Integral Representation: Multiply Superharmonic Functions

Abstract

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by

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Ph. D. Mathematics

For $i = 1, \ldots, n$, let Ω_i be a harmonic space of Brelot (viz. satisfying Axioms I, II, III, IV) with positive potential. Let $M = M^+ - M^+$ where M^+ is the cone of positive multiply superharmonic functions on $\Omega = \prod_{i=1}^{n} \Omega_i$. An Hausdorff locally convex topology γ is defined on M and it is shown that M^+ has a compact metrizable base A with respect to γ. Thus there is an integral representation for the elements of M in terms of a signed Radon measure on A, carried by the extreme points of A.

Some results for tensor products of general ordered Hausdorff locally convex topological vector spaces are given. One of these results is applied in another approach to integral representation for the elements of M which involves duality theory.

Finally the nature of the extreme points of the base A is discussed.

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Introduction

Let Ω be a connected Hausdorff space which is locally compact but not compact. Assume that Ω has a countable base and that there is defined on Ω a system of harmonic functions satisfying the four axioms of M. Brelot which are described in Chapter I. If in addition there is a positive potential (Def. 1.2, p. 3) on Ω, then Ω is called a harmonic space of Brelot with positive potential. Let S^+ be the cone of positive superharmonic functions (Def. 1.1, p. 2) on Ω, and $S = S^+ - S^+$. Then it is well known that with respect to a certain topology τ on S, the convex cone S^+ has a compact metrizable base B, and that if $s \in S^+$, then there exists a unique Radon measure μ on B, carried by the extreme points of B, $\mathcal{E}(B)$, such that if $x \in \Omega$, $s(x) = \int_B v(x)\, d\mu(v)$ [2, p. 26], [8, pp. 503-507].

For $i = 1, \ldots, n$, let Ω_i be a harmonic space of Brelot with positive potential satisfying Axioms I, II, III, IV. Consider the product space $\Omega = \prod_{i=1}^{n} \Omega_i$, the convex cone M^+ of positive multiply superharmonic functions on Ω (Def. 1.4, p. 9), and the real vector space $M = M^+ - M^+$. One can ask whether there is an integral representation for the elements of M^+ as there is in the case of superharmonic functions of one variable. Such a representation if it exists would have a number of important applications, in particular, in the study of holomorphic functions of several complex variables and in probability. There have been two partial answers to this question.

In 1966, K.Gowrisankaran [6], proved that the cone MH^+ of positive multiply harmonic functions on Ω (Def. 1.3, p.9) has a compact metrizable base for the topology of uniform convergence on compact sets. The author also showed that MH^+ is a lattice in its own order, which is the natural order. From this it follows that the elements of MH^+ have a unique integral representation in terms of a Radon measure on the base of MH^+, carried by the extreme points of the base.

In 1968, R.Cairoli [4], using probabilistic methods, showed that the elements of a certain class, $\mathfrak{H}$, of multiply superharmonic functions of two variables, have a unique integral representation. The elements of $\mathfrak{H}$ are of the form:

> $v \in \mathfrak{H}$ if $v=v_1+v_2+v_3+v_4$ where v_1 is a multiply harmonic function; v_2 is harmonic in the first variable and a potential in the second variable; v_3 is a potential in the first variable and harmonic in the second variable; v_4 is a potential in both variables separately.

In this thesis we shall consider the problem of integral representation of multiply superharmonic functions on the product of harmonic spaces of Brelot with positive potential. In Chapter II we will show that the cone M^+ does have a compact metrizable base A for a certain Hausdorff locally convex topology γ on M. We take a rather different approach to proving this result in that we first show that A is precompact, then that M^+ itself is complete. Once A is compact metrizable, it then follows easily that the elements of M have an integral representation in terms of a signed Radon measure on A, carried by $\mathcal{E}(A)$. Whether this integral representation is unique has not yet been determined and at the moment seems to be quite difficult, even in the simplest case. The nature of the extreme points of the base A is also a matter of interest. In Chapter II we give a partial answer to this question which is taken up again in Chapter IV.

Let $\Omega = \prod_{i=1}^{n} \Omega_i$, where Ω_i is a harmonic space of Brelot with positive potential. If $S_i = S_i^+ - S_i^+$, S_i^+ the cone of positive superharmonic functions on Ω_i for each i, then the space $\mathcal{S} = S_1 \otimes \ldots \otimes S_n$ can be considered as a subspace of M. In Chapter IV we study the relationship between the two spaces $\mathcal{S}$ and M by applying some results from the theory of duality between two Hausdorff locally convex topological vector spaces. In particular we use a theorem proved in Chapter III to obtain the following result: $\widehat{(\mathcal{S}, \pi)}$, the completion of $\mathcal{S}$ with respect to a certain topology π on $\mathcal{S}$, contains the space M. Furthermore, the set

$$Q = \{ \sum_{j=1}^{m} s_1^j \otimes \ldots \otimes s_n^j \mid s_i^j \in S_i^+ \text{ for all } i, j \}$$

is a convex cone in M^+ with the properties that

(1) $\widehat{(\mathcal{S}, \pi)} = \overline{\overline{Q} - \overline{Q}}$.

(2) $\overline{Q}$ has a compact metrizable base C where C is the closed convex hull of $B_1 \otimes \ldots \otimes B_n$, B_i a compact metrizable base of S_i^+ for each i.

(3) $(Q-Q)$ is dense in M.

We also show that the extreme points of C are precisely the elements in $\mathcal{S}$ of the form $b_1 \otimes \ldots \otimes b_n$ where b_i is an extreme point in B_i for each i. Going back to the question of the extreme points of the base A for the cone M^+ we show, by considering an example, that although $\mathcal{E}(C) \subset \mathcal{E}(A)$, the reverse inclusion is not necessarily true.

In Chapter III there are some minor results in the theory of tensor products of ordered topological vector spaces, the main result being Theorem 3.2 which is applied in Chapter IV.

The result in Chapters II and IV are believed to be original unless explicitly stated otherwise. Although the results in Chapter III are also believed to be original, there is some connection with the works of Hustad [9, p. 83], and Peressini and Sherbert [12, pp. 182-185].

I would like to thank Professor K. Gowrisankaran for his guidance and many helpful suggestions in preparing this thesis. I would also like to thank my family and friends for their encouragement, in particular, Mr. P. Roberts and Professor J. Lambek.

CHAPTER I

In this chapter we will give a brief summary of the integral representation result for superharmonic functions of one variable on a space of the type we consider in this thesis. For a detailed account the reader is referred to the paper, Axiomatique des Fonctions Harmoniques et Surharmoniques dans un Espace Localement Compact, by M. Brelot [2]. For a more general result in this direction (viz. without the Axiom IV)see R.M. Hervé [8]. We will also describe the properties of multiply harmonic and multiply superharmonic functions which will be used in the following chapters.

Section 1: Axioms I, II, III, IV

Let Ω be a connected, Hausdorff space which is locally compact but not compact. To each open subset $\omega \subset \Omega$ there is assigned a vector space of real valued continuous functions on ω, called harmonic functions, satisfying the following axioms.

Axiom I. Let ω, δ be open subsets of Ω and $\delta \subset \omega$. Then a function harmonic on ω is harmonic on δ. If f is a continuous real valued function on ω which is harmonic on a neighborhood of each point in ω, then f is harmonic on ω.

Axiom II. A nonempty, open, relatively compact subset ω is called a regular open set if each continuous function f on the boundary of ω, $\partial\omega$, has a unique continuous extension to the closure of ω, $\bar{\omega}$, denoted by H_f^ω, which is harmonic on ω, further satisfying the condition that if $f \geq 0$, then $H_f^\omega \geq 0$.

Axiom II requires the existence of a base of regular open domains for Ω.

If f is a continuous real valued function on $\partial\omega$, ω a regular open set, and $x \in \omega$, then the mapping $\ell : f \to H_f^\omega(x)$ is a positive linear functional on the space of continuous real valued functions on $\partial\omega$, a compact set. Hence ℓ defines a positive Radon measure on $\partial\omega$ which we will denote by $d\rho_x^\omega$. Then $\int f \, d\rho_x^\omega = H_f^\omega(x)$.

Axiom III. If ω is an open connected set in Ω, and $\mathcal{H}$ is an increasing directed family of harmonic functions on ω, then the upper envelope of $\mathcal{H}$ is either harmonic on ω or identical to $+\infty$.

Definition 1.1. If ω is an open subset of Ω, then an extended real valued function v on ω is superharmonic if

i) $v > -\infty$

ii) v is lower semi-continuous

iii) for each regular domain $\delta \subset \bar{\delta} \subset \omega$, $x \in \delta$, $v(x) \geq \int v \, d\rho_x^\delta$.

iv) $v \not\equiv +\infty$ on any connected component of ω.

Axiom IV. A regular domain $\omega \in \Omega$ is completely determining if for every pair of positive superharmonic functions on Ω, v_1, v_2, harmonic on ω, the condition $v_1 = v_2$ on the complement of ω, $C(\omega)$, implies $v_1 \equiv v_2$ on Ω.

Axiom IV requires the existence of a countable base of completely determining regular domains for Ω.

If v is a positive superharmonic function on an open subset ω, then v has a harmonic minorant on ω. For if $h \equiv 0$ on ω, then h is harmonic on ω and $h(x) \leq v(x)$ for each $x \in \omega$. It can be shown that if a superharmonic function has a harmonic minorant, then it has a greatest harmonic minorant.

Definition 1.2. A potential on an open subset ω is a positive superharmonic function on ω with greatest harmonic minorant equal to zero.

We assume the existence of a positive potential on the space Ω. If such a potential does not exist, one can show that all the positive superharmonic functions are proportional to each other.

A space Ω, connected, Hausdorff, locally compact but not compact, for which there is defined a system of harmonic functions satisfying the above four axioms and having a positive potential will be called a harmonic space of Brelot with positive potential.

Section 2: Properties of Superharmonic Functions and the Vector Space S

1) If v_1, v_2 are superharmonic functions on an open subset ω, then $\inf(v_1, v_2)$, $\lambda_1 v_1$, $\lambda_1 v_1 + \lambda_2 v_2$ $(\lambda_1, \lambda_2 \geq 0)$ are superharmonic on ω.

2) If $\mathcal{H}$ is an increasing directed family of superharmonic functions on an open subset ω, then the upper envelope of $\mathcal{H}$ is either superharmonic or $\equiv +\infty$ on each connected component of ω.

3) If v is a positive superharmonic function on a domain ω, then $v > 0$ or $v \equiv 0$ on ω.

4) If v is a superharmonic function on an open subset ω, $x \in \omega$, and $\{\delta_n\}$ is a sequence of regular domains such that $\overline{\delta}_n \subset \delta_{n-1} \subset \omega$, $\{x\} = \cap \delta_n$ then $\int v \, d\rho_x^{\delta_n} \nearrow v(x)$.

5) If v is a superharmonic function on Ω, and ω a regular domain $\subset \Omega$, then the function $\overline{v} \equiv v$ on $C(\omega)$ and equal to $\int v \, d\rho_x^{\omega}$ for each $x \in \omega$, is superharmonic on ω.

6) If there is a positive potential on Ω, then there is a positive finite continuous potential on Ω. In addition, if ω is any regular domain, the existence of a positive potential on Ω implies that there is a positive superharmonic function on Ω which is not harmonic on ω.

Let S^+ be the set of positive superharmonic functions on Ω. By Property 1, page 3, S^+ is a convex cone. That is, $S^+ + S^+ \subset S^+$, and $\lambda S^+ \subset S^+$ for $\lambda \geq 0$. The fact that $S^+ \cap \{-S^+\} = \{0\}$ follows from the definition of S^+. We define an equivalence relation on the pairs of elements of S^+ in the following way:

let $v_1, v_2, v_3, v_4 \in S^+$; then $(v_1, v_2) \sim (v_3, v_4)$

if $v_1 + v_4 = v_2 + v_3$.

Let S be the set of equivalence classes thus formed and define addition and scalar multiplication in the usual way, i.e.,

$$\lambda[(v_1, v_2)] = [(\lambda v_1, \lambda v_2)] \quad (\lambda > 0)$$

$$\lambda[(v_1, v_2)] = [(-\lambda v_2, -\lambda v_1)] \quad \text{if } \lambda < 0$$

$$[(v_1, v_2)] + [(v_3, v_4)] = [(v_1 + v_3, v_2 + v_4)].$$

Then S is a real vector space, and if we make the identification of S^+ with $\{[(v,0)] \mid v \in S^+\}$, then $S = S^+ - S^+$. The cone S^+ defines a partial ordering on S which is called the generic order. If $v_1, v_2 \in S$, then v_1 is less than or equal to v_2 for the generic order, denoted by $v_1 \propto v_2$, if $v_2 = v_1 + w$, where $w \in S^+$. If $v_1 \propto v_2$, then $v_1(x) \leq v_2(x)$ for each $x \in \Omega$. The converse of this is not necessarily true. The cone S^+ is a lattice for the natural order. Also, we have the following theorem.

Theorem 1.1. The cone S^+ is a lattice for the generic order.

Section 3: Integral Representation for Elements of S.

