the function substitutivity equality axioms, and some concluding remarks are given in Section 9. Each section begins with an overview, which briefly outlines and surveys the contents of that section. All proofs have been placed in an Appendix.

Actually the work of [1, 2] is quite related to what is done here, and deserves greater comparison. Unfortunately, comparison is not straight forward. In [1, 2], the notion of *redundancy*, a somewhat

different notion of “generalized subsumption”, is introduced. This notion is based on a clause ordering which is well-founded on ground clauses, and is basically an equality generalization of the notion of *redundancy* introduced for Ordered Resolution (see eg. [36]). The system based on *strict superposition* [1, 2], Bachmair and Ganzinger present is refutation complete with “redundancy deletion”, and *contextual term rewriting* [1, 2], (the latter essentially corresponding to the “normalization techniques” we describe). Further, the given proof of this completeness, based on the construction of a model for a given *saturated* [1, 2] set of consistent clauses via a canonical rewrite system, is certainly much briefer than the proof-theoretical argument we provide for the completeness of our system. Finally, the modified superposition based calculus presented builds in all the the equality clauses and is arguably of much greater practical significance, and hence, theoretical interest, than $EPC_{<}$.

Still, the approaches taken and results correspondingly obtained simply differ. Unlike those of the modified superposition rules of [1, 2], the deduction rules for $EPC_{<}$ do not enjoy a straight forward *lifting property*, and this is a primary complication as it thwarts use of relatively brief semantic arguments of the sort found in [1]. More importantly, Σ_0 -subsumption, our generalized notion of subsumption, and the notion of redundancy, as introduced in, for example, [1] are independent.

For example, if C_1 and C_2 are the clauses $\{x \neq a, a \neq d\}$ and $\{x \neq a, x \neq d\}$, respectively, (where x is a variable and a and d denote distinct ground terms), then, in general, neither is C_1 redundant in $\{C_2\}$, nor is C_2 redundant in $\{C_1\}$. However both $C_1 \Sigma_0$ -subsumes C_2 and $C_2 \Sigma_0$ -subsumes C_1 .

Further, the notion of redundancy (and its associated deletion) is intended to be well founded (with respect to an introduced clause ordering which is total on ground clauses), while no such assumption is imposed on Σ_0 -subsumption. The semantic characterization we provide for Σ_0 -subsumption indicates that in some sense, Σ_0 -subsumption is the *correct* generalization for subsumption over the *built-in* theory Σ_0 , and since redundancy is not a generalization of Σ_0 -subsumption, redundancy can not, in the same sense, be the *correct* generalization for subsumption over the equality axioms. Finally, it is perhaps worth pointing out that it is unclear in [1, 2] to what extent the non-local condition of redundancy can be effectively established. In contrast, the Σ_0 -subsumption condition is local, (like subsumption), and methods for effective detection are available.

Part 2.

COCOLOG [8, 9, 7, 38] has been systematically developed for the deterministic case. In applications to Discrete Event System(s) (DES) problems, *uncontrollable events* have been largely treated as *exogenous inputs*, whose contribution towards the problem of effective COCOLOG controller *synthesis* was to be largely absorbed into the task of controller *verification*. However, for many DES problems, the formulation of a non-deterministic model seems more natural and more straightforward, or at least easier, than a deterministic counterpart. In particular, allowing non-determinism often allows significantly more compact FSM representations. Further, it may be argued that hybrid systems are essentially non-deterministic. Thus the definition of predicates suitable for control analysis, synthesis and specification, for non-deterministic finite state machines, as well aggregation methodologies for such, may be seen as ground work for future developments in Hybrid-COCOLOG. In this part of thesis (in Chapters 10 and 12), we introduce some analogues of the basic primitives of COCOLOG, suitable for the generalization to non-deterministic FSMs. In particular, a primitive corresponding to the notion of winning reachability is introduced.

State aggregation methods have been proposed as an approach to hybrid systems [32, 11, 12, 14, 40, 23], and large scale discrete systems, where such aggregation is offered as a basis for hierarchical control [13, 42, 41, 43, 40, 3, 5, 4]. The state aggregation technique of [13, 42, 41, 43, 40], principally involves the partitioning of a state space $\mathcal{X}$ of a finite state machine into partition elements or *blocks* $B_1, \dots, B_k$. Higher level dynamics are established in terms of the notion of *dynamical consistency* (DC), which, rather informally here, holds of an order pair $\langle B_i, B_j \rangle$ of blocks if for any state $x \in B_i$, there is a control u_x which will steer the state *directly* into B_j . That is, under the associated control, the first departure from B_i will occur in B_j .

The authors of [13, 42, 41, 43] go on to consider a specific class of partitions: the *in-block controllable* (IBC) partitions, for which there is a lattice structure in terms of which a *hierarchical control policy* is described.

In [3], variants of the notions of DC and IBC are introduced; *ST-DC* and *ST-IBC*, which relativize the notions DC and IBC to given source and target subsets of the state space, and so generalize these notions. The ST-IBC partitions are again shown form a lattice, and the associated results generalize those of [13, 42, 41, 43]. In [11, 12, 14, 40, 23] partitions of continuous state spaces are considered as an aggregation approach to hybrid systems, wherein essentially the partitioning is used to abstract a FSM from a continuous dynamical system. The higher level dynamics are once again described essentially in terms of the notion of DC, and a continuous analogue to the notion of IBC

is considered. A state aggregation method has been proposed for discrete event finite automata in [5, 4] as an alternative approach to hierarchical supervisory control.

In terms of the notion of winning reachability, we propose (in Chapters 11 and 13) and (in the subsequent chapters) analyse some state aggregation techniques for hierarchical control of (non-deterministic) FSMs, possibly suitable for a non-deterministic formulation of Hierarchical-COCOLOG. In particular, in Chapter 11 the notion of a *centered cover* of (the state space of) an FSM is introduced, in terms of which the higher-level *meta-machine* is defined. The notion of centered cover is proposed as a suitable generalization of partition, and plays a fundamental role in what follows. The motivating intuition of a centered cover comes from hybrid-systems: very informally, a member of a centered cover may be thought of representing a *mode*, and the center of such a member as a region of controllable stability within that mode which defines a set of *essentially equivalent* set-points, each mode being associated with such a set.

For a given (centered) cover $\mathfrak{B}$, the states of the (fully observed case) meta-machine correspond to the members or (centered) *blocks* of $\mathfrak{B}$, and the dynamics are specified essentially by saying that a (non-deterministic) transition $B_i \xrightarrow{\alpha} B_j$, between (centered) *blocks* B_i and B_j exists whenever B_j belongs to a minimal subset $\mathcal{J} \subseteq \mathfrak{B}$, such that every state in the center of B_i can be directly winningly steered to the center of some block in $\mathcal{J}$. As a very simple example, a three member cover of a 6 state FSM, and its associated meta-machine are illustrated in figure 0.1. In this cover, the center of each of the blocks (I, II and III) is just the block itself.

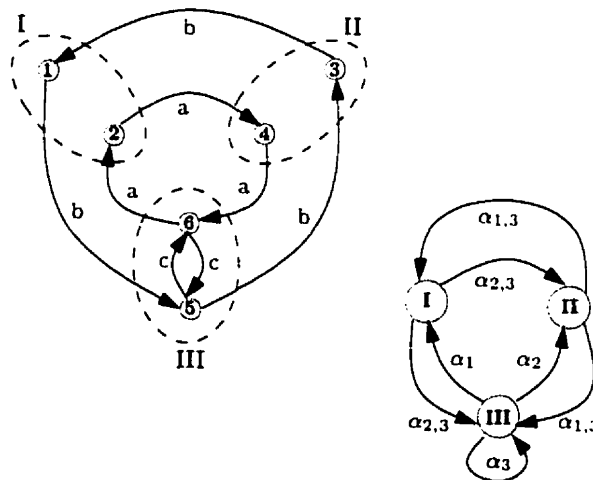


FIGURE 0.1. A Cover of a 6 state FSM and its corresponding meta-machine

Several classes of centered-covers are introduced. By construction, state-to-state *controllability* (ie. winning reachability), of the meta-machine guarantees *center-to-center controllability* of the base FSM. In Chapter 14 *Weakly-quasi-convex* (WQC) covers are introduced and shown to lift center-to-center controllability of the base FSM to state-to-state controllability of the associated meta-machine. *Quasi-convex* (QC) covers are then introduced as a special subclass of the WQC covers, and (in Chapter 15) shown to possess lattice theoretic properties, (they form upper semi-lattices). These results may be seen as a proper generalization of the corresponding theory in [13, 42, 41, 43, 40, 3]. Somewhat surprisingly, in one of the lattices, a *more abstract* machine, (in the sense of the associated ordering) may have more states than the machine which “refines” it.

Unfortunately, the specification of these classes is fairly complex, and the corresponding computation of such covers potentially very demanding. *Regular* covers are introduced (in Chapter 16) partially in response to this situation. Regular covers are shown to similarly lift center-to-center controllability of the base FSM to state-to-state controllability of the associated meta-machine. However the criterion for cover regularity is vastly relaxed in contrast to the criteria for WQC and QC covers. This simplification is gained essentially by abandoning the lattice type properties, (similar to those established for IBC and ST-IBC partitions). Thus regularity is not preserved under any known variant, (two such operations are introduced in this thesis), of *chained union*, (see, eg. [13]).

However, though state-to-state controllability of base FSM was not a nominal assumption for any of the foregoing, (rather, center-to-center controllability was), since the meta-machine associated with a regular cover is (state-to-state) controllable, there is reasonable motivation to consider yet another class of covers, this class appropriate for controllable FSMs. Thus (in Chapter 16) *complete center controllable* (CCC) covers are introduced. In a complete center controllable on an FSM, every state belongs to some center, and every block is center controllable, informally, in the sense that for any two states x_1 and x_2 in the center of a block, it is possible to winningly steer x_1 to x_2 within the block. This class of covers does possess all of the “usual” lattice properties; the CCC covers once again forming a complete lattice. There is no shortage of CCC covers of a controllable machine: in fact, given a controllable FSM, for any partition of its state space there is at least one CCC cover consisting of blocks, one such for each member of the partition, the center of that block being the partition member. Some examples are given. In particular, Chapter 17 contains some examples of regular covers of the state space of a so called transfer line example. Chapter 18 contains a brief summary and an indication of related work for possible future development.

Part 1

$PC_{<}$, $EPC_{<}$, and the Equality Predicate and Subsumption in Resolution-Based Automatic-Theorem-Proving: the First Four Axioms

CHAPTER 1

Some Formal Preliminaries

We assume familiarity with the standard nomenclature and folklore of first-order clause-based resolution/refutation. In particular, we assume familiarity with *ordinary binary merge resolution* (OBMR): where a clause is simply regarded as a (finite) set of literals, (the clause representing a disjunction of those literals), and for any clauses A and B , (such that the variables of B are disjoint from those of A)¹, the set of (OBMR-)resolvents, $OBMR(A, B)$, of A and B are all those clauses C such that for some non-empty subsets $\emptyset \neq A' \subseteq A$ and $\emptyset \neq B' \subseteq B$, the set $A' \cup \{\neg L \mid L \in B'\}$ unifies, to L_0 , say, with most general unifier θ , and $C \equiv A\theta \setminus \{L_0\} \cup B\theta \setminus \{\neg L_0\}$.

By standard definition, a clause is a *tautology* precisely when it contains a pair of oppositely signed but otherwise identical literals. A clause C *subsumes* a clause D precisely when there is some substitution θ , such that $C\theta \subseteq D$. With regards to OBMR, we assume familiarity with what we shall term the **principle of subsumption** (for OBMR)²:

Suppose Γ_B is an OBMR-deduction of a clause B , from a set of clauses $S \cup \{D\}$, in which D occurs, and C subsumes D . Then there is an OBMR-deduction $\Gamma_{B'}$ of a clause B' , where B' subsumes B , from $S \cup \{C\}$, in which at least one instance of C occurs, but in which no instance of D occurs, and where, if, (for Γ either Γ_B or $\Gamma_{B'}$),

$F(\Gamma)$ denotes the underlying graph of Γ , (forget the clauses which label the nodes), regarded as a poset, (so that $n < n'$ if n' is a parent node of n), and for any node n of Γ , $\Gamma(n)$ denotes the clause which labels n in Γ ,

¹Otherwise, A or B is replaced by some alphabetic variant, so as to make their sets of variables disjoint, with the (OBMR-)resolvents defined only up to alphabetic equivalence.

²Though this is a fairly straight forward or even obvious result, and the formulation here is our own, a reasonable reference is [15].

there is an order preserving embedding $j: F(\Gamma_{B'}) \rightarrow F(\Gamma_B)$,
 such that for any node n of $\Gamma_{B'}$, $\Gamma_{B'}(n)$ subsumes $\Gamma_B(j(n))$.

Of course, in practice, a clause is (typically) represented by a repetition-free list of literals, and by an *ordered clause*, we simply mean such a list. Similarly, by the *left-most literal* of an ordered clause C , which satisfies some property, $\mathfrak{P}$, we simply mean the first literal of C which satisfies $\mathfrak{P}$.

For any (ordered) clause C , $|C|$ will denote the underlying set of literals of C .

By a *clause ordering function*, o , we simply mean any function o which orders the literals of any given (unordered) clause, so that, for any clause C , $o(C)$ is an ordered clause whose underlying set of literals, $|o(C)|$, is C . By a *parameterized clause ordering function*, o , we simply mean any clause ordering function which takes additional arguments, (so that, o has arity $n + 1$ for some $n \geq 0$, and for any additional arguments $a_1, \dots, a_n$, of the appropriate sort, for any unordered clause C , $o(\vec{a}, C)$ is an ordered clause whose underlying set of literals is C).

For any, possibly ordered, clauses A and B , whenever A and/or B is ordered, for economy, by the set expressions $A \cup B$, $A \setminus B$, $A \subseteq B$, we shall simply mean the unordered clauses $|A| \cup |B|$, $|A| \setminus |B|$, and the condition that $|A| \subseteq |B|$, respectively. Similarly, for any literal L , by $L \in A$, we mean $L \in |A|$. We shall also say that $A = B$, *modulo ordering*, to indicate that $|A| = |B|$.

Of course, for any (possibly ordered) clause A , and any literal L , L *occurs in* A when $L \in A$. For any clause A , and any predicate symbol P , (we will say that), P *occurs positively in* A if for some tuple, $\vec{t} \equiv t_1, \dots, t_n$, of terms, (where $n \geq 0$ is the arity of P), the literal $P(\vec{t})$ occurs in A . Similarly, P *occurs negatively in* A if for some tuple, $\vec{t}$, of terms, the literal $\neg P(\vec{t})$ occurs in A . A predicate symbol P *occurs in* a clause A if P occurs positively or negatively in A . We may refer to a literal $P(\vec{t})$ ($\neg P(\vec{t})$) as a *positive* (*negative*) P *instance*. (respectively).

For any clause A , A is *pure-positive*, or simply *positive*, if A contains no negatively signed literal. A is *mixed* if A contains at least one negatively signed literal, and *pure-negative*, or simply *negative* if A contains no positively signed literal. For any clause A , and any predicate symbol P , A is *pure-positive with respect to* (or *pure-positive in*) P , if P occurs at most positively in A , (that is, P does not occur negatively in A). A is *mixed with respect to* (or *mixed in*) P , if P occurs negatively in A , and A is *pure-negative with respect to* (or *pure-negative in*) P , if P does not occur positively in A .

DEFINITION 1.0.1. For any sequences Δ , and Δ' , let Δ, Δ' denote the concatenation of the sequences Δ , and Δ' , and let $|\Delta|$ denote the underlying set of Δ , (thus, for example, if Δ is a sequence of clauses, $|\Delta|$ denotes the set of those clauses which occur in the sequence, Δ).

DEFINITION 1.0.2. (Proof Trees)

Let $\mathfrak{R}$ be a set of (sound) deduction rules, such that each rule $r \in \mathfrak{R}$ is a function of a fixed finite arity k , mapping each k -tuple $\vec{C}$ of clauses lying in the domain of r to a set of clauses $r(\vec{C})$.

Let D be a clause, and S be a set of clauses. Then by a $(\mathfrak{R})$ -proof tree of D from S , we mean a finite labelled tree Π such that:

- i. each node of Π is labelled with some unique clause,
and the root node of Π is labelled with D .
- ii. the label C associated with any leaf node of Π is a member of S , and
- iii. if $n_1, \dots, n_k$ are the sequenced parent nodes of a node $n \in \Pi$, labelled with $C_1, \dots, C_k$ and C respectively, then for some $r \in \mathfrak{R}$ of arity k , $C \in r(C_1, \dots, C_k)$.

We will also say such a Π is a *proof tree of D from S in $\mathfrak{R}$* , or simply that Π is an $\mathfrak{R}$ -deduction of D (from S).

DEFINITION 1.0.3. We will say a node $n \in \Pi$ *lies (strictly) below* $n' \in \Pi$, if n' is an (strict) ancestor of n in Π , or, equivalently, if n lies within the branch segment which travels from the root node of Π to terminate at (but does not include) n' . If the context prevents ambiguity, we will also say a clause D *occurs below* a clause C , if there is a node $n \in \Pi$ with label D which lies below the node $n' \in \Pi$ which is being implicitly specified by the label C . If a clause C is associated to a leaf node $n \in \Pi$, we will say that (with respect to Π) C is a *leaf clause*.

If a clause C is associated to a node $n \in \Pi$, and at least one of the parent nodes of n is a leaf node, we will say that (with respect to Π), C is a *leaf resolvent*.

DEFINITION 1.0.4. (Ξ -unsatisfiable)

Given a set of clauses Ξ , we will say a set S of clauses is Ξ -unsatisfiable if $\Xi \cup S$ is an unsatisfiable set of clauses.

DEFINITION 1.0.5. (leaf-index)

Given a deduction Π of a clause D from (or refutation of) a set S , for any $C \in S$, let us say that

C has *leaf-index* k in Π , if C occurs at a leaf node in Π exactly $k \geq 0$ times. We will denote the leaf-index of a clause C in a deduction Π by $k_{\Pi}(C)$.

CHAPTER 2

$PC_{<}$, A System For Binary Clausal Refutation-Resolution

2.0. Overview

In this chapter the binary resolution/refutation system $PC_{<}$ is introduced, and various results for it are presented.

In Section 2.1 the deduction rules for $PC_{<}$ are presented (in Subsection 2.1.1), soundness noted, and (in Subsection 2.1.2), some technical notions are introduced for later use.

In Section 2.2 some technical results for *lifting* are presented. These culminate in Lemma 2.2.4, which, essentially, states the sense in which lifting holds for $PC_{<}$.

Section 2.3 simply contains Theorem 2.3.1, which states the refutation completeness of $PC_{<}$.

In Section 2.4 some more technical results are obtained, (analogous to those of Section 2.2, but), this time for subsumption in $PC_{<}$, which culminate in Lemma 2.4.4. (and Corollary 2.4.1).

In Section 2.5 ($PC_{<-}$) *deduction sequences* are introduced, and in this framework, the refutation completeness of $PC_{<}$ with subsumption (and tautology) deletion is formulated and established.

Section 2.6 consists of a relatively informal discussion about deduction/deletion strategies, with Proposition 2.6.1 its main result. The section addresses the more traditional setting of refutation completeness with subsumption (and tautology) deletion *within the context* of a given deduction/deletion strategy, and illustrates (with Proposition 2.6.1), how the main results of Section 2.5 may be usefully interpreted in such a setting.

Some Comments:

The principal complication, in particular for lifting (addressed in Section 2.2, and the analogous condition for ‘subsumption’ in place of ‘substitution instance’, addressed in Section 2.4), in $PC_{<}$,

over the more typical, (and relatively trivial), situation as found in, say, OBM, is the left-most (negative instance of the $<$ -maximal predicate symbol) restriction of $PC_{<}$. Were this restriction dropped, (there would no longer be any reason to take clauses as ordered, and moreover), it is not hard to see that the deduction rules would lift directly: that is, it *would* be the case that

whenever A is a substitution instance of A' , (A' subsumes A),

B is a substitution instance of B' , (B' subsumes B),

and $C \in PC_{<}(A, B)$,

there exists a $C' \in PC_{<}(A', B')$ such that C is a substitution instance of C' ,

(C' subsumes C , respectively).

The restriction invalidates this property, and it is precisely because such “direct lifting” *does not hold* in $PC_{<}$, that much of the complication arises. In particular, if this property held, then the need for the technical notions introduced in Subsection 2.1.2 of Section 2.1 vanishes.

Nevertheless, in principle at least, the left-most restriction in $PC_{<}$ is desirable, (because, for example, it helps further reduce the number of potential resolvents), and in principle at least, it is worth while to accept the extra “burden of proof”, in order to obtain the desired results for the unadulterated system $PC_{<}$.

2.1. Predicate Colouring ($PC_{<}$):

The Deduction Rules and Some Technical Apparatus

2.1.1. The Deduction Rules

The basic idea behind $PC_{<}$ is to restrict ordinary binary merge resolution in a number of ways. We first restrict resolution between clauses to resolution on a maximal predicate symbol. In other words, we totally order the predicate symbols, allow resolution between two clauses only if their maximal predicate symbol is the same, and then allow resolution only on literals of that predicate name. Thus, in passing to the empty clause, we eliminate literals, predicate name by predicate name. Further, given that two clauses have the same maximal predicate symbol, we only allow resolution between them if (at least) one of them is pure-positive with respect to that predicate name. (This “at least one parent must be pure-positive (with respect to the maximal predicate name)” restriction, is similar to the requirement that *electrons* be pure-positive in *hyper-resolution*, [33].)

Finally, given that these restrictions on resolution are satisfied, we further demand that we merge

resolve only on (unifying sub-clauses of the non-pure-positive clause which contain) the left-most negative literal whose predicate name is mutually maximal. (This literal occurs in the clause which is not pure-positive with respect to the “active” predicate symbol). This deduction system we tentatively entitle *predicate colouring*, and denote it by $PC_{<}$. (The subscript, $<$, in the name ‘ $PC_{<}$ ’, is to indicate a particular (strict) total ordering on (unsigned) predicate symbols, and so we actually intend here the definition of a class of (restricted) ordinary binary merge resolution systems, where the ordering, $<$, is a parameter. The class is also (implicitly) parameterized by the ordering functions o , as seen below.

DEFINITION 2.1.1. ($PC_{<}$ -resolution)

Let $<$ be any strict linear order on predicate names.

Then for any ordered clauses A and B , such that A and B share no variables in common, and any unordered clause C , C is an *unordered $PC_{<}$ -resolvent* of A and B if:

- 1) the $<$ -maximal predicate symbol P , say, occurring in A is the $<$ -maximal predicate symbol which occurs in B ,
- 2) A is pure-positive with respect to P , (B is pure-positive with respect to P),
and B is mixed with respect to P , (A is mixed with respect to P),
- 3) when L_0 , say, denotes the left-most negative P instance, occurring in B , (occurring in A),
there are subclauses $\emptyset \neq A' \subseteq A$ and $L_0 \in B' \subseteq B$, ($L_0 \in A' \subseteq A$ and $\emptyset \neq B' \subseteq B$),
and a most general substitution θ which unifies $A' \cup \{\neg L \mid L \in B'\}$, ($\{\neg L \mid L \in A'\} \cup B'$),
(to the singleton $\{\neg L_0\theta\}$), and
- 4) $C \equiv A\theta \setminus \{\neg L_0\theta\} \mid B\theta \setminus \{L_0\theta\}$, ($C \equiv A\theta \setminus \{L_0\theta\} \mid B\theta \setminus \{\neg L_0\theta\}$, respectively).

For any ordered clauses A and B sharing no variables in common, let $|PC_{<}(A, B)|$ denote the set of all unordered $PC_{<}$ -resolvents C , of A and B . If A and B do have variables in common, let $|PC_{<}(A, B)|$ denote $|PC_{<}(A', B')|$, where A' and B' are some variable disjoint alphabetic variants of A and B respectively.¹

Let o be any fixed parameterized clause ordering function, so that for any ordered clauses A and B , and any unordered clause C , $o(A, B, C)$ is an ordered clause C' such that $|C'| = C$.

¹Thus unordered $PC_{<}$ -resolvents are only defined up to alphabetic equivalence.

Then for any clauses A and B , let $PC_{<}(A, B)$ denote the collection of ordered clauses:

$\{o(A, B, C) \mid C \in |PC_{<}(A, B)|\}$. (Thus, $PC_{<}(A, B)$ is the set $|PC_{<}(A, B)|$, where each clause in the set has been endowed with some (arbitrary) ordering.)

Soundness of the deduction rules for $PC_{<}$ is obvious, (since, for example, for any clauses A and B , clearly, $|PC_{<}(A, B)| \subseteq OBM R(A, B)$).

2.1.2. Some Technical Apparatus for $PC_{<}$

DEFINITION 2.1.2. Given a clause B , with $<$ -maximal predicate symbol P , say, (P possibly '='), B mixed with respect to P , and a finite collection $\mathfrak{C}$ of clauses, such that $\forall C \in \mathfrak{C}$, C is pure-positive with respect to P , let us say that D is $PC_{<}^{+1}$ -deducible from $\{B\} \cup \mathfrak{C}$, if D is pure-positive with respect to P , (or has $<$ -maximal predicate symbol $Q < P$), and there exist sequences (for some $k \geq 1$), $C_0, \dots, C_{k-1}$, and $D_0, \dots, D_{k-1}, D_k$, such that:

$$i. D_0 \equiv B \text{ and } D_k \equiv D.$$

$$ii. \forall i: 0 \leq i \leq k-1, C_i \text{ has } <\text{-maximal predicate symbol } P; C_i \in \mathfrak{C}, \text{ and}$$

$$iii. \forall i: 0 \leq i \leq k-1, D_{i+1} \in PC_{<}^1(D_i, C_i).$$

(Thus a D which is $PC_{<}^{+1}$ -deducible from $\{B\} \cup \mathfrak{C}$ is a deduction-ordering maximal clause obtained from B by $PC_{<}$ -resolutions with clauses in $\mathfrak{C}$, which has $<$ -maximal predicate symbol less than or equal to P , and is pure-positive with respect to P .)

Further, let us say that Π is a $PC_{<}^{+1}$ -deduction of D from $\{B\} \cup \mathfrak{C}$, if Π is a ("linear"-) $PC_{<}$ -deduction of depth k , (for some $k \geq 1$), with $k+1$ leaves $B \equiv D_0, C_0, \dots, C_{k-1}$, and k non-leaf nodes (labelled by) $D_1, \dots, D_k$, as above, and as illustrated in Figure 2.1 below.

Note that since $\mathfrak{C}$ is finite by assumption, and each $PC_{<}$ -resolution eliminates at least one (in particular, the leftmost) negative P literal, clearly there are at most finitely many clauses D $PC_{<}^{+1}$ -deducible from $\{B\} \cup \mathfrak{C}$, and every associated $PC_{<}^{+1}$ -deduction Π (of D from $\{B\} \cup \mathfrak{C}$), can be of depth at most n , where n is the number of distinct negative P instances which occur in B .

Also note that in any $PC_{<}^{+1}$ -deduction Π of D from $\{B\} \cup \mathfrak{C}$, (as above), B occurs exactly once, that occurrence as a leaf, and every resolvent in Π has exactly one parent which is a leaf in $\mathfrak{C}$.

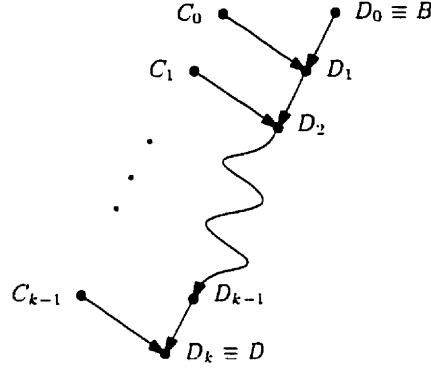


FIGURE 2.1. A Linear Deduction

DEFINITION 2.1.3. Given a $PC_{<}$ -refutation Π of (or a $PC_{<}$ -deduction of a clause D from) a set S , let us say that a node $n \in \Pi$, with label C_n , say, is $PC_{<}$ -initial if:

- i. n is a leaf, or
- ii. for some parent node n' , say, of n , the clause $C_{n'}$, labelling n' is mixed in its $<$ -maximal predicate symbol P , say, (P possibly '='), while C_n is pure-positive with respect to P .

Note that if Π is a $PC_{<}^{+1}$ -deduction, (of a clause D from $\{B\} \cup \mathcal{E}$, say) as in Definition 2.1.2, then the root node of Π , (labelled by D) is $PC_{<}$ -initial. In particular, if Π is any $PC_{<}$ -refutation (of an unsatisfiable set S), the root node of Π , (labelled by the empty clause) is $PC_{<}$ -initial.

OBSERVATION 2.1.1. If n , with label C_n , say, is any non-leaf node in a $PC_{<}$ -refutation of (or a $PC_{<}$ -deduction of a clause D from) a set S , then:

- i) n has exactly one deduction-ordering maximal (proper) ancestor n' , with label $C_{n'}$, say, such that n' is $PC_{<}$ -initial and $C_{n'}$ is mixed with respect to its $<$ -maximal predicate symbol, P (possibly '='), say, and
- ii) there is a finite sequence of nodes $m_1, \dots, m_k$, (for some $k \geq 1$) with labels $C_{m_1}, \dots, C_{m_k}$, say, such that for $j: 1 \leq j \leq k$, C_{m_j} has $<$ -maximal predicate symbol P and is pure-positive with respect to P , and such that the fragment Γ (say,) of Π , rooted at n and pruned in $\{m_1, \dots, m_k\}$, is a depth k ("linear") $PC_{<}$ -deduction of C_n from $\{C_{n'}\} \cup \{C_{m_1}, \dots, C_{m_k}\}$, with $k+1$ leaves, (labelled by) $C_{n'}, C_{m_1}, \dots, C_{m_k}$, and k non-leaf nodes.

In particular, if C_n is pure-positive with respect to P , then the fragment Γ , above, is a $PC_{<}^{+1}$ -deduction of C_n from $\{C_{n'}, C_{m_1}, \dots, C_{m_k}\}$.

DEFINITION 2.1.4. For any $\Pi, n, C_n, n', C_{n'}, m_1, \dots, m_k, C_{m_1}, \dots, C_{m_k}$ as in Observation 2.1.1, above, let us say n (and that C_n) has $PC_{<}$ -initial *principal ancestor* n' , ($C_{n'}$) and *ancillary parents* $m_1, \dots, m_k$, ($C_{m_1}, \dots, C_{m_k}$).

OBSERVATION 2.1.2. Note that (in $PC_{<}$), every ancillary parent (being pure-positive in its $<$ -maximal predicate symbol) is $PC_{<}$ -initial.

Also, if Π is any non-trivial $PC_{<}$ -deduction of a clause D , where D (at the root), is $PC_{<}$ -initial, (in Π), then Π may be *decomposed* into a $PC_{<}^{+1}$ -deduction Π' , of D , from principal ancestor B , (say), and ancillary parents $C_0, \dots, C_{k-1}$, (say, for some $k \geq 1$), a (possibly trivial) $PC_{<}$ -deduction Π_B of B , and (possibly trivial), $PC_{<}$ -deductions, Π_{C_i} , of C_i , for $i: 0 \leq i \leq k-1$, respectively.

2.2. Lifting in $PC_{<}$

LEMMA 2.2.1. Let B and $\mathfrak{C}$ be as in Definition 2.1.2, and suppose B' is any re-ordering of (in particular, the negative P literals in) B . Then, if Π is any $PC_{<}^{+1}$ -deduction of D from $\{B\} \cup \mathfrak{C}$, there is a $PC_{<}^{+1}$ -deduction Π' of D' from $\{B'\} \cup \mathfrak{C}$, where D' is some re-ordering of D , and such that $\forall C \in \mathfrak{C}, k_{\Pi'}(C) = k_{\Pi}(C)$.

Of course, for any re-ordering D' of D , D' subsumes D .

LEMMA 2.2.2. Suppose $B \in PC_{<}^1(A, D)$, for some clauses A , B , and D , such that A has $<$ -maximal predicate symbol P and is pure-positive with respect to P , (P possibly ' $=$ '). (and D has $<$ -maximal predicate symbol P and is mixed with respect to P).

Suppose also that for some clauses A' and D' , (modulo clause ordering), A is a substitution instance of A' , and D is a substitution instance, $D'\theta_D$, (for some substitution θ_D , say), of D' , such that with L_1 , say, the left-most negative P instance in D' , $L_1\theta_D$ occurs as the left-most negative P instance in D .

Then there exists $B' \in PC_{<}^1(A', D')$ such that (modulo ordering), B is a substitution instance of B' .

LEMMA 2.2.3. Let B be a clause with $<$ -maximal predicate symbol P , say, (P possibly '='), B mixed with respect to P , and suppose that (for some $k \geq 1$), $C_0, \dots, C_{k-1}$, (the C_i not necessarily distinct), and $B = D_0, \dots, D_{k-1}$, $D_k = D$, represents a $PC_{<}^{+1}$ -deduction Π ,

(as in Definition 2.1.2). Suppose B is (modulo ordering), a substitution instance of some clause B' , and that $C'_0, \dots, C'_{k-1}$, (the C'_i not necessarily distinct), is a sequence of (not necessarily distinct) clauses such that $\forall i: 0 \leq i \leq k-1$, C_i is (modulo ordering) a substitution instance of C'_i .

Then there exists a $PC_{<}^{+1}$ -deduction Π' , of some clause D' , from principal ancestor B' and ancillary parents $\{C'_0, \dots, C'_{k-1}\}$, where D is (modulo ordering) a substitution instance of D' , and such that $\forall C' \in \{C'_0, \dots, C'_{k-1}\}$, $k_{\Pi'}(C') = ||\{i: 0 \leq i \leq k-1; C'_i = C'\}||$.

LEMMA 2.2.4. Suppose Π is a $PC_{<}$ -deduction of a clause A , from a set S , and (A , at) the root is $PC_{<}$ -initial. Suppose $\mathfrak{L}$ denotes the set of leaf nodes n of Π , and for each $n \in \mathfrak{L}$, let B_n denote the clause (in Π) at n . Suppose that for each $n \in \mathfrak{L}$, B'_n is a clause, of which (modulo ordering), B_n is a substitution instance, and that S' denotes the set comprised of the clauses B'_n , (as n ranges over the leaf nodes of Π).

Then there is a $PC_{<}$ -deduction Π' of a clause A' , $PC_{<}$ -initial (in Π'), of which (modulo ordering), A is a substitution instance, from S' , such that $\forall B' \in S'$, $k_{\Pi'}(B') = ||\{n \in \mathfrak{L}: B'_n = B'\}||$.

Obviously then, the total number $\kappa(\Pi')$ of leaf nodes of Π' , must be equal to the total number $\kappa(\Pi)$, of leaf nodes of Π .

2.3. Refutation Completeness of $PC_{<}$

THEOREM 2.3.1. $PC_{<}$ is refutation complete.

2.4. Subsumption and $PC_{<}$

LEMMA 2.4.1. Suppose $B \in PC_{<}^1(A, D)$, for some clauses A , B , and D , such that A has $<$ -maximal predicate symbol P and is pure-positive with respect to P , (P possibly '='), (and D has $<$ -maximal predicate symbol P and is mixed with respect to P). Suppose also that for some clauses A' and D' , A' subsumes A , and D' subsumes D , and that either:

- i) D' is pure-positive with respect to P , or

- ii) with θ_D some substitution such that $D'\theta_D \subseteq D$, and L_1 say, the left-most negative P instance in D' , $L_1\theta_D$ occurs as the left-most negative P instance in D .

Then, in case (i), D' subsumes B , and in case (ii), either A' subsumes B .

or there exists $B' \in PC_{<}^1(A', D')$ such that B' subsumes B .

LEMMA 2.4.2. Suppose Π is a $PC_{<}^{+1}$ -deduction of a clause D , from ancillary parents in $\mathcal{C}$, (say), and principal ancestor B , B with $<$ -maximal predicate symbol (and mixed with respect to) P , (say, P possibly '='). Then if B' subsumes B and B' is pure-positive with respect to P , B' subsumes D .

LEMMA 2.4.3. Let B be a clause with $<$ -maximal predicate symbol P , say, (P possibly '='), B mixed with respect to P , and suppose that (for some $k \geq 1$), $C_0, \dots, C_{k-1}$, (the C_i not necessarily distinct), and $B = D_0, \dots, D_{k-1}, D_k = D$, represents a $PC_{<}^{+1}$ -deduction Π , (as in Definition 2.1.2). Suppose B' subsumes B , and that $C'_0, \dots, C'_{k-1}$, is a sequence of (not necessarily distinct) clauses such that $\forall i: 0 \leq i \leq k-1, C'_i$ subsumes C_i .

Then either:

- i. B' subsumes D ,
- ii. C'_i subsumes D , for some $i: 0 \leq i \leq k-1$, such that (modulo ordering) $C'_i \neq C_i$, or
- iii. there exists a $PC_{<}^{+1}$ -deduction Π' , of a clause D' , from principal ancestor B' and ancillary parents in $\{C'_0, \dots, C'_{k-1}\}$, where D' subsumes D , and such that $\forall C' \in \{C'_0, \dots, C'_{k-1}\}, \kappa_{\Pi'}(C') \leq ||\{i: 0 \leq i \leq k-1: C'_i = C'\}||$.

LEMMA 2.4.4. Suppose Π is a $PC_{<}$ -deduction of a clause A , from a set S , and (A , at) the root is $PC_{<}$ -initial. Suppose $\mathcal{L}$ denotes the set of leaf nodes n of Π , and for each $n \in \mathcal{L}$, let B_n denote the clause (in Π) at n . Suppose that for each $n \in \mathcal{L}$, B'_n is a clause which subsumes B_n , and that S' denotes the set comprised of the clauses B'_n , (as n ranges over the leaf nodes of Π).

Then there is a $PC_{<}$ -deduction Π' of a clause A' , $PC_{<}$ -initial (in Π'), which subsumes A , from S' , such that $\forall B' \in S', \kappa_{\Pi'}(B') \leq ||\{n \in \mathcal{L}: B'_n = B'\}||$.

Obviously then, the total number $\kappa(\Pi')$ of leaf nodes of Π' , must be less than or equal to the total number $\kappa(\Pi)$, of leaf nodes of Π .

As the empty clause, $\square$ is necessarily $PC_{<}$ -initial, we also immediately have the following:

COROLLARY 2.4.1. Suppose Π is a $PC_{<}$ -refutation of S , that $\mathcal{L}$ denotes the set of leaf nodes n of Π , and for each $n \in \mathcal{L}$, B_n denotes the clause (in Π) at n . Suppose that for each $n \in \mathcal{L}$, B'_n is a clause which subsumes B_n , and that S' denotes the set comprised of the clauses B'_n (as n ranges over the leaf nodes of Π).

Then there is a $PC_{<}$ -refutation Π' of S' , such that $\forall B' \in S', \kappa_{\Pi'}(B') \leq ||\{n \in \mathcal{L}: B'_n = B'\}||$.

Again, obviously, the total number $\kappa(\Pi')$ of leaf nodes of Π' , must be less than or equal to the total number $\kappa(\Pi)$, of leaf nodes of Π .

2.5. Refutation Completeness Under Subsumption (and Tautology) Deletion

DEFINITION 2.5.1. For any finite non-empty set S of clauses, by a $PC_{<}$ -deduction sequence from S , we shall mean any sequence Δ of clauses $C_0, C_1, \dots$ such that either for some $k \geq 0$, $C_k = \square$, or:

- 1) $\forall C \in S, \exists i \geq 0$, such that modulo alphabetic equivalence, $C_i = C$, and
- 2) $\forall i \geq 0, \forall j \geq 0, \forall C \in PC_{<}^1(C_i, C_j), \exists k \geq 0$, such that modulo alphabetic equivalence, $C_k = C$.

By a *partial* $PC_{<}$ -deduction sequence from S , we mean any (possibly empty) initial (consecutive) subsequence $C_1, C_2, \dots, C_k$, (for some $k \geq 0$), of some $PC_{<}$ -deduction sequence from S , $\Delta = C_1, C_2, \dots$.

DEFINITION 2.5.2. For any finite non-empty set S of clauses, by a *subsumption deleted* $PC_{<}$ -deduction sequence from S , we shall mean any sequence Δ of clauses $C_0, C_1, \dots$, such that:

- 1) $\forall C \in S, \exists i \geq 0$, such that C_i subsumes C , and
- 2) $\forall i \geq 0, \forall j \geq 0, \forall C \in PC_{<}^1(C_i, C_j), \exists k \geq 0$, such that C_k subsumes C .

LEMMA 2.5.1. Let $\Delta \equiv C_0, C_1, \dots$, be a subsumption deleted $PC_{<}$ -deduction sequence from S , (for some finite non-empty set of clauses S), and let S' be any finite non-empty set of clauses such that, for every clause $C \in S'$, C_i subsumes C , for some $i \geq 0$.

If Π is any $PC_{<}$ -deduction of (a clause) D , from S' , in which D , (at the root) is $PC_{<}$ -initial, (in Π), then for some $n \geq 0$, C_n subsumes D .

THEOREM 2.5.1. ($PC_{<}$ is refutation complete under subsumption deletion.)

Given any finite unsatisfiable set S of clauses, for any subsumption deleted $PC_{<}$ -deduction sequence from S , $\Delta \equiv C_0, C_1, \dots$, it is the case that for some $n \geq 0$, C_n is the empty clause, $\square$.

LEMMA 2.5.2. If D is a $PC_{<}$ -resolvent of a clause B with a tautology C , then either:

- i) D is a tautology, or
- ii) B subsumes D .

DEFINITION 2.5.3. For any finite non-empty set S of clauses, by a *subsumption and tautology deleted $PC_{<}$ -deduction sequence from S* , we shall mean any sequence Δ of clauses $C_0, C_1, \dots$, such that:

- 1) $\forall C \in S$, either C is a tautology, or $\exists i \geq 0$, such that C_i subsumes C , and
- 2) $\forall i \geq 0, \forall j \geq 0, \forall C \in PC_{<}^1(C_i, C_j)$, either C is a tautology, or $\exists k \geq 0$, such that C_k subsumes C .

LEMMA 2.5.3. Let S be a finite non-empty set of clauses. Given a non-empty subsumption and tautology deleted $PC_{<}$ -deduction sequence from S , $\Delta \equiv C_0, C_1, \dots$, there exists a subsumption deleted $PC_{<}$ -deduction sequence from S , $\Delta' \equiv C'_0, C'_1, \dots$, such that $\forall j \geq 0$, either:

- a) $\exists i \geq 0$ such that $C'_j = C_i$, or
- b) C'_j is a tautology.

COROLLARY 2.5.1. ($PC_{<}$ is refutation complete under subsumption and tautology deletion.)

Given any finite unsatisfiable set S of clauses, for any subsumption and tautology deleted $PC_{<}$ -deduction sequence from S , $\Delta \equiv C_0, C_1, \dots$, it is the case that for some $n \geq 0$, C_n is the empty clause, $\square$.

2.6. Generating Deduction Sequences with Deletion

Refutation completeness under subsumption deletion, (for a resolution system $\mathfrak{R}$), is normally formulated in the context of (a particular deletion strategy, within) a particular deduction strategy. Our approach, via Definition 2.5.1, and Definition 2.5.3,

(and their $EPC_{<}$ analogues, found in the next section), was to deal with the issue of refutation completeness with subsumption deletion, in fashion relatively free of the context of a specific deduction/deletion strategy.

There are of course, various strategies for generating (sound) $(PC_{<-})$ deduction sequences, (with or without deletion), which are guaranteed to eventually deduce the empty clause, from any initially given finite unsatisfiable set of clauses, S . This statement is more generally true of almost any (finitary, first order) refutation complete (clausal, or clause-like) resolution system $\mathcal{R}$, (where, for instance, for any finite set, X , of "clauses", the set of $\mathcal{R}$ -resolvents of the "clauses" in X , $\mathcal{R}(X)$, say, is finite).

But rather than going through the rather lengthy² formulation required to define the appropriate generalized notions and specify exactly which systems the following discussion is amenable to, in the remainder of this section, we will simply use the terms 'tautology', and 'subsumption', 'deduction', 'clause', 'resolvent', ' $\square$ ', et cetera, somewhat informally, and leave the obvious generality (and, for different systems, the necessary modifications) of our remarks for the reader to observe. Because some of the material we will present (particularly that describing the basic deduction strategies) is folklore, presented only to provide a conceptual backdrop for what we wish to convey, and because (in our opinion) the content of this section is relatively trivial, this section will be presented quite informally.

The most basic deduction strategy is the "shell method", where, essentially:

With S_0 denoting the given initial given finite clause set, inductively, for any $i \geq 0$, one successively constructs $S_{i+1} \equiv S_i \cup \mathcal{R}(S_i)$, where, (for each $i \geq 0$) $\mathcal{R}(S_i)$ denotes the set of all resolvents generated from (pairs of, -in those cases where $\mathcal{R}$ consists of only binary deduction rules), clauses in S_i .

Of course, it is intended that the construction of the successor shells, $S_{i+1}, \dots$, may be stopped once $\square \in S_i$. Also, of course, by refutation completeness, by definition, for any (finite) unsatisfiable S , there is an $\mathcal{R}$ -deduction of the empty clause, $\square$, from $S_0 \equiv S$, and if C has a depth k deduction from S , clearly $C \in S_k$, so there is some finite $n \geq 0$ such that $\square \in S_n$.

²It would be, at least relative to the intended content of this section.

In the context of the shell method strategy³, the property of refutation completeness under subsumption (and tautology) deletion is naturally formulated in terms of claiming that in passing from S_i to S_{i+1} , we can first “thin” S_i to any $S'_i \subseteq S_i$ such that for any clause $C \in S_i$, (either C is a tautology, or), there is some $C' \in S'_i$, such that C' subsumes C , (and then take $S_{i+1} \equiv S'_i \cup \mathcal{R}(S'_i)$).

Presumably better than the shell method strategy, is a “best-first” strategy, where, for example, essentially:

a sequential data base of clauses is dynamically maintained, where with S_0 denoting the given initial given finite clause set, the data base initially contains S_0 ,

a “resolution table” indicating which resolutions have (or have not yet) been performed, is dynamically maintained,

there is a positive integer “fitness function”, $f(\cdot)$, which assigns to each clause, C , a fitness, $f(C) \geq 0$, such that for any integer $n \geq 0$, any initially given finite set S of clauses, there are at most finitely many⁴ clauses C deducible from S with $f(C) \leq n$.

and at each stage, a clause C such that a resolution with a clause C' , (where all resolvents between C and C' are generated), which precedes C in the data base, has not yet been performed, for which $f(C)$ is minimal (amongst such clauses), is selected from the data base, together with some C' as above, all resolvents of C and C' are generated and each resolvent that is not, modulo alphabetic equivalence, already in the data base, is added to the data base, and finally the resolution table is updated.

(Usually other things are going on here, as well. For example, clauses may be assigned integer names, and, as well, the names of the parent clauses from which they were resolved. In this way, at any stage, for any clause C in the data-base, it is a relatively easy matter to construct the deduction tree of C , and thereby determine, for any clause C' in the data base, if C' was an ancestor of C . Also, for proof retrieval, for instance, sometimes deleted clauses aren't actually deleted from the

³It is in this context, that Sibert [35] shows the refutation completeness of his system under (his) subsumption deletion. However his subsumption relation is little more than the “classical” one, and, (basically because of an absence of a “left-most literal restriction”), unlike our rules, his deduction rules (directly) lift with respect to subsumption.

⁴For example, $f(C)$ might be $f_1(C) + f_2(C)$, where $f_1(\cdot)$ is positive and monotonically increasing with respect to the clause's length, and $f_2(\cdot)$ is positive and monotonically increasing with the maximum of the (syntactic) lengths of the terms occurring in the clause.

data base, but merely marked/tagged as such.)

The procedure is, of course, expected to stop once the empty clause has been deduced.

Again, by refutation completeness, by definition, for any (finite) unsatisfiable S , there is an $\mathfrak{R}$ -deduction of the empty clause, $\square$, from $S_0 \equiv S$. Since clearly any clause C in the data base has a deduction from S , by our assumption on the fitness function $f(\cdot)$, for any pair of clauses C' and C in the data base, eventually we will (either deduce the empty clause, or) resolve C against C' . Thus, if C has a depth k deduction from S , we will eventually (unless at some stage before, we have the empty clause in the data base), have C in the data base, and so, (if S is unsatisfiable), at some stage we will have deduced the empty clause.

In this context, the question of refutation completeness under subsumption (and tautology) deletion leads naturally to the notions of "forward" versus "backward" (subsumption-) deletion.

The property of refutation completeness under forward subsumption (and tautology) deletion, is then (formulated as) the assertion that the above procedure will eventually deduce the empty clause, when it is modified so that,

where before:

"... all resolvents of C and C' are generated and each resolvent that is not, modulo alphabetic equivalence, already in the data base, is added ..."

now:

"... all resolvents of C and C' are generated and each resolvent that is not (a tautology and is not) subsumed by some clause already in the data base, is added ..."

A (best-first) deduction/deletion strategy involving backward (subsumption) deletion is any such (best-first) strategy as above, where, however, one is allowed to delete clauses from the data base, if they are subsumed by some newly deduced (or introduced) clause, and typically, providing some extra criterion is satisfied, this criterion specified by the strategy. Depending on the particular deletion strategy, when backward subsumption is detected, one may also be allowed to delete additional clauses. For example, when a new clause C subsumes a clause C' already in the data base and the criterion is satisfied, (depending on the particular deletion strategy), one may also be allowed to delete from the data base, all clauses, D , other than C , which have C' as an ancestor, (according

to their recorded deduction tree).

The criterion stipulates that the subsuming clause is somehow better⁵ than the clause subsumed, and, as we later show, may play an important role in the preservation of refutation completeness. The criterion for backward (subsumption) deletion specified in the context of a particular best-first deduction strategy, (in application to a specific resolution system $\mathfrak{R}$), together with the description of which clauses should be deleted, can be referred to as that strategy's *backward deletion policy*.

Refutation completeness under (restricted) backward subsumption deletion of course asserts that some such strategy, (when applied to some specific system), involving backward deletion is refutation complete, (in the obvious sense), and (in this best-first context), a system (really, the specific strategy should also be given), is said to be refutation complete under both forward and (restricted) backward subsumption (as well as tautology) deletion if any such strategy involving backward deletion can be given, such that when modified so that forward subsumption (and tautology) deletion is present, the resulting strategy remains refutation complete.

Evidently without any deletion involved, both the shell method as well as any best-first deduction strategy (as above), (when applied to $PC_{<}$), for any given initial (finite) clause set S , naturally yield a $(PC_{<-})$ deduction sequence, Δ , as in Definition 2.5.1, such that for any clause C , $C \in |\Delta|$ if and only if the strategy deduced C at some point. Specifically:

For the shell method, let Δ_0 be some sequencing of S_0 , for $i \geq 0$, let Δ_{i+1} denote some sequencing of those clauses in $S_{i+1} \setminus S_i$, and let $\Delta \equiv \Delta_0, \Delta_1, \dots$.

For a best-first strategy (as above), C_i , the i^{th} clause in Δ , is the i^{th} clause in sequence of clauses given in the data base at any stage where the data base contains i clauses.

Moreover, (evidently) any implementation of the shell method (without deletion), can, without loss of efficiency, be equivalently described in terms of the sequential construction of Δ , as above, (where one still keeps track of which shell (set) each $C_i \in |\Delta|$ belongs). Here at any stage i , the sequence $\Delta_0, \Delta_1, \dots, \Delta_i$, is the current data base, which is now sequentially constructed.

In terms of any such sequential description of the shell method deduction strategy, the notions of forward subsumption and backward subsumption deletion are naturally formulated:

⁵For example, the criterion might be that the subsuming clause is shorter than the one subsumed.

At any stage, $i \geq 0$, we have a finite repetition free sequence: $\Delta_0, \Delta_1, \dots, \Delta_i \equiv C_1, C_2, \dots, C_m$, say, (for some $m \geq 0$), where $S_i = |\Delta_0, \Delta_1, \dots, \Delta_i|$.

In passing from S_i to $S'_i \subseteq S_i$, clauses may be sequentially deleted, so that if S'_i has $k \geq 0$ fewer clauses than S_i , then there is a sequence: $S_i \equiv T_0 \supseteq T_1 \supseteq \dots \supseteq T_k \equiv S'_i$, of clause sets T_j , such that, $(\forall 0 \leq j < k)$, T_j contains exactly T_{j+1} together with one clause not in T_{j+1} .

In each transition from T_j to T_{j+1} , some clause, $C_l \in S_i \cap T_j$ is subsumed by some other (distinct) clause $C_n \in S_i \cap T_j$, and C_l is the clause deleted, (so that $T_j = T_{j+1} \cup \{C_l\}$). If $l < n$, then the deletion is an instance of backward subsumption deletion, and otherwise, (if $l > n$), the deletion is an instance of forward subsumption deletion.

We now take the meaning of “the shell method with forward subsumption (and tautology) deletion”. to be clear, (as a description of a class of deduction/deletion strategies).

Also, modulo the backward deletion policy, we take as clear the meaning of “the shell method with (forward subsumption (and tautology) deletion, and), (restricted) backward deletion”.

We have not shown that $PC_{<}$ -deduction is refutation complete under (general) subsumption (and tautology) deletion in the context of the shell method deduction strategy⁶. and certainly we have not shown refutation completeness in the context of an arbitrary best-first deduction/deletion strategy involving backward deletion.

It is easy to see however that for any initial set of clauses S , the deduction sequence, Δ , naturally generated by both the⁷ shell method with (only) forward subsumption (and tautology) deletion, and any best-first deduction strategy with (only) forward subsumption (and tautology) deletion, is a subsumption (and tautology) deleted ($PC_{<}$ -) deduction sequence from S , as in Definition 2.5.2, (or as in Definition 2.5.3, respectively). Thus, we may apply Theorem 2.5.1. (or Corollary 2.5.1), to conclude that any of these deduction/deletion strategies (employing only forward subsumption and tautology deletion), is refutation complete.

Our current framework, (of subsumption (and tautology) deleted ($PC_{<}$ -) deduction sequences),

⁶I don't know whether or not this is true. With respect to the approach taken in this paper, it seems clear that for any system in which subsumption directly lifts (with respect to the deduction rules, such as Sibert's [35]), the result, (though not immediate from Theorem 2.5.1 or Corollary 2.5.1), has a fairly straight forward proof, involving, essentially, that subsumption deletion can not increase the depth of a refutation. Unfortunately however, this is not the case for $PC_{<}$.

⁷Strictly speaking, this should instead be: “... naturally generated by *any* ... deletion, as well as any best-first deduction ...”, because of the choice involved in the sequencing of the clauses

however also allows us to say something positive about refutation completeness of these systems with some backward deletion.

The key idea is to ensure that the binary relation, $\supseteq$, say, defined by $C \supseteq C'$ when C' subsumes C and C' satisfies the additional criterion for the backward subsumption of C , is a well-founded pre-order relation⁸, in the obvious sense that, for any infinite sequence of clauses: $C_0, C_1, \dots$ such that $\forall i \geq 0, C_i \supseteq C_{i+1}$, there exist indices $0 \leq j < i$, such that $C_i \supseteq C_j$.

(Further, as part of the deletion policy, the deletion of C by the backward subsumption of C by C' , should not take place if $C' \supseteq C$, but rather, in such a case, C' should be deleted, by the forward subsumption of C' by C . The reader will see how all this is used in Proposition 2.6.1, below.)

OBSERVATION 2.6.1. Subsumption alone is not well founded. (as a pre-order).

EXAMPLE 2.6.1. For example⁹, consider the sequence of clauses: $R_0, R_1, R_2, \dots$, where for some binary predicate symbol P , (for instance, $P = '='$), and variables $x_1, x_2, \dots$.

$R_0 \equiv P(x_1, x_1)$, and for all $i \geq 1$, $R_i \equiv P(x_1, x_2) \mid P(x_2, x_3) \mid \dots \mid P(x_{2^i-1}, x_{2^i}) \mid P(x_{2^i}, x_1)$.

(Thus R_i contains 2^i distinct variables; 2^i literals, and describes a "necklace" with 2^n "beads".)

Evidently, $R_1 \equiv P(x_1, x_2) \mid P(x_2, x_1)$ subsumes R_0 , and more generally,

$\forall i \geq 0$, R_{i+1} subsumes R_i . For if $i \geq 1$, then with θ the substitution defined by:

$$\forall j: 1 \leq j \leq 2^i, \quad \theta: x_j \mapsto y_j, \quad \text{and} \quad \forall j: 2^i + 1 \leq j \leq 2^{i+1}, \quad \theta: x_j \mapsto y_{j-2^i}.$$

(for some new variables $y_1, \dots, y_{2^i}$), we have:

$$\begin{aligned} R_{i+1}\theta &= (P(x_1, x_2) \mid P(x_2, x_3) \mid \dots \mid P(x_{2^i+1-1}, x_{2^i+1}) \mid P(x_{2^i+1}, x_1))\theta. \\ &= P(y_1, y_2) \mid \dots \mid P(y_{2^i-1}, y_{2^i}) \mid P(y_{2^i}, y_1), \end{aligned}$$

which (is alphabetically equivalent to R_i , and hence) subsumes R_i .

Further, (it is easy to see that), for no $0 \leq j < i$, is it the case that R_j subsumes R_i .

However, if, (for any clauses C and C'), $C \supseteq C'$ is defined to be the condition that:

C' subsumes C and the length (the number of literals) of C' is no greater than that of C ,

then it is not hard to show that $\supseteq$ is a pre-order, and in any infinite sequence $C_0, C_1, \dots$ such

⁸A pre-order is a binary relation, $\sqsubseteq$, satisfying the axioms: $x \sqsubseteq x$ (reflexivity), and $x \sqsubseteq y \wedge y \sqsubseteq z \Rightarrow x \sqsubseteq z$ (transitivity). That is, a pre-order is a partial order, $\sqsubseteq$, except possibly not satisfying the anti-symmetry axiom: $x \sqsubseteq y \wedge y \sqsubseteq x \Rightarrow x = y$.

⁹The same example also shows that similarly, Σ_0 -subsumption, introduced in the next section, is not well founded.

that $\forall i \geq 0, C_i \supseteq C_{i+1}$, there exist indices $0 \leq j < i$, such that $C_i \supseteq C_j$. That is, then $\supseteq$ is a well-founded pre-order¹⁰.

For $PC_{<}$, (and $EPC_{<}$, introduced in the next section), probably better is the following pre-order.

Let $P_0, P_1, \dots, P_k$, denote the predicate symbols occurring in a given refutation problem, ordered so that $\forall k': 0 \leq k' < k, P_{k'} < P_{k'+1}$.

For any clause C and any predicate symbol P , let $\#(C, P, +)$ denote the number of (distinct) literals in C which are positive instances of the predicate symbol P , and let $\#(C, P, -)$ denote the number of (distinct) literals in C which are negative instances of the predicate symbol P .

Let $\supseteq$ be defined by: (for any clauses C and C'), $C \supseteq C'$ if (and only if) C' subsumes C and either:

- i. there exists $k': 0 \leq k' \leq k$ such that:
for all $k'': k' < k'' \leq k, \#(C', P_{k''}, -) = \#(C, P_{k''}, -)$ and $\#(C', P_{k''}, +) = \#(C, P_{k''}, +)$,
and either $\#(C', P_{k'}, -) < \#(C, P_{k'}, -)$ or,
 $\#(C', P_{k'}, -) = \#(C, P_{k'}, -)$ and $\#(C', P_{k'}, +) < \#(C, P_{k'}, +)$, or
- ii. $\forall k': 0 \leq k' \leq k, \#(C', P_{k'}, +) = \#(C, P_{k'}, +)$ and $\#(C', P_{k'}, -) = \#(C, P_{k'}, -)$.
(from which it clearly follows that C and C' are of equal length).

It is not hard to show that with this definition for $\supseteq$, $\supseteq$ is a well-founded pre-order¹¹.

Thus we have at least one “nice”¹² well-founded pre-order $\supseteq$.

However, for the following proposition, any well-founded pre-order $\supseteq$,

(such that if $C \supseteq C'$, then C' subsumes C), will do.

PROPOSITION 2.6.1. Let $\supseteq$ be any well-founded pre-order on clauses such that whenever $C \supseteq C'$, C' subsumes C . Let $\mathfrak{S}$ be any shell method or best-first deduction strategy (as described above), for $PC_{<}$, with forward subsumption (and tautology) deletion, and with (restricted) backward

¹⁰It may also be shown to be such, (though it's somewhat trickier), if $C \supseteq C'$ is defined to be the condition that: the length of C' is no greater than that of C , and $C' \Sigma_0$ -subsumes C , where Σ_0 -subsumption is the generalized subsumption relation introduced in the next section.

¹¹It may also be shown to be a well-founded pre-order, when, in its definition (just given), ‘subsumes’, is replaced by ‘ Σ_0 -subsumes’, which gives us a well-founded pre-order, well-suited for $EPC_{<}$, (Σ_0 -subsumes and $EPC_{<}$ subjects of the next section).

¹²At least because deduction in $PC_{<}$ is partially goal directed: -“reduce the maximal predicate symbol”, and in an obvious (though here at least, informal) sense, the last example given for a well-founded pre-order $\supseteq$, “meshes well” with this directed deduction.

(subsumption) deletion, where the criterion for the backward deletion of (an established) clause C by a (newly deduced) clause C' is given by $C \supseteq C'$, and, in which case, providing (and only if) it is not the case that $C' \supseteq C$, the backward deletion policy allows the deletion of C .

Then $\mathcal{S}$ is refutation complete.

CHAPTER 3

Σ_0 , Σ_0 -subsumption and Σ_0 -tautology

3.0. Overview

The purpose of this chapter is to introduce the logical theory Σ_0 , and the notions of Σ_0 -tautology and Σ_0 -subsumption which are extensions of the classical notions of tautology and subsumption. We also obtain some results which indicate that these generalizations are (in an obvious sense) *natural* for the theory Σ_0 . We also establish some equivalences for alternative formulations intended to more directly support reasonably efficient¹ computational implementation.

Section 3.1 introduces Σ_0 , the set the equality clauses which will be *built-in*.

Section 3.2 is divided into subsections. Subsection 3.2.1 provides some background and motivation for the generalization of the notions of tautology and subsumption, and as well, is a very brief overview for Section 3.2. In Subsection 3.2.2 some formal preliminaries are presented. In Subsection 3.2.3 the notions of Σ_0 -tautology (Definition 3.2.17), and Σ_0 -subsumption (Definition 3.2.18), are finally introduced, and some results for these notions are presented. In particular, Theorem 3.2.2 establishes a (desirable) semantic characterization for Σ_0 -tautology.

In Section 3.3 we introduce the notion of weak completeness, and obtain (in Theorem 3.3.3) a partial (semantic) characterization for Σ_0 -subsumption in terms of Σ_0 -tautology. Soundness of Σ_0 -subsumption is given in Theorem 3.3.4.

¹Of course, by “reasonably efficient” here, we do not mean “polynomial”.

3.1. Σ_0

DEFINITION 3.1.1. (Equality clause)

A clause C is an *equality clause* if C contains only literals based on the equality predicate '='.

DEFINITION 3.1.2. For any $n \geq 0$ and any n -tuples of terms, $\vec{s} \equiv s_1, \dots, s_n$ and $\vec{t} \equiv t_1, \dots, t_n$, let $\neq(\vec{s}, \vec{t})$ denote the (sub-) clause $\neq(s_1, t_1) \mid \dots \mid \neq(s_i, t_i) \mid \dots \mid \neq(s_n, t_n)$.

DEFINITION 3.1.3. A literal L is a *trivial equality (literal)* if L is of the form $=(t, t)$, (for some term t). Similarly, a literal L is a *trivial inequality (literal)* if L is of the form $\neq(t, t)$, (for some term t). A clause C is a *trivial inequality clause* if every literal which occurs in C is a trivial inequality.

The axiom schemata for the binary equality predicate are:

$$\begin{aligned}
 x &= x && \text{(reflexivity)} \\
 x = y &\Rightarrow y = x && \text{(symmetry)} \\
 x = y &\Rightarrow (y = z \Rightarrow x = z) && \text{(transitivity)} \\
 \vec{x} = \vec{y} &\Rightarrow (P(\vec{x}) \Rightarrow P(\vec{y})) (\forall P) && \text{(predicate substitutivity)} \\
 \vec{x} = \vec{y} &\Rightarrow f(\vec{x}) = f(\vec{y}) (\forall f) && \text{(function substitutivity)}
 \end{aligned}$$

where $\vec{x} \equiv x_1, \dots, x_n$ and $\vec{y} \equiv y_1, \dots, y_n$ are each lists of n distinct variables, which share no variable in common, where $n \geq 1$ is the arity of P (or f), and further, where $\vec{x} = \vec{y}$ denotes the conjunction $x_1 = y_1 \wedge \dots \wedge x_i = y_i \wedge \dots \wedge x_n = y_n$.

(Alternatively, but logically equivalently, the schemata for the predicate substitutivity and function substitutivity may be expressed as:

$$x = y \Rightarrow (P(\vec{u}, x, \vec{v}) \Rightarrow P(\vec{u}, y, \vec{v})) (\forall P) \quad \text{and} \quad x = y \Rightarrow f(\vec{u}, x, \vec{v}) = f(\vec{u}, y, \vec{v}) (\forall f),$$

respectively.)

Using the fairly standard notation of representing a clause by separating the clause's members (or sub-sets) with the symbol '|' (to represent logical 'or'), the above equality axiom schemata may be expressed respectively in clausal form as:

DEFINITION 3.1.4. (The Equality axiom clauses)

$=(x, x)$	(QR)
$\neq(x, y) \mid =(y, x)$	(QS)
$\neq(x, y) \mid \neq(y, z) \mid =(x, z)$	(QT)
$\neq(\vec{x}, \vec{y}) \mid \neg P(\vec{x}) \mid P(\vec{y})$	(QP)
$\neq(\vec{x}, \vec{y}) \mid =(f(\vec{x}), f(\vec{y}))$	(Qf)

where again, $\vec{x} \equiv x_1, \dots, x_n$ and $\vec{y} \equiv y_1, \dots, y_n$ are each lists of n distinct variables, which share no variable in common, and $n \geq 1$ is the arity of P (or f).

DEFINITION 3.1.5. Let Σ_0 denote the collection consisting of the clauses QR , QS , QT , together with the clauses of the form QP , (for the various predicate symbols P).

3.2. Σ_0 -tautology and Σ_0 -subsumption

3.2.1. Motivational Introduction and Overview

By standard definition, a clause is a tautology precisely when it contains a pair of pair of oppositely signed but otherwise identical literals. In resolution refutation systems such as OBMR and $EPC_{<}$, the latter introduced in the next section, such clauses can be deleted (without loss of refutation completeness). When the clausal axioms (or axiom schemes) QR , QS , QT , and QP are *built-in* as in $EPC_{<}$, any clause containing a substitution instance of (ie. subsumed by) one of these (axiom) clauses may, for similar reasons, also be deleted. In other words, the notion of tautology may be extended.

Moreover, (in $EPC_{<}$), any substitution instance of a (ordinary binary merge) resolvent D of a clause C with, for example, the axiom of transitivity for '=', QT , (or QP , or QS) can also be deleted, and we are naturally led to extend the notion of subsumption, in some way such that in this extended sense, such a C subsumes (such a) D .

For a trivial example, $=(a(), b())$ subsumes, in this extended sense, $\neq(b(), c()) \mid =(a(), c())$, the latter being a resolvent from the former with QT .

Certainly, $QT \models =(a(), b()) \Rightarrow \neq(b(), c()) \mid =(a(), c())$.

The motivation for the generalization (or extension) of the notion of subsumption is the knowledge that more clauses may be deleted than would otherwise be, without threat to completeness. Certainly, the verification for a generalized subsumption, of one clause by another, must be computationally feasible in the sense that over the course of solving some problem, one expects to need to perform such a verification a larger number of times, and we would like an automatic-theorem-prover to be able to solve problems in reasonable amounts of time.

Of course, it is also not essential (or even necessarily expedient) for refutation completeness, that our generalized subsumption relation be complete. In other words, we are *not* insisting that if

$\Sigma_0 \models C \Rightarrow D$, then C subsumes, in the generalized sense, D .

In this section we will introduce our extensions, Σ_0 -tautology and Σ_0 -subsumption, of tautology and subsumption respectively, and in Theorem 3.2.2 and Theorem 3.3.3, we obtain their (partial) “semantic” characterization.

The “standard” test for the condition that clause D subsumes a clause C , is to first negate C , to produce a set $\mathcal{U}$ of unit (length) clauses whose literals are obtained as the negations of those of C , (the “or” turns into an “and”), where the variables of C have been “frozen” into new (Skolem) constants. Then one attempts to reduce D to the empty clause via resolutions where the resolvent has as one of its parent clauses some member of $\mathcal{U}$.

The basic/initial idea for testing the generalized subsumption is essentially the modification of the “standard” subsumption verification is as follows: one “inflates” the set of ground units $\mathcal{U}$, via the set of positive (ground) equality facts in $\mathcal{U}$, by applying the clausal axioms of Σ_0 completely over the (finite) set of (ground) terms occurring in $\mathcal{U}$.

For example, in the verification that, in the extended sense, A subsumes B ,

when $A \equiv \neg(a(), b())$, and $B \equiv \neg(b(), y) \mid \neg(a(), y)$, before we search for a unit refutation of A via the set of unit clauses $\{\neg(b(), \bar{y}), \neg(a(), \bar{y})\}$, (as we would do in the “standard” subsumption checking process), we first augment this set of units with the unit clause $\neg(a(), b())$.

This “naive approach” would be valid in $\text{EPC}_<$ in the sense that it would not threaten (theoretical) completeness. But it would generate computational difficulties by creating a combinatorial explosion (much like “naive resolution-refutation” itself). This could happen, for instance, because of

the presence of a “large” number of inequality literals within the clause whose subsumed status is being checked, and the presence of literals whose predicate are of “large” arity. For example, via transitivity, 5 stated non-trivial equalities may give rise to 10 more equalities (modulo symmetry!), and possibly the equivalence of 6 objects, in the presence of which, a single literal whose predicate is of effective substitutional arity 4, may give rise to 1295 others. Such a situation would result in a large amount of temporary representational construction, and potentially in an exorbitantly expensive unit resolution-refutation based subsumption verification.

In (3) of Theorem 3.2.1, and (6) of Theorem 3.2.3, we present characterizations for Σ_0 -tautology and Σ_0 -subsumption upon which “reasonably efficient” corresponding verification procedures may be based. Sketches of these algorithms, together with examples of Σ_0 -subsumption and Σ_0 -tautology in terms of these characterizations, are given in Appendix H.

3.2.2. Some Formal Preliminaries

DEFINITION 3.2.1. (Substitution extensions)

Given substitutions σ , and θ , we will say that σ is an *extension* of θ if there exists a substitution σ' such that $\sigma = \theta\sigma'$. In such a case, σ *factors through* θ .

DEFINITION 3.2.2. (Compatibility of substitutions)

Let $\Theta \equiv \{\theta_i\}_{i \in N}$, be a finite set of substitutions. Then the substitutions in Θ are *compatible* if there exists a substitution σ , such that $\forall \theta_i \in \Theta$, σ is an extension of θ_i .

In other words, $\forall \theta_i \in \Theta$, $\exists \sigma_i$, : $\sigma = \theta_i \sigma_i$.

DEFINITION 3.2.3. For any (unordered) clause C , let the *symmetric closure* of C denote the (unordered) clause C' which is minimal such that:

- i. $C \subseteq C'$
- ii. for any equality literal $=(s, t) \in C$, $=(t, s) \in C'$, and
- iii. for any inequality literal $\neq(s, t) \in C$, $\neq(t, s) \in C'$.

We will say a clause C is *symmetrically closed* if the symmetric closure of C is simply C .

Evidently, a clause C is symmetrically closed exactly when ii.) and iii.), above, hold for C in place of C' . Also, clearly, if C is symmetrically closed then the symmetric closure of C is simply C , and the symmetric closure of any clause C is symmetrically closed.

DEFINITION 3.2.4. For any clause A

- 1) let A^+ denote the set of all positive equality literals in A ,
 - 2) let A^- denote the set of all inequality literals in A , and
 - 3) let A^0 denote the set of all literals of A which have some predicate symbol $P \neq '='$.
- (That is, $A^0 \equiv A \setminus (A^+ \cup A^-)$.)

DEFINITION 3.2.5. For non-equality literals $L_1 \equiv \delta P(\vec{s})$ and $L_2 \equiv \delta P(\vec{t})$ of the same sign, δ , and predicate symbol, $P \neq '='$, (where $\vec{s}$ and $\vec{t}$ are n -tuples of terms, where $n \geq 0$ is the arity of P), let $\langle L_1 | L_2 \rangle$ denote the pure negative equality clause $\neq(\vec{s}, \vec{t})$.

DEFINITION 3.2.6. The Operation: $\overline{(\cdot)}$

For t a term, a sequence of terms, a set of terms, a collection of sets of terms, a literal, or a clause, (et cetera), we define a purely formal operation $\overline{(\cdot)} : t \mapsto \overline{(t)}$, which (*formally*) *freezes* or *grounds* the variables within an expression, substituting, for each variable, x , a uniquely corresponding formal constant symbol, $\bar{x}$. This operation may be defined recursively as follows.

For t a variable x ,	$\overline{(\cdot)} : x \mapsto \bar{x}$.
For t a term $f(a_1, \dots, a_k)$,	$\overline{(\cdot)} : f(a_1, \dots, a_k) \mapsto f(\overline{(a_1)}, \dots, \overline{(a_k)})$.
For t a sequence $\{t_1, \dots, t_k\}$ of terms,	$\overline{(\cdot)} : \{t_1, \dots, t_k\} \mapsto \{\overline{(t_1)}, \dots, \overline{(t_k)}\}$.
For t a set α of terms,	$\overline{(\cdot)} : \alpha \mapsto \{\overline{(t)} : t \in \alpha\}$.
For t a collection $\mathcal{T}$ of sets of terms,	$\overline{(\cdot)} : \mathcal{T} \mapsto \{\overline{(\alpha)} : \alpha \in \mathcal{T}\}$.
For t a positive literal $P(a_1, \dots, a_k)$,	$\overline{(\cdot)} : P(a_1, \dots, a_k) \mapsto P(\overline{(a_1)}, \dots, \overline{(a_k)})$.
For t a negative literal $\neg P(a_1, \dots, a_k)$,	$\overline{(\cdot)} : \neg P(a_1, \dots, a_k) \mapsto \neg P(\overline{(a_1)}, \dots, \overline{(a_k)})$.
For t a clause $L_1 \mid \dots \mid L_m$,	$\overline{(\cdot)} : L_1 \mid \dots \mid L_m \mapsto \overline{(L_1)} \mid \dots \mid \overline{(L_m)}$.

We will typically simply write $\bar{t}$ instead of $\overline{(t)}$, though it is the latter which should be understood as intended. For any t (for which $\bar{t}$ is defined), t is *ground* if and only if $\bar{t} = t$.

For any term (or clause, et cetera) X which contains no barred variables, if $\bar{X} \neq X$, we shall say $\bar{X}$ is a *barred* or (*formally*) *frozen* term (clause, et cetera).

For any barred term (or clause, et cetera) X , $\bar{X} = X$.

DEFINITION 3.2.7. The Operation: $(\cdot)$

In a completely similar fashion as for $(\cdot)$, we define a purely formal operation $(\cdot)$. For a term, a sequence of terms, a set of terms, a collection of sets of terms, a literal, or a clause, (et cetera), t , let $\underline{t}$ denote the *unbaring* or *thawing* of all the barred or frozen (formally ground) variables $\bar{x} \in t \mapsto x \in \underline{t}$, so that if $t = \bar{s}$, (for some term, sequence of terms, substitution, literal, clause, et cetera, s), then $\underline{t} = s$.

We will typically simply write $\underline{t}$ instead of (t) , though it is the latter which should be understood as intended. For any term (or clause, et cetera) X , we shall say $\underline{X}$ is an *unbarred* or *thawed* term (clause, et cetera).

Note that a clause is unbarred if and only if no barred variables occur in it.

We intend here that $(\cdot)$ and $(\cdot)$ be defined in such a way that, before the introduction of $(\cdot)$, all clauses may have been considered as unbarred, and that now, through $(\cdot)$, we have introduced a new class of clauses, namely the barred clauses. Further, we intend that now every clause will either be a barred, or an unbarred clause.

DEFINITION 3.2.8. $\mathcal{T}_{(\cdot)}$

Given a clause A all of whose literals are inequality literals, (that is, a pure-negative equality clause), let $\mathcal{T}_A$ denote the set of non-singleton equivalence classes of (top-level) terms occurring in A , generated as the transitive closure of the set of equalities expressed by the negation of A , when all the terms occurring in A are treated as if they are ground terms.

Note that if a (pure-negative equality) clause A is unbarred then $\overline{\mathcal{T}_A} = \mathcal{T}_{\overline{A}}$, and $\mathcal{T}_A = (\overline{\mathcal{T}_A})$.

$\mathcal{T}_{\overline{A}}$ is the set of non-singleton equivalence classes of (top-level) (ground) terms occurring in A , generated as the transitive closure of the set of equalities expressed by the negation of A .

DEFINITION 3.2.9. $\mathcal{T}_{(\cdot)}(\cdot)$

Given a pure-negative equality clause A and a positive equality clause B , let $\mathcal{T}_A(B)$ be B' , where B' is the " $\mathcal{T}_A$ -inflation" of B , obtained as follows.

Begin by letting B' be empty. Then examine the literals of B one by one. For each literal $=(s, t)$ in B , determine whether or not s is a member of some (necessarily unique) equivalence class α_s of $\mathcal{T}_A$ (defined as specified by Definition 3.2.8). Similarly, determine the existence of α_t . If both α_s and α_t exist, for each pair s', t' , such that $s' \in \alpha_s$ and $t' \in \alpha_t$, introduce the literal $=(s', t')$ into B' , if it is not already present in B' . If only one, say α_s , exists, then for each $s' \in \alpha_s$, we

introduce the literal $\neg(s', t)$ into B' . Finally, if neither α_s nor α_t exists, then simply place the literal $\neg(s, t)$ into B' . (For an example, see $\mathcal{T}_D^-(D^+)$ in Example H.0.1, on page 355.)

More generally, if B is any clause, let $\mathcal{T}_A(B)$ denote the “ $\mathcal{T}_A$ -inflation” of B , obtained as the minimal clause $B' \supseteq B$ such that if $L \in B'$ and t is any top level argument of L ,

(so that for some $n \geq 1$ and for some terms $s, s_1, \dots, s_n$, L is $\neg(s, t)$, $\neg(t, s)$, $\neg(s, t)$, $\neg(t, s)$, $P(\vec{s})$ or $\neg P(\vec{s})$, where $\vec{s} \equiv s_1, \dots, s_n$, and for some $i: 1 \leq i \leq n$, $s_i = t$),

and if $t \in \alpha$, for (necessarily unique) $\alpha \in \mathcal{T}_A$, then, for every $t' \in \alpha$, $L[t'/t] \in B'$,

whenever $L[t'/t]$ denotes the result of substituting t' for a top level occurrence of t .

For any L , $L[t'/t]$, $t', t \in \alpha \in \mathcal{T}_A$ as above, we may say that $L[t'/t]$ is a $\mathcal{T}_A$ -equivalent of L .

(or simply that $L[t'/t]$ is $\mathcal{T}_A$ -equivalent to L).

DEFINITION 3.2.10. $\overline{\mathcal{T}_A}(\cdot)$

Given a pure-negative equality clause A and any clause B , let $\overline{\mathcal{T}_A}(B)$ be defined in the same way as was $\mathcal{T}_A(B)$, except now with $\overline{\mathcal{T}_A}$ in place of $\mathcal{T}_A$.

Note that $\overline{\mathcal{T}_A}(\overline{B})$ is equivalent to $\overline{\mathcal{T}_A(B)}$, and as well, to $\mathcal{T}_A(\overline{B})$.

DEFINITION 3.2.11. $M(\cdot, \cdot)$

Given two clauses C and D be two clauses, neither of which contain any equality (or inequality) literals, let $M(C, D)$ be a sequence $[c_1, \dots, c_k]$, of “columns” c_i , corresponding to the distinct literals L_i in $C \equiv L_1, \dots, L_k$, as follows. For each (non-equality) literal L_i in C , let c_i be the “column” of pure-inequality clauses, $\langle L_i | L' \rangle$, one clause per (ground) literal L' , of the same predicate name and sign, of $\overline{D}$.

DEFINITION 3.2.12. $\| \cdot \|$

For any clause C , let $\|C\|$ denote the number of (distinct) literals in C , that is,

$\|C\|$ is the length of C . For any finite list of terms $\vec{t} \equiv t_1, \dots, t_k$, (for some $k \geq 0$), let $\|\vec{t}\| = k$.

DEFINITION 3.2.13. Given a pure-negative equality clause D , and a pure-negative equality clause I , let $\delta_0(D, I)$ represent the set of those literals $\neg(s, t) \in I$ such that $s \neq t$, and such that for no $\alpha \in \mathcal{T}_D$ is it the case that $\{s, t\} \subseteq \alpha$.

In the case that $\delta_0(D, I)$ is a singleton, $\{\neg(s, t)\}$, (say, for some terms s and t),

let $\delta(D, I)$ denote the singleton $\{\neg(s, t)\}$.

If $\delta_0(D, I)$ is empty, let $\delta(D, I)$ denote the set $\{=(s, t) \mid \neq(s, t) \in I\}$.

Otherwise, let $\delta(D, I) \equiv \emptyset$.

DEFINITION 3.2.14. Given a pure-negative equality clause D , and a clause C , let $\Delta(D, C)$ denote $\mathcal{T}_D(E)$, where $E \equiv \bigcup \{\delta(D, \neq(\vec{s}, \vec{t})) \mid \{\neg P(\vec{s}), P(\vec{t})\} \subseteq C^0\}$.

DEFINITION 3.2.15. For any clause C , let $cl(C)$ denote the symmetric closure of $\mathcal{T}_{C^-}(C^+) \cup \Delta(C^-, C^0) \cup \mathcal{T}_{C^-}(C^0) \cup \mathcal{T}_{C^-}(C^-)$.

OBSERVATION 3.2.1. The following observations are either trivial or easy. For any clauses C and C' , and any pure-negative equality clauses A , and A' :

- a) $C \subseteq \mathcal{T}_A(C)$
- b) $\mathcal{T}_A(C) = \mathcal{T}_{\mathcal{T}_A(A)}(C)$
- c) $\mathcal{T}_A(\mathcal{T}_A(C)) = \mathcal{T}_A(C)$
- d) $\mathcal{T}_A(C \cup C') = \mathcal{T}_A(C) \cup \mathcal{T}_A(C')$, and $\mathcal{T}_A(\Box) = \Box$
- e) $A \subseteq A'$ and $C \subseteq C' \Rightarrow \mathcal{T}_A(C) \subseteq \mathcal{T}_{A'}(C')$
- f) $\overline{cl(C)} = cl(\overline{C})$, and if $C = \underline{C}$ (C is an ordinary (unbarred) clause), $\underline{cl(C)} = cl(C)$
- g) $C \subseteq cl(C)$
- h) $cl(cl(C)) = cl(C)$
- i) $C \subseteq C' \Rightarrow cl(C) \subseteq cl(C')$
- j) if F is a trivial inequality clause, $cl(C \cup F) = cl(C) \cup F$
- k) for any substitution θ , $\mathcal{T}_{A\theta}(C\theta) = \mathcal{T}_A(C)\theta$
- l) for any substitution θ , $cl(C)\theta \subseteq cl(C\theta)$
- m) for any substitution θ , $C\theta \subseteq \mathcal{T}_{(C')^-}(C')$ $\Leftrightarrow \mathcal{T}_{C^-}(C)\theta \subseteq \mathcal{T}_{(C')^-}(C')$
- n) for any substitution θ , if C is an equality clause, then (modulo ordering), $cl(C)\theta = cl(C\theta)$

DEFINITION 3.2.16. For any clause A , let $\neg A$ denote the set $\{\neg \vec{L} \mid L \in A\}$.
(of ground unit clauses obtained by the formal negation of A).

3.2.3. Σ_0 -tautology and Σ_0 -subsumption

DEFINITION 3.2.17. A clause C is a Σ_0 -tautology if $cl(C)$ either contains oppositely signed but otherwise identical literals, (that is, is a tautology), or contains a trivial equality literal.

THEOREM 3.2.1. The following are equivalent:

- 1) C is a Σ_0 -tautology.
- 2) Either:
 - i) C contains literals $P(\vec{s})$; $\neg P(\vec{t})$, $P \neq '='$, such that $\forall \neq(a, b) \in \neq(\vec{s}, \vec{t})$, either $a = b$, or $\exists \alpha \in \mathcal{T}_{C^-} : \{a, b\} \subseteq \alpha$, or
 - ii) for some equality literal $=(s, t) \in C$ either $s = t$, or for some $\alpha \in \mathcal{T}_{C^-}$, $\{s, t\} \subseteq \alpha$.
- 3) If M denotes the set $\{\neg L \mid L \in C^+\} \cup \{\neq(\vec{s}, \vec{t}) \mid P(\vec{s}) : \neg P(\vec{t}) \in C^0\}$, of pure-inequality clauses, there exists some $A \in M$, whose naturally induced set $\mathcal{T}_A$ of equivalence classes is such that for each $\beta \in \mathcal{T}_A$ there is an $\alpha_\beta \in \mathcal{T}_{C^-}$ with $\beta \subseteq \alpha_\beta$.
- 4) $\overline{C}$ is a Σ_0 -tautology.
- 5) $cl(C)$ is a Σ_0 -tautology.

COROLLARY 3.2.1. (Σ_0 -tautology is preserved under substitution)

Let C be a Σ_0 -tautology. Then for any substitution θ , $C\theta$ is a Σ_0 -tautology.

THEOREM 3.2.2. C is a Σ_0 -tautology if and only if $\Sigma_0 \models C$.

COROLLARY 3.2.2. If C is a Σ_0 -tautology, and C subsumes D , then D is a Σ_0 -tautology.

OBSERVATION 3.2.2. For any clause C , $\Sigma_0 \models (\forall)C \Leftrightarrow (\forall)cl(C)$.

DEFINITION 3.2.18. C Σ_0 -subsumes D if for some substitution θ , and some trivial inequality clause, F , $C\theta \subseteq cl(D) \cup F$.

The following observation is trivial.

OBSERVATION 3.2.3. If clauses A' and B' are alphabetic variants of clauses A and B , respectively, then A Σ_0 -subsumes B implies A' Σ_0 -subsumes B' .

THEOREM 3.2.3. Given clauses C and D , the following are equivalent:

- 1) C Σ_0 -subsumes D ,

- 2) for some substitution θ , and some *ground* trivial inequality clause F' , $C\theta \subseteq cl(\overline{D}) \cup F'$,
- 3) $cl(C)$ Σ_0 -subsumes $cl(D)$,
- 4) $cl(C)$ Σ_0 -subsumes D ,
- 5) C Σ_0 -subsumes $cl(D)$,
- 6) With $S_1(C^+, D)$ denoting the collection of most general substitutions θ such that, modulo symmetry, $C^+\theta \subseteq \overline{\mathcal{T}_{D^-}(D^+)} \cup \Delta(\overline{D^-}, \overline{D^0})$,

$S_2(C^-, \mathcal{T}_{D^-})$ denoting the collection of most general substitutions θ such that for all inequality literals $\neq(a, b) \in C^-$, either $a\theta = b\theta$, or there exists some equivalence class $\alpha \in \overline{\mathcal{T}_{D^-}}$, such that $\{a\theta, b\theta\} \subseteq \alpha$, and

$S_3(C^0, D^0, \mathcal{T}_{D^-})$ denoting the collection of most general substitutions θ such that there is a path γ through $M(C^0, D^0)$, (selecting from each column c_j , one member $\gamma(j) \equiv c_{j,i}$, say), such that $\forall j, 1 \leq j \leq \|C^0\|$, for all $\neq(s, \bar{t}) \in \gamma(j)$, either $s\theta = \bar{t}$, or there exists some equivalence class $\alpha \in \overline{\mathcal{T}_{D^-}}$, such that $\{s\theta, \bar{t}\} \subseteq \alpha$,

(where, if a column of $M(C^0, D^0)$ is empty then $S_3(C^0, D^0, \mathcal{T}_{D^-})$ is empty),

$\exists \theta_1 \in S_1(C^+, D)$, $\exists \theta_2 \in S_2(C^-, \mathcal{T}_{D^-})$; $\exists \theta_3 \in S_3(C^0, D^0, \mathcal{T}_{D^-})$, such that $\theta_1, \theta_2, \theta_3$ are compatible.

- 7) There exists a substitution θ such that all of the following conditions hold:

(i.) modulo symmetry, $C^+\theta \subseteq \overline{\mathcal{T}_{D^-}(D^+)} \cup \Delta(\overline{D^-}, \overline{D^0})$.

(ii.) For all (inequality) literals $\neq(a, b) \in C^-$, either $a\theta = b\theta$, or there exists some equivalence class $\alpha \in \overline{\mathcal{T}_{D^-}}$, such that $\{a\theta, b\theta\} \subseteq \alpha$.

(iii.) There is a path γ through $M(C^0, D^0)$, (selecting from each column c_j , one member $\gamma(j) \equiv c_{j,i}$, say), such that for all $j : 1 \leq j \leq \|C^0\|$, for all $\neq(s, \bar{t}) \in \gamma(j)$, either $s\theta = \bar{t}$, or there exists some equivalence class $\alpha \in \overline{\mathcal{T}_{D^-}}$, such that $\{s\theta, \bar{t}\} \subseteq \alpha$.

Definition 3.2.11, together with cases (6) and (7), (case (6) in particular), of Theorem 3.2.3 allow the formulation of a reasonably efficient² algorithm for computing the Σ_0 -subsumption relation. This algorithm, an example, and some motivation for the algorithm is given in Appendix H.

²By “reasonably efficient”, we of course do not mean “polynomial cost”.

THEOREM 3.2.4. (Σ_0 -subsumption Extends Subsumption)

For clauses C and D , if C subsumes D then C Σ_0 -subsumes D .

The example (Example H.0.1) given in Appendix H, shows that the relation of Σ_0 -subsumption is in fact a proper extension of that of subsumption.

LEMMA 3.2.1. For any substitution θ , if C Σ_0 -subsumes D , then C Σ_0 -subsumes $D\theta$.

LEMMA 3.2.2. (Transitivity of Σ_0 -subsumption)

If B Σ_0 -subsumes C , and C Σ_0 -subsumes D , then B Σ_0 -subsumes D .

3.3. Weak Completeness of Subsumption and Σ_0 -subsumption

Recall that a clause C *subsumes* a clause D if there is a substitution instance θ such that $C\theta \subseteq D$. Of course if C subsumes D , then C logically implies D , in standard notation, $\emptyset \models C \Rightarrow D$. That is, subsumption is sound with respect to semantic entailment with the empty theory. (and hence, with any theory).

However, the converse is not true in general. For example, it is easy to see that, while:

$$\emptyset \models (P(x, a) \mid \neg P(b, x)) \Rightarrow (P(a, a) \mid \neg P(b, b)),$$

(for some constants a , and b), there is no substitution, θ , such that

$$(P(x, a) \mid \neg P(b, x))\theta \subseteq (P(a, a) \mid \neg P(b, b)).$$

Thus subsumption is not (logically) complete. But subsumption is “weakly complete” (or “linearly complete”), in the following sense.

If $\emptyset \models C \Rightarrow D$, then we know by the refutation completeness of (say, ordinary, binary-merge) resolution that there is some finite set $\mathcal{C}$ of ground instances of C , such that with $\Sigma \equiv \mathcal{C} \cup \neg D$, Σ is unsatisfiable.

Now, if there exists a (ground) unsatisfiable Σ as above, such that $\mathcal{C}$ contains only one (ground) instance of C , then (it is easy to see that) either D is a *tautology*, (in which case, precisely, D contains oppositely signed but otherwise identical literals), or C does subsume D .

We make this notion formal:

DEFINITION 3.3.1. Let Σ be a set of clauses (comprising a clausal theory), and R , and T , be binary, and unary, respectively, (syntactic) relations on clauses. Then R is *weakly complete*, (or *linearly complete*) with respect to T and the theory Σ if: whenever there exists a substitution θ , (possibly assigning some or all variables to formally *frozen* variables), such that $C\theta$ is ground³ and $\Sigma \cup \{C\theta\} \cup \neg D \models \perp$, then either $T(D)$ or $R(C, D)$ holds.

For now, (in light of the preceding comments), we simply note here that subsumption is weakly complete with respect to tautology and the empty theory, $\emptyset$.

With respect to $\Sigma_0 \equiv \{QR, QS, QT, QP\}$, (where QP stands for the appropriate set of clauses), the joint generalization of subsumption to Σ_0 -subsumption, and tautology to Σ_0 -tautology, is natural, in the sense that, as we will show, Σ_0 -subsumption is weakly complete with respect to Σ_0 -tautology and Σ_0 .

We need a few Lemmas:

LEMMA 3.3.1. (Structure Lemma for OBM)

Suppose Π is an OBM refutation proof (tree) for a finite unsatisfiable set of clauses Σ , in which a ground clause C occurs (as the label of a node $n \in \Pi$). Then there is an OBM refutation proof Π' , say, of Σ in which occurs only a single instance of the clause C , (there is only one node in Π' labelled with the clause C). Further, the proof (Π') may be constructed so that this single instance of $C \equiv L_1, L_2, \dots, L_k$, (say, for positive integer k), is resolved only against (OBM Σ deducible) clauses C_i , for $i = 1, 2, \dots, k$, each of which has an OBM deduction Γ_i , (from Σ), which contains no clause subsumed by L_i , and where C_i unifies with $\neg L_i$, (so that each C_i (OBM-) resolves away a single literal (L_i) of C , leaving no residue).

LEMMA 3.3.2. (for Σ_0 -subsumption is weakly complete)

Suppose $\Sigma_0 \cup \neg D$ is satisfiable. Then:

- i) If L is a ground positive equality unit clause, $L \equiv (s_0, t_0)$, say, then $\Sigma_0 \cup \neg D \cup \{L\}$ is unsatisfiable if and only if $L \in \mathcal{T}_{\overline{D}^-}(\overline{D}^+) \cup \Delta(\overline{D}^-, \overline{D}^0)$.
- ii) If L is a ground inequality unit clause, $L \equiv \neq(s_0, t_0)$, say, then $\Sigma_0 \cup \neg D \cup \{L\}$ is unsatisfiable if and only if $s_0 = t_0$, or $L \in \mathcal{T}_{\overline{D}^-}(\overline{D}^-)$.

³ $C\theta$ may contain frozen variables, since any such is considered a ground term.

- iii) If L is a ground non-equality unit clause, $L \equiv P(\vec{a})$, (or $L \equiv \neg P(\vec{a})$), say, then $\Sigma_0 \cup \neg D \cup \{L\}$ is unsatisfiable if and only if $L \in \mathcal{T}_{\overline{D}}(\overline{D^0})$.

THEOREM 3.3.1. Σ_0 -subsumption is weakly complete, with respect to Σ_0 -tautology and the clausal theory Σ_0 .

In fact, it is easy to see the following:

THEOREM 3.3.2. (Partial characterization of subsumption)

For clauses C and D , D not a tautology. C subsumes D if and only if for some ground substitution instance $C\theta$ of C , $\{C\theta\} \cup \neg D$ is unsatisfiable.

In analogy, we have the following:

THEOREM 3.3.3. (Partial characterization of Σ_0 -subsumption)

For clauses C and D , D not a Σ_0 -tautology, C Σ_0 -subsumes D if and only if for some ground substitution instance $C\theta$ of C , $\Sigma_0 \cup \neg D \cup \{C\theta\}$ is unsatisfiable.

Clearly if C is a tautology, and C subsumes D , then D is also a tautology.

We also state the following analogue:

COROLLARY 3.3.1. If C is a Σ_0 -tautology and C Σ_0 -subsumes D , then D is a Σ_0 -tautology.

THEOREM 3.3.4. (Soundness of Σ_0 -subsumption)

Suppose for clauses C and D , that C Σ_0 -subsumes D . Then $\Sigma_0 \models (\forall)C \Rightarrow (\forall)D$.

CHAPTER 4

$EPC_{<}$

4.0. Overview

In this chapter the binary resolution/refutation system $EPC_{<}$ is introduced, and various results for it are presented.

In Section 4.1 the deduction rules for $EPC_{<}$ are presented, together with some technical notions. The section is divided into subsections. In Subsection 4.1.1 the deduction rules for the “above-equality fragment” are introduced (in Definition 4.1.1). In Subsection 4.1.2 the (initial formulation of the) deduction rules for the “equality fragment” are introduced (in Definition 4.1.5). In addition, a few other notions are introduced, and some observations are made. In Subsection 4.1.3 some technical notions, (analogous to those for $PC_{<}$ of Subsection 2.1.2), are introduced for later use.

In Section 4.2 some technical results for “lifting” and “grounding” in the above-equality fragment (of $EPC_{<}$) are presented, the major result (for lifting, in the above-equality fragment) being Lemma 4.2.5.

Section 4.3 similarly contains technical results for subsumption in the above-equality fragment of $EPC_{<}$, (the main result here being Lemma 4.3.4), and some more technical results for grounding in the above-equality fragment.

Similarly, in Section 4.4, technical results for lifting and grounding in the equality fragment are presented, while Section 4.5 contains results for grounding and subsumption in the equality fragment. The “big” result of Section 4.5 is clearly Lemma 4.5.10.

Some Comments:

As will be seen, the deduction rules, in particular those for the equality fragment are quite “syntactically bulky”. The rules in this section are in fact, later, given some reformulation (in Section 8.4). Despite their lack of elegance, their formulation (as below), allows and facilitates certain results. In particular, the equality fragment deduction rules were formulated in terms of “choice functions”, essentially to allow/facilitate lifting. As was the case for $PC_{<}$, the “left-most restriction”, again complicates the task for the proof theory, (though, as the reader will see, here the situation is in any event, already more complicated).

4.1. $EPC_{<}$: The Deduction Rules and Some Preliminaries

$EPC_{<}$ is a binary resolution system which is based on, and is a modification of, $PC_{<}$. The primary motivation (and achieved design goal), for the modification (of the deduction rules) is to produce a system which is refutation complete with Σ_0 -tautology and Σ_0 -subsumption deletion, (within deduction/deletion strategy contexts of the same sort as those for which we have shown the refutation completeness of $PC_{<}$ under tautology and subsumption deletion).

As in $PC_{<}$, in $EPC_{<}$, we again totally order the predicate symbols, now, in such a fashion that the equality predicate symbol, ‘=’, is minimal with respect to this ordering. As in $PC_{<}$, in $EPC_{<}$, resolution between two clauses is again restricted to those cases where the maximal predicate symbol occurring in each clause is the same, and then allow resolution only on literals of that predicate name. Again, further, given that two clauses have the same maximal predicate symbol, we only allow resolution between them if (at least) one of them is pure-positive with respect to that predicate name, and we *merge resolve* only on (sub-clauses (of the non-pure-positive clause) which contain) the left-most negative literal whose predicate name is mutually maximal. As before, the subscript, $<$, in the name “ $EPC_{<}$ ”, is to indicate a particular total ordering on (unsigned) predicate symbols, and so actually, we intend here the definition of a class of (restricted) ordinary binary merge resolution systems, where the ordering, $<$, is a parameter, and the class is also (implicitly) parameterized by the ordering functions o , seen shortly.

4.1.1. The Deduction Rules: The Above-equality fragment

DEFINITION 4.1.1. ($EPC_{<}$ -resolution, -The Above-equality Fragment)

Let $<$ be any strict linear order on predicate names in which ‘=’ is minimal.

Then for any ordered clauses A and B , such that A and B share no variables in common and at

least one of A and B has a $<$ -maximal predicate symbol $P \neq '='$, and any unordered clause C .
 C is an *unordered EPC_<-resolvent* of A and B if:

- 1) the $<$ -maximal predicate symbol P , say, occurring in A is the $<$ -maximal predicate symbol which occurs in B .
- 2) A is pure-positive with respect to P , and B is mixed with respect to P .
 (B is pure-positive with respect to P , and A is mixed with respect to P),
- 3) when L_0 , say, denotes the left-most negative P instance.
 (of the form $\neg P(\vec{t})$, for some $arity(P)$ -ary term tuple $\vec{t}$), occurring in B . (occurring in A),
 there are non-empty sub-clauses $\emptyset \neq A' \subseteq A$: $L_0 \in B' \subseteq B$, ($L_0 \in A' \subseteq A$: $\emptyset \neq B' \subseteq B$),
 and for some $i > 0$, $j > 0$, and $arity(P)$ -ary term tuples $\vec{s}_1, \dots, \vec{s}_i$ and $\vec{t}_1, \dots, \vec{t}_j$, say,
 such that $A' \equiv \{P(\vec{s}_1), \dots, P(\vec{s}_i)\}$ and $B' \equiv \{\neg P(\vec{t}_1), \dots, \neg P(\vec{t}_j)\}$,
 ($B' \equiv \{P(\vec{s}_1), \dots, P(\vec{s}_i)\}$ and $A' \equiv \{\neg P(\vec{t}_1), \dots, \neg P(\vec{t}_j)\}$, respectively),
- 4) $C \equiv \bigcup \{ \neg(\vec{s}_{i'}, \vec{t}_{j'}) \mid 1 \leq i' \leq i; 1 \leq j' \leq j \} \cup A \setminus A' \cup B \setminus B'$.

For any ordered clauses A and B sharing no variables in common, and such that at least one of A and B has a $<$ -maximal predicate symbol $P \neq '='$, let $|EPC_{<}(A, B)|$ denote the set of all unordered EPC_<-resolvents C , of A and B . If A and B do have variables in common, let $|EPC_{<}(A, B)|$ denote $|EPC_{<}(A', B')|$, where A' and B' are some variable disjoint alphabetic variants of A and B respectively.¹

Let o be any fixed parametrized clause ordering function, so that for any ordered clauses A and B , and any unordered clause C , $o(A, B, C)$ is an ordered clause C' such that $|C'| = C$.

Then for any clauses A and B , such that at least one of A and B has a $<$ -maximal predicate symbol $P \neq '='$, let $EPC_{<}(A, B)$ denote the collection of ordered clauses:

$\{o(A, B, C) \mid C \in |EPC_{<}(A, B)|\}$. (Thus, $EPC_{<}(A, B)$ is the set $|EPC_{<}(A, B)|$, where each clause in the set has been endowed with some (arbitrary) ordering.)

Soundness here is easily seen.

Note that in the formation of the resolvent, $o(C)$, where

$$C \equiv \bigcup \{ \neg(\vec{s}_{i'}, \vec{t}_{j'}) \mid 1 \leq i' \leq i; 1 \leq j' \leq j \} \cup A \setminus A' \cup B \setminus B', \text{ as above,}$$

¹Thus unordered EPC_<-resolvents are only defined up to alphabetic equivalence.

no (partial) unifying substitution is applied. However, we will see later, (in Section 8.4), that if θ is a most general unifying substitution of β , where β is *any* subset of $\{\{s, t\} \mid \neq(s, t) \in \neq(\vec{s}_i, \vec{t}_{j'})\}, 1 \leq i' \leq i; 1 \leq j' \leq j\}$, (θ unifying each element $\{s, t\} \in \beta$ to a singleton, $\{s\theta\} = \{t\theta\}$), then we can take $o(D \setminus F(D))$ in place of $o(C)$, (as the resolvent), where $D \equiv \bigcup \{\neq(\vec{s}_i, \vec{t}_{j'})\theta \mid 1 \leq j' \leq j\} \cup \bigcup \{\neq(\vec{s}_i, \vec{t}_1)\theta \mid 2 \leq i' \leq i\} \cup A\theta \setminus A'\theta \cup B\theta \setminus B'\theta$, and where $F(D)$ denotes the set of trivial inequalities in D .

4.1.2. The Deduction Rules: The Equality fragment

DEFINITION 4.1.2. If A is an equality clause, let $QR_{full}(A)$ denote either:

- i. $A\sigma \setminus A^-\sigma$, where σ is the most general substitution such that $s\sigma = t\sigma$, for each inequality $\neq(s, t) \in A^-$, if $A^- \neq \emptyset$, and this substitution exists, and
- ii. A , otherwise.

If $\mathcal{C}$ is a finite collection of equality clauses, let $QR_{full}(\mathcal{C})$ denote the set $\{QR_{full}(C) : C \in \mathcal{C}\}$.

Note that (modulo ordering), $QR_{full}(A) = A$, whenever A is a positive equality clause, and that if A is a (non-empty, pure) inequality clause, $QR_{full}(A) \neq A$ if and only if $QR_{full}(A) = \square$, the empty clause.

OBSERVATION 4.1.1. For any clause A , $A \Sigma_0$ -subsumes $\square$ if and only if $QR_{full}(A) = \square$.

DEFINITION 4.1.3. Let o denote some arbitrary fixed (clause ordering) function, which associates to each (unordered) clause C , (regarded as an unordered set of literals), C equipped with some ordering. For an (ordered) equality clause A , let $QR_{first}(A)$ denote the least set X of (ordered) clauses such that:

- i. $A \in X$, equipped with its original ordering,
- ii. if $A' \in X$, A' mixed, and for $\neq(s, t)$ denoting the leftmost inequality in A' , the most general substitution σ such that $s\sigma = t\sigma$ exists, then $A'\sigma \setminus \{\neq(s, t)\sigma\} \in X$, equipped with the ordering, $o(A'\sigma \setminus \{\neq(s, t)\sigma\})$.

It is intended that $QR_{first}(\cdot)$ is functional, so that when $QR_{first}(A)$ is computed more than once, its clauses are always ordered in the same way.

Note that since closure condition (ii) above can introduce, for each clause $A' \in X$, finitely many (in fact, at most one) new clauses in X , and since every clause so introduced, must contain strictly fewer inequalities than A' , $QR_{first}(A)$ must be finite. Also, (it is easy to see that),

$$QR_{full}(A) = \square \text{ if and only if } \square \in QR_{first}(A).$$

Also, it is easy to see the following:

OBSERVATION 4.1.2. For any equality clause A , (modulo ordering), $QR_{full}(A) \in QR_{first}(A)$.

Further, it is easy to see that:

OBSERVATION 4.1.3. If A is a ground equality clause containing at least one non-trivial inequality literal, then there is an $A' \in QR_{first}(A)$ such that the left-most inequality in A' is non-trivial.

DEFINITION 4.1.4. Given an equality clause E , by a *symmetric choice function* on E , we shall mean a function $c: E \rightarrow \{0, 1\}$, such that, (for any term t), for any $q(t, t) \in E$, for q either $=$ or $\neq$, $c(q(t, t)) = 0$.

Intuitively, a symmetric choice function c on E , will be used to indicate that, for q either $=$ or $\neq$, a literal $q(s, t)$, for which $c(q(s, t)) = 1$ is considered reversed, that is, as $q(t, s)$, (while if $c(q(s, t)) = 0$, the literal is considered unchanged).

DEFINITION 4.1.5. (The Equality Rules, Initial Formulation)

Let o be any fixed parametrized clause ordering function, so that for any ordered clauses A and B , and any unordered clause C , $o(A, B, C)$ is an ordered clause C' such that $|C'| = C$. Given non-empty (ordered) equality clauses A and D , (with disjoint variables)²,

1.a) If A is positive, and D is mixed (contains inequality literals), and D^- denotes the (non-empty) set of inequality literals in D , let $EPC_{+,-}(A, D)$ denote the set of unordered clauses B such that:

- i. there exists a non-empty subset $A_R \subseteq A$, a non-empty subset $D_R \subseteq D^-$, the left-most inequality literal, $\neq(s_0, t_0)$, say, of D in D_R , and a (possibly empty) subset $D_S \subseteq D^- \setminus D_R$, such that for some symmetric choice function c on $A_R \cup D_R \cup D_S$, with $c(\neq(s_0, t_0)) = 0$, the most general substitution σ , say, unifying:

²Otherwise, as usual, let A and D be replaced with variable disjoint alphabetic variants.

$$\begin{aligned} & \{ \{=(s', t') \mid =(s', t') \in A_R; c(=(s', t')) = 0\} \cup \{=(t', s') \mid =(s', t') \in A_R; c(=(s', t')) = 1\} \cup \\ & \quad \{=(s', t') \mid \neq(s', t') \in D_R; c(\neq(s', t')) = 0\} \cup \{=(t', s') \mid \neq(s', t') \in D_R; c(\neq(s', t')) = 1\}, \\ & \quad \{s' \mid =(s', t') \in A_R; c(=(s', t')) = 0\} \cup \{t' \mid =(s', t') \in A_R; c(=(s', t')) = 1\} \cup \\ & \quad \{s' \mid \neq(s', t') \in D_S; c(\neq(s', t')) = 0\} \cup \{t' \mid \neq(s', t') \in D_S; c(\neq(s', t')) = 1\} \}. \end{aligned}$$

exists, σ unifying the two sets in the above doubleton set to the singletons $\{=(s, r)\}$ and $\{s\}$, respectively, for some terms s and r , say, and

$$\begin{aligned} \text{ii. } B & \equiv A\sigma \setminus \{=(r, s), =(s, r)\} \cup D\sigma \setminus (\{\neq(r, s), \neq(s, r)\} \cup D_S\sigma) \cup \\ & \quad \{\neq(r, t'\sigma) \mid \neq(s', t') \in D_S; \neq(s', t')\sigma \notin \{\neq(r, s), \neq(s, r)\}; c(\neq(s', t')) = 0\} \cup \\ & \quad \{\neq(s'\sigma, r) \mid \neq(s', t') \in D_S; \neq(s', t')\sigma \notin \{\neq(r, s), \neq(s, r)\}; c(\neq(s', t')) = 1\}. \end{aligned}$$

1.b) Otherwise, let $EPC_{+,-}(A, D) \equiv \emptyset$.

2) (Given any non-empty equality clauses A and D),

Let $EPC_{-}^1(A, D)$ denote the set $\bigcup \{EPC_{+,-}(QR_{full}(A), D') \mid D' \in QR_{first}(D)\}$.

Let $EPC_{-}(A, D)$ be $\{\square\}$, if $QR_{full}(D)$ is the empty clause, $\square$, (or, equivalently, $\square \in QR_{first}(D)$), and otherwise, the set $\{o(A, D, C) \mid C \in EPC_{-}^1(A, D)\}$.

3.a) If A and D , are both positive, let $EPC_{+,+}(A, D)$ denote the set of clauses B such that:

i. there exists a non-empty subset $A_R \subseteq A$, a non-empty subset $D_R \subseteq D$, such that for some symmetric choice function c on $A_R \cup D_R$, the most general substitution σ , say, unifying:

$$\begin{aligned} & \{ \{=(r', s') \in A_R \mid c(=(r', s')) = 0\} \cup \{=(s', r') \mid =(r', s') \in A_R; c(=(r', s')) = 1\} \cdot \\ & \quad \{=(s', t') \in D_R \mid c(=(s', t')) = 0\} \cup \{=(t', s') \mid =(s', t') \in D_R; c(=(s', t')) = 1\} \cdot \\ & \quad \{s' \mid =(r', s') \in A_R; c(=(r', s')) = 0\} \cup \{s' \mid =(s', r') \in A_R; c(=(s', r')) = 1\} \cup \\ & \quad \{s' \mid =(s', t') \in D_R; c(=(s', t')) = 0\} \cup \{s' \mid =(t', s') \in D_R; c(=(t', s')) = 1\} \}, \end{aligned}$$

exists, σ unifying the three sets in the above triple-ton set to the singletons $\{=(r, s)\}$,

$\{=(s, t)\}$, and $\{s\}$, respectively, for some terms r , s , and t , say, and

$$\text{ii. } B \equiv A\sigma \setminus \{=(r, s), =(s, r)\} \cup D\sigma \setminus \{=(s, t), =(t, s)\} \cup \{=(r, t)\},$$

3.b) Otherwise, let $EPC_{+,+}(A, D) \equiv \emptyset$.

4) (Given any non-empty equality clauses A and D),

Let $EPC_{+}(A, D) \equiv \{o(A, D, C) \mid C \in EPC_{+,+}(QR_{full}(A), QR_{full}(D))\}$.

5) Let $EPC_{<}^1(A, D) \equiv EPC_{-}(A, D) \cup EPC_{-}(D, A) \cup EPC_{+}(A, D)$.

Note that $EPC_{+,-}(A, D)$ is always empty when A is not a positive equality clause, and unlike $EPC_{+,+}(-, -)$, $EPC_{+,-}(-, -)$ is non-symmetric. Similarly, $EPC_{+}(-, -)$ is symmetric, while $EPC_{-}(-, -)$ is not.

It is straight forward to check that (as was the case for the above-equality fragment, again), the $EPC_{<}$ deduction rules for the equality fragment are sound.

DEFINITION 4.1.6. Given $B \in EPC_{<}^1(A, D)$, for some equality clauses A and D ,

1) let us say that B is an *EPC₊-resolvent* of A and D , if $B \in EPC_{+}(A, D)$.

Otherwise, ($B \in EPC_{-}(A, D) \cup EPC_{-}(D, A)$, and) we will say that B is an *EPC₋-resolvent* of A and D .

2) let us say that A occurs as (or is) a *QR-positive parent* of B , if either $B \in EPC_{+}(A, D)$, or (B is a *EPC₋-resolvent* of A and D , and) $B \in EPC_{-}(A, D)$.

Otherwise, ($B \in EPC_{-}(D, A)$, and), A occurs as (or is) the *negative parent* of B .

DEFINITION 4.1.7. By an *EPC_<-deduction* of a clause C from a collection of clauses $\mathcal{C}$ we of course mean a binary proof tree Π where:

- i. every leaf node of Π is labelled by some clause of $\mathcal{C}$.
- ii. every offspring node n' of parent nodes n and m in Π , is labelled with a clause $C_{n'} \in EPC_{<}^1(C_n, C_m)$, where C_n and C_m are the clause labels of n and m respectively, and
- iii. the root node of Π is labelled by C .

By an *equality EPC_<-deduction* of a clause C from a collection of clauses $\mathcal{C}$, we mean a $EPC_{<}$ -deduction Π of C from $\mathcal{C}$, where every leaf clause of Π is an equality clause E (in $\mathcal{C}$).

Note that if Π is an equality $EPC_{<}$ -deduction of a clause C from a collection of clauses $\mathcal{C}$, then C , as well as every other clause (labelling a node) in Π , is an equality clause.

DEFINITION 4.1.8. Given an equality $EPC_{<}$ -deduction Π , we will say that a clause (instance) C , labelling some node n of Π , is *QR-positive* (or *occurs QR-positively*) at n in (or with respect to) Π , if either:

- i. C is a QR -positive parent of some clause B . (so that B labels the offspring node of n and some co-parent node n' , with label C' say, and $B \in EPC_{<}^1(C, C')$, where C occurs as a QR -positive parent of B), or

- ii. n is the root node of Π , and either C is a positive (equality) clause, or $QR_{full}(C) \neq C$.

Otherwise, we will say C is *negative* (or *occurs negatively*) at n (in, or with respect to, Π).

Clearly, if (an equality clause) A labels the root of any equality deduction Π , A is QR -positive (at the root) in Π if and only if A is QR -positive in the trivial deduction (of A).

In light of this, we will also simply say that an equality clause A is *QR -positive*, if it is QR -positive in the trivial deduction. Thus if a context, of some node labelled by A , in some $EPC_{<}$ -deduction Π , has been given, then the former definition will be intended for " A is QR -positive...", but otherwise, we shall take the trivial deduction for the context, by default. A context will be explicitly given in any ambiguous situations, unless the context of the trivial deduction is intended.

We now also make the following:

OBSERVATION 4.1.4. If A is QR -positive (with respect to the trivial deduction), and A' subsumes A , then A' is QR -positive.

OBSERVATION 4.1.5. For any QR -positive equality clause, C , and any substitution, σ , σ is a most general substitution such that $(C\sigma)^-$ contains only trivial inequalities, if and only if

σ is a most general substitution such that $(cl(C)\sigma)^-$ contains only trivial inequalities.

DEFINITION 4.1.9. We will say that an equality $EPC_{<}$ -deduction Π is:

- 1) an EPC_+ -deduction, if every non-leaf clause in Π is a EPC_+ -resolvent of its parents.
- 2) an EPC_- -deduction, if every non-leaf clause in Π is a EPC_- -resolvent of its parents.

DEFINITION 4.1.10. Given a set of (QR -positive equality) clauses $\mathfrak{C}$,

let $EPC_+^1(\mathfrak{C})$ denote the set $\bigcup\{EPC_+(C, C') \mid C, C' \in \mathfrak{C}\} \cup \mathfrak{C}$.

Inductively, for any integer $k \geq 1$, let $EPC_+^{k+1}(\mathfrak{C})$ denote $EPC_+^1(EPC_+^k(\mathfrak{C}))$,

and let $EPC_+(\mathfrak{C})$ denote $\bigcup\{EPC_+^k(\mathfrak{C}) \mid k \geq 1\}$.

Thus $EPC_+(\mathfrak{C})$ denotes $\mathfrak{C}$, together with the set of all clauses which have EPC_+ -deductions from $\mathfrak{C}$, (these are the clauses which are *EPC_+ -deducible from $\mathfrak{C}$*).

Note that if, for some set of clauses $\mathcal{C}$, $C \in EPC_+(\mathcal{C}) \setminus \mathcal{C}$, then C must be a positive equality clause.

In OBMR it is never necessary to resolve a clause against itself in the process of generating resolvents in order to obtain a refutation proof. This situation does not prevail with respect to the $EPC_{<}$ -resolution rules (for the equality fragment) introduced. As a simple situation where we must allow $EPC_{<}$ -resolutions between a clause and itself, consider the following set of clauses (labelled 1-4):

- $$\begin{array}{ll} (1) \neq(SK1(), SK2()) & (2) \neq(SK2(), SK3()) \\ (3) \neq(SK3(), SK1()) & (4) =(x, a1()) \mid =(x, a2()) \end{array}$$

These express the inconsistent situation that each of three distinct things equals at least one of two objects. Yet, the only $EPC_{<}$ -resolution possible is between the last clause and itself. Thus resolvents must be obtained from the sets $EPC_+(A, A)$, for QR -positive equality clauses A .

A simple example set showing that we must (also) obtain resolvents from $EPC_-(A, A)$, for QR -positive equality clauses A , is given by the following set of clauses (labelled 1-2):

- $$(1) \neq(x, c()) \mid =(f(x), x) \qquad (2) \neq(f(f(c())), c())$$

(for some constant $c()$).

It is not hard to see that this pair of clauses, taken together with the reflexivity axiom, QR , is unsatisfiable, but to obtain the refutation, it is necessary to obtain the resolvent:

$$=(f(f(c())), c()) \in EPC_-((1), (1)).$$

Thus in order to guarantee refutation completeness, we must allow $EPC_{<}$ -resolution of QR -positive equality clauses against themselves. In fact this is sufficient to guarantee completeness, as we shall later show. Of course, if A has $<$ -maximal predicate symbol P and $P \neq '='$, then

$EPC_{<}^1(A, A) = \emptyset$, since no such clause A can be both pure-positive and mixed, with respect to P .

4.1.3. Some Preliminaries

OBSERVATION 4.1.6. If clauses A' and B' are alphabetic variants of clauses A and B , respectively, then for all $C \in EPC_{<}^1(A, B)$ there exists $C' \in EPC_{<}^1(A', B')$ such that C' is an alphabetic variant of C .

DEFINITION 4.1.11. Given a clause B , with $<$ -maximal predicate symbol, P , say, $P \neq '='$, B mixed with respect to P , and a finite collection $\mathfrak{C}$ of clauses, such that $\forall C \in \mathfrak{C}$, C is pure-positive with respect to P , let us say that D is $EPC_{<}^{+1}$ -deducible from $\{B\} \cup \mathfrak{C}$, if D is pure-positive with respect to P , (or has $<$ -maximal predicate symbol ' Q ' $< P$), and there exist sequences (for some $k \geq 1$), $C_0, \dots, C_{k-1}$, and $D_0, \dots, D_{k-1}, D_k$, such that:

- i.) $D_0 \equiv B$ and $D_k \equiv D$,
- ii.) $\forall i: 0 \leq i \leq k-1$. C_i has $<$ -maximal predicate symbol P , and $C_i \in \mathfrak{C}$, and
- iii.) $\forall i: 0 \leq i \leq k-1$. $D_{i+1} \in EPC_{<}^{-1}(D_i, C_i)$.

(Thus a D which is $EPC_{<}^{+1}$ -deducible from $\{B\} \cup \mathfrak{C}$ is a deduction-ordering maximal clause obtained from B by $EPC_{<}$ -resolutions with clauses in $\mathfrak{C}$, which has $<$ -maximal predicate symbol less than or equal to P , and is pure-positive with respect to P .)

Further, let us say that Π is an $EPC_{<}^{+1}$ -deduction of D from $\{B\} \cup \mathfrak{C}$, if Π is a ("linear")- $EPC_{<}$ -deduction of depth k , (for some $k \geq 1$), with $k+1$ leaves $B \equiv D_0, C_0, \dots, C_{k-1}$, and k non-leaf nodes $D_1, \dots, D_k$, as above, and as illustrated in Figure 2.1, page 23.

DEFINITION 4.1.12. Given a mixed equality clause B , and a finite collection $\mathfrak{C}$ of equality clauses, let us say that D is $EPC_{<}^{+1}$ -deducible from $\{B\} \cup \mathfrak{C}$, if D is QR -positive (in Π), and there exist sequences (for some $k \geq 1$), $C_0, \dots, C_{k-1}$, and $D_0, \dots, D_{k-1}, D_k$, such that:

- i.) $D_0 \equiv B$ and $D_k \equiv D$,
- ii.) $\forall i: 0 \leq i \leq k-1$. $C_i \in \mathfrak{C}$, and
- iii.) $\forall i: 0 \leq i \leq k-1$. $D_{i+1} \in EPC_{<}^{-1}(C_i, D_i) \cap EPC_{<}^{-}(C_i, D_i)$.

(so that C_i is the QR -positive parent, while D_i is the negative parent of D_{i+1}).

Further, let us say that Π is an $EPC_{<}^{+1}$ -deduction of D from (equality clauses) $\{B\} \cup \mathfrak{C}$, if Π is an $EPC_{<}^{-}$ -deduction of depth k , (for some $k \geq 1$), with $k+1$ leaves $B \equiv D_0, C_0, \dots, C_{k-1}$, and k non-leaf nodes $D_1, \dots, D_k$, as above, and as illustrated in Figure 2.1, page 23.

Note that since $\mathfrak{C}$ is finite by assumption, and each $EPC_{<}$ -resolution eliminates at least one (in particular, the left-most) negative P (P possibly ' $=$ ') literal, clearly there are at most finitely many clauses D $EPC_{<}^{+1}$ -deducible from $\{B\} \cup \mathfrak{C}$, and every associated $EPC_{<}^{+1}$ -deduction Π (of D from $\{B\} \cup \mathfrak{C}$), can be of depth at most n , where n is the number of distinct negative P

instances which occur in B .

Also note that in any $\text{EPC}_{<}^{+1}$ -deduction Π of D from $\{B\} \cup \mathcal{C}$, (as above), B occurs exactly once, that occurrence as a leaf, and every resolvent in Π has exactly one parent which is a leaf in $\mathcal{C}$.

DEFINITION 4.1.13. Let us say that a $\text{EPC}_{<}$ -deduction Π is *above-equality*, if there is at most one node of Π which is labelled by an equality clause.

Note that if there is a (necessarily unique) node in an above-equality $\text{EPC}_{<}$ -deduction Π , which is labelled by an equality clause, that node is the root node of Π . Also, note that if Π is a $\text{EPC}_{<}^{+1}$ -deduction from $\{B\} \cup \mathcal{C}$, where B has with $<$ -maximal predicate symbol, $P \neq '='$, as in Definition 4.1.11, then $k_{\Pi}(B) = 1$, and Π is above-equality.

DEFINITION 4.1.14. Given a $\text{EPC}_{<}$ -refutation Π of, (or a $\text{EPC}_{<}$ -deduction of a clause D from), a set $\mathcal{S}$, Π above-equality, let us say that a node $n \in \Pi$, with label C_n , say, is *EPC_<-initial* (in Π), if:

- i. n is a leaf, or
- ii. for some parent node n' , say, of n , the clause $C_{n'}$, labelling n' is mixed in its $<$ -maximal predicate symbol P , say, while C_n is pure-positive with respect to P , (or has $<$ -maximal predicate symbol ' Q ' $<$ P).

If a clause C labels a $\text{EPC}_{<}$ -initial node n in a $\text{EPC}_{<}$ -deduction Π , we will say that C is *EPC_<-initial*, at n in Π , or simply that C is $\text{EPC}_{<}$ -initial in Π , if the context makes it clear at which node $n \in \Pi$, we are considering C .

Note that if Π is an above-equality $\text{EPC}_{<}^{+1}$ -deduction, (of a clause D from $\{B\} \cup \mathcal{C}$, say) as in Definition 4.1.11, then the root node of Π , (labelled by D) is $\text{EPC}_{<}$ -initial. Also, if Π is a $\text{EPC}_{<}$ -deduction of a (possibly empty) equality clause, E (say), and Π is above-equality, then the root node of Π , (labelled by E) is $\text{EPC}_{<}$ -initial. Further, it is not hard to see that if Π is any above-equality $\text{EPC}_{<}$ -deduction, then any node n in Π which is labelled by a clause C which is pure-positive with respect to its $<$ -maximal predicate symbol P , is $\text{EPC}_{<}$ -initial.

OBSERVATION 4.1.7. If n , with label C_n , say, is any non-leaf node in a $\text{EPC}_{<}$ -deduction of a clause D from a set $\mathcal{S}$, which is above-equality, then:

- i) n has exactly one deduction-ordering maximal (proper) ancestor n' , with label $C_{n'}$, say, such that n' is $\text{EPC}_{<}$ -initial and $C_{n'}$ is mixed with respect to its $<$ -maximal predicate symbol P , $P \neq '='$, and
- ii) there is a finite sequence of nodes $m_1, \dots, m_k$, (for some $k \geq 1$) with labels $C_{m_1}, \dots, C_{m_k}$, say, such that for $j: 1 \leq j \leq k$, C_{m_j} has $<$ -maximal predicate symbol P and is pure-positive with respect to P , and such that the fragment Γ (say,) of Π , rooted at n and pruned in $\{m_1, \dots, m_k\}$, is a depth k ("linear"-) $\text{EPC}_{<}$ -deduction of C_n from $\{C_{n'}\} \cup \{C_{m_1}, \dots, C_{m_k}\}$, with $k+1$ leaves, (labelled by) $C_{n'}, C_{m_1}, \dots, C_{m_k}$, and k non-leaf nodes.

In particular, if C_n is pure-positive with respect to P , then the fragment Γ , above, is a (above-equality) $\text{EPC}_{<}^{+1}$ -deduction.

DEFINITION 4.1.15. For any $\Pi, n, C_n, n', C_{n'}, m_1, \dots, m_k, C_{m_1}, \dots, C_{m_k}$ as in Observation 4.1.7, above, let us say n (and that C_n) has *$\text{EPC}_{<}$ -initial principal ancestor n' , ($C_{n'}$)* and *ancillary parents $m_1, \dots, m_k$, ($C_{m_1}, \dots, C_{m_k}$)*.

OBSERVATION 4.1.8. Note that (in $\text{EPC}_{<}$), every ancillary parent, (being pure-positive in its $<$ -maximal predicate symbol, $P \neq '='$), is $\text{EPC}_{<}$ -initial. Also, if Π is any non-trivial above-equality $\text{EPC}_{<}$ -deduction of a clause D , where D (at the root), is $\text{EPC}_{<}$ -initial (in Π), then Π may be *decomposed* into a $\text{EPC}_{<}^{+1}$ -deduction Π' of D , from principal ancestor B , (say), and ancillary parents $C_0, \dots, C_{k-1}$, (say, for some $k \geq 1$), a (possibly trivial) $\text{EPC}_{<}$ -deduction Π_B of B , and (possibly trivial) $\text{EPC}_{<}$ -deductions, Π_{C_i} , of C_i , for $i: 0 \leq i \leq k-1$, respectively.

4.2. Lifting and Grounding in the Above-Equality Fragment of $\text{EPC}_{<}$

LEMMA 4.2.1. Let B be mixed with respect to its $<$ -maximal predicate symbol P , for some $P \neq '='$, and suppose B' is any re-ordering of, (in particular, the negative P -instances in), B . Then if Π is any (above-equality) $\text{EPC}_{<}^{+1}$ -deduction of D from $\{B\} \cup \mathfrak{C}$, there is an (above-equality) $\text{EPC}_{<}^{+1}$ -deduction Π' of D' from $\{B'\} \cup \mathfrak{C}$, where D' is some re-ordering of D , and such that $\forall C \in \mathfrak{C}, k_{\Pi'}(C) = k_{\Pi}(C)$.

Of course, any re-ordering D' of D , subsumes D .

The $\text{EPC}_{<}$ -deduction analogue of Lemma 2.2.2 is:

LEMMA 4.2.2. Suppose $B \in \text{EPC}_{<}^1(A, D)$, for some clauses A , B , and D , such that A has $<$ -maximal predicate symbol P and is pure-positive with respect to P , $P \neq '='$, (and D has $<$ -maximal predicate symbol P and is mixed with respect to P).

Suppose also that (modulo clause ordering), A is a substitution instance of A' , and D is a substitution instance, $D'\theta_D$, of D' , (for some substitution θ_D , say), and with L_0 say, the left-most negative P instance in D' , $L_0\theta_D$ occurs as the left-most negative P instance in D . Then there exists $B' \in \text{EPC}_{<}^1(A', D')$ such that (modulo ordering), B is a substitution instance of B' .

We have the following elementary lemma for “grounding”, in the above-equality fragment.

LEMMA 4.2.3. Suppose $B \in \text{EPC}_{<}^1(A, D)$, for some clauses A , B , and D , where D , or A , has $<$ -maximal predicate symbol $P \neq '='$, and let B_g be a ground substitution instance of B .

Then there exist ground substitution instances A' and D' of A and D , respectively, and a (ground clause) $B'_g \in \text{EPC}_{<}^1(A', D')$, such that $B'_g \subseteq B_g$.

The (above-equality) $\text{EPC}_{<}$ -deduction analogue of Lemma 2.2.3) is:

LEMMA 4.2.4. Let B be a clause with $<$ -maximal predicate symbol P , say, $P \neq '='$,

B mixed with respect to P , and suppose that (for some $k \geq 1$), $C_0, \dots, C_{k-1}$,

(the C_i not necessarily distinct), and $B = D_0, \dots, D_{k-1}, D_k = D$, represents a (above equality) $\text{EPC}_{<}^{+1}$ -deduction Π , (as in Definition 4.1.11). Suppose B is (modulo ordering), a substitution instance of some clause B' , and that $C'_0, \dots, C'_{k-1}$, (the C'_i not necessarily distinct), is a sequence of (not necessarily distinct) clauses such that $\forall i: 0 \leq i \leq k-1$, C_i is (modulo ordering) a substitution instance of C'_i .

Then there exists a $\text{EPC}_{<}^{+1}$ -deduction Π' , of some clause D' , from principal ancestor B' and ancillary parents $\{C'_0, \dots, C'_{k-1}\}$, where D is (modulo ordering) a substitution instance of D' , and such that $\forall C' \in \{C'_0, \dots, C'_{k-1}\}$, $k_{\Pi'}(C') = \|\{i \mid 0 \leq i \leq k-1; C'_i = C'\}\|$.

The $\text{EPC}_{<}$ -deduction analogue of Lemma 2.2.4 is:

LEMMA 4.2.5. Suppose Π is an above-equality $\text{EPC}_{<}$ -deduction of a clause A , from a set $\mathfrak{S}$, and (A, at) the root is $\text{EPC}_{<}$ -initial. Suppose $\mathcal{L}$ denotes the set of leaf nodes n of Π , and for

each $n \in \mathcal{L}$, let B_n denote the clause (in Π) at n . Suppose that for each $n \in \mathcal{L}$, B'_n is a clause, of which (modulo ordering), B_n is a substitution instance, and that $\mathfrak{S}'$ denotes the set comprised of the clauses B'_n , (as n ranges over the leaf nodes of Π).

Then there is an above-equality $\text{EPC}_{<}$ -deduction Π' of a clause A' , $\text{EPC}_{<}$ -initial (in Π'), of which (modulo ordering), A is a substitution instance, from $\mathfrak{S}'$, such that

$$\forall B' \in \mathfrak{S}', \kappa_{\Pi'}(B') = ||\{n \in \mathcal{L} \mid B'_n = B'\}||.$$

(Obviously then, the total number $\kappa(\Pi')$ of leaf nodes of Π' , must be equal to the total number $\kappa(\Pi)$, of leaf nodes of Π .)

COROLLARY 4.2.1. Suppose Π is an above-equality $\text{EPC}_{<}$ -deduction of a (possibly empty) equality clause E from $\mathfrak{S}$, that $\mathcal{L}$ denotes the set of leaf nodes n of Π , and for each $n \in \mathcal{L}$, B_n denotes the clause (in Π) at n . Suppose that for each $n \in \mathcal{L}$, B'_n is a clause, of which (modulo ordering), B_n is a substitution instance, and that $\mathfrak{S}'$ denotes the set comprised of the clauses B'_n , (as n ranges over the leaf nodes of Π).

Then there is an above-equality $\text{EPC}_{<}$ -deduction, Π' of a (equality) clause E' , of which (modulo ordering), E is a substitution instance, from $\mathfrak{S}'$, such that

$$\forall B' \in \mathfrak{S}', \kappa_{\Pi'}(B') = ||\{n \in \mathcal{L} \mid B'_n = B'\}||.$$

4.3. Subsumption in the Above-Equality Fragment of $\text{EPC}_{<}$

The following lemma is a direct analogue, for the above-equality fragment of $\text{EPC}_{<}$ -deduction, of Lemma 2.4.1 for (subsumption in) $\text{PC}_{<}$ -deduction. For this fragment, there is a tight analogy, and results for $\text{PC}_{<}$ -deduction carry over nicely.

LEMMA 4.3.1. Suppose $B \in \text{EPC}_{<}^1(A, D)$, for some clauses A , B , and D , such that A has $<$ -maximal predicate symbol P and is pure-positive with respect to P , $P \neq '='$, (and D has $<$ -maximal predicate symbol P and is mixed with respect to P). Suppose also that (for some clauses A' and D'), A' subsumes A , and D' subsumes D , and that either:

- (i) D' is pure-positive with respect to P , or
- (ii) with θ_D some substitution such that $D'\theta_D \subseteq D$, and L_0 say, the left-most negative P instance in D' , $L_0\theta_D$ occurs as the left-most negative P instance in D .

Then, in case (i), D' subsumes B , and in case (ii), either A' subsumes B , or there exists $B' \in EPC_{<}^1(A', D')$ such that B' subsumes B .

The above-equality $EPC_{<}$ -deduction analogue of the $PC_{<}$ -deduction Lemma 2.4.2 is:

LEMMA 4.3.2. Suppose Π is a (above equality) $EPC_{<}^{+1}$ -deduction of a clause D , from ancillary parents in $\mathfrak{C}$, (say), and principal ancestor B , B with $<$ -maximal predicate symbol (and mixed with respect to) P , say, $P \neq '='$.

Then if B' subsumes B and B' (has $<$ -maximal predicate symbol $< P$ or) is pure-positive with respect to P , B' subsumes D .

The above-equality $EPC_{<}$ -deduction analogue of the $PC_{<}$ -deduction Lemma 2.4.3 is:

LEMMA 4.3.3. Let B be a clause with $<$ -maximal predicate symbol P , say, $P \neq '='$, B mixed with respect to P , and suppose that (for some $k \geq 1$), $C_0, \dots, C_{k-1}$, (the C_i not necessarily distinct), and $B = D_0, \dots, D_{k-1}$, $D_k = D$, represents a $EPC_{<}^{+1}$ -deduction Π , (as in Definition 4.1.11). Suppose B' subsumes B , and that $C'_0, \dots, C'_{k-1}$, is a sequence of (not necessarily distinct) clauses such that $\forall i: 0 \leq i \leq k-1$, C'_i subsumes C_i .

Then either:

- i. B' subsumes D ,
- ii. C'_i subsumes D , for some $i: 0 \leq i \leq k-1$, such that (modulo ordering) $C'_i \neq C_i$, or
- iii. there exists a $EPC_{<}^{+1}$ -deduction Π' , of a clause D' , from principal ancestor B' and ancillary parents in $\{C'_0, \dots, C'_{k-1}\}$, where D' subsumes D , and such that $\forall C' \in \{C'_0, \dots, C'_{k-1}\}$, $k_{\Pi'}(C') \leq ||\{i \mid 0 \leq i \leq k-1; C'_i = C'\}||$.

The above-equality $EPC_{<}$ -deduction analogue of the $PC_{<}$ -deduction Lemma 2.4.4 is:

LEMMA 4.3.4. Suppose Π is an above-equality $EPC_{<}$ -deduction of a clause A , from a set $\mathfrak{S}$, and (A, at) the root is $EPC_{<}$ -initial. Suppose $\mathfrak{L}$ denotes the set of leaf nodes n of Π ,

and for each $n \in \mathfrak{L}$, let B_n denote the clause (in Π) at n . Suppose that for each $n \in \mathfrak{L}$,

B'_n is a clause which subsumes B_n , and that $\mathfrak{S}'$ denotes the set comprised of the clauses B'_n , (as n ranges over the leaf nodes of Π).

Then there is an above-equality $EPC_{<}$ -deduction Π' of a clause A' , $EPC_{<}$ -initial (in Π'),

which subsumes A , from $\mathfrak{S}'$, such that $\forall B' \in \mathfrak{S}', k_{\Pi'}(B') \leq ||\{n \in \mathfrak{L} \mid B'_n = B'\}||$.

(Obviously then, the total number $\kappa(\Pi')$ of leaf nodes of Π' , must be less than or equal to the total number $\kappa(\Pi)$, of leaf nodes of Π .)

We have the following analogue of (both Corollary 2.4.1, and), Corollary 4.2.1:

COROLLARY 4.3.1. Suppose Π is an above-equality $\text{EPC}_{<}$ -deduction of a (possibly empty) equality clause E from $\mathfrak{S}$, that $\mathfrak{L}$ denotes the set of leaf nodes n of Π , and for each $n \in \mathfrak{L}$, B_n denotes the clause (in Π) at n . Suppose that for each $n \in \mathfrak{L}$, B'_n is a clause which subsumes B_n , and that $\mathfrak{S}'$ denotes the set comprised of the clauses B'_n , (as n ranges over the leaf nodes of Π).

Then there is an above-equality $\text{EPC}_{<}$ -deduction, Π' of a (equality) clause E' which subsumes E , from $\mathfrak{S}'$, such that $\forall B' \in \mathfrak{S}', k_{\Pi'}(B') \leq ||\{n \in \mathfrak{L} \mid B'_n = B'\}||$.

We now obtain a couple of analogous results for “grounding”, in the above-equality fragment.

LEMMA 4.3.5. Suppose Π is an above-equality $\text{EPC}_{<}$ -deduction of a clause A , from a set $\mathfrak{S}$, and (A, at) the root is $\text{EPC}_{<}$ -initial. Suppose $\mathfrak{L}$ denotes the set of leaf nodes n of Π , and for each $n \in \mathfrak{L}$, let B_n denote the clause (in Π) at n . Suppose that A_g is a ground substitution instance of A .

Then for each $n \in \mathfrak{L}$, there is a ground clause B'_n , which is a substitution instance of B_n , such that, with $\mathfrak{S}'$ denoting $\{B'_n \mid n \in \mathfrak{L}\}$, there is an above-equality $\text{EPC}_{<}$ -deduction Π' of a (ground) clause A' , $\text{EPC}_{<}$ -initial (in Π'), from $\mathfrak{S}'$, such that $A' \subseteq A_g$, and such that $\forall B' \in \mathfrak{S}', k_{\Pi'}(B') \leq ||\{n \in \mathfrak{L} \mid B'_n = B'\}||$.

COROLLARY 4.3.2. Suppose Π is an above-equality $\text{EPC}_{<}$ -deduction of an equality clause E , from a set $\mathfrak{S}$. Suppose $\mathfrak{L}$ denotes the set of leaf nodes, n of Π , and for each $n \in \mathfrak{L}$, let B_n denote the clause (in Π) at n . Let E_g be a ground substitution instance of E .

Then for each $n \in \mathfrak{L}$, there is a ground substitution instance, B'_n , of B_n , such that, with $\mathfrak{S}'$ denoting $\{B'_n \mid n \in \mathfrak{L}\}$, there is an above-equality $\text{EPC}_{<}$ -deduction Π' of a (ground) equality clause E' , from $\mathfrak{S}'$, where $E' \subseteq E_g$, and such that $\forall B' \in \mathfrak{S}', k_{\Pi'}(B') \leq ||\{n \in \mathfrak{L} \mid B'_n = B'\}||$.

DEFINITION 4.3.1. For any above-equality $\text{EPC}_{<}$ -deduction, Π , of an equality clause E , and any substitution, σ , such that $E\sigma$ is ground, let $\downarrow(\Pi, \sigma)$ denote some ground above-equality $\text{EPC}_{<}$ -deduction, Π' , of some ground equality clause $E' \subseteq E\sigma$, as described, and whose existence is guaranteed, in Corollary 4.3.2.

We now return to the consideration of the equality fragment.

4.4. Lifting and Grounding in the Equality Fragment of $\text{EPC}_{<}$

LEMMA 4.4.1. Suppose $B \in \text{EPC}_{+,-}(A, D)$, for some non-empty equality clauses A (positive) and D (mixed). Suppose further that (modulo clause ordering), A is a substitution instance of A' , and D is a substitution instance, $D'\rho_D$, of D' , and with L the left-most inequality in D' , $L\rho_D$ occurs as the left-most inequality in D , (for some substitution, ρ_D). Then there exists $B' \in \text{EPC}_{+,-}(A', D')$ such that (modulo ordering), B is a substitution instance of B' .

We introduce some lemmas for “grounding”, in the equality fragment.

LEMMA 4.4.2. Let A , B and D , be ground equality clauses such that $B \in \text{EPC}_{+,-}(A, D)$. Suppose that A' is A , modulo clause ordering, and D' is a ground equality clause exactly of the form $D \mid F$, where F contains only (ground) trivial inequality literals. That is, D' is ordered so that D , (as an ordered clause), occurs as an initial sub-sequence of D' , and the only literals in D' which follow D are those trivial inequalities in F .

Then there exists a (ground clause) $B' \in \text{EPC}_{+,-}(A', D')$, such that $B' \subseteq B \cup F$.

LEMMA 4.4.3. Suppose $B \in \text{EPC}_{+,-}(A, D)$, for some clauses A , B , and D , and let B_g be a ground substitution instance of B . Then there exist ground substitution instances A' and D' , of A and D , respectively, and a (ground clause) $B' \in \text{EPC}_{-}^1(A', D')$, such that $B' \subseteq B_g$.

COROLLARY 4.4.1. Suppose $B \in \text{EPC}_{-}^1(A, D)$, for some clauses A , B , and D , and let B_g be a ground substitution instance of B . Then there exist ground substitution instances A' and D' of A and D , respectively, and a (ground clause) $B' \in \text{EPC}_{-}^1(A', D')$, such that $B' \subseteq B_g \cup F$, for some ground trivial inequality clause F .

LEMMA 4.4.4. Suppose $B \in \text{EPC}_{+,+}(A, D)$, for some non-empty (positive) equality clauses A and D . Suppose further that (modulo clause ordering), A is a substitution instance of A' , and D is a substitution instance of D' . Then there exists $B' \in \text{EPC}_{+,+}(A', D')$ such that (modulo ordering), B is a substitution instance of B' .

LEMMA 4.4.5. Suppose $B \in \text{EPC}_+(A, D)$, for some non-empty equality clauses A and D . Suppose further that (modulo clause ordering), A is a substitution instance of A' , and D is a substitution instance of D' . Then there exists $B' \in \text{EPC}_+(A', D')$ such that (modulo ordering), B is a substitution instance of B' .

COROLLARY 4.4.2. Suppose there is a depth k EPC_+ -deduction Π , say, of an (QR -positive equality) clause B from a set of QR -positive equality clauses, $\mathcal{C}$. Suppose further that $\mathcal{C}'$ is a set of QR -positive equality clauses, such that for every $C \in \mathcal{C}$, there is a $C' \in \mathcal{C}'$ such that, (modulo ordering), C is a substitution instance of C' . Then there is a depth $k' \leq k$ EPC_+ -deduction Π' , of a (QR -positive equality) clause B' from $\mathcal{C}'$, such that (modulo ordering), B is a substitution instance of B' .

The following is immediate.

COROLLARY 4.4.3. Suppose $\mathcal{C}$ and $\mathcal{C}'$ are collections of QR -positive equality clauses, such that for every $C \in \mathcal{C}$, there is a $C' \in \mathcal{C}'$ such that, (modulo ordering), C is a substitution instance of C' . Then for each $D \in \text{EPC}_+(\mathcal{C})$, there is a $D' \in \text{EPC}_+(\mathcal{C}')$, such that (modulo ordering), D is a substitution instance of D' .