If $\mathfrak{B}$ is the countable base of completely determining regular domains for Ω, then each couple (ω, x), $\omega \in \mathfrak{B}$, $x \in \omega$, defines a linear functional on S in the following way. If $s \in S$, then $s = s_1 - s_2$ where $s_1, s_2 \in S^+$. Let $(\omega, x)(s) = \int s_1 \, d\rho_x^{\omega} - \int s_2 \, d\rho_x^{\omega}$. Let Σ be the set of all finite linear combinations of linear functionals of this form, and let τ be the weakest topology on S such that the elements of Σ are continuous. The topology τ is locally convex by definition, and by means of Property 4, page 4, one can show that it is Hausdorff.

If ω_0 is a fixed element of $\mathfrak{B}$ and x_0 is a fixed point in ω_0, then since (ω_0, x_0) is a strictly positive linear functional on S, $B = \{s \in S^+ \mid \int s\, d\rho_{x_0}^{\omega_0} = 1\}$ is a base for S^+. (Recall that a set A is a base for a cone K, if $A = \{x \in K \mid f(x) = 1\}$ where f is a strictly positive linear functional on K.) Suppose that X is a countable dense subset of Ω and $x_0 \in X$. If τ' is the weakest topology on S such that linear functionals of the form (ω, x), $\omega \in \mathfrak{B}$, $x \in \omega \cap X$, are continuous, then τ' is an Hausdorff topology on S which is metrizable since it is defined by a countable number of seminorms. One has the following result for the base B.

<u>Theorem 1.2.</u> The base B is compact and metrizable for the topology τ' on S, and τ' coincides on S^+ with τ.

With the help of Theorems 1.1, 1.2 we can now prove the integral representation result for superharmonic functions of one variable.

<u>Theorem 1.3.</u> If $s \in S^+$, then there exists a unique Radon measure μ on the base B, carried by the extreme points of B, $\mathcal{E}(B)$, such that for each $x \in \Omega$, $s(x) = \int_B v(x)\, d\mu(v)$. In addition, if ν is a positive measure on B, then $\int_B v(x)\, d\nu(v) \in S^+$.

Proof: We will prove only the first statement. Since B is a compact metrizable set for the topology τ on S, and S^+ is a lattice in its own order, Choquet's Integral Representation Theorem [13, p.70] applies. That is, if $s \in B$, then $\exists$ a unique Radon measure μ on B, carried by $\mathcal{E}(B)$, such that, if ℓ is a continuous linear functional on S^+, then $\ell(s) = \int_B \ell(v)\, d\mu(v)$. Now, if $\omega \in \mathcal{B}$, $x \in \omega$, then (ω, x) is a continuous linear functional on S^+ for the topology τ. Thus

$$\int s \, d\rho_x^{\omega} = \int_B [\int v \, d\rho_x^{\omega}]\, d\mu(v).$$

If $x \in \Omega$ and $\{\omega_n\}$ is a sequence of elements of $\mathcal{B}$, $\overline{\omega}_n \subset \omega_{n-1}$, $x \in \omega_n$, $\forall n$, and decreasing to $\{x\}$, then

$$\lim_n \int s \, d\rho_x^{\omega_n} = \lim_n \int_B [\int v \, d\rho_x^{\omega_n}]\, d\mu(v).$$

Since the integrand on the right is monotonically increasing with n, by the Monotone Convergence Theorem, the limit of the integrals is equal to

$$\int_B \lim_n [\int v \, d\rho_x^{\omega_n}]\, d\mu(v) = \int_B v(x)\, d\mu(v).$$

By Property 4, page 4,

$$\lim_n \int s \, d\rho_x^{\omega_n} = s(x),$$

therefore

$$s(x) = \int_B v(x)\, d\mu(v).$$

If $s \in S^+$, then there is a unique $b \in B$, $\lambda \geq 0$, such that $s = \lambda b$. Hence there is a unique Radon measure μ representing s. Similarly if $s \in S$, then s can be written as $\lambda_1 s_1 - \lambda_2 s_2$, where $s_1, s_2 \in S^+$ and $\lambda_1, \lambda_2 \geq 0$. Thus each element in S has a representation in terms of a signed Radon measure on B, carried by $\mathcal{E}(B)$.

Section 4: Multiply Harmonic and Multiply Superharmonic Functions

For each i, $i = 1, \ldots, n$, let Ω_i be a harmonic space of Brelot with positive potential. We will now denote by Ω, the product space $\prod_{i=1}^{n} \Omega_i$, and we will use the following notation:

$\mathcal{B}_i$; a countable base of completely determining regular domains for Ω_i.

S_i^+; the cone of positive superharmonic functions on Ω_i.

$S_i = S_i^+ - S_i^+$.

Σ_i; the linear span of $\{(\omega, x) \mid \omega \in \mathcal{B}_i, \ x \in \omega\}$.

τ_i; the weakest topology on S_i such that the linear functionals in Σ_i are continuous.

B_i; the compact metrizable base for S_i^+ given by Theorem 1.2.

(ω_i^0, x_i^0); the linear functional in Σ_i which generates B_i. That is,

$$B_i = \{s \in S_i^+ \mid \int s \, d\rho_{x_i^0}^{\omega_i^0} = 1\}.$$

Definition 1.3. A real valued continuous function h on an open subset $\omega \subset \Omega$ is multiply harmonic on ω if it is harmonic in each variable separately (i.e., if all variables but one are fixed, the resulting function is harmonic).

Let MH^+ be the set of positive multiply harmonic functions on Ω.

Definition 1.4. If ω is an open subset of Ω, v an extended real valued function on ω, then v is multiply superharmonic on ω if

i) $v > -\infty$

ii) v is lower semi-continuous

iii) v is superharmonic in each variable separately, or $\equiv +\infty$.

iv) $v \not\equiv +\infty$ on any connected component of ω.

Let M^+ be the set of positive multiply superharmonic functions on Ω.

As might be expected, multiply harmonic and multiply superharmonic functions have properties similar to those of harmonic and superharmonic functions. We shall give only a partial list of these properties, in particular, those which will be used in the following chapters.

1) If $m_1, m_2 \in M^+$, then so are, $\inf(m_1, m_2)$, $\lambda_1 m_1$, $\lambda_1 m_1 + \lambda_2 m_2$ where $\lambda_1, \lambda_2 \geq 0$.

2) If $m_1, m_2 \in MH^+$, then $m_1, m_2 \in M^+$, and $(\lambda_1 m_1 + \lambda_2 m_2)$ is multiply harmonic for all real values, λ_1, λ_2.

3) Let $MH^{+}(\omega)$ be the cone of positive multiply harmonic functions on ω an open subset of Ω. Then if $x_0 \in \omega$, $\{m \in MH^{+}(\omega) \mid m(x_0) = 1\}$ is a compact metrizable base for $MH^{+}(\omega)$ [6 , pp.45-47].

4) If $m \in MH^{+}(\omega)$, then $m > 0$ or $m \equiv 0$ on ω, provided ω is connected.

5) If $m \in M^{+}$, and for $i = 1, \ldots, n$, $\omega_i \in \mathcal{B}_i$, $x_i \in \omega_i$, then the multiple integral,

$$\int \cdots \int m(\sigma_1, \ldots, \sigma_n)\, d\rho_{x_1}^{\omega_1}(\sigma_1) \cdots d\rho_{x_n}^{\omega_n}(\sigma_n)$$

exists and is equal to any of the iterated integrals.

6) If x_1 is a fixed point in ω_1, ω_1 a fixed element in $\mathcal{B}_1$, $m \in M^{+}$, then $\int m(\sigma_1, \ldots, \sigma_n)\, d\rho_{x_1}^{\omega_1}(\sigma_1)$ is a positive multiply superharmonic function on $\prod_{i=2}^{n} \Omega_i$. This result is true for any fixed $x_i \in \omega_i$, ω_i a fixed element of $\mathcal{B}_i$.

We will give a proof of this result for the case of two variables. The proof carries over directly to finitely many variables. Let $h(y) = \int m(x, y)\, d\rho_{x_1}^{\omega_1}(x)$. Clearly $h \geq 0$. Let $\{y_n\}$ be any sequence of points in Ω_2 with $\lim_n y_n = y$. Then $\liminf_n h(y_n) = \liminf_n \int m(x, y_n)\, d\rho_{x_1}^{\omega_1}(x) \geq \int \liminf_n m(x, y_n)\, d\rho_{x_1}^{\omega_1}$ by Fatou's Lemma. Since m is itself lower semicontinuous, $\liminf_n m(x, y_n) \geq m(x, y)$ and we have $\liminf_n h(y_n) \geq h(y)$. Therefore h is lower semicontinuous on Ω_2. Now suppose y_0 is a fixed point in Ω_2 and δ is a regular domain with $y_0 \in \delta$. Then

$$\int h(y)\, d\rho_{y_0}^{\delta}(y) = \iint m(x,y)\, d\rho_{x_1}^{\omega_1}(x)\, d\rho_{y_0}^{\delta}(y) = \iint m(x,y)\, d\rho_{y_0}^{\delta}(y)\, d\rho_{x_1}^{\omega_1}(x)$$

$$\leq \int m(x, y_0)\, d\rho_{x_1}^{\omega_1}(x) = h(y_0)$$

since $m(x, \ )$ is a superharmonic function on Ω_2 for each fixed $x \in \Omega_1$. If $h(y)$ is not finite on an everywhere dense subset of Ω_2, then $m \equiv +\infty$ on some connected component of $\Omega_1 \times \Omega_2$ and this is not possible since $m \in M^+$. Hence h is a positive superharmonic function on Ω_2.

7) Let $x \in \Omega$, $x = (x_1, \ldots, x_n)$. For each i, let $\{\omega_i^p\}$ be a sequence of neighborhoods of x_i, $\omega_i^p \in \mathcal{B}_i$ $\forall p$, $\overline{\omega}_i^p \subset \omega_i^{p-1}$, and decreasing to $\{x_i\}$. If $\mathfrak{F}$ is the collection of all sets of the form $(\omega_1^{p_1} \times \omega_2^{p_2} \times \ldots \times \omega_n^{p_n})$, then $\mathfrak{F}$ is a countable decreasing directed family of neighborhoods of x. If m is a multiply superharmonic function on Ω, then following the filter $\mathfrak{F}$

$$\int \cdots \int m(\sigma_1, \ldots, \sigma_n)\, d\rho_{x_1}^{\omega_1^{p_1}}(\sigma_1) \ldots d\rho_{x_n}^{\omega_n^{p_n}}(\sigma_n) \nearrow m(x_1, \ldots, x_n).$$

As before we will prove this for $\Omega = \Omega_1 \times \Omega_2$. Let $x \in \Omega$, $x = (x_1, x_2)$, m a multiply superharmonic function on Ω. Then $m(\cdot, x_2)$ is either a superharmonic function on Ω_1 or $m(\ , x_2) \equiv +\infty$. Let $\{\omega_1^n\}$ be a sequence of neighborhoods of x_1, $\omega_1^n \in \mathcal{B}_1$ $\forall n$, $\overline{\omega}_1^n \subset \omega_1^{n-1}$, and decreasing to $\{x_1\}$. Then

1) $$\int m(x_1, x_2)\, d\rho_{x_1}^{\omega_1^n}(x) \nearrow m(x_1, x_2).$$

This follows from Property 4, page 4, if $m(\ , x_2)$ is superharmonic.

It is immediate if $m(\ ,x_2) \equiv +\infty$. Now, $P_n(y) = \int m(x,y)\, d\rho_{x_1}^{\omega_1^n}(x)$ is a superharmonic function on Ω_2 by Property 6, page 10. If $\{\omega_2^m\}$ is a sequence of neighborhoods of x_2, $\omega_2^m \in \mathcal{B}_2\ \forall m$, $\overline{\omega}_2^m \subset \omega_2^{m-1}$, and decreasing to $\{x_2\}$, then for each m,

$$2) \qquad \int P_n(y)\, d\rho_{x_2}^{\omega_2^m}(y) \nearrow P_n(x_2).$$

If $m(x_1,x_2) = +\infty$, and N is any positive integer, then from (1) above $\exists n$ such that $N < \int m(x_1,x_2)\, d\rho_{x_1}^{\omega_1^n}(x) = P_n(x_2)$. From (2) above $\exists m$ depending on n such that

$$N < \int P_n(y)\, d\rho_{x_2}^{\omega_2^m}(y) = \iint m(x,y)\, d\rho_{x_1}^{\omega_1^n}(x)\, d\rho_{x_2}^{\omega_2^m}(y).$$

Now if $\omega_1 \in \mathcal{B}_1$, $\overline{\omega}_1 \subset \omega_1^n$, $\omega_2 \in \mathcal{B}_2$, $\overline{\omega}_2 \subset \omega_2^m$, $x_1 \in \omega_1$, $x_2 \in \omega_2$, then

$$\iint m(x,y)\, d\rho_{x_1}^{\omega_1}(x)\, d\rho_{x_2}^{\omega_2}(y) \geq \iint m(x,y)\, d\rho_{x_1}^{\omega_1^n}(x)\, d\rho_{x_2}^{\omega_2^m}(x).$$

Therefore

$$\lim_{\mathfrak{F}} \iint m(x,y)\, d\rho_{x_1}^{\omega_1^{p_1}}(x)\, d\rho_{x_2}^{\omega_2^{p_2}}(y) > N,$$

and since N was any positive integer, we have the desired result. If $m(x_1,x_2) < +\infty$, and $\epsilon > 0$, then from (1) above $\exists n$ such that

$$\left| \int m(x,y)\, d\rho_{x_1}^{\omega_1^n}(x) - m(x_1,x_2) \right| < \epsilon/2,$$

and from (2) above $\exists m$ depending on n such that

$$\left| \iint m(x,y)\, d\rho_{x_1}^{\omega_1^n}(x)\, d\rho_{x_2}^{\omega_2^m}(y) - \int m(x,y)\, d\rho_{x_1}^{\omega_1^n}(x) \right| < \epsilon/2.$$

Therefore

$$\left| \iint m(x,y)\, d\rho_{x_1}^{\omega_1^n}(x)\, d\rho_{x_2}^{\omega_2^m}(y) - m(x_1,x_2) \right| < \epsilon.$$

As above if $\omega_1 \in \mathcal{B}_1$, $x_1 \in \omega_1 \subset \overline{\omega}_1 \subset \omega_1{}^n$, $\omega_2 \in \mathcal{B}_2$, $x_2 \in \omega_2 \subset \overline{\omega}_2 \subset \omega_2{}^m$, then

$$\iint m(x, y)\, d\rho_{x_1}^{\overline{\omega}_1}(x)\, d\rho_{x_2}^{\omega_2}(y) \geq \iint m(x, y)\, d\rho_{x_1}^{\omega_1^n}(x)\, d\rho_{x_2}^{\omega_2^m}(y).$$

This, together with the fact that

$$\iint m(x, y)\, d\rho_{x_1}^{\omega_1}(x)\, d\rho_{x_2}^{\omega_2}(y) \leq m(x_1, x_2)$$

for any $\omega_1 \in \mathcal{B}_1$, $\omega_2 \in \mathcal{B}_2$, $x_1 \in \omega_1$, $x_2 \in \omega_2$, implies that following the filter $\mathfrak{F}$,

$$\iint m(x, y)\, d\rho_{x_1}^{\omega_1^{p_1}}(x)\, d\rho_{x_2}^{\omega_2^{p_2}}(y) \nearrow m(x_1, x_2).$$

The fact that these integrals form an increasing directed set is due to the increasing nature of the limit in Property 4, page 4. One should also note that if $x_1, x_2, m, \{\omega_i{}^p\}$, $i=1, 2$ are as above, then it is true that

$$\iint m(\sigma_1, \sigma_2)\, d\rho_{x_1}^{\omega_1^p}(\sigma_1)\, d\rho_{x_2}^{\omega_2^p}(\sigma_2) \nearrow m(x_1, x_2).$$

We will use Properties 5, 6, 7 above extensively in the following chapters. Whenever we refer to a point $x \in \Omega$ and the product filter $\mathfrak{F}$ of neighborhoods of x, we mean the filter $\mathfrak{F}$ as described in Property 7 above. Let us denote an arbitrary element, $\omega_1^{p_1} \times \ldots \times \omega_n{}^{p_n}$, of $\mathfrak{F}$, simply by the symbol ω^p, unless it is necessary to refer to a particular factor of ω^p, say $\omega_i{}^{p_i}$ for some $i = 1, \ldots, n$. Using this notation, we will then write

$$\int \cdots \int m(\sigma_1, \ldots, \sigma_n)\, d\rho_{x_1}^{\omega_1^{p_1}}(\sigma_1) \ldots d\rho_{x_n}^{\omega_n^{p_n}}(\sigma_n)$$

as $\int m\, d\rho_x^{\omega^p}$, and Property 7 above can be rewritten as $\lim_{\mathfrak{F}} \int m\, d\rho_x^{\omega^p} = m(x)$.

CHAPTER II

As in Chapter I, $\Omega = \prod_{i=1}^{n} \Omega_i$, where Ω_i is a harmonic space of Brelot with positive potential for each i, (viz. a harmonic space satisfying the Axioms I, II, III, IV), and M^+ is the set of positive multiply superharmonic functions on Ω. By Property 1, page 9, M^+ is a convex cone, since it is obvious that $M^+ \cap \{-M^+\} = \{0\}$. We define an equivalence relation on the pairs of elements of M^+ whereby $(m_1, m_2) \sim (m_3, m_4)$ if $m_1+m_4 = m_2+m_3$ $(m_i \in M^+,\ i=1,2,3,4)$. Let M be the resulting set of equivalence classes. Then $M = M^+ - M^+$ under the identification of M^+ with $\{[(m, 0)] \mid m \in M^+\}$. Let us first consider a topology γ on M. We will show that γ is Hausdorff and locally convex. We will use the notation introduced in Chapter I in the following.

If $T = \Sigma_1 \otimes \ldots \otimes \Sigma_n$, then T can be considered as a set of linear functionals on M. For if $m \in M^+$, and for $i = 1, \ldots, n$, $(\omega_i, x_i) \in \Sigma_i$, then we let

$$[(\omega_1, x_1) \otimes \ldots \otimes (\omega_n, x_n)](m) = \int \ldots \int m(\sigma_1, \ldots, \sigma_n)\, d\rho_{x_1}^{\omega_1}(\sigma_1) \ldots d\rho_{x_n}^{\omega_n}(\sigma_n).$$

Since this integral exists by Property 5, page 10, this defines a linear functional on M^+ which can be extended to M. The elements of T are finite linear combinations of functionals of this form.

Definition 2.1. The topology γ is the weakest topology on M such that elements of T are continuous.

Theorem 2.1. The topology γ is Hausdorff and locally convex.

Proof: That γ is locally convex is clear from its definition. If $m_1, m_2 \in M^+$, $m_1 \neq m_2$, then $\exists x \in \Omega$ such that $m_1(x) \neq m_2(x)$. If $\mathfrak{F}$ is the product filter of neighborhoods of x, then, since $\lim_{\omega^p \in \mathfrak{F}} \int m_i \, d\rho_x^{\omega^p} = m_i(x)$, i=1,2, $\exists$ some $\omega^p \in \mathfrak{F}$ such that $\int m_1 \, d\rho_x^{\omega^p} \neq \int m_2 \, d\rho_x^{\omega^p}$. Hence the topology γ is Hausdorff.

Lemma 2.1. The cone M^+ has a base.

Proof: For each i, i=1, . . . , n, $B_i = \{s \in S_i^+ \mid \int s \, d\rho_{x_i^0}^{\omega_i^0} = 1\}$ is a base for the cone S_i^+. Let $z_0 = [(\omega_1^0, x_1^0) \otimes \ldots \otimes (\omega_n^0, x_n^0)]$, then $z_0 \in T$. For each i, if $s \in S_i^+$, $s \neq 0$, then $\int s \, d\rho_{x_i^0}^{\omega_i^0} > 0$. Therefore, as a result of Property 6, page 10, if $m \in M^+$, $m \neq 0$, then $z_0(m) > 0$. Since z_0 is a strictly positive linear functional on M, the set $A = \{m \in M^+ \mid z_0(m) = 1\}$ is a base for M^+.

Notation: In the considerations that follow A will stand for the above base for a fixed choice of (ω_i^0, x_i^0), i = 1, . . . , n.

Let us now consider the topology γ on A. We will show that A is compact and metrizable. The following results will be stated for n variables but the proofs will be given only for the case of two variables. The methods in the proofs carry over directly to finitely many variables except for Theorem 2.4 which will be proved by induction. We are restricting our attention to two variables because, in the case of three or more variables, the notation becomes very complex and the method of the proof is clouded.

Lemma 2.2. If $z \in T$, $z = [(\omega_1, x_1) \otimes \ldots \otimes (\omega_n, x_n)]$ where $\omega_i \in \mathcal{B}_i$, $x_i \in \omega_i$ for each i, then $\exists n, N > 0 \ni$ for each $m \in A$, $n \le z(m) \le N$.

Proof: Let $z = [(\omega_1, x_1) \otimes (\omega_2, x_2)]$. If $m \in A$, then

$$\iint m(\sigma_1, \sigma_2)\, d\rho_{x_1}^{\omega_1{}^o_o}(\sigma_1)\, d\rho_{x_2}^{\omega_2{}^o_o}(\sigma_2) = 1.$$

Let $m'(\sigma_2) = \int m(\sigma_1, \sigma_2)\, d\rho_{x_1}^{\omega_1{}^o_o}(\sigma_1)$, then $m' \in B_2$. Since B_2 is τ_2 compact and (ω_2, x_2) is a τ_2 strictly positive continuous linear functional on $S_2{}^+ \supset B_2$, $\exists K, k > 0$ such that for each $s \in B_2$, $k \le \int s\, d\rho_{x_2}^{\omega_2} \le K$. Hence,

$$0 < k \le \int m'\, d\rho_{x_2}^{\omega_2}$$

$$= \iint m(\sigma_1, \sigma_2)\, d\rho_{x_1}^{\omega_1{}^o_o}(\sigma_1)\, d\rho_{x_2}^{\omega_2}(\sigma_2)$$

$$= \iint m(\sigma_1, \sigma_2)\, d\rho_{x_2}^{\omega_2}(\sigma_2)\, d\rho_{x_1}^{\omega_1{}^o_o}(\sigma_1) \qquad \text{(by Fubini's Theorem)}$$

$$\le K.$$

Note that k, K do not depend on $m \in A$. Now, since $B_1 = \{s \in S_1{}^+ \mid \int s\, d\rho_{x_1}^{\omega_1{}^o_o} = 1\}$ is τ_1 compact in $S_1{}^+$, any set of the form $\{s \in S_1{}^+ \mid a \le \int s\, d\rho_{x_1}^{\omega_1{}^o_o} \le b;\ a, b \ge 0\}$, is also τ_1 compact. If $m''(\sigma_1) = \int m(\sigma_1, \sigma_2)\, d\rho_{x_2}^{\omega_2}(\sigma_2)$, then $m'' \in \{s \in S_1{}^+ \mid k \le \int s\, d\rho_{x_1}^{\omega_1{}^o_o} \le K\}$. Since (ω_1, x_1) is a strictly positive τ_1 continuous linear functional, $\exists\, \ell, L > 0$ such that $\ell \le \int m''\, d\rho_{x_1}^{\omega_1} \le L$. Therefore

$$0 < \ell \leq \iint m(\sigma_1, \sigma_2)\, d\rho_{x_2}^{\omega_2}(\sigma_2)\, d\rho_{x_1}^{\omega_1}(\sigma_1)$$

$$= \iint m(\sigma_1, \sigma_2)\, d\rho_{x_1}^{\omega_1}(\sigma_1)\, d\rho_{x_2}^{\omega_2}(\sigma_2) \qquad \text{(by Fubini's Theorem)}$$

$$\leq L,$$

where ℓ, L are independent of $m \in A$.

Theorem 2.2. The base A for the cone M^+ is bounded.

Proof: If $z \in T$, then $z = \sum_{j=1}^{m} \lambda_j z_j$ where for each j, $z_j = [(\omega_1^j, x_1^j)\otimes \ldots \otimes(\omega_n^j, x_n^j)]$, $\omega_i^j \in \mathcal{B}_i$, $x_i^j \in \omega_i^j$ for each i. By Lemma 2.2, for each j, $\exists N_j > 0$, such that if $m \in A$, then $|z_j(m)| \leq N_j$. Then $\forall m \in A$, $|z(m)| \leq \sum_{j=1}^{m} |\lambda_j| N_j$, hence A is bounded in the γ topology.

Corollary 2.1. If $z \in T$, $\exists N > 0$, $\ni \forall m \in M^+$ $|z(m)| \leq N \cdot z_0(m)$, where as before $z_0 = [(\omega_1^0, x_1^0)\otimes \ldots \otimes(\omega_n^0, x_n^0)]$.

Proof: If $z \in T$, then by Theorem 2.2, $\exists N > 0 \ni |z(a)| \leq N$, $\forall a \in A$. If $m \in M^+$, then $m = \lambda a$, $\lambda \geq 0$, $a \in A$. Hence $|z(m)| = \lambda |z(a)| \leq \lambda N = N \cdot z_0(m)$.

Corollary 2.2. Let $z \in T$ such that $z = [(\omega_1, x_1)\otimes \ldots \otimes(\omega_n, x_n)]$ where $\omega_i \in \mathcal{B}_i$, $x_i \in \omega_i$ for each i. If $C = \{m \in M^+ | z(m) = 1\}$, then C is a γ-bounded set.

Proof: Since $\forall m \in M^+$, $m \neq 0$, $z(m) > 0$, C is also a base for the cone M^+. If $c \in C$, then $c = \lambda a$, $\lambda > 0$, $a \in A$, and $z_0(c) = \lambda$. From Lemma 2.2, $\exists N, n > 0 \ni \forall a \in A$, $0 < n \leq z(a) \leq N$, or $0 < 1/N \leq 1/z(a) \leq 1/n$. Since $c \in C$

implies that $z(c)=1$, for each λ such that $\lambda a = c \in C$, $1 = z(c) = \lambda z(a)$. Hence $\lambda = 1/z(a)$, and $1/N \leq \lambda \leq 1/n$. Therefore $\forall c \in C$, $1/N \leq z_0(c) \leq 1/n$, and z_0 is bounded on C, by some constant $K>0$.