4.5. Subsumption in the Equality Fragment of $\text{EPC}_{<}$

LEMMA 4.5.1. Suppose $B \in \text{EPC}_{+,-}(A, D)$, for some non-empty equality clauses A (positive) and D (mixed). Suppose also that (for some clauses A' and D'), A' subsumes A , and D' subsumes D , and that either:

- (i) D' is positive, or
- (ii) with ρ_D some substitution such that $D'\rho_D \subseteq D$, and L_1 say, the left-most inequality in D' , $L_1\rho_D$ occurs as the left-most inequality in D .

Then, in case (i), D' subsumes B , and in case (ii), either A' subsumes B and A is not a substitution instance of A' , or there exists $B' \in EPC_{+,-}(A', D')$ such that B' subsumes B .

LEMMA 4.5.2. Suppose $B \in EPC_{+,+}(A, D)$, for some non-empty (positive) equality clauses A and D . Suppose further that (for some positive equality clauses A' and D'), A' subsumes A , and D' subsumes D . Then either A' subsumes B , or D' subsumes B , or there exists $B' \in EPC_{+,+}(A', D')$ such that B' subsumes B .

LEMMA 4.5.3. Suppose $B \in EPC_{+}(A, D)$, for some non-empty equality clauses A and D . Suppose further that A' subsumes A , and D' subsumes D . Then either A' subsumes B , or D' subsumes B , or there exists $B' \in EPC_{+}(A', D')$ such that B' subsumes B .

COROLLARY 4.5.1. Suppose there is a depth k EPC_{+} -deduction Π , say, of an (QR -positive equality) clause B from a set of QR -positive equality clauses, $\mathcal{C}$. Suppose further that $\mathcal{C}'$ is a set of QR -positive equality clauses, such that for every $C \in \mathcal{C}$, there is a $C' \in \mathcal{C}'$ such that C' subsumes C . Then there is a depth $k' \leq k$ EPC_{+} -deduction Π' , of a (QR -positive equality) clause B' from $\mathcal{C}'$, such that B' subsumes B .

The following is immediate.

COROLLARY 4.5.2. Suppose $\mathcal{C}$ and $\mathcal{C}'$ are collections of QR -positive equality clauses, such that for every $C \in \mathcal{C}$, there is a $C' \in \mathcal{C}'$ such that C' subsumes C . Then for each $D \in EPC_{+}(\mathcal{C})$, there is a $D' \in EPC_{+}(\mathcal{C}')$ such that D' subsumes D .

LEMMA 4.5.4. Suppose C is a positive equality clause. Then $QR_{full}(B)$ subsumes C , for some (equality) clause B , if and only if, for some substitution θ , and some trivial inequality clause F , $B\theta \subseteq C \cup F$.

COROLLARY 4.5.3. Suppose C is a positive equality clause, and $QR_{full}(B\sigma)$ subsumes C , for some substitution σ , and some (equality) clause B . Then $QR_{full}(B)$ subsumes C .

We also have the following:

COROLLARY 4.5.4. Suppose C is a positive equality clause, $QR_{full}(B)$ subsumes C , and A subsumes B , for some (equality) clauses A and B . Then $QR_{full}(A)$ subsumes C .

DEFINITION 4.5.1. For a clause A , let $\chi(A)$ denote the sum of the decrements of sizes of the equivalence classes α in τ_{A-} . That is, let $\chi(A) = \sum \{ \|\alpha\| - 1 \mid \alpha \in \tau_{A-} \}$.

We leave the reader to convince her/himself of the following:

OBSERVATION 4.5.1. If $\chi(A) = n$, for a ground clause A , then in order to (EPC_{-}) resolve away all non-trivial inequality literals, it is necessary to do at least n (EPC_{-}) resolutions. (with positive equality clauses).

LEMMA 4.5.5. Suppose that C is a ground positive equality clause, and τ is a (finite) set of (finite) equivalence classes of ground terms, ($\tau = \tau_{B-}$, for some clause B , say). Suppose also that for every pair of distinct (ground) terms s and t , such that, for some $\alpha \in \tau$, $\{s, t\} \subseteq \alpha$, there is a (ground) positive equality clause $E_{s,t}$, such that $E_{s,t} \subseteq C \cup \{=(s, t), =(t, s)\}$, and $\emptyset \neq \{=(s, t), =(t, s)\} \cap E_{s,t}$.

Then if E is any ground equality clause such that for any non-trivial inequality $\neq(s', t') \in E$, there is some $\alpha \in \tau$, such that $\{s', t'\} \subseteq \alpha$, and $\chi(E) = n > 0$, then from E together with the $E_{s,t}$ above, there is a depth n $\text{EPC}_{<}^{+1}$ -deduction of a clause G , such that $G \subseteq C \cup E \setminus E^- \cup F'$, for some trivial inequality clause F' .

(Of course, if $n = 0$, then $E \subseteq C \cup E \setminus E^- \cup F'$, for some trivial inequality clause F' , already.)

LEMMA 4.5.6. Let Π denote a depth k $\text{EPC}_{<}^{+1}$ -deduction of an (ground, QR -positive) equality clause D from ground equality clauses $\{B\} \cup \mathcal{C}$. (So Π is a depth k EPC_{-} -deduction of a QR -positive (equality) clause D from a ground (mixed, equality clause) B , and a collection of ground (QR -positive equality) clauses $\mathcal{C}$, such that B occurs (at a leaf) negatively in Π , and each $C \in \mathcal{C}$ occurs QR -positively in Π).

Suppose also that $B' \subseteq B \cup F$, for some ground trivial inequality clause F .

Then either:

- i) $QR_{full}(B')$ subsumes $QR_{full}(D)$, or
- ii) there is a finite collection $\mathcal{C}'$ of (QR -positive equality) clauses C' , each $C' \in \mathcal{C}'$ having an EPC_{+} -deduction $\Pi_{C'}$ from $\mathcal{C}$, and a $\text{EPC}_{<}^{+1}$ -deduction Π' , with depth $k' \leq k$, of some (QR -positive equality) clause D' from $\{B'\} \cup \mathcal{C}'$, such that $QR_{full}(D')$ subsumes $QR_{full}(D)$.

We now obtain an analogous result for grounding, in the equality fragment, which will guarantee a certain kind of grounding.

LEMMA 4.5.7. Let Π denote a depth k $\text{EPC}_{<}^{+1}$ -deduction of a (QR -positive equality) clause D from equality clauses $\{B\} \cup \mathcal{C}$.

(So Π is a depth k EPC_{-} -deduction of a QR -positive (equality) clause D from a (mixed, equality clause) B , and a collection of (QR -positive equality) clauses $\mathcal{C}$, such that B occurs (at a leaf) negatively in Π , and each $C \in \mathcal{C}$ occurs QR -positively in Π).

Let D_g be a QR -positive ground substitution instance of D .

Then there is a finite collection $\mathcal{C}''$ of (QR -positive equality) clauses C'' , each $C'' \in \mathcal{C}''$ having an EPC_{+} -deduction $\Pi_{C''}$ from $\mathcal{C}$, a finite collection $\mathcal{C}'$, of QR -positive ground substitution instances of clauses in $\mathcal{C}''$, a ground substitution instance, B' , of B , and a $\text{EPC}_{<}^{+1}$ -deduction Π' , with depth $k' \leq k$, of some ground (QR -positive equality) clause D' from $\{B'\} \cup \mathcal{C}'$, such that $QR_{full}(D') \subseteq QR_{full}(D_g)$.

DEFINITION 4.5.2. For any $\text{EPC}_{<}^{+1}$ -deduction, Π , of a QR -positive equality clause E , and any substitution, σ , such that $E\sigma$ is ground and QR -positive, let $\downarrow(\Pi, \sigma)$ denote some ground $\text{EPC}_{<}^{+1}$ -deduction, Π' , of some ground QR -positive equality clause E' , such that $QR_{full}(E') \subseteq QR_{full}(E\sigma)$, as described, and whose existence is guaranteed, in Lemma 4.5.7.

LEMMA 4.5.8. Suppose B is a ground equality clause, and (for some clause B' and substitution θ), $B'\theta \subseteq B$, and further, that $B'\theta$ contains at least one non-trivial inequality literal. Then there exists a $B'' \in QR_{first}(B')$, containing a (non-trivial) left-most inequality $\neq(s'', t'')$, say, such that for some substitution ρ , $B''\rho \subseteq B$, and $\neq(s'', t'')\rho$ is non-trivial.

LEMMA 4.5.9. Let Π denote a depth k $\text{EPC}_{<}^{+1}$ -deduction of an (ground, QR -positive) equality clause D from ground equality clauses $\{B\} \cup \mathcal{C}$. (So Π is a depth k EPC_{-} -deduction of a QR -positive (equality) clause D from a ground (mixed, equality clause) B , and a collection of ground (QR -positive equality) clauses $\mathcal{C}$, such that B occurs (at a leaf) negatively in Π , and each $C \in \mathcal{C}$ occurs QR -positively in Π). Let $\mathcal{C}'$ be a collection of QR -positive equality clauses, such that, for each $C \in \mathcal{C}$, there is some $C' \in \mathcal{C}'$ such that C' subsumes C .

Suppose also that B' subsumes B .

(Note as a special case of this, B is, modulo ordering, a substitution instance of B').

Then either:

- i) for some $C' \in \mathfrak{C}'$, $QR_{full}(C')$ subsumes $QR_{full}(D)$, or
- ii) $QR_{full}(B')$ subsumes $QR_{full}(D)$, or
- iii) there is a finite collection $\mathfrak{C}''$ of (QR -positive equality) clauses C'' , each $C'' \in \mathfrak{C}''$ having an EPC_+ -deduction $\Pi_{C''}$ from $\mathfrak{C}'$, and a $EPC_{<}^{+1}$ -deduction Π' , with depth $k' \leq k$, of some (QR -positive equality) clause D' from $\{B'\} \cup \mathfrak{C}''$, such that $QR_{full}(D')$ subsumes $QR_{full}(D)$.

Further, in the special case that for each $C \in \mathfrak{C}$, there is some $C' \in \mathfrak{C}'$ such that C is (modulo ordering) a substitution instance of C' , case (i) above can be dropped, (that is, either (ii) or (iii) above, must hold).

LEMMA 4.5.10. Let Π denote a depth k $EPC_{<}^{+1}$ -deduction of an (QR -positive) equality clause D from equality clauses $\{B\} \cup \mathfrak{C}$. (So Π is a depth k EPC_- -deduction of a QR -positive (equality) clause D from a (mixed, equality clause) B , and a collection of (QR -positive equality) clauses, $\mathfrak{C}$, such that B occurs (at a leaf) negatively in Π , and each $C \in \mathfrak{C}$ occurs QR -positively in Π).

Let $\mathfrak{C}'$ be a collection of QR -positive equality clauses, such that, for each $C \in \mathfrak{C}$, there is some $C' \in \mathfrak{C}'$ such that C' subsumes C . Suppose also that B' subsumes B . (Note as a special case of this, B is, modulo ordering, a substitution instance of B').

Then either:

- i) for some $C' \in EPC_+(\mathfrak{C}')$, $QR_{full}(C')$ subsumes $QR_{full}(D)$, or
- ii) $QR_{full}(B')$ subsumes $QR_{full}(D)$, or
- iii) there is a finite collection $\mathfrak{C}''$ of (QR -positive equality) clauses C'' , each $C'' \in \mathfrak{C}''$ having an EPC_+ -deduction $\Pi_{C''}$ from $\mathfrak{C}'$, and a $EPC_{<}^{+1}$ -deduction Π' , with depth $k' \leq k$, of some (QR -positive equality) clause D' from $\{B'\} \cup \mathfrak{C}''$, such that $QR_{full}(D')$ subsumes $QR_{full}(D)$.

Further, in the special case that for each $C \in \mathfrak{C}$, there is some $C' \in \mathfrak{C}'$ such that C is (modulo ordering) a substitution instance of C' , case (i) above can be dropped, (that is, either (ii) or (iii) above, must hold).

CHAPTER 5

Refutation Completeness for $\text{EPC}_{<}$

5.0. Overview

This chapter is brief, and is devoted simply to demonstrating the refutation completeness of $\text{EPC}_{<}$.

The proof is done in stages. First, (in Section 5.1), the relationship between $\text{EPC}_{<}$ and $\text{PC}_{<}$ in the above-equality fragment is examined, and ultimately, Corollary 5.1.1 is obtained.

Second, (in Section 5.2), the relationship between $\text{EPC}_{<}$ and $\text{PC}_{<}$ in the equality fragment is (very briefly), examined, and Lemma 5.2.1 is obtained.

Finally, these results are combined to obtain the desired result, Theorem 5.2.1.

5.1. The Above-Equality Fragment

It is convenient to introduce the following:

DEFINITION 5.1.1. Let $C \vdash_{QR} D$ denote the condition that, there is some substitution θ , and some (possibly empty) set F , comprised entirely of trivial inequality literals, such that, (modulo clause ordering), $C\theta = D \cup F$.

PROPOSITION 5.1.1. $\vdash_{QR}$ is transitive. That is, if $B \vdash_{QR} C$; $C \vdash_{QR} D$, then $B \vdash_{QR} D$.

We note the following:

OBSERVATION 5.1.1. Suppose $B \in \text{PC}_{<}^1(A, D)$, for some clauses A , B , and D , such that A (and also D) has $<$ -maximal predicate symbol P , $P \neq '='$.

Then there exists a clause $B' \in \text{EPC}_{<}^1(A, D)$, such that $B' \vdash_{QR} B$.

We now present some technical lemmas.

The following is an analogue of Lemma 2.2.2 and of Lemma 4.2.2:

LEMMA 5.1.1. Suppose $B \in \text{EPC}_{<}^1(A, D)$, for some clauses A , B , and D , such that A has $<$ -maximal predicate symbol P and is pure-positive with respect to P , $P \neq '='$, (and D has $<$ -maximal predicate symbol P and is mixed with respect to P). Suppose also that $A' \vdash_{QR} A$, $D' \vdash_{QR} D$, and more specifically, modulo clause ordering, $D'\theta_D \subseteq D \cup F_D$, for some F_D , comprised entirely of trivial inequality literals, (and for some substitution θ_D , say), and with L_0 say, the left-most negative P instance in D' , $L_0\theta_D$ occurs as the left-most negative P instance in D . Then there exists $B' \in \text{EPC}_{<}^1(A', D')$ such that $B' \vdash_{QR} B$.

The following is an analogue of Lemma 2.2.3 and Lemma 4.2.4.

LEMMA 5.1.2. Let B be a clause with $<$ -maximal predicate symbol P , say, $P \neq '='$, B mixed with respect to P , and suppose that (for some $k \geq 1$), $C_0, \dots, C_{k-1}$, (the C_i not necessarily distinct), and $B = D_0, \dots, D_{k-1}$, $D_k = D$, represents a (above equality) $\text{EPC}_{<}^{+1}$ -deduction Π , (as in Definition 4.1.11). Suppose that (for some clause B'), $B' \vdash_{QR} B$, and that $C'_0, \dots, C'_{k-1}$, (the C'_i not necessarily distinct), is a sequence of (not necessarily distinct) clauses such that $\forall i: 0 \leq i \leq k-1$, $C'_i \vdash_{QR} C_i$. Then there exists a $\text{EPC}_{<}^{+1}$ -deduction Π' , of some clause D' , from principal ancestor B' and ancillary parents $\{C'_0, \dots, C'_{k-1}\}$, where $D' \vdash_{QR} D$, and such that $\forall C' \in \{C'_0, \dots, C'_{k-1}\}$ $k_{\Pi'}(C') = \|\{i \mid 0 \leq i \leq k-1 : C'_i = C'\}\|$.

An analogue of Lemma 2.2.4 and of Lemma 4.2.5 is:

LEMMA 5.1.3. Suppose Π is an above-equality $\text{EPC}_{<}$ -deduction of a clause A , from a set $\mathfrak{S}$, and (A , at) the root is $\text{EPC}_{<}$ -initial. Suppose $\mathcal{L}$ denotes the set of leaf nodes n of Π , and for each $n \in \mathcal{L}$, let B_n denote the clause (in Π) at n . Suppose that for each $n \in \mathcal{L}$, B'_n is a clause such that $B'_n \vdash_{QR} B_n$, and that $\mathfrak{S}'$ denotes the set comprised of the clauses B'_n , (as n ranges over the leaf nodes of Π). Then there is a $\text{EPC}_{<}$ -deduction Π' of a clause A' , $\text{EPC}_{<}$ -initial (in Π'), where $A' \vdash_{QR} A$, from $\mathfrak{S}'$, such that $\forall B' \in \mathfrak{S}'$, $k_{\Pi'}(B') = \|\{n \in \mathcal{L} \mid B'_n = B'\}\|$.

(Obviously then, the total number $\kappa(\Pi')$ of leaf nodes of Π' , must be equal to the total number $\kappa(\Pi)$, of leaf nodes of Π .)

Having these technical lemmas in hand, the following major result is easy.

LEMMA 5.1.4. Suppose Π is a $\text{PC}_{<}$ -deduction in which no equality literal is resolved upon, of a clause A , from a set $\mathfrak{S}$, and (A, at) the root is $\text{PC}_{<}$ -initial.

Then there is an above-equality $\text{EPC}_{<}$ -deduction Π' of a clause A' , $\text{EPC}_{<}$ -initial (in Π'), where $A' \vdash_{QR} A$, from $\mathfrak{S}$, such that $\forall C \in \mathfrak{S}, k_{\Pi'}(C) = k_{\Pi}(C)$.

Lemma 5.1.4 immediately yields the following:

COROLLARY 5.1.1. Suppose Π is a $\text{PC}_{<}$ -deduction in which no equality literal is resolved upon, of an (possibly empty) equality clause A , from a set $\mathfrak{S}$.

Then there is an above-equality $\text{EPC}_{<}$ -deduction Π' of an equality clause A' , where $A' \vdash_{QR} A$, from $\mathfrak{S}$, such that $\forall C \in \mathfrak{S}, k_{\Pi'}(C) = k_{\Pi}(C)$.

5.2. The Equality Fragment, and the Combined Result

LEMMA 5.2.1. Suppose $C \in \text{PC}_{<}^1(A, B)$, for some equality clauses A , B , and C .

Then (C is an equality clause, and) there exists $C' \in \text{EPC}_{<}^1(A, B)$, such that,

(modulo clause ordering), $C' \subseteq C$.

THEOREM 5.2.1. ($\text{EPC}_{<}$ is refutation complete, weak version.)

Let S be any unsatisfiable set of clauses. Then there is a $\text{EPC}_{<}$ -refutation of $S \cup \{=(x, x)\}$.

We obtain a strong version of this theorem, as Corollary 7.2.7, in Subsection 7.2.6.



CHAPTER 6

Σ_0 -subsumption and $\text{EPC}_{<}$

6.0. Overview

This (relatively short) chapter contains some technical results for Σ_0 -subsumption in $\text{EPC}_{<}$, which are essentially restricted analogues of those results in Section 2.4. There are two sections.

Section 6.1, addresses Σ_0 -subsumption in the above-equality fragment (of $\text{EPC}_{<}$), with main results Lemma 6.1.3 and Corollary 6.1.1.

Section 6.2, addresses Σ_0 -subsumption in the equality fragment, and culminates in Lemma 6.2.11. Additionally, Section 6.1 contains a number of (technically useful) smaller results, pertaining to Σ_0 -subsumption and “ QR_{first} ”. (or “ QR_{full} ”).

Unfortunately, the proof provided for Lemma 6.1.3, (in the Appendix) is rather lengthy. However, in this case at least, it was not clear (to the author) that “Lemma modularization” was warranted in terms of (Lemma) re-usability, and anyway, there are already plenty of “technical Lemmas” to contend with.

6.1. Σ_0 -subsumption in the Above-Equality Fragment

LEMMA 6.1.1. Suppose C Σ_0 -subsumes D , and $B \in \text{EPC}_{<}^1(A, D)$, for some clauses A , B , C , and D , such that D has $<$ -maximal predicate symbol P and is pure-positive with respect to P , $P \neq '='$. Then (C is pure-positive with respect to P , and) either C Σ_0 -subsumes B , or there exists $B' \in \text{EPC}_{<}^1(A, C)$ such that B' Σ_0 -subsumes B .

DEFINITION 6.1.1. For any clause A , let $cl_0(A)$ denote the symmetric closure of $\mathcal{T}_A-(A)$.

DEFINITION 6.1.2. For clause A and B , $A \Sigma_0\text{-subsumes}_0 B$ if there exists a substitution θ , and a trivial inequality clause F , such that $A\theta \subseteq \text{cl}_0(B) \cup F$.

LEMMA 6.1.2. Let B be a clause with $<$ -maximal predicate symbol P , say, $P \neq '='$, B mixed with respect to P . Let $\mathcal{C}$ be a finite collection of clauses such that $\forall C \in \mathcal{C}$, C has $<$ -maximal predicate symbol P , and is pure-positive with respect to P . Suppose B' is a clause and $B' \Sigma_0\text{-subsumes}_0 B$.

Then if Π is any $\text{EPC}_{<}^{+1}$ -deduction of D from $\{B\} \cup \mathcal{C}$, either:

- i. $B' \Sigma_0\text{-subsumes}_0 D$, or
- ii. there is an $\text{EPC}_{<}^{+1}$ -deduction Π' of some clause D' from $\{B'\} \cup \mathcal{C}$, such that $D' \Sigma_0\text{-subsumes}_0 D$, and such that further, $\forall C \in \mathcal{C}$, $k_{\Pi'}(C) \leq k_{\Pi}(C)$.

LEMMA 6.1.3. Let $C \neq D$, $C \Sigma_0\text{-subsume } D$, (for clauses C and D). Suppose Π denotes an above-equality $\text{EPC}_{<}$ -deduction of an equality clause E , where D labels at least one leaf of Π , (that is, $k_{\Pi}(D) \geq 1$). Then there exists a finite forest $\mathfrak{F}$ of $\text{EPC}_{<}$ -deductions Π' , where:

- a) $\forall \Pi' \in \mathfrak{F}$, Π' contains exactly one equality clause, $E_{\Pi'}$, say, (necessarily at the root of Π').
- b) $\Sigma_0 \cup \{E_{\Pi'} : \Pi' \in \mathfrak{F}\} \models E$, and
- c) $\forall \Pi' \in \mathfrak{F}$,
 - i) $k_{\Pi'}(C') \leq k_{\Pi}(C')$, for any clause $C' \notin \{C, D\}$ labelling a leaf in Π' ,
 - ii) $k_{\Pi'}(C) \leq k_{\Pi}(C) + 1$, and
 - iii) $k_{\Pi'}(D) \leq k_{\Pi}(D) - 1$.

COROLLARY 6.1.1. Suppose Π is an above-equality $\text{EPC}_{<}$ -deduction of an equality clause E , and that $\mathcal{L}$ denotes the set of leaf nodes of Π .

For any leaf node $n \in \mathcal{L}$, let C_n denote the clause labelling n in Π , and suppose that, to each $n \in \mathcal{L}$, is associated a clause B_n , such that $B_n \Sigma_0\text{-subsumes } C_n$.

Then there exists a finite forest $\mathfrak{F}$ of above-equality $\text{EPC}_{<}$ -deductions Π' , of equality clauses, $E_{\Pi'}$, say, such that:

- a) $\Sigma_0 \cup \{E_{\Pi'} : \Pi' \in \mathfrak{F}\} \models E$, and
- b) $\forall \Pi' \in \mathfrak{F}$, for any clause B , $k_{\Pi'}(B) \leq ||\{n \in \mathcal{L} \mid B_n = B\}||$.

6.2. Σ_0 -subsumption in the Equality Fragment

DEFINITION 6.2.1. Given (ordered) equality clauses A and A' , we will say that A equals (or is) A' , *modulo symmetry*, (and write $A = A'$, *modulo symmetry*), if either both A and A' are the empty clause, or: (for some $n \geq 1$, for some terms $s_1, \dots, s_n, t_1, \dots, t_n, s'_1, \dots, s'_n, t'_1, \dots, t'_n$), A is $q_1(s_1, t_1) \mid \dots \mid q_n(s_n, t_n)$, A' is $q_1(s'_1, t'_1) \mid \dots \mid q_n(s'_n, t'_n)$, (where, for each $i: 1 \leq i \leq n$, q_i is either $=$ or $\neq$), and for each $i: 1 \leq i \leq n$, either $q_i(s_i, t_i)$ is $q_i(s'_i, t'_i)$, or $q_i(s_i, t_i)$ is $q_i(t'_i, s'_i)$.

By $A \subseteq A'$, *modulo symmetry*, we of course only mean that, for any terms s and t , for q either $=$ or $\neq$, if $q(s, t) \in A$ then either $q(s, t) \in A'$ or $q(t, s) \in A'$.

Similarly, an assertion of equality between A and A' under both the qualifiers, *modulo symmetry* and *modulo (clause) ordering*, will mean only that, *modulo symmetry*, both $A \subseteq A'$, and $A' \subseteq A$.

An assertion of equality between two sets $\mathfrak{A}$ and $\mathfrak{A}'$ of (ordered) clauses, qualified by either *modulo symmetry*, *modulo (clause) ordering*, or both, will of course mean that each element in $\mathfrak{A}$ is equal to, modulo the given qualification, some element in $\mathfrak{A}'$, and vice versa.

The following observation is easy:

OBSERVATION 6.2.1. If A and B are both positive equality clauses, then A Σ_0 -subsumes B if and only if there is a substitution θ such that for every literal $=(s, t) \in A$, either $=(s, t)\theta \in B$, or $=(t, s)\theta \in B$, (that is, such that $A\theta \subseteq B$, modulo symmetry).

Thus, restricted to positive equality clauses, Σ_0 -subsumption is just subsumption, except that equality is treated as symmetric.

The following observations are also easy:

OBSERVATION 6.2.2. For any (ordered) clauses A , A' , B , and B' , if, modulo both symmetry and ordering, $A' = A$ and $B' = B$, then:

1. $QR_{full}(A) = QR_{full}(A')$, modulo both ordering and symmetry.
2. A Σ_0 -subsumes B implies that A' Σ_0 -subsumes B' .

OBSERVATION 6.2.3. For any (ordered) equality clauses A , A' , B , and B' , if modulo symmetry, $A' = A$ and $B' = B$, then:

1. For any clause $C \in EPC_{+,-}(A, B)$, there is a clause $C' \in EPC_{+,-}(A', B')$ such that, modulo both ordering and symmetry, $C = C'$.
2. For any clause $C \in EPC_{+,+}(A, B)$, there is a clause $C' \in EPC_{+,+}(A', B')$ such that, modulo both ordering and symmetry, $C = C'$.
3. $EPC_+(A, B) = EPC_+(A', B')$, modulo both ordering and symmetry.

LEMMA 6.2.1. If B is a positive equality clause, then for any (equality) clause A , if $QR_{full}(A) \Sigma_0$ -subsumes B , then $QR_{full}(A)$ is positive.

LEMMA 6.2.2. If B is a positive equality clause, and A is any clause, then $A \Sigma_0$ -subsumes B if and only if (A is an equality clause, and) $QR_{full}(A) \Sigma_0$ -subsumes B .

It is also useful to note that:

LEMMA 6.2.3. If B is any equality clause, and A is any clause, then $A \Sigma_0$ -subsumes B if (A is an equality clause, and) for some $A' \in QR_{first}(A)$, $A' \Sigma_0$ -subsumes B .

COROLLARY 6.2.1. If B is any equality clause, and A is any clause, then $A \Sigma_0$ -subsumes B if (A is an equality clause, and) $QR_{full}(A) \Sigma_0$ -subsumes B .

LEMMA 6.2.4. Suppose $C \in EPC_-(A, B)$, for some (ordered) equality clauses A (QR -positive), and B , (mixed), and that, for some clause A' , $A' \Sigma_0$ -subsumes A . Then either $A' \Sigma_0$ -subsumes C , or there exists a $C' \in EPC_-(A', B)$ such that $C' \Sigma_0$ -subsumes C .

LEMMA 6.2.5. Suppose $C \in EPC_+(A, B)$, for some (ordered QR -positive) equality clauses A and B , and that, for some clauses A' and B' , $A' \Sigma_0$ -subsumes A , and $B' \Sigma_0$ -subsumes B . Then either:

- a) $A' \Sigma_0$ -subsumes C ,
- b) $B' \Sigma_0$ -subsumes C , or
- c) there exists a $C' \in EPC_+(A', B')$ such that $C' \Sigma_0$ -subsumes C .

COROLLARY 6.2.2. Suppose there is a depth k EPC_+ -deduction Π , say, of an (QR -positive equality) clause B from a set of QR -positive equality clauses, $\mathfrak{C}$. Suppose further that $\mathfrak{C}'$ is a set of QR -positive equality clauses, such that for every $C \in \mathfrak{C}$, there is a $C' \in \mathfrak{C}'$ such that $C' \Sigma_0$ -subsumes C . Then there is a depth $k' \leq k$ EPC_+ -deduction Π' , of a (QR -positive equality) clause B' from $\mathfrak{C}'$, such that $B' \Sigma_0$ -subsumes B .

The following is immediate.

COROLLARY 6.2.3. Suppose $\mathfrak{C}$ and $\mathfrak{C}'$ are collections of QR -positive equality clauses.

such that for every $C \in \mathfrak{C}$, there is a $C' \in \mathfrak{C}'$ such that $C' \Sigma_0$ -subsumes C .

Then for each $D \in \text{EPC}_+(\mathfrak{C})$, there is a $D' \in \text{EPC}_+(\mathfrak{C}')$ such that $D' \Sigma_0$ -subsumes D .

COROLLARY 6.2.4. Suppose $\mathfrak{C}$ and $\mathfrak{C}'$ are collections of QR -positive equality clauses.

such that for every $C \in \mathfrak{C}$, there exists a $C' \in \mathfrak{C}'$ such that either $C' \Sigma_0$ -subsumes C , or

$C' \Sigma_0$ -subsumes $QR_{full}(C)$. Then for each $D \in \text{EPC}_+(\mathfrak{C})$, there is a $D' \in \text{EPC}_+(\mathfrak{C}')$ such that $D' \Sigma_0$ -subsumes $QR_{full}(D)$.

We need an slight modification of Lemma 4.5.8.

LEMMA 6.2.6. Suppose B is a ground equality clause, and

(for some clause B' and substitution θ), *modulo symmetry*, $B'\theta \subseteq B$, and further,

that $B'\theta$ contains at least one non-trivial inequality literal.

Then there exists a $B'' \in QR_{first}(B')$, containing a (non-trivial) left-most inequality $\neq(s'', t'')$, say, such that for some substitution ρ , *modulo symmetry*, $B''\rho \subseteq B$, and $\neq(s'', t'')\rho$ is non-trivial.

LEMMA 6.2.7. Let Π denote a depth k $\text{EPC}_<^{+1}$ -deduction of an (ground, QR -positive) equality clause D from ground equality clauses $\{B\} \cup \mathfrak{C}$. (So Π is a depth k EPC_- -deduction of a QR -positive (equality) clause D from a ground (mixed, equality clause) B , and a collection of ground (QR -positive equality) clauses $\mathfrak{C}$, such that B occurs (at a leaf) negatively in Π , and each $C \in \mathfrak{C}$ occurs QR -positively in Π). Let $\mathfrak{C}'$ be a collection of QR -positive equality clauses, such that, for each $C \in \mathfrak{C}$, there is some $C' \in \mathfrak{C}'$, such that $C' \Sigma_0$ -subsumes C . Suppose also that (B' is a clause, and), for some substitution, θ , *modulo symmetry*, $B'\theta \subseteq B$. Then either:

- i) for some $C' \in \mathfrak{C}'$, $QR_{full}(C') \Sigma_0$ -subsumes $QR_{full}(D)$, or

- ii) $QR_{full}(B')$ Σ_0 -subsumes $QR_{full}(D)$, or
- iii) there is a finite collection $\mathcal{C}''$ of (QR -positive equality) clauses C'' , each $C'' \in \mathcal{C}''$ having an EPC_+ -deduction $\Pi_{C''}$ from $\mathcal{C}'$, and an $\text{EPC}_<^{+1}$ -deduction Π' , with depth $k' \leq k$, of some (QR -positive equality) clause D' from $\{B'\} \cup \mathcal{C}''$, such that $QR_{full}(D')$ Σ_0 -subsumes $QR_{full}(D)$.

LEMMA 6.2.8. Let Π denote a depth k $\text{EPC}_<^{+1}$ -deduction of an (ground QR -positive) equality clause D from ground equality clauses $\{B\} \cup \mathcal{C}$. (So Π is a depth k EPC_- -deduction of a QR -positive (equality) clause D from a ground (mixed, equality clause) B , and a collection of ground (QR -positive equality) clauses $\mathcal{C}$, such that B occurs (at a leaf) negatively in Π , and each $C \in \mathcal{C}$ occurs QR -positively in Π). Suppose also that B' is a ground clause, and B' Σ_0 -subsumes B .

Then either:

- i) there is a $G' \in \text{EPC}_+(\mathcal{C} \cup \{B'\})$ such that G' Σ_0 -subsumes $QR_{full}(D)$, or
- ii) there is a finite collection $\mathcal{C}'$ of (QR -positive equality) clauses C' , each $C' \in \mathcal{C}'$ having an EPC_+ -deduction $\Pi_{C'}$ from $\mathcal{C}$, and an $\text{EPC}_<^{+1}$ -deduction Π' , with depth $k' \leq k$, of some (QR -positive equality) clause D' from $\{B'\} \cup \mathcal{C}'$, such that there is a $G' \in \text{EPC}_+(\mathcal{C} \cup \{D'\})$ such that G' Σ_0 -subsumes $QR_{full}(D)$.

The reader may have noticed from the proof, that under the hypothesis of Lemma 6.2.8, we could have actually shown (the somewhat more informative result) that either:

- i) B' Σ_0 -subsumes $QR_{full}(D)$, or
- ii) there is an EPC_+ -deduction Π_0 from $\{B'\} \cup \mathcal{C}$, of a (positive equality clause) G' , where $k_{\Pi_0}(B') = 1$, and G' Σ_0 -subsumes $QR_{full}(D)$, or
- iii) there is a finite collection $\mathcal{C}'$ of (QR -positive equality) clauses C' , each $C' \in \mathcal{C}'$ having an EPC_+ -deduction $\Pi_{C'}$ from $\mathcal{C}$, and such that there is an $\text{EPC}_<^{+1}$ -deduction Π' , with depth $k' \leq k$, of some (QR -positive equality) clause D' from $\{B'\} \cup \mathcal{C}'$, such that either:
 - iii.i) D' Σ_0 -subsumes $QR_{full}(D)$, or
 - iii.ii) there is an EPC_+ -deduction Π_0 from $\{D'\} \cup \mathcal{C}$, of a (positive equality clause) G' , where $k_{\Pi_0}(D') = 1$, and G' Σ_0 -subsumes $QR_{full}(D)$.

Of course, we selected the simpler statement because, for our purposes, it is enough, and we wanted to avoid making the proof unduly complicated. We leave the proof of this statement as an exercise for the zealous reader. We only mention in passing that “improvements” are similarly possible for the following two lemmas.

LEMMA 6.2.9. Let Π denote a depth k $\text{EPC}_{<}^{+1}$ -deduction of an (ground QR -positive) equality clause D from ground equality clauses $\{B\} \cup \mathcal{C}$. (So Π is a depth k EPC_- -deduction of a QR -positive (equality) clause D from a ground (mixed, equality clause) B , and a collection of ground (QR -positive equality) clauses $\mathcal{C}$, such that B occurs (at a leaf) negatively in Π , and each $C \in \mathcal{C}$ occurs QR -positively in Π). Let $\mathcal{C}'$ be a collection of ground QR -positive equality clauses, such that, for each $C \in \mathcal{C}$, there is some $C' \in \mathcal{C}'$ such that $C' \Sigma_0$ -subsumes C .

Suppose also that B' is a ground clause, and $B' \Sigma_0$ -subsumes B . Then either:

- i) there is a $G' \in \text{EPC}_+(\mathcal{C}' \cup \{B'\})$ such that $G' \Sigma_0$ -subsumes $QR_{full}(D)$, or
- ii) there is a finite collection $\mathcal{C}''$ of (QR -positive equality) clauses C'' , each $C'' \in \mathcal{C}''$ having an EPC_+ -deduction $\Pi_{C''}$ from $\mathcal{C}'$, and an $\text{EPC}_{<}^{+1}$ -deduction Π' , with depth $k' \leq k$, of some (QR -positive equality) clause D' from $\{B'\} \cup \mathcal{C}''$, such that there is a $G' \in \text{EPC}_+(\mathcal{C}' \cup \{D'\})$ such that $G' \Sigma_0$ -subsumes $QR_{full}(D)$.

LEMMA 6.2.10. Let Π denote a depth k $\text{EPC}_{<}^{+1}$ -deduction of an (QR -positive) equality clause D from equality clauses $\{B\} \cup \mathcal{C}$.

(So Π is a depth k EPC_- -deduction of a QR -positive (equality) clause D from a (mixed, equality clause) B , and a collection of (QR -positive equality) clauses $\mathcal{C}$, such that B occurs (at a leaf) negatively in Π , and each $C \in \mathcal{C}$ occurs QR -positively in Π).

Let $\mathcal{C}'$ be a collection of QR -positive equality clauses, such that, for each $C \in \mathcal{C}$, there is some $C' \in \mathcal{C}'$ such that $C' \Sigma_0$ -subsumes C .

Suppose also that (B' is a clause, and), $B' \Sigma_0$ -subsumes B . Then either:

- i) there is a $G' \in \text{EPC}_+(\mathcal{C}' \cup \{B'\})$ such that $G' \Sigma_0$ -subsumes $QR_{full}(D)$, or
- ii) there is a finite collection $\mathcal{C}''$ of (QR -positive equality) clauses C'' , each $C'' \in \mathcal{C}''$ having an EPC_+ -deduction $\Pi_{C''}$ from $\mathcal{C}'$, and an $\text{EPC}_{<}^{+1}$ -deduction Π' , with depth $k' \leq k$, of some (QR -positive equality) clause D' from $\{B'\} \cup \mathcal{C}''$, such that there is a $G' \in \text{EPC}_+(\mathcal{C}' \cup \{D'\})$ such that $G' \Sigma_0$ -subsumes $QR_{full}(D)$.

LEMMA 6.2.11. Let B be a (mixed) equality clause and suppose that (for some $k \geq 1$), $C_0, \dots, C_{k-1}$, (the C_i not necessarily distinct QR -positive equality clauses), and $B = D_0, \dots, D_{k-1}, D_k = D$, represents an $\text{EPC}_<^{+1}$ -deduction (which is also an EPC_- -deduction), Π .

(of a QR -positive clause, D). Suppose that $C'_0, \dots, C'_{k-1}$, is a sequence of (not necessarily distinct QR -positive equality) clauses such that

$\forall i: 0 \leq i \leq k-1, C'_i, \Sigma_0$ -subsumes C_i . Then either:

1. C'_0, Σ_0 -subsumes D , or
2. there exists a $D_1' \in \text{EPC}_-(C'_0, B)$, such that either:
 - i. there is a $G' \in \text{EPC}_+(\{D_1', C'_0, \dots, C'_{k-1}\})$, such that $G' \Sigma_0$ -subsumes $QR_{full}(D)$,
or
 - ii. $k > 1$ and there is a finite collection $\mathfrak{C}''$ of (QR -positive equality) clauses C'' ,
each $C'' \in \mathfrak{C}''$ having an EPC_+ -deduction $\Pi_{C''}$ from $\{C'_1, \dots, C'_{k-1}\}$, and an
 $\text{EPC}_<^{+1}$ -deduction Π' , with depth $k' \leq k-1$, of some (QR -positive equality) clause
 D' from $\{D_1'\} \cup \mathfrak{C}''$, (that is, from principal ancestor D_1' and ancillary parents $\mathfrak{C}''$),
such that there is a (QR -positive equality clause) $G' \in \text{EPC}_+(\{D', C'_1, \dots, C'_{k-1}\})$
such that $G' \Sigma_0$ -subsumes $QR_{full}(D)$.

CHAPTER 7

Σ_0 -tautology versus Σ_0 -subsumption, and Refutation Completeness of $EPC_{<}$ Under Deletion

7.0. Overview

This chapter is composed of three sections.

Section 7.1 is very short, establishing a useful relationship between Σ_0 -tautology and Σ_0 -subsumption in Theorem 7.1.1.

Section 7.2 is longer, and contains the (long awaited) establishment of the refutation completeness of $EPC_{<}$ with (or under), Σ_0 -subsumption (and Σ_0 -tautology) deletion.

Section 7.2 is divided into subsections. In Subsection 7.2.1, ($EPC_{<}$ -) *deduction sequences* are introduced. Subsections 7.2.2, 7.2.3, 7.2.4, and 7.2.5, are each very short, establishing technical results for the above-equality fragment, the positive equality fragment, the negative equality fragment, and the equality fragment, respectively, (the single result in Subsection 7.2.5 being a synthesis of the respective results of Subsections 7.2.3 and 7.2.4).

In Subsection 7.2.6 “everything” comes together in Theorem 7.2.1, (and Corollary 7.2.2), (wherein the respective results of Subsections 7.2.2 and 7.2.5 have been synthesized), and the break-water/watershed is there established. The further introduction of Σ_0 -tautology deletion is then broached with Lemma 7.2.5, from which Corollaries 7.2.3 through 7.2.7 follow quite easily.

Corollary 7.2.7 is the strong version of Theorem 5.2.1, establishing the refutation completeness of $EPC_{<}$, in the appropriate sense. Corollary 7.2.5 may be considered to be (at least, one of) the principal result (s) established in this work.

The results of Section 7.2 are of course, intended to sit in rough analogy with those (for $PC_{<}$) of 2.5.

Finally, in explicit analogy (by reference) to what was written (for $PC_{<}$) in Section 2.6, Section 7.3 briefly acknowledges the discrepancy between the formulation given for the (completeness) results of Section 7.2 and the “more traditional” setting of refutation completeness with deletion. (where it is presented within the context of a given deduction/deletion strategy), and offers Proposition 7.3.1), which (in analogy with Proposition 2.6.1 for $PC_{<}$), illustrates how the main results of Section 7.2 may be (usefully) interpreted in such a setting.

7.1. Σ_0 -tautology versus Σ_0 -subsumption

LEMMA 7.1.1. For an equality clause A , if A is a Σ_0 -tautology, and $A' \in QR_{first}(A)$, then A' is a Σ_0 -tautology.

COROLLARY 7.1.1. For an equality clause A , if A is a Σ_0 -tautology, then so is $QR_{full}(A)$.

It is not hard to see that if D is an (ordinary, binary, merge-) resolvent of a clause B with a tautology C , then either D is also a tautology, or B subsumes D .

An analogous result holds in $EPC_{<}$.

THEOREM 7.1.1. If D is a $EPC_{<}$ -resolvent of a clause B with a Σ_0 -tautology C , then either:

- i) D is a Σ_0 -tautology, or
- ii) B Σ_0 -subsumes D .

7.2. Refutation Completeness of $EPC_{<}$ Under Deletion

7.2.1. Preliminaries

DEFINITION 7.2.1. For any finite non-empty set S of clauses, by a Σ_0 -subsumption deleted $EPC_{<}$ -deduction sequence from S , we shall mean any sequence Δ of clauses $C_0, C_1, \dots$, such that:

- 1) $\forall C \in S. \exists i \geq 0$, such that C_i Σ_0 -subsumes C , and
- 2) $\forall i \geq 0. \forall j \geq 0. \forall C \in EPC_{<}^1(C_i, C_j). \exists k \geq 0$, such that C_k Σ_0 -subsumes C .

DEFINITION 7.2.2. For any finite non-empty set S of clauses, by a refined Σ_0 -subsumption deleted $EPC_{<}$ -deduction sequence from S , we shall mean any sequence of clauses, Δ , such that:

- 1) $\forall C \in S$, if C is the empty clause, $\square$, then $\square \in |\Delta|$, and otherwise, $\exists C' \in |\Delta|$, such that C' Σ_0 -subsumes C , and
- 2) $\forall A \in |\Delta|. \forall B \in |\Delta|. \forall C \in EPC_{<}^1(A, B)$, if $C = \square$, then $\square \in |\Delta|$, and otherwise, $\exists C' \in |\Delta|$, such that C' Σ_0 -subsumes C .

OBSERVATION 7.2.1. For any finite non-empty set S of clauses,
if Δ is any refined Σ_0 -subsumption deleted $EPC_{<}$ -deduction sequence from S ,
then Δ is a Σ_0 -subsumption deleted $EPC_{<}$ -deduction sequence from S .

DEFINITION 7.2.3. For any finite non-empty set S of clauses,
by a Σ_0 -subsumption and Σ_0 -tautology deleted $EPC_{<}$ -deduction sequence from S ,
we shall mean any sequence Δ of clauses $C_0, C_1, \dots$, such that:

- 1) $\forall C \in S$, either C is a Σ_0 -tautology, or $\exists i \geq 0$, such that $C_i \Sigma_0$ -subsumes C , and
- 2) $\forall i \geq 0, \forall j \geq 0, \forall C \in EPC_{<}^1(C_i, C_j)$, either C is a Σ_0 -tautology,
or $\exists k \geq 0$, such that $C_k \Sigma_0$ -subsumes C .

DEFINITION 7.2.4. For any finite non-empty set S of clauses,
by a *refined* Σ_0 -subsumption and Σ_0 -tautology deleted $EPC_{<}$ -deduction sequence from S ,
we shall mean any sequence of clauses, Δ , such that:

- 1) $\forall C \in S$, if C is the empty clause, $\square$, then $\square \in |\Delta|$, and otherwise,
either C is a Σ_0 -tautology, or $\exists C' \in |\Delta|$, such that $C' \Sigma_0$ -subsumes C , and
- 2) $\forall A \in |\Delta|, \forall B \in |\Delta|, \forall C \in EPC_{<}^1(A, B)$, if $C = \square$, then $\square \in |\Delta|$, and otherwise,
either C is a Σ_0 -tautology, or $\exists C' \in |\Delta|$, such that $C' \Sigma_0$ -subsumes C .

OBSERVATION 7.2.2. For any finite non-empty set S of clauses,
if Δ is any refined Σ_0 -subsumption and Σ_0 -tautology deleted $EPC_{<}$ -deduction sequence from S ,
then Δ is a Σ_0 -subsumption and Σ_0 -tautology deleted $EPC_{<}$ -deduction sequence from S .

7.2.2. The Above-Equality Fragment

LEMMA 7.2.1. Let Δ be a Σ_0 -subsumption deleted $EPC_{<}$ -deduction sequence from S ,
for some finite non-empty set of clauses S .

Let $\mathfrak{F}$ be a finite collection of above-equality $EPC_{<}$ -deductions Π , of equality clauses, E_Π ,
such that for each clause C which labels a leaf of any $\Pi \in \mathfrak{F}$, there is some $C' \in |\Delta|$,
such that $C' \Sigma_0$ -subsumes C .

Then there is a finite collection of equality clauses $\mathfrak{E}_0 \subseteq |\Delta|$, such that, $\forall \Pi \in \mathfrak{F}, \Sigma_0 \cup \mathfrak{E}_0 \models E_\Pi$.

7.2.3. The Positive Equality Fragment

LEMMA 7.2.2. Let Δ be a Σ_0 -subsumption deleted $EPC_{<}$ -deduction sequence from S ,
for some finite non-empty set of clauses S .

Let Π be an EPC_+ -deduction of a (QR -positive, equality) clause, E , say,
such that for any (QR -positive, equality) clause, C , which labels a leaf of Π ,
there is a (QR -positive, equality) clause $C' \in |\Delta|$ such that $C' \Sigma_0$ -subsumes C .

Then there is a (QR -positive equality) clause $E' \in |\Delta|$, such that $E' \Sigma_0$ -subsumes E .

The following is immediate.

COROLLARY 7.2.1. Let Δ be a Σ_0 -subsumption deleted $\text{EPC}_{<}$ -deduction sequence from S , for some finite non-empty set of clauses S .

Let $\mathcal{C}$ be a finite collection of QR -positive equality clauses, such that for all $C \in \mathcal{C}$, there exists a $C' \in |\Delta|$, such that $C' \Sigma_0$ -subsumes C .

Then for all $B \in \text{EPC}_+(\mathcal{C})$, there exists a $B' \in |\Delta|$ such that $B' \Sigma_0$ -subsumes B .

7.2.4. The Negative Equality Fragment

LEMMA 7.2.3. For some finite non-empty set, S , of clauses,

let Δ be a Σ_0 -subsumption deleted $\text{EPC}_{<}$ -deduction sequence from S .

Let B be a mixed equality clause, and let $\mathcal{C}$ be a finite collection of QR -positive equality clauses.

Let Π denote a $\text{EPC}_{<}^{+1}$ -deduction of a (QR -positive equality) clause D , from principal ancestor B and ancillary parents in $\mathcal{C}$.

Suppose further, that there is a $B' \in |\Delta|$, and a finite collection $\mathcal{C}' \subseteq |\Delta|$,

such that $B' \Sigma_0$ -subsumes B , and such that, $\forall C \in \mathcal{C}$, $\exists C' \in \mathcal{C}'$, such that $C' \Sigma_0$ -subsumes C .

Then there exists a $D' \in |\Delta|$, such that $D' \Sigma_0$ -subsumes $QR_{full}(D)$.

7.2.5. The Equality Fragment (Synthesis)

LEMMA 7.2.4. For some finite non-empty set S of clauses, let Δ be a Σ_0 -subsumption deleted $\text{EPC}_{<}$ -deduction sequence from S .

Let $\mathcal{E}$ be a finite collection of equality clauses such that,

$\forall E \in \mathcal{E}$, there exists an $E' \in |\Delta|$, such that $E' \Sigma_0$ -subsumes E .

Suppose Π is an equality $\text{EPC}_{<}$ -deduction of a QR -positive equality clause D , from $\mathcal{E}$.

Then there exists a $D' \in |\Delta|$, such that $D' \Sigma_0$ -subsumes $QR_{full}(D)$.

7.2.6. Synthesis and Result

THEOREM 7.2.1. ($\text{EPC}_{<}$ is refutation complete under Σ_0 -subsumption deletion, weak version.)

Let S be any finite unsatisfiable set of clauses, such that $\Sigma_0 \subseteq S$.

If Δ is any Σ_0 -subsumption deleted $\text{EPC}_{<}$ -deduction sequence from S ,

then $\exists D \in |\Delta|$ such that $D \Sigma_0$ -subsumes $\square$.

COROLLARY 7.2.2. ($\text{EPC}_{<}$ is refutation complete under refined Σ_0 -subsumption deletion, weak version.)

Let S be any finite unsatisfiable set of clauses, such that $\Sigma_0 \subseteq S$.

If Δ is any refined Σ_0 -subsumption deleted $\text{EPC}_{<}$ -deduction sequence from S , then $\square \in |\Delta|$.

LEMMA 7.2.5. Let S be a finite non-empty set of clauses.

Given a non-empty Σ_0 -subsumption and Σ_0 -tautology deleted $\text{EPC}_{<}$ -deduction sequence from S ,

$\Delta \equiv C_0, C_1, \dots$, there exists a Σ_0 -subsumption deleted $\text{EPC}_{<}$ -deduction sequence from S ,

$\Delta' \equiv C'_0, C'_1, \dots$, such that $\forall j \geq 0$, either:

a) $\exists i \geq 0$ such that $C'_j = C_i$, or

b) C'_j is a Σ_0 -tautology.

COROLLARY 7.2.3. (EPC_< is refutation complete under Σ_0 -subsumption and Σ_0 -tautology deletion, weak version.)

Given any finite unsatisfiable set S of clauses, such that $\Sigma_0 \subseteq S$,

if Δ is any Σ_0 -subsumption and Σ_0 -tautology deleted EPC_<-deduction sequence from S ,

then $\exists D \in |\Delta|$ such that D Σ_0 -subsumes $\square$.

We now also have:

COROLLARY 7.2.4. (EPC_< is refutation complete under refined Σ_0 -subsumption and Σ_0 -tautology deletion, weak version.)

Given any finite unsatisfiable set S of clauses, such that $\Sigma_0 \subseteq S$,

if Δ is any refined Σ_0 -subsumption and Σ_0 -tautology deleted EPC_<-deduction sequence from S ,

then $\square \in |\Delta|$.

Corollaries 7.2.3 and 7.2.4 can be immediately strengthened to:

COROLLARY 7.2.5. (EPC_< is refutation complete under Σ_0 -subsumption and Σ_0 -tautology deletion, strong version.)

Given any finite Σ_0 -unsatisfiable set S of clauses,

1) if Δ is any Σ_0 -subsumption and Σ_0 -tautology deleted EPC_<-deduction sequence from S ,
then $\exists D \in |\Delta|$ such that D Σ_0 -subsumes $\square$.

2) if Δ is any refined Σ_0 -subsumption and Σ_0 -tautology deleted EPC_<-deduction sequence from S ,
then $\square \in |\Delta|$.

Similarly, Theorem 7.2.1, and Corollary 7.2.2, can be strengthened.

COROLLARY 7.2.6. (EPC_< is refutation complete under Σ_0 -subsumption deletion, strong version.)

Given any finite Σ_0 -unsatisfiable set S of clauses,

1) if Δ is any Σ_0 -subsumption deleted EPC_<-deduction sequence from S ,
then $\exists D \in |\Delta|$ such that D Σ_0 -subsumes $\square$.

2) if Δ is any refined Σ_0 -subsumption deleted EPC_<-deduction sequence from S ,
then $\square \in |\Delta|$.

We also, of course, have the following strengthening of Theorem 5.2.1.

COROLLARY 7.2.7. (EPC_< is refutation complete, strong version.)

Given any finite Σ_0 -unsatisfiable set, S of clauses, there is a EPC_<-refutation of S .

7.3. Generating Deduction Sequences with Deletion

With regard to deduction/deletion strategies, everything in Section 2.6 for PC_< and subsumption, carries over for EPC_< and Σ_0 -subsumption, without complications.

In particular, we have:

PROPOSITION 7.3.1. Let $\supseteq$ be any well-founded pre-order on clauses such that whenever $C \supseteq C'$, C' Σ_0 -subsumes C . Let $\mathcal{S}$ be any shell method or best-first deduction strategy (as described in Section 2.6), for $\text{EPC}_<$, with forward Σ_0 -subsumption (and Σ_0 -tautology) deletion, and with (restricted) backward (Σ_0 -subsumption) deletion, where the criterion for the backward deletion of (an established) clause C by a (newly deduced) clause C' is given by $C \supseteq C'$, and, in which case, providing (and only if) it is not the case that $C' \supseteq C$, the backward deletion policy allows the deletion of C . Then $\mathcal{S}$ is (Σ_0 -) refutation complete.

CHAPTER 8

Clause Reduction, Refutation Completeness of $EPC_{<}$ With Reduction Under Deletion, and Refining the Deduction Rules

8.0. Overview

This chapter is composed of four sections.

In Section 8.1 some technical results are presented, which will allow the justification of certain *clause reduction procedures* in the above-equality fragment. This section is divided into subsections. In Subsection 8.1.1 *QR-factoring* is introduced, and some results, analogous to results found in Sections 4.2 and 4.3, are obtained for QR-factoring in the above-equality fragment (of $EPC_{<}$). Analogously, in Subsection 8.1.2, notions of *equivalence* and *demodulation* are introduced, and some results, (again, analogous to results found in Sections 4.2 and 4.3), are obtained for the notion of equivalence introduced. In Subsection 8.1.3 some notions of *reduction* (for the above-equality fragment of $EPC_{<}$) are introduced, are the major results of Subsections 8.1.1 and 8.1.2 are synthesized to produce a sequence of technical results culminating in Corollary 8.1.8. (an obvious mimicry of Corollary 6.1.1.78).

In Section 8.2, $(\mathcal{E}-)$ reduct Σ_0 -subsumption, an extension of Σ_0 -subsumption, is introduced, and the results of Section 7.2 are recast for the admission/introduction of *reduction* (into the $EPC_{<}$ -deduction process). This Section is again divided into Subsections. In Subsection 8.2.1 the extension of Σ_0 -subsumption, $(\mathcal{E}-)$ reduct Σ_0 -subsumption, and correspondingly modified (reduced $EPC_{<}$) *deduction sequences* are introduced. Subsections 8.2.2 and 8.2.3 are short, and treat the above-equality and equality fragments, respectively. Subsection 8.2.4 simply lays out the appropriate analogues of the main results of Subsection 7.2.6, (page 88). The principal results of Subsections 8.2.2 and 8.2.3 are synthesized in Corollaries 8.2.3 and 8.2.4. Lemma 8.2.3 then prepares the way for the further introduction of Σ_0 -tautology deletion, (just as Lemma 7.2.5 did in Subsection 7.2.6), and Corollaries 8.2.5 and 8.2.6 follow.

Section 8.3 is a brief revisiting of the subject of reduction. Here, *repf's* (generalizing *replacement factors*), and *folding* (a generalization of a special case of *Replacement Resolution*), are introduced and some observations are made.

Section 8.4 addresses, (at least partially) the issue of refining/narrowing the deduction rules of $\text{EPC}_{<}$. In particular, in Subsection 8.4.1 through a short sequence of transformations, a reformulation of the equality fragment deduction rules is given, (in Definition 8.4.8), which is “narrower” (though not necessarily any more elegant) than the initial formulation, but for which the associated proof theory, (for example “lifting”), would have been more problematic.

Finally, Section 8.5, briefly addresses the issue of interpreting, now the completeness results, (in particular, Corollaries 8.2.5 and 8.2.6), of Section 8.2, within the context of some particular given deduction/deletion strategy.

8.1. Clause Reduction: I

8.1.1. QR-factoring, (in the Above-Equality Fragment)

DEFINITION 8.1.1. For any clause A , we will let $F(A)$ denote the set of trivial inequalities in A .

DEFINITION 8.1.2. For any clauses A and A' , we will say that A' is a *QR-factor* of A if, for some (possibly empty) $I \subseteq A^-$, a most general substitution, θ , say, which unifies s and t for each $\neq(s, t) \in I$, (so that $I\theta$ contains only trivial inequality literals), exists, and modulo ordering, $A' = A\theta \setminus F(A\theta)$.

OBSERVATION 8.1.1. For any clauses A and A' ,

A' is a QR-factor of a subclause of A if and only if A' is a subclause of a QR-factor of A .

Here of course, by “subclause”, we mean the condition that:

(an ordered clause) B is a subclause of (an ordered clause) C , if (and only if), $B \subseteq C$.

In analogy with Lemma 4.2.2, (or Lemma 2.2.2), we have:

LEMMA 8.1.1. Suppose $B \in \text{EPC}_{<}^1(A, D)$, for some clauses A , B , and D , such that A has $<$ -maximal predicate symbol P and is pure-positive with respect to P , $P \neq '='$, (and D has $<$ -maximal predicate symbol P and is mixed with respect to P). Suppose also that A' is a QR-factor of A , and D' is a QR-factor, $D\theta_D \setminus F(D\theta_D)$, (modulo ordering), of D , (for some $I_D \subseteq D^-$, and some substitution θ_D , say, such that θ_D is a most general substitution such that every literal in $I_D\theta_D$ is a trivial inequality), and with L_0 say, the left-most negative P instance in D , $L_0\theta_D$ occurs as the left-most negative P instance in D' .

Then there exists $B' \in \text{EPC}_{<}^1(A', D')$, such that B' is a QR-factor of some $B'' \subseteq B$.

The following is an analogue to Lemma 4.3.1.

COROLLARY 8.1.1. Suppose $B \in \text{EPC}_{<}^1(A, D)$, for some clauses A , B , and D , such that A has $<$ -maximal predicate symbol P and is pure-positive with respect to P , $P \neq '='$, (and D has $<$ -maximal predicate symbol P and is mixed with respect to P). Suppose that A'' and D'' are subclauses of A and D , respectively, (that is, $A'' \subseteq A$ and $D'' \subseteq D$), and with L_0 say, the left-most negative P instance in D , either:

- (i) D'' is pure-positive with respect to P , or
- (ii) L_0 occurs as the left-most negative P instance in D'' .

Suppose also that A' is a QR-factor of A'' , and D' is a QR-factor, $D''\theta_{D''} \setminus F(D''\theta_{D''})$, (modulo ordering), of D'' ,

(for some $I_{D''} \subseteq D''^-$, and some substitution $\theta_{D''}$, say, such that $\theta_{D''}$ is a most general substitution such that every literal in $I_{D''}\theta_{D''}$ is a trivial inequality), and $L_0\theta_{D''}$ occurs as the left-most negative P instance in D' .

Then, in case (i), D' is a QR-factor of some $B'' \subseteq B$, and in case (ii), either A' is a QR-factor of some $B'' \subseteq B$, or there exists $B' \in \text{EPC}_{<}^1(A', D')$ such that B' is a QR-factor of some $B'' \subseteq B$.

The (above-equality) QR-factoring analogue of Lemma 4.3.2 is:

LEMMA 8.1.2. Suppose Π is an (above-equality) $\text{EPC}_{<}^{+1}$ -deduction of a clause D , from ancillary parents in $\mathcal{C}$, (say), and principal ancestor B . B with $<$ -maximal predicate symbol (and mixed with respect to) P , say, $P \neq '='$.

Then if B' is a QR-factor of a subclause of B and B' (has $<$ -maximal predicate symbol $< P$ or) is pure-positive with respect to P . B' is a QR-factor of a subclause of D .

The (above-equality) QR-factoring analogue of Lemma 4.3.3 is:

LEMMA 8.1.3. Let B be a clause with $<$ -maximal predicate symbol P , say, $P \neq '='$. B mixed with respect to P , and suppose that (for some $k \geq 1$), $C_0, \dots, C_{k-1}$, (the C_i not necessarily distinct), and $B = D_0, \dots, D_{k-1}$, $D_k = D$, represents an (above-equality) $\text{EPC}_{<}^{+1}$ -deduction Π , (as in Definition 4.1.11). Suppose B' is a QR-factor of a subclause of B , and that $C'_0, \dots, C'_{k-1}$, (the C'_i not necessarily distinct), is a sequence of (not necessarily distinct) clauses such that $\forall i: 0 \leq i \leq k-1$, C'_i is a QR-factor of a subclause of C_i .

Then either:

- i. B' is a QR-factor of a subclause of D .
- ii. C'_i is a QR-factor of a subclause of D , for some $i: 0 \leq i \leq k-1$, such that (modulo ordering) $C'_i \neq C_i$, or
- iii. there exists a $\text{EPC}_{<}^{+1}$ -deduction Π' , of some clause D' , from principal ancestor B' and ancillary parents $\{C'_0, \dots, C'_{k-1}\}$, where D' is a QR-factor of a subclause of D , and such that $\forall C' \in \{C'_0, \dots, C'_{k-1}\} \quad k_{\Pi'}(C') = ||\{i \mid 0 \leq i \leq k-1; C'_i = C'\}||$.

The (above-equality) QR-factoring analogue of Lemma 4.3.4 is:

LEMMA 8.1.4. Suppose Π is an above-equality $\text{EPC}_{<}$ -deduction of a clause A , from a set $\mathcal{S}$, and (A , at) the root is $\text{EPC}_{<}$ -initial. Suppose $\mathcal{L}$ denotes the set of leaf nodes n of Π , and for each $n \in \mathcal{L}$, let B_n denote the clause (in Π) at n . Suppose that for each $n \in \mathcal{L}$, B'_n is a QR-factor of a subclause of B_n , and that $\mathcal{S}'$ denotes the set comprised of the clauses B'_n , (as n ranges over the leaf nodes of Π).

Then there is an above-equality $\text{EPC}_{\leq}$ -deduction Π' of a clause A' , $\text{EPC}_{\leq}$ -initial (in Π'), which is a QR-factor of a subclause of A , from $\mathfrak{S}'$, such that $\forall B' \in \mathfrak{S}', k_{\Pi'}(B') \leq ||\{n \in \mathfrak{L} \mid B'_n = B'\}||$.

The (above-equality) QR-factoring analogue of Corollary 4.3.1:

COROLLARY 8.1.2. Suppose Π is an above-equality $\text{EPC}_{\leq}$ -deduction of a (possibly empty) equality clause E from $\mathfrak{S}$, that $\mathfrak{L}$ denotes the set of leaf nodes n of Π , and for each $n \in \mathfrak{L}$, B_n denotes the clause (in Π) at n . Suppose that for each $n \in \mathfrak{L}$, B'_n is a QR-factor of a subclause of B_n , and that $\mathfrak{S}'$ denotes the set comprised of the clauses B'_n , (as n ranges over the leaf nodes of Π).

Then there is an above-equality $\text{EPC}_{\leq}$ -deduction, Π' of a (equality) clause E' which is a QR-factor of a subclause of E , from $\mathfrak{S}'$, such that $\forall B' \in \mathfrak{S}', k_{\Pi'}(B') \leq ||\{n \in \mathfrak{L} \mid B'_n = B'\}||$.

8.1.2. Demodulation, (in the Above-Equality Fragment)

DEFINITION 8.1.3. For any set, $\mathfrak{E}$ of positive equality unit clauses, let $\text{Terms}(\mathfrak{E})$ denote the collection of all terms s and t , such that $\text{E}(s, t) \in \mathfrak{E}$.
(so $s \in \text{Terms}(\mathfrak{E})$ if and only if for some term t , either $\text{E}(s, t) \in \mathfrak{E}$ or $\text{E}(t, s) \in \mathfrak{E}$).

DEFINITION 8.1.4. For any set, $\mathfrak{E}$ of positive equality unit clauses, let $cl_+(\mathfrak{E})$ denote the least set X such that:

- i) $\mathfrak{E} \subseteq X$.
- ii) if $\text{E}(s, t) \in X$, then $\text{E}(t, s) \in X$.
- iii) if $\text{E}(r, s); \text{E}(s', t) \in X$, and the most general unifier, θ , say, of s and s' exists, then $\text{E}(r\theta, t\theta) \in X$.

Evidently, (for any set, $\mathfrak{E}$, of positive equality units),
 $\mathfrak{E} \subseteq cl_+(\mathfrak{E})$, $\text{Terms}(\mathfrak{E}) \subseteq \text{Terms}(cl_+(\mathfrak{E}))$, and if for any terms s and t , $s \sim t$ denotes the condition that $\text{E}(s, t) \in cl_+(\mathfrak{E})$, then $\sim$ is an equivalence relation on $\text{Terms}(cl_+(\mathfrak{E}))$.

Also, for any set, $\mathfrak{E}$, of positive equality units, for any finite subset, $X \subseteq cl_+(\mathfrak{E})$,
(evidently) there exists a finite subset $\mathfrak{E}' \subseteq \mathfrak{E}$, such that $X \subseteq cl_+(\mathfrak{E}')$.

DEFINITION 8.1.5. Let $\mathfrak{E}$ be any set of positive equality unit clauses,

- 1) For any terms s and t , t is a *flat demodulant* of s , (with respect to, or via $\mathfrak{E}$), if either $s = t$, $\exists L \in \mathfrak{E}$ such that L subsumes $\text{E}(s, t)$, or $\exists L \in \mathfrak{E}$ such that L subsumes $\text{E}(t, s)$.
- 2) For any literals, L and L' , L' is a *flat demodulant* of L , (with respect to, or via $\mathfrak{E}$), if (for some predicate symbol P , possibly '=', of arity $k \geq 0$, and for some not necessarily distinct terms $s_1, \dots, s_k$ and some not necessarily distinct terms $t_1, \dots, t_k$),
 $L = P(s_1, \dots, s_k)$ and $L' = P(t_1, \dots, t_k)$, or $L = \neg P(s_1, \dots, s_k)$ and $L' = \neg P(t_1, \dots, t_k)$,
(so L and L' have the same sign and predicate name),
and further, $\forall i: 0 \leq i \leq k$, t_i is a flat demodulant of s_i ,

- 3) For any unordered clauses C and C' , C' is a *flat demodulant* of C , (with respect to $\mathfrak{E}$), if there is a surjection $f: C \rightarrow C'$, (mapping each literal $L \in C$ to some literal $L' \in C'$), such that, $\forall L \in C$, $f(L)$ is a flat demodulant of L .
- 4) For ordered clauses C and C' , C' is a *flat demodulant* of C , (with respect to, or via $\mathfrak{E}$), if, (C' is repetition free, that is, contains no duplicate literals, and), when the ordering is dropped, (and C and C' are regarded as their underlying sets of literals), C' is a flat demodulant of C .
- 5) For terms, literals, or clauses X and X' , (we will also say that)
 X is an (*above-equality, flat*) $\mathfrak{E}$ -*demodulant* of X' , if X is a flat demodulant of X' , with respect to $\mathfrak{E}$.

DEFINITION 8.1.6. A (finite $\mathfrak{E}$ parameterized) *flat demodulation function* is any function f , such that for any finite set $\mathfrak{E}$ of positive equality unit clauses, and any (unordered) clause C ,
 $f(\mathfrak{E}, C) \equiv \{f(\mathfrak{E}, L) \mid L \in C\}$ is an unordered flat demodulant of C , with respect to $\mathfrak{E}$.

DEFINITION 8.1.7. We will say that a (finite $\mathfrak{E}$ parameterized) flat demodulation function f is *well-founded*, if there is some well-founded total order, $\succsim_f$, on terms, such that for any finite set $\mathfrak{E}$, of positive equality unit clauses, for any (signed) literal, $\mp P(t_1 \dots t_n)$,
 $(n \geq 0$ the arity of the predicate symbol P , possibly '='),
 $f(\mathfrak{E}, \mp P(t_1 \dots t_n)) = \mp P(f(\mathfrak{E}, t_1), \dots, f(\mathfrak{E}, t_n))$, (sign preserving),
and such that for any term t , $t \succsim_f f(\mathfrak{E}, t)$.

DEFINITION 8.1.8. For any set $\mathfrak{E}$ of positive equality units, and any terms, literals, or clauses, X and X' , let $X \succsim_{\mathfrak{E}} X'$ denote the condition that X' is a flat demodulant of X , with respect to $\mathfrak{E}$.

Some observations:

Evidently, for any set $\mathfrak{E}$ of positive equality units, and any terms, literals, or clauses, X and X' ,
 $X \succsim_{\mathfrak{E}} X'$ if and only if for some finite $\mathfrak{E}' \subseteq \mathfrak{E}$, $X \succsim_{\mathfrak{E}'} X'$.

Evidently, $\succsim_{\mathfrak{E}}$ is a reflexive and symmetric relation on terms, and on literals, but only a reflexive relation on clauses.

Further, $\succsim_{cl+(\mathfrak{E})}$ is an equivalence relation on terms, and on literals, and (in general, only) a reflexive and transitive relation on clauses.

Also, evidently, if $\mathfrak{E}' \subseteq \mathfrak{E}$ then $\succsim_{\mathfrak{E}} \subseteq \succsim_{\mathfrak{E}'}$.

Finally, we point out that if X and X' are both terms, or both literals, then $X \succsim_{\emptyset} X'$ if and only if $X = X'$, while for (unordered or repetition free ordered) clauses C and C' , $C \succsim_{\emptyset} C'$ if and only if modulo clause ordering, $C = C'$.

DEFINITION 8.1.9. For any set, $\mathfrak{E}$ of positive equality units:

- 1) for terms, (or literals), X and X' , we will say that X and X' are $\mathfrak{E}$ -*equivalent*, and write $X \sim_{\mathfrak{E}} X'$, if $X \succsim_{cl+(\mathfrak{E})} X'$, and
- 2) for any clauses A and B , we will say that A and B are $\mathfrak{E}$ -*equivalent*, and write $A \sim_{\mathfrak{E}} B$, if $\forall L \in A$, $\exists L' \in B$, such that $L \sim_{\mathfrak{E}} L'$, and $\forall L' \in B$, $\exists L \in A$, such that $L' \sim_{\mathfrak{E}} L$.

Some observations:

Evidently, $\sim_{\mathfrak{E}}$ is an equivalence relation on terms or literals, and (as relations on terms or literals),

$$\sim_{\mathfrak{E}} = \sim_{cl_+(\mathfrak{E})} = \sim_{cl_+(\mathfrak{E})}$$

Also, (it is easy to see that) $\sim_{\mathfrak{E}}$ is an equivalence relation on clauses.

Clearly if $\mathfrak{E} \subseteq \mathfrak{E}'$ then $\sim_{\mathfrak{E}} \subseteq \sim_{\mathfrak{E}'}$, and once again, we point out that

if X and X' are both terms, or both literals, then $X \sim_{\emptyset} X'$ if and only if $X = X'$,
while for clauses C and C' , $C \sim_{\emptyset} C'$ if and only if modulo clause ordering, $C = C'$.

Actually, it may be shown that: (for any set, $\mathfrak{E}$ of positive equality units), $\sim_{\mathfrak{E}}$ denotes the least equivalence relation on terms, (literals, clauses), such that $X \sim_{\mathfrak{E}} X'$, whenever $X \succsim_{\mathfrak{E}} X'$, (that is, whenever X' is an $\mathfrak{E}$ -demodulant of X), for any terms, (literals, clauses, respectively), X and X' . In other words, as relations on terms or literals, $\sim_{\mathfrak{E}}$ is the transitive closure of $\succsim_{\mathfrak{E}}$, while, as relations on clauses, $\sim_{\mathfrak{E}}$ is the transitive and symmetric closure of $\succsim_{\mathfrak{E}}$. For our purposes here, however, this characterization, though desirable, is unnecessary, (still, the reader might find its proof an enjoyable exercise).

OBSERVATION 8.1.2. Let $\mathfrak{E}$ be any set $\mathfrak{E}$ of positive equality units clauses.

Then for any terms, literals, or clauses, X and X' ,

$X \sim_{\mathfrak{E}} X'$ if and only if for some finite $\mathfrak{E}' \subseteq \mathfrak{E}$, $X \sim_{\mathfrak{E}'} X'$.

Obviously:

OBSERVATION 8.1.3. For terms, literals, or clauses X and X' , (and any set of positive equality units $\mathfrak{E}$), if X is an (above-equality) $\mathfrak{E}$ -demodulant of X' , then $X \sim_{\mathfrak{E}} X'$.

OBSERVATION 8.1.4. For any clauses A , A' , B , B' , C , and C' , if $A \sim_{\mathfrak{E}} A'$ and $B \sim_{\mathfrak{E}} B'$,

C is (some ordering of) $A \cup B$, and C' is (some ordering of) $A' \cup B'$, then $C \sim_{\mathfrak{E}} C'$.

OBSERVATION 8.1.5. For any clauses A and A' ,

A' is $\mathfrak{E}$ -equivalent to a subclause of A if and only if

A' is a subclause of a clause which is $\mathfrak{E}$ -equivalent to A .

Somewhat less clearly, we also have:

OBSERVATION 8.1.6. For any clause A , and any (finite set of positive equality units), $\mathfrak{E}$,

A' is an $\mathfrak{E}$ -demodulant of a subclause of A if and only if

A' is a subclause of an $\mathfrak{E}$ -demodulant of A .

In analogy with Lemma 4.2.2, (or Lemma 2.2.2), we have:

LEMMA 8.1.5. Suppose $B \in \text{EPC}_{<}^1(A, D)$, for some clauses A , B , (A', D') , and D , such that A has $<$ -maximal predicate symbol P and is pure-positive with respect to P , $P \neq '='$, (and D has $<$ -maximal predicate symbol P and is mixed with respect to P).

Suppose also that $A' \sim_{\mathfrak{E}} A$ and $D' \sim_{\mathfrak{E}} D$, and with L_0 and L'_0 , say, the left-most negative P instance in D and D' respectively, $L_0 \sim_{\mathfrak{E}} L'_0$.

Then there exists $B' \in \text{EPC}_{<}^1(A', D')$, such that B' is $\mathfrak{E}$ -equivalent to some $B'' \subseteq B$.

The following is an analogue to Lemma 4.3.1.

COROLLARY 8.1.3. Suppose $B \in \text{EPC}_{<}^1(A, D)$, for some clauses A , B , and D , such that A has $<$ -maximal predicate symbol P and is pure-positive with respect to P , $P \neq '='$, (and D has $<$ -maximal predicate symbol P and is mixed with respect to P). Suppose that A'' and D'' are subclauses of A and D , respectively, (that is, $A'' \subseteq A$ and $D'' \subseteq D$), and with L_0 , say, the left-most negative P instance in D , either:

- (i) D'' is pure-positive with respect to P or
- (ii) L_0 occurs as the left-most negative P instance in D'' .

Suppose also that $A' \sim_{\mathcal{E}} A''$, and $D' \sim_{\mathcal{E}} D''$, and with L'_0 , say, the left-most negative P instance in D' , $L_0 \sim_{\mathcal{E}} L'_0$.

Then, in case (i), $D' \sim_{\mathcal{E}} B''$, for some $B'' \subseteq B$, and in case (ii), either $A' \sim_{\mathcal{E}} B''$, for some $B'' \subseteq B$, or there exists $B' \in \text{EPC}_{<}^1(A', D')$ such that $B' \sim_{\mathcal{E}} B''$, for some $B'' \subseteq B$.

The (above-equality) demodulation analogue of Lemma 4.3.2 is:

LEMMA 8.1.6. Suppose Π is an (above-equality) $\text{EPC}_{<}^{+1}$ -deduction of a clause D , from ancillary parents in $\mathcal{C}$, (say), and principal ancestor B , B with $<$ -maximal predicate symbol (and mixed with respect to) P , say, $P \neq '='$.

Then, if B' is $\mathcal{E}$ -equivalent to a subclause of B and B' (has $<$ -maximal predicate symbol $< P$ or) is pure-positive with respect to P , B' is a $\mathcal{E}$ -equivalent to a subclause of D .

The (above-equality) demodulation analogue of Lemma 4.3.3 is:

LEMMA 8.1.7. Let B be a clause with $<$ -maximal predicate symbol P , say, $P \neq '='$, B mixed with respect to P , and suppose that (for some $k \geq 1$), $C_0, \dots, C_{k-1}$, (the C_i not necessarily distinct), and $B = D_0, \dots, D_{k-1}$, $D_k = D$, represents an (above-equality) $\text{EPC}_{<}^{+1}$ -deduction Π , (as in Definition 4.1.11). Suppose B' is $\mathcal{E}$ -equivalent to a subclause of B , and that $C'_0, \dots, C'_{k-1}$, (the C'_i not necessarily distinct), is a sequence of (not necessarily distinct) clauses such that $\forall i: 0 \leq i \leq k-1$, C'_i is $\mathcal{E}$ -equivalent to a subclause of C_i .

Then either:

- i. B' is $\mathcal{E}$ -equivalent to a subclause of D ,
- ii. C'_i is $\mathcal{E}$ -equivalent to a subclause of D , for some $i: 0 \leq i \leq k-1$, such that (modulo ordering) $C'_i \neq C_i$, or
- iii. there exists a $\text{EPC}_{<}^{+1}$ -deduction Π' , of some clause D' , from principal ancestor B' and ancillary parents $\{C'_0, \dots, C'_{k-1}\}$, where D' is $\mathcal{E}$ -equivalent to a subclause of D , and such that $\forall C' \in \{C'_0, \dots, C'_{k-1}\} \quad k_{\Pi'}(C') = \|\{i \mid 0 \leq i \leq k-1; C'_i = C'\}\|$.

The (above-equality) demodulation analogue of Lemma 4.3.4 is:

LEMMA 8.1.8. Suppose Π is an above-equality $\text{EPC}_{<}$ -deduction of a clause A , from a set $\mathcal{S}$, and (A , at) the root is $\text{EPC}_{<}$ -initial. Suppose $\mathcal{L}$ denotes the set of leaf nodes n of Π , and for each

$n \in \mathcal{L}$, let B_n denote the clause (in Π) at n . Suppose that for each $n \in \mathcal{L}$, B'_n is $\mathcal{E}$ -equivalent to a subclause of B_n , and that $\mathcal{S}'$ denotes the set comprised of the clauses B'_n , (as n ranges over the leaf nodes of Π).

Then there is an above-equality $\text{EPC}_{<}$ -deduction Π' of a clause A' , $\text{EPC}_{<}$ -initial (in Π'), which is $\mathcal{E}$ -equivalent to a subclause of A , from $\mathcal{S}'$, such that

$$\forall B' \in \mathcal{S}', k_{\Pi'}(B') \leq ||\{n \in \mathcal{L} \mid B'_n = B'\}||.$$

The (above-equality) demodulation analogue of Corollary 4.3.1:

COROLLARY 8.1.4. Suppose Π is an above-equality $\text{EPC}_{<}$ -deduction of a (possibly empty) equality clause E from $\mathcal{S}$, that $\mathcal{L}$ denotes the set of leaf nodes n of Π , and for each $n \in \mathcal{L}$, B_n denotes the clause (in Π) at n . Suppose that for each $n \in \mathcal{L}$, B'_n is $\mathcal{E}$ -equivalent to a subclause of B_n , and that $\mathcal{S}'$ denotes the set comprised of the clauses B'_n , (as n ranges over the leaf nodes of Π).

Then there is an above-equality $\text{EPC}_{<}$ -deduction, Π' of a (equality) clause E' which is $\mathcal{E}$ -equivalent to a subclause of E , from $\mathcal{S}'$, such that $\forall B' \in \mathcal{S}', k_{\Pi'}(B') \leq ||\{n \in \mathcal{L} \mid B'_n = B'\}||$.

8.1.3. Reduction: Synthesis

DEFINITION 8.1.10. Let $\mathcal{E}$ denote a (possibly empty) set of positive equality unit clauses.

By a $(\mathcal{E}\text{-})\text{reduct}$ of a clause, C , we shall mean any clause C_k ,

where $k \geq 0$, and where there is a finite $(\mathcal{E}\text{-})\text{reduction sequence}$:

$$C = C_0, \dots, C_k, \text{ where } \forall i: 0 \leq i < k, \text{ either } C_{i+1} \text{ is a QR-factor of } C_i, \text{ or } C_i \sim_{\mathcal{E}} C_{i+1}.$$

We will let $C \succ_{\mathcal{E}} C'$ denote the condition that C' is a $(\mathcal{E}\text{-})\text{reduct}$ of C .

(We have parenthesised $\mathcal{E}$ in, for example, “ $(\mathcal{E}\text{-})\text{reduct}$ ”, to indicate, that in contexts that make it clear which set $\mathcal{E}$, is involved, we may drop the prefix, $\mathcal{E}$.)

Evidently, $\succ_{\mathcal{E}}$ defines a reflexive, transitive (pre-order) relation on clauses, and is a proper extension of $\sim_{\mathcal{E}}$.

In practice, the relation “ $C_i \sim_{\mathcal{E}} C_{i+1}$ ”, may instead be replaced by the relation “ $C_i \succ_{\mathcal{E}} C_{i+1}$ ”, where $\succ_{\mathcal{E}} \subseteq \sim_{\mathcal{E}}$, is some well-founded proper sub-relation of $\sim_{\mathcal{E}}$. In particular, it may well be the case that if $C_i \succ_{\mathcal{E}} C_{i+1}$, then for some $\mathcal{E}' \subseteq \text{cl}_+(\mathcal{E})$, $C_i \succ_{\mathcal{E}'} C_{i+1}$, (that is, C_{i+1} will be a flat demodulant of C_i).

In practice, further, the well-founded partial order relation $\succ$, may be based upon some *demodulation function* $\hat{\rho}$, so that $A \succ B$ will hold precisely when, modulo ordering, $\hat{\rho}(A) = B$, and $\hat{\rho}$ in turn, may be based upon some well-founded total (or partial) order, $\succ_{\rho}$, on terms, (typically), (not preserved by substitution, but), such that $s \succ_{\rho} t$, if t occurs as a sub-term of s . The relation $s \succ_{\rho} t$, may again involve a *demodulation function*, ρ , say, which maps terms to (provably equal) terms, and is such that, for any term t , $t \succ_{\rho} \rho(t)$.

The first point here is that any such *reduct* (or *reduction sequence*) formed in this way, (that is, when in the definition, $\sim_{\mathfrak{E}}$, is replaced by some $\succ \subseteq \sim_{\mathfrak{E}}$), will remain a reduct (or reduction sequence, respectively) in the sense that we have formulated.

The second point is that, in the context of any such restricted notion of reduct (and reduction sequence), (with $C \succ_{\mathfrak{E}} C'$ denoting the condition that C' is such an $\mathfrak{E}$ -reduct of C), then, (it is easy to see that), $\succ_{\mathfrak{E}}$ becomes a well-founded ordering relation on clauses. We do not restrict ourselves to the case when $\succ_{\mathfrak{E}}$ is well-founded, simply because we do not require well-foundedness here. Nevertheless, for later use, we define and then make some observations about the following special cases of $\succ_{\mathfrak{E}}$.

DEFINITION 8.1.11. Let $\mathfrak{E}$ denote a (possibly empty) set of positive equality unit clauses.

By a $(\mathfrak{E})$ -*fd-reduct* of a clause, C , we shall mean any clause C_k ,

where $k \geq 0$, and where there is a finite $(\mathfrak{E})$ -*fd-reduction sequence*:

$C = C_0, \dots, C_k$, where $\forall i: 0 \leq i < k$, either C_{i+1} is a QR-factor of C_i , or $C_i \succ_{\mathfrak{E}} C_{i+1}$.

We will let $C \succ_{fd, \mathfrak{E}} C'$ denote the condition that C' is a $(\mathfrak{E})$ -fd-reduct of C .

(Once again, we have parenthesised $\mathfrak{E}$ in, for example, " $(\mathfrak{E})$ -fd-reduct", to indicate, that in contexts that make it clear which set $\mathfrak{E}$, is involved, we may drop the prefix. $\mathfrak{E}$.)

Evidently, once again, $\succ_{fd, \mathfrak{E}}$ defines a reflexive, transitive (pre-order) relation on (both ordered and unordered) clauses, and (this time), is a proper extension of $\succ_{\mathfrak{E}}$.

DEFINITION 8.1.12. Let ρ be a (finite $\mathfrak{E}$ parameterized) flat demodulation function.

(as in Definition 8.1.6), and let $\mathfrak{E}$ denote a (possibly empty) set of positive equality unit clauses.

By a $(\mathfrak{E})$ - ρ -*reduct* of a clause, C , we shall mean any clause C_k ,

where $k \geq 0$, and where there is a finite $(\mathfrak{E})$ - ρ -*reduction sequence*: $C = C_0, \dots, C_k$,

where $\forall i: 0 \leq i < k$, either C_{i+1} is a QR-factor of C_i , or there is a finite subset $\mathfrak{E}_i \subseteq \mathfrak{E}$, such that, (C_i is repetition free, and modulo ordering), $\rho(\mathfrak{E}_i, C_i) = C_{i+1}$.

We will let $C \succ_{\rho, \mathfrak{E}} C'$ denote the condition that C' is a $(\mathfrak{E})$ - ρ -reduct of C .

Evidently, for any flat demodulation function, ρ , (as in Definition 8.1.6),

and any (possibly empty) set of positive equality unit clauses, $\mathfrak{E}$, any $\mathfrak{E}$ - ρ -reduction sequence is an $\mathfrak{E}$ -fd-reduction sequence, and any $\mathfrak{E}$ - ρ -reduct is an $\mathfrak{E}$ -fd-reduct.

Also, again, $\succ_{\rho, \mathfrak{E}}$ defines a reflexive, transitive (pre-order) relation on clauses.

OBSERVATION 8.1.7. Let $\succ_{\rho}$ denote some well-founded total order for a well-ordered flat demodulation function, ρ , (as in Definition 8.1.7).

That is, $\succ_{\rho}$ is such that for any term t , and any finite set of positive equality units $\mathfrak{E}'$, $t \succ_{\rho} \rho(\mathfrak{E}', t)$. Then $\succ_{\rho}$ extends to a well-founded total order relation, $\widehat{\succ}_{\rho}$, say, on literals, so that for any finite set of positive equality units $\mathfrak{E}$, and any literals L and L' , if $\rho(\mathfrak{E}, L) = L'$, then $L \widehat{\succ}_{\rho} L'$.

OBSERVATION 8.1.8. If ρ is a well-ordered flat demodulation function, then for any set $\mathfrak{E}$ of positive equality units, $\succ_{\rho, \mathfrak{E}}$ defines a partial order relation on unordered clauses.

OBSERVATION 8.1.9. For any well-founded flat demodulation function, ρ , (as in Definition 8.1.7), and any set $\mathfrak{E}$ of positive equality unit clauses, $\succ_{\rho, \mathfrak{E}}$ is well-founded, as a pre-order on ordered clauses, and as a partial order on unordered clauses.

We now return to the more general setting, (where well-foundedness is not a hypothesis).

OBSERVATION 8.1.10. Let $\mathcal{E}$ be any set $\mathcal{E}$ of positive equality units clauses.

Then for any clauses, C and C' ,

$C \succ_{\mathcal{E}} C'$ if and only if for some finite $\mathcal{E}' \subseteq \mathcal{E}$, $C \succ_{\mathcal{E}'} C'$.

DEFINITION 8.1.13. Let Σ_Q denote the collection comprised of Σ_0 , together with the various functional substitution axioms for equality, $\neq(\vec{x}, \vec{y}) \mid = (f(\vec{x}), f(\vec{y}))$,

(where $\vec{x}$ and $\vec{y}$ denote disjoint n -tuples of distinct variables, where $n \geq 1$, is the arity of f).

OBSERVATION 8.1.11. For any set $\mathcal{E}$, of positive equality unit clauses, and any clauses C and C' , if $C \succ_{\mathcal{E}} C'$, then $\mathcal{E} \cup \Sigma_Q \models (\forall)C \Leftrightarrow (\forall)C'$.

From Corollary 8.1.2, we (easily) obtain:

COROLLARY 8.1.5. Let $\mathcal{E}$ denote a (possibly empty) set of positive equality unit clauses.

Suppose Π is an above-equality $\text{EPC}_{<}$ -deduction of a (possibly empty) equality clause E'' from $\mathcal{S}$, where, for some equality clause E , $\mathcal{E} \cup \Sigma_Q \cup \{E''\} \models E$.

Suppose that $\mathcal{L}$ denotes the set of leaf nodes n of Π , and for each $n \in \mathcal{L}$, B_n denotes the clause (in Π) at n . Finally, suppose that for each $n \in \mathcal{L}$, B'_n is a QR-factor of a subclause of B_n , and that $\mathcal{S}'$ denotes the set comprised of the clauses B'_n , (as n ranges over the leaf nodes of Π).

Then there is an above-equality $\text{EPC}_{<}$ -deduction, Π' of an equality clause E' , from $\mathcal{S}'$, such that $\mathcal{E} \cup \Sigma_Q \cup \{E'\} \models E$, and such that $\forall B' \in \mathcal{S}', k_{\Pi'}(B') \leq ||\{n \in \mathcal{L} \mid B'_n = B'\}||$.

Similarly, from Corollary 8.1.4, we obtain:

COROLLARY 8.1.6. Let $\mathcal{E}$ denote a (possibly empty) set of positive equality unit clauses.

Suppose Π is an above-equality $\text{EPC}_{<}$ -deduction of a (possibly empty) equality clause E'' from $\mathcal{S}$, where, for some equality clause E , $\mathcal{E} \cup \Sigma_Q \cup \{E''\} \models E$.

Suppose that $\mathcal{L}$ denotes the set of leaf nodes n of Π , and for each $n \in \mathcal{L}$, B_n denotes the clause (in Π) at n . Finally, suppose that for each $n \in \mathcal{L}$, B'_n is $\mathcal{E}$ -equivalent to a subclause of B_n , and that $\mathcal{S}'$ denotes the set comprised of the clauses B'_n , (as n ranges over the leaf nodes of Π).

Then there is an above-equality $\text{EPC}_{<}$ -deduction, Π' of an equality clause E' , from $\mathcal{S}'$, such that $\mathcal{E} \cup \Sigma_Q \cup \{E'\} \models E$, and such that $\forall B' \in \mathcal{S}', k_{\Pi'}(B') \leq ||\{n \in \mathcal{L} \mid B'_n = B'\}||$.

From Corollaries 8.1.5 and 8.1.6 we obtain:

COROLLARY 8.1.7. Let $\mathcal{E}$ denote a (possibly empty) set of positive equality unit clauses.

Suppose Π is an above-equality $\text{EPC}_{<}$ -deduction of a (possibly empty) equality clause E'' from $\mathcal{S}$, where, for some equality clause E , $\mathcal{E} \cup \Sigma_Q \cup \{E''\} \models E$.

Suppose that $\mathcal{L}$ denotes the set of leaf nodes n of Π , and for each $n \in \mathcal{L}$, B_n denotes the clause (in Π) at n . Finally, suppose that for each $n \in \mathcal{L}$, there is associated a clause B'_n , such that $B_n \succ_{\mathcal{E}} B'_n$, and that $\mathcal{S}'$ denotes the set comprised of the clauses B'_n , (as n ranges over the leaf nodes of Π).

Then there is an above-equality $\text{EPC}_{<}$ -deduction, Π' of a equality clause E' , from $\mathcal{S}'$, such that $\mathcal{E} \cup \Sigma_Q \cup \{E'\} \models E$, and such that $\forall B' \in \mathcal{S}', k_{\Pi'}(B') \leq ||\{n \in \mathcal{L} \mid B'_n = B'\}||$.

From Corollaries 6.1.1 and 8.1.7 we obtain:

COROLLARY 8.1.8. Let $\mathfrak{E}$ denote a (possibly empty) set of positive equality unit clauses. Suppose Π is an above-equality $\text{EPC}_{<}$ -deduction of a (possibly empty) equality clause E from $\mathfrak{S}$. Suppose that $\mathfrak{L}$ denotes the set of leaf nodes n of Π , and for each $n \in \mathfrak{L}$, B_n denotes the clause (in Π) at n . Finally, suppose that for each $n \in \mathfrak{L}$, there are associated clauses B'_n , and B''_n , such that $B'_n \Sigma_0$ -subsumes B''_n , and $B_n \succ_{\mathfrak{E}} B''_n$, and that $\mathfrak{S}'$ denotes the set comprised of the clauses B'_n , (as n ranges over the leaf nodes of Π).

Then there is a finite forest $\mathfrak{F}$ of above-equality $\text{EPC}_{<}$ -deductions Π' , of equality clauses, $E_{\Pi'}$, say, such that:

- a) $\mathfrak{E} \cup \Sigma_Q \cup \{E_{\Pi'} : \Pi' \in \mathfrak{F}\} \models E$, and
- b) $\forall \Pi' \in \mathfrak{F}$, for any clause B' , $k_{\Pi'}(B') \leq ||\{n \in \mathfrak{L} \mid B'_n = B'\}||$.

8.2. Refutation Completeness of $\text{EPC}_{<}$ With Reduction, Under Deletion

In this section, we introduce “reduction” into the results of Section 7.2.

8.2.1. Preliminaries

DEFINITION 8.2.1. Let EQF denote $\Sigma_Q \setminus \Sigma_0$. That is, EQF denotes the set of clauses representing the axioms of functional substitutivity for equality.

DEFINITION 8.2.2. For any set $\mathfrak{E}$ of positive equality unit clauses, and any clauses B and C , B ($\mathfrak{E}$ -)reduct Σ_0 -subsumes C if there is some clause A such that: $C \succ_{\mathfrak{E}} A$ and $B \Sigma_0$ -subsumes A .

(Once again, have parenthesised $\mathfrak{E}$, this time in “($\mathfrak{E}$ -)reduct Σ_0 -subsumes”, to indicate, that the prefix, $\mathfrak{E}$ may be dropped in contexts that make it clear which set $\mathfrak{E}$, is involved.)

OBSERVATION 8.2.1. Let $\mathfrak{E}$ be any set $\mathfrak{E}$ of positive equality units clauses. Then for any clauses, C and C' , C $\mathfrak{E}$ -reduct Σ_0 -subsumes C' if and only if for some finite $\mathfrak{E}' \subseteq \mathfrak{E}$, C $\mathfrak{E}'$ -reduct Σ_0 -subsumes C' .

OBSERVATION 8.2.2. If $\mathfrak{E}$ is any set of positive equality units clauses, for any clauses C' and C , if $C' \Sigma_0$ -subsumes C , then C' $\mathfrak{E}$ -reduct Σ_0 -subsumes C .

Thus ‘ $\mathfrak{E}$ -reduct Σ_0 -subsumes’ is a generalization of ‘ Σ_0 -subsumes’.

It is easy to see that ‘ $\mathfrak{E}$ -reduct Σ_0 -subsumes’ is, in fact, a proper generalization of ‘ Σ_0 -subsumes’.

OBSERVATION 8.2.3. Suppose E is an equality clause, $\mathfrak{E}$ is a set of positive equality units clauses, and E' $\mathfrak{E}$ -reduct Σ_0 -subsumes E . Then E' is an equality clause, and $\mathfrak{E} \cup \Sigma_Q \cup \{E'\} \models E$.

EXAMPLE 8.2.1. Suppose $\mathfrak{E} \equiv \{=(f(b), a)\}$, and consider the clauses:

$C_1 \equiv P(a)$, $C_2 \equiv \neg(x, b) \mid P(f(x)) \mid \neg(g(x), g(c))$, and $C_3 \equiv \neg(c, b) \mid P(f(c))$.
 (where x is a variable, a, b , and c are some (pairwise) distinct ground terms).

Evidently, $C_2 \succ_{\mathcal{E}} P(a) \mid \neg(g(b), g(c))$, and so, clearly, C_1 $\mathcal{E}$ -reduct Σ_0 -subsumes C_2 .

Also, evidently, C_2 Σ_0 -subsumes C_3 , and so C_2 $\mathcal{E}$ -reduct Σ_0 -subsumes C_3 .

However, the only clause C' such that $C_3 \succ_{\mathcal{E}} C'$, is, modulo symmetry and ordering, C_3 itself, and since C_1 does not Σ_0 -subsume C_3 , C_1 does not $\mathcal{E}$ -reduct Σ_0 -subsumes C_3 .

Thus, the relation of $\mathcal{E}$ -reduct Σ_0 -subsumption, is not, in general, transitive. Because transitivity is desirable, this situation provides obvious motivation for pursuing some reformulation. However, such will not be explored here. Rather, we shall content ourselves with the non-transitive relation of $\mathcal{E}$ -reduct Σ_0 -subsumption, as we have defined it, and simply record the observation.

OBSERVATION 8.2.4. $\mathcal{E}$ -reduct Σ_0 -subsumption is not (in general) transitive.

DEFINITION 8.2.3. For any finite non-empty set S of clauses, by a Σ_0 -subsumption deleted reduced $EPC_{<}$ -deduction sequence from S , we shall mean any sequence, Δ , of clauses such that:

- 1) $\forall C \in S$, if $C \in EQF$, then $\exists C' \in |\Delta|$, such that $C' \Sigma_0$ -subsumes C , and otherwise, for some set of positive equality units $\mathcal{E} \subseteq |\Delta|$, $\exists C' \in |\Delta|$, such that $C' \mathcal{E}$ -reduct Σ_0 -subsumes C , and
- 2) $\forall A \in |\Delta|, \forall B \in |\Delta|, \forall C \in EPC_{<}^1(A, B)$, if A or B is an equality clause, then $\exists C' \in |\Delta|$, such that $C' \Sigma_0$ -subsumes C , and otherwise, for some set of positive equality units $\mathcal{E} \subseteq |\Delta|$, $\exists C' \in |\Delta|$, such that $C' \mathcal{E}$ -reduct Σ_0 -subsumes C .

DEFINITION 8.2.4. For any finite non-empty set S of clauses, by a refined Σ_0 -subsumption deleted reduced $EPC_{<}$ -deduction sequence from S , we shall mean any sequence, Δ , of clauses such that:

- 1) $\forall C \in S$, if C is the empty clause, $\square$, then $\square \in |\Delta|$, and otherwise, if $C \in EQF$, then $\exists C' \in |\Delta|$, such that $C' \Sigma_0$ -subsumes C , and otherwise, for some set of positive equality units $\mathcal{E} \subseteq |\Delta|$, $\exists C' \in |\Delta|$, such that $C' \mathcal{E}$ -reduct Σ_0 -subsumes C , and
- 2) $\forall A \in |\Delta|, \forall B \in |\Delta|, \forall C \in EPC_{<}^1(A, B)$, if $C = \square$, then $\square \in |\Delta|$, and otherwise, if A or B is an equality clause, then $\exists C' \in |\Delta|$, such that $C' \Sigma_0$ -subsumes C , and otherwise, for some set of positive equality units $\mathcal{E} \subseteq |\Delta|$, $\exists C' \in |\Delta|$, such that $C' \mathcal{E}$ -reduct Σ_0 -subsumes C .

OBSERVATION 8.2.5. For any finite non-empty set S of clauses, if Δ is any refined Σ_0 -subsumption deleted reduced $EPC_{<}$ -deduction sequence from S , then Δ is a Σ_0 -subsumption deleted reduced $EPC_{<}$ -deduction sequence from S .

DEFINITION 8.2.5. For any finite non-empty set S of clauses, by a Σ_0 -subsumption and Σ_0 -tautology deleted reduced $EPC_{<}$ -deduction sequence from S , we shall mean any sequence, Δ , of clauses such that:

- 1) $\forall C \in S$, either C is a Σ_0 -tautology, or, if $C \in EQF$, then $\exists C' \in |\Delta|$, such that $C' \Sigma_0$ -subsumes C , and otherwise, for some set of positive equality units $\mathcal{E} \subseteq |\Delta|$, $\exists C' \in |\Delta|$, such that $C' \mathcal{E}$ -reduct Σ_0 -subsumes C , and

- 2) $\forall A \in |\Delta|. \forall B \in |\Delta|. \forall C \in EPC_{\leq}^1(A, B)$, either C is a Σ_0 -tautology, or,
 if A or B is an equality clause, then $\exists C' \in |\Delta|$, such that C' Σ_0 -subsumes C ,
 and otherwise, for some set of positive equality units $\mathcal{E} \subseteq |\Delta|$, $\exists C' \in |\Delta|$,
 such that C' $\mathcal{E}$ -reduct Σ_0 -subsumes C .

DEFINITION 8.2.6. For any finite non-empty set S of clauses, by a *refined Σ_0 -subsumption and Σ_0 -tautology deleted reduced $EPC_{\leq}$ -deduction sequence from S* , we shall mean any sequence, Δ , of clauses such that:

- 1) $\forall C \in S$, if $C = \square$, then $\square \in |\Delta|$, and otherwise, either C is a Σ_0 -tautology, or,
 if $C \in EQF$, then $\exists C' \in |\Delta|$, such that C' Σ_0 -subsumes C , and otherwise, for some set
 of positive equality units $\mathcal{E} \subseteq |\Delta|$, $\exists C' \in |\Delta|$, such that C' $\mathcal{E}$ -reduct Σ_0 -subsumes C , and
- 2) $\forall A \in |\Delta|. \forall B \in |\Delta|. \forall C \in EPC_{\leq}^1(A, B)$,
 if $C = \square$, then $\square \in |\Delta|$, and otherwise, either C is a Σ_0 -tautology, or,
 if A or B is an equality clause, then $\exists C' \in |\Delta|$, such that C' Σ_0 -subsumes C ,
 and otherwise, for some set of positive equality units $\mathcal{E} \subseteq |\Delta|$, $\exists C' \in |\Delta|$,
 such that C' $\mathcal{E}$ -reduct Σ_0 -subsumes C .

OBSERVATION 8.2.6. For any finite non-empty set S of clauses, if Δ is any refined Σ_0 -subsumption and Σ_0 -tautology deleted reduced $EPC_{\leq}$ -deduction sequence from S , then Δ is a Σ_0 -subsumption and Σ_0 -tautology deleted reduced $EPC_{\leq}$ -deduction sequence from S .

8.2.2. The Above-Equality Fragment

LEMMA 8.2.1. Let $\Delta \equiv C_0, C_1, \dots$ be a Σ_0 -subsumption deleted reduced $EPC_{\leq}$ -deduction sequence from S , for some finite non-empty set of clauses S . Let $\mathcal{E} \subseteq |\Delta|$ be a collection of positive equality units. Let $\mathfrak{F}$ be a finite collection of above-equality $EPC_{\leq}$ -deductions Π , of equality clauses, E_{Π} , such that for each clause C which labels a leaf of any $\Pi \in \mathfrak{F}$, there is some $C' \in |\Delta|$, such that C' $\mathcal{E}$ -reduct Σ_0 -subsumes C .

Then there is a finite collection of equality clauses $\mathcal{E}_0 \subseteq |\Delta|$, such that, $\forall \Pi \in \mathfrak{F}$, $\mathcal{E} \cup \Sigma_Q \cup \mathcal{E}_0 \models E_{\Pi}$.

COROLLARY 8.2.1. Let Δ be a Σ_0 -subsumption deleted reduced $EPC_{\leq}$ -deduction sequence from S , for some finite non-empty set of clauses S . Let $\mathcal{E} \subseteq |\Delta|$ be a collection of positive equality units. Let $\mathfrak{F}$ be a finite collection of above-equality $EPC_{\leq}$ -deductions Π , of equality clauses, E_{Π} , such that for each clause C which labels a leaf of any $\Pi \in \mathfrak{F}$, there is some $C' \in |\Delta|$, such that C' $\mathcal{E}$ -reduct Σ_0 -subsumes C .

Then there is a finite collection of equality clauses $\mathcal{E}_0 \subseteq |\Delta|$, such that, $\forall \Pi \in \mathfrak{F}$, $\Sigma_Q \cup \mathcal{E}_0 \models E_{\Pi}$.

COROLLARY 8.2.2. Let Δ be a Σ_0 -subsumption deleted reduced $EPC_{\leq}$ -deduction sequence from S , for some finite non-empty set of clauses S . Let $\mathcal{E} \subseteq |\Delta|$ be a collection of positive equality units. Let $\mathfrak{F}$ be a finite collection of above-equality $EPC_{\leq}$ -deductions Π , of equality clauses, E_{Π} , such that for each clause C which labels a leaf of any $\Pi \in \mathfrak{F}$, $C \in S$.

Then there is a finite collection of equality clauses $\mathcal{E}_0 \subseteq |\Delta|$, such that, $\forall \Pi \in \mathfrak{F}$, $\Sigma_Q \cup \mathcal{E}_0 \models E_{\Pi}$.

8.2.3. The Equality Fragment

PROPOSITION 8.2.1. Let Δ be a (refined) Σ_0 -subsumption deleted reduced $\text{EPC}_{<}$ -deduction sequence from S , for some finite non-empty set of clauses, S , and let $\mathfrak{E} \subseteq |\Delta|$ be any finite non-empty set of equality clauses which occur in Δ .

Then there exists a (refined, respectively) Σ_0 -subsumption deleted $\text{EPC}_{<}$ -deduction sequence from $\mathfrak{E}$, Δ' , such that $\mathfrak{E} \subseteq |\Delta'| \subseteq |\Delta|$.

Because of Proposition 8.2.1, evidently, all of the results:

Lemma 7.2.2, Corollary 7.2.1, Lemma 7.2.3, Lemma 7.2.4 of Section 7.2, remain valid, when “ Σ_0 -subsumption deleted *reduced* $\text{EPC}_{<}$ -deduction sequence” is substituted for “ Σ_0 -subsumption deleted $\text{EPC}_{<}$ -deduction sequence”.

In particular, (and for example), the modification of Lemma 7.2.4:

LEMMA 8.2.2. For some finite non-empty set S of clauses, let Δ , be a Σ_0 -subsumption deleted reduced $\text{EPC}_{<}$ -deduction sequence from S . Let $\mathfrak{E}$ be a finite collection of equality clauses such that $\forall E \in \mathfrak{E}$, there exists an $E' \in |\Delta|$, such that $E' \Sigma_0$ -subsumes E . Suppose Π is an equality $\text{EPC}_{<}$ -deduction of a QR -positive equality clause D , from $\mathfrak{E}$.

Then there exists a $D' \in |\Delta|$, such that $D' \Sigma_0$ -subsumes $QR_{full}(D)$.

8.2.4. Synthesis and Result

The modification of Theorem 7.2.1 is:

COROLLARY 8.2.3. (Reduced $\text{EPC}_{<}$ is refutation complete under Σ_0 -subsumption deletion)

Let S be any finite Σ_0 -unsatisfiable set of clauses, such that $EQF \subseteq S$.

If Δ is any Σ_0 -subsumption deleted reduced $\text{EPC}_{<}$ -deduction sequence from S ,

then $\exists D \in |\Delta|$ such that $D \Sigma_0$ -subsumes $\square$.

(The analogue of Corollary 7.2.2 is:)

COROLLARY 8.2.4. (Reduced $\text{EPC}_{<}$ is refutation complete under refined Σ_0 -subsumption deletion)

Let S be any finite Σ_0 -unsatisfiable set of clauses, such that $EQF \subseteq S$.

If Δ is any refined Σ_0 -subsumption deleted reduced $\text{EPC}_{<}$ -deduction sequence from S ,

then $\square \in |\Delta|$.

LEMMA 8.2.3. Let S be a finite non-empty set of clauses. Given a non-empty Σ_0 -subsumption and Σ_0 -tautology deleted reduced $\text{EPC}_{<}$ -deduction sequence from S , $\Delta \equiv C_0, C_1, \dots$, there exists a Σ_0 -subsumption deleted reduced $\text{EPC}_{<}$ -deduction sequence from S , $\Delta' \equiv C'_0, C'_1, \dots$, such that $\forall j \geq 0$, either:

- a) $\exists i \geq 0$ such that $C'_j = C_i$, or
- b) C'_j is a Σ_0 -tautology.

Corollary 7.2.3 becomes:

COROLLARY 8.2.5. (Reduced $EPC_{<}$ is ref. comp. under Σ_0 -subsumption and Σ_0 -tautology del.)
 Let S be any finite Σ_0 -unsatisfiable set of clauses, such that $EQF \subseteq S$.
 If Δ is any Σ_0 -subsumption and Σ_0 -tautology deleted reduced $EPC_{<}$ -deduction sequence from S ,
 then $\exists D \in |\Delta|$ such that $D \Sigma_0$ -subsumes $\square$.

Corollary 7.2.4 becomes:

COROLLARY 8.2.6. (Red. $EPC_{<}$ is ref. comp. under refined Σ_0 -subsumption and Σ_0 -tautology del.)
 Let S be any finite Σ_0 -unsatisfiable set of clauses, such that $EQF \subseteq S$.
 If Δ is any refined Σ_0 -subsumption and Σ_0 -tautology deleted reduced $EPC_{<}$ -deduction sequence from S , then $\square \in |\Delta|$.

8.3. Clause Reduction: II

Some comments:

Ordinary *replacement factoring* [20] is the process of replacing each clause C with a factor $C\theta$, whenever there exists such a substitution θ such that $C\theta$ is a proper subset of C .
 Such a factor, $C\theta$, is a *replacement factor* of C .

The generalized notion of subsumption made possible by the building in the set of axioms Σ_0 , does yield a feasible algorithm which soundly identifies (some cases of) logical entailment between two clauses assuming Σ_0 , (Theorem 3.3.4). Since this entailment is, except in equality free clause environments, a strengthening of empty theory entailment, (which is ordinary subsumption), we may indeed obtain a feasible improvement of (ordinary) replacement factoring.

An algorithm proposed for the detection of this strengthened replacement factoring, (Σ_0 -repf-ing, introduced below), is presented in Appendix H.

A *replacement resolvent* [20] is a resolvent clause C which properly subsumes one of its parent clauses. *Replacement Resolution* [20] is a reduction strategy where clauses are replaced with replacement resolvents for these clauses, when they exist.

Again, because (the relation of) Σ_0 -subsumption is a proper extension of (that of) subsumption, “ Σ_0 -generalized forms” of replacement resolution and replacement resolvents can obviously be formulated. However, because the efficacy, or even viability, of backward Σ_0 -subsumption deletion is not known in general, (as opposed to when a criterion for backward deletion requires a well-founded pre-order relation, $\supseteq$ to hold, as in Proposition 7.3.1), we restrict our attention to (the “ Σ_0 -generalization” of) *folding*, a special case of replacement resolution.

DEFINITION 8.3.1. By an Σ_0 -repf of a clause, C , we shall mean any (ordering of any) subset, $C' \subseteq C$, such that $C \Sigma_0$ -subsumes C' .

For any set $\mathfrak{E}$ of positive equality units, by an $\mathfrak{E}$ -repf of a clause C , we shall mean any Σ_0 -repf of some C'' such that $C \succ_{fd, \mathfrak{E}} C''$.

Thus, (for any set $\mathcal{E}$ of positive equality units), C' is an $\mathcal{E}$ -repf of a clause C , precisely when, for some clause C'' such that $C \succ_{fd, \mathcal{E}} C''$, C' is any (ordering of any) subset, $C' \subseteq C''$, such that $C'' \Sigma_0$ -subsumes C' . Evidently, any clause C has a Σ_0 -repf, and since, (for any set of positive equality units $\mathcal{E}$), for any clause C , $C \succ_{fd, \mathcal{E}} C$, any clause C has a $\mathcal{E}$ -repf.

Note that if (C' is an $\mathcal{E}$ -repf of a clause C , and) C'' is any clause such that $C' \subseteq C''$, $C'' \Sigma_0$ -subsumes C' , and $C \succ_{fd, \mathcal{E}} C''$, then clearly, (as $C' \subseteq C''$), $C' \Sigma_0$ -subsumes C'' , and, (as $C \succ_{fd, \mathcal{E}} C''$), $C \succ_{\mathcal{E}} C''$, so $C' \mathcal{E}$ -reduct Σ_0 -subsumes C .

Thus C' is an $\mathcal{E}$ -repf of a clause C , when, in a suitably (and rather) restricted sense, $C' \mathcal{E}$ -reduct Σ_0 -subsumes C .

The (notion of) Σ_0 -repf is the obvious generalization of "replacement factor", namely, when we replace "subsumption" with " Σ_0 -subsumption".

Thus if C' is a replacement factor of C , then C' is a Σ_0 -repf of C . It is easy to see that the converse is not true, and so the notion of Σ_0 -repf is a proper generalization of that of replacement factor.

If C' is a replacement factor of C , then ($C' \subseteq C$ and for some substitution θ , $C\theta \subseteq C'$ and so), C' is subsumption equivalent to C .

Also, it is easy to show¹ that if (for some clauses C , C' , and C''), $C'' \subseteq C' \subseteq C$, then C'' is a replacement factor of C if and only if C' is a replacement factor of C and C'' is a replacement factor of C' .

Similarly:

OBSERVATION 8.3.1. For any clauses C , C' , and C'' ,

1. If C' is a Σ_0 -repf of C , then C and C' are Σ_0 -subsumption equivalent.
2. If $C'' \subseteq C' \subseteq C$, then C'' is a Σ_0 -repf of C if and only if C' is a Σ_0 -repf of C and C'' is a Σ_0 -repf of C' .

If C' is a replacement factor of a tautology, C , then (C subsumes C' , and so) C' is a tautology. Similarly:

OBSERVATION 8.3.2. (For any clauses C and C'), if C' is a Σ_0 -repf of a Σ_0 -tautology C , then C' is a Σ_0 -tautology.

As is the case for replacement factoring, " Σ_0 -repf-ing" does not lift directly, in the sense that, it is *not* the case that:

if C'' is a replacement factor of $C\theta$, (for some clause C and some substitution θ),
then there is a replacement factor C' of C such that C'' is $C'\theta$,

(though, any replacement factor of C must subsume C''),

and similarly, it is *not* the case that:

¹A completely analogous argument, (the only modification essentially required being the substitution of 'subsumes' for ' Σ_0 -subsumes'), to that given for proof for part 2 of Observation 8.3.1, below, (the proof in the Appendix), will suffice.

if C'' , is a Σ_0 -repf of $C\theta$, (for some clause C and some substitution θ),
 then there is a Σ_0 -repf C' of C , such that,
 (even modulo ordering and symmetry), C'' is $C'\theta$.
 (though any Σ_0 - repf of C must Σ_0 -subsume C'').

A simple counter example in the equality fragment which works for both,
 (replacement factoring, and Σ_0 -repf-ing), is given by:

$$C \equiv =(x, b) \mid =(a, y), \text{ and } \theta \equiv [a/x].$$

(where a and b denote some non-variable terms which do not contain either of the variables x, y).

Here C is a minimal Σ_0 -repf (replacement factor) of itself, while $=(a, b)$ is clearly a Σ_0 -repf
 (and replacement factor) of $C\theta = =(a, b) \mid =(a, y)$.

Other similarities also pertain. For example, (though it can be shown that any two minimal replacement factors of any clause C , are alphabetically equivalent),
 minimal replacement factors are not unique, (for example, consider the clause $=(x, a) \mid =(y, b)$).

Similarly, minimal Σ_0 -repf's are not unique. (but now the non-uniqueness is more serious).

An example is given by taking C to be the ground clause: $\neq(a, b) \mid \neq(a, c) \mid \neq(b, c)$.

Then (it is not hard to see that), $C' \equiv \neq(a, b) \mid \neq(a, c)$, and $C'' \equiv \neq(a, b) \mid \neq(b, c)$,

are both minimal Σ_0 -repf's of C .

Of course, for any clause C , and any substitution θ , if $C' \subseteq C$ and C'' is a minimal Σ_0 -repf of $C'\sigma$, then $C'' \subseteq C\sigma$, and no Σ_0 -repf of $C\sigma$ is properly contained in C'' .

OBSERVATION 8.3.3. For any clauses C and D ,
 if C' is any Σ_0 -repf of C , and D' is any Σ_0 -repf of D ,
 then $C \Sigma_0$ -subsumes D if and only if $C' \Sigma_0$ -subsumes D' .

Thus we don't improve our " Σ_0 -subsumption-thinning" by "normalizing" clauses into, say, minimal Σ_0 -repf's. That process, however, can be used advantageously to reduce the number of (EPC_<-) resolvents, particularly in the equality fragment. We discuss this subject in the next section.

Ordinary subsumption may be utilized as a vehicle for a certain class of replacement resolution as follows. Let C^k denote the clause which obtained from C by changing the sign (or polarity) of the k^{th} literal, C_k of C . Suppose there exists some current clause B such that B does not subsume C , but B subsumes C^k . Then (by the soundness of subsumption, $B \Rightarrow C^k$ and so), evidently $B \wedge C \Rightarrow C \setminus \{C_k\}$. Since $C \setminus \{C_k\}$ is also a proper subset of C , $C \setminus \{C_k\}$ is a replacement resolvent of C .

We call such a process *folding*. It is a special case of replacement resolution, and a weak generalization of the "replacement principle" of Robinson [34]. There one tested to see if a unit clause subsumed an oppositely signed literal of a clause. The viability of folding rests upon an efficient algorithm for subsumption, (or perhaps more precisely, an algorithm which is efficient at determining when subsumption is not possible).

This special case of replacement resolution may again be generalized via replacing ordinary subsumption with Σ_0 -subsumption in the definition of folding. In other words, given a pair of clauses B, C , such that B Σ_0 -folds into C , on L in C , we may evidently deduce the “resolvent” $C \setminus \{L\}$, which, trivially subsumes C . We can deduce $C \setminus \{L\}$, because of the soundness of Σ_0 -subsumption. (That is, since if B Σ_0 -subsumes C , then $\Sigma_0 \models B \Rightarrow C$, and given that $B \Rightarrow (C \setminus \{L\} \cup \{\neg L\})$, from B and C , we may clearly obtain $C \setminus \{L\}$).

For example, $P(s, t) \mid C$ Σ_0 -folds into $\neg(s, r) \mid P(r, t) \mid C$, since $P(s, t) \mid C$ Σ_0 -subsumes $\neg(s, r) \mid P(r, t) \mid C$. Hence we may replace $\neg(s, r) \mid P(r, t) \mid C$ with the clause $P(r, t) \mid C$.

It is probably worth while here to point out the special cases of folding and Σ_0 -folding clauses into themselves.

For Σ_0 -folding clauses into themselves, the basic idea is simply to modify the obvious procedure for finding Σ_0 -repf's, (as described in Appendix H), as follows.

Instead of successively testing if $C \equiv L_1 \mid \dots \mid L_k$ Σ_0 -subsumes $C \setminus \{L_i\}$, (for $i: 1 \leq i \leq k$), we now test to see if C Σ_0 -subsumes $\{\neg L_i\} \cup C \setminus \{L_i\}$.

If we obtain a positive answer, (for some $i: 1 \leq i \leq k$), then we may (soundly) obtain $C \setminus \{L_i\}$ as a Σ_0 -folding of C into itself, (and we may replace C with $C \setminus \{L_i\}$).

Evidently, if $C \setminus \{L_i\}$ is a Σ_0 -repf of C , then (since C Σ_0 -subsumes $C \setminus \{L_i\}$, it follows that C Σ_0 -subsumes $\{\neg L_i\} \cup C \setminus \{L_i\}$, and so), $C \setminus \{L_i\}$ is a Σ_0 -folding of C into itself, and so by simply changing the Σ_0 -subsumption tests in the procedure for finding (minimal) Σ_0 -repf's as described above, we may obtain an extension of the relation “is a Σ_0 -repf of”. Example H.0.5, given in Appendix H, shows that this is in fact a proper extension.

Exactly analogously, by modifying the corresponding procedure for obtaining replacement factors, so that, instead of successively testing if $C \equiv L_1 \mid \dots \mid L_k$ subsumes $C \setminus \{L_i\}$, (for $i: 1 \leq i \leq k$), we now test to see if C subsumes $\{\neg L_i\} \cup C \setminus \{L_i\}$, (and, upon a positive answer, obtain $C \setminus \{L_i\}$ as a folding of C into itself), we get a kind of generalization for replacement factoring.

For if $C \setminus \{L_i\}$ is a replacement factor of C , then (since C subsumes $C \setminus \{L_i\}$, it follows that C subsumes $\{\neg L_i\} \cup C \setminus \{L_i\}$, and so), $C \setminus \{L_i\}$ is a folding of C into itself, and so by simply changing the subsumption tests in the procedure for finding (minimal) replacement factors as described above, we obtain a procedure for an extension of the relation “is a replacement factor of”. That this is a proper extension, is shown by Example H.0.5, given in Appendix H. This same example evidently also serves as example of a clause, C , (where $cl(C) = C$ and), which has no proper Σ_0 -repf, but for which there exists $C' \subsetneq C$ such that C' is a Σ_0 -folding of C into itself.

8.4. Refining the Deduction Rules

As $C' \subseteq C$, whenever C' is a Σ_0 -repf of a clause C , C' Σ_0 -subsumes C , and evidently Corollaries 7.2.6 and 7.2.5 say, in particular, that for a given initial unsatisfiable set S of clauses, we can first “normalize” the clauses of S , by replacing each clause $C \in S$ by some (for example, minimal)

Σ_0 -repf C' of C , and similarly “normalize” each resolvent generated.

Corollaries 8.2.3, 8.2.4, 8.2.5, and 8.2.6 evidently further inform us that if $EQF \subseteq S$, and $\mathcal{E}_0 \subseteq S$, is a collection of positive equality unit clauses, we can first “normalize” the clauses of S , as before, except instead of replacing each clause $C \in S \setminus EQF$ by some (for example, minimal) Σ_0 -repf C' of C , we can replace any such C by some (for example, minimal) $\mathcal{E}_0$ -repf, C' , of C , and similarly. if $\mathcal{E} \subseteq S'$ is a collection of positive equality units contained in the current (deduced) clause set S' , then whenever a clause C is generated from non-equality parents, instead of simply “normalizing” C , to some (for example, minimal) Σ_0 -repf, we can “normalize” C to some (for example, minimal) $\mathcal{E}$ -repf, C' , of C , (before augmenting the current clause set by the new clause, if C' is not Σ_0 -subsume'd by any clause already in the current clause set).

Perhaps most usefully, this says that, in the process of “normalizing” a resolvent, C , of non-equality parents, we can QR-factor C as much as possible, thus reducing the number of (inequality) literals, which is otherwise bound to grow large because of their generation via the above-equality deduction rule.

Further, we can utilize flat demodulation, to assist in this process of inequality elimination. More specifically, suppose (C was generated from non-equality parents, and) C' is a QR-factor of C , such that for no inequality literal, $\neq(s', t') \in C'$, does s' unify with t' , but for some $\neq(s, t) \in C'$, there is a flat-demodulant, $\neq(s'', t'')$ of $\neq(s, t)$, (via $\mathcal{E} \subseteq S'$), such that s'' unifies with t'' , with most general unifying substitution θ , say. Then we can replace C by any $\mathcal{E}$ -repf of $C'\theta \setminus \{\neq(s, t)\theta\}$.

QR-factoring, and the process of reducing the number of inequality literals more generally, involves the application of a substitution, and thus can, in turn, promote the possibility of reducing the resolvent's length yet again, by creating further proper repf's.

In particular, suppose C is any (EPC_<-) resolvent of non-equality parents A and B , where, modulo ordering, C is of the form

$$\bigcup \{ \neq(\vec{s}_{i'}, \vec{t}_{j'}) \mid 1 \leq i' \leq i; 1 \leq j' \leq j \} \cup A \setminus A' \cup B \setminus B', \text{ (as in Definition 4.1.1).}$$

Then if θ is a most general unifying substitution of β , where β is *any* subset of $\{ \{s, t\} \mid \neq(s, t) \in \neq(\vec{s}_{i'}, \vec{t}_{j'}) , 1 \leq i' \leq i; 1 \leq j' \leq j \}$.

(θ unifying each element $\{s, t\} \in \beta$ to a singleton, $\{s\theta\} = \{t\theta\}$),

and $D_1 \equiv \bigcup \{ \neq(\vec{s}_{i'}, \vec{t}_{j'})\theta \mid 1 \leq i' \leq i; 1 \leq j' \leq j \} \cup (A \setminus A')\theta \cup (B \setminus B')\theta$,

then $D_1 \setminus F(D_1)$ is a QR-factor of C .

If $D_2 \equiv \bigcup \{ \neq(\vec{s}_{i'}, \vec{t}_{j'})\theta \mid 1 \leq j' \leq j \} \cup \bigcup \{ \neq(\vec{s}_{i'}, \vec{t}_{i'})\theta \mid 2 \leq i' \leq i \} \cup (A \setminus A')\theta \cup (B \setminus B')\theta$,

then evidently D_2 is a Σ_0 -repf of D_1 , and so ($D_2 \subseteq D_1$ and $D_1 \Sigma_0$ -subsumes D_2 ,

so $D_2 \setminus F(D_2) \subseteq D_1 \setminus F(D_1)$ and $D_1 \setminus F(D_1) \Sigma_0$ -subsumes $D_2 \setminus F(D_2)$, and so),

$D_2 \setminus F(D_2)$ is a Σ_0 -repf of $D_1 \setminus F(D_1)$, and hence $D_2 \setminus F(D_2)$ is a $\mathcal{E}_0$ -repf of C .

Letting $D \equiv \bigcup \{ \neq(\vec{s}_{i'}, \vec{t}_{j'})\theta \mid 1 \leq j' \leq j \} \cup \bigcup \{ \neq(\vec{s}_{i'}, \vec{t}_{i'})\theta \mid 2 \leq i' \leq i \} \cup A\theta \setminus A'\theta \cup B\theta \setminus B'\theta$,

clearly $D \setminus F(D) \subseteq D_2 \setminus F(D_2)$, so, trivially, $D \setminus F(D)$ subsumes $D_2 \setminus F(D_2)$, and evidently, since D soundly follows from A and B ,

(consider the associated resolvent from parents $A\theta$ $B\theta$, instead of A and B , respectively),

we can (at the very least), take $o(D \setminus F(D))$ in place of $o(C)$, (as the corresponding resolvent).

Of course, Corollaries 8.2.3, 8.2.4, 8.2.5, and 8.2.6, actually say much more. Suppose that our current clause set is S' , (S' corresponding to the set of clauses in some initial fragment of some (refined) Σ_0 -subsumption (and Σ_0 -tautology) deleted (reduced) $\text{EPC}_{<}$ -deduction sequence from S , where $EQF \subseteq S$), and we have just obtained a $\text{EPC}_{<}$ -resolvent C from some non-equality parents, $A; B \in S'$.

Suppose further that $\mathcal{E} \subseteq S'$ is a collection of positive equality units, and we want to know whether or not we need to (eventually) augment S' by C , (or some $\mathcal{E}$ -repl of C). The corollaries assert that we can delete C if there is any $C' \in S'$, such that $C' \mathcal{E}$ -reduct Σ_0 -subsumes C , that is, if there is any $C' \in S'$, such that $C' \Sigma_0$ -subsumes C'' , where C'' is some (arbitrary) $\mathcal{E}$ -reduct of C . Thus we are offered the intriguing possibility of utilizing a deletion strategy based on a (generalized) subsumption relation which becomes stronger as the number of deduced positive equality units grows, (that is, the deletion relation becomes stronger as the deduction/refutation search proceeds).

We shall not pursue this issue here. Rather, we return to an examination of the deduction rules in the equality fragment. We return for a few quick comments on deletion using $\mathcal{E}$ -reduct Σ_0 -subsumption in Section 8.5.

8.4.1. Tightening the Equality Rules

Evidently, C' is a Σ_0 -repl of a positive equality clause C exactly when $C' \subseteq C$ and for some substitution, θ , modulo symmetry, $C\theta \subseteq C'$, that is, when, modulo symmetry, C' is a replacement factor of C .

OBSERVATION 8.4.1. For any QR -positive equality clause C , if C' is a Σ_0 -repl of C , then C' is QR -positive and further, $QR_{full}(C)$ and $QR_{full}(C')$ are subsumption equivalent.

Note, with $C' \subseteq C$ as hypothesised in the preceding observation, it need *not* be the case that modulo alphabetic equivalence and ordering, $QR_{full}(C') = QR_{full}(C)$.

EXAMPLE 8.4.1. A simple example is given by taking:

$$C' \equiv =(a, x) \mid \neq(x, b) \quad \text{and} \quad C \equiv =(a, x) \mid \neq(x, b) \mid =(z, b) \mid \neq(z, y).$$

where a and b are ground terms, and x , y , and z are variables. Clearly $C' \subseteq C$, and

$$C[a/z, a/y] = =(a, x) \mid \neq(x, b) \mid =(a, b) \mid \neq(a, a) \subseteq cl(C') \cup F, \text{ where } F \equiv \{\neq(a, a)\},$$

so $C \Sigma_0$ -subsumes C' . Here, modulo alphabetic equivalence, (and ordering),

$$QR_{full}(C') = \{=(a, b)\} \text{ and } QR_{full}(C) = \{=(a, b), =(z, b)\}.$$

Rather informally, by *narrowing*², we shall mean any such process whereby the number of resolvents may be reduced, without loss of refutation completeness. Thinning via Σ_0 -subsumption, the set of resolvents $\text{EPC}_{<}(\mathcal{A}, D)$,

²The term seems to be generally associated only with paramodulation, but, anyway, the rule $\text{EPC}_{+,-}(\cdot, \cdot)$, (and hence, also, $\text{EPC}_{<}(\cdot, \cdot)$), does involve a restricted amount of paramodulation.

(of equality clauses A and D , and more generally, $\text{EPC}_{<}^1(A, D)$, for any clauses A and D), (down to a minimal subset $X \subseteq \text{EPC}_{<}(A, D)$, ($X \subseteq \text{EPC}_{<}^1(A, D)$), such that for each $C \in \text{EPC}_{<}(A, D)$, (for each $C \in \text{EPC}_{<}^1(A, D)$), there is some $C' \in X$ such that $C' \Sigma_0$ -subsumes C), seems to provide, (in $\text{EPC}_{<}$), the greatest contribution to narrowing. (Of course, we have seen that, (in $\text{EPC}_{<}$), such “ Σ_0 -subsumption thinning” does not destroy refutation completeness.)

Nevertheless, the $\text{EPC}_{<}(\cdot, \cdot)$ rule we formulated as part of the deductive machinery for the equality fragment, is particularly unattractive, in terms of the potential number of resolvents generated. Thus, we seek here to “refine”, in particular, the deduction rule, $\text{EPC}_{<}(\cdot, \cdot)$, (and $\text{EPC}_{+,-}(\cdot, \cdot)$), so as to reduce the number of resolvents generated.

Recall that for any non-empty equality clauses A and D ,

$\text{EPC}_{<}(A, D)$ was defined as:

$\{\square\}$, if $QR_{full}(D)$ is the empty clause, $\square$,

and otherwise, for some parameterized clause ordering function o ,

$\{o(A, D, C) \mid C \in \text{EPC}_{<}^1(A, D)\}$,

where $\text{EPC}_{<}^1(A, D) \equiv \bigcup \{\text{EPC}_{+,-}(QR_{full}(A), D') \mid D' \in QR_{first}(D)\}$,

and $\text{EPC}_{+,-}(A, D)$ was defined as:

a) if A is positive, and D is mixed,

the collection of all (unordered) clauses B such that:

there exists a non-empty subset $A_R \subseteq A$, a non-empty subset $D_R \subseteq D^-$, the left-most inequality literal, $\neq(s_0, t_0)$, say, of D in D_R , and a (possibly empty) subset $D_S \subseteq D^- \setminus D_R$, such that for some symmetric choice function c on $A_R \cup D_R \cup D_S$, with $c(\neq(s_0, t_0)) = 0$, the most general substitution σ , say, unifying:

$$\begin{aligned} & \{ \{=(s', t') \mid =(s', t') \in A_R; c(=(s', t')) = 0\} \cup \{=(t', s') \mid =(s', t') \in A_R; c(=(s', t')) = 1\} \cup \\ & \quad \{=(s', t') \mid \neq(s', t') \in D_R; c(\neq(s', t')) = 0\} \cup \{=(t', s') \mid \neq(s', t') \in D_R; c(\neq(s', t')) = 1\}, \\ & \quad \{s' \mid =(s', t') \in A_R; c(=(s', t')) = 0\} \cup \{t' \mid =(s', t') \in A_R; c(=(s', t')) = 1\} \cup \\ & \quad \{s' \mid \neq(s', t') \in D_S; c(\neq(s', t')) = 0\} \cup \{t' \mid \neq(s', t') \in D_S; c(\neq(s', t')) = 1\} \}, \end{aligned}$$

exists, σ unifying the two sets in the above doubleton set to the singletons $\{=(s, r)\}$ and $\{s\}$, respectively, for some terms s and r , say, and

$$\begin{aligned} B \equiv & A\sigma \setminus \{=(r, s), =(s, r)\} \cup D\sigma \setminus (\{\neq(r, s), \neq(s, r)\} \cup D_S\sigma) \cup \\ & \{\neq(r, t'\sigma) \mid \neq(s', t') \in D_S; \neq(s', t')\sigma \notin \{\neq(r, s), \neq(s, r)\}; c(\neq(s', t')) = 0\} \cup \\ & \{\neq(s'\sigma, r) \mid \neq(s', t') \in D_S; \neq(s', t')\sigma \notin \{\neq(r, s), \neq(s, r)\}; c(\neq(s', t')) = 1\}, \end{aligned}$$

b) and otherwise, $\emptyset$.

As first approach to a “narrowing” of $\text{EPC}_{+,-}(\cdot, \cdot)$,

DEFINITION 8.4.1. For equality clauses, A non-empty positive, and D non-empty mixed, let $\mathcal{U}(A, D)$ be the collection of all (unordered) clauses B such that:

- i. there exists a non-empty subset $A_R \subseteq A$, a non-empty subset $D_R \subseteq D^-$, the left-most inequality literal, $\neq(s_0, t_0)$, say, of D in D_R , such that for some symmetric choice function c on $A_R \cup D_R$, with $c(\neq(s_0, t_0)) = 0$, the most general substitution σ , say, unifying:

$$\{=(s', t') \mid =(s', t') \in A_R; c(=(s', t')) = 0\} \cup \{=(t', s') \mid =(s', t') \in A_R; c(=(s', t')) = 1\} \cup$$

$$\{=(s', t') \mid \neq(s', t') \in D_R; c(\neq(s', t')) = 0\} \cup \{=(t', s') \mid \neq(s', t') \in D_R; c(\neq(s', t')) = 1\},$$
 exists, σ unifying the above set to the singleton $\{=(s, r)\}$, for some terms s and r , say.
- ii. there exists a (possibly empty) subset $D'_S \subseteq D^- \sigma \setminus \{\neq(s, r), \neq(r, s)\}$, such that for some symmetric choice function c' on D'_S , the most general substitution ρ , say, unifying:

$$\{s\} \cup \{s' \mid \neq(s', t') \in D'_S; c'(\neq(s', t')) = 0\} \cup \{t' \mid \neq(s', t') \in D'_S; c'(\neq(s', t')) = 1\},$$
 exists, ρ unifying the above set to the singleton $\{s\rho\}$.
- iii. neither $A\sigma\rho$ nor $D\sigma\rho$ is a Σ_0 -tautology, (so, in particular, $s\rho \neq r\rho$), and
- iv. $B \equiv A\sigma\rho \setminus \{=(r, s)\rho, =(s, r)\rho\} \cup D\sigma\rho \setminus (\{\neq(r, s)\rho, \neq(s, r)\rho\} \cup D'_S\rho) \cup$

$$(D'_S\rho \setminus \{\neq(r, s)\rho, \neq(s, r)\rho\})[r\rho/s\rho],$$

where $(D'_S\rho \setminus \{\neq(r, s)\rho, \neq(s, r)\rho\})[r\rho/s\rho]$ denotes $D'_S\rho \setminus \{\neq(r, s)\rho, \neq(s, r)\rho\}$, with all top-level occurrences of (the term) $s\rho$ in $D'_S\rho \setminus \{\neq(r, s)\rho, \neq(s, r)\rho\}$,

(which appear directly as arguments of a literal), replaced by $r\rho$.

PROPOSITION 8.4.1. For any non-empty equality clauses, A positive, and D mixed, for any $C \in \text{EPC}_{+,-}(A, D)$, either:

- a) C is a Σ_0 -tautology,
- b) D Σ_0 -subsumes C ,
- c) A Σ_0 -subsumes C , or
- d) there exists $C' \in \mathcal{U}(A, D)$ such that C' Σ_0 -subsumes C .

DEFINITION 8.4.2. Let $\mathfrak{SR}_1(\cdot, \cdot, \cdot)$ be some function that assigns to each triple $\langle A, D, \neq(s_0, t_0) \rangle$, a pair of unordered clauses, $\langle A', D' \rangle$, with components $p_1(\langle A', D' \rangle) = A'$ and $p_2(\langle A', D' \rangle) = D'$, where:

- i. if A is a non-empty (unordered) positive equality clause,
 D is a non-empty (unordered) mixed equality clause,
 $\neq(s_0, t_0) \in D$, and $\neq(s_0, t_0)$ is non-trivial, then
 A' is (the underlying set of literals of) some minimal (non-empty) Σ_0 -repf of A , and
 D' is (the underlying set of literals of) some minimal Σ_0 -repf of D such that $(s_0, t_0) \in D'$,
- ii. and otherwise, $A' = A$ and $D' = D$.

DEFINITION 8.4.3. For equality clauses, A non-empty positive, and D non-empty mixed, let $\mathfrak{V}(A, D)$ be the collection of all (unordered) clauses B such that:

- i. there exists a non-empty subset $A_R \subseteq A$, a non-empty subset $D_R \subseteq D^-$, the left-most inequality literal, $\neq(s_0, t_0)$, say, of D in D_R , such that for some symmetric choice function c on $A_R \cup D_R$, with $c(\neq(s_0, t_0)) = 0$, the most general substitution σ , say, unifying:

$$\{=(s', t') \mid =(s', t') \in A_R; c(=(s', t')) = 0\} \cup \{=(t', s') \mid =(s', t') \in A_R; c(=(s', t')) = 1\} \cup$$

$$\{=(s', t') \mid \neq(s', t') \in D_R; c(\neq(s', t')) = 0\} \cup \{=(t', s') \mid \neq(s', t') \in D_R; c(\neq(s', t')) = 1\},$$
exists, σ unifying the above set to the singleton $\{=(s, r)\}$, for some terms s and r , say,
- ii. $s \neq r$, and $\mathfrak{S}\mathfrak{R}_1(A\sigma, D\sigma, \neq(s, r)) = \langle A', D' \rangle$, say,
- iii. there exists a (possibly empty) subset $D'_S \subseteq (D')^- \setminus \{\neq(s, r), \neq(r, s)\}$, such that for some symmetric choice function c' on D'_S , the most general substitution ρ , say, unifying:

$$\{s\} \cup \{s' \mid \neq(s', t') \in D'_S; c'(\neq(s', t')) = 0\} \cup \{t' \mid \neq(s', t') \in D'_S; c'(\neq(s', t')) = 1\},$$
exists, ρ unifying the above set to the singleton $\{s\rho\}$,
- iv. $A'\rho \cap \{=(s, r)\rho, =(r, s)\rho\} \neq \emptyset$, and neither $A'\rho$ nor $D'\rho$ is a Σ_0 -tautology.
(so, in particular, $s\rho \neq r\rho$), and
- v. $B \equiv A'\rho \setminus \{=(r, s)\rho, =(s, r)\rho\} \cup D'\rho \setminus (\{\neq(r, s)\rho, \neq(s, r)\rho\} \cup D'_S\rho) \cup$
 $(D'_S\rho \setminus \{\neq(r, s)\rho, \neq(s, r)\rho\})[r\rho/s\rho],$

where $(D'_S\rho \setminus \{\neq(r, s)\rho, \neq(s, r)\rho\})[r\rho/s\rho]$ denotes $D'_S\rho \setminus \{\neq(r, s)\rho, \neq(s, r)\rho\}$, with all top-level occurrences of (the term) $s\rho$ in $D'_S\rho \setminus \{\neq(r, s)\rho, \neq(s, r)\rho\}$, (which appear directly as arguments of a literal), replaced by $r\rho$.

PROPOSITION 8.4.2. For any non-empty equality clauses, A positive, and D mixed, for any $C \in \text{EPC}_{+,-}(A, D)$, either:

- a) C is a Σ_0 -tautology.
- b) D Σ_0 -subsumes C .
- c) A Σ_0 -subsumes C , or
- d) there exists $C' \in \mathfrak{V}(A, D)$ such that C' Σ_0 -subsumes C .

DEFINITION 8.4.4. Let $\mathfrak{S}\mathfrak{R}_2(\cdot)$ be some function that assigns to each unordered clause A , (the underlying set of literals of) some minimal Σ_0 -repf of A .

DEFINITION 8.4.5. For equality clauses, A non-empty positive, and D non-empty mixed, let $\mathfrak{W}(A, D)$ be the collection of all (unordered) clauses B such that:

- i. there exists a non-empty subset $A_R \subseteq A$, a non-empty subset $D_R \subseteq D^-$, the left-most inequality literal, $\neq(s_0, t_0)$, say, of D in D_R , such that for some symmetric choice function c on $A_R \cup D_R$, with $c(\neq(s_0, t_0)) = 0$, the most general substitution σ , say, unifying:

$$\{=(s', t') \mid =(s', t') \in A_R; c(=(s', t')) = 0\} \cup \{=(t', s') \mid =(s', t') \in A_R; c(=(s', t')) = 1\} \cup$$

$$\{=(s', t') \mid \neq(s', t') \in D_R; c(\neq(s', t')) = 0\} \cup \{=(t', s') \mid \neq(s', t') \in D_R; c(\neq(s', t')) = 1\},$$
exists, σ unifying the above set to the singleton $\{=(s, r)\}$, for some terms s and r , say,
- ii. $s \neq r$, and $\mathfrak{S}\mathfrak{R}_1(A\sigma, D\sigma, \neq(s, r)) = \langle A', D' \rangle$, say,

- iii. there exists a (possibly empty) subset $D'_S \subseteq (D')^- \setminus \{\neq(s, r), \neq(r, s)\}$, such that for some symmetric choice function c' on D'_S , the most general substitution ρ , say, unifying: $\{s\} \cup \{s' \mid \neq(s', t') \in D'_S; c'(\neq(s', t')) = 0\} \cup \{t' \mid \neq(s', t') \in D'_S; c'(\neq(s', t')) = 1\}$, exists, ρ unifying the above set to the singleton $\{s\rho\}$.
- iv. $A'\rho \cap \{=(s, r)\rho, =(r, s)\rho\} \neq \emptyset$, and neither $A'\rho$ nor $D'\rho$ is a Σ_0 -tautology. (so, in particular, $s\rho \neq r\rho$),
- v. $\mathfrak{S}\mathfrak{R}_1(A'\rho, D'\rho, \neq(s, r)\rho) = \langle A'', D'' \rangle$, say, $D''_S \equiv D'_S\rho \cap D'' \setminus \{\neq(r, s)\rho, \neq(s, r)\rho\}$,
- vi. $A'' \cap \{=(s, r)\rho, =(r, s)\rho\} \neq \emptyset$, and
- vii. $B = \mathfrak{S}\mathfrak{R}_2(B')$, where $B' \equiv A'' \setminus \{=(r, s)\rho, =(s, r)\rho\} \cup D'' \setminus (\{\neq(r, s)\rho, \neq(s, r)\rho\} \cup D''_S) \cup D''_S[r\rho/s\rho]$. (where $D''_S[r\rho/s\rho]$ denotes D''_S , with all top-level occurrences of (the term) $s\rho$ in D''_S . (which appear directly as arguments of a literal), replaced by $r\rho$).