If $z' \in T$, then by Corollary 2.1, $\exists N'>0$ such that $\forall m \in M^+$, $|z'(m)| \leq N' \cdot z_0(m)$. Since $C \subset M^+$, $\forall c \in C$, $|z'(c)| \leq N' \cdot z_0(c) \leq N' \cdot K$. Hence C is γ-bounded.

Proposition 2.1. In the γ topology on M, a bounded set is also precompact.

Proof: If two Hausdorff locally convex topological vector spaces E, F are in duality, then one can show that a $\sigma(E,F)$ bounded set is $\sigma(E,F)$ precompact. [14, page 50] The proof of Proposition 2.1 is exactly the same as the proof of this result and will be given here for the sake of completeness.

If $\mathcal{V}$ is a subbasis of neighborhoods of zero for a topology τ on a locally convex Hausdorff topological vector space E, then a set $D \subset E$ is τ-precompact if for every $V \in \mathcal{V}$, $\exists d_1, \ldots, d_n$, all elements of D, such that $D \subset \bigcup_{i=1}^{n} (d_i + V)$ [14, page 50]. If $\mathcal{V}$ is the collection of all sets of the form $\{m \in M \mid |z(m)| \leq \epsilon;\ z \in T,\ \epsilon > 0\}$, then $\mathcal{V}$ is a subbasis for the topology γ on M. Suppose $D \subset M$ is bounded and $V \in \mathcal{V}$, $V = \{m \in M \mid |z'(m)| < 1\}$, where $z' \in T$. Since D is bounded, the image of D under z', $z'(D)$, is a bounded set of real numbers. Therefore, there is a finite number of closed intervals, $I_1, \ldots, I_p$, each of diameter less than 1, such that $z'(D) \cap I_i \neq \phi$ for each i, and $z'(D) \subset \bigcup_{i=1}^{p} I_i$. Now for each i, choose some $d_i \in D$ such that $z'(d_i) \in I_i$. If $d \in D \cap (z')^{-1}(I_i)$, then $|z'(d) - z'(d_i)| < 1$ and $d \in d_i + V$. Since $D \subset \bigcup_{i=1}^{p} (z')^{-1}(I_i)$, we have that $D \subset \bigcup_{i=1}^{p} (d_i + V)$, and D is precompact.

As a result of Theorem 2.2 and Proposition 2.1, we can conclude that the base A is precompact. To show that the cone M^+ is complete we will consider a Cauchy net $\{v_j\}_{j \in J} \subset M^+$ and from this net, construct a function $\tilde{v}$ on Ω. The method used in constructing $\tilde{v}$ is a modification of one used by Avanissian in [1, page 32]. We will prove in the following lemmas and theorems that $\tilde{v} \in M^+$ and that the net $\{v_j\}_{j \in J}$ converges to $\tilde{v}$ in the γ topology.

Definition 2.1. Let $\{v_j\}_{j \in J}$ be a Cauchy net in M^+ and $\omega_i \in \mathcal{B}_i$, $i = 1, \ldots, n$. Let x_i be an arbitrary element of ω_i and $x = (x_1, \ldots, x_n)$. Then we define

$$v(\omega_1 \times \ldots \times \omega_n)(x) = \lim_j \int \ldots \int v_j(\sigma_1, \ldots, \sigma_n)\, d\rho_{x_1}^{\omega_1}(\sigma_1) \ldots d\rho_{x_n}^{\omega_n}(\sigma_n).$$

Lemma 2.3. The real valued function $v(\omega_1 \times \ldots \times \omega_n)$ defined on $\omega_1 \times \ldots \times \omega_n$ is a positive multiply harmonic function on $\omega_1 \times \ldots \times \omega_n$.

Proof: As stated in the introduction, in [6, pp.45-47], the author shows that the cone of positive multiply harmonic functions on $(\omega_1 \times \ldots \times \omega_n)$, denoted by $MH^+(\omega_1 \times \ldots \times \omega_n)$, has a compact base D (for the topology of uniform convergence on compact sets). The base $D = \{h \in MH^+(\omega_1 \times \ldots \times \omega_n \mid h(x_1^0, \ldots, x_n^0) = 1\}$ where $(x_1^0, \ldots, x_n^0)$ is a fixed point in $\omega_1 \times \ldots \times \omega_n$.

Let $(x_1, \ldots, x_n) = x \in \omega_1 \times \ldots \times \omega_n$. Then for each j, let $h_j(x) =$ $\int \ldots \int v_j(\sigma_1, \ldots, \sigma_n)\, d\rho_{x_1}^{\omega_1}(\sigma_1) \ldots d\rho_{x_n}^{\omega_n}(\sigma_n)$. The function $h_j \in MH^+(\omega_1 \times \ldots \times \omega_n)$: $\{v_j\}_{j\in J}$ γ-Cauchy in M^+ implies the net $\{h_j\}_{j\in J}$ converges pointwise in $\omega_1 \times \ldots \times \omega_n$. Therefore $\exists N > 0$ and an index j' such that for each index $j > j'$, $0 \leq h_j(x_1^0, \ldots, x_n^0) \leq N$. The base D being compact implies that the set $D' = \{h \in MH^+(\omega_1 \times \ldots \times \omega_n) \mid h(x_1^0, \ldots, x_n^0) \leq N\}$ is also compact. Since the net $\{h_j\}_{j > j'} \subset D'$, $\{h_j\}_{j\in J}$ has an accumulation point h in $MH^+(\omega_1 \times \ldots \times \omega_n)$. However, the fact that $\{h_j\}_{j\in J}$ converges pointwise in $\omega_1 \times \ldots \times \omega_n$ implies that h is actually the limit of the net $\{h_j\}_{j\in J}$. Therefore

$$\lim_j h_j = h = v(\omega_1 \times \ldots \times \omega_n) \in MH^+(\omega_1 \times \ldots \times \omega_n).$$

The lemma is proved.

If $x \in \Omega$, $x = (x_1, \ldots, x_n)$ and for each i, $\omega_i, \delta_i \in \mathcal{B}_i$, $x_i \in \delta_i \subset \overline{\delta}_i \subset \omega_i$, then $v \in M^+$ implies that $\int \ldots \int v(\sigma_1, \ldots, \sigma_n)\, d\rho_{x_1}^{\omega_1}(\sigma_1) \ldots d\rho_{x_n}^{\omega_n}(\sigma_n) \leq$

$$\int \ldots \int v(\sigma_1, \ldots, \sigma_n)\, d\rho_{x_1}^{\delta_1}(\sigma_1) \ldots d\rho_{x_n}^{\delta_n}(\sigma_n),$$

by Property 7, page 11. Therefore, if $\{v_j\}_{j\in J}$ is a Cauchy net in M^+, $x \in \Omega$, and $\mathfrak{F}$ is the product filter of neighborhoods of x described previously, $\{v(\omega^p)(x)\}_{\omega^p \in \mathfrak{F}}$ is a countable increasing directed family of positive real numbers and hence has a limit, finite or equal to $+\infty$.

<u>Definition 2.2</u>. If $\{v_j\}_{j\in J}$ is a Cauchy net in M^+, $x \in \Omega$, $\mathfrak{F}$ the product filter of neighborhoods of x, then $\tilde{v}(x) = \lim_{\omega^p \in \mathfrak{F}} v(\omega^p)(x)$.

Lemma 2.4. $\tilde{v}$ is a nonnegative, extended real valued, lower semi-continuous function on Ω.

Proof: From the definition $\tilde{v}$ is clearly a nonnegative extended real valued function. Let $\alpha \geq 0$ and let $x \in \Omega_1 \times \Omega_2$, $x = (x_1, x_2)$ be such that $\tilde{v}(x) > \alpha \geq 0$. Since $\tilde{v}(x) = \lim_{\omega^p \in \mathfrak{F}} v(\omega^p)(x)$, $\exists$ some $\omega^p \in \mathfrak{F}$ such that $v(\omega^p)(x) > \alpha$. By Lemma 2.3, $v(\omega^p)$ is a multiply harmonic function on ω^p, hence continuous. Therefore $\exists\, \delta$, an open subset of $\Omega_1 \times \Omega_2$, $x \in \delta \subset \bar{\delta} \subset \omega^p$ such that if $z \in \delta$, then $v(\omega^p)(z) > \alpha$. Now if $\mathfrak{F}_z$ is the product filter of neighborhoods of $z \in \delta$, $\mathfrak{F}_z$ as previously described, then $\exists\, \omega_z{}^k \in \mathfrak{F}_z$ such that $\omega_z{}^k \subset \omega^p$. Then we have that $v(\omega_z{}^k)(z) \geq v(\omega^p)(z) > \alpha$ and hence $\tilde{v}(z) \geq v(\omega_z{}^k)(z) > \alpha$ for $z \in \delta$. This shows that $\{x \,|\, \tilde{v}(x) > \alpha\}$ is open for every $\alpha \geq 0$. However $\tilde{v} \geq 0$, hence $\{x \,|\, \tilde{v}(x) > -\beta,\ \beta > 0\}$ is the whole space. Therefore $\tilde{v}$ is a lower semi-continuous function on $\Omega_1 \times \Omega_2$.

Lemma 2.5. Let k be a fixed integer, $1 \leq k \leq n$. Then

$$\tilde{v}(x_1, \ldots, x_k, \ldots, x_n) \geq \int \tilde{v}(x_1, \ldots, x_{k-1}, \sigma_k, x_{k+1}, \ldots, x_n)\, d\rho_{x_k}^{\omega_k}(\sigma_k),$$

where x_i is a fixed point in Ω_i, ω_k any regular domain in Ω_k such that $x_k \in \omega_k$, and σ_k varies over Ω_k.

Proof: Let $x = (x_1, x_2) \in \Omega_1 \times \Omega_2$ be fixed, and let ω_1 be any regular domain in Ω_1, $x_1 \in \omega_1$. We will show that

$$\tilde{v}(x_1, x_2) \geq \int \tilde{v}(\sigma_1, x_2)\, d\rho_{x_1}^{\omega_1}(\sigma_1).$$

The product filter $\mathfrak{F}$ of neighborhoods of (x_1, x_2) was formed by considering two sequences $\{\omega_1{}^n\}, \{\omega_2{}^n\}$ of neighborhoods of x_1, x_2 respectively, where for each i, $\omega_i{}^n \in \mathcal{B}_i$ for each n, $\bar{\omega}_i{}^n \subset \omega_i{}^{n-1}$, and $\bigcap_n \omega_i{}^n = \{x_i\}$. $\mathfrak{F}$ was the collection of all elements of the form $\omega_1{}^n \times \omega_2{}^m$. Let us consider the subfilter $\mathfrak{F}'$ of $\mathfrak{F}$ consisting only of the elements $\omega_1{}^n \times \omega_2{}^n$. Then $\mathfrak{F}'$ is in fact a sequence, and since $\lim_{\omega^p \in \mathfrak{F}} v(\omega^p)(x_1, x_2) = \tilde{v}(x_1, x_2)$, we have that $\lim_{\omega^p \in \mathfrak{F}'} v(\omega^p)(x_1, x_2) = \tilde{v}(x_1, x_2)$ as well. For the remainder of the proof we will consider only those $\omega^p \in \mathfrak{F}'$ and we will write $\lim_{\omega^p \in \mathfrak{F}'} v(\omega^p)(x_1, x_2)$ as $\lim_p v(\omega^p)(x_1, x_2)$ where it is to be understood that $\omega^p \in \mathfrak{F}'$. Let N be a positive integer such that $n > N$ implies $\bar{\omega}_1{}^n \subset \omega_1$. Then for each $n > N$,

$$
\begin{aligned}
v(\omega^n)(x_1, x_2) &= v(\omega_1{}^n \times \omega_2{}^n)(x_1, x_2) \\
&= \lim_j \iint v_j(\sigma_1, \sigma_2)\, d\rho_{x_1}^{\omega_1^n}(\sigma_1)\, d\rho_{x_2}^{\omega_2^n}(\sigma_2) \\
&= \lim_j \iint v_j(\sigma_1, \sigma_2)\, d\rho_{x_2}^{\omega_2^n}(\sigma_2)\, d\rho_{x_1}^{\omega_1^n}(\sigma_1)
\end{aligned}
$$

(by Fubini's Theorem)

$$
\geq \lim_j \iint v_j(\sigma_1, \sigma_2)\, d\rho_{x_2}^{\omega_2^n}\, d\rho_{x_1}^{\omega_1}(\sigma_1),
$$

since $\bar{\omega}_1{}^n \subset \omega_1$. Let n be fixed, and $n > N$. Let $k_j(\sigma_1) = \int v_j(\sigma_1, \sigma_2)\, d\rho_{x_2}^{\omega_2^n}(\sigma_2)$. Then for each j, $k_j \in S_1{}^+$ by Property 6, page 10, and since the net $\{v_j\}_{j \in J}$

is Cauchy, $\{k_j\}_{j\in J}$ is a τ_1-Cauchy net in S_1^+. However, S_1^+ having a τ_1-compact base implies that S_1^+ is τ_1-complete. Therefore $\exists k \in S_1^+$ such that $\lim_j k_j = k$ and hence $\lim_j \int k_j(\sigma_1)\, d\rho_{x_1}^{\omega_1}(\sigma_1) = \int k(\sigma_1)\, d\rho_{x_1}^{\omega_1}(\sigma_1)$. Now let $\{\omega^q\}$ be a decreasing sequence of neighborhoods of σ_1, $\omega^q \in \mathcal{B}_1$, for each q, $\overline{\omega}^q \subset \omega^{q-1}$, and $\bigcap_q \omega^q = \{\sigma_1\}$. Since $k \in S_1^+$, by Property 4, page 4, $k(\sigma_1) = \lim_q \int k(\sigma)\, d\rho_{\sigma_1}^{\omega^q}(\sigma)$. Also, for each q, $\int k(\sigma)\, d\rho_{\sigma_1}^{\omega^q}(\sigma) = \lim_j \int k_j(\sigma)\, d\rho_{\sigma_1}^{\omega^q}(\sigma)$ since $\lim_j k_j = k$ in the τ_1 topology. Then from above we have that

$$
\begin{aligned}
& v(\omega_1^n \times \omega_2^n)(x_1, x_2) \\
& \geq \lim_j \iint v_j(\sigma_1, \sigma_2)\, d\rho_{x_2}^{\omega_2^n}(\sigma_2)\, d\rho_{x_1}^{\omega_1}(\sigma_1) \\
& = \lim_j \int k_j(\sigma_1)\, d\rho_{x_1}^{\omega_1}(\sigma_1) \\
& = \int k(\sigma_1)\, d\rho_{x_1}^{\omega_1}(\sigma_1) \\
& = \int [\lim_q \int k(\sigma)\, d\rho_{\sigma_1}^{\omega^q}(\sigma)]\, d\rho_{x_1}^{\omega_1}(\sigma_1) \\
& = \int [\lim_q \lim_j \int k_j(\sigma)\, d\rho_{\sigma_1}^{\omega^q}(\sigma)]\, d\rho_{x_1}^{\omega_1}(\sigma_1) \\
& = \int [\lim_q \lim_j \iint v_j(\sigma\ , \sigma_2)\, d\rho_{x_2}^{\omega_2^n}(\sigma_2)\, d\rho_{\sigma_1}^{\omega^q}(\sigma)]\, d\rho_{x_1}^{\omega_1}(\sigma_1) \\
& = \int [\lim_q \lim_j \iint v_j(\sigma, \sigma_2)\, d\rho_{\sigma_1}^{\omega^q}(\sigma)\, d\rho_{x_2}^{\omega_2^n}(\sigma_2)]\, d\rho_{x_1}^{\omega_1}(\sigma_1),
\end{aligned}
$$

the last equality being a result of Fubini's Theorem.

Therefore, for any $n > N$, we have

$$v(\omega_1^n \times \omega_2^n)(x_1, x_2)$$

$$\geq \int [\lim_q \lim_j \iint v_j(\sigma, \sigma_2)\, d\rho_{\sigma_1}^{\omega^q}(\sigma)\, d\rho_{x_2}^{\omega_2^n}(\sigma_2)]\, d\rho_{x_1}^{\omega_1}(\sigma_1),$$

hence,

$$\tilde{v}(x_1, x_2) = \lim_n v(\omega^n)(x_1, x_2)$$

$$\geq \lim_n \int [\lim_q \lim_j \iint v_j(\sigma, \sigma_2)\, d\rho_{\sigma_1}^{\omega^q}(\sigma)\, d\rho_{x_2}^{\omega_2^n}(\sigma_2)]\, d\rho_{x_1}^{\omega_1}(\sigma_1).$$

Now, if

$$f_n(\sigma_1) = \lim_q \lim_j \iint v_j(\sigma, \sigma_2)\, d\rho_{\sigma_1}^{\omega^q}(\sigma)\, d\rho_{x_2}^{\omega_2^n}(\sigma_2),$$

then $\{f_n\}$ is an increasing sequence of measurable functions in σ_1, and $\lim_n f_n(\sigma_1) = \tilde{v}(\sigma_1, x_2)$. Then by the Monotone Convergence Theorem we can interchange the limit and the integral and we have

$$\tilde{v}(x_1, x_2) \geq \int \tilde{v}(\sigma_1, x_2)\, d\rho_{x_1}^{\omega_1}(\sigma_1).$$

The proof is complete.

Theorem 2.3. The function $\tilde{v}$ constructed above belongs to M^+.

Proof: We have already shown that $\tilde{v} \geq 0$, hyperharmonic in each variable separately. We now have to show that $\tilde{v} \not\equiv +\infty$ [6 , p.33].

Suppose on the contrary, $\tilde{v} \equiv +\infty$. Let $\omega_1 \subset \Omega_1$, $\omega_2 \subset \Omega_2$ be regular domains and let $N > 0$. For every $x = (x_1, x_2)$ on $\partial\omega_1 \times \partial\omega_2$, we can find a neighborhood of the form $\delta_1 \times \delta_2$ such that δ_i is regular in Ω_i and $v(\delta_1 \times \delta_2)(x_1, x_2) > 2N$. We can find a neighborhood V_x of $x = (x_1, x_2)$ such that $v(\delta_1 \times \delta_2)(y) > 3N/2$ for $y \in V_x$ and $\overline{V_x} \subset \delta_1 \times \delta_2$. Now, since $\iint v_j(\sigma_1, \sigma_2)\, d\rho_{x_1}^{\delta_1}(\sigma_1)\, d\rho_{x_2}^{\delta_2}(\sigma_2)$ converges locally uniformly to $v(\delta_1 \times \delta_2)(x_1, x_2)$, for j following the Cauchy filter J, $\exists$ a subfilter $J'(x)$ such that for each $j \in J'(x)$ we have $\iint v_j(\sigma_1, \sigma_2)\, d\rho_{y_1}^{\delta_1}(\sigma_1)\, d\rho_{y_2}^{\delta_2}(\sigma_2) > N$ for $(y_1, y_2) \in V_x$. Furthermore

$$v_j(y_1, y_2) \geq \iint v_j(\sigma_1, \sigma_2)\, d\rho_{y_1}^{\delta_1}(\sigma_1)\, d\rho_{y_2}^{\delta_2}(\sigma_2) > N$$

for $(y_1, y_2) \in V_x$, $j \in J'(x)$. For every $x \in \partial\omega_1 \times \partial\omega_2$ we choose a neighborhood by the above process, and since $\partial\omega_1 \times \partial\omega_2$ is compact, we may assume that $\bigcup_{i=1}^{m} V_{x_i}$ covers $\partial\omega_1 \times \partial\omega_2$, and let $J' = \bigcap_{i=1}^{m} J'(x_i)$. Then J' is a subfilter of J and for every $j \in J'$, for all $y \in \partial\omega_1 \times \partial\omega_2$, we have $v_j(y) > N$. This reasoning holds good for all $N > 0$. Hence

$$\lim_j \Big(\inf_{y \in \partial\omega_1 \times \partial\omega_2} v_j(y) \Big)$$

exists and is equal to $+\infty$. However, this last conclusion is a contradiction since for every j,

$$\Big[\inf_{y \in \partial\omega_1 \times \partial\omega_2} v_j(y) \Big] \cdot \rho_{x_1}^{\omega_1}(\partial\omega_1)\, \rho_{x_2}^{\omega_2}(\partial\omega_2) \leq \iint v_j(\sigma_1, \sigma_2)\, d\rho_{x_1}^{\omega_1}(\sigma_1)\, d\rho_{x_2}^{\omega_2}(\sigma_2),$$

and $\lim_j [\inf_{y \in \partial\omega_1 \times \partial\omega_2} v_j(y)] \leq \lim_j \iint v_j(\sigma_1, \sigma_2)\, d\rho_{x_1}^{\omega_1}(\sigma_1)\, d\rho_{x_2}^{\omega_2}(\sigma_2) < +\infty$.

We conclude therefore that $\tilde{v} \not\equiv +\infty$.

Theorem 2.4. M^+ is γ complete.

Proof: Let $\omega_i \in \mathfrak{B}_i$, $x_i \in \omega_i$, $i=1,\ldots,n$. We will show that if $z = [(\omega_1, x_1) \otimes \ldots \otimes (\omega_n, x_n)]$, then $\lim_j z(v_j) = z(\tilde{v})$. If z is an arbitrary element of T, then z is a finite linear combination of functionals of this form. Hence, we can conclude then that the net $\{v_j\}_{j \in J}$ converges to $\tilde{v}$ in the γ topology.

Case 1 Functions of two variables

Let $x_1 \in \omega \in \mathfrak{B}_1$, $y_1 \in \delta \in \mathfrak{B}_2$, and $z = [(\omega, x_1) \otimes (\delta, y_1)]$. If $V_j'(y) = \int v_j(x, y)\, d\rho_{x_1}^{\omega}(x)$, then $\{V_j'\}_{j \in J}$ is a τ_2 Cauchy net in S_2^+. S_2^+ being τ_2 complete implies that $\exists V' \in S_2^+$ such that $\lim_j V_j' = V'$. Let $W(y) = \int \tilde{v}(x, y)\, d\rho_{x_1}^{\omega}(x)$. If $W(y) = V'(y)$, $y \in \Omega_2$, then

$$
\begin{aligned}
z(\tilde{v}) &= \iint \tilde{v}(x, y)\; d\rho_{x_1}^{\omega}(x)\; d\rho_{y_1}^{\delta}(y) \\
&= \int V'(y)\; d\rho_{y_1}^{\delta}(y) \\
&= \lim_j \int V_j'(y)\; d\rho_{y_1}^{\delta}(y) \qquad \text{(by definition of } \tau_2\text{-convergence)} \\
&= \lim_j \iint v_j(x, y)\; d\rho_{x_1}^{\omega}(x)\; d\rho_{y_1}^{\delta}(y) \\
&= \lim_j z(v_j).
\end{aligned}
$$

Suppose $\exists y_0 \in \Omega_2 \quad W(y_0) < V'(y_0)$. If $\{\delta^m\}$ is a sequence of neighborhoods of y_0 such that $\bar{\delta}^m \subset \delta^{m-1}$, $\delta^m \in \mathcal{B}_2 \; \forall m$, and $\cap \delta^m = \{y_0\}$, then $\exists m$ such that $m' \geq m$ implies that

$$W(y_0) < \int V'(y)\, d\rho_{y_0}^{\delta^{m'}}(y).$$

Let m be fixed, and let $V_j^m(x) = \int v_j(x, y)\, d\rho_{y_0}^{\delta^m}(y)$. Then $\{V_j^m\}_{j \in J}$ is a τ_1 Cauchy net in S_1^+ which is τ_1 complete. Hence $\exists V^2 \in S_1^+$, such that $\lim_j V_j^m = V^2$. From above,

$$\begin{aligned}
\int \tilde{v}(x, y_0)\, d\rho_{x_1}^{\omega}(x) &= W(y_0) \\
&< \int V'(y)\, d\rho_{y_0}^{\delta^m}(y) \\
&= \lim_j \int V_j'(y)\, d\rho_{y_0}^{\delta^m}(y) \\
&= \lim_j \iint v_j(x, y)\, d\rho_{x_1}^{\omega}(x)\, d\rho_{y_0}^{\delta^m} \\
&= \lim_j \iint v_j(x, y)\, d\rho_{y_0}^{\delta^m}(y)\, d\rho_{x_1}^{\omega}(x) \quad \text{(by Fubini's Theorem)} \\
&= \lim_j \int V_j^m(x)\, d\rho_{x_1}^{\omega}(x) \\
&= \int V^2(x)\, d\rho_{x_1}^{\omega}(x).
\end{aligned}$$

Since $\tilde{v}(\ , y_0)$, $V^2 \in S_1^+$,

$$\int \tilde{v}(x, y_0)\, d\rho_{x_1}^{\omega}(x) < \int V^2(x)\, d\rho_{x_1}^{\omega}(x)$$

implies that $\exists x_0 \in \partial\omega$ such that $\tilde{v}(x_0, y_0) < V^2(x_0)$. Suppose $\{\omega^q\}$ is a sequence of neighborhoods of x_0, decreasing to $\{x_0\}$, $\omega^q \in \mathcal{B}_1$ $\forall q$, $\bar{\omega}^q \subset \omega^{q-1}$. Then,

$$\begin{aligned}\tilde{v}(x_0, y_0) &< V^2(x_0)\\ &= \lim_q \int V^2(x)\, d\rho_{x_0}^{\omega^q}(x)\\ &= \lim_q \lim_j \int V_j^m(x)\, d\rho_{x_0}^{\omega^q}(x)\\ &= \lim_q \lim_j \iint v_j(x, y)\, d\rho_{y_0}^{\delta^m}(y)\, d\rho_{x_0}^{\omega^q}(x).\end{aligned}$$

Now, if $m' > m$, then for each $x \in \Omega_1$,

$$\int v_j(x, y)\, d\rho_{y_0}^{\delta^{m'}}(y) \geq \int v_j(x, y)\, d\rho_{y_0}^{\delta^m}(y),$$

hence for each $m' > m$,

$$\tilde{v}(x_0, y_0) < \lim_q \lim_j \iint v_j(x, y)\, d\rho_{y_0}^{\delta^{m'}}(y)\, d\rho_{x_0}^{\omega^q}(x),$$

and therefore

$$\tilde{v}(x_0, y_0) < \lim_m \Big[\lim_q \lim_j \iint v_j(x, y)\, d\rho_{y_0}^{\delta^m}(y)\, d\rho_{x_0}^{\omega^q}(x)\Big].$$

However the right side of this inequality is precisely $\tilde{v}(x_0, y_0)$. This contradiction implies that $W(y_0) \geq V'(y_0)$.