PROPOSITION 8.4.3. For any non-empty equality clauses, A positive, and D mixed, for any $C \in \text{EPC}_{+,-}(A, D)$, either:

- a) C is a Σ_0 -tautology.
- b) D Σ_0 -subsumes C ,
- c) A Σ_0 -subsumes C . or
- d) there exists $C' \in \mathfrak{W}(A, D)$ such that C' Σ_0 -subsumes C .

DEFINITION 8.4.6. Let $\text{tlength}(\cdot)$ denote the function which assigns to each term t , a length, $\text{tlength}(t) \in \mathbb{N}$, as follows:

$$\begin{aligned} \text{tlength}(t) &= 1, & \text{if for some variable, } x, \quad t \equiv x. \text{ and} \\ \text{tlength}(t) &= 2 + \sum_{i=1}^n \text{tlength}(t_i), & \text{if for some } n \geq 0, \text{ some function symbol } f \text{ of arity } n, \\ & & \text{and some terms } t_1, \dots, t_n, \quad t \equiv f(t_1, \dots, t_n). \end{aligned}$$

DEFINITION 8.4.7.

- 1) For any literal L , let $\text{flip}(L)$ denote:
 - $\neq(t, s)$, if L is an inequality, $L \equiv \neq(s, t)$, where $\text{tlength}(s) < \text{tlength}(t)$, and
 - L , otherwise.
- 2) For any (unordered) clause C , let $\text{flip}(C)$ denote (the unordered clause) $\{\text{flip}(L) \mid L \in C\}$.

DEFINITION 8.4.8. For any equality clauses A and D ,

- 1) let $\widehat{\text{EPC}}_{+,-}(A, D)$ denote:
 - i. $\{o(\text{flip}(C)) \mid C \in \mathfrak{W}(A, D)\}$, where o is some (arbitrary) clause ordering function, if A is positive, and D is mixed, and
 - ii. $\emptyset$, otherwise.
- 2) let $\widehat{\text{EPC}}_-(A, D)$ denote:
 - i. $\{\square\}$, if $QR_{\text{full}}(D)$ is the empty clause, $\square$, and

ii. $\bigcup \{ \widehat{\text{EPC}}_{+,-}(\text{QR}_{full}(A), D') \mid D' \in \text{QR}_{first}(D) \}, \text{ otherwise.}$

The motivation behind the introduction of $\text{flip}(\cdot)$, is, obviously, to reduce the potential number of resolvents of the clauses generated and other clauses. In particular, suppose the left-most inequality literal of a clause, C , is $\neq(x, t)$, where x is a variable and t is some non-variable term, and C' is (some ordering of) $\text{flip}(C)$.

Though there are obvious counter-examples, because of special status given to the left-most term in the left-most inequality, it seems reasonable to suppose that for any clause A , the number of clauses in $\widehat{\text{EPC}}_{+,-}(A, C')$ will, on average, be fewer than in $\widehat{\text{EPC}}_{+,-}(A, C)$, and more generally, that for any clauses A and C , if C' is (some ordering of) $\text{flip}(C)$, the number of clauses in $\widehat{\text{EPC}}_{+,-}(A, C')$ will, on average, be fewer than in $\widehat{\text{EPC}}_{+,-}(A, C)$.

A second "justification" may exist, this based on the possibility that one might be able to show that it is (on average), beneficial to replace terms of greater length with terms of lesser length, (so that $\text{length}(\tau\rho) < \text{length}(s\rho)$, in the contribution $D_S''[r\rho/s\rho]$, as described in Definition 8.4.5), rather than conversely. At any rate, hard justifications are not required here, and it suffices that introducing $\text{flip}(\cdot)$ seems reasonable enough, and anyway, (as we shall show shortly), is not problematical.

Based along the same lines of reasoning, (as for the introduction of $\text{flip}(\cdot)$), we should also presumably choose, in Definition 8.4.8, for $o(\cdot)$, a clause ordering function that would at least, place $\neq(s', t')$ to the left of $\neq(s, t)$, if s and t are both variables, and at least one of s' and t' is a non-variable term.

PROPOSITION 8.4.4. For any equality clauses A and D , for any $C \in \text{EPC}_{-}(A, D)$, either:

- a) C is a Σ_0 -tautology.
- b) D Σ_0 -subsumes C .
- c) A Σ_0 -subsumes C , or
- d) there exists $C' \in \widehat{\text{EPC}}_{-}(A, D)$ such that C' Σ_0 -subsumes C .

Thus if we interpret EPC_{-} as $\widehat{\text{EPC}}_{-}$ in $\text{EPC}_{<}$, (that is, for any equality clauses A and D , take $\text{EPC}_{<}^1(A, D) \equiv \widehat{\text{EPC}}_{-}(A, D) \cup \widehat{\text{EPC}}_{-}(D, A) \cup \text{EPC}_{+}(A, D)$), evidently, for any non-empty set of clauses, S , any (refined) Σ_0 -subsumption (and Σ_0 -tautology) deleted (reduced) $\text{EPC}_{<}$ -deduction sequence from S , remains a (refined) Σ_0 -subsumption (and Σ_0 -tautology) deleted (reduced) $\text{EPC}_{<}$ -deduction sequence from S , (respectively), and evidently, Corollaries 7.2.6, 7.2.5, 8.2.3, 8.2.4, 8.2.5, and 8.2.6 all remain true, (when we take $\widehat{\text{EPC}}_{-}$ as the definition for EPC_{-}).

Note that, in particular, $\widehat{\text{EPC}}_{+,-}(A, D)$, says that in the creation of a resolvent C , the subset of inequality literals of the appropriate substitution instance of D , when regarded as a graph (whose vertices are terms, and connected by an edge when a corresponding inequality literal is in the set), can first be pruned down to a minimal collection of spanning trees, (whose transitive closure is that

of the original graph), before, in particular, the deletion of the literal of the inequality literal resolved upon.

The equality deduction rules can obviously be further narrowed.
(in such a fashion that the appropriate analogue of Proposition 8.4.4 remains true, and so),
without invalidating the major results on refutation completeness under deletion,
but the specification and verification of such refinements are exercises of a rather technical nature,
and we content ourselves, here with having simply indicated how,
(through the mechanism of taking Σ_0 -repf's), such narrowing may be achieved.

Of course, the process of replacing a clause with a Σ_0 -repf, (or just a replacement factor³),
in the initial given set of clauses, during the formation of a resolvent⁴, and/or after the clause is
generated as a resolvent, is sound and applicable (refutation-completeness preserving), in any clausal
resolution-refutation system which is refutation complete with subsumption (or even, sub-clause)
deletion.

8.5. $\mathcal{E}$ -reduct Σ_0 -subsumption and Generating Deduction Sequences with Deletion

In Section 8.4, we mentioned the possibility of deletion based on some (decision) procedure $XS(\cdot, \cdot)$,
such that, if, with the obvious notation that:

$Set(\Sigma_0\text{-subsumes})$ denotes the collection of all pairs, $\langle C', C \rangle$, such that $C' \Sigma_0\text{-subsumes } C$,

$Set(XS)$ denotes the collection of all pairs, $\langle C', C \rangle$, such that $XS(C', C)$, and

$Set(\mathcal{E}\text{-reduct } \Sigma_0\text{-subsumes})$ denotes the collection of all pairs, $\langle C', C \rangle$, such that

$C' \mathcal{E}\text{-reduct } \Sigma_0\text{-subsumes } C$, where $\mathcal{E}$ is a set of positive equality units,

then whenever $\mathcal{E}$ is “(implied by) the current set” of positive equality units,

$Set(\Sigma_0\text{-subsumes}) \subset Set(XS) \subseteq Set(\mathcal{E}\text{-reduct } \Sigma_0\text{-subsumes})$.

There are a few possibilities here for XS , which might prove fruitful. Presumably, we want XS to
correspond to a relation which is decidable in a natural way, that is, not some ad hoc computational
pruning of some only semi-decidable⁵ relation.

Conceptually, (and rather informally), a natural candidate for XS is described by saying that:
whereas in the normal subsumption test, where in the test for the subsumption of a clause C by
a clause C' , one searches for a linear unit (OBMR-) refutation of C' , where the set of units is
given by the logical negation of C , and, in the test of the Σ_0 -subsumes of C by C' , again one
basically searches for a linear unit (OBMR-) refutation of C' , where now the set of units is given
by the logical negation of $cl(C)$, together with the reflexivity axiom for equality, $=(x, x)$, the test
for $XS(C', C)$ is (can be) a search for a linear unit (OBMR-) refutation of C' , where now the set
of units is given by the logical negation of $cl(C)$, together with the reflexivity axiom for equality,
 $=(x, x)$, and, as well, the (symmetric closure of the current) set of positive equality unit clauses, $\mathcal{E}$.

³Clearly for refutation problems not involving the equality predicate, a Σ_0 -repf is simply a replacement factor.

⁴This needs to be done in a sufficiently careful fashion. For example, note that, in Definition 8.4.5,

D' is some minimal Σ_0 -repf of $D\sigma$ such that $\neq(s, r) \in D'$.

⁵The word problem for (some) finitely representable groups is undecidable.

That is, in this scheme, basically one does not allow interaction between the unit equality clauses in $\mathcal{E}$ and the transitivity axiom for equality, thus ignoring the (potentially infinite) transitive closure of $\mathcal{E}$.

Independent of the particular choice for the further generalization of Σ_0 -subsumes, XS , however, we can make some observations about refutation completeness in the context of deduction/deletions strategies with “ XS -deletion”.

As before, in the context of either a shell-method or a best first deduction strategy (Σ_0 -tautology deletion and) *forward* deletion via any procedure, XS , for detecting (some fragment of) $\mathcal{E}$ -reduct Σ_0 -subsumption (for “moving” $\mathcal{E}$), (the deletion defined in the obvious way with respect to Definition 8.2.4, (and 8.2.6), -that is, so that in applying the forward deletion, one is left with the right kind of deduction sequence, so that we can apply Corollary 8.2.5, or Corollary 8.2.6), would, evidently, preserve refutation completeness, as it would even in the presence of any well-founded backward Σ_0 -subsumption, (given by a well-founded pre-order relation, $\succeq$ as before), or even, “well-founded backward $\mathcal{E}$ -reduct Σ_0 -subsumption”.

Similarly, by Observation 8.1.9, there is no problem with “back-normalization”, (of non-equality clauses, say, in the presence of a newly deduced positive equality unit), providing that we restrict ourselves in the “back-normalization” of an (established) clause C , to a (new) clause C' , to those $\mathcal{E}$ -repf's C' of C , which are Σ_0 -repf's (of not just any clause C'' such that $C \succ_{fd,\mathcal{E}} C''$, but rather), of some clause C'' such that $C \succ_{\rho,\mathcal{E}} C''$, for some well-founded flat demodulation function, ρ .

A minor complication is that our proof of Proposition 7.3.1, (like that of Proposition 2.6.1), assumed $\succeq$, was a (well-founded) pre-order, and, more specifically, relied on the transitivity of $\succeq$.

However, $\mathcal{E}$ -reduct Σ_0 -subsumption is not transitive, (Observation 8.2.4), and if the procedure which identifies (some subset of) the $\mathcal{E}$ -reduct Σ_0 -subsumption condition, (for possibly “moving” $\mathcal{E}$), can not be guaranteed to be transitive, (and we still want to backward delete with respect to some such $\mathcal{E}$ -reduct Σ_0 -subsumption condition), then instead of taking the criterion for the backward deletion of a clause C by a clause C' to be given by a pre-order relation, $C \succeq C'$, as before, the simplest fix⁶ is simply to take as criterion an irreflexive binary relation, $\triangleright$, say, which is well-founded simply as such, (that is, simply such that there can exist no infinite chains: $C_0 \triangleright C_1 \triangleright \dots$).

For example, corresponding to the two candidates for $\succeq$ given in Section 2.6, we have as two possible candidates for $\triangleright$:

$C \triangleright C'$ if (C' $\mathcal{E}$ -reduct Σ_0 -subsumes C and) the length of C' is (strictly) less than that of C .

and, presumably better,

$C \triangleright C'$ if (C' $\mathcal{E}$ -reduct Σ_0 -subsumes C and), (with ‘ $=$ ’ $= P_0, P_1, \dots, P_k$, denoting the predicate symbols occurring in a given refutation problem, ordered so that $\forall k': 0 \leq k' < k, P_{k'} < P_{k'+1}$),

⁶With this change, “the second case”, (the situation in which “transitivity” must be used), in the proof of Proposition 7.3.1, (or in the proof of Proposition 2.6.1), simply does not exist.

there exists $k' : 0 \leq k' \leq k$ such that:

for all $k'' : k' < k'' \leq k$, $\#(C', P_{k''}, -) = \#(C, P_{k''}, -)$ and $\#(C', P_{k''}, +) = \#(C, P_{k''}, +)$,

and either $\#(C', P_{k'}, -) < \#(C, P_{k'}, -)$ or,

$\#(C', P_{k'}, -) = \#(C, P_{k'}, -)$ and $\#(C', P_{k'}, +) < \#(C, P_{k'}, +)$.

Clearly either of these specifies such a well-founded relation, $\triangleright$.

CHAPTER 9

A Few Words on the Function Substitutivity Equality Axioms and Conclusions

The contents of this paper are, in the author's opinion, primarily of theoretical interest. $EPC_{<}$ is a clausal resolution style system, which is refutation complete with respect to Σ_0 -subsumption and Σ_0 -tautology deletion, where these relations are natural generalizations of subsumption and tautology associated to the *built-in* equality axioms, Σ_0 . Let me briefly say something about the possibility of further modifying the deductions rules, so that we may similarly, build-in Σ_Q , the full equality axiom set.

The practical significance of $EPC_{<}$ is severely compromised by the fact that the equality axioms for functional substitution have not been built-in. In particular, the equality fragment can likely be handled more efficiently. One could of course, simply utilize $EPC_{<}$ for the treatment of the above-equality fragment¹, and some other, more efficient technique for the equality fragment. But surely a better answer is possible.

The condition naturally corresponding to the “fully extended subsumption relation”, (“ Σ_Q -subsumption”):

Does there exists a ground substitution instance, $C\sigma$, of a clause C , such that

$\Sigma_Q \cup \{C\sigma\} \cup \neg D$ is unsatisfiable?

is in fact decidable², for any clauses C and D , (though possibly computationally hard).

The condition naturally corresponding to the “fully extended tautology relation”:

Is it the case that $\Sigma_Q \models C$?

is of course, also decidable, for any clause C .

Initially, however, I had felt that the existence of a resolution/refutation-style deduction system, $\mathcal{R}$ (akin to $EPC_{<}$), for which refutation completeness is preserved with respect to “ Σ_Q -subsumption”

¹Still, evidently, the “generalized subsumption condition”, (Σ_0 -subsumption) for the deletion of an $EPC_{<}$ -resolvent of non-equality parents can be yet further weakened, (to “ Σ_Q -subsumption”).

²I have only recently discovered this. I haven't yet bothered to write up a proper proof, and the proof I have is rather involved, so I won't try to present the argument here.

deletion, was rather unlikely. In hindsight, it now seems to me quite possible that a suitable modification of $EPC_{<}$ may be shown to be just such an $\mathcal{R}$.

Informally, one such possible modification of $EPC_{<}$ is essentially motivated simply by the idea that an equality literal $=(r, s)$ of a positive equality clause C , should implicitly represent each equality $=(t[r/x], t[s/x])$, where t is term in which the variable x occurs. The resolution (term unification and replacement) in the EPC_{+} -resolution rule then becomes paramodulation in a “corresponding” rule, (where *factoring* is to be done so as to take the “implicit representation” into account), while the EPC_{-} -resolution rule is modified, so that, in particular, the unification eliminating the left-most inequality of the mixed parent, is replaced by a unification eliminating a left-most inequality literal in a disagreement set, [17, 16], (of the positive equality literal and the left-most inequality of the mixed parent), thus reducing the “disagreement” between the two arguments of the left-most inequality literal.

Whether or not such an $\mathcal{R}$, (more rigorously specified, of course), can be shown to be refutation complete with “ Σ_Q -subsumption” deletion, (presumably, in a fashion along the lines of, and in extension to, the fashion used here for $EPC_{<}$ and Σ_0 -subsumption), remains an intriguing question. In light of the verifications done in this paper, the task of doing the requisite verifications might be relatively straight forward. On the other hand, this task might further highlight the need for greater and more flexible modularization (or a polymorphization?), of the proof theory developed here. Worse, it could be that there does not exist a binary clausal resolution-style system $\mathcal{R}$, which is refutation complete with “ Σ_Q -subsumption” deletion, (in the manner that $EPC_{<}$ is refutation complete with Σ_0 -subsumption deletion).³

Independent of whether or not any such (ordered-clause representation based) system $\mathcal{R}$, exists at all, perhaps it would be more expedient to look for such an $\mathcal{R}$, outside of the framework of clausal representation. In the author’s opinion, in terms of developing a “ Σ_Q -subsumption” based refutation system, (in particular, for the equality fragment), abandoning the clausal representation framework, (for one where in particular, instead of a positive equality literal in a “clause”, is an expression which may represent multiple term equalities), is likely to prove most fruitful. Still, (generalized) subsumption based mechanisms might not be the most efficient approach.

³Restricting the deletion to that of positive (equality) clauses, (in which case Σ_0 -subsumption for example, amounts to little more than subsumption), as would in effect be done, for example in “subsumption based” *hyper-resolution*, [33], is of course another scenario, which is not that intended here.

Result Dependencies

For what it may be worth, the explicit “result” dependencies are represented diagrammatically in Figure 9.1. That is, the diagram is a graph whose vertices correspond to results (propositions, theorems, observations, corollaries, or lemmas) which have either been given proof (in the Appendix), or been explicitly referenced in the proof of some other result, or both.

A black directed edge from result “*A*” to result “*B*” occurs when in the proof given (in the Appendix), for *B*, an explicit appeal to result *A* has been made. A grey directed edge from result “*A*” to result “*B*” occurs when in the proof given for *B*, we have cited the proof supplied for *A*, (as something analogous, or from which the proof of *B* may be constructed).⁴

Node Index for Figure 9.1:

Node:	Result	Page	Node:	Result	Page
1	Lemma 2.2.1	(24)	2	Lemma 2.2.2	(24)
3	Lemma 2.2.3	(25)	4	Lemma 2.2.4	(25)
5	Theorem 2.3.1	(25)	6	Lemma 2.4.1	(25)
7	Lemma 2.4.2	(26)	8	Lemma 2.4.3	(26)
9	Lemma 2.4.4	(26)	10	Lemma 2.5.1	(27)
11	Theorem 2.5.1	(28)	12	Lemma 2.5.2	(28)
13	Lemma 2.5.3	(28)	14	Corollary 2.5.1	(28)
15	Proposition 2.6.1	(35)	16	Observation 3.2.1	(45)
17	Theorem 3.2.1	(46)	18	Corollary 3.2.1	(46)
19	Theorem 3.2.2	(46)	20	Corollary 3.2.2	(46)
21	Observation 3.2.2	(46)	22	Theorem 3.2.3	(46)
23	Theorem 3.2.4	(48)	24	Lemma 3.2.1	(48)
25	Lemma 3.2.2	(48)	26	Lemma 3.3.1	(49)
27	Lemma 3.3.2	(49)	28	Theorem 3.3.1	(50)
29	Theorem 3.3.2	(50)	30	Theorem 3.3.3	(50)
31	Corollary 3.3.1	(50)	32	Theorem 3.3.4	(50)

⁴Unfortunately, there may be a few mistakes in the graph. It was largely mechanically generated, (-I hacked a small program to pull explicit references from my .tex files), with some “manual correction”, and I haven’t thoroughly checked that all of the “black edges” drawn truly ought to be.

RESULT DEPENDENCIES

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47	Lemma 4.3.5	(66)	48	Corollary 4.3.2	(66)
49	Lemma 4.4.1	(67)	50	Lemma 4.4.2	(67)
51	Lemma 4.4.3	(67)	52	Corollary 4.4.1	(67)
53	Lemma 4.4.4	(68)	54	Lemma 4.4.5	(68)
55	Corollary 4.4.2	(68)	56	Lemma 4.5.1	(68)
57	Lemma 4.5.2	(69)	58	Lemma 4.5.3	(69)
59	Corollary 4.5.1	(69)	60	Lemma 4.5.4	(69)
61	Corollary 4.5.3	(69)	62	Corollary 4.5.4	(69)
63	Lemma 4.5.5	(70)	64	Lemma 4.5.6	(70)
65	Lemma 4.5.7	(71)	66	Lemma 4.5.8	(71)
67	Lemma 4.5.9	(71)	68	Lemma 4.5.10	(72)
69	Proposition 5.1.1	(73)	70	Observation 5.1.1	(73)
71	Lemma 5.1.1	(74)	72	Lemma 5.1.2	(74)
73	Lemma 5.1.3	(74)	74	Lemma 5.1.4	(75)
75	Corollary 5.1.1	(75)	76	Lemma 5.2.1	(75)
77	Theorem 5.2.1	(75)	78	Lemma 6.1.1	(77)
79	Lemma 6.1.2	(78)	80	Lemma 6.1.3	(78)
81	Corollary 6.1.1	(78)	82	Lemma 6.2.1	(80)
83	Lemma 6.2.2	(80)	84	Lemma 6.2.3	(80)
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105	Corollary 7.2.2	(88)	106	Observation 7.2.2	(87)

107	Lemma 7.2.5	(88)	108	Corollary 7.2.3	(89)
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137	Corollary 8.1.6	(100)	138	Corollary 8.1.7	(100)
139	Corollary 8.1.8	(101)	140	Observation 8.2.1	(101)
141	Observation 8.2.2	(101)	142	Observation 8.2.3	(101)
143	Observation 8.2.5	(102)	144	Lemma 8.2.1	(103)
145	Corollary 8.2.1	(103)	146	Corollary 8.2.2	(103)
147	Proposition 8.2.1	(104)	148	Lemma 8.2.2	(104)
149	Corollary 8.2.3	(104)	150	Corollary 8.2.4	(104)
151	Observation 8.2.6	(103)	152	Lemma 8.2.3	(104)
153	Corollary 8.2.5	(105)	154	Corollary 8.2.6	(105)
155	Observation 8.3.1	(106)	156	Observation 8.3.2	(106)
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163	Observation 2.1.2	(24)	164	Observation 4.5.1	(70)
165	Corollary 4.5.2	(69)	166	Corollary 4.4.3	(68)
167	Corollary 7.2.1	(88)	168	Corollary 6.2.3	(81)
169	Observation 6.2.1	(79)	170	Observation 6.2.2	(79)
171	Observation 6.2.3	(79)	172	Observation 4.1.2	(55)
173	Observation 4.1.6	(59)	174	Observation 4.1.3	(55)
175	Observation 3.2.3	(46)			

RESULT DEPENDENCIES

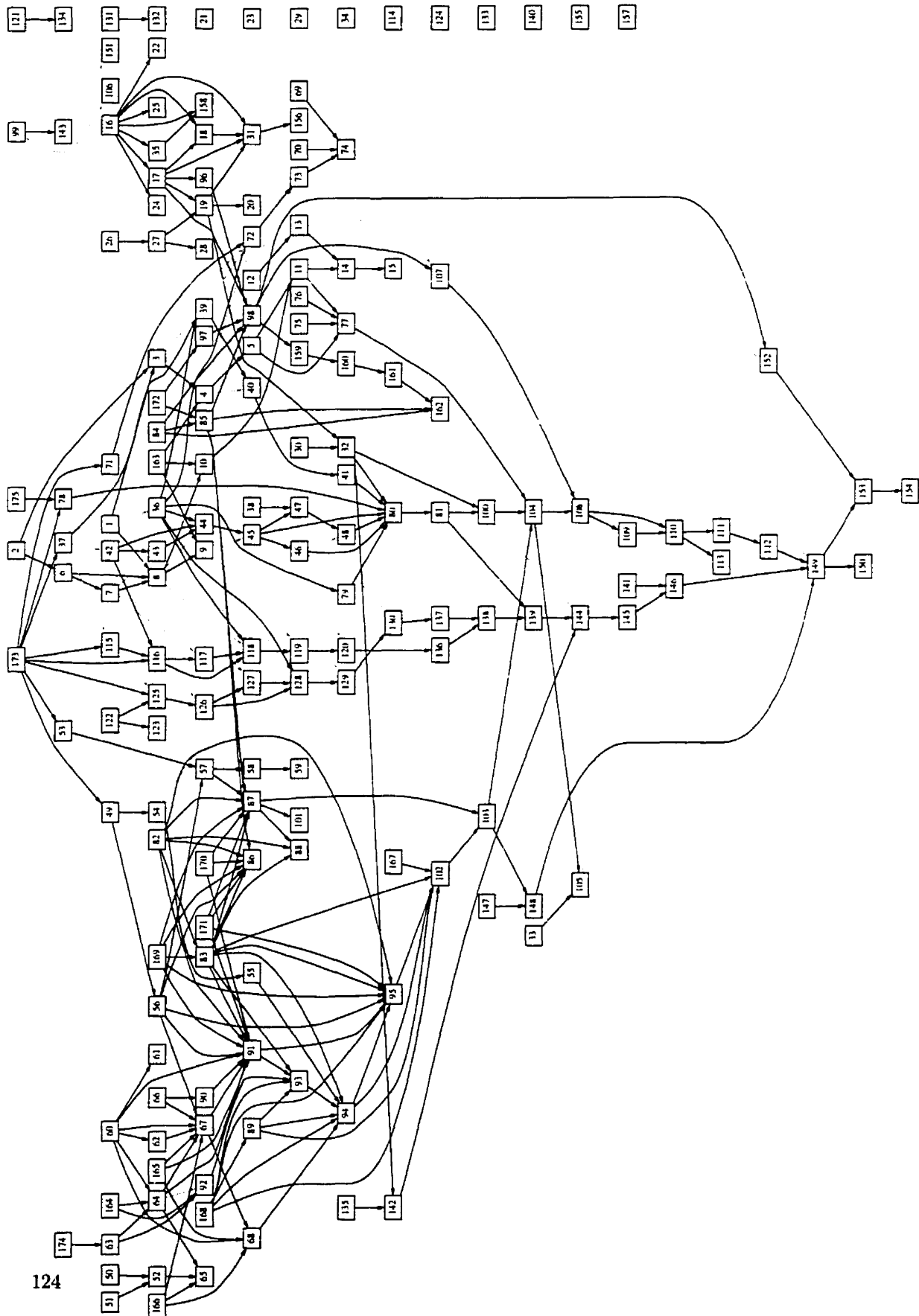


FIGURE 9.1. Result Dependencies

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Part 2

Aggregation Techniques for the Control of Non-Deterministic Finite State Machines

CHAPTER 10

Preliminaries and Winning Reachability

10.1. Some Notational Preliminaries

DEFINITION 10.1.1. By a *labelled transition system* (LTS), $\mathcal{G}$, we mean a triple $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, where $\mathcal{X}$ and $\mathcal{U}$ are finite sets, $\mathcal{X} \neq \emptyset$, and $\mathcal{E} \subseteq \mathcal{X} \times \mathcal{U} \times \mathcal{X}$.

DEFINITION 10.1.2. By an *non-deterministic finite state machine* (FSM), $\mathcal{M}$, we mean an LTS, $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, taken together with an *output function*, $\eta: \mathcal{X} \rightarrow \mathcal{Y}$, where $\mathcal{Y}$ is a finite set of *outputs*. The sets $\mathcal{X}$ and $\mathcal{U}$ will be said to consist respectively, of the *states* and the *inputs* of the FSM $\mathcal{M}$, and the relation $\mathcal{E}$ is the *transition relation* of $\mathcal{M}$.

DEFINITION 10.1.3. For any integers $N \geq k \geq 0$, let $[k, N]$ denote $\{n \mid k \leq n \leq N\}$, and let $[k, N)$ denote $\{n \mid k \leq n < N\}$.

DEFINITION 10.1.4. $\mathcal{N} \equiv \mathcal{N}(|\mathcal{X}|)$ will denote the set of integers $[0, N_\infty]$, where $N_\infty \equiv N_\infty(|\mathcal{X}|)$ is some positive integer, termed a *toy infinity*.

10.2. Winning Reachability

In analogy with the formulation of the *reachability predicate* of COCOLOG, we formulate:

DEFINITION 10.2.1. *Winning (or Robust) Reachability*

For any LTS, $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, let the *winning reachability predicate*, $R \subseteq \mathcal{X} \times 2^{\mathcal{X}} \times \mathcal{N}$, (of $\mathcal{G}$), be defined by the sentence:

$$\forall s_1 \in \mathcal{X}, \forall S \subseteq \mathcal{X}, \forall n \in \mathcal{N}.$$

$$\left[\begin{array}{l} R(s_1, S, n) \\ \Leftrightarrow \\ \left[\begin{array}{l} n = \infty \wedge \exists n' \in [0, n). R(s_1, S, n') \\ \vee \\ n = 0 \wedge s_1 \in S \\ \vee \\ \left[\begin{array}{l} n \in [1, \infty) \\ \wedge \\ \exists u \in \mathcal{U}. \\ \left[\begin{array}{l} \exists s \in \mathcal{X}. [\mathcal{E}(s_1, u, s) \wedge R(s, S, n-1)] \\ \wedge \\ \forall s \in \mathcal{X}. [\mathcal{E}(s_1, u, s) \Rightarrow \exists n' \in [0, n). R(s, S, n')] \end{array} \right] \end{array} \right] \end{array} \right] \end{array} \right].$$

More generally, for any non-empty subset $B \subseteq \mathcal{X}$,

let the *B-constrained winning reachability predicate*, $R_B \subseteq B \times 2^B \times \mathcal{N}$, (of $\mathcal{G}$), be defined by the sentence:

$$\forall s_1 \in B, \forall S \subseteq B, \forall n \in \mathcal{N}.$$

$$\left[\begin{array}{l} R_B(s_1, S, n) \\ \Leftrightarrow \\ \left[\begin{array}{l} n = N_\infty \wedge \exists n' \in [0, n). R_B(s_1, S, n') \\ \vee \\ n = 0 \wedge s_1 \in S \\ \vee \\ \left[\begin{array}{l} n \in [1, N_\infty) \\ \wedge \\ \exists u \in \mathcal{U}. \\ \left[\begin{array}{l} \exists s \in B. [\mathcal{E}(s_1, u, s) \wedge R_B(s, S, n-1)] \\ \wedge \\ \forall s \in \mathcal{X}. [\mathcal{E}(s_1, u, s) \Rightarrow [s \in B \wedge \exists n' \in [0, n). R_B(s, S, n')]] \end{array} \right] \end{array} \right] \end{array} \right] \end{array} \right]$$

Of course, neither of the above expressions are, as they stand, first order. However, since $\mathcal{X}$ is always finite, for any non-empty subset $B \subseteq \mathcal{X}$ and any subset $S \subseteq B$, we can express a first order sentence specifying the binary predicate $R_B(\cdot, S, \cdot) \subseteq B \times \mathcal{N}$, (and a first order sentence specifying the binary predicate $R(\cdot, S, \cdot) \subseteq \mathcal{X} \times \mathcal{N}$). In what follows, we shall utilize a number of expressions that are naturally regarded as being higher order, but since everything here is finite, any such expression can be equivalently¹ expressed as something first order, in fact, as something propositional.

Non-deterministic analogues for μ -reachability (see [ref]) can also be easily formulated.

To paraphrase the definition of the constrained reachability predicate, for a state $s \in \mathcal{X}$, a subset $B \subseteq \mathcal{X}$, and a set $S \subseteq \mathcal{X}$, $R_B(s, S, N_\infty)$ holds just when $R_B(s, S, n')$, for some $n': 0 \leq n' < N_\infty$, while $R_B(s, S, 0)$ holds just when $s \in S$. For any $n \in \mathcal{N} \setminus \{0, N_\infty\}$, $R_B(s, S, n)$ holds just when there is some $u \in \mathcal{U}$, such that there is some resulting state s' , such that $s' \in B$ and $R_B(s', S, n-1)$, and further, for any possible resulting state s' , $s' \in B$ and $R_B(s', S, n')$, for some $n': 0 \leq n' < n$.

Intuitively, the winning reachability predicate captures precisely the notion of winning reachability: $R(s, S, n)$ holds when n is the minimal number of steps, or *moves* required to reach the goal set S , assuming that we are starting from the initial state s , and given that an *ideal opponent* is responsible for choosing the actual successor states, (to resolve non-determinacy), in such a way as to delay as long as possible, the achievement of S . In other words, for given s and S , if n is minimal such that $R(s, S, n)$ holds, then n is the shortest time within which arrival at S can be guaranteed. If n is not minimal among those n such that $R(s, S, n)$, $R(s, S, n)$ holds if there exists a length n

¹This may involve eliminating certain (predicate) arguments by introducing new corresponding predicates, (that is, arguments into names), but such may be regarded as only an implementation issue. In fact, the current (model verification based) implementation of COCOLOG allows the manipulation of "higher order objects", such as lists of lists.

path to S , and we are not forced to accept longer paths.

The B -constrained winning reachability predicate differs only in that, in following a winning trajectory:

$s \equiv s_0, s_1, \dots, s_k \in S$, where $\forall i: 0 \leq i < k$, there exists $n_i: 0 < n_i \leq N_\infty$, such that $R_B(s_i, S, n_i)$, (and $R_B(s_k, S, 0)$), and $n_i > n_{i+1}$, where $n_k \equiv 0$,

each (successive) state s_i must lie in B , and a winning strategy must ensure that this will be so.

OBSERVATION 10.2.1. For any LTS, $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$,

$R \subseteq \mathcal{X} \times 2^{\mathcal{X}} \times \mathcal{N}$ is the same as $R_B \subseteq B \times 2^B \times \mathcal{N}$, when $B \subseteq \mathcal{X}$ is taken to be $\mathcal{X}$.

That is:

$$\forall s \in \mathcal{X}, \forall S \subseteq \mathcal{X}, \forall n \in \mathcal{N}. [R_{\mathcal{X}}(s, S, n) \Leftrightarrow R(s, S, n)].$$

For convenience and brevity, we introduce the following notation.

DEFINITION 10.2.2. For an LTS, $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, for any $s_1, s_2 \in \mathcal{X}$, any $S \subseteq \mathcal{X}$, and any $u_1 \in \mathcal{U}$, let:

$$\mathcal{U}(s_1, s_2) \equiv \{u \in \mathcal{U} \mid \mathcal{E}(s_1, u, s_2)\}.$$

$$\mathcal{X}(s_1, -) \equiv \{s \in \mathcal{X} \mid \exists u \in \mathcal{U}. \mathcal{E}(s_1, u, s)\}, \quad \mathcal{X}(-, s_2) \equiv \{s \in \mathcal{X} \mid \exists u \in \mathcal{U}. \mathcal{E}(s, u, s_2)\}.$$

$$\mathcal{X}(S, -) \equiv \{s \in \mathcal{X} \mid \exists s' \in S, \exists u \in \mathcal{U}. \mathcal{E}(s', u, s)\}, \quad \mathcal{X}(-, S) \equiv \{s \in \mathcal{X} \mid \exists s' \in S, \exists u \in \mathcal{U}. \mathcal{E}(s, u, s')\}.$$

$$\mathcal{U}(s_1, -) \equiv \{u \in \mathcal{U} \mid \exists s \in \mathcal{X}. \mathcal{E}(s_1, u, s)\}, \quad \mathcal{U}(-, s_2) \equiv \{u \in \mathcal{U} \mid \exists s \in \mathcal{X}. \mathcal{E}(s, u, s_2)\}.$$

$$\mathcal{U}(S, -) \equiv \{u \in \mathcal{U} \mid \exists s \in S, \exists s' \in \mathcal{X}. \mathcal{E}(s, u, s')\}, \quad \mathcal{U}(-, S) \equiv \{u \in \mathcal{U} \mid \exists s \in \mathcal{X}, \exists s' \in S. \mathcal{E}(s, u, s')\}.$$

$$\mathcal{E}(s_1, u_1, -) \equiv \{s \in \mathcal{X} \mid \mathcal{E}(s_1, u_1, s)\}, \quad \mathcal{E}(-, u_1, s_2) \equiv \{s \in \mathcal{X} \mid \mathcal{E}(s, u_1, s_2)\}.$$

$$\mathcal{E}(S, u_1, -) \equiv \{s \in \mathcal{X} \mid \exists s' \in S. \mathcal{E}(s', u_1, s)\}, \quad \mathcal{E}(-, u_1, S) \equiv \{s \in \mathcal{X} \mid \exists s' \in S. \mathcal{E}(s, u_1, s')\}.$$

$$\mathcal{E}^\circ(S, u_1, -) \equiv \mathcal{E}(S, u_1, -) \cup (S \setminus \mathcal{E}(-, u_1, \mathcal{X})), \quad \mathcal{E}^\circ(-, u_1, S) \equiv \mathcal{E}(-, u_1, S) \cup (S \setminus \mathcal{E}(-, u_1, \mathcal{X})).$$

$$\mathcal{E}^\forall(S, u_1, -) \equiv \{s \in \mathcal{X} \mid \emptyset \neq \mathcal{E}(-, u_1, s) \subseteq S\}, \text{ and } \mathcal{E}^\forall(-, u_1, S) \equiv \{s \in \mathcal{X} \mid \emptyset \neq \mathcal{E}(s, u_1, -) \subseteq S\}.$$

Note that $\forall u \in \mathcal{U}, \forall s_1 \in \mathcal{X}, [u \in \mathcal{U}(s_1, -) \Rightarrow \exists s \in \mathcal{E}(s_1, u, -)]$.

Evidently, (for any $s_1, s_2 \in \mathcal{X}$, any $S \subseteq \mathcal{X}$, and any $u_1 \in \mathcal{U}$),

$\mathcal{U}(s_1, s_2) = \mathcal{U}(s_1, -) \cap \mathcal{U}(-, s_2)$ is just the collection of transition labels, $a \in \mathcal{U}$, such that a labels a transition from s_1 to s_2 in the LTS $\mathcal{G}$.

$\mathcal{X}(s_1, -) = \bigcup_{u \in \mathcal{U}} \mathcal{E}(s_1, u, -)$ is the collection of *posterior* (or *children*) states of s_1 , (alternatively, the *state-fan-out from* s_1), while $\mathcal{X}(-, s_2) = \bigcup_{u \in \mathcal{U}} \mathcal{E}(-, u, s_2)$ is the collection of *anterior* (or *parent*) states of s_2 , (alternatively, the *state-fan-in to* s_2).

$\mathcal{X}(S, -) = \bigcup_{s \in S} \mathcal{X}(s, -)$ is the collection of *posterior* (or *children*) states of S ,

(alternatively, the *state-fan-out from* S), while $\mathcal{X}(-, S) = \bigcup_{s \in S} \mathcal{X}(-, s)$ is the collection of *anterior* (or *parent*) states of S , (alternatively, the *state-fan-in to* S).

$\mathcal{U}(s_1, -)$ is the set of inputs accepted at s_1 , while $\mathcal{U}(-, s_2)$ is the set of inputs which may result in s_2 .

$\mathcal{U}(S, -) = \bigcup_{s \in S} \mathcal{U}(s, -)$ is the collection of inputs which are accepted at a state $s \in S$, while $\mathcal{U}(-, S) = \bigcup_{s \in S} \mathcal{U}(-, s)$ is the collection of inputs which may result in a state $s \in S$.

$\mathcal{E}(s_1, u_1, -)$ is the set of possible posterior states, given the input u_1 at the state s_1 , that is, the u_1 -*posterior* states of s_1 ,

(alternatively, the u_1 -*state-fan-out from* s_1), while $\mathcal{E}(-, u_1, s_2)$ is the set of possible anterior states s , such that s_2 is a posterior state, (ie. the set of possible anterior states to s_2), given the input u_1 , that is, the u_1 -*anterior* states of s_2 , (alternatively, the u_1 -*state-fan-in to* s_2).

$\mathcal{E}(S, u_1, -) = \bigcup_{s \in S} \mathcal{E}(s, u_1, -)$ is the collection of u_1 -posterior states of some state s of S , (alternatively, the u_1 -*state-fan-out from* S), while $\mathcal{E}(-, u_1, S) = \bigcup_{s \in S} \mathcal{E}(-, u_1, s)$ is the collection of states u_1 -anterior to some $s \in S$, (alternatively, the u_1 -*state-fan-in to* S).

$\mathcal{E}^\circ(S, u_1, -)$ is just the collection of states u_1 -posterior to S , together with those states of S at which u_1 is not accepted, and is thus, in an obvious sense, the set of possible outcome states, given the input u_1 at some state $s \in S$.

Similarly, $\mathcal{E}^\circ(-, u_1, S)$ is just the collection of states u_1 -anterior to S , together with those states of S at which u_1 is not accepted, and is thus, in an obvious sense, the set of possible previous states, given the input u_1 and a current state $s \in S$.

$\mathcal{E}^\forall(S, u_1, -)$ is the set of states s potentially resulting from u_1 , (ie. $s \in \mathcal{E}(\mathcal{X}, u_1, -)$) such that every state s' which is u_1 -anterior to s belongs to S , while $\mathcal{E}^\forall(-, u_1, S)$ is the set of states $s \in \mathcal{E}(-, u_1, \mathcal{X})$, (at which u_1 is defined) such that every state s' which is u_1 -posterior to s belongs to S .

Also, for example, in terms of this notation, (for any $B \subseteq \mathcal{X}$), the B -constrained winning reachability predicate, $R_B \subseteq B \times 2^B \times \mathcal{N}$ may equivalently be defined by the expression:

$$\forall s_1 \in B, \forall S \subseteq B, \forall n \in \mathcal{N}.$$

$$\left[\begin{array}{c} R_B(s_1, S, n) \\ \Leftrightarrow \\ \left[\begin{array}{c} n = N_\infty \wedge \exists n' \in [0, n). R_B(s_1, S, n') \\ \vee \\ n = 0 \wedge s_1 \in S \\ \vee \\ \left[\begin{array}{c} n \in [1, N_\infty) \\ \wedge \\ \exists u \in \mathcal{U}. \\ \left[\begin{array}{c} \exists s \in \mathcal{E}(s_1, u, -) \cap B. R_B(s, S, n-1) \\ \wedge \\ \forall s \in \mathcal{E}(s_1, u, -). [s \in B \wedge \exists n' \in [0, n). R_B(s, S, n')]] \end{array} \right] \end{array} \right] \end{array} \right] \end{array} \right]$$

DEFINITION 10.2.3. For an LTS, $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, for any non-empty $B \subseteq \mathcal{X}$, any $S \subseteq B$, and any $n \in \mathcal{N}$, let:

$$R_B(-, S, n) \equiv \{s \in B \mid R_B(s, S, n)\}$$

OBSERVATION 10.2.2. Evidently, (for any LTS, $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, for any non-empty $B \subseteq \mathcal{X}$).

$$\forall S \subseteq B, \forall n \in \mathcal{N}.$$

$$R_B(-, S, n) = \begin{cases} \bigcup_{n' \in [0, n)} R_B(-, S, n') & \text{if } n = N_\infty \\ S & \text{if } n = 0 \\ B \cap \bigcup_{u \in \mathcal{U}} \left(\begin{array}{c} \mathcal{E}(-, u, R_B(-, S, n-1)) \\ \cap \\ \mathcal{E}^\forall(-, u, \bigcup_{n' \in [0, n)} R_B(-, S, n')) \end{array} \right) & \text{if } n \in [1, N_\infty) \end{cases}.$$

A small LTS and some of its (winning) reachability sets are illustrated in figure 10.1. In the figure, all edges are assumed to share some common label, α .

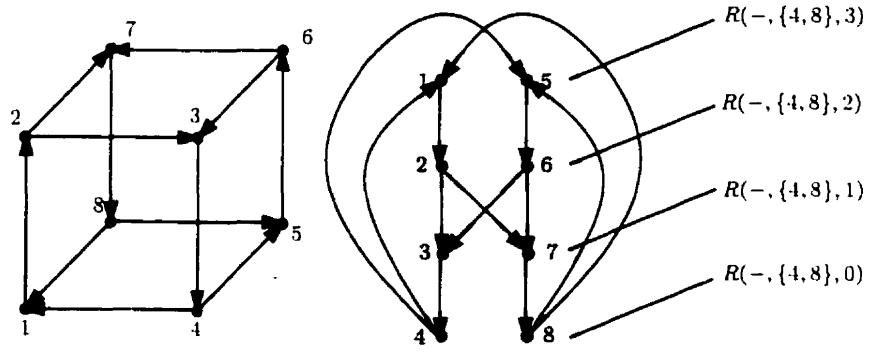


FIGURE 10.1. A small LTS and some of its (winning) reachability sets.



CHAPTER 11

Covers and (FOC) Meta-Machines

11.1. Centered Covers and In-Boundaries

DEFINITION 11.1.1. For any (non-deterministic) FSM $\mathcal{M}$, with LTS, $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, for any non-empty subset $B \subseteq \mathcal{X}$, a *center* of B is any non-empty subset $B_0 \subseteq B$.

A *centered* subset (or *centered block*) $B \subseteq \mathcal{X}$, is simply a non-empty subset $B \subseteq \mathcal{X}$ equipped with a center. We denote the center of a centered subset, B , by C_B .

A *centered cover* of $\mathcal{M}$ is simply a cover $\mathfrak{B} \equiv \{B_1, \dots, B_K\}$ of $\mathcal{X}$.

(i.e. $\forall i, j \in [1, K] \dots \emptyset \neq B_i \subseteq \mathcal{X}$ and $\mathcal{X} = \bigcup_{i \in [1, K]} B_i$),

where each block B_i is equipped with a center, $C_{B_i} \subseteq B_i$, and the centers are disjoint,

(i.e. for any $B, B' \in \mathfrak{B}$, $B \neq B' \Rightarrow C_B \cap C_{B'} = \emptyset$).

DEFINITION 11.1.2. For any FSM $\mathcal{M}$, with LTS, $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, and any centered subset $B \subseteq \mathcal{X}$, let $\mathfrak{H}(B)$, the *hull* of B , denote the set

$$\{s \in B \mid \exists n \in \mathcal{N}. R_B(s, C_B, n)\} = B \cap R_B(-, C_B, \mathcal{N}_\infty).$$

Evidently, for any FSM $\mathcal{M}$, with LTS, $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, and any centered subset $B \subseteq \mathcal{X}$, the hull of B is simply the collection of those members of B which are “robustly B -constrained steerable to C_B ”. Note that always

$$\emptyset \neq C_B \subseteq \mathfrak{H}(B) \subseteq B,$$

(and in particular, if $C_B = B$, then $\mathfrak{H}(B) = B$ also).

DEFINITION 11.1.3. For any centered cover $\mathfrak{B} \subseteq 2^{\mathcal{X}}$ of an FSM $\mathcal{M}$, with LTS, $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, for any $B \in \mathfrak{B}$, and any collection $\beta \subseteq \mathfrak{B}$,

1. let $\partial_{in}^0(B)$, the *simple-in-boundary* of B , denote the set

$$\{s \in B \mid \exists s' \in \mathcal{X} \setminus B, \exists u \in \mathcal{U}. \mathcal{E}(s', u, s)\} = B \cap \mathcal{X}(\mathcal{X} \setminus B, -).$$

2. let $\partial(B)$, the *in-boundary* of B , denote the set

$$\left(\partial_{in}^0(B) \cup \bigcup (\mathfrak{B} \setminus \{B\}) \right) \cap \mathfrak{H}(B) .$$

3. let $\partial(\beta)$ denote

$$\bigcup_{B \in \beta} \partial(B) .$$

11.2. Meta-Machines (FOC)

DEFINITION 11.2.1. The Meta-Machine: Fully Observed Case (FOC)

Let $\mathfrak{B} \equiv \{B_1, \dots, B_K\}$ be a centered cover of an FSM $\mathcal{M}$, with LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$.

For any blocks $B_1, B_2 \in \mathfrak{B}$, there is an input α_β labelling a directed edge from B_1 to B_2 in the (non-deterministic) *FOC meta-machine* whenever $B_2 \in \beta$ and $\beta \subseteq \mathfrak{B}$ is a minimal subset of $\mathfrak{B}$ such that $\forall s \in C_{B_1}, \exists n \in \mathcal{N}. R_{B_1 \cup \partial(\beta)}(s, \partial(\beta), n)$.

Figures 11.1 and 11.2 illustrate some of these ideas. In figure 11.1, a 23 state LTS is equipped with a 4 block centered cover. (consisting of blocks $B_1, \dots, B_4$), and the centers, hull, and in-boundary of the blocks are indicated. In figure 11.2, the corresponding meta-machine is depicted.

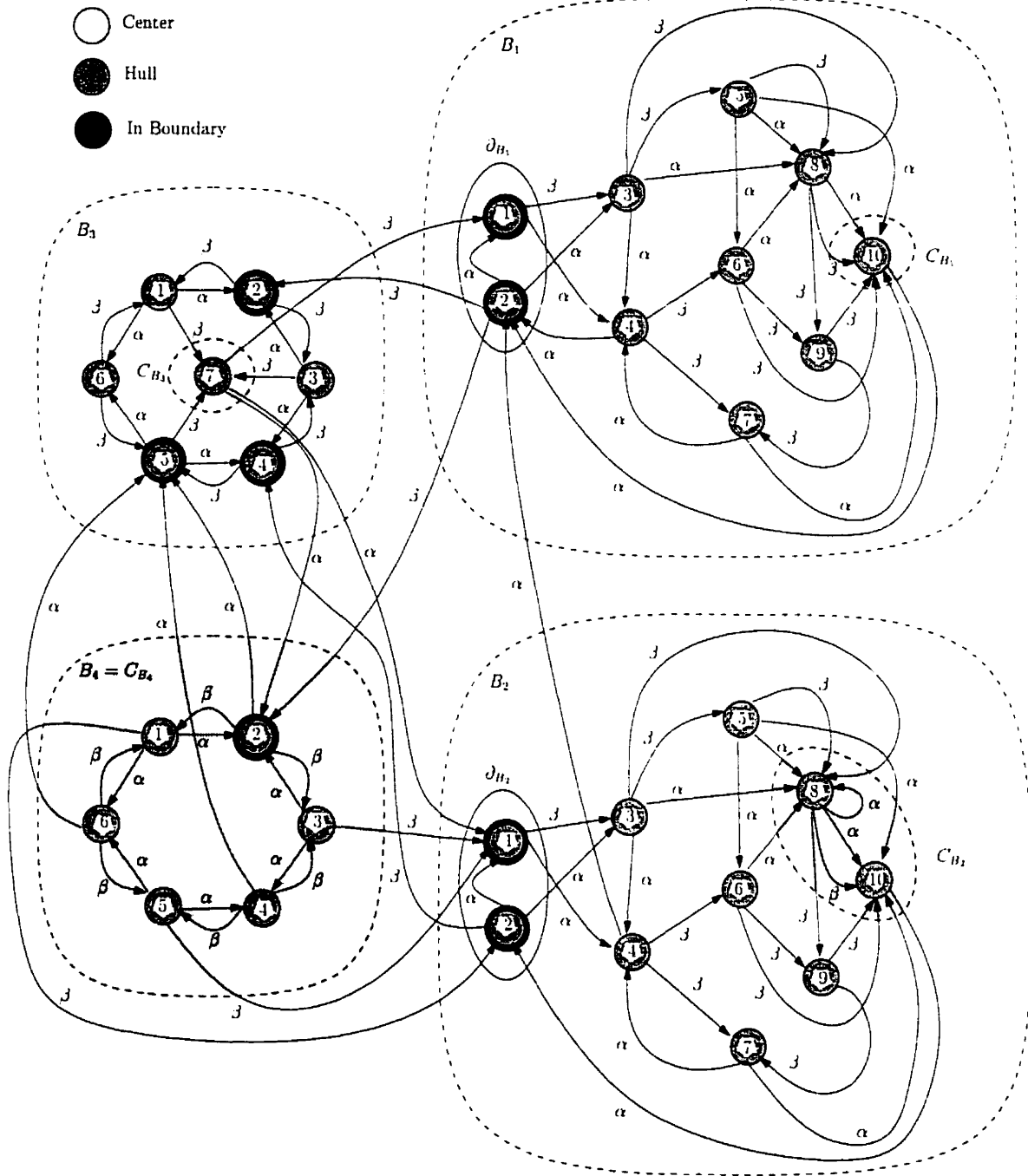
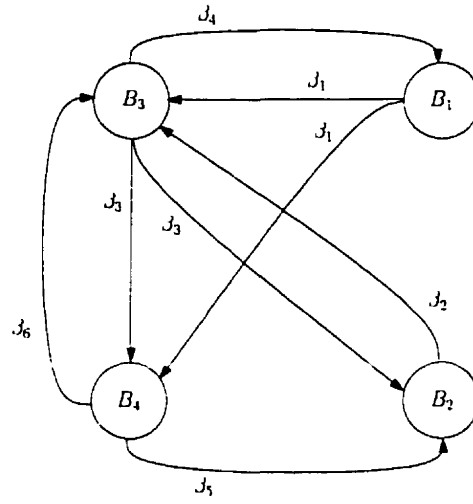


FIGURE 11.1. A centered covering of a 23 state LTS.



$$\forall B_i, \forall B_j, [j \neq i \Rightarrow \exists k \geq 0, R(B_i, B_j, k)]$$

FIGURE 11.2. The meta-machine for the LTS of figure 11.1.

CHAPTER 12

Observability

It is reasonable to formulate the following *set collapsibility* predicate.

DEFINITION 12.0.2. For a (non-deterministic) FSM, $\mathcal{M} \equiv \langle \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle, \eta: \mathcal{X} \rightarrow \mathcal{Y} \rangle$, let the *set collapsibility* predicate, $SColl \subseteq 2^{\mathcal{X}} \times \mathcal{N}$ be defined by the following sentence:

$$\forall S \subseteq \mathcal{X}, \forall n \in \mathcal{N}. \left[\begin{array}{l} SColl(S, n) \\ \Leftrightarrow \\ \left[\begin{array}{l} n = N_{\infty} \wedge \exists n' \in [0, n). SColl(S, n') \\ \vee \\ n = 0 \wedge \forall s, t \in S. [s \neq t \Rightarrow \eta(s) \neq \eta(t)] \\ \vee \\ \left[\begin{array}{l} n \in [1, N_{\infty}) \\ \wedge \\ \forall y \in \mathcal{Y}, \exists u \in \mathcal{U}, \exists n' \in [0, n). SColl(\mathcal{E}^{\circ}(\eta^{-1}(y) \cap S, u, -), n') \end{array} \right] \end{array} \right] \end{array} \right].$$

DEFINITION 12.0.3. A (non-deterministic) FSM, $\mathcal{M} \equiv \langle \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle, \eta: \mathcal{X} \rightarrow \mathcal{Y} \rangle$, is *current state observable* if

$$\exists n \in \mathcal{N}. SColl(\mathcal{X}, n).$$

The *differentiated* FSM's are an interesting restricted class of FSM's.

DEFINITION 12.0.4. An FSM $\mathcal{M}$, with LTS, $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, and output function, $\eta: \mathcal{X} \rightarrow \mathcal{Y}$, say, is (*output*) *differentiated* if

$$\forall u \in \mathcal{U}, \forall s, s', s'' \in \mathcal{X}. [\mathcal{E}(s, u, s') \wedge \mathcal{E}(s, u, s'') \wedge s' \neq s'' \Rightarrow \eta(s') \neq \eta(s'')].$$

OBSERVATION 12.0.1. Evidently, if an FSM is differentiated, then the size of the FSM's *CSE predicate*

(defined as in Basic-COCOLOG [ref]) is monotone with respect to its evolution,
(under a sequence of admissible input-output pairs).

In particular, once the CSE set is a singleton, it remains a singleton.

For a differentiated FSM, it is reasonable to formulate the following *collapsibility* predicate.

DEFINITION 12.0.5. For a differentiated FSM, $\mathcal{M} \equiv \langle \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle, \eta: \mathcal{X} \rightarrow \mathcal{Y} \rangle$, let the *collapsibility* predicate, $Coll \subseteq \mathcal{X} \times \mathcal{X} \times \mathcal{N}$ be defined by the following sentence:

$$\forall s_1, s_2 \in \mathcal{X}, \forall n \in \mathcal{N}.$$

$$\left[\begin{array}{c} Coll(s_1, s_2, n) \\ \Leftrightarrow \\ \left[\begin{array}{c} n = N_\infty \wedge \exists n' \in [0, n). Coll(s_1, s_2, n') \\ \vee \\ n = 0 \wedge (s_1 = s_2 \vee \eta(s_1) \neq \eta(s_2)) \\ \vee \\ \left[\begin{array}{c} n \in [1, N_\infty) \\ \wedge \\ \left[\begin{array}{c} s_1 \neq s_2 \wedge \eta(s_1) = \eta(s_2) \\ \Rightarrow \\ \exists u \in \mathcal{U}, \forall s_1', s_2' \in \mathcal{E}^\circ(\{s_1, s_2\}, u, -), \exists n' \in [0, n). Coll(s_1', s_2', n') \end{array} \right] \end{array} \right] \end{array} \right] \end{array} \right].$$

PROPOSITION 12.0.1. For any differentiated FSM $\mathcal{M}$, with LTS, $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, and output function, $\eta: \mathcal{X} \rightarrow \mathcal{Y}$, say, $\mathcal{M}$ is current state observable if and only if

$$\forall s_1, s_2 \in \mathcal{X}, \exists n \in \mathcal{N}. Coll(s_1, s_2, n).$$

It is (again) useful to define ("block"-) restricted versions.

DEFINITION 12.0.6. For a (non-deterministic) FSM, $\mathcal{M} \equiv \langle \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle, \eta: \mathcal{X} \rightarrow \mathcal{Y} \rangle$, and any non-empty $B \subseteq \mathcal{X}$,

1) let the *B-constrained set collapsibility* predicate, $SColl_B \subseteq 2^B \times \mathcal{N}$, be defined by the sentence:

$$\forall S \subseteq B, \forall n \in \mathcal{N}.$$

$$\left[\begin{array}{l} SColl_B(S, n) \\ \Leftrightarrow \\ \left[\begin{array}{l} n = N_\infty \wedge \exists n' \in [0, n). SColl_B(S, n') \\ \vee \\ n = 0 \wedge \forall s, t \in S. [s \neq t \Rightarrow \eta(s) \neq \eta(t)] \\ \vee \\ \left[\begin{array}{l} n \in [1, N_\infty) \\ \wedge \\ \forall y \in \mathcal{Y}. \exists u \in \mathcal{U}. \left[\begin{array}{l} \mathcal{E}^\circ(\eta^{-1}(y) \cap S, u, -) \subseteq B \\ \wedge \\ \exists n' \in [0, n). SColl(\mathcal{E}^\circ(\eta^{-1}(y) \cap S, u, -), n') \end{array} \right] \end{array} \right] \end{array} \right] \end{array} \right]$$

2) let the B -constrained collapsibility predicate, $Coll_B \subseteq B \times B \times \mathcal{N}$, be defined by the sentence:

$$\forall s_1, s_2 \in B, \forall n \in \mathcal{N}.$$

$$\left[\begin{array}{l} Coll_B(s_1, s_2, n) \\ \Leftrightarrow \\ \left[\begin{array}{l} n = N_\infty \wedge \exists n' \in [0, n). Coll_B(s_1, s_2, n') \\ \vee \\ n = 0 \wedge (s_1 = s_2 \vee \eta(s_1) \neq \eta(s_2)) \\ \vee \\ \left[\begin{array}{l} n \in [1, N_\infty) \\ \wedge \\ \left[\begin{array}{l} s_1 \neq s_2 \wedge \eta(s_1) = \eta(s_2) \\ \Rightarrow \\ \exists u \in \mathcal{U}, \forall s_1', s_2' \in \mathcal{E}^\circ(\{s_1, s_2\}, u, -). \left[\begin{array}{l} \{s_1', s_2'\} \subseteq B \\ \wedge \\ \exists n' \in [0, n). Coll(s_1', s_2', n') \end{array} \right] \end{array} \right] \end{array} \right] \end{array} \right] \end{array} \right]$$

PROPOSITION 12.0.2. For any differentiated FSM, $\mathcal{M} \equiv \langle \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle, \eta: \mathcal{X} \rightarrow \mathcal{Y} \rangle$, for any non-empty subset $B \subseteq \mathcal{X}$, for any $S \subseteq B$,

$$\exists n \in \mathcal{N}. SColl_B(S, n) \Leftrightarrow \forall s_1, s_2 \in S, \exists n \in \mathcal{N}. Coll_B(s_1, s_2, n).$$

DEFINITION 12.0.7. For any FSM $\mathcal{M}$, $\mathcal{M} \equiv \langle \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle, \eta: \mathcal{X} \rightarrow \mathcal{Y} \rangle$, for any centered subset $B \subseteq \mathcal{X}$, B is *center observable* if

$$\exists n \in \mathcal{N}. SColl_{\mathfrak{H}(B)}(C_B, n) .$$

In the special case that $C_B = B$, (and so $\mathfrak{H}(B) = B$), we may say that B is *in-block observable* when B is center observable.

For any centered cover $\mathfrak{C}$ of $\mathcal{M}$, $\mathfrak{C}$ is *center observable* if $\forall B \in \mathfrak{C}$, B is center observable.

PROPOSITION 12.0.3. For any differentiated FSM, $\mathcal{M} \equiv \langle \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle, \eta: \mathcal{X} \rightarrow \mathcal{Y} \rangle$, for any centered subset $B \subseteq \mathcal{X}$, B is center observable if and only if

$$\forall s_1, s_2 \in C_B. \exists n \in \mathcal{N}. Coll_{\mathfrak{H}(B)}(s_1, s_2, n) .$$

Notice that in principle, a hierarchical control policy can now be established for a partially observed *differentiated* FSM, $\mathcal{M} \equiv \langle \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle, \eta: \mathcal{X} \rightarrow \mathcal{Y} \rangle$, with centered cover $\mathfrak{C}$, under the rather restrictive hypothesis that *localization* (collapsing the CSE set to a singleton) can be done *block-wise*: that is, the initial CSE set may be taken as a subset of the center of a block which is center observable.

Namely, one initially does an *in-block localization*, to obtain a singleton CSE, which remains a singleton under system evolution, (since $\mathcal{M}$ is differentiated), and so (once the CSE has been reduced to a singleton within the original block), one can directly utilize the FOC meta-machine, to compute trajectories, in a hierarchical fashion.

Note that, (conceptually), the center observability of a block B demands that whenever the CSE set is initially a subset of C_B we can find a (possibly output dependent) *localization sequence* of inputs, which “*coherently*” necessarily eventually collapses the CSE set under system evolution, without CSE states straying outside of $\mathfrak{H}(B)$.

By coherence, here we mean that (for any $y \in \mathcal{Y}$), a candidate input $u \in \mathcal{U}$, (in for example Definition 12.0.6 (1)), must (uniformly) (necessarily staying inside $\mathfrak{H}(B)$) push the estimate set forward to a set, $S' = \mathcal{E}^\circ(\eta^{-1}(y) \cap S, u, -)$, which is necessarily, (i.e. what ever the observed output), *collapsible* in fewer inputs than the original.

Of course, this is a very demanding requirement, and it is certainly unreasonable to expect that, in general, one can find a non-trivial, *center observable* (centered) cover $\mathfrak{C}$, for a given FSM, $\mathcal{M} \equiv \langle \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle, \eta: \mathcal{X} \rightarrow \mathcal{Y} \rangle$.

In the following chapter, we re-address reachability, specifically in the case where the FSM is not differentiated, and for those cases when exact (block-wise) localization (that is, resulting in a singleton CSE set) is not possible.

CHAPTER 13

Coherent Reachability (or CSE-set Reachability)

13.1. Coherent Reachability

In the case when the FSM is either non-differentiated, or exact (mode-wise) localization is not possible, reachability and state-estimation can no longer be de-coupled.

DEFINITION 13.1.1. For any FSM, $\mathcal{M} \equiv \langle \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle, \eta: \mathcal{X} \rightarrow \mathcal{Y} \rangle$, let the *coherent reachability* predicate, $R^{Coh} \subseteq 2^{\mathcal{X}} \times 2^{2^{\mathcal{X}}} \times \mathcal{N}$ be defined by the sentence:

$$\forall S \subseteq \mathcal{X}, \forall \kappa \subseteq 2^{\mathcal{X}}, \forall n \in \mathcal{N}.$$

$$\left[\begin{array}{l} R^{Coh}(S, \kappa, n) \\ \Leftrightarrow \\ \left[\begin{array}{l} n = N_{\infty} \wedge \exists n' \in [0, n). R^{Coh}(S, \kappa, n') \\ \vee \\ n = 0 \wedge \exists S' \in \kappa. S \subseteq S' \\ \vee \\ \left[\begin{array}{l} n \in [1, N_{\infty}) \\ \wedge \\ \exists y \in \mathcal{Y}, \exists u \in \mathcal{U}. R^{Coh}(\mathcal{E}^{\circ}(\eta^{-1}(y) \cap S, u, -), \kappa, n-1) \\ \wedge \\ \forall y \in \mathcal{Y}. \left[\begin{array}{l} \exists n' \in [0, n). R^{Coh}(\eta^{-1}(y) \cap S, \kappa, n') \\ \vee \\ \exists u \in \mathcal{U}, \exists n' \in [0, n). R^{Coh}(\mathcal{E}^{\circ}(\eta^{-1}(y) \cap S, u, -), \kappa, n') \end{array} \right] \end{array} \right] \end{array} \right] \end{array} \right]$$

More generally, for any non-empty subset $B \subseteq \mathcal{X}$, let the *B-constrained coherent reachability* predicate, $R_B^{Coh} \subseteq 2^B \times 2^{2^B} \times \mathcal{N}$ be defined by the sentence:

$$\forall S \subseteq B, \forall \kappa \subseteq 2^B, \forall n \in \mathcal{N}.$$

$$\left[\begin{array}{c} R_B^{Coh}(S, \kappa, n) \\ \Leftrightarrow \\ \left[\begin{array}{c} n = N_\infty \wedge \exists n' \in [0, n). R_B^{Coh}(S, \kappa, n') \\ \vee \\ n = 0 \wedge \exists S' \in \kappa. S \subseteq S' \\ \vee \\ \left[\begin{array}{c} n \in [1, N_\infty) \\ \wedge \\ \exists y \in \mathcal{Y}, \exists u \in \mathcal{U}. R^{Coh}(\mathcal{E}^\circ(\eta^{-1}(y) \cap S, u, -), \kappa, n-1) \\ \wedge \\ \forall y \in \mathcal{Y}. \left[\begin{array}{c} \exists n' \in [0, n). R_B^{Coh}(\eta^{-1}(y) \cap S, \kappa, n') \\ \vee \\ \exists u \in \mathcal{U}. \\ \left[\begin{array}{c} \mathcal{E}^\circ(\eta^{-1}(y) \cap S, u, -) \subseteq B \\ \wedge \\ \exists n' \in [0, n). R_B^{Coh}(\mathcal{E}^\circ(\eta^{-1}(y) \cap S, u, -), \kappa, n') \end{array} \right] \end{array} \right] \end{array} \right] \end{array} \right] \end{array} \right]$$

OBSERVATION 13.1.1. For any FSM, $\mathcal{M} \equiv \langle \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle, \eta: \mathcal{X} \rightarrow \mathcal{Y} \rangle$.

$R^{Coh} \subseteq 2^{\mathcal{X}} \times 2^{2^{\mathcal{X}}} \times \mathcal{N}$ is simply $R_B^{Coh} \subseteq 2^B \times 2^{2^B} \times \mathcal{N}$ when $B \subseteq \mathcal{X}$ is taken to be $\mathcal{X}$.

That is:

$$\forall S \subseteq \mathcal{X}, \forall \kappa \subseteq 2^{\mathcal{X}}, \forall n \in \mathcal{N}. [R_{\mathcal{X}}^{Coh}(S, \kappa, n) \Leftrightarrow R^{Coh}(S, \kappa, n)].$$

OBSERVATION 13.1.2. For any FSM, $\mathcal{M} \equiv \langle \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle, \eta: \mathcal{X} \rightarrow \mathcal{Y} \rangle$, and any non-empty subset $B \subseteq \mathcal{X}$,

$$\forall S_1, S_2 \subseteq B, \forall \kappa_1, \kappa_2 \subseteq 2^B, \forall n \in \mathcal{N}.$$

$$[S_2 \subseteq S_1 \wedge \kappa_1 \subseteq \kappa_2 \Rightarrow [R_B^{Coh}(S_1, \kappa_1, n) \Rightarrow R_B^{Coh}(S_2, \kappa_2, n)]] .$$

The following notation is helpful.

DEFINITION 13.1.2. For any set X , and any pre-order¹ $\sqsubseteq \subseteq X \times X$, and any set $Y \subseteq X$,

let $\uparrow_{\sqsubseteq}(Y)$ denote

¹A pre-order is a reflexive and transitive binary relation, like a partial order, but a pre-order need not be anti-symmetric.

$$\{y \in Y \mid \forall x \in Y. y \sqsubseteq x \Rightarrow y = x\}$$

(that is, $\uparrow_{\sqsubseteq}(Y)$ denotes the collection of the $\sqsubseteq$ -maximal members of Y), and similarly,

let $\downarrow_{\sqsubseteq}(Y)$ denote

$$\{y \in Y \mid \forall x \in Y. x \sqsubseteq y \Rightarrow x = y\}$$

(that is, $\downarrow_{\sqsubseteq}(Y)$ denotes the collection of the $\sqsubseteq$ -minimal members of Y).

Let $\sqsubseteq_{2^X} \subseteq 2^X \times 2^X$ denote the relation defined by

$$\forall Y_1, Y_2 \subseteq X. [Y_1 \sqsubseteq_{2^X} Y_2 \Leftrightarrow [\forall x_1 \in Y_1, \exists x_2 \in Y_2. x_1 \sqsubseteq x_2]] .$$

OBSERVATION 13.1.3. For any set X , and any pre-order $\sqsubseteq \subseteq X \times X$, and any set $Y \subseteq X$,

both $\uparrow_{\sqsubseteq}(Y)$ and $\downarrow_{\sqsubseteq}(Y)$ are non-empty whenever Y is non-empty,

and $\sqsubseteq_{2^X} \subseteq 2^X \times 2^X$ is a pre-order.

OBSERVATION 13.1.4. For any FSM, $\mathcal{M} \equiv \langle \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle, \eta: \mathcal{X} \rightarrow \mathcal{Y} \rangle$, and any non-empty subset $B \subseteq \mathcal{X}$,

$$\forall S_1, S_2 \subseteq B, \forall \kappa_1, \kappa_2 \subseteq 2^B, \forall n \in \mathcal{N}.$$

$$[S_2 \subseteq S_1 \wedge \kappa_1 \sqsubseteq_{2^B} \kappa_2 \Rightarrow [R_B^{Coh}(S_1, \kappa_1, n) \Rightarrow R_B^{Coh}(S_2, \kappa_2, n)]] .$$

PROPOSITION 13.1.1. For any FSM, $\mathcal{M} \equiv \langle \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle, \eta: \mathcal{X} \rightarrow \mathcal{Y} \rangle$, and any non-empty subset $B \subseteq \mathcal{X}$,

$$\forall S_1 \subseteq B, \forall \kappa_1 \subseteq 2^B.$$

$$\left[\exists \kappa \subseteq 2^B. \left[\begin{array}{c} \exists n \in \mathcal{N}. R_B^{Coh}(S_1, \kappa, n) \\ \wedge \\ \forall S \in \kappa, \exists n \in \mathcal{N}. R_B^{Coh}(S, \kappa_1, n) \end{array} \right] \Rightarrow \exists n \in \mathcal{N}. R_B^{Coh}(S_1, \kappa_1, n) \right] .$$

DEFINITION 13.1.3. For any FSM, $\mathcal{M} \equiv \langle \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle, \eta: \mathcal{X} \rightarrow \mathcal{Y} \rangle$, and any centered subset $B \subseteq \mathcal{X}$, let $\mathfrak{H}^{Coh}(B)$, (the *coherent hull* of B) denote the set

$$\uparrow_{\sqsubseteq}(\{S \subseteq B \mid \exists n \in \mathcal{N}. R_B^{Coh}(S, \{C_B\}, n)\}) .$$

Conceptually, for any FSM, $\mathcal{M} \equiv \langle \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle, \eta: \mathcal{X} \rightarrow \mathcal{Y} \rangle$, and any centered subset $B \subseteq \mathcal{X}$, the coherent hull of B is simply the collection of those maximal subsets of B which are “robustly B -constrained coherently steerable into C_B ”.

OBSERVATION 13.1.5. For any FSM, $\mathcal{M} \equiv \langle \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle, \eta: \mathcal{X} \rightarrow \mathcal{Y} \rangle$, and any centered subset $B \subseteq \mathcal{X}$,

the coherent hull of B is a non-empty collection of non-empty subsets of B .

Further,

$$\exists S \in \mathfrak{H}^{Coh}(B). \emptyset \neq C_B \subseteq S \subseteq B,$$

(and in particular, if $C_B = B$, then $\mathfrak{H}^{Coh}(B) = \{B\}$).

Proof. For any centered subset $B \subseteq \mathcal{X}$, C_B is a non-empty subset of B such that

$$\exists n \in \mathcal{N}. R_B^{Coh}(C_B, \{C_B\}, n), \text{ (since } R_B^{Coh}(C_B, \{C_B\}, 0) \text{)}.$$

Thus, (by Observation 13.1.3),

$$\mathfrak{H}^{Coh}(B) \equiv \uparrow_{\subseteq} (\{S \subseteq B \mid \exists n \in \mathcal{N}. R_B^{Coh}(S, \{C_B\}, n)\}) \text{ is nonempty.}$$

Further, evidently $\forall S \in \mathfrak{H}^{Coh}(B)$, S is non-empty, (since $\emptyset \subsetneq C_B$).

and there exists $S \in \mathfrak{H}^{Coh}(B)$ such that $C_B \subseteq S$. ■

DEFINITION 13.1.4. For any FSM, $\mathcal{M} \equiv \langle \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle, \eta: \mathcal{X} \rightarrow \mathcal{Y} \rangle$, any centered subset $B \subseteq \mathcal{X}$.

(with center, C_B), and any $\kappa_0 \subseteq 2^{C_B}$, κ_0 , (a collection of subsets of the center, C_B).

is a *coherent center* of B if

$$\kappa_0 \in \downarrow_{\subseteq} \left(\downarrow_{\subseteq_{2^{C_B}}} (\{ \kappa \subseteq 2^{C_B} \mid \forall S \in \mathfrak{H}^{Coh}(B). \exists n \in \mathcal{N}. R_B^{Coh}(S, \kappa, n) \}) \right).$$

A centered subset $B \subseteq \mathcal{X}$, is *coherently centered* if B is equipped with a (designated) coherent center of B , which, (for any coherently centered B), will be denoted by C_B^{Coh} .

A *coherently centered cover* of $\mathcal{M}$ is simply a centered cover $\mathcal{C} \equiv \{B_1, \dots, B_K\}$ of $\mathcal{X}$.

where each centered block B_i is coherently centered.

OBSERVATION 13.1.6. For any FSM, $\mathcal{M} \equiv \langle \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle, \eta: \mathcal{X} \rightarrow \mathcal{Y} \rangle$, and any centered subset $B \subseteq \mathcal{X}$,

$$\begin{aligned} & \downarrow_{\subseteq} \left(\downarrow_{\subseteq_{2^{C_B}}} (\{ \kappa \subseteq 2^{C_B} \mid \forall S \in \mathfrak{H}^{Coh}(B). \exists n \in \mathcal{N}. R_B^{Coh}(S, \kappa, n) \}) \right) \\ & = \\ & \downarrow_{\subseteq_{2^{C_B}}} (\{ \uparrow_{\subseteq}(\kappa) \mid \kappa \subseteq 2^{C_B} \wedge \forall S \in \mathfrak{H}^{Coh}(B). \exists n \in \mathcal{N}. R_B^{Coh}(S, \kappa, n) \}) \end{aligned}$$

OBSERVATION 13.1.7. For any FSM, $\mathcal{M} \equiv \langle \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle, \eta: \mathcal{X} \rightarrow \mathcal{Y} \rangle$, and any centered subset $B \subseteq \mathcal{X}$,

a *coherent center* of B exists, and any coherent center is a non-empty collection of non-empty subsets of C_B . Further, (for any coherently centered B), $C_B^{Coh} \subseteq_{2^{C_B}} \{C_B\}$.

Proof. Since, (by the definition of $\mathfrak{H}^{Coh}(B)$),

$\{C_B\} \subseteq 2^{C_B}$ such that $\forall S \in \mathfrak{H}^{coh}(B), \exists n \in \mathcal{N}. R_B^{coh}(S, \{C_B\}, n)$,
 $\{C_B\} \in \{\kappa \subseteq 2^{C_B} \mid \forall S \in \mathfrak{H}^{coh}(B), \exists n \in \mathcal{N}. R_B^{coh}(S, \kappa, n)\}$.

Thus, (by two applications of Observation 13.1.3),

$$\downarrow \subseteq \left(\downarrow_{\subseteq_{2^{(\cdot)}}}, \left(\{\kappa \subseteq 2^{C_B} \mid \forall S \in \mathfrak{H}^{coh}(B), \exists n \in \mathcal{N}. R_B^{coh}(S, \kappa, n)\} \right) \right)$$

is non-empty, (so a coherent center of B exists).

Moreover, for any $\kappa \in \downarrow_{\subseteq_{2^{(\cdot)}}} \left(\{\kappa \subseteq 2^{C_B} \mid \forall S \in \mathfrak{H}^{coh}(B), \exists n \in \mathcal{N}. R_B^{coh}(S, \kappa, n)\} \right)$

$\kappa \subseteq_{2^{(\cdot)}} \{C_B\}$, (and so, in particular, $C_B^{coh} \subseteq_{2^{(\cdot)}} \{C_B\}$).

For any coherent center $\kappa \subseteq 2^{C_B} \forall S \in \mathfrak{H}^{coh}(B), \exists n \in \mathcal{N}. R_B^{coh}(S, \kappa, n)$.

By Observation 13.1.5, $\exists S \in \mathfrak{H}^{coh}(B). C_B \subseteq S$.

By Observation 13.1.2, therefore, $\exists n \in \mathcal{N}. R_B^{coh}(C_B, \kappa, n)$.

It follows that κ must contain at least one non-empty set, and so can not be empty, and further.

(by Observation 13.1.6, we must have $\kappa = \uparrow_{\subseteq}(\kappa)$ and so κ),

can not contain any empty members. ■

13.2. Coherent-In-Boundaries and CC Meta-Machines

DEFINITION 13.2.1. For any FSM, $\mathcal{M} \equiv \langle \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle, \eta: \mathcal{X} \rightarrow \mathcal{Y} \rangle$, and any cover, $\mathfrak{C} \equiv \{B_1, \dots, B_K\}$ of $\mathcal{X}$, for any centered block $B \in \mathfrak{C}$, let $\partial^{coh}(B)$, (the *coherent-in-boundary* of B), denote the set

$$\uparrow_{\subseteq} \left(\{ \alpha \subseteq \partial(B) \mid \exists n \in \mathcal{N}. R_B^{coh}(\alpha, \{C_B\}, n) \} \right).$$

For any collection $\mathcal{J} \subseteq \mathfrak{C}$, of centered blocks of $\mathfrak{C}$, let $\partial^{coh}(\mathcal{J})$ denote

$$\bigcup_{B \in \mathcal{J}} \partial^{coh}(B).$$

OBSERVATION 13.2.1. For any FSM, $\mathcal{M} \equiv \langle \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle, \eta: \mathcal{X} \rightarrow \mathcal{Y} \rangle$, and any cover, $\mathfrak{C} \equiv \{B_1, \dots, B_K\}$ of $\mathcal{X}$, for any coherently centered block $B \in \mathfrak{C}$, (with coherent center C_B^{coh}),

$$\{ \alpha \subseteq \partial(B) \mid \exists n \in \mathcal{N}. R_B^{coh}(\alpha, C_B^{coh}, n) \} = \{ \alpha \subseteq \partial(B) \mid \exists n \in \mathcal{N}. R_B^{coh}(\alpha, \{C_B\}, n) \}.$$

Proof. Let $LHS \equiv \{ \alpha \subseteq \partial(B) \mid \exists n \in \mathcal{N}. R_B^{coh}(\alpha, C_B^{coh}, n) \}$, and

$RHS \equiv \{ \alpha \subseteq \partial(B) \mid \exists n \in \mathcal{N}. R_B^{coh}(\alpha, \{C_B\}, n) \}$.

Let $\alpha \in LHS$. Then $\alpha \subseteq \partial(B)$ and $\exists n \in \mathcal{N}. R_B^{coh}(\alpha, C_B^{coh}, n)$.

Since, (by Observation 13.1.7), $\forall C \in C_B^{coh}, C_B^{coh} \subseteq_{2^{(\cdot)}} \{C_B\}$,

by Observation 13.1.4, $\exists n \in \mathcal{N}. R_B^{coh}(\alpha, \{C_B\}, n)$. Therefore, $\alpha \in RHS$.

Conversely, let $\alpha \in RHS$. Then $\alpha \subseteq \partial(B)$ and $\exists n \in \mathcal{N}. R_B^{coh}(\alpha, \{C_B\}, n)$.

But by Observation 13.1.5, $\exists S \in \mathfrak{H}^{coh}(B). C_B \subseteq S$.

Since $S \in \mathfrak{H}^{coh}(B)$, (and C_B^{coh} is a coherent center of B), $\exists n \in \mathcal{N}. R_B^{coh}(S, C_B^{coh}, n)$.

Therefore, (by Observation 13.1.2), $\exists n \in \mathcal{N} . R_B^{Coh}(C_B, C_B^{Coh}, n)$.

Thus, since $\exists n \in \mathcal{N} . R_B^{Coh}(\alpha, \{C_B\}, n)$, (by Proposition 13.1.1), $\exists n \in \mathcal{N} . R_B^{Coh}(\alpha, C_B^{Coh}, n)$, so $\alpha \in LHS$. ■

COROLLARY 13.2.1. For any FSM, $\mathcal{M} \equiv \langle \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle, \eta: \mathcal{X} \rightarrow \mathcal{Y} \rangle$, and any coherently centered subset $B \subseteq \mathcal{X}$, (with coherent center C_B^{Coh}),

$$\forall \alpha \subseteq \partial(B) .$$

$$\left[\begin{array}{c} \alpha \in \partial^{Coh}(B) \\ \Leftrightarrow \\ \alpha \in \uparrow_{\subseteq} (\{ \alpha \subseteq \partial(B) \mid \exists n \in \mathcal{N} . R_B^{Coh}(\alpha, C_B^{Coh}, n) \}) \end{array} \right] .$$

Proof. By Observation 13.2.1,

$$\{ \alpha \subseteq \partial(B) \mid \exists n \in \mathcal{N} . R_B^{Coh}(\alpha, C_B^{Coh}, n) \} = \{ \alpha \subseteq \partial(B) \mid \exists n \in \mathcal{N} . R_B^{Coh}(\alpha, \{C_B\}, n) \} .$$

Therefore,

$$\begin{aligned} & \uparrow_{\subseteq} (\{ \alpha \subseteq \partial(B) \mid \exists n \in \mathcal{N} . R_B^{Coh}(\alpha, C_B^{Coh}, n) \}) \\ &= \\ & \uparrow_{\subseteq} (\{ \alpha \subseteq \partial(B) \mid \exists n \in \mathcal{N} . R_B^{Coh}(\alpha, \{C_B\}, n) \}) \\ &= \\ & \partial^{Coh}(B) \end{aligned}$$
■

DEFINITION 13.2.2. The Meta-Machine: Coherence Case

Let $\mathcal{C} \equiv \{B_1, \dots, B_K\}$ be a coherently centered cover of an FSM, $\mathcal{M} \equiv \langle \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle, \eta: \mathcal{X} \rightarrow \mathcal{Y} \rangle$. For any blocks $B_1, B_2 \in \mathcal{C}$, there is an input α_β labelling a directed edge from B_1 to B_2 in the (non-deterministic) *CC meta-machine* whenever $B_2 \in \beta$, and $\beta \subseteq \mathcal{C}$ is a minimal subset of $\mathcal{C}$ such that $\forall C \in C_{B_1}^{Coh}, \exists n \in \mathcal{N} . R_{B_1 \cup \bigcup \partial^{Coh}(\beta)}^{Coh}(C, \partial^{Coh}(\beta), n)$.

We now return to the full observation case to examine the non-deterministic analogues of *in-block controllable* [13, 42, 41, 43] partitions, and the *partition lattice* [13, 42, 41, 43] results for deterministic finite state machines.

CHAPTER 14

Controllability of the FOC Meta-Machine and Quasi-Convexity

14.1. Out-Boundaries and Betweenness

Suppose we have a (non-deterministic) FSM, $\mathcal{M}$ with LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, and a non-empty collection of non-empty subsets $\mathcal{C} \subseteq 2^{\mathcal{X}}$, such that $\forall C, C' \in \mathcal{C}, \forall s \in C, \exists k \geq 0, R(s, C', k)$, and we want a sufficient condition that, for some centered cover $\mathfrak{B} \subseteq 2^{\mathcal{X}}$, such that $\forall B \in \mathfrak{B}, C_B \in \mathcal{C}$, the FOC meta-machine, (with LTS $\mathcal{G}^{\mathfrak{B}} \equiv \langle \mathfrak{B}, \mathcal{U}^{\mathfrak{B}}, \mathcal{E}^{\mathfrak{B}} \rangle$), is controllable, that is, $\forall B, B' \in \mathfrak{B}, \exists k \geq 0, R^{FOC(\mathfrak{B})}(B, \{B'\}, k)$, where $R^{FOC(\mathfrak{B})} \subseteq \mathfrak{B} \times 2^{\mathfrak{B}} \times \mathcal{N}$ is the winning reachability predicate (of the LTS, $\mathcal{G}^{\mathfrak{B}}$), of the (FOC) meta-machine.

In other words, we have a collection, $\mathcal{C}$, of candidate centers of a given FSM, $\mathcal{M}$, such that $\mathcal{M}$ is center-to-center (winning) controllable, and we would like to know, for a given centered cover $\mathfrak{B} \subseteq 2^{\mathcal{X}}$, of $\mathcal{M}$, such that $\forall B \in \mathfrak{B}, C_B \in \mathcal{C}$, whether the corresponding (FOC) meta-machine, $\mathcal{M}^{FOC(\mathfrak{B})}$, is controllable, without actually testing the reachability of the constructed meta-machine.

We work up to a theorem which gives a weak but sufficient condition on the centered cover $\mathfrak{B}$ of an FSM, $\mathcal{M}$, for the controllability of the meta-machine, $\mathcal{M}^{FOC(\mathfrak{B})}$.

DEFINITION 14.1.1. For any FSM $\mathcal{M}$ with LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, for any $s \in \mathcal{X}$, and any $S \subseteq \mathcal{X}$,

1. let $\mu R(s, S, -)$ denote the least $n \in \mathcal{N}$ such that $R(s, S, n)$, if any such $n \in \mathcal{N}$ exists, otherwise, let $\mu R(s, S, -)$ denote N_{∞} ,
2. let $fwm_{\mathcal{G}}(s, S)$ denote the set

$$\{a \in \mathcal{U}(s, -) \mid \forall s' \in \mathcal{E}(s, a, -), \exists k' \in [0, \mu R(s, S, -)) . R(s', S, k')\} .$$

(In other words, $fwm_{\mathcal{G}}(s, S)$ denotes the set of first winning moves/inputs for steering s into S). and

3. for any non-empty $S' \subseteq \mathcal{X}$,
let $\mu R(S', S, -)$ denote the least $n \in \mathcal{N}$ such that $\forall s' \in S', \mu R(s', S, -) \leq n$.
That is, $\mu R(S', S, -) = \max\{\mu R(s', S, -) \mid s' \in S'\}$.

OBSERVATION 14.1.1. For any centered cover $\mathfrak{B}$ of a FSM with LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, for any $B \in \mathfrak{B}$, $B' \in \mathfrak{B} \setminus \{B\}$, and any $s \in B'$,

$$\exists k \in \mathcal{N} . R(s, C_B, k) \Leftrightarrow \exists k \in \mathcal{N} . R(s, \partial(B), k)$$

DEFINITION 14.1.2. For any centered block $B \subseteq \mathcal{X}$, of an (FSM with) LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, let the *simple-out-boundary* of B be denoted by $\partial_{out}^0(B)$, and be defined to be the set

$$\{s \in B \mid \exists u \in \mathcal{U}, \exists s' \in \mathcal{X} \setminus B . \mathcal{E}(s, u, s')\} = B \cap \mathcal{X}(-, \mathcal{X} \setminus B) .$$

DEFINITION 14.1.3. For any centered block $B \subseteq \mathcal{X}$, of an (FSM with) LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, let the *external-simple-out-boundary* of B be denoted by $\partial_{out}^0(B)$, and be defined to be the set

$$\{s \in \mathcal{X} \setminus B \mid \exists u \in \mathcal{U}, \exists s' \in B . \mathcal{E}(s', u, s)\} = \mathcal{X}(B) \setminus B .$$

OBSERVATION 14.1.2. Evidently, for any centered block $B \subseteq \mathcal{X}$, of an (FSM with) LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, and any $s \in \mathcal{X}$,

$$s \in \partial_{out}^0(B) \Leftrightarrow [s \notin B \wedge \exists u \in \mathcal{U}, \exists s' \in \partial_{out}^0(B) . \mathcal{E}(s', u, s)] .$$

DEFINITION 14.1.4. For any LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, and $B \subseteq \mathcal{X}$, for any $s \in B$, any $S \subseteq B$, and any $S' \subseteq \mathcal{X}$,

- 1) let $WR_B^{\mathcal{G}}(s, S')$, or simply $WR_B(s, S')$, denote $\exists n \in \mathcal{N} . R_{B \cup S'}(s, S', n)$,
- 2) let $WR_B^{\mathcal{G}}(S, S')$, or simply $WR_B(S, S')$, denote $\forall s \in S . WR_B^{\mathcal{G}}(s, S')$, and
- 3) let $WR_B^{\mathcal{G}}(-, S')$, or simply $WR_B(-, S')$, denote the collection $\{s \in \mathcal{X} \mid WR_B^{\mathcal{G}}(s, S')\}$.

When $B = \mathcal{X}$, we will write $WR^{\mathcal{G}}(\cdot, \cdot)$, or simply $WR(\cdot, \cdot)$, (instead of $WR_{\mathcal{X}}^{\mathcal{G}}(\cdot, \cdot)$, or $WR_{\mathcal{X}}(\cdot, \cdot)$, respectively).

DEFINITION 14.1.5. For any LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, for any $s \in \mathcal{X}$, any $S \subseteq \mathcal{X}$, let $SBTWN(s, S)$ denote the least set $X \subseteq 2^{\mathcal{X}}$ such that:

- 1) if $WR(s, S)$, and $s \notin S$, then for any $u \in fwm(s, S)$, $\mathcal{E}(s, u, -) \in X$, and
- 2) for any $S' \in X$, for any $s' \in S'$,
if $s' \notin S$, then for any $u \in fwm(s', S)$, $\mathcal{E}(s', u, -) \cup S' \setminus \{s'\} \in X$.

For any $s \in \mathcal{X}$ any $S \subseteq \mathcal{X}$, and any $S' \subseteq \mathcal{X}$,

S' is *strictly between* s and S if $S' \in SBTWN(s, S)$.

OBSERVATION 14.1.3. For any LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, for any $s \in \mathcal{X}$, and any $S \subseteq \mathcal{X}$,

1. $SBTWN(s, S) \neq \emptyset \Leftrightarrow WR(s, S)$,
2. $\forall S' \subseteq \mathcal{X}. [S' \in SBTWN(s, S) \Rightarrow [WR(s, S) \Leftrightarrow WR(s, S') \wedge WR(S', S)]]$,
3. $\forall S' \in SBTWN(s, S). [S' \neq \emptyset \wedge \mu R(s, S', -) \leq \mu R(s, S, -) \wedge \mu R(S', S, -) < \mu R(s, S, -)]$,
and
4. $WR(s, S) \Leftrightarrow \left[s \in S \vee \exists S' \subseteq \mathcal{X}. \left[WR(s, S') \wedge WR(S', S) \wedge S' \in SBTWN(s, S) \right] \right]$

OBSERVATION 14.1.4. For any block $B \in \mathfrak{B}$, of a centered cover $\mathfrak{B}$ of a FSM with LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, for any $s \in B$, and any $B' \in \mathfrak{B} \setminus \{B\}$,

if $C_{B'} \cap B = \emptyset$, then

$$WR(s, C_{B'}) \Leftrightarrow \exists S \subseteq \mathfrak{D}_{out}^0(B). [WR_B(s, S) \wedge WR(S, C_{B'}) \wedge S \in SBTWN(s, C_{B'})] ,$$

and otherwise,

either $s \in C_{B'}$ or

$$WR(s, C_{B'}) \Leftrightarrow \exists S \subseteq \mathfrak{D}_{out}^0(B) \cup \partial(B'). [WR_B(s, S) \wedge WR(S, C_{B'}) \wedge S \in SBTWN(s, C_{B'})] .$$

14.2. Still More Boundaries

DEFINITION 14.2.1. For any block $B \in \mathfrak{B}$, of a centered cover $\mathfrak{B}$ of a FSM with LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, let the *weak-in-boundary* of B be denoted by $\partial_{in}^w(B)$, and be defined to be the set

$$\left\{ s \in B \left| \begin{array}{l} \exists B' \in \mathfrak{B} \setminus \{B\}. s \in \mathfrak{D}_{out}^0(B') \\ \wedge \\ WR_B(s, \mathfrak{D}_{out}^0(B)) \\ \vee \\ \exists B' \in \mathfrak{B}. \left[\begin{array}{l} C_{B'} \cap B \neq \emptyset \\ \wedge \\ WR(s, C_{B'}) \end{array} \right] \end{array} \right. \right\} = \left(\begin{array}{l} B \cap \bigcup_{B' \in \mathfrak{B} \setminus \{B\}} \mathfrak{D}_{out}^0(B') \\ \cap \\ WR_B(-, \mathfrak{D}_{out}^0(B)) \\ \cup \\ \bigcup_{\substack{B' \in \mathfrak{B} \setminus \{B\} \\ C_{B'} \cap B \neq \emptyset}} WR_B(-, C_{B'}) \end{array} \right) .$$

DEFINITION 14.2.2. For any block $B \in \mathfrak{B}$, of a centered cover $\mathfrak{B}$ of a FSM with LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, let the *external-strong-out-boundary* of B be denoted by $\mathfrak{D}_{out}^s(B)$, and be defined to be the set

$$\downarrow \subseteq \left(\begin{array}{c} \{ S \subseteq \partial_{out}^0(B) \mid \exists s \in C_B \cup \partial_{in}^w(B). WR_B(s, S) \} \\ \cup \\ \left\{ \begin{array}{l} S \subseteq \partial_{out}^0(B) \\ \wedge \\ B' \in \mathfrak{B} \setminus \{B\} \\ \wedge \\ C_{B'} \cap B \neq \emptyset \\ \wedge \\ S \cup \partial(B') \mid \exists s \in C_B \cup \partial_{in}^w(B). \\ \left[\begin{array}{l} s \in C_{B'} \\ \vee \\ \left[\begin{array}{l} WR(s, C_{B'}) \\ \wedge \\ \exists S' \subseteq \partial(B'). \\ [WR_B(s, S \cup S') \wedge S \cup S' \in SBWT.N(s, C_{B'})] \end{array} \right] \end{array} \right] \end{array} \right\} \end{array} \right) .$$

OBSERVATION 14.2.1. For any block $B \in \mathfrak{B}$, of a centered cover $\mathfrak{B}$ of a FSM with LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, for any $s \in C_B \cup \partial_{in}^w(B)$, and any $B' \in \mathfrak{B} \setminus \{B\}$.

$$WR(s, C_{B'})$$

$$\Leftrightarrow$$

$$\exists S' \subseteq \mathcal{X}. \exists S \in \partial_{out}^s(B).$$

$$\left[\begin{array}{l} S \subseteq S' \cup \partial(B') \wedge WR_B(s, S') \wedge WR(S', C_{B'}) \\ \wedge \\ [s \in C_{B'} \wedge S' \subseteq \partial(B')] \vee S' \in SBTWN(s, C_{B'}) \\ \wedge \\ \forall s' \in S' \setminus \partial(B'), \forall B'' \in \mathfrak{B} \setminus \{B\}. [s' \in B'' \Rightarrow s' \in \partial_{in}^w(B'')] \end{array} \right] .$$

Proof. Let $s \in C_B \cup \partial_{in}^w(B)$, for some block $B \in \mathfrak{B}$, of a centered cover $\mathfrak{B}$ of a FSM with LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, and let $B' \in \mathfrak{B} \setminus \{B\}$.

($\Leftarrow$): If $\exists S' \subseteq \mathcal{X}. [WR_B(s, S') \wedge WR(S', C_{B'})]$,

then clearly, $\exists S' \subseteq \mathcal{X} \cdot [WR(s, S') \wedge WR(S', C_{B'})]$, and so $WR(s, C_{B'})$.

($\Rightarrow$): Conversely, suppose $WR(s, C_{B'})$.

There are two cases to consider. Either:

1. $B \cap C_{B'} = \emptyset$, or
2. $B \cap C_{B'} \neq \emptyset$.

In Case 1, evidently $s \notin C_{B'}$, and we can pursue first-winning moves, steering s to $C_{B'}$, until we reach a set $S' \subseteq \mathfrak{D}_{out}^0(B)$, such that $WR_B(s, S')$, $WR(S', C_{B'})$, and $S' \in SBTWN(s, C_{B'})$.

Further, evidently, for any $s' \in S'$, (and so, in particular, for any $s' \in S' \setminus \partial(B')$), and any $B'' \in \mathfrak{B} \setminus \{B\}$ such that $s' \in B''$, since $WR(s', C_{B'})$, either $WR_{B''}(s', \mathfrak{D}_{out}^0(B''))$, or $C_{B'} \cap B'' \neq \emptyset$, (and, $WR(s', C_{B'})$), and thus $s' \in \partial_{in}^w(B'')$.

Finally, since by assumption, $s \in C_B \cup \partial_{in}^w(B)$, evidently there exists some $S \in \mathfrak{D}_{out}^s(B)$ such that $S \subseteq S' \cup \partial(B')$, as required.

In case 2, if $s \in C_{B'}$, then (since $s \in C_B \cup \partial_{in}^w(B)$),

evidently there exists some $S \in \mathfrak{D}_{out}^s(B)$ such that $S \subseteq \partial(B')$.

Taking $S' \equiv B \cap C_{B'}$, evidently $s \in S'$, and $S' \subseteq \partial(B')$, and so $S \subseteq S' \cup \partial(B')$, $WR_B(s, S')$, and $WR(S', C_{B'})$. Further, since $S' \subseteq \partial(B')$, the condition:

$$\forall s' \in S' \setminus \partial(B'), \forall B'' \in \mathfrak{B} \setminus \{B\}, s' \in B'' \Rightarrow s' \in \partial_{in}^w(B'')$$

is vacuously true.

Otherwise, (if $s \notin C_{B'}$), again evidently we can pursue first-winning moves, steering s to $C_{B'}$, until we reach a set $S' \subseteq \mathfrak{D}_{out}^0(B) \cup \partial(B')$, such that $WR_B(s, S')$, $WR(S', C_{B'})$, and $S' \in SBTWN(s, C_{B'})$, and where for any $s' \in S' \setminus \partial(B')$, and any $B'' \in \mathfrak{B} \setminus \{B\}$ such that $s' \in B''$, $s' \in \partial_{in}^w(B'')$.

For if $s \notin \partial(B')$, it is easy to see that such an S' exists, (again, by pursuing first-winning moves, steering s to $C_{B'}$, we must eventually arrive in such a set S').

Otherwise, if $s \in \partial(B')$, then we may take $S' \equiv \mathcal{E}(s, u, -)$, for some $u \in fwm_{B'}(s, C_{B'})$, that is, where u is some first-winning move for B' -constrained steering s to $C_{B'}$.

(which must exist since $s \in \partial(B')$). Then evidently, for any $s' \in S' \equiv \mathcal{E}(s, u, -)$,

either $s' \in \mathfrak{D}_{out}^0(B)$, (and for any $B'' \in \mathfrak{B} \setminus \{B\}$ such that $s' \in B''$, $s' \in \partial_{in}^w(B'')$),

or ($s' \in B \cap \mathfrak{H}(B')$, and so), $s' \in \partial(B')$. Thus, in this case, again, $S' \subseteq \mathfrak{D}_{out}^0(B) \cup \partial(B')$,

and evidently $WR_B(s, S')$, $WR(S', C_{B'})$, and $S' \in SBTWN(s, C_{B'})$.

Since $S' \subseteq \mathfrak{D}_{out}^0(B) \cup \partial(B')$, such that $WR_B(s, S')$, $WR(S', C_{B'})$, and $S' \in SBTWN(s, C_{B'})$, evidently, taking $S_1 \equiv S' \cap \mathfrak{D}_{out}^0(B)$, and $S_2 \equiv S' \cap \partial(B')$, we have $S_1 \subseteq \mathfrak{D}_{out}^0(B)$, $S_2 \subseteq \partial(B')$, $WR_B(s, S_1 \cup S_2)$, and $S_1 \cup S_2 \in SBTWN(s, C_{B'})$, and since by assumption, $s \in C_B \cup \partial_{in}^w(B)$ and $WR(s, C_{B'})$, again, evidently there exists some $S \in \mathfrak{D}_{out}^s(B)$ such that $S \subseteq S' \cup \partial(B') = S_1 \cup \partial(B')$, as required. ■

14.3. Weak-Quasi-Convexity

DEFINITION 14.3.1. A block B of centered cover $\mathfrak{B}$, of an (FSM with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, is *weakly-quasi-convex* if:

- 1) $\partial_{in}^w(B) \subseteq \mathfrak{H}(B)$, and
- 2) $\forall S \in \mathfrak{S}_{out}^*(B) . WR_B(C_B, S)$.

A centered cover $\mathfrak{B}$, of an (FSM with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, is *weakly-quasi-convex* if for every B in $\mathfrak{B}$, B is weakly-quasi-convex.

OBSERVATION 14.3.1. For any weakly-quasi-convex block $B \in \mathfrak{B}$, of a centered cover $\mathfrak{B}$ of a FSM with LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, for any block $B' \in \mathfrak{B} \setminus \{B\}$ and any $s \in C_B \cup \partial_{in}^w(B)$,

$$\begin{aligned} & WR(s, C_{B'}) \\ \Rightarrow \\ & \exists S \in \mathfrak{S}_{out}^*(B) . \\ & \left[\begin{array}{l} WR_B(C_B, S) \wedge WR(S, C_{B'}) \\ \wedge \\ [s \in C_{B'} \wedge S \subseteq \partial(B')] \vee \forall s' \in S \setminus \partial(B') . \mu R(s', C_{B'}, -) < \mu R(s, C_{B'}, -) \\ \wedge \\ \forall s' \in S \setminus \partial(B') , \forall B'' \in \mathfrak{B} \setminus \{B\} . [s' \in B'' \Rightarrow s' \in \partial_{in}^w(B'')] \end{array} \right] . \end{aligned}$$

Proof. Let $s \in C_B \cup \partial_{in}^w(B)$, for some weakly-quasi-convex block $B \in \mathfrak{B}$, of a centered cover $\mathfrak{B}$ of a FSM with LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, let $B' \in \mathfrak{B} \setminus \{B\}$, and suppose $WR(s, C_{B'})$.

Then by Observation 14.2.1, there some $S \in \mathfrak{S}_{out}^*(B)$, such that for some $S' \subseteq \mathcal{X}$,

$$\begin{aligned} & S \subseteq S' \cup \partial(B') \wedge WR_B(s, S') \wedge WR(S', C_{B'}) \\ & \wedge \\ & [s \in C_{B'} \wedge S' \subseteq \partial(B')] \vee S' \in SBTWN(s, C_{B'}) \\ & \wedge \\ & \forall s' \in S' \setminus \partial(B') , \forall B'' \in \mathfrak{B} \setminus \{B\} . [s' \in B'' \Rightarrow s' \in \partial_{in}^w(B'')] \end{aligned}$$

Since B is weakly-quasi-convex, and $S \in \mathfrak{S}_{out}^*(B)$, it is immediate that $WR_B(C_B, S)$.

Since $WR(S', C_{B'})$, and $S \subseteq S' \cup \partial(B')$, (and certainly $WR(\partial(B'), C_{B'})$), $WR(S, C_{B'})$ follows directly.

If $s \in C_{B'}$, then since $S' \subseteq \partial(B')$, and $S \subseteq S' \cup \partial(B')$, evidently, $S \subseteq \partial(B')$.

Alternatively, if $s \notin C_{B'}$, then, since $S' \in SBTWN(s, C_{B'})$, $\forall s' \in S' . \mu R(s', C_{B'}, -) < \mu R(s, C_{B'}, -)$, and, again since $S \subseteq S' \cup \partial(B')$, clearly it follows that $\forall s' \in S \setminus \partial(B') . \mu R(s', C_{B'}, -) < \mu R(s, C_{B'}, -)$.

Finally, since $\forall s' \in S' \setminus \partial(B'), \forall B'' \in \mathfrak{B} \setminus \{B\} . [s' \in B'' \Rightarrow s' \in \partial_{in}^w(B'')]$,
 and, again, since $S \subseteq S' \cup \partial(B')$, clearly it follows that
 $\forall s' \in S \setminus \partial(B'), \forall B'' \in \mathfrak{B} \setminus \{B\} . [s' \in B'' \Rightarrow s' \in \partial_{in}^w(B'')]$. ■

THEOREM 14.3.1. Suppose $\mathfrak{B}$, is a weakly-quasi-convex centered cover of a FSM with LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$. Then

$$\forall B, B' \in \mathfrak{B}, \forall s \in C_B \cup \partial_{in}^w(B) . [\exists k \in \mathcal{N} . R(s, C_{B'}, k) \Rightarrow \exists k \in \mathcal{N} . R^{FOC(\mathfrak{B})}(B, \{B'\}, k)]$$

Proof. We prove the result by induction on $\mu R(s, C_{B'}, -)$.

Let $B, B' \in \mathfrak{B}$ and $s \in C_B \cup \partial_{in}^w(B)$, and suppose $\exists k \in \mathcal{N} . R(s, C_{B'}, k)$.

If $B = B'$, then trivially, $\exists k \in \mathcal{N} . R^{FOC(\mathfrak{B})}(B, \{B'\}, k)$. ($k = 0$), and so we may take $B \neq B'$.

By Observation 14.3.1,

$$\exists S \in \mathfrak{S}_{out}^s(B) . \left[\begin{array}{l} WR_B(C_B, S) \wedge WR(S, C_{B'}) \\ \wedge \\ [s \in C_{B'} \wedge S \subseteq \partial(B')] \vee \forall s' \in S \setminus \partial(B') . \mu R(s', C_{B'}, -) < \mu R(s, C_{B'}, -) \\ \wedge \\ \forall s' \in S \setminus \partial(B') . \forall B'' \in \mathfrak{B} \setminus \{B\} . [s' \in B'' \Rightarrow s' \in \partial_{in}^w(B'')] \end{array} \right] .$$

If $\mu R(s, C_{B'}, -) = 0$, then $s \in C_{B'}$, and so $S \subseteq \partial(B')$. Since $WR_B(C_B, S)$, and $S \subseteq \partial(B')$,
 evidently ($\forall s \in C_B . \exists k \in \mathcal{N} . R_B(s, \partial(B'), k)$, and),

$\mathcal{J} \equiv \{B'\}$ is a minimal subset of $\mathfrak{B}$ such that $\forall s \in C_B, \exists n \in \mathcal{N} . R_{B \cup \partial(\mathcal{J})}(s, \partial(\mathcal{J}), n)$.

and so there is an input $\alpha_{\{B'\}}$ from B to B' , and to *only* B' , in the FOC meta-machine.

Thus $\exists k \in \mathcal{N} . R^{FOC(\mathfrak{B})}(B, \{B'\}, k)$, ($k = 1$).

Now suppose the result for all $B, B' \in \mathfrak{B}$ and $s \in B$ such that

(either $\nexists k \in \mathcal{N} . R(s, C_{B'}, k)$, or) $0 \leq \mu R(s, C_{B'}, -) < N < N_\infty$ (IH),

and that now $B, B' \in \mathfrak{B}$, distinct, $s \in C_B \cup \partial_{in}^w(B)$, and $\mu R(s, C_{B'}, -) = N$.

Since $\mu R(s, C_{B'}, -) = N > 0$, $s \notin C_{B'}$, so $\forall s' \in S \setminus \partial(B') . \mu R(s', C_{B'}, -) < \mu R(s, C_{B'}, -) = N$,
 and for any $B'' \in \mathfrak{B} \setminus \{B\} . [s' \in B'' \Rightarrow s' \in \partial_{in}^w(B'')]$.

Thus, for any $s' \in S \setminus \partial(B')$, for any $B'' \in \mathfrak{B} \setminus \{B\}$ such that $s' \in B''$, (since B'' is weakly-quasi-convex), $s' \in \mathfrak{H}(B'')$, and by (IH) there exists $k' \in \mathcal{N} . R^{FOC(\mathfrak{B})}(B'', \{B'\}, k')$.

For any $s' \in S \setminus \partial(B')$, let $B(s') \in \mathfrak{B}$ be some block such that $s' \in B(s')$, and let

$$\beta' \equiv \{B(s') \mid s' \in S \setminus \partial(B')\} \cup \{B'\} ,$$

if there exists some $s' \in S \cap \partial(B')$, and otherwise, let

$$\beta' \equiv \{B(s') \mid s' \in S\} ,$$

Then since $WR_B(C_B, S)$, evidently $WR_B(C_B, \partial(\beta))$, and thus there exists some $\beta \subseteq \beta' \subseteq \mathfrak{B}$, such that $B \xrightarrow{\alpha_\beta} \beta$ is a directed edge of $\mathcal{M}^{FOC(\mathfrak{B})}$, (that is, such that the input α_β takes B to a $B'' \in \mathfrak{B}$ exactly when $B'' \in \beta$). Since $\forall B'' \in \beta, \exists k' \in \mathcal{N}. R^{FOC(\mathfrak{B})}(B'', \{B'\}, k')$, (and since $R^{FOC(\mathfrak{B})}(B, \beta, 1)$), evidently $\exists k \in \mathcal{N}. R^{FOC(\mathfrak{B})}(B, \{B'\}, k)$. ■

THEOREM 14.3.2. Suppose $\mathfrak{B}$, is a centered cover of a FSM with LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$. Then

$$\forall B, B' \in \mathfrak{B}. [\exists k \in \mathcal{N}. R^{FOC(\mathfrak{B})}(B, \{B'\}, k) \Rightarrow \forall s \in C_B, \exists k \in \mathcal{N}. R(s, C_{B'}, k)]$$

Proof. We prove the result by induction on $\mu R^{FOC(\mathfrak{B})}(B, \{B'\}, -)$.

If $\mu R^{FOC(\mathfrak{B})}(B, \{B'\}, -) = 0$, then $B = B'$, so $C_B = C_{B'}$.

Thus, trivially, $\forall s \in C_B, \exists k \in \mathcal{N}. R(s, C_{B'}, k)$ ($k = 0$).

Now suppose the result for all $B, B' \in \mathfrak{B}$, such that (either $\mu R^{FOC(\mathfrak{B})}(B, \{B'\}, k) > 0$, or $0 \leq \mu R^{FOC(\mathfrak{B})}(B, \{B'\}, -) < N$ (IH), and that now $B, B' \in \mathfrak{B}$, and $\mu R^{FOC(\mathfrak{B})}(B, \{B'\}, -) = N$).

By the definition of (constrained) winning reachability predicate,

since $\mu R^{FOC(\mathfrak{B})}(B, \{B'\}, -) = N > 0$, $B \neq B'$, and (since $R^{FOC(\mathfrak{B})}(B, \{B'\}, \mathcal{N})$), there exists $\beta \subseteq \mathfrak{B}$ and some a FOC meta-machine input $\alpha_\beta \in \text{fwm}_{\mathcal{G}\mathfrak{B}}(B, \{B'\}) \subseteq \mathcal{U}^{\mathfrak{B}}(B, -) \subseteq \mathcal{E}^{\mathfrak{B}}$ such that $\mathcal{E}^{\mathfrak{B}}(B, \alpha_\beta, -)$ is non-empty, and $\forall B'' \in \mathcal{E}^{\mathfrak{B}}(B, \alpha_\beta, -), \exists k' \in [0, N). R^{FOC(\mathfrak{B})}(B'', \{B'\}, k')$.

Thus $\forall B'' \in \mathcal{E}^{\mathfrak{B}}(B, \alpha_\beta, -), \mu R^{FOC(\mathfrak{B})}(B'', \{B'\}, -) < N$, and so by (IH),

$\forall B'' \in \mathcal{E}^{\mathfrak{B}}(B, \alpha_\beta, -), \forall s' \in C_{B''}, \exists k' \in \mathcal{N}. R(s', C_{B'}, k')$.

But since $\alpha_\beta \in \mathcal{U}^{\mathfrak{B}}(B, -)$, by definition of the (LTS of the) FOC meta-machine, $\mathcal{E}^{\mathfrak{B}}(B, \alpha_\beta, -) = \beta$, and $\forall s \in C_B, \exists k \in \mathcal{N}. R_{B \cup \partial(\beta)}(s, \partial(\beta), k)$.

Thus,

(since for any $\beta \subseteq \mathfrak{B}$, for any $s'' \in \partial(\beta)$, it must be the case that for some $B'' \in \beta, s'' \in \partial(B'')$, and so $\exists k \in \mathcal{N}. R_{B''}(s'', C_{B''}, k)$),

$\forall s \in C_B, \exists k \in \mathcal{N}. R(s, C(\beta), k)$, where $C(\beta)$ denotes $\bigcup_{B'' \in \beta} C_{B''}$.

Evidently, (since $\forall B'' \in \beta, \forall s' \in C_{B''}, \exists k' \in \mathcal{N}. R(s', C_{B'}, k')$),

it follows that $\forall s \in C_B, \exists k \in \mathcal{N}. R(s, C_{B'}, k)$, as required. ■

COROLLARY 14.3.1. Suppose $\mathfrak{B}$, is a weakly-quasi-convex centered cover of a FSM with LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$. Then

$$\forall B, B' \in \mathfrak{B}, [WR(C_B, C_{B'}) \Leftrightarrow \exists k \in \mathcal{N}. R^{FOC(\mathfrak{B})}(B, \{B'\}, k)].$$

Proof. Let $B, B' \in \mathfrak{B}$.

Suppose $WR(C_B, C_{B'})$. Let $s \in C_B$. Then since $WR(C_B, C_{B'})$, $\exists k \in \mathcal{N}. R(s, C_{B'}, k)$,

and so by Theorem 14.3.1, $\exists k \in \mathcal{N}. R^{FOC(\mathfrak{B})}(B, \{B'\}, k)$.

Conversely, suppose that $\exists k \in \mathcal{N}. R^{FOC(\mathfrak{B})}(B, \{B'\}, k)$, and let $s \in C_B$.

Then by Theorem 14.3.2 $\exists k \in \mathcal{N} . R(s, C_{B'}, k)$, and so $WR(C_B, C_{B'})$. ■

COROLLARY 14.3.2. Suppose $\mathfrak{B}$, is a weakly-quasi-convex centered cover of a FSM, $\mathcal{M}$, with LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$. Then

$$\forall B, B' \in \mathfrak{B} . WR(C_B, C_{B'}) \Leftrightarrow \forall B, B' \in \mathfrak{B} , \exists k \in \mathcal{N} . R^{FOC(\mathfrak{B})}(B, \{B'\}, k) .$$

That is the FOC meta-machine $\mathcal{M}^{\mathfrak{B}}$ is controllable if and only if $\mathcal{M}$ is center-to-center controllable.

14.4. Quasi-Convex Convers

We now formulate a restricted class of the weakly-quasi-convex covers, the *quasi-convex* covers, for which there will be an associated lattice.

DEFINITION 14.4.1. Given an (FSM $\mathcal{M}$ with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, a subset $S \subseteq \mathcal{X}$ is *stable* if

$$\forall s \in S . \exists u \in \mathcal{U}(s, -) . \mathcal{E}(s, u, -) \subseteq S .$$

DEFINITION 14.4.2. For any block $B \in \mathfrak{B}$, of a centered cover $\mathfrak{B}$ of an (FSM with), LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, let the *strong-center-out-boundary* of B be denoted by $\partial_{center-out}^s(B)$, and be defined to be the set

$$\downarrow \subseteq (\{ S \subseteq (B \setminus C_B) \cup \partial_{out}^0(B) \mid \exists s \in C_B . WR_{C_B}(s, S) \}) .$$

OBSERVATION 14.4.1. For any block $B \in \mathfrak{B}$, of a centered cover $\mathfrak{B}$ of a FSM with LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, for any $S \subseteq \partial_{out}^0(B)$,

$$\exists s \in B . WR_B(s, S) \Leftrightarrow \exists s \in \partial_{out}^0(B) . WR_B(s, S) .$$

Proof. Since $\partial_{out}^0(B) \subseteq B$, ($\Leftarrow$) is trivial.

Conversely, suppose $s \in B$, $S \subseteq \partial_{out}^0(B)$, and $WR_B(s, S)$.

If $s \in \partial_{out}^0(B)$, the result is immediate.

Otherwise, since $S \subseteq \partial_{out}^0(B)$, $S \cap B = \emptyset$, while $s \in B$, and $WR_B(s, S)$. Thus evidently there exists a non-empty set $S' \subseteq \partial_{out}^0(B)$, such that $S' \in SBTWN(s, S)$, and such that $WR_B(s, S')$, and $WR_B(S', S)$,

(since a “winning strategy” from s into S must eventually lead to states s' , for each of which there exists an input $u_{s'} \in \mathcal{U}$ such that $\mathcal{E}(s', u_{s'}, -) \subseteq S$).

Since for any $s' \in S'$, $s' \in \partial_{out}^0(B)$ and $WR_B(s', S)$, $\exists s \in \partial_{out}^0(B) . WR_B(s, S)$. ■

OBSERVATION 14.4.2. For any (FSM $\mathcal{M}$ with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, for any centered block $B \subseteq \mathcal{X}$, and any $S \subseteq \mathcal{X}$,

$$\begin{aligned}
 S \subseteq \partial_{out}^0(B), \exists s \in \partial_{out}^0(B). WR_B(s, S) \\
 \Leftrightarrow \\
 S \subseteq \partial_{out}^0(B), \exists s \in \partial_{out}^0(B), \exists u \in \mathcal{U}(s, -). \mathcal{E}(s, u, -) \subseteq S
 \end{aligned}$$

DEFINITION 14.4.3. For any block $B \in \mathfrak{B}$, of a centered cover $\mathfrak{B}$ of a FSM with LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, let the *simplified-strong-out-boundary* of B be denoted by $\tilde{\partial}_{simplified-out}^s(B)$, and be defined to be the set

$$\downarrow \subseteq \left(\begin{array}{c} \{ S \subseteq \partial_{out}^0(B) \mid \exists s \in \partial_{out}^0(B). WR_B(s, S) \} \\ \cup \\ \{ S \subseteq \mathcal{X} \mid S \cap B \neq \emptyset \wedge S \cap C_B = \emptyset \wedge S \text{ is stable} \} \end{array} \right).$$

OBSERVATION 14.4.3. For any block $B \in \mathfrak{B}$, of a centered cover $\mathfrak{B}$ of a FSM with LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, $\tilde{\partial}_{simplified-out}^s(B)$ is the set

$$\downarrow \subseteq \left(\begin{array}{c} \{ \mathcal{E}(s, u, -) \mid s \in \partial_{out}^0(B), u \in \mathcal{U}(s, -). \mathcal{E}(s, u, -) \cap B = \emptyset \} \\ \cup \\ \{ S \subseteq \mathcal{X} \mid S \cap B \neq \emptyset \wedge S \cap C_B = \emptyset \wedge S \text{ is stable} \} \end{array} \right).$$

DEFINITION 14.4.4. A block B of a centered cover $\mathfrak{B}$, of an (FSM with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, is *quasi-convex* if:

- 1) $\mathfrak{H}(B) = B$,
- 2) $\forall S \in \tilde{\partial}_{simplified-out}^s(B), \exists s \in C_B. WR_B(s, S)$,
- 3) $\forall S \in \tilde{\partial}_{center-out}^s(B). WR_B(C_B, S)$, and
- 4) C_B is stable.

A centered cover $\mathfrak{B}$, of an (FSM with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, is *quasi-convex* if for every B in $\mathfrak{B}$, B is quasi-convex.

OBSERVATION 14.4.4. For any quasi-convex (centered) block $B \subseteq \mathcal{X}$, of an LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, any set $S_0 \subseteq \mathcal{X}$ such that $S_0 \supseteq B$, and any set $S \subseteq \mathcal{X} \setminus C_B$,

$$\exists s \in C_B. WR_{S_0}^{\mathcal{G}}(s, S) \Leftrightarrow WR_{S_0}^{\mathcal{G}}(C_B, S).$$

Proof. Since C_B is necessarily non-empty, $\Leftarrow$, is trivial.

Conversely, suppose $s \in C_B$ and $WR_{S_0}(s, S)$. Since $S \subseteq \mathcal{X} \setminus C_B$, $s \notin S$, and evidently, there exists a set $S' \subseteq (B \setminus C_B) \cup \partial_{out}^0(B)$,

(reached by following first-winning moves steering s , out of C_B , towards S), such that $WR_{C_B}(s, S')$, and $WR_{S_0}(S', S)$.

Evidently there exists $S'' \in \mathfrak{S}_{center-out}^s(B)$ such that $S'' \subseteq S'$, and so $WR_{S_0}(S'', S)$. Since B is quasi-convex, $WR_B(C_B, S'')$, and thus (as $B \subseteq S_0$), $WR_{S_0}(C_B, S)$. ■

OBSERVATION 14.4.5. If a centered cover $\mathfrak{B}$, of an (FSM with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, is quasi-convex, then $\mathfrak{B}$ is weakly-quasi-convex.

Proof. Suppose $\mathfrak{B}$ is quasi-convex, and let $B \in \mathfrak{B}$. Then B is quasi-convex.

Thus $\mathfrak{H}(B) = B$, and since necessarily, $\partial_{in}^w(B) \subseteq B$, clearly $\partial_{in}^w(B) \subseteq \mathfrak{H}(B)$.

Now let $S \in \mathfrak{S}_{out}^s(B)$. We will show that $WR_B(C_B, S)$, and hence, that B is weakly-quasi-convex. Since $B \in \mathfrak{B}$ was arbitrary, it will follow that $\mathfrak{B}$ is weakly-quasi-convex.

Since $S \in \mathfrak{S}_{out}^s(B)$, either,

case (1): $S \subseteq \mathfrak{S}_{out}^0(B)$ and $\exists s \in C_B \cup \partial_{in}^w(B)$, $WR_B(s, S)$, or

case (2): $S = S_0 \cup \partial(B')$, for some $S_0 \subseteq \mathfrak{S}_{out}^0(B)$, and some $B' \in \mathfrak{B} \setminus \{B\}$ such that

$$\begin{array}{c} C_{B'} \cap B \neq \emptyset \\ \wedge \\ \exists s \in C_B \cup \partial_{in}^w(B). \\ \left[\begin{array}{c} s \in C_{B'} \\ \vee \\ \left[\begin{array}{c} WR(s, C_{B'}) \\ \wedge \\ \exists S' \subseteq \partial(B'). \\ [WR_B(s, S_0 \cup S') \wedge S_0 \cup S' \in SBWTN(s, C_{B'})] \end{array} \right] \end{array} \right] \end{array} .$$

In case (1), (since $C_B \cup \partial_{in}^w(B) \subseteq B$), there exists $s \in B$ such that $WR_B(s, S)$,

and since $S \subseteq \mathfrak{S}_{out}^0(B)$, by Observation 14.4.1, $\exists s \in \mathfrak{S}_{out}^0(B)$, $WR_B(s, S)$.

Thus, evidently, there exists $S' \in \mathfrak{S}_{simplified-out}^s(B)$, such that $S' \subseteq S$, and since B is quasi-convex, $\exists s' \in C_B$, $WR_B(s', S')$.

Since $S' \subseteq S \subseteq \mathfrak{S}_{out}^0(B)$, $C_B \cap S' = \emptyset$, and so by Observation 14.4.4, $WR_B(C_B, S')$.

Since $S' \subseteq S$, $WR_B(C_B, S)$, as required.

In case (2), since B' is quasi-convex, $C_{B'}$ is stable. As $B \neq B'$, $C_{B'} \cap C_B = \emptyset$,

and since $C_{B'} \cap B \neq \emptyset$, evidently, there exists $S' \in \mathfrak{S}_{simplified-out}^s(B)$ such that $S' \subseteq C_{B'}$.

Since B is quasi-convex, $\exists s' \in C_B$, $WR_B(s', S')$.

Since $S' \subseteq C_{B'}$, $C_B \cap S' = \emptyset$, and again, by Observation 14.4.4, $WR_B(C_B, S')$.

Since $S' \subseteq C_{B'}$, $WR(C_B, C_{B'})$ so it follows that $WR(C_B, \partial(B'))$, and hence, (since $\partial(B') \subseteq S$), that $WR_B(C_B, S)$, as required. ■

CHAPTER 15

Lattices of Centered Covers

There is a natural partial order relation and sup-lattice structure on the centered covers of an FSM.

DEFINITION 15.0.5. For any (FSM $\mathcal{M}$ with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, a collection $\mathfrak{C}$, of centered blocks of $\mathcal{X}$, (so that for each $B \in \mathfrak{C}$, B is a centered block, $B \subseteq \mathcal{X}$), is *covering* if $\forall s \in \mathcal{X} . \exists B \in \mathfrak{C} . s \in B$.

Thus a covering collection $\mathfrak{C}$ of centered blocks is a centered cover just when the centers, $C_B, C_{B'}$, of any distinct blocks $B, B' \in \mathfrak{C}$, are disjoint.

Let $Coverings(\mathcal{X})$ denote the set of all covering collections of centered blocks of $\mathcal{X}$.

Let $Covers(\mathcal{X})$ denote the collection of all centered covers of $\mathcal{X}$.

Let $Blocks(\mathcal{X})$ denote the collection of all centered blocks of $\mathcal{X}$.

OBSERVATION 15.0.6. Evidently, for any LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, every centered cover is covering, and for every covering collection, $\mathfrak{C}$, of centered blocks,

$$\mathfrak{C} \in 2^{Blocks(\mathcal{X})} \quad \text{and} \quad Covers(\mathcal{X}) \subseteq Coverings(\mathcal{X}) \subseteq 2^{Blocks(\mathcal{X})} .$$

DEFINITION 15.0.6. Given an (FSM with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, and two centered blocks, $B_1, B_2 \subseteq \mathcal{X}$, let $B_1 \Subset B_2$ denote the condition:

$$B_1 \subseteq B_2 \wedge C_{B_1} \subseteq C_{B_2} .$$

OBSERVATION 15.0.7. For any (FSM with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$,

$\Subset \subseteq Blocks(\mathcal{X}) \times Blocks(\mathcal{X})$ is a partial order (and thus, a pre-order), relation on $Blocks(\mathcal{X})$.

OBSERVATION 15.0.8. Given an (FSM with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, $\Subset_{2(\cdot)} \subseteq 2^{Blocks(\mathcal{X})} \times 2^{Blocks(\mathcal{X})}$ is

1. a pre-order on $Coverings(\mathcal{X})$,
(that is, when restricted to $Coverings(\mathcal{X}) \times Coverings(\mathcal{X}) \subseteq 2^{Blocks(\mathcal{X})} \times 2^{Blocks(\mathcal{X})}$), and
2. a partial order on $Covers(\mathcal{X})$,

(that is, when restricted to $Covers(\mathcal{X}) \times Covers(\mathcal{X}) \subseteq 2^{Blocks(\mathcal{X})} \times 2^{Blocks(\mathcal{X})}$).

DEFINITION 15.0.7. For any centered blocks $B, B' \in Blocks(\mathcal{X})$ of an (FSM $\mathcal{M}$ with) LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$.

- 1) let $\mathbb{Q}_0(B, B')$ denote the condition that $B \cap B' \neq \emptyset$,
- 2) let $\mathbb{Q}_1(B, B')$ denote the condition that either $C_B \cap B' \neq \emptyset$ or $B \cap C_{B'} \neq \emptyset$. (or both), and
- 3) let $\mathbb{Q}_2(B, B')$ denote the condition that $C_B \cap C_{B'} \neq \emptyset$.

OBSERVATION 15.0.9. For any centered blocks $B, B' \in Blocks(\mathcal{X})$ of an (FSM $\mathcal{M}$ with) LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, clearly,

$$\mathbb{Q}_2(B, B') \Rightarrow \mathbb{Q}_1(B, B') \Rightarrow \mathbb{Q}_0(B, B').$$

DEFINITION 15.0.8. For any centered blocks $B_1, B_2 \in Blocks(\mathcal{X})$ of an (FSM $\mathcal{M}$ with) LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, such that $\mathbb{Q}_1(B_1, B_2)$, let $B_1 \uplus B_2$ denote the centered block with states $B_1 \cup B_2$ and center $C_{B_1 \uplus B_2} \equiv C_{B_1} \cup C_{B_2}$.

OBSERVATION 15.0.10. For any (FSM $\mathcal{M}$ with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, for any quasi-convex blocks $B_1, B_2 \in Blocks(\mathcal{X})$, if $\mathbb{Q}_1(B_1, B_2)$ then $B_1 \uplus B_2$ is quasi-convex.

Proof. It suffices to show that :

- 1) $\mathfrak{H}(B_1 \uplus B_2) = B_1 \uplus B_2$,
- 2) $\forall S \in \mathfrak{S}_{simplified-out}^s(B_1 \uplus B_2), \exists s \in C_{B_1 \uplus B_2}. WR_{B_1 \uplus B_2}(s, S)$.
- 3) $\forall S \in \mathfrak{S}_{center-out}^s(B_1 \uplus B_2). WR_{B_1 \uplus B_2}(C_{B_1 \uplus B_2}, S)$, and
- 4) $C_{B_1 \uplus B_2}$ is stable.

For 1): Let $s \in B_1 \uplus B_2$. Then for some index $i \in \{1, 2\}$ $s \in B_i$. Since (for $i \in \{1, 2\}$), B_i is quasi-convex, $\mathfrak{H}(B_i) = B_i$. Thus $WR_{B_i}(s, C_{B_i})$, and since $B_i \subseteq B_1 \cup B_2$ and $C_{B_i} \subseteq C_{B_1} \cup C_{B_2} = C_{B_1 \uplus B_2}$, it follows that $WR_{B_1 \uplus B_2}(s, C_{B_1 \uplus B_2})$. Thus $\mathfrak{H}(B_1 \uplus B_2) = B_1 \uplus B_2$.

For 2): Let $S \in \mathfrak{S}_{simplified-out}^s(B_1 \uplus B_2)$. Then, by Observation 14.4.3, evidently either

- 2i) $S = \mathcal{E}(s, u, -)$, for some $s \in \partial_{out}^0(B_1 \uplus B_2)$ and some $u \in \mathcal{U}(s, -)$ such that $\mathcal{E}(s, u, -) \cap (B_1 \uplus B_2) = \emptyset$, or
- 2ii) $S \subseteq \mathcal{X}$ such that $S \cap B_1 \uplus B_2 \neq \emptyset$, $S \cap C_{B_1 \uplus B_2} = \emptyset$, and S is stable.

In case 2i): Evidently $\partial_{out}^0(B_1 \uplus B_2) \subseteq \partial_{out}^0(B_1) \cup \partial_{out}^0(B_2)$. Thus, for some index $j \in \{1, 2\}$, $s \in \partial_{out}^0(B_j)$.

Since $u \in \mathcal{U}(s, -)$ is such that $\mathcal{E}(s, u, -) \cap (B_1 \cup B_2) = \emptyset$, it follows that (since $B_j \subseteq B_1 \cup B_2$), $\mathcal{E}(s, u, -) \cap B_j = \emptyset$.

Thus, again by Observation 14.4.3, evidently there exists some $S' \in \mathfrak{S}_{simplified-out}^s(B_j)$, such that $S' \subseteq S$.

Since B_j is quasi-convex, $\exists s \in C_{B_j} \cdot WR_{B_j}(s, S')$. As $C_{B_j} \subseteq C_{B_1 \uplus B_2}$, $B_j \subseteq B_1 \cup B_2$, and $S' \subseteq S$, it follows that $\exists s \in C_{B_1 \uplus B_2} \cdot WR_{B_1 \uplus B_2}(s, S)$, as required.

In case 2ii), since $S \cap B_1 \uplus B_2 \neq \emptyset$, for some index $j \in \{1, 2\}$, $S \cap B_j \neq \emptyset$, and since $S \cap C_{B_1 \uplus B_2} = \emptyset$, (and $C_{B_j} \subseteq C_{B_1 \uplus B_2}$), clearly $S \cap C_{B_j} = \emptyset$. As $S \subseteq \mathcal{X}$ is stable, it follows that there is some $S' \in \mathfrak{S}_{simplified-out}^s(B_j)$, such that $S' \subseteq S$.

As B_j is quasi-convex, $\exists s \in C_{B_j} \cdot WR_{B_j}(s, S')$, and again, (since $C_{B_j} \subseteq C_{B_1 \uplus B_2}$, $B_j \subseteq B_1 \cup B_2$, and $S' \subseteq S$), it follows that $\exists s \in C_{B_1 \uplus B_2} \cdot WR_{B_1 \uplus B_2}(s, S)$, as required.

For 3): Let $S \in \mathfrak{S}_{center-out}^s(B_1 \uplus B_2)$. Then S is minimal such that

$S \subseteq (B_1 \uplus B_2 \setminus C_{B_1 \uplus B_2}) \cup \mathfrak{S}_{out}^0(B_1 \uplus B_2)$ and for some $s \in C_{B_1 \uplus B_2} \cdot WR_{C_{B_1 \uplus B_2}}(s, S)$.

As $s \in C_{B_1 \uplus B_2} = C_{B_1} \cup C_{B_2}$, for some index $j \in \{1, 2\}$, $s \in C_{B_j}$, and $WR_{C_{B_1 \cup B_2}}(s, S)$.

As $S \subseteq (B_1 \uplus B_2 \setminus C_{B_1 \uplus B_2}) \cup \mathfrak{S}_{out}^0(B_1 \uplus B_2)$, $S \subseteq (B_1 \cup B_2) \setminus (C_{B_1} \cup C_{B_2}) \cup \mathfrak{S}_{out}^0(B_1 \uplus B_2)$.

Thus, $S \subseteq (B_1 \cup B_2) \setminus C_{B_j} \cup \mathfrak{S}_{out}^0(B_1 \uplus B_2)$.

Thus evidently, as $C_{B_j} \subseteq B_j \subseteq B_1 \cup B_2$ and $(B_1 \cup B_2) \cap \mathfrak{S}_{out}^0(B_1 \uplus B_2) = \emptyset$.

$S \subseteq (B_1 \cup B_2 \cup \mathfrak{S}_{out}^0(B_1 \uplus B_2)) \setminus C_{B_j}$,

and so there exists $S' \subseteq ((B_j \setminus C_{B_j}) \cup \mathfrak{S}_{out}^0(B_j))$, such that $WR_{C_{B_j}}(s, S')$, $S' \setminus S \subseteq C_{B_1 \cup B_2} \cdot WR_{C_{B_1 \cup B_2}}(S' \setminus S, S)$, and $S' \in SBTWN(s, S)$.

As $s \in C_{B_j}$, and $S' \subseteq ((B_j \setminus C_{B_j}) \cup \mathfrak{S}_{out}^0(B_j))$, such that $WR_{C_{B_j}}(s, S')$.

there exists $S'' \in \mathfrak{S}_{center-out}^s(B_j)$, such that $S'' \subseteq S'$.

Since B_j is quasi-convex, $WR_{B_j}(C_{B_j}, S'')$, and so, (as $B_j \subseteq B_1 \cup B_2$, and $S'' \subseteq S'$), $WR_{B_1 \cup B_2}(C_{B_j}, S')$.

Since $(S' \setminus S \subseteq C_{B_1 \cup B_2})$ and, $WR_{C_{B_1 \cup B_2}}(S' \setminus S, S)$, it follows that $WR_{B_1 \cup B_2}(S' \setminus S, S)$.

and hence that $WR_{B_1 \cup B_2}(C_{B_j}, S)$.

Let $j' \in \{1, 2\} \setminus \{j\}$ denote the other index.

If $C_{B_{j'}} \cap B_j \neq \emptyset$, let $s' \in C_{B_{j'}} \cap B_j$.

Since B_j is quasi-convex, $\mathfrak{H}(B_j) = B_j$, so $s' \in \mathfrak{H}(B_j)$, and so $WR_{B_j}(s', C_{B_j})$.

Thus (as $WR_{B_1 \cup B_2}(C_{B_j}, S)$), $WR_{B_1 \cup B_2}(s', S)$.

As $s' \in C_{B_{j'}}$, $B_{j'} \subseteq B_1 \cup B_2$ and $S \subseteq \mathcal{X} \setminus C_{B_{j'}}$, by Observation 14.4.4, it follows that $WR_{B_1 \cup B_2}(C_{B_{j'}}, S)$.

Thus if $C_{B_{j'}} \cap B_j \neq \emptyset$, then $WR_{B_1 \cup B_2}(C_{B_j} \cup C_{B_{j'}}, S)$, that is, $WR_{B_1 \cup B_2}(C_{B_j \uplus B_{j'}}, S)$, as required.

Otherwise, $C_{B_{j'}} \cap B_j = \emptyset$ and since by hypothesis, $\mathbb{Q}_1(B_1, B_2)$, $C_{B_j} \cap B_{j'} \neq \emptyset$.

Since B_j is quasi-convex, C_{B_j} is stable, and since $C_{B_j} \cap B_j = \emptyset$, (and $C_{B_j} \subseteq B_j$), $C_{B_j} \cap C_{B_{j'}} = \emptyset$.

Thus, evidently, there exists some $S' \in \mathfrak{S}_{simplified-out}^s(B_{j'})$ such that $S' \subseteq C_{B_{j'}}$.

As $B_{j'}$ is quasi-convex, there exists $s' \in C_{B_{j'}}$ such that $WR_{B_{j'}}(s', S')$, and hence, (as $S' \subseteq C_{B_{j'}}$),

such that $WR_{B_{j'}}(s', C_{B_j})$.

Thus, again, (as $WR_{B_1 \cup B_2}(C_{B_j}, S)$), $WR_{B_1 \cup B_2}(s', S)$, and as $s' \in C_{B_{j'}}$, $B_{j'} \subseteq B_1 \cup B_2$, and $S \subseteq \mathcal{X} \setminus C_{B_{j'}}$, by Observation 14.4.4, it follows that $WR_{B_1 \cup B_2}(C_{B_{j'}}, S)$.

Thus, again, $WR_{B_1 \cup B_2}(C_{B_j} \cup C_{B_{j'}}, S)$, that is, $WR_{B_1 \cup B_2}(C_{B_j \uplus B_{j'}}, S)$, as required.

For 4): Since B_1 and B_2 are quasi-convex, both C_{B_1} and C_{B_2} are stable.

As stability is (evidently) preserved under union, $C_{B_j \uplus B_{j'}} = C_{B_j} \cup C_{B_{j'}}$, is stable. ■

DEFINITION 15.0.9. For any (FSM $\mathcal{M}$ with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, for any $\mathfrak{C} \in \text{Coverings}(\mathcal{X})$, and $i \in \{1, 2\}$,

1) let $\text{confl}_i(\mathfrak{C})$ denote

$$\mathfrak{C} \cup \{B \uplus B' \mid B, B' \in \mathfrak{C}; \mathbb{Q}_i(B, B')\} \text{, and}$$

2) let $\text{Confl}_i(\mathfrak{C})$ denote

$$\uparrow_{\mathbb{E}_2(\cdot)}(\text{confl}_i(\mathfrak{C})) \text{.}$$

OBSERVATION 15.0.11. For any (FSM $\mathcal{M}$ with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, and for $j \in \{1, 2\}$.

- i. $\text{Coverings}(\mathcal{X})$ is a finite $\vee$ -semi-lattice, where $\vee$ is union, $(\cup)$, and the partial order is inclusion, $(\subseteq)$.
- ii. $\text{confl}_j(\cdot)$ defines a function from $\text{Coverings}(\mathcal{X})$ into itself, which is inflationary (and monotonic) with respect to inclusion,
- iii. $\text{Confl}_j(\cdot)$ defines a function from $\text{Coverings}(\mathcal{X})$ into itself, which is inflationary (and monotonic) with respect to the pre-order $\mathbb{E}_2(\cdot)$.

OBSERVATION 15.0.12. For any (FSM $\mathcal{M}$ with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, for $j \in \{1, 2\}$, for any $\mathfrak{C} \in \text{Coverings}(\mathcal{X})$, there exists a $\subseteq$ -minimal $\mathfrak{C}' \in \text{Coverings}(\mathcal{X})$, such that

- i. $\mathfrak{C} \subseteq \mathfrak{C}'$,
- ii. $\text{confl}_j(\mathfrak{C}') = \mathfrak{C}'$,
($\mathfrak{C}'$ is the least fixed-point of $\text{confl}_j(\cdot)$ which contains $\mathfrak{C}$), and further,
- iii. $\uparrow_{\mathbb{E}_2(\cdot)}(\mathfrak{C}') \in \text{Covers}(\mathcal{X})$
(that is, $\uparrow_{\mathbb{E}_2(\cdot)}(\mathfrak{C}')$ is a centered cover of $\mathcal{X}$), and
- iv. $\uparrow_{\mathbb{E}_2(\cdot)}(\mathfrak{C}')$ can be calculated as $\mathfrak{C}_n$, where $n \geq 0$ is the least index $i \geq 0$, such that $\mathfrak{C}_{i+1} = \mathfrak{C}_i$, in the sequence

$$\mathfrak{C}_0, \dots, \mathfrak{C}_i, \mathfrak{C}_{i+1}, \dots$$

where $\mathfrak{C}_0 \equiv \uparrow_{\mathbb{E}_2(\cdot)}(\mathfrak{C})$ and $\forall i \geq 0$, $\mathfrak{C}_{i+1} \equiv \text{Confl}_j(\mathfrak{C}_i)$.

DEFINITION 15.0.10. For any (FSM $\mathcal{M}$ with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, for $j \in \{1, 2\}$, for any centered covers $\mathfrak{B}_1, \mathfrak{B}_2$ of $\mathcal{X}$, let $\mathfrak{B}_1 \uplus_j \mathfrak{B}_2$ denote

$$\uparrow_{\in_{2(\cdot)}}(\mathfrak{C}'),$$

where $\mathfrak{C}'$ is the least fixed-point of $\text{confl}_j(\cdot)$ which contains $\mathfrak{B}_1 \cup \mathfrak{B}_2$.

OBSERVATION 15.0.13. For any (FSM $\mathcal{M}$ with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, $\text{Covers}(\mathcal{X})$ is a finite $\vee$ -semi-lattice, where $\vee$ is $\uplus_2$, and the partial order is $\in_{2(\cdot)}$.

DEFINITION 15.0.11. A covering collection $\mathfrak{B} \in \text{Coverings}(\mathcal{X})$, of (an FSM, $\mathcal{M}$, with) LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, is *center separated*, or simply *separated*, if

$$\forall B, B' \in \mathfrak{B}. [B \neq B' \Rightarrow C_B \cap B' = \emptyset].$$

OBSERVATION 15.0.14. Clearly, (for any LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$), any center separated $\mathfrak{B} \in \text{Coverings}(\mathcal{X})$ is a cover, (that is, $\mathfrak{B} \in \text{Covers}(\mathcal{X})$).

DEFINITION 15.0.12. For any (FSM $\mathcal{M}$ with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, let $\text{SepCovers}(\mathcal{X})$ denote the collection of all center separated covers of $\mathcal{X}$.

OBSERVATION 15.0.15. For any (FSM $\mathcal{M}$ with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, $\text{SepCovers}(\mathcal{X})$ is a finite $\vee$ -semi-lattice, where $\vee$ is $\uplus_1$, and the partial order is $\in_{2(\cdot)}$.

DEFINITION 15.0.13. For any (FSM $\mathcal{M}$ with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$,

- 1) let $QCC(\mathcal{X})$ denote the collection of all quasi-convex covers of $\mathcal{X}$, and
- 2) let $QCSC(\mathcal{X})$ denote the collection of all quasi-convex separated covers of $\mathcal{X}$,
(that is, $QCSC(\mathcal{X}) \equiv \text{SepCovers}(\mathcal{X}) \cap QCC(\mathcal{X})$).

OBSERVATION 15.0.16. For any (FSM $\mathcal{M}$ with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$,

- 1) $QCC(\mathcal{X})$ is a finite sub- $\vee$ -semi-lattice of $\text{Covers}(\mathcal{X})$,
(where $\vee$ is $\uplus_2$, and the partial order is $\in_{2(\cdot)}$), and
- 2) $QCSC(\mathcal{X})$ is a finite sub- $\vee$ -semi-lattice of $\text{SepCovers}(\mathcal{X})$,
(where $\vee$ is $\uplus_1$, and the partial order is $\in_{2(\cdot)}$).