Suppose $W(y_0) > V'(y_0)$. If $\{\delta^m\}$ is again a sequence of neighborhoods of y_0, decreasing to $\{y_0\}$, $\bar{\delta}^m \subset \delta^{m-1}$, $\delta^m \in \mathcal{B}_2$ for each m, then $W(y_0) > V'(y_0) \geq \int V'(y)\, d\rho_{y_0}^{\delta^m}(y)$ for each m. Let α and β be real numbers such that $W(y_0) > \alpha > \beta > V'(y_0)$. Let

$$V_j^{2,m}(x) = \int v_j(x, y)\, d\rho_{y_0}^{\delta^m}(y).$$

As before $\{V_j^{2,m}\}$ is a Cauchy net in S_1^+ and hence converges to some $V^{2,m} \in S_1^+$. From above we have for each m,

$$\beta > \int V'(y)\, d\rho_{y_0}^{\delta^m}(y)$$

$$= \lim_j \iint v_j(x, y)\, d\rho_{x_1}^{\omega}(x)\, d\rho_{y_0}^{\delta^m}(y)$$

$$= \lim_j \iint v_j(x, y)\, d\rho_{y_0}^{\delta^m}(y)\, d\rho_{x_1}^{\omega}(x)$$

$$= \lim_j \int V_j^{2,m}(x)\, d\rho_{x_1}^{\omega}(x)$$

$$= \int V^{2,m}(x)\, d\rho_{x_1}^{\omega}(x).$$

As $m \to +\infty$, $\int V^{2,m}(x)\, d\rho_{x_1}^{\omega}(x)$ is monotonically increasing since for a fixed j, $\int v_j(x, y)\, d\rho_{y_0}^{\delta^m}(y)$ is monotonically increasing with m. Hence $\beta \geq \lim_m \int V^{2,m}(x)\, d\rho_{x_1}^{\omega}(x)$. Now, if $\{\omega^q\}$ is a sequence of neighborhoods of $x \in \partial\omega$, decreasing to $\{x\}$, $\omega^q \in \mathcal{B}_1$, $\overline{\omega}^q \subset \omega^{q-1}$ for each q, then,

$$V^{2,m}(x) = \lim_q \int V^{2,m}(\sigma)\, d\rho_x^{\omega^q}(\sigma).$$

Hence,

$$\beta \geq \lim_m \int V^{2,m}(x)\, d\rho_{x_1}^{\omega}(x)$$

$$= \lim_m \int [\lim_q \int V^{2,m}(\sigma)\, d\rho_x^{\omega^q}(\sigma)]\, d\rho_{x_1}^{\omega}(x)$$

$$= \lim_{m} \int [\lim_{q} \lim_{j} \int V_j^{2, m}(\sigma)\, d\rho_x^{\omega^q}(\sigma)]\, d\rho_{x_1}^{\omega}(x)$$

$$= \lim_{m} \int \lim_{q} \lim_{j} \iint v_j(\sigma, y)\, d\rho_{y_0}^{\delta^m}(y)\, d\rho_x^{\omega^q}(\sigma)]\, d\rho_{x_1}^{\omega}(x).$$

Since,

$$\lim_{m} \lim_{q} \lim_{j} \iint v_j(\sigma, y)\, d\rho_{y_0}^{\delta^m}(y)\, d\rho_x^{\omega^q}(\sigma)$$

$$= \lim_{m} \lim_{q} \lim_{j} \iint v_j(\sigma, y)\, d\rho_x^{\omega^q}(\sigma)\, d\rho_{y_0}^{\delta^m}(y)$$

$$= \tilde{v}(x, y_0),$$

we can apply the Dominated Convergence Theorem, and we have,

$$\alpha > \beta \geq \int \tilde{v}(x, y_0)\, d\rho_{x_1}^{\omega}(x).$$

However, α was chosen so that

$$\alpha < W(y_0) = \int \tilde{v}(x, y_0)\, d\rho_{x_1}^{\omega}(x).$$

This contradiction implies that for all $y \in \Omega_2$, $W(y) = V'(y)$, and therefore that $\lim_{j} z(v_j) = z(\tilde{v})$.

Case 2: $n > 2$

Now let $z = [(\omega_1, x_1) \otimes \ldots \otimes (\omega_n, x_n)]$, $\omega_i \in \mathcal{B}_i$, $x_i \in \omega_i$, for each i. We assume that the theorem is true for $k = n-1$. That is, if $\Omega' = \prod_{i=1}^{n-1} \Omega_i$, the cone $(M')^+$ of positive multiply superharmonic functions on Ω' is complete for the corresponding γ' topology.

Let $W(\sigma_n) = \int \cdots \int \tilde{v}(\sigma_1, \ldots, \sigma_n)\, d\rho_{x_1}^{\omega_1}(\sigma_1) \cdots d\rho_{x_{n-1}}^{\omega_{n-1}}(\sigma_{n-1})$, $V_j'(\sigma_n) = \int \cdots \int v_j(\sigma_1, \ldots, \sigma_n)\, d\rho_{x_1}^{\omega_1}(\sigma_1) \cdots d\rho_{x_{n-1}}^{\omega_{n-1}}(\sigma_{n-1})$. As before $\{V_j'\}$ is a τ_n Cauchy net in S_n^+ which is τ_n complete. Hence $\exists\, V' \in S_n^+$ such that $\lim_j V_j' = V'$. We will show that for each $\sigma_n \in \Omega_n$, $W(\sigma_n) = V'(\sigma_n)$.

Suppose for some $\sigma_n \in \Omega_n$, $W(\sigma_n) < V'(\sigma_n)$. Let $\{\delta^m\}$ be a sequence of neighborhoods of σ_n, decreasing to $\{\sigma_n\}$, $\overline{\delta}^m \subset \delta^{m-1}$, $\delta^m \in \mathcal{B}_n$ for each m. Then $\exists\, m \ni m' > m$ implies

$$W(\sigma_n) < \int V'(r)\, d\rho_{\sigma_n}^{\delta^{m'}}(r) = \lim_j \int V_j'(r) d\rho_{\sigma_n}^{\delta^{m'}}(r).$$

Let $V_j^m(\sigma_1, \ldots, \sigma_{n-1}) = \int v_j(\sigma_1, \ldots, \sigma_{n-1}, r)\, d\rho_{\sigma_n}^{\delta^m}(r)$. Then for each j, V_j^m is a positive multiply superharmonic function on $\Omega' = \prod_{i=1}^{n-1} \Omega_i$, and since $\{v_j\}_{j \in J}$ is a Cauchy net in M^+, $\{V_j^m\}_{j \in J}$ is a Cauchy net in $(M')^+$. By assumption $(M')^+$ is complete, hence $\exists\, V^2 \in (M')^+ \ni \lim_j V_j^m = V^2$. Then we have

$$\begin{aligned}
W(\sigma_n) &< \lim_j \int V_j'(r)\, d\rho_{\sigma_n}^{\delta^m}(r) \\
&= \lim_j \int \Big[\int \cdots \int v_j(\sigma_1, \ldots, \sigma_{n-1}, r)\, d\rho_{x_1}^{\omega_1}(\sigma_1) \cdots d\rho_{x_{n-1}}^{\omega_{n-1}}(\sigma_{n-1})\Big] d\rho_{\sigma_n}^{\delta^m}(r) \\
&= \lim_j \int \cdots \int \Big[\int v_j(\sigma_1, \ldots, \sigma_{n-1}, r)\, d\rho_{\sigma_n}^{\delta^m}(r)\Big] d\rho_{x_1}^{\omega_1}(\sigma_1) \cdots d\rho_{x_{n-1}}^{\omega_{n-1}}(\sigma_{n-1}) \\
&\qquad \text{(by Fubini's Theorem)} \\
&= \lim_j \int \cdots \int V_j^m(\sigma_1, \ldots, \sigma_{n-1})\, d\rho_{x_1}^{\omega_1}(\sigma_1) \cdots d\rho_{x_{n-1}}^{\omega_{n-1}}(\sigma_{n-1}) \\
&= \int \cdots \int V^2(\sigma_1, \ldots, \sigma_{n-1})\, d\rho_{x_1}^{\omega_1}(\sigma_1) \cdots d\rho_{x_{n-1}}^{\omega_{n-1}}(\sigma_{n-1}).
\end{aligned}$$

Since $W(\sigma_n) = \int \cdots \int \tilde{v}(\sigma_1, \ldots, \sigma_{n-1}, \sigma_n) d\rho_{x_1}^{\omega_1}(\sigma_1) \ldots d\rho_{x_{n-1}}^{\omega_{n-1}}(\sigma_{n-1})$, the above inequality implies that $\exists\, (\sigma_1, \ldots, \sigma_{n-1}) \in \partial\omega_1 \times \ldots \times \partial\omega_{n-1}$ such that $\tilde{v}(\sigma_1, \ldots, \sigma_{n-1}, \sigma_n) < V^2(\sigma_1, \ldots, \sigma_{n-1})$. If $\{\gamma_i{}^p\}$ is a sequence of neighborhoods of σ_i, decreasing to $\{\sigma_i\}$, $\overline{\gamma}_i{}^p \subset \overline{\gamma}_i{}^{p-1}$, $\gamma_i{}^p \in \mathcal{B}_i$ for each i, then $\exists$ some $p' \ni p > p'$ implies

$$\tilde{v}(\sigma_1, \ldots, \sigma_{n-1}, \sigma_n) < \int \cdots \int V^2(r_1, \ldots, r_{n-1}) d\rho_{\sigma_1}^{\gamma_1{}^p}(r_1) \ldots d\rho_{\sigma_{n-1}}^{\gamma_{n-1}{}^p}(r_{n-1})$$

$$= \lim_j \int \cdots \int V_j^2(r_1, \ldots, r_{n-1}) d\rho_{\sigma_1}^{\gamma_1{}^p}(r_1) \ldots d\rho_{\sigma_{n-1}}^{\gamma_{n-1}{}^p}(r_{n-1}).$$

Hence

$$\tilde{v}(\sigma_1, \ldots, \sigma_{n-1}, \sigma_n) < \lim_p \lim_j \int \cdots \int V_j^2(r_1, \ldots, r_{n-1})\, d\rho_{\sigma_1}^{\gamma_1{}^p}(r_1) \ldots d\rho_{\sigma_{n-1}}^{\gamma_{n-1}{}^p}(r_{n-1})$$

$$= \lim_p \lim_j \int \cdots \int \left[\int v_j(r_1, \ldots, r_{n-1}, r_n) d\rho_{\sigma_n}^{\delta^m}(r)\right] d\rho_{\sigma_1}^{\gamma_1{}^p}(r_1) \ldots d\rho_{\sigma_{n-1}}^{\gamma_{n-1}{}^p}(r_{n-1})$$

This inequality holds for all $m' > m$, since the right side is monotonically increasing with m. Hence it holds for the limit, and we have

$$\tilde{v}(\sigma_1, \ldots, \sigma_n) < \lim_m \lim_p v(\gamma_1{}^p \times \ldots \times \gamma_{n-1}{}^p \times \delta^m)(\sigma_1, \ldots, \sigma_{n-1}, \sigma_n)$$

$$= \tilde{v}(\sigma_1, \ldots, \sigma_n).$$

This contradiction implies $W(\sigma_n) \geq V'(\sigma_n)$ for all $\sigma_n \in \Omega_n$.