DEFINITION 15.0.14. For any finite set S , let $\sharp(S)$ denote the cardinality of S .

In [13, 42, 41, 43], partitions of $\mathcal{X}$ are considered for a deterministic (FSM $\mathcal{M}$ with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$.

There, partitions of $\mathcal{X}$ form a finite lattice with $\vee$ -operation *chained union* and partial order relation $\preceq$, where for any partitions Π and Π' of $\mathcal{X}$, $\Pi \preceq \Pi'$ precisely when

$$\forall S \in \Pi, \exists S' \in \Pi'. S \subseteq S'.$$

Note that in this situation, if $\Pi \preceq \Pi'$, for some partitions Π and Π' of $\mathcal{X}$, then it is also the case that

$$\forall S' \in \Pi', \exists S \in \Pi. S \subseteq S' \text{ and } \sharp(\Pi') \leq \sharp(\Pi).$$

In words, this essentially means that when $\Pi \preceq \Pi'$, every member of Π' is some “coarsening” of some member of Π .

Though evidently this ordering property does not hold in (the $\mathbb{U}_2$ -semi-lattice) $Covers(\mathcal{X})$, it does hold in (the $\vee$ -semi-lattice) $SepCovers(\mathcal{X})$.

OBSERVATION 15.0.17. If $\mathcal{C}, \mathcal{C}' \in SepCovers(\mathcal{X})$, for an (FSM $\mathcal{M}$ with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, and $\mathcal{C} \in_{2(\cdot)} \mathcal{C}'$, then

$$\forall B' \in \mathcal{C}', \exists B \in \mathcal{C}. B \in B'.$$

Proof. Suppose $\mathcal{C} \in_{2(\cdot)} \mathcal{C}'$.

Then evidently there exists a function $f: \mathcal{C} \rightarrow \mathcal{C}'$ such that for each $B \in \mathcal{C}$, $B \in f(B)$.

Let $\mathfrak{B} \equiv \{f(B) \mid B \in \mathcal{C}\}$.

Then $\mathfrak{B} \subseteq \mathcal{C}'$, and since $\mathcal{C} \in Covers(\mathcal{X})$, evidently $\mathfrak{B} \in Covers(\mathcal{X})$.

(In fact, as $\mathfrak{B} \subseteq \mathcal{C}'$ and $\mathcal{C}' \in SepCovers(\mathcal{X})$, evidently $\mathfrak{B} \in SepCovers(\mathcal{X})$.)

Suppose $B' \in \mathcal{C}' \setminus \mathfrak{B}$. Then, as $\mathfrak{B} \in Covers(\mathcal{X})$, $C_{B'} \cap B'' \neq \emptyset$, for some $B'' \in \mathfrak{B}$.

But since $\mathcal{C}' \in SepCovers(\mathcal{X})$, $\mathfrak{B} \cup \{B'\} \subseteq \mathcal{C}'$, must be center separated, and so (since $C_{B'} \cap B'' \neq \emptyset$, for some $B'' \in \mathfrak{B}$), $B' = B''$, for some $B'' \in \mathfrak{B}$, which contradicts our choice of $B' \in \mathcal{C}' \setminus \mathfrak{B}$.

Thus $\mathcal{C}' \subseteq \mathfrak{B}$, and so $\mathcal{C}' = \mathfrak{B}$.

Therefore, for any $B' \in \mathcal{C}'$, there exists some $B \in \mathcal{C}$ such that $B' = f(B)$, and hence, such that $B \in B'$. ■

COROLLARY 15.0.1. If $\mathcal{C}, \mathcal{C}' \in SepCovers(\mathcal{X})$, for an (FSM $\mathcal{M}$ with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, and $\mathcal{C} \in_{2(\cdot)} \mathcal{C}'$, then

$$\sharp(\mathcal{C}') \leq \sharp(\mathcal{C}).$$

Proof. Suppose $\mathcal{C} \in_{2(\cdot)} \mathcal{C}'$.

Then by Observation 15.0.17, for any $B' \in \mathcal{C}'$, there exists some $B \in \mathcal{C}$ such that $B \in B'$.

Thus there exists a function $f: \mathcal{C}' \rightarrow \mathcal{C}$ such that for each $B' \in \mathcal{C}'$, $f(B') \in B'$.

If for some $B'_1, B'_2 \in \mathcal{C}'$, $f(B'_1) = f(B'_2)$, then $f(B'_1) = f(B'_2) \in B'_1$, and $f(B'_1) = f(B'_2) \in B'_2$.

Thus, in particular, $C_{f(B'_1)} \subset C_{B'_1} \cap C_{B'_2}$, and so $C_{B'_1} \cap C_{B'_2} \neq \emptyset$.

Since $\mathcal{C}$ is, by hypothesis, center separated, it follows that $B'_1 = B'_2$.

Thus $f: \mathcal{C}' \rightarrow \mathcal{C}$ is 1-1, and so $\sharp(\mathcal{C}') \leq \sharp(\mathcal{C})$. ■

OBSERVATION 15.0.18. Suppose $\mathcal{C}, \mathcal{C}' \in QCC(\mathcal{X})$, for an (FSM $\mathcal{M}$ with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, and $\mathcal{C} \in_{2(1)} \mathcal{C}'$. Then if the FOC meta-machine $\mathcal{M}^{\mathcal{C}}$ is controllable, the FOC meta-machine $\mathcal{M}^{\mathcal{C}'}$ is controllable.

Proof. Suppose $\mathcal{C} \in_{2(1)} \mathcal{C}'$, and the FOC meta-machine $\mathcal{M}^{\mathcal{C}}$ is controllable.

Since $\mathcal{C}$ and $\mathcal{C}'$ are quasi-convex, by Observation 14.4.5, $\mathcal{C}$ and $\mathcal{C}'$ are weakly-quasi-convex.

Since the FOC meta-machine $\mathcal{M}^{\mathcal{C}}$ is controllable, by Corollary 14.3.2, $\mathcal{M}$ is center-to-center controllable with respect to the cover $\mathcal{C} \in QCC(\mathcal{X})$.

Again by Corollary 14.3.2, in order to show that the FOC meta-machine $\mathcal{M}^{\mathcal{C}'}$ is controllable, it suffices to show that $\mathcal{M}$ is center-to-center controllable with respect to the cover $\mathcal{C}' \in QCC(\mathcal{X})$.

Let $B'_1, B'_2 \in \mathcal{C}'$, distinct. Then since by assumption, $\mathcal{C}'$ is a cover and $B'_1 \neq B'_2$, it follows that $C_{B'_1} \cap C_{B'_2} = \emptyset$. Since $\mathcal{C}$ is a cover, there exists $B_1 \in \mathcal{C}$ such that $B_1 \cap C_{B'_1} \neq \emptyset$. Similarly, there exists $B_2 \in \mathcal{C}$ such that $B_2 \cap C_{B'_2} \neq \emptyset$.

Since $\mathcal{M}$ is center-to-center controllable with respect to the cover $\mathcal{C} \in QCC(\mathcal{X})$, in particular, we have that $WR(C_{B_1}, C_{B_2})$.

Since $B'_1, B'_2 \in \mathcal{C}' \in QCC(\mathcal{X})$, B'_1 and B'_2 are quasi-convex, and so $C_{B'_1}$ and $C_{B'_2}$ are stable.

Let $i \in \{1, 2\}$ denote either index. If $C_{B_i} \cap C_{B'_i} \neq \emptyset$, then since $B_i \in \mathcal{C}$, $B'_i \in \mathcal{C}'$, and $\mathcal{C} \in_{2(1)} \mathcal{C}'$, evidently $B_i \in B'_i$, and so $C_{B_i} \subseteq C_{B'_i}$.

Otherwise, if $C_{B_i} \cap C_{B'_i} = \emptyset$, then since $C_{B'_i}$ is stable, there is some $s_i \in C_{B_i}$ and some $S_i \in \mathfrak{S}_{simplified-out}^s(B_i)$ such that $S_i \subseteq C_{B'_i}$ and $WR_{B_i}(s_i, S_i)$, and hence such that $WR_{B_i}(s_i, C_{B'_i})$. Thus, by Observation 14.4.4, $WR_{B_i}(C_{B_i}, C_{B'_i})$.

There are four cases to now consider.

Case 1: If $C_{B_1} \cap C_{B'_1} \neq \emptyset$ and $C_{B_2} \cap C_{B'_2} \neq \emptyset$, then $B_1 \in B'_1$ and $B_2 \in B'_2$. In this case, evidently $B_1 \neq B_2$, because if $B_1 = B_2$, then $\emptyset \neq C_{B_1} = C_{B_2} \subseteq C_{B'_1} \cap C_{B'_2}$, which contradicts the condition that $C_{B'_1} \cap C_{B'_2} = \emptyset$.

Since $WR(C_{B_1}, C_{B_2})$ and $C_{B_2} \subseteq C_{B'_2}$, evidently $WR(C_{B_1}, C_{B'_2})$, and since $C_{B_1} \subseteq C_{B'_1}$, (and $C_{B'_1} \cap C_{B'_2} = \emptyset$), again by Observation 14.4.4, it follows that $WR(C_{B'_1}, C_{B'_2})$.

Case 2: If $C_{B_1} \cap C_{B'_1} \neq \emptyset$ and $C_{B_2} \cap C_{B'_2} = \emptyset$, then $B_1 \in B'_1$ and $WR_{B_2}(C_{B_2}, C_{B'_2})$. If $B_1 = B_2$, then $WR_{B_1}(C_{B_1}, C_{B'_2})$, and so certainly $WR(C_{B_1}, C_{B'_2})$.

Since $WR_{B_2}(C_{B_2}, C_{B'_2})$, (and $WR(C_{B_1}, C_{B_2})$), it again follows that $WR(C_{B_1}, C_{B'_2})$. Thus, in either case, $WR(C_{B_1}, C_{B'_2})$.

Since (as $B_1 \in B'_1$), $C_{B_1} \subseteq C_{B'_1}$, and $C_{B'_1} \cap C_{B'_2} = \emptyset$, again by Observation 14.4.4, it follows that $WR(C_{B'_1}, C_{B'_2})$.

Case 3: Suppose $C_{B_1} \cap C_{B'_1} = \emptyset$ and $C_{B_2} \cap C_{B'_2} \neq \emptyset$.

Then $B_2 \subseteq B'_2$, and since $B_1 \cap C_{B'_1} \neq \emptyset$, there exists some $s'_1 \in C_{B'_1} \cap B_1$.

Since $B_1 \in \mathcal{C} \in QCC(\mathcal{C})$ is quasi-convex, $s'_1 \in \mathfrak{H}(B_1)$, and so $WR_{B_1}(s'_1, C_{B_1})$.

Since again, we have that $WR(C_{B_1}, C_{B_2})$, it follows that $WR(s'_1, C_{B_2})$,

and hence, (as $B_2 \subseteq B'_2$ and so $C_{B_2} \subseteq C_{B'_2}$), that $WR(s'_1, C_{B'_2})$.

Since $s'_1 \in C_{B'_1}$ and $C_{B'_1} \cap C_{B'_2} = \emptyset$, again by Observation 14.4.4, it follows that $WR(C_{B'_1}, C_{B'_2})$.

Case 4: Suppose $C_{B_1} \cap C_{B'_1} = \emptyset$, and $C_{B_2} \cap C_{B'_2} = \emptyset$.

Then $WR_{B_2}(C_{B_2}, C_{B'_2})$, and just as in case (3) above, there exists some $s'_1 \in C_{B'_1} \cap B_1$ such that $WR(s'_1, C_{B_2})$.

Thus, evidently again it follows that $WR(s'_1, C_{B'_2})$, and again, since $s'_1 \in C_{B'_1}$ and $C_{B'_1} \cap C_{B'_2} = \emptyset$, by Observation 14.4.4, it follows that $WR(C_{B'_1}, C_{B'_2})$.

Thus, in all cases $WR(C_{B'_1}, C_{B'_2})$, and since B'_1 and B'_2 were chosen arbitrarily as distinct members of $\mathcal{C}'$, $\mathcal{M}$ is center-to-center controllable with respect to the cover $\mathcal{C}' \in QCC(\mathcal{X})$, as required. ■

OBSERVATION 15.0.19. For any (FSM $\mathcal{M}$ with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, and any centered cover $\mathcal{C} \in \text{Covers}(\mathcal{X})$,

- 1) If $\forall B \in \mathcal{C}. C_B = B$, then $\mathcal{C}$ is a centered partition of $\mathcal{X}$.
- 2) If $\mathcal{C}$ is a centered partition of $\mathcal{X}$, then $\mathcal{C} \in \text{SepCovers}(\mathcal{X})$, (that is, $\mathcal{C}$ is separated).

EXAMPLE 15.0.1. Let Π be any IBC partition [13, 42, 41, 43], of a deterministic (FSM $\mathcal{M}$ with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, or any ST-IBC partition [3] of a deterministic *T-co-accessible* deterministic (FSM $\mathcal{M}$ with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, (that is, every state $s \in \mathcal{X}$ is *T-co-accessible* [3]), for some $S, T \subseteq \mathcal{X}$, and assume that each block $B \in \Pi$ is stable. Then if $\mathcal{C}_\Pi$ denotes the centered cover obtained from Π by assigning to each block $B \in \Pi$, the center $C_B \equiv B$. $\mathcal{C}_\Pi \in QCSC(\mathcal{X})$.

DEFINITION 15.0.15. For any (FSM $\mathcal{M}$ with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$,

- 1a) a centered block $B \subseteq \mathcal{X}$ is *fully centered* if $C_B = B$, and
- 1b) a centered cover $\mathfrak{B}$ of $\mathcal{X}$ is *fully centered* if for each block $B \in \mathfrak{B}$ is fully centered.
- 2a) let $FC(\mathcal{X})$ denote the collection of all fully centered covers of $\mathcal{X}$, and
- 2b) let $FCQCC(\mathcal{X})$ denote the collection of all fully centered quasi-convex covers of $\mathcal{X}$, (that is, $FCQCC(\mathcal{X}) \equiv FC(\mathcal{X}) \cap QCC(\mathcal{X})$).

OBSERVATION 15.0.20. For any (FSM $\mathcal{M}$ with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, if $\mathfrak{B}$ is a fully centered cover of $\mathcal{X}$, then $\mathfrak{B}$ is a partition of $\mathcal{X}$, and thus is center separated. Further, $FCQCC(\mathcal{X}) = FC(\mathcal{X}) \cap QCSC(\mathcal{X})$, and $FCQCC(\mathcal{X})$ is a finite sub- $\vee$ -semi-lattice of $QCSC(\mathcal{X})$, (where $\vee$ is Ψ_1 , and the partial order is $\subseteq_{2(\cdot)}$), on which $\Psi_1 = \Psi_2$.

Proof. (sketch) Since any fully centered cover of $\mathcal{X}$ is center separated, $FCQCC(\mathcal{X}) = FC(\mathcal{X}) \cap QCSC(\mathcal{X})$.

If $B, B' \in \text{Blocks}(\mathcal{X})$ are fully centered, then evidently

$$\mathbb{Q}_2(B, B') \Leftrightarrow \mathbb{Q}_1(B, B') \Leftrightarrow \mathbb{Q}_0(B, B') .$$

Thus evidently, for any collection $\mathcal{C} \subseteq \text{Blocks}(\mathcal{X})$ of fully centered blocks, $\text{confl}_1(\mathcal{C}) = \text{confl}_2(\mathcal{C})$, so $\text{Confl}_1(\mathcal{C}) = \text{Confl}_2(\mathcal{C})$. Hence if $\mathfrak{B}_1, \mathfrak{B}_2 \in \text{FC}(\mathcal{X})$ are fully centered, $\mathfrak{B}_1 \uplus \mathfrak{B}_2 = \mathfrak{B}_1 \uplus_2 \mathfrak{B}_2$, and further, evidently, each block in $\mathfrak{B}_1 \uplus \mathfrak{B}_2$ is fully centered, and so $\mathfrak{B}_1 \uplus \mathfrak{B}_2 \in \text{FC}(\mathcal{X})$. Thus, if further $\mathfrak{B}_1, \mathfrak{B}_2 \in \text{QCC}(\mathcal{X})$, $\mathfrak{B}_1 \uplus \mathfrak{B}_2 \in \text{FCQCC}(\mathcal{X})$. ■

Recall that a block B of a cover $\mathfrak{B}$, of an (FSM with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, is quasi-convex if:

- 1) $\mathfrak{H}(B) = B$,
- 2) $\forall S \in \mathfrak{S}_{\text{simplified-out}}^s(B), \exists s \in C_B . WR_B(s, S)$,
- 3) $\forall S \in \mathfrak{S}_{\text{center-out}}^s(B) . WR_B(C_B, S)$, and
- 4) C_B is stable.

and that $\mathfrak{B}$ is quasi-convex if for every B in $\mathfrak{B}$, B is quasi-convex.

If $\mathfrak{B}$ is fully centered, then evidently conditions (1) and (2) (above), are trivially satisfied. If $\mathfrak{B}$ is fully centered, then for any block $B \in \mathfrak{B}$, evidently

$$\begin{aligned} \mathfrak{S}_{\text{center-out}}^s(B) &= \downarrow_{\subseteq} (\{ S \subseteq (B \setminus C_B) \cup \mathfrak{S}_{\text{out}}^0(B) \mid \exists s \in C_B . WR_{C_B}(s, S) \}) \\ &= \downarrow_{\subseteq} (\{ S \subseteq \mathfrak{S}_{\text{out}}^0(B) \mid \exists s \in \mathfrak{S}_{\text{out}}^0(B) . WR_B(s, S) \}) \\ &= \downarrow_{\subseteq} (\{ \mathcal{E}(s, u, -) \mid s \in \mathfrak{S}_{\text{out}}^0(B) . u \in \mathcal{U}(s, -) . \mathcal{E}(s, u, -) \cap B = \emptyset \}) \end{aligned}$$

Thus, if $\mathfrak{B}$ is fully centered, evidently condition (3) becomes condition (1'):

$$\forall S \in \downarrow_{\subseteq} (\{ \mathcal{E}(s, u, -) \mid s \in \mathfrak{S}_{\text{out}}^0(B) , u \in \mathcal{U}(s, -) . \mathcal{E}(s, u, -) \cap B = \emptyset \}) . WR_B(B, S) .$$

Condition (4), becomes the condition (2'): B is stable.



CHAPTER 16

Regular, Complete and Center Controllable Covers

16.1. Regular Covers

We have seen that quasi-convex and weakly-quasi-convex covers, $\mathfrak{B}$, have the property that if a FSM $\mathcal{M}$ with LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, is center-to-center controllable, then the FOC meta-machine $\mathcal{M}^{FOC(\mathfrak{B})}$ is controllable. That is, if

$$\forall B, B' \in \mathfrak{B}. WR(C_B, C_{B'})$$

then

$$\forall B, B' \in \mathfrak{B}. WR^{FOC(\mathfrak{B})}(B, \{B'\}) .$$

Unfortunately, besides simply being quite complicated, the requirements for quasi-convexity (or weak-quasi-convexity) are computationally demanding in their verification, and moreover, they are simply rather restrictive.

However, many of these constraints are required, and imposed, for the sake of the lattice properties, these deemed desirable. For the sake of simplicity and greater amenability then, we now turn our back on these lattice theoretic properties, and in this chapter we introduce *regular covers* as a much less demanding class of covers which have the above “low to high controllability” property.

DEFINITION 16.1.1. For any block $B \in \mathfrak{B}$, of a centered cover $\mathfrak{B}$ of a FSM with LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, let the *external-meta-out-boundary* of B be denoted by $\mathfrak{B}_0^{FOC(\mathfrak{B})}(B)$, and be defined to be the set

$$\downarrow \subseteq \left(\begin{array}{c} \{ \{B'\} \mid B' \in \mathfrak{B} \setminus \{B\}; \exists s \in B. s \in \partial(B') \} \\ \cup \\ \{ \beta \subseteq \mathfrak{B} \setminus \{B\} \mid \exists s \in \partial_{out}^0(B), \exists u \in \mathcal{U}(s). \mathcal{E}(s, u, -) \subseteq (\mathcal{X} \setminus B) \cap \bigcup \beta \} \end{array} \right) .$$

OBSERVATION 16.1.1. Suppose $\mathfrak{B}$ is a centered cover of a FSM with LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, such that for each $B \in \mathfrak{B}$, $\mathfrak{H}(B) = B$. Then for any block $B \in \mathfrak{B}$,

$$\mathfrak{B}_0^{FOC(\mathfrak{B})}(B) \subseteq \{ \beta \subseteq \mathfrak{B} \setminus \{B\} \mid \exists s \in B. WR_B(s, \partial(\beta)) \} .$$

Proof.(Sketch) If $B' \in \mathfrak{B} \setminus \{B\}$ and for some $s \in B$, $s \in \partial(B')$, then trivially $WR_B(s, \partial(B'))$.

If $\beta \subseteq \mathfrak{B} \setminus \{B\}$ and for some $s \in \partial_{out}^0(B)$, there exists $u \in \mathcal{U}(s)$ such that $\mathcal{E}(s, u, -) \subseteq (\mathcal{X} \setminus B) \cap \bigcup \beta$, then, since $\mathfrak{H}(B') = B'$, $\partial(B') = \partial_{in}^0(B') \cup (B' \cap \bigcup (\mathfrak{B} \setminus \{B'\}))$, for every $B' \in \beta$, and so evidently, $\exists s \in B. WR_B(s, \partial(\beta))$. ■

DEFINITION 16.1.2. A centered cover $\mathfrak{B}$ of a FSM with LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$ is *regular* if for every $B \in \mathfrak{B}$, for every $\beta \in \mathfrak{B}_0^{FOC(\mathfrak{B})}(B)$, $WR_B(C_B, \partial(\beta))$.

The next proposition addresses the generality of regular covers vis a vi quasi-convex covers.

PROPOSITION 16.1.1. Every quasi-convex cover is regular.

Proof. Let $\mathfrak{B}$ be a quasi-convex cover of a FSM $\mathcal{M}$ with LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, and let $B \in \mathfrak{B}$.

Suppose $B' \in \mathfrak{B} \setminus \{B\}$ and $B \cap \partial(B') \neq \emptyset$. Then since (B' is quasi-convex), $\mathfrak{H}(B') = B'$ and $C_{B'} \cap C_B = \emptyset$, evidently there exists some $S \in \mathfrak{B}_{simplified-out}^s(B)$ such that $S \subseteq B' \cap \partial_{out}^0(B)$. Thus (as $S \subseteq \partial_{out}^0(B)$, $S \cap B = \emptyset$, so $S \cap C_B = \emptyset$, and so, since B is quasi-convex), $WR_B(C_B, S)$. Thus as ($S \subseteq B' \cap \partial_{out}^0(B)$: $\mathfrak{H}(B') = B'$, and so), $S \subseteq \partial(B')$, it follows that $WR_B(C_B, \partial(B'))$.

Suppose $\beta \subseteq \mathfrak{B} \setminus \{B\}$ and for some $s \in \partial_{out}^0(B)$, and some $u \in \mathcal{U}(s)$, $\mathcal{E}(s, u, -) \subseteq (\mathcal{X} \setminus B) \cap \bigcup \beta$. Then evidently there is some $S \in \mathfrak{B}_{simplified-out}^s(B)$ such that $S \subseteq \mathcal{E}(s, u, -)$ and $S \subseteq \partial_{out}^0(B)$. (and so $S \cap B = \emptyset$). Thus, (as $S \cap B = \emptyset$, $S \cap C_B = \emptyset$, and so, since B is quasi-convex), $WR_B(C_B, S)$. As $S \subseteq \partial_{out}^0(B) \cap \bigcup \beta$, (and $\mathfrak{B}$ is quasi-convex), $S \subseteq \partial(\beta)$, and so $WR_B(C_B, \partial(\beta))$.

Thus evidently, for every $\beta \in \mathfrak{B}_0^{FOC(\mathfrak{B})}(B)$, we have $WR_B(C_B, \partial(\beta))$.

Since $B \in \mathfrak{B}$ was arbitrary, $\mathfrak{B}$ is regular. ■

LEMMA 16.1.1. Suppose $\mathfrak{B}$ is a regular centered cover of a FSM $\mathcal{M}$ with LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, and $\forall B, B' \in \mathfrak{B}. WR(C_B, C_{B'})$. (ie. $\mathcal{M}$ is center-to-center controllable). Then

$$\forall B, B' \in \mathfrak{B}, \forall s \in B. [WR(s, C_{B'}) \Rightarrow WR^{FOC(\mathfrak{B})}(B, \{B'\})] .$$

Proof. Evidently, may assume $B \neq B'$. By induction on $\mu = \mu R(s, C_{B'}, -)$.

If $\mu = 0$ then $s \in B \cap C_{B'}$, and therefore, $s \in \partial(B')$. Thus $\beta_1 \equiv \{B'\} \in \mathfrak{B}_0^{FOC(\mathfrak{B})}(B)$, and so, (since $\mathfrak{B}$ is regular, and $\partial(\beta_1) = \partial(B')$), $WR_B(C_B, \partial(B'))$. Thus, evidently, $WR^{FOC(\mathfrak{B})}(B, \{B'\})$.

Assume the result for $0 \leq \mu < N$ and that now $\mu = N$. Since $WR(s, C_{B'})$, there exists $a \in fwm(s, C_{B'})$. Consider $S_1 \equiv \mathcal{E}(s, a, -)$.

If $S_1 \cap B \neq \emptyset$, then evidently we have $\mu R(s', C_{B'}, -) < N$, for some $s' \in S_1 \cap B$, and thus by the induction hypothesis, $WR^{FOC(\mathfrak{B})}(B, \{B'\})$.

Otherwise, let $\beta_1 \subseteq \mathfrak{B}$ be a minimal collection such that $S_1 \subseteq \bigcup \beta_1$, and for each $s' \in S_1$, let $B(s') \in \beta_1$ denote a member of β_1 such that $s' \in B(s')$. Evidently $\forall s' \in S_1$, ($B(s') \neq B$ and), $\mu R(s', C_{B'}, -) < N$, and so by the induction hypothesis, $WR^{FOC(\mathfrak{B})}(B(s'), \{B'\})$. But clearly there exists some $\beta \in \mathfrak{B}_0^{FOC(\mathfrak{B})}(B)$ such that $\beta \subseteq \beta_1$, and since $\mathfrak{B}$ is regular, $WR_B(C_B, \partial(\beta))$. Since $\beta \subseteq \beta_1$, it follows that $WR^{FOC(\mathfrak{B})}(B, \beta_1)$, and

therefore (since for every $B' \in \beta_1$ there is some $s' \in S_1$ such that $B' = B(s')$),
 $WR^{FOC(\mathfrak{B})}(B, \{B'\})$. ■

COROLLARY 16.1.1. Suppose $\mathfrak{B}$ is a regular centered cover of a FSM $\mathcal{M}$ with LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, and $\forall B, B' \in \mathfrak{B}. WR(C_B, C_{B'})$, (ie. $\mathcal{M}$ is center-to-center controllable). Then

$$\forall B, B' \in \mathfrak{B}. WR^{FOC(\mathfrak{B})}(B, \{B'\}) \text{ , (ie. the meta-machine } \mathcal{M}^{FOC(\mathfrak{B})} \text{ is controllable).}$$

Proof. Let $B, B' \in \mathfrak{B}$, and $s \in C_B$. Then $WR(s, C_{B'})$, and so by Lemma 16.1.1,
 $WR^{FOC(\mathfrak{B})}(B, \{B'\})$. ■

COROLLARY 16.1.2. Suppose $\mathfrak{B}$ is a regular centered cover of a FSM $\mathcal{M}$ with LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$.

Then $\mathcal{M}$ is center-to-center controllable, (ie. $\forall B, B' \in \mathfrak{B}. WR(C_B, C_{B'})$),

if and only if

$$\mathcal{M}^{FOC(\mathfrak{B})} \text{ is controllable, (ie. } \forall B, B' \in \mathfrak{B}. WR^{FOC(\mathfrak{B})}(B, \{B'\}) \text{).}$$

Proof. $\Rightarrow$: from Corollary 16.1.1.

$\Leftarrow$: from Theorem 14.3.2. ■

The following existence result for regular covers asserts that given a “center-to-center controllable” set $\mathcal{C}$ of candidate centers, there exists at least one regular cover $\mathfrak{B}$ with centers $\mathcal{C}$.

PROPOSITION 16.1.2. Suppose Π_Y is a partition of a non-empty subset $Y \subseteq \mathcal{X}$ of the state set $\mathcal{X}$ of an LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, such that $\forall C, C' \in \Pi_Y. WR(C, C')$.

Then the maximal centered cover with centers Π_Y is regular.

Proof. Let $\mathfrak{B}$ be the maximal centered cover with centers Π_Y . Then each block $B \in \mathfrak{B}$ corresponds uniquely to some $C \in \Pi_Y$, with $B = \mathcal{X}$ and $C_B = C$. Since $\forall C, C' \in \Pi_Y. WR(C, C')$,
 (and $B = \mathcal{X}$ for each $B \in \mathfrak{B}$), evidently $\forall B, B' \in \mathfrak{B}. WR_B(C_B, C_{B'})$.
 Since, for any $B, B' \in \mathfrak{B}$, $C_{B'} \subseteq B$, evidently further, $C_{B'} \subseteq \partial(B')$, and so $WR_B(C_B, \partial(\{B'\}))$.
 Thus evidently, for every $\beta \in \mathfrak{B}_0^{FOC(\mathfrak{B})}(B)$, $WR_B(C_B, \partial(\beta))$, and so $\mathfrak{B}$ is regular. ■

OBSERVATION 16.1.2. For any partition Π_Y of a non-empty subset $Y \subseteq \mathcal{X}$ of the state set $\mathcal{X}$ of a FSM $\mathcal{M}$ with LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, such that $\forall C, C' \in \Pi_Y. WR(C, C')$, the FOC meta-machine $\mathcal{M}^{FOC(\mathfrak{B}_0)}$ corresponding to the maximal centered cover, $\mathfrak{B}_0$, with centers Π_Y , is (deterministic and) complete: that is, $\forall B, B' \in \mathfrak{B}_0, WR_B(C_B, \partial(\{B'\}))$.

Proof. This is clear from the proof of Proposition 16.1.2. ■

Corollary 16.1.1 tells us that if $\mathfrak{B}$ is regular, then the FOC meta-machine $\mathcal{M}^{FOC(\mathfrak{B})}$ is actually (state-to-state) controllable. We are thus motivated to now examine yet other properties of covers. We formulate the notions of cover *center controllability* and *completeness* in the following section.

16.2. Complete and Center Controllable Covers

DEFINITION 16.2.1. A block $B \in \mathfrak{B}$ of centered cover $\mathfrak{B}$ of a FSM with LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$ is *center controllable* if for any pair of states x and y in the center of B , x is B -constrained winning reachable to y . In other words, if

$$\forall x, y \in C_B. WR_B(x, \{y\}).$$

The cover $\mathfrak{B}$ is *center controllable* if B is center controllable for every block $B \in \mathfrak{B}$.

DEFINITION 16.2.2. A centered cover $\mathfrak{B}$ of a FSM with LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$ is *center complete*, or simply *complete* if every state belongs to the center of some block. That is, if

$$\mathcal{X} = \bigcup_{B \in \mathfrak{B}} C_B.$$

EXAMPLE 16.2.1. A complete center controllable cover of (the LTS of) a small FSM is illustrated in figure 16.1. The cover illustrated, consisting of two members B_1 and B_2 , with centers as indicated, is evidently both complete and center controllable. For example, in order to steer from state 2 to state 5 while staying in block B_2 , an input b is first applied. If the state resulting is 3, then a second application of the input b drives the state “home” to 5. If the state resulting (from the first application of the input b) is instead 4, then an application of input c drives the state to 3, and then input b may be applied to drive the state home to 5 as before. Note that in this example B_1 is in fact a minimal subset B of the state space $\{1, 2, 3, 4, 5\}$, for which there exists a state $x \neq 1$, such that $WR_B(1, \{x\})$. Similarly B_2 is a minimal subset $B \subseteq \{1, 2, 3, 4, 5\}$ for which there exists a state $x \neq 5$, such that $WR_B(5, \{x\})$ and $WR_B(x, \{5\})$. Thus this example attempts to also illustrate a situation where controllability requirements might disallow non-trivial partitions, though (centered) covers prevail.

PROPOSITION 16.2.1. If $\mathfrak{B}$ is a complete center controllable cover of a FSM $\mathcal{M}$ with LTS $\langle \mathcal{Y}, \mathcal{U}, \mathcal{E} \rangle$, and $\forall B, B' \in \mathfrak{B}. WR(C_B, C_{B'})$, (ie. $\mathcal{M}$ is center-to-center controllable), then

$$\forall B, B' \in \mathfrak{B}. WR^{FOC(\mathfrak{B})}(B, \{B'\}), \text{ (ie. the meta-machine } \mathcal{M}^{FOC(\mathfrak{B})} \text{ is controllable).}$$

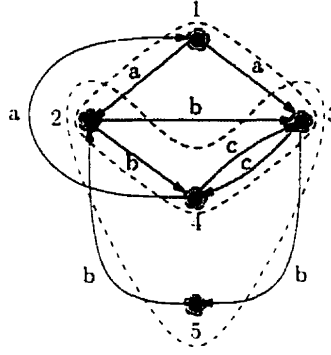
Proof. Let $B, B' \in \mathfrak{B}$.

If $B = B'$, then trivially, $WR^{FOC(\mathfrak{B})}(B, \{B'\})$, and so evidently, may assume $B \neq B'$.

By induction on $\rho = \rho(B, B') = \min\{\mu R(s, C_{B'}, -) \mid s \in C_B\}$.

If $\rho = 0$ then evidently $C_B \cap C_{B'} \neq \emptyset$, which contradicts the assumption that $B \neq B'$.

Assume the result for $0 \leq \rho < N$ and that now $\rho = N$. Let $s \in C_B$ be a center point of B such that $\mu R(s, C_{B'}, -) = N$. Since $WR(s, C_{B'})$, there exists $a \in fwm(s, C_{B'})$. Consider $S_1 \equiv \mathcal{E}(s, a, -)$.



Complete Center Controllable Cover: $\{B_1, B_2\}$.

where:

$$\begin{aligned} B_1 &= \{1, 2, 3, 4\}, & B_2 &= \{2, 3, 4, 5\}, \\ C_{B_1} &= \{1, 4\} & \text{and} & & C_{B_2} &= \{2, 3, 5\}. \end{aligned}$$

FIGURE 16.1. A Complete Center Controllable Cover of a 5 state FSM

If $S_1 \cap C_B \neq \emptyset$, then evidently we have $\mu R(s', C_{B'}, -) < N$, for some $s' \in S_1 \cap C_B$, but this contradicts the assumption that $\rho(B, B') = N$. Thus we may assume $S_1 \cap C_B = \emptyset$.

Let $\beta_1 \subseteq \mathfrak{B}$ be a minimal collection such that $S_1 \subseteq \bigcup_{B'' \in \beta_1} C_{B''}$.

and for each $s' \in S_1$, let $B(s') \in \beta_1$ denote a member of β_1 such that $s' \in C_{B(s')}$.

Such a β_1 obviously exists since $\mathfrak{B}$ is center complete. Evidently $\forall s' \in S_1$, ($B(s') \neq B$ and), $\mu R(s', C_{B'}, -) < N$, so $\rho(B(s'), B') < N$, and so by the induction hypothesis, $WR^{FOC(\mathfrak{B})}(B(s'), \{B'\})$.

But since $s \in C_B$, B is center controllable, and evidently $\emptyset \neq S_1 = \mathcal{E}(s, a, -) \subseteq \partial(\beta_1)$.

clearly $WR_B(C_B, \partial(\beta_1))$. Thus $WR^{FOC(\mathfrak{B})}(B, \beta_1)$, and therefore

(since for every $B' \in \beta_1$ there is some $s' \in S_1$ such that $B' = B(s')$), $WR^{FOC(\mathfrak{B})}(B, \{B'\})$. ■

COROLLARY 16.2.1. Suppose $\mathfrak{B}$ is a complete center controllable cover of a FSM $\mathcal{M}$ with LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$. Then $\mathcal{M}$ is center-to-center controllable, (ie. $\forall B, B' \in \mathfrak{B}$. $WR(C_B, C_{B'})$).

if and only if

$$\mathcal{M}^{FOC(\mathfrak{B})} \text{ is controllable, (ie. } \forall B, B' \in \mathfrak{B}. WR^{FOC(\mathfrak{B})}(B, \{B'\})).$$

Proof. $\Rightarrow$: from Proposition 16.2.1.

$\Leftarrow$: from Theorem 14.3.2. ■

COROLLARY 16.2.2. Suppose $\mathfrak{B}$ is a complete center controllable cover of a FSM $\mathcal{M}$ with LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$. Then $\mathcal{M}$ is controllable, (ie. $\forall s, s' \in \mathcal{X}$. $WR(s, \{s'\})$),

if and only if

$$\mathcal{M}^{FOC(\mathfrak{B})} \text{ is controllable, (ie. } \forall B, B' \in \mathfrak{B}. WR^{FOC(\mathfrak{B})}(B, \{B'\})).$$

Proof. By Corollary 16.2.1, it suffices to show that

$\mathcal{M}$ is controllable, (ie. $\forall s, s' \in \mathcal{X}. WR(s, \{s'\})$),

if and only if

$\mathcal{M}$ is center-to-center controllable, (ie. $\forall B, B' \in \mathfrak{B}. WR(C_B, C_{B'})$).

If $\forall s, s' \in \mathcal{X}. WR(s, \{s'\})$, then clearly $\forall B, B' \in \mathfrak{B}. WR(C_B, C_{B'})$,
so if $\mathcal{M}$ is controllable, then $\mathcal{M}$ is center-to-center controllable.

Conversely, suppose $\mathcal{M}$ is center-to-center controllable, and let $s, s' \in \mathcal{X}$.

Since $\mathfrak{B}$ is complete, there exist $B, B' \in \mathfrak{B}$, such that $s \in C_B$ and $s' \in C_{B'}$.

Since $\mathcal{M}$ is center-to-center controllable, $WR(s, C_{B'})$.

Finally since $\mathfrak{B}$ is center controllable, $WR_{B'}(C_{B'}, \{s'\})$, and so evidently, $WR(s, \{s'\})$.

Since $s, s' \in \mathcal{X}$, were chosen arbitrarily, evidently $\mathcal{M}$ is controllable. ■

Note that if a cover $\mathfrak{B}$ of a FSM $\mathcal{M}$ with LTS $\langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, is only complete, but not center controllable, then, even if for every cover member $B \in \mathfrak{B}$, $\mathfrak{H}(B) = B$, the center-to-center controllability of $\mathcal{M}$ is not sufficient to guarantee the controllability of the meta-machine $\mathcal{M}^{FOC(\mathfrak{B})}$. Indeed, controllability of the meta-machine $\mathcal{M}^{FOC(\mathfrak{B})}$ may fail even if $\mathcal{M}$ is in fact controllable. Figure 16.2 illustrates a complete cover, $\mathfrak{B}$, of a 6 state controllable LTS, comprised of only three block members, (I, II and III), each of which is fully centered. While the LTS is evidently controllable, the corresponding meta-machine, also depicted, is clearly not controllable, (block III is not winningly reachable from either of blocks I or II).

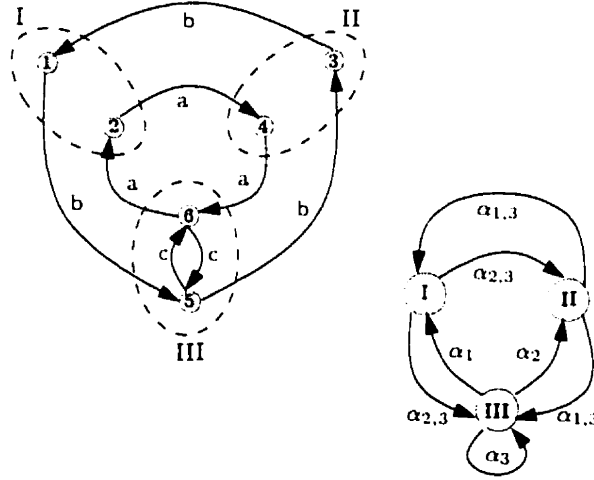


FIGURE 16.2. A Complete Cover of a 6 state FSM and its corresponding (non-controllable) meta-machine

DEFINITION 16.2.3. For any (FSM $\mathcal{M}$ with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$,
let $CCCC(\mathcal{X})$ denote the collection of all complete center controllable (CCC) covers of $\mathcal{X}$.

OBSERVATION 16.2.1. For any (FSM $\mathcal{M}$ with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$,

- 1) $CCCC(\mathcal{X})$ is a finite sub- $\vee$ -semi-lattice of $Covers(\mathcal{X})$,
(where $\vee$ is $\cup_2$, and the partial order is $\subseteq_2(\cdot)$), and in fact

- 2) $CCCC(\mathcal{X})$ forms a complete lattice, (where $\bigvee$ is $\mathbb{U}_2$, and the partial order is $\subseteq_{2(\cdot)}$), with top element the cover consisting of a single fully centered block which contains all of $\mathcal{X}$, and bottom element the cover consisting of the (fully centered) singleton blocks.

Proof. (Sketch): If $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are two complete center controllable covers of $\mathcal{X}$, then clearly $\mathfrak{B}_1 \mathbb{U}_2 \mathfrak{B}_2$ is 1) complete, and 2) center controllable, and hence, CCC. ■

OBSERVATION 16.2.2. If $\mathfrak{C}, \mathfrak{C}' \in CCCC(\mathcal{X})$, for an (FSM $\mathcal{M}$ with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, and $\mathfrak{C} \subseteq_{2(\cdot)} \mathfrak{C}'$, then

$$\forall B' \in \mathfrak{C}', \exists B \in \mathfrak{C}. B \subseteq B'.$$

Proof. Suppose $\mathfrak{C} \subseteq_{2(\cdot)} \mathfrak{C}'$.

Then evidently there exists a function $f: \mathfrak{C} \rightarrow \mathfrak{C}'$ such that for each $B \in \mathfrak{C}$, $B \subseteq f(B)$.

Let $\mathfrak{B} \equiv \{f(B) \mid B \in \mathfrak{C}\}$. Then $\mathfrak{B} \subseteq \mathfrak{C}'$, and since $\mathfrak{C}, \mathfrak{C}' \in CCCC(\mathcal{X})$, evidently $\mathfrak{B} \in CCCC(\mathcal{X})$.

Suppose $B' \in \mathfrak{C}' \setminus \mathfrak{B}$. Then, as $\mathfrak{B} \in CCCC(\mathcal{X})$, $C_{B'} \cap C_{B''} \neq \emptyset$, for some $B'' \in \mathfrak{B}$.

But since $\mathfrak{C}' \in CCCC(\mathcal{X})$, (and $B', B'' \in \mathfrak{C}'$), $B' = B''$, for some $B'' \in \mathfrak{B}$, which contradicts our choice of $B' \in \mathfrak{C}' \setminus \mathfrak{B}$. Thus $\mathfrak{C}' \subseteq \mathfrak{B}$, and so $\mathfrak{C}' = \mathfrak{B}$.

Therefore, for any $B' \in \mathfrak{C}'$, there exists some $B \in \mathfrak{C}$ such that $B' = f(B)$, and hence, such that $B \subseteq B'$. ■

COROLLARY 16.2.3. If $\mathfrak{C}, \mathfrak{C}' \in CCCC(\mathcal{X})$, for an LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, and $\mathfrak{C} \subseteq_{2(\cdot)} \mathfrak{C}'$, then

$$\sharp(\mathfrak{C}') \leq \sharp(\mathfrak{C}).$$

Proof. Suppose $\mathfrak{C} \subseteq_{2(\cdot)} \mathfrak{C}'$.

Then by Observation 16.2.2, for any $B' \in \mathfrak{C}'$, there exists some $B \in \mathfrak{C}$ such that $B \subseteq B'$. Thus there exists a function $f: \mathfrak{C}' \rightarrow \mathfrak{C}$ such that for each $B' \in \mathfrak{C}'$, $f(B') \subseteq B'$.

If for some $B'_1, B'_2 \in \mathfrak{C}'$, $f(B'_1) = f(B'_2)$, then $f(B'_1) = f(B'_2) \subseteq B'_1$, and $f(B'_1) = f(B'_2) \subseteq B'_2$.

Thus, in particular, $C_{f(B'_1)} \subseteq C_{B'_1} \cap C_{B'_2}$, and so $C_{B'_1} \cap C_{B'_2} \neq \emptyset$.

It follows that $B'_1 = B'_2$. Thus $f: \mathfrak{C}' \rightarrow \mathfrak{C}$ is 1-1, and so $\sharp(\mathfrak{C}') \leq \sharp(\mathfrak{C})$. ■

OBSERVATION 16.2.3. Suppose $\mathfrak{C}, \mathfrak{C}' \in CCCC(\mathcal{X})$, for an FSM $\mathcal{M}$ with LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, and $\mathfrak{C} \subseteq_{2(\cdot)} \mathfrak{C}'$. Then if the FOC meta-machine $\mathcal{M}^{\mathfrak{C}}$ is controllable, the FOC meta-machine $\mathcal{M}^{\mathfrak{C}'}$ is controllable.

Proof. Since $\mathfrak{C}$ and $\mathfrak{C}'$ are CCC, by Corollary 16.2.2,

$\mathcal{M}^{\mathfrak{C}}$ is controllable if and only if $\mathcal{M}$ is controllable if and only if $\mathcal{M}^{\mathfrak{C}'}$ is controllable. ■

DEFINITION 16.2.4. For any (FSM $\mathcal{M}$ with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, let $FCCCC(\mathcal{X})$ denote the collection of all fully centered complete center controllable covers of $\mathcal{X}$, that is, $FCCCC(\mathcal{X}) \equiv FC(\mathcal{X}) \cap CCCC(\mathcal{X})$.

OBSERVATION 16.2.4. For any (FSM $\mathcal{M}$ with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, and any cover $\mathfrak{C} \in CCCC(\mathcal{X})$, $\mathfrak{C}$ is fully centered if and only if $\mathfrak{C}$ is a partition if and only if $\mathfrak{C}$ is center separated.

In particular, $FCCCC(\mathcal{X}) = FC(\mathcal{X}) \cap CCCC(\mathcal{X}) = SepCovers(\mathcal{X}) \cap CCCC(\mathcal{X})$.

EXAMPLE 16.2.2. Let Π be any IBC partition [13, 42, 41, 43], of a deterministic (FSM $\mathcal{M}$ with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$. Then if $\mathfrak{C}_\Pi$ denotes the centered cover obtained from Π by assigning to each block $B \in \Pi$, the full center $C_B \equiv B$, $\mathfrak{C}_\Pi \in FCCCC(\mathcal{X})$. In other words, the fully centered CCC covers correspond exactly to the IBC-partitions [13, 42, 41, 43] when $\mathcal{M}$ is deterministic.

OBSERVATION 16.2.5. For any (FSM $\mathcal{M}$ with) LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$, $FCCCC(\mathcal{X})$ is a finite sub-lattice of $CCCC(\mathcal{X})$, (where $\vee$ is $\uplus_2$, and the partial order is $\leq_{2^{\mathcal{X}}}$), on which $\uplus_1 = \uplus_2$.

Proof. (sketch) If $B, B' \in Blocks(\mathcal{X})$ are fully centered, then evidently

$$\mathbb{Q}_2(B, B') \Leftrightarrow \mathbb{Q}_1(B, B') \Leftrightarrow \mathbb{Q}_0(B, B') .$$

Thus evidently, for any collection $\mathfrak{C} \subseteq Blocks(\mathcal{X})$ of fully centered blocks, $confl_1(\mathfrak{C}) = confl_2(\mathfrak{C})$, so $Confl_1(\mathfrak{C}) = Confl_2(\mathfrak{C})$. Hence if $\mathfrak{B}_1, \mathfrak{B}_2 \in FC(\mathcal{X})$ are fully centered covers.

$\mathfrak{B}_1 \uplus_1 \mathfrak{B}_2 = \mathfrak{B}_1 \uplus_2 \mathfrak{B}_2$, and further, evidently, each block in $\mathfrak{B}_1 \uplus_1 \mathfrak{B}_2$ is fully centered, and so $\mathfrak{B}_1 \uplus_1 \mathfrak{B}_2 \in FC(\mathcal{X})$. Thus, if further $\mathfrak{B}_1, \mathfrak{B}_2 \in CCCC(\mathcal{X})$, $\mathfrak{B}_1 \uplus_1 \mathfrak{B}_2 \in FCCCC(\mathcal{X})$. ■

Evidently for any controllable LTS $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \rangle$,

for any partition Π of $\mathcal{X}$, there is a complete center controllable cover $\mathfrak{C}_\Pi \in CCCC(\mathcal{X})$, such that there is a bijection $f: \Pi \xrightarrow{\sim} \mathfrak{C}_\Pi$, where for any $B \in \Pi$, $B = C_{f(B)}$.

For instance, one such $\mathfrak{C}_\Pi$ is clearly obtained by taking, for each $B \in \Pi$, $f(B)$ to be the centered block, (with center $C_{f(B)} = B$ and) consisting of $\mathcal{X}$ itself.

This cover may be considered to be the trivial CCC cover (CCCC) with centers Π .

Less trivially, evidently there must exist ($\leq_{2^{\mathcal{X}}}$ -) minimal CCCC's with centers Π .

Such is simply a CCCC $\mathfrak{C}$ with centers Π , such that for no block $B \in \mathfrak{C}$, is there a proper subset $B' \subsetneq B$ where $C_B \subseteq B'$, and such that $\forall s, s' \in C_B$, $WR_{B'}(s, \{s'\})$.

CHAPTER 17

Some Regular Covers for a Transfer Line

17.1. Exogenous Inputs and Discrete Event Systems

There are natural translations of (1) deterministic FSMs with exogenous inputs, and (2) discrete event finite automata into non-deterministic FSMs.

(1) Consider a deterministic FSM $\mathcal{M} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{Y}, \Phi : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{X}, \eta : \mathcal{X} \rightarrow \mathcal{Y} \rangle$, where $\mathcal{U} = \mathcal{U}_c \cup \mathcal{V}$, $\mathcal{U}_c$ consisting of the set of available control inputs, and $\mathcal{V}$ consisting of the set of possible exogenous inputs. The exogenous inputs are supposed to represent external disturbances to the FSM which may occur, in an alternating, two-player fashion, between consecutive the issuance of inputs from the controller.

From the FSM $\mathcal{M}$ we may construct a non-deterministic FSM model $\mathcal{M}' \equiv \langle \mathcal{G}, \eta \rangle$, where $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}_c, \mathcal{E} \rangle$, and $\mathcal{E} \subseteq \mathcal{X} \times \mathcal{U}_c \times \mathcal{X}$ is the collection of triples $\langle x, u, z \rangle$, for $x, z \in \mathcal{X}$ and $u \in \mathcal{U}_c$, such that there exist some $y \in \mathcal{X}$ and some $v \in \mathcal{V}$, such that $\Phi(x, u) = y$, and $\Phi(y, v) = z$.

This mapping is illustrated in Figure 17.1 below.

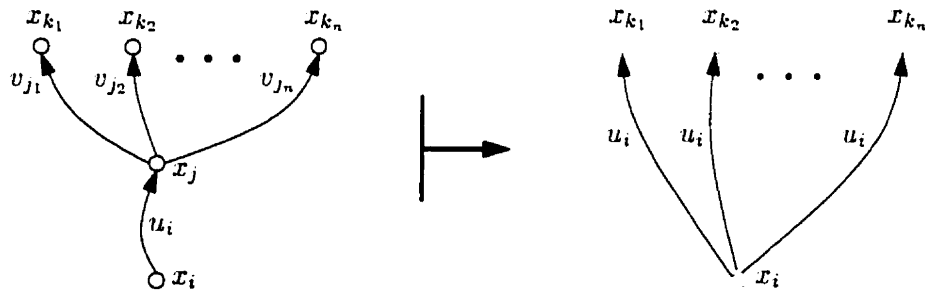


FIGURE 17.1. Transforming a deterministic FSM with exogenous inputs into a non-deterministic FSM.

(2) Consider a discrete event finite automata $\mathcal{G} \equiv \langle \mathcal{X}, \mathcal{U}, \mathcal{E} \subseteq \mathcal{X} \times \mathcal{U} \times \mathcal{X} \rangle$, where $\mathcal{U} = \mathcal{U}_c \cup \mathcal{V}$, $\mathcal{U}_c$ consisting of the set of *controllable* events, and $\mathcal{V}$ consisting of the *uncontrollable* events.

From the discrete event finite automata $\mathcal{G}$ we may construct a non-deterministic FSM

$\mathcal{M}' \equiv \langle \mathcal{G}', \eta' \rangle$, where $\mathcal{G}' \equiv \langle \mathcal{X}, \mathcal{U}_c \cup \{e\}, \mathcal{E}' \rangle$ and $\eta': \mathcal{X} \rightarrow \mathcal{X}$ is the identity on $\mathcal{X}$, and where $\mathcal{E}' \subseteq \mathcal{X} \times \mathcal{U}_c \cup \{e\} \times \mathcal{X}$ is the collection of triples $\langle x, u, z \rangle$, for $x, z \in \mathcal{X}$ and $u \in \mathcal{U}_c \cup \{e\}$, such that either:

$$u \in \mathcal{U}_c \text{ and } \langle x, u, z \rangle \in \mathcal{E},$$

$$u \in \mathcal{U}_c, \text{ for some } y \in \mathcal{X}, \langle x, u, y \rangle \in \mathcal{E}, \text{ and for some } v \in \mathcal{V}, \langle x, v, z \rangle \in \mathcal{E}, \text{ or finally,}$$

$$u = e \text{ and for some } v \in \mathcal{V}, \langle x, v, z \rangle \in \mathcal{E}.$$

The newly introduced input e represents the control input corresponding to non-deterministic result of *blocking all controllable edges* at a given node.

This mapping is illustrated in Figure 17.2 below.

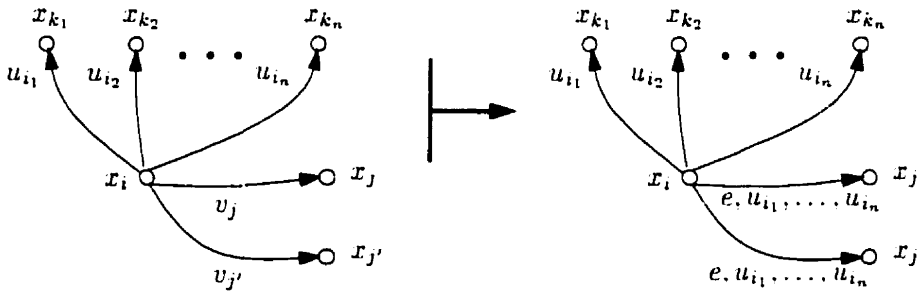


FIGURE 17.2. Transforming a discrete event finite automata into a non-deterministic FSM.

17.2. Example: Some Regular Covers for a Transfer Line

In [4], an aggregation method for the treatment of discrete event finite automata is presented and illustrated with a *transfer line* example. The transfer line consists of three machines (or work stations), $M1$, $M2$ and $M3$, three buffers, $B1$, $B2$ and $B3$, and a testing unit, TU , as illustrated in figure 17.3 below, taken from [4]. Each machine and each buffer can hold at most one *token* at any time. The flow of a token into the testing unit or any of the three machines can be controlled (blocked), but the flow of a token, (out of the testing unit or any of the three machines and), into a buffer cannot be prevented.

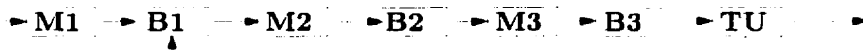


FIGURE 17.3. The transfer line

The *state space* considered for the example problem consists of septuples $\langle m_1, m_2, m_3, m_4, b_1, b_2, b_3 \rangle$, where $m_1, \dots, m_4 \in \{I, W\}$, and $b_1, \dots, b_3 \in \{0, 1\}$. With the testing unit identified with a fourth

machine, $m_i = I$ (W) corresponds to the condition that the i^{th} machine is *idle*, (*working*, respectively), while $b_j = 0$ (1) corresponds to the condition that the j^{th} buffer is empty (full, respectively).

Figure 17.4 below, is an abstract depiction of the *controllable subspace* for the example, with minor modification, also taken from [4]. Edges labeled 1,3,5 and 7 are controllable, (blockable), while edges labeled 2,4,6,8 and 9 are not.

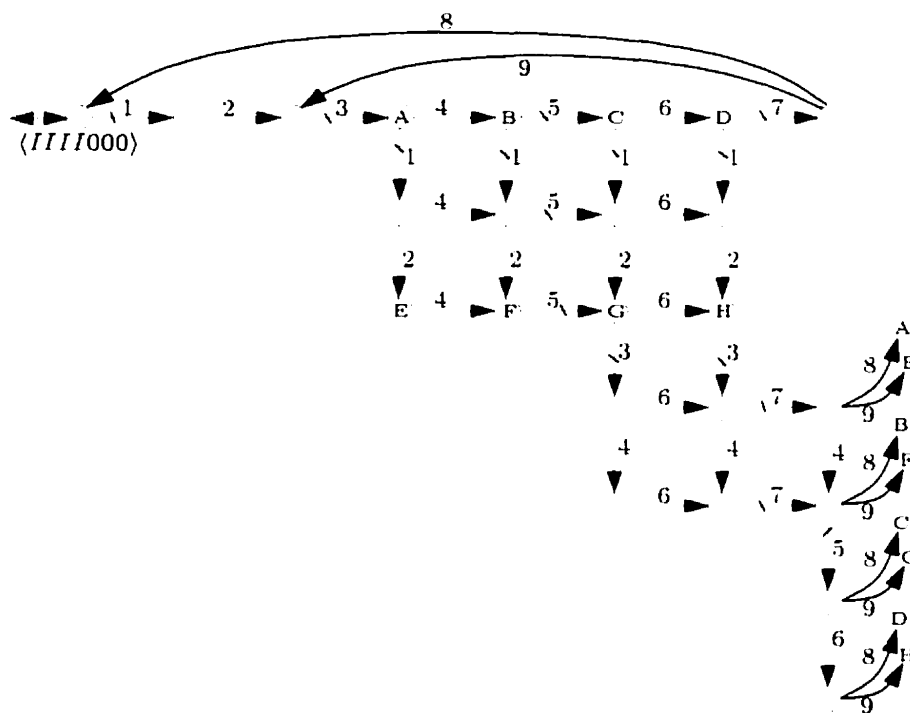


FIGURE 17.4. The transfer line: the controllable subspace.

As described above, a non-deterministic finite state-machine, $\mathcal{M}$, may be computed, in terms of which issues of controllability may be addressed. Figures 17.5, through 17.10 illustrate several possible regular covers for $\mathcal{M}$. To some extent these figures indicate the flexibility of regular covers, and some of the possible tradeoffs involved. For Figures 17.5, 17.7, and 17.10 the covers are also center-to-center controllable, (and hence the corresponding meta-machines are controllable), while Figures 17.6, 17.8, and 17.9, depict covers which are regular, but not center-to-center controllable.

More specifically, Figures 17.5 and 17.7, in contrast with Figures 17.6 and 17.8, respectively, indicate how (meta-machine) controllability can be lost if a center, (namely, that of block I, in Figures 17.5 and 17.6, and that of block IV, in Figures 17.7 and 17.8), is not chosen sufficiently small. Figure 17.9, in contrast to Figures 17.5 and 17.7, shows how (center-to-center) controllability may be lost if cover members, and their associated centers, are made too small. Finally Figure 17.10, in contrast with Figure 17.9, shows how controllability may be recovered, in this case by allowing blocks to

overlap, (in contrast, the covers depicted in Figures 17.5 through 17.9 are all partitions), by augmenting the center of a block (I) with a state (F) which formerly belonged to the center of another block (II), and finally redefining the second block and its associated center, (making the state G belong to the center to block II). Note that in Figure 17.10, in order for points in the in-boundary of block II to be steered within the block to its center, (consisting of the state G), they must pass through state F, and thus, while remaining in block II, must be steered through the center of block I.

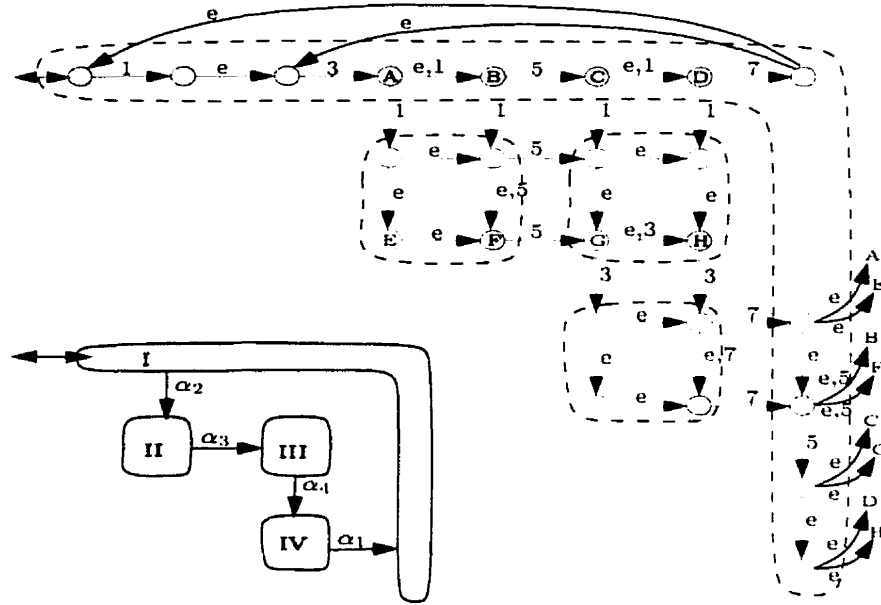


FIGURE 17.5. A regular center-to-center controllable cover for the transfer line. Example 1

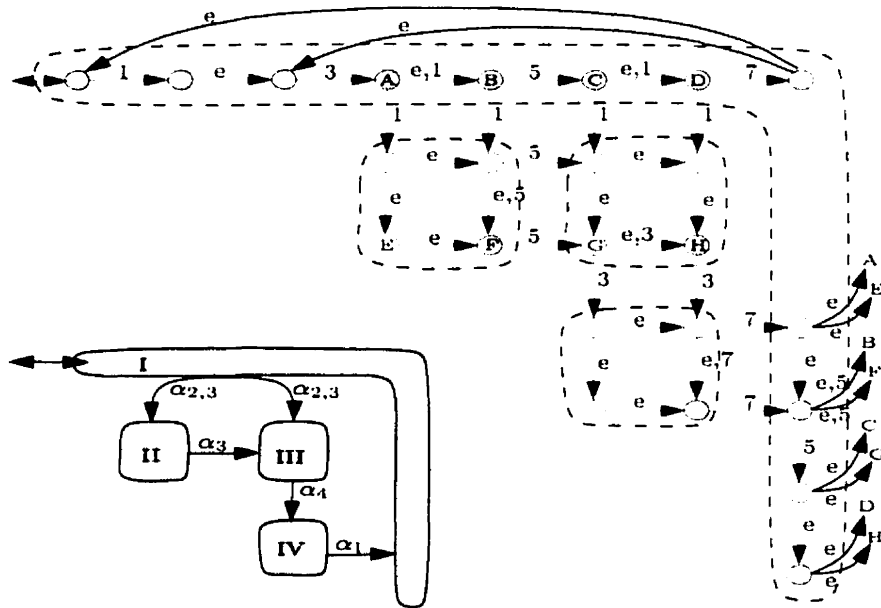


FIGURE 17.6. A regular cover for the transfer line. Example 2

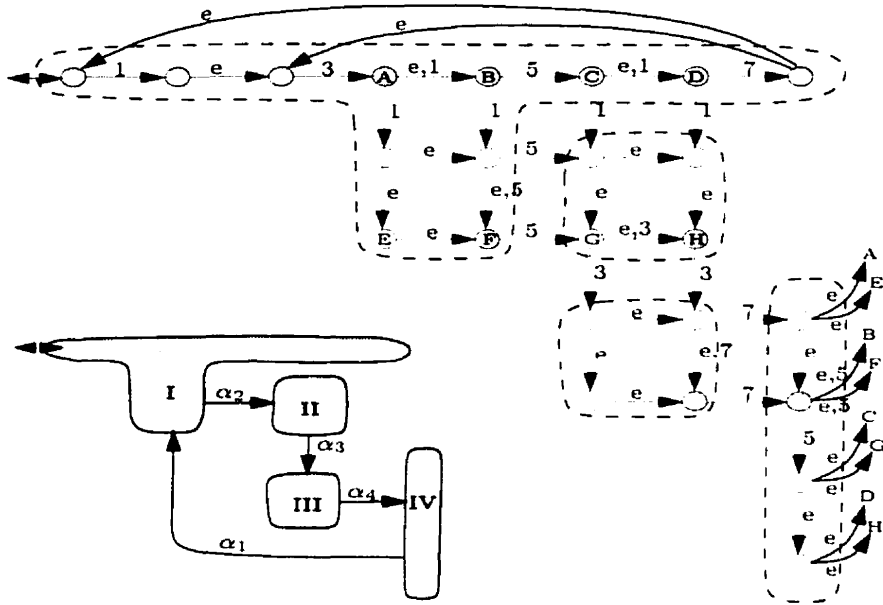


FIGURE 17.7. A regular center-to-center controllable cover for the transfer line. Example 3

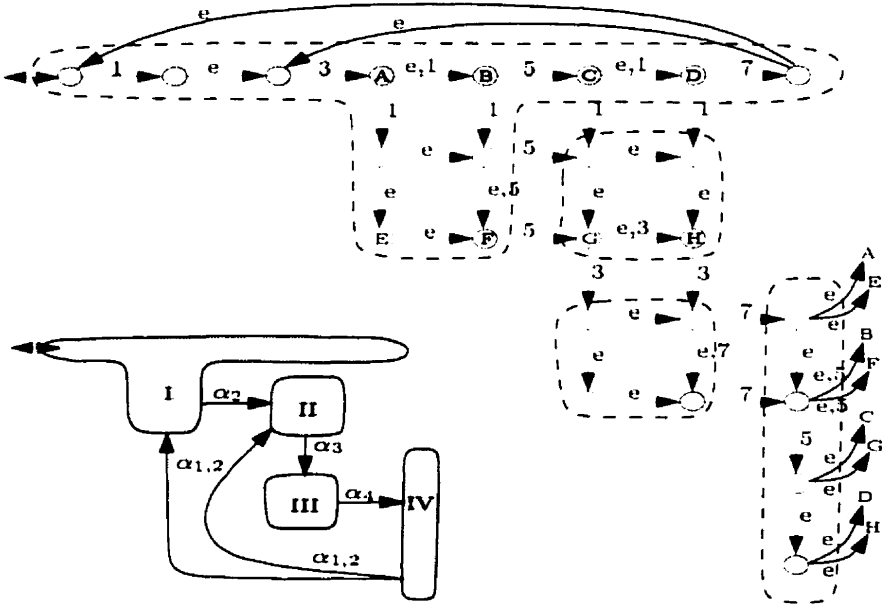


FIGURE 17.8. A regular cover for the transfer line. Example 4

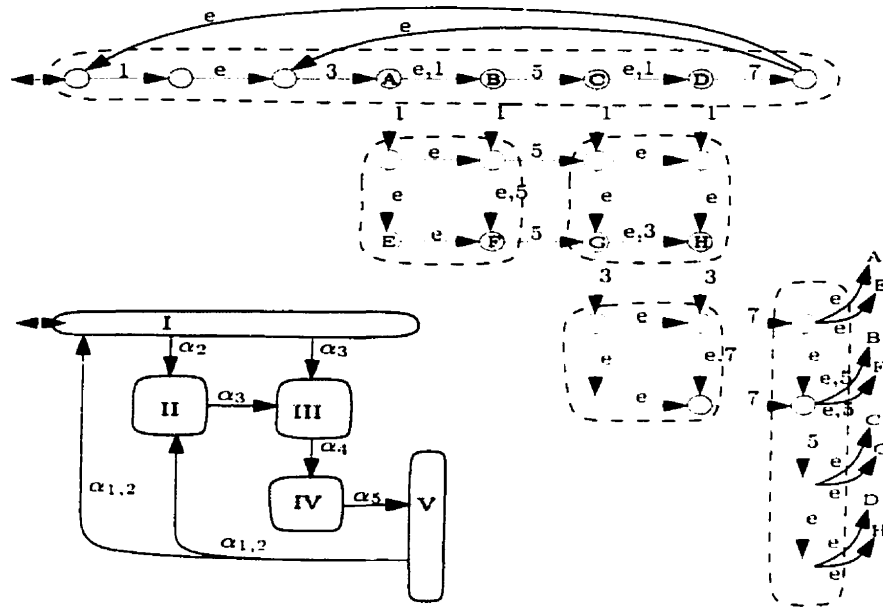


FIGURE 17.9. A regular cover for the transfer line. Example 5

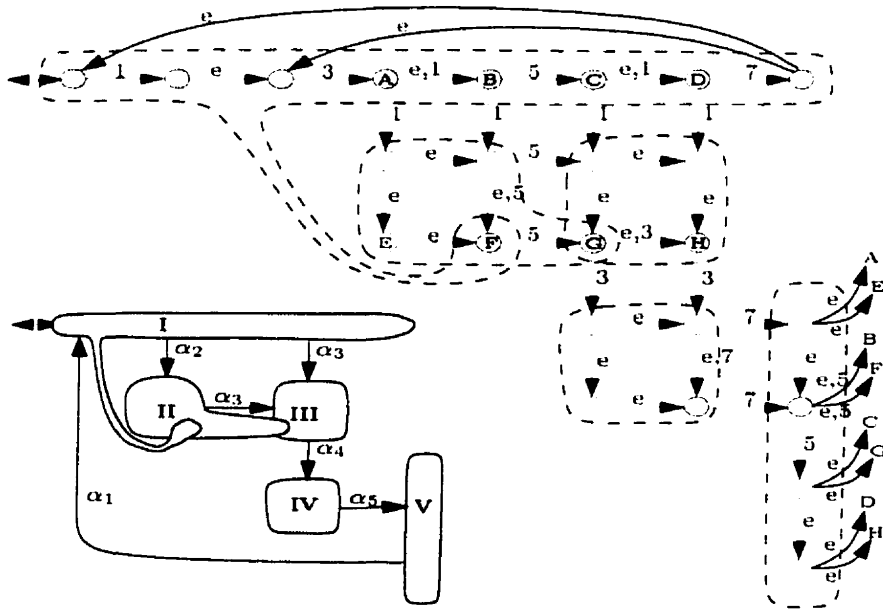


FIGURE 17.10. A regular center-to-center controllable cover for the transfer line. Example 6

CHAPTER 18

Concluding Remarks

Thus, we have formalized some primitives corresponding to the notions of (winning, or robust) reachability, observability, and coupled “reachability with state estimation”, or *coherent reachability*, for FSMs. We have proposed aggregation techniques in terms of *centered covers*, whereby a corresponding meta-level FSM may be constructed, and by construction, its control may be utilized in a hierarchical fashion for the (primarily, center-to-center) controllability of the original FSM. This abstraction process may clearly be iterated, and a multiple-level control hierarchy correspondingly constructed. Specific classes of centered covers have been proposed for the so called *fully observed case*¹. In particular, *quasi-convex* covers have the property that *center-to-center controllability* of the base FSM is sufficient to guarantee state-to-state controllability of the abstracted FSM, and additionally, preservation properties under certain lattice theoretic operations. However the criterion for quasi-convexity is rather demanding. This criterion is greater relaxed in the criterion for *regularity*. Though regular covers also have the property that center-to-center controllability of the base FSM is sufficient to guarantee state-to-state controllability of the abstracted FSM, they are not closed under the analogous lattice theoretical operations. *Complete center controllable* covers have been introduced and proposed for the abstraction of state-to-state controllable FSMs. The proposed machine abstraction with respect to such covers preserves controllability, and moreover, the class is closed with respect to all of the “standard” lattice operations.

Future work along these lines may include some incorporation of the notions introduced here into modified formulations of COCOLOG, (Macro-COCOLOG, Hierarchical-COCOLOG, and possibly also Hybrid-COCOLOG), for the generalization to the non-deterministic case. Perhaps somewhat more concretely, future work might also include a methodological treatment of some “reasonably generic” hybrid system control problem, along the lines of the methods presented herein: specifically, the assignment of *centers* to *modes*, and the calculation of corresponding *in-boundaries*, (to create a discrete meta-level FSM, the local or mode-wise possibly continuous control being directed according to the computed optimal non-deterministic transitions of the FSM). Though here, exact computation of *in-boundaries* may prove infeasible, it seems quite plausible that, at least for non-optimal control, conservative estimates may suffice and also be computationally accessible. Finally, and not unrelated to the last project, perhaps some of the ideas presented here could find application in the solution to a real-world control problem, where the non-determinism (within the model) arises as a result of sensor

¹“So called” because this case includes the situation where the FSM is differentiated and the initial state is either known or may be determined via state estimation.

uncertainty, centers possibly corresponding to regions of high sensor acuity and correspondingly relatively high degrees of control.

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APPENDIX A

$PC_{<}$, A System For Binary Clausal Refutation-Resolution

A.1. Lifting in $PC_{<}$

For Lemma 2.2.1:

Proof: Let $L_1, \dots, L_n$, denote the distinct negative P instances of B indexed in the order in which they occur in B . Let $B = D_0, C_0, \dots, C_{k-1}$, as leaves, and $D_1, \dots, D_k = D$, as non-leaves, indicate the $PC_{<}^{+1}$ -deduction Π of D , as in Definition 2.1.2, (for some $k \geq 1$).

The C_i 's (for $0 \leq i \leq k-1$), need not be distinct, but we may assume that no distinct C_i 's share a variable in common, and that no C_i shares a variable with B . Let C_i' denote the non-empty subset of C_i which is resolved upon.

As D is obtained in its $PC_{<}^{+1}$ -deduction, Π , substitution instances of the L_i , (for $1 \leq i \leq n$), are inherited, resolved away, and unified (identified). Here, we are assuming that in the $PC_{<}^{+1}$ -deduction, identical literals will be immediately identified, (so that duplicates are disposed of).

In order to keep track of the inheritance, let us denote a surviving substitution instance of L_i in D_j by $L_i^{(j)}$, and further associate to each surviving (substitution instance, $L_i^{(j)}$, of), L_i in D_j (for $0 \leq j \leq k$), a set of original indices, $\alpha(i, j)$, where (for each $i: 1 \leq i \leq n$ and each $j: 0 \leq j \leq k$), $\alpha(i, j)$ is defined as follows:

$j = 0$: $\alpha(i, j)$ is the set $\{i\}$,

$j > 0$: for each L_i surviving in D_j , $\alpha(i, j) \equiv \bigcup \{ \alpha(i', j-1) : L_{i'}^{(j)} = L_i^{(j)} \}$

and otherwise, (if L_i does not survive in D_j), let $\alpha(i, j) \equiv \emptyset$.

Now it is not difficult to check that $\forall j: 0 \leq j \leq k, \forall i: 1 \leq i \leq n, \forall i': 1 \leq i' \leq n$,

$\alpha(i, j) \neq \emptyset$ if and only if L_i survives in D_j , (as $L_i^{(j)}$), and

$\alpha(i, j) \cap \alpha(i', j) \neq \emptyset$ implies $\alpha(i, j) = \alpha(i', j)$ and $L_{i'}^{(j)} = L_i^{(j)}$.

Further, whenever $\alpha(i, j) \neq \emptyset$, $i \in \alpha(i, j)$, and $i' \in \alpha(i, j) \Rightarrow L_{i'}^{(j)} = L_i^{(j)} \in D_j$.

Thus L_i survives in D_j if and only if $i \in \bigcup \{ \alpha(i', j) : 1 \leq i' \leq n \}$.

At each stage $j: 0 \leq j \leq k$, for any L_i surviving in D_j , $\alpha(i, j)$ is just the set of original indices i' for which a substitution instance, $L_{i'}^{(j)}$, of $L_{i'} \in B$ has (been been inherited from D_0 , and) survived in D_j , and such that $L_{i'}^{(j)} = L_i^{(j)}$. More simply stated, $\alpha(i, j)$ is just the set of original indices i' for which (a substitution instance of) the literal $L_{i'}$ has survived in stage j , and is now

represented by $L_i^{(j)}$.

It is also not difficult to see that $\forall i: 1 \leq i \leq n, \forall j: 0 \leq j < k, \alpha(i, j+1) \neq \emptyset \Rightarrow \alpha(i, j) \subseteq \alpha(i, j+1)$. Thus, if also $1 \leq i' \leq n$ for some $i' \neq i$, and $\alpha(i', j) \cap \alpha(i, j+1) \neq \emptyset$, then since either $\alpha(i, j+1) = \alpha(i', j+1)$, or $\alpha(i, j+1) \cap \alpha(i', j+1) = \emptyset$, it easily follows that $\alpha(i', j) \subseteq \alpha(i, j+1)$. Also, $\alpha(i, k) = \emptyset, \forall i: 1 \leq i \leq n$, while $\forall j: 0 \leq j \leq k-1 \exists i: 1 \leq i \leq n$ such that $\alpha(i, j) \neq \emptyset$.

For each $i: 1 \leq i \leq n$, let $j(i)$ denote the integer $j: 0 \leq j \leq k-1$ such that $\alpha(i, j) \neq \emptyset$ and $\alpha(i, j+1) = \emptyset$. That is, L_i is resolved away at the $j(i) + 1^{th}$ stage.

Clearly as i ranges over $1 \leq i \leq n$, $j(i)$ ranges over the entire set of $j: 0 \leq j \leq k-1$.

For each stage $j: 0 \leq j \leq k-1$ let $\beta(j)$ denote: $\bigcup\{\alpha(i, j): 1 \leq i \leq n\} \setminus \bigcup\{\alpha(i, j+1): 1 \leq i \leq n\}$. That is, $\beta(j)$ denotes the set of indices i , such that L_i is resolved away at the $j + 1^{th}$ step.

Clearly $j \neq j'$ if and only if $\beta(j) \neq \beta(j')$, and $\forall j: 0 \leq j \leq k-1, \beta(j) \neq \emptyset$.

As sets ordered under inclusion, the $\alpha(i, j)$, (for $1 \leq i \leq n; 0 \leq j \leq k-1$), form a poset, whose maximal elements form a partition of the set $\{1, \dots, n\}$ of original indices. Evidently, $i \in \beta(j(i))$, and $\{\beta(j): 0 \leq j \leq k-1\}$ is also a partition of the set $\{1, \dots, n\}$ of original indices.

D is just (some re-ordering of some alphabetic variant of),

$B\sigma \setminus \{L_i\sigma: 1 \leq i \leq n\} \cup C_0 \setminus C_0'\sigma \cup \dots \cup C_{k-1}\sigma \setminus C_{k-1}'\sigma$,

where σ is the most general unifier of: $\{\neg L_i: i \in \beta(0)\} \cup C_0' \dots \cup \{\neg L_i: i \in \beta(k-1)\} \cup C_{k-1}'$, (each set member of this set being unified under σ to a singleton).

For some re-ordering B' of B , construct a $PC_{<}^{+1}$ -deduction from B' , as follows:
as each (substitution instance of) L_i becomes left-most, resolve away $\{L_{i'}: i' \in \beta(j(i))\}$ against (the non-empty subset $C_{j(i)}'$ in) $C_{j(i)}$.

At each stage, the entire set of (inherited, substitution instances of) L_i such that $i \in \beta(j)$, (for some $j: 0 \leq j \leq k-1$), are resolved away. There must be exactly k stages, (the last of the substitution instances of the L_i being resolved away exactly in the k^{th} stage), and distinct stages necessarily involve distinct β 's, and hence distinct $j(i)$'s.

At the point when all (inherited substitution instances of) L_i 's have been resolved away, (at the k^{th} stage), evidently we will have a $PC_{<}^{+1}$ -deduction Π' , from B' , and $C_0, \dots, C_{k-1}$, (so that $\forall C \in \mathcal{C}, k_{\Pi'}(C) = k_{\Pi}(C)$), again, of (some ordering of some alphabetic variant of), $B\sigma \setminus \{L_i\sigma: 1 \leq i \leq n\} \cup C_0 \setminus C_0'\sigma \cup \dots \cup C_{k-1}\sigma \setminus C_{k-1}'\sigma$, with σ as before. ■

For Lemma 2.2.2:

Proof: Let A, A', B, D , and D' as in the statement of the lemma, (and without loss of generality, assume no two share a common variable).

By assumption, there is some substitution θ_A , (which we may assume, has domain contained in the set of variables of A'), such that (modulo clause ordering) $A'\theta_A = A$.

Clearly A' must be pure-positive with respect to P and have $<$ -maximal predicate symbol P , and D' is mixed with respect to P , with $<$ -maximal predicate symbol P . By the assumed separation of

variables, evidently (we may assume the domain of θ_D is a subset of the variables of D , and), there is a substitution θ , say, (for example, the composite, $\theta_A\theta_D$), such that (modulo clause ordering) $A'\theta = A$ and $D'\theta = D$, (and $L_1\theta$ occurs as the left-most inequality in D).

For some non-empty subsets $A_R \subseteq A$, and $D_R \subseteq D$, (say), the left-most negative P instance $L_1\theta$, in D occurring in D_R , $A_R \cup \{\neg L: L \in D_R\}$ unifies, with a most general unifying substitution σ , say, to $\{L_0\}$, say.

$B \equiv A\sigma \setminus \{L_0\} \cup D\sigma \setminus \{\neg L_0\}$. Trivially, $A'\theta\sigma = A\sigma$, $D'\theta\sigma = D\sigma$, $\neg L_0 \in D'\theta\sigma$, and $L_0 \in A'\theta\sigma$.

Let $A'_R \equiv \{L \in A': L\theta\sigma = L_0\}$, and let $D'_R \equiv \{L \in D': L\theta\sigma = \neg L_0\}$.

Then clearly $\emptyset \neq A'_R \subseteq A'$, and $\emptyset \neq D'_R \subseteq D'$, and the left-most negative P instance, L_1 , in D' occurs in D'_R .

$\theta\sigma$ unifies $A'_R \cup \{\neg L: L \in D'_R\}$ to $\{L_0\}$, so $A'_R \cup \{\neg L: L \in D'_R\}$ unifies, with most general unifying substitution φ (say), to $\{L_0'\}$, (say), for some literal L_0' such that, for some substitution ρ , $L_0'\rho = L_0$, where further, $\theta\sigma = \varphi\rho$.

Let $B' \equiv A'\varphi \setminus \{L_0'\} \cup D'\varphi \setminus \{\neg L_0'\}$. Then evidently, (for some ordering), $B' \in PC_{<}^1(A', D')$.

Clearly $A'\varphi\rho \setminus A'_R\varphi\rho \subset (A'\varphi \setminus A'_R\varphi)\rho$.

Conversely, suppose $L'' \in (A'\varphi \setminus A'_R\varphi)\rho$. Then $L'' = L'\rho$ for some $L' \in A'\varphi \setminus A'_R\varphi$.

But $A'_R\varphi = \{L\varphi \in A'\varphi: L\varphi\rho = L_0\}$, and $A'_R\varphi\rho = \{L_0\}$. Thus, $L'' = L'\rho$ for some $L' \in A'\varphi$ such that $L'\rho \neq L_0$, and so, $L'' \in A'\varphi\rho \setminus A'_R\varphi\rho$.

Thus $(A'\varphi \setminus A'_R\varphi)\rho = A'\varphi\rho \setminus A'_R\varphi\rho$. Similarly, $(D'\varphi \setminus D'_R\varphi)\rho = D'\varphi\rho \setminus D'_R\varphi\rho$.

Thus $B'\rho = (A'\varphi \setminus \{L_0'\})\rho \cup (D'\varphi \setminus \{\neg L_0'\})\rho = (A'\varphi \setminus A'_R\varphi)\rho \cup (D'\varphi \setminus D'_R\varphi)\rho$
 $= A'\varphi\rho \setminus A'_R\varphi\rho \cup D'\varphi\rho \setminus D'_R\varphi\rho = A'\theta\sigma \setminus \{L_0\} \cup D'\theta\sigma \setminus \{\neg L_0\}$
 $= A\sigma \setminus \{L_0\} \cup D\sigma \setminus \{\neg L_0\} = B$, (modulo ordering). ■

For Lemma 2.2.3:

Proof: By induction on the depth k , of the $PC_{<}^{+1}$ -deduction Π .

If $k = 1$, then $D \in PC_{<}^1(C_0, B)$ and D must be pure-positive with respect to P .

As B is a substitution instance of B' , (modulo symmetry), $B'\theta = B$ for some substitution θ .

By Lemma 2.2.1, we may assume that, with L the left-most negative P instance in B' , $L\theta$ occurs as the left-most negative P instance in B .

By Lemma 2.2.2, there exists $D' \in PC_{<}^1(C'_0, B')$ such that D is a substitution instance of D' . (Clearly D' must be pure-positive with respect to P , and this is a $PC_{<}^{+1}$ -deduction Π' of D' from $\{B'\} \cup \{C'_0\}$, such that $k_{\Pi'}(C'_0) = 1 \leq ||\{0\}||$, and we're done.)

Assume the result for all $k: 1 \leq k < N$ and that now $k = N$.

We may proceed as in the case for $k = 1$:

By Lemma 2.2.1, we may assume that, with L the left-most negative P instance in B' , $L\theta$ occurs as the left-most negative P instance in B .

By Lemma 2.2.2, (as $D_1 \in PC_{<}^1(C_0, B)$), there exists $D_1' \in PC_{<}^1(C'_0, B')$ such that D_1 is a

substitution instance of D_1' .

By applying the induction hypothesis to D_1 and $C_1, \dots, C_{k-1}$, being substitution instances of D_1' and $C'_1, \dots, C'_{k-1}$, respectively, in the $PC_{<}^{+1}$ -deduction Π'' represented by $C_1, \dots, C_{k-1}$, and $D_1, \dots, D_{k-1}$, $D_k = D$, there is a $PC_{<}^{+1}$ -deduction Π''' of some clause D'' , from principal ancestor D_1' and ancillary parents in $\{C'_1, \dots, C'_{k-1}\}$, where D is a substitution instance of D'' , and such that $\forall C' \in \{C'_1, \dots, C'_{k-1}\}$, $k_{\Pi'''}(C') = ||\{i: 1 \leq i \leq k-1; C'_i = C'\}||$.

Taking D' to be D'' and Π' to be $D_1' \in PC_{<}^1(C'_0, B')$ followed by Π''' , Π' is a $PC_{<}^{+1}$ -deduction from principal ancestor B' and ancillary parents in $\{C'_0, \dots, C'_{k-1}\}$, of D' , where D is a substitution instance of D' , and such that

$$\forall C' \in \{C'_0, \dots, C'_{k-1}\}, k_{\Pi'}(C') = ||\{i: 0 \leq i \leq k-1; C'_i = C'\}||. \quad \blacksquare$$

For Lemma 2.2.4.

Proof (Sketch): We prove the result by induction on the total number of leaf nodes, $\kappa(\Pi) \equiv \kappa$, say, in Π .

If $\kappa = 1$, then $A \in S$, and Π consists of a single (leaf!) node, n , labelled by A . Thus $A = B_n$, and the result is trivially obtained by taking A' to be B'_n and Π' to be the single node deduction of A' .

Now suppose the result is true for all κ such that $0 \leq \kappa < K$, we will show the result for $\kappa(\Pi) = K$.

Evidently, (or, by Observation 2.1.2), Π may be decomposed into a $PC_{<}^{+1}$ -deduction Δ , of A from some $PC_{<}$ -initial (in Π !) principal ancestor B and ($PC_{<}$ -initial) ancillary parents $C_0, \dots, C_{k-1}$, (for some $k > 1$), together with (possibly trivial, that is containing only one node) $PC_{<}$ -deductions $\Pi_B, \Pi_{C_0}, \dots, \Pi_{C_{k-1}}$, of B and $C_0, \dots, C_{k-1}$, respectively.

For H denoting either B or one of the C_i , clearly the total number of leaves, $\kappa(\Pi_H)$, of Π_H is strictly less than K . Let $\mathfrak{L}_H$ denote the set of leaf nodes of Π which belong, (as leaves) to the fragment Π_H .

By the induction hypothesis there is a $PC_{<}$ -deduction $\Pi_{H'}$, from S' , of a clause H' , (say), $PC_{<}$ -initial (in $\Pi_{H'}$), of which H is a substitution instance, such that $\forall B' \in S', k_{\Pi_{H'}}(B') = ||\{n \in \mathfrak{L}_H: B'_n = B'\}||$.

Thus we have $PC_{<}$ -deductions $\Pi_{B'}, \Pi_{C'_0}, \dots, \Pi_{C'_{k-1}}$, of clauses B' and $C'_0, \dots, C'_{k-1}$, of which B , and $C_0, \dots, C_{k-1}$ are substitution instances, as above, for H corresponding to B , and $C_0, \dots, C_{k-1}$, respectively.

By Lemma 2.2.3, there exists a $PC_{<}^{+1}$ -deduction Δ' , of a clause A' , from principal ancestor B' and ancillary parents $\{C'_0, \dots, C'_{k-1}\}$, where A is a substitution instance of A' , and such that $\forall C' \in \{C'_0, \dots, C'_{k-1}\}, k_{\Pi'}(C') = ||\{i: 0 \leq i \leq k-1; C'_i = C'\}||$.

Evidently, by gluing the $PC_{<}$ -deductions $\Pi_{B'}$ of B' and each $\Pi_{C'_i}$ of C'_i , (for $i: 0 \leq i \leq k-1$), to an appropriate leaf node (whose label is the deduced clause), of Δ' , (as needed), we obtain a Π' as desired. ■

A.2. Refutation Completeness of $PC_{<}$

For Theorem 2.3.1:

Proof: We use the fact that $PC_{<}$ -deduction lifts, as in either Lemma 2.2.4. That is, we will first prove that if S is any finite unsatisfiable set of *ground* clauses, then there exists a $PC_{<}$ -refutation of S . For an arbitrary finite unsatisfiable S we then obtain a (ground) $PC_{<}$ -refutation Π' , of of a finite set S' , of ground substitution instances of clauses in S , and then lift, (by Lemma 2.2.4), Π' to obtain the desired $PC_{<}$ -refutation of S .

We prove the ground case by induction on the number, n , of distinct unsigned literals occurring in the (ground) set S .

The case $n = 0$ is trivial, (for then S is comprised only of the empty clause $\square$, and the trivial proof: " $\square$ " is then a $PC_{<}$ -refutation of S).

Suppose we have proved the result (for the ground case), for all $n: 0 \leq n < N$, and that now S is an unsatisfiable set of ground clauses containing (exactly) N distinct literals, modulo literal sign. Clearly we may assume that no clause of S is a tautology.

Let L_0 denote a positive literal occurring in a clause in S , such that the predicate symbol, P_0 , say, of L_0 is $<$ -minimal amongst the predicate symbols P occurring in (literals of) S . Let $\mathcal{C}_1$ denote the set comprised of those clauses of S which contain L_0 (as a positive literal), and let $\mathcal{C}_2$ denote $S \setminus \mathcal{C}_1$.

Let $\mathcal{C}_1'$, denote the set of clauses obtained from $\mathcal{C}_1$ by deleting from each clause, the literal L_0 , and let $\mathcal{C}_2'$ denote the clauses of $\mathcal{C}_2$, which do not contain the literal $\neg L_0$. Let S' denote $\mathcal{C}_1' \cup \mathcal{C}_2'$.

If there is a truth assignment (for the literals which occur in S'), v , which satisfies S' , by extending v to assign L_0 to *false*, evidently we obtain a truth assignment satisfying S . Since this is impossible by assumption, S' is unsatisfiable.

By the induction hypothesis, then, (since clearly the number of distinct (modulo sign) literals in S' must be some $n < N$), there is a $PC_{<}$ -refutation, Π_1' say, of S' .

If now $\mathcal{C}_1'$ denotes $\mathcal{C}_1$ with all (positive) instances of L_0 (in each clause), "boxed", (instead of removed), (and S' denotes $\mathcal{C}_1' \cup \mathcal{C}_2'$), (and boxed literals are regarded as "invisible" literals which simply "tag along"), it is not difficult to see that the " $PC_{<}$ -refutation", Π_1' , is either a $PC_{<}$ -deduction of the empty clause, (and no "boxed" literal occurs in a clause in Π_1'), or a "boxed literal containing" $PC_{<}$ -deduction of the boxed literal L_0 .

In the former case, Π_1' is already a $PC_{<}$ -refutation of S , and we are done.

In the latter case, by "unboxing" the literal L_0 in Π_1' (it is easy to see that), we a (possibly trivial, corresponding to the case when L_0 is a unit clause in S), $PC_{<}$ -deduction, Π_1 , say, of L_0 .

By a completely similar argument, (considering the literal $\neg L_0$ as we did for L_0), there is (either a $PC_{<}$ -refutation of S , in which case we are again done, or), a “boxed literal containing” $PC_{<}$ -deduction, Π_2' , say, of the boxed literal $\neg L_0$.

A $PC_{<}$ resolution in Π_2' may become illegal when the literal $\neg L_0$ is unboxed, precisely because the unboxed $\neg L_0$ lies as the left-most negative instance of P_0 , while the resolution (in Π_2' , where $\neg L_0$ was boxed, and hence “invisible”), occurred on a negative instance of P_0 which is no longer left-most.

But (clearly), by $PC_{<}$ -resolving with the unit L_0 , we may $PC_{<}$ -resolve away each such $\neg L_0$ (including that instance at the root), as it becomes left-most. By “splicing in” the $PC_{<}$ -deduction of L_0 at each such point, evidently we now obtain a $PC_{<}$ -refutation of S , as required. ■

A.3. Subsumption and $PC_{<}$

For Lemma 2.4.1:

Proof:

As in the proof for Lemma 2.2.2, let A , A' , B , D , and D' as in the statement of the lemma, (and without loss of generality, assume no two share a common variable), and let θ be a substitution such that $A'\theta \subseteq A$ and $D'\theta \subseteq D$, (and, in case (ii), $L_1\theta$ occurs as the left-most inequality in D).

Clearly the $<$ -maximal predicate symbol occurring in D' must be less than or equal to P , the $<$ -maximal predicate symbol occurring in A' must be less than or equal to P , and A' is pure-positive with respect to P .

For some non-empty subsets $A_R \subseteq A$, and $D_R \subseteq D$, the left-most negative P instance, $L_1\theta$, in D occurring in D_R , $A_R \cup \{\neg L : L \in D_R\}$ unifies, with a most general unifying substitution σ , say, to $\{L_0\}$, say, and $B \equiv A\sigma \setminus \{L_0\} \cup D\sigma \setminus \{\neg L_0\}$.

Trivially, $A'\theta\sigma \subseteq A\sigma$, and so $A'\theta\sigma \setminus \{L_0\} \subseteq A\sigma \setminus \{L_0\}$, while $D'\theta\sigma \subseteq D\sigma$, and so $D'\theta\sigma \setminus \{\neg L_0\} \subseteq D\sigma \setminus \{\neg L_0\}$.

In case (i), clearly $\neg L_0 \notin D'\theta\sigma$. Thus $D'\theta\sigma \subseteq D\sigma \setminus \{\neg L_0\} \subseteq B$, and so D' subsumes B .

In case (ii), as $L_1\theta \in D_R$, $L_1\theta\sigma = \neg L_0$, and so $\neg L_0 \in D'\theta\sigma$.

If $L_0 \notin A'\theta\sigma$, then $A'\theta\sigma \subseteq A\sigma \setminus \{L_0\} \subseteq B$, and so A' subsumes B .

Otherwise, consider the substitution instance $A'' \equiv A'\theta\sigma$ of A' , and the substitution instance D'' , taken to be $D'\theta\sigma$ of D' , ordered so that $\neg L_0 (= L_1\theta\sigma)$, occurs as the left-most negative P instance in D'' . We have $A'' \setminus \{L_0\} \subseteq A\sigma \setminus \{L_0\}$, and $D'' \setminus \{\neg L_0\} \subseteq D\sigma \setminus \{\neg L_0\}$.

Let $A''_R \equiv \{L_0\}$, and $D''_R \equiv \{\neg L_0\}$. Then $\emptyset \neq A''_R \subseteq A''$, and $\emptyset \neq D''_R \subseteq D''$.

Take $B'' \equiv A'' \setminus \{L_0\} \cup D'' \setminus \{\neg L_0\}$. Then clearly $B'' \in PC_{<}^1(A'', D'')$, and $B'' \subseteq B$.

By lifting, (Lemma 2.2.2), there exists a $B' \in PC_{<}^1(A', D')$ such that B'' is a substitution instance of B' . Evidently B' subsumes B . ■

For Lemma 2.4.2:

Proof: By induction on the depth k , of the $PC_{<}^{+1}$ -deduction Π .

If $k = 1$, then for some $C \in \mathfrak{C}$, $D \in PC_{<}^1(C, B)$. As B' is pure-positive with respect to P , by Lemma 2.4.1, (case (i)), B' subsumes D .

Assume the result for all $k: 1 \leq k < N$ and that now $k = N$.

Let $C_0, \dots, C_{k-1}$, (the $C_i \in \mathfrak{C}$ not necessarily distinct), and $B = D_0, \dots, D_{k-1}$, $D_k = D$, represent Π , (as in Definition 2.1.2).

$D_1 \in PC_{<}^1(C_0, B)$, and, as in the case for $k = 1$, above, by Lemma 2.4.1, case (i), B' subsumes D_1 . By applying the induction hypothesis to B' subsuming D_1 , in the (depth $k - 1$) $PC_{<}^{+1}$ -deduction Π' represented by $C_1, \dots, C_{k-1}$, and $D_1, \dots, D_{k-1}$, $D_k = D$, B' subsumes D . ■

For Lemma 2.4.3:

Proof: By induction on the depth k , of the $PC_{<}^{+1}$ -deduction Π .

If $k = 1$, then $D \in PC_{<}^1(C_0, B)$ and D must be pure-positive with respect to P .

As B' subsumes B , $B'\theta \subseteq B$ for some substitution θ .

If B' is pure-positive with respect to P , then by Lemma 2.4.2, (or Lemma 2.4.1, (case (i))), B' subsumes D . (So i. holds, and we're done.)

Otherwise, by Lemma 2.2.1, we may assume that with L the left-most negative P instance in B' , $L\theta$ occurs as the left-most negative P instance in B .

If (modulo ordering) $C'_0 = C_0$, by Lemma 2.4.1, (case (ii)), there exists $D' \in PC_{<}^1(C'_0, B')$ such that D' subsumes D .

Otherwise, by Lemma 2.4.1, (case (ii)), either C'_0 subsumes D , (in which case ii. holds, and we're done), or (again), there exists $D' \in PC_{<}^1(C'_0, B')$ such that D' subsumes D .

(Clearly D' must be pure-positive with respect to P , and this is a $PC_{<}^{+1}$ -deduction Π' of D' from $\{B'\} \cup \{C'_0\}$, such that $k_{\Pi'}(C'_0) = 1 \leq ||\{0\}||$. Thus iii. holds, and we're done.)

Assume the result for all $k: 1 \leq k < N$ and that now $k = N$.

We may proceed as in the case for $k = 1$:

If B' is pure-positive with respect to P , then by Lemma 2.4.2, B' subsumes D .

Otherwise, by Lemma 2.2.1, we may assume that with L the left-most negative P instance in B' , $L\theta$ occurs as the left-most negative P instance in B . If (modulo ordering) $C'_0 = C_0$, by Lemma 2.4.1, (case (ii)), there exists $D' \in PC_{<}^1(C'_0, B')$ such that D' subsumes D . Otherwise, by Lemma 2.4.1, (case (ii)), (as $D_1 \in PC_{<}^1(C_0, B)$), either C'_0 subsumes D , or (again) there exists $D'_1 \in PC_{<}^1(C'_0, B')$ such that D'_1 subsumes D_1 .

By applying the induction hypothesis to D'_1 subsuming D_1 , and $C'_1, \dots, C'_{k-1}$ subsuming $C_1, \dots, C_{k-1}$, respectively, in the $PC_{<}^{+1}$ -deduction Π'' represented by $C_1, \dots, C_{k-1}$, and $D_1, \dots, D_{k-1}$, $D_k = D$, either:

i'. D'_1 subsumes D , or

- ii'. C'_i subsumes D , for some $i: 1 \leq i \leq k-1$, such that (modulo ordering) $C'_i \neq C_i$, or
- iii'. there is a $PC_{<}^{+1}$ -deduction Π''' of some clause D'' , from principal ancestor D_1' and ancillary parents in $\{C'_1, \dots, C'_{k-1}\}$, where D'' subsumes D , and such that $\forall C' \in \{C'_1, \dots, C'_{k-1}\}$, $k_{\Pi'''}(C') \leq ||\{i: 1 \leq i \leq k-1; C'_i = C''\}||$.

In case i', taking D' to be D_1' clearly D' must be pure-positive with respect to P , and $D' \in PC_{<}^{+1}(C'_0, B')$ is a $PC_{<}^{+1}$ -deduction Π' of D' from principal ancestor B' and ancillary parents in $\{C'_0, \dots, C'_{k-1}\}$, such that $\forall C' \in \{C'_0, \dots, C'_{k-1}\}$, $k_{\Pi'}(C') \leq ||\{i: 0 \leq i \leq k-1; C'_i = C''\}||$.

In case ii'., we have ii., directly.

In case iii'., taking D' to be D'' and Π' to be $D_1' \in PC_{<}^{+1}(C'_0, B')$ followed by Π''' . Π' is a $PC_{<}^{+1}$ -deduction from principal ancestor B' and ancillary parents in $\{C'_0, \dots, C'_{k-1}\}$, where D' subsumes D , and such that $\forall C' \in \{C'_0, \dots, C'_{k-1}\}$ $k_{\Pi'}(C') \leq ||\{i: 0 \leq i \leq k-1; C'_i = C''\}||$. ■

For Lemma 2.4.4:

Proof (Sketch): We prove the result by induction on the total number of leaf nodes, $\kappa(\Pi) \equiv \kappa$, say, in Π .

If $\kappa = 1$, then $A \in S$, and Π consists of a single (leaf!) node, n , labelled by A . Thus $A = B_n$, and the result is trivially obtained by taking A' to be B'_n and Π' to be the single node deduction of A' .

Now suppose the result is true for all κ such that $0 \leq \kappa < K$, we will show the result for $\kappa(\Pi) = K$.

Evidently, (or, by Observation 2.1.2), Π may be decomposed into a $PC_{<}^{+1}$ -deduction Δ , of A from some $PC_{<}$ -initial (in Π !) principal ancestor B and ($PC_{<}$ -initial) ancillary parents $C_0, \dots, C_{k-1}$, (for some $k > 1$), together with (possibly trivial, that is containing only one node) $PC_{<}$ -deductions $\Pi_B, \Pi_{C_0}, \dots, \Pi_{C_{k-1}}$, of B and $C_0, \dots, C_{k-1}$, respectively.

For H denoting either B or one of the C_i , clearly the total number of leaves, $\kappa(\Pi_H)$, of Π_H is strictly less than K . Let $\mathfrak{L}_H$ denote the set of leaf nodes of Π which belong, (as leaves) to the fragment Π_H . By the induction hypothesis there is a $PC_{<}$ -deduction $\Pi_{H'}$, from S' , of a clause H' , (say), $PC_{<}$ -initial (in $\Pi_{H'}$), which subsumes H , such that $\forall B' \in S'$, $k_{\Pi_{H'}}(B') \leq ||\{n \in \mathfrak{L}_H: B'_n = B'\}||$.

Thus we have $PC_{<}$ -deductions $\Pi_{B'}, \Pi_{C'_0}, \dots, \Pi_{C'_{k-1}}$, of clauses B' subsuming B , and C'_0 subsuming $C_0, \dots, C'_{k-1}$ subsuming C_{k-1} , as above, for H corresponding to B , and $C_0, \dots, C_{k-1}$, respectively.

By Lemma 2.4.3, either:

- i. B' subsumes A ,
- ii. C'_i subsumes A , for some $i: 0 \leq i \leq k-1$, such that (modulo ordering) $C'_i \neq C_i$, or
- iii. there exists a $PC_{<}^{+1}$ -deduction Δ' , of a clause A' , from principal ancestor B' and ancillary parents in $\{C'_0, \dots, C'_{k-1}\}$, where A' subsumes A , and such that $\forall C' \in \{C'_0, \dots, C'_{k-1}\}$, $k_{\Pi'}(C') \leq ||\{i: 0 \leq i \leq k-1; C'_i = C''\}||$.

In case i., or case ii., taking Π' to be $\Pi_{B'}$, or $\Pi_{C'_i}$, respectively, evidently suffices.

In case iii., evidently, by gluing the $PC_{<}$ -deductions $\Pi_{B'}$ of B' and each $\Pi_{C'_i}$ of C'_i , (for $i: 0 \leq i \leq k-1$), to an appropriate leaf node (whose label is the deduced clause), of Δ' (as needed), we obtain a Π' as desired. ■

A.4. Refutation Completeness of $PC_{<}$ Under Subsumption (and Tautology) Deletion

For Lemma 2.5.1:

Proof: By induction on the depth $m \geq 0$, of Π .

If $m = 0$, then $D \in S'$, and so, trivially, for some $n \geq 0$, C_n subsumes D .

Suppose the result for all $0 \leq m < M$, and that now $m = M$.

(Since $m > 0$, by Observation 2.1.2), Π_D can be decomposed into a $PC_{<}^{+1}$ -deduction Γ , of D from some $PC_{<}$ -initial (in Π !) principal ancestor B , (say), and some ($PC_{<}$ -initial) ancillary parents $C'_0, \dots, C'_{k-1}$, (for some $k > 1$), together with (possibly trivial, that is containing only one node) $PC_{<}$ -deductions $\Pi_B, \Pi_{C'_0}, \dots, \Pi_{C'_{k-1}}$, of B and $C'_0, \dots, C'_{k-1}$, respectively.

For H denoting either B or one of the C'_i , H is $PC_{<}$ -initial., (in particular, at the root in Π_H), and the deduction depth of H , (that is, the depth of Π_H), is strictly less than M . Thus by the induction hypothesis, there exists $n' \geq 0$ such that $C_{n'}$ subsumes H .

Thus there is a sequence $n'(0), \dots, n'(k-1), n'(k)$, of indices $n' \geq 0$, such that $C_{n'(k)}$ subsumes B , and $\forall i: 0 \leq i \leq k-1$, $C_{n'(i)}$ subsumes C'_i .

Γ has depth k , and of course $k \leq M$.

If $k < M$, then, by the induction hypothesis, (applied to the depth k $PC_{<}^{+1}$ -deduction Γ of D from $\{B, C'_0, \dots, C'_{k-1}\}$), there exists $n \geq 0$ such that C_n subsumes D .

Otherwise, ($k = M$, and), by Lemma 2.4.3, either:

- i. $C_{n'(k)}$ subsumes D ,
- ii. $C_{n'(i)}$ subsumes D , for some $i: 0 \leq i \leq k-1$, such that (modulo ordering) $C_{n'(i)} \neq C'_i$, or
- iii. there exists a $PC_{<}^{+1}$ -deduction Γ' , of a clause D' , from principal ancestor $C_{n'(k)}$ and ancillary parents in $\{C_{n'(0)}, \dots, C_{n'(k-1)}\}$, where D' subsumes D , and such that $\forall C' \in \{C_{n'(0)}, \dots, C_{n'(k-1)}\}, k_{\Pi'}(C') \leq \|\{i: 0 \leq i \leq k-1; C_{n'(i)} = C'\}\|$.

In case i., or case ii., the result is immediate.

In case iii., evidently, (by considering the leaf index of each clause involved, it is not hard to see that), Γ' has depth $k' \leq k = M$.

Again, if $k' < M$, again, by the induction hypothesis, there is an $n \geq 0$ such that C_n subsumes D' , and so, (since D' subsumes D , by the transitivity of subsumption), such that C_n subsumes D .

Otherwise, $k' = k = M > 0$. We also have, in particular, for some $C' \in \{C_{n'(0)}, \dots, C_{n'(k-1)}\}$, some $B'_1 \in PC_{<}^1(C', C_{n'(k)})$, such that either:

- a) B'_1 is D' , (if $k' = 1$), or
- b) (if $k' > 1$), there is a depth $k' - 1$ $PC_{<}^{+1}$ -deduction, Γ'' , say, of D' from principal ancestor B'_1 and ancillary parents in $\{C_{n'(0)}, \dots, C_{n'(k-1)}\}$.

Evidently, (by condition 2 of Definition 2.5.2), there exists an $n' \geq 0$ such that $C_{n'}$ subsumes B'_1 .

In case (a), it follows, (again by the transitivity of subsumption), that there exists an $n \geq 0$ such that C_n subsumes D .

In case (b), by again applying the induction hypothesis,

(this time to the depth $k' - 1 < M$ $PC_{<}$ -deduction Γ'' , of D' from $\{B'_1, C_{n'(0)}, \dots, C_{n'(k-1)}\}$), it again follows that there exists an $n \geq 0$ such that C_n subsumes D' , and hence, such that C_n subsumes D . ■

For Theorem 2.5.1:

Proof: We have already shown (Theorem 2.3.1), that $PC_{<}$ is refutation complete. Hence, since, by assumption, S is unsatisfiable, we know there exists a $PC_{<}$ -refutation Π , say, of S . Of course, Π is $PC_{<}$ -deduction of the empty clause, from S .

As the empty clause, $\square$, at the root of Π is necessarily $PC_{<}$ -initial, it easily follows from Lemma 2.5.1, that there exists $n \geq 0$ such that C_n subsumes $\square$.

As only the empty clause subsumes $\square$, it follows that there exists $n \geq 0$ such that $C_n = \square$. ■

For Lemma 2.5.2:

Proof: This is an easy exercise. ■

For Lemma 2.5.3:

Proof (sketch): Since S is finite, and Δ is a non-empty subsumption and tautology deleted $PC_{<}$ -deduction sequence from S , there exists a $k_0 \geq 1$ such that for all $C \in S$, either C is a tautology, or $\exists i: 0 \leq i \leq k_0$, such that C_i subsumes C .

Let S' denote the collection of those $C \in S$ such that for no $i: 0 \leq i \leq k_0$, is it the case that C_i subsumes C . Then, of course, S' is a (possibly empty) finite subset of S , and for all $C \in S'$, C is a tautology.

Let $\Delta_0 \equiv C_0, C_1, \dots, C_{k_0}$.

Let Δ_1 be some, (possibly empty) sequencing of the clauses in S' .

For $n > 0$, n even, let Δ_n denote the sequence $C_{(n/2)k_0+1}, \dots, C_{((n+2)/2)k_0}$.

For $n > 1$, n odd, let

$$X_n \equiv \{C \in PC_{<}^1(A, B) \mid A \in |\Delta_0, \dots, \Delta_{n-1}| \setminus \{C_0, \dots, C_{((n+1)/2)k_0}\}; B \in |\Delta_0, \dots, \Delta_{n-1}|\} \\ \cup \{C \in PC_{<}^1(A, B) \mid A; B \in \{C_0, \dots, C_{((n+1)/2)k_0}\}; C \text{ is a tautology}\}.$$

Let $Y_n \equiv \{C \in X_n \mid \text{for no } A \in |\Delta_0, \dots, \Delta_{n-1}| \text{ does } A \text{ subsume } C\}$.

Finally, (for $n > 1$, n odd), let Δ_n denote a (possibly empty) sequencing of the clauses in Y_n .

Evidently, $\forall n \geq 0$, Δ_n is a finite sequence.

Let Δ' denote the concatenation $\Delta_0, \Delta_1, \dots$, (and still, let $\Delta' \equiv C'_0, C'_1, \dots$).

Evidently, $|\Delta| \subseteq |\Delta'|$, and $\forall C \in S$, $\exists C' \in |\Delta'|$, such that C' subsumes C .

We claim (1): If $A; B \in |\Delta'|$, and $C \in PC_{\leq}^1(A, B)$,
then $\exists C' \in |\Delta'|$ such that C' subsumes C .

Thus, given claim (1), evidently Δ' is a subsumption deleted $PC_{\leq}$ -deduction sequence from S .
(as required).

We also claim (2): For any $C \in |\Delta'| \setminus |\Delta|$, C is a tautology.

Given claim (2), evidently, Δ' satisfies the additional property stated in the lemma, and we're done.

Justification of claim (1):

Recall $|\Delta| \subseteq |\Delta'|$.

If $A; B \in |\Delta|$, then

(since Δ is a subsumption and tautology deleted $PC_{\leq}$ -deduction sequence from S),

if C is not a tautology, $\exists C' \in |\Delta|$ such that C' subsumes C .

Hence, in this case, $\exists C' \in |\Delta'|$ such that C' subsumes C . (as claimed).

If (A and B are in $|\Delta|$ and), C is a tautology, then for some odd $n \geq 1$,

we have both $A; B \in \{C_0, \dots, C_{((n+1)/2)k_0}\}$, and $C \in X_n$.

Thus, (for some odd $n \geq 1$), we have $C \in Y_n = |\Delta_n| \subseteq |\Delta'|$, unless there exists a
 $C' \in |\Delta_0, \dots, \Delta_{n-1}|$, such that C' subsumes C .

But in both situations, (if $A; B \in \Delta$), $\exists C' \in |\Delta'|$ such that C' subsumes C . (as claimed).

If $A \in |\Delta|$ and $B \in |\Delta'| \setminus |\Delta|$, (and $C \in PC_{\leq}^1(A, B)$),

then there exist $n_1 \geq 0$, odd, such that $B \in \Delta_{n_1}$.

and $n_2 \geq 0$, odd, such that $A \in \{C_0, \dots, C_{((n_2+1)/2)k_0}\}$.

Let $n \equiv \max\{n_1 + 2, n_2\}$.

Then (n is odd, and), $B \in |\Delta_0, \dots, \Delta_{n-1}| \setminus \{C_0, \dots, C_{((n+1)/2)k_0}\}$, and $A \in |\Delta_0, \dots, \Delta_{n-1}|$.

(since $\{C_0, \dots, C_{((n+1)/2)k_0}\} \subseteq |\Delta_0| \cup |\Delta_2| \cup \dots \cup |\Delta_{n-1}|$).

From this it follows that $C \in X_n$, from which in turn, as before, it follows that
(either $C \in |\Delta_n| \subseteq |\Delta'|$, or), $\exists C' \in |\Delta'|$ such that C' subsumes C , (as claimed).

The case when $B \in |\Delta|$ and $A \in |\Delta'| \setminus |\Delta|$, follows from the symmetry of $PC_{\leq}$.

The final case is when both $A \in |\Delta'| \setminus |\Delta|$, and $B \in |\Delta'| \setminus |\Delta|$.

Again, there exist $n_1 \geq 0$, odd, such that $A \in \Delta_{n_1}$, and $n_2 \geq 0$, odd, such that $B \in \Delta_{n_2}$.

Let $n \equiv \max\{n_1 + 2, n_2 + 2\}$.

Then (n is odd, and), $A; B \in |\Delta_0, \dots, \Delta_{n-1}| \setminus \{C_0, \dots, C_{((n+1)/2)k_0}\}$, and it follows, as before,
that (either $C \in |\Delta_n| \subseteq |\Delta'|$, or), $\exists C' \in |\Delta'|$ such that C' subsumes C , (as claimed).

This completes our demonstration of claim (1).

Justification of claim (2):

Of course, for any $C \in |\Delta'|$, $C \in |\Delta_n|$, for some $n \geq 0$, and if $C \in |\Delta_n|$, for an even $n \geq 0$, then, (since, for n even, $|\Delta_n| \subseteq |\Delta|$), $C \in |\Delta|$.

Thus, (in order to demonstrate claim (2)), it suffices to show that for every odd $n \geq 0$, $\forall C \in |\Delta_n| \setminus |\Delta|$, C is a tautology.

Since $|\Delta_n| \setminus |\Delta| \subseteq |\Delta_n| \setminus \{C_0, \dots, C_{((n+1)/2)k_0}\}$, it suffices to show that for every odd $n \geq 0$, $\forall C \in |\Delta_n| \setminus \{C_0, \dots, C_{((n+1)/2)k_0}\}$, C is a tautology.

We establish this by showing, by induction on $m \geq 0$, that, $(\forall m \geq 0)$, $\forall C \in |\Delta_{2m+1}| \setminus \{C_0, \dots, C_{(m+1)k_0}\}$, C is a tautology.

If $m = 0$, the statement is immediate, since, (by construction), $|\Delta_1|$ consists only of tautologies.

Now suppose the result for all $0 \leq m < N$, and that now $m = N$.

Let $C \in |\Delta_{2m+1}| \setminus \{C_0, \dots, C_{(m+1)k_0}\}$.

Then $C \in Y_{2m+1}$, so for no $C' \in |\Delta_0, \dots, \Delta_{2m}|$, does C' subsume C , and either

$(C \in PC_{<}^1(A, B)$, for some $A, B \in \{C_0, \dots, C_{(m+1)k_0}\}$ and)

C is a tautology, in which case we're done, or

$C \in PC_{<}^1(A, B)$, for some $A \in |\Delta_0, \dots, \Delta_{2m}| \setminus \{C_0, \dots, C_{(m+1)k_0}\}$, and some $B \in |\Delta_0, \dots, \Delta_{2m}|$.

As $|\Delta_{2m}| = \{C_{m \cdot k_0 + 1}, \dots, C_{(m+1)k_0}\} \subseteq \{C_0, \dots, C_{(m+1)k_0}\}$.

$$\begin{aligned} |\Delta_0, \dots, \Delta_{2m}| \setminus \{C_0, \dots, C_{(m+1)k_0}\} &= (|\Delta_0, \dots, \Delta_{2m-1}| \cup |\Delta_{2m}|) \setminus \{C_0, \dots, C_{(m+1)k_0}\} \\ &= |\Delta_0, \dots, \Delta_{2m-1}| \setminus \{C_0, \dots, C_{(m+1)k_0}\}, \end{aligned}$$

and $A \in |\Delta_0, \dots, \Delta_{2m-1}| \setminus \{C_0, \dots, C_{(m+1)k_0}\}$.

By the induction hypothesis, for all $m': 0 \leq m' < m$, $\forall C \in |\Delta_{2m'+1}| \setminus \{C_0, \dots, C_{(m'+1)k_0}\}$, C is a tautology.

But $|\Delta_0, \dots, \Delta_{2m-1}| \setminus \{C_0, \dots, C_{(m+1)k_0}\} =$

$$\begin{aligned} &|\Delta_0| \setminus \{C_0, \dots, C_{(m+1)k_0}\} \cup |\Delta_1| \setminus \{C_0, \dots, C_{(m+1)k_0}\} \cup \dots \cup |\Delta_{2(m-1)+1}| \setminus \{C_0, \dots, C_{(m+1)k_0}\} \subseteq \\ &\emptyset \cup |\Delta_1| \setminus \{C_0, \dots, C_{k_0}\} \cup \emptyset \cup \dots \cup |\Delta_{2(m-1)+1}| \setminus \{C_0, \dots, C_{m \cdot k_0}\}. \end{aligned}$$

Thus A is a tautology. By Lemma 2.5.2, then, either B subsumes C , or C is a tautology.

But, since by assumption, for no $C' \in |\Delta_0, \dots, \Delta_{2m}|$, does C' subsume C , and $B \in |\Delta_0, \dots, \Delta_{2m}|$, the former case is impossible. Thus C is a tautology.

This completes the inductive argument, and our demonstration of claim (2). ■

For Corollary 2.5.1:

Proof: By Lemma 2.5.3, there exists a subsumption deleted $PC_{<}$ -deduction sequence from S , $\Delta' \equiv C'_0, C'_1, \dots$, such that $\forall j \geq 0$, either:

- a) $\exists i \geq 0$ such that $C'_j = C_i$, or

b) C'_j is a tautology.

Since, by assumption, S is unsatisfiable, by Theorem 2.5.1, it is the case that for some $n \geq 0$, C'_n is the empty clause, $\square$.

Thus, since, clearly, $\square$ is not a tautology, there exists an $i \geq 0$, such that $\square = C_i$. ■

A.5. Generating Deduction Sequences with Deletion

For Proposition 2.6.1:

Proof (sketch): Let S be any finite unsatisfiable set of clauses.

Let $\Delta^* \equiv C_0, C_1, \dots$ denote the (possibly infinite) sequence of clauses generated by $\mathfrak{S}$, on the input S , (those generated clauses which were not forward subsumption (or tautology) deleted, and so were entered, at some point, in the data base), where each clause C_i in Δ is either marked or not, and is marked if and only if C_i was, (at some point) backward subsumed by some (in turn, possibly marked) clause, C_j , for some $j > i$.

From Δ^* we shall construct a subsumption and tautology deleted ($PC_{<-}$) deduction sequence, Δ , from S , such that $\forall C \in |\Delta|$, C occurs unmarked in the sequence Δ^* .

Since (by assumption), S is unsatisfiable, by Corollary 2.5.1, $\square \in |\Delta|$, so it follows that for some $j \geq 0$, $C_j = \square$, (and occurs unmarked in Δ^*), and hence, clearly, at some point, $\mathfrak{S}$ deduced the empty clause, $\square$.

To construct Δ :

For any $i \geq 0$, if $C_i \in |\Delta^*|$ is a marked clause, there exists some $i' > i$ such that $C_i \supseteq C_{i'}$ and C_i was deleted, by $\mathfrak{S}$, because of the backward subsumption of C_i by $C_{i'}$.

If, in turn, $C_{i'}$ is marked, there exists some $i'' > i'$ such that $C_{i'} \supseteq C_{i''}$ and $C_{i'}$ was deleted, by $\mathfrak{S}$, because of the backward subsumption of $C_{i'}$ by $C_{i''}$, and so on.

Since (by assumption) $\supseteq$ is well-founded, any such sequence $C_i, C_{i'}, C_{i''}, \dots$, must eventually terminate, in some necessarily unmarked (for otherwise the sequence could be extended), clause $C_{i^{(n)}}$, or there must exist $j < k$, such that $C_{i^{(k)}} \supseteq C_{i^{(j)}}$.

In the first case, let $f(i)$ denote the index $i^{(n)}$.

In the second case, we have

$$C_{i^{(j)}} \supseteq C_{i^{(j+1)}} \supseteq \dots \supseteq C_{i^{(k-1)}} \supseteq C_{i^{(k)}} \supseteq C_{i^{(j)}} ,$$

so by the (assumed) transitivity of $\supseteq$, in particular, we have $C_{i^{(j+1)}} \supseteq C_{i^{(j)}}$.

But (by our assumed backward deletion policy), this contradicts our assumption that $C_{i^{(j)}}$ was deleted by $\mathfrak{S}$ because of its backward subsumption by $C_{i^{(j+1)}}$.

Thus only the first case may apply.

For any $i \geq 0$ such that $C_i \in |\Delta^*|$ is unmarked, let $f(i)$ denote i .

Thus we have a function, f , mapping each index i such that C_i is unmarked (in Δ^*), to i , and mapping each index i such that C_i is marked (in Δ^*), to some index $f(i)$,

such that $C_{f(i)}$ is unmarked (in Δ^*), and $C_i \supseteq C_{f(i)}$.

Clearly, $\forall i \geq 0$, $C_{f(i)}$ occurs as an unmarked clause in Δ^* , and subsumes C_i .

Let $\Delta \equiv C_{f(0)}, C_{f(1)}, C_{f(2)}, \dots$,

(or this sequence with any arbitrary number of repetitions removed).

Then, clearly every clause $C \in |\Delta|$, occurs as an unmarked clause in the sequence Δ^* .

for every $C' \in |\Delta^*|$, there is some $C \in |\Delta|$ such that C subsumes C' ,

and (it is not hard to check that), Δ is indeed a subsumption and tautology deleted ($PC_{<-}$) deduction sequence, from S .

(For:

i. if $C \in S$ and C is not a tautology, then, evidently, for some C' which subsumes C ,

we have $C' \in |\Delta^*|$, and so there is some $C'' \in |\Delta|$ such that C'' subsumes C' , and so

(by transitivity of subsumption) such that C'' subsumes C , and

ii. for any $A; B \in |\Delta|$, A and B occur unmarked in $|\Delta^*|$, and so for any $C \in PC_{<}^1(A, B)$ which is not a tautology, there exists a $C' \in |\Delta^*|$ such that C' subsumes C , and, as in the case for (i.), above, there is some $C'' \in |\Delta|$ such that C'' subsumes C , as required.) ■

APPENDIX B

Σ_0 , Σ_0 -subsumption and Σ_0 -tautology

B.1. Σ_0 -tautology and Σ_0 -subsumption

For Observation 3.2.1:

Proof (sketch): a) through f) are trivial, and, clearly, a) implies g).

For h): Since

$$\begin{aligned} \mathcal{T}_{\mathcal{T}_C-(C^-)}(\mathcal{T}_C-(C^+ \cup \Delta(C^-, C^0))) &= \mathcal{T}_C-(\mathcal{T}_C-(C^+ \cup \Delta(C^-, C^0))) = \mathcal{T}_C-(C^+ \cup \Delta(C^-, C^0)) \\ &= \mathcal{T}_C-(C^+) \cup \mathcal{T}_C-(\Delta(C^-, C^0)) = \mathcal{T}_C-(C^+) \cup \mathcal{T}_C-(\mathcal{T}_C-(\bigcup\{\delta(C^-, \neq(\vec{s}, \vec{t})) \dots\})) \\ &= \mathcal{T}_C-(C^+) \cup \mathcal{T}_C-(\bigcup\{\delta(C^-, \neq(\vec{s}, \vec{t})) \dots\}) = \mathcal{T}_C-(C^+) \cup \Delta(C^-, C^0), \end{aligned}$$

and $\mathcal{T}_{\mathcal{T}_C-(C^-)}(\mathcal{T}_C-(C^0)) = \mathcal{T}_C-(\mathcal{T}_C-(C^0)) = \mathcal{T}_C-(C^0)$, and $\mathcal{T}_{\mathcal{T}_C-(C^-)}(\mathcal{T}_C-(C^-)) = \mathcal{T}_C-(C^-)$,

and since, modulo symmetry, $cl(C) = \mathcal{T}_C-(C^+) \cup \Delta(C^-, C^0) \cup \mathcal{T}_C-(C^0) \cup \mathcal{T}_C-(C^-)$.

(and the negative equality part of $cl(C)$ is, modulo symmetry, $\mathcal{T}_C-(C^-)$, et cetera).

clearly any possible literal $L \in cl(cl(C)) \setminus cl(C)$ must occur, modulo symmetry.

in $\Delta(\mathcal{T}_C-(C^-), \mathcal{T}_C-(C^0)) \setminus \Delta(C^-, C^0)$.

But similarly, it is not hard to see that $\Delta(\mathcal{T}_C-(C^-), \mathcal{T}_C-(C^0)) = \Delta(C^-, C^0)$.

Therefore, (since both $cl(cl(C))$ and $cl(C)$ are symmetrically closed), $cl(cl(C)) = cl(C)$.

i) is also straight forward. In particular, since $C \subseteq C'$, $C^- \subseteq (C')^-$, and for any pure-negative equality clause I , $\delta_0(C^-, I) \supseteq \delta_0((C')^-, I)$, and so (evidently) $\delta(C^-, I) \subseteq \delta((C')^-, I)$.

Thus (as $C^0 \subseteq (C')^0$, evidently) $\Delta(C^-, C^0) \subseteq \Delta((C')^-, (C')^0)$.

j) is clear, from the definition of $cl(C)$.

k): $L' \in \mathcal{T}_A(C)\theta$ if and only if $L' = L''\theta$, for some literal $L'' \in \mathcal{T}_A(C)$,

if and only if $L' = L''\theta$, for some $\mathcal{T}_A$ -equivalent L'' of some literal $L \in (C)$,

if and only if $L' = L''\theta$, for some literal $L'' \equiv \delta Q(t''_1, \dots, t''_n)$, (say), with the same sign δ , and predicate symbol Q

(of arity $n \geq 0$), as some literal $L \equiv \delta Q(t_1, \dots, t_n) \in C$, such that for each $i: 1 \leq i \leq n$ either $t_i = t''_i$, or for some $\alpha \in \mathcal{T}_A$, $\{t_i, t''_i\} \subseteq \alpha$,

if and only if $L' \equiv \delta Q(t'_1, \dots, t'_n)$, (say), has the same sign δ , and predicate symbol Q (of arity $n \geq 0$), as some literal $L\theta \equiv \delta Q(t_1\theta, \dots, t_n\theta) \in C\theta$, such that for each $i: 1 \leq i \leq n$ either $t_i\theta = t'_i$,

or for some $\beta \in \mathcal{T}_{A\theta}$, $\{t_i\theta, t'_i\} \subseteq \beta$,
 if and only if L' is $\mathcal{T}_{A\theta}$ -equivalent to some literal in $C\theta$,
 if and only if $L' \in \mathcal{T}_{A\theta}(C\theta)$.

l): Let $L' \in cl(C)\theta$. Then, modulo symmetry,
 $L' \in \mathcal{T}_{C-}(C^+)\theta \cup \Delta(C^-, C^0)\theta \cup \mathcal{T}_{C-}(C^0)\theta \cup \mathcal{T}_{C-}(C^-)\theta$.

If $L' \in \mathcal{T}_{C-}(C^+)\theta \cup \mathcal{T}_{C-}(C^0)\theta \cup \mathcal{T}_{C-}(C^-)\theta$, then, from k),
 $L' \in \mathcal{T}_{C-\theta}(C^+\theta) \cup \mathcal{T}_{C-\theta}(C^0\theta) \cup \mathcal{T}_{C-\theta}(C^-\theta) \subseteq cl(C\theta)$.

Otherwise, $L' \in \Delta(C^-, C^0)\theta \setminus (\mathcal{T}_{C-}(C^+)\theta \cup \mathcal{T}_{C-}(C^0)\theta \cup \mathcal{T}_{C-}(C^-)\theta)$.

Thus, L' is $L\theta$, for some literal L $\mathcal{T}_{C-}$ -equivalent to some literal $=(s, t)$, for some literals $P(\vec{s}) : \neg P(\vec{t})$ in C^0 , $\neq(s, t) \in \neq(\vec{s}, \vec{t})$ and $\forall \neq(s_i, t_i) \in \neq(\vec{s}, \vec{t}) \setminus \{\neq(s, t)\}$, either $s_i = t_i$ or $\exists \alpha_i \in \mathcal{T}_{C-} : \{s_i, t_i\} \subseteq \alpha_i$.

Hence there are literals $P(\vec{s})\theta; \neg P(\vec{t})\theta$ in $C^0\theta$, such that $\neq(s\theta, t\theta) \in \neq(\vec{s}, \vec{t})\theta$ and
 $\forall \neq(s_i\theta, t_i\theta) \in \neq(\vec{s}, \vec{t})\theta \setminus \{\neq(s\theta, t\theta)\}$, either $s_i\theta = t_i\theta$ or $\exists \beta_i \in \mathcal{T}_{C-\theta} : \{s_i\theta, t_i\theta\} \subseteq \beta_i$.

Thus $\neq(s\theta, t\theta) \in \Delta(C^-\theta, C^0\theta) \subseteq cl(C\theta)$. Since L is $\mathcal{T}_{C-}$ -equivalent to $=(s, t)$,
 (it is easy to see that) $L\theta$ is $\mathcal{T}_{C-\theta}$ -equivalent to $\neq(s\theta, t\theta)$. Since $\Delta(C^-\theta, C^0\theta)$ is $\mathcal{T}_{C-\theta}$ -equivalence
 closed, it must be the case that $L' \equiv L\theta \in \Delta(C^-\theta, C^0\theta) \subseteq cl(C\theta)$.

Thus, we have shown that, for any L' , $L' \in cl(C)\theta$ implies $L' \in cl(C\theta)$, and hence that,
 $cl(C)\theta \subseteq cl(C\theta)$.

m): Suppose that for some substitution θ , (modulo ordering), $\mathcal{T}_{C-}(C)\theta \subseteq \mathcal{T}_{(C')-}(C')$.
 Then, since (by (a), above), $C \subseteq \mathcal{T}_{C-}(C)$, clearly $C\theta \subseteq \mathcal{T}_{C-}(C)\theta$, and so $C\theta \subseteq \mathcal{T}_{(C')-}(C')$.

Conversely, suppose that, for some substitution θ , $C\theta \subseteq \mathcal{T}_{(C')-}(C')$.
 Then, clearly, $(C\theta)^- \subseteq (\mathcal{T}_{(C')-}(C'))^- = \mathcal{T}_{(C')-}((C')^-)$, and since, clearly, $C^-\theta = (C\theta)^-$,

$$\begin{aligned} \mathcal{T}_{C-}(C)\theta &= \mathcal{T}_{(C\theta)-}(C\theta) \quad (\text{by (k), above}), \\ &\subseteq \mathcal{T}_{\mathcal{T}_{(C')-}((C')^-)}(\mathcal{T}_{(C')-}(C')) \quad (\text{by (e), above}), \\ &= \mathcal{T}_{(C')-}(\mathcal{T}_{(C')-}(C')) \quad (\text{by (b), above}), \\ &= \mathcal{T}_{(C')-}(C') \quad (\text{by (c), above}). \end{aligned}$$

n) Suppose C is an equality clause.

Then modulo symmetry, (and ordering), $cl(C) = \mathcal{T}_{C-}(C)$, and so, (modulo symmetry and ordering),
 $cl(C)\theta = \mathcal{T}_{C-}(C)\theta$.

Since, (by (k), above), $\mathcal{T}_{C-}(C)\theta = \mathcal{T}_{C-\theta}(C\theta)$, and, clearly, $C^-\theta = (C\theta)^-$, evidently, (modulo
 symmetry and ordering), $cl(C)\theta = \mathcal{T}_{(C\theta)-}(C\theta) = cl(C\theta)$. ■

For Theorem 3.2.1:

Proof (sketch):

1) $\Leftrightarrow$ 4) is trivial in light of $\overline{cl(C)} = cl(\overline{C})$ (by Observation 3.2.1 f), above).

1) $\Leftrightarrow$ 5) is trivial in light of $cl(cl(C)) = cl(C)$ (by Observation 3.2.1 h), above).

1) $\Leftrightarrow$ 2) is also quite clear, and involves only an unravelling of the definitions. We leave this as an exercise for the reader.

2) $\Rightarrow$ 3): If i), C contains literals $P(\vec{s}); \neg P(\vec{t})$, $P \neq '='$, such that $\forall \neq(a, b) \in \neq(\vec{s}, \vec{t})$, either $a = b$, or $\exists \alpha \in \mathcal{T}_{C^-} : \{a, b\} \subseteq \alpha$, clearly, (since $\mathcal{T}_{C^-}$ is a collection of disjoint equivalence classes), it must be the case that $\forall \beta \in \mathcal{T}_{\neq(\vec{s}, \vec{t})} : \exists \alpha_\beta \in \mathcal{T}_{C^-} : \beta \subseteq \alpha_\beta$, and 3) holds.

If ii), for some equality literal $=(s, t) \in C$, $s = t$, or for some $\alpha \in \mathcal{T}_{C^-} : \{s, t\} \subseteq \alpha$, then $\neq(s, t) \in M$ and if $s = t$, $\mathcal{T}_{\neq(s, t)}$ is empty, and otherwise, $\mathcal{T}_{\neq(s, t)} = \{\{s, t\}\}$, and for some $\alpha \in \mathcal{T}_{C^-} : \{s, t\} \subseteq \alpha$. In either case, 3) follows directly.

3) $\Rightarrow$ 2): Suppose for some $A \in \{\neg L \mid L \in C^+\}$, (that) $\forall \beta \in \mathcal{T}_A : \exists \alpha_\beta \in \mathcal{T}_{C^-} : \beta \subseteq \alpha_\beta$. Then for some literal $=(s, t) \in C^+$, A is $\neq(s, t)$, and either $\mathcal{T}_A$ is empty, in which case $s = t$, or $\mathcal{T}_{\neq(s, t)} = \{\{s, t\}\}$, and $\exists \alpha \in \mathcal{T}_{C^-} : \{s, t\} \subseteq \alpha$. In either case, 2) (case (ii)), follows directly.

Suppose for some $A \in \{\neq(\vec{s}, \vec{t}) \mid P(\vec{s}); \neg P(\vec{t}) \in C^0\}$, (that), $\forall \beta \in \mathcal{T}_A : \exists \alpha_\beta \in \mathcal{T}_{C^-} : \beta \subseteq \alpha_\beta$. Then $\forall \neq(a, b) \in A$, either $a = b$, or $\exists \beta \in \mathcal{T}_A : \{a, b\} \subseteq \beta$, (and so either $a = b$, or $\exists \alpha \in \mathcal{T}_{C^-} : \{a, b\} \subseteq \beta \subseteq \alpha$), and 2) (case (i)) follows directly. ■

For Corollary 3.2.1:

Proof: From Observation 3.2.1, (l), $cl(C)\theta \subseteq cl(C\theta)$, and clearly if C is a tautology, then so is $cl(C)\theta$, and hence so is $cl(C\theta)$. From Theorem 3.2.1 5), above, then, $C\theta$ is a Σ_0 -tautology. ■

For Theorem 3.2.2:

Proof: $\Sigma_0 \models C$ if and only if $\Sigma_0 \cup \neg C$ is unsatisfiable, (which, in turn, is the case) if and only if $\Sigma_0 \cup \neg C^+ \cup \neg C^0 \cup \neg C^-$ is unsatisfiable.

It is easy to see that $\Sigma_0 \cup \neg C^-$, (being composed of only Σ_0 together with a set of positive equality units) must be satisfiable. By using completeness (that is, the refutation completeness of ordinary, binary, merge-resolution), and examining proof structure (as is done in the proof of Lemma 3.3.2, -see later), it is not hard to make the following observations:

- i) $\Sigma_0 \cup \neg C^+ \cup \neg C^0 \cup \neg C^-$ is unsatisfiable if and only if either $\Sigma_0 \cup \neg C^- \cup \neg C^0$ or $\Sigma_0 \cup \neg C^- \cup \neg C^+$ is unsatisfiable.
(This is because no clause with a (OBMR) ancestor in $\neg C^0$ can resolve with any (negative equality unit) clause in $\neg C^+$, and every clause with an ancestor in $\neg C^+$, must be a pure-inequality clause, and hence not resolvable with any (non-equality unit) clause in $\neg C^0$.)
- ii) $\Sigma_0 \cup \neg C^- \cup \neg C^+$ is unsatisfiable if and only if for some literal $=(s, t) \in C^+$, $\Sigma_0 \cup \neg C^- \cup \{\neq(s, t)\}$ is unsatisfiable.

(A pure-inequality clause can not be resolved against any clause in $\neg C^+$, and see the comment, following i), above.)

- iii) $\Sigma_0 \cup \neg C^- \cup \neg C^0$ is unsatisfiable if and only if for some literals $P(\vec{s})$ and $\neg P(\vec{t})$ in C^0 , $\Sigma_0 \cup \neg C^- \cup \{\neq(\vec{s}, \vec{t})\}$ is unsatisfiable.

(In light of the comment, following i), above, and further, observe that since the only clause containing the predicate symbol P in $\Sigma_0 \cup \neg C^- \cup \neg C^0$, are either units in $\neg C^0$, or (instances of) QP in Σ_0 , any clause with resolution ancestor $L \in \neg C^0$, has at most one other ancestor $L' \in \neg C^0$ and L' must have the same predicate symbol and opposite sign as L .)

Thus (carrying on), $\Sigma_0 \cup \neg C^- \cup \neg C^+$ is unsatisfiable if and only if for some literal $=(s, t) \in C^+$, $\Sigma_0 \cup \neg C^- \cup \{\neq(s, t)\}$ is unsatisfiable, (by (ii), above), which, (it is not hard to see), is the case if and only if either $s = t$, or for some $\alpha \in \mathcal{T}_{C^-}$, $\{s, t\} \subseteq \alpha$.

Also, $\Sigma_0 \cup \neg C^- \cup \neg C^0$ is unsatisfiable if and only if for some literals $P(\vec{s})$ and $\neg P(\vec{t})$ in C^0 , $\Sigma_0 \cup \neg C^- \cup \{\neq(\vec{s}, \vec{t})\}$ is unsatisfiable (by (iii), above), which, (it is not hard to see), is the case if and only if there are literals $P(\vec{s})$ and $\neg P(\vec{t})$ in C^0 , such that $\forall \neq(a, b) \in \neq(\vec{s}, \vec{t})$, either $a = b$, or $\exists \alpha \in \mathcal{T}_{C^-} : \{a, b\} \subseteq \alpha$.

Theorem 3.2.1 (2), now establishes the equivalence. ■

For Corollary 3.2.2:

Proof: If C subsumes D , then $\emptyset \models (\forall)C \Rightarrow (\forall)D$. (hence $\Sigma_0 \models (\forall)C \Rightarrow (\forall)D$), and since C is a Σ_0 -tautology, $\Sigma_0 \models (\forall)C$.

Thus $\Sigma_0 \models (\forall)D$, and hence, (by Theorem 3.2.2), D is a Σ_0 -tautology. ■

For Observation 3.2.2:

Proof (sketch): It is trivial to see that $\Sigma_0 \models \neg C \Leftrightarrow \neg cl(C)$. ■

For Theorem 3.2.3:

Proof:

1) $\Rightarrow$ 2): Since C Σ_0 -subsumes D , there exists a substitution θ such that $C\theta \subseteq cl(D) \cup F$, where F is some trivial inequality clause. Let ρ be the substitution such that $(cl(D) \cup F)\rho = \overline{cl(D) \cup F}$. (so, $\rho: Vars(cl(D) \cup F) \rightarrow \{\bar{x} \mid x \in Vars(cl(D) \cup F)\}$ $\rho: x \mapsto \bar{x}$).

Then with $\varphi \equiv \theta\rho$, $C\varphi \subseteq \overline{cl(D) \cup F} = \overline{cl(D)} \cup \bar{F} = cl(\bar{D}) \cup \bar{F}$, and each literal in $\bar{F}$ is a ground trivial inequality.

2) $\Rightarrow$ 1): For each literal $L \in C$, $L\theta \in cl(\bar{D}) \cup F'$.

Let φ denote the substitution : $x \in Vars(C) \mapsto \theta(x)$.

If $L\theta \in F'$ then L is of the form $\neq(s, t)$ for some (unbarred) terms s and t , with $s\theta = t\theta$.

Hence, $s\varphi = \underline{s\theta} = \underline{t\theta} = t\varphi$.

If $L\theta \in cl(\bar{D})$, then $L\varphi = \underline{L\theta} \in cl(D)$.

Thus for every literal $L \in C$, $L\varphi \in cl(D) \cup F$, where $F \equiv \underline{F'}$, consists only of trivial inequalities.

1) $\Rightarrow$ 3): By definition, if C Σ_0 -subsumes D , there is some substitution θ such that $C\theta \subseteq cl(D) \cup F$, where F is a trivial inequality clause.

Thus by d), h), i) and j) of Observation 3.2.1, above, $cl(C\theta) \subseteq cl(D) \cup F$.

But (by (l) of Observation 3.2.1), $cl(C)\theta \subseteq cl(C\theta)$. Thus $cl(C)$ Σ_0 -subsumes $cl(D)$.

3) $\Leftrightarrow$ 4): This is trivial, given that $cl(cl(D)) = cl(D)$, (by h) of Observation 3.2.1)..

1) $\Leftrightarrow$ 5): Is again trivial, given $cl(cl(D)) = cl(D)$.

2) $\Rightarrow$ 7): We must show there exists a substitution θ such that:

- i) modulo symmetry, $C^+\theta \subseteq \overline{\mathcal{T}_{D^-}(\overline{D^+})} \cup \Delta(\overline{D^-}, \overline{D^0})$,
- ii) for all literals $\neq(a, b) \in C^-$ either $a\theta = b\theta$ or $\exists \alpha \in \overline{\mathcal{T}_{D^-}}$ such that $\{a\theta, b\theta\} \subseteq \alpha$, and
- iii) there is a path γ through $M(C^0, D^0)$, (selecting from each column C_j , one member $\gamma(j) = c_{j,i}$, say), such that for all $j: 1 \leq j \leq \|C^0\|$, for all $\neq(s, \bar{t}) \in \gamma(j)$, either $s\theta = \bar{t}$, or $\exists \alpha \in \overline{\mathcal{T}_{D^-}}: \{s\theta, \bar{t}\} \subseteq \alpha$.

Since $C\theta \subseteq cl(\overline{D}) \cup F'$, (where F' is some ground trivial inequality clause),

modulo symmetry, $C\theta \subseteq \mathcal{T}_{D^-}(\overline{D^+}) \cup \Delta(\overline{D^-}, \overline{D^0}) \cup \mathcal{T}_{D^-}(\overline{D^0}) \cup \mathcal{T}_{D^-}(\overline{D^-}) \cup F'$.

(by definition of cl). Thus, (since $\mathcal{T}_{D^-}(\overline{D^+}) = \overline{\mathcal{T}_{D^-}(\overline{D^+})}$), clearly, modulo symmetry,

$C^+\theta \subseteq \overline{\mathcal{T}_{D^-}(\overline{D^+})} \cup \Delta(\overline{D^-}, \overline{D^0})$, (and thus (i) holds).

Similarly, modulo symmetry, $C^0\theta \subseteq \overline{\mathcal{T}_{D^-}(\overline{D^0})}$, and, modulo symmetry, $C^-\theta \subseteq \overline{\mathcal{T}_{D^-}(\overline{D^-})} \cup F'$.

Given any literal $\neq(a, b) \in C^-$, since, modulo symmetry, $C^-\theta \subseteq \overline{\mathcal{T}_{D^-}(\overline{D^-})} \cup F'$, clearly,

modulo symmetry, $\neq(a, b)\theta \in \mathcal{T}_{D^-}(\overline{D^-}) \cup F'$.

Thus either $a\theta = b\theta$ (if $\neq(a, b)\theta \in F'$), in which case we have (ii) directly,

or else, modulo symmetry, $\neq(a, b)\theta \in \mathcal{T}_{D^-}(\overline{D^-})$,

and hence for some inequality $\neq(s, t) \in \overline{D^-}$, $\neq(a, b)\theta$ is $\mathcal{T}_{D^-}$ -equivalent to $\neq(s, t)$.

If ($a\theta \neq b\theta$ and) $s = t$, then there must exist $\alpha \in \mathcal{T}_{D^-}$ such that $\{a\theta, b\theta, s\} \subseteq \alpha$, and otherwise,

$\exists \alpha \in \mathcal{T}_{D^-}$ such that $\{s, t\} \subseteq \alpha$, and $\{a\theta, s\} \subseteq \alpha$; $\{b\theta, t\} \subseteq \alpha$, and hence,

such that $\{a\theta, b\theta\} \subseteq \{a\theta, b\theta, s, t\} \subseteq \alpha$. In either case, we have (ii).

Since $C^0\theta \subseteq \overline{\mathcal{T}_{D^-}(\overline{D^0})}$, for each literal $L_j \in C^0\theta$, there is a $\mathcal{T}_{D^-}$ -equivalent literal $L'_i, \in \overline{D^0}$.

With δ (denoting) the sign of L_j , and P the predicate symbol of L_j , we have $L_j \equiv \delta P(\vec{s}\theta)$; $L'_i \equiv \delta P(\vec{t})$, for some $\vec{s}\theta$ and $\vec{t}$ such that for all $\neq(s\theta, \bar{t}) \in \neq(\vec{s}\theta, \vec{t})$, either $s\theta = \bar{t}$, or $\exists \alpha \in \mathcal{T}_{D^-}$ such that $\{s\theta, \bar{t}\} \subseteq \alpha$. Thus (iii) holds.

7) $\Rightarrow$ 2): Condition ii) says that if $a\theta \neq b\theta$, $\exists \alpha \in \mathcal{T}_{D^-}: \{a\theta, b\theta\} \subseteq \alpha$. But (it is easy to see that) this says that if $a\theta \neq b\theta$, $\neq(a\theta, b\theta) \in \mathcal{T}_{D^-}(\overline{D^-})$. Thus clearly ii) implies $C^-\theta \subseteq \mathcal{T}_{D^-}(\overline{D^-}) \cup F'$, for some trivial inequality clause F' .

Condition iii) says that for each literal $L_j \in C^0\theta$, there is a $\mathcal{T}_{D^-}$ -equivalent literal $L'_i \in \overline{D^0}$. But then clearly $C^0\theta \subseteq \mathcal{T}_{D^-}(\overline{D^0}) = \overline{\mathcal{T}_{D^-}(D^0)}$.

These inclusions, together with the inclusion of condition i) (of 7)), yield 2).

6) $\Leftrightarrow$ 7):

Assuming (6), $\exists \theta_1 \in S_1(C^+, D)$, $\exists \theta_2 \in S_2(C^-, \mathcal{T}_{D^-})$, and $\exists \theta_3 \in S_3(C^0, D^0, \mathcal{T}_{D^-})$, such that θ_1 , θ_2 , and θ_3 are compatible. Therefore, there exists θ such that θ is an extension of each of θ_1 , θ_2 , θ_3 , and for some substitutions ψ_1 , ψ_2 , and ψ_3 , (say), $\theta = \theta_1\psi_1 = \theta_2\psi_2 = \theta_3\psi_3$. Since $\theta = \theta_1\psi_1$, and $\theta_1 \in S_1(C^+, D)$, (i.) holds. Since $\theta = \theta_2\psi_2$, and $\theta_2 \in S_2(C^-, \mathcal{T}_{D^-})$, (ii.) holds. Finally, since $\theta = \theta_3\psi_3$, and $\theta_3 \in S_3(C^0, D^0, \mathcal{T}_{D^-})$, (iii.) holds.

Conversely, suppose there is some substitution θ for which conditions (i.), (ii.), and (iii.) hold. Then, since $S_1(C^+, D) \equiv \{\theta : \theta \text{ is a most general substitution such that } C^+\theta \subseteq \overline{\mathcal{T}_{D^-}(D^+)} \cup \Delta(\overline{D^-}, \overline{D^0})\}$, and θ is a substitution such that (i.) holds, $\exists \theta_1 \in S_1(C^+, D)$ such that for some substitution ψ_1 , $\theta = \theta_1\psi_1$. Similarly, since $S_2(C^-, \mathcal{T}_{D^-}) \equiv \{\theta : \theta \text{ is a most general substitution such that for all (inequality) literals } \neq(a, b) \in C^-, \text{ either } a\theta = b\theta, \text{ or there exists some equivalence class } \alpha \in \overline{\mathcal{T}_{D^-}}, \text{ such that } \{a\theta, b\theta\} \subseteq \alpha\}$, and θ is a substitution such that (ii.) holds, $\exists \theta_2 \in S_2(C^-, \mathcal{T}_{D^-})$ such that for some substitution ψ_2 , $\theta = \theta_2\psi_2$. Similarly, $\exists \theta_3 \in S_3(C^0, D^0, \mathcal{T}_{D^-})$, such that for some substitution ψ_3 , $\theta = \theta_3\psi_3$. Since θ_1 , θ_2 , and θ_3 have a common extension (θ), they are compatible. ■

For Theorem 3.2.4:

Proof:

If C subsumes D , then there exists a substitution θ such that $C\theta \subseteq D$. But thus, since $D \subseteq cl(D)$, there is a substitution θ such that $C\theta \subseteq cl(D)$, and so C Σ_0 -subsumes D . ■

For Lemma 3.2.1:

Proof: If C Σ_0 -subsumes D , then for some substitution φ , and some trivial inequality clause F , $C\varphi \subseteq cl(D) \cup F$. But clearly $F\theta$ is a trivial inequality clause, and by Observation 3.2.1. (l), $cl(D)\theta \subseteq cl(D\theta)$, so $C\varphi\theta \subseteq cl(D\theta) \cup F\theta$, and C Σ_0 -subsumes $D\theta$. ■

For Lemma 3.2.2:

Proof: If B Σ_0 -subsumes C , and C Σ_0 -subsumes D , then there are substitutions θ_1 and θ_2 , and trivial inequality clauses F_1 and F_2 , such that $B\theta_1 \subseteq cl(C) \cup F_1$ and $C\theta_2 \subseteq cl(D) \cup F_2$.

Thus (by Observation 3.2.1, (h) and (j)), $cl(C\theta_2) \subseteq cl(D) \cup F_2$, and so (by Observation 3.2.1, (l)), $B\theta_1\theta_2 \subseteq cl(C)\theta_2 \cup F_1\theta_2 \subseteq cl(C\theta_2) \cup F_1\theta_2 \subseteq cl(D) \cup F_1\theta_2 \cup F_2$.

As $F_1\theta_2$ is (clearly) a trivial inequality clause, so is $F_1\theta_2 \cup F_2$, and so B Σ_0 -subsumes D . ■

B.2. Weak Completeness of Subsumption and Σ_0 -subsumption

For Lemma 3.3.1:

Proof: The lemma is quite obvious. By the “principle of subsumption”, (for OBM), for each literal L_i of C , there is an OBM refutation proof Π_i in which L_i occurs at least once, and “takes the place of” C , and every other clause L_i subsumes.

By “masking” the literals L_i in Π_i , it is easy to see that we may obtain from Π_i an OBM deduction, Γ_i , of a clause C_i , such that C_i unifies with $\neg L_i$, and where Γ_i contains no clause subsumed by L_i . By successively resolving the literals L_i of C away with the C_i , we may obtain the proof Π' . ■

For Lemma 3.3.2:

Proof: We first prove (ii).

Suppose $\Sigma_0 \cup \neg D \cup \{ \neq(s_0, t_0) \}$ is unsatisfiable, and $s_0 \neq t_0$.

Then there is a finite set Σ_0' of ground instances of clauses in Σ_0 , such that $\Sigma_0' \cup \neg D \cup \{ \neq(s_0, t_0) \}$ is unsatisfiable, and (by refutation completeness of OBM) a (propositional, that is, substitution free) refutation Π of $\Sigma_0' \cup \neg D \cup \{ \neq(s_0, t_0) \}$, in which $\neq(s_0, t_0)$ occurs. (by soundness, and since $\Sigma_0' \cup \neg D$ is satisfiable). By Lemma 3.3.1, there is an OBM deduction Γ' , from $\Sigma_0' \cup \neg D$ of $\neq(s_0, t_0)$.

Note that the only positive equality literals in Σ_0' occur in substitution instances of QR , QS , and QT , and the only positive equality literals in $\neg D$ are those which occur (as unit clauses) in $\neg D^-$. Therefore, (it is not hard to see that) the only positive equality units $\neq(s_0, t_0)$ deducible from $\Sigma_0' \cup \neg D$ must either be instances of QR , (so that $s_0 = t_0$), or equalities established in $\mathcal{T}_{D^-}$. Thus if $s_0 \neq t_0$, then $\neq(s_0, t_0) \in \mathcal{T}_{D^-}(\overline{D^-})$.

The converse direction is trivial.

For (i):

$(\Leftarrow)$: is trivial.

$(\Rightarrow)$: Since $\Sigma_0 \cup \neg D \cup \{ \neq(s_0, t_0) \}$ is unsatisfiable, (but $\Sigma_0 \cup \neg D$ is), clearly $s_0 \neq t_0$. As in the case for (ii), above, there must exist a propositional OBM deduction Γ' , from $\Sigma_0' \cup \neg D$, where Σ_0' is a set of ground instances of clauses in Σ_0 , of $\neq(s_0, t_0)$.

The only clauses in $\Sigma_0' \cup \neg D$ which contain inequality literals, are those instances of QS , QT , and QP , in Σ_0' , and those unit clauses in $\neg D^+$.

The only clauses in $\Sigma_0' \cup \neg D$ which contain non-equality literals are those instances of QP in Σ_0' , and those units in $\neg D^0$.

Finally, the only clauses in $\Sigma_0' \cup \neg D$ which contain positive equality literals, are those instances of QR , QS , and QT in Σ_0' , and those units in $\neg D^-$.

Because of this last fact, and the fact that both QS , and QT , (instances of which comprise the only non-unit clauses in $\Sigma_0' \cup \neg D$ which contain a positive equality literal), contain at most one positive equality literal, it is easy to see that any clause which is OBM deducible from $\Sigma_0' \cup \neg D$ can contain at most one positive equality literal.

Further, in light of the previous observations, the deduction of each such deducible clause which contains a (unique!) positive equality involves only positive equality units in $\neg D^-$, and instances of QR , QS , and QT .

An inequality unit in $\neg D^+$, can resolve only with one of these clauses, leaving a pure-negative equality clause behind.

A pure-negative equality clause can in turn resolve only with one of these clauses deducible from $\neg D^-$, and instances of QR , QS , and QT , once again leaving a pure-negative equality clause behind.

Thus the clauses in $\neg D^+$ are “isolated” from instances of QP and units in $\neg D^0$, in the sense that no offspring clause of a unit in $\neg D^+$, can resolve with any offspring of any instance of QP , or any offspring of any clause in $\neg D^0$.

Therefore the deduction Γ' of $\neq(s_0, t_0)$ is either a deduction from $\neg D^+$, $\neg D^-$, and instances of QR , QS , and QT in Σ_0' , or a deduction from $\neg D^0$, $\neg D^-$, and clauses in Σ_0' .

It is not difficult to see that, in the former case, $\neq(s_0, t_0) \in \mathcal{T}_{\overline{D^-}}(\overline{D^+})$,

and in the latter case, $\neq(s_0, t_0) \in \Delta(\overline{D^-}, \overline{D^0})$.

Hence, $\neq(s_0, t_0) \in \mathcal{T}_{\overline{D^-}}(\overline{D^+}) \cup \Delta(\overline{D^-}, \overline{D^0})$.

For (iii):

($\Leftarrow$): is again trivial.

($\Rightarrow$): Again, as in the case for (ii) above, there must exist a propositional OBMIR deduction Γ' , from $\Sigma_0' \cup \neg D$, where Σ_0' is a set of ground instances of clauses in Σ_0 , of $\neg L$.

Suppose $\neg L$ is $P(\vec{a})$. Note that the only literals which are positive instances of P which occur in Σ_0' are instances of QP , and the only instances of P which occur in $\neg D$, occur as unit clauses in $\neg D^0$.

From this, and in light of the proof for (i) above, it is not difficult to see that we must have $L \in \mathcal{T}_{\overline{D^-}}(\overline{D^0})$.

(Reasoning as in the proof for (i), in fact, it is easy to see that the instances of QP , for any predicate symbol P , are “isolated” from the instances of QP for a distinct symbol P' , as well as from the inequality units of $\neg D^+$.)

The situation for $\neg L \equiv \neg P(\vec{a})$ is exactly similar. ■

For Theorem 3.3.1:

Proof: Suppose that for some ground instance $C\theta$ of C , $\Sigma_0 \cup \neg D \cup \{C\theta\}$ is unsatisfiable.

If $\Sigma_0 \cup \neg D$ is unsatisfiable, ($\Sigma_0 \models D$, and), D is a Σ_0 -tautology.

Suppose $\Sigma_0 \cup \neg D$ is satisfiable.

Since $\Sigma_0 \cup \neg D \cup \{C\theta\}$ is unsatisfiable, for each literal $L \in C\theta$, $\Sigma_0 \cup \neg D \cup \{L\}$ is unsatisfiable.

By Lemma 3.3.2, for each positive equality literal $L \in C^+\theta$, $L \in \mathcal{T}_{\overline{D^-}}(\overline{D^+}) \cup \Delta(\overline{D^-}, \overline{D^0})$,

for each inequality $L \equiv \neq(s_0, t_0) \in C^-\theta$, either $s_0 = t_0$, or $L \in \mathcal{T}_{\overline{D^-}}(\overline{D^-})$,

and for each non-equality literal $L \in C^0\theta$, $L \in \mathcal{T}_{\overline{D^-}}(\overline{D^0})$.

Clearly then, $C^+\theta \subseteq \mathcal{T}_{\overline{D}^-}(\overline{D}^+) \cup \Delta(\overline{D}^-, \overline{D}^0)$, $C^0\theta \subseteq \mathcal{T}_{\overline{D}^-}(\overline{D}^0)$, and $C^-\theta \subseteq \mathcal{T}_{\overline{D}^-}(\overline{D}^-) \cup F$, where F is some trivial inequality clause. Hence C Σ_0 -subsumes D . ■

For Theorem 3.3.2:

Proof: ($\Rightarrow$): If C subsumes D , there exists a substitution φ such that $C\varphi \subseteq D$. Let θ_0 denote the substitution which “freezes” the variables in D , (sending $x \mapsto \bar{x}$). Then $C\varphi\theta_0 \subseteq D\theta_0 \equiv \overline{D}$. Therefore, letting $\theta \equiv \varphi\theta_0$, we have $C\theta$ ground, and clearly $\{C\theta\} \cup \neg D$ is unsatisfiable.

($\Leftarrow$): Suppose that for some substitution θ , $C\theta$ is ground and $\{C\theta\} \cup \neg D$ is unsatisfiable. Then for each $L \in C\theta$ $\{L\} \cup \neg D$ is unsatisfiable. Since D is not a tautology, $\neg D$ is satisfiable, and so $\neg L \in \neg D$, and hence $L \in \overline{D}$. Therefore $C\theta \subseteq \overline{D}$. Because, (clearly) we can assume that the variables in C are disjoint from those in D , that the domain of θ is a subset of the (set of) variables in C , and that neither C nor D contains any formally frozen variables, we can see that we can “thaw”: $\underline{C}\theta = C\underline{\theta}$, and $\underline{C}\theta \subseteq D$, whence, with $\varphi \equiv \underline{\theta}$, $C\varphi \subseteq D$, and C subsumes D . ■

For Theorem 3.3.3:

Proof: Since we have already shown that Σ_0 -subsumption is weakly complete with respect to Σ_0 -tautology and Σ_0 , it remains only to show the “only if” direction. That is, it remains to show that if C Σ_0 -subsumes D , then there is a substitution θ such that $C\theta$ is ground, (possibly containing “frozen” variables occurring in $\neg D$), and such that $\Sigma_0 \cup \neg D \cup \{C\theta\}$ is unsatisfiable.

If C Σ_0 -subsumes D , then for some substitution θ , and some trivial inequality clause F , $C^+\theta \subseteq \mathcal{T}_{\overline{D}^-}(\overline{D}^+) \cup \Delta(\overline{D}^-, \overline{D}^0)$, $C^0\theta \subseteq \mathcal{T}_{\overline{D}^-}(\overline{D}^0)$, and $C^-\theta \subseteq \mathcal{T}_{\overline{D}^-}(\overline{D}^-) \cup F$.

Clearly $C\theta$ is ground, and $\{C\theta\} \cup \neg(\mathcal{T}_{\overline{D}^-}(\overline{D}^+) \cup \Delta(\overline{D}^-, \overline{D}^0) \cup \mathcal{T}_{\overline{D}^-}(\overline{D}^0) \cup \mathcal{T}_{\overline{D}^-}(\overline{D}^-) \cup F)$ is unsatisfiable.

Also, it is easy to see that $\Sigma_0 \models D \Leftrightarrow \mathcal{T}_{\overline{D}^-}(\overline{D}^+) \cup \Delta(\overline{D}^-, \overline{D}^0) \cup \mathcal{T}_{\overline{D}^-}(\overline{D}^0) \cup \mathcal{T}_{\overline{D}^-}(\overline{D}^-) \cup F$ (and that, hence $\Sigma_0 \models \neg D \Leftrightarrow \neg(\mathcal{T}_{\overline{D}^-}(\overline{D}^+) \cup \Delta(\overline{D}^-, \overline{D}^0) \cup \mathcal{T}_{\overline{D}^-}(\overline{D}^0) \cup \mathcal{T}_{\overline{D}^-}(\overline{D}^-) \cup F)$).

Therefore, $\Sigma_0 \cup \{C\theta\} \cup \neg D$ is unsatisfiable. ■

For Corollary 3.3.1:

Proof: Since C Σ_0 -subsumes D , either D is a Σ_0 -tautology, or there is a substitution θ such that $\Sigma_0 \cup \neg D \cup \{C\theta\}$ is unsatisfiable. But (since C is a Σ_0 -tautology), $\Sigma_0 \models C$ and hence $\Sigma_0 \cup \neg D$ is unsatisfiable. Thus $\Sigma_0 \models D$, so (by Theorem 3.2.2), D is a Σ_0 -tautology.

(Alternatively: Since C Σ_0 -subsumes D , there is a substitution θ and a trivial inequality clause F such that $C\theta \subseteq cl(D) \cup F$. Thus (by Observation 3.2.1, (h) and (j)), $cl(C\theta) \subseteq cl(D) \cup F$.

Since C is a Σ_0 -tautology, by Corollary 3.2.1, $C\theta$ is a Σ_0 -tautology.

Thus evidently, $cl(D)$ is a Σ_0 -tautology, and so, by Theorem 3.2.1, D is a Σ_0 -tautology.) ■

For Theorem 3.3.4:

Proof: If D is a Σ_0 -tautology, then (by Theorem 3.2.2), $\Sigma_0 \models D$, and so, trivially, $\Sigma_0 \models C \Rightarrow D$.

Otherwise, by Theorem 3.3.3, there is some ground substitution instance $C\theta$ of C , such that $\Sigma_0 \cup \neg D \cup \{C\theta\}$ is unsatisfiable. Thus $\Sigma_0 \cup \{C\theta\} \models D$, and since $\{C\} \models C\theta$, $\Sigma_0 \cup \{C\} \models D$, and so $\Sigma_0 \models (\forall)C \Rightarrow (\forall)D$. ■

APPENDIX C

EPC_<

C.1. EPC_<: The Deduction Rules and Some Preliminaries

C.1.1. The Deduction Rules: The Equality fragment

For Observation 4.1.1:

Proof: A Σ_0 -subsumes $\Box$, if and only if there is some substitution, θ , and some trivial inequality clause, F , such that $A\theta \subseteq cl(\Box) \cup F$.

Since, (clearly), $cl(\Box) = \Box$, A Σ_0 -subsumes $\Box$ if and only if there is some substitution, θ , and some trivial inequality clause, F , such that $A\theta \subseteq \cup F$, and, evidently, this occurs, if and only if $QR_{full}(A) = \Box$. ■

For Observation 4.1.4:

Proof: Since A is QR -positive, either A is positive, or $QR_{full}(A) \neq A$.

If A is positive, then since A' subsumes A , clearly A' must also be positive, and hence, QR -positive.

If $QR_{full}(A) \neq A$, there is a substitution σ such that for each inequality literal $\neq(s, t) \in A$, $s\sigma = t\sigma$. But if A' subsumes A , there is a substitution θ such that $A'\theta \subseteq A$. Thus for any inequality literal $\neq(s', t') \in A'$, it must be the case that $s\theta\sigma = t\theta\sigma$, and so, either A' is positive or $QR_{full}(A') \neq A'$, so A' is QR -positive. ■

For Observation 4.1.5:

Proof: Let σ be a most general substitution such that $(C\sigma)^-$ contains only trivial inequalities.

Since $(C\sigma)^-$ contains only trivial inequalities, (modulo ordering), $cl(C\sigma) = \tau_{(C\sigma)^-}(C\sigma) = C\sigma$.

Thus, since, (by (n) of Observation 3.2.1), (modulo ordering), $cl(C)\sigma = cl(C\sigma)$, it follows that (modulo ordering), $cl(C)\sigma = C\sigma$, and hence, clearly, that $(cl(C)\sigma)^-$ contains only trivial inequalities.

Thus if σ' denotes any most general substitution such that $(cl(C)\sigma')^-$ contains only trivial inequalities, there is some substitution, θ' , say, such that $\sigma = \sigma'\theta'$.

But since $(cl(C)\sigma')^-$ contains only trivial inequalities, and $C \subseteq cl(C)$, clearly, $(C\sigma')^-$ contains only trivial inequalities, and so, (since σ is most general such that $(C\sigma)^-$ contains only trivial inequalities), there is some substitution, θ , say, such that $\sigma' = \sigma\theta$.

Now let ρ be any substitution such that $(cl(C)\rho)^-$ contains only trivial inequalities. Then (since σ' is most general such that $(cl(C)\sigma')^-$ contains only trivial inequalities), there is some substitution, φ , such that $\rho = \sigma'\varphi = \sigma\theta\varphi$.

Thus, if ρ be is substitution such that $(cl(C)\rho)^-$ contains only trivial inequalities, there is some substitution λ , ($\lambda = \theta\varphi$), such that $\rho = \sigma\lambda$, and (since $(cl(C)\sigma)^-$ contains only trivial inequalities), thus, σ is a most general substitution such that $(cl(C)\sigma)^-$ contains only trivial inequalities.

Similarly, if ρ is any substitution such that $(C\rho)^-$ contains only trivial inequalities, then (since σ is most general such that $(C\sigma)^-$ contains only trivial inequalities), there is some substitution, φ , such that $\rho = \sigma\varphi = \sigma'\theta'\varphi$.

Thus, if ρ be is substitution such that $(C\rho)^-$ contains only trivial inequalities, there is some substitution λ , ($\lambda = \theta'\varphi$), such that $\rho = \sigma'\lambda$, and (since $(C\sigma')^-$ contains only trivial inequalities), thus, σ' is a most general substitution such that $(C\sigma')^-$ contains only trivial inequalities. ■

C.1.2. Some Preliminaries

Observations 4.1.6, 4.1.7, and 4.1.8 are left for the reader to check.

C.2. Lifting and Grounding in the Above-Equality Fragment of $\text{EPC}_{<}$

For Lemma 4.2.1:

Proof (Sketch): Let Π denote a $\text{EPC}_{<}^{+1}$ -deduction of D from $\{B\} \cup \mathcal{C}$. Since each resolution in Π involves eliminating one or more negative P instances by resolving against a clause, pure-positive in P , of $\mathcal{C}$, it is easy to see that Π may be trivially reorganized to yield a $\text{EPC}_{<}^{+1}$ -deduction Π' of (some permutation of) D from $\{B'\} \cup \mathcal{C}$, such that for each $C \in \mathcal{C}$, $k_{\Pi'}(C) = k_{\Pi}(C)$.

Alternatively, a proof may be constructed directly from that given for Lemma 2.2.1. (here substitutions (instantiations) need not be considered (to occur), and the description of D must be modified, in the obvious way). ■

For Lemma 4.2.2:

Proof: (sketch) Let A , A' , B , D , and D' , as in the statement of the lemma, (and assume that A , A' , D , and D' , share (pairwise) no variable in common, which we can do by Observation 4.1.6). By assumption, there is some substitution θ_A , (which we may assume, has domain contained in the set of variables of A'), such that (modulo clause ordering) $A'\theta_A = A$.

Clearly A' must be pure-positive with respect to P and have $<$ -maximal predicate symbol P , and D' is mixed with respect to P , with $<$ -maximal predicate symbol P . By the assumed separation of variables, evidently (we may assume the domain of θ_D is a subset of the variables of D , and), there is a substitution θ , say, (for example, the composite, $\theta_A\theta_D$), such that (modulo clause ordering) $A'\theta = A$ and $D'\theta = D$, (and $L_0\theta$ occurs as the left-most inequality in D).

For some non-empty subsets $A_R \subseteq A$, and $D_R \subseteq D$, (say), the left-most negative P instance, $L_0\theta$, of D occurring in D_R , (and some term tuples $\vec{s}_1, \dots, \vec{s}_i$ and $\vec{t}_1, \dots, \vec{t}_j$),
 $A_R \equiv \{P(\vec{s}_1), \dots, P(\vec{s}_i)\}$; $D_R \equiv \{\neg P(\vec{t}_1), \dots, \neg P(\vec{t}_j)\}$, and (modulo clause ordering),

$$B = \bigcup_{\substack{1 \leq i' \leq i \\ 1 \leq j' \leq j}} \neg(\vec{s}_{i'}, \vec{t}_{j'}) \cup A \setminus A_R \cup D \setminus D_R.$$

Let $A'_R \equiv \{L \in A' \mid L\theta \in A_R\}$, and let $D'_R \equiv \{L \in D' \mid L\theta \in D_R\}$.

Then $L_0 \in D'_R$, and clearly $D'_R\theta = D'\theta \cap D_R = D \cap D_R = D_R \neq \emptyset$, and $\emptyset \neq D'_R \subseteq D'$.

Similarly, $A'_R\theta = A'\theta \cap A_R = A \cap A_R = A_R \neq \emptyset$, and $\emptyset \neq A'_R \subseteq A'$.

Let $A'_R \equiv \{P(\vec{q}_1), \dots, P(\vec{q}_k)\}$, say, and $D'_R \equiv \{\neg P(\vec{r}_1), \dots, \neg P(\vec{r}_l)\}$, say. Clearly if we let

$$B' \equiv \bigcup_{\substack{1 \leq k' \leq k \\ 1 \leq l' \leq l}} \neg(\vec{q}_{k'}, \vec{r}_{l'}) \cup A' \setminus A'_R \cup D' \setminus D'_R, \text{ then } B' \in \text{EPC}_{<}^1(A', D').$$

$$\text{Evidently, } B'\theta = \bigcup_{\substack{1 \leq k' \leq k \\ 1 \leq l' \leq l}} \neg(\vec{q}_{k'}\theta, \vec{r}_{l'}\theta) \cup (A' \setminus A'_R)\theta \cup (D' \setminus D'_R)\theta.$$

By the choice of A'_R , $\{\vec{q}_{k'}\theta \mid 1 \leq k' \leq k\} = \{\vec{s}_{i'} \mid 1 \leq i' \leq i\}$, and $(A' \setminus A'_R)\theta = A \setminus A_R$.

Similarly, by the choice of D'_R , $\{\vec{r}_{l'}\theta \mid 1 \leq l' \leq l\} = \{\vec{t}_{j'} \mid 1 \leq j' \leq j\}$, and $(D' \setminus D'_R)\theta = D \setminus D_R$.

Thus (modulo ordering), $B'\theta = B$. ■

For Lemma 4.2.3:

Proof: Let A , B , D , and D_g , as in the statement of the lemma.

(and, as usual, assume that A and D share no variable in common).

Since $\text{EPC}_{<}^1(A, D) \neq \emptyset$, both A and D have the same $<$ -maximal predicate symbol.

$P \neq '='$, and without loss of generality, (since $\text{EPC}_{<}^1(\cdot, \cdot)$ is symmetric), we may assume that A is pure-positive with respect to P , (and D is mixed with respect to P).

For some non-empty subsets $A_R \subseteq A$, and $D_R \subseteq D$, (say), the left-most negative P instance, L_0 , say, of D occurring in D_R , (and some term tuples $\vec{s}_1, \dots, \vec{s}_i$ and $\vec{t}_1, \dots, \vec{t}_j$),

$A_R \equiv \{P(\vec{s}_1), \dots, P(\vec{s}_i)\}$; $D_R \equiv \{\neg P(\vec{t}_1), \dots, \neg P(\vec{t}_j)\}$, and (modulo clause ordering),

$$B = \bigcup_{\substack{1 \leq i' \leq i \\ 1 \leq j' \leq j}} \neg(\vec{s}_{i'}, \vec{t}_{j'}) \cup A \setminus A_R \cup D \setminus D_R.$$

Since B_g is a substitution instance of B , for some substitution θ , (modulo clause ordering),

$$B_g = \bigcup_{\substack{1 \leq i' \leq i \\ 1 \leq j' \leq j}} \neg(\vec{s}_{i'}, \vec{t}_{j'})\theta \cup (A \setminus A_R)\theta \cup (D \setminus D_R)\theta.$$

Since B_g is ground, evidently, so are $A_R\theta$, $D_R\theta$, $(A \setminus A_R)\theta$, and $(D \setminus D_R)\theta$, and hence, (since $A\theta = (A \setminus A_R)\theta \cup A_R\theta$, and $D\theta = (D \setminus D_R)\theta \cup D_R\theta$), both $A\theta$ and $D\theta$ are ground.

Let A' be an arbitrary ordering of $A\theta$, and let D' be any ordering of $D\theta$ such that the left-most literal of D' lies in $D_R\theta$. Let B'_g be (some ordering of) $\bigcup_{\substack{1 \leq i' \leq i \\ 1 \leq j' \leq j}} \neg(\vec{s}_{i'}, \vec{t}_{j'})\theta \cup A\theta \setminus A_R\theta \cup D\theta \setminus D_R\theta$.

Then, evidently, A' and D' are ground substitution instances of A and D , respectively. $B'_g \in \text{EPC}_{<}^1(A', D')$, and $B'_g \subseteq B_g$, as required. ■

For Lemma 4.2.4:

Proof (Sketch): Again, one can follow the outline of the proof given for Lemma 2.2.3. That is, one can argue by an induction on the depth of Π , this time appealing to Lemma 4.2.2 and Lemma 4.2.1, in place of (their $\text{PC}_{<}$ -deduction analogues), Lemma 2.2.2 and Lemma 2.2.1, respectively. ■

For Lemma 4.2.5:

Proof (Sketch): A proof is essentially provided by that given for Lemma 2.2.4, with (“ $\text{EPC}_{<}$ ” substituted for “ $\text{PC}_{<}$ ”, and), Lemma 4.2.4, substituted for Lemma 2.2.3. Then one merely has to observe that the Π' so constructed, is above-equality. ■

For Corollary 4.2.1:

Proof: As the root clause E of an above-equality $\text{EPC}_{<}$ -deduction of an equality clause E is necessarily $\text{EPC}_{<}$ -initial, by Lemma 4.2.5, there is an above-equality $\text{EPC}_{<}$ -deduction Π' of a clause E' , $\text{EPC}_{<}$ -initial (in Π'), of which E is a substitution instance, from $\mathfrak{S}'$, such that $\forall B' \in \mathfrak{S}', k_{\Pi'}(B') = ||\{n \in \mathfrak{L} \mid B'_n = B'\}||$.

Since E is a substitution instance of E' , and E is an equality clause, evidently, E' is also an equality clause. ■

C.3. Subsumption in the Above-Equality Fragment of $\text{EPC}_{<}$

For Lemma 4.3.1:

Proof: The proof is a modification of that given for Lemma 4.2.2.
(We could have used Lemma 4.2.2, but without significant advantage.)

As in the proof for Lemma 4.2.2, let A , A' , B , D , and D' , as in the statement of the lemma, (and without loss of generality, assume no two share a common variable), and let θ be a substitution such that $A'\theta \subseteq A$ and $D'\theta \subseteq D$, (and, in case (ii), $L_0\theta$ occurs as the left-most inequality in D).

Clearly the $<$ -maximal predicate symbol occurring in D' must be less than or equal to P .

For some non-empty subsets $A_R \subseteq A$, and $D_R \subseteq D$, (say), the left-most negative P instance, $L_0\theta$, of D occurring in D_R , (and some term tuples $\vec{s}_1, \dots, \vec{s}_i$ and $\vec{t}_1, \dots, \vec{t}_j$),

$A_R \equiv \{P(\vec{s}_1), \dots, P(\vec{s}_i)\}$; $D_R \equiv \{\neg P(\vec{t}_1), \dots, \neg P(\vec{t}_j)\}$, and

$B \equiv \bigcup_{\substack{1 \leq i' \leq i \\ 1 \leq j' \leq j}} \neg(\vec{s}_{i'}, \vec{t}_{j'}) \cup A \setminus A_R \cup D \setminus D_R$.

Let $D'_R \equiv \{L \in D' \mid L\theta \in D_R\}$. Clearly, $D'_R\theta = D'\theta \cap D_R$.

In case (i), $D'_R = \emptyset$, so $D'\theta \subseteq D \setminus D_R \subseteq B$, and so D' subsumes B .

In case (ii), $L_0\theta$ occurs as the left-most negative P instance in D , and hence $L_0\theta \in D_R$. Thus $L_0 \in D'_R$, and clearly $\emptyset \neq D'_R \subseteq D'$.

Let $A'_R \equiv \{L \in A' \mid L\theta \in A_R\}$. Then $A'_R\theta = A'\theta \cap A_R$, and $A'_R \subseteq A'$.

If $A'_R = \emptyset$, then $A'\theta \subseteq A \setminus A_R \subseteq B$, and so A' subsumes B .

Otherwise, $A'_R \equiv \{P(\tilde{q}_1), \dots, P(\tilde{q}_k)\}$, say, and $D'_R \equiv \{\neg P(\tilde{r}_1), \dots, \neg P(\tilde{r}_l)\}$, say. Clearly if we let

$$B' \equiv \bigcup_{\substack{1 \leq k' \leq k \\ 1 \leq l' \leq l}} \neg(\tilde{q}_{k'}, \tilde{r}_{l'}) \cup A' \setminus A'_R \cup D' \setminus D'_R, \text{ then } B' \in \text{EPC}_{<}^1(A', D').$$

We claim that B' subsumes B .

$$\text{Evidently, } B'\theta \equiv \bigcup_{\substack{1 \leq k' \leq k \\ 1 \leq l' \leq l}} \neg(\tilde{q}_{k'}\theta, \tilde{r}_{l'}\theta) \cup (A' \setminus A'_R)\theta \cup (D' \setminus D'_R)\theta.$$

By the choice of A'_R , $\{\tilde{q}_{k'}\theta \mid 1 \leq k' \leq k\} \subseteq \{\tilde{s}_{i'} \mid 1 \leq i' \leq i\}$, and $(A' \setminus A'_R)\theta \subseteq A \setminus A_R$.

Similarly, by the choice of D'_R , $\{\tilde{r}_{l'}\theta \mid 1 \leq l' \leq l\} \subseteq \{\tilde{t}_{j'} \mid 1 \leq j' \leq j\}$, and $(D' \setminus D'_R)\theta \subseteq D \setminus D_R$. Thus $B'\theta \subseteq B$, and hence, B' subsumes B . ■

For Lemma 4.3.2:

Proof: A proof is that given for Lemma 2.4.2, with (“ $\text{EPC}_{<}$ ” substituted for “ $\text{PC}_{<}$ ”, and), Lemma 4.3.1 substituted for (its analogue) Lemma 2.4.1. ■

For Lemma 4.3.3:

Proof (sketch): A proof is that given for Lemma 2.4.3, with (“ $\text{EPC}_{<}$ ” substituted for “ $\text{PC}_{<}$ ”, and), Lemma 4.3.2, Lemma 4.3.1, and Lemma 4.2.1 substituted for (their $\text{PC}_{<}$ -deduction analogues), Lemma 2.4.2, Lemma 2.4.1, and Lemma 2.2.1, respectively. ■

For Lemma 4.3.4:

Proof: A proof is that given for Lemma 2.4.4, with “(above-equality) $\text{EPC}_{<}^{+1}$ ” substituted for “ $\text{PC}_{<}^{+1}$ -deduction”, “ $\text{EPC}_{<}$ -initial” substituted for “ $\text{PC}_{<}$ -initial”, “above-equality $\text{EPC}_{<}$ -deduction” substituted for “ $\text{PC}_{<}$ -deduction”, and Lemma 4.3.3, substituted for Lemma 2.4.3. ■

For Corollary 4.3.1:

Proof: As the root clause E of an above-equality $\text{EPC}_{<}$ -deduction of an equality clause E is necessarily $\text{EPC}_{<}$ -initial, by Lemma 4.3.4, there is an above-equality $\text{EPC}_{<}$ -deduction Π' of a clause E' , $\text{EPC}_{<}$ -initial (in Π'), which subsumes E , from $\mathfrak{S}'$, and such that $\forall B' \in \mathfrak{S}', k_{\Pi'}(B') \leq ||\{n \in \mathcal{L} \mid B'_n = B'\}||$.

Since E' subsumes E , and E is an equality clause, evidently, E' is also an equality clause. ■

For Lemma 4.3.5:

Proof: By induction on the total number of leaf nodes, $\kappa \equiv \kappa(\Pi) \geq 1$ of Π .

If $\kappa = 1$, then Π is trivial, consisting of only one node, n_0 , say. n_0 is labelled by A , so B_{n_0} is A , $A \in \mathfrak{S}$, and $\mathfrak{L} \equiv \{n_0\}$.

Letting both B'_{n_0} and A' be A_g , clearly, B'_{n_0} is a substitution instance of B_{n_0} , and taking Π' to be the trivial deduction of (that is, consisting only of one node labelled by) A' , evidently suffices.

Now suppose the result for all κ : $1 \leq \kappa < N$, and that $\kappa = \kappa(\Pi) = N$.

Evidently there is some node n_1 in Π with (deduction) depth 1, with labelling clause B say and with leaf parents, n_0 and n'_0 , say, labelled by some non-equality clauses C and D , say, in $\mathfrak{S}$.

Of course, $B \in EPC_{<}^1(C, D)$.

Letting Π^{n_1} denote the fragment of Π obtained by (top-) pruning Π at n_1 ,

(so that now n_1 is a leaf of Π^{n_1}), evidently Π^{n_1} is an above-equality $EPC_{<}$ -deduction of A , $EPC_{<}$ -initial (in Π^{n_1}), from $\mathfrak{S} \cup \{B\}$, with leaves $\mathfrak{L}' \equiv \mathfrak{L} \setminus \{n_0, n'_0\} \cup \{n_1\}$, a node $n \in \mathfrak{L} \setminus \{n_0, n'_0\}$ labelled (in Π^{n_1}), by B_n , n_1 labelled by B , and $\kappa(\Pi^{n_1}) = N - 1 < N$.

By induction, there is a finite collection of ground clauses, $\mathfrak{S}'' \equiv \{B'_n \mid n \in \mathfrak{L} \setminus \{n_0, n'_0\}\} \cup \{B'\}$, where $\forall n \in \mathfrak{L} \setminus \{n_0, n'_0\}$, B'_n is a ground substitution instance of B_n , and B' is a ground substitution instance of B , and there is an above-equality $EPC_{<}$ -deduction Π'' of a (ground) clause A'' , $EPC_{<}$ -initial (in Π''), from $\mathfrak{S}''$, such that $A'' \subseteq A_g$, and such that

$$(C.3.C.1) \quad \forall B''' \notin \{B'\}, \quad k_{\Pi''}(B''') \leq ||\{n \in \mathfrak{L} \setminus \{n_0, n'_0\} \mid B'_n = B'''\}||, \text{ and}$$

$$(C.3.C.2) \quad k_{\Pi''}(B') \leq ||\{n \in \mathfrak{L} \setminus \{n_0, n'_0\} \mid B'_n = B'\}|| + 1.$$

Now if $k_{\Pi''}(B') = 0$, let B'_{n_0} and $B'_{n'_0}$ be defined to be any ground substitution instances of A and D , respectively.

Then clearly, $\forall B'''$, $k_{\Pi''}(B''') \leq ||\{n \in \mathfrak{L} \mid B'_n = B'''\}||$.

Thus, (if $k_{\Pi''}(B') = 0$), taking $\mathfrak{S}' \equiv \{B'_n \mid n \in \mathfrak{L}\}$, and Π'' for Π' , and A'' for A' , clearly suffices.

Thus we may assume that $k_{\Pi''}(B') \geq 1$.

Let $\mathfrak{L}''$ denote the set of leaves of Π'' , and suppose that for any $n \in \mathfrak{L}''$, B''_n denotes the (ground) clause which labels the leaf, n in Π'' .

Let $m_0 \in \mathfrak{L}''$ denote some chosen leaf whose label, B''_{m_0} , is B' .

Since $B \in EPC_{<}^1(C, D)$, and B' is a ground substitution instance of B , by Lemma 4.2.3, there exist ground substitution instances C' and D' of C and D , respectively, and a (ground clause) $B'' \in EPC_{<}^1(C', D')$, such that $B'' \subseteq B'$.

Evidently C' and D' are non-equality clauses, and $C' \neq D'$.

For any $n \in \mathfrak{L}'' \setminus \{m_0\}$, let $B'''_n \equiv B''_n$, and let $B'''_{m_0} \equiv B''$.

In this way, clearly, to each leaf node $n \in \mathfrak{L}''$, (of Π''), we have associated a ground clause, B'''_n , such that B'''_n subsumes the clause, B''_n , which labels n in Π'' .

Let $\mathfrak{S}''' \equiv \{B'''_n \mid n \in \mathfrak{L}''\}$.

By Lemma 4.3.4, (applied to Π'' , and at most, the single proper subsumption instance of B'' subsuming B' at m_0), there is a EPC_<-deduction Π''' of a clause A' , EPC_<-initial (in Π'''), which subsumes A'' , from $\mathfrak{S}'''$, such that

$$(C.3.C.3) \quad \forall B''' \in \mathfrak{S}''', \quad k_{\Pi'''}(B''') \leq ||\{n \in \mathfrak{L}'' \mid B'''_n = B'''\}||.$$

Since $\mathfrak{S}'''$ contains only ground clauses, A' is ground, so, (as A' subsumes A''), $A' \subseteq A''$, and thus, (since $A'' \subseteq A_g$), $A' \subseteq A_g$.

Now for any $B''' \notin \{B''\}$, evidently,

$$(C.3.C.4) \quad ||\{n \in \mathfrak{L}'' \mid B'''_n = B'''\}|| = ||\{n \in \mathfrak{L}'' \setminus \{m_0\} \mid B''_n = B'''\}||.$$

In contrast,

$$(C.3.C.5) \quad ||\{n \in \mathfrak{L}'' \mid B'''_n = B''\}|| = ||\{n \in \mathfrak{L}'' \setminus \{m_0\} \mid B''_n = B''\}|| + 1.$$

Since $B''_{m_0} = B'$, for any $B''' \notin \{B', B''\}$, evidently,

$$(C.3.C.6) \quad ||\{n \in \mathfrak{L}'' \mid B'''_n = B'''\}|| = ||\{n \in \mathfrak{L}'' \mid B''_n = B'''\}|| \\ = k_{\Pi''}(B''') \leq ||\{n \in \mathfrak{L} \setminus \{n_0, n'_0\} \mid B'_n = B'''\}||.$$

If $B' \neq B''$, then, from (C.3.C.4), (since $B''_{m_0} = B'$), and (C.3.C.2),

$$(C.3.C.7) \quad ||\{n \in \mathfrak{L}'' \mid B'''_n = B'\}|| = ||\{n \in \mathfrak{L}'' \setminus \{m_0\} \mid B''_n = B'\}|| \\ = ||\{n \in \mathfrak{L}'' \mid B''_n = B'\}|| - 1 = k_{\Pi''}(B') - 1 \leq ||\{n \in \mathfrak{L} \setminus \{n_0, n'_0\} \mid B'_n = B'\}||.$$

Also, (if $B' \neq B''$), from (C.3.C.5), (since $B''_{m_0} = B' \neq B''$), and (C.3.C.1),

$$(C.3.C.8) \quad ||\{n \in \mathfrak{L}'' \mid B'''_n = B''\}|| = ||\{n \in \mathfrak{L}'' \setminus \{m_0\} \mid B''_n = B''\}|| + 1 \\ = ||\{n \in \mathfrak{L}'' \mid B''_n = B''\}|| + 1 = k_{\Pi''}(B'') + 1 \leq ||\{n \in \mathfrak{L} \setminus \{n_0, n'_0\} \mid B'_n = B''\}|| + 1.$$

On the other hand, if $B' = B''$, then, from (C.3.C.5), (since $B''_{m_0} = B'$), and (C.3.C.2),

$$(C.3.C.9) \quad ||\{n \in \mathfrak{L}'' \mid B'''_n = B'\}|| = ||\{n \in \mathfrak{L}'' \setminus \{m_0\} \mid B''_n = B'\}|| + 1 \\ = ||\{n \in \mathfrak{L}'' \mid B''_n = B'\}|| = k_{\Pi''}(B') \leq ||\{n \in \mathfrak{L} \setminus \{n_0, n'_0\} \mid B'_n = B'\}|| + 1.$$

Thus, if $B' \neq B''$, from (C.3.C.3), (C.3.C.6), (C.3.C.7), and (C.3.C.8), we have:

$$\forall B''' \notin \{B''\}, \quad k_{\Pi'''}(B''') \leq ||\{n \in \mathfrak{L} \setminus \{n_0, n'_0\} \mid B'_n = B'''\}||, \text{ and}$$

$$k_{\Pi'''}(B'') \leq ||\{n \in \mathfrak{L} \setminus \{n_0, n'_0\} \mid B'_n = B''\}|| + 1,$$

Similarly, if $B' = B''$, from (C.3.C.3), (C.3.C.6), and (C.3.C.9), we also have:

$$\forall B''' \notin \{B''\}, \quad k_{\Pi'''}(B''') \leq ||\{n \in \mathfrak{L} \setminus \{n_0, n'_0\} \mid B'_n = B'''\}||, \text{ and}$$

$$k_{\Pi'''}(B'') \leq ||\{n \in \mathfrak{L} \setminus \{n_0, n'_0\} \mid B'_n = B''\}|| + 1,$$

In any case,

$$(C.3.C.10) \quad \forall B''' \notin \{B''\}, \quad k_{\Pi'''}(B''') \leq ||\{n \in \mathcal{L} \setminus \{n_0, n'_0\} \mid B'_n = B'''\}\|, \text{ and}$$

$$(C.3.C.11) \quad k_{\Pi'''}(B'') \leq ||\{n \in \mathcal{L} \setminus \{n_0, n'_0\} \mid B'_n = B''\}\| + 1.$$

Now if $k_{\Pi'''}(B'') = 0$, then clearly, $\forall B'''$, $k_{\Pi'''}(B''') \leq ||\{n \in \mathcal{L} \setminus \{n_0, n'_0\} \mid B'_n = B'''\}\|$.

Further, as Π''' is leafed in $\mathfrak{S}''' = \{B'''_n \mid n \in \mathcal{L}''\}$,

if $k_{\Pi'''}(B'') = 0$, then Π''' is leafed in $\{B'''_n \mid n \in \mathcal{L}'' \setminus \{m_0\}\}$. Since

$$\{B'''_n \mid n \in \mathcal{L}'' \setminus \{m_0\}\} = \{B''_n \mid n \in \mathcal{L}'' \setminus \{m_0\}\} \subseteq \mathfrak{S}'' = \{B'_n \mid n \in \mathcal{L} \setminus \{n_0, n'_0\}\} \cup \{B'\},$$

Π''' is leafed in $\{B'_n \mid n \in \mathcal{L} \setminus \{n_0, n'_0\}\} \cup \{B'\}$.

Since (if $k_{\Pi'''}(B'') = 0$), $k_{\Pi'''}(B') \leq ||\{n \in \mathcal{L} \setminus \{n_0, n'_0\} \mid B'_n = B'\}\|$.

Thus, either $k_{\Pi'''}(B') = 0$, or $B' = B'_n$, for some $n \in \mathcal{L} \setminus \{n_0, n'_0\}$.

At any rate, (if $k_{\Pi'''}(B'') = 0$), Π''' is leafed in $\{B'_n \mid n \in \mathcal{L} \setminus \{n_0, n'_0\}\}$.

Thus, like before, (if $k_{\Pi'''}(B'') = 0$), letting B'_{n_0} and $B'_{n'_0}$ be defined to be any ground substitution instances of A and D , respectively, clearly, $(\forall B''', k_{\Pi'''}(B''') \leq ||\{n \in \mathcal{L} \mid B'_n = B'''\}\|$, and), taking $\mathfrak{S}' \equiv \{B'_n \mid n \in \mathcal{L}\}$, and Π''' for Π' , suffices.

Thus we may now assume that $k_{\Pi'''}(B'') \geq 1$.

Let B'_{n_0} and $B'_{n'_0}$ be defined to be C' and D' respectively.

Let Π' be the EPC_<-deduction which results from affixing the 1-step EPC_<-deduction $B'' \in \text{EPC}_{<}^1(C', D')$, to some leaf node of Π''' whose labelling clause is B'' .

Then Π' is an above-equality EPC_<-deduction of A' , EPC_<-initial (at the root, in Π'), and $\forall n \in \mathcal{L}$, we have an associated B'_n , which is a ground substitution instance of B_n .

Further, evidently,

$$(C.3.C.12) \quad \forall B''' \notin \{B'', C', D'\}, \quad k_{\Pi'}(B''') = k_{\Pi'''}(B''').$$

Also, if $B'' \notin \{C', D'\}$,

$$(C.3.C.13) \quad k_{\Pi'}(B'') = k_{\Pi'''}(B'') - 1.$$

Thus, if $B'' \notin \{C', D'\}$, from (C.3.C.10), and (C.3.C.11) we get:

$$(C.3.C.14) \quad \forall B''' \notin \{C', D'\}, \quad k_{\Pi'}(B''') \leq ||\{n \in \mathcal{L} \setminus \{n_0, n'_0\} \mid B'_n = B'''\}\|.$$

Also, if $B'' \notin \{C', D'\}$, since $C' \neq D'$, evidently:

$$(C.3.C.15) \quad k_{\Pi'}(C') = k_{\Pi'''}(C') + 1, \text{ and}$$

$$(C.3.C.16) \quad k_{\Pi'}(D') = k_{\Pi'''}(D') + 1.$$

Thus, again by (C.3.C.10) and (C.3.C.11), (if $B'' \notin \{C', D'\}$),

$$(C.3.C.17) \quad k_{\Pi'}(C') \leq ||\{n \in \mathfrak{L} \setminus \{n_0, n'_0\} \mid B'_n = C'\}|| + 1, \text{ and}$$

$$(C.3.C.18) \quad k_{\Pi'}(D') \leq ||\{n \in \mathfrak{L} \setminus \{n_0, n'_0\} \mid B'_n = D'\}|| + 1.$$

Since, clearly,

$$\forall B''' \notin \{C', D'\}, ||\{n \in \mathfrak{L} \setminus \{n_0, n'_0\} \mid B'_n = B'''\}|| = ||\{n \in \mathfrak{L} \mid B'_n = B'''\}||,$$

while (since $C' \neq D'$),

$$||\{n \in \mathfrak{L} \setminus \{n_0, n'_0\} \mid B'_n = C'\}|| = ||\{n \in \mathfrak{L} \mid B'_n = C'\}|| - 1, \text{ and}$$

$$||\{n \in \mathfrak{L} \setminus \{n_0, n'_0\} \mid B'_n = D'\}|| = ||\{n \in \mathfrak{L} \mid B'_n = D'\}|| - 1,$$

from (C.3.C.14), (C.3.C.17), and (C.3.C.18), (if $B'' \notin \{C', D'\}$), we get, respectively:

$$(C.3.C.19) \quad \forall B''' \notin \{C', D'\}, \quad k_{\Pi'}(B''') \leq ||\{n \in \mathfrak{L} \mid B'_n = B'''\}||,$$

$$(C.3.C.20) \quad k_{\Pi'}(C') \leq ||\{n \in \mathfrak{L} \mid B'_n = C'\}||, \text{ and}$$

$$(C.3.C.21) \quad k_{\Pi'}(D') \leq ||\{n \in \mathfrak{L} \mid B'_n = D'\}||.$$

Thus, if $B'' \notin \{C', D'\}$, we have

$$(C.3.C.22) \quad \forall B''', \quad k_{\Pi'}(B''') \leq ||\{n \in \mathfrak{L} \mid B'_n = B'''\}||,$$

as required.

Now suppose $B'' \in \{C', D'\}$. Assume, without loss of generality, (because of the symmetry of the situation), that $B'' = C'$.

We still have (C.3.C.12), which simplifies to:

$$(C.3.C.23) \quad \forall B''' \notin \{C', D'\}, \quad k_{\Pi'}(B''') = k_{\Pi'''}(B''').$$

Clearly also, (since $C' = B''$, and in constructing Π' from Π''' , we have lost one leaf with label B'' , and gained one with label C'),

$$(C.3.C.24) \quad k_{\Pi'}(C') = k_{\Pi'''}(C'), \text{ while}$$

$$(C.3.C.25) \quad k_{\Pi'}(D') = k_{\Pi'''}(D') + 1.$$

Since $C' = B''$, while $D' \neq C' = B''$, by (C.3.C.10) and (C.3.C.11), from (C.3.C.23), (C.3.C.24), and (C.3.C.25), respectively, we get:

$$(C.3.C.26) \quad \forall B''' \notin \{C', D'\}, \quad k_{\Pi'}(B''') \leq ||\{n \in \mathfrak{L} \setminus \{n_0, n'_0\} \mid B'_n = B'''\}||,$$

$$(C.3.C.27) \quad k_{\Pi'}(C') \leq ||\{n \in \mathcal{L} \setminus \{n_0, n'_0\} \mid B'_n = C'\}|| + 1, \text{ and}$$

$$(C.3.C.28) \quad k_{\Pi'}(D') \leq ||\{n \in \mathcal{L} \setminus \{n_0, n'_0\} \mid B'_n = D'\}|| + 1.$$

That is, (C.3.C.14), (C.3.C.17), and (C.3.C.18), remain valid, and inequations (C.3.C.19), (C.3.C.20), (C.3.C.21), and hence, (C.3.C.22), follow as before. Thus, again, taking $\mathfrak{S}' \equiv \{B'_n \mid n \in \mathcal{L}\}$, and Π''' for Π' , suffices. ■

For Corollary 4.3.2:

Proof: The argument is like that given for Corollary 4.3.1. with minor modifications:

As the root clause E of an above-equality $\text{EPC}_{<}$ -deduction of an equality clause E is necessarily $\text{EPC}_{<}$ -initial, by Lemma 4.3.5, there is a collection $\mathfrak{S}' \equiv \{B'_n \mid n \in \mathcal{L}\}$, where $\forall n \in \mathcal{L}$, B'_n is a ground substitution instance of B_n , and an above-equality $\text{EPC}_{<}$ -deduction Π' of a (ground) clause E' , $\text{EPC}_{<}$ -initial (in Π'), from $\mathfrak{S}'$, such that $E' \subseteq E_g$, and such that

$$\forall B' \in \mathfrak{S}', \quad k_{\Pi'}(B') \leq ||\{n \in \mathcal{L} \mid B'_n = B'\}||.$$

Since $E' \subseteq E_g$, and E_g is (obviously), an equality clause, so is E' . ■

C.4. Lifting and Grounding in the Equality Fragment of $\text{EPC}_{<}$

For Lemma 4.4.1:

Proof: Let A , A' , B , D , and D' , as in the statement of the lemma, (and assume that A , A' , D , and D' , share (pairwise) no variable in common, which we can do by Observation 4.1.6).

By assumption, there is some substitution ρ_A , (which we may assume, has domain contained in the set of variables of A'), such that (modulo clause ordering) $A'\rho_A = A$. Clearly A' must be positive, and D' is mixed. By the assumed separation of variables, evidently there is a substitution ρ , say, (for example, the composite, $\rho_A\rho_D$), such that (modulo clause ordering) $A'\rho = A$, $D'\rho = D$, and with L the left-most inequality in D' , $L\rho$ occurs as the left-most inequality in D .

Let D^- denote the inequality literals of D .

There is some non-empty subset $A_R \subseteq A$, some non-empty subset $D_R \subseteq D^-$, the left-most inequality of D in D_R , and a (possibly empty) subset $D_S \subseteq D^- \setminus D_R$, such that for some symmetric choice function c on $A_R \cup D_R \cup D_S$, the most general substitution σ , say, unifying:

$$\begin{aligned} & \{ \{=(s', t') \mid =(s', t') \in A_R; c(=(s', t')) = 0\} \cup \{=(t', s') \mid =(s', t') \in A_R; c(=(s', t')) = 1\} \cup \\ & \quad \{=(s', t') \mid \neq(s', t') \in D_R; c(\neq(s', t')) = 0\} \cup \{=(t', s') \mid \neq(s', t') \in D_R; c(\neq(s', t')) = 1\}, \\ & \{s' \mid =(s', t') \in A_R; c(=(s', t')) = 0\} \cup \{t' \mid =(s', t') \in A_R; c(=(s', t')) = 1\} \cup \\ & \quad \{s' \mid \neq(s', t') \in D_S; c(\neq(s', t')) = 0\} \cup \{t' \mid \neq(s', t') \in D_S; c(\neq(s', t')) = 1\} \}, \end{aligned}$$

exists, σ unifying the two sets in the above doubleton set to the singletons $\{=(s, r)\}$ and $\{s\}$, say, respectively, and

$$\begin{aligned} B & \equiv A\sigma \setminus \{=(r, s), =(s, r)\} \cup D\sigma \setminus (\{\neq(r, s), \neq(s, r)\} \cup D_S\sigma) \cup \\ & \quad \{\neq(r, t'\sigma) \mid \neq(s', t') \in D_S; \neq(s', t')\sigma \notin \{\neq(r, s), \neq(s, r)\}; c(\neq(s', t')) = 0\} \cup \end{aligned}$$

$$\{\neq(s'\sigma, r) \mid \neq(s', t') \in D_S; \neq(s', t')\sigma \notin \{\neq(r, s), \neq(s, r)\}; c(\neq(s', t')) = 1\}.$$

By assumption. $A'\rho = A$, (modulo clause ordering), and hence $A'\rho\sigma = A\sigma$.

Evidently $\{=(s, r), =(r, s)\} \cap A'\rho\sigma \supseteq A_R\sigma \neq \emptyset$.

Let $A'_R \equiv \{L \in A' \mid L\rho\sigma \in \{=(s, r), =(r, s)\}\}$. Clearly $\emptyset \neq A'_R \subseteq A'$, and

$$\{=(s, r), =(r, s)\} \supseteq A'_R\rho\sigma = A'\rho\sigma \cap \{=(s, r), =(r, s)\} = A\sigma \cap \{=(s, r), =(r, s)\} \supseteq A_R\sigma.$$

Also, $A_R \subseteq A'_R\rho$.

We similarly have $D'\rho = D$, (modulo clause ordering), and hence $D'\rho\sigma = D\sigma$.

Let $D'_R \equiv \{L \in D' \mid L\rho\sigma \in \{\neq(s, r), \neq(r, s)\}\}$. Clearly $\emptyset \neq D'_R \subseteq D'^-$, and

$$D'_R\rho\sigma = D'\rho\sigma \cap \{\neq(s, r), \neq(r, s)\} = D\sigma \cap \{\neq(s, r), \neq(r, s)\} \subseteq \{\neq(s, r), \neq(r, s)\}.$$

Also, $D_R \subseteq D'_R\rho$, and with L the left-most inequality in D' , L is in D'_R , (since $L\rho$ occurs as the left-most inequality in D , and thus occurs in D_R , so $L\rho\sigma \in \{\neq(s, r), \neq(r, s)\}$).

Let $D'_S \equiv \{L \in D'^- \setminus D'_R \mid L\rho\sigma \in D_S\sigma\}$. Clearly, $D'_S \subseteq D'^- \setminus D'_R$, and $D'_S\rho\sigma \subseteq D_S\sigma$.

so for any literal $\neq(s'', t'') \in D'_S$, either $s''\rho\sigma = s$, or $t''\rho\sigma = s$.

In the other direction, if $L \in D_S\sigma$, then L is $\neq(s', t')\sigma$, for some $\neq(s', t') \in D_S \subseteq D^- \setminus D_R$.

Now if L is $\neq(s'', t'')\rho\sigma$, for some $\neq(s'', t'') \in D'^- \setminus D'_R$, then $L \in D'_S\rho\sigma$.

Otherwise, for all $\neq(s'', t'') \in D'^-$ such that $L = \neq(s'', t'')\rho\sigma$,

(and there is at least one such, such that $\neq(s'', t'')\rho = \neq(s', t')$), $\neq(s'', t'') \in D'_R$. In particular,

$L \in \{\neq(s, r), \neq(r, s)\}$. Therefore, $D'_S\rho\sigma \setminus \{\neq(s, r), \neq(r, s)\} = D_S\sigma \setminus \{\neq(s, r), \neq(r, s)\}$.

Let c' be the choice function on $A'_R \cup D'_R \cup D'_S$ defined separately on the (disjoint) sets A'_R , D'_R , and D'_S , as follows:

$$\begin{aligned} \text{On } A'_R: \quad &=(s', t') \in A'_R \mapsto 0, \quad \text{if } =(s', t')\rho\sigma = =(s, r), \text{ and} \\ &=(s', t') \in A'_R \mapsto 1, \quad \text{otherwise. (if } =(s', t')\rho\sigma = =(r, s)). \end{aligned}$$

$$\begin{aligned} \text{On } D'_R: \quad &\neq(s', t') \in D'_R \mapsto 0, \quad \text{if } \neq(s', t')\rho\sigma = \neq(s, r), \text{ and} \\ &\neq(s', t') \in D'_R \mapsto 1, \quad \text{otherwise. (if } \neq(s', t')\rho\sigma = \neq(r, s)). \end{aligned}$$

We define $c': D'_S \rightarrow \{0, 1\}$ by stages:

let $D'_S \equiv \bigcup_{k': 0 \leq k' \leq k} D''_{k'}$, where $D''_0 = \{L \in D'_S \mid L\rho \in D_S\}$, and $\forall k': 0 \leq k' < k$,

$D''_{k'+1} = D''_{k'} \cup \{\neq(s'_{k'+1}, t'_{k'+1})\}$ where the $\neq(s'_{k'+1}, t'_{k'+1})$ are chosen (distinct),

so that $D'_S = D''_0 \cup \{\neq(s'_1, t'_1), \dots, \neq(s'_k, t'_k)\}$.

Define $c': D''_0 \rightarrow \{0, 1\}$, by

$$\neq(s', t') \mapsto c(\neq(s', t')\rho), \text{ and}$$

$$c': D''_{k'+1} \rightarrow \{0, 1\} \quad (\forall k': 0 \leq k' < k), \text{ by}$$

$$\neq(s', t') \in D''_k \mapsto c'(\neq(s', t')), \text{ (as defined on } D''_k);$$

$$\neq(s'_{k'+1}, t'_{k'+1}) \mapsto 0, \text{ in case (a) or case (c), and}$$

$$1, \text{ in case (b) or case (d),}$$

where: case (a) applies if both:

- i) there is $\neq(s', t') \in D_S$ such that $\neq(s', t')\sigma = \neq(s'_{k'+1}, t'_{k'+1})\rho\sigma$ and $c(\neq(s', t')) = 0$, and
- ii) for no $\neq(s'', t'') \in D''_{k'}$ such that $\neq(s'', t'')\rho\sigma = \neq(s'_{k'+1}, t'_{k'+1})\rho\sigma$ is it the case that $c'(\neq(s'', t'')) = 0$,

case (b) applies if case (a) does not apply and

- i) there is $\neq(s', t') \in D_S$ such that $\neq(s', t')\sigma = \neq(s'_{k'+1}, t'_{k'+1})\rho\sigma$ and $c(\neq(s', t')) = 1$, and
- ii) for no $\neq(s'', t'') \in D''_{k'}$ such that $\neq(s'', t'')\rho\sigma = \neq(s'_{k'+1}, t'_{k'+1})\rho\sigma$ is it the case that $c'(\neq(s'', t'')) = 1$,

case (c) applies if neither case (a) nor case (b) apply, and there is $\neq(s', t') \in D_S$ such that $\neq(s', t')\sigma = \neq(s'_{k'+1}, t'_{k'+1})\rho\sigma$ and $c(\neq(s', t')) = 0$, and

case (d) applies if none of cases (a) though (c) apply.

Some observations are now appropriate.

1. Obviously, the cases (a) through (d) above, are mutually exclusive and exhaustive.

Thus $c': D''_k \rightarrow \{0, 1\}$ is a well defined choice function on $D'_S = D''_k$.

2. In all cases (a) through (d) above, there exists $\neq(s', t') \in D_S$ such that $\neq(s', t')\sigma = \neq(s'_{k'+1}, t'_{k'+1})\rho\sigma$ and $c(\neq(s', t')) = c'(\neq(s'_{k'+1}, t'_{k'+1}))$.

For this, the only case to check is case (d): If none of (a) through (c) hold, then in particular, for all $\neq(s', t') \in D_S$ such that $\neq(s', t')\sigma = \neq(s'_{k'+1}, t'_{k'+1})\rho\sigma$, (it is the case that) $c(\neq(s', t')) = 1$. But as $\neq(s'_{k'+1}, t'_{k'+1}) \in D'_S$, (by the definition of D'_S), there is some $\neq(s', t') \in D_S$ such that $\neq(s', t')\sigma = \neq(s'_{k'+1}, t'_{k'+1})\rho\sigma$, and so, such that also, $c(\neq(s', t')) = 1$.

3. As a consequence of 2., and the definition of c' on D''_0 , for all $\neq(s'', t'') \in D'_S$, there exists $\neq(s', t') \in D_S$ such that $\neq(s', t')\sigma = \neq(s'', t'')\rho\sigma$ and $c(\neq(s', t')) = c'(\neq(s'', t''))$.

4. As a consequence of 3., it is easy to check that, for every literal $\neq(s'', t'') \in D'_S$.

$$c'(\neq(s'', t'')) = 0 \Rightarrow s''\rho\sigma = s \quad \text{and} \quad c'(\neq(s'', t'')) = 1 \Rightarrow t''\rho\sigma = s.$$

(For example, if $c'(\neq(s'', t'')) = 1$, as there exists $\neq(s', t') \in D_S$ such that $\neq(s', t')\sigma = \neq(s'', t'')\rho\sigma$ and $c(\neq(s', t')) = 1$, it follows that $t'\sigma = t''\rho\sigma = s$.)

Now that the choice function c' on $A'_R \cup D'_R \cup D'_S$ has been defined, observe that $\rho\sigma$ unifies the set: $S' \equiv$

$$\begin{aligned} & \{ \{=(s', t') \mid =(s', t') \in A'_R; c'(\neq(s', t')) = 0\} \cup \{=(t', s') \mid =(s', t') \in A'_R; c'(\neq(s', t')) = 1\} \cup \\ & \quad \{=(s', t') \mid \neq(s', t') \in D'_R; c'(\neq(s', t')) = 0\} \cup \{=(t', s') \mid \neq(s', t') \in D'_R; c'(\neq(s', t')) = 1\}, \\ & \{s' \mid =(s', t') \in A'_R; c'(\neq(s', t')) = 0\} \cup \{t' \mid =(s', t') \in A'_R; c'(\neq(s', t')) = 1\} \cup \\ & \quad \{s' \mid \neq(s', t') \in D'_S; c'(\neq(s', t')) = 0\} \cup \{t' \mid \neq(s', t') \in D'_S; c'(\neq(s', t')) = 1\} \}, \end{aligned}$$

to the set $\{ \{=(s, r)\}, \{s\} \}$.

Thus there is a most general unifier, σ' , say, of S' , so that $S'\sigma' = \{\{(s_0, t_0)\}, \{s_0\}\}$ for some terms s_0 and t_0 , say, and (there is) a substitution θ , such that $\rho\sigma = \sigma'\theta$, and $\{(s_0, t_0)\theta\} = \{(s, r)\}$.

Let $B' \equiv A'\sigma' \setminus \{\{(r_0, s_0)\}, \{(s_0, r_0)\}\} \cup D'\sigma' \setminus (\{\neq(r_0, s_0), \neq(s_0, r_0)\} \cup D'_S\sigma') \cup$
 $\{\neq(r_0, t'\sigma') \mid \neq(s', t') \in D'_S; \neq(s', t')\sigma' \notin \{\neq(r_0, s_0), \neq(s_0, r_0)\}; c'(\neq(s', t')) = 0\} \cup$
 $\{\neq(s'\sigma', r_0) \mid \neq(s', t') \in D'_S; \neq(s', t')\sigma' \notin \{\neq(r_0, s_0), \neq(s_0, r_0)\}; c'(\neq(s', t')) = 1\}.$

Then $B' \in \text{EPC}_{+,-}(A', D')$.

We claim (1): $(A'\sigma' \setminus \{\{(r_0, s_0)\}, \{(s_0, r_0)\}\})\theta = A\sigma \setminus \{\{(r, s)\}, \{(s, r)\}\}.$

In order to see this, first observe that:

$$A\sigma \setminus \{\{(r, s)\}, \{(s, r)\}\} = A'\rho\sigma \setminus \{\{(r, s)\}, \{(s, r)\}\} = A'\sigma'\theta \setminus \{\{(r_0, s_0)\theta\}, \{(s_0, r_0)\theta\}\}.$$

Thus $(\supseteq)$ is trivial.

For $(\subseteq)$: Let $L \in (A'\sigma' \setminus \{\{(r_0, s_0)\}, \{(s_0, r_0)\}\})\theta$.

Then trivially, $L \in A'\sigma'\theta = A'\rho\sigma$, and L is $L'\theta$, for some $L' \in A'\sigma' \setminus \{\{(r_0, s_0)\}, \{(s_0, r_0)\}\}.$

Here L' is $\{(s', t')\sigma'$ for some $\{(s', t') \in A'\}$, such that $\{(s', t')\sigma' \notin \{\{(r_0, s_0)\}, \{(s_0, r_0)\}\}.$

Thus L is $\{(s', t')\sigma'\theta = \{(s', t')\rho\sigma$, for some $\{(s', t') \in A'\}$, such that

$$\{(s', t')\sigma' \notin \{\{(r_0, s_0)\}, \{(s_0, r_0)\}\}.$$

Note that if $\{(s', t') \in A'_R$, then either $\{(s', t')\sigma' = \{(s_0, r_0)\}$, (and $c'(\{(s', t')\}) = 0$), or $\{(s', t')\sigma' = \{(r_0, s_0)\}$, (and $c'(\{(s', t')\}) = 1$). In either case, $\{(s', t')\sigma' \in \{\{(r_0, s_0)\}, \{(s_0, r_0)\}\}.$

Since $\{(s', t')\sigma' \notin \{\{(r_0, s_0)\}, \{(s_0, r_0)\}\}$, it follows that $\{(s', t') \notin A'_R$, and so (by the choice of A'_R), $\{(s', t')\rho\sigma \notin \{\{(r, s)\}, \{(s, r)\}\}.$ Thus $L \notin \{\{(r, s)\}, \{(s, r)\}\}.$ so $L \in A'\rho\sigma \setminus \{\{(r, s)\}, \{(s, r)\}\}$, that is, $L \in A\sigma \setminus \{\{(r, s)\}, \{(s, r)\}\}.$

Similarly, we claim (2):

$$(D'\sigma' \setminus (\{\neq(r_0, s_0), \neq(s_0, r_0)\} \cup D'_S\sigma'))\theta = D\sigma \setminus (\{\neq(r, s), \neq(s, r)\} \cup D_S\sigma).$$

In order to see this, first observe that:

$$\begin{aligned} D\sigma \setminus (\{\neq(r, s), \neq(s, r)\} \cup D_S\sigma) &= D\sigma \setminus (\{\neq(r, s), \neq(s, r)\} \cup D'_S\rho\sigma) \\ &= D'\rho\sigma \setminus (\{\neq(r, s), \neq(s, r)\} \cup D'_S\rho\sigma) = D'\sigma'\theta \setminus (\{\neq(r_0, s_0)\theta, \neq(s_0, r_0)\theta\} \cup D'_S\sigma'\theta) \end{aligned}$$

(The first equality above is due to the previously observed result that

$$D'_S\rho\sigma \setminus \{\neq(r, s), \neq(s, r)\} = D_S\sigma \setminus \{\neq(r, s), \neq(s, r)\}, \text{ and so}$$

$$D'_S\rho\sigma \cup \{\neq(r, s), \neq(s, r)\} = D_S\sigma \cup \{\neq(r, s), \neq(s, r)\}.)$$

Thus $(\supseteq)$ is trivial.

For $(\subseteq)$: let $L \in (D'\sigma' \setminus (\{\neq(r_0, s_0), \neq(s_0, r_0)\} \cup D'_S\sigma'))\theta$.

Then trivially $L \in D'\sigma'\theta = D'\rho\sigma = D\sigma$. If L is a positive equality literal,

then, as D'_S contains only inequality literals, $L \in D'\sigma'\theta \setminus (\{\neq(r_0, s_0)\theta, \neq(s_0, r_0)\theta\} \cup D'_S\sigma'\theta)$, (and so $L \in D\sigma \setminus (\{\neq(r, s), \neq(s, r)\} \cup D_S\sigma)$, and we are done).

Otherwise, L is $L'\theta$, for some $L' \in D'\sigma' \setminus (\{\neq(r_0, s_0), \neq(s_0, r_0)\} \cup D'_S\sigma')$, where L' is $\neq(s', t')\sigma'$, for some $\neq(s', t') \in D'$, such that $\neq(s', t')\sigma' \notin \{\neq(r_0, s_0), \neq(s_0, r_0)\} \cup D'_S\sigma'$.

Thus L is $\neq(s', t')\sigma'\theta$, which is $\neq(s', t')\rho\sigma$. Note that if $\neq(s', t') \in D'_R$, then either: $\neq(s', t')\sigma'$ is $\neq(s_0, r_0)$. (and $c'(\neq(s', t')) = 0$), or $\neq(s', t')\sigma'$ is $\neq(r_0, s_0)$, (and $c'(\neq(s', t')) = 1$). In either case, $\neq(s', t')\sigma' \in \{\neq(r_0, s_0), \neq(s_0, r_0)\}$.

Since $\neq(s', t')\sigma' \notin \{\neq(r_0, s_0), \neq(s_0, r_0)\} \cup D'_S\sigma'$, it follows that $\neq(s', t') \notin D'_R$, and so, by the choice of D'_R , $\neq(s', t')\rho\sigma \notin \{\neq(r, s), \neq(s, r)\}$, that is, $L \notin \{\neq(r, s), \neq(s, r)\}$.

As $\neq(s', t')\sigma' \notin D'_S\sigma'$, clearly, $\neq(s', t') \notin D'_S$.

By the choice of D'_S , therefore, (as $\neq(s', t') \in D' \setminus D'_R$), $\neq(s', t')\rho\sigma = L \notin D_S\sigma$.

Thus $L \notin \{\neq(r, s), \neq(s, r)\} \cup D_S\sigma$, and so $L \in D\sigma \setminus (\{\neq(r, s), \neq(s, r)\} \cup D_S\sigma)$.

Finally, we claim (3):

$$3A. \{ \neq(r_0, t''\sigma') \mid \neq(s'', t'') \in D'_S; \neq(s'', t'')\sigma' \notin \{\neq(r_0, s_0), \neq(s_0, r_0)\}; c'(\neq(s'', t'')) = 0 \} \theta = \{ \neq(r, t'\sigma) \mid \neq(s', t') \in D_S; \neq(s', t')\sigma \notin \{\neq(r, s), \neq(s, r)\}; c(\neq(s', t')) = 0 \}.$$

$$3B. \{ \neq(s''\sigma', r_0) \mid \neq(s'', t'') \in D'_S; \neq(s'', t'')\sigma' \notin \{\neq(r_0, s_0), \neq(s_0, r_0)\}; c'(\neq(s'', t'')) = 1 \} \theta = \{ \neq(s'\sigma, r) \mid \neq(s', t') \in D_S; \neq(s', t')\sigma \notin \{\neq(r, s), \neq(s, r)\}; c(\neq(s', t')) = 1 \}.$$

($\subseteq$): 3A. If $L \in LHS$, then L is $\neq(r_0, t''\sigma')\theta$, where there is some inequality $\neq(s'', t'') \in D'_S$ such that $\neq(s'', t'')\sigma' \notin \{\neq(r_0, s_0), \neq(s_0, r_0)\}; c'(\neq(s'', t'')) = 0$.

As $r_0\theta = r$; $t''\sigma'\theta = t''\rho\sigma$, L is $\neq(r, t''\rho\sigma)$.

As $\neq(s'', t'') \in D'_S$, it follows that $\neq(s'', t'') \notin D'_R$; $\neq(s'', t'')\rho\sigma \in D_S\sigma$.

As $\neq(s'', t'') \notin D'_R$, by the choice of D'_R , $\neq(s'', t'')\rho\sigma \notin \{\neq(s, r), \neq(r, s)\}$.

As $\neq(s'', t'') \in D'_S$, also, there exists $\neq(s', t') \in D_S$: $\neq(s', t')\sigma = \neq(s'', t'')\rho\sigma$; $c(\neq(s', t')) = 0$.

Here, (as $\neq(s'', t'')\rho\sigma \notin \{\neq(s, r), \neq(r, s)\}$), we also have that $\neq(s', t')\sigma \notin \{\neq(s, r), \neq(r, s)\}$, and so $\neq(r, t'\sigma) \in RHS$. But $\neq(r, t'\sigma)$ is $\neq(r, t''\rho\sigma)$, which is L , so $L \in RHS$.

($\subseteq$) for 3B. is exactly analogous (replacing 0 with 1).

($\supseteq$): 3A. If $L \in RHS$ then L is $\neq(r, t'\sigma)$, where there is some inequality $\neq(s', t') \in D_S$ such that $\neq(s', t')\sigma \notin \{\neq(r, s), \neq(s, r)\}$ and $c(\neq(s', t')) = 0$.

There is some inequality $\neq(s'', t'') \in D'$ such that $\neq(s'', t'')\rho = \neq(s', t')$,

(and hence, such that $\neq(s'', t'')\rho\sigma = \neq(s', t')\sigma \in D_S\sigma \setminus \{\neq(r, s), \neq(s, r)\}$).

Evidently (by the choice of D'_R), $\neq(s'', t'') \notin D'_R$, and so $\neq(s'', t'') \in D'_S$.

Thus, by the definition of c' , (in particular, on D''_0 , as evidently, $\neq(s'', t'') \in D''_0$),

$$c'(\neq(s'', t'')) = c(\neq(s'', t''))\rho = c(\neq(s', t')) = 0.$$

If it were the case that $\neq(s'', t'')\sigma' \in \{\neq(r_0, s_0), \neq(s_0, r_0)\}$, then it would follow that $\neq(s'', t'')\sigma'\theta = \neq(s'', t'')\rho\sigma \in \{\neq(r, s), \neq(s, r)\}$. Thus (as this can not be the case), $\neq(s'', t'')\sigma' \notin \{\neq(r_0, s_0), \neq(s_0, r_0)\}$.

It follows that $\neq(r_0, t''\sigma')\theta \in LHS$, but $\neq(r_0, t''\sigma')\theta$ is $\neq(r, t''\sigma'\theta)$, which is $\neq(r, t''\rho\sigma)$, which is which is $\neq(r, t'\sigma)$, which is L , (so $L \in LHS$).

($\supseteq$) for 3B. is exactly similar (replacing 0 with 1).

From claims (1) through (3), evidently $B'\theta = B$, and B is a substitution instance of B' . ■

For Lemma 4.4.2:

Proof: We leave this as an easy exercise for the reader. ■

For Lemma 4.4.3:

Proof (sketch): As $B \in \text{EPC}_{+,-}(\mathcal{A}, D)$, $\mathcal{A}$ is positive, non-empty, and B is mixed, non-empty.

Further, there exists a non-empty subset $\mathcal{A}_R \subseteq \mathcal{A}$, a non-empty subset $D_R \subseteq D^-$, the left-most inequality literal, $\neq(s_0, t_0)$, say, of D in D_R , and a (possibly empty) subset $D_S \subseteq D^- \setminus D_R$, such that for some symmetric choice function c on $\mathcal{A}_R \cup D_R \cup D_S$, with $c(\neq(s_0, t_0)) = 0$.

the most general substitution σ , say, unifying:

$$\begin{aligned} & \{ \{=(s', t') \mid =(s', t') \in \mathcal{A}_R; c(=(s', t')) = 0\} \cup \{=(t', s') \mid =(s', t') \in \mathcal{A}_R; c(=(s', t')) = 1\} \cup \\ & \quad \{ \neq(s', t') \mid \neq(s', t') \in D_R; c(\neq(s', t')) = 0\} \cup \{ \neq(t', s') \mid \neq(s', t') \in D_R; c(\neq(s', t')) = 1\} \} \\ & \{s' \mid =(s', t') \in \mathcal{A}_R; c(=(s', t')) = 0\} \cup \{t' \mid =(s', t') \in \mathcal{A}_R; c(=(s', t')) = 1\} \cup \\ & \quad \{s' \mid \neq(s', t') \in D_S; c(\neq(s', t')) = 0\} \cup \{t' \mid \neq(s', t') \in D_S; c(\neq(s', t')) = 1\} \}. \end{aligned}$$

exists, σ unifying the two sets in the above doubleton set to the singletons $\{=(s, r)\}$ and $\{s\}$, respectively, for some terms s and r , say, and, modulo ordering,

$$\begin{aligned} B & \equiv \mathcal{A}\sigma \setminus \{=(r, s), =(s, r)\} \cup D\sigma \setminus (\{\neq(r, s), \neq(s, r)\} \cup D_S\sigma) \cup \\ & \quad \{\neq(r, t'\sigma) \mid \neq(s', t') \in D_S; \neq(s', t')\sigma \notin \{\neq(r, s), \neq(s, r)\}; c(\neq(s', t')) = 0\} \cup \\ & \quad \{\neq(s'\sigma, r) \mid \neq(s', t') \in D_S; \neq(s', t')\sigma \notin \{\neq(r, s), \neq(s, r)\}; c(\neq(s', t')) = 1\}. \end{aligned}$$

As B_g is a ground substitution instance of B , for some substitution, θ , up to clause ordering,

$$\begin{aligned} B_g & \equiv (\mathcal{A}\sigma \setminus \{=(r, s), =(s, r)\})\theta \cup (D\sigma \setminus (\{\neq(r, s), \neq(s, r)\} \cup D_S\sigma))\theta \cup \\ & \quad \{\neq(r, t'\sigma)\theta \mid \neq(s', t') \in D_S; \neq(s', t')\sigma \notin \{\neq(r, s), \neq(s, r)\}; c(\neq(s', t')) = 0\} \cup \\ & \quad \{\neq(s'\sigma, r)\theta \mid \neq(s', t') \in D_S; \neq(s', t')\sigma \notin \{\neq(r, s), \neq(s, r)\}; c(\neq(s', t')) = 1\}. \end{aligned}$$

Evidently, for some substitution ρ , both $\mathcal{A}\sigma\theta\rho$ and $D\sigma\theta\rho$ are ground.

Since, $\neq(s, r) \in D\sigma$, $\neq(s, r)\theta\rho \in D\sigma\theta\rho$.

Let A' denote any ordering of $A\sigma\theta$, and Let D' denote any ordering of $D\sigma\theta$ such that the left-most inequality occurring in D' is $\neq(s, r)\theta\rho$.

Then obviously, A' and D' are ground substitution instances of A and D , respectively, and (hence), A' is positive, non-empty, while B' is mixed and non-empty.

Let $A'_R \equiv A_R\sigma\theta\rho$, $D'_R \equiv D_R\sigma\theta\rho$, $D'_S \equiv D_S\sigma\theta\rho \setminus D_R\sigma\theta\rho$.

Then evidently A'_R is a non-empty subset of A' , D'_R is a non-empty subset of D'^- , and $D'_S \subseteq D'^- \setminus D'_R$.

Let $c' : A'_R \cup D'_R \cup D'_S \rightarrow \{0, 1\}$, be defined as follows.

On $A'_R \cup D'_R$: For any $L' \in A'_R \cup D'_R$, $c'(L') = 0$ if there exists $L \in A_R \cup D_R$, such that $L\sigma\theta\rho = L'$ and $c(L) = 0$, and otherwise, $c'(L') = 1$.

On D'_S : For any $L' \in D'_S$, $c'(L') = 0$ if there exists $L \in D_S$, such that $L\sigma\theta\rho = L'$ and $c(L) = 0$, and otherwise, $c'(L') = 1$.

Then c' is a symmetric choice function on $A'_R \cup D'_R \cup D'_S$, with $c'(\neq(s, r)\theta\rho) = 0$, (since $\neq(s_0, t_0) \in D_R$, $c(\neq(s_0, t_0)) = 0$, and $\neq(s_0, t_0)\sigma\theta\rho = \neq(s, r)\theta\rho$).

Now by the definition of $c' : A'_R \cup D'_R \cup D'_S \rightarrow \{0, 1\}$, since A_R and A'_R necessarily contain only positive equality literals, while D_R and D'_R contain only inequality literals,

for any $L' \in A'_R$, (D'_R), if $c'(L') = 0$, then there exists $L \in A_R$, (D_R), such that $L\sigma\theta\rho = L'$ and $c(L) = 0$, (respectively).

Similarly, since $A'_R = A_R\sigma\theta\rho$ and $D'_R = D_R\sigma\theta\rho$, for any $L' \in A'_R$, (D'_R), if $c'(L') = 1$,

(since, clearly, there is some $L \in A_R$, (D_R), such that $L\sigma\theta\rho = L'$ and $c(L) \neq 0$),

there exists $L \in A_R$, (D_R), such that $L\sigma\theta\rho = L'$ and $c(L) = 1$, (respectively).

Also, (by the definition of $c' : A'_R \cup D'_R \cup D'_S \rightarrow \{0, 1\}$, and since $D'_S \subseteq D_S\sigma\theta\rho$),

for any $L' \in D'_S$, if $c'(L') = 0$, then there exists $L \in D_S$ such that $L\sigma\theta\rho = L'$ and $c(L) = 0$,

and if $c'(L') = 1$,

(since, clearly, there exists $L \in D_S$, such that $L\sigma\theta\rho = L'$ and $c(L) \neq 0$, it must be the case that),

there exists $L \in D_S$, such that $L\sigma\theta\rho = L'$ and $c(L) = 1$.

As, evidently,

$\{L\sigma \mid L \in A_R; c(L) = 0\} \subseteq \{=(s, r)\}$ and $\{L\sigma \mid L \in A_R; c(L) = 1\} \subseteq \{=(r, s)\}$,

$\{L\sigma \mid L \in D_R; c(L) = 0\} \subseteq \{\neq(s, r)\}$ and $\{L\sigma \mid L \in D_R; c(L) = 1\} \subseteq \{\neq(r, s)\}$, and

$\{s'\sigma \mid \neq(s', t') \in D_S; c(\neq(s', t')) = 0, \text{ or } \neq(t', s') \in D_S; c(\neq(t', s')) = 1\} \subseteq \{s\}$,

evidently,

$\{L \in A'_R \mid c'(L) = 0\} \subseteq \{=(s, r)\theta\rho\}$ and $\{L \in A'_R \mid c'(L) = 1\} \subseteq \{=(r, s)\theta\rho\}$,

$\{L \in D'_R \mid c'(L) = 0\} \subseteq \{\neq(s, r)\theta\rho\}$ and $\{L \in D'_R \mid c'(L) = 1\} \subseteq \{\neq(r, s)\theta\rho\}$, and

$\{s'' \mid \neq(s'', t'') \in D'_S; c(\neq(s'', t'')) = 0, \text{ or } \neq(t'', s'') \in D'_S; c(\neq(t'', s'')) = 1\} \subseteq \{s\theta\rho\}$.

Thus, (it is not hard to see that), the most general substitution unifying:

$$\begin{aligned} & \{ \{=(s'', t'') \in A'_R \mid c'(\neq(s'', t'')) = 0\} \cup \{=(t'', s'') \mid =(s'', t'') \in A'_R; c'(\neq(s'', t'')) = 1\} \cup \\ & \quad \{=(s'', t'') \mid \neq(s'', t'') \in D'_R; c'(\neq(s'', t'')) = 0\} \cup \{=(t'', s'') \mid \neq(s'', t'') \in D'_R; c'(\neq(s'', t'')) = 1\} \cdot \\ & \quad \{s'' \mid =(s'', t'') \in A'_R; c'(\neq(s'', t'')) = 0\} \cup \{t'' \mid =(s'', t'') \in A'_R; c'(\neq(s'', t'')) = 1\} \cup \\ & \quad \{s'' \mid \neq(s'', t'') \in D'_S; c'(\neq(s'', t'')) = 0\} \cup \{t'' \mid \neq(s'', t'') \in D'_S; c'(\neq(s'', t'')) = 1\} \}, \end{aligned}$$

exists, and the empty substitution, which unifies the two sets in the above doubleton set to the singletons $\{=(s, r)\theta\rho\}$ and $\{s\theta\rho\}$, respectively,

Evidently, if B' is defined to be (some ordering of),

$$\begin{aligned} & A' \setminus \{=(r, s)\theta\rho, =(s, r)\theta\rho\} \cup D' \setminus (\{\neq(r, s)\theta\rho, \neq(s, r)\theta\rho\} \cup D'_S) \cup \\ & \quad \{\neq(r\theta\rho, t'') \mid \neq(s'', t'') \in D'_S \setminus \{\neq(r, s)\theta\rho, \neq(s, r)\theta\rho\}; c'(\neq(s'', t'')) = 0\} \cup \\ & \quad \{\neq(s'', r\theta\rho) \mid \neq(s'', t'') \in D'_S \setminus \{\neq(r, s)\theta\rho, \neq(s, r)\theta\rho\}; c'(\neq(s'', t'')) = 1\}, \end{aligned}$$

then $B' \in \text{EPC}_{+,-}(A', D')$.

It remains only to check that $B' \subseteq B_g$. But,

$$\begin{aligned} \text{(C.4.C.1)} \quad A' \setminus \{=(r, s)\theta\rho, =(s, r)\theta\rho\} &= A\sigma\theta\rho \setminus \{=(r, s)\theta\rho, =(s, r)\theta\rho\} \\ &\subseteq (A\sigma \setminus \{=(r, s), =(s, r)\})\theta\rho = (A\sigma \setminus \{=(r, s), =(s, r)\})\theta \subseteq B_g. \end{aligned}$$

(the last equality holding since $(A\sigma \setminus \{=(r, s), =(s, r)\})\theta \subseteq B_g$ is ground), and

$$\begin{aligned} \text{(C.4.C.2)} \quad D' \setminus (\{\neq(r, s)\theta\rho, \neq(s, r)\theta\rho\} \cup D'_S) &= D\sigma\theta\rho \setminus (\{\neq(r, s)\theta\rho, \neq(s, r)\theta\rho\} \cup D_S\sigma\theta\rho) \\ &\subseteq (D\sigma \setminus (\{\neq(r, s), \neq(s, r)\} \cup D_S\sigma))\theta\rho = (D\sigma \setminus (\{\neq(r, s), \neq(s, r)\} \cup D_S\sigma))\theta \subseteq B_g. \end{aligned}$$

(the first equality holding since $D'_S = D_S\sigma\theta\rho \setminus D'_R$ and $D'_R \subseteq \{\neq(r, s)\theta\rho, \neq(s, r)\theta\rho\}$, and the last equality holding since $(D\sigma \setminus (\{\neq(r, s), \neq(s, r)\} \cup D_S\sigma))\theta \subseteq B_g$ is ground).

Further, if $\neq(s'', t'') \in D'_S \setminus \{\neq(r, s)\theta\rho, \neq(s, r)\theta\rho\}$,

then, (as observed earlier), there is an inequality $\neq(s', t') \in D_S$, such that $\neq(s', t')\sigma\theta\rho = \neq(s'', t'')$, and $c(\neq(s', t')) = c'(\neq(s'', t''))$.

Since $\neq(s'', t'') \notin \{\neq(r, s)\theta\rho, \neq(s, r)\theta\rho\}$, $\neq(s', t')\sigma\theta\rho \notin \{\neq(r, s)\theta\rho, \neq(s, r)\theta\rho\}$.

It follows that $\neq(s', t')\sigma \notin \{\neq(r, s), \neq(s, r)\}$.

(For if $\neq(s', t')\sigma \in \{\neq(r, s), \neq(s, r)\}$, then, clearly, $\neq(s', t')\sigma\theta\rho \in \{\neq(r, s)\theta\rho, \neq(s, r)\theta\rho\}$.)

If $c'(\neq(s'', t'')) = 0$, then $c(\neq(s', t')) = 0$, and $\neq(r, t')\sigma\theta \in B_g$.

As B_g is ground, so is $\neq(r, t')\sigma\theta$, and so $\neq(r, t')\sigma\theta = \neq(r, t')\sigma\theta\rho = \neq(r\theta\rho, t'')$, (since $\neq(s', t')\sigma\theta\rho = \neq(s'', t'')$, and hence, $t'\sigma\theta\rho = t''$). It follows that $\neq(r\theta\rho, t'') \in B_g$.

Similarly, if $c'(\neq(s'', t'')) = 1$, then $c(\neq(s', t')) = 1$, and $\neq(s'\sigma, r)\theta \in B_g$.

Again, as B_g is ground, so is $\neq(s'\sigma, r)\theta$, and so $\neq(s'\sigma, r)\theta = \neq(s'\sigma, r)\theta\rho = \neq(s'', r\theta\rho)$, (since $\neq(s', t')\sigma\theta\rho = \neq(s'', t'')$, and hence, $s'\sigma\theta\rho = s''$). It follows that $\neq(s'', r\theta\rho) \in B_g$.

Thus, evidently,

$$\begin{aligned}
 (C.4.C.3) \quad & \{ \neq(r\theta\rho, t'') \mid \neq(s'', t'') \in D'_S \setminus \{ \neq(r, s)\theta\rho, \neq(s, r)\theta\rho \}; c'(\neq(s'', t'')) = 0 \} \cup \\
 & \{ \neq(s'', r\theta\rho) \mid \neq(s'', t'') \in D'_S \setminus \{ \neq(r, s)\theta\rho, \neq(s, r)\theta\rho \}; c'(\neq(s'', t'')) = 1 \} \\
 & \subseteq \{ \neq(r, t'\sigma)\theta \mid \neq(s', t') \in D_S; \neq(s', t')\sigma \notin \{ \neq(r, s), \neq(s, r) \}; c(\neq(s', t')) = 0 \} \cup \\
 & \{ \neq(s'\sigma, r)\theta \mid \neq(s', t') \in D_S; \neq(s', t')\sigma \notin \{ \neq(r, s), \neq(s, r) \}; c(\neq(s', t')) = 1 \} \subseteq B_g.
 \end{aligned}$$

From inclusions (C.4.C.1), (C.4.C.2), and (C.4.C.3), evidently $B' \subseteq B_g$, as required. ■

For Corollary 4.4.1:

Proof: Since $B \in \text{EPC}_{-}^{-1}(\mathcal{A}, D)$, either:

- 1) both B and $QR_{full}(D)$ are the empty clause, $\square$, or
- 2) ($QR_{full}(\mathcal{A})$ is a non-empty and positive, and), $B \in \text{EPC}_{-}^{-1}(\mathcal{A}, D) \cap \text{EPC}_{-}^{+}(\mathcal{A}, D)$.

In case (1), as $QR_{full}(D) = \square$, there is a substitution, σ , such that $D\sigma$ contains only trivial inequality literals, and evidently, there is some ground substitution instance, $D\theta$, of D , which contains only trivial inequality literals.

Let D' denote (some ordering of), $D\theta$, and let $\mathcal{A}'$ denote any ground substitution instance of $\mathcal{A}$. Let B' be $\square$.

Then $QR_{full}(D') = \square$, and so $\square \in \text{EPC}_{-}^{-1}(\mathcal{A}', D')$, that is, $B' \in \text{EPC}_{-}^{-1}(\mathcal{A}', D')$.

The only ground substitution instance, B_g , of $\square$ is $\square$ itself, and $B' = \square \subseteq \square = B_g$, (so we're done).

In case (2): As $B \in \text{EPC}_{-}^{-1}(\mathcal{A}, D) \cap \text{EPC}_{-}^{+}(\mathcal{A}, D)$, $B \in \text{EPC}_{+,-}(\mathcal{A}'', D'')$, for some non-empty, mixed $D'' \in QR_{first}(D)$, and where $\mathcal{A}''$ is positive and denotes $QR_{full}(\mathcal{A})$.

By Lemma 4.4.3, there exist ground substitution instances $\mathcal{A}''_0$ and D''_0 of $\mathcal{A}''$ and D'' respectively, and a ground clause $B'' \in \text{EPC}_{+,-}(\mathcal{A}''_0, D''_0)$, such that $B'' \subseteq B_g$.

As $\mathcal{A}'' = QR_{full}(\mathcal{A})$, (and is positive), $\mathcal{A}''$ is (some ordering of) $\mathcal{A}\sigma_1 \setminus \mathcal{A}^{-}\sigma_1$, for some substitution σ_1 , such that $\mathcal{A}^{-}\sigma_1$ contains only trivial inequality literals.

As $\mathcal{A}''_0$ is a ground substitution instance of $\mathcal{A}''$, $\mathcal{A}''_0$ is (some ordering of) $\mathcal{A}\sigma_1\theta_1 \setminus \mathcal{A}^{-}\sigma_1\theta_1$, for some substitution θ_1 , (such that $\mathcal{A}\sigma_1\theta_1 \setminus \mathcal{A}^{-}\sigma_1\theta_1$ is ground), and, clearly, $\mathcal{A}^{-}\sigma_1\theta_1$ contains only trivial inequality literals.

(Of course, since $\mathcal{A}^{-}$ is the collection of all inequality literals in $\mathcal{A}$
 $\mathcal{A}\sigma_1\theta_1 \setminus \mathcal{A}^{-}\sigma_1\theta_1 = (\mathcal{A}\sigma_1 \setminus \mathcal{A}^{-}\sigma_1)\theta_1 = (\mathcal{A} \setminus \mathcal{A}^{-})\sigma_1\theta_1$).

As $D'' \in QR_{first}(D)$, D'' is some ordering of $D\sigma_2 \setminus D_0\sigma_2$, for some substitution σ_2 , and some $D_0 \subseteq D$, and where $D_0\sigma_2$ contains only trivial inequality literals.

As D''_0 is a ground substitution instance of D'' , D''_0 is some ordering of $(D\sigma_2 \setminus D_0\sigma_2)\theta_2$, for some substitution θ_2 , (such that $(D\sigma_2 \setminus D_0\sigma_2)\theta_2$ is ground).

Let A' be some ordering of any ground substitution instance, $A\sigma_1\theta_1\rho_1$, of $A\sigma_1\theta_1$.

Then clearly, $A'^- = A'^-\sigma_1\theta_1\rho_1$ contains only ground trivial inequalities,

and $QR_{full}(A')$ is some ordering of $A' \setminus A'^-$.

As, modulo ordering, $A''_0 = A\sigma_1\theta_1 \setminus A'^-\sigma_1\theta_1$, is ground, modulo ordering,

$A''_0 = A\sigma_1\theta_1\rho_1 \setminus A'^-\sigma_1\theta_1\rho_1 = A' \setminus A'^-$,

and so, modulo ordering, $A''_0 = QR_{full}(A')$.

Let D' be some ground substitution instance, $D\sigma_2\theta_2\rho_2$, of $D\sigma_2\theta_2$, ordered so that D''_0

(which equals, modulo ordering, $(D\sigma_2 \setminus D_0\sigma_2)\theta_2$, which equals, as D''_0 is ground, $(D\sigma_2 \setminus D_0\sigma_2)\theta_2\rho_2$, and is, at any rate, contained in D'), as ordered, occurs as the first string of literals in D' .

Since $D' \subseteq D\sigma_2\theta_2\rho_2 \subseteq (D\sigma_2 \setminus D_0\sigma_2)\theta_2\rho_2 \cup D_0\sigma_2\theta_2\rho_2 \subseteq D''_0 \cup D_0\sigma_2\theta_2\rho_2$,

and since (as $D_0\sigma$ contains only trivial inequality literals, clearly),

$D_0\sigma_2\theta_2\rho_2$ contains only trivial inequality literals, evidently D' is exactly of the form $D''_0 \mid F$, where F contains only ground trivial inequality literals.

Thus, (as $B'' \in \text{EPC}_{+,-}(A''_0, D''_0)$), by Lemma 4.4.2, there exists a (ground clause).

$B' \in \text{EPC}_{+,-}(QR_{full}(A'), D')$, such that $B' \subseteq B'' \cup F$.

As $B'' \subseteq B_g$, $B' \subseteq B_g \cup F$, and as $D' \in QR_{first}(D')$, $B' \in \text{EPC}_{-}^1(A', D')$. ■

For Lemma 4.4.4:

Proof: Let A , B , A' , D , and D' , as in the statement of the lemma, (and assume that A , A' , D , and D' share (pairwise) no variable in common, which we can do by Observation 4.1.6).

By assumption, there is some substitution ρ_A , such that (modulo clause ordering), $A'\rho_A = A$, and some substitution ρ_D , such that $D'\rho_D = D$.

By the assumed separation of variables, evidently there is a substitution ρ , say,

(for example, the composite, $\rho_A\rho_D$), such that (modulo clause ordering) $A'\rho = A$, and $D'\rho = D$.

There is some non-empty subset $A_R \subseteq A$, and some non-empty subset $D_R \subseteq D$, such that for some symmetric choice function c on $A_R \cup D_R$, the most general substitution σ , say, unifying:

$$\begin{aligned} & \{ \{=(r', s') \mid =(r', s') \in A_R; c(=(r', s')) = 0\} \cup \{=(s', r') \mid =(r', s') \in A_R; c(=(r', s')) = 1\}, \\ & \{=(s', t') \mid =(s', t') \in D_R; c(=(s', t')) = 0\} \cup \{=(t', s') \mid =(s', t') \in D_R; c(=(s', t')) = 1\}, \\ & \{s' \mid =(r', s') \in A_R; c(=(r', s')) = 0\} \cup \{s' \mid =(s', r') \in A_R; c(=(s', r')) = 1\} \cup \\ & \{s' \mid =(s', t') \in D_R; c(=(s', t')) = 0\} \cup \{s' \mid =(t', s') \in D_R; c(=(t', s')) = 1\} \}, \end{aligned}$$

exists, σ unifying the three sets in the above tripleton set to the sets $\{=(r, s)\}$, $\{=(s, t)\}$, and $\{s\}$, respectively, for some terms r , s , and t , say, and

$$B \equiv A\sigma \setminus \{=(r, s), =(s, r)\} \cup D\sigma \setminus \{=(s, t), =(t, s)\} \cup \{=(r, t)\}.$$

As $A'\rho = A$, $A'\rho\sigma = A\sigma$, so $\{=(r, s), =(s, r)\} \cap A'\rho\sigma \supset A_R\sigma \neq \emptyset$.

Let $A'_R \equiv \{L \in A' \mid L\rho\sigma \in \{=(r, s), =(s, r)\}\}$.

Clearly $\emptyset \neq A'_R \subseteq A'$, and $A'_R\rho\sigma \subseteq \{=(r, s), =(s, r)\}$.

As $D'\rho = D$, $D'\rho\sigma = D\sigma$, so $\{=(s, t), =(t, s)\} \cap D'\rho\sigma \supset D_R\sigma \neq \emptyset$.

Let $D'_R \equiv \{L \in D' \mid L\rho\sigma \in \{=(s, t), =(t, s)\}\}$.

Then $\emptyset \neq D'_R \subseteq D'$, and $D'_R \rho \sigma \subseteq \{=(s, t), =(t, s)\}$.

Let c' be the choice function on $A'_R \cup D'_R$ defined separately on the (disjoint) sets A'_R , and D'_R , as follows:

On A'_R : $=(r', s') \in A'_R \mapsto 0$, if $=(r', s') \rho \sigma = =(r, s)$, and
 $=(r', s') \in A'_R \mapsto 1$, otherwise, (if $=(r', s') \rho \sigma = =(s, r)$).

On D'_R : $=(s', t') \in D'_R \mapsto 0$, if $=(s', t') \rho \sigma = =(s, t)$, and
 $=(s', t') \in D'_R \mapsto 1$, otherwise, (if $=(s', t') \rho \sigma = =(t, s)$).

Evidently, $\rho \sigma$ unifies the set $S' \equiv$

$$\begin{aligned} & \{ \{=(r', s') \mid =(r', s') \in A'_R; c'((r', s')) = 0\} \cup \{=(s', r') \mid =(r', s') \in A'_R; c'((r', s')) = 1\} \} \\ & \{=(s', t') \mid =(s', t') \in D'_R; c'((s', t')) = 0\} \cup \{=(t', s') \mid =(s', t') \in D'_R; c'((s', t')) = 1\} \} \\ & \{s' \mid =(r', s') \in A'_R; c'((r', s')) = 0\} \cup \{s' \mid =(s', r') \in A'_R; c'((s', r')) = 1\} \cup \\ & \{s' \mid =(s', t') \in D'_R; c'((s', t')) = 0\} \cup \{s' \mid =(t', s') \in D'_R; c'((t', s')) = 1\} \} \end{aligned}$$

to the set $\{=(r, s)\}, \{=(s, t)\}, \{s\}$.

Thus there is a most general unifier, σ' , say, of S' , so that $S'\sigma' = \{=(r_0, s_0)\}, \{=(s_0, t_0)\}, \{s_0\}$, for some terms r_0 , s_0 and t_0 , say, and (there is) a substitution θ , such that $\rho \sigma = \sigma' \theta$, and $r_0 \theta = r$, $s_0 \theta = s$, and $t_0 \theta = t$.

Let $B' \equiv A'\sigma' \setminus \{=(r_0, s_0), =(s_0, r_0)\} \cup D\sigma' \setminus \{=(s_0, t_0), =(t_0, s_0)\} \cup \{=(r_0, t_0)\}$.

Then $B' \in EPC_{+,+}(A', D')$.

We claim (1): $(A'\sigma' \setminus \{=(r_0, s_0), =(s_0, r_0)\})\theta = A\sigma \setminus \{=(r, s), =(s, r)\}$.

In order to see this, first observe that:

$$A\sigma \setminus \{=(r, s), =(s, r)\} = A'\rho \sigma \setminus \{=(r, s), =(s, r)\} = A'\sigma' \theta \setminus \{=(r_0, s_0)\theta, =(s_0, r_0)\theta\}.$$

Thus $(\supseteq)$ is trivial.

For $(\subseteq)$: Let $L \in (A'\sigma' \setminus \{=(r_0, s_0), =(s_0, r_0)\})\theta$.

Then trivially, $L \in A'\sigma' \theta = A'\rho \sigma$, and L is $L'\theta$, for some $L' \in A'\sigma' \setminus \{=(r_0, s_0), =(s_0, r_0)\}$.

Here L' is $=(s', t')\sigma'$ for some $=(s', t') \in A'$, such that $=(s', t')\sigma' \notin \{=(r_0, s_0), =(s_0, r_0)\}$.

Thus L is $=(s', t')\sigma' \theta = =(s', t')\rho \sigma$, for some $=(s', t') \in A'$, such that $=(s', t')\sigma' \notin \{=(r_0, s_0), =(s_0, r_0)\}$.

Note that if $=(s', t') \in A'_R$, then either $=(s', t')\sigma' = =(s_0, r_0)$, (and $c'((s', t')) = 0$), or $=(s', t')\sigma' = =(r_0, s_0)$, (and $c'((s', t')) = 1$). In either case, $=(s', t')\sigma' \in \{=(r_0, s_0), =(s_0, r_0)\}$.

Since $=(s', t')\sigma' \notin \{=(r_0, s_0), =(s_0, r_0)\}$, it follows that $=(s', t') \notin A'_R$, and so (by the choice of A'_R), $=(s', t')\rho \sigma \notin \{=(r, s), =(s, r)\}$. Thus $L \notin \{=(r, s), =(s, r)\}$, so $L \in A'\rho \sigma \setminus \{=(r, s), =(s, r)\}$, that is, $L \in A\sigma \setminus \{=(r, s), =(s, r)\}$.

Similarly, it may be shown (in an exactly similar fashion) that (2):

$$(D'\sigma' \setminus \{=(s_0, t_0), =(t_0, s_0)\})\theta = D\sigma \setminus \{=(s, t), =(t, s)\}.$$

Since also, $=(r, t) = =(r_0, t_0)\theta$, it follows from (1) and (2) that $B'\theta = B$. ■

For Lemma 4.4.5:

Proof (sketch): Let A , B , A' , D , and D' , as in the statement of the lemma, (and assume that A , A' , D , and D' share (pairwise) no variable in common, as before.)

There is a substitution ρ , say, such that (modulo clause ordering) $A'\rho = A$.

Now $\text{EPC}_+(A, D) = \text{EPC}_{+,+}(QR_{full}(A), QR_{full}(D))$.

Letting A^- denote the inequality literals of A , $QR_{full}(A) = A\varphi \setminus A^-\varphi$, is positive, where φ is some (most general) substitution such that $s\varphi = t\varphi$ for each inequality $\neq(s, t) \in A^-$.

As (modulo clause ordering), $A'\rho = A$, it immediately follows that $A'\rho\varphi = A\varphi$.

Thus $\rho\varphi$ is a substitution which renders every inequality of A' into a trivial one, it follows that there must a most general substitution (which unifies the terms of each inequality of A'), ψ , say, and there exists a substitution λ such that $\rho\varphi = \psi\lambda$, (substitutions being applied from left to right). Further, (with A'^- denoting the inequality literals of A'), $QR_{full}(A') = A'\psi \setminus A'^-\psi$.

Now (since only positive equality literals occur in $A'\psi \setminus A'^-\psi$, while only inequality literals occur in $A'^-\psi\lambda$), $(A'\psi \setminus A'^-\psi)\lambda = A'\psi\lambda \setminus A'^-\psi\lambda$. Also, (since $A'\rho\varphi \setminus A'^-\rho\varphi$ is positive, and $A^-\varphi$ is purely negative, and $A\varphi \setminus A^-\varphi$ is positive, and $A'^-\rho\varphi$ is purely negative), $A'\rho\varphi \setminus A'^-\rho\varphi = A\varphi \setminus A^-\varphi$.

Thus $QR_{full}(A')\lambda = A'\psi\lambda \setminus A'^-\psi\lambda = A'\rho\varphi \setminus A'^-\rho\varphi = QR_{full}(A)$.

Hence $QR_{full}(A)$ is a substitution instance of $QR_{full}(A')$.

Similarly, $QR_{full}(D)$ is a substitution instance of $QR_{full}(D')$.

The result now follows from Lemma 4.4.1. ■

For Corollary 4.4.2:

Proof: By induction on the depth $k \geq 0$ of the EPC_+ -deduction Π .

The case $k = 0$: Here $B \in \mathfrak{C}$. Hence there is a $B' \in \mathfrak{C}'$, such that

B is a substitution instance of B' . That is, there is depth 0 EPC_+ -deduction of a clause B' from $\mathfrak{C}'$, such that B is a substitution instance of B' .

Suppose the result for all $0 \leq k < N$, and that now $k = N$.

$B \in \text{EPC}_+(A, D)$, for some A and D , having EPC_+ -deductions Π_A and Π_D , of depths $k_A < k$ and $k_D < k$, say, respectively.

By applying the induction hypothesis, (twice), there are EPC_+ -deductions $\Pi_{A'}$ and $\Pi_{D'}$, of depths $k_{A'} \leq k_A$ and $k_{D'} \leq k_D$, of clauses A' and D' , respectively, from $\mathfrak{C}'$, such that A is a substitution instance of A' , and D is a substitution instance of D' .

By Lemma 4.4.5, there is a $B' \in \text{EPC}_+(A', D')$, such that B is a substitution instance of B' .

The required depth $k' \leq k$ EPC_+ -deduction Π' , of B' from $\mathcal{C}'$ may now be obtained by following $\Pi_{A'}$ and $\Pi_{D'}$ by the EPC_+ -resolvent, $B' \in \text{EPC}_+(\mathcal{A}', D')$. ■

Corollary 4.4.3 follows immediately from Corollary 4.4.2.

C.5. Subsumption in the Equality Fragment of $\text{EPC}_<$

For Lemma 4.5.1:

Proof: As in the proof for Lemma 4.4.1, let A , A' , B , D , and D' as in the statement of the lemma, (and without loss of generality, assume no two share a common variable),

and let ρ be a substitution such that $A'\rho \subseteq A$ and $D'\rho \subseteq D$, where now, if A is a substitution instance of A' , we choose ρ so that also $A'\rho = A$,

(and, in case (ii), $L_1\rho$ occurs as the left-most inequality in D).

Clearly A' must be positive. Let D^- denote the negative inequality literals of D .

There is some non-empty subset $A_R \subseteq A$, some non-empty subset $D_R \subseteq D^-$, the left-most inequality of D , $L_1\rho$, in D_R , and a (possibly empty) subset $D_S \subseteq D^- \setminus D_R$, such that for some symmetric choice function c on $A_R \cup D_R \cup D_S$, the most general substitution σ , say, unifying:

$$\begin{aligned} & \{ \{=(s', t') \mid =(s', t') \in A_R; c(=(s', t')) = 0\} \cup \{=(t', s') \mid =(s', t') \in A_R; c(=(s', t')) = 1\} \cup \\ & \quad \{=(s', t') \mid \neq(s', t') \in D_R; c(\neq(s', t')) = 0\} \cup \{=(t', s') \mid \neq(s', t') \in D_R; c(\neq(s', t')) = 1\} \} \\ & \{s' \mid =(s', t') \in A_R; c(=(s', t')) = 0\} \cup \{t' \mid =(s', t') \in A_R; c(=(s', t')) = 1\} \cup \\ & \quad \{s' \mid \neq(s', t') \in D_S; c(\neq(s', t')) = 0\} \cup \{t' \mid \neq(s', t') \in D_S; c(\neq(s', t')) = 1\} \} \end{aligned}$$

exists, σ unifying the two sets in the above doubleton set to the singletons $\{=(s, r)\}$ and $\{s\}$, say, respectively, and

$$\begin{aligned} B \equiv & A\sigma \setminus \{=(r, s), =(s, r)\} \cup D\sigma \setminus (\{\neq(r, s), \neq(s, r)\} \cup D_S\sigma) \cup \\ & \{ \neq(r, t'\sigma) \mid \neq(s', t') \in D_S; \neq(s', t')\sigma \notin \{=(r, s), =(s, r)\}; c(\neq(s', t')) = 0 \} \cup \\ & \{ \neq(s'\sigma, r) \mid \neq(s', t') \in D_S; \neq(s', t')\sigma \notin \{=(r, s), =(s, r)\}; c(\neq(s', t')) = 1 \} \end{aligned}$$

By assumption, $A'\rho \subseteq A$, so $A'\rho\sigma \subseteq A\sigma$, and hence

$$A'\rho\sigma \setminus \{=(r, s), =(s, r)\} \subseteq A\sigma \setminus \{=(r, s), =(s, r)\}.$$

Similarly, $D'\rho\sigma \subseteq D\sigma$, and $D'\rho\sigma \setminus (\{\neq(r, s), \neq(s, r)\} \cup D_S\sigma) \subseteq D\sigma \setminus (\{\neq(r, s), \neq(s, r)\} \cup D_S\sigma)$.

In case (i), clearly $D'\rho\sigma \cap (\{\neq(r, s), \neq(s, r)\} \cup D_S\sigma) = \emptyset$.

Thus $D'\rho\sigma \subseteq D\sigma \setminus (\{\neq(r, s), \neq(s, r)\} \cup D_S\sigma) \subseteq B$, and so D' subsumes B .

In case (ii), as $L_1\rho \in D_R$, we have $L_1\rho\sigma \in D'\rho\sigma \cap D_R\sigma$.

If $A'\rho\sigma \cap \{=(r, s), =(s, r)\} = \emptyset$, then A is not a substitution instance of A' ,

(for otherwise, $A'\rho = A$, and so $A'\rho\sigma \cap \{=(r, s), =(s, r)\} = A\sigma \cap \{=(r, s), =(s, r)\} \supseteq A_R\sigma \neq \emptyset$),

and $A'\rho\sigma \subseteq A\sigma \setminus \{=(r, s), =(s, r)\} \subseteq B$, and so A' subsumes B .

Otherwise, consider the substitution instances $A'' \equiv A'\rho\sigma$ and $D'' \equiv D'\rho\sigma$, of clauses A' and D' , respectively, where D'' is ordered so that $L_1\rho\sigma$ occurs as the left-most inequality in D'' .

We will show there exists a $B'' \in \text{EPC}_{+,-}(A'', D'')$ such that $B'' \subseteq B$, and hence, such that B'' subsumes B . By lifting, (Lemma 4.4.1), it will follow that there exists $B' \in \text{EPC}_{+,-}(A', D')$ such that B' subsumes B'' , and hence, subsumes B .

Let $A''_R \equiv A'\rho\sigma \cap \{=(r, s), =(s, r)\}$. Clearly $A''_R \subseteq A''$, and by hypothesis, $A''_R \neq \emptyset$. Also $A''_R \subseteq \{=(r, s), =(s, r)\}$.

Let $D''_R \equiv D'\rho\sigma \cap D_R\sigma$, and let $D''_S \equiv D'\rho\sigma \cap D_S\sigma \setminus \{\neq(r, s), \neq(s, r)\}$. Then $\emptyset \neq D''_R \subseteq (D'')^-$, the left-most inequality of D'' , ($L_1\rho\sigma$), is in D''_R , and $D''_R \subseteq \{\neq(r, s), \neq(s, r)\}$. Also, $D''_S \subseteq (D'')^- \setminus D''_R$, and $D''_S \cap \{\neq(r, s), \neq(s, r)\} = \emptyset$.

Let $c': A''_R \cup D''_R \cup D''_S \rightarrow \{0, 1\}$, defined on:

A''_R by:

$$\begin{aligned} =(r, s) \in A''_R &\mapsto 1, \text{ and} \\ =(s, r) \in A''_R &\mapsto 0. \end{aligned}$$

D''_R by:

$$\begin{aligned} \neq(r, s) \in D''_R &\mapsto 1, \text{ and} \\ \neq(s, r) \in D''_R &\mapsto 0, \end{aligned}$$

D''_S by:

$$\begin{aligned} \neq(s'', t'') \in D''_S &\mapsto 1, \text{ if there is } \neq(s', t') \in D_S \text{ such that } \neq(s', t')\sigma = \neq(s'', t'') \text{ and} \\ &\quad c(\neq(s', t')) = 1, \text{ and} \\ \neq(s'', t'') \in D''_S &\mapsto 0, \text{ otherwise, (there is } \neq(s', t') \in D_S \text{ such that } \neq(s', t')\sigma = \neq(s'', t'') \\ &\quad \text{and } c(\neq(s', t')) = 0). \end{aligned}$$

Evidently, the empty substitution unifies:

$$\begin{aligned} &\{ \{=(s'', t'') \in A''_R \mid c'(\neq(s'', t'')) = 0\} \cup \{=(t'', s'') \mid =(s'', t'') \in A''_R; c'(\neq(s'', t'')) = 1\} \cup \\ &\quad \{=(s'', t'') \mid \neq(s'', t'') \in D''_R; c'(\neq(s'', t'')) = 0\} \cup \\ &\quad \{=(t'', s'') \mid \neq(s'', t'') \in D''_R; c'(\neq(s'', t'')) = 1\} \}. \end{aligned}$$

$$\{s'' \mid =(s'', t'') \in A''_R; c'(\neq(s'', t'')) = 0\} \cup \{t'' \mid =(s'', t'') \in A''_R; c'(\neq(s'', t'')) = 1\} \cup$$

$$\{s'' \mid \neq(s'', t'') \in D''_S; c'(\neq(s'', t'')) = 0\} \cup \{t'' \mid \neq(s'', t'') \in D''_S; c'(\neq(s'', t'')) = 1\} \},$$

to $\{ \{=(s, r)\}, \{s\} \}$,

(and the empty substitution is the most general unifier).

Now let $B'' \equiv A'' \setminus \{=(r, s), =(s, r)\} \cup D'' \setminus (\{\neq(r, s), \neq(s, r)\} \cup D''_S) \cup$
 $\{\neq(r, t'') \mid \neq(s'', t'') \in D''_S; c'(\neq(s'', t'')) = 0\} \cup$
 $\{\neq(s'', r) \mid \neq(s'', t'') \in D''_S; c'(\neq(s'', t'')) = 1\}.$

Clearly $B'' \in \text{EPC}_{+, -}(A'', D'')$.

Also, $A'' \setminus \{=(r, s), =(s, r)\} \subseteq A\sigma \setminus \{=(r, s), =(s, r)\}$,

$D'' \setminus (\{\neq(r, s), \neq(s, r)\} \cup D''_S) \subseteq D\sigma \setminus (\{\neq(r, s), \neq(s, r)\} \cup D_S\sigma)$,

(since $\{\neq(r, s), \neq(s, r)\} \cup D''_S = \{\neq(r, s), \neq(s, r)\} \cup (D'' \cap D_S\sigma)$, and so

$$D'' \setminus (\{\neq(r, s), \neq(s, r)\} \cup D''_S) = D'' \setminus \{\neq(r, s), \neq(s, r)\} \cap D'' \setminus (D'' \cap D_S\sigma) =$$

$$D'' \setminus \{\neq(r, s), \neq(s, r)\} \cap D'' \setminus D_S\sigma = D'' \setminus (\{\neq(r, s), \neq(s, r)\} \cup D_S\sigma) \subseteq D\sigma \setminus (\{\neq(r, s), \neq(s, r)\} \cup D_S\sigma),$$

and (it is easy to show that),

$$\begin{aligned} \{ \neq(r, t'') \mid \neq(s'', t'') \in D''_S; c'(\neq(s'', t'')) = 0 \} \cup \{ \neq(s'', r) \mid \neq(s'', t'') \in D''_S; c'(\neq(s'', t'')) = 1 \} \subseteq \\ \{ \neq(r, t'\sigma) \mid \neq(s', t') \in D_S; \neq(s', t')\sigma \notin \{\neq(r, s), \neq(s, r)\}; c(\neq(s', t')) = 0 \} \cup \\ \{ \neq(s'\sigma, r) \mid \neq(s', t') \in D_S; \neq(s', t')\sigma \notin \{\neq(r, s), \neq(s, r)\}; c(\neq(s', t')) = 1 \}. \end{aligned}$$

Thus $B'' \subseteq B$, as required. ■

For Lemma 4.5.2:

Proof: As usual, (or, at least, as in the proof for Lemma 4.5.1), let A , A' , B , D , and D' as in the statement of the lemma, (and without loss of generality, assume no two share a common variable), and let ρ be a substitution such that $A'\rho \subseteq A$ and $D'\rho \subseteq D$.

Clearly A' and D' must be must be positive.

There is some non-empty subset $A_R \subseteq A$, and some non-empty subset $D_R \subseteq D$, such that for some symmetric choice function c on $A_R \cup D_R$, the most general substitution σ , say, unifying:

$$\begin{aligned} \{ \{=(r', s') \mid =(r', s') \in A_R; c(=(r', s')) = 0\} \cup \{=(s', r') \mid =(r', s') \in A_R; c(=(r', s')) = 1\}, \\ \{=(s', t') \mid =(s', t') \in D_R; c(=(s', t')) = 0\} \cup \{=(t', s') \mid =(s', t') \in D_R; c(=(s', t')) = 1\}, \\ \{s' \mid =(r', s') \in A_R; c(=(r', s')) = 0\} \cup \{s' \mid =(s', r') \in A_R; c(=(s', r')) = 1\} \cup \\ \{s' \mid =(s', t') \in D_R; c(=(s', t')) = 0\} \cup \{s' \mid =(t', s') \in D_R; c(=(t', s')) = 1\} \}. \end{aligned}$$

exists, σ unifying the three sets in the above tripleton set to the sets $\{=(r, s)\}$, $\{=(s, t)\}$, and $\{s\}$, respectively, for some terms r , s , and t , say, and

$$B \equiv A\sigma \setminus \{=(r, s), =(s, r)\} \cup D\sigma \setminus \{=(s, t), =(t, s)\} \cup \{=(r, t)\}.$$

As $A'\rho \subseteq A$, $A'\rho\sigma \subseteq A\sigma$.

If $\{=(r, s), =(s, r)\} \cap A'\rho\sigma = \emptyset$, $A'\rho\sigma \subseteq A\sigma \setminus \{=(r, s), =(s, r)\} \subseteq B$, and so A' subsumes B .

Similarly, $D'\rho\sigma \subseteq D\sigma$, and if $\{=(s, t), =(t, s)\} \cap D'\rho\sigma = \emptyset$, $D'\rho\sigma \subseteq D\sigma \setminus \{=(s, t), =(t, s)\} \subseteq B$, and so D' subsumes B .

Thus we may assume that $A'\rho\sigma \cap \{=(r, s), =(s, r)\} \neq \emptyset$, and $D'\rho\sigma \cap \{=(s, t), =(t, s)\} \neq \emptyset$.

Consider the substitution instances $A'' \equiv A'\rho\sigma$ and $D'' \equiv D'\rho\sigma$ of clauses A' and D' , respectively. We will show there exists a $B'' \in \text{EPC}_{+, +}(A'', D'')$ such that $B'' \subseteq B$, and hence, such that B'' subsumes B . By lifting, (Lemma 4.4.4), it will follow that there exists $B' \in \text{EPC}_{+, +}(A', D')$ such that B' subsumes B'' , and hence, subsumes B .

We have $A'' \setminus \{=(r, s), =(s, r)\} \subseteq A\sigma \setminus \{=(r, s), =(s, r)\}$, and
 $D'' \setminus \{=(s, t), =(t, s)\} \subseteq D\sigma \setminus \{=(s, t), =(t, s)\}$.

Let $A''_R \equiv A'\rho\sigma \cap \{=(r, s), =(s, r)\}$.

Then $\emptyset \neq A''_R \subseteq A''$, and $A''_R \subseteq \{=(r, s), =(s, r)\}$.

Similarly, let $D''_R \equiv D'\rho\sigma \cap \{=(s, t), =(t, s)\}$. Then $\emptyset \neq D''_R \subseteq D''$, and $D''_R \subseteq \{=(s, t), =(t, s)\}$.

Let c' be the choice function on $A''_R \cup D''_R$ defined separately on the (disjoint) sets A''_R , and D''_R as follows:

On A''_R : $=(r'', s'') \in A''_R \mapsto 0$, if $=(r'', s'') = =(r, s)$, and
 $=(r'', s'') \in A''_R \mapsto 1$, otherwise, (if $=(r'', s'') = =(s, r)$).

On D''_R : $=(s'', t'') \in D''_R \mapsto 0$, if $=(s'', t'') = =(s, t)$, and
 $=(s'', t'') \in D''_R \mapsto 1$, otherwise, (if $=(s'', t'') = =(t, s)$).

Evidently, the empty substitution unifies the set $S' \equiv$

$$\begin{aligned} & \{ \{=(r'', s'') \in A''_R \mid c'(\=(r'', s'')) = 0\} \cup \{=(s'', r'') \mid \=(r'', s'') \in A''_R; c'(\=(r'', s'')) = 1\} \\ & \{=(s'', t'') \in D''_R \mid c'(\=(s'', t'')) = 0\} \cup \{=(t'', s'') \mid \=(s'', t'') \in D''_R; c'(\=(s'', t'')) = 1\} \\ & \{s'' \mid \=(r'', s'') \in A''_R; c'(\=(r'', s'')) = 0\} \cup \{s'' \mid \=(s'', r'') \in A''_R; c'(\=(s'', r'')) = 1\} \cup \\ & \{s'' \mid \=(s'', t'') \in D''_R; c'(\=(s'', t'')) = 0\} \cup \{s'' \mid \=(t'', s'') \in D''_R; c'(\=(t'', s'')) = 1\} \}. \end{aligned}$$

to the set $\{ \{=(r, s)\}, \{=(s, t)\}, \{s\} \}$, (and this is the most general unifier).

Let $B'' \equiv A'' \setminus \{=(r, s), =(s, r)\} \cup D'' \setminus \{=(s, t), =(t, s)\} \cup \{=(r, t)\}$.

Then $B'' \in \text{EPC}_{+,+}(A'', D'')$, and $B'' \subseteq B$, as required. ■

For Lemma 4.5.3:

Proof (sketch): The proof is similar to that given for Lemma 4.4.5.

Just as in that proof,

(except the lines:

“As (modulo clause ordering), $A'\rho = A$, it immediately follows that $A'\rho\varphi = A\varphi$ ”.

“ $A'\rho\varphi \setminus A'^-\rho\varphi = A\varphi \setminus A^-\varphi$ ”, and

“Thus $QR_{full}(A')\lambda = A'\psi\lambda \setminus A'^-\psi\lambda = A'\rho\varphi \setminus A'^-\rho\varphi = QR_{full}(A)$ ”.

should here be replaced with:

“As $A'\rho \subseteq A$, it immediately follows that $A'\rho\varphi \subseteq A\varphi$ ”.

“ $A'\rho\varphi \setminus A'^-\rho\varphi \subseteq A\varphi \setminus A^-\varphi$ ”, and

“Thus $QR_{full}(A')\lambda = A'\psi\lambda \setminus A'^-\psi\lambda = A'\rho\varphi \setminus A'^-\rho\varphi \subseteq QR_{full}(A)$ ”.

respectively),

we can show that $QR_{full}(A')$ subsumes $QR_{full}(A)$,

(instead of “ $QR_{full}(A)$ is a substitution instance of $QR_{full}(A')$ ”),

and that, similarly, $QR_{full}(D')$ subsumes $QR_{full}(D)$.

The result now follows from Lemma 4.5.2,

(instead of, as in the proof of Lemma 4.4.5, from Lemma 4.4.1). ■

For Corollary 4.5.1:

Proof: By induction on the depth $k \geq 0$ of the EPC₊-deduction Π .

The case $k = 0$: Here $B \in \mathcal{C}$. Hence there is a $B' \in \mathcal{C}'$, such that B' subsumes B . That is, there is depth 0 EPC₊-deduction of a clause B' from $\mathcal{C}'$, such that B' subsumes B .

Suppose the result for all $0 \leq k < N$, and that now $k = N$.

$B \in \text{EPC}_+(A, D)$, for some A and D , having EPC₊-deductions Π_A and Π_D , of depths $k_A < k$ and $k_D < k$, say, respectively.

By applying the induction hypothesis, (twice), there are EPC₊-deductions $\Pi_{A'}$ and $\Pi_{D'}$, of depths $k_{A'} \leq k_A$ and $k_{D'} \leq k_D$, of clauses A' and D' , respectively, from $\mathcal{C}'$, such that A' subsumes A , and D' subsumes D .

By Lemma 4.5.3, either A' subsumes B , or D' subsumes B , or there exists $B' \in \text{EPC}_+(A', D')$ such that B' subsumes B .

In the first case, take Π' to be $\Pi_{A'}$ and B' to be A' , in the second case, take Π' to be $\Pi_{D'}$ and B' to be D' , and in the last case, take Π' to be the result of following $\Pi_{A'}$ and $\Pi_{D'}$ by the EPC₊-resolvent, $B' \in \text{EPC}_+(A', D')$.

Then Π' is a depth $k' \leq k$ EPC₊-deduction Π' , of a clause B' from $\mathcal{C}'$, where B' subsumes B , as required. ■

Corollary 4.5.2 follows immediately from Corollary 4.5.1.

For Lemma 4.5.4:

Proof: If for some substitution θ , $B\theta \subseteq C \cup F$, (where F consists only of trivial inequalities), then, since C is positive, there is a substitution σ such that $s\sigma = t\sigma$, for each inequality $\neq(s, t) \in B$. Thus, there is a most general such substitution σ , and $QR_{full}(B) = B\sigma \setminus B^-\sigma = (B \setminus B^-)\sigma$, (B^- denoting the inequalities of B), is positive. Further, for some substitution ρ , $\theta = \sigma\rho$, so $QR_{full}(B)\rho = (B \setminus B^-)\theta \subseteq C$, and so $QR_{full}(B)$ subsumes C .

Conversely, suppose $QR_{full}(B)$ subsumes C . $QR_{full}(B)$ is either B , or $(B \setminus B^-)\sigma$, where σ is a most general substitution such that $s\sigma = t\sigma$, for all inequalities $\neq(s, t) \in B^-$.

If $QR_{full}(B) = B$, B subsumes C , so (trivially), there is a substitution θ such that $B\theta \subseteq C \cup F$.

Otherwise, there is a substitution ρ such that $(B \setminus B^-)\sigma\rho \subseteq C$, and thus, such that $B\sigma\rho \subseteq C \cup B^-\sigma\rho$. But $B^-\sigma\rho$ is a trivial inequality clause F , and so there is a substitution θ , (namely, $\sigma\rho$), such that $B\theta \subseteq C \cup F$, (for some trivial inequality clause, F). ■

For Corollary 4.5.3:

Proof: If $QR_{full}(B\sigma)$ subsumes C , by Lemma 4.5.4, there is some substitution θ , and a trivial inequality clause F , such that $B\sigma\theta \subseteq C \cup F$. But $\sigma\theta$ is some substitution ρ , say, so by Lemma 4.5.4, $QR_{full}(B)$ subsumes C . ■

For Corollary 4.5.4:

Proof: If $QR_{full}(B)$ subsumes C , by Lemma 4.5.4, there is some substitution θ , and a trivial inequality clause F , such that $B\theta \subseteq C \cup F$. But if A subsumes B , then for some substitution σ , $A\sigma \subseteq B$, so $A\sigma\theta \subseteq B\theta \subseteq C \cup F$, and since $\sigma\theta$ is a substitution ρ , say, again by Lemma 4.5.4, $QR_{full}(A)$ subsumes C . ■

Observation 4.5.1 is left as an easy induction for the reader.

For Lemma 4.5.5:

Proof (sketch): By induction on $\chi(E) = n > 0$.

If $n = 1$, then for some distinct terms s' and t' , $\emptyset \neq \{\neq(s', t'), \neq(t', s')\} \cap E$, and $E^- \subseteq \{\neq(s', t'), \neq(t', s')\} \cup F'$, for some trivial inequality clause F' .

Clearly, there is $G \in EPC_-(E_{s', t'}, E)$ such that $G \subseteq C \cup E \setminus E^- \cup F'$, (for some trivial inequality clause F'), and this is a depth 1 $EPC_{<}^{+1}$ -deduction of G .

Now suppose the result for all $1 \leq n < N$. If $\chi(E) = N$, then (by Observation 4.1.3), for some $E' \in QR_{first}(E)$, and some distinct terms s' and t' , we must have $\neq(t', s')$ occurring as the left-most inequality in E' .

Let G_1 denote the EPC_- -resolvent of $E_{s', t'}$ and E , in $EPC_{+, -}(E_{s', t'}, E')$, which (first eliminates some "left-most" trivial inequalities, via $QR_{first}(\cdot)$, and eliminates $\neq(s', t')$ and $\neq(t', s')$, and), replaces all occurrences of t' remaining in $E'^- \setminus \{\neq(s', t'), \neq(t', s'), \neq(t', t')\}$, via s' .

Then $G_1 \subseteq C \cup E \setminus E^- \cup (E^- \setminus \{\neq(s', t'), \neq(t', s'), \neq(t', t')\})[s'/t'] \cup F'$,

(for some trivial inequality clause F'), where, for a clause A , the notation $A[s'/t']$ represents the uniform replacement of all top-level, (that is, occurring directly as arguments of a literal of A), occurrences of t' in $A \setminus \{\neq(t', t')\}$, by the term s' .

Obviously G_1 is a (ground) equality clause. Consider any non-trivial inequality $\neq(s'', t'') \in G_1$.

If $\neq(s'', t'') \in E$, then by hypothesis, there is some $\alpha \in \tau$, such that $\{s'', t''\} \subseteq \alpha$.

Otherwise, $s'' = s'$ and $\neq(s'', t'') = \neq(s', t'') = \neq(t', t'')[s'/t']$,

(or $t'' = s'$, and $\neq(s'', t'') = \neq(s'', s') = \neq(s'', t')[s'/t']$,

-but this case may be treated with essentially the same argument),

where $\neq(t', t'')$ is some non-trivial inequality in E .

Since, by assumption, both $\neq(s', t')$ and $\neq(t', t'')$ are non-trivial inequalities in E , there is $\alpha' \in \tau$, such that $\{s', t'\} \subseteq \alpha'$, and $\alpha'' \in \tau$, such that $\{t', t''\} \subseteq \alpha''$.

But since $t' \in \alpha' \cap \alpha'' \neq \emptyset$, (and these are equivalence classes), $\alpha' = \alpha''$.

Thus there is $\alpha' \in \tau$, such that $\{s', t', t''\} \subseteq \alpha'$, and hence, such that $\{s'', t''\} \subseteq \alpha'$.

(and similarly in the case that $t'' = s'$, and $\neq(s'', t'') = \neq(s'', s') = \neq(s'', t')[s'/t']$, above).

With n denoting $\chi(G_1)$, n must be $N - 1$.

For with $\alpha \in \tau_{E^-}$, such that $\{s', t'\} \subseteq \alpha$, either $\|\alpha\| = 2$, (and $\alpha = \{s', t'\}$), or $\|\alpha\| > 2$.

In the former case, the only non-trivial inequalities of E containing as an argument either s' or t' are in $\{\neq(s', t'), \neq(t', s')\}$, and the substitution of s' for t' is ineffectual. In this case $\tau_{G_1^-}$ is $\tau_{E^-} \setminus \{\alpha\}$, and clearly, $\chi(G_1) = N - 1$.

In the latter case, either s' or t' must survive in some non-trivial inequality in $E^- \setminus \{\neq(s', t'), \neq(t', s')\}$, and the substitution of s' for t' amounts, in $\tau_{G_1^-}$, to simply the removal of t' from α , where it resided, in τ_{E^-} , and again, $\chi(G_1) = N - 1$.

Thus $1 \leq n < N$, (if $n = 0$, then $N = 1$, which conflicts with our assumption that $1 < N$), and we may apply the induction hypothesis to G_1 , to obtain a depth n (that is, a depth $N - 1$), $EPC_{<}^{+1}$ -deduction of a clause G , from E together with the $E_{s,t}$ above, such that $G \subseteq C \cup E \setminus E^- \cup F'$, for some trivial inequality clause F' . Preceding this depth $N - 1$ $EPC_{<}^{+1}$ -deduction by taking G_1 as the $EPC_{<}$ -resolvent of $E_{s',t'}$ and E , we obtain the required depth N $EPC_{<}^{+1}$ -deduction. ■

For Lemma 4.5.6:

Proof (sketch): Since D is ground and QR -positive, D must be positive, except possibly, for some trivial inequality literals.

There exist sequences $C_0, \dots, C_{k-1}$, and $D_0, \dots, D_{k-1}, D_k$, such that:

- i.) $D_0 \equiv B$ and $D_k \equiv D$,
- ii.) $\forall i: 0 \leq i \leq k-1, C_i \in \mathfrak{C}$, and
- iii.) $\forall i: 0 \leq i \leq k-1, D_{i+1} \in EPC_{<}(C_i, D_i) \cap EPC_{<}^+(C_i, D_i)$, (so that C_i is the QR -positive parent, while D_i is the negative parent of $EPC_{<}$ -resolvent D_{i+1}).

Let e_i denote the (positive equality) literal of C_i which is resolved away in the resolution responsible for $D_{i+1} \in EPC_{<}(C_i, D_i) \cap EPC_{<}^+(C_i, D_i)$, (and suppose B^- denotes the inequalities of B).

It is not hard to see that D must be of the form:

$$C_0 \setminus (F_0 \cup \{e_0\}) \cup \dots \cup C_{k-1} \setminus (F_{k-1} \cup \{e_{k-1}\}) \cup B \setminus B^- \cup F,$$

where F , and for each $i: 0 \leq i \leq k-1$, F_i , consist only of trivial inequalities.

(The F_i represent the inequalities in C_i which are eliminated, under " QR_{full} ", so that the $C_i \setminus F_i$ are positive, for each $i: 0 \leq i \leq k-1$.)

$$QR_{full}(D) \text{ is of the form } C_0 \setminus (F_0 \cup \{e_0\}) \cup \dots \cup C_{k-1} \setminus (F_{k-1} \cup \{e_{k-1}\}) \cup B \setminus B^-.$$

Also, evidently it must be the case that, for each equivalence class α in the set of equivalence classes, τ_{B^-} , (which is $\tau_{B^- \setminus F}$, since trivial inequalities never contribute), $\alpha \subseteq \beta$, for some equivalence class β , generated from equalities in $\{e_0, e_1, \dots, e_{k-1}\}$.

It is also not hard to see that, for every non-trivial (ground) inequality $\neq(s', t')$, such that, for some $\alpha \in \tau_{B^-}$, $\{s', t'\} \subseteq \alpha$, there is a positive equality clause $E_{s',t'}$, which is $EPC_{<}^+$ -deducible from $\{C_0, C_1, \dots, C_{k-1}\}$, such that $\emptyset \neq \{(s', t'), (t', s')\} \cap E_{s',t'}$, and

$$E_{s',t'} \subseteq C_0 \setminus (F_0 \cup \{e_0\}) \cup \dots \cup C_{k-1} \setminus (F_{k-1} \cup \{e_{k-1}\}) \cup \{(s', t'), (t', s')\},$$

Since $B' \subseteq B \cup F$, (where F consists only of trivial inequalities), clearly, $(B')^- \subseteq B^- \cup F$. Thus, for each equivalence class α' in the set of equivalence classes, $\tau_{(B')^-}$, $\alpha' \subseteq \alpha$, for some $\alpha \in \tau_{B^-}$.

In particular, for every non-trivial inequality $\neq(s', t') \in (B')^-$, there is some $\alpha \in \tau_{B^-}$, such that $\{s', t'\} \subseteq \alpha$.

Now if $\chi(B') > 1$, (of Definition 4.5.1), then with $k' > 1$ denoting $\chi(B')$, by Lemma 4.5.5, from B' together with the $E_{s', t'}$ above, there is a depth k' $\text{EPC}_{<}^{+1}$ -deduction Π' of a clause G , such that $G \subseteq C_0 \setminus (F_0 \cup \{e_0\}) \cup \dots \cup C_{k-1} \setminus (F_{k-1} \cup \{e_{k-1}\}) \cup B' \setminus (B')^- \cup F'$, for some trivial inequality clause F' .

On the other hand, if $\chi(B') = 0$, we know B' is already such a clause G , as above.

But since $B' \subseteq B \cup F$, clearly $k' = \chi(B') \leq \chi(B) \leq k$, the last inequality by Observation 4.5.1. Taking G for the required D' , (if $\chi(B') > 0$), it remains only to show that $QR_{full}(G)$ subsumes $QR_{full}(D)$.

But as D is assumed QR -positive, $QR_{full}(D)$ is positive. Hence by Lemma 4.5.4, $QR_{full}(G)$ subsumes $QR_{full}(D)$ if (and only if), for some substitution θ , $G\theta \subseteq QR_{full}(D) \cup F'$, for some trivial inequality clause F' . As G is ground, the condition is simply $G \subseteq QR_{full}(D) \cup F$, (for some trivial inequality clause F').

Recall, $QR_{full}(D)$ is $C_0 \setminus (F_0 \cup \{e_0\}) \cup \dots \cup C_{k-1} \setminus (F_{k-1} \cup \{e_{k-1}\}) \cup B \setminus B^-$. Since $B' \subseteq B \cup F$, $B' \setminus (B')^- \subseteq B \setminus B^-$, so it follows that $G \subseteq QR_{full}(D) \cup F'$, (for some trivial inequality clause F' , and we're done). ■

For Lemma 4.5.7:

Proof: By induction on the depth $k \geq 1$.

We first make some general observations.

Of course, if D_g be a QR -positive ground substitution instance of D ,

then ($QR_{full}(D_g)$ is positive, and as D_g is ground), the only inequalities in D_g are ground and trivial.