Suppose $W(\sigma_n) > V'(\sigma_n)$ for some $\sigma_n \in \Omega_n$. Let $\{\omega_n^{\,p}\}$ be a sequence of neighborhoods of σ_n, decreasing to $\{\sigma_n\}$, $\overline{\omega}_n^{\,p} \subset \omega_n^{\,p-1}$, $\omega_n^{\,p} \in \mathcal{B}_n$ for each p. Let α be a real number such that $W(\sigma_n) > \alpha > V'(\sigma_n)$. Then for each p,

$$\alpha > \int V'(r)\, d\rho_{\sigma_n}^{\omega_n^{p}}(r)$$

$$= \lim_j \int V_j'(r)\, d\rho_{\sigma_n}^{\omega_n^{p}}(r)$$

$$= \lim_j \int [\int \cdots \int v_j(\sigma_1, \ldots, \sigma_{n-1}, r)\, d\rho_{x_1}^{\omega_1}(\sigma_1) \ldots d\rho_{x_{n-1}}^{\omega_{n-1}}(\sigma_{n-1})]\, d\rho_{\sigma_n}^{\omega_n^{p}}(r).$$

Let $h_j^{\,p}(\sigma_1, \ldots, \sigma_{n-1}) = \int v_j(\sigma_1, \ldots, \sigma_{n-1}, r)\, d\rho_{\sigma_n}^{\omega_n^{p}}(r)$. Then $\{h_j^{\,p}\}_{j \in J}$ is a Cauchy net of positive multiply superharmonic functions on $\Omega_1 \times \ldots \times \Omega_{n-1}$. By assumption, $\exists\, h^p \in (M')^+$ such that $\lim_j h_j^{\,p} = h^p$. Then from above we have that by means of Fubini's Theorem that,

$$\alpha > \lim_j \int \cdots \int h_j^{\,p}(\sigma_1, \ldots, \sigma_{n-1})\, d\rho_{x_1}^{\omega_1}(\sigma_1) \ldots d\rho_{x_{n-1}}^{\omega_{n-1}}(\sigma_{n-1})$$

$$= \int \cdots \int h^p(\sigma_1, \ldots, \sigma_{n-1})\, d\rho_{x_1}^{\omega_1}(\sigma_1) \ldots d\rho_{x_{n-1}}^{\omega_{n-1}}(\sigma_{n-1}),$$

for each p. Now let $\{\omega_i^{\,q}\}$ be a sequence of neighborhoods of σ_i, $i = 1, \ldots, n-1$, such that $\bigcap_q \omega_i^{\,q} = \{\sigma_i\}$, $\overline{\omega}_i^{\,q} \subset \omega_i^{\,q-1}$, $\omega_i^{\,q} \in \mathcal{B}_i$ for each i. Then

$$h^p(\sigma_1, \ldots, \sigma_{n-1}) = \lim_q \int \cdots \int h^p(r_1, \ldots, r_{n-1})\, d\rho_{\sigma_1}^{\omega_1^{q}}(r_1) \ldots d\rho_{\sigma_{n-1}}^{\omega_{n-1}^{q}}(\sigma_{n-1}).$$

Hence,

$$\alpha \geq \int \cdots \int h^{p}(\sigma_1, \ldots, \sigma_{n-1})\, d\rho_{x_1}^{\omega_1}(\sigma_1) \cdots d\rho_{x_{n-1}}^{\omega_{n-1}}(\sigma_{n-1})$$

$$= \int \cdots \int [\lim_{q} \int \cdots \int h^{p}(r_1, \ldots, r_{n-1})\, d\rho_{\sigma_1}^{\omega_1^{q}}(r_1) \cdots d\rho_{\sigma_{n-1}}^{\omega_{n-1}^{q}}(r_{n-1})]\, d\rho_{x_1}^{\omega_1}(\sigma_1) \cdots d\rho_{x_{n-1}}^{\omega_{n-1}}(\sigma_{n-1})$$

$$= \int \cdots \int [\lim_{q} \lim_{j} \int \cdots \int h_j^{p}(r_1, \ldots, r_{n-1})\, d\rho_{\sigma_1}^{\omega_1^{q}}(r_1) \cdots d\rho_{\sigma_{n-1}}^{\omega_{n-1}^{q}}(r_{n-1})]\, d\rho_{x_1}^{\omega_1}(\sigma_1) \cdots d\rho_{x_{n-1}}^{\omega_{n-1}}(\sigma_{n-1})$$

$$= \int \cdots \int [\lim_{q} \lim_{j} \int \cdots \int v_j(r_1, \ldots, r_n)\, d\rho_{\sigma_n}^{\omega_n^{p}}(r_n)\, d\rho_{\sigma_1}^{\omega_1^{q}}(r_1) \cdots d\rho_{\sigma_{n-1}}^{\omega_{n-1}^{q}}(r_{n-1})]\, d\rho_{x_1}^{\omega_1}(\sigma_1) \cdots d\rho_{x_{n-1}}^{\omega_{n-1}}(\sigma_{n-1}).$$

This last inequality is true for each p, hence true for the limit over p. As in Part 1, by means of the Dominated Convergence Theorem we can interchange the limit and integral, and we have,

$$\alpha \geq \int \cdots \int [\lim_{p} \lim_{q} v(\omega_1^{q} \times \cdots \times \omega_{n-1}^{q} \times \omega_n^{p})(\sigma_1, \ldots, \sigma_{n-1}, \sigma_n)]\, d\rho_{x_1}^{\omega_1}(\sigma_1) \cdots d\rho_{x_{n-1}}^{\omega_{n-1}}(\sigma_{n-1})$$

$$= \int\int \tilde{v}(\sigma_1, \ldots, \sigma_n)\, d\rho_{x_1}^{\omega_1}(\sigma_1) \cdots d\rho_{x_{n-1}}^{\omega_{n-1}}(\sigma_{n-1}).$$

This is a contradiction to the way in which α was chosen, hence $W(\sigma_n) = V'(\sigma_n)$ for all $\sigma_n \in \Omega_n$. This implies therefore that $\lim_j z(v_j) = z(\tilde{v})$, and M^+ is γ complete.

Theorem 2.5. The base A for M^+ is compact and metrizable for the γ topology on M.

Proof: Since M^+ is complete, it is closed. Hence $A \subset M^+$ is closed, since it is the intersection of M^+ with a closed hyperplane. Therefore A is complete, and since we have already shown A is precompact, we conclude that A is compact.

If X_i is a countable dense subset of Ω_i for i=1, . . . , n, then $\mathfrak{T}'$, the rational linear span of the set $\{[(\omega_1, x_1) \otimes \ldots \otimes (\omega_n, x_n) \mid \omega_i \in \mathcal{B}_i, x_i \in \omega_i \cap X_i \ \forall i\}$, is a countable subset of T. If γ' is the weakest topology on M such that elements of $\mathfrak{T}$ are continuous, then γ' is an Hausdorff metrizable topology on M. Since A is γ-compact and γ' is weaker than γ, γ' coincides with γ on A. Hence A is γ-metrizable.

The integral representation result for elements of M follows now by a straightforward application of the Choquet Integral Representation Theorem [13 ,p.19] .

Theorem 2.6. If $m \in M$, then $\exists$ a signed Radon measure μ on A, carried by $\mathcal{E}(A)$, such that $\forall x \in \Omega$, $m(x) = \int_A v(x)\, d\mu(v)$.

Proof: Let $m \in A$. Since A is compact and metrizable, $\exists$ a Radon measure μ on A, carried by $\mathcal{E}(A)$, such that, if ℓ is a continuous linear functional on M^+, then

$$\ell(m) = \int_A \ell(v)\, d\mu(v).$$

Suppose $x = (x_1, \ldots, x_n) \in \Omega$ and for each i, $\{\omega_i^p\}$ is a sequence of neighborhoods of x_i, decreasing to $\{x_i\}$, $\overline{\omega}_i^p \subset \omega_i^{p-1}$, $\omega_i^p \in \mathcal{B}_i$ for each i. Then for each p,

$$\int \cdots \int m(\sigma_1, \ldots, \sigma_n)\, d\rho_{x_1}^{\omega_1^p}(\sigma_1) \ldots d\rho_{x_n}^{\omega_n^p}(\sigma_n)$$

is a continuous linear functional on the elements m in M. Hence for each p,

$$\int \cdots \int m(\sigma_1, \ldots, \sigma_n)\, d\rho_{x_1}^{\omega_1^p}(\sigma_1) \ldots d\rho_{x_n}^{\omega_n^p}(\sigma_n) =$$

$$\int_A \left[\int \cdots \int v(\sigma_1, \ldots, \sigma_n)\, d\rho_{x_1}^{\omega_1^p}(\sigma_1) \ldots d\rho_{x_n}^{\omega_n^p}(\sigma_n)\right] d\mu(v).$$

Taking the limit as $p \to +\infty$, by Property 7, page 11, we have

$$m(x_1, \ldots, x_n) = \lim_p \int_A \left[\int \cdots \int v(\sigma_1, \ldots, \sigma_n) d\rho_{x_1}^{\omega_1^p}(\sigma_1) \ldots d\rho_{x_n}^{\omega_n^p}(\sigma_n)\right] d\mu(v).$$

In this last integral, the integrand is an increasing sequence of integrable functions with respect to p, so by the Monotone Convergence Theorem, we can interchange the limit and the integral. Hence $m(x) =$

$$m(x_1, \ldots, x_n) = \int_A \lim_p \left[\int \cdots \int v(\sigma_1, \ldots, \sigma_n) d\rho_{x_1}^{\omega_1^p}(\sigma_1) \ldots d\rho_{x_n}^{\omega_n^p}(\sigma_n)\right] d\mu(v)$$

$$= \int_A v(x)\, d\mu(v).$$

Now if $m \in M$, then $m = \lambda_1 a_1 - \lambda_2 a_2$, $\lambda_1, \lambda_2 \geq 0$; $a_1, a_2 \in A$. Hence $\exists$ a signed Radon measure μ on A carried by $\mathcal{E}(A)$ such that if $x \in \Omega$

$$m(x) = \int_A v(x)\, d\mu(v).$$

There are two questions which one should consider next:

1) What are the extreme points of A?

2) Is there a unique Radon measure on A representing a given function $m \in M^+$?

We give below a partial answer to the first question, and we will continue the discussion of this question in Chapter IV.

Theorem 2.7. If b_i is an extreme point of the base B_i for the cone S_i^+, $i=1,\ldots,n$, then the function $b_i \otimes \ldots \otimes b_n \in \mathcal{E}(A)$ where for $x = (x_1, \ldots, x_n) \in \Omega$, $b_1 \otimes \ldots \otimes b_n(x) = b_1(x_1) \cdot \ldots \cdot b_n(x_n)$.

Proof: Let us consider $\Omega = \Omega_1 \times \Omega_2$, $b_i \in \mathcal{E}(B_i)$, $i=1,2$, and $m \in M^+$ such that $(b_1 \otimes b_2 - m) \in M^+$. Then for each $(x,y) \in \Omega$, $m(x,y) \leq b_1 \otimes b_2(x,y)$. Also, since $(b_1 \otimes b_2 - m) \in M^+$, if y is a fixed element in Ω_2, then $(b_2(y) \cdot b_1 - m(\ ,y)) \in S_1^+$. Since $b_1 \in \mathcal{E}(B_1)$, $m(\ ,y) = c_y \cdot b_2(y) \cdot b_1$, where c_y is a constant depending on y. Similarly one can show that for a fixed $x \in \Omega_1$, $m(x,\ ) = c_x \cdot b_1(x) \cdot b_2$, c_x a constant depending on x, since $b_2 \in \mathcal{E}(B_2)$. Now, if $(x,y) \in \Omega$, $c_x \cdot b_1(x) \cdot b_2(y) = m(x,y) = c_y \cdot b_1(x) \cdot b_2(y)$. Since $b_1(x)\, b_2(y) > 0$ for all $(x,y) \in \Omega$, we have that $c_x = c_y$ for all

$(x, y) \in \Omega$. Now, if $c_{y_1} \neq c_{y_2}$ for some $y_1, y_2 \in \Omega_2$, $y_1 \neq y_2$, then for $x \in \Omega_1$, since $(x, y_1), (x, y_2) \in \Omega = \Omega_1 \times \Omega_2$, $c_{y_1} = c_x = c_{y_2}$. Hence the function m is a constant multiple of $b_1 \otimes b_2$. One deduces easily from this the fact that $b_1 \otimes b_2 \in \mathcal{E}(A)$.

CHAPTER III

In this chapter we give a few results of general interest for ordered Hausdorff locally convex topological vector spaces. One of these, Theorem 3.2 will be used in Chapter IV.

Let (E, τ) be an Hausdorff locally convex topological vector space. A set $K \subset E$ is a convex cone if (i) $K+K \subset K$, (ii) $\lambda K \subset K$, $\lambda \geq 0$, (iii) $K \cap \{-K\} = \{0\}$. If K is a convex cone in E then K defines a partial ordering on E whereby if $x, y \in E$, $x \leq y$ if and only if $(y-x) \in K$. K is then called the positive cone.

Definition 3.1. Let K be a convex cone in E.

(i) K generates E if $E = K - K$.

(ii) B is a base for K if $B = \{x \in K \mid f(x) = 1\}$ where f is a strictly positive linear functional on E.

(iii) If τ is a topology on E for which (E, τ) is an Hausdorff locally convex topological vector space and K is τ-closed, then (E, τ, K) is an ordered Hausdorff locally convex topological vector space with convex cone K.

If $x, y \in E$, K a convex cone in E and $x \leq y$ (i.e. $(y-x) \in K$), then the set $I = \{z \in E \mid x \leq z \leq y\}$ is an order interval in E and is denoted by $[x, y]$.

Definition 3.2. If $x_0 \in K$ such that $E = \bigcup_{n=1}^{\infty} n[-x_0, x_0]$, then x_0 is an order unit in E.

If $E = \overline{K-K}$, K a convex cone in E, and E' is the continuous dual of E, then $K' = \{f \in E' \mid f(x) \geq 0,\ x \in K\}$ is a convex cone in E' called the dual cone. The ordering defined on E' by K' is called the dual ordering. In 1962, O.Hustad [9, p.83] proved the following result:

Let (E, τ) be an Hausdorff locally convex topological vector space with K a closed set contained in E and $K+K \subset K$, $\lambda K \subset K$, $\lambda \geq 0$. Then K' is a convex cone which is $\sigma(E', E)$ locally compact if and only if K has an order unit and every K-positive linear functional on E is continuous.

We give a somewhat similar result in Theorem 3.1. Let (E, τ, K) be an ordered Hausdorff locally convex topological vector space with convex cone K. Suppose E, F are in duality and that τ is compatible with the duality. Finally let $Q = \{y \in F \mid \langle x, y \rangle \geq 0,\ x \in K\}$.

<u>Theorem 3.1</u>. The convex cone K is $\sigma(E, F)$ locally compact if and only if $\exists\, y_0 \in F$ such that $y_0^{-1}(1) \cap K$ is $\sigma(E, F)$ complete and for each $y \in F$, there exists $n > 0$ such that $(ny_0 - y), (y + ny_0) \in Q$. (We remark here that if Q is itself a convex cone then such an element y_0 in F is an order unit for the ordering on F defined by Q.)

<u>Proof</u>: Since K is a τ-closed convex set, K is $\sigma(E, F)$ closed [14, p.34]. Then, the convex cone K has a $\sigma(E, F)$ compact base if and only if K is $\sigma(E, F)$ locally compact [11, p.188]. Suppose K has

a $\sigma(E,F)$ compact base B. Without loss of generality we may assume that $B = \{x \in K \mid \langle x_0, y_0\rangle = 1\}$ where $y_0 \in F$. We will show that y_0 has the desired properties. Clearly $y_0^{-1}(1) \cap K$ is $\sigma(E,F)$ complete. If $y \in F$, then there exists $n > 0$ such that $|\langle b, y\rangle| \leq n$ for all $b \in B$, since B is $\sigma(E,F)$ compact. Then, if $x \in K$, $-n\langle x, y_0\rangle \leq \langle x, y\rangle \leq n\langle x, y_0\rangle$. Therefore $(ny_0 - y), (y + ny_0) \in Q$.

Suppose $y_0 \in F$ such that $y_0^{-1}(1) \cap K$ is $\sigma(E,F)$ complete and for each $y \in F$ there is some $n > 0$ such that $(ny_0 - y), (y + ny_0) \in Q$. Clearly y_0 is nonnegative on K. If $x \in K$, then $\langle x, y_0\rangle = 0 \Rightarrow \langle x, y\rangle = 0$ for all $y \in F$. Hence $x = 0$. Therefore y_0 is a strictly positive linear functional on E and $B = \{x \in K \mid \langle x, y_0\rangle = 1\}$ is a base for K. The first assumption on y_0 states that B is $\sigma(E,F)$ complete. The second assumption on y_0 states that each y in F is bounded on B. Then B is a $\sigma(E,F)$ bounded set and hence $\sigma(E,F)$ precompact [14, p. 50]. This implies finally that B is $\sigma(E,F)$ compact and therefore that K is $\sigma(E,F)$ locally compact.

We will consider next, two ordered Hausdorff locally convex topological vector spaces $(E_1, \tau_1, K_1), (E_2, \tau_2, K_2)$ having convex cones K_1, K_2 respectively. Consider $E_1 \otimes E_2$, the tensor product of E_1, E_2, and the set $P \subset E_1 \otimes E_2$, $P = \{\sum_{i=1}^{m} x_i \otimes y_i \mid x_i \in K_1,\ y_i \in K_2 \text{ for each } i\}$. Peressini and Sherbert in [12, p. 183] show that P, which clearly satisfies the conditions $P + P \subset P$; $\lambda P \subset P$, $\lambda \geq 0$, is a convex cone if there is a strictly positive linear functional on E_i, i=1 or i=2. They also state that P generates $E_1 \otimes E_2$ if K_i generates E_i, i=1, 2. The projective topology on $E_1 \otimes E_2$, denoted by $E_1 \otimes_\pi E_2$, is the finest locally convex topology such that the canonical map $\varphi: E_1 \times E_2 \to E_1 \otimes E_2$ is continuous. The completion of $E_1 \otimes_\pi E_2$ is denoted by $E_1 \hat{\otimes}_\pi E_2$.

The following results are stated for the two spaces (E_1, τ_1, K_1), (E_2, τ_2, K_2), however they are true for finitely many variables and the proofs carry over directly. In the following P is the set described above.

<u>Proposition 3.1</u>. Suppose $E_i = K_i - K_i$, i=1, 2, and that the continuous dual E_i' has an order unit x_i' for the dual ordering. Then $x_1' \otimes x_2'$ is an order unit in $(E_1 \otimes_\pi E_2)'$ for the ordering given by P'.

<u>Proof</u>: Let $B_i = \{x \in K_i \mid x_i'(x) = 1\}$. Then B_i is a base for K_i which is $\sigma(E_i, E_i')$ bounded since x_i' is an order unit. Hence B_i is τ_i bounded. Then $B_1 \times B_2$ is bounded in the product topology on $E_1 \times E_2$, and hence $B_1 \otimes B_2$ is bounded in $E_1 \otimes_\pi E_2$. Now, if $z' \in (E_1 \otimes_\pi E_2)'$, then $\exists M > 0$ such that $|z'(B_1 \otimes B_2)| \le M$.

Therefore, if $b_1 \otimes b_2 \in B_1 \otimes B_2$, then $|z'(b_1 \otimes b_2)| \le M x_1' \otimes x_2'(b_1 \otimes b_2)$. If $p \in P$, then $p = \sum_{i=1}^{n} \alpha_i b_1^i \otimes b_2^i$, where $\alpha_i \ge 0$, $b_1^i \in B_1$, $b_2^i \in B_2$ for each i. Hence $p \in P$ implies $|z'(p)| \le M x_1' \otimes x_2'(p)$. Therefore $x_1' \otimes x_2'$ is an order unit in $(E_1 \otimes_\pi E_2)'$ for the ordering given by the dual cone P'.

Theorem 3.2. If $E = E_1 \hat{\otimes}_\pi E_2$, where $E_i = K_i - K_i$, and K_i has a compact base B_i, $i=1,2$, then

(i) $\overline{P}$ is a convex cone in E

(ii) $\overline{P}$ has a compact base $B = \overline{co}\, B_1 \otimes B_2$, the closed convex hull of $B_1 \otimes B_2$

(iii) $E = \overline{\overline{P} - \overline{P}}$.

Proof: (i) Clearly $\overline{P} + \overline{P} \subset \overline{P}$, and if $\lambda \geq 0$, $\lambda \overline{P} \subset \overline{P}$. Suppose $z \in E_1 \hat{\otimes}_\pi E_2$ and $z = \overline{P} \cap \{-\overline{P}\}$. Since K_i has a compact B_i we assume without loss of generality that $B_i = \{x \in K_i \,|\, x_i'(x) = 1\}$ where $x_i' \in E_i'$. Then by Theorem 3.1, x_i' is an order unit in E_i' for the ordering given by K_i'. Now, if $z_0' = x_1' \otimes x_2'$, $z \in \overline{P} \cap \{-\overline{P}\}$, $z_0'(z) = 0$. Since z_0' is order unit in $(E_1 \otimes_\pi E_2)'$ by Proposition 3.1, and since $(E_1 \otimes_\pi E_2)' = (E_1 \hat{\otimes}_\pi E_2)'$, for any $z' \in (E_1 \hat{\otimes}_\pi E_2)'$ $\exists$ $M > 0$ such that $|z'(p)| \leq M z_0'(p)$ for all $p \in P$. Hence $z'(z) = 0$ for all $z' \in (E_1 \otimes_\pi E_2)'$ and $z = 0$. Therefore $\overline{P} \cap \{-\overline{P}\} = \{0\}$ and $\overline{P}$ is a convex cone.

(ii) The set $B = \{z \in \overline{P} \,|\, z_0'(z) = 1\}$ is a base for $\overline{P}$ since z_0' is a strictly positive linear functional on $E_1 \hat{\otimes}_\pi E_2$. Clearly $co\, B_1 \otimes B_2 \subset B$. If $z \in B \cap P$, then $z_0'(z) = 1$ and $z = \sum_{i=1}^{n} \alpha_i b_1^i \otimes b_2^i$, where $\alpha_i \geq 0$, $b_1^i \in B_1$, $b_2^i \in B_2$ for each i. This implies $\sum_{i=1}^{n} \alpha_i = 1$, and hence $z \in co\, B_1 \otimes B_2$. Let $z \in \overline{P}$ and $z_0'(z) = 1$. Then there is a Cauchy net $\{z_\alpha\}_{\alpha \in A} \subset P$ such that $\lim_\alpha z_\alpha = z$. Now for each α, $z_\alpha = \sum_{i=1}^{n^\alpha} \beta_i b_1^i \otimes b_2^i$, where

$\beta_i \geq 0$, $b_1^i \in B_1$, $b_2^i \in B_2$ for each i and for each α. Since $z_0'(z) = 1$, $\lim_\alpha z_0'(z_\alpha) = 1$, and hence $\lim_\alpha(\sum_{i=1}^{n_\alpha} \beta_i) = 1$. Let $w_\alpha = z_\alpha/(\sum_{i=1}^{n_\alpha} \beta_i)$ for each α. Then $\lim_\alpha w_\alpha = \lim_\alpha z_\alpha / \lim_\alpha (\sum_{i=1}^{n_\alpha} \beta_i) = z$, and $w_\alpha \in \text{co}\, B_1 \otimes B_2$ for each α. Hence $z \in \overline{\text{co}}\, B_1 \otimes B_2$. Since B_1, B_2 are compact in E_1, E_2 respectively, $B_1 \otimes B_2$ is compact in $E = E_1 \otimes_\pi E_2$. However, E being a complete space implies that $B = \overline{\text{co}}\, B_1 \otimes B_2$ is also compact [14, p. 60].

(iii) Since $E_1 \otimes E_2 = P - P$, $E = E_1 \hat{\otimes}_\pi E_2 = \overline{P - P} \subset \overline{\overline{P} - \overline{P}} \subset E$. Hence $E = \overline{\overline{P} - \overline{P}}$.

CHAPTER IV

In this chapter we will consider another approach to the question of integral representation for multiply superharmonic functions.

Let Ω be a harmonic space of Brelot (viz. a harmonic space satisfying Axioms I, II, III, IV) with positive potential. In Chapter I, we defined a topology τ on $S = S^{+}-S^{+}$, where S^{+} is the cone of positive superharmonic functions on Ω. The topology τ was defined as the weakest topology on S such that elements of the set Σ are continuous. Recall that Σ is the linear span of the set of all linear functionals on S of the form (ω, x), $\omega \in \mathcal{B}$, a base of completely determining regular domains for Ω, $x \in \omega$. We would like to consider the topology $\bar{\tau}$ as a weak topology on S with respect to some duality. It can happen that for some $z_i \in \Sigma$, $i=1, \ldots, n$, $\sum_{i=1}^{n} \alpha_i z_i(s) = 0$ for all $s \in S$ where the α_i's are real but not all zero. For example, let $\Omega = (0, 1)$, the open unit interval in R', $\mathcal{B} = \{(a, b) \mid 0 < a < 1, 0 < b < 1; a, b$ rational numbers$\}$, $\omega = (1/4, 3/4)$. Since the harmonic functions on this space Ω are linear functions, for each $s \in S$, $x \in \omega$, $(\omega, x)(s) = mx + b$, where m, b depend only on s and ω. If $x_1 = 1/3$, $x_2 = 1/2$, $x_3 = 2/3$, $\alpha_1 = 1/2$, $\alpha_2 = -1$, $\alpha_3 = 1/2$, then $\sum_{i=1}^{3} \alpha_i = 0$, $\sum_{i=1}^{3} \alpha_i x_i = 0$. Then for all $s \in S$,

$$\sum_{i=1}^{3} \alpha_i (\omega, x_i)(s) = m \sum_{i=1}^{3} \alpha_i x_i + b \sum_{i=1}^{3} \alpha_i = 0.$$

We will see below that this difficulty can easily be avoided, however, it is interesting to note that if $\Omega \subset R^n$, $n \geq 2$, Ω an open domain with positive potential (for the solutions of the Laplace equation), then this problem does not occur. We will show briefly why this is so, since it does not play a part in what is to follow.

Let $\Omega \subset R^n$, $n \geq 2$, Ω an open domain with positive potential. Suppose $\mathcal{B}$ is a countable base of open spheres in Ω. Then $\mathcal{B}$ is a countable base of completely determining regularopen domains for Ω. Let $(\omega_1, x_1), (\omega_2, x_2) \in \Sigma$ such that either $x_1 \neq x_2$ or $x_1 = x_2$ but $\omega_1 \neq \omega_2$. If $x_1 \neq x_2$, let $y \in \Omega - \omega_1 \cup \omega_2$ such that y is not on the perpendicular bisector of the line segment joining x_1 and x_2. If $\Omega \subset R^2$, then the function $v(x) = \log(|x-y|)^{-1}$ for all $x \in \Omega$ is an element of S such that $(\omega_1, x_1)(v) \neq (\omega_2, x_2)(v)$. If $\Omega \subset R^n$, $n > 2$, then the function $v(x) = 1/\|x-y\|^{n-2}$ for $x \in \Omega$ is an element of S^+ and $(\omega_1, x_1)(v) \neq (\omega_2, x_2)(v)$. If $x_1 = x_2$, but $\omega_1 \neq \omega_2$, then $\exists y \in \partial\omega_1$, $y \notin \partial\omega_2$. Let $\delta \in \mathcal{B}$ such that $\delta \cap \partial\omega_2 = \phi$ and $y \in \delta$. Since there is a positive potential on Ω we can choose a function $v \in S^+$ which is not harmonic on δ. The function $\bar{v}$, identical to v on $C(\delta)$, the complement of δ, and equal to $\int v \, d\rho_x^{\delta}$ for $x \in \delta$, is an element of S^+, and $(v - \