Further, if, for some clause D' , D' is ground, and $QR_{full}(D') \subseteq QR_{full}(D_g)$, then,

(as D_g is QR -positive, $QR_{full}(D_g)$ is positive, and hence, so is $QR_{full}(D')$, and so), D' is QR -positive.

If $k = 1$ then $D \in \text{EPC}_{<}^{-1}(C, B) \cap \text{EPC}_{<}^{-1}(C, B)$, for some $C \in \mathfrak{C}$.

(and D_g is a ground substitution instance of D).

By Corollary 4.4.1, there exist ground substitution instances C' and B' of C and B , respectively, and a (ground clause) $D' \in \text{EPC}_{<}^{-1}(C', B')$, such that $D' \subseteq D_g \cup F$,

for some finite F containing only ground trivial inequalities.

As D' , D_g and F are ground, (and $D' \subseteq D_g \cup F$), clearly, $QR_{full}(D') \subseteq QR_{full}(D_g)$.

Taking $\mathfrak{C}'' \equiv \{C\}$, $\mathfrak{C}' \equiv \{C'\}$, and Π' to be the depth 1 $\text{EPC}_{<}^{+1}$ -deduction, $D' \in \text{EPC}_{<}^{-1}(C', B')$, evidently suffices.

Suppose the result for all k : $1 \leq k < N$, and that now, Π has depth $k = N$.

Let $C_0, \dots, C_{k-1}$, (the $C_i \in \mathfrak{C}$ not necessarily distinct), and $B = D_0, \dots, D_{k-1}$, $D_k = D$, represent the $\text{EPC}_{<}^{+1}$ -deduction Π , (as in Definition 4.1.11).

Evidently, $C_1, \dots, C_{k-1}$ and $D_1, \dots, D_{k-1}, D_k = D$, represents a depth $k-1$ $\text{EPC}_{<}^{+1}$ -deduction, Π'' , say.

By induction, there is a finite collection $\mathfrak{C}^{iv}$ of (QR -positive equality) clauses C^{iv} , each $C^{iv} \in \mathfrak{C}^{iv}$ having an EPC_+ -deduction $\Pi_{C^{iv}}$ from $\mathfrak{C}$, a finite collection $\mathfrak{C}'''$, of QR -positive ground substitution instances of clauses in $\mathfrak{C}^{iv}$, a ground substitution instance, D_1' , of D_1 , and a $\text{EPC}_{<}^{+1}$ -deduction Π''' , with depth $k'' \leq k-1$, of some ground (QR -positive equality) clause D'' , from $\{D_1'\} \cup \mathfrak{C}'''$, such that $QR_{full}(D'') \subseteq QR_{full}(D_g)$.

Since $D_1 \in \text{EPC}_{<}^{-1}(C_0, B) \cap \text{EPC}_{<}^{-1}(C_0', B')$ and D_1' is a ground substitution instance of D_1 , by Corollary 4.4.1, there exist ground substitution instances C_0' and B' of C_0 and B , respectively, and a (ground clause) $D_1'' \in \text{EPC}_{<}^{-1}(C_0', B')$, such that $D_1'' \subseteq D_1' \cup F$, for some finite F containing only ground trivial inequalities.

Since Π''' is a depth k'' $\text{EPC}_{<}^{+1}$ -deduction of D'' , from $\{D_1'\} \cup \mathfrak{C}'''$, (every clause in $\{D_1'\} \cup \mathfrak{C}'''$ is ground, and $D_1'' \subseteq D_1' \cup F$), by Lemma 4.5.6, either:

- i) $QR_{full}(D_1'')$ subsumes $QR_{full}(D'')$, or
- ii) there is a finite collection $\mathfrak{C}^v$ of (QR -positive equality) clauses C^v , each $C^v \in \mathfrak{C}^v$ having an EPC_+ -deduction Π_{C^v} from $\mathfrak{C}'''$, and a $\text{EPC}_{<}^{+1}$ -deduction Π^{iv} , with depth $k''' \leq k''$, of some (QR -positive equality) clause D''' from $\{D_1''\} \cup \mathfrak{C}^v$, such that $QR_{full}(D''') \subseteq QR_{full}(D'')$.

In case (i), (since D_1'' and D'' are ground, so are $QR_{full}(D_1'')$ and $QR_{full}(D'')$ and thus), we have $QR_{full}(D_1'') \subseteq QR_{full}(D'')$. Since D'' is QR -positive, evidently, so is D_1'' . Also, since $QR_{full}(D'') \subseteq QR_{full}(D_g)$, we have $QR_{full}(D_1'') \subseteq QR_{full}(D_g)$. Thus, taking $\mathfrak{C}'' \equiv \{C_0\}$, $\mathfrak{C}' \equiv \{C_0'\}$, $D' \equiv D_1''$, and Π' to be the depth 1 $\text{EPC}_{<}^{+1}$ -deduction, $D_1'' \in \text{EPC}_{<}^{-1}(C_0', B')$, evidently suffices.

In case (ii):

As each $C^v \in \mathfrak{C}^v$ has an EPC_+ -deduction Π_{C^v} from $\mathfrak{C}'''$, and each $C''' \in \mathfrak{C}'''$ is a (QR -positive) ground substitution instance of some (QR -positive equality) clause in $\mathfrak{C}^{iv}$, evidently, each $C^v \in \mathfrak{C}^v$ is a ground QR -positive equality clause.

Also, by "lifting", (by repeated application of Corollary 4.4.3), there is a finite collection, $\mathfrak{C}^{vi}$ of (QR -positive equality) clauses C^{vi} , such that for each $C^v \in \mathfrak{C}^v$, there is some $C^{vi} \in \mathfrak{C}^{vi}$, such that C^v is a substitution instance of C^{vi} , and such that each $C^{vi} \in \mathfrak{C}^{vi}$ has an EPC_+ -deduction from $\mathfrak{C}^{iv}$.

As each $C^{iv} \in \mathfrak{C}^{iv}$ has an EPC_+ -deduction from $\mathfrak{C}$, evidently, each $C^{vi} \in \mathfrak{C}^{vi}$ has an EPC_+ -deduction from $\mathfrak{C}$.

Let $\mathfrak{C}'' \equiv \mathfrak{C}^{vi} \cup \{C_0\}$, and let $\mathfrak{C}' \equiv \mathfrak{C}^v \cup \{C_0'\}$.

Then evidently, $\mathfrak{C}''$ is a finite collection of (QR -positive equality) clauses C'' , each $C'' \in \mathfrak{C}''$ having an EPC_+ -deduction $\Pi_{C''}$ from $\mathfrak{C}$, and $\mathfrak{C}'$ is a finite collection of QR -positive ground substitution instances of clauses in $\mathfrak{C}''$.

Take D' be D''' , and let Π' denote the result of affixing (the root node of) the depth 1 $\text{EPC}_{<}^{-1}$ -deduction, $D_1'' \in \text{EPC}_{<}^{-1}(C_0', B')$, to (the leaf node labelled by) D_1'' , in Π^{iv} .

Then evidently, Π' is a $\text{EPC}_{<}^{+1}$ -deduction, with depth $k' \leq k'' + 1 \leq k$, of a (QR -positive equality) clause D' from $\{B'\} \cup \mathfrak{C}'$, such that $QR_{full}(D')$ subsumes $QR_{full}(D'')$.

Since B' is ground, and every clause of $\mathfrak{C}'$ is ground, it follows that D' is ground.

Thus, $QR_{full}(D') \subseteq QR_{full}(D'')$, and so, (since, as before, $QR_{full}(D'') \subseteq QR_{full}(D_g)$), we have $QR_{full}(D') \subseteq QR_{full}(D_g)$. ■

For Lemma 4.5.8:

Proof: By induction on n , where n is the number of inequalities $\neq(s', t') \in B'$ such that $\neq(s', t')\theta$ is trivial.

If $n = 0$, then the left-most inequality $\neq(s'', t'')$, say, of B' (exists, and), is such that $\neq(s'', t'')\theta$ is non-trivial. As $B' \in QR_{first}(B')$, (ordered as originally), taking B' for B'' , and θ for ρ , suffices.

Suppose the result for all $0 \leq n < N$, and that now $n = N$.

If the left-most inequality, $\neq(s', t')$, say, of B' , is such that $\neq(s', t')\theta$ is non-trivial, then again, taking B' for B'' , and θ for ρ , suffices.

Otherwise, $\neq(s', t')\theta$ is trivial, (so θ unifies s' and t'), so the most general substitution σ , say, which unifies $\{s', t'\}$, exists, and for some substitution θ' , we have $\sigma\theta' = \theta$.

As $\neq(s', t')$ is left-most in B' , we have some $B''' \in QR_{first}(B')$, where B''' is some ordering of $B'\sigma \setminus \{\neq(s', t')\sigma\}$.

For each $\neq(s''', t''') \in B'''$ such that $\neq(s''', t''')\theta'$ is trivial, there is some $\neq(s', t') \in B'$ such that $\neq(s', t')\sigma = \neq(s''', t''')$, and (hence), such that also, $\neq(s', t')\theta = \neq(s', t')\sigma\theta' = \neq(s''', t''')\theta'$ is trivial.

Thus the number of inequalities $\neq(s''', t''') \in B'''$ such that $\neq(s''', t''')\theta'$ is trivial, is strictly less than N .

Also, clearly, $B'''\theta' \subseteq B'\sigma\theta' = B'\theta \subseteq B$.

Thus by the induction hypothesis, there is some $B'' \in QR_{first}(B''')$ containing a (non-trivial) left-most inequality $\neq(s'', t'')$, say, such that for some substitution ρ , $B''\rho \subseteq B$, and $\neq(s'', t'')\rho$ is non-trivial.

But since $B''' \in QR_{first}(B')$ and $B'' \in QR_{first}(B''')$, $B'' \in QR_{first}(B')$, and so we're done. ■

For Lemma 4.5.9:

Proof (sketch): We prove the Lemma by induction on the depth k .

Lemma 4.5.6 shows the Lemma for the case when B' is ground, and when $\mathfrak{C}'$ is $\mathfrak{C}$.

As in the proof of Lemma 4.5.6, D must be of the form:

$$C_0 \setminus (F_0 \cup \{e_0\}) \cup \dots \cup C_{k-1} \setminus (F_{k-1} \cup \{e_{k-1}\}) \cup B \setminus B^- \cup F,$$

where F , and for each $i: 0 \leq i \leq k-1$, F_i , consist only of trivial inequalities.

(The F_i represent the inequalities in C_i which are eliminated, under " QR_{full} ", so that the $C_i \setminus F_i$, are positive, for each $i: 0 \leq i \leq k-1$.)

$$QR_{full}(D) \text{ is of the form } C_0 \setminus (F_0 \cup \{e_0\}) \cup \dots \cup C_{k-1} \setminus (F_{k-1} \cup \{e_{k-1}\}) \cup B \setminus B^-.$$

Since B' subsumes B , for some substitution θ , we have $B'\theta \subseteq B$.

If $B'\theta$ contains only trivial inequality literals, then $B'\theta \subseteq B \setminus B^- \cup F \subseteq QR_{full}(D) \cup F$, for some F containing only trivial inequalities, and so, by Lemma 4.5.4, $QR_{full}(B')$ subsumes $QR_{full}(D)$. (that is, case (ii) holds).

Thus we may assume $B'\theta$ contains at least one non-trivial inequality. By Lemma 4.5.8, then, there exists a $B'' \in QR_{first}(B')$, containing a (non-trivial) left-most inequality $\neq(s'', t'')$, say, such that for some substitution ρ , $B''\rho \subseteq B$, and $\neq(s'', t'')\rho$ is non-trivial.

Let $\neq(s, r)$ denote this non-trivial inequality, $\neq(s'', t'')\rho \in B''\rho \subseteq B$.

Evidently, by Lemma 4.5.6, (re-ordering B), and Corollary 4.5.2, (or Corollary 4.4.3, as appropriate), we may assume $\neq(s, r)$ occurs as the left-most inequality in B .

As Π is a depth k $EPC_{<}^{+1}$ -deduction of D from ground equality clauses $\{B\} \cup \mathcal{C}$, there exist sequences $C_0, \dots, C_{k-1}$, and $D_0, \dots, D_{k-1}, D_k$, such that:

- i.) $D_0 \equiv B$ and $D_k \equiv D$.
- ii.) $\forall i: 0 \leq i \leq k-1, C_i \in \mathcal{C}$, and
- iii.) $\forall i: 0 \leq i \leq k-1, D_{i+1} \in EPC_{-}(C_i, D_i)$, (so that C_i is the QR -positive parent, while D_i is the negative parent of EPC_{-} -resolvent D_{i+1}).

In particular, for some QR -positive $C_0 \in \mathcal{C}$, $D_1 \in EPC_{-}(C_0, B)$.

Thus $D_1 \in EPC_{+,-}(QR_{full}(C_0), B''')$, for some $B''' \in QR_{first}(B)$.

But as (we are assuming) the non-trivial inequality $\neq(s, r)$ occurs as the left-most inequality in B , B''' must be none other than B . Let C denote $QR_{full}(C_0)$.

By hypothesis, there is some $C'_0 \in \mathcal{C}'$ such that C'_0 subsumes C_0 . By Corollary 4.5.4,

(taking the C in the Corollary equal to $QR_{full}(B)$, and B to be our current C_0).

$QR_{full}(C'_0)$ subsumes C , (which is $QR_{full}(C_0)$). Let C' denote $QR_{full}(C'_0)$.

If C_0 is actually (modulo ordering) a substitution instance of C'_0 , then (it is easy to see that), $QR_{full}(C_0)$ is actually (modulo ordering) a substitution instance of $QR_{full}(C'_0)$, that is, C is a substitution instance of C' .

As the left-most inequality $\neq(s'', t'')$, of $B'' \in QR_{first}(B')$ (exists, and) has image (maps) under ρ to $\neq(s, r)$, which occurs left-most in B , $B''\rho \subseteq B$, and $D_1 \in EPC_{+,-}(C, B)$, by Lemma 4.5.1, case (ii), either:

I. C' subsumes D_1 , or

II. there exists $D_1' \in EPC_{+,-}(C', B'')$ such that D_1' subsumes D_1 ,

and II. holds, (that is, case I. can be omitted), if C is a substitution instance of C' , (hence, if C_0 is a substitution instance of C'_0).

In case II., since $B'' \in QR_{first}(B')$, and $C' = QR_{full}(C'_0)$, $D_1' \in EPC_{-}(C'_0, B')$.

We now do an induction on the depth k .

Suppose $k = 1$. Then $D = D_1$.

In case I., clearly (i) (of this Lemma) holds, and we're done.

In case II., as D_1' subsumes D_1 , by Corollary 4.5.4,

(taking the C in the Corollary equal to $QR_{full}(B)$, and B to be our current D),
 $QR_{full}(D_1')$ subsumes $QR_{full}(D)$.

Take D' to be D_1' , and $\mathcal{C}''$ to be $\mathcal{C}'$. Then evidently the deduction of $D_1' \in \text{EPC}_{<}(C'_0, B')$ is a depth 1 $\text{EPC}_{<}^{+1}$ -deduction Π' , of a (QR -positive equality) clause D' from $\{B'\} \cup \mathcal{C}''$, such that $QR_{full}(D')$ subsumes $QR_{full}(D)$, as required.

Suppose the result for all $1 \leq k < N$, and that now $k = N$.

In case I., there is some substitution φ such that $C'\varphi \subseteq D_1$. Since C' is positive, (it is not hard to see that), $C'\varphi \subseteq D_1 \setminus D_1^- \subseteq B \setminus B^- \subseteq QR_{full}(D)$. Thus C' subsumes $QR_{full}(D)$, and clearly (i) (of this Lemma) holds, and we're done.

Thus we may assume case II. applies.

As the sequences $C_1, \dots, C_{k-1}$ and $D_1, \dots, D_{k-1}, D_k \equiv D$ represent a depth $k - 1$ $\text{EPC}_{<}^{+1}$ -deduction of D from ground equality clauses $\{D_1\} \cup \mathcal{C}$, (and D_1' subsumes D_1), by induction hypothesis, either:

- i'. for some $C' \in \mathcal{C}'$, $QR_{full}(C')$ subsumes $QR_{full}(D)$, or
- ii'. $QR_{full}(D_1')$ subsumes $QR_{full}(D)$, or
- iii'. there is a finite collection $\mathcal{C}'''$ of (QR -positive equality) clauses C''' , each $C''' \in \mathcal{C}'''$ having an EPC_+ -deduction $\Pi_{C'''}$ from $\mathcal{C}'$, and a $\text{EPC}_{<}^{+1}$ -deduction Π'' , with depth $k'' \leq k - 1$, of some (QR -positive equality) clause D' from $\{D_1'\} \cup \mathcal{C}'''$, such that $QR_{full}(D')$ subsumes $QR_{full}(D)$.

where case (i') need not be considered when we are in the special case that for each $C \in \mathcal{C}$, there is some $C' \in \mathcal{C}'$ such that C is (modulo ordering) a substitution instance of C' .

In case (i'), we have case (i), (of the Lemma), directly, and so there is nothing further required here.

In case (ii'), evidently, D_1' must again be QR -positive, and again taking D' to be D_1' , and $\mathcal{C}''$ to be $\mathcal{C}'$, the deduction of $D' \in \text{EPC}_{<}(C'_0, B')$ is a depth $1 \leq k$ $\text{EPC}_{<}^{+1}$ -deduction Π' , of a (QR -positive equality) clause D' from $\{B'\} \cup \mathcal{C}''$, such that $QR_{full}(D')$ subsumes $QR_{full}(D)$, as required.

In case (iii'), taking $\mathcal{C}''$ to be $\mathcal{C}''' \cup \{C'_0\}$, and preceding the depth $k'' \leq k - 1$ $\text{EPC}_{<}^{+1}$ -deduction Π'' , of D' from $\{D_1'\} \cup \mathcal{C}'''$, by the resolution $D_1' \in \text{EPC}_{<}(C'_0, B')$, evidently we have a finite collection $\mathcal{C}''$ of (QR -positive equality) clauses C'' , each $C'' \in \mathcal{C}''$ having an EPC_+ -deduction $\Pi_{C''}$ from $\mathcal{C}'$, and a $\text{EPC}_{<}^{+1}$ -deduction Π' , with depth $k' \leq k$, of a (QR -positive equality) clause D' from $\{B'\} \cup \mathcal{C}''$, such that $QR_{full}(D')$ subsumes $QR_{full}(D)$, as required. ■

For Lemma 4.5.10:

Proof (sketch): Lemma 4.5.9 shows the Lemma for the cases where B and $\mathcal{C}$, (and hence D), are ground.

Let σ denote the most general substitution which unifies the two terms of each inequality of D , (as in Definition 4.1.2), and let ρ be the substitution which assigns to each variable, x , in $D\sigma$, its associated formal constant, $\bar{x}$. Evidently $D\sigma\rho$ is ground, and $D\sigma\rho = \overline{D\sigma}$.

Let Π_1 denote $\downarrow(\Pi, \sigma\rho)$, (of Definition 4.5.2). Then there is a finite collection $\mathcal{C}^{iv} \subseteq EPC_+(\mathcal{C})$, (of QR -positive equality clauses, C^{iv}), a finite collection $\mathcal{C}'''$, of QR -positive ground substitution instances of clauses in $\mathcal{C}^{iv}$, and a ground substitution instance, B'' , of B , such that Π_1 is a depth $k'' \leq k$, $EPC_{<}^{+1}$ -deduction of some ground (QR -positive equality) clause D'' from $\{B''\} \cup \mathcal{C}'''$, where $QR_{full}(D'') \subseteq QR_{full}(D\sigma\rho)$.

As B' subsumes B , (clearly) B' subsumes B'' .

Since $\mathcal{C}^{iv} \subseteq EPC_+(\mathcal{C})$, by Corollary 4.5.2, for each $C^{iv} \in \mathcal{C}^{iv}$, there is a $C^v \in EPC_+(\mathcal{C}')$, such that C^v subsumes C^{iv} , and in the special case that for each $C \in \mathcal{C}$, there is some $C' \in \mathcal{C}'$ such that C is (modulo ordering) a substitution instance of C' , by Corollary 4.4.3, for each $C^{iv} \in \mathcal{C}^{iv}$, there is a $C^v \in EPC_+(\mathcal{C}')$, such that C^{iv} is a substitution instance of C^v .

Thus, clearly, there is a finite $\mathcal{C}^v \subseteq EPC_+(\mathcal{C}')$, (of QR -positive equality clauses, C^v), such that for each $C^{iv} \in \mathcal{C}^{iv}$, there is a $C^v \in \mathcal{C}^v$, such that C^v subsumes C^{iv} , and in the special case, (that for each $C \in \mathcal{C}$, there is some $C' \in \mathcal{C}'$ such that C is a substitution instance of C'), (we may assume that), for each $C^{iv} \in \mathcal{C}^{iv}$, there is a $C^v \in \mathcal{C}^v$, such that, in fact, C^{iv} is a substitution instance of C^v .

Since, for each $C''' \in \mathcal{C}'''$, there exists a $C^{iv} \in \mathcal{C}^{iv}$ such that C''' is a ground substitution instance of C^{iv} , evidently, for each $C''' \in \mathcal{C}'''$, there exists a $C^v \in \mathcal{C}^v$, such that C^v subsumes C''' , and in the special case, for each $C''' \in \mathcal{C}'''$, there exists a $C^v \in \mathcal{C}^v$, such that, in fact, C''' is a substitution instance of C^v .

By applying Lemma 4.5.9, (the result for “the ground case” which we have proven), (to Π_1), we know that either:

- i') for some $C^v \in \mathcal{C}^v$, $QR_{full}(C^v)$ subsumes $QR_{full}(D'')$, or
- ii') $QR_{full}(B')$ subsumes $QR_{full}(D'')$, or
- iii') there is a finite collection $\mathcal{C}''$ of (QR -positive equality) clauses C'' , each $C'' \in \mathcal{C}''$ having an EPC_+ -deduction $\Pi_{C''}$ from $\mathcal{C}^v$, and a $EPC_{<}^{+1}$ -deduction Π' , with depth $k' \leq k''$, of some (QR -positive equality) clause D' from $\{B'\} \cup \mathcal{C}''$, such that $QR_{full}(D')$ subsumes $QR_{full}(D'')$.

Further, in the special case that for each $C''' \in \mathcal{C}'''$, there is some $C^v \in \mathcal{C}^v$, such that C''' is (modulo ordering) a substitution instance of C^v , case (i') above can be dropped, (that is, either (ii') or (iii') above, must hold).

Thus, in the special case that for each $C \in \mathcal{C}$, there is some $C' \in \mathcal{C}'$ such that C is a substitution instance of C' , case (i') above can be dropped, (that is, either (ii') or (iii') above, must hold).

But suppose $QR_{full}(A)$ subsumes $QR_{full}(D'')$, where A is either some $C^v \in \mathcal{C}^v$, B' , or D' , as in cases (i'), (ii'), and (iii'), respectively, above.

Since $QR_{full}(D'') \subseteq QR_{full}(D\sigma\rho)$, clearly, $QR_{full}(A)$ subsumes $QR_{full}(D\sigma\rho)$.

(Since $QR_{full}(D\sigma\rho)$ is positive), By Lemma 4.5.4, we know there is a substitution θ , such that $A\theta \subseteq QR_{full}(D\sigma\rho) \cup F$, for some trivial inequality clause F .

But $D\sigma\rho$ is only $\overline{D\sigma}$, where σ is the most general substitution unifying the terms of each inequality of D , as in Definition 4.1.2. Thus $QR_{full}(D\sigma\rho) = QR_{full}(\overline{D\sigma}) = \overline{QR_{full}(D\sigma)} = \overline{QR_{full}(D)}$.

Since A can not contain any formally frozen variables, it follows that there is a substitution θ' such that $A\theta' \subseteq QR_{full}(D) \cup \underline{F}$. Clearly also $\underline{F}$ can consist only of trivial inequalities.

Thus again by Lemma 4.5.4, $QR_{full}(A)$ subsumes $QR_{full}(D)$.

Hence, in cases (i'), (ii'), and (iii'), we have

- i'') for some $C^v \in \mathcal{C}^v$, $QR_{full}(C^v)$ subsumes $QR_{full}(D)$, or
- ii'') $QR_{full}(B')$ subsumes $QR_{full}(D)$, or
- iii'') there is a finite collection $\mathcal{C}'' \subseteq \text{EPC}_+(\mathcal{C}^v)$, (of QR -positive equality clauses C''), and a $\text{EPC}_{<}^{+1}$ -deduction Π' , with depth $k' \leq k''$, of some (QR -positive equality) clause D' from $\{B'\} \cup \mathcal{C}''$, such that $QR_{full}(D')$ subsumes $QR_{full}(D)$.

respectively.

In case (i''), (as $\mathcal{C}^v \subseteq \text{EPC}_+(\mathcal{C}')$), clearly we have (i), and in case (ii''), we have (ii) directly.

Finally, in case (iii''), (again, as $\mathcal{C}^v \subseteq \text{EPC}_+(\mathcal{C}')$), evidently each $C'' \in \mathcal{C}''$ has an EPC_+ -deduction $\Pi_{C''}$ from $\mathcal{C}'$, and since $k'' \leq k$, we have (iii). ■

APPENDIX D

Refutation Completeness for $\text{EPC}_{<}$

D.1. The Above-Equality Fragment

For Proposition 5.1.1:

Proof: $\exists \theta_1 : B\theta_1 = C \cup F_1$, for some F_1 consisting only of trivial inequalities, and $\exists \theta_2 : C\theta_2 = D \cup F_2$, for some F_2 consisting only of trivial inequalities.

Letting $\theta \equiv \theta_1\theta_2$; $F \equiv F_1\theta_2 \cup F_2$, we have F comprised only of trivial inequalities, and $B\theta = B\theta_1\theta_2 = C\theta_2 \cup F_1\theta_2 = D \cup F_1\theta_2 \cup F_2 = D \cup F$. ■

For Observation 5.1.1:

Proof: An easy exercise from the definitions. ■

For Lemma 5.1.1:

Proof: (sketch) We modify the proof presented for Lemma 4.2.2.

Let A , A' , B , D , and D' , as in the statement of the lemma, (and assume that A , A' , D , and D' , share (pairwise) no variable in common, which we can do by Observation 4.1.6).

By assumption, there is some substitution θ_A , (which we may assume, has domain contained in the set of variables of A'), such that (modulo clause ordering) $A'\theta_A = A \cup F_A$, for some F_A comprised entirely of trivial inequality literals.

Clearly A' must be pure-positive with respect to P and have $<$ -maximal predicate symbol P , and D' is mixed with respect to P , with $<$ -maximal predicate symbol P . By the assumed separation of variables, evidently (we may assume the domain of θ_D is a subset of the variables of D , and), there is a substitution θ , say, (for example, the composite, $\theta_A\theta_D$), such that (modulo clause ordering) $A'\theta = A \cup F_1$ and $D'\theta = D \cup F_2$, for some F_1 and F_2 consisting of only trivial inequalities, (and $L_0\theta$ occurs as the left-most inequality in D).

For some non-empty subsets $A_R \subseteq A$, and $D_R \subseteq D$, (say), the left-most negative P instance, $L_0\theta$, of D occurring in D_R , (and some term tuples $\vec{s}_1, \dots, \vec{s}_i$ and $\vec{t}_1, \dots, \vec{t}_j$), $A_R \equiv \{P(\vec{s}_1), \dots, P(\vec{s}_i)\}$; $D_R \equiv \{\neg P(\vec{t}_1), \dots, \neg P(\vec{t}_j)\}$, and

$$B \equiv \bigcup_{\substack{1 \leq i' \leq i \\ 1 \leq j' \leq j}} \neq(\vec{s}_{i'}, \vec{t}_{j'}) \cup A \setminus A_R \cup D \setminus D_R.$$

Let $A'_R \equiv \{L \in A' \mid L\theta \in A_R\}$, and let $D'_R \equiv \{L \in D' \mid L\theta \in D_R\}$.

Then $L_0 \in D'_R$, and clearly $D'_R\theta = D'\theta \cap D_R = D \cap D_R = D_R \neq \emptyset$, and $\emptyset \neq D'_R \subseteq D'$.

Similarly, $A'_R\theta = A'\theta \cap A_R = A \cap A_R = A_R \neq \emptyset$, and $\emptyset \neq A'_R \subseteq A'$.

Let $A'_R \equiv \{P(\vec{q}_1), \dots, P(\vec{q}_k)\}$, say, and $D'_R \equiv \{\neg P(\vec{r}_1), \dots, \neg P(\vec{r}_l)\}$, say. Clearly if we let

$$B' \equiv \bigcup_{\substack{1 \leq k' \leq k \\ 1 \leq l' \leq l}} \neq(\vec{q}_{k'}, \vec{r}_{l'}) \cup A' \setminus A'_R \cup D' \setminus D'_R, \text{ then } B' \in \text{EPC}_{<}^1(A', D').$$

$$\text{Evidently, } B'\theta \equiv \bigcup_{\substack{1 \leq k' \leq k \\ 1 \leq l' \leq l}} \neq(\vec{q}_{k'}\theta, \vec{r}_{l'}\theta) \cup (A' \setminus A'_R)\theta \cup (D' \setminus D'_R)\theta.$$

By the choice of A'_R , $\{\vec{q}_{k'}\theta \mid 1 \leq k' \leq k\} = \{\vec{s}_{i'} \mid 1 \leq i' \leq i\}$, and $(A' \setminus A'_R)\theta = A \setminus A_R \cup F_1$.

Similarly, by the choice of D'_R , $\{\vec{r}_{l'}\theta \mid 1 \leq l' \leq l\} = \{\vec{t}_{j'} \mid 1 \leq j' \leq j\}$, and

$$(D' \setminus D'_R)\theta = D \setminus D_R \cup F_2.$$

Thus (modulo ordering), $B'\theta = B \cup F_1 \cup F_2$, and since $F_1 \cup F_2$ consists of only trivial inequalities, $B' \vdash_{QR} B$. ■

For Lemma 5.1.2:

Proof (Sketch): As was the case for the proof of Lemma 4.2.4, again, one can follow the outline of the proof given for Lemma 2.2.3. That is, one can argue by an induction on the depth of Π , this time appealing to Lemma 5.1.1 and Lemma 4.2.1, in place of (their $\text{PC}_{<}$ -deduction analogues), Lemma 2.2.2 and Lemma 2.2.1, respectively. ■

For Lemma 5.1.3:

Proof (Sketch): A proof is that given for Lemma 2.2.4, with (“ $\text{EPC}_{<}$ ” substituted for “ $\text{PC}_{<}$ ”, and), Lemma 5.1.2, substituted for Lemma 2.2.3. ■

For Lemma 5.1.4:

Proof (sketch): By induction on the depth $d \geq 0$ say, of Π .

If $d = 0$, then $A \in \mathfrak{S}$, Π consists only of the root with label A , and the result is trivial.

Suppose the result for all $0 \leq d < N$, and that now $d = N$.

Let $\mathfrak{L}$ denote the set of leaf nodes n of Π , and let $\mathfrak{L}_1$ denote the set of nodes of Π which occur with depth 1, (having both parent nodes in $\mathfrak{L}$). Of course, $\mathfrak{L} \cap \mathfrak{L}_1 = \emptyset$.

For each $n \in \mathfrak{L} \cup \mathfrak{L}_1$, let B_n denote the clause (in Π) at n .

Let $\mathfrak{S}_1$ denote the set $\{B_n : n \in \mathfrak{L}_1\}$.

Pruning the branches of Π in $\mathfrak{L}_1 \cup \mathfrak{L}$, results in a depth $N - 1$ $\text{PC}_{<}$ -deduction Π_1 , say, of A , from $\mathfrak{S}_1 \cup \mathfrak{S}$, in which A , (at the root) is $\text{PC}_{<}$ -initial.

Let $\mathfrak{L}_0$ denote the collection of those leaf nodes of Π which remain (as leaf nodes) in Π_1 .

Of course, $\mathfrak{L}_0 \subseteq \mathfrak{L}$, $\mathfrak{L}_0 \cap \mathfrak{L}_1 = \emptyset$, and (by renaming nodes as necessary, we may assume that),

$\mathcal{L}_0 \cup \mathcal{L}_1$ is the set of leaf nodes of Π_1 .

By induction hypothesis, there is an above-equality $\text{EPC}_{<}$ -deduction Π_1'' of a clause A'' , $\text{EPC}_{<}$ -initial (in Π_1''), where $A'' \vdash_{QR} A$, from $\mathcal{S}_1 \cup \mathcal{S}$, such that $\forall B \in \mathcal{S}_1 \cup \mathcal{S}, k_{\Pi_1''}(B) = k_{\Pi_1}(B)$.

For each $n \in \mathcal{L}_1$, $B_n \in \text{PC}_{<}^1(A_n, D_n)$, for some clauses A_n and D_n of $\mathcal{S}$, (which together label the parent nodes of n in Π).

Let $\mathfrak{T}$ denote the set of triplets $\{\langle B_n, A_n, D_n \rangle \mid n \in \mathcal{L}_1\}$, (for B_n , A_n , and D_n as above).

By Observation 5.1.1, for each $\langle B_n, A_n, D_n \rangle \in \mathfrak{T}$, there exists a clause $B'_n \in \text{EPC}_{<}^1(A_n, D_n)$, such that $B'_n \vdash_{QR} B_n$.

Let $\mathfrak{T}'$ denote the set of triplets $\{\langle B'_n, A_n, D_n \rangle \mid n \in \mathcal{L}_1\}$, (taking one $\langle B'_n, A_n, D_n \rangle$, as above, for each $\langle B_n, A_n, D_n \rangle \in \mathfrak{T}$).

Let $\mathcal{S}_1'$ denote the set $\{B'_n \mid n \in \mathcal{L}_1\}$,

For each node $n \in \mathcal{L}_0$, let B'_n be defined by B_n , (the clause which labels the leaf node n in Π).

By Lemma 5.1.3, there is a $\text{EPC}_{<}$ -deduction Π'_1 , say, of a clause A' , $\text{EPC}_{<}$ -initial (in Π'_1), where $A' \vdash_{QR} A''$, from $\mathcal{S}_1' \cup \mathcal{S}$, such that

$$\forall B' \in \mathcal{S}_1' \cup \mathcal{S}, k_{\Pi'_1}(B') = |\{n \in \mathcal{L}_0 \cup \mathcal{L}_1 \mid B'_n = B'\}|.$$

Since $\vdash_{QR}$ is transitive, (Proposition 5.1.1), $A' \vdash_{QR} A$.

Let $\mathcal{L}'$ denote the set of leaf nodes of Π'_1 , and for each $n' \in \mathcal{L}'$, let $C_{n'}$, say, denote the clause labelling n' in Π'_1 .

The above equality establishes the existence of a bijection $f: \mathcal{L}_0 \cup \mathcal{L}_1 \rightarrow \mathcal{L}'$, between the leaf nodes of Π_1 and Π'_1 , such that for all $n \in \mathcal{L}_0 \cup \mathcal{L}_1$, $C_{f(n)} = B'_n$.

Let $n' \in \mathcal{L}'$, and let n denote $f^{-1}(n')$. Either

- (i) $n \in \mathcal{L}_0$ or
- (ii) $n \in \mathcal{L}_1$.

In case (i), $C_{n'} = B'_n = B_n$, and in case (ii), $C_{n'} = B'_n$, $\langle B'_n, A_n, D_n \rangle \in \mathfrak{T}'$, and $\langle B_n, A_n, D_n \rangle \in \mathfrak{T}$.

Let Π' denote the $\text{EPC}_{<}$ -deduction obtained by gluing to each $n' \in \mathcal{L}'$ such that $n \in \mathcal{L}_1$, where $n = f^{-1}(n')$, the depth 1 $\text{EPC}_{<}$ -deduction $B'_n \in \text{EPC}_{<}^1(A_n, D_n)$.

Evidently Π' is an above-equality $\text{EPC}_{<}$ -deduction of A' , which is $\text{EPC}_{<}$ -initial (in Π'), where $A' \vdash_{QR} A$, from $\mathcal{S}$, such that $\forall C \in \mathcal{S}, k_{\Pi'}(C) = k_{\Pi}(C)$. ■

For Corollary 5.1.1:

Proof: If Π is a $\text{PC}_{<}$ -deduction, in which no equality literal is resolved upon, of an *equality* clause A , then A (at the root) is necessarily $\text{PC}_{<}$ -initial.

Also, if (A is an equality clause, and), $A' \vdash_{QR} A$, for some clause A' , then A' must also be an equality clause.

(Of course, if Π' is any above-equality $\text{EPC}_{<}$ -deduction of an equality clause A' , then A' is necessarily $\text{EPC}_{<}$ -initial in Π' .) ■

D.2. The Equality Fragment, and the Combined Result

For Lemma 5.2.1:

Proof: Since $C \in \text{PC}_{<}^1(A, B)$, either A is (pure-) positive and B is mixed, or B is positive and A is mixed.

If A is (pure-) positive and B is mixed, then for some non-empty subset $A_R \subseteq A$, and some non-empty subset $B_R \subseteq B$, such that B_R contains the left-most inequality literal, $\neq(s_0, t_0)$, say, of B , there is a most general unifier, θ , of $A_R \cup \{=(s', t') \mid \neq(s', t') \in B_R\}$, to (the singleton) $\{(s, t)\}$, say, and C is, up to clause ordering, $A\theta \setminus \{(s, t)\} \cup B\theta \setminus \{\neq(s, t)\}$.

Taking B_S to be the empty set, (we then have $B_S \subseteq B^- \setminus B_R$), and c to be the symmetric choice function on $A_R \cup B_R \cup B_S$, which maps every literal to 0. (we then have $c(\neq(s_0, t_0)) = 0$), θ (as defined above) is then a most general substitution unifying:

$$\begin{aligned} & \{ \{=(s', t') \mid =(s', t') \in A_R; c(=(s', t')) = 0\} \cup \{=(t', s') \mid =(s', t') \in A_R; c(=(s', t')) = 1\} \cup \\ & \quad \{=(s', t') \mid \neq(s', t') \in B_R; c(\neq(s', t')) = 0\} \cup \{=(t', s') \mid \neq(s', t') \in B_R; c(\neq(s', t')) = 1\} \}, \\ & \{ s' \mid =(s', t') \in A_R; c(=(s', t')) = 0 \} \cup \{ t' \mid =(s', t') \in A_R; c(=(s', t')) = 1 \} \cup \\ & \quad \{ s' \mid \neq(s', t') \in B_S; c(\neq(s', t')) = 0 \} \cup \{ t' \mid \neq(s', t') \in B_S; c(\neq(s', t')) = 1 \} \}. \end{aligned}$$

which is simply:

$$\{ \{=(s', t') \mid =(s', t') \in A_R\} \cup \{=(s', t') \mid \neq(s', t') \in B_R\}, \{ s' \mid =(s', t') \in A_R \} \}.$$

and θ unifies the two sets in the above doubleton set to the singletons $\{(s, t)\}$ and $\{s\}$, respectively.

Thus, with $C' \equiv A\theta \setminus \{(s, t), =(t, s)\} \cup B\theta \setminus \{\neq(s, t), \neq(t, s)\}$, $C' \in \text{EPC}_{+,-}(A, B)$. Clearly, $C' \subseteq C$.

Also (as A is positive, $A = QR_{first}(A)$, and as $B \in QR_{first}(B)$), $\text{EPC}_{+,-}(A, B) \subseteq \text{EPC}_{-}(A, B) \subseteq \text{EPC}_{<}^1(A, B)$, so $C' \in \text{EPC}_{<}^1(A, B)$.

The case when B is positive and A is mixed is similar, (the major difference being only that now, $C' \in \text{EPC}_{+,-}(B, A)$). ■

For Theorem 5.2.1:

Proof (sketch): Since S is unsatisfiable, and $\text{PC}_{<}$ is refutation complete, (Theorem 2.3.1), there is a $\text{PC}_{<}$ -deduction Π , say, of the empty clause, $\square$, from S , where, we may assume, ' $=$ ' is minimal with respect to the predicate ordering relation, $<$.

For each branch β of Π , let E_β , denote the upper-most (when, in Π , the empty clause is at the bottom of, and the leaves are upper-most), (possibly empty), equality clause occurring along β , (so that E_β contains only positive and negative instances of the equality predicate, ' $=$ ', while any proper ancestor of E_β in Π , contains at least one non-equality literal).

Let $\mathfrak{E}$ denote the collection of all the E_β .

Evidently, either $\mathfrak{E}$ consists only of the empty clause, $\square$, (and this is the case precisely when no equality literals are resolved upon, or even occur, in Π), or $\square \notin \mathfrak{E} \neq \emptyset$, and $\mathfrak{E}$ contains at least two members.

In any case, (since $\text{PC}_{<}$ -deduction is sound), $\mathfrak{E}$ is an unsatisfiable set of equality clauses.

For each clause $E \in \mathfrak{E}$, let Π_E denote a $\text{PC}_{<}$ -deduction of E from S , which occurs as a fragment of Π , (this fragment rooted at a node labelled by E).

By Corollary 5.1.1, for each $E \in \mathfrak{E}$, there is an equality clause E' such that $E' \vdash_{QR} E$, and an above-equality $\text{EPC}_{<}$ -deduction $\Pi'_{E'}$ of E' , from S , such that $\forall C \in S, k_{\Pi'_{E'}}(C) = k_{\Pi_E}(C)$.

Let $\mathfrak{E}'$ denote the collection $\{E' \mid E \in \mathfrak{E}\}$.

As $\mathfrak{E}$ is unsatisfiable, evidently $\mathfrak{E}' \cup \{=(x, x)\}$ is also unsatisfiable.

Again, as $\text{PC}_{<}$ is refutation complete, there is a $\text{PC}_{<}$ -refutation Λ say, of $\mathfrak{E}' \cup \{=(x, x)\}$.

But then by Lemma 5.2.1. and Theorem 2.5.1. (which asserts the refutation completeness of $\text{PC}_{<}$ under subsumption deletion), it follows that there is a $\text{EPC}_{<}$ -refutation, Λ' , say, of $\mathfrak{E}' \cup \{=(x, x)\}$.

By gluing the $\text{EPC}_{<}$ -deductions $\Pi'_{E'}$, onto Λ' , as appropriate, (at each leaf node, n , of Λ' , which is labelled by a clause $E' \in \mathfrak{E}' \setminus S$, $\Pi'_{E'}$ is affixed), evidently we obtain a $\text{EPC}_{<}$ -refutation of $S \cup \{=(x, x)\}$, as required. ■

APPENDIX E

Σ_0 -subsumption and $\text{EPC}_{<}$

E.1. Σ_0 -subsumption in the Above-Equality Fragment

For Lemma 6.1.1:

Proof: Let A , B , C , and D , as in the statement of the lemma, (and assume that the variables of A , ($\text{Vars}(A)$) are disjoint from those of C , as well as from those of D , which we can clearly do by Observation 4.1.6 and Observation 3.2.3).

As D is pure-positive with respect to P , (with $<$ -maximal predicate symbol $P \neq '=')$, A has $<$ -maximal predicate symbol P and is mixed with respect to P . As $C \Sigma_0$ -subsumes D , there is a substitution θ , and a trivial inequality clause F , such that (the domain of θ is a subset of the set of variables occurring in C , and) $C\theta \subseteq \text{cl}(D) \cup F$. Clearly C must also be pure-positive with respect to P , and have $<$ -maximal predicate symbol less than or equal to P .

For some non-empty subsets $A_R \subseteq A$, and $D_R \subseteq D$, (say), the left-most negative P instance, L , say, of A occurring in A_R , (and some term tuples $\vec{s}_1, \dots, \vec{s}_i$ and $\vec{t}_1, \dots, \vec{t}_j$),

$$A_R \equiv \{\neg P(\vec{s}_1), \dots, \neg P(\vec{s}_i)\}; D_R \equiv \{P(\vec{t}_1), \dots, P(\vec{t}_j)\}, \text{ and}$$

$$B \equiv \bigcup_{\substack{1 \leq i' \leq i \\ 1 \leq j' \leq j}} \neg P(\vec{s}_{i'}, \vec{t}_{j'}) \cup A \setminus A_R \cup D \setminus D_R.$$

Now evidently, modulo symmetry,

$$\begin{aligned} \text{cl}(D) &= \mathcal{T}_{D^-}(D_R) \cup \mathcal{T}_{D^-}(D \setminus D_R) \cup \Delta(D^-, D^0) \\ &= \mathcal{T}_{D^-}(D_R) \cup \mathcal{T}_{D^-}(D \setminus D_R) \cup \Delta(D^-, D^0 \setminus D_R) \cup \Delta(D^-, D^0 \cap D_R) \cup \\ &\quad \bigcup \{\Delta(D^-, \{L, L'\}) : L \in D^0 \setminus D_R; L' \in D_R\}. \end{aligned}$$

But since D_R consists only of positive P literals, and D is pure-positive with respect to P ,

$$\Delta(D^-, D^0 \cap D_R) \cup \bigcup \{\Delta(D^-, \{L, L'\}) : L \in D^0 \setminus D_R; L' \in D_R\} = \emptyset.$$

Thus, (modulo symmetry), $\text{cl}(D) = \mathcal{T}_{D^-}(D_R) \cup \mathcal{T}_{D^-}(D \setminus D_R) \cup \Delta(D^-, D^0 \setminus D_R)$.

Since, (modulo symmetry), $\text{cl}(D \setminus D_R) = \mathcal{T}_{D^-}(D \setminus D_R) \cup \Delta(D^-, D^0 \setminus D_R)$, (as $(D \setminus D_R)^0 = D^0 \setminus D_R$), clearly it follows that if $L \in C\theta \setminus \mathcal{T}_{D^-}(D_R)$ then $L \in \text{cl}(D \setminus D_R) \cup F$.

Thus if $C\theta \cap \mathcal{T}_{D^-}(D_R) = \emptyset$, $C \Sigma_0$ -subsumes $D \setminus D_R$, and hence $C \Sigma_0$ -subsumes B .

If $C\theta \cap \mathcal{T}_{D^-}(D_R) \neq \emptyset$, let $C_R \equiv \{L \in C : L\theta \in \mathcal{T}_{D^-}(D_R)\}$, and let $C_R \equiv \{P(\vec{r}_1), \dots, P(\vec{r}_k)\}$, say.

Clearly if we let $B' \equiv \bigcup_{\substack{1 \leq i' \leq i \\ 1 \leq k' \leq k}} \neq (\vec{s}_{i'}, \vec{r}_{k'}) \cup A \setminus A_R \cup C \setminus C_R$, then $B' \in \text{EPC}_<^1(A, C)$.

We claim that $B' \Sigma_0$ -subsumes B .

(Since, by assumption, $\text{Vars}(A) \cap \text{Vars}(C) = \emptyset$, and $\text{domain}(\theta) \subseteq \text{Vars}(C)$),

$$B'\theta \equiv \bigcup_{\substack{1 \leq i' \leq i \\ 1 \leq k' \leq k}} \neq (\vec{s}_{i'}, \vec{r}_{k'}\theta) \cup A \setminus A_R \cup (C \setminus C_R)\theta.$$

By the choice of C_R , for all $k': 1 \leq k' \leq k$, there exists $j': 1 \leq j' \leq j$ such that $\vec{r}_{k'}\theta$ is $\mathcal{T}_D$ -equivalent to $\vec{t}_{j'}$, and for each literal $L \in (C \setminus C_R)\theta$, $L \in \text{cl}(D \setminus D_R) \cup F$. As $D^- \subseteq D \setminus D_R \subseteq B$, it follows that ($\text{cl}(D \setminus D_R) \subseteq \text{cl}(B)$ and) $B'\theta \subseteq \text{cl}(B) \cup F$, and hence, that $B' \Sigma_0$ -subsumes B . ■

For Lemma 6.1.2:

Proof (sketch): We prove the result by induction on the depth k of Π .

Now for some $k \geq 1$, $D \equiv D_k$, $D_0 \equiv B$, for some clauses $C_0, \dots, C_{k-1} \in \mathcal{C}$, and $D_1, \dots, D_{k-1}$, such that $\forall i: 0 \leq i \leq k-1$, ($\text{Vars}(C_i) \cap \text{Vars}(D_0) = \emptyset$ and), $D_{i+1} \in \text{EPC}_<^1(D_i, C_i)$.

In particular, we have ($\text{Vars}(C_0) \cap \text{Vars}(B) = \emptyset$ and), $D_1 \equiv \bigcup_{\substack{1 \leq i' \leq i \\ 1 \leq j' \leq j}} \neq (\vec{s}_{i'}, \vec{t}_{j'}) \cup B \setminus B_R \cup C_0 \setminus C_{0R}$.

for some non-empty subsets $C_{0R} \subseteq C_0$, and $B_R \subseteq B$, (say), the left-most negative P instance of B occurring in B_R , where, (for some term tuples $\vec{s}_1, \dots, \vec{s}_i$ and $\vec{t}_1, \dots, \vec{t}_j$), $B_R \equiv \{\neg P(\vec{s}_1), \dots, \neg P(\vec{s}_i)\}$, and $C_{0R} \equiv \{P(\vec{t}_1), \dots, P(\vec{t}_j)\}$.

Since $B' \Sigma_0$ -subsumes B , there exists a substitution θ ,

(which we may assume has domain in $\text{Vars}(B)$, and so $\forall x \in \text{Vars}(C_0)$, $\theta: x \mapsto x$),

and a trivial inequality clause F , such that $B'\theta \subseteq \text{cl}_0(B) \cup F$.

That is, modulo symmetry, $B'\theta \subseteq \mathcal{T}_{B-}(B) \cup F$.

Evidently $\mathcal{T}_{B-}(B) = \mathcal{T}_{B-}(B_R) \cup \mathcal{T}_{B-}(B \setminus B_R)$.

Also, (as no inequality literals of, or inherited from, B are resolved away in the formation of D), evidently $B^- \subseteq D_1^-$. Now (for any $k \geq 1$), either:

- A. $B'\theta \cap \mathcal{T}_{B-}(B_R) = \emptyset$, or
- B. $B'\theta \cap \mathcal{T}_{B-}(B_R) \neq \emptyset$.

Case A: Suppose $B'\theta \cap \mathcal{T}_{B-}(B_R) = \emptyset$.

Then, modulo symmetry, $B'\theta \subseteq \mathcal{T}_{B-}(B \setminus B_R) \cup F \subseteq \mathcal{T}_{B-}(D_1) \cup F \subseteq \mathcal{T}_{D_1-}(D_1) \cup F$, and hence $B' \Sigma_0$ -subsumes D_1 .

Case B: Otherwise, let $B'_R \equiv \{L \in B': L\theta \in \mathcal{T}_{B-}(B_R)\}$, and for $B'_R \equiv \{\neg P(\vec{s}'_1), \dots, \neg P(\vec{s}'_k)\}$, say, let

$$D'_1 \equiv \bigcup_{\substack{1 \leq k' \leq k \\ 1 \leq j' \leq j}} \neq (\vec{s}'_{k'}, \vec{t}_{j'}) \cup B' \setminus B'_R \cup C_0 \setminus C_{0R}.$$

It is easy to see that $D'_1 \Sigma_0$ -subsumes D_1 ,

(where D'_1 subsumes $\bigcup_{\substack{1 \leq k' \leq k \\ 1 \leq j' \leq j}} \neq (\vec{s}'_{k'}, \vec{t}_{j'}) \cup (B' \setminus B'_R)\theta \cup (C_0 \setminus C_{0R})$, which in turn subsumes

D_1 , since each $\vec{s}'_{k'}\theta$ is $\mathcal{T}_{B-}$ -equivalent to some $\vec{s}_{i'}$, and, modulo symmetry, $(B' \setminus B'_R)\theta \subseteq \mathcal{T}_{B-}(B \setminus B_R) \cup F \subseteq \mathcal{T}_{D_1-}(D_1) \cup F$).

Suppose $k = 1$.

In case (A), since $D_1 \equiv D$, clearly (i) holds, and we are done.

In case (B), as $D_1 \equiv D$ is pure-positive with respect to P , by assumption, (and $D'_1 \Sigma_0$ -subsumes $_0 D_1$), it follows that D'_1 must also be pure-positive with respect to P , and the left-most literal of B' which is a negative instance of P must lie in B'_R . Thus, the above $\text{EPC}_{<}^+$ -resolution of B' with C_0 to yield D'_1 is a (depth 1) $\text{EPC}_{<}^+$ -deduction Π' , (say), from $\{B'\} \cup \mathcal{C}$.

Clearly, $\forall C \in \mathcal{C} \ k_{\Pi'}(C) \leq k_{\Pi}(C)$, (and so (ii) holds), and we're done.

Now suppose the result for all $1 \leq k' < k$, and that now Π has depth $k > 1$.

Evidently ($D \equiv D_k \neq D_1$, and) the fragment, Δ , say, of Π , obtained by pruning Π at D_1 (pruning off the top two leaves, labelled by B and C_0) is an $\text{EPC}_{<}^+$ -deduction of D from $\{D_1\} \cup \mathcal{C}$, with depth $1 \leq k - 1 < k$.

In case (A), since $B' \Sigma_0$ -subsumes $_0 D_1$, by hypothesis, either:

- A.i. $B' \Sigma_0$ -subsumes $_0 D$, or
- A.ii. there is an $\text{EPC}_{<}^+$ -deduction Δ' of some clause D' from $\{B'\} \cup \mathcal{C}$, such that $D' \Sigma_0$ -subsumes $_0 D$, and such that further, $\forall C \in \mathcal{C}, k_{\Delta'}(C) \leq k_{\Delta}(C)$.

Case (A.i) directly yields (i), while case (A.ii), recalling the construction of Δ from Π , clearly yields (ii), (and we're done).

In case (B), since $D'_1 \Sigma_0$ -subsumes $_0 D_1$, by the induction hypothesis, either:

- B.i. $D'_1 \Sigma_0$ -subsumes $_0 D$, or
- B.ii. there is an $\text{EPC}_{<}^+$ -deduction Δ' of some clause D' from $\{D'_1\} \cup \mathcal{C}$, such that $D' \Sigma_0$ -subsumes $_0 D$, and such that further, $\forall C \in \mathcal{C}, k_{\Delta'}(C) \leq k_{\Delta}(C)$.

In case (B.i), just as in case (B) for when $k = 1$ (above), the given $\text{EPC}_{<}^+$ -resolution of B' with C_0 to yield D'_1 is a (depth 1) $\text{EPC}_{<}^+$ -deduction Π' , from $\{B'\} \cup \mathcal{C}$, where $\forall C \in \mathcal{C} \ k_{\Pi'}(C) \leq k_{\Pi}(C)$. Thus, again ((ii) holds and), we're done.

Case (B.ii) is the only situation where the "left-most negative P-instance" restriction must factor into our argument.

If $D'_1 \notin \text{EPC}_{<}^+(B', C'_0)$, let B'' denote any permutation of B' such that if L denotes the left-most literal in B'' which is a negative instance of the predicate P , then $L\theta \in T_{B^-}(B_R)$.

Now, (taking $B''_R \equiv B'_R$, and D'_1 as before), $D'_1 \in \text{EPC}_{<}^+(B'', C_0)$, and (recalling the construction of Δ from Π), the $\text{EPC}_{<}^+$ -deduction Δ' of (B.ii) immediately yields an $\text{EPC}_{<}^+$ -deduction Π'' , (say), of D' from $\{B''\} \cup \mathcal{C}$, such that $D' \Sigma_0$ -subsumes $_0 D$, and such that further, $\forall C \in \mathcal{C}, k_{\Pi''}(C) \leq k_{\Pi}(C)$.

By Lemma 4.2.1, (since B'' is a re-ordering of B'), therefore, there is a $\text{EPC}_{<}^{+1}$ -deduction Π' , of D' from $\{B'\} \cup \mathcal{C}$, such that $D' \Sigma_0\text{-subsumes}_0 D$, and such that further, $\forall C \in \mathcal{C}$, $k_{\Pi'}(C) \leq k_{\Pi}(C)$, (that is, (ii) holds), and we are now indeed done. ■

For Lemma 6.1.3:

Proof: We prove the result by induction on the total number of leaf nodes, $\kappa(\Pi) \equiv \kappa$, say, of Π .

If $\kappa = 1$, then Π consists of a single node labelled by $E \equiv D$, and D must be an equality clause. Since $C \Sigma_0\text{-subsumes}$ D , C must also be an equality clause. By the soundness of Σ_0 -subsumption. (Theorem 3.3.4), $\Sigma_0 \cup \{C\} \models D$. Setting $\mathfrak{F} = \{\Pi'\}$, where Π' consists of a single node labelled by C , clearly suffices.

Suppose the result is true for all κ : $1 \leq \kappa \leq K$, and that now $\kappa = K$. Let m denote a leaf node of Π labelled by D . Let n denote the deduction-ordering maximal node of Π for which m is an ancestor, and the clause C_n , labelling n , is such that $C \Sigma_0\text{-subsumes}$ C_n .

If $n \neq m$, then by letting Π' denote the result of pruning Π at n , (so that n , labelled by C_n , is now a leaf of Π'), and by applying the induction hypothesis, clearly we obtain an $\mathfrak{F}$ as required. Thus we may assume the clause label of no (proper) descendent of m is Σ_0 -subsume'd by C .

Since $C \Sigma_0\text{-subsumes}$ D , there is a substitution θ , and a trivial inequality clause F , such that, modulo symmetry,

$$C\theta \subseteq cl_0(D) \cup \Delta(D^-, D^0) \cup F.$$

We now use induction on the number of (positive equality) literals, $\epsilon(C, D) \equiv \epsilon$, say, in $C\theta \cap \Delta(D^-, D^0) \setminus cl_0(D)$, where, here, the set difference is supposed to disregard the distinction between literals which are equal modulo symmetry.

Suppose $\epsilon = 0$. Then, modulo symmetry, $C\theta \subseteq cl_0(D) \cup F$. Thus $C \Sigma_0\text{-subsumes}_0 D$.

Let P_D, P_C , denote the $<$ -maximal predicate symbol in D, C , respectively. $P_D \neq '='$, for otherwise D is an equality clause, and $\kappa = 1$, a contradiction. As $C \Sigma_0\text{-subsumes}$ D , $P_C \leq P_D$.

If $P_C < P_D$, let p denote the deduction-ordering minimal node, such that m is an (proper) ancestor of p , and the clause label C_p of p in Π , has $<$ -maximal predicate symbol P_C .

It is easy to see that the only literals resolved upon in the formation of C_p (from D), must have $<$ -maximal predicate symbol greater than P_C , and for some substitution θ' , $C\theta' \subseteq cl_0(C_p) \cup F'$, modulo symmetry, for some trivial inequality clause F' . Thus $C \Sigma_0\text{-subsumes}$ C_p . This contradicts the assumption that no proper descendent of m has its label Σ_0 -subsume'd by C .

Thus we may restrict our attention to the case (for $\kappa = K$, and $\epsilon = 0$) when $'=' \neq P_C = P_D = P$, say. Either:

- A) D is pure-positive with respect to P , or,
- B) D is mixed with respect to P .

In case A):

Suppose that in Π , D , at m , is resolved against A , (say), at m' , to obtain the resolvent B , at m_1 , say. Then A must also have $<$ -maximal predicate symbol P , and be mixed with respect to P .

By Lemma 6.1.1, either C Σ_0 -subsumes B , or $\exists B' \in \text{EPC}_{<}^1(A, C)$ such that B' Σ_0 -subsumes B . Since the possibility that C Σ_0 -subsumes B contradicts the assumption that no proper descendent of m has its label Σ_0 -subsume'd by C , the latter is true.

Let Π^{m_1} denote the result of pruning Π at m_1 , so that now m_1 , with label B , is a leaf of Π^{m_1} .

By applying the induction hypothesis, (for B' , B , and Π^{m_1} , in place of C , D , and Π , respectively), evidently we obtain an $\mathfrak{F}'$, such that by affixing the depth 1 $\text{EPC}_{<}$ -deduction of B' from A and C , to some single leaf node labelled by B' (if such exists) in Π'' , for each $\Pi'' \in \mathfrak{F}'$, we obtain an $\mathfrak{F}$ as required.

In case B):

Let m_1 denote an $\text{EPC}_{<}$ -initial node in Π , with label B , say, such that m_1 is the root of an $\text{EPC}_{<}^{+1}$ -deduction Γ , say, (which occurs as a fragment of Π), of B , from principal ancestor D (at m), and ancillary parents, $C_1, C_2, \dots, C_k$, say, (each pure-positive with respect to P), at nodes $n_1, n_2, \dots, n_k$, (respectively), say, (for some integer $k \geq 1$). Let Π_{m_1} denote the fragment of Π which is rooted at m_1 and is an $\text{EPC}_{<}$ -deduction of B , and for each $i: 1 \leq i \leq k$,

let Π_{n_i} denote the fragment of Π which is rooted at n_i and is an $\text{EPC}_{<}$ -deduction of C_i .

(Thus Π_{m_1} is composed of Γ , together with the Π_{n_i}).

Again, let Π^{m_1} denote the result of pruning Π at m_1 , so that now m_1 , with label B , is a leaf of Π^{m_1} . Let $\mathfrak{C} \equiv \{C_1, \dots, C_k\}$.

By Lemma 6.1.2, either C Σ_0 -subsumes $_0$ B , (and hence C Σ_0 -subsumes B), or there is an $\text{EPC}_{<}^{+1}$ -deduction Γ' of a clause B' , from C and $\mathfrak{C}$, such that B' Σ_0 -subsumes $_0$ B , and such that $\forall C \in \mathfrak{C}, \kappa_{\Gamma'}(C) \leq \kappa_{\Gamma}(C)$.

But the former possibility again contradicts the assumption that no proper descendent of m has its label Σ_0 -subsume'd by C . Thus there is an $\text{EPC}_{<}^{+1}$ -deduction Γ' of a clause B' , from (principal ancestor) C and (ancillary ancestors in) $\mathfrak{C}$, such that B' Σ_0 -subsumes $_0$ B , and such that $\forall C \in \mathfrak{C}, \kappa_{\Gamma'}(C) \leq \kappa_{\Gamma}(C)$.

Let Π'_{m_1} denote the $\text{EPC}_{<}$ -deduction of B' , which results from affixing to each ancillary leaf, (of Γ' which is labelled by a), $C' \in \mathfrak{C}$, some Π_{n_i} , as appropriate, so that for any clause C' , $\kappa_{\Pi'_{m_1}}(C') \leq \kappa_{\Pi_{m_1}}(C')$.

Again, by applying the induction hypothesis,

(for B' and B , and Π^{m_1} , in place of C , D , and Π , respectively),

evidently we obtain an $\mathfrak{F}'$, such that by affixing Π'_{m_1} to some single leaf node labelled by B' (if such exists) in Π'' , for each $\Pi'' \in \mathfrak{F}'$, we obtain an $\mathfrak{F}$ as required.

Now suppose that we have proved the result for all C , D , and Π , as in the Lemma's hypothesis, such that (either $\kappa(\Pi) < K$, or) $\kappa(\Pi) = K$, and $0 \leq \epsilon(C, D) < \nu$, (for some $\nu \geq 1$), and that now, C , D and Π are as in the Lemma's hypothesis, D labels a leaf node $m \in \Pi$, $\kappa(\Pi) = K$,

and $\epsilon \equiv \epsilon(C, D) = \nu$.

Here we may also assume that C does not subsume D , since otherwise, (clearly), $\epsilon = 0$. Recall that we may assume that the label of no (proper) descendent of m is Σ_0 -subsume'd by C .

Let ρ be a substitution which assigns to each variable x in E the formal constant (or "frozen variable"), $\bar{x}$. Then evidently, $E\rho = \bar{E}$ and $E\rho$ is ground.

$\downarrow(\Pi, \rho)$, (of Definition 4.3.1), is an above-equality $\text{EPC}_{<}$ -deduction of a clause E' , where $E' \subseteq E\rho$, such that $\kappa(\downarrow(\Pi, \rho)) \leq K$.

I. Suppose that for no substitution instance, $D\gamma$, of D , is it the case that $k_{\downarrow(\Pi, \rho)}(D\gamma) \geq 1$.

Then, (as every leaf clause of $\downarrow(\Pi, \rho)$ is a substitution instance of some leaf clause of Π), evidently, by Corollary 4.2.1, $\downarrow(\Pi, \rho)$ can be lifted to a (above-equality $\text{EPC}_{<}$ -deduction), Π' , say, such that taking $\mathfrak{F} \equiv \{\Pi'\}$ suffices, (for a), b) and c) of this Lemma).

More syntactically:

Let $\mathfrak{L}$ denote the set of leaf nodes of Π , and let $B: n \in \mathfrak{L} \mapsto B_n$, such that $\forall n \in \mathfrak{L}$, B_n is the clause labelling n in Π , (so, in particular, $B_m = D$).

Then, (as guaranteed by Corollary 4.3.2), there is a mapping $B': n \in \mathfrak{L} \mapsto B'_n$, such that $\forall n \in \mathfrak{L}$, B'_n is a ground substitution instance of B_n , and further, $\forall B'$, $k_{\downarrow(\Pi, \rho)}(B') \leq ||\{n \in \mathfrak{L} \mid B'_n = B'\}||$.

If for no substitution instance, $D\gamma$, of D , is it the case that $k_{\downarrow(\Pi, \rho)}(D\gamma) \geq 1$, then, in particular, $k_{\downarrow(\Pi, \rho)}(B'_m) = 0$, and clearly $\kappa(\downarrow(\Pi, \rho)) < \kappa(\Pi) = K$.

More over, if $\mathfrak{L}'$ denotes the set of leaf nodes of $\downarrow(\Pi, \rho)$, and for each $n' \in \mathfrak{L}'$, $G_{n'}$ denotes the clause which labels the leaf n' in $\downarrow(\Pi, \rho)$, then, evidently, there is an injective function $f: \mathfrak{L}' \rightarrow \mathfrak{L} \setminus \{m\}$, such that, $\forall n' \in \mathfrak{L}'$, $G_{n'} = B'_{f(n')}$.

Thus, $\forall n' \in \mathfrak{L}'$, $B_{f(n')}$, (the clause which labels $f(n')$ in Π), is a clause of which $G_{n'}$ is a substitution instance.

By Corollary 4.2.1, there is a lifting, Π' , of $\downarrow(\Pi, \rho)$, such that Π' is an above-equality $\text{EPC}_{<}$ -deduction of an equality clause E'' of which E' is a substitution instance, and such that

$$\begin{aligned} \forall A, \quad k_{\Pi''}(A) &= ||\{n' \in \mathfrak{L}' \mid A = B_{f(n')}\}|| \\ &\leq ||\{n \in \mathfrak{L} \setminus \{m\} \mid A = B_n\}|| \quad (\text{since } f \text{ is injective}) \\ &\leq k_{\Pi}(A), \end{aligned}$$

while, in particular,

$$\begin{aligned} k_{\Pi''}(D) &= ||\{n' \in \mathfrak{L}' \mid D = B_{f(n')}\}|| \\ &\leq ||\{n \in \mathfrak{L} \setminus \{m\} \mid D = B_n\}|| \\ &\leq k_{\Pi}(D) - 1, \quad (\text{since } B_m = D). \end{aligned}$$

As E' is a substitution instance of E'' , and $E' \subseteq E\rho = \bar{E}$, clearly, for some substitution θ , $E''\theta \subseteq \bar{E}$. Since (evidently), E'' cannot contain any formally ground variables, E'' subsumes E , and so $\{E''\} \models E$.

Thus, if for no (ground) substitution instance, $D\gamma$, of D , is it the case that $k_{\mathbf{I}(\Pi, \rho)}(D\gamma) \geq 1$, then taking $\mathfrak{F} \equiv \{\Pi'\}$, (Π' as above), evidently suffices, (for a), b) and c) of this Lemma).

Thus we may assume that:

II. there is some (ground) substitution instance, $D\gamma$, of D , such that $k_{\mathbf{I}(\Pi, \rho)}(D\gamma) \geq 1$.

Since $C \Sigma_0$ -subsumes D , for some (ground) substitution instance $C\varphi$ of C , $C\varphi \subseteq cl(D\gamma) \cup F$, for some ground set of trivial inequalities F , (and so $C\varphi \Sigma_0$ -subsumes $D\gamma$).

If $\kappa(\downarrow(\Pi, \rho)) < K$, or if $\epsilon(C\varphi, D\gamma) < \nu$, we may apply the induction hypothesis, (on κ , or on ϵ , for $\kappa = K$), to infer the existence of a finite family $\mathfrak{F}'$ of $\text{EPC}_{<}$ -deductions Π'' , where:

- a') $\forall \Pi'' \in \mathfrak{F}'$, Π'' contains exactly one (necessarily here, ground) equality clause, $E_{\Pi''}$, say, (that at the root of Π''), and
- b') $\Sigma_0 \cup \{E_{\Pi''} : \Pi'' \in \mathfrak{F}'\} \models E'$, and
- c') $\forall \Pi'' \in \mathfrak{F}'$,
 - i) $\forall C' \notin \{C\varphi, D\gamma\}$, $k_{\Pi''}(C') \leq k_{\mathbf{I}(\Pi, \rho)}(C')$;
 - ii) $k_{\Pi''}(C\varphi) \leq k_{\mathbf{I}(\Pi, \rho)}(C\varphi) + 1$; and
 - iii) $k_{\Pi''}(D\gamma) \leq k_{\mathbf{I}(\Pi, \rho)}(D\gamma) - 1$.

Note that as $E' \subseteq E\rho = \bar{E}$, if $\Sigma_0 \cup \{E_{\Pi''} : \Pi'' \in \mathfrak{F}'\} \models E'$, then $\Sigma_0 \cup \{E_{\Pi''} : \Pi'' \in \mathfrak{F}'\} \models \bar{E}$.

Also note that $\Sigma_0 \cup \{E_{\Pi''} : \Pi'' \in \mathfrak{F}'\} \models \bar{E}$ if and only if $\Sigma_0 \cup \neg\bar{E} \cup \{E_{\Pi''} : \Pi'' \in \mathfrak{F}'\} \models \perp$, which (in turn) implies $\Sigma_0 \cup \neg E \cup \{E_{\Pi''} : \Pi'' \in \mathfrak{F}'\} \models \perp$, which implies $\Sigma_0 \cup \{E_{\Pi''} : \Pi'' \in \mathfrak{F}'\} \models E$.

Thus, by lifting, (by Corollary 4.2.1), it is not hard to check that), evidently we may obtain from $\mathfrak{F}'$, a finite family $\mathfrak{F}$ of $\text{EPC}_{<}$ -deductions Π , satisfying a), b), and c), as required.

Otherwise, suppose $\kappa(\downarrow(\Pi, \rho)) = K$ and $\epsilon(C\varphi, D\gamma) = \nu$.

Let $\langle a, b \rangle \in C\varphi \cap \Delta(D^{-}\gamma, D^0\gamma) \setminus cl_0(D\gamma)$;

and let $P(\vec{a})$, and $\neg P(\vec{b}) \in D\gamma$, for some tuples $\vec{a} \equiv a_1, \dots, a_J$; $\vec{b} \equiv b_1, \dots, b_J$, (where $1 \leq J \in \mathbb{N}$ is the arity of $P \neq '='$), such that $\forall j: 1 \leq j \leq J$, either:

- i. $a_j = b_j$, or
- ii. $\{a_j, b_j\} \subseteq \alpha$, for some $\alpha \in \mathcal{T}_{D\gamma^-}$, or
- iii. $\{a_j, b_j\} = \{a, b\}$,

and such that iii. occurs, for at least one $j: 1 \leq j \leq J$, say j_0 .

Let $D' \equiv P(\vec{a}) \mid \neg P(\vec{b}) \mid D\gamma^-$. Then $D' \subseteq D\gamma$, so D' subsumes $D\gamma$.

If $D' \neq D\gamma$, by Corollary 4.3.1, there exists a (here, necessarily ground), above-equality $\text{EPC}_{<}$ -deduction Π_0 of an equality clause E_0 such that E_0 subsumes E' , (and so $E_0 \subseteq E' \subseteq \bar{E}$), and such that (!):

- a) $\forall C' \notin \{D', D\gamma\}$, $k_{\Pi_0}(C') \leq k_{\mathbf{I}(\Pi, \rho)}(C')$;
- b) $k_{\Pi_0}(D') \leq k_{\mathbf{I}(\Pi, \rho)}(D') + 1$, and

$$c) \quad k_{\Pi_0}(D\gamma) \leq k_{\downarrow(\Pi, \rho)}(D\gamma) - 1.$$

Otherwise, (if $D' = D\gamma$), let $\Pi_0 \equiv \downarrow(\Pi, \rho)$, and $E_0 \equiv E'$.

In this case, obviously, $\forall C'$, $k_{\Pi_0}(C') = k_{\downarrow(\Pi, \rho)}(C')$.

Clearly (in either case), $\kappa(\Pi_0) \leq \kappa(\downarrow(\Pi, \rho)) = K$.

If $k_{\Pi_0}(D') = 0$, or if $C\varphi = D'$, then evidently we can lift, (by Corollary 4.2.1), Π_0 , to obtain a Π' , such that $\mathfrak{F} \equiv \{\Pi'\}$ satisfies the stated requirements.

Otherwise, let m_0 denote a node of Π_0 labelled by D' .

Let n_0 , labelled by C_{n_0} , say, denote the co-parent node of m_0 (labelled by D') in Π_0 , with offspring node n_0' , with label $C_{n_0'}$, say. Let m , with label C_m , denote the first node below, (in Π_0), (the deduction-ordering maximal node which is a descendent of) n_0' , for which the literal $P(\vec{a})$, inherited from D' , is being resolved upon, with co-parent clause C_l , (at node l , say), and resolvent $C_{m'}$, (at m' , say).

As C_{n_0} is resolved against D' , (in Π_0), C_{n_0} must be of the form $G \cup \{P(\vec{b}'_{i'}) : 1 \leq i' \leq i\}$, (for some $i \geq 1$), where the $<$ -maximal predicate symbol of G is less than or equal to P , and G is pure-positive with respect to P . Also, we may assume $G \cap \{P(\vec{b}'_{i'}) : 1 \leq i' \leq i\} = \emptyset$, so $C_{n_0} \neq G$. The offspring, $C_{n_0'}$, must be of the form $D\gamma^- \mid G \mid P(\vec{a}) \mid I'$, where $I' \equiv \bigcup \{\neq(\vec{b}'_{i'}, \vec{b}) : 1 \leq i' \leq i\}$. Note, evidently we may assume $G \neq C_{n_0'}$, and that $C_{n_0'} \neq C_{n_0}$.

C_m must be of the form $G' \mid P(\vec{a}) \cup \{P(\vec{a}'_{j'}) : 1 \leq j' \leq j\} \mid D\gamma^- \mid I'$, (for some $j \geq 0$), pure-positive with respect to P . P $<$ -maximal in C_m , (of course, new inequality literals, et cetera, may appear in G' , but assume $P(\vec{a}) \notin G'$, and $G' \cap \{P(\vec{a}'_{j'}) : 1 \leq j' \leq j\} = \emptyset$). Evidently, we may also assume $C_m \neq C_{n_0}$.

C_l must be of the form $\{\neg P(\vec{a}''_{k'}) : 1 \leq k' \leq k\} \mid K'$, (for some $k \geq 1$), with $<$ -maximal predicate symbol P , the left-most negative instance of P in C_l occurring in $\{\neg P(\vec{a}''_{k'}) : 1 \leq k' \leq k\}$, (and for some K' such that $\{\neg P(\vec{a}''_{k'}) : 1 \leq k' \leq k\} \cap K' = \emptyset$), and $C_{m'}$ must be of the form $K' \mid G' \mid D\gamma^- \mid I' \mid I''$, where $I'' \equiv \bigcup \{\neq(\vec{a}''_{k'}, \vec{a}) : 1 \leq k' \leq k\} \cup \bigcup \{\neq(\vec{a}''_{k'}, \vec{a}'_{j'}) : 1 \leq k' \leq k, 1 \leq j' \leq j\}$.

Let $\Gamma_{C_{n_0}}$ be the fragment of Π_0 which is the $\text{EPC}_{<}$ -deduction of C_{n_0} , ($\Gamma_{C_{n_0}}$ rooted at n_0), and similarly let Γ_{C_l} denote the fragment of Π_0 , (rooted at l), which is the $\text{EPC}_{<}$ -deduction of C_l . Let Γ denote the fragment of Π_0 , which is rooted at m , and pruned at n_0' , so that Γ "takes $C_{n_0'}$ to C_m ".

The situation is given pictorially in Figure E.1.
where, (modulo clause ordering):

$$\begin{aligned} D' &\equiv \neg P(\vec{b}) \mid P(\vec{a}) \mid D\gamma^- \\ C_{n_0} &\equiv G \cup \{P(\vec{b}'_{i'}) : 1 \leq i' \leq i\} \\ C_{n_0'} &\equiv G \cup \{P(\vec{a})\} \cup D\gamma^- \mid I', \text{ (where } I' \equiv \bigcup \{\neq(\vec{b}'_{i'}, \vec{b}) : 1 \leq i' \leq i\} \text{)} \\ C_m &\equiv G' \mid P(\vec{a}) \cup \{P(\vec{a}'_{j'}) : 1 \leq j' \leq j\} \mid D\gamma^- \mid I' \\ C_l &\equiv \{\neg P(\vec{a}''_{k'}) : 1 \leq k' \leq k\} \mid K' \end{aligned}$$

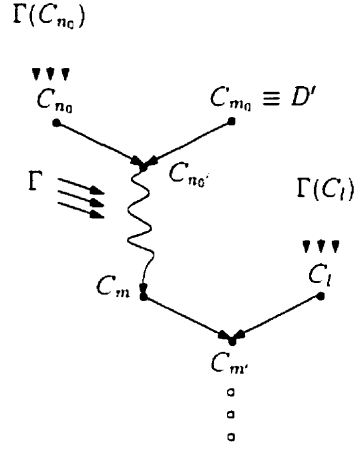


FIGURE E.1

$$C_{m'} \equiv K' \mid G' \mid D\gamma^- \mid I' \mid I''.$$

Now since the (relative) ordering of positive literals in a clause is irrelevant with respect to $\text{EPC}_{<}$ -deduction, observe that we could have “delayed” resolving C_{n_0} with D' .

Note that in “the application of Γ to $C_{n_0'}$ ”, neither $P(\bar{a})$, nor any literal in $D\gamma^- \mid I'$ is resolved upon, (and of course, no clause in Γ is an equality clause, so Γ is above-equality).

Also, since C_m is pure-positive with respect to P , C_m is $\text{EPC}_{<}$ -initial in Γ .

Since $G \subseteq C_{n_0'}$, (G subsumes $C_{n_0'}$, and), Lemma 4.3.4, there is an $\text{EPC}_{<}$ -deduction Γ' , (in which G takes the place of $C_{n_0'}$ at n_0'), of a clause $C_{m'}$ such that $C_{m'} \subseteq C_m$; where (!!):

- i) $\forall C' \notin \{G, C_{n_0'}\}, \quad k_{\Gamma'}(C') \leq k_{\Gamma}(C'),$
- ii) $1 \leq k_{\Gamma'}(G) \leq k_{\Gamma}(G) + 1,$ and
- iii) $k_{\Gamma'}(C_{n_0'}) \leq k_{\Gamma}(C_{n_0'}) - 1.$

Clearly $\kappa(\Gamma') \leq \kappa(\Gamma).$

Let Π_{0C_m} denote the fragment of Π_0 which is rooted at m , (and constitutes an $\text{EPC}_{<}$ -deduction of C_m).

Let $\Pi_0^{C_m}$ denote the fragment of Π_0 which is pruned at m , (and constitutes an $\text{EPC}_{<}$ -deduction of E_0).

Again (as $C_{m'} \subseteq C_m$), by Lemma 4.3.4, (or by Corollary 4.3.1), there is an above-equality $\text{EPC}_{<}$ -deduction $\Pi_0'^{C_{m'}}$, say, of an equality clause $E_0' \subseteq E_0$, such that:

- i) $\forall C' \notin \{C_{m'}, C_m\}, \quad k_{\Pi_0', C_{m'}}(C') \leq k_{\Pi_0, C_m}(C'),$
- ii) $1 \leq k_{\Pi_0', C_{m'}}(C_{m'}) \leq k_{\Pi_0, C_m}(C_{m'}) + 1,$ and
- iii) $k_{\Pi_0', C_{m'}}(C_m) \leq k_{\Pi_0, C_m}(C_m) - 1.$

Let $\Pi_0^{C_{n_0'}}$ denote the fragment of Π_0 which is pruned at n_0' ,
(which constitutes an $\text{EPC}_{<}$ -deduction of E_0).

Evidently $\Pi_0^{C_{n_0'}}$ denotes the result of affixing Γ (which is rooted at m) to some single leaf of $\Pi_0^{C_m}$, which is labelled by C_m , (the leaf being m).

Let $\Pi_0^{C_{n_0'}}[\Gamma'/\Gamma]$ denote the result of affixing Γ' to some single leaf of $\Pi_0^{C_m'}$, which is labelled by C_m' . To simplify notation, let us denote $\Pi_0^{C_{n_0'}}[\Gamma'/\Gamma]$ by Π_0^G .

Evidently, Π_0^G is an $\text{EPC}_{<}$ -deduction, in which only the root node is an equality clause, that clause being E_0' . We have $E_0' \subseteq E_0$. Further, we claim (1):

- 1.i) $\forall C' \notin \{G, C_{n_0'}\}, \quad k_{\Pi_0^G}(C') \leq k_{\Pi_0^{C_{n_0'}}}(C'),$
- 1.ii) $k_{\Pi_0^G}(G) \leq k_{\Pi_0^{C_{n_0'}}}(G) + 1$
- 1.iii) $k_{\Pi_0^G}(C_{n_0'}) \leq k_{\Pi_0^{C_{n_0'}}}(C_{n_0'}) - 1$

Proof of claim (1):

Since $\Pi_0^{C_{n_0'}}$ denotes the result of affixing Γ to $\Pi_0^{C_m}$, (at m), evidently

$$\forall C' \notin \{C_m\}, \quad k_{\Pi_0^{C_{n_0'}}}(C') = k_{\Pi_0^{C_m}}(C') + k_{\Gamma}(C'), \text{ and } k_{\Pi_0^{C_{n_0'}}}(C_m) = k_{\Pi_0^{C_m}}(C_m) + k_{\Gamma}(C_m) - 1$$

while similarly,

$$\forall C' \notin \{C_m'\}, \quad k_{\Pi_0^G}(C') = k_{\Pi_0^{C_m'}}(C') + k_{\Gamma'}(C'), \text{ and } k_{\Pi_0^G}(C_m') = k_{\Pi_0^{C_m'}}(C_m') + k_{\Gamma'}(C_m') - 1$$

Thus, showing 1.i) is equivalent to showing that: $\forall C' \notin \{G, C_{n_0'}\},$

$$\text{if } C' \notin \{C_m, C_m'\}, \text{ (then) } k_{\Pi_0^{C_{n_0'}}}(C') + k_{\Gamma'}(C') \leq k_{\Pi_0^{C_m}}(C') + k_{\Gamma}(C').$$

Here, if $C' \notin \{C_m, C_m'\}, \quad k_{\Pi_0^{C_{n_0'}}}(C') \leq k_{\Pi_0^{C_m}}(C'),$ and if $C' \notin \{G, C_{n_0'}\}, \quad k_{\Gamma'}(C') \leq k_{\Gamma}(C').$

Hence if $C' \notin \{G, C_{n_0'}, C_m, C_m'\}, \quad k_{\Pi_0^{C_{n_0'}}}(C') + k_{\Gamma'}(C') \leq k_{\Pi_0^{C_m}}(C') + k_{\Gamma}(C'),$

and hence $k_{\Pi_0^G}(C') \leq k_{\Pi_0^{C_{n_0'}}}(C'),$

while if $C' = C_m \notin \{G, C_{n_0'}, C_m'\}, \quad k_{\Pi_0^{C_{n_0'}}}(C_m) = k_{\Pi_0^{C_m}}(C_m) + k_{\Gamma}(C_m) - 1,$

$$k_{\Pi_0^G}(C_m) = k_{\Pi_0^{C_m'}}(C_m) + k_{\Gamma'}(C_m); \quad k_{\Pi_0^{C_m'}}(C_m) \leq k_{\Pi_0^{C_m}}(C_m) - 1.$$

Thus,

$$\begin{aligned} k_{\Pi_0^G}(C_m) &= k_{\Pi_0^{C_m'}}(C_m) + k_{\Gamma'}(C_m) \leq k_{\Pi_0^{C_m}}(C_m) - 1 + k_{\Gamma'}(C_m) \\ &\leq k_{\Pi_0^{C_m}}(C_m) - 1 + k_{\Gamma}(C_m) = k_{\Pi_0^{C_{n_0'}}}(C_m), \end{aligned}$$

(since under the assumption $C_m \notin \{G, C_{n_0'}\}, \quad k_{\Gamma'}(C_m) \leq k_{\Gamma}(C_m)$).

Similarly, if $C' = C_m' \notin \{G, C_{n_0'}, C_m\}, \quad k_{\Pi_0^{C_{n_0'}}}(C_m') = k_{\Pi_0^{C_m}}(C_m') + k_{\Gamma}(C_m'),$

$$k_{\Pi_0^G}(C_m') = k_{\Pi_0^{C_m'}}(C_m') + k_{\Gamma'}(C_m') - 1; \quad k_{\Pi_0^{C_m'}}(C_m') \leq k_{\Pi_0^{C_m}}(C_m') + 1,$$

(and $k_{\Gamma'}(C_m') \leq k_{\Gamma}(C_m')$).

$$\begin{aligned} \text{Thus, } k_{\Pi_0 G}(C_m') &= k_{\Pi_0', C_m'}(C_m') + k_{\Gamma'}(C_m') - 1 \leq k_{\Pi_0 C_m}(C_m') + k_{\Gamma'}(C_m') \\ &\leq k_{\Pi_0 C_m}(C_m') + k_{\Gamma}(C_m') = k_{\Pi_0' C_{n_0'}}(C_m'). \end{aligned}$$

Finally, if $C' = C_m = C_m' \notin \{G, C_{n_0'}\}$,

$$k_{\Pi_0' C_{n_0'}}(C_m) = k_{\Pi_0 C_m}(C_m) + k_{\Gamma}(C_m) - 1, \quad k_{\Pi_0 G}(C_m) = k_{\Pi_0', C_m'}(C_m) + k_{\Gamma'}(C_m) - 1,$$

and $k_{\Pi_0', C_m'}(C_m) = k_{\Pi_0 C_m}(C_m)$. (taking $\Pi_0'^{C_m'} \equiv \Pi_0^{C_m}$), (and $k_{\Gamma'}(C_m) \leq k_{\Gamma}(C_m)$).

$$\text{Thus, } k_{\Pi_0 G}(C_m) = k_{\Pi_0', C_m'}(C_m) + k_{\Gamma'}(C_m) - 1 \leq k_{\Pi_0 C_m}(C_m) + k_{\Gamma}(C_m) - 1 = k_{\Pi_0' C_{n_0'}}(C_m).$$

Thus we have shown that $\forall C' \notin \{G, C_{n_0'}\}, \quad k_{\Pi_0 G}(C') \leq k_{\Pi_0' C_{n_0'}}(C')$.

(that is, we have shown 1.i).

For 1.ii), similarly:

$$\text{If } G \notin \{C_m, C_m'\}, \text{ then } k_{\Pi_0', C_m'}(G) \leq k_{\Pi_0 C_m}(G); \quad k_{\Pi_0' C_{n_0'}}(G) = k_{\Pi_0 C_m}(G) + k_{\Gamma}(G),$$

$$\text{and } k_{\Pi_0 G}(G) = k_{\Pi_0', C_m'}(G) + k_{\Gamma'}(G), \quad (\text{and } k_{\Gamma'}(G) \leq k_{\Gamma}(G) + 1).$$

$$\text{Thus, } k_{\Pi_0 G}(G) = k_{\Pi_0', C_m'}(G) + k_{\Gamma'}(G) \leq k_{\Pi_0 C_m}(G) + k_{\Gamma}(G) + 1 = k_{\Pi_0' C_{n_0'}}(G) + 1,$$

(that is, 1.ii), as claimed).

$$\text{If } G = C_m \neq C_m', \text{ then } k_{\Pi_0', C_m'}(G) \leq k_{\Pi_0 C_m}(G) - 1; \quad k_{\Pi_0' C_{n_0'}}(G) = k_{\Pi_0 C_m}(G) + k_{\Gamma}(G) - 1,$$

$$\text{and } k_{\Pi_0 G}(G) = k_{\Pi_0', C_m'}(G) + k_{\Gamma'}(G), \quad (\text{and } k_{\Gamma'}(G) \leq k_{\Gamma}(G) + 1).$$

$$\text{Thus, } k_{\Pi_0 G}(G) = k_{\Pi_0', C_m'}(G) + k_{\Gamma'}(G) \leq k_{\Pi_0 C_m}(G) + k_{\Gamma}(G) \leq k_{\Pi_0' C_{n_0'}}(G) + 1,$$

(that is, 1.ii), as claimed).

$$\text{If } G = C_m' \neq C_m, \text{ then } k_{\Pi_0', C_m'}(G) \leq k_{\Pi_0 C_m}(G) + 1; \quad k_{\Pi_0' C_{n_0'}}(G) = k_{\Pi_0 C_m}(G) + k_{\Gamma}(G),$$

$$\text{and } k_{\Pi_0 G}(G) = k_{\Pi_0', C_m'}(G) + k_{\Gamma'}(G) - 1, \quad (\text{and } k_{\Gamma'}(G) \leq k_{\Gamma}(G) + 1).$$

$$\text{Thus, } k_{\Pi_0 G}(G) = k_{\Pi_0', C_m'}(G) + k_{\Gamma'}(G) - 1 \leq k_{\Pi_0 C_m}(G) + k_{\Gamma}(G) + 1 \leq k_{\Pi_0' C_{n_0'}}(G) + 1,$$

(that is, 1.ii), as claimed).

$$\text{Finally, if } G = C_m' = C_m, \text{ then } k_{\Pi_0', C_m'}(G) = k_{\Pi_0 C_m}(G), \quad (\text{taking } \Pi_0'^{C_m'} \equiv \Pi_0^{C_m});$$

$$k_{\Pi_0' C_{n_0'}}(G) = k_{\Pi_0 C_m}(G) + k_{\Gamma}(G) - 1, \text{ and } k_{\Pi_0 G}(G) = k_{\Pi_0', C_m'}(G) + k_{\Gamma'}(G) - 1,$$

$$(\text{and } k_{\Gamma'}(G) \leq k_{\Gamma}(G) + 1).$$

$$\text{Thus, } k_{\Pi_0 G}(G) = k_{\Pi_0', C_m'}(G) + k_{\Gamma'}(G) - 1 \leq k_{\Pi_0 C_m}(G) + k_{\Gamma}(G) \leq k_{\Pi_0' C_{n_0'}}(G) + 1,$$

(that is, 1.ii), as claimed).

In any event, 1.ii) holds.

For 1.iii), similarly:

If $C_{n_0'} \notin \{C_m, C_m'\}$, then $k_{\Pi_0', C_m'}(C_{n_0'}) \leq k_{\Pi_0', C_m}(C_{n_0'})$;

$$k_{\Pi_0', C_{n_0'}}(C_{n_0'}) = k_{\Pi_0', C_m}(C_{n_0'}) + k_{\Gamma'}(C_{n_0'}); \quad k_{\Pi_0', G}(C_{n_0'}) = k_{\Pi_0', C_m'}(C_{n_0'}) + k_{\Gamma'}(C_{n_0'}),$$

$$(\text{and } k_{\Gamma'}(C_{n_0'}) \leq k_{\Gamma}(C_{n_0'}) - 1).$$

$$\text{Thus, } k_{\Pi_0', G}(C_{n_0'}) = k_{\Pi_0', C_m'}(C_{n_0'}) + k_{\Gamma'}(C_{n_0'}) \leq k_{\Pi_0', C_m}(C_{n_0'}) + k_{\Gamma}(C_{n_0'}) - 1 = k_{\Pi_0', C_{n_0'}}(C_{n_0'}) - 1.$$

(that is, 1.iii), as claimed).

If $C_{n_0'} = C_m \neq C_m'$, then $k_{\Pi_0', C_m'}(C_m) \leq k_{\Pi_0', C_m}(C_m) - 1$;

$$k_{\Pi_0', C_{n_0'}}(C_m) = k_{\Pi_0', C_m}(C_m) + k_{\Gamma}(C_m) - 1, \text{ and } k_{\Pi_0', G}(C_m) = k_{\Pi_0', C_m'}(C_m) + k_{\Gamma'}(C_m),$$

$$(\text{and } k_{\Gamma'}(C_m) \leq k_{\Gamma}(C_m) - 1).$$

$$\text{Thus, } k_{\Pi_0', G}(C_m) = k_{\Pi_0', C_m'}(C_m) + k_{\Gamma'}(C_m) \leq k_{\Pi_0', C_m}(C_m) + k_{\Gamma}(C_m) - 2 \leq k_{\Pi_0', C_{n_0'}}(C_m) - 1.$$

(that is, 1.iii), as claimed).

If $C_{n_0'} = C_m' \neq C_m$, then $k_{\Pi_0', C_m'}(C_m') \leq k_{\Pi_0', C_m}(C_m') + 1$;

$$k_{\Pi_0', C_{n_0'}}(C_m') = k_{\Pi_0', C_m}(C_m') + k_{\Gamma}(C_m'), \text{ and } k_{\Pi_0', G}(C_m') = k_{\Pi_0', C_m'}(C_m') + k_{\Gamma'}(C_m') - 1,$$

$$(\text{and } k_{\Gamma'}(C_m') \leq k_{\Gamma}(C_m') - 1).$$

$$\text{Thus, } k_{\Pi_0', G}(C_m') = k_{\Pi_0', C_m'}(C_m') + k_{\Gamma'}(C_m') - 1 \leq k_{\Pi_0', C_m}(C_m') + k_{\Gamma}(C_m') - 1 \leq k_{\Pi_0', C_{n_0'}}(C_m') - 1.$$

(that is, 1.iii), as claimed).

Finally, if $G = C_m' = C_m$, then $k_{\Pi_0', C_m'}(C_m) = k_{\Pi_0', C_m}(C_m)$, (taking $\Pi_0'^{C_m'} \equiv \Pi_0^{C_m}$);

$$k_{\Pi_0', C_{n_0'}}(C_m) = k_{\Pi_0', C_m}(C_m) + k_{\Gamma}(C_m) - 1, \text{ and } k_{\Pi_0', G}(C_m) = k_{\Pi_0', C_m'}(C_m) + k_{\Gamma'}(C_m) - 1,$$

$$(\text{and } k_{\Gamma'}(C_m) \leq k_{\Gamma}(C_m) - 1).$$

$$\text{Thus, } k_{\Pi_0', G}(C_m) = k_{\Pi_0', C_m'}(C_m) + k_{\Gamma'}(C_m) - 1 \leq k_{\Pi_0', C_m}(C_m) + k_{\Gamma}(C_m) - 2 \leq k_{\Pi_0', C_{n_0'}}(C_m) - 1.$$

(that is, 1.iii), as claimed).

In any event, 1.iii) holds.

This completes the proof of claim (1).

Suppose that $k_{\Gamma'}(G) = 0$. We claim (2) that:

$$2.i) \quad \forall C' \notin \{D'\}, \quad k_{\Pi_0', G}(C') \leq k_{\Pi_0'}(C');$$

$$2.ii) \quad k_{\Pi_0', G}(D') \leq k_{\Pi_0'}(D') - 1$$

If we can show this claim, then evidently Π_0^G is a $\text{EPC}_{<}$ -deduction, in which occurs exactly one equality clause, (at the root), E_0' , where $E_0' \subseteq E_0 \subseteq E' \subseteq \overline{E}$, such that:

$$i. \quad \forall C' \notin \{D', D\Gamma\}, \quad k_{\Pi_0', G}(C') \leq k_{\Pi_0'}(C') \leq k_{\Pi_0'}(C'),$$

$$ii. \quad k_{\Pi_0', G}(D') \leq k_{\Pi_0'}(D') - 1 \leq k_{\Pi_0'}(D'), \text{ and}$$

$$iii. \quad k_{\Pi_0', G}(D\Gamma) \leq k_{\Pi_0'}(D\Gamma) \leq k_{\Pi_0'}(D\Gamma) - 1,$$

an hence such that:

- i. $\forall C' \notin \{D\Gamma\}, \quad k_{\Pi_0 \sigma}(C') \leq k_{\Pi, \rho}(C'),$ and
- ii. $k_{\Pi_0 \sigma}(D\Gamma) \leq k_{\Pi, \rho}(D\Gamma) - 1.$

By lifting Π_0^G , as before, evidently we may obtain, (from Π_0^G), a Π' , such that with $\mathfrak{F} \equiv \{\Pi'\}$, $\mathfrak{F}$ satisfies the requirements a), b) and c) of the lemma. Thus, (once we have shown claim (2)), we may assume $k_{\Gamma'}(G) \geq 1$.

Proof of claim (2):

If $k_{\Gamma'}(G) = 0$, then:

- i) $\forall C' \notin \{C_{n_0'}\}, \quad k_{\Gamma'}(C') \leq k_{\Gamma}(C');$
- ii) $k_{\Gamma'}(C_{n_0'}) \leq k_{\Gamma}(C_{n_0'}) - 1.$

By examining the proof of claim (1), (and using i) and ii) above, in place of (!)), it is easy to see that under the hypothesis that $k_{\Gamma'}(G) = 0$, claim (1) can be strengthened to:

- 1' .i) $\forall C' \notin \{C_{n_0'}\}, \quad k_{\Pi_0 \sigma}(C') \leq k_{\Pi_0 \sigma_{n_0'}}(C');$
- 1' .ii) $k_{\Pi_0 \sigma}(C_{n_0'}) \leq k_{\Pi_0 \sigma_{n_0'}}(C_{n_0'}) - 1.$

Now by the definition of $\Pi_0^{C_{n_0'}}$, (see figure E.1):

- i) $\forall C' \notin \{C_{n_0'}, D'\}, \quad k_{\Pi_0}(C') = k_{\Pi_0 \sigma_{n_0'}}(C') + k_{\Gamma_{C_{n_0}}}(C');$
- ii) $k_{\Pi_0}(D') = k_{\Pi_0 \sigma_{n_0'}}(D') + k_{\Gamma_{C_{n_0}}}(D') + 1;$
- iii) $k_{\Pi_0}(C_{n_0'}) = k_{\Pi_0 \sigma_{n_0'}}(C_{n_0'}) + k_{\Gamma_{C_{n_0}}}(C_{n_0'}) - 1.$

By i) above, $\forall C' \notin \{C_{n_0'}, D'\}, \quad k_{\Pi_0}(C') = k_{\Pi_0 \sigma_{n_0'}}(C') + k_{\Gamma_{C_{n_0}}}(C').$

But then by 1' .i), $k_{\Pi_0 \sigma}(C') \leq k_{\Pi_0 \sigma_{n_0'}}(C') = k_{\Pi_0}(C') - k_{\Gamma_{C_{n_0}}}(C') \leq k_{\Pi_0}(C').$
(since $k_{\Gamma_{C_{n_0}}}(C') \geq 0$).

Also, if $C' = C_{n_0'}$, then $k_{\Pi_0 \sigma}(C') \leq k_{\Pi_0 \sigma_{n_0'}}(C') - 1$ (by 1' .ii)),
 $= k_{\Pi_0}(C') - k_{\Gamma_{C_{n_0}}}(C')$ (by iii) above),
 $\leq k_{\Pi_0}(C'),$ (since $k_{\Gamma_{C_{n_0}}}(C') \geq 0$).

Thus $\forall C' \notin \{D'\}, \quad k_{\Pi_0 \sigma}(C') \leq k_{\Pi_0}(C'),$ (as required, for 2.i)).

For 2.ii): $D' \notin \{C_{n_0'}\}$, since, for example, by construction $C_{n_0'}$ is pure-positive with respect to P , while D' is not), and hence,

$$\begin{aligned} k_{\Pi_0 \sigma}(D') &\leq k_{\Pi_0 \sigma_{n_0'}}(D') \quad (\text{by 1' .i) }), \\ &\leq k_{\Pi_0}(D') - k_{\Gamma_{C_{n_0}}}(D') - 1 \quad (\text{by ii) above}), \\ &\leq k_{\Pi_0}(D') - 1, \quad (\text{as required for 2.ii) }). \end{aligned}$$

This completes the proof of claim (2).

Now suppose $k_{\Gamma'}(G) \geq 1$.

Now since Γ' “takes G to C_m' ”, where $C_m' \subseteq C_m$, and since the presence of the positive literals $\{P(\vec{b}_{i'}) : 1 \leq i' \leq i\}$ is innocuous to the $\text{EPC}_{<}$ -deduction involved in Γ , Γ could have been “applied directly to C_{n_0} ”, (instead of first resolving C_{n_0} with D').

More precisely, if some (leaf) instance of G in Γ' is replaced with $C_{n_0} = G \cup \{P(\vec{b}_{i'}) : 1 \leq i' \leq i\}$, (it is not hard to see that) we obtain an $\text{EPC}_{<}$ -deduction Γ'' , say, of a clause

$C_m'' \subseteq C_m' \cup \{P(\vec{b}_{i'}) : 1 \leq i' \leq i\}$, such that $\kappa(\Gamma'') = \kappa(\Gamma') \leq \kappa(\Gamma)$, and where further:

- i) $\forall C' \notin \{G, C_{n_0}\}, \quad k_{\Gamma''}(C') \leq k_{\Gamma'}(C')$;
- ii) $1 \leq k_{\Gamma''}(C_{n_0}) \leq k_{\Gamma'}(C_{n_0}) + 1$;
- iii) $k_{\Gamma''}(G) \leq k_{\Gamma'}(G) - 1$.

If $\{P(\vec{b}_{i'}) : 1 \leq i' \leq i\} \cap C_m'' = \emptyset$, then $C_m'' \subseteq C_m' \subseteq C_m$.

Thus if $C_m \neq C_m''$, as $C_m'' \subseteq C_m$, (and $\Pi_0^{C_m}$ is above-equality), by Lemma 4.3.4, there is an $\text{EPC}_{<}$ -deduction $\Pi_0^{C_m}[C_m''/C_m] \equiv \Delta$, say, containing exactly one equality clause (necessarily at the root), E_0'' , say, such that $E_0'' \subseteq E_0 \subseteq E' \subseteq \bar{E}$, and:

- i) $\forall C' \notin \{C_m, C_m''\}, \quad k_{\Delta}(C') \leq k_{\Pi_0^{C_m}}(C')$;
- ii) $1 \leq k_{\Delta}(C_m'') \leq k_{\Pi_0^{C_m}}(C_m'') + 1$;
- iii) $k_{\Delta}(C_m) \leq k_{\Pi_0^{C_m}}(C_m) - 1$.

Otherwise, if $C_m = C_m''$, taking $\Delta \equiv \Pi_0^{C_m}$, we have $\forall C', \quad k_{\Delta}(C') = k_{\Pi_0^{C_m}}(C')$.

Let Π_0' denote the result of affixing Γ'' to a leaf labelled by C_m'' in Δ , and then affixing $\Gamma_{C_{n_0}}$ to a leaf labelled by C_{n_0} , (in the intermediate result, -recall $k_{\Gamma''}(C_{n_0}) \geq 1$). Clearly the Π_0' is an $\text{EPC}_{<}$ -deduction, containing exactly one equality clause, that clause being E_0'' , (and $E_0'' \subseteq E_0 \subseteq E' \subseteq \bar{E}$).

Again we claim (3):

- 3.i) $\forall C' \notin \{D'\}, \quad k_{\Pi_0'}(C') \leq k_{\Pi_0}(C')$;
- 3.ii) $k_{\Pi_0'}(D') \leq k_{\Pi_0}(D') - 1$.

As before, given this claim, evidently we may (therefore) lift Π_0' to obtain a Π' , so that taking $\mathfrak{F} \equiv \{\Pi'\}$ meets the requirements a), b), and c) of the lemma.

To see 3.i) and 3.ii):

Note that, by the construction of Π_0' , if $C_m'' \neq C_{n_0}$,

- i) $\forall C' \notin \{C_m'', C_{n_0}\}, \quad k_{\Pi_0'}(C') = k_{\Delta}(C') + k_{\Gamma''}(C') + k_{\Gamma_{C_{n_0}}}(C')$,
- ii) $k_{\Pi_0'}(C_m'') = k_{\Delta}(C_m'') + k_{\Gamma''}(C_m'') + k_{\Gamma_{C_{n_0}}}(C_m'') - 1$, and
- iii) $k_{\Pi_0'}(C_{n_0}) = k_{\Delta}(C_{n_0}) + k_{\Gamma''}(C_{n_0}) + k_{\Gamma_{C_{n_0}}}(C_{n_0}) - 1$,

and otherwise, (if $C_m'' = C_{n_0}$),

- i) $\forall C' \notin \{C_{n_0}\}, \quad k_{\Pi_0'}(C') = k_{\Delta}(C') + k_{\Gamma''}(C') + k_{\Gamma_{C_{n_0}}}(C')$;
- ii) $k_{\Pi_0'}(C_{n_0}) = k_{\Delta}(C_{n_0}) + k_{\Gamma''}(C_{n_0}) + k_{\Gamma_{C_{n_0}}}(C_{n_0}) - 2$.

Now recall: if $C_m \neq C_m''$,

- i) $\forall C' \notin \{C_m, C_m''\}, \quad k_\Delta(C') \leq k_{\Pi_0 C_m}(C')$;
- ii) $1 \leq k_\Delta(C_m'') \leq k_{\Pi_0 C_m}(C_m'') + 1$;
- iii) $\leq k_\Delta(C_m) \leq k_{\Pi_0 C_m}(C_m) - 1$,

and otherwise, (if $C_m = C_m''$), $\forall C', \quad k_\Delta(C') = k_{\Pi_0 C_m}(C')$.

Also recall that $C_m \neq C_{n_0}$.

Thus, (putting these together, it is not difficult to see that), it follows that (α):

- i) $\forall C' \notin \{C_m, C_{n_0}\}, \quad k_{\Pi_0'}(C') \leq k_{\Pi_0 C_m}(C') + k_{\Gamma''}(C') + k_{\Gamma_{C_{n_0}}}(C')$,
- ii) $k_{\Pi_0'}(C_m) \leq k_{\Pi_0 C_m}(C_m) + k_{\Gamma''}(C_m) + k_{\Gamma_{C_{n_0}}}(C_m) - 1$;
- iii) $k_{\Pi_0'}(C_{n_0}) \leq k_{\Pi_0 C_m}(C_{n_0}) + k_{\Gamma''}(C_{n_0}) + k_{\Gamma_{C_{n_0}}}(C_{n_0}) - 1$.

Also, by the definitions of Π_0 , Π_{0C_m} , and $\Pi_0^{C_m}$, (see figure E.1),

- i) $\forall C' \notin \{C_m\}, \quad k_{\Pi_0}(C') = k_{\Pi_0 C_m}(C') + k_{\Pi_{0C_m}}(C')$, and
- ii) $k_{\Pi_0}(C_m) = k_{\Pi_0 C_m}(C_m) + k_{\Pi_{0C_m}}(C_m) - 1$.

By definitions of Π_{0C_m} , Γ , and $\Gamma_{C_{n_0}}$, (see figure E.1),

- i) $\forall C' \notin \{C_{n_0}', D'\}, \quad k_{\Pi_{0C_m}}(C') = k_\Gamma(C') + k_{\Gamma_{C_{n_0}}}(C')$.
- ii) $k_{\Pi_{0C_m}}(D') = k_\Gamma(D') + k_{\Gamma_{C_{n_0}}}(D') + 1$, and
- iii) $k_{\Pi_{0C_m}}(C_{n_0}') = k_\Gamma(C_{n_0}') + k_{\Gamma_{C_{n_0}}}(C_{n_0}') - 1$.

Thus (β):

- i) $\forall C' \notin \{C_{n_0}', D', C_m\}, \quad k_{\Pi_0}(C') = k_{\Pi_0 C_m}(C') + k_\Gamma(C') + k_{\Gamma_{C_{n_0}}}(C')$.
(clearly $D' \notin \{C_m, C_{n_0}'\}$, as C_m and C_{n_0}' are pure-positive with respect to P , while D' is not, and),
- ii) if $C' = C_m \neq C_{n_0}'$, $k_{\Pi_0}(C') = k_{\Pi_0 C_m}(C') + k_\Gamma(C') + k_{\Gamma_{C_{n_0}}}(C') - 1$,
- iii) if $C' = C_{n_0}' \neq C_m$, $k_{\Pi_0}(C') = k_{\Pi_0 C_m}(C') + k_\Gamma(C') + k_{\Gamma_{C_{n_0}}}(C') - 1$,
- iv) if $C' = C_{n_0}' = C_m$, $k_{\Pi_0}(C') = k_{\Pi_0 C_m}(C') + k_\Gamma(C') + k_{\Gamma_{C_{n_0}}}(C') - 2$, and
- v) $k_{\Pi_0}(D') = k_{\Pi_0 C_m}(D') + k_\Gamma(D') + k_{\Gamma_{C_{n_0}}}(D') + 1$.

Also recall (!!):

- i) $\forall C' \notin \{G, C_{n_0}'\}, \quad k_{\Gamma'}(C') \leq k_\Gamma(C')$,
- ii) $1 \leq k_{\Gamma'}(G) \leq k_\Gamma(G) + 1$, and
- iii) $k_{\Gamma'}(C_{n_0}') \leq k_\Gamma(C_{n_0}') - 1$,

together with

- i) $\forall C' \notin \{G, C_{n_0}\}, \quad k_{\Gamma''}(C') \leq k_{\Gamma'}(C')$;
- ii) $1 \leq k_{\Gamma''}(C_{n_0}) \leq k_{\Gamma'}(C_{n_0}) + 1$;
- iii) $k_{\Gamma''}(G) \leq k_{\Gamma'}(G) - 1$.

Putting these together, together with the facts $C_{n_0} \neq G \neq C_{n_0'}$; $C_{n_0} \neq C_{n_0'}$, it is easy to see that (γ) :

- i) $\forall C' \notin \{C_{n_0}, C_{n_0'}\}, \quad k_{\Gamma''}(C') \leq k_{\Gamma}(C'),$
- ii) $k_{\Gamma''}(C_{n_0}) \leq k_{\Gamma}(C_{n_0}) + 1,$ and
- iii) $k_{\Gamma''}(C_{n_0'}) \leq k_{\Gamma}(C_{n_0'}) - 1.$

Now putting inequalities (α) , and (γ) together, and using $C_m \neq C_{n_0} \neq C_{n_0'}$, we get. (δ) :

- i) $\forall C' \notin \{C_m, C_{n_0'}\}, \quad k_{\Pi_0'}(C') \leq k_{\Pi_0 C_m}(C') + k_{\Gamma}(C') + k_{\Gamma_{C_{n_0}}}(C'),$
- ii) $k_{\Pi_0'}(C_m) \leq k_{\Pi_0 C_m}(C_m) + k_{\Gamma}(C_m) + k_{\Gamma_{C_{n_0}}}(C_m) - 1,$ and
- iii) $k_{\Pi_0'}(C_{n_0'}) \leq k_{\Pi_0 C_m}(C_{n_0'}) + k_{\Gamma}(C_{n_0'}) + k_{\Gamma_{C_{n_0}}}(C_{n_0'}) - 1.$

in the case when $C_m \neq C_{n_0'}$, and. (δ') :

- i) $\forall C' \notin \{C_m\}, \quad k_{\Pi_0'}(C') \leq k_{\Pi_0 C_m}(C') + k_{\Gamma}(C') + k_{\Gamma_{C_{n_0}}}(C'),$ and
- ii) $k_{\Pi_0'}(C_m) \leq k_{\Pi_0 C_m}(C_m) + k_{\Gamma}(C_m) + k_{\Gamma_{C_{n_0}}}(C_m) - 2.$

when $C_m = C_{n_0'}$.

Now 3.i) and 3.ii) follow easily from (β) , (δ) , and (δ') , (for 3.ii), recall that $D' \notin \{C_m, C_{n_0'}\}$. This completes the proof of claim (3).

Thus we have disposed of the case when $\{P(\vec{b}_{i'}) : 1 \leq i' \leq i\} \cap C_m'' = \emptyset$. and we may assume $\emptyset \neq \{P(\vec{b}_{i'}) : 1 \leq i' \leq i\} \cap C_m''$.

Let F denote $C_m'' \cap \{P(\vec{b}_{i'}) : 1 \leq i' \leq i\} \cup C_m'' \cap (\{P(\vec{a})\} \cup \{P(\vec{a}_{j'}) : 1 \leq j' \leq j\})$.

Then $F \neq \emptyset$, and (since $C_m'' \subseteq C_m' \cup \{P(\vec{b}_{i'}) : 1 \leq i' \leq i\}$).

$$C_m'' \setminus F \subseteq C_m' \setminus F \subseteq C_m \setminus F \subseteq G' \mid (D\gamma)^- \mid I'.$$

Let H denote $K' \mid C_m'' \setminus F \mid I''' \mid I''$,

where $I''' \equiv \bigcup \{ \neq(\vec{a}_{k'}, \vec{c}) : 1 \leq k' \leq k; P(\vec{c}) \in (\{P(\vec{a})\} \cup \{P(\vec{a}_{j'} : 1 \leq j' \leq j\}) \cap C_m'' \}$.

and $I'' \equiv \bigcup \{ \neq(\vec{a}_{k'}, \vec{b}_{i'}) : 1 \leq k' \leq k, 1 \leq i' \leq i, : P(\vec{b}_{i'}) \in F \}$.

Clearly H is an $\text{EPC}_{<}$ -resolvent of C_l , (ordered as in Π_0), and C_m'' . Also, clearly $I''' \subseteq I''$, and $H \subseteq K' \mid G' \mid (D\gamma)^- \mid I' \mid I'' \mid I''$.

Let $\Pi_{0C_m'}$ denote the fragment of Π_0 which is rooted at m' , and is an $\text{EPC}_{<}$ -deduction of C_m' , (see figure E.1).

Suppose $\neq(a, b) \notin C_m'$, (or $\{\neq(a, b), \neq(b, a)\} \cap C_m' = \emptyset$).

Since no equality literal can be resolved upon in Π_0 , (or in $\Pi_{0C_m'}$), it is not hard to see that $\neq(a, b) \notin D'$.

Conversely, it is not hard to see that if we add the ground literal $\neq(a, b)$ to D' at m_0 , in Π_0 , (and in $\Pi_{0C_m'}$), and pass the literal downwards through Π_0 , (and $\Pi_{0C_m'}$), as required by the rules of $\text{EPC}_{<}$ -deduction, we obtain from Π_0 , (and from $\Pi_{0C_m'}$), a (ground) $\text{EPC}_{<}$ -deduction Π_1 . (Δ) ,

say, of a (ground) clause E_1 , ($C_{m'}'$), say, where $E_1 \subseteq E_0 \cup \{\neq(a, b)\}$,
(and $C_{m'}' \subseteq C_{m'} \cup \{\neq(a, b)\}$), and where $\kappa(\Pi_1) = \kappa(\Pi_0)$, (and $\kappa(\Delta) = \kappa(\Pi_{0C_{m'}})$, respectively),
and E_1 is the only equality clause in Π_1 .

By this construction, Δ occurs as a top fragment of Π_1 , ("top" in the obvious sense that, as a fragment of Π_1 , each leaf of Δ is a leaf of Π_1), and further:

- i) $\forall C' \notin \{D', D' \cup \{\neq(a, b)\}\}$, $k_{\Pi_1}(C') = k_{\Pi_0}(C')$, and $k_{\Delta}(C') = k_{\Pi_{0C_{m'}}}(C')$;
- ii) $k_{\Pi_1}(D' \cup \{\neq(a, b)\}) = k_{\Pi_0}(D' \cup \{\neq(a, b)\}) + 1$,
and $k_{\Delta}(D' \cup \{\neq(a, b)\}) = k_{\Pi_{0C_{m'}}}(D' \cup \{\neq(a, b)\}) + 1$, and
- iii) $k_{\Pi_1}(D') = k_{\Pi_0}(D') - 1$, and $k_{\Delta}(D') = k_{\Pi_{0C_{m'}}}(D') - 1$.

Since no equality literal can be resolved upon in Π_0 , (or in $\Pi_{0C_{m'}}$), (it is easy to see that) in fact, $C_{m'}' = C_{m'} \cup \{\neq(a, b)\}$, and $E_1 = E_0 \cup \{\neq(a, b)\}$.

On the other hand, if $\neq(a, b) \in C_{m'}$, then (because no equality literal can be resolved upon in Π_0), it is easy to see that $\neq(a, b) \in E_0$. In this case, let $\Delta \equiv \Pi_{0C_{m'}}$, $C_{m'}' \equiv C_{m'}$, $\Pi_1 \equiv \Pi_0$, and $E_1 \equiv E_0$. Then, of course, we have $\forall C'$, $k_{\Pi_1}(C') = k_{\Pi_0}(C')$, and $k_{\Delta}(C') = k_{\Pi_{0C_{m'}}}(C')$.

Note that, by construction, $C_{m'}' = C_{m'}$ if and only if $\neq(a, b) \in C_{m'}$.

Let $\Pi_1^{C_{m'}'}$ denote the fragment of Π_1 which is obtained by pruning Π_1 at the root node of Δ , (thus removing the fragment Δ from Π_1 , making $C_{m'}'$ the label of a leaf in $\Pi_1^{C_{m'}'}$). Then $\Pi_1^{C_{m'}'}$ is an EPC_<-deduction containing exactly one equality clause, namely E_1 , (which occurs at the root).

Further, it is not hard to see that, $k_{\Pi_1^{C_{m'}'}}(C_{m'}') \geq 1$, and
if $C_{m'}' \neq C_{m'}$, then $\forall C' \notin \{C_{m'}, C_{m'}'\}$, $k_{\Pi_1^{C_{m'}'}}(C') = k_{\Pi_0^{C_{m'}}}(C')$.
 $k_{\Pi_1^{C_{m'}'}}(C_{m'}') = k_{\Pi_0^{C_{m'}}}(C_{m'}') + 1$; $k_{\Pi_1^{C_{m'}'}}(C_{m'}) = k_{\Pi_0^{C_{m'}}}(C_{m'}) - 1$.
while if $C_{m'}' = C_{m'}$, $\forall C'$, $k_{\Pi_1^{C_{m'}'}}(C') = k_{\Pi_0^{C_{m'}}}(C')$.

Also, clearly $\kappa(\Pi_0) = \kappa(\Pi_1) = \kappa(\Pi_1^{C_{m'}'}) + \kappa(\Delta) - 1$, and $\kappa(\Delta) = \kappa(\Pi_{0C_{m'}}) > 1$,
so $\kappa(\Pi_1^{C_{m'}'}) \leq \kappa(\Pi_0) - 1 < K$.

We claim (4) that $H \Sigma_0$ -subsumes $C_{m'}'$.

To see claim (4):

Recall that $C_{m'} = K' \mid G' \mid (D\gamma)^- \mid I' \mid I''$, where $I' = \bigcup \{\neq(\vec{b}'_{i'}, \vec{b}): 1 \leq i' \leq i\}$;
 $I'' = \bigcup \{\neq(\vec{a}'_{k'}, \vec{a}): 1 \leq k' \leq k\} \cup \bigcup \{\neq(\vec{a}'_{k'}, \vec{a}'_{j'}): 1 \leq k' \leq k; 1 \leq j' \leq j\}$.

Thus $C_{m'}' = K' \mid G' \mid (D\gamma)^- \mid I' \mid I'' \mid \neq(a, b)$.

Since $H \subseteq K' \mid G' \mid (D\gamma)^- \mid I' \mid I'' \mid I^{iv}$, clearly to show that $H \Sigma_0$ -subsumes $C_{m'}'$, it suffices to show that $I^{iv} \Sigma_0$ -subsumes $(D\gamma)^- \mid I' \mid I'' \mid \neq(a, b)$.

Recall the choice of $P(\vec{a})$, $\neg P(\vec{b}) \in D\gamma$, and $\neq(a, b) \in C\varphi \cap \Delta(D^-\gamma, D^0\gamma) \setminus cl_0(D\gamma)$,
where $\vec{a} \equiv a_1, \dots, a_J$; $\vec{b} \equiv b_1, \dots, b_J$; and $\forall j: 1 \leq j \leq J$, either:

- i. $a_j = b_j$, or
 - ii. $\{a_j, b_j\} \subseteq \alpha$, for some $\alpha \in \mathcal{T}_{D\gamma^-}$, or
 - iii. $\{a_j, b_j\} = \{a, b\}$,
- (iii. occurring, for at least one $j: 1 \leq j \leq J$, say j_0).

Now it is easy to see that $\forall j: 1 \leq j \leq J$, either:

- i'. $a_j = b_j$, or
- ii'. $\{a_j, b_j\} \subseteq \alpha'$, for some $\alpha' \in \mathcal{T}_{(D\gamma^- \cup \{\neg(a,b)\})}$.

To show that $I^{iv} \Sigma_0$ -subsumes $(D\gamma)^- \mid I' \mid I'' \mid \neg(a,b)$, it will suffice to show that for any literal $\neg((\vec{a}'_{k'})_j, (\vec{b}'_{i'})_j) \in I^{iv}$,

(for some $k': 1 \leq k' \leq k$, and some $i': 1 \leq i' \leq i$, where $P(\vec{b}'_{i'}) \in F$, and any $j: 1 \leq j \leq J$),
 $(\vec{a}'_{k'} \equiv (\vec{a}'_{k'})_1, \dots, (\vec{a}'_{k'})_J; \vec{b}'_{i'} \equiv (\vec{b}'_{i'})_1, \dots, (\vec{b}'_{i'})_J)$.

either:

- a) $(\vec{a}'_{k'})_j = (\vec{b}'_{i'})_j$, or
- b) $\{(\vec{a}'_{k'})_j, (\vec{b}'_{i'})_j\} \subseteq \alpha$, for some $\alpha \in \mathcal{T}_{(D\gamma^- \cup \{\neg(a,b)\}) \cup I' \cup I''}$.

Let $\neg((\vec{a}'_{k'})_j, (\vec{b}'_{i'})_j)$ be such a literal.

Recall $\neg((\vec{a}'_{k'})_j, a_j) \in I''$. Thus either $(\vec{a}'_{k'})_j = a_j$, or there is some $\beta \in \mathcal{T}_{I''}$ such that $\{(\vec{a}'_{k'})_j, a_j\} \subseteq \beta$.

Also, $\neg((\vec{b}'_{i'})_j, b_j) \in I'$, and so either $(\vec{b}'_{i'})_j = b_j$, or for some $\beta' \in \mathcal{T}_{I'}$, $\{(\vec{b}'_{i'})_j, b_j\} \subseteq \beta'$.

But either $a_j = b_j$, or $\{a_j, b_j\} \subseteq \alpha'$, for some $\alpha' \in \mathcal{T}_{(D\gamma^- \cup \{\neg(a,b)\})}$.

Since for every $\{\alpha', \alpha''\} \subseteq \mathcal{T}_{I'} \cup \mathcal{T}_{I''} \cup \mathcal{T}_{(D\gamma^- \cup \{\neg(a,b)\})}$, (α' and α'' not necessarily distinct), if $\alpha' \cap \alpha'' \neq \emptyset$, there is an (equivalence class) $\alpha \in \mathcal{T}_{(D\gamma^- \cup \{\neg(a,b)\}) \cup I' \cup I''}$ such that $\alpha' \cup \alpha'' \subseteq \alpha$, it is now easy to see that either a) or b) (above) must hold. This completes the justification of claim (4), and so $H \Sigma_0$ -subsumes $C_{m'}'$.

Let Γ'_H denote the $\text{EPC}_{<}$ -deduction of H , which is the result of first affixing $\Gamma_{C_{n_0}}$ to a leaf of Γ'' labelled by C_{n_0} , (recall $k_{\Gamma''}(C_{n_0}) \geq 1$), obtaining a $\text{EPC}_{<}$ -deduction of $C_{m''}$, and then following this, and the $\text{EPC}_{<}$ -deduction Γ_{C_l} of C_l , by the resolvent H , of $C_{m''}$ and C_l , (see Figure E.2. below).

Note that $\kappa(\Gamma'_H) = \kappa(\Gamma'') + \kappa(\Gamma_{C_{n_0}}) + \kappa(\Gamma_{C_l}) - 1$, and

- i) $\forall C' \notin \{C_{n_0}\}, k_{\Gamma'_H}(C') = k_{\Gamma_{C_{n_0}}}(C') + k_{\Gamma''}(C') + k_{\Gamma_{C_l}}(C')$, and
- ii) $k_{\Gamma'_H}(C_{n_0}) = k_{\Gamma_{C_{n_0}}}(C_{n_0}) + k_{\Gamma''}(C_{n_0}) + k_{\Gamma_{C_l}}(C_{n_0}) - 1$.

Recalling inequalities (γ), we also have: $\forall C' \notin \{C_{n_0'}\}, k_{\Gamma'_H}(C') \leq k_{\Gamma_{C_{n_0}}}(C') + k_{\Gamma''}(C') + k_{\Gamma_{C_l}}(C')$;
 $k_{\Gamma'_H}(C_{n_0'}) \leq k_{\Gamma_{C_{n_0}}}(C_{n_0'}) + k_{\Gamma''}(C_{n_0'}) + k_{\Gamma_{C_l}}(C_{n_0'}) - 1$.

Also, evidently, (see figure E.1), $\forall C' \notin \{C_{n_0'}, D'\}, k_{\Pi_0 C_{m'}}(C') = k_{\Gamma_{C_{n_0}}}(C') + k_{\Gamma''}(C') + k_{\Gamma_{C_l}}(C')$;

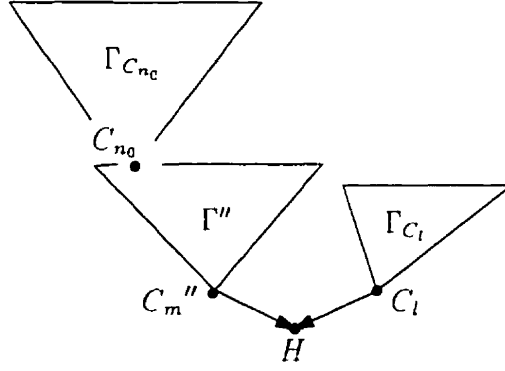


FIGURE E.2

$$k_{\Pi_0 C_m'}(D') = k_{\Gamma_{C_{n_0}}}(D') + k_{\Gamma}(D') + k_{\Gamma_{C_l}}(D') + 1;$$

$$k_{\Pi_0 C_m'}(C_{n_0}') = k_{\Gamma_{C_{n_0}}}(C_{n_0}') + k_{\Gamma}(C_{n_0}') + k_{\Gamma_{C_l}}(C_{n_0}') - 1.$$

Putting these last two sets of inequalities together, we get:

$$\forall C' \notin \{C_{n_0}', D'\}, \quad k_{\Gamma_H}(C') \leq k_{\Pi_0 C_m'}(C').$$

$$\text{Also, } k_{\Gamma_H}(C_{n_0}') \leq k_{\Gamma_{C_{n_0}}}(C_{n_0}') + k_{\Gamma}(C_{n_0}') + k_{\Gamma_{C_l}}(C_{n_0}') - 1 \leq k_{\Pi_0 C_m'}(C_{n_0}'), \text{ but}$$

$$k_{\Gamma_H}(D') \leq k_{\Gamma_{C_{n_0}}}(D') + k_{\Gamma}(D') + k_{\Gamma_{C_l}}(D') \leq k_{\Pi_0 C_m'}(D') - 1.$$

$$\text{Thus, } (\epsilon): \forall C' \notin \{D'\}, \quad k_{\Gamma_H}(C') \leq k_{\Pi_0 C_m'}(C'); \quad k_{\Gamma_H}(D') \leq k_{\Pi_0 C_m'}(D') - 1.$$

Suppose $H = C_{m'}'$.

Let Π_2 , say, denote the result of affixing Γ_H to a leaf with label $C_{m'}'$ in $\Pi_1^{C_{m'}'}$.

Then clearly Π_2 is a $\text{EPC}_{<}$ -deduction, containing exactly one equality clause, namely E_1 .

$$\text{Now } \forall C' \notin \{C_{m'}'\}, \quad k_{\Pi_2}(C') = k_{\Gamma_H}(C') + k_{\Pi_1^{C_{m'}'}}(C');$$

$$k_{\Pi_2}(C_{m'}') = k_{\Gamma_H}(C_{m'}') + k_{\Pi_1^{C_{m'}'}}(C_{m'}') - 1.$$

$$\text{Recall that if } C_{m'}' \neq C_{m'}, \quad \forall C' \notin \{C_{m'}, C_{m'}'\}, \quad k_{\Pi_1^{C_{m'}'}}(C') = k_{\Pi_0^{C_{m'}}}(C'),$$

$$k_{\Pi_1^{C_{m'}'}}(C_{m'}') = k_{\Pi_0^{C_{m'}}}(C_{m'}') + 1; \quad k_{\Pi_1^{C_{m'}'}}(C_{m'}) = k_{\Pi_0^{C_{m'}}}(C_{m'}) - 1,$$

$$\text{while if } C_{m'}' = C_{m'}, \quad \forall C', \quad k_{\Pi_1^{C_{m'}'}}(C') = k_{\Pi_0^{C_{m'}}}(C').$$

Putting these (conditional) equalities together, we get:

$$\forall C' \notin \{C_{m'}'\}, \quad k_{\Pi_2}(C') = k_{\Gamma_H}(C') + k_{\Pi_0^{C_{m'}}}(C'); \quad k_{\Pi_2}(C_{m'}') = k_{\Gamma_H}(C_{m'}') + k_{\Pi_0^{C_{m'}}}(C_{m'}') - 1.$$

By definition of Π_0 , $\Pi_0^{C_{m'}}$, and $\Pi_0^{C_{m'}}$, (see figure E.1),

$$\forall C' \notin \{C_{m'}'\}, \quad k_{\Pi_0}(C') = k_{\Pi_0^{C_{m'}}}(C') + k_{\Pi_0^{C_{m'}}}(C'), \text{ and}$$

$$k_{\Pi_0}(C_{m'}) = k_{\Pi_0^{C_{m'}}}(C_{m'}) + k_{\Pi_0^{C_{m'}}}(C_{m'}) - 1.$$

Thus, $\forall C', \quad k_{\Pi_2}(C') = k_{\Gamma'_H}(C') + k_{\Pi_0}(C') - k_{\Pi_0 C_{m'}}(C')$.

Finally, this last inequality, and (ϵ) , yields:

$$\forall C' \notin \{D'\}, \quad k_{\Pi_2}(C') \leq k_{\Pi_0}(C'); \quad k_{\Pi_2}(D') \leq k_{\Pi_0}(D') - 1.$$

Thus, in the case when $H = C_{m'}$, $\mathfrak{F}'' \equiv \{\Pi_2\}$ is a finite (singleton, in this case), collection of $\text{EPC}_{<}$ -deductions Π'' , such that:

- a') $\forall \Pi'' \in \mathfrak{F}''$, Π'' contains a single equality clause, $E_{\Pi''}$,
(necessarily occurring at the root of Π''), (in this case, $E_{\Pi''} = E_{\Pi_2} = E_1$).
- b') $\Sigma_0 \cup \{E_{\Pi''} : \Pi'' \in \mathfrak{F}''\} \models E_1$, (in this case, $\{E_1\} \models E_1$), and
- c') $\forall \Pi'' \in \mathfrak{F}''$, $\forall C' \notin \{D'\}$, $k_{\Pi''}(C') \leq k_{\Pi_0}(C')$; $k_{\Pi''}(D') \leq k_{\Pi_0}(D') - 1$.
(as in this case, $\forall C' \notin \{D'\}$, $k_{\Pi_2}(C') \leq k_{\Pi_0}(C')$; $k_{\Pi_2}(D') \leq k_{\Pi_0}(D') - 1$).

Remember $E_1 \subseteq E_0 \cup \{\neq(a, b)\} \subseteq E' \cup \{\neq(a, b)\} \subseteq \bar{E} \cup \{\neq(a, b)\}$.

For the case when $H \neq C_{m'}$, we will use the induction hypothesis to produce a such an $\mathfrak{F}''$.
(as above).

Now suppose $H \neq C_{m'}$.

Since H Σ_0 -subsumes $C_{m'}$, and $\kappa(\Pi_1^{C_{m'}}) < K$, and $k_{\Pi_1^{C_{m'}}}(C_{m'}) \geq 1$.

(and $\Pi_1^{C_{m'}}$ contains exactly one equality clause, E_1), by the induction hypothesis, from $\Pi_1^{C_{m'}}$, we may obtain a finite family $\mathfrak{F}'''$ of $\text{EPC}_{<}$ -deductions Π''' , such that:

- a') $\forall \Pi''' \in \mathfrak{F}'''$, Π''' contains exactly one equality clause, $E_{\Pi'''}$, say,
(necessarily occurring at the root),
- b') $\Sigma_0 \cup \{E_{\Pi'''} : \Pi''' \in \mathfrak{F}'''\} \models E_1$, and
- c') $\forall \Pi''' \in \mathfrak{F}'''$.
 - i) $\forall C' \notin \{H, C_{m'}\}$, $k_{\Pi'''}(C') \leq k_{\Pi_1^{C_{m'}}}(C')$.
 - ii) $k_{\Pi'''}(H) \leq k_{\Pi_1^{C_{m'}}}(H) + 1$, and
 - iii) $k_{\Pi'''}(C_{m'}) \leq k_{\Pi_1^{C_{m'}}}(C_{m'}) - 1$.

Now, for each $\Pi''' \in \mathfrak{F}'''$, let Π'' denote the result of affixing Γ'_H to some leaf node of Π''' with label H , if there is such a node. (that is, if $k_{\Pi'''}(H) \geq 1$), and otherwise, let $\Pi'' \equiv \Pi'''$.

Let $\mathfrak{F}''$ denote the resulting collection of Π'' 's. (one Π'' for each Π''' in $\mathfrak{F}'''$).

Then clearly $\mathfrak{F}''$ is a finite collection of $\text{EPC}_{<}$ -deductions, Π'' , each of which contains exactly one equality clause, $E_{\Pi''}$, say, and $\Sigma_0 \cup \{E_{\Pi''} : \Pi'' \in \mathfrak{F}''\} \models E_1$.

We will now show that for any $\Pi'' \in \mathfrak{F}''$, $\forall C' \notin \{D'\}$, $k_{\Pi''}(C') \leq k_{\Pi_0}(C')$; $k_{\Pi''}(D') \leq k_{\Pi_0}(D') - 1$.

Let us first consider a $\Pi'' \in \mathfrak{F}''$, which was obtained from a $\Pi''' \in \mathfrak{F}'''$, where $k_{\Pi'''}(H) = 0$, and so $(\Pi'' = \Pi''')$, and $k_{\Pi''}(H) = 0$, and $\Pi'' \in \mathfrak{F}'''$.

Thus $\forall C' \notin \{C_{m'}\}$, $k_{\Pi''}(C') \leq k_{\Pi_1^{C_{m'}}}(C')$; $k_{\Pi''}(C_{m'}) \leq k_{\Pi_1^{C_{m'}}}(C_{m'}) - 1$.

As in the case for $H = C_{m'}$ above, recall that:

if $C_{m'}' \neq C_{m'}$, $\forall C' \notin \{C_{m'}, C_{m'}'\}$, $k_{\Pi_1 C_{m'}'}(C') = k_{\Pi_0 C_{m'}}(C')$,
 $k_{\Pi_1 C_{m'}'}(C_{m'}') = k_{\Pi_0 C_{m'}}(C_{m'}) + 1$; $k_{\Pi_1 C_{m'}'}(C_{m'}) = k_{\Pi_0 C_{m'}}(C_{m'}) - 1$,
 while if $C_{m'}' = C_{m'}$, $\forall C'$, $k_{\Pi_1 C_{m'}'}(C') = k_{\Pi_0 C_{m'}}(C')$.

Again, combining these, we obtain:

$$\forall C' \notin \{C_{m'}'\}, k_{\Pi''}(C') \leq k_{\Pi_0 C_{m'}}(C'); \quad k_{\Pi''}(C_{m'}) \leq k_{\Pi_0 C_{m'}}(C_{m'}) - 1.$$

Again, using: $\forall C' \notin \{C_{m'}'\}$, $k_{\Pi_0}(C') = k_{\Pi_0 C_{m'}}(C') + k_{\Pi_0 C_{m'}}(C')$, and

$$k_{\Pi_0}(C_{m'}) = k_{\Pi_0 C_{m'}}(C_{m'}) + k_{\Pi_0 C_{m'}}(C_{m'}) - 1,$$

we further obtain: $\forall C'$, $k_{\Pi''}(C') \leq k_{\Pi_0}(C') - k_{\Pi_0 C_{m'}}(C')$.

Since $\forall C'$, $k_{\Pi_0 C_{m'}}(C') \geq 0$, and $k_{\Pi_0 C_{m'}}(D') \geq 1$, we thus obtain the result that,

$$\forall C' \notin \{D'\}, k_{\Pi''}(C') \leq k_{\Pi_0}(C'); \quad k_{\Pi''}(D') \leq k_{\Pi_0}(D') - 1.$$

Now consider a $\Pi'' \in \mathfrak{F}''$, obtained from a $\Pi''' \in \mathfrak{F}'''$, where $k_{\Pi'''}(H) \geq 1$. (and clearly, having treated the previous case, we may assume that either $\Pi'' \notin \mathfrak{F}'''$, or $k_{\Pi''}(H) \geq 1$).

Thus suppose Π'' was obtained from $\Pi''' \in \mathfrak{F}'''$, where:

$$\forall C' \notin \{H, C_{m'}'\}, k_{\Pi'''}(C') \leq k_{\Pi_1 C_{m'}'}(C'); :$$

$$1 \leq k_{\Pi'''}(H) \leq k_{\Pi_1 C_{m'}'}(H) + 1, \text{ and } k_{\Pi'''}(C_{m'}') \leq k_{\Pi_1 C_{m'}'}(C_{m'}') - 1.$$

By construction, then $\forall C' \notin \{C_{m'}'\}$, $k_{\Pi''}(C') \leq k_{\Pi_1 C_{m'}'}(C') + k_{\Gamma'_H}(C')$:

$$k_{\Pi''}(C_{m'}') \leq k_{\Pi_1 C_{m'}'}(C_{m'}') + k_{\Gamma'_H}(C_{m'}') - 1.$$

As before, we may then obtain:

$$\forall C' \notin \{C_{m'}'\}, k_{\Pi''}(C') \leq k_{\Pi_0 C_{m'}}(C') + k_{\Gamma'_H}(C') ; \quad k_{\Pi''}(C_{m'}) \leq k_{\Pi_0 C_{m'}}(C_{m'}) + k_{\Gamma'_H}(C_{m'}) - 1,$$

and then obtain: $\forall C'$, $k_{\Pi''}(C') \leq k_{\Pi_0}(C') - k_{\Pi_0 C_{m'}}(C') + k_{\Gamma'_H}(C')$.

Finally, recalling (ε), $\forall C' \notin \{D'\}$, $k_{\Gamma'_H}(C') \leq k_{\Pi_0 C_{m'}}(C') ; \quad k_{\Gamma'_H}(D') \leq k_{\Pi_0 C_{m'}}(D') - 1$,

we obtain our result: $\forall C' \notin \{D'\}$, $k_{\Pi''}(C') \leq k_{\Pi_0}(C') ; \quad k_{\Pi''}(D') \leq k_{\Pi_0}(D') - 1$.

Thus, for any $\Pi'' \in \mathfrak{F}''$, $\forall C' \notin \{D'\}$, $k_{\Pi''}(C') \leq k_{\Pi_0}(C') ; \quad k_{\Pi''}(D') \leq k_{\Pi_0}(D') - 1$, as promised.

Recalling (!):

$\forall C' \notin \{D', D\gamma\}$, $k_{\Pi_0}(C') \leq k_{\chi(\Pi, \rho)}(C') ; \quad k_{\Pi_0}(D') \leq k_{\chi(\Pi, \rho)}(D') + 1$, and $k_{\Pi_0}(D\gamma) \leq k_{\chi(\Pi, \rho)}(D\gamma) - 1$,
 if $D' \neq D\gamma$,

and that, if $D' = D\gamma$, we have ($\Pi_0 \equiv \downarrow(\Pi, \rho)$), and) $\forall C'$, $k_{\Pi_0}(C') = k_{\chi(\Pi, \rho)}(C')$,

(clearly), it follows that for any $\Pi'' \in \mathfrak{F}''$, $\forall C' \notin \{D\gamma\}$, $k_{\Pi''}(C') \leq k_{\chi(\Pi, \rho)}(C')$, and

$k_{\Pi''}(D\gamma) \leq k_{\chi(\Pi, \rho)}(D\gamma) - 1$, (independent of whether or not $H = C_{m'}'$, or $D' = D\gamma$).

Recall $E_1 \subseteq E_0 \cup \{\neq(a, b)\} \subseteq E' \cup \{\neq(a, b)\} \subseteq \overline{E} \cup \{\neq(a, b)\}$.

Now we will use the induction hypothesis again, to produce a finite family $\mathfrak{G}''$ of $\text{EPC}_{<}$ -deductions Π'' , such that:

- a'') each $\Pi'' \in \mathfrak{G}''$ contains exactly one equality clause, $E_{\Pi''}$, say,
- b'') $\Sigma_0 \cup \{E_{\Pi''} : \Pi'' \in \mathfrak{G}''\} \models \{=(a, b)\} \cup \bar{E}$, and such that
- c'') $\forall \Pi'' \in \mathfrak{G}'', \forall C' \notin \{C\varphi, D\gamma\}, k_{\Pi''}(C') \leq k_{\mathfrak{I}(\Pi, \rho)}(C'),$
 $k_{\Pi''}(C\varphi) \leq k_{\mathfrak{I}(\Pi, \rho)}(C\varphi) + 1; k_{\Pi''}(D\gamma) \leq k_{\mathfrak{I}(\Pi, \rho)}(D\gamma) - 1.$

By taking $\mathfrak{F}' \equiv \mathfrak{F}'' \cup \mathfrak{G}''$, evidently we obtain a finite collection of $\text{EPC}_{<}$ -deductions Π' , such that:

- a) each $\Pi' \in \mathfrak{F}'$ contains exactly one equality clause, $E_{\Pi'}$, say,
- b) $\Sigma_0 \cup \{E_{\Pi'} : \Pi' \in \mathfrak{F}'\} \models \bar{E}$, and
- c) $\forall \Pi' \in \mathfrak{F}', \forall C' \notin \{C\varphi, D\gamma\}, k_{\Pi'}(C') \leq k_{\mathfrak{I}(\Pi, \rho)}(C'),$
 $k_{\Pi'}(C\varphi) \leq k_{\mathfrak{I}(\Pi, \rho)}(C\varphi) + 1; k_{\Pi'}(D\gamma) \leq k_{\mathfrak{I}(\Pi, \rho)}(D\gamma) - 1.$

Again, by lifting $\mathfrak{F}'$, as before, evidently we obtain an $\mathfrak{F}$ satisfying the requirements a), b), and c) of the statement of the lemma.

To construct the $\mathfrak{G}''$:

Recall that we are assuming we have proved the result, (stated in the lemma) for any C, D , and Π as in the hypothesis of the lemma, (whenever $\kappa(\Pi) < K$, or) whenever $\kappa(\Pi) = K$, and $0 \leq \epsilon(C, D) < \nu$. Further recall, $\kappa(\downarrow(\Pi, \rho)) = K$, $k_{\mathfrak{I}(\Pi, \rho)}(D\gamma) \geq 1$, $C\varphi \subseteq cl(D\gamma) \subseteq cl(D\gamma) \cup F$, (for some ground set of trivial inequalities F), $\epsilon(C\varphi, D\gamma) = \nu \geq 1$, and $=(a, b) \in C\varphi \cap \Delta(D^- \gamma, D^0 \gamma) \setminus cl_0(D\gamma)$.

Thus, letting $C_0 \equiv C\varphi \setminus \{=(a, b)\}$, clearly $\epsilon(C_0, D\gamma) = \nu - 1 < \nu$, and $C\varphi \subseteq cl(D\gamma) \subseteq cl(D\gamma) \cup F$, (for some ground set of trivial inequalities F), (hence, C_0 Σ_0 -subsumes $D\gamma$). By the induction hypothesis (on ν), there is a finite collection $\mathfrak{G}'''$ of $\text{EPC}_{<}$ -deductions Π''' , where:

- a''') $\forall \Pi''' \in \mathfrak{G}'''$, Π''' contains exactly one equality clause, $E_{\Pi'''}$, say,
- b''') $\Sigma_0 \cup \{E_{\Pi'''} : \Pi''' \in \mathfrak{G}'''\} \models \bar{E}$, and
- c''') $\forall \Pi''' \in \mathfrak{G}''', \forall C' \notin \{C_0, D\gamma\}, k_{\Pi'''}(C') \leq k_{\mathfrak{I}(\Pi, \rho)}(C'),$
 $k_{\Pi'''}(C_0) \leq k_{\mathfrak{I}(\Pi, \rho)}(C_0) + 1; k_{\Pi'''}(D\gamma) \leq k_{\mathfrak{I}(\Pi, \rho)}(D\gamma) - 1.$

Now for each $\Pi''' \in \mathfrak{G}'''$, let us associate an $\text{EPC}_{<}$ -deduction Π'' , as follows.

If $k_{\Pi'''}(C_0) \leq k_{\mathfrak{I}(\Pi, \rho)}(C_0)$, let $\Pi'' \equiv \Pi'''$. Otherwise, (if $k_{\Pi'''}(C_0) = k_{\mathfrak{I}(\Pi, \rho)}(C_0) + 1$), let Π'' denote the result of replacing C_0 at some leaf of Π''' with $C\varphi$, and passing the introduced literal $=(a, b)$ downwards, as required by the rules of $\text{EPC}_{<}$ -deduction. Let $\mathfrak{G}''$ be the collection of these Π'' , (one Π'' for each $\Pi''' \in \mathfrak{G}'''$).

It is easy to see that with $\mathfrak{G}''$ constructed in this way, $\mathfrak{G}''$ is a finite collection of $\text{EPC}_{<}$ -deductions Π'' , such that:

- a'') $\forall \Pi'' \in \mathfrak{G}''$, Π'' contains exactly one equality clause, $E_{\Pi''}$, say,
- b'') $\Sigma_0 \cup \{E_{\Pi''} : \Pi'' \in \mathfrak{G}''\} \models \bar{E} \cup \{=(a, b)\}$, and
- c'') $\forall \Pi'' \in \mathfrak{G}'', \forall C' \notin \{C\varphi, D\gamma\}, k_{\Pi''}(C') \leq k_{\mathfrak{I}(\Pi, \rho)}(C');$

$$k_{\Pi''}(C\varphi)k_{\Pi,\rho}(C\varphi) + 1; k_{\Pi''}(D\gamma)k_{\Pi,\rho}(D\gamma) - 1,$$

as required. This completes the proof of the lemma. $\blacksquare$

For Corollary 6.1.1:

Proof (sketch): By induction on $\nu \equiv \nu(\Pi) \geq 0$, where $\nu(\Pi)$ is defined to be the number of leaf nodes n of Π such that B_n is other than C_n . That is, $\nu(\Pi) = ||\{n \in \mathcal{L} \mid C_n \neq B_n\}||$.

If $\nu(\Pi) = 0$, the result is trivial, (for we may take $\mathfrak{F}$ to be the singleton $\{\Pi\}$).

Now suppose the result for all $0 \leq \nu < N$, and that now $\nu = N$.

Let n_0 denote some leaf node of Π such that $C_{n_0} \neq B_{n_0}$. For each $n \in \mathcal{L}$ such that $n \neq n_0$, let B'_n be B_n , and let B'_{n_0} be C_{n_0} .

By the induction hypothesis, there exists a finite forest $\mathfrak{F}'$ of above-equality $\text{EPC}_{<}$ -deductions Π'' , of equality clauses $E_{\Pi''}$, say, such that:

- a') $\Sigma_0 \cup \{E_{\Pi''} : \Pi'' \in \mathfrak{F}'\} \models E$, and
- b') $\forall \Pi'' \in \mathfrak{F}'$, for any clause B' , $k_{\Pi''}(B') \leq ||\{n \in \mathcal{L} \mid B'_n = B'\}||$.

For any $\Pi'' \in \mathfrak{F}'$, if $B' \notin \{B_{n_0}, C_{n_0}\}$, then for any $n \in \mathcal{L}$, $B' = B'_n$ if and only if $B' = B_n$.

For if $B' = B_n$, (and $B' \notin \{B_{n_0}, C_{n_0}\}$), then $B_n \notin \{B_{n_0}, C_{n_0}\}$, so $n \neq n_0$, so $B_n = B'_n$, and hence $B' = B'_n$. Conversely, if $B' = B'_n$, (and $B' \notin \{B_{n_0}, C_{n_0}\}$), then $B'_n \notin \{B_{n_0}, C_{n_0}\}$, whence $n \neq n_0$, so $B'_n = B_n$, and hence $B' = B_n$.

Thus (by b') above), for any $\Pi'' \in \mathfrak{F}'$, if $B' \notin \{B_{n_0}, C_{n_0}\}$, $k_{\Pi''}(B') \leq ||\{n \in \mathcal{L} \mid B_n = B'\}||$.

Also, (for any $\Pi'' \in \mathfrak{F}'$), $\{n \in \mathcal{L} \mid B'_n = C_{n_0}\} = \{n \in \mathcal{L} \mid B_n = C_{n_0}\} \cup \{n_0\}$,
(and $n_0 \notin \{n \in \mathcal{L} \mid B_n = C_{n_0}\}$, as $B_{n_0} \neq C_{n_0}$),

while $\{n \in \mathcal{L} \mid B'_n = B_{n_0}\} = \{n \in \mathcal{L} \mid B_n = B_{n_0}\} \setminus \{n_0\}$,
(and $n_0 \in \{n \in \mathcal{L} \mid B_n = B_{n_0}\}$).

Thus, (by b'), for any $\Pi'' \in \mathfrak{F}'$, $k_{\Pi''}(C_{n_0}) \leq ||\{n \in \mathcal{L} \mid B'_n = C_{n_0}\}|| = ||\{n \in \mathcal{L} \mid B_n = C_{n_0}\}|| + 1$,

while $k_{\Pi''}(B_{n_0}) \leq ||\{n \in \mathcal{L} \mid B'_n = B_{n_0}\}|| = ||\{n \in \mathcal{L} \mid B_n = B_{n_0}\}|| - 1$.

Now for each $\Pi'' \in \mathfrak{F}'$, either:

1. $k_{\Pi''}(C_{n_0}) = 0$, or
2. $k_{\Pi''}(C_{n_0}) \geq 0$.

In case 1, evidently, for any clause B' , $k_{\Pi''}(B') \leq ||\{n \in \mathcal{L} \mid B_n = B'\}||$.

In this case, let $\mathfrak{F}_{\Pi''}$ denote the singleton, $\{\Pi''\}$.

In case 2, since, by hypothesis, B_{n_0} Σ_0 -subsumes C_{n_0} , by Lemma 6.1.3, there exists a finite forest $\mathfrak{F}_{\Pi''}$, of above-equality $\text{EPC}_{<}$ -deductions Π' , of equality clauses $E_{\Pi'}$, say, where:

- a'') $\Sigma_0 \cup \{E_{\Pi'} \mid \Pi' \in \mathfrak{F}_{\Pi''}\} \models E_{\Pi''}$, and

- b'') $\forall \Pi' \in \mathfrak{F}_{\Pi''}$,
- i. $\forall B' \notin \{B_{n_0}, C_{n_0}\}$, $k_{\Pi'}(B') \leq k_{\Pi''}(B')$,
 - ii. $k_{\Pi'}(B_{n_0}) \leq k_{\Pi''}(B_{n_0}) + 1$, and
 - iii. $k_{\Pi'}(C_{n_0}) \leq k_{\Pi''}(C_{n_0}) - 1$.

Thus, (in case 2, by b''), and the previous inequalities established for $k_{\Pi''}(B_{n_0})$, $k_{\Pi''}(C_{n_0})$, and $k_{\Pi''}(B')$, when $B' \notin \{B_{n_0}, C_{n_0}\}$,

$\forall \Pi' \in \mathfrak{F}_{\Pi''}$, for any clause B' , $k_{\Pi'}(B') \leq \|\{n \in \mathcal{L} \mid B_n = B'\}\|$.

In either case, (1 or 2), then, $\forall \Pi' \in \mathfrak{F}_{\Pi''}$, for any clause B' , $k_{\Pi'}(B') \leq \|\{n \in \mathcal{L} \mid B_n = B'\}\|$.

Let $\mathfrak{F}$ denote $\bigcup_{\Pi'' \in \mathfrak{F}'} \mathfrak{F}_{\Pi''}$.

Then $\mathfrak{F}$ is a finite family of above-equality $\text{EPC}_{<}$ -deductions Π' , of equality clauses $E_{\Pi'}$, say, such that:

- a) $\Sigma_0 \cup \{E_{\Pi'} \mid \Pi' \in \mathfrak{F}\} \models \Sigma_0 \cup \{E_{\Pi''} \mid \Pi'' \in \mathfrak{F}'\} \models E$, and
- b) $\forall \Pi' \in \mathfrak{F}$, for any clause B' , $k_{\Pi'}(B') \leq \|\{n \in \mathcal{L} \mid B_n = B'\}\|$,

as required. ■

E.2. Σ_0 -subsumption in the Equality Fragment

Observations 6.2.1, 6.2.2, and 6.2.3 are easy.

For Lemma 6.2.1:

Proof: If $QR_{full}(A)$ Σ_0 -subsumes B , then there exists a substitution θ such that, modulo symmetry, $QR_{full}(A)\theta \subseteq B \cup F$, for some set F of trivial inequalities. Thus $QR_{full}(A)\theta$ can contain no non-trivial inequality literals.

If $QR_{full}(A)$ is not positive, then $QR_{full}(A)$ must be A . But then $A\theta$ contains no non-trivial inequality literals. It follows that either A is positive, or that $A \neq QR_{full}(A)$, and $QR_{full}(A)$ is positive. In either case, $QR_{full}(A)$ is positive. ■

For Lemma 6.2.2:

Proof: If A Σ_0 -subsumes B , then (since B is an equality clause), modulo symmetry, for some substitution θ , $A\theta \subseteq \mathcal{T}_{B^-}(B) \cup F$, for some trivial inequality clause F . Clearly A must be an equality clause.

Since B is positive, $\mathcal{T}_{B^-}(B) = B$, so, modulo symmetry, $A\theta \subseteq B \cup F$. Thus, θ is a substitution such that $s\theta = t\theta$, for any inequality $\neq(s, t) \in B$. It follows that, with σ denoting the most general substitution which unifies the arguments of each inequality of A , (as in Definition 4.1.2), that $QR_{full}(A) = A\sigma \setminus A^-\sigma$, and for some substitution ρ , $\theta = \sigma\rho$.

Thus $QR_{full}(A)\rho = (A\sigma \setminus A^-\sigma)\rho = A\sigma\rho \setminus A^-\sigma\rho = A\theta \setminus A^-\theta$. Hence, modulo symmetry, $QR_{full}(A)\rho \subseteq B$, (since $QR_{full}(A)\rho = A\theta \setminus A^-\theta$ is positive), and so (clearly), $QR_{full}(A) \Sigma_0$ -subsumes B .

Conversely, if $(A$ is an equality clause, and) $QR_{full}(A) \Sigma_0$ -subsumes B , (for a positive equality clause B), then by Lemma 6.2.1, $QR_{full}(A)$ is positive, and so (by Observation 6.2.1), there is a substitution θ such that, modulo symmetry, $QR_{full}(A)\theta \subseteq B$.

But (as $QR_{full}(A)$ is a positive equality clause), $QR_{full}(A)$ is just $A\sigma \setminus A^-\sigma$, for some (possibly identity) substitution such that $A^-\sigma$ contains only trivial inequalities. Thus, modulo symmetry, $(A\sigma \setminus A^-\sigma)\theta = A\sigma\theta \setminus A^-\sigma\theta \subseteq B$, so, modulo symmetry, $A\sigma\theta \subseteq B \cup A^-\sigma\theta$.

As $A^-\sigma\theta$ contains only trivial inequalities, clearly $A \Sigma_0$ -subsumes B . ■

For Lemma 6.2.3:

Proof: Let $A' \in QR_{first}(A)$, and suppose $A' \Sigma_0$ -subsumes B , for some equality clauses A and B . Since $A' \in QR_{first}(A)$, (it is to see that) A' is (modulo clause ordering), just $A\sigma \setminus I\sigma$, where I is some (possibly empty) set of inequalities of A and σ is a most general substitution unifying the two arguments of every inequality in I .

Since, by assumption, $A' \Sigma_0$ -subsumes B , for some substitution θ , we have, $(A\sigma \setminus I\sigma)\theta \subseteq cl(B) \cup F$, (modulo symmetry), for some set of trivial inequalities F .

But as each inequality in $I\sigma$ is trivial, each inequality in $I\sigma\theta$ is trivial, and (modulo symmetry), $A\sigma\theta \subseteq (A\sigma \setminus I\sigma)\theta \cup I\sigma\theta \subseteq cl(B) \cup F'$, where $F' \equiv F \cup I\sigma\theta$ is a (finite) set of trivial inequalities. Thus, (as $\sigma\theta$ is some substitution, ρ), $A \Sigma_0$ -subsumes B . ■

For Corollary 6.2.1:

Proof: The corollary follows immediately from Lemma 6.2.3, as $QR_{full}(A) \in QR_{first}(A)$. (Observation 4.1.2). ■

For Lemma 6.2.4:

Proof: Since $C \in EPC_-(A, B)$, either:

- i. both C and $QR_{full}(B)$ are the empty clause $\square$, or
- ii. ($QR_{full}(A)$ is positive, and) $C \in EPC_{+,-}(QR_{full}(A), B')$, for some $B' \in QR_{first}(B)$.

In case i.: Clearly $\square \in EPC_-(A', B)$, and since any clause, (in particular, $\square$), Σ_0 -subsumes itself, we are done.

In case ii.: As $A' \Sigma_0$ -subsumes A , (it follows that A' is an equality clause, and that), $A' \Sigma_0$ -subsumes $QR_{full}(A)$.

Since $QR_{full}(A)$ is positive, by Lemma 6.2.2 and Lemma 6.2.1, then, $QR_{full}(A')$ is positive and $QR_{full}(A') \Sigma_0$ -subsumes $QR_{full}(A)$. By Observation 6.2.1, for some A'' which equals $QR_{full}(A')$ modulo symmetry, A'' subsumes $QR_{full}(A)$.

By Lemma 4.5.1, (case (ii)), either A'' subsumes C or there is a $C'' \in \text{EPC}_{+,-}(A'', B')$ such that C'' subsumes C .

If A'' subsumes C , then $A'' \Sigma_0$ -subsumes C , so by Observation 6.2.2 (2), $QR_{full}(A') \Sigma_0$ -subsumes C . Thus, by Corollary 6.2.1, $A' \Sigma_0$ -subsumes C , (and we're done).

Otherwise, there is $C'' \in \text{EPC}_{+,-}(A'', B')$ such that C'' subsumes C , and so by Observation 6.2.3 (1), there is $C' \in \text{EPC}_{+,-}(QR_{full}(A'), B')$, such that C' is equal to C'' , modulo both ordering and symmetry. Since C'' subsumes C , $C'' \Sigma_0$ -subsumes C , and again by Observation 6.2.2 (2), $C' \Sigma_0$ -subsumes C . But since $B' \in QR_{first}(B)$, $C' \in \text{EPC}_-(A', B)$, (and again, we're done). ■

For Lemma 6.2.5:

Proof: Since $C \in \text{EPC}_+(A, B)$, (both $QR_{full}(A)$ and $QR_{full}(B)$ are positive, and) $C \in \text{EPC}_{+,+}(QR_{full}(A), QR_{full}(B))$.

As $A' \Sigma_0$ -subsumes A , and $B' \Sigma_0$ -subsumes B , it follows that A' and B' are equality clauses, and that), $A' \Sigma_0$ -subsumes $QR_{full}(A)$ and $B' \Sigma_0$ -subsumes $QR_{full}(B)$.

Since $QR_{full}(A)$ and $QR_{full}(B)$ are positive, by Lemma 6.2.2 and Lemma 6.2.1, then, both $QR_{full}(A')$ and $QR_{full}(B')$ are positive and $QR_{full}(A') \Sigma_0$ -subsumes $QR_{full}(A)$, while $QR_{full}(B') \Sigma_0$ -subsumes $QR_{full}(B)$.

By Observation 6.2.1, for some A'' and B'' , which, modulo symmetry, equal $QR_{full}(A')$ and $QR_{full}(B')$, respectively, A'' subsumes $QR_{full}(A)$, while B'' subsumes $QR_{full}(B)$.

By Lemma 4.5.2, either:

- a') A'' subsumes C ,
- b') B'' subsumes C , or
- c') there is a $C'' \in \text{EPC}_{+,+}(A'', B'')$ such that C'' subsumes C .

In case (a'), $A'' \Sigma_0$ -subsumes C , so by Observation 6.2.2 (2), $QR_{full}(A') \Sigma_0$ -subsumes C . Thus, by Corollary 6.2.1, $A' \Sigma_0$ -subsumes C , (and we have (a)).

Similarly, in case (b'), $B' \Sigma_0$ -subsumes C , (and we have (b)).

In case (c'), there is a $C'' \in \text{EPC}_{+,+}(A'', B'')$ such that $C'' \Sigma_0$ -subsumes C . By Observation 6.2.3 (2), there is $C' \in \text{EPC}_{+,+}(QR_{full}(A'), QR_{full}(B''))$, (and hence, $C' \in \text{EPC}_+(A', B')$), such that C' is equal to C'' , modulo both ordering and symmetry, and so (again by Observation 6.2.2 (2)), $C' \Sigma_0$ -subsumes C , (and we have (c)). ■

For Corollary 6.2.2:

Proof: Just as the proof for Corollary 4.5.1, by a simple induction on the depth $k \geq 0$ of the EPC_+ -deduction Π .

Of course, for QR -positive B if B' is any clause such that $B' \Sigma_0$ -subsumes B , then $B' \Sigma_0$ -subsumes $QR_{full}(B)$, and $QR_{full}(B)$ is positive (since B is QR -positive), so by Lemma 6.2.2, $QR_{full}(B') \Sigma_0$ -subsumes $QR_{full}(B)$, so by Lemma 6.2.1, $QR_{full}(B')$ is positive, and so B' is

QR -positive.

The case $k = 0$: Here $B \in \mathcal{C}$. Hence there is a $B' \in \mathcal{C}'$, such that $B' \Sigma_0$ -subsumes B . That is, there is depth 0 EPC_+ -deduction of a clause B' from $\mathcal{C}'$, such that $B' \Sigma_0$ -subsumes B .

Suppose the result for all $0 \leq k < N$, and that now $k = N$.

$B \in EPC_+(A, D)$, for some A and D , having EPC_+ -deductions Π_A and Π_D , of depths $k_A < k$ and $k_D < k$, say, respectively.

By applying the induction hypothesis, (twice), there are EPC_+ -deductions $\Pi_{A'}$ and $\Pi_{D'}$, of depths $k_{A'} \leq k_A$ and $k_{D'} \leq k_D$, of clauses A' and D' , respectively, from $\mathcal{C}'$, such that $A' \Sigma_0$ -subsumes A , and $D' \Sigma_0$ -subsumes D .

By Lemma 6.2.5, either $A' \Sigma_0$ -subsumes B , or $D' \Sigma_0$ -subsumes B , or there exists $B' \in EPC_+(A', D')$ such that $B' \Sigma_0$ -subsumes B .

In the first case, take Π' to be $\Pi_{A'}$ and B' to be A' , in the second case, take Π' to be $\Pi_{D'}$ and B' to be D' , and in the last case, take Π' to be the result of following $\Pi_{A'}$ and $\Pi_{D'}$ by the EPC_+ -resolvent, $B' \in EPC_+(A', D')$.

Then Π' is a depth $k' \leq k$ EPC_+ -deduction Π' , of a clause B' from $\mathcal{C}'$, where $B' \Sigma_0$ -subsumes B , as required. ■

Corollary 6.2.3 follows immediately from Corollary 6.2.2.

For Corollary 6.2.4:

Proof (Sketch): Let $C \in \mathcal{C}$. Then there is a $C' \in \mathcal{C}'$ such that either $C' \Sigma_0$ -subsumes C , or $C' \Sigma_0$ -subsumes $QR_{full}(C)$. But if $C' \Sigma_0$ -subsumes C , then $C' \Sigma_0$ -subsumes $QR_{full}(C)$. Thus, under our hypotheses, for every $C \in \mathcal{C}$, there exists a $C' \in \mathcal{C}'$ such that $C' \Sigma_0$ -subsumes $QR_{full}(C)$.

Let $D \in EPC_+(\mathcal{C})$.

If $D \in \mathcal{C}$, then there is a $C' \in \mathcal{C}'$ such that $C' \Sigma_0$ -subsumes $QR_{full}(D)$, (and we're done).

Otherwise, $D \in EPC_+(\mathcal{C}) \setminus \mathcal{C}$, so D is positive and $QR_{full}(D)$ is D .

Further, evidently, $D \in EPC_+(\mathcal{C}'')$, where $\mathcal{C}''$ is the set $\{QR_{full}(C) \mid C \in \mathcal{C}\}$.

For every $C'' \in \mathcal{C}''$, there exists a $C' \in \mathcal{C}'$ such that $C' \Sigma_0$ -subsumes C'' . Thus, by Corollary 6.2.3, there is a $D' \in EPC_+(\mathcal{C}')$ such that $D' \Sigma_0$ -subsumes D . Since D is $QR_{full}(D)$, we are done. ■

For Lemma 6.2.6:

Proof: The proof resembles that for Lemma 4.5.8.

By induction on n , where n is the number of inequalities $\neq(s', t') \in B'$ such that $\neq(s', t')\theta$ is trivial.

If $n = 0$, then the left-most inequality $\neq(s'', t'')$, say, of B' (exists, and), is such that $\neq(s'', t'')\theta$ is non-trivial. As $B' \in QR_{first}(B')$, (ordered as originally), taking B' for B'' , and θ for ρ , suffices.

Suppose the result for all $0 \leq n < N$, and that now $n = N$.

If the left-most inequality, $\neq(s', t')$, say, of B' , is such that $\neq(s', t')\theta$ is non-trivial, then again, taking B' for B'' , and θ for ρ , suffices.

Otherwise, $\neq(s', t')\theta$ is trivial, (so θ unifies s' and t'), so the most general substitution σ , say, which unifies $\{s', t'\}$, exists, and for some substitution θ' , we have $\sigma\theta' = \theta$.

As $\neq(s', t')$ is left-most in B' , we have some $B''' \in QR_{first}(B')$, where B''' is some ordering of $B'\sigma \setminus \{\neq(s', t')\sigma\}$.

For each $\neq(s''', t''') \in B'''$ such that $\neq(s''', t''')\theta'$ is trivial,

there is some $\neq(s', t') \in B'$ such that $\neq(s', t')\sigma = \neq(s''', t''')$, and (hence), such that also,

$\neq(s', t')\theta = \neq(s', t')\sigma\theta' = \neq(s''', t''')\theta'$ is trivial.

Thus the number of inequalities $\neq(s''', t''') \in B'''$ such that $\neq(s''', t''')\theta'$ is trivial, is strictly less than N .

Also, clearly, $B'''\theta' \subseteq B'\sigma\theta' = B'\theta$, and since, modulo symmetry, $B'\theta \subseteq B$, (it is the case that), modulo symmetry, $B'''\theta' \subseteq B$.

Thus by the induction hypothesis, there is some $B'' \in QR_{first}(B''')$ containing a (non-trivial) left-most inequality $\neq(s'', t'')$, say, such that for some substitution ρ , modulo symmetry, $B''\rho \subseteq B$, and $\neq(s'', t'')\rho$ is non-trivial.

But since $B''' \in QR_{first}(B')$ and $B'' \in QR_{first}(B''')$, $B'' \in QR_{first}(B')$, and so we're done. ■

For Lemma 6.2.7:

Proof: The proof is similar to that of Lemma 4.5.9. We prove the Lemma by induction on the depth k .

As in the proof of Lemma 4.5.9, D must be of the form:

$$C_0 \setminus (F_0 \cup \{e_0\}) \cup \dots \cup C_{k-1} \setminus (F_{k-1} \cup \{e_{k-1}\}) \cup B \setminus B^- \cup F,$$

where F , and for each $i: 0 \leq i \leq k-1$, F_i , consist only of trivial inequalities.

(where the F_i represent the inequalities in C_i which are eliminated, under " QR_{full} ", so that the $C_i \setminus F_i$, are positive, for each $i: 0 \leq i \leq k-1$.)

$$QR_{full}(D) \text{ is of the form } C_0 \setminus (F_0 \cup \{e_0\}) \cup \dots \cup C_{k-1} \setminus (F_{k-1} \cup \{e_{k-1}\}) \cup B \setminus B^-.$$

B' is equal, modulo symmetry, to some B^{iv} , such that $B^{iv}\theta \subseteq B$.

If $B'\theta$ contains only trivial inequality literals, then $B^{iv}\theta$ contains only trivial inequality literals, and $B^{iv}\theta \subseteq B \setminus B^- \cup F \subseteq QR_{full}(D) \cup F$, for some F containing only trivial inequalities, and so, by Lemma 4.5.4, $QR_{full}(B^{iv})$ subsumes $QR_{full}(D)$. Hence, $QR_{full}(B^{iv}) \Sigma_0$ -subsumes $QR_{full}(D)$, and hence, by Observation 6.2.2 (2), $QR_{full}(B') \Sigma_0$ -subsumes $QR_{full}(D)$, (that is, case (ii) holds).

Thus we may assume $B'\theta$ contains at least one non-trivial inequality. Of course, under this assumption, $B^{iv}\theta$ and (therefore) B also contain at least one non-trivial inequality, and hence, (since B is ground), $QR_{full}(B) \neq \square$.

By Lemma 6.2.6, there exists a $B'' \in QR_{first}(B')$, containing a (non-trivial) left-most inequality $\neq(s'', t'')$, say, such that for some substitution ρ , $B''\rho \subseteq B$, modulo symmetry, and $\neq(s'', t'')\rho$ is non-trivial.

Let $\neq(s, r)$ denote this non-trivial inequality, $\neq(s'', t'')\rho \in B''\rho$. Then either $\neq(s, r) \in B$ or $\neq(r, s) \in B$.

Evidently, by Lemma 4.5.6, (re-ordering B), and Corollary 4.5.2, we may assume that either $\neq(s, r)$ or $\neq(r, s)$, occurs as the left-most inequality in B .

As Π is a depth k $EPC_{<}^{+1}$ -deduction of D from ground equality clauses $\{B\} \cup \mathcal{C}$, there exist sequences $C_0, \dots, C_{k-1}$, and $D_0, \dots, D_{k-1}, D_k$, such that:

- i.) $D_0 \equiv B$ and $D_k \equiv D$,
- ii.) $\forall i: 0 \leq i \leq k-1, C_i \in \mathcal{C}$, and
- iii.) $\forall i: 0 \leq i \leq k-1, D_{i+1} \in EPC_{-}(C_i, D_i)$, (so that C_i is the QR -positive parent, while D_i is the negative parent of EPC_{-} -resolvent D_{i+1}).

In particular, for some QR -positive $C_0 \in \mathcal{C}$, $D_1 \in EPC_{-}(C_0, B)$.

Thus $D_1 \in EPC_{+,-}(QR_{full}(C_0), B'')$, for some $B'' \in QR_{first}(B)$.

But as (we are assuming) either the non-trivial inequality $\neq(s, r)$ or the non-trivial inequality $\neq(r, s)$ occurs as the left-most inequality in B , B'' must be none other than B .

Let C denote (the positive equality clause), $QR_{full}(C_0)$. By hypothesis, there is some $C'_0 \in \mathcal{C}'$ such that $C'_0 \Sigma_0$ -subsumes C_0 . Since $C'_0 \Sigma_0$ -subsumes C_0 , it follows that $C'_0 \Sigma_0$ -subsumes C , and hence, (by Lemma 6.2.2), $QR_{full}(C'_0) \Sigma_0$ -subsumes C . Let C' denote $QR_{full}(C'_0)$.

Since $C' \Sigma_0$ -subsumes C , (and C is positive, it follows, by Lemma 6.2.1, that C' is positive, and), by Observation 6.2.1, there is a clause C'' such that $C'' = C'$ modulo symmetry, and such that C'' subsumes C .

Let B^v denote a clause which is equal, modulo symmetry, to B'' , and is such that $B^v\rho \subseteq B$, so that the left-most inequality of B^v is either $\neq(s'', t'')$ or $\neq(t'', s'')$, and has image (maps) under ρ to the left-most inequality in B . (which is either $\neq(s, r)$ or $\neq(r, s)$).

As the left-most inequality of B^v (exists, and) has image (maps) under ρ to the left-most inequality in B , $B^v\rho \subseteq B$, and $D_1 \in EPC_{+,-}(C, B)$, by Lemma 4.5.1, case (ii), either:

- I. C'' subsumes D_1 , or
- II. there exists $D_1'' \in EPC_{+,-}(C'', B^v)$ such that D_1'' subsumes D_1 .

In case I., evidently $C' \Sigma_0$ -subsumes D_1 .

In case II., there exists a substitution, θ' , say, such that $D_1''\theta' \subseteq D_1$. Since $C'' = C'$ modulo symmetry, and $B^v = B''$, modulo symmetry, by Observation 6.2.3 (1), there exists $D_1' \in EPC_{+,-}(C', B'')$ such that D_1' equals D_1'' , modulo both symmetry and ordering. Thus, evidently, there exists $D_1' \in EPC_{+,-}(C', B'')$ such that for some substitution θ' , $D_1'\theta' \subseteq D_1$, modulo symmetry. As $B'' \in QR_{first}(B')$, and $C' = QR_{full}(C'_0)$, $D_1' \in EPC_{-}(C'_0, B')$.

We now do an induction on the depth k .

Suppose $k = 1$. Then $D = D_1$, and D_1 is QR -positive, and so $QR_{full}(D_1)$ is positive.

In case I., clearly (i) (of this lemma) holds, and we're done.

In case II., since for some substitution θ' , $D_1'\theta' \subseteq D_1$, modulo symmetry, evidently, $D_1' \Sigma_0$ -subsumes D_1 . Hence $D_1' \Sigma_0$ -subsumes $QR_{full}(D_1)$. Since $QR_{full}(D_1)$ is positive, by Lemma 6.2.2, $QR_{full}(D_1') \Sigma_0$ -subsumes $QR_{full}(D_1)$.

Take D' to be D_1' , and $\mathcal{C}''$ to be $\mathcal{C}'$. Then evidently the deduction of $D_1' \in \text{EPC}_-(C'_0, B')$ is a depth 1 $\text{EPC}_<^{+1}$ -deduction Π' , of a (QR -positive equality) clause D' from $\{B'\} \cup \mathcal{C}''$, such that $QR_{full}(D') \Sigma_0$ -subsumes $QR_{full}(D)$, as required.

Suppose the result for all $1 \leq k < N$, and that now $k = N$.

In case I., there is some substitution φ such that $C''\varphi \subseteq D_1$. Since C'' is positive, (it is not hard to see that), $C'\varphi \subseteq D_1 \setminus D_1^- \subseteq B \setminus B^- \subseteq QR_{full}(D)$. Thus C'' subsumes $QR_{full}(D)$, so $C' \Sigma_0$ -subsumes $QR_{full}(D)$, and clearly (i) (of this lemma) holds, and we're done.

Thus we may assume case II. applies.

As the sequences $C_1, \dots, C_{k-1}$ and $D_1, \dots, D_{k-1}, D_k \equiv D$ represent a depth $k-1$ $\text{EPC}_<^{+1}$ -deduction of D from ground equality clauses $\{D_1\} \cup \mathcal{C}$. (and modulo symmetry, $D_1'\theta' \subseteq D_1$), by induction hypothesis, either:

- i'. for some $C' \in \mathcal{C}'$, $QR_{full}(C') \Sigma_0$ -subsumes $QR_{full}(D)$, or
- ii'. $QR_{full}(D_1') \Sigma_0$ -subsumes $QR_{full}(D)$, or
- iii'. there is a finite collection $\mathcal{C}'''$ of (QR -positive equality) clauses C''' , each $C''' \in \mathcal{C}'''$ having an EPC_+ -deduction $\Pi_{C'''}$ from $\mathcal{C}'$, and an $\text{EPC}_<^{+1}$ -deduction Π'' , with depth $k'' \leq k-1$, of some (QR -positive equality) clause D' from $\{D_1'\} \cup \mathcal{C}'''$, such that $QR_{full}(D') \Sigma_0$ -subsumes $QR_{full}(D)$.

In case (i'), we have case (i), (of the Lemma), directly, and so there is nothing further required here.

In case (ii'), evidently, D_1' must again be QR -positive, and again taking D' to be D_1' , and $\mathcal{C}''$ to be $\mathcal{C}'$, the deduction of $D' \in \text{EPC}_-(C'_0, B')$ is a depth $1 \leq k$ $\text{EPC}_<^{+1}$ -deduction Π' , of a (QR -positive equality) clause D' from $\{B'\} \cup \mathcal{C}''$, such that $QR_{full}(D') \Sigma_0$ -subsumes $QR_{full}(D)$, as required.

In case (iii'), taking $\mathcal{C}''$ to be $\mathcal{C}''' \cup \{C'_0\}$, and preceding the depth $k'' \leq k-1$ $\text{EPC}_<^{+1}$ -deduction Π'' , of D' from $\{D_1'\} \cup \mathcal{C}'''$, by the resolution $D_1' \in \text{EPC}_-(C'_0, B')$, evidently we have a finite collection $\mathcal{C}''$ of (QR -positive equality) clauses C'' , each $C'' \in \mathcal{C}''$ having an EPC_+ -deduction $\Pi_{C''}$ from $\mathcal{C}'$, and an $\text{EPC}_<^{+1}$ -deduction Π' , with depth $k' \leq k$, of a (QR -positive equality) clause D' from $\{B'\} \cup \mathcal{C}''$, such that $QR_{full}(D') \Sigma_0$ -subsumes $QR_{full}(D)$, as required. ■

For Lemma 6.2.8:

Proof (sketch): The proof is similar to that for the analogous Lemma 4.5.6.

Since D is ground and QR -positive, D must be positive, except possibly, for some trivial inequality literals.

There exist sequences $C_0, \dots, C_{k-1}$ and $D_0, \dots, D_{k-1}, D_k$ such that:

- i.) $D_0 \equiv B$ and $D_k \equiv D$,
- ii.) $\forall i: 0 \leq i \leq k-1, C_i \in \mathcal{C}$, and
- iii.) $\forall i: 0 \leq i \leq k-1, D_{i+1} \in \text{EPC}_-(C_i, D_i)$, (so that C_i is the QR -positive parent, while D_i is the negative parent of EPC_- -resolvent D_{i+1}).

Let e_i denote the (positive equality) literal of C_i which is resolved away in the resolution responsible for $D_{i+1} \in \text{EPC}_-(C_i, D_i)$, and suppose B^- denotes the inequalities of B .

It is not hard to see that D must be of the form:

$$C_0 \setminus (F_0 \cup \{e_0\}) \cup \dots \cup C_{k-1} \setminus (F_{k-1} \cup \{e_{k-1}\}) \cup B \setminus B^- \cup F,$$

where F , and for each $i: 0 \leq i \leq k-1, F_i$, consist only of trivial inequalities.

(The F_i represent the inequalities in C_i which are eliminated, under " QR_{full} ", so that the $C_i \setminus F_i$ are positive, for each $i: 0 \leq i \leq k-1$.)

Further, every (trivial) inequality $\neq(t, t) \in F$, must have been inherited from B , so $F \subseteq B$.

$QR_{full}(D)$ is of the form $C_0 \setminus (F_0 \cup \{e_0\}) \cup \dots \cup C_{k-1} \setminus (F_{k-1} \cup \{e_{k-1}\}) \cup B \setminus B^-$.

Also, evidently it must be the case that, for each equivalence class α in the set of equivalence classes, $\mathcal{T}_{B^-}$, (which is $\mathcal{T}_{B^- \setminus F}$, since trivial inequalities never contribute), $\alpha \subseteq \beta$, for some equivalence class β , generated from equalities in $\{e_0, e_1, \dots, e_{k-1}\}$.

It is also not hard to see that, for every non-trivial (ground) inequality $\neq(s', t')$, such that, for some $\alpha \in \mathcal{T}_{B^-}$, $\{s', t'\} \subseteq \alpha$, there is a positive equality clause $E_{s', t'}$, which is EPC_+ -deducible from $\{C_0, C_1, \dots, C_{k-1}\}$, such that

$$E_{s', t'} \subseteq C_0 \setminus (F_0 \cup \{e_0\}) \cup \dots \cup C_{k-1} \setminus (F_{k-1} \cup \{e_{k-1}\}) \cup \{(s', t'), (t', s')\},$$

and $\emptyset \neq \{(s', t'), (t', s')\} \cap E_{s', t'}$.

Since B' Σ_0 -subsumes B , there is a substitution θ such that modulo symmetry, $B'\theta \subseteq cl(B) \cup F$, for some trivial inequality clause F . Since B is an equality clause, $cl(B)$ is just $\mathcal{T}_{B^-}(B)$. Since B' is ground, modulo symmetry, $B' \subseteq cl(B) \cup F$, so, modulo symmetry, $B' \subseteq \mathcal{T}_{B^-}(B) \cup F$, and, modulo symmetry, $(B')^- \subseteq \mathcal{T}_{B^-}(B^-)$. Thus, for each equivalence class α' in the set of equivalence classes, $\mathcal{T}_{(B')^-}$, $\alpha' \subseteq \alpha$, for some $\alpha \in \mathcal{T}_{B^-}$. In particular, for every non-trivial inequality $\neq(s', t') \in (B')^-$, there is some $\alpha \in \mathcal{T}_{B^-}$, such that $\{s', t'\} \subseteq \alpha$.

Now if $\chi(B') > 1$, (of Definition 4.5.1), then by Lemma 4.5.5, with $k' > 1$ denoting $\chi(B')$, from B' together with the $E_{s', t'}$ above, there is a depth k' $\text{EPC}_<^{+1}$ -deduction Π' of a clause G , such that $G \subseteq C_0 \setminus (F_0 \cup \{e_0\}) \cup \dots \cup C_{k-1} \setminus (F_{k-1} \cup \{e_{k-1}\}) \cup B' \setminus (B')^- \cup F'$, for some trivial inequality clause F' .

On the other hand, if $\chi(B') = 0$, we know B' is already such a clause G , as above.

But since, modulo symmetry, $B' \subseteq \mathcal{T}_{B^-}(B) \cup F$, clearly $k' = \chi(B') \leq \chi(B) \leq k$, the last inequality by Observation 4.5.1. Taking G for the required D' , (if $\chi(B') > 0$), it remains only to show that there is a (equality clause) $G' \in \text{EPC}_+(\mathcal{C} \cup \{G\})$ such that $G' \Sigma_0$ -subsumes $QR_{full}(D)$.

As $G \subseteq C_0 \setminus (F_0 \cup \{e_0\}) \cup \dots \cup C_{k-1} \setminus (F_{k-1} \cup \{e_{k-1}\}) \cup B' \setminus (B')^- \cup F'$, for some trivial inequality clause F' , and (recall), $QR_{full}(D)$ is $C_0 \setminus (F_0 \cup \{e_0\}) \cup \dots \cup C_{k-1} \setminus (F_{k-1} \cup \{e_{k-1}\}) \cup B \setminus B^-$, clearly, other than trivial inequalities, the only literals $L \in G$ which might not lie in $QR_{full}(D)$, are those positive equality literals $L \in B' \setminus (B')^- \setminus B$.

As $B' \subseteq \mathcal{T}_{B^-}(B) \cup F$, every positive equality literal $L \in B'$ is $\mathcal{T}_{B^-}$ -equivalent to some positive equality literal $L' \in B$. That is, for each positive equality literal, $=(s', t')$, say, in B' , there is some positive equality literal $=(s, t)$, say, such that $=(s, t) \in B$, (or else $=(t, s) \in B$), and such that for some $\alpha \in \mathcal{T}_{B^-}$, $\{s, s'\} \subseteq \alpha$, and for some $\alpha' \in \mathcal{T}_{B^-}$, $\{t, t'\} \subseteq \alpha'$.

By EPC_+ -resolving on those literals $=(s', t') \in B' \setminus (B')^- \setminus B$, with appropriate $E_{s', s}$ and $E_{t', t}$ as necessary, such that $=(s, t) \in B$ (or $=(t, s) \in B$), (it is not hard to see that) eventually one may obtain a clause G' such that, modulo symmetry,

$$G' \subseteq C_0 \setminus (F_0 \cup \{e_0\}) \cup \dots \cup C_{k-1} \setminus (F_{k-1} \cup \{e_{k-1}\}) \cup B \setminus (B)^- \cup F'.$$

Since, clearly, $G' \Sigma_0$ -subsumes $QR_{full}(D)$, there is indeed $G' \in \text{EPC}_+(\mathcal{C} \cup \{G\})$ such that $G' \Sigma_0$ -subsumes $QR_{full}(D)$. ■

For Lemma 6.2.9:

Proof: Lemma 6.2.8 justifies this Lemma in the case when $\mathcal{C}'$ is $\mathcal{C}$.

By Lemma 6.2.8, under the hypotheses, either:

- i') there is a (equality clause) $G'' \in \text{EPC}_+(\mathcal{C} \cup \{B'\})$ such that $G'' \Sigma_0$ -subsumes $QR_{full}(D)$, or
- ii') there is a finite collection $\mathcal{C}'''$ of (QR -positive equality) clauses C''' , each $C''' \in \mathcal{C}'''$ having an EPC_+ -deduction $\Pi_{C'''}$ from $\mathcal{C}$, and an $\text{EPC}_<^{+1}$ -deduction Π'' , with depth $k'' \leq k$, of some (QR -positive equality) clause D'' from $\{B'\} \cup \mathcal{C}'''$, such that there is a (equality clause) $G'' \in \text{EPC}_+(\mathcal{C} \cup \{D''\})$ such that $G'' \Sigma_0$ -subsumes $QR_{full}(D)$.

In case (i'): Since there is $G'' \in \text{EPC}_+(\mathcal{C} \cup \{B'\})$ such that $G'' \Sigma_0$ -subsumes $QR_{full}(D)$, by Corollary 6.2.3, there is a clause $G' \in \text{EPC}_+(\mathcal{C}' \cup \{B'\})$ such that $G' \Sigma_0$ -subsumes G'' , and hence, (by transitivity of Σ_0 -subsumption), such that $G' \Sigma_0$ -subsumes $QR_{full}(D)$, (and we have (i), so here we're done).

In case (ii'): Again by (repeated application of) Corollary 6.2.3, there is a finite collection $\mathcal{C}^{iv}$ of (QR -positive equality) clauses C^{iv} , each $C^{iv} \in \mathcal{C}^{iv}$ having an EPC_+ -deduction $\Pi_{C^{iv}}$ from $\mathcal{C}'$, such that for each $C''' \in \mathcal{C}'''$, there is a $C^{iv} \in \mathcal{C}^{iv}$, such that $C^{iv} \Sigma_0$ -subsumes C''' .

Now by Lemma 6.2.7, (applied to the depth $k'' \leq k$ $\text{EPC}_<^{+1}$ -deduction Π'' , of the QR -positive equality clause D'' from $\{B'\} \cup \mathcal{C}'''$), either:

- i'') for some $C^{iv} \in \mathcal{C}^{iv}$, $QR_{full}(C^{iv}) \Sigma_0$ -subsumes $QR_{full}(D'')$, or
- ii'') $QR_{full}(B') \Sigma_0$ -subsumes $QR_{full}(D'')$, or

iii'') there is a finite collection $\mathcal{C}''$ of (QR -positive equality) clauses C'' , each $C'' \in \mathcal{C}''$ having an EPC_+ -deduction from $\mathcal{C}^{iv}$, and an $EPC_{<}^{+1}$ -deduction Π' , with depth $k' \leq k''$, of some (QR -positive equality) clause D' from $\{B'\} \cup \mathcal{C}''$, such that $QR_{full}(D') \Sigma_0$ -subsumes $QR_{full}(D'')$.

In each of the cases (i''), (ii''), (iii''), above, we have $QR_{full}(A) \Sigma_0$ -subsumes $QR_{full}(D'')$, for some clause A . Specifically, A is some $C^{iv} \in \mathcal{C}^{iv}$ in case (i''), A is B' in case (ii''), and A is D' (for some D' as in (iii'') above), in case (iii'').

As D'' is QR -positive, $QR_{full}(D'')$ is positive, and so by Lemma 6.2.2, $A \Sigma_0$ -subsumes $QR_{full}(D'')$.

By our hypothesis (of (ii')), there is a (equality clause) $G'' \in EPC_+(\mathcal{C} \cup \{D''\})$ such that $G'' \Sigma_0$ -subsumes $QR_{full}(D)$.

As $QR_{full}(D)$ is positive, by Lemma 6.2.2, $QR_{full}(G'') \Sigma_0$ -subsumes $QR_{full}(D)$.

By Corollary 6.2.4, there is a $G' \in EPC_+(\mathcal{C}' \cup \{A\})$, such that $G' \Sigma_0$ -subsumes $QR_{full}(G'')$, and hence, (by the transitivity of Σ_0 -subsumes), such that $G' \Sigma_0$ -subsumes $QR_{full}(D)$.

In Case (i''): A is some $C^{iv} \in \mathcal{C}^{iv}$, (such that $QR_{full}(C^{iv}) \Sigma_0$ -subsumes $QR_{full}(D'')$). Thus $A \in EPC_+(\mathcal{C}')$, and so evidently, $G' \in EPC_+(\mathcal{C}')$, and hence $G' \in EPC_+(\mathcal{C}' \cup \{B'\})$, so we have (i), (and here we're done).

In Case (ii''): A is B' , and we have (i) immediately, (so we're done).

In Case (iii''): There is a depth $k' \leq k'' \leq k$ $EPC_{<}^{+1}$ -deduction Π' , of a (QR -positive equality) clause D' from $\{B'\} \cup \mathcal{C}''$, such that $QR_{full}(D') \Sigma_0$ -subsumes $QR_{full}(D'')$, and A is D' .

Since for every $C^{iv} \in \mathcal{C}^{iv}$, (it is the case that), $C^{iv} \in EPC_+(\mathcal{C}')$, evidently, each $C'' \in \mathcal{C}''$ has an EPC_+ -deduction $\Pi_{C''}$ from $\mathcal{C}'$.

Thus Π' is a depth $k' \leq k$ $EPC_{<}^{+1}$ -deduction of a (QR -positive equality) clause D' , from $\{B'\} \cup \mathcal{C}''$, such that there is a $G' \in EPC_+(\mathcal{C}' \cup \{D'\})$ such that $G' \Sigma_0$ -subsumes $QR_{full}(D)$, as required for (ii), (and so we're done). ■

For Lemma 6.2.10:

Proof: Lemma 6.2.9 justifies this Lemma in the case when everything is ground.

The proof of this lemma bears a strong resemblance to that of Lemma 6.2.9.

(and some semblance to that of Lemma 4.5.10).

Let σ denote the most general substitution which unifies the two terms of each inequality of D , (as in Definition 4.1.2), and let ρ be the substitution which assigns to each variable, x , in $D\sigma$, its associated formal constant, $\bar{x}$. Evidently $D\sigma\rho$ is ground, and $D\sigma\rho = \overline{D\sigma}$.

Let Π_1 denote $\downarrow(\Pi, \sigma\rho)$, (of Definition 4.5.2). Then there is a finite collection $\mathcal{C}^{iv} \subseteq EPC_+(\mathcal{C})$, (of QR -positive equality clauses, C^{iv}), a finite collection $\mathcal{C}'''$, of QR -positive ground substitution instances of clauses in $\mathcal{C}^{iv}$, and a ground substitution instance, B'' , of B , such that Π_1 is a depth $k'' \leq k$, $EPC_{<}^{+1}$ -deduction of some ground (QR -positive equality) clause D'' from $\{B''\} \cup \mathcal{C}'''$.

where $QR_{full}(D'') \subseteq QR_{full}(D\sigma\rho)$.

Since we are assuming B' Σ_0 -subsumes B , B' Σ_0 -subsumes B'' , so for some substitution σ' , $B'\sigma' \subseteq \mathcal{T}_{B''} - (B'') \cup F''$, for some ground trivial inequality clause F'' . Let B''' denote the ground substitution instance, $B'\sigma'$, of B' . Of course, B''' Σ_0 -subsumes B'' .

Since $\mathfrak{C}^{iv} \subseteq \text{EPC}_+(\mathfrak{C})$, by Corollary 6.2.3, for each $C^{iv} \in \mathfrak{C}^{iv}$, there is a $C^{vi} \in \text{EPC}_+(\mathfrak{C}')$, such that C^{vi} Σ_0 -subsumes C^{iv} . Thus, clearly, there is a finite $\mathfrak{C}^{vi} \subseteq \text{EPC}_+(\mathfrak{C}')$, (of QR -positive equality clauses, C^{vi}), such that for each $C^{iv} \in \mathfrak{C}^{iv}$, there is a $C^{vi} \in \mathfrak{C}^{vi}$, such that C^{vi} Σ_0 -subsumes C^{iv} .

Since, for each $C''' \in \mathfrak{C}'''$, there exists a $C^{iv} \in \mathfrak{C}^{iv}$ such that C''' is a ground substitution instance of C^{iv} . evidently, for each $C''' \in \mathfrak{C}'''$, there exists a $C^{vi} \in \mathfrak{C}^{vi}$, such that C^{vi} Σ_0 -subsumes C''' , and hence, also some (associated) substitution θ such that $C^{vi}\theta \subseteq \mathcal{T}_{C'''} - (C''') \cup F'''$, for some ground trivial inequality clause F''' . Let C^v denote the ground substitution instance, $C^{vi}\theta$, of C^{vi} , and let $\mathfrak{C}^v$ denote the collection of the C^v , so that for each $C''' \in \mathfrak{C}'''$, there is some (ground) $C^v \in \mathfrak{C}^v$ such that C^v Σ_0 -subsumes C''' .

(Also, since every $C''' \in \mathfrak{C}'''$ is ground and QR -positive, evidently, every $C^v \in \mathfrak{C}^v$ is QR -positive.)

Now, by applying, Lemma 6.2.9, the result for the ground case which we have proven, we know that either:

- i') there is a $G'' \in \text{EPC}_+(\mathfrak{C}^v \cup \{B'''\})$ such that G'' Σ_0 -subsumes $QR_{full}(D'')$, or
- ii') there is a finite collection $\mathfrak{C}^{vii}$ of (QR -positive equality) clauses C^{vii} , each $C^{vii} \in \mathfrak{C}^{vii}$ having an EPC_+ -deduction $\Pi'_{C^{vii}}$ from $\mathfrak{C}^v$, and such that there is an $\text{EPC}_<^{+1}$ -deduction Π_1' , with depth $k''' \leq k'' \leq k$, of some (QR -positive equality) clause D''' from $\{B'''\} \cup \mathfrak{C}^{vii}$, such that there is a $G'' \in \text{EPC}_+(\mathfrak{C}^v \cup \{D'''\})$ such that G'' Σ_0 -subsumes $QR_{full}(D'')$.

In case(i'), there is an EPC_+ -deduction Π_0' , say of G'' from $\mathfrak{C}^v \cup \{B'''\}$.

By ("thawing" and), lifting Π_0' ,

(the leaf B''' to B' , and the leaves in $\mathfrak{C}^v$ to their corresponding clauses in $\mathfrak{C}^{vi}$),

via Corollary 4.4.2, we obtain an EPC_+ -deduction Π_0 , say, of a (QR -positive equality) clause G' such that G'' is a substitution instance of G' , from $\mathfrak{C}^{vi} \cup \{B'\}$.

Since $\mathfrak{C}^{vi} \subseteq \text{EPC}_+(\mathfrak{C}')$, evidently $G' \in \text{EPC}_+(\mathfrak{C}' \cup \{B'\})$.

Since G' subsumes G'' , and G'' Σ_0 -subsumes $QR_{full}(D'')$, it follows that G' Σ_0 -subsumes $QR_{full}(D'')$.

Since $QR_{full}(D'') \subseteq QR_{full}(D\sigma\rho)$, G' Σ_0 -subsumes $QR_{full}(D\sigma\rho)$.

But $D\sigma\rho$ is only $\overline{D\sigma}$, where σ is the most general substitution unifying the terms of each inequality of D , as in Definition 4.1.2. Thus $QR_{full}(D\sigma\rho) = QR_{full}(\overline{D\sigma}) = \overline{QR_{full}(D\sigma)} = \overline{QR_{full}(D)}$.

Since G' can not contain any formally frozen variables, it follows that G' Σ_0 -subsumes $QR_{full}(D)$.

Thus there is a (equality clause) $G' \in \text{EPC}_+(\mathfrak{C}' \cup \{B'\})$ such that G' Σ_0 -subsumes $QR_{full}(D)$, (and we have (i)).

In case(ii'), by ("thawing" and) lifting, via Corollary 4.4.2, the EPC_+ -deduction $\Pi'_{C^{vii}}$ of C^{vii} from $\mathfrak{C}^v$,

(the leaves in $\mathfrak{C}^v$ to their corresponding clauses in $\mathfrak{C}^{vi}$), for each $C^{vii} \in \mathfrak{C}^{vii}$, we obtain a finite

collection $\mathfrak{C}^{viii}$ of (QR -positive equality) clauses C^{viii} , each $C^{viii} \in \mathfrak{C}^{viii}$ (corresponding to a $C^{vii} \in \mathfrak{C}^{vii}$, the C^{vii} being a substitution instance of the C^{viii} , and) having an EPC_+ -deduction $\Pi_{C^{viii}}$ from $\mathfrak{C}^{vi}$.

Since $\mathfrak{C}^{vi} \subseteq EPC_+(\mathfrak{C}')$, evidently, $\mathfrak{C}^{viii} \subseteq EPC_+(\mathfrak{C}')$.

By (“thawing” and) lifting, via (the special “substitution instance” case of) Lemma 4.5.10, Π_1' , (the leaf B''' to B' and the leaves $C^{vii} \in \mathfrak{C}^{vii}$ to their corresponding $C^{viii} \in \mathfrak{C}^{viii}$), either:

1. $QR_{full}(B')$ subsumes $QR_{full}(D''')$, or
2. there is a finite collection $\mathfrak{C}''$ of (QR -positive equality) clauses C'' , each $C'' \in \mathfrak{C}''$ having an EPC_+ -deduction from $\mathfrak{C}^{viii}$, and an $EPC_{<}^{+1}$ -deduction Π' , with depth $k' \leq k''' \leq k'' \leq k$, of some (QR -positive equality) clause D' from $\{B'\} \cup \mathfrak{C}''$, such that $QR_{full}(D')$ subsumes $QR_{full}(D''')$.

The situation is akin to one which occurred in the proof of Lemma 6.2.9, and we take a similar approach.

In both the cases (1,2), above, we have $QR_{full}(A) \Sigma_0$ -subsumes $QR_{full}(D''')$, for some clause A . Specifically, A is B' in case (1), and A is D' (for some D' as in (2) above), in case (2).

As D''' is QR -positive, $QR_{full}(D''')$ is positive, and so by by Lemma 6.2.2, $A \Sigma_0$ -subsumes $QR_{full}(D''')$.

By our hypothesis (of (ii')), there is a $G'' \in EPC_+(\mathfrak{C}'' \cup \{D'''\})$ such that $G'' \Sigma_0$ -subsumes $QR_{full}(D''')$.

Since D'' is QR -positive, $QR_{full}(D'')$ is positive, and so, $QR_{full}(G'') \Sigma_0$ -subsumes $QR_{full}(D'')$.

By Corollary 6.2.4, there is a $G' \in EPC_+(\mathfrak{C}^{vi} \cup \{A\})$, such that $G' \Sigma_0$ -subsumes $QR_{full}(G'')$, and hence, (by the transitivity of Σ_0 -subsumes), such that $G' \Sigma_0$ -subsumes $QR_{full}(D'')$.

Since $\mathfrak{C}^{vi} \subseteq EPC_+(\mathfrak{C}')$, evidently, $G' \in EPC_+(\mathfrak{C}' \cup \{A\})$.

Again, since $QR_{full}(D'') \subseteq QR_{full}(D\sigma\rho)$, $G' \Sigma_0$ -subsumes $QR_{full}(D\sigma\rho)$.

But, again, $D\sigma\rho$ is only $\overline{D\sigma}$, where σ is the most general substitution unifying the terms of each inequality of D , as in Definition 4.1.2.

Thus $QR_{full}(D\sigma\rho) = QR_{full}(\overline{D\sigma}) = \overline{QR_{full}(D\sigma)} = \overline{QR_{full}(D)}$.

Since (in either case), G' can not contain any formally frozen variables, it follows that $G' \Sigma_0$ -subsumes $QR_{full}(D)$.

Returning now, to the individual consideration of the cases:

In case (1): A is B' , (so there is a $G' \in EPC_+(\mathfrak{C}' \cup \{B'\})$, such that $G' \Sigma_0$ -subsumes $QR_{full}(D)$), and we have (i) immediately, (so we're done).

In case (2): There is a a depth $k' \leq k''' \leq k'' \leq k$ $EPC_{<}^{+1}$ -deduction Π' , of a (QR -positive equality) clause D' from $\{B'\} \cup \mathfrak{C}''$, such that $QR_{full}(D') \Sigma_0$ -subsumes $QR_{full}(D''')$, and A is D' .

Since $\mathfrak{C}^{viii} \subseteq \text{EPC}_+(\mathfrak{C}^{vi})$ and $\mathfrak{C}^{vi} \subseteq \text{EPC}_+(\mathfrak{C}')$,
 (and each $C'' \in \mathfrak{C}''$ has an EPC_+ -deduction from $\mathfrak{C}^{viii}$),
 evidently, each $C'' \in \mathfrak{C}''$ has an EPC_+ -deduction $\Pi_{C''}$ from $\mathfrak{C}'$.

Thus, (each $C'' \in \mathfrak{C}''$ has an EPC_+ -deduction $\Pi_{C''}$ from $\mathfrak{C}'$ and), Π' is a depth $k' \leq k$
 $\text{EPC}_<^{+1}$ -deduction of a (QR -positive equality) clause D' from $\{B'\} \cup \mathfrak{C}''$, such that there is
 a $G' \in \text{EPC}_+(\mathfrak{C}' \cup \{D'\})$ where $G' \Sigma_0$ -subsumes $QR_{full}(D)$, as required for (ii),
 (and so we're done). ■

For Lemma 6.2.11:

Proof: The proof is somewhat similar to that for Lemma 6.2.7.

As $D_1 \in \text{EPC}_-^+(C_0, B) \cap \text{EPC}_-(C_0, B)$, $QR_{full}(C_0)$ is positive, and
 $D_1 \in \text{EPC}_{+,-}(QR_{full}(C_0), B')$, for some (mixed) $B' \in QR_{first}(B)$.

Let C denote (the positive equality clause), $QR_{full}(C_0)$. As $C'_0 \Sigma_0$ -subsumes C_0 , it follows that
 $C'_0 \Sigma_0$ -subsumes C , and hence, (by Lemma 6.2.2), $QR_{full}(C'_0) \Sigma_0$ -subsumes C .

Let C' denote $QR_{full}(C'_0)$.

Since $C' \Sigma_0$ -subsumes C , (and C is positive, it follows, by Lemma 6.2.1, that C' is positive, and),
 by Observation 6.2.1, there is a clause C'' such that $C'' = C'$ modulo symmetry, and such that
 C'' subsumes C .

As $D_1 \in \text{EPC}_{+,-}(C, B')$, by Lemma 4.5.1, case (ii), either:

I. C'' subsumes D_1 , or

II. there exists $D_1'' \in \text{EPC}_{+,-}(C'', B')$ such that D_1'' subsumes D_1 .

In case I., evidently $C' \Sigma_0$ -subsumes D_1 . C' is $QR_{full}(C'_0)$, and since for any equality clause
 A , $A \Sigma_0$ -subsumes $QR_{full}(A)$, by transitivity of Σ_0 -subsumption, $C'_0 \Sigma_0$ -subsumes D_1 .

In case II., there exists a substitution, θ' , say, such that $D_1''\theta' \subseteq D_1$.

Since $C'' = C'$ modulo symmetry, by Observation 6.2.3 (1), there exists

$D_1' \in \text{EPC}_{+,-}(C', B')$ such that D_1' equals D_1'' , modulo both symmetry and ordering. Thus,
 evidently, there exists $D_1' \in \text{EPC}_{+,-}(C', B')$ such that for some substitution θ' , $D_1'\theta' \subseteq D_1$,
 modulo symmetry, and hence, such that $D_1' \Sigma_0$ -subsumes D_1 . As $B' \in QR_{first}(B)$, and
 $C' = QR_{full}(C'_0)$, $D_1' \in \text{EPC}_-(C'_0, B)$.

We now consider the cases when (the depth), $k = 1$, and $k > 1$, separately.

As case (a), (above), is immediately dispensed with, (for each value of k , we may assume case (b),
 as well as the results, above, which hold in case (b).

Suppose $k = 1$. Then $D = D_1$, and D_1 is QR -positive, and so $QR_{full}(D_1)$ is positive.

In case I., clearly (1) (of this lemma) holds, (and we're done).

In case II., since $D_1' \Sigma_0$ -subsumes D , $D_1' \Sigma_0$ -subsumes $QR_{full}(D)$. Of course,
 $D_1' \in \text{EPC}_+(\{D_1', C'_0, \dots, C'_{k-1}\})$, so, case (2.i) (of this lemma) holds, (and we're done).

Suppose $1 < k$. Then D_1 is mixed, $C_1, \dots, C_{k-1}$, and $D_1 = D_1, \dots, D_{k-1}, D_k = D$, represents a (depth $k-1$) $\text{EPC}_{<}^{+1}$ -deduction, Π'' , (of the QR -positive clause, D). Also, $C'_1, \dots, C'_{k-1}$ is a sequence of (not necessarily distinct QR -positive equality clauses) clauses such that $\forall i: 1 \leq i \leq k-1, C'_i \Sigma_0\text{-subsumes } C_i$.

In case I., as $C' \Sigma_0\text{-subsumes } D_1$, by Lemma 6.2.10, either

- i') there is a (equality clause) $G' \in \text{EPC}_+(\{C', C'_1, \dots, C'_{k-1}\})$ such that $G' \Sigma_0\text{-subsumes } QR_{full}(D)$, or
- ii') there is a finite collection $\mathfrak{C}''$ of (QR -positive equality) clauses C'' , each $C'' \in \mathfrak{C}''$ having an EPC_+ -deduction $\Pi_{C''}$ from $\{C'_1, \dots, C'_{k-1}\}$, and an $\text{EPC}_{<}^{+1}$ -deduction Π' , with depth $k' \leq k-1$, of some (QR -positive equality) clause D' from $\{C'\} \cup \mathfrak{C}''$, (with principal ancestor C'), such that there is a (equality clause) $G' \in \text{EPC}_+(\{D', C'_1, \dots, C'_{k-1}\})$ such that $G' \Sigma_0\text{-subsumes } QR_{full}(D)$.

However, C' is positive, so case (ii') can not apply. Thus there is a (equality clause) $G'' \in \text{EPC}_+(\{C', C'_1, \dots, C'_{k-1}\})$ such that $G'' \Sigma_0\text{-subsumes } QR_{full}(D)$.

Since $C'_0 \Sigma_0\text{-subsumes } C'$, by Corollary 6.2.3, there is a $G' \in \text{EPC}_+(\{C'_0, \dots, C'_{k-1}\}) \subseteq \text{EPC}_+(\{D_1', C'_0, \dots, C'_{k-1}\})$, such that $G' \Sigma_0\text{-subsumes } G''$, and hence, (by transitivity), such that $G' \Sigma_0\text{-subsumes } QR_{full}(D)$.

Thus we have case (2.i), (and we're done). Thus we may assume case II. applies.

As $D_1' \Sigma_0\text{-subsumes } D_1$, again by Lemma 6.2.10, either

- i') there is a (equality clause) $G' \in \text{EPC}_+(\{D_1', C'_1, \dots, C'_{k-1}\})$ such that $G' \Sigma_0\text{-subsumes } QR_{full}(D)$, or
- ii') there is a finite collection $\mathfrak{C}''$ of (QR -positive equality) clauses C'' , each $C'' \in \mathfrak{C}''$ having an EPC_+ -deduction $\Pi_{C''}$ from $\{C'_1, \dots, C'_{k-1}\}$, and an $\text{EPC}_{<}^{+1}$ -deduction Π' , with depth $k' \leq k-1$, of some (QR -positive equality) clause D' from $\{D_1'\} \cup \mathfrak{C}''$, (with principal ancestor D_1'), such that there is a (equality clause) $G' \in \text{EPC}_+(\{D', C'_1, \dots, C'_{k-1}\})$ such that $G' \Sigma_0\text{-subsumes } QR_{full}(D)$.

Since, trivially, $\text{EPC}_+(\{D_1', C'_1, \dots, C'_{k-1}\}) \subseteq \text{EPC}_+(\{D_1', C'_0, \dots, C'_{k-1}\})$, case (i'), immediately yields case (2.i), (and we're done).

Case (ii') is just a restatement of case (2.ii), (so we're done). ■



APPENDIX F

Σ_0 -tautology versus Σ_0 -subsumption, and Refutation Completeness of $\text{EPC}_<$ Under Deletion

F.1. Σ_0 -tautology versus Σ_0 -subsumption

For Lemma 7.1.1:

Proof: Since, by hypothesis, A is a Σ_0 -tautology, (by (2) of Theorem 3.2.1), either:

- a) for some equality literal $\text{=(s, t)} \in A$, $s = t$, or
- b) for some (non-trivial) equality literal $\text{=(s, t)} \in A$, $\{s, t\} \subseteq \alpha$, for some $\alpha \in \mathcal{T}_{A^-}$, or
- c) for some literals $P(\vec{s})$; $\neg P(\vec{t})$ in A , $P \neq '='$, $\forall \neq(a, b) \in \neq(\vec{s}, \vec{t})$, either $a = b$,
or $\exists \alpha \in \mathcal{T}_{A^-} : \{a, b\} \subseteq \alpha$.

Since, by hypothesis, A is an equality clause, possibility c) is ruled out, and so either a) or b) holds of A .

Since $A' \in QR_{first}(A)$, (it is to see that), modulo ordering, A' is just $A\sigma \setminus I\sigma$ where I is some (possibly empty) set of inequalities of A and σ is a most general substitution unifying the two arguments of every inequality in I .

If a) holds of A clearly there must also be a trivial equality in A' , and so a) holds of A' , and so (by 2) of Theorem 3.2.1), A' must be a Σ_0 -tautology.

If b) holds of A similarly, it is easy to see that for some positive equality literal $\text{=(s, t)}\sigma \in A\sigma$, either:

- i) $s\sigma = t\sigma$, or
- ii) $\{s\sigma, t\sigma\} \subseteq \alpha'$, for some $\alpha' \in \mathcal{T}_{A\sigma^-}$.

In either case, clearly A' contains the equality $\text{=(s}\sigma, t\sigma)$.

In case (i), then clearly a) holds of A' , and A' is a Σ_0 -tautology.

In case (ii), since every inequality in $I\sigma$ is trivial, $\mathcal{T}_{A\sigma^-} = \mathcal{T}_{(A\sigma \setminus I\sigma)^-}$, and so $\{s\sigma, t\sigma\} \subseteq \alpha'$, for some $\alpha' \in \mathcal{T}_{A'^-}$. Thus again, (by 2) of Theorem 3.2.1), A' is a Σ_0 -tautology. ■

For Corollary 7.1.1:

Proof: By Observation 4.1.2, $QR_{full}(A) \in QR_{first}(A)$. ■

For Theorem 7.1.1:

Proof (sketch): Since C is a Σ_0 -tautology, (by 2) of Theorem 3.2.1), either:

- a) for some equality literal $=(s, t) \in C$, $s = t$, or
- b) for some (non-trivial) equality literal $=(s, t) \in C$, $\{s, t\} \subseteq \alpha$, for some $\alpha \in \mathcal{T}_{C-}$, or
- c) for some literals $P(\vec{s}); \neg P(\vec{t})$ in C , $P \neq '='$, $\forall \neq(a, b) \in \neq(\vec{s}, \vec{t})$, either $a = b$, or $\exists \alpha \in \mathcal{T}_{C-} : \{a, b\} \subseteq \alpha$.

Since D is a $EPC_{<}$ -resolvent of B and C , either:

- I) For some predicate symbol P , $P \neq '='$, both B and C have $<$ -maximal predicate symbol P , exactly one of B or C is pure-positive in P , and the $EPC_{<}$ -resolution producing the resolvent $D \in EPC_{<}^1(B, C)$ was done on the leftmost negatively signed literal with predicate symbol P in the non-pure-positive (with respect to P) parent, or
- II) B and C are equality clauses and:
 - II.i) $D \in EPC_+(B, C)$, or
 - II.ii) $D \in EPC_-(B, C)$, or
 - II.iii) $D \in EPC_-(C, B)$.

Case I): Resolution occurs on a non-equality literal of C , and so if (C is a Σ_0 -tautology because of) either a), or b), it is easy to see that D is a Σ_0 -tautology. Thus we may restrict our attention to the case where neither a), nor b) hold of C , but c) prevails, and so, there are literals $P'(\vec{s}); \neg P'(\vec{t})$ in C , for some $P' \neq '='$, such that $\forall \neq(a, b) \in \neq(\vec{s}, \vec{t})$, either $a = b$, or $\exists \alpha \in \mathcal{T}_{C-} : \{a, b\} \subseteq \alpha$.

Either:

- I.i) B is pure-positive with respect to P , and C is mixed in P , or
- I.ii) C is pure-positive with respect to P , and B is mixed in P .

Case I.i): For some non-empty subsets $B_R \subseteq B$, and $C_R \subseteq C$, (say), the left-most negative P instance, L , say, of C occurring in C_R , (and some term tuples $\vec{s}_1, \dots, \vec{s}_i$ and $\vec{t}_1, \dots, \vec{t}_j$), $B_R \equiv \{P(\vec{s}_1), \dots, P(\vec{s}_i)\}$; $C_R \equiv \{\neg P(\vec{t}_1), \dots, \neg P(\vec{t}_j)\}$, we have

$$D \equiv \bigcup_{1 \leq i' \leq i, 1 \leq j' \leq j} \neq(\vec{s}_{i'}, \vec{t}_{j'}) \cup B \setminus B_R \cup C \setminus C_R.$$

Clearly $C^- \subseteq D^-$.

If P' and P are distinct, then $P'(\vec{s}) \in D$ and $\neg P'(\vec{t}) \in D$, and so clearly D is a Σ_0 -tautology, (by criterion c), applied to D).

Otherwise, $P' = P$, and so $P(\vec{s}) \in C \setminus C_R \subseteq D$.

If $\neg P(\vec{t}) \in D$, then, (since $C^- \subseteq D^-$), again clearly D is a Σ_0 -tautology, (by criterion c), applied to D).

Otherwise, $\neg P(\vec{t}) \in C_R$. Again, $P(\vec{s}) \in D$ and (by c)), $P(\vec{s})$ and $P(\vec{t})$ are $\mathcal{T}_{C-}$ -equivalent, and (since $C^- \subseteq D^-$), hence $\mathcal{T}_D$ -equivalent.

But, since $\bigcup_{1 \leq i' \leq i} \neq(\vec{s}_{i'}, \vec{t}) \subseteq D$, each literal $P(\vec{s}_{i'}) \in B_R$ is $\mathcal{T}_D$ -equivalent to $P(\vec{t})$, and hence, $\mathcal{T}_D$ -equivalent to $P(\vec{s})$.

As $P(\vec{s}) \in D$, and $B \setminus B_R \subseteq D$, it follows that $B \subseteq cl(D)$, and hence, that $B \Sigma_0$ -subsumes D .

Case I.ii): For some non-empty subsets $B_R \subseteq B$, and $C_R \subseteq C$. (say), the left-most negative P instance, L , say, of B occurring in B_R , (and some term tuples $\vec{s}_1, \dots, \vec{s}_i$ and $\vec{t}_1, \dots, \vec{t}_j$), $B_R \equiv \{\neg P(\vec{s}_1), \dots, \neg P(\vec{s}_i)\}$; $C_R \equiv \{P(\vec{t}_1), \dots, P(\vec{t}_j)\}$, we have

$$D \equiv \bigcup_{\substack{1 \leq i' \leq i \\ 1 \leq j' \leq j}} \neq(\vec{s}_{i'}, \vec{t}_{j'}) \cup B \setminus B_R \cup C \setminus C_R.$$

Again, clearly $C^- \subseteq D^-$.

Again, if P' and P are distinct, then $P'(\vec{s}) \in D$ and $\neg P'(\vec{t}) \in D$, and so clearly D is a Σ_0 -tautology, (by criterion c), applied to D).

Otherwise, $P' = P$. But this (together with assumption c)), contradicts our assumption that C is pure-positive with respect to P .

The argument for case I) is now complete.

Case II): Suppose B and C are equality clauses. Then clearly possibility c) for the Σ_0 -tautology C is ruled out, and hence either a) or b) applies.

Case II.i): Suppose $D \in \text{EPC}_+(B, C)$. Then both $QR_{full}(B)$, and $QR_{full}(C)$ are non-empty positive equality clauses, and $D \in \text{EPC}_{+,+}(QR_{full}(B), QR_{full}(C))$.

Let (us denote) $B' \equiv QR_{full}(B)$, and $C' \equiv QR_{full}(C)$.

By Corollary 7.1.1, C' is a Σ_0 -tautology, and since C' is positive, a) must hold for C' .

So for some term t , $=(t, t) \in C'$.

If the $(\text{EPC}_{+,+})$ -resolution between B' and C' ,

(which is responsible for generating $D \in \text{EPC}_{+,+}(B', C')$), does not eliminate the literal $=(t, t)$, clearly, (for some substitution θ , $=(t\theta, t\theta) \in D$, and), D is a Σ_0 -tautology, (under condition a)).

However, if the resolution does eliminate the literal $=(t, t)$ of C' , then, for some substitution θ , (and some term s), D is of the form: $\{=(s\theta, t\theta)\} \cup B'\theta \setminus \{=(s\theta, t\theta), =(t\theta, s\theta)\} \cup C'\theta \setminus \{=(t, t)\theta\}$, and so (modulo symmetry), $B'\theta \subseteq D$, and so, $B' \Sigma_0$ -subsumes D . By Corollary 6.2.1, it follows that, again, $B \Sigma_0$ -subsumes D .

Case II.ii): Suppose $D \in \text{EPC}_-(B, C)$. Then either D is $\square$ and $QR_{full}(C)$ is the empty clause, $\square$, or $D \in \text{EPC}_{+,-}(QR_{full}(B), C')$ for some $C' \in QR_{first}(C)$.

Since (by hypothesis) C is a Σ_0 -tautology, (it is easy to see that) the former case is impossible.

In the latter case, by Lemma 7.1.1, C' is a Σ_0 -tautology, (and so either a) or b) must hold of the equality clause C'). Let $B' \equiv QR_{full}(B)$. Thus, we are assuming that $D \in \text{EPC}_{+,-}(B', C')$ for some Σ_0 -tautology C' , and we need to show either that D is a Σ_0 -tautology, or that $B' \Sigma_0$ -subsumes D .

Clearly $B' \equiv QR_{full}(B)$ must be a positive equality clause.

Further:

- i'. there exists a non-empty subset $B'_R \subseteq B'$, a non-empty subset $C'_R \subseteq C'^-$, the left-most inequality literal, $\neq(s_0, t_0)$, say, of C' in C'_R , and a (possibly empty) subset $C'_S \subseteq C'^- \setminus C'_R$, such that for some symmetric choice function c on $B'_R \cup C'_R \cup C'_S$, with $c(\neq(s_0, t_0)) = 0$, the most general substitution σ , say, unifying:

$$\begin{aligned} & \{ \{=(s', t') \mid =(s', t') \in B'_R; c(=(s', t')) = 0\} \cup \{=(t', s') \mid =(s', t') \in B'_R; c(=(s', t')) = 1\} \cup \\ & \quad \{=(s', t') \mid \neq(s', t') \in C'_R; c(\neq(s', t')) = 0\} \cup \{=(t', s') \mid \neq(s', t') \in C'_R; c(\neq(s', t')) = 1\} \}, \\ & \{s' \mid =(s', t') \in B'_R; c(=(s', t')) = 0\} \cup \{t' \mid =(s', t') \in B'_R; c(=(s', t')) = 1\} \cup \\ & \quad \{s' \mid \neq(s', t') \in C'_S; c(\neq(s', t')) = 0\} \cup \{t' \mid \neq(s', t') \in C'_S; c(\neq(s', t')) = 1\} \}. \end{aligned}$$

exists, σ unifying the two sets in the above doubleton set to the singletons $\{=(s, r)\}$ and $\{s\}$, respectively, for some terms s and r , say, and

- ii'. up to clause ordering,

$$\begin{aligned} D & \equiv B'\sigma \setminus \{=(r, s), =(s, r)\} \cup C'\sigma \setminus (\{\neq(r, s), \neq(s, r)\} \cup C'_S\sigma) \cup \\ & \quad \{\neq(r, t'\sigma) \mid \neq(s', t') \in C'_S; \neq(s', t')\sigma \notin \{\neq(r, s), \neq(s, r)\}; c(\neq(s', t')) = 0\} \cup \\ & \quad \{\neq(s'\sigma, r) \mid \neq(s', t') \in C'_S; \neq(s', t')\sigma \notin \{\neq(r, s), \neq(s, r)\}; c(\neq(s', t')) = 1\} \}. \end{aligned}$$

If a) holds of C' (then for some term t , $=(t, t) \in C'$), then,

(since $C'_S \subseteq C'^-\sigma$ consists only of inequality literals, it follows that $=(t\sigma, t\sigma) \notin C'_S\sigma$, and),

clearly, $=(t\sigma, t\sigma) \in C'\sigma \setminus (\{\neq(r, s), \neq(s, r)\} \cup C'_S\sigma) \subseteq D$. So a) holds of D .

and (again, by 2) of Theorem 3.2.1), D is a Σ_0 -tautology.

Otherwise, b) holds of C' , and it is easy to see that, hence, either a) or b) must hold of $C'\sigma$. But again, if a) holds of $C'\sigma$, D is a Σ_0 -tautology. Thus we may assume that b) holds of $C'\sigma$.

Let $=(s_1, t_1) \in C'\sigma$ such that $s_1 \neq t_1$, and for some $\beta \in \mathcal{T}_{C'\sigma-}$, $\{s_1, t_1\} \subseteq \beta$.

Clearly $=(s_1, t_1) \in D$.

Now (since either $\neq(r, s) \in C'\sigma$, or $\neq(s, r) \in C'\sigma$, it is easy to see that), either $\{s, r\} \subseteq \beta$, or $\{s, r\} \cap \beta = \emptyset$.

If $\{s, r\} \cap \beta = \emptyset$, then $\beta \in \mathcal{T}_{C'\sigma- \setminus \{\neq(r, s), \neq(s, r)\}}$. Also, every non-trivial inequality literal $\neq(s'', t'')$ in $C'_S\sigma$ has one of its (top-level) terms equal to s and hence, for the $\beta' \in \mathcal{T}_{C'\sigma-}$ such that $\{s'', t''\} \subseteq \beta'$, we have $\beta \cap \beta' = \emptyset$.

Thus, (it is not hard to see that) in fact, (if $\{s, r\} \cap \beta = \emptyset$), $\beta \in \mathcal{T}_{C'\sigma- \setminus (\{\neq(r, s), \neq(s, r)\} \cup C'_S\sigma)}$.

Since $C'\sigma \setminus (\{\neq(r, s), \neq(s, r)\} \cup C'_S\sigma) \subseteq D$, $\exists \alpha \in \mathcal{T}_D$ such that $\{s_1, t_1\} \subseteq \beta \subseteq \alpha$, and hence condition b) holds of D and so D is a Σ_0 -tautology.

If $\{s, r\} \subseteq \beta$, then $\{s_1, t_1, s, r\} \subseteq \beta$.

[We first make an observation]. Let A be any equality clause such that:

A1. $=(s_1, t_1) \in A$,

A2. $\{\neq(r, s), \neq(s, r)\} \cap A \neq \emptyset$, and further,

A3. $\{s_1, t_1, s, r\} \subseteq \beta$, for some $\beta \in \mathcal{T}_A$.

For any such A , (by the definition of $\mathcal{T}_{(\cdot)}$, and recalling that, by assumption, $s_1 \neq t_1$, it is not difficult to see that), either:

- i. $s_1 = s$ and $t_1 = r$, or
- ii. $s_1 = s$ and $\{t_1, r\} \subseteq \beta''$ or
- iii. $\{s_1, s\} \subseteq \beta'$ and $t_1 = r$, or
- iv. $\{s_1, s\} \subseteq \beta'$ and $\{t_1, r\} \subseteq \beta''$ or
- v. $s_1 = r$ and $t_1 = s$, or
- vi. $s_1 = r$ and $\{t_1, s\} \subseteq \beta''$ or
- vii. $\{s_1, r\} \subseteq \beta'$ and $t_1 = s$, or
- viii. $\{s_1, r\} \subseteq \beta'$ and $\{t_1, s\} \subseteq \beta''$, or
- ix. $\{s_1, t_1\} \subseteq \beta'$,

for some $\beta', \beta'' \in \mathcal{T}_{A - \{\neg(r, s), \neg(s, r)\}}$.

In the last case (ix.), since $\models(s_1, t_1) \in A$, and hence, $\models(s_1, t_1) \in A \setminus \{\neg(r, s), \neg(s, r)\}$, clearly $A \setminus \{\neg(r, s), \neg(s, r)\}$ is a Σ_0 -tautology (by condition b)).

In any other event, (cases i. - viii., above), clearly $\models(s, r)$ is $\mathcal{T}_{A - \{\neg(r, s), \neg(s, r)\}}$ -equivalent either to $\models(s_1, t_1)$, or to $\models(t_1, s_1)$, (and $\models(r, s)$ is also $\mathcal{T}_{A - \{\neg(r, s), \neg(s, r)\}}$ -equivalent either to $\models(s_1, t_1)$, or to $\models(t_1, s_1)$).

[Having made this observation, we proceed].

Let C'' denote (the clause)

$$C'\sigma \setminus (\{\neg(r, s), \neg(s, r)\} \cup C'_S\sigma) \cup \\ \{\neg(r, t'\sigma) \mid \neg(s', t') \in C'_S; \neg(s', t')\sigma \notin \{\neg(r, s), \neg(s, r)\}; c(\neg(s', t')) = 0\} \cup \\ \{\neg(s'\sigma, r) \mid \neg(s', t') \in C'_S; \neg(s', t')\sigma \notin \{\neg(r, s), \neg(s, r)\}; c(\neg(s', t')) = 1\}.$$

Obviously $C'' \subseteq D$.

[We now separate out a special case.]

If $r = s$ or if every literal, $\neg(s'\sigma, t'\sigma)$, in $C'_S\sigma$ is such that $\neg(s'\sigma, t'\sigma) \in \{\neg(r, s), \neg(s, r)\}$, then $C'' = C'\sigma \setminus \{\neg(r, s), \neg(s, r)\}$.

$C'\sigma$ is such a clause A as above, (satisfying A1 through A3 above), and hence,

(since $C'' = C'\sigma \setminus \{\neg(r, s), \neg(s, r)\}$), either C'' is a Σ_0 -tautology or $\models(s, r)$ is $\mathcal{T}_{C''}$ -equivalent either to $\models(s_1, t_1)$, or to $\models(t_1, s_1)$.

As $C'' \subseteq D$, it follows (respectively), that either D is a Σ_0 -tautology, or $\models(s, r)$ is $\mathcal{T}_D$ -equivalent either to $\models(s_1, t_1)$, or to $\models(t_1, s_1)$.

In the former case, we're done.

In the latter case, since $\models(s_1, t_1) \in D$, it follows that $\models(s, r) \in cl(D)$, and since $B'\sigma \setminus \{\models(r, s), \models(s, r)\} \subseteq D \subseteq cl(D)$, (it follows that) $B'\sigma \subseteq cl(D)$, (modulo symmetry), and so,

B' Σ_0 -subsumes D . By Corollary 6.2.1, it follows that, again, B Σ_0 -subsumes D , (and we're done).

[Having treated the special case, we proceed to treat what remains.]

Otherwise, of course, $r \neq s$ and there exists $\neq(s', t') \in C'_S$ such that $\neq(s', t')\sigma \notin \{\neq(r, s), \neq(s, r)\}$.

Let C''' denote

$$\begin{aligned} C'\sigma \setminus (C'_S\sigma \setminus \{\neq(r, s), \neq(s, r)\}) \cup \\ \{\neq(r, t'\sigma) \mid \neq(s', t') \in C'_S; \neq(s', t')\sigma \notin \{\neq(r, s), \neq(s, r)\}; c(\neq(s', t')) = 0\} \cup \\ \{\neq(s'\sigma, r) \mid \neq(s', t') \in C'_S; \neq(s', t')\sigma \notin \{\neq(r, s), \neq(s, r)\}; c(\neq(s', t')) = 1\}. \end{aligned}$$

Then $C''' \setminus \{\neq(r, s), \neq(s, r)\} \subseteq C''$, and $\{\neq(r, s), \neq(s, r)\} \cap C''' \supseteq \{\neq(r, s), \neq(s, r)\} \cap C'\sigma \neq \emptyset$.

Further, $C'\sigma$ and C''' differ only to the extent that some (top-level) instances of the term s in inequalities of $C'\sigma$ have been replaced by the term r , in C''' .

(though $\{\neq(r, s), \neq(s, r)\} \cap C''' \neq \emptyset$ and $\{\neq(r, s), \neq(s, r)\} \cap C'\sigma \neq \emptyset$).

Thus, (by, for example, considering the inequalities as indicating the edges in a graph whose nodes are the terms, it is not difficult to see that), $\mathcal{T}_{C'\sigma} = \mathcal{T}_{C'''}$.

It is now easy to see that C''' also satisfies A1 through A3 above.

Since $C''' \setminus \{\neq(r, s), \neq(s, r)\} \subseteq C''$, it once again follows that either C'' is a Σ_0 -tautology or $=(s, r)$ is $\mathcal{T}_{C''}$ -equivalent either to $=(s_1, t_1)$, or to $=(t_1, s_1)$, and from this, just as before, we may conclude that either D is a Σ_0 -tautology, or B Σ_0 -subsumes D .

Case II.iii) : Suppose $D \in \text{EPC}_-(C, B)$. Then either D is $\square$ and $QR_{full}(B)$ is the empty clause, $\square$, or $D \in \text{EPC}_{+,-}(QR_{full}(C), B')$ for some $B' \in QR_{first}(B)$.

In the first case, that $QR_{full}(B) = \square$, there exists a substitution θ such that each literal of $B\theta$ is a trivial inequality. Thus, trivially, B Σ_0 -subsumes D , where D is $\square$.

Now suppose $D \in \text{EPC}_{+,-}(QR_{full}(C), B')$ for some $B' \in QR_{first}(B)$.

Let (us denote) $C' \equiv QR_{full}(C)$.

By Corollary 7.1.1, C' is a Σ_0 -tautology, and since C' is positive, a) must hold for C' .

So for some term t , $=(t, t) \in C'$.

If the $(\text{EPC}_{+,-})$ resolution between C' and B' ,

(which is responsible for generating $D \in \text{EPC}_{+,-}(C', B')$),

does not eliminate the literal $=(t, t)$, clearly, (for some substitution σ , $=(t\sigma, t\sigma) \in D$, and),

D is a Σ_0 -tautology, (under condition a)).

However, if the resolution does eliminate the literal $=(t, t)$ of C' , then,

(from the definitions, it is easy to see that), for some non-empty subsets $C'_R \subseteq C'$, and $B'_R \subseteq B'$,

some possibly empty subset $B'_S \subseteq B' \setminus B'_R$, some symmetric choice function c on $C'_R \cup B'_R \cup B'_S$,

some substitution σ , and some term s , (such that $t\sigma = s$), up to clause ordering,

$$\begin{aligned} D \equiv C'\sigma \setminus \{=(s, s)\} \cup B'\sigma \setminus (\{\neq(s, s)\} \cup B'_S\sigma) \cup \\ \{\neq(s, t'\sigma) \mid \neq(s', t') \in B'_S; \neq(s', t')\sigma \notin \{\neq(s, s)\}; c(\neq(s', t')) = 0\} \cup \\ \{\neq(s'\sigma, s) \mid \neq(s', t') \in B'_S; \neq(s', t')\sigma \notin \{\neq(s, s)\}; c(\neq(s', t')) = 1\}, \end{aligned}$$

which is $C'\sigma \setminus \{=(s, s)\} \cup B'\sigma \setminus (\{\neq(s, s)\} \cup B'_S\sigma) \cup B'_S\sigma \setminus \{=(s, s)\}$,

which is $C'\sigma \setminus \{=(s, s)\} \cup B'\sigma \setminus \{\neq(s, s)\}$.

Thus, clearly, $B'\sigma \setminus \{\neq(s, s)\} \subseteq D$, and so, (trivially), $B' \Sigma_0$ -subsumes D . By Lemma 6.2.3, again, it follows that, $B \Sigma_0$ -subsumes D . This completes the argument for case II). ■

F.2. Refutation Completeness of $\text{EPC}_{<}$ Under Deletion

F.2.1. Preliminaries

For Observation 7.2.1:

Proof: This is trivial, (the indexing is only a matter of aesthetics, and), since $\square \Sigma_0$ -subsumes C , for any clause C . ■

For Observation 7.2.2:

Proof: The same as that given for Observation 7.2.1. ■

F.2.2. The Above-Equality Fragment

For Lemma 7.2.1:

Proof: We first introduce some convenient notation.

For any non-empty finite collection, $\mathfrak{F}$, of $\text{EPC}_{<}$ -deductions, let $\nu\kappa(\mathfrak{F})$ denote the maximum number of leaf nodes which occur in a $\Pi \in \mathfrak{F}$. For any $\text{EPC}_{<}$ -deduction Π , $\kappa(\Pi)$ will denote, as usual, the total number of leaf nodes in Π .

Obviously, for any non-empty finite collection, $\mathfrak{F}$, $\nu\kappa(\mathfrak{F}) \geq 1$, and $\nu\kappa(\mathfrak{F}) = 1$ precisely when every $\Pi \in \mathfrak{F}$ is trivial, that is, consisting of a single node which is both a leaf and the root of Π .

In this case we may take $\mathfrak{E}_0$ to be a collection of $E_{\Pi'} \in |\Delta|$, such that $E_{\Pi'} \Sigma_0$ -subsumes E_{Π} , picking one $E_{\Pi'}$ for each $\Pi \in \mathfrak{F}$. Then, since, by the soundness of Σ_0 -subsumption, (Theorem 3.3.4), $\Sigma_0 \cup \{E_{\Pi'}\} \models E_{\Pi}$, clearly, $\forall \Pi \in \mathfrak{F}$, $\Sigma_0 \cup \mathfrak{E}_0 \models E_{\Pi}$.

We claim (1): Given any $\mathfrak{F}$ as in the hypothesis, (of this lemma), if $\nu\kappa(\mathfrak{F}) > 1$, then there is a finite collection, $\mathfrak{F}'$, of above-equality $\text{EPC}_{<}$ -deductions Π' , of equality clauses $E_{\Pi'}$, such that:

- i) for each C' which labels a leaf of any $\Pi' \in \mathfrak{F}'$, $\exists C'' \in |\Delta|$, such that $C'' \Sigma_0$ -subsumes C' ,
- ii) $\forall \Pi \in \mathfrak{F}$, $\Sigma_0 \cup \{E_{\Pi'} \mid \Pi' \in \mathfrak{F}'\} \models E_{\Pi}$, and
- iii) $\nu\kappa(\mathfrak{F}') < \nu\kappa(\mathfrak{F})$.

Justification of claim (1):

By assumption, for each clause C which labels a leaf of any $\Pi \in \mathfrak{F}$, there is some $C' \in |\Delta|$, such that C' Σ_0 -subsumes C .

For each $\Pi \in \mathfrak{F}$, let $\mathfrak{L}_\Pi$ denote the set of leaf nodes of Π , and for each $n \in \mathfrak{L}_\Pi$, let $C(\Pi)_n$ denote the clause which labels n in Π .

Also, (for each $\Pi \in \mathfrak{F}$), to each $n \in \mathfrak{L}_\Pi$, let there be associated a clause $B(\Pi)_n \in |\Delta|$, such that $B(\Pi)_n$ Σ_0 -subsumes $C(\Pi)_n$.

By Corollary 6.1.1, for each $\Pi \in \mathfrak{F}$, there is a finite forest $\mathfrak{F}_\Pi$ of above-equality $\text{EPC}_<$ -deductions Π' , of equality clauses, $E_{\Pi'}$, say, such that:

- a) $\Sigma_0 \cup \{E_{\Pi'} : \Pi' \in \mathfrak{F}_\Pi\} \models E_\Pi$, and
- b) $\forall \Pi' \in \mathfrak{F}_\Pi$, for any clause B , $k_{\Pi'}(B) \leq ||\{n \in \mathfrak{L}_\Pi \mid B(\Pi)_n = B\}||$.

Evidently, (for each $\Pi \in \mathfrak{F}$), for each $\Pi' \in \mathfrak{F}_\Pi$, the leaf clauses of Π' are members of $|\Delta|$, and $\kappa(\Pi') \leq \kappa(\Pi)$.

For any non-trivial $\Pi' \in \mathfrak{F}_\Pi$, there is a node $n_{\Pi'}$, say, with deduction depth 1, such that there is a pair of leaves of Π' which are parents of $n_{\Pi'}$, in Π' , with $(\text{EPC}_<-)$ resolvent $D(n_{\Pi'})$, say, ($D(n_{\Pi'})$ labelling $n_{\Pi'}$ in Π').

Since $|\Delta|$ is a Σ_0 -subsumption deleted $\text{EPC}_<$ -deduction sequence (from S), for every such $D(n_{\Pi'})$, there is some $D'(n_{\Pi'}) \in |\Delta|$, such that $D'(n_{\Pi'})$ Σ_0 -subsumes $D(n_{\Pi'})$.

For any non-trivial $\Pi' \in \mathfrak{F}_\Pi$, let $\text{prune}(\Pi')$ denote the $\text{EPC}_<$ -deduction (of $E_{\Pi'}$), which is the fragment of Π' , where the two parent nodes of (the depth 1 node), $n_{\Pi'}$ have been pruned away. If $\Pi' \in \mathfrak{F}_\Pi$ is trivial, let $\text{prune}(\Pi')$ denote Π' itself.

Then with $\mathfrak{F}'_\Pi \equiv \{\text{prune}(\Pi') \mid \Pi' \in \mathfrak{F}_\Pi\}$, $\mathfrak{F}'_\Pi$ is a finite collection of above-equality $\text{EPC}_<$ -deductions Π'' , of equality clauses, $E_{\Pi''}$, say, such that $\forall \Pi'' \in \mathfrak{F}'_\Pi$, any clause, C''_n , which labels a leaf, n , of Π'' , is Σ_0 -subsumed by a clause in $|\Delta|$, and such that $\Sigma_0 \cup \{E_{\Pi''} : \Pi'' \in \mathfrak{F}'_\Pi\} \models E_\Pi$.

Further, for every $\Pi'' \in \mathfrak{F}'_\Pi$, clearly $\kappa(\Pi'') < \kappa(\Pi)$.

Finally, let $\mathfrak{F}'$ denote $\bigcup_{\Pi \in \mathfrak{F}} \mathfrak{F}'_\Pi$. Then, evidently, $\mathfrak{F}'$ is a finite collection, of above-equality $\text{EPC}_<$ -deductions Π' , of equality clauses $E_{\Pi'}$, such that:

- i) for each C' which labels a leaf of any $\Pi' \in \mathfrak{F}'$, $\exists C'' \in |\Delta|$, such that C'' Σ_0 -subsumes C' ,
- ii) $\forall \Pi \in \mathfrak{F}$, $\Sigma_0 \cup \{E_{\Pi'} \mid \Pi' \in \mathfrak{F}'\} \models E_\Pi$, and
- iii) $\nu\kappa(\mathfrak{F}') < \nu\kappa(\mathfrak{F})$,

as required. This completes our justification of claim (1).

Thus, from claim (1), (and the fact that the result holds if $\nu\kappa(\mathfrak{F}') = 1$), the result now follows by a simple induction on $\nu\kappa = \nu\kappa(\mathfrak{F}) \geq 1$. ■

F.2.3. The Positive Equality Fragment

For Lemma 7.2.2:

Proof: By induction on the (deduction) depth $k \geq 0$, say, of Π .

When $k = 0$, Π consists of a single node labelled by E , and since this node is a leaf, by hypothesis, there is a clause $E' \in |\Delta|$, such that $E' \Sigma_0$ -subsumes E .

(Of course, since E has an EPC_+ -deduction, E is a QR -positive equality clause, and, since $E' \Sigma_0$ -subsumes E , E' is necessarily an QR -positive equality clause).

Now suppose the result for all $0 \leq k < N$, and that the depth, k of Π is N .

The last EPC_+ -resolvent in Π , is $E \in EPC_+^1(E_1, E_2)$, for some QR -positive equality clauses, E_1 and E_2 , each having an EPC_+ -deduction Π_{E_1} , and Π_{E_2} , (such that, for each clause C which labels a leaf of either Π_{E_1} or Π_{E_2} , there is a $C' \in |\Delta|$ such that $C' \Sigma_0$ -subsumes C), of depths $k' < k$, and $k'' < k$, respectively.

By induction hypothesis, there exist (QR -positive equality clauses), $E_1' : E_2' \in |\Delta|$, such that $E_1' \Sigma_0$ -subsumes E_1 , and $E_2' \Sigma_0$ -subsumes E_2 .

By Lemma 6.2.5 either:

- a) $E_1' \Sigma_0$ -subsumes E .
- b) $E_2' \Sigma_0$ -subsumes E , or
- c) there exists an $E'' \in EPC_+(E_1', E_2')$ such that $E'' \Sigma_0$ -subsumes E .

In case (a) or (b), clearly there is a QR -positive equality clause $E' \in |\Delta|$, such that $E' \Sigma_0$ -subsumes E .

In case (c), since Δ is a Σ_0 -subsumption deleted $EPC_{\leq}$ -deduction sequence (from S , for some S), there exists an $E' \in |\Delta|$, such that $E' \Sigma_0$ -subsumes E'' , and hence, (by the transitivity of Σ_0 -subsumption), such that $E' \Sigma_0$ -subsumes E . (Again, E' is necessarily a QR -positive equality clause). ■

Corollary 7.2.1 follows immediately from Lemma 7.2.2.

F.2.4. The Negative Equality Fragment

For Lemma 7.2.3:

Proof: By induction on the depth $k \geq 1$ of Π .

To avoid repetition, we first obtain some results which are uniform with respect to the two cases, $k = 1$ and $k > 1$, of the parameter, $k \geq 1$.

We claim (1): to show there exists a $D' \in |\Delta|$, such that $D' \Sigma_0$ -subsumes $QR_{full}(D)$, it will suffice to show:

- (α) for some finite $S' \subseteq |\Delta|$, there exists some clause $G' \in \text{EPC}_+(S')$, such that $G' \Sigma_0$ -subsumes $QR_{full}(D)$.

Justification of claim (1):

Let S' be a finite subset of $|\Delta|$, and $G' \in \text{EPC}_+(S')$, such that $G' \Sigma_0$ -subsumes $QR_{full}(D)$. Then since $S' \subseteq |\Delta|$, by Corollary 7.2.1, there is a $D' \in |\Delta|$ such that $D' \Sigma_0$ -subsumes G' . Since $G' \Sigma_0$ -subsumes $QR_{full}(D)$, by the transitivity of Σ_0 -subsumption, $D' \Sigma_0$ -subsumes $QR_{full}(D)$, as required, (and so (α) suffices).

This completes our justification of claim (1).

Now by Lemma 6.2.10, either:

- i) there is a (equality clause) $G' \in \text{EPC}_+(\mathfrak{C}' \cup \{B'\})$ such that $G' \Sigma_0$ -subsumes $QR_{full}(D)$, or
- ii) there is a finite collection $\mathfrak{C}''$ of (QR -positive equality) clauses C'' , each $C'' \in \mathfrak{C}''$ having an EPC_+ -deduction $\Pi_{C''}$ from $\mathfrak{C}'$, and a $\text{EPC}_<^{+1}$ -deduction Π' , with depth $k' \leq k$, of some (QR -positive equality) clause D'' from principal ancestor B' and ancillary parents in $\mathfrak{C}''$, such that there is a (QR -positive equality clause), $G' \in \text{EPC}_+(\mathfrak{C}' \cup \{D''\})$, such that $G' \Sigma_0$ -subsumes $QR_{full}(D)$.

We pursue case (i):

In case (i), since $\mathfrak{C}' \cup \{B'\} \subseteq |\Delta|$, we have condition (α) immediately.

We pursue case (ii):

In case (ii), let, (as in case (ii), above), $\mathfrak{C}''$ be a finite collection of QR -positive equality clauses C'' , each $C'' \in \mathfrak{C}''$ having an EPC_+ -deduction $\Pi_{C''}$ from $\mathfrak{C}'$, D'' and G' be QR -positive equality clauses, and Π' be a $\text{EPC}_<^{+1}$ -deduction with depth $k' \leq k$, of D'' from $\{B'\} \cup \mathfrak{C}''$, (that is, from principal ancestor B' and ancillary parents $\mathfrak{C}''$), such that $G' \in \text{EPC}_+(\mathfrak{C}' \cup \{D''\})$ and $G' \Sigma_0$ -subsumes $QR_{full}(D)$.

We now claim (2): to show there exists a $D' \in |\Delta|$, such that $D' \Sigma_0$ -subsumes $QR_{full}(D)$, it will suffice to show:

(β) for some finite $S' \subseteq |\Delta|$, there exists some clause $G^{iv} \in EPC_+(S')$,
such that $G^{iv} \Sigma_0$ -subsumes $QR_{full}(D'')$.

Justification of claim (2):

Let S' be a finite subset of $|\Delta|$, and $G^{iv} \in EPC_+(S')$, such that $G^{iv} \Sigma_0$ -subsumes $QR_{full}(D'')$.
Then since $S' \subseteq |\Delta|$, by Corollary 7.2.1, there is a $G^v \in |\Delta|$ such that $G^v \Sigma_0$ -subsumes G^{iv} ,
and hence, (by the transitivity of Σ_0 -subsumption), such that $G^v \Sigma_0$ -subsumes $QR_{full}(D'')$.

Now since $G' \in EPC_+(\mathfrak{C}' \cup \{D''\})$, by Corollary 6.2.4, there exists a $G^{vi} \in EPC_+(\mathfrak{C}' \cup \{G^v\})$,
such that $G^{vi} \Sigma_0$ -subsumes $QR_{full}(G')$.

But as D is QR -positive, $QR_{full}(D)$ is positive, and hence, (as $G' \Sigma_0$ -subsumes $QR_{full}(D)$),
by Lemma 6.2.2, $QR_{full}(G') \Sigma_0$ -subsumes $QR_{full}(D)$.

Thus, by the transitivity of Σ_0 -subsumption, $G^{vi} \Sigma_0$ -subsumes $QR_{full}(D)$.

Since, ($G^{vi} \in EPC_+(\mathfrak{C}' \cup \{G^v\})$ and), $\mathfrak{C}' \cup \{G^v\} \subseteq |\Delta|$, we have condition (α).
(and since (α) suffices, so does (β)).

This completes our justification of claim (2).

Now, (in case (ii)), let $C''_0, \dots, C''_{k'-1}$ and $B' \equiv D''_0, \dots, D''_{k'-1}, D''_{k'} \equiv D''$.

(where $\forall j: 0 \leq j \leq k' - 1, C''_j \in \mathfrak{C}''$), represent Π' (as in Definition 4.1.12).

Since, $\forall j: 0 \leq j \leq k' - 1, C''_j \in \mathfrak{C}'' \subseteq EPC_+(\mathfrak{C}')$, by Corollary 7.2.1, for each C''_j ,
there is a $C'''_j \in |\Delta|$, such that $C'''_j \Sigma_0$ -subsumes C''_j .

Let $C'''_0, \dots, C'''_{k'-1}$, denote a sequence of clauses such that

$\forall j: 0 \leq j \leq k' - 1, C'''_j \in |\Delta|$ and $C'''_j \Sigma_0$ -subsumes C''_j .

By Lemma 6.2.11, (applied to Π' and the C'''_j which Σ_0 -subsume the respective C''_j), either:

ii.1. $C'''_0 \Sigma_0$ -subsumes D'' , or

ii.2. there exists a $D_1''' \in EPC_-(C'''_0, B')$, such that either:

ii.2.a. there is a $G'' \in EPC_+(\{D_1''', C'''_0, \dots, C'''_{k'-1}\})$, such that $G'' \Sigma_0$ -subsumes $QR_{full}(D'')$,
or

ii.2.b. $k > 1$ and there is a finite collection $\mathfrak{C}^{iv}$ of (QR -positive equality) clauses C^{iv} ,
each $C^{iv} \in \mathfrak{C}^{iv}$ having an EPC_+ -deduction $\Pi_{C^{iv}}$ from $\{C'''_1, \dots, C'''_{k'-1}\}$,
and a $EPC_{<}^{+1}$ -deduction Π'' , with depth $k'' \leq k' - 1$, of some
(QR -positive equality) clause D''' from $\{D_1'''\} \cup \mathfrak{C}^{iv}$,
(that is, from principal ancestor D_1''' and ancillary parents $\mathfrak{C}^{iv}$),
such that there is a (QR -positive equality clause) $G'' \in EPC_+(\{D''', C'''_1, \dots, C'''_{k'-1}\})$,
such that $G'' \Sigma_0$ -subsumes $QR_{full}(D'')$.

In case (ii.1), since $C'''_0 \Sigma_0$ -subsumes D'' , $C'''_0 \Sigma_0$ -subsumes $QR_{full}(D'')$,
and since $C'''_0 \in EPC_+(\{C'''_0\})$ and $\{C'''_0\} \subseteq |\Delta|$, we have condition (β).

(Alternatively, since $G' \in \text{EPC}_+(\mathfrak{C}' \cup \{D''\})$ and G' Σ_0 -subsumes $QR_{full}(D)$, by Corollary 6.2.3, and the transitivity of Σ_0 -subsumption, it follows that there exists a $G'' \in \text{EPC}_+(\mathfrak{C}' \cup \{C'''_0\})$ such that G'' Σ_0 -subsumes $QR_{full}(D)$. Since $\mathfrak{C}' \cup \{C'''_0\} \subseteq |\Delta|$, once again, we have condition (α) .)

In case (ii.2.a), let $D_1''' \in \text{EPC}_-(C'''_0, B')$, and $G'' \in \text{EPC}_+(\{D_1''', C'''_0, \dots, C'''_{k'-1}\})$, such that G'' Σ_0 -subsumes $QR_{full}(D'')$.

Since $C'''_0; B' \in |\Delta|$,

(and $|\Delta|$ is a Σ_0 -subsumption deleted $\text{EPC}_<$ -deduction sequence from S),

there is a $G''' \in |\Delta|$, such that G''' Σ_0 -subsumes D_1''' .

By Corollary 6.2.3, there is a $G^{iv} \in \text{EPC}_+(\{G''', C'''_0, \dots, C'''_{k'-1}\})$, such that G^{iv} Σ_0 -subsumes G'' , and hence, (by the transitivity of Σ_0 -subsumption), such that G^{iv} Σ_0 -subsumes $QR_{full}(D'')$. Since $\{G''', C'''_0, \dots, C'''_{k'-1}\} \subseteq |\Delta|$, we have condition (β) .

As case (ii.2.b) can apply only if $k > 1$, we hold off our “uniform” analysis of this case, and now return to separately examine the cases $k = 1$ and $k > 1$, for the induction.

If $k = 1$, then only cases (i), (ii.1), and (ii.2.a) may apply, and since (we have shown that) either condition (α) or condition (β) holds in each of these cases, the result holds, and we are done.

Now suppose the result for all $1 \leq k < N$, and that the depth of Π is $k = N$.

The only case needing consideration which remains is case (ii.2.b).

Let, (as in case ((ii.2.b))), $D_1''' \in \text{EPC}_-(C'''_0, B')$, $\mathfrak{C}^{iv}$ be a finite collection of QR -positive equality clauses C^{iv} , each $C^{iv} \in \mathfrak{C}^{iv}$ having an EPC_+ -deduction $\Pi_{C^{iv}}$ from $\{C'''_1, \dots, C'''_{k'-1}\}$,

D''' and G'' be QR -positive equality clauses,

and Π'' be a $\text{EPC}_<^{+1}$ -deduction with depth $k'' \leq k' - 1$, of D''' from $\{D_1'''\} \cup \mathfrak{C}^{iv}$.

(that is, from principal ancestor D_1''' and ancillary parents $\mathfrak{C}^{iv}$).

such that $G'' \in \text{EPC}_+(\{D''', C'''_1, \dots, C'''_{k'-1}\})$ and G'' Σ_0 -subsumes $QR_{full}(D'')$.

As in case (ii.2.a), since $C'''_0; B' \in |\Delta|$, there is a $G''' \in |\Delta|$, such that G''' Σ_0 -subsumes D_1''' .

As $\{C'''_1, \dots, C'''_{k'-1}\} \subseteq |\Delta|$, by (repeated application of) Corollary 7.2.1, there is a finite set,

$\mathfrak{C}^v \subseteq |\Delta|$ such that for each $C^{iv} \in \mathfrak{C}^{iv}$, there is some $C^v \in \mathfrak{C}^v$, such that C^v Σ_0 -subsumes C^{iv} .

By induction, (applied to Π'') there is a $D^{iv} \in |\Delta|$, such that D^{iv} Σ_0 -subsumes $QR_{full}(D''')$.

By Corollary 6.2.4, it now follows that there exists a $G^{iv} \in \text{EPC}_+(\{D^{iv}, C'''_1, \dots, C'''_{k'-1}\})$ such that G^{iv} Σ_0 -subsumes $QR_{full}(G'')$.

But as D'' is QR -positive, $QR_{full}(D'')$ is positive, and hence, (as G'' Σ_0 -subsumes $QR_{full}(D'')$), by Lemma 6.2.2, $QR_{full}(G'')$ Σ_0 -subsumes $QR_{full}(D'')$.

Thus, by the transitivity of Σ_0 -subsumption, G^{iv} Σ_0 -subsumes $QR_{full}(D'')$.

Since $\{D^{iv}, C'''_1, \dots, C'''_{k'-1}\} \subseteq |\Delta|$, once again, we have condition (β) . ■

F.2.5. The Equality Fragment (Synthesis)

For Lemma 7.2.4:

Proof: By induction on the depth $k \geq 0$ of Π .

If $k = 0$, then $D \in \mathfrak{E}$, so, by hypothesis, there is some $E' \in |\Delta|$, such that $E' \Sigma_0$ -subsumes D , and hence, such that $E' \Sigma_0$ -subsumes $QR_{full}(D)$.

Suppose the result for all $0 \leq k < N$, and that now the depth of Π is $k = N$.

Either:

- i. $D \in EPC_{-}(A, B)$, and both D and $QR_{full}(B)$ are the empty clause, $\square$, or
- ii. $D \in EPC_{-}^+(C, B) \cap EPC_{-}(C, B)$, or
- iii. $D \in EPC_{+}^+(C, G)$,

where: A is some equality clause, with a depth $k_A < k$ $EPC_{<}$ -deduction, Π_A , (say), from $\mathfrak{E}$.
 B is a mixed equality clause, with a depth $k_B < k$ $EPC_{<}$ -deduction, Π_B , from $\mathfrak{E}$, and
 C and G are QR -positive equality clauses, with $EPC_{<}$ -deductions Π_C and Π_G , from $\mathfrak{E}$, of depths $k_C < k$ and $k_G < k$, respectively.

In case (i), as $QR_{full}(B)$ is $\square$, B is QR -positive. As $k_B < k$, by induction, there exists $B' \in |\Delta|$, such that $B' \Sigma_0$ -subsumes $QR_{full}(B)$, that is, such that $B' \Sigma_0$ -subsumes $\square$.

Thus, there exists $B' \in |\Delta|$, such that $B' \Sigma_0$ -subsumes $QR_{full}(D)$.

In case (ii), $QR_{full}(B) \neq \square$ and evidently, Π can be decomposed into a $EPC_{<}^{+1}$ -deduction, Γ , and a finite collection $\{\Pi_{C_0}, \dots, \Pi_{C_{k'-1}}\}$ of $EPC_{<}^{+1}$ -deductions, for some $k': 1 \leq k' \leq k$, where for $i: 0 \leq i \leq k' - 1$, C_i is QR -positive, Π_{C_i} is a $EPC_{<}^{+1}$ -deduction of C_i , from $\mathfrak{E}$, of depth k_i , say, where $k_i < k$, and Γ is a $EPC_{<}^{+1}$ -deduction of D , from ancillary parents $C_0, \dots, C_{k'-1}$, and principal ancestor, some mixed $B_0 \in \mathfrak{E}$.

By induction, for each $i: 0 \leq i \leq k' - 1$, there is a $C'_i \in \Delta$, such that $C'_i \Sigma_0$ -subsumes $QR_{full}(C_i)$.

By replacing each ancillary ancestor leaf C_i , in Γ with $QR_{full}(C_i)$, (for $i: 0 \leq i \leq k' - 1$), from Γ , we evidently obtain a $EPC_{<}^{+1}$ -deduction Γ' , where Γ' is a (depth k') $EPC_{<}^{+1}$ -deduction of D , from ancillary parents $QR_{full}(C_0), \dots, QR_{full}(C_{k'-1})$, and principal ancestor $B_0 \in \mathfrak{E}$.

By assumption, there exists $B'_0 \in |\Delta|$, such that $B'_0 \Sigma_0$ -subsumes B_0 .

Thus, by Lemma 7.2.3, there exists a $D' \in |\Delta|$, such that $D' \Sigma_0$ -subsumes $QR_{full}(D)$.

In case (iii), (as $k_C < k$ and $k_G < k$), by induction there exist $C': G' \in |\Delta|$, such that $C' \Sigma_0$ -subsumes $QR_{full}(C)$ and $G' \Sigma_0$ -subsumes $QR_{full}(G)$.

As $D \in EPC_{+}^+(C, G)$, $D \in EPC_{+}^+(QR_{full}(C), QR_{full}(G))$, and so, by Lemma 6.2.5, either:

- a) $C' \Sigma_0$ -subsumes D ,

- b) G' Σ_0 -subsumes D , or
- c) there exists an $D'' \in EPC_+(C', G')$ such that D'' Σ_0 -subsumes D .

Further, D is positive, and so $QR_{full}(D)$ is simply D .

In case (a) or (b), clearly there exists a $D' \in |\Delta|$, such that D' Σ_0 -subsumes $QR_{full}(D)$.

In case (c), (since Δ is a Σ_0 -subsumption deleted $EPC_{<}$ -deduction sequence from S), there exists a $D' \in |\Delta|$ such that D' Σ_0 -subsumes D'' . Hence, (by transitivity of Σ_0 -subsumption), there exists a $D' \in |\Delta|$ such that D' Σ_0 -subsumes D , that is there exists a $D' \in |\Delta|$ such that D' Σ_0 -subsumes $QR_{full}(D)$. ■

F.2.6. Synthesis and Result

For Theorem 7.2.1:

Proof: We have already shown, (in Theorem 5.2.1), that if S is any unsatisfiable set of clauses, then there is a $EPC_{<}$ -refutation Π , say, of $S \cup \{=(x, x)\}$.

Since $\{=(x, x)\} \subseteq \Sigma_0 \subseteq S$, Π is a $EPC_{<}$ -refutation of S .

Of course, Π is $EPC_{<}$ -deduction of the empty clause, from S .

(As the proof of Theorem 5.2.1), For each branch β of Π , let E_β , denote the upper-most (when, in Π , the empty clause is at the bottom, and the leaves are upper-most), (possibly empty), equality clause occurring along β .

(So that E_β contains only positive and negative instances of the equality predicate, $' = '$, while any proper ancestor of E_β in Π , contains at least one non-equality literal.)

Let $\mathfrak{E}$ denote the collection of all the E_β .

Evidently, either $\mathfrak{E}$ consists only of the empty clause, $\square$,

(and this is the case precisely when no equality literals are resolved upon, or even occur, in Π),

or $\square \notin \mathfrak{E} \neq \emptyset$, and $\mathfrak{E}$ contains at least two members. In any case,

(since $EPC_{<}$ -deduction is Σ_0 -sound), $\mathfrak{E} \cup \Sigma_0$ is an unsatisfiable set of equality clauses.

For each clause $E \in \mathfrak{E}$, let Π_E denote a $EPC_{<}$ -deduction of E from S , which occurs as a fragment of Π , (this fragment rooted at a node labelled by E).

Let $\mathfrak{F}$ denote the finite collection $\{\Pi_E \mid E \in \mathfrak{E}\}$.

By Lemma 7.2.1, there is a finite collection of equality clauses $\mathfrak{E}_0 \subseteq |\Delta|$,

such that $\forall E \in \mathfrak{E}, \Sigma_0 \cup \mathfrak{E}_0 \models E$.

In particular, since $\mathfrak{E} \cup \Sigma_0$ is unsatisfiable, so is $\Sigma_0 \cup \mathfrak{E}_0$.

Thus, by (again), Theorem 5.2.1, there is a $EPC_{<}$ -deduction Π_0 , say, of the empty clause, from $\Sigma_0 \cup \mathfrak{E}_0$.

Since, by assumption, $\Sigma_0 \subseteq S$,
 (and Δ is a Σ_0 -subsumption deleted $EPC_{<}$ -deduction sequence from S),
 for each $A \in \Sigma_0$, there is some clause $A' \in |\Delta|$, such that $A' \Sigma_0$ -subsumes A .
 Of course, as $\mathcal{C}_0 \subseteq |\Delta|$, trivially, for each $A \in \mathcal{C}_0$, there is some $A' \in |\Delta|$,
 such that $A' \Sigma_0$ -subsumes A , (for instance, A).

Thus, (as the empty clause, $\square$, is QR -positive), by Lemma 7.2.4, there exists a $D \in |\Delta|$,
 such that $D \Sigma_0$ -subsumes $\square$. ■

For Corollary 7.2.2:

Proof: Let Δ be a refined Σ_0 -subsumption deleted $EPC_{<}$ -deduction sequence from S .
 Then since Δ is also (simply) a Σ_0 -subsumption deleted $EPC_{<}$ -deduction sequence from S ,
 by Theorem 7.2.1, there exists a $D \in |\Delta|$, such that $D \Sigma_0$ -subsumes $\square$.

Since if $D \Sigma_0$ -subsumes $\square$, it must be the case that $QR_{full}(D) = \square$, (this is Observation 4.1.1),
 there exists a $D \in |\Delta|$, such that $QR_{full}(D) = \square$.

Letting $C \in |\Delta|$, be any clause, (for instance, D), we have $\square \in EPC_{<}(C, D)$.

Thus, (since $C; D \in |\Delta|$ and Δ is a refined Σ_0 -subsumption deleted $EPC_{<}$ -deduction sequence
 from S), $\square \in |\Delta|$. ■

For Lemma 7.2.5:

Proof:

With 'EPC' substituted for 'PC', ' Σ_0 -subsumes' substituted for 'subsumes',

(and ' Σ_0 -subsume' and ' Σ_0 -subsumption' substituted for 'subsumes' and 'subsumption', respec-
 tively), ' Σ_0 -tautology' substituted for 'tautology'.

and Theorem 7.1.1 substituted for Lemma 2.5.2, the proof is the same as that given for Lemma
 2.5.3. ■

For Corollary 7.2.3:

Proof: Since S is unsatisfiable, (clearly), Δ is non-empty. By Lemma 7.2.5, there exists a
 Σ_0 -subsumption deleted $EPC_{<}$ -deduction sequence from S , Δ' , such that $\forall C' \in |\Delta'|$, either:

- a) $C' \in |\Delta|$, or
- b) C' is a Σ_0 -tautology.

Since, by assumption, $\Sigma_0 \subseteq S$ and S is unsatisfiable, by Theorem 7.2.1, it is the case that,
 $\exists D \in |\Delta'|$ such that $D \Sigma_0$ -subsumes $\square$.

Clearly, D is not a Σ_0 -tautology, so $D \in |\Delta|$. ■

For Corollary 7.2.4:

Proof:

Substituting ' Σ_0 -subsumption and Σ_0 -tautology' in place of ' Σ_0 -subsumption',
 and 'Corollary 7.2.3' in place of 'Theorem 7.2.1',
 the proof is the same as that given for Corollary 7.2.2. ■

For Corollary 7.2.5:

Proof (sketch): Since, every $A \in \Sigma_0$ is a Σ_0 -tautology, (evidently),
 if Δ is any Σ_0 -subsumption and Σ_0 -tautology deleted $\text{EPC}_{<}$ -deduction sequence from S ,
 (for any S), then
 Δ is also a Σ_0 -subsumption and Σ_0 -tautology deleted $\text{EPC}_{<}$ -deduction sequence from $S \cup \Sigma_0$.

Similarly, (it is not hard to check that, for any S),
 any refined Σ_0 -subsumption and Σ_0 -tautology deleted $\text{EPC}_{<}$ -deduction sequence from S ,
 is a refined Σ_0 -subsumption and Σ_0 -tautology deleted $\text{EPC}_{<}$ -deduction sequence from $S \cup \Sigma_0$.

Let S be a finite set of clauses such that $\Sigma_0 \cup S$ is unsatisfiable.

Let S' denote $S \cup \Sigma_0$, then (clearly), S' is unsatisfiable, and $\Sigma_0 \subseteq S'$.

1) and 2) now follow immediately from Corollary 7.2.3, and Corollary 7.2.4, respectively. ■

For Corollary 7.2.6:

Proof: Evidently, (for any S),
 every Σ_0 -subsumption deleted $\text{EPC}_{<}$ -deduction sequence from S ,
 is also a Σ_0 -subsumption and Σ_0 -tautology deleted $\text{EPC}_{<}$ -deduction sequence from S .
 Similarly, (it is easy to check that, for any S),
 every refined Σ_0 -subsumption deleted $\text{EPC}_{<}$ -deduction sequence from S ,
 is also a refined Σ_0 -subsumption and Σ_0 -tautology deleted $\text{EPC}_{<}$ -deduction sequence from S .

Thus 1) and 2) follow directly from 1) and 2), respectively, of Corollary 7.2.5. ■

For Corollary 7.2.7:

Proof: Evidently, (S is not empty, and),
 there is a refined Σ_0 -subsumption deleted $\text{EPC}_{<}$ -deduction sequence from S ,
 $\Delta \equiv C_0, C_1, \dots$, say, such that for all $i: i \geq 0$, either:
 $C_i \in S$, or there exist $j; k: 0 \leq j < i; 0 \leq k < i$, such that $C_i \in \text{EPC}_{<}^1(C_j, C_k)$.

By Corollary 7.2.6 (2), $\square \in |\Delta|$.

Since (clearly) for any $C \in |\Delta|$, there is a $\text{EPC}_{<}$ -deduction of C from S ,
 there is a $\text{EPC}_{<}$ -deduction of $\square$ from S . ■

F.3. Generating Deduction Sequences with Deletion

For Proposition 7.3.1:

Proof (sketch): Let S be any finite set of clauses such that $S \cup \Sigma_0$ is unsatisfiable.

Let $\Delta^* \equiv C_0, C_1, \dots$ denote the (possibly infinite) sequence of clauses generated by $\mathfrak{S}$, on the input S , (those generated clauses which were not forward Σ_0 -subsumption (or Σ_0 -tautology) deleted, and so were entered, at some point, in the data base), where each clause C_i in Δ is either marked or not, and is marked if and only if C_i was, (at some point) backward Σ_0 -subsumed by some (in turn, possibly marked) clause, C_j , for some $j > i$.

From Δ^* we shall construct a refined Σ_0 -subsumption and Σ_0 -tautology deleted (EPC_<-) deduction sequence, Δ , from S , such that $\forall C \in |\Delta|$, C occurs unmarked in the sequence Δ^* .

Since (by assumption), $S \cup \Sigma_0$ is unsatisfiable, by Corollary 7.2.5 (2), $\square \in |\Delta|$, so it follows that for some $j \geq 0$, $C_j = \square$, (and occurs unmarked in Δ^*), and hence, clearly, at some point, $\mathfrak{S}$ deduced the empty clause, $\square$.

To construct Δ :

For any $i \geq 0$, if $C_i \in |\Delta^*|$ is a marked clause, there exists some $i' > i$ such that $C_i \supseteq C_{i'}$ and C_i was deleted, by $\mathfrak{S}$, because of the backward Σ_0 -subsumption of C_i by $C_{i'}$.

If, in turn, $C_{i'}$ is marked, there exists some $i'' > i'$ such that $C_{i'} \supseteq C_{i''}$ and $C_{i'}$ was deleted, by $\mathfrak{S}$, because of the backward Σ_0 -subsumption of $C_{i'}$ by $C_{i''}$, and so on.

Since (by assumption) $\supseteq$ is well-founded, any such sequence $C_i, C_{i'}, C_{i''}, \dots$, must eventually terminate, in some necessarily unmarked (for otherwise the sequence could be extended), clause $C_{i^{(n)}}$, or there must exist $j < k$, such that $C_{i^{(k)}} \supseteq C_{i^{(j)}}$.

In the first case, let $f(i)$ denote the index $i^{(n)}$.

In the second case, we have

$$C_{i^{(j)}} \supseteq C_{i^{(j+1)}} \supseteq \dots \supseteq C_{i^{(k-1)}} \supseteq C_{i^{(k)}} \supseteq C_{i^{(j)}}.$$

so by the (assumed) transitivity of $\supseteq$, in particular, we have $C_{i^{(j+1)}} \supseteq C_{i^{(j)}}$.

But (by our assumed backward deletion policy), this contradicts our assumption that $C_{i^{(j)}}$ was deleted by $\mathfrak{S}$ because of its backward Σ_0 -subsumption by $C_{i^{(j+1)}}$.

Thus only the first case may apply.

For any $i \geq 0$ such that $C_i \in |\Delta^*|$ is unmarked, let $f(i)$ denote i .

Thus we have a function, f , mapping each index i such that C_i is unmarked (in Δ^*), to i , and mapping each index i such that C_i is marked (in Δ^*), to some index $f(i)$, such that $C_{f(i)}$ is unmarked (in Δ^*), and $C_i \supseteq C_{f(i)}$.

Clearly, $\forall i \geq 0$, $C_{f(i)}$ occurs as an unmarked clause in Δ^* , and Σ_0 -subsumes C_i .

Let $\Delta \equiv C_{f(0)}, C_{f(1)}, C_{f(2)}, \dots$,

(or this sequence with any arbitrary number of repetitions removed).

Then, clearly every clause $C \in |\Delta|$, occurs as an unmarked clause in the sequence Δ^* ,

for every $C' \in |\Delta^*|$, there is some $C \in |\Delta|$ such that C Σ_0 -subsumes C' .

and (it is not hard to check that), Δ is indeed a refined Σ_0 -subsumption and Σ_0 -tautology deleted (EPC_<-) deduction sequence, from S .

(For:

- i. If $C \in S$: if $C = \square$, then $\square \in |\Delta^*|$, unmarked, and so $\square \in |\Delta|$,
and otherwise, if C is not a Σ_0 -tautology, then, evidently, for some C' which Σ_0 -subsumes C ,
we have $C' \in |\Delta^*|$, and so there is some $C'' \in |\Delta|$ such that C'' Σ_0 -subsumes C' ,
and so (by transitivity of Σ_0 -subsumption) such that C'' Σ_0 -subsumes C , and
- ii. for any $A; B \in |\Delta|$, A and B occur unmarked in $|\Delta^*|$, and so for any $C \in \text{EPC}_{<}^1(A, B)$,
if $C = \square$, then $\square \in |\Delta^*|$, unmarked, and so $\square \in |\Delta|$,
and otherwise, C is not a Σ_0 -tautology, there exists a $C' \in |\Delta^*|$ such that C' Σ_0 -subsumes C ,
and, as in the case for (i.), above, there is some $C'' \in |\Delta|$ such that C'' Σ_0 -subsumes C ,
as required.) ■

APPENDIX G

Clause Reduction, Refutation Completeness of $\text{EPC}_{<}$ With Reduction Under Deletion, and Refining the Deduction Rules

G.1. Clause Reduction: I

G.1.1. QR-factoring, (in the Above-Equality Fragment)

For Observation 8.1.1:

Proof: Let A' be a QR-factor of some clause B , where $B \subseteq A$.

Then, modulo ordering, $A' = B\theta \setminus F(B\theta)$, where θ is the most general substitution such that $I\theta$ contains only trivial inequality literals, for some $I \subseteq B^-$.

Since $B \subseteq A$, clearly, $B^- \subseteq A^-$. Thus $I \subseteq A^-$.

Also, since $B \subseteq A$, clearly, $B\theta \subseteq A\theta$. Thus $B\theta \setminus F(B\theta) \subseteq A\theta$.

Since $B\theta \setminus F(B\theta)$ contains no trivial inequalities, (while $F(A\theta)$ contains only such), evidently,
 $B\theta \setminus F(B\theta) \subseteq A\theta \setminus F(A\theta)$.

Let A'' be an ordering of $A\theta \setminus F(A\theta)$. Then evidently A'' is a QR-factor of A , and $A' \subseteq A''$.

Conversely, suppose A'' is a QR-factor of A , and $A' \subseteq A''$.

Then, modulo ordering, $A'' = A\theta \setminus F(A\theta)$, where θ is the most general substitution such that $I\theta$ contains only trivial inequality literals, for some $I \subseteq A^-$.

Further, since A'' is a QR-factor (of A), A'' contains no trivial inequalities, so (as $A' \subseteq A''$), neither does A' .

Let B be (some ordering of) $I \cup \{L \in A \mid L\theta \in A'\}$.

Then, evidently, $I \subseteq B \subseteq A$, (so $I \subseteq B^-$), $A' \subseteq B\theta$ and moreover, modulo ordering,

$A' = B\theta \setminus F(B\theta)$. Thus A' is a QR-factor of $B \subseteq A$. ■

For Lemma 8.1.1:

Proof: (sketch) We obtain the proof as a modification of that given for Lemma 4.2.2.

Let A , A' , B , D , and D' , as in the statement of the lemma, and assume that A and D , (and that A' and D'), share no variable in common, which we can do by Observation 4.1.6. By assumption, there is some $I_A \subseteq A^-$, and some substitution θ_A , (which we may assume, has domain contained in the set of variables of A'), such that θ_A is a most general substitution such that $I_A\theta_A$ contains only trivial inequality literals, and such that (modulo clause ordering) $A' = A\theta_A \setminus F(A\theta_A)$.

Clearly A' must be pure-positive with respect to P and have $<$ -maximal predicate symbol P , and D' is mixed with respect to P , with $<$ -maximal predicate symbol P .

By the assumed separation of variables, evidently (we may assume the domain of θ_D is a subset of the variables of D , and), there is a substitution θ , say, (for example, the composite, $\theta_A\theta_D$), such that (modulo clause ordering) $A' = A\theta \setminus F(A\theta)$, $D' = D\theta \setminus F(D\theta)$, and θ is a most general substitution such that every literal in $I_A\theta \cup I_B\theta$ is a trivial inequality, (and $L_0\theta$ occurs as the left-most inequality in D').

Further, evidently we may assume that for any term s occurring in A , $s\theta = s\theta_A$, while for any term t occurring in D , $t\theta = t\theta_D$, (and, hence, that for any literal $L \in A$, $L\theta = L\theta_A$, while for any literal $L \in D$, $L\theta = L\theta_D$), and finally, that if $L \in (F(A\theta) \cap D\theta) \cup (A\theta \cap F(D\theta))$, then L is a *ground* trivial inequality, and $L \in F(A\theta) \cap F(D\theta)$.

For some non-empty subsets $A_R \subseteq A$, and $D_R \subseteq D$, (say), the left-most negative P instance, $L_0\theta$, of D occurring in D_R , (and some term tuples $\vec{s}_1, \dots, \vec{s}_i$ and $\vec{t}_1, \dots, \vec{t}_j$), $A_R \equiv \{P(\vec{s}_1), \dots, P(\vec{s}_i)\}$; $D_R \equiv \{\neg P(\vec{t}_1), \dots, \neg P(\vec{t}_j)\}$, and (modulo clause ordering),

$$B = \bigcup_{\substack{1 \leq i' \leq i \\ 1 \leq j' \leq j}} \neg(\vec{s}_{i'}, \vec{t}_{j'}) \cup A \setminus A_R \cup D \setminus D_R.$$

Let $I \equiv I_A \cup I_B$. Then θ is a most general substitution such that every literal in I is a trivial inequality, and evidently, $I \subseteq B^-$.

Let B'' be (some ordering of)

$$\bigcup_{\substack{1 \leq i' \leq i \\ 1 \leq j' \leq j}} \neg(\vec{s}_{i'}, \vec{t}_{j'}) \cup \{L \in A \setminus A_R \mid L\theta \notin A_R\theta\} \cup \{L \in D \setminus D_R \mid L\theta \notin D_R\theta\}.$$

Then evidently $I \subseteq B'' \subseteq B$, and

$$\begin{aligned} B''\theta \setminus F(B''\theta) &= \left(\bigcup_{\substack{1 \leq i' \leq i \\ 1 \leq j' \leq j}} \neq(\tilde{s}_{i'}, \tilde{t}_{j'})\theta \cup A\theta \setminus A_R\theta \cup D\theta \setminus D_R\theta \right) \setminus (F(A\theta) \cup F(B\theta)) \\ &= \bigcup_{\substack{1 \leq i' \leq i \\ 1 \leq j' \leq j}} \neq(\tilde{s}_{i'}, \tilde{t}_{j'})\theta \cup (A\theta \setminus A_R\theta) \setminus F(A\theta) \cup (D\theta \setminus D_R\theta) \setminus F(D\theta). \end{aligned}$$

Let $A'_R \equiv A_R\theta$, and let $D'_R \equiv D_R\theta$.

Then $L_0\theta \in D'_R$, and clearly $\emptyset \neq A'_R \subseteq A'$, and $\emptyset \neq D'_R \subseteq D'$.

Let $A'_R \equiv \{P(\tilde{q}_1), \dots, P(\tilde{q}_k)\}$, say, and $D'_R \equiv \{\neg P(\tilde{r}_1), \dots, \neg P(\tilde{r}_l)\}$, say.

Clearly if we let (modulo ordering)

$$B' \equiv \bigcup_{\substack{1 \leq k' \leq k \\ 1 \leq l' \leq l}} \neq(\tilde{q}_{k'}, \tilde{r}_{l'}) \cup A' \setminus A'_R \cup D' \setminus D'_R,$$

then, (for some ordering), $B' \in \text{EPC}_{<}^1(A', D')$.

Evidently, $\{\tilde{q}_1, \dots, \tilde{q}_k\} = \{\tilde{s}_1\theta, \dots, \tilde{s}_i\theta\}$, $\{\tilde{r}_1, \dots, \tilde{r}_l\} = \{\tilde{t}_1\theta, \dots, \tilde{t}_j\theta\}$, and (modulo ordering),

$$\begin{aligned} B' &= \bigcup_{\substack{1 \leq i' \leq i \\ 1 \leq j' \leq j}} \neq(\tilde{s}_{i'}, \tilde{t}_{j'})\theta \cup (A\theta \setminus F(A\theta)) \setminus A_R\theta \cup (D\theta \setminus F(D\theta)) \setminus D_R\theta \\ &= \bigcup_{\substack{1 \leq i' \leq i \\ 1 \leq j' \leq j}} \neq(\tilde{s}_{i'}, \tilde{t}_{j'})\theta \cup (A\theta \setminus A_R\theta) \setminus F(A\theta) \cup (D\theta \setminus D_R\theta) \setminus F(D\theta) \\ &= B''\theta \setminus F(B''\theta). \end{aligned}$$

Thus, (as $I \subseteq B'' \subseteq B$, and θ is a most general substitution such that I contains only trivial inequalities), B' is a QR-factor of $B'' \subseteq B$. ■

For Corollary 8.1.1:

Proof: Let A , A' , A'' , B , D , D' , and D'' , as in the statement of the lemma,

(and assume that A , and D , (and that A' and D'), share no variable in common, which we can do by Observation 4.1.6).

Since $A'' \subseteq A$ and $D'' \subseteq D$, then it is easy to see,

(for example by inspecting the proof given for Lemma 4.3.1, wherein, we are in the special case that the substitution θ , is the empty substitution),

that in case (i), $D'' \subseteq B$,

while in case (ii), either $A'' \subseteq B$, or there exists $B''' \in \text{EPC}_{<}^1(A'', D'')$, such that $B''' \subseteq B$.

Thus, in case (i), clearly D' is a QR-factor of some $B'' \subseteq B$, (B'' here being D''), as required.

In case (ii), similarly, if $A'' \subseteq B$, then clearly, A' is a QR-factor of some $B'' \subseteq B$, (B'' here being A'').

On the other hand, suppose there exists $B''' \in \text{EPC}_{<}^1(A'', D'')$, such that $B''' \subseteq B$. Then by Lemma 8.1.1, there exists $B' \in \text{EPC}_{<}^1(A', D')$, such that B' is a QR-factor of some $B'' \subseteq B'''$.

Since $B''' \subseteq B$, clearly $B'' \subseteq B$, and so B' is a QR-factor of some $B'' \subseteq B$.

Thus in case (ii), either A' is a QR-factor of some $B'' \subseteq B$, or there exists $B' \in \text{EPC}_{<}^1(A', D')$ such that B' is a QR-factor of some $B'' \subseteq B$, as required. ■

For Lemma 8.1.2:

Proof: Again, a proof is that given for Lemma 2.4.2, with (“EPC_<” substituted for “PC_<”, “is a QR-factor of a subclause of” substituted for “subsumes”, and), Corollary 8.1.1 substituted for (its analogue) Lemma 2.4.1. ■

For Lemma 8.1.3:

Proof (sketch): Again, a proof is that given for Lemma 2.4.3, with (“EPC_<” substituted for “PC_<”, “is a QR-factor of a subclause of” substituted for “subsumes”, and), Lemma 8.1.2, Corollary 8.1.1, and Lemma 4.2.1 substituted for (the PC_<-deduction analogues), Lemma 2.4.2, Lemma 2.4.1, and Lemma 2.2.1, respectively. ■

For Lemma 8.1.4:

Proof: Again, a proof is that given for Lemma 2.4.4, with “(above-equality) EPC_<⁺¹” substituted for “PC_<⁺¹-deduction”, “EPC_<-initial” substituted for “PC_<-initial”, “above-equality EPC_<-deduction” substituted for “PC_<-deduction”, “is a QR-factor of a subclause of” substituted for “subsumes”, (and “which is a QR-factor of a subclause of” substituted for “subsuming”), and Lemma 8.1.3, substituted for Lemma 2.4.3. ■

For Corollary 8.1.2:

Proof: The argument is the same as that given for Corollary 4.3.1, with the following modifications: replace “Lemma 4.3.4” with “Lemma 8.1.4”, and replace “subsumes” with “is a QR-factor of a subclause of”. ■

G.1.2. Demodulation, (in the Above-Equality Fragment)

For Observation 8.1.2:

Proof: We first examine the case when X and X' are both terms, or both literals.

Since $X \sim_{\mathfrak{E}} X'$, $X \succsim_{cl_+(\mathfrak{E})} X'$.

Hence for some finite $\mathfrak{E}'' \subseteq cl_+(\mathfrak{E})$, we have $X \succsim_{\mathfrak{E}''} X'$.

But then, since $\mathfrak{E}''$ is finite, there is some finite $\mathfrak{E}' \subseteq \mathfrak{E}$, such that $\mathfrak{E}'' \subseteq cl_+(\mathfrak{E}')$.

Since $\mathfrak{E}'' \subseteq cl_+(\mathfrak{E}')$ and $X \succsim_{\mathfrak{E}''} X'$, it follows that $X \succsim_{cl_+(\mathfrak{E}')} X'$, and hence, $X \sim_{\mathfrak{E}'} X'$.

Conversely, if, for some (finite) $\mathfrak{E}' \subseteq \mathfrak{E}$, $X \sim_{\mathfrak{E}'} X'$, then, clearly, $X \sim_{\mathfrak{E}} X'$.

Now suppose that A and B are clauses, and that $A \sim_{\mathfrak{E}} B$.

Then $\forall L \in A, \exists L' \in B$, such that $L \sim_{\mathfrak{E}} L'$, and $\forall L \in B, \exists L' \in A$, such that $L \sim_{\mathfrak{E}} L'$.

By the result for literals already shown, evidently, for each $L \in A$, there exists some finite $\mathfrak{E}_L \subseteq \mathfrak{E}$, such that for some $L' \in B$, $L \sim_{\mathfrak{E}_L} L'$, and similarly, for each $L \in B$, there exists some finite $\mathfrak{E}_L \subseteq \mathfrak{E}$, such that for some $L' \in A$, $L \sim_{\mathfrak{E}_L} L'$.

Let $\mathfrak{E}' \equiv \bigcup_{L \in A \cup B} \mathfrak{E}_L$.

Then evidently, $\mathfrak{E}'$ is a finite subset of $\mathfrak{E}$, and (since $\forall L \in A \cup B, \mathfrak{E}_L \subseteq \mathfrak{E}'$),

$\forall L \in A, \exists L' \in B$, such that $L \sim_{\mathfrak{E}'} L'$, and $\forall L \in B, \exists L' \in A$, such that $L \sim_{\mathfrak{E}'} L'$.

Thus $A \sim_{\mathfrak{E}'} B$.

Conversely, if, for some (finite) $\mathfrak{E}' \subseteq \mathfrak{E}$, $A \sim_{\mathfrak{E}'} B$, then

$\forall L \in A, \exists L' \in B$, such that $L \sim_{\mathfrak{E}'} L'$, and $\forall L \in B, \exists L' \in A$, such that $L \sim_{\mathfrak{E}'} L'$.

Thus, clearly,

$\forall L \in A, \exists L' \in B$, such that $L \sim_{\mathfrak{E}} L'$, and $\forall L \in B, \exists L' \in A$, such that $L \sim_{\mathfrak{E}} L'$.

and so $A \sim_{\mathfrak{E}} B$. ■

For Observation 8.1.4:

Proof: Let $L \in C$. Then either $L \in A$ or $L \in B$.

In the former case, since $A \sim_{\mathfrak{E}} A' \subseteq C'$, there is some $L' \in C'$ such that $L \sim_{\mathfrak{E}} L'$.

Similarly, in the latter case, since $B \sim_{\mathfrak{E}} B' \subseteq C'$, there is some $L' \in C'$ such that $L \sim_{\mathfrak{E}} L'$.

Conversely, let $L \in C'$. Then either $L \in A'$ or $L \in B'$.

In the former case, since $A' \sim_{\mathfrak{E}} A \subseteq C$, there is some $L' \in C$ such that $L \sim_{\mathfrak{E}} L'$.

Similarly, in the latter case, since $B' \sim_{\mathfrak{E}} B \subseteq C$, there is some $L' \in C$ such that $L \sim_{\mathfrak{E}} L'$. ■

For Observation 8.1.5:

Proof: This is basically obvious.

Let $A' \sim_{\mathfrak{E}} B$, for some $B \subseteq A$.

Let A'' be some ordering of $A \setminus B \cup A'$.

Then clearly $A' \subseteq A''$, and

(since $A \setminus B \sim_{\mathcal{E}} A \setminus B$, $B \sim_{\mathcal{E}} A'$, and (modulo ordering), $A = A \setminus B \cup B$, while $A'' = A \setminus B \cup A'$), by Observation 8.1.4, $A \sim_{\mathcal{E}} A''$.

Conversely, suppose $A' \subseteq A''$, where $A'' \sim_{\mathcal{E}} A$.

Let $B \equiv \{L \in A \mid \exists L' \in A' : L' \sim_{\mathcal{E}} L\}$.

Then obviously $B \subseteq A$, and it will suffice to show that $A' \sim_{\mathcal{E}} B$.

Clearly, $\forall L \in B$, $\exists L' \in A'$, such that $L \sim_{\mathcal{E}} L'$.

Thus it will suffice to show that $\forall L' \in A' \exists L \in B$ such that $L \sim_{\mathcal{E}} L'$.

Let $L' \in A'$. Since $A' \subseteq A'' \sim_{\mathcal{E}} A$, there exists $L \in A$ such that $L' \sim_{\mathcal{E}} L$. But then $L \in B$. ■

For Observation 8.1.6:

Proof: Let A' be an $\mathcal{E}$ -demodulant of some clause B , where $B \subseteq A$.

Then there is a surjection $f: B \rightarrow A'$, such that $\forall L \in B$, $f(L)$ is an $\mathcal{E}$ -demodulant of L .

Let $g: A \rightarrow A' \cup A \setminus B$, be defined by:

for $L \in B$, $g(L) = f(L)$, while for $L \in A \setminus B$, $g(L) = L$.

Then clearly g is a surjective function mapping A onto $A' \cup A \setminus B$, such that $\forall L \in A$, $g(L)$ is an $\mathcal{E}$ -demodulant of L .

Let A'' be an ordering of $A' \cup A \setminus B$. Then clearly, A'' is an $\mathcal{E}$ -demodulant of A , and $A' \subseteq A''$.

Conversely, suppose A'' is a $\mathcal{E}$ -demodulant of A , and $A' \subseteq A''$.

Then there is a surjection $f: A \rightarrow A''$, such that $\forall L \in A$, $f(L)$ is an $\mathcal{E}$ -demodulant of L .

Let B be (some ordering of) $\{L \in A \mid f(L) \in A'\}$, and let $g: B \rightarrow A'$, be defined by $\forall L \in B$, $g(L) = f(L)$.

Then evidently, $B \subseteq A$, and $g: B \rightarrow A'$, is a surjection, such that $\forall L \in A$, $f(L)$ is an $\mathcal{E}$ -demodulant of L .

Thus A' is an $\mathcal{E}$ -demodulant of $B \subseteq A$. ■

For Lemma 8.1.5:

Proof: (sketch) The proof is a modification of that given for Lemma 8.1.1.

Let A , A' , B , D , and D' , as in the statement of the lemma, (and assume that A and D and that A' and D' share no variable in common, which we can do by Observation 4.1.6).

Since, by assumption, $A' \sim_{\epsilon} A$ and $D' \sim_{\epsilon} D$, clearly A' must be pure-positive with respect to P and have $<$ -maximal predicate symbol P , and D' is mixed with respect to P , with $<$ -maximal predicate symbol P .

For some non-empty subsets $A_R \subseteq A$, and $D_R \subseteq D$, (say), the left-most negative P instance, $L_0\theta$, of D occurring in D_R , (and some term tuples $\vec{s}_1, \dots, \vec{s}_i$ and $\vec{t}_1, \dots, \vec{t}_j$),

$A_R \equiv \{P(\vec{s}_1), \dots, P(\vec{s}_i)\}$; $D_R \equiv \{\neg P(\vec{t}_1), \dots, \neg P(\vec{t}_j)\}$, and (modulo clause ordering),

$$B = \bigcup_{\substack{1 \leq i' \leq i \\ 1 \leq j' \leq j}} \neg(\vec{s}_{i'}, \vec{t}_{j'}) \cup A \setminus A_R \cup D \setminus D_R.$$

Let B'' be (some ordering of)

$$\bigcup_{\substack{1 \leq i' \leq i \\ 1 \leq j' \leq j}} \neg(\vec{s}_{i'}, \vec{t}_{j'}) \cup \{L \in A \setminus A_R \mid \nexists L' \in A_R : L' \sim_{\epsilon} L\} \cup \{L \in D \setminus D_R \mid \nexists L' \in D_R : L' \sim_{\epsilon} L\}.$$

Then clearly $B'' \subseteq B$.

Let $A'_R \equiv \{L' \in A' \mid \exists L \in A_R : L \sim_{\epsilon} L'\}$, and let $D'_R \equiv \{L' \in D' \mid \exists L \in D_R : L \sim_{\epsilon} L'\}$.

Then $L'_0 \in D'_R$, and clearly $\emptyset \neq A'_R \subseteq A'$, and $\emptyset \neq D'_R \subseteq D'$.

Let $A'_R \equiv \{P(\vec{q}_1), \dots, P(\vec{q}_k)\}$, say, and $D'_R \equiv \{\neg P(\vec{r}_1), \dots, \neg P(\vec{r}_l)\}$, say.

Clearly if we let (modulo ordering)

$$B' \equiv \bigcup_{\substack{1 \leq k' \leq k \\ 1 \leq l' \leq l}} \neg(\vec{q}_{k'}, \vec{r}_{l'}) \cup A' \setminus A'_R \cup D' \setminus D'_R, \text{ then, (for some ordering), } B' \in \text{EPC}_{<}^1(A', D').$$

In order complete the proof, obviously, it will suffice to show that $B' \sim_{\epsilon} B''$.

By (two applications of) Observation 8.1.4, to show $B' \sim_{\epsilon} B''$, evidently, it will suffice to show that

- 1) $\bigcup_{\substack{1 \leq k' \leq k \\ 1 \leq l' \leq l}} \neg(\vec{q}_{k'}, \vec{r}_{l'}) \sim_{\epsilon} \bigcup_{\substack{1 \leq i' \leq i \\ 1 \leq j' \leq j}} \neg(\vec{s}_{i'}, \vec{t}_{j'}).$
- 2) $A' \setminus A'_R \sim_{\epsilon} \{L \in A \setminus A_R \mid \nexists L' \in A_R : L' \sim_{\epsilon} L\}$, and
- 3) $D' \setminus D'_R \sim_{\epsilon} \{L \in D \setminus D_R \mid \nexists L' \in D_R : L' \sim_{\epsilon} L\}.$

To show 1):

We claim that $A_R \sim_{\epsilon} A'_R$.

For let $L \in A_R$. Then $L \in A$, and (since $A \sim_{\epsilon} A'$), hence there exists $L' \in A'$ such that $L \sim_{\epsilon} L'$. But then $L' \in A'_R$.

Conversely, let $L' \in A'_R$. Then, (by the definition of A'_R), there exists $L \in A_R$ such that $L \sim_{\epsilon} L'$.

Thus $A_R \sim_{\epsilon} A'_R$.

Hence, $\forall i': 1 \leq i' \leq i, \exists k': 1 \leq k' \leq k$ such that $P(\vec{s}_{i'}) \sim_{\epsilon} P(\vec{q}_{k'})$,

and $\forall k': 1 \leq k' \leq k, \exists i': 1 \leq i' \leq i$ such that $P(\vec{s}_{i'}) \sim_{\epsilon} P(\vec{q}_{k'})$.

Hence, $\forall i': 1 \leq i' \leq i, \exists k': 1 \leq k' \leq k$ such that $\vec{s}_{i'} \sim_{\epsilon} \vec{q}_{k'}$,

and $\forall k': 1 \leq k' \leq k, \exists i': 1 \leq i' \leq i$ such that $\vec{s}_{i'} \sim_{\epsilon} \vec{q}_{k'}$,

(where for any tuples $\vec{a} \equiv a_1, \dots, a_n$ and $\vec{b} \equiv b_1, \dots, b_n$, by $\vec{a} \sim_{\epsilon} \vec{b}$,

we mean that for each $n': 1 \leq n' \leq n, a_{n'} \sim_{\epsilon} b_{n'}$).

Similarly, $D_R \sim_{\epsilon} D'_R$, and $\forall j': 1 \leq j' \leq j, \exists l': 1 \leq l' \leq l$ such that $\vec{t}_{j'} \sim_{\epsilon} \vec{r}_{l'}$,

and $\forall l': 1 \leq l' \leq l, \exists j': 1 \leq j' \leq j$ such that $\vec{t}_{j'} \sim_{\epsilon} \vec{r}_{l'}$.

Thus $\forall i': 1 \leq i' \leq i, \forall j': 1 \leq j' \leq j, \exists k': 1 \leq k' \leq k, \exists l': 1 \leq l' \leq l, \neq(\vec{s}_{i'}, \vec{t}_{j'}) \sim_{\epsilon} \neq(\vec{q}_{k'}, \vec{r}_{l'})$,

and $\forall k': 1 \leq k' \leq k, \forall l': 1 \leq l' \leq l, \exists i': 1 \leq i' \leq i, \exists j': 1 \leq j' \leq j, \neq(\vec{s}_{i'}, \vec{t}_{j'}) \sim_{\epsilon} \neq(\vec{q}_{k'}, \vec{r}_{l'})$.

It follows that $\bigcup_{\substack{1 \leq k' \leq k \\ 1 \leq l' \leq l}} \neq(\vec{q}_{k'}, \vec{r}_{l'}) \sim_{\epsilon} \bigcup_{\substack{1 \leq i' \leq i \\ 1 \leq j' \leq j}} \neq(\vec{s}_{i'}, \vec{t}_{j'})$, (that is, (1) holds), as claimed.

For 2):

Let $L' \in A' \setminus A'_R$.

Since $L' \notin A'_R$, (but $L' \in A'$), (from the choice of A'_R it follows that), $\nexists L'' \in A_R$ such that $L'' \sim_{\epsilon} L'$.

But since $L' \in A'$ and $A' \sim_{\epsilon} A$, there exists $L \in A$ such that $L \sim_{\epsilon} L'$.

Thus there exists some $L \in A \setminus A_R$, such that $L \sim_{\epsilon} L'$.

If there exists some $L'' \in A_R : L'' \sim_{\epsilon} L$, then, clearly, there exists some $L'' \in A_R$ such that $L'' \sim_{\epsilon} L'$, a contradiction. Therefore, $\nexists L'' \in A_R : L'' \sim_{\epsilon} L$.

Thus there exists some $L \in \{L \in A \setminus A_R \mid \nexists L' \in A_R : L' \sim_{\epsilon} L\}$, such that $L \sim_{\epsilon} L'$.

Conversely, let $L \in \{L \in A \setminus A_R \mid \nexists L' \in A_R : L' \sim_{\epsilon} L\}$.

Then $L \in A \setminus A_R$, and there does not exist any $L'' \in A_R$ such that $L'' \sim_{\epsilon} L$.

Since $L \in A$ and $A \sim_{\epsilon} A'$, there is some $L' \in A'$, such that $L \sim_{\epsilon} L'$.

If $\exists L'' \in A_R$ such that $L'' \sim_{\epsilon} L'$, then, (since $L \sim_{\epsilon} L'$), $\exists L'' \in A_R$ such that $L'' \sim_{\epsilon} L$, a contradiction.

Thus $L' \notin A'_R$, and so $L' \in A' \setminus A'_R$, and so, there exists $L' \in A' \setminus A'_R$ such that $L \sim_{\epsilon} L'$.

Thus $A' \setminus A'_R \sim_{\epsilon} \{L \in A \setminus A_R \mid \nexists L' \in A_R : L' \sim_{\epsilon} L\}$, (that is, (2) holds), as claimed.

An argument for (3) can, (evidently), be made in an exactly analogous fashion to that just given for (2). ■

For Corollary 8.1.3:

Proof: A proof may be obtained as a straight forward modification of that given for Corollary 8.1.1. -specifically, in that proof, replace: “is a QR-factor of some” with “is $\mathfrak{E}$ -equivalent to some”, and “Lemma 8.1.1” with “Lemma 8.1.5”. ■

For Lemma 8.1.6:

Proof: Again, a proof is that given for Lemma 2.4.2, with “ $\text{EPC}_{<}$ ” substituted for “ $\text{PC}_{<}$ ”, “is $\mathfrak{E}$ -equivalent to a subclause of” substituted for “subsumes”, and, Corollary 8.1.3 substituted for (its analogue) Lemma 2.4.1. ■

For Lemma 8.1.7:

Proof (sketch): Again, a proof is that given for Lemma 2.4.3, with “ $\text{EPC}_{<}$ ” substituted for “ $\text{PC}_{<}$ ”,

“is $\mathfrak{E}$ -equivalent to a subclause of” substituted for “subsumes”, and,

Lemma 8.1.6, Corollary 8.1.3, and Lemma 4.2.1 substituted for (the $\text{PC}_{<}$ -deduction analogues).

Lemma 2.4.2, Lemma 2.4.1, and Lemma 2.2.1, respectively. ■

For Lemma 8.1.8:

Proof: Again, a proof is that given for Lemma 2.4.4, with

“(above-equality) $\text{EPC}_{<}^{+1}$ ” substituted for “ $\text{PC}_{<}^{+1}$ -deduction”,

“ $\text{EPC}_{<}$ -initial” substituted for “ $\text{PC}_{<}$ -initial”,

“above-equality $\text{EPC}_{<}$ -deduction” substituted for “ $\text{PC}_{<}$ -deduction”,

“is $\mathfrak{E}$ -equivalent to a subclause of” substituted for “subsumes”,

(and “which is $\mathfrak{E}$ -equivalent to a subclause of” substituted for “subsuming”),

and Lemma 8.1.7, substituted for Lemma 2.4.3. ■

For Corollary 8.1.4:

Proof: The argument is the same as that given for Corollary 4.3.1, with the following modifications: replace “Lemma 4.3.4” with “Lemma 8.1.8”,

and replace “subsumes” with “is $\mathfrak{E}$ -equivalent to a subclause of”. ■

G.1.3. Reduction: Synthesis

For Observation 8.1.7:

Proof (sketch): For any literals L and L' , let $L \widehat{\succ}_\rho L'$ if:

- i. $P_{L'} < P_L$, where $P_{L'}$ and P_L denote the predicate symbols of L' and L respectively, or else
- ii. $P_{L'} = P_L$, and the sign of L' is positive, while that of L is negative, or else

- iii. L both share the same predicate symbol, P , say, of arity, $n \geq 0$, (say), and both L and L' have the same sign, δ , and $L \equiv \delta P(t_1, \dots, t_n)$, $L' \equiv \delta P(t'_1, \dots, t'_n)$, and there exists $i: 1 \leq i \leq n$, such that for all $j: 1 \leq j < i$, $t_j = t'_j$, but $t_i \neq t'_i$, and $t_i \succ_\rho t'_i$, or else
- iv. $L = L'$.

Then it is easy to check that $\widehat{\succ}_\rho$ is a total order on literals.

Clearly any two literals are comparable. Reflexivity is trivial.

For transitivity, we suppose $L_1 \widehat{\succ}_\rho L_2$ and $L_2 \widehat{\succ}_\rho L_3$, and establish $L_1 \widehat{\succ}_\rho L_3$.

The only interesting case is when both $L_1 \widehat{\succ}_\rho L_2$ and $L_2 \widehat{\succ}_\rho L_3$, are due to condition (iii), above.

Then $L_1 \equiv \delta P(r_1, \dots, r_n)$, $L_2 \equiv \delta P(s_1, \dots, s_n)$, and $L_3 \equiv \delta P(t_1, \dots, t_n)$, say,

and there exists $i_1: 1 \leq i_1 \leq n$, such that for all $j: 1 \leq j < i_1$, $r_j = s_j$, but $r_{i_1} \neq s_{i_1}$, and $r_{i_1} \succ_\rho s_{i_1}$,

and $i_2: 1 \leq i_2 \leq n$, such that for all $j: 1 \leq j < i_2$, $s_j = t_j$, but $s_{i_2} \neq t_{i_2}$, and $s_{i_2} \succ_\rho t_{i_2}$.

Let $i_3 \equiv \min\{i_1, i_2\}$.

Then clearly, $1 \leq i_3 \leq n$, and if $i_1 \neq i_2$, for all $j: 1 \leq j < i_3$, $r_j = t_j$, but $r_{i_3} \neq t_{i_3}$, and $r_{i_3} \succ_\rho t_{i_3}$.

But alternatively, if $i_1 = i_2 = i_3$, then we have for all $j: 1 \leq j < i_3$, $r_j = s_j = t_j$, $r_{i_3} \neq s_{i_3}$, and $r_{i_3} \succ_\rho s_{i_3}$, $s_{i_3} \neq t_{i_3}$, and $s_{i_3} \succ_\rho t_{i_3}$.

By transitivity, thus $r_{i_3} \succ_\rho t_{i_3}$. Further, $r_{i_3} \neq t_{i_3}$, for otherwise $t_{i_3} \succ_\rho s_{i_3}$, which, by anti-symmetry, yields $s_{i_3} = t_{i_3}$, a contradiction.

Thus, again, for all $j: 1 \leq j < i_3$, $r_j = t_j$, but $r_{i_3} \neq t_{i_3}$, and $r_{i_3} \succ_\rho t_{i_3}$, and evidently, again by condition (iii) above, $L_1 \widehat{\succ}_\rho L_3$ as required.

For anti-symmetry, we assume $L_1 \widehat{\succ}_\rho L_2$ and $L_2 \widehat{\succ}_\rho L_1$, and establish $L_1 = L_2$. Again the only interesting case is when both are due to condition (iii) above, and the verification is similar.

(Here, only "the case $i_1 = i_2$ " is possible, and again, anti-symmetry forces equality).

For any finite set of positive equality units $\mathfrak{E}$, and any literals L and L' , if $\rho(\mathfrak{E}, L) = L'$, then with $L \equiv \delta P(t_1, \dots, t_n)$, say, we have $L' = \rho(\mathfrak{E}, L) = \delta P(\rho(\mathfrak{E}, t_1), \dots, \rho(\mathfrak{E}, t_n))$, and since $\forall i: 1 \leq i \leq n$, $t_i \succ_\rho \rho(\mathfrak{E}, t_i)$, clearly, (by conditions (iii) or (iv), above), $L \widehat{\succ}_\rho L'$.

That $\widehat{\succ}_\rho$ is well-founded now follows from the fact that (we assume) there are only finitely many predicate symbols, each with their own fixed finite arity, and so any infinite descending chain: $L_0 \widehat{\succ}_\rho L_1 \widehat{\succ}_\rho \dots$, of literals, the sequence $L_0, L_1, L_2, \dots$ has an infinite "tail", (the original sequence, with finitely many initial elements removed), $\mathfrak{LS}$, say, where for some $N \geq 0$, $\mathfrak{LS} \equiv L_N, L_{N+1}, L_{N+1}, \dots$, such that $\forall i \geq N$, $L_i = \delta P(t^i_1, \dots, t^i_n)$. That is, every literal in

this tail $\mathfrak{L}\mathfrak{S}$, shares the same sign δ and predicate symbol P , of some arity $n \geq 0$.
Further, for all $i \geq N$, the relation $L_i \hat{\succ}_\rho L_{i+1}$, is satisfied just by condition (iii) or (iv) above.

The case when $n = 0$ may now be immediately dispensed with, since, if $n = 0$, there exists $N \geq 0$, such that $\forall i \geq N, L_i = \delta P()$.

Otherwise, $\mathfrak{L}\mathfrak{S}$, naturally induces n infinite sequences: $\mathfrak{TS}_1, \dots, \mathfrak{TS}_n$, say, where $\forall j: 1 \leq j \leq n$, $\mathfrak{TS}_j \equiv t_j^N, t_j^{N+1}, t_j^{N+2}, \dots$.

Since, for any $i \geq N$, the relation $L_i \hat{\succ}_\rho L_{i+1}$, holds by condition (iii) or (iv), evidently, for any $i \geq N$, $t_i^1 \succ_\rho t_{i+1}^1$, and by the well-foundedness of $\succ_\rho$, there exists $N_1 \geq N$ such that $\forall i \geq N_1, t_i^1 = t_{i+1}^1$.

Similarly, there exists $N_2 \geq N_1$ such that $\forall i \geq N_2, t_i^2 = t_{i+1}^2$, and so on, so that finally, there exists $N_n \geq N_{n-1} \geq \dots \geq N_1 \geq N$ such that $\forall j: 1 \leq j \leq n, \forall i \geq N_n, t_i^j = t_{i+1}^j$.

Thus, $\forall i \geq N_n, L_i = L_{N_n}$. ■

For Observation 8.1.8

Proof (sketch): As ρ is a flat demodulation function, we have already observed that for any set $\mathfrak{E}$ of positive equality units, $\succ_{\rho, \mathfrak{E}}$ defines a reflexive, transitive (pre-order) relation on clauses.

To show anti-symmetry, suppose C and C' are unordered clauses such that, $C \succ_{\rho, \mathfrak{E}} C'$ and $C' \succ_{\rho, \mathfrak{E}} C$.

Since $C \succ_{\rho, \mathfrak{E}} C'$, there is a finite ($\mathfrak{E}$ -) ρ -reduction sequence:

$C = C_0, \dots, C_k = C'$, where $\forall i: 0 \leq i < k$, either C_{i+1} is a QR-factor of C_i , or there is a finite subset $\mathfrak{E}_i \subseteq \mathfrak{E}$, such that $\rho(\mathfrak{E}_i, C_i) = C_{i+1}$. In either case, $(\forall i: 0 \leq i < k), \|C_{i+1}\| \leq \|C_i\|$, that is, C_{i+1} can have at most as many literals as does C_i . Further, if C_{i+1} is a QR-factor of C_i , then $\|C_{i+1}\| < \|C_i\|$.

Thus $\|C'\| \leq \|C\|$, and $\|C'\| < \|C\|$ unless each of the steps in the reduction sequence was, for some finite $\mathfrak{E}' \subseteq \mathfrak{E}$, an (injective) application of $\rho(\mathfrak{E}', \cdot)$.

Similarly, since $C' \succ_{\rho, \mathfrak{E}} C$, $\|C\| \leq \|C'\|$, and so $\|C\| = \|C'\| = n \geq 0$, say, and there is a finite sequence $C = C_0, \dots, C_k = C'$, such that $\forall i: 0 \leq i < k, \|C_i\| = n$ and there is a finite subset $\mathfrak{E}_i \subseteq \mathfrak{E}$, such that $C_{i+1} = \rho(\mathfrak{E}_i, C_i)$.

Of course, if $n = 0$, then $C = C' = \square$, and so we're done.

Otherwise, let $C \equiv L_{0,1} \mid L_{0,2} \mid \cdots \mid L_{0,n}$, say.

Then $C_1 \equiv L_{1,1} \mid L_{1,2} \mid \cdots \mid L_{1,n}$, where $\forall j: 1 \leq j \leq n$, $L_{1,j} = \rho(\mathfrak{E}_0, L_{0,j})$,

and more generally, $\forall i: 0 < i \leq k$, $C_i \equiv L_{i,1} \mid L_{i,2} \mid \cdots \mid L_{i,n}$, where $\forall j: 1 \leq j \leq n$, $L_{i,j} = \rho(\mathfrak{E}_{i-1}, L_{i-1,j})$.

Let $\widehat{\succ}_\rho$, be a well-founded total order relation on literals, such that for any finite set of positive equality units $\mathfrak{E}'$, and any literals L and L' , if $\rho(\mathfrak{E}', L) = L'$, then $L \widehat{\succ}_\rho L'$, as in Observation 8.1.7.

Then $\forall j: 1 \leq j \leq n$. $\forall i: 0 \leq i < k$, (since $L_{i+1,j} = \rho(\mathfrak{E}_i, L_{i,j})$), $L_{i,j} \widehat{\succ}_\rho L_{i+1,j}$.

By transitivity, then, $\forall j: 1 \leq j \leq n$. $L_{0,j} \widehat{\succ}_\rho L_{k,j}$.

Evidently, the composite map: $f_1 \equiv \rho(\mathfrak{E}_{k-1}, \rho(\mathfrak{E}_{k-2}, \cdots \rho(\mathfrak{E}_0, \cdot) \cdots))$ is a bijection from the literals C to the literals of C' , such that $\forall j: 1 \leq j \leq n$, $f_1(L_{0,j}) = L_{k,j}$.

Thus there is a bijection $f_1: C \rightarrow C'$, such that $\forall L \in C$, $L \widehat{\succ}_\rho f_1(L)$.

Similarly, there is a bijection $f_2: C' \rightarrow C$, such that $\forall L' \in C'$, $L' \widehat{\succ}_\rho f_2(L')$.

Let g denote the composite $f_2 \circ f_1: C \rightarrow C$. Then g is a bijection mapping a finite set to itself, and it is easy to show that for any such g there is some finite $n \geq 1$, such that the n -fold iterated composition of g with itself, g^n , is the identity function on C .

But g^n factors as $h \circ f_1$, where $h = g^{n-1} \circ f_2$, (incidentally, g^0 is the identity function on C), and since $h: C' \rightarrow C$ is the composition of functions f such that, for any literal $L \in \text{dom}(f)$, $L \widehat{\succ}_\rho f(L)$, by transitivity of $\widehat{\succ}_\rho$, it follows that for all $L' \in C'$, $L' \widehat{\succ}_\rho h(L')$.

Thus, for any $L \in C$, $L \widehat{\succ}_\rho f_1(L)$, and $f_1(L) \widehat{\succ}_\rho h(f_1(L)) = L$, so by anti-symmetry, $L = f_1(L)$. Thus $C' = f_1(C) = C$. ■

For Observation 8.1.9:

Proof: We first consider $\succ_{\rho, \mathfrak{E}}$ as the pre-order on unordered clauses.

Suppose $C_0 \succ_{\rho, \mathfrak{E}} C_1 \succ_{\rho, \mathfrak{E}} \cdots$, is any infinite chain. We must show that for some $n \geq 1$, there exists $m: 0 \leq m < n$, such that $C_n \succ_{\rho, \mathfrak{E}} C_m$.

Evidently we may assume that the sequence $C_0, C_1, C_2, \dots$ is repetition free modulo ordering, -that is, for no $0 \leq i < j$ is it the case that modulo ordering, $C_i = C_j$, (for otherwise, $C_j \succ_{\rho, \mathfrak{E}} C_i$, and we're done).

Since $\forall i \geq 0$, $C_i \succ_{\rho, \mathfrak{E}} C_{i+1}$,

$\forall i \geq 0$, there exists $k_i \geq 0$ and a finite sequence $C_i = C_{i,0}, C_{i,1}, \dots, C_{i,k_i} = C_{i+1}$,
 such that for all $j: 0 \leq j < k_i$, either $C_{i,j+1}$ is a QR-factor of $C_{i,j}$,
 or there is a finite subset $\mathfrak{E}_{i,j} \subseteq \mathfrak{E}$, such that $\rho(\mathfrak{E}_{i,j}, C_{i,j}) = C_{i,j+1}$.

Thus, by inserting the (possibly empty) finite sequence: $C_{i,1}, \dots, C_{i,k_i-1}$ between C_i and C_{i+1} ,
 for each $i \geq 0$, and by relabelling, we obtain an infinite sequence:

$\mathfrak{S} \equiv B_0, B_1, B_2, \dots$, say, such that for all $i \geq 0$, either B_{i+1} is a QR-factor of B_i ,
 or there is a finite subset $\mathfrak{E}_i \subseteq \mathfrak{E}$, such that $\rho(\mathfrak{E}_i, B_i) = B_{i+1}$,

Further, evidently there is a strictly monotone increasing function $f: \mathbb{N} \rightarrow \mathbb{N}$, such that for each
 $i \geq 0$, $f(i) \geq i$, $B_{f(i)} = C_i$, and for any $p \geq 0$, there exists $i_p \geq 0$ such that for any $i \geq i_p$,
 $f(i) \geq p$.

Moreover, the infinite "tail" subsequence $B_{f(i_p)}, B_{f(i_p)+1}, B_{f(i_p)+2}, \dots$ is, in the same way,
 associated to the infinite "tail" (subsequence) $C_{i_p}, C_{i_p+1}, C_{i_p+2}, \dots$. In particular, for all $i \geq i_p$,
 C_i belongs in the sequence $B_{f(i_p)}, B_{f(i_p)+1}, B_{f(i_p)+2}, \dots$.

Whenever B_{i+1} is a QR-factor of B_i , $\|B_{i+1}\| < \|B_i\|$.

that is, the length (number of literals) of B_{i+1} is less than the length of B_i .

and alternatively, whenever, for some finite subset $\mathfrak{E}_i \subseteq \mathfrak{E}$, $\rho(\mathfrak{E}_i, B_i) = B_{i+1}$, $\|B_{i+1}\| \leq \|B_i\|$.

Thus the length of the clauses in the sequence $\mathfrak{S}$ is monotonically decreasing, and there can only be
 finitely many indices $i \geq 0$ such that B_{i+1} is a QR-factor of B_i , and more generally, only finitely
 many indices $i \geq 0$ such that $\|B_{i+1}\| < \|B_i\|$.

Thus there is some $o \geq 0$, and some $p \geq 0$ such that for all $i \geq p$, $\|B_i\| = \|B_{i+1}\| = o$.

and there is some finite subset $\mathfrak{E}_i \subseteq \mathfrak{E}$, such that $\rho(\mathfrak{E}_i, B_i) = B_{i+1}$.

The case $o = 0$ is trivial, (for if $B_i = B_{i+1} = \dots = \square$, then, in particular, $C_{i_p} = C_{i_p+1} = \square$).

For any $i \geq p$, let $B_i \equiv L_{i,1} \mid L_{i,2} \mid \dots \mid L_{i,o}$.

Evidently, for any $i \geq p$, as a function mapping the set of literals of B_i to the set of literals of
 B_{i+1} , $\rho(\mathfrak{E}_i, \cdot)$ is bijective.

Thus the sequence $B_p, B_{p+1}, B_{p+2}, \dots$ naturally induces o infinite literal sequences:

$\mathfrak{L}\mathfrak{S}_1, \dots, \mathfrak{L}\mathfrak{S}_o$, say, where $\forall j: 1 \leq j \leq o$, $\mathfrak{L}\mathfrak{S}_j \equiv L_j^0, L_j^1, L_j^2, \dots$, say,

where $L_j^0 \equiv L_{p,j}$, and $\forall i \geq 0$, $L_j^{i+1} \equiv \rho(\mathfrak{E}_{p+i}, L_j^i)$.

Further, $(\forall j: 1 \leq j \leq o)$, $\forall i \geq 0$, $L_j^i \in B_{p+i}$, and $\forall i \geq 0$, $\{L_j^i \mid 1 \leq j \leq o\} = \{L \in B_{p+i}\}$.

Evidently, $\forall j: 1 \leq j \leq o$, there is some sign, δ_j , say, and some predicate symbol P_j , (possibly equality), of arity $a_j \geq 0$, say, such that $\forall i \geq 0$, there exist terms $t^{j,i}_1, \dots, t^{j,i}_{a_j}$, such that $L_j^i = \delta_j P_j(t^{j,i}_1, \dots, t^{j,i}_{a_j})$.

Further, since $(\forall j: 1 \leq j \leq o, \forall i \geq 0)$, $L_j^{i+1} = \rho(\mathfrak{E}_{p+i}, L_j^i)$, evidently,

$\forall k: 1 \leq k \leq a_j$, $t^{j,i+1}_k = \rho(\mathfrak{E}_{p+i}, t^{j,i}_k)$, and for each $j: 1 \leq j \leq o$,

$\mathfrak{L}\mathfrak{S}_j$ naturally induces a_j infinite term sequences: $\mathfrak{T}\mathfrak{S}_{j,1}, \dots, \mathfrak{T}\mathfrak{S}_{j,a_j}$, say, where

$\forall k: 1 \leq k \leq a_j$, $\mathfrak{T}\mathfrak{S}_{j,k} \equiv t^{j,0}_k, t^{j,1}_k, t^{j,2}_k, \dots$, and where, $\forall i \geq 0$, $t^{j,i+1}_k = \rho(\mathfrak{E}_{p+i}, t^{j,i}_k)$.

Let $\succsim_\rho$ denote some well-founded total order for ρ , that is, such that for any term t ,

and any finite set of positive equality units $\mathfrak{E}'$, $t \succsim_\rho \rho(\mathfrak{E}', t)$.

Then, $\forall j: 1 \leq j \leq o$, $\forall k: 1 \leq k \leq a_j$, $\forall i \geq 0$, since $t^{j,i+1}_k = \rho(\mathfrak{E}_{p+i}, t^{j,i}_k)$,

we have $t^{j,i}_k \succsim_\rho t^{j,i+1}_k$, and since $\succsim_\rho$ is well-founded, the infinite chain: $t^{j,0}_k \succsim_\rho t^{j,1}_k \succsim_\rho \dots$,

must terminate. That is, there must be some $n_{j,k} \geq 0$, such that $\forall i \geq n_{j,k}$, $t^{j,i+1}_k = t^{j,i}_k$.

Let $n' = \max\{n_{j,k} \mid 1 \leq j \leq o; 1 \leq k \leq a_j\}$.

Then $\forall j: 1 \leq j \leq o$, $\forall k: 1 \leq k \leq a_j$, $\forall i \geq n'$, we have $t^{j,i+1}_k = t^{j,i}_k$,

so $\forall i \geq n'$ $\forall j: 1 \leq j \leq o$, we have $L_j^{i+1} = L_j^i$.

Since $\forall i \geq 0$, $\{L_j^i \mid 1 \leq j \leq o\} = \{L \in B_{p+i}\}$, it follows that $\forall i \geq n'$, modulo ordering,

$B_{p+i+1} = B_{p+i}$. That is, $\forall i \geq n'$, $B_{p+i} = B_{p+n'}$ so $\forall i \geq n' + p$, $B_i = B_{p+n'}$.

Let $n = i_{p+n'} + 1$. Then $\forall i \geq n - 1$, modulo ordering, $C_i = B_{p+n'}$, so, in particular,

modulo ordering, $C_n = C_{n-1}$.

For $\succsim_{\rho, \mathfrak{E}}$ as a partial order on clauses, (though minor simplifications are possible), we just showed above that if $C_0 \succsim_{\rho, \mathfrak{E}} C_1 \succsim_{\rho, \mathfrak{E}} \dots$, is any infinite chain (of unordered clauses), there actually exists some $n \geq 1$, such that $\forall i \geq n$, (modulo ordering), $C_i = C_n$.

Since any infinite chain $C_0 \succsim_{\rho, \mathfrak{E}} C_1 \succsim_{\rho, \mathfrak{E}} \dots$, of unordered clauses trivially gives rise to some chain of ordered clauses, (order each clause arbitrarily), where the ordered chain is the same except for the fact that, each of its clauses is ordered, we know that there exists some $n \geq 1$, such that $\forall i \geq n$, $C_i = C_n$. ■

For Observation 8.1.10:

Proof (sketch): Suppose $C \succsim_{\mathfrak{E}} C'$. Then there is a finite $\mathfrak{E}$ -reduction sequence:

$C = C_0, \dots, C_k = C'$, where $\forall i: 0 \leq i < k$, either C_{i+1} is a QR-factor of C_i , or $C_i \sim_{\mathfrak{E}} C_{i+1}$.

Let $J \equiv \{i: 0 \leq i < k \mid C_{i+1} \text{ is not a QR-factor of } C_i\}$.

Then clearly, $\forall j \in J, 0 \leq j < k$, and $\forall j \in J, C_j \sim_{\mathfrak{E}} C_{j+1}$.

By Observation 8.1.2, $\forall j \in J$, there exists some finite $\mathfrak{E}_j \subseteq \mathfrak{E}$, such that $C_j \sim_{\mathfrak{E}_j} C_{j+1}$.

Let $\mathfrak{E}' \equiv \bigcup_{j \in J} \mathfrak{E}_j$.

Then evidently, $\mathfrak{E}'$ is a finite subset of $\mathfrak{E}$, and, (since $\forall j \in J, \mathfrak{E}_j \subseteq \mathfrak{E}'$), $\forall j \in J, C_j \sim_{\mathfrak{E}'} C_{j+1}$.

Thus, evidently, $C = C_0, \dots, C_k = C'$ is a $\mathfrak{E}'$ -reduction sequence, and $C \succ_{\mathfrak{E}'} C'$.

Conversely, if, for some (finite) $\mathfrak{E}' \subseteq \mathfrak{E}$, $C \succ_{\mathfrak{E}'} C'$,

then there is a finite $\mathfrak{E}$ -reduction sequence:

$C = C_0, \dots, C_k = C'$, where $\forall i: 0 \leq i < k$, either C_{i+1} is a QR-factor of C_i , or $C_i \sim_{\mathfrak{E}'} C_{i+1}$.

Thus clearly, $\forall i: 0 \leq i < k$, either C_{i+1} is a QR-factor of C_i , or $C_i \sim_{\mathfrak{E}} C_{i+1}$.

and hence, $C \succ_{\mathfrak{E}} C'$. ■

For Observation 8.1.11:

Proof (sketch): Suppose C' is a QR-factor of C .

Then evidently, $\Sigma_Q \cup \{C\} \models C'$, and $\Sigma_Q \cup \{C'\} \models C$.

Thus, trivially, $\mathfrak{E} \cup \Sigma_Q \cup \{C\} \models C'$, and $\mathfrak{E} \cup \Sigma_Q \cup \{C'\} \models C$.

Thus, $\mathfrak{E} \cup \Sigma_Q \models (\forall)C \Leftrightarrow (\forall)C'$.

Suppose $C \sim_{\mathfrak{E}} C'$.

Then evidently, $\mathfrak{E} \cup \Sigma_Q \cup \{C\} \models C'$, and $\mathfrak{E} \cup \Sigma_Q \cup \{C'\} \models C$.

and so, again, $\mathfrak{E} \cup \Sigma_Q \models (\forall)C \Leftrightarrow (\forall)C'$.

Now (since, for some $k \geq 0$, $C = C_0, \dots, C_k = C'$,

where $\forall i: 0 \leq i < k$, either C_{i+1} is a QR-factor of C_i , or $C_i \sim_{\mathfrak{E}} C_{i+1}$),

the result follows by a trivial induction on $k \geq 0$. ■

For Corollary 8.1.5:

Proof: Under the given hypotheses, by Corollary 8.1.2, there is an above-equality EPC_<-deduction, Π' of an equality clause E' , from $\mathfrak{S}'$, such that

$\forall B' \in \mathfrak{S}', k_{\Pi'}(B') \leq ||\{n \in \mathfrak{L} \mid B'_n = B'\}||$,

where E' is a QR-factor of a subclause, E''' , say, of E'' .

Evidently, since E' is a QR-factor of E''' , $\Sigma_Q \cup \{E'\} \models E'''$.

As $E''' \subseteq E''$, $\{E'''\} \models E''$, so $\Sigma_Q \cup \{E'\} \models E''$.

Thus, clearly, $\mathfrak{E} \cup \Sigma_Q \cup \{E'\} \models E''$.

Since, (by hypothesis), $\mathfrak{E} \cup \Sigma_Q \cup \{E''\} \models E$, evidently therefore, $\mathfrak{E} \cup \Sigma_Q \cup \{E'\} \models E$. ■

For Corollary 8.1.6:

Proof: Under the given hypotheses, by Corollary 8.1.4, there is an above-equality $\text{EPC}_{\prec}$ -deduction Π' of an equality clause E' , from $\mathfrak{S}'$, such that

$$\forall B' \in \mathfrak{S}', k_{\Pi'}(B') \leq ||\{n \in \mathfrak{L} \mid B'_n = B'\}||,$$

where E' is $\mathfrak{E}$ -equivalent to a subclause, E''' , say, of E'' .

Evidently, since $E' \sim_{\mathfrak{E}} E'''$, $\mathfrak{E} \cup \Sigma_Q \cup \{E'\} \models E'''$.

As $E''' \subseteq E''$, $\{E'''\} \models E''$, so $\mathfrak{E} \cup \Sigma_Q \cup \{E'\} \models E''$.

Since, (by hypothesis), $\mathfrak{E} \cup \Sigma_Q \cup \{E''\} \models E$, evidently therefore, $\mathfrak{E} \cup \Sigma_Q \cup \{E'\} \models E$. ■

For Corollary 8.1.7:

Proof (sketch): Since, for each $n \in \mathfrak{L}$, $B_n \succ_{\mathfrak{E}} B'_n$, for each $n \in \mathfrak{L}$, there is a $k \geq 0$ and a reduction sequence $B_n = C_0, \dots, C_k = B'_n$.

Let us represent, in the natural way, any given clause labelling, $A: n \in \mathfrak{L} \mapsto A_n$, of the set of leaf nodes, $\mathfrak{L}$, by an K -vector $\vec{A}$, where K is the size of $\mathfrak{L}$.

That is, choose some fixed consecutive enumeration of the nodes in $\mathfrak{L}$, from 1 to K , and if the i^{th} node in this enumeration is n , then A_n is the i^{th} component in $\vec{A}$.

Clearly, from any K -vector $\vec{A}$, we can retrieve the original labelling $A: n \in \mathfrak{L} \mapsto A_n$.

We shall also refer to any such $\vec{A}$, as a *labelling*.

Given a labelling, $\vec{A}$, of length K , let us denote, (for $i: 1 \leq i \leq K$), the i^{th} component of $\vec{A}$, by $[A]_i$.

Let us say that a (re-)labelling $\vec{A}'$ of a labelling $\vec{A}$ is of type 1, if, for each $i: 1 \leq i \leq K$, $[A']_i$ is a QR-factor of $[A]_i$, and of type 2, if, for each $i: 1 \leq i \leq K$, $[A]_i \sim_{\mathfrak{E}} [A']_i$.

We shall also say that $\vec{A}$ to $\vec{A}'$ is a *re-labelling step* of type 1, (type 2), if $\vec{A}'$ is a re-labelling of $\vec{A}$, of type 1, (type 2) respectively.

Evidently, we can find a labelling sequence:

$\vec{B} = \vec{B}_0, \dots, \vec{B}_N = \vec{B}'$, (for some $N \geq 0$), such that,
for any $j: 0 \leq j < N$, $\vec{B}_{j+1}$ is a type 1, or a type 2, re-labelling of $\vec{B}_j$.

[In particular, for instance, there is such a labelling sequence where for any $j: 0 \leq j < N$, there is at most one index $i: 1 \leq i \leq K$, such that $[\vec{B}_{j+1}]_i \neq [\vec{B}_j]_i$,

(that is, such that the i^{th} component of $\tilde{B}_{j+1}$ does not equal the i^{th} component of $\tilde{B}_j$).

Also, since re-labelling steps $\tilde{A}$ to $\tilde{A}'$, and $\tilde{A}'$ to $\tilde{A}''$,
 (that is, $\tilde{A}''$ is a re-labelling of $\tilde{A}'$, which in turn is a re-labelling of $\tilde{A}$),
 both of type 1 (or both of type 2), obviously compose to a single re-labelling step of type 1,
 (type 2, respectively), there is a labelling sequence, as above, where the re-labelling steps have alternating types.]

By Corollary 8.1.5 and Corollary 8.1.6, if $\tilde{A}$ to $\tilde{A}'$ is a re-labelling step of either type 1 or type 2, and Π is an above-equality $\text{EPC}_{<}$ -deduction of a (possibly empty) equality clause E'' , from $\{[\tilde{A}]_i \mid 1 \leq i \leq K\}$, where, for some equality clause E , $\mathfrak{E} \cup \Sigma_Q \cup \{E''\} \models E$, and $\forall C, k_{\Pi}(C) \leq ||\{i: 1 \leq i \leq K \mid [\tilde{A}]_i = C\}||$, then there is an above-equality $\text{EPC}_{<}$ -deduction, Π' of an equality clause E' , from $\{[\tilde{A}']_i \mid 1 \leq i \leq K\}$, such that $\mathfrak{E} \cup \Sigma_Q \cup \{E'\} \models E$, and such that $\forall C, k_{\Pi'}(C) \leq ||\{i: 1 \leq i \leq K \mid [\tilde{A}']_i = C\}||$, (that is, such that $\forall C, k_{\Pi'}(C) \leq ||\{n \in \mathfrak{L} \mid \tilde{A}'_n = C\}||$).

Now the result follows by a simple induction on the re-labelling sequence length, $N \geq 0$:

The case for $N = 0$ is clearly trivial.

Thus, suppose the result for all N' : $0 \leq N' < N$, and that now $N' = N$.

We have the re-labelling sequence

$\tilde{B} = \tilde{B}_0, \dots, \tilde{B}_{N-1}$, where $\tilde{B}_{N-1}$ to $\tilde{B}_N = \tilde{B}'$, is a re-labelling step of type 1 or 2.

By induction, there is an above-equality $\text{EPC}_{<}$ -deduction, Π''' of an equality clause E''' ,

from $\{[\tilde{B}_{N-1}]_i \mid 1 \leq i \leq K\}$, such that $\mathfrak{E} \cup \Sigma_Q \cup \{E'''\} \models E$, and such that

$\forall C, k_{\Pi'''}(C) \leq ||\{i: 1 \leq i \leq K \mid [\tilde{B}_{N-1}]_i = C\}||$.

Since $\tilde{B}_{N-1}$ to $\tilde{B}'$, is a re-labelling step of type 1 or 2, there is an above-equality $\text{EPC}_{<}$ -deduction,

Π' of an equality clause E' , from $\{[\tilde{B}']_i \mid 1 \leq i \leq K\}$, such that $\mathfrak{E} \cup \Sigma_Q \cup \{E'\} \models E$,

and such that $\forall C, k_{\Pi'}(C) \leq ||\{i: 1 \leq i \leq K \mid [\tilde{B}']_i = C\}||$,

(that is, such that $\forall C, k_{\Pi'}(C) \leq ||\{n \in \mathfrak{L} \mid \tilde{B}'_n = C\}||$). ■

For Corollary 8.1.8:

Proof (sketch): Let $\mathfrak{S}''$ denote the collection $\{B''_n \mid n \in \mathfrak{L}\}$.

Then $\forall n \in \mathfrak{L}, B_n \succ_{\mathfrak{E}} B''_n$.

As Π is an above-equality $\text{EPC}_{<}$ -deduction of an equality clause, E , from $\mathfrak{S}$, trivially, Π is an above-equality $\text{EPC}_{<}$ -deduction of an equality clause E'' , (say, where in fact, E'' is E), where, (for some equality clause E), $\mathfrak{E} \cup \Sigma_Q \cup \{E''\} \models E$.

Thus, by Corollary 8.1.7, there is an above-equality $\text{EPC}_{<}$ -deduction, Π'' of a equality clause, $E_{\Pi''}$, say, from $\mathfrak{S}''$, such that $\mathfrak{E} \cup \Sigma_Q \cup \{E_{\Pi''}\} \models E$, and such that $\forall B'' \in \mathfrak{S}'', k_{\Pi''}(B'') \leq ||\{n \in \mathfrak{L} \mid B''_n = B''\}||$.

Let $\mathfrak{L}''$ denote the set of leaf nodes of Π'' , and for any node $n \in \mathfrak{L}''$, let C''_n denote the clause which labels n'' in Π'' .

Evidently there is an injection $j: \mathfrak{L}'' \rightarrow \mathfrak{L}$, such that, for any node $n \in \mathfrak{L}''$, $C''_n = B''_{j(n)}$, and, by renaming nodes in $\mathfrak{L}''$ as necessary, evidently we may assume that j is an inclusion.

Thus we may assume that $\mathfrak{L}'' \subseteq \mathfrak{L}$ and that, for any $n \in \mathfrak{L}''$, B''_n is the clause which labels n in Π'' .

Clearly, to each $n \in \mathfrak{L}''$, is associated a clause B'_n , such that $B'_n \Sigma_0$ -subsumes B''_n .

Thus, by Corollary 6.1.1, there exists a finite forest $\mathfrak{F}$ of above-equality $\text{EPC}_{<}$ -deductions Π' , of equality clauses, $E_{\Pi'}$, say, such that:

- a') $\Sigma_0 \cup \{E_{\Pi'} : \Pi' \in \mathfrak{F}\} \models E_{\Pi''}$, and
- b') $\forall \Pi' \in \mathfrak{F}$, for any clause B' , $k_{\Pi'}(B') \leq ||\{n \in \mathfrak{L}'' \mid B'_n = B'\}||$.

Since $\Sigma_0 \subseteq \Sigma_Q$, and $\mathfrak{E} \cup \Sigma_Q \cup \{E_{\Pi''}\} \models E$, from (a'), evidently,

$\mathfrak{E} \cup \Sigma_Q \cup \{E_{\Pi'} : \Pi' \in \mathfrak{F}\} \models E$. (this is (a)).

Also, (since $\mathfrak{L}'' \subseteq \mathfrak{L}$), from (b'), clearly

$\forall \Pi' \in \mathfrak{F}$, for any clause B' , $k_{\Pi'}(B') \leq ||\{n \in \mathfrak{L} \mid B'_n = B'\}||$. (this is (b)). ■

G.2. Refutation Completeness of $\text{EPC}_{<}$ With Reduction, Under Deletion

G.2.1. Preliminaries

For Observation 8.2.1:

Proof: (In light of 8.1.10), this is trivial. ■

For Observation 8.2.2:

Proof: Since $C \succ_{\epsilon} C'$, and $C' \Sigma_0$ -subsumes C , $C' \mathfrak{E}$ -reduct Σ_0 -subsumes C . ■

For Observation 8.2.3:

Proof: Since $E' \mathfrak{E}$ -reduct Σ_0 -subsumes E there is some clause E'' such that $E \succ_{\epsilon} E''$ and $E' \Sigma_0$ -subsumes E'' .

Since $E' \Sigma_0$ -subsumes E'' , by the soundness of Σ_0 -subsumption, (Theorem 3.3.4), it follows that $\Sigma_0 \cup \{E'\} \models E''$.

Since $E \succ_{\mathfrak{E}} E''$, by Observation 8.1.11, $\mathfrak{E} \cup \Sigma_Q \models (\forall)E \Leftrightarrow (\forall)E''$.

Thus $\mathfrak{E} \cup \Sigma_Q \cup \{E''\} \models E$,

and since $\Sigma_0 \subseteq \Sigma_Q$, evidently, it follows that

$\mathfrak{E} \cup \Sigma_Q \cup \{E'\} \models E$.

Of course, since $E \succ_{\mathfrak{E}} E''$, and E is an equality clause, clearly, so is E'' , and further, since $E' \Sigma_0$ -subsumes E'' , clearly, E' must also be an equality clause. ■

For Observation 8.2.5:

Proof: As it was for Observation 7.2.1, this is trivial, since $\square \Sigma_0$ -subsumes C , for any clause C , and hence, also, clearly, $\square \mathfrak{E}$ -reduct Σ_0 -subsumes C , for any clause C . ■

For Observation 8.2.6:

Proof: The same as that given for Observation 8.2.5. ■

G.2.2. The Above-Equality Fragment

For Lemma 8.2.1:

Proof (sketch): The proof may be obtained from that given for Lemma 7.2.1, by the following modifications. Replace (everywhere):

“ Σ_0 -subsumes” with “ $\mathfrak{E}$ -reduct Σ_0 -subsumes”,

“by the soundness of Σ_0 -subsumption, (Theorem 3.3.4)” with “by Observation 8.2.3”,

“ Σ_0 ” with “ $\mathfrak{E} \cup \Sigma_Q$ ”,

“let there be associated a clause $B(\Pi)_n \in |\Delta|$, such that $B(\Pi)_n \Sigma_0$ -subsumes $C(\Pi)_n$ ” with

“let there be associated a clause $B(\Pi)_n \in |\Delta|$, such that $B(\Pi)_n \mathfrak{E}$ -reduct Σ_0 -subsumes $C(\Pi)_n$, and also a clause $A(\Pi)_n$, such that $B(\Pi)_n \Sigma_0$ -subsumes $A(\Pi)_n$, and $C(\Pi)_n \succ_{\mathfrak{E}} A(\Pi)_n$ ”

“Corollary 6.1.1” with “Corollary 8.1.8”, and

“ Σ_0 -subsumption deleted $\text{EPC}_{<}$ -deduction sequence” with

“ Σ_0 -subsumption deleted reduced $\text{EPC}_{<}$ -deduction sequence”. ■

For Corollary 8.2.1:

Proof: By Lemma 8.2.1, there is a finite collection of equality clauses $\mathfrak{E}_0' \subseteq |\Delta|$, such that,

$\forall \Pi \in \mathfrak{F}, \mathcal{E} \cup \Sigma_Q \cup \mathcal{E}_0' \models E_\Pi$.

But by compactness (for first order logic), (and since $\mathfrak{F}$ is finite), it follows that there is a finite $\mathcal{E}' \subseteq \mathcal{E}$, such that $\forall \Pi \in \mathfrak{F}, \mathcal{E}' \cup \Sigma_Q \cup \mathcal{E}_0' \models E_\Pi$.

Since $\mathcal{E} \subseteq |\Delta|$, trivially, $\mathcal{E}' \subseteq |\Delta|$, and taking $\mathcal{E}' \cup \mathcal{E}_0'$ for $\mathcal{E}_0$ evidently suffices. ■

For Corollary 8.2.2:

Proof: Since Δ is a Σ_0 -subsumption deleted reduced $\text{EPC}_{<}$ -deduction sequence from S , then (by Observation 8.2.2, evidently, for all $C \in S$, there exists some $C' \in |\Delta|$, such that $C' \mathcal{E}$ -reduct Σ_0 -subsumes C .

Thus for each clause C which labels a leaf of any $\Pi \in \mathfrak{F}$, there exists some $C' \in |\Delta|$, such that $C' \mathcal{E}$ -reduct Σ_0 -subsumes C .

By Corollary 8.2.1, it follows that there is a finite collection of equality clauses $\mathcal{E}_0 \subseteq |\Delta|$, such that $\forall \Pi \in \mathfrak{F}, \Sigma_Q \cup \mathcal{E}_0 \models E_\Pi$. ■

G.2.3. The Equality Fragment

For Proposition 8.2.1:

Proof: $\Delta \equiv C_0, C_1, \dots$ be a (refined) Σ_0 -subsumption deleted reduced $\text{EPC}_{<}$ -deduction sequence from S , and let $\mathcal{E} \subseteq |\Delta|$ be some finite non-empty set of equality clauses.

Since Δ is a (refined) Σ_0 -subsumption deleted reduced $\text{EPC}_{<}$ -deduction sequence, (from S), for any equality clauses $A; B \in |\Delta|$

$\forall C \in \text{EPC}_{<}^1(A, B), \exists C' \in |\Delta|$, such that $C' \Sigma_0$ -subsumes C ,

(and in the “refined” case:

$\forall C \in \text{EPC}_{<}^1(A, B)$, if $C = \square$, then $\square \in |\Delta|$, and otherwise, $\exists C' \in |\Delta|$, such that $C' \Sigma_0$ -subsumes C .)

For any equality clauses $A; B \in |\Delta|$, let $\Delta(A, B)$, denote a finite subset of $|\Delta|$, such that

$\forall C \in \text{EPC}_{<}^1(A, B), \exists C' \in \Delta(A, B)$, such that $C' \Sigma_0$ -subsumes C ,

(and in the “refined” case, such that:

$\forall C \in \text{EPC}_{<}^1(A, B)$, if $C = \square$, then $\square \in \Delta(A, B)$, and otherwise, $\exists C' \in \Delta(A, B)$, such that $C' \Sigma_0$ -subsumes C .)

[Since $\text{EPC}_{<}^1(A, B)$ is always finite, evidently, such a $\Delta(A, B)$ always exists.]

Let Δ_0 be some sequencing of the clauses in $\mathcal{E}$.

Then $\emptyset \neq |\Delta_0| \subseteq |\Delta|$.

For $n \geq 1$, let Δ_n be a sequencing of the clauses in $\bigcup \{ \Delta(A, B) \mid A : B \in |\Delta_0, \dots, \Delta_{n-1}| \}$.

Evidently, for all $n \geq 0$, Δ_n is a finite sequence.

Let Δ' be the concatenation: $\Delta_0, \Delta_1, \dots$.

Then evidently Δ' is a (refined) Σ_0 -subsumption deleted $EPC_{<}$ -deduction sequence from $\mathfrak{E}$, such that $\mathfrak{E} \subseteq |\Delta'| \subseteq |\Delta|$. ■

For Lemma 8.2.2:

Proof: Let S , Δ , $\mathfrak{E}$, Π , and D , be as in the hypothesis.

Let $\mathfrak{E}'$ be a finite subset of $|\Delta|$, such that $\forall E \in \mathfrak{E}$, there exists an $E' \in \mathfrak{E}'$, such that $E' \Sigma_0$ -subsumes E .

Of course, as Π is an equality $EPC_{<}$ -deduction from $\mathfrak{E}$, $\mathfrak{E}$ is non-empty, and so $\mathfrak{E}'$ is non-empty.

By Proposition 8.2.1, there is a Σ_0 -subsumption deleted $EPC_{<}$ -deduction sequence from $\mathfrak{E}'$, Δ' , such that $\mathfrak{E}' \subseteq |\Delta'| \subseteq |\Delta|$.

Clearly $\mathfrak{E}$ is a finite collection of equality clauses such that $\forall E \in \mathfrak{E}$, there exists an $E' \in |\Delta'|$, such that $E' \Sigma_0$ -subsumes E .

Thus, by Lemma 7.2.4, there exists a $D' \in |\Delta'|$, such that $D' \Sigma_0$ -subsumes $QR_{full}(D)$.

As $|\Delta'| \subseteq |\Delta|$, it follows that there exists $D' \in |\Delta|$, such that $D' \Sigma_0$ -subsumes $QR_{full}(D)$. ■

G.2.4. Synthesis and Result

For Corollary 8.2.3:

Proof: A proof is given by that of Theorem 7.2.1, with the following modifications:

Substitute:

‘Corollary 7.2.7’ for ‘Theorem 5.2.1’,

‘if $\Sigma_0 \cup S$ is unsatisfiable’ for ‘if S is any unsatisfiable set of clauses’,

‘say, of S ’ for ‘say, of $S \cup \{=(x, x)\}$. Since $\{=(x, x)\} \subseteq \Sigma_0 \subseteq S$, Π is a $EPC_{<}$ -refutation of S ’

‘Corollary 8.2.2’ for ‘Lemma 8.2.1’,

‘ $\forall E \in \mathfrak{E} \Sigma_Q \cup \mathfrak{E}_0 \models E$ ’ for ‘ $\forall E \in \mathfrak{E} \Sigma_0 \cup \mathfrak{E}_0 \models E$ ’,

‘since $\mathfrak{E} \cup \Sigma_0$ is unsatisfiable, and $\Sigma_0 \subseteq \Sigma_Q$, $\Sigma_Q \cup \mathfrak{E}_0$ is unsatisfiable’ for

‘since $\mathfrak{E} \cup \Sigma_0$ is unsatisfiable, so is $\Sigma_0 \cup \mathfrak{E}_0$ ’,

‘Corollary 7.2.7, (since $\Sigma_Q = \Sigma_0 \cup EQF$)’ for ‘Theorem 5.2.1’,

‘from $EQF \cup \mathfrak{E}_0$ ’ for ‘from $\Sigma_0 \cup \mathfrak{E}_0$ ’,

'EQF $\subseteq S$ ' for ' $\Sigma_0 \subseteq S$ ',
 ' Σ_0 -subsumption deleted reduced EPC $_{<}$ -deduction sequence from S ' for
 ' Σ_0 -subsumption deleted EPC $_{<}$ -deduction sequence from S ',
 'for each $A \in EQF$ ' for 'for each $A \in \Sigma_0$ ',
 and finally, 'Lemma 8.2.2' for 'Lemma 7.2.4'. ■

For Corollary 8.2.4:

Proof: A proof is given by that for Corollary 7.2.2, with the following modifications. Substitute:

'refined Σ_0 -subsumption deleted reduced EPC $_{<}$ -deduction sequence' for
 'refined Σ_0 -subsumption deleted EPC $_{<}$ -deduction sequence'. (twice).
 ' Σ_0 -subsumption deleted reduced EPC $_{<}$ -deduction sequence' for
 ' Σ_0 -subsumption deleted EPC $_{<}$ -deduction sequence'.
 and 'Corollary 8.2.3' for 'Theorem 7.2.1'. ■

For Lemma 8.2.3:

Proof (sketch):

A proof may be constructed from that for Lemma 2.5.3, by some editing.

In order to facilitate the specification of this task, it is convenient to introduce some notational shorthand.

For any clauses A , B , and C , and any sets of clauses $\mathfrak{X}$ and $\mathfrak{Y}$,

let $COV(\mathfrak{X}, \mathfrak{Y}, A, B, C)$ denote:

if A or B is an equality clause, then $\exists C' \in \mathfrak{Y}$, such that $C' \Sigma_0$ -subsumes C ,
 and otherwise, for some set of positive equality units $\mathfrak{E} \subseteq \mathfrak{X}$, $\exists C' \in \mathfrak{Y}$,
 such that $C' \mathfrak{E}$ -reduct Σ_0 -subsumes C

and let $ECOV(\mathfrak{X}, B, C)$ denote:

if $C \in EQF$, then $B \Sigma_0$ -subsumes C , and otherwise, for some set of positive
 equality units $\mathfrak{E} \subseteq \mathfrak{X}$, $B \mathfrak{E}$ -reduct Σ_0 -subsumes C

With these abbreviations in hand, a proof is given by that for Lemma 2.5.3, with the following modifications. Substitute:

' Σ_0 -subsumption and Σ_0 -tautology deleted reduced EPC $_{<}$ -deduction sequence' for
 'subsumption and tautology deleted PC $_{<}$ -deduction sequence', (everywhere),
 ' Σ_0 -subsumption deleted reduced EPC $_{<}$ -deduction sequence' for
 'subsumption deleted PC $_{<}$ -deduction sequence', (everywhere),
 ' Σ_0 -tautology for 'tautology', (everywhere),
 (and ' Σ_0 -tautology's' for 'tautologies'),
 'EPC' for 'PC', (everywhere),

'or $\exists i: 0 \leq i \leq k_0$, such that, $ECOV(|\Delta|, C_i, C)$ ' for
 'or $\exists i: 0 \leq i \leq k_0$, such that C_i subsumes C ',
 'for no $i: 0 \leq i \leq k_0$, is it the case that, $ECOV(|\Delta|, C_i, C)$ ' for
 'for no $i: 0 \leq i \leq k_0$, is it the case that C_i subsumes C ',
 'does A Σ_0 -subsume C ' for 'does A subsume C ',
 'and $\forall C \in S$, $\exists C' \in |\Delta'|$, such that, $ECOV(|\Delta|, C', C)$,
 (and hence, since $|\Delta| \subseteq |\Delta'|$, such that $ECOV(|\Delta'|, C', C)$)' for
 'and $\forall C \in S$, $\exists C' \in |\Delta'|$, such that C' subsumes C ',
 'then, $COV(|\Delta|, |\Delta'|, A, B, C)$, (and hence, since $|\Delta| \subseteq |\Delta'|$, $COV(|\Delta'|, |\Delta'|, A, B, C)$)' for
 'then $\exists C' \in |\Delta'|$ such that C' subsumes C ',
 ' $COV(|\Delta|, |\Delta|, A, B, C)$ ' for
 ' $\exists C' \in |\Delta|$ such that C' subsumes C ',
 (and throughout the rest of the justification of claim (1)),
 ' $COV(|\Delta|, |\Delta'|, A, B, C)$ ' for ' $\exists C' \in |\Delta'|$ such that C' subsumes C ',
 'unless there exists a $C' \in |\Delta_0, \dots, \Delta_{n-1}|$, such that C' Σ_0 -subsumes C ,
 (but if there is such a C' , clearly, $COV(|\Delta|, |\Delta'|, A, B, C)$)' for
 'unless there exists a $C' \in |\Delta_0, \dots, \Delta_{n-1}|$, such that C' subsumes C ',
 'does C' Σ_0 -subsume C ' for 'does C' subsume C ', (everywhere),
 'Theorem 7.1.1' for 'Lemma 2.5.2',
 and finally, 'either B Σ_0 -subsumes C , or C is a Σ_0 -tautology' for
 'either B subsumes C , or C is a tautology'. ■

For Corollary 8.2.5:

Proof: Since $\Sigma_0 \cup S$ is unsatisfiable, (clearly), Δ is non-empty. By Lemma 8.2.3,
 there exists a Σ_0 -subsumption deleted reduced $EPC_{\leq}$ -deduction sequence from S , Δ' ,
 such that $\forall C' \in |\Delta'| \setminus |\Delta|$, C' is a Σ_0 -tautology.

Since, by assumption, $EQF \subseteq S$ and $\Sigma_0 \cup S$ is unsatisfiable, by Corollary 8.2.3, it is the case that,
 $\exists D \in |\Delta'|$ such that D Σ_0 -subsumes $\square$.

Clearly, D is not a Σ_0 -tautology, so $D \in |\Delta|$. ■

For Corollary 8.2.6:

Proof: Substituting:

' Σ_0 -subsumption and Σ_0 -tautology deleted reduced' in place of ' Σ_0 -subsumption deleted',
 and 'Corollary 8.2.5' in place of 'Theorem 7.2.1',
 the proof is the same as that given for Corollary 7.2.2. ■

G.3. Clause Reduction: II

For Observation 8.3.1:

Proof: For 1: If C' is a Σ_0 -repf of C , then $C' \subseteq C$, so clearly C' Σ_0 -subsumes C , and C Σ_0 -subsumes C' .

For 2: If C' is a Σ_0 -repf of C , and C'' is a Σ_0 -repf of C' , then C Σ_0 -subsumes C' , and C' Σ_0 -subsumes C'' .

Thus (by transitivity of Σ_0 -subsumes, it follows that), C Σ_0 -subsumes C'' , so (since $C'' \subseteq C$), C'' is a Σ_0 -repf of C .

Conversely, if C'' is a Σ_0 -repf of C , then C Σ_0 -subsumes C'' .

Thus (since $C'' \subseteq C'$), C Σ_0 -subsumes C' , and (since $C' \subseteq C$), C' Σ_0 -subsumes C'' .

Thus C' is a Σ_0 -repf of C and C'' is a Σ_0 -repf of C' . ■

For Observation 8.3.2:

Proof: If C' is a Σ_0 -repf of C , then C Σ_0 -subsumes C' , and so, by Corollary 3.3.1, C' is a Σ_0 -tautology. ■

For Observation 8.3.3:

Proof: Since C' Σ_0 -subsumes C and vice-versa, (by the transitivity of Σ_0 -subsumes), clearly, C Σ_0 -subsumes D if and only if C' Σ_0 -subsumes D .

Similarly, since D' Σ_0 -subsumes D and vice-versa, clearly, C' Σ_0 -subsumes D if and only if C' Σ_0 -subsumes D' . ■

G.4. Refining the Deduction Rules

G.4.1. Tightening the Equality Rules

For Observation 8.4.1:

Proof: Since, by assumption, C' is a Σ_0 -repf of C , $C' \subseteq C$, and C Σ_0 -subsumes C' .

Since C is QR -positive, $QR_{full}(C)$ is positive, and (modulo ordering), $QR_{full}(C) = C\sigma \setminus (C\sigma)^-$, where σ is some most general substitution such that $(C\sigma)^-$ contains only trivial inequalities.

Evidently, $C\sigma = QR_{full}(C) \cup (C\sigma)^-$.

Since $C' \subseteq C$, clearly, $C'\sigma \subseteq C\sigma$, $(C'\sigma)^- \subseteq (C\sigma)^-$, and $(C'\sigma)^-$ contains only trivial inequalities.

Thus C' is QR -positive, so $QR_{full}(C')$ is positive, and $QR_{full}(C') = C'\sigma' \setminus (C'\sigma')^-$,

where σ' is some most general substitution such that $(C'\sigma')^-$ contains only trivial inequalities.

Evidently, $C'\sigma' = QR_{full}(C') \cup (C'\sigma')^-$.

By Observation 4.1.5, σ' is a most general substitution such that $(cl(C')\sigma')^-$ contains only trivial inequalities.

Since $(C'\sigma)^-$ contains only trivial inequalities, for some substitution, θ' , $\sigma = \sigma'\theta'$.

As $C \Sigma_0$ -subsumes C' , there is some substitution, φ , and some F consisting only of trivial inequalities, such that $C\varphi \subseteq cl(C') \cup F$.

Thus $C\varphi\sigma' \subseteq cl(C')\sigma' \cup F'$, where $F' = F\sigma'$ clearly, contains only trivial inequalities, and so $(C\varphi\sigma')^-$ contains only trivial inequalities.

Since σ is most general such that $(C\sigma)^-$ contains only trivial inequalities, there is some substitution, θ , such that $\varphi\sigma' = \sigma\theta$.

Thus, $C\sigma\theta \subseteq cl(C')\sigma' \cup F'$, where F' contains only trivial inequalities.

Since $C'\sigma'$ contains only trivial inequalities, $cl(C'\sigma') = C'\sigma'$, and since $C'\sigma' = QR_{full}(C') \cup (C'\sigma')^-$, it follows that $cl(C'\sigma') = QR_{full}(C') \cup (C'\sigma')^-$.

Since, (by (n), of Observation 3.2.1), $cl(C')\sigma' = cl(C'\sigma')$, evidently, $cl(C')\sigma' = QR_{full}(C') \cup (C'\sigma')^-$.

Since $C\sigma = QR_{full}(C) \cup (C\sigma)^-$, clearly, $C\sigma\theta = QR_{full}(C)\theta \cup (C\sigma)^-\theta$.

Thus, $QR_{full}(C)\theta \cup (C\sigma)^-\theta = C\sigma\theta \subseteq QR_{full}(C') \cup (C'\sigma')^- \cup F'$.

and so $QR_{full}(C)\theta \subseteq QR_{full}(C') \cup F''$, where $F'' = (C'\sigma')^- \cup F'$ contains only trivial inequalities.

But since $QR_{full}(C)$ is positive, so is $QR_{full}(C)\theta$.

Thus $QR_{full}(C)\theta \subseteq QR_{full}(C')$, (and $QR_{full}(C)$ subsumes $QR_{full}(C')$).

On the other hand, since $C' \subseteq C$, clearly, $C'\sigma'\theta' \subseteq C\sigma'\theta'$, and so

$C'\sigma'\theta' \subseteq C\sigma'\theta' = C\sigma = QR_{full}(C) \cup (C\sigma)^-$.

But since $C'\sigma' = QR_{full}(C') \cup (C'\sigma')^-$, clearly, $C'\sigma'\theta' = QR_{full}(C')\theta' \cup (C'\sigma')^-\theta'$.

and so $QR_{full}(C')\theta' \cup (C'\sigma')^-\theta' \subseteq QR_{full}(C) \cup (C\sigma)^-$.

Thus $QR_{full}(C')\theta' \subseteq QR_{full}(C) \cup (C\sigma)^-$.

As $QR_{full}(C')$ is positive, so is $QR_{full}(C')\theta'$, and so $QR_{full}(C')\theta' \subseteq QR_{full}(C)$.

(and $QR_{full}(C')$ subsumes $QR_{full}(C)$). ■

For Proposition 8.4.1:

Proof: Suppose $C \in \text{EPC}_{+,-}(\mathcal{A}, D)$, for some non-empty equality clauses, $\mathcal{A}$ positive, and D mixed.

Then there exists a non-empty subset $\mathcal{A}_R \subseteq \mathcal{A}$, a non-empty subset $D_R \subseteq D^-$, the left-most inequality literal, $\neq(s_0, t_0)$, say, of D in D_R , and a (possibly empty) subset $D_S \subseteq D^- \setminus D_R$, such that for some symmetric choice function c on $\mathcal{A}_R \cup D_R \cup D_S$, with $c(\neq(s_0, t_0)) = 0$,

the most general substitution σ , say, unifying:

$$\begin{aligned} & \{ \{=(s', t') \mid =(s', t') \in \mathcal{A}_R; c(=(s', t')) = 0\} \cup \{=(t', s') \mid =(s', t') \in \mathcal{A}_R; c(=(s', t')) = 1\} \cup \\ & \quad \{=(s', t') \mid \neq(s', t') \in D_R; c(\neq(s', t')) = 0\} \cup \{=(t', s') \mid \neq(s', t') \in D_R; c(\neq(s', t')) = 1\}, \\ & \quad \{s' \mid =(s', t') \in \mathcal{A}_R; c(=(s', t')) = 0\} \cup \{t' \mid =(s', t') \in \mathcal{A}_R; c(=(s', t')) = 1\} \cup \\ & \quad \{s' \mid \neq(s', t') \in D_S; c(\neq(s', t')) = 0\} \cup \{t' \mid \neq(s', t') \in D_S; c(\neq(s', t')) = 1\} \}, \end{aligned}$$

exists, σ unifying the two sets in the above doubleton set to the singletons $\{=(s, r)\}$ and $\{s\}$, respectively, for some terms s and r , say, and

$$B \equiv \mathcal{A}\sigma \setminus \{=(r, s), =(s, r)\} \cup D\sigma \setminus (\{\neq(r, s), \neq(s, r)\} \cup D_S\sigma) \cup$$

$$\begin{aligned} & \{ \neq(r, t'\sigma) \mid \neq(s', t') \in D_S; \neq(s', t')\sigma \notin \{ \neq(r, s), \neq(s, r) \}; c(\neq(s', t')) = 0 \} \cup \\ & \{ \neq(s'\sigma, r) \mid \neq(s', t') \in D_S; \neq(s', t')\sigma \notin \{ \neq(r, s), \neq(s, r) \}; c(\neq(s', t')) = 1 \}. \end{aligned}$$

Let $A'_R \equiv \{ L \in A \mid L\sigma \in \{=(s, r), =(r, s)\} \}$, and

$$D'_R \equiv \{ L \in D \mid L\sigma \in \{ \neq(s, r), \neq(r, s) \} \}.$$

Then $A_R \subseteq A'_R$, $D_R \subseteq D'_R$, and as required, $\emptyset \neq A'_R \subseteq A$, $\emptyset \neq D'_R \subseteq D^-$, and the left-most inequality literal, $\neq(s_0, t_0)$, of D is in D_R .

Let $c'': A'_R \cup D'_R \rightarrow \{0, 1\}$, be defined by:

$$c'': L \in A_R \mapsto c(L).$$

$$c'': =(s'', t'') \in A'_R \setminus A_R \mapsto 0, \text{ if } s''\sigma = s, \text{ and}$$

$$c'': =(s'', t'') \in A'_R \setminus A_R \mapsto 1, \text{ otherwise, and}$$

$$c'': L \in D_R \mapsto c(L),$$

$$c'': \neq(s'', t'') \in D'_R \setminus D_R \mapsto 0, \text{ if } s''\sigma = s, \text{ and}$$

$$c'': \neq(s'', t'') \in D'_R \setminus D_R \mapsto 1, \text{ otherwise.}$$

Evidently, if $c'': =(s'', t'') \in A'_R \setminus A_R \mapsto 1$, then $s''\sigma \neq s$, and (since $=(s'', t'') \in A'_R$), since $=(s'', t'')\sigma \in \{=(s, r), =(r, s)\}$, it must be the case that $t''\sigma = s$, and $s''\sigma = r$.

Thus, (evidently), if $c'': =(s'', t'') \in A'_R \mapsto 0$, then $=(s'', t'')\sigma = =(s, r)$.

while if $c'': =(s'', t'') \in A'_R \mapsto 1$, then $=(s'', t'')\sigma = =(r, s)$.

Similarly, if $c'': \neq(s'', t'') \in D'_R \setminus D_R \mapsto 1$, then $t''\sigma = s$, and $s''\sigma = r$, and

if $c'': \neq(s'', t'') \in D'_R \mapsto 0$, then $\neq(s'', t'')\sigma = \neq(s, r)$.

while if $c'': \neq(s'', t'') \in D'_R \mapsto 1$, then $\neq(s'', t'')\sigma = \neq(r, s)$.

Evidently c'' is a symmetric choice function on $A'_R \cup D'_R$, such that $c''(\neq(s_0, t_0)) = 0$.

Let $\mathfrak{S}$ denote the set

$$\begin{aligned} & \{=(s', t') \mid =(s', t') \in A'_R; c''(=(s', t')) = 0\} \cup \{=(t', s') \mid =(s', t') \in A'_R; c''(=(s', t')) = 1\} \cup \\ & \{ \neq(s', t') \mid \neq(s', t') \in D'_R; c''(\neq(s', t')) = 0\} \cup \{ \neq(t', s') \mid \neq(s', t') \in D'_R; c''(\neq(s', t')) = 1\}. \end{aligned}$$

Since, (clearly), σ unifies $\mathfrak{S}$, to the singleton $\{=(s, r)\}$, there is a most general substitution, σ' , such that σ' unifies $\mathfrak{S}$ to the singleton $\{=(s'', r'')\}$, where $s'' = s_0\sigma'$ and $r'' = t_0\sigma'$.

and for some substitution, θ , say, $\sigma = \sigma'\theta$. (and $=(s_0, t_0)\sigma'\theta = =(s'', r'')\theta = =(s, r)$).

$$\text{Let } D'_S \equiv \{ L \in D^- \mid L\sigma \in D_S\sigma \} \setminus \{ \neq(s'', r''), \neq(r'', s'') \},$$

(then $D'_S \subseteq D^- \sigma' \setminus \{ \neq(s'', r''), \neq(r'', s'') \}$ as required),

and let $c': D'_S \rightarrow \{0, 1\}$, be defined by, (for any $L' \in D'_S$),

$$\begin{aligned} c'(L') &= 0, \text{ if there exists } L'' \in D_S, \text{ and some } L \in D^-, \text{ such that } L' = L\sigma', \\ &\quad L\sigma = L''\sigma, \text{ and } c(L'') = 0, \text{ and} \\ &= 1, \text{ otherwise.} \end{aligned}$$

Clearly, since for any $L' \in D'_S$ there is some $L \in D^-$ and some $L'' \in D_S$ such that $L' = L\sigma'$ and $L\sigma = L''\sigma$, (and for any $L'' \in D_S$ either $c(L'') = 0$ or $c(L'') = 1$), for any $L' \in D'_S$ there is some

$L \in D^-$ and some $L'' \in D_S$ such that $L' = L\sigma'$, $L\sigma = L''\sigma$, and $c(L'') = c'(L')$.

Thus for any $L' \in D'_S$ there is some $L \in D^-$ and some $L'' \in D_S$ such that $L'\theta = L\sigma'\theta$, $L\sigma'\theta = L''\sigma'\theta$, and $c(L'') = c'(L')$.

Thus for any $L' \in D'_S$ there is some $L'' \in D_S$ such that $L'\theta = L''\sigma'\theta$ and $c(L'') = c'(L')$.

Thus $\{s' \mid \neg(s', t') \in D'_S; c'(\neg(s', t')) = 0\} \cup \{t' \mid \neg(s', t') \in D'_S; c'(\neg(s', t')) = 1\}\theta =$
 $\{s'\theta \mid \neg(s', t') \in D'_S; c'(\neg(s', t')) = 0\} \cup \{t'\theta \mid \neg(s', t') \in D'_S; c'(\neg(s', t')) = 1\} \subseteq$
 $\{s'\sigma'\theta \mid \neg(s', t') \in D_S; c(\neg(s', t')) = 0\} \cup \{t'\sigma'\theta \mid \neg(s', t') \in D_S; c(\neg(s', t')) = 1\}.$

and so

$$\begin{aligned} \{s\} &= \{s_0\}\sigma = \{s_0\}\sigma'\theta = \{s''\}\theta \\ &\subseteq (\{s''\} \cup \{s' \mid \neg(s', t') \in D'_S; c'(\neg(s', t')) = 0\} \cup \{t' \mid \neg(s', t') \in D'_S; c'(\neg(s', t')) = 1\})\theta \\ &= (\{s_0\sigma'\} \cup \{s' \mid \neg(s', t') \in D'_S; c'(\neg(s', t')) = 0\} \cup \{t' \mid \neg(s', t') \in D'_S; c'(\neg(s', t')) = 1\})\theta \\ &\subseteq \{s_0\sigma'\theta\} \cup \{s'\sigma'\theta \mid \neg(s', t') \in D_S; c(\neg(s', t')) = 0\} \cup \{t'\sigma'\theta \mid \neg(s', t') \in D_S; c(\neg(s', t')) = 1\} \\ &= (\{s_0\} \cup \{s' \mid \neg(s', t') \in D_S; c(\neg(s', t')) = 0\} \cup \{t' \mid \neg(s', t') \in D_S; c(\neg(s', t')) = 1\})\sigma'\theta \\ &= (\{s_0\} \cup \{s' \mid \neg(s', t') \in D_S; c(\neg(s', t')) = 0\} \cup \{t' \mid \neg(s', t') \in D_S; c(\neg(s', t')) = 1\})\sigma \\ &= \{s\} \end{aligned}$$

so θ unifies $\{s''\} \cup \{s' \mid \neg(s', t') \in D'_S; c'(\neg(s', t')) = 0\} \cup \{t' \mid \neg(s', t') \in D'_S; c'(\neg(s', t')) = 1\}$ to $\{s\}$.

Thus there is a most general substitution, ρ , say, which unifies

$$\{s''\} \cup \{s' \mid \neg(s', t') \in D'_S; c'(\neg(s', t')) = 0\} \cup \{t' \mid \neg(s', t') \in D'_S; c'(\neg(s', t')) = 1\},$$

to $\{s''\rho\}$, and for some substitution, θ' , say, $\theta = \rho\theta'$.

$$(\text{and } =(s_0, t_0)\sigma'\rho\theta' = =(s_0, t_0)\sigma'\theta = =(s'', r'')\theta = =(s'', r'')\rho\theta' = =(s, r)).$$

Since $\sigma = \sigma'\theta$, and $\theta = \rho\theta'$, clearly, $\sigma = \sigma'\rho\theta'$.

If neither $A\sigma'\rho$ nor $D\sigma'\rho$ is a Σ_0 -tautology, then letting

$$\begin{aligned} B' \equiv & A\sigma'\rho \setminus \{=(r'', s'')\rho, =(s'', r'')\rho\} \cup D\sigma'\rho \setminus (\{\neg(r'', s'')\rho, \neg(s'', r'')\rho\} \cup D'_S\rho) \cup \\ & (D'_S\rho \setminus \{\neg(r'', s'')\rho, \neg(s'', r'')\rho\})[r''\rho/s''\rho], \end{aligned}$$

evidently, $B' \in \mathcal{U}(A, D)$.

But if $A\sigma'\rho$ is a Σ_0 -tautology, then (since $A\sigma = A\sigma'\rho\theta'$), so is $A\sigma$.

Similarly, if $D\sigma'\rho$ is a Σ_0 -tautology, then (since $D\sigma = D\sigma'\rho\theta'$), so is $D\sigma$.

It is easy to see that, (for example, by examine the proof of Theorem 7.1.1, if $A\sigma$ is a Σ_0 -tautology, (as in case II.iii of that proof), then either B is a Σ_0 -tautology, or D Σ_0 -subsumes B , and if $D\sigma$ is a Σ_0 -tautology, (as in case II.ii of that proof), then either B is a Σ_0 -tautology, or A Σ_0 -subsumes B .

Thus if either $A\sigma'\rho$ or $D\sigma'\rho$ is a Σ_0 -tautology, either B is a Σ_0 -tautology, (that is, (a)), D Σ_0 -subsumes B , ((b)), or A Σ_0 -subsumes B , ((c)).

Otherwise, we have $B' \in \mathcal{U}(A, D)$, we may assume that $r \neq s$, (for otherwise, clearly, $A\sigma$ is a Σ_0 -tautology), and it will suffice to show that B' Σ_0 -subsumes B , (for then will will have (d)).

But as (up to ordering),

$$\begin{aligned} B' &= A\sigma'\rho \setminus \{=(r'', s'')\rho, =(s'', r'')\rho\} \cup D\sigma'\rho \setminus (\{\neq(r'', s'')\rho, \neq(s'', r'')\rho\} \cup D'_S\rho) \cup \\ &\quad (D'_S\rho \setminus \{\neq(r'', s'')\rho, \neq(s'', r'')\rho\})[r''\rho/s''\rho], \\ B'\theta' &= (A\sigma'\rho \setminus \{=(r'', s'')\rho, =(s'', r'')\rho\})\theta' \cup (D\sigma'\rho \setminus (\{\neq(r'', s'')\rho, \neq(s'', r'')\rho\} \cup D'_S\rho))\theta' \cup \\ &\quad ((D'_S\rho \setminus \{\neq(r'', s'')\rho, \neq(s'', r'')\rho\})[r''\rho/s''\rho])\theta'. \end{aligned}$$

Now, since $\sigma = \sigma'\rho\theta'$, and $=(s'', r'')\rho\theta' = =(s, r)$, by the definition of A'_R , D'_R , σ' , D'_S , $s'' = s_0\sigma'$ and $r'' = t_0\sigma'$,

$$\begin{aligned} (A\sigma'\rho \setminus \{=(r'', s'')\rho, =(s'', r'')\rho\})\theta' &\subseteq (A\sigma' \setminus \{=(r'', s''), =(s'', r'')\})\rho\theta' \\ &= A\sigma'\rho\theta' \setminus \{=(r'', s'')\rho\theta', =(s'', r'')\rho\theta'\} \\ &= A\sigma \setminus \{=(r, s), =(s, r)\} \end{aligned}$$

Similarly,

$$\begin{aligned} (D\sigma'\rho \setminus (\{\neq(r'', s'')\rho, \neq(s'', r'')\rho\} \cup D'_S\rho))\theta' &\subseteq (D\sigma' \setminus (\{\neq(r'', s''), \neq(s'', r'')\} \cup D'_S))\rho\theta' \\ &= (D\sigma' \setminus (\{\neq(r'', s''), \neq(s'', r'')\} \cup \{L \in D^- \mid L\sigma \in D_S\sigma\}\sigma'))\rho\theta' \\ &= D\sigma'\rho\theta' \setminus (\{\neq(r'', s''), \neq(s'', r'')\}\rho\theta' \cup \{L \in D^- \mid L\sigma \in D_S\sigma\}\sigma'\rho\theta') \\ &= D\sigma \setminus (\{\neq(r, s), \neq(s, r)\} \cup D_S\sigma) \end{aligned}$$

Further, suppose $L'' \in ((D'_S\rho \setminus \{\neq(r'', s'')\rho, \neq(s'', r'')\rho\})[r''\rho/s''\rho])\theta'$.

Then $L'' = (L'[r''\rho/s''\rho])\theta'$, for some $L' \in D'_S\rho \setminus \{\neq(r'', s'')\rho, \neq(s'', r'')\rho\}$.

Since $L' \in D'_S\rho \setminus \{\neq(r'', s'')\rho, \neq(s'', r'')\rho\}$, for some terms s' and t' , $\neq(s', t') \in D'_S$.

$L' = \neq(s', t')\rho$, and either $s'\rho = s''\rho$ or $t'\rho = s''\rho$.

Further $\neq(s', t')\rho \notin \{\neq(r'', s'')\rho, \neq(s'', r'')\rho\}$.

If L'' is non-trivial, then $s'\rho \neq t'\rho$, and exactly one of s' and t' becomes $s''\rho$, under the application of ρ .

If $c'(\neq(s', t')) = 0$, then $s'\rho = s''\rho$, and (if L'' is non-trivial),

$$\begin{aligned} L'' &= (L'[r''\rho/s''\rho])\theta' = (\neq(s', t')\rho[r''\rho/s''\rho])\theta' = \neq(r''\rho, t'\rho)\theta' \\ &= \neq(r''\rho\theta', t'\rho\theta') = \neq(r, t'\rho\theta') = \neq(r, t'\theta). \end{aligned}$$

As $L'\theta = \neq(s', t')\rho\theta' = \neq(s', t')\theta \in D'_S\theta$, and for any literal $\neq(s', t') \in D'_S$, there is some $L'' \in D_S$ such that $\neq(s', t')\theta = L''\sigma'\theta$ and $c(L'') = c'(\neq(s', t'))$, there is some literal $L'' \in D_S$ such that $\neq(s'\theta, t'\theta) = L''\sigma$ and $c(L'') = 0$.

Thus $s'\theta = s$, and unless $t'\theta = r$,

$$\neq(r, t'\theta) \in \{\neq(r, t'\sigma) \mid \neq(s', t') \in D_S; \neq(s', t')\sigma \notin \{\neq(r, s), \neq(s, r)\}; c(\neq(s', t')) = 0\} \subseteq B.$$

That is, $L'' \in B$.

But if $t'\theta = r$, then $\neq(s', t')\theta = \neq(s, r)$. As $\neq(s', t') \in D'_S$, $\neq(s', t') = \neq(s''', t''')\sigma'$ for some terms s''' and t''' , and some literal, $\neq(s''', t''') \in D^-$.

Thus $\neq(s', t')\theta = \neq(s''', t''')\sigma'\theta = \neq(s''', t''')\sigma = \neq(s, r)$, and so $\neq(s''', t''') \in D_R$.

Thus $\neq(s', t') = \neq(s''', t''')\sigma' \in \{\neq(r'', s''), \neq(s'', r'')\}$.

As this clearly contradicts the assumption that $\neq(s', t')\rho \notin \{\neq(r'', s'')\rho, \neq(s'', r'')\rho\}$, it follows that $t'\theta \neq r$, and $L'' \in B$.

Similarly, if (L'' is non-trivial and) $c'(\neq(s', t')) = 1$, then $L'' \in B$.

Thus, for any $L'' \in ((D'_S \rho \setminus \{\neq(r'', s'')\rho, \neq(s'', r'')\rho\})[r''\rho/s''\rho])\theta'$,

L'' is either a trivial inequality, or $L'' \in B$.

Thus, for some set F consisting only of trivial inequalities, $B'\theta' \subseteq B \cup F$.

and so B' Σ_0 -subsumes B . ■

For Proposition 8.4.2:

Proof: For any equality clauses, A non-empty positive, and D non-empty mixed, let $B \in \text{EPC}_{+,-}(A, D)$.

Then, (as in the proof of Proposition 8.4.1), there exists a non-empty subset $A_R \subseteq A$, a non-empty subset $D_R \subseteq D^-$, the left-most inequality literal, $\neq(s_0, t_0)$, say, of D in D_R , and a (possibly empty) subset $D_S \subseteq D^- \setminus D_R$,

such that for some symmetric choice function c on $A_R \cup D_R \cup D_S$, with $c(\neq(s_0, t_0)) = 0$,

the most general substitution σ , say, unifying:

$$\begin{aligned} & \{ \{=(s', t') \mid =(s', t') \in A_R; c(=(s', t')) = 0\} \cup \{=(t', s') \mid =(s', t') \in A_R; c(=(s', t')) = 1\} \cup \\ & \quad \{=(s', t') \mid \neq(s', t') \in D_R; c(\neq(s', t')) = 0\} \cup \{=(t', s') \mid \neq(s', t') \in D_R; c(\neq(s', t')) = 1\} \} \\ & \quad \{s' \mid =(s', t') \in A_R; c(=(s', t')) = 0\} \cup \{t' \mid =(s', t') \in A_R; c(=(s', t')) = 1\} \cup \\ & \quad \{s' \mid \neq(s', t') \in D_S; c(\neq(s', t')) = 0\} \cup \{t' \mid \neq(s', t') \in D_S; c(\neq(s', t')) = 1\} \}. \end{aligned}$$

exists, σ unifying the two sets in the above doubleton set to the singletons $\{=(s, r)\}$ and $\{s\}$, respectively, for some terms s and r , say, and

$$\begin{aligned} B & \equiv A\sigma \setminus \{=(r, s), =(s, r)\} \cup D\sigma \setminus (\{\neq(r, s), \neq(s, r)\} \cup D_S\sigma) \cup \\ & \quad \{\neq(r, t'\sigma) \mid \neq(s', t') \in D_S; \neq(s', t')\sigma \notin \{\neq(r, s), \neq(s, r)\}; c(\neq(s', t')) = 0\} \cup \\ & \quad \{\neq(s'\sigma, r) \mid \neq(s', t') \in D_S; \neq(s', t')\sigma \notin \{\neq(r, s), \neq(s, r)\}; c(\neq(s', t')) = 1\}. \end{aligned}$$

Let $\emptyset \neq A'_R \subseteq A$, $\neq(s_0, t_0) \in D'_R \subseteq D^-$, $c': A'_R \cup D'_R \rightarrow \{0, 1\}$, $\sigma, \sigma', s'' = s_0\sigma', r'' = t_0\sigma', \theta$, where $\sigma = \sigma'\theta$,

$$D'_S = \{L \in D^- \mid L\sigma \in D_S\sigma\} \setminus \{\neq(s'', r''), \neq(r'', s'')\} \subseteq D^-\sigma' \setminus \{\neq(s'', r''), \neq(r'', s'')\},$$

$$c': D'_S \rightarrow \{0, 1\},$$

ρ , unifying $\{s''\} \cup \{s' \mid \neq(s', t') \in D'_S; c'(\neq(s', t')) = 0\} \cup \{t' \mid \neq(s', t') \in D'_S; c'(\neq(s', t')) = 1\}$,

θ' , such that $\theta = \rho\theta'$, (and hence, such that $\sigma = \sigma'\rho\theta'$),

and finally,

$$B' = A\sigma'\rho \setminus \{=(r'', s'')\rho, =(s'', r'')\rho\} \cup D\sigma'\rho \setminus (\{\neq(r'', s'')\rho, \neq(s'', r'')\rho\} \cup D'_S\rho) \cup (D'_S\rho \setminus \{\neq(r'', s'')\rho, \neq(s'', r'')\rho\})[r''\rho/s''\rho],$$

all be defined as in the proof of Proposition 8.4.1.

If $r'' = s''$, then, clearly, $r = r''\theta = s''\theta = s$, and, as in the proof of Proposition 8.4.1, $(=(r, r) \in A\sigma)$, and so $A\sigma$ is a Σ_0 -tautology, and it follows that either B is a Σ_0 -tautology, or D Σ_0 -subsumes B .

Thus, we may assume that $r'' \neq s''$.

$$\text{Let } \langle A', D' \rangle = \mathfrak{SR}_1(A\sigma', D\sigma', \neq(s'', r'')).$$

Since $D'_S \subseteq D^-\sigma' \setminus \{\neq(s'', r''), \neq(r'', s'')\} = (D\sigma')^- \setminus \{\neq(s'', r''), \neq(r'', s'')\}$, and $D' \subseteq D\sigma'$, evidently $\{\neq(s', t') \in D'_S \cap D' \mid s'\rho \neq t'\rho\}$ is a (possibly empty) subset, D''_S , say,

$$D''_S \subseteq (D')^- \setminus \{\neq(s'', r''), \neq(r'', s'')\},$$

$c' \upharpoonright D''_S$ is a symmetric choice function, c''' , say, on D''_S ,

the most general substitution ρ' , say, unifying:

$$\{s''\} \cup \{s' \mid \neq(s', t') \in D''_S : c'''(\neq(s', t')) = 0\} \cup \{t' \mid \neq(s', t') \in D''_S : c'''(\neq(s', t')) = 1\}.$$

exists, ρ' unifying the above set to the singleton $\{s''\rho'\}$,

and since, clearly,

$$\begin{aligned} & \{s''\} \cup \{s' \mid \neq(s', t') \in D''_S : c'''(\neq(s', t')) = 0\} \cup \{t' \mid \neq(s', t') \in D''_S : c'''(\neq(s', t')) = 1\} \\ & \subseteq \{s''\} \cup \{s' \mid \neq(s', t') \in D'_S : c'(\neq(s', t')) = 0\} \cup \{t' \mid \neq(s', t') \in D'_S : c'(\neq(s', t')) = 1\}. \end{aligned}$$

(ρ unifies the subset, and), for some substitution, φ , $\rho = \rho'\varphi$.

Since $A' \subseteq A\sigma'$ and $D' \subseteq D\sigma'$, clearly, $A'\rho' \subseteq A\sigma'\rho'$, and $D'\rho' \subseteq D\sigma'\rho'$.

Thus if either $A'\rho'$ or $D'\rho'$ is a Σ_0 -tautology, either $A\sigma'\rho'$ or $D\sigma'\rho'$ is a Σ_0 -tautology.

But then either $A\sigma = A\sigma'\rho'\varphi\theta'$ or $D\sigma = D\sigma'\rho'\varphi\theta'$ is a Σ_0 -tautology, and again, as in the proof of Proposition 8.4.1, either B is a Σ_0 -tautology, A Σ_0 -subsumes B , or D Σ_0 -subsumes B .

Thus, we may assume that neither $A'\rho'$ nor $D'\rho'$ is a Σ_0 -tautology.

$$\text{Let } B'' \equiv A'\rho' \setminus \{=(r'', s'')\rho', =(s'', r'')\rho'\} \cup D'\rho' \setminus (\{\neq(r'', s'')\rho', \neq(s'', r'')\rho'\} \cup D''_S\rho') \cup (D''_S\rho' \setminus \{\neq(r'', s'')\rho', \neq(s'', r'')\rho'\})[r''\rho'/s''\rho'].$$

$$\text{If } A'\rho' \cap \{=(s'', r'')\rho', =(r'', s'')\rho'\} = \emptyset,$$

then, clearly, $A'\rho' \subseteq B''$, and so A' subsumes B'' .

As A' is a Σ_0 -repf of $A\sigma'$, $A\sigma'$ Σ_0 -subsumes A' , and so $A\sigma'$ Σ_0 -subsumes B'' .

Thus, since, clearly, A subsumes $A\sigma'$, it follows that A Σ_0 -subsumes B'' .

We will show that B'' Σ_0 -subsumes B' . Since B' Σ_0 -subsumes B , the result will follow.

Clearly,

$$B''\varphi \equiv (A'\rho' \setminus \{=(r'', s'')\rho', =(s'', r'')\rho'\})\varphi \cup (D'\rho' \setminus (\{\neq(r'', s'')\rho', \neq(s'', r'')\rho'\} \cup D''_S\rho'))\varphi \cup$$

$$((D_S''\rho' \setminus \{\neq(r'', s'')\rho', \neq(s'', r'')\rho'\})[r''\rho'/s''\rho'])\varphi.$$

We claim that:

- 1) $(A'\rho' \setminus \{=(r'', s'')\rho', =(s'', r'')\rho'\})\varphi \subseteq A'\rho \setminus \{=(r'', s'')\rho, =(s'', r'')\rho\},$
- 2) For some F consisting only of trivial inequality literals,
 $(D'\rho' \setminus (\{\neq(r'', s'')\rho', \neq(s'', r'')\rho'\} \cup D_S''\rho'))\varphi \subseteq D'\rho \setminus (\{\neq(r'', s'')\rho, \neq(s'', r'')\rho\} \cup D_S'\rho) \cup F:$
- 3) $(D_S''\rho' \setminus \{\neq(r'', s'')\rho', \neq(s'', r'')\rho'\})[r''\rho'/s''\rho']\varphi \subseteq (D_S'\rho \setminus \{\neq(r'', s'')\rho, \neq(s'', r'')\rho\})[r''\rho/s''\rho].$

For 1):

Let $L \in (A'\rho' \setminus \{=(r'', s'')\rho', =(s'', r'')\rho'\})\varphi.$

Then $L = L'\sigma'\rho'\varphi$, for some $L' \in A$, such that $L'\sigma' \in A'$ and

$L'\sigma'\rho' \in A'\rho' \setminus \{=(r'', s'')\rho', =(s'', r'')\rho'\}.$

Obviously, (as $\rho'\varphi = \rho$), $L \in A\rho$. If $L \in \{=(r'', s'')\rho, =(s'', r'')\rho\},$

then $L\theta' \in \{=(r'', s'')\rho\theta', =(s'', r'')\rho\theta'\} = \{=(r'', s'')\theta, =(s'', r'')\theta\} = \{=(r, s), =(s, r)\}.$

Thus $L'\sigma'\rho'\varphi\theta' = L'\sigma'\rho\theta' = L'\sigma'\theta = L'\sigma \in \{=(r, s), =(s, r)\}.$

But then (as $L'\sigma \in \{=(r, s), =(s, r)\}$), $L' \in A'_R$, and $L'\sigma' \in \{=(r'', s''), =(s'', r'')\}.$

As this (clearly), contradicts the condition that $L'\sigma'\rho' \in A'\rho' \setminus \{=(r'', s'')\rho', =(s'', r'')\rho'\},$

we may conclude that $L \notin \{=(r'', s'')\rho, =(s'', r'')\rho\},$

and hence that $L \in A'\rho \setminus \{=(r'', s'')\rho, =(s'', r'')\rho\}.$

For 2):

Let $L \in (D'\rho' \setminus (\{\neq(r'', s'')\rho', \neq(s'', r'')\rho'\} \cup D_S''\rho'))\varphi.$

Then $L = L'\sigma'\rho'\varphi = L'\sigma'\rho$, for some $L' \in D^-$,

such that $L'\sigma' \in D'$ and $L'\sigma'\rho' \in D'\rho' \setminus (\{\neq(r'', s'')\rho', \neq(s'', r'')\rho'\} \cup D_S''\rho').$

Obviously, (as $\rho'\varphi = \rho$), $L \in D'\rho.$

If $L \in \{\neq(r'', s'')\rho, \neq(s'', r'')\rho\},$

then $L\theta' \in \{\neq(r'', s'')\rho\theta', \neq(s'', r'')\rho\theta'\} = \{\neq(r'', s'')\theta, \neq(s'', r'')\theta\} = \{\neq(r, s), \neq(s, r)\}.$

Thus $L'\sigma'\rho'\varphi\theta' = L'\sigma'\rho\theta' = L'\sigma'\theta = L'\sigma \in \{\neq(r, s), \neq(s, r)\}.$

But then (as $L'\sigma \in \{\neq(r, s), \neq(s, r)\}$), $L' \in D'_R$, and $L'\sigma' \in \{\neq(r'', s''), \neq(s'', r'')\}.$

As this (clearly), contradicts the condition that $L'\sigma'\rho' \in D'\rho' \setminus (\{\neq(r'', s'')\rho', \neq(s'', r'')\rho'\} \cup D_S''\rho').$

we may conclude that $L \notin \{\neq(r'', s'')\rho, \neq(s'', r'')\rho\}.$

If $L \in D_S'\rho$, then

$$\begin{aligned} L\theta' \in D_S'\rho\theta' &= D_S'\theta \\ &= (\{L \in D^- \mid L\sigma \in D_S\sigma\}\sigma' \setminus \{\neq(s'', r''), \neq(r'', s'')\})\theta \\ &\subseteq \{L \in D^- \mid L\sigma \in D_S\sigma\}\sigma\theta = \{L \in D^- \mid L\sigma \in D_S\sigma\}\sigma = D_S\sigma. \end{aligned}$$

Thus $L'\sigma'\rho'\varphi\theta' = L'\sigma'\rho\theta' = L'\sigma'\theta = L'\sigma \in D_S\sigma.$

Since further, (as shown above), we may assume that
 $L'\sigma = L'\theta' \notin \{\neq(r, s), \neq(s, r)\}, \quad L'\sigma' \notin \{\neq(r'', s''), \neq(s'', r'')\}.$

But then (as $D'_S = \{L \in D^- \mid L\sigma \in D_S\sigma\}\sigma' \setminus \{\neq(s'', r''), \neq(r'', s'')\}\}, \quad L'\sigma' \in D'_S.$

As $L'\sigma' \in D'$, then, unless $L'\sigma'\rho$ is a trivial inequality,
 it follows that $L'\sigma' \in D''_S$, and hence that $L'\sigma'\rho' \in D''_S\rho'.$

As this clearly contradicts the condition that $L'\sigma'\rho' \in D'\rho' \setminus (\{\neq(r'', s'')\rho', \neq(s'', r'')\rho'\} \cup D''_S\rho')$,
 we may conclude that either $L = L'\sigma'\rho$ is a trivial inequality, or $L \notin D'_S\rho.$

Thus we may conclude that, for some F consisting of only trivial inequality literals.

$$L \in D'\rho \setminus (\{\neq(r'', s'')\rho, \neq(s'', r'')\rho\} \cup D'_S\rho) \cup F,$$

(and hence that, for some F consisting of only trivial inequality literals.

$$(D'\rho' \setminus (\{\neq(r'', s'')\rho', \neq(s'', r'')\rho'\} \cup D''_S\rho'))\varphi \subseteq D'\rho \setminus (\{\neq(r'', s'')\rho, \neq(s'', r'')\rho\} \cup D'_S\rho) \cup F).$$

For 3):

Let $LHS \equiv (D''_S\rho' \setminus \{\neq(r'', s'')\rho', \neq(s'', r'')\rho'\})[r''\rho'/s''\rho']\varphi$, and

$$RHS \equiv (D'_S\rho \setminus \{\neq(r'', s'')\rho, \neq(s'', r'')\rho\})[r''\rho/s''\rho].$$

Let $L \in LHS.$

Then $L = (L'\rho'[r''\rho'/s''\rho'])\varphi$, for some $L' \in D''_S$ such that $L'\rho' \in D''_S\rho' \setminus \{\neq(r'', s'')\rho', \neq(s'', r'')\rho'\}.$

Since $L' \in D''_S$, $L' = \neq(s', t')$, for some terms s and t , such that $s'\rho \neq t'\rho$, and either:

- a) $s'\rho' = s''\rho' \neq t'\rho'.$
- b) $t'\rho' = s''\rho' \neq s'\rho'.$ or
- c) $s'\rho' = s''\rho' = t'\rho'.$

But if c), then $s'\rho = s'\rho'\varphi = t'\rho'\varphi = t'\rho$, which contradicts the condition that $s'\rho \neq t'\rho.$

Thus, either a) or b) holds, and correspondingly, $L'\rho'[r''\rho'/s''\rho']$ is either:

- a) $\neq(r''\rho', t'\rho'),$ or
- b) $\neq(s'\rho', r''\rho'),$

and (correspondingly), $L = (L'\rho'[r''\rho'/s''\rho'])\varphi$ is either:

- a) $\neq(r''\rho, t'\rho),$ or
- b) $\neq(s'\rho, r''\rho),$

Since $L \in D''_S \subseteq D'_S$, for some $L'' \in D^-$, $L' = L''\sigma'$, and thus, if $L'\rho \in \{\neq(r'', s'')\rho, \neq(s'', r'')\rho\},$
 then

$$\begin{aligned} L''\sigma &= L''\sigma'\theta = L'\theta = L'\rho\theta' \in \{\neq(r'', s'')\rho\theta', \neq(s'', r'')\rho\theta'\} = \{\neq(r'', s'')\theta, \neq(s'', r'')\theta\} \\ &= \{\neq(r, s), \neq(s, r)\}, \end{aligned}$$

and hence $L'' \in D'_R$ and $L' = L''\sigma' \in \{\neq(r'', s''), \neq(s'', r'')\}.$

But then $L'\rho' \in \{\neq(r'', s'')\rho', \neq(s'', r'')\rho'\},$ and as this contradicts the condition that

$L'\rho' \in D''_S\rho' \setminus \{\neq(r'', s'')\rho', \neq(s'', r'')\rho'\},$ we may conclude that $L'\rho \notin \{\neq(r'', s'')\rho, \neq(s'', r'')\rho\}.$

As $L' \in D_S''$, $L'\rho \in D_S'\rho \setminus \{\neq(r'', s'')\rho, \neq(s'', r'')\rho\}$, and so $L'\rho[r''\rho/s''\rho] \in RHS$.

As $L' = \neq(s', t') \in D_S''$, $s'\rho \neq t'\rho$, and in case (a),
 since $s'\rho' = s''\rho' \neq t'\rho'$, $s'\rho = s'\rho'\varphi = s''\rho'\varphi = s''\rho \neq t'\rho$,
 while in case (b), since $t'\rho' = s''\rho' \neq s'\rho'$, $t'\rho = t'\rho'\varphi = s''\rho'\varphi = s''\rho \neq s'\rho$.

Thus (correspondingly), $L'\rho[r''\rho/s''\rho]$ is either:

- a) $\neq(r''\rho, t'\rho)$, or
- b) $\neq(s'\rho, r''\rho)$.

Thus, (in either case). $L = (L'\rho[r''\rho/s''\rho])\varphi = L'\rho[r''\rho/s''\rho] \in RHS$.

From 1), 2), and 3), and since $A' \subseteq A\sigma'$, $D' \subseteq D\sigma'$, and $D_S'' \subseteq D_S'$, evidently, for some F consisting only of trivial inequality literals,

$$\begin{aligned} B''\varphi &\subseteq A'\rho \setminus \{=(r'', s'')\rho, =(s'', r'')\rho\} \cup D'\rho \setminus (\{\neq(r'', s'')\rho, \neq(s'', r'')\rho\} \cup D_S'\rho) \cup F \cup \\ &\quad ((D_S''\rho \setminus \{\neq(r'', s'')\rho, \neq(s'', r'')\rho\})[r''\rho/s''\rho]) \\ &\subseteq A\sigma'\rho \setminus \{=(r'', s'')\rho, =(s'', r'')\rho\} \cup D\sigma'\rho \setminus (\{\neq(r'', s'')\rho, \neq(s'', r'')\rho\} \cup D_S'\rho) \cup \\ &\quad (D_S'\rho \setminus \{\neq(r'', s'')\rho, \neq(s'', r'')\rho\})[r''\rho/s''\rho] \cup F \\ &= B' \cup F, \end{aligned}$$

and thus $B'' \Sigma_0$ -subsumes B' , as required. ■

For Proposition 8.4.3:

Proof: Suppose $C \in \text{EPC}_{+,-}(A, D)$, for some equality clauses, A positive, and D mixed.

By Proposition 8.4.2, either:

C is a Σ_0 -tautology, $D \Sigma_0$ -subsumes C , $A \Sigma_0$ -subsumes C , or
 there exists $B' \in \mathfrak{W}(A, D)$ such that $B' \Sigma_0$ -subsumes C .

Evidently it sufficed to assume there exists $B' \in \mathfrak{W}(A, D)$ such that $B' \Sigma_0$ -subsumes C .

Since $B' \in \mathfrak{W}(A, D)$,

- i. there exists a non-empty subset $A_R \subseteq A$, a non-empty subset $D_R \subseteq D^-$, the left-most inequality literal, $\neq(s_0, t_0)$, say, of D in D_R , such that for some symmetric choice function c on $A_R \cup D_R$, with $c(\neq(s_0, t_0)) = 0$, the most general substitution σ , say, unifying:
 $\{=(s', t') \mid =(s', t') \in A_R; c(=(s', t')) = 0\} \cup \{=(t', s') \mid =(s', t') \in A_R; c(=(s', t')) = 1\} \cup$
 $\{=(s', t') \mid \neq(s', t') \in D_R; c(\neq(s', t')) = 0\} \cup \{=(t', s') \mid \neq(s', t') \in D_R; c(\neq(s', t')) = 1\},$
 exists, σ unifying the above set to the singleton $\{=(s, r)\}$, for some terms s and r , say,
- ii. $s \neq r$, and $\mathfrak{GR}_1(A\sigma, D\sigma, \neq(s, r)) = \langle A', D' \rangle$, say,
- iii. there exists a (possibly empty) subset $D_S' \subseteq (D')^- \setminus \{\neq(s, r), \neq(r, s)\}$, such that for some symmetric choice function c' on D_S' , the most general substitution ρ , say, unifying:
 $\{s\} \cup \{s' \mid \neq(s', t') \in D_S'; c'(\neq(s', t')) = 0\} \cup \{t' \mid \neq(s', t') \in D_S'; c'(\neq(s', t')) = 1\},$
 exists, ρ unifying the above set to the singleton $\{s\rho\}$,

- iv. $A'\rho \cap \{(s, r)\rho, =(r, s)\rho\} \neq \emptyset$, and neither $A'\rho$ nor $D'\rho$ is a Σ_0 -tautology, (so, in particular, $s\rho \neq r\rho$), and
- v. $B' \equiv A'\rho \setminus \{(r, s)\rho, =(s, r)\rho\} \cup D'\rho \setminus (\{\neq(r, s)\rho, \neq(s, r)\rho\} \cup D'_S\rho) \cup (D'_S\rho \setminus \{\neq(r, s)\rho, \neq(s, r)\rho\})[r\rho/s\rho]$,

Let $\langle A'', D'' \rangle \equiv \mathfrak{S}\mathfrak{R}_1(A'\rho, D'\rho, \neq(s, r)\rho)$, and $D''_S \equiv D'_S\rho \cap D'' \setminus \{\neq(r, s)\rho, \neq(s, r)\rho\}$.

If $A'' \cap \{(s, r)\rho, =(r, s)\rho\} \neq \emptyset$, then clearly $A'' \subseteq B'$, and since, evidently, (as $A \Sigma_0$ -subsumes $A\sigma \Sigma_0$ -subsumes $A' \Sigma_0$ -subsumes $A'\rho \Sigma_0$ -subsumes A''), $A \Sigma_0$ -subsumes A'' , it follows that $A \Sigma_0$ -subsumes B' .

Since $B' \Sigma_0$ -subsumes C , then, $A \Sigma_0$ -subsumes C , (and so we have (c)).

Otherwise, let $C'' \equiv A'' \setminus \{(r, s)\rho, =(s, r)\rho\} \cup D'' \setminus (\{\neq(r, s)\rho, \neq(s, r)\rho\} \cup D''_S) \cup D''_S[r\rho/s\rho]$, and let $C' \equiv \mathfrak{S}\mathfrak{R}_2(C'')$.

Evidently, then, $C' \in \mathfrak{W}(A, D)$.

Since $A'' \subseteq A'\rho$, clearly $A'' \setminus \{(r, s)\rho, =(s, r)\rho\} \subseteq A'\rho \setminus \{(r, s)\rho, =(s, r)\rho\}$.

Since $D'' \subseteq D'\rho$, $D'_S\rho \subseteq D'\rho$, and $D''_S = D'_S\rho \cap D'' \setminus \{\neq(r, s)\rho, \neq(s, r)\rho\}$,

evidently $\{\neq(r, s)\rho, \neq(s, r)\rho\} \cup D''_S = \{\neq(r, s)\rho, \neq(s, r)\rho\} \cup D'_S\rho \cap D''$ and

$$\begin{aligned} D'' \setminus (\{\neq(r, s)\rho, \neq(s, r)\rho\} \cup D''_S) &= D'\rho \cap D'' \setminus (\{\neq(r, s)\rho, \neq(s, r)\rho\} \cup D'_S\rho \cap D'') \\ &= (D'\rho \cap D'' \setminus D'_S\rho \cap D'') \setminus \{\neq(r, s)\rho, \neq(s, r)\rho\} \\ &= (D'\rho \setminus D'_S\rho) \cap D'' \setminus \{\neq(r, s)\rho, \neq(s, r)\rho\} \\ &\subseteq (D'\rho \setminus D'_S\rho) \setminus \{\neq(r, s)\rho, \neq(s, r)\rho\} \\ &= D'\rho \setminus (\{\neq(r, s)\rho, \neq(s, r)\rho\} \cup D'_S\rho). \end{aligned}$$

$$\begin{aligned} \text{Further, } D''_S[r\rho/s\rho] &= (D'_S\rho \cap D'' \setminus \{\neq(r, s)\rho, \neq(s, r)\rho\})[r\rho/s\rho] \\ &\subseteq (D'_S\rho \setminus \{\neq(r, s)\rho, \neq(s, r)\rho\})[r\rho/s\rho]. \end{aligned}$$

Thus $C'' \subseteq B'$, so $C'' \Sigma_0$ -subsumes B' , and since $B' \Sigma_0$ -subsumes C , $C'' \Sigma_0$ -subsumes C .

As C' is a Σ_0 -repf of C'' , $C' \subseteq C''$, and so, evidently, $C' \Sigma_0$ -subsumes C . ■

For Proposition 8.4.4:

Proof: Since $C \in \text{EPC}_-(A, D)$, either:

- i) $C = QR_{full}(D) = \square$, or
- ii) $C \in \text{EPC}_{+,-}(QR_{full}(A), D')$, for some $D' \in QR_{first}(D)$.

In case (i), clearly, we have $\square \in \widehat{\text{EPC}}_-(A, D)$, (and as $\square \Sigma_0$ -subsumes $\square$, we have (d)).

In case (ii), evidently $QR_{full}(A) \equiv A'$, say, is non-empty positive, and there is some non-empty mixed $D' \in QR_{first}(D)$ such that C is (some ordering of an unordered clause) in $\text{EPC}_{+,-}(A', D')$.

Thus by Proposition 8.4.3, either:

- a') C is a Σ_0 -tautology,

- b') $D' \Sigma_0$ -subsumes C ,
- c') $A' \Sigma_0$ -subsumes C , or
- d') there exists $C'' \in \mathfrak{M}(A', D')$ such that $C'' \Sigma_0$ -subsumes C .

Case (a') is just case (a).

In case (b'), since $D' \in QR_{first}(D)$, by Lemma 6.2.3, $D \Sigma_0$ -subsumes C , and so we have (b).

Similarly, in case (c'), since $A' = QR_{full}(A)$, by Corollary 6.2.1, $A \Sigma_0$ -subsumes C , and so we have (c).

In case (d'), evidently we have, modulo symmetry and ordering, $C' \equiv C''$,

for some C' in $\widehat{EPC}_{+,-}(A', D') \subseteq \widehat{EPC}_{-}(A, D)$, and so

(since clearly $C' \Sigma_0$ -subsumes C'' , and $C'' \Sigma_0$ -subsumes C , by transitivity of Σ_0 -subsumption), there is some $C' \in \widehat{EPC}_{-}(A, D)$ such that $C' \Sigma_0$ -subsumes C , and we have (d). ■

APPENDIX H

Some Examples and Algorithms

An Example of Σ_0 -subsumption

EXAMPLE H.0.1. In terms of the characterization (6) of Theorem 3.2.3, for Σ_0 -subsumption, as an example of Σ_0 -subsumption, consider C to be the clause:

$$P(x, b(), y) \mid P(f(y), z, a()) \mid \neg P(x, c(), y) \mid Q(z) \mid =(c(), y)$$

and D to be the clause:

$$P(c(), c(), s) \mid P(v, c(), a()) \mid \neg P(b(), b(), g(w, b())) \mid Q(b()) \mid =(b(), s) \mid =(d(), s) \mid \neq(b(), c()) \mid \\ \neq(v, f(t)) \mid \neq(s, t) \mid \neq(s, g(w, b())) .$$

Evidently, C does not subsume D .

In this case C^+ is $=(c(), y)$.

C^0 is $P(x, b(), y) \mid P(f(y), z, a()) \mid \neg P(x, c(), y) \mid Q(z)$,

and C^- is the empty clause $\square$.

D^+ is $=(b(), s) \mid =(d(), s)$,

D^0 is $P(c(), c(), s) \mid P(v, c(), a()) \mid \neg P(b(), b(), g(w, b())) \mid Q(b())$,

and D^- is $\neq(b(), c()) \mid \neq(v, f(t)) \mid \neq(s, t) \mid \neq(s, g(w, b()))$.

D^- provides us with the following collection of equivalence classes,

$\overline{\mathcal{T}_{D^-}}$ is $\{\{b(), c()\}, \{\bar{v}, f(\bar{t})\}, \{\bar{s}, \bar{t}, g(\bar{w}, b())\}\}$.

$\mathcal{T}_{D^-}$ is $\{\{b(), c()\}, \{v, f(t)\}, \{s, t, g(w, b())\}\}$.

We obtain $\mathcal{T}_{D^+}(D^+)$ as:

$$=(b(), s) \mid =(c(), s) \mid =(b(), t) \mid =(b(), g(w, b())) \mid =(c(), t) \mid =(c(), g(w, b())) \mid \\ =(d(), s) \mid =(d(), t) \mid =(d(), g(w, b()))$$

$\overline{\mathcal{T}_{D-}(D^+)}$ is:

$$\begin{aligned} &=(b(), \bar{s}) \mid =(c(), \bar{s}) \mid =(b(), \bar{t}) \mid =(b(), g(\bar{w}, b())) \mid =(c(), \bar{t}) \mid =(c(), g(\bar{w}, b())) \mid \\ &=(d(), \bar{s}) \mid =(d(), \bar{t}) \mid =(d(), g(\bar{w}, b())) \end{aligned}$$

From (recall) $C^0 \equiv P(x, b(), y) \mid P(f(y), z, a()) \mid \neg P(x, c(), y) \mid Q(z)$, and

$\overline{D^0} \equiv P(c(), c(), \bar{s}) \mid P(\bar{v}, c(), a()) \mid \neg P(b(), b(), g(\bar{w}, b())) \mid Q(b())$, we obtain for $M(C^0, D^0)$, the “matrix” $\langle M_1, M_2, M_3, M_4 \rangle$, with columns M_1 through M_4 (respectively corresponding to the 4 literals of C^0):

$$M_1: \left\langle \begin{array}{l} \neq(x, c()) \mid \neq(b(), c()) \mid \neq(y, \bar{s}) \\ \neq(x, \bar{v}) \mid \neq(b(), c()) \mid \neq(y, a()) \end{array} \right\rangle$$

$$M_2: \left\langle \begin{array}{l} \neq(f(y), c()) \mid \neq(z, c()) \mid \neq(a(), \bar{s}) \\ \neq(f(y), \bar{v}) \mid \neq(z, c()) \mid \neq(a(), a()) \end{array} \right\rangle$$

$$M_3: \left\langle \neq(x, b()) \mid \neq(c(), b()) \mid \neq(y, g(\bar{w}, b())) \right\rangle$$

$$M_4: \left\langle \neq(z, b()) \right\rangle$$

One can verify that here $S_1(C^+, D) = \{[\bar{s}/y], [\bar{t}/y], [g(\bar{w}, b())/y]\}$.

$S_2(C^-, \mathcal{T}_{D-}) = \{\square\}$, (where $\square$ denotes the empty substitution).

$S_3(C^0, D^0, \mathcal{T}_{D-})$ is $\{[b()/x, \bar{t}/y, b()/z], [c()/x, \bar{t}/y, b()/z], [b()/x, \bar{t}/y, c()/z], [c()/x, \bar{t}/y, c()/z]\}$.

Since, for example, $[\bar{t}/y]$, $\square$, and $[b()/x, \bar{t}/y, b()/z]$ are compatible.

$C \Sigma_0$ -subsumes D . (end of Example).

An Algorithm for Σ_0 -subsumption Verification

In the following, we will utilize the notation introduced in Subsection 3.2, (on page 39).

$\Pi_0(C, D)$ (return 1 if C Σ_0 -subsumes D ; 0 otherwise) $\langle$

```

   $s_1 := S_1(C^+, D);$ 
  while  $[\sigma \in s_1] \langle$ 
     $s_2 := S_2(C^- \sigma, \mathcal{T}_{D^-});$ 
    while  $[\sigma' \in s_2] \langle$ 
      if  $\exists [\sigma'' \in S_3(C^0 \sigma \sigma', D^0, \mathcal{T}_{D^-})]$  break;
      else  $s_2 := s_2 \setminus \{\sigma'\}; \rangle$ 
    if  $\exists [\sigma' \in s_2]$  break;
    else  $s_1 := s_1 \setminus \{\sigma\}; \rangle$ 
  if  $\exists [\sigma \in s_1]$  return(1);
  else return(0);  $\rangle$ 
```

Subroutine for Π_0 : Algorithm for computing $S_1(C^+, D)$

$\Pi_1(C^+, D)$ (returns $S_1(C^+, D)$) $\langle$

```

   $out := \{\};$ 
   $B := C^+;$ 
  if  $\exists [= (b_1, b_2) \in B] \langle$ 
     $F := \overline{\mathcal{T}_{D^-}(D^+)} \cup \overline{\Delta(D^-, D^0)};$ 
    while  $[= (f_1, f_2) \in F] \langle$ 
       $s_1 := MGU(\{\{b_1, f_1\}, \{b_2, f_2\}\}) \cup MGU(\{\{b_1, f_2\}, \{b_2, f_1\}\})$ 
      while  $[\sigma \in s_1] \langle$ 
         $B' := B \setminus \{=(b_1, b_2)\};$ 
         $s_2 := \Pi_1(B' \sigma, D^+, \mathcal{T}_{D^-})$ 
         $s_3 := \{\sigma \sigma' : \sigma' \in s_2\}$ 
        while  $[\theta \in s_3] \langle$ 
           $out := S\_AUGMENT(out, \theta);$ 
           $s_3 := s_3 \setminus \{\theta\}; \rangle$ 
         $s_1 := s_1 \setminus \{\sigma\}; \rangle$ 
       $F := F \setminus \{=(f_1, f_2)\}; \rangle$ 
    else return( $\{\{\}\}$ );  $\rangle$ 
```

In the above, $[]$ denotes the empty substitution.

In the above, for a (finite) set α of (finite) sets of terms, $MGU(\alpha)$ returns either

- i. the singleton $\{\sigma\}$, if σ exists, where σ is the most general substitution such that $\forall \beta \in \alpha$, $\beta\sigma$ is a singleton,
(that is σ is the most general simultaneous unifier of the collections in α), or
- ii. the empty set $\{\}$, otherwise.

For a set of substitutions Θ and a substitution θ , $S\_AUGMENT(\Theta, \theta)$ returns some set of substitutions Θ' , such that

- i. $\Theta' \subset \Theta$, and
- ii. $\forall \sigma \in \Theta \cup \{\theta\}$, $\exists \sigma' \in \Theta' : \sigma$ factors through (is an extension of) σ' .

Subroutine for Π_0 : Algorithm for computing $S_2(C^-, \mathcal{T}_{D-})$

```

 $\Pi_2(C^-, \mathcal{T}_{D-})$  (returns  $S_2(C^-, \mathcal{T}_{D-})$ )  $\langle$ 
   $out := \{\};$ 
   $C := C^-;$ 
  while  $[\neq(s, t) \in C]$   $\langle$ 
    if  $\exists [\theta \in MGU(\{\{s, t\}\})]$   $\langle$ 
       $out1 := \Pi_2(C^- \theta \setminus \{\neq(s, t)\theta\}, \mathcal{T}_{D-})$ 
      while  $[\sigma \in out1]$   $\langle$ 
         $out := S\_AUGMENT(out, \theta\sigma);$ 
         $out1 := out1 \setminus \{\sigma\};$   $\rangle$ 
       $C := C \setminus \{\neq(s, t)\};$   $\rangle$ 
   $out1 := \Pi_4(\mathcal{T}_{C-}, \mathcal{T}_{D-});$ 
  while  $[\sigma \in out1]$   $\langle$ 
     $out := S\_AUGMENT(out, \sigma);$ 
     $out1 := out1 \setminus \{\sigma\};$   $\rangle$ 
  return( $out$ );  $\rangle$ 

```

Subroutine for Π_2 :

Algorithm for computing a most general subset of $\{\sigma' : \forall \beta \in \mathcal{T}_{C-}, \exists \alpha \in \overline{\mathcal{T}_{D-}} : \beta\sigma' \subseteq \alpha\}$

```

 $\Pi_4(\mathcal{T}_{C-}, \mathcal{T}_{D-})$   $\langle$ 
   $out := \{\};$ 
  if  $\exists [\beta \in MAXML(\mathcal{T}_{C-})]$   $\langle$ 
    if  $\exists [t \in \beta : t = \bar{t}]$  (ie. if there is a ground term  $t$  in  $\beta$ )  $\langle$ 
      if  $\exists [\alpha \in \overline{\mathcal{T}_{D-}} : t \in \alpha]$   $\langle$ 
         $out2 := \{\tau : \beta\tau \subseteq \alpha\};$ 
        while  $[\tau \in out2]$   $\langle$ 
           $out1 := \Pi_4(\mathcal{T}_{C-} \setminus \{\beta\} \tau, \mathcal{T}_{D-});$ 
          while  $[\sigma \in out1]$   $\langle$ 
             $out := S\_AUGMENT(out, \tau\sigma);$ 
             $out1 := out1 \setminus \{\sigma\};$   $\rangle$ 
           $out2 := out2 \setminus \{\tau\};$   $\rangle$ 
        return( $out$ );  $\rangle$ 
      else return( $\{\}$ );  $\rangle$ 
    else  $\langle$ 
       $\mathcal{T} := \overline{\mathcal{T}_{D-}};$ 
      while  $[\alpha \in \mathcal{T}]$   $\langle$ 
         $out2 := \{\tau : \beta\tau \subseteq \alpha\};$ 

```



```

        while  $[\tau \in out2]$   $\langle$ 
             $out1 := \Pi_4(\mathcal{T}_{C-} \setminus \{\beta\} \tau, \mathcal{T}_{D-});$ 
            while  $[\sigma \in out1]$   $\langle$ 
                 $out := S\_AUGMENT(out, \tau\sigma);$ 
                 $out1 := out1 \setminus \{\sigma\};$   $\rangle$ 
             $out2 := out2 \setminus \{\tau\};$   $\rangle$ 
             $\mathcal{T} := \mathcal{T} \setminus \{\alpha\};$   $\rangle$ 
        return(out);  $\rangle$   $\rangle$ 
    else return( $\{\{\}\}$ );  $\rangle$ 

```

In the above, for any set (or list) α of sets, $MAXML(\alpha)$ denotes the subset β of α comprised of those members of α with maximal cardinality (amongst those sets in α).

Subroutine for Π_0 : Algorithm for computing $S_3(C^0, D^0, \mathcal{T}_{D-})$

```

 $\Pi_3(C^0, D^0, \mathcal{T}_{D-})$  (returns  $S_3(C^0, D^0, \mathcal{T}_{D-})$ )  $\langle$ 
     $out := \Pi_5(M(C^0, D^0), \mathcal{T}_{D-});$ 
    return(out)  $\rangle$ 

```

Subroutine for Π_3 :

```

 $\Pi_5(M, \mathcal{T}_{D-})$   $\langle$ 
     $out := \{\};$ 
    if  $\exists [V \in MINML(M)]$   $\langle$ 
        if  $\exists [d \in V]$   $\langle$ 
             $V' := V;$ 
            while  $[d \in V']$   $\langle$ 
                 $s1 := S_2(d, \mathcal{T}_{D-});$ 
                while  $[\sigma \in s1]$   $\langle$ 
                     $out1 := \Pi_5(M \setminus \{V\} \sigma, \mathcal{T}_{D-});$ 
                    while  $[\theta \in out1]$   $\langle$ 
                         $out := S\_AUGMENT(out, \sigma\theta);$ 
                         $out1 := out1 \setminus \{\theta\};$   $\rangle$ 
                     $s1 := s1 \setminus \{\sigma\};$   $\rangle$ 
                 $V' := V' \setminus \{d\};$   $\rangle$ 
            else return( $\{\}$ );  $\rangle$ 
        else return( $\{\{\}\}$ );  $\rangle$ 

```

In the above, for any set (or list) α of sets, $MINML(\alpha)$ denotes the subset β of α comprised of those members of α with minimal cardinality (amongst those sets in α).

Sketch of an Algorithm for Σ_0 -tautology Verification

The determination of whether or not a clause is an ordinary tautology, (or contains an instance of QR, QS, or QT, modulo symmetry), is straightforward. The algorithm for Σ_0 -tautology detection may first test for these. Given that the clause C in question has not already been so determined to be a Σ_0 -tautology, the test may proceed (essentially) as follows.

The variables of C are frozen, and assume now, for the remainder of this argument, that $C \equiv \overline{C}$, $C^+ \equiv \overline{C^+}$, $C^0 \equiv \overline{C^0}$, and $C^- \equiv \overline{C^-}$.

We construct the set $\mathcal{T}$ of equivalence clauses over the (ground) terms occurring as arguments to predicates in C^- , which is naturally induced by the equalities obtained via negating C^- . We construct a single column matrix M , whose entries are pure-inequality clauses. These clauses are obtained as follows: (1) each positive equality literal in C^+ is negated to yield a unit clause, and (2) $\langle L_1, L_2 \rangle$ is a member of the column, for each pair of positively literals L_1 and L_2 , sharing a common predicate symbol, P (say. $P \neq '='$), and such that $\{L_1, \neg L_2\} \subseteq C^0$.

C is then a Σ_0 -tautology if, for some c_i of M , the naturally induced set $\mathcal{T}_{c_i}$ of equivalence classes is such that for each $\beta \in \mathcal{T}_{c_i}$, there is an $\alpha_\beta \in \mathcal{T}$, such that $\beta \subseteq \alpha_\beta$.

(Actually, in an implementation, the problem may be simply passed to a sub-routine of the routine for Σ_0 -subsumption verification).

As motivation for the final part of the algorithm for Σ_0 -tautology verification sketched above, observe that we are essentially testing the condition

$$\Sigma_0 \models (\neg C^-) \Rightarrow \overline{C^+} \mid \overline{C^0}$$

via our algorithm for Σ_0 -subsumption verification.

EXAMPLE H.0.2. As a simple example to illustrate these ideas, consider the clause

$C: \neq(x, b()) \mid \neg P(f(y), b()) \mid P(f(y), x)$. To determine that this clause is a Σ_0 -tautology, the algorithm described above would proceed as follows: C is partitioned into sub-clauses C^+ , C^0 , and C^- , where $C^+ = \square$, (the empty sub-clause), $C^0 = \neg P(f(\overline{y}), b()) \mid P(f(\overline{y}), \overline{x})$, and $C^- = \neq(\overline{x}, b())$. $\mathcal{T}$ is computed from C^- to be simply the singleton $\{\{\overline{x}, b()\}\}$, and M is computed to be the matrix with one column M_1 , where M_1 is:

$$c_1 : \neq(f(\overline{y}), f(\overline{y})) \mid \neq(b(), \overline{x})$$

There is only one entry in M_1 , namely c_1 , and $\mathcal{T}_{c_1}$ is computed as the singleton: $\{\{b(), \overline{x}\}\}$. The only member β in this set is $\{b(), \overline{x}\}$, and there is indeed an $\alpha_\beta \in \mathcal{T}$, namely $\{\overline{x}, b()\}$, such that $\beta \subseteq \alpha_\beta$. Thus, C is determined to be a Σ_0 -tautology.

Sketch of an Algorithm for Σ_0 -repf-ing

If $C \equiv L_1, L_2, \dots, L_k$, say, and $C' \subsetneq C$ is a (proper) Σ_0 -repf of C , then C Σ_0 -subsumes C' , and for some $i: 1 \leq i \leq k$, $C' \subseteq C \setminus \{L_i\}$.

Therefore, (C' Σ_0 -subsumes $C \setminus \{L_i\}$, and so by Lemma 3.2.2), for some $i: 1 \leq i \leq k$, C Σ_0 -subsumes $C \setminus \{L_i\}$.

Further, since $C \setminus \{L_i\} \subseteq C$, $C \setminus \{L_i\}$ Σ_0 -subsumes C' .

Thus $C \setminus \{L_i\}$ is a (proper) Σ_0 -repf of C and C' is a Σ_0 -repf of $C \setminus \{L_i\}$.

Of course, (because "is a Σ_0 -repf of" is transitive), any Σ_0 -repf of $C \setminus \{L_i\}$ is also a Σ_0 -repf of C .

Thus the most obvious way to search for a minimal Σ_0 -repf of a clause $C \equiv L_1, L_2, \dots, L_k$, say, is simply to test, for each $i: 1 \leq i \leq k$, successively, whether or not C Σ_0 -subsumes $C \setminus \{L_i\}$.

If (Case 1): some $i: 1 \leq i \leq k$, such that C Σ_0 -subsumes $C \setminus \{L_i\}$, is found, then $C \setminus \{L_i\}$ is a (proper) Σ_0 -repf of C , and for any minimal Σ_0 -repf, C' such that $C' \subseteq C \setminus \{L_i\}$, (and there is at least one), either $C' = C \setminus \{L_i\}$, (modulo ordering), or C' is now a (proper) Σ_0 -repf of $C \setminus \{L_i\}$.

Otherwise, (if no such $i: 1 \leq i \leq k$ is found),

(Case 2): C has no proper Σ_0 -repf's. (and is a minimal Σ_0 -repf of itself).

Thus, in Case 1, by applying the same process recursively to C' , we clearly obtain a minimal Σ_0 -repf of C , as desired.

Note that for a clause C of length k , an algorithm based directly on this sketch would of course do up to k Σ_0 -subsumption tests before determining C has no proper Σ_0 -repf., and would do some $i: 1 \leq i \leq k$ such tests before finding a "success": where C Σ_0 -subsumes $C' \equiv C \setminus \{L_i\}$. (C' then of length $k - 1$), and (in recursion) at most $k - 1$ such "successes" are possible.

This is certainly reasonable from a computational point of view.

(bounded by k^2 Σ_0 -subsumption tests), but it may be possible to do somewhat better.

Of course, by similar reasoning, an entirely analogous algorithm (to that sketched above) may be used to compute ordinary minimal replacement factors, (replacing ' Σ_0 -subsumes' above with 'subsumes').

For replacement factoring, it is not hard to show that if C' is a minimal replacement factor of a clause C , then ($C' \subseteq C$ and), there is a substitution θ such that not only is it the case that $C\theta \subseteq C'$, but also, regarded as a map sending literals to literals, (that is, between clauses), θ is the identity on C' , and equivalently, regarded as map sending terms to terms, θ is the identity on the variables occurring in C' .

Thus, if C' is a minimal replacement factor of C , then C' is some subset of C such that, if:

- i) for any clause A , $Sppt(A)$ denotes the set of (free) variables occurring in (literals of), A , and
- ii) for any clause B , $Fz_{Sppt(A)}(B)$ denotes the result of formally "freezing" every variable of $Sppt(A)$ in B , (that is, applying the substitution $x \in Sppt(A) \mapsto \bar{x}$, which sends each variable $x \in Sppt(A)$ to its unique formal (Skolem) constant, $\bar{x}$, to the clause B).

then $Fz_{Sppt(C')}(C \setminus C')$ subsumes $Fz_{Sppt(C')}(C')$.

By so reducing the number of free variables in the "antecedent clause", (here $Fz_{Sppt(C')}(C \setminus C')$, as opposed to simply C), of the subsumption test, the cost of the corresponding subsumption test may be reduced.

However, there is a catch: it is no longer sufficient to (iteratively) restrict the "subsumption tests" to the cases where C' is $C \setminus \{L\}$, for some $L \in C$, as before. In fact, it is not in general sufficient to freeze all variables *except* those in a single literal L .

For example:

EXAMPLE H.0.3. Consider

$$C \equiv P(x, y) \mid P(y, u) \mid P(u, v) \mid P(v, z) \mid P(x, y') \mid P(y', u') \mid P(u', v') \mid P(v', z).$$

C has a (in fact, minimal) replacement factor (with the substitution $[y/y', u/u', v/v']$),

$$C' \equiv P(x, y) \mid P(y, u) \mid P(u, v) \mid P(v, z),$$

(but no replacement factoring is possible if all but two of the variables are initially frozen).

Quite analogous to the situation described above for minimal replacement factors, it may also be shown that if C' is a minimal Σ_0 -repf of a clause C , then

($C' \subseteq C$, so $cl(C') \subseteq cl(C)$, -by (i) of Observation 3.2.1, page 45, and),

there is a substitution θ such that not only is it the case that $cl(C)\theta \subseteq cl(C') \cup F$, for some trivial inequality clause F , (-by (4) of Theorem 3.2.3, page 46), but also, regarded as a map sending literals to literals, (that is, between clauses), θ is the identity on its "abbreviated image". $cl(C)\theta \setminus F(cl(C)\theta) \subseteq cl(C')$, where $F(cl(C)\theta)$ denotes the collection of the trivial inequalities in $cl(C)\theta$.

From this it follows that there is a substitution θ , such that, regarded as map sending terms to terms, θ is the identity on the variables occurring in $cl(C)\theta \setminus F(cl(C)\theta) \subseteq cl(C')$, and $C' \subseteq C$ is in fact minimal such that $cl(C)\theta \setminus F(cl(C)\theta) \subseteq cl(C')$.

Despite this, however, it is *not* in general true that if C' is a minimal Σ_0 -repf of C , then

($C' \subseteq C$ and) there is a substitution θ , such that, not only is it the case that $C\theta \subseteq cl(C') \cup F$, for some trivial inequality clause F , but also, (regarded as map sending terms to terms), θ is the identity on the variables occurring in C' .

For example:

EXAMPLE H.0.4. Consider

$$C \equiv \neq(d(), h(y)) \mid P(x) \mid \neq(d(), f(x)) \mid \neq(f(x), h(e())) \mid \neq(d(), f(b())) \mid P(b()),$$

$$\text{and } C' \equiv \neq(d(), f(x)) \mid \neq(f(x), h(e())) \mid \neq(d(), f(b())) \mid P(b()).$$

Then clearly $C' \subsetneq C$, and

$$C[b()/x, e()/y] = \neq(d(), h(e())) \mid P(b()) \mid \neq(d(), f(b())) \mid \neq(f(b()), h(e())) \mid P(b()),$$

while (modulo symmetry),

$$\begin{aligned} cl(C') = & \neq(d(), f(x)) \mid \neq(d(), h(e())) \mid \neq(f(x), h(e())) \mid \neq(f(b()), f(x)) \mid \\ & \neq(f(b()), h(e())) \mid \neq(d(), f(b())) \mid P(b()). \end{aligned}$$

and $C[b()/x, e()/y] \subseteq cl(C')$, so C' is a proper Σ_0 -repf of C .

But if θ is any substitution such that $C\theta \subseteq cl(C') \cup F$, for some trivial inequality clause F , then $\theta: x \mapsto b()$ and $\theta: y \mapsto e()$ are forced.

In other words $[b()/x, e()/y]$ is the unique most general substitution θ such that $C\theta \subseteq cl(C') \cup F$, for some

trivial inequality clause F .

Further, $C'[b()/x, e()/y] = \neg(d(), f(b())) \mid \neg(f(b()), h(e())) \mid P(b()) \neq C'$,

(so $\theta \equiv [b()/x, e()/y]$ is not the identity on C' .)

(In fact $C \cap C'[b()/x, e()/y] = C' \cap C'[b()/x, e()/y] = \neg(d(), f(b())) \mid P(b())$, and $C'[b()/x, e()/y] \not\subseteq C'$.)

Finally, further note that (it is not hard to show that), (C' has no proper Σ_0 -repf, and so),

C' is in fact a minimal Σ_0 -repf of C .

(When $C'' \equiv \neg(f(x), h(e())) \mid \neg(d(), f(b())) \mid P(b())$,

for C' to Σ_0 -subsume C'' , we must have a substitution θ such that $\neg(d(), f(x))\theta \in cl(C'') \cup F$, for some trivial inequality clause F . Only $\neg(d(), f(x))\theta \in cl(C'') = C''$ is possible, and $\theta: x \mapsto b()$ is forced.

But $C'[b()/x] \equiv \neg(d(), f(b())) \mid \neg(f(b()), h(e())) \mid P(b()) \not\subseteq cl(C'') \cup F$,

for any trivial inequality clause F .

Similarly, for each of the cases when C'' is taken as $\neg(d(), f(x)) \mid \neg(d(), f(b())) \mid P(b())$,

$\neg(d(), f(x)) \mid \neg(f(x), h(e())) \mid P(b())$, or $\neg(d(), f(x)) \mid \neg(f(x), h(e())) \mid \neg(d(), f(b()))$,

evidently it is not the case that C' Σ_0 -subsumes C'' .)

Thus, although a "reasonable" algorithm for the detection of Σ_0 -repf's, B , for a given (input) clause C , based on the "freezing" of those variables which occur in the candidate subsets B , (entirely analogous to one which exists for finding replacement factors), can obviously still be formulated, such an algorithm, would not in general find all possible Σ_0 -repf's, (that is, would not be complete), and (despite what ever merits such might still possess), will, (for the sake of brevity, at least), not be presented here.

Still, it is perhaps worth while to conceptually distinguish the "trivial case" of (proper) Σ_0 -repf's. Here we mean those situations where a clause C has a proper subclause $C' \subsetneq C$ such that, for some trivial inequality clause $F \subseteq C$, $C \subseteq cl(C') \cup F$, that is, when the substitution justifying the Σ_0 -subsumption may be taken to be the identity substitution.

Of course this case alone (together with the preservation of refutation completeness by Σ_0 -repf-ing), informs us that we may always delete trivial inequality literals from any clause C . By assuming the process of (immediate) trivial inequality deletion, the trivial proper Σ_0 -repf's, can be described simply as those cases when there is a $C' \subsetneq C$, such that $C \subseteq cl(C')$.

Evidently, if for some $L \in C$, $C \subseteq cl(C \setminus \{L\})$, then $L \in cl(C \setminus \{L\})$, and $\{\neg L\} \cup C \setminus \{L\}$ is a Σ_0 -tautology.

Of course, if C is a Σ_0 -tautology, then (by, for example, Corollary 7.2.5, page 89), C may be deleted.

Otherwise, (evidently, $C \setminus \{L\}$ is not a Σ_0 -tautology, and),

if (for some $L \in C$), $\{\neg L\} \cup C \setminus \{L\}$ is a Σ_0 -tautology, then (by Corollary 3.2.1, page 46),

$\{\neg \bar{L}\} \cup \bar{C} \setminus \{\bar{L}\}$ is a Σ_0 -tautology, so (by Theorem 3.2.2, page 46), $\Sigma_0 \models \{\neg \bar{L}\} \cup \bar{C} \setminus \{\bar{L}\}$.

Thus, evidently, $\Sigma_0 \cup \{\bar{C}\} \models \bar{C} \setminus \{\bar{L}\}$, so $\Sigma_0 \cup \{\bar{C}\} \cup \neg(\bar{C} \setminus \{\bar{L}\})$, is unsatisfiable, and so

(by Theorem 3.3.3, page 50), $\overline{C}$ Σ_0 -subsumes $\overline{C \setminus \{L\}}$.

But (since $\overline{C}$ is ground), this means $\overline{C} \subseteq cl(\overline{C \setminus \{L\}})$,
and from this we may conclude $C \subseteq cl(C \setminus \{L\})$.

Thus, if C is not a Σ_0 -tautology, for any $L \in C$,
 $C \subseteq cl(C \setminus \{L\})$, if and only if $\{\neg L\} \cup C \setminus \{L\}$ is a Σ_0 -tautology,
and evidently, the trivial cases of Σ_0 -repf-ing can be found by successively testing
if $\{\neg L_i\} \cup C \setminus \{L_i\}$ is a Σ_0 -tautology, (for $i: 1 \leq i \leq k$, when $C \equiv L_1 \mid \cdots \mid L_k$, say).

EXAMPLE H.0.5. Consider $C \equiv P(y, z) \mid P(z, z) \mid \neg P(x, y)$.

Suppose $C' \subsetneq C$ and there exists θ such that $C\theta \subseteq C'$.

Evidently the only possibility is $C' \equiv P(z, z) \mid \neg P(x, y)$, and $\theta: P(y, z) \mapsto P(z, z)$ is forced,
and so $\theta: y \mapsto z$ is forced.

But then $\theta: \neg P(x, y) \mapsto \neg P(\theta(x), z)$, and $C\theta = P(z, z) \mid \neg P(\theta(x), z) \not\subseteq C$.

However, C subsumes $C' \equiv \neg P(y, z) \mid P(z, z) \mid \neg P(x, y)$.

(For with $\theta \equiv [z/y, y/x]$, we get $C\theta = P(z, z) \mid \neg P(y, z) \subseteq C'$).

Therefore we have $C'' \equiv P(z, z) \mid \neg P(x, y)$ as a "generalized replacement factor" of C .

Document Log:

Manuscript Version 0 — 21 January 1998
Typeset by $\text{\LaTeX}$ — 21 January 1998

TOM MACKLING

CENTER FOR INTELLIGENT MACHINES, MCGILL UNIVERSITY, 3480 UNIVERSITY ST., MONTRÉAL (QUÉBEC) H3A 2A7,
CANADA, *Tel.* : (514) 934-3822
E-mail address: `fafnir@cim.mcgill.ca`

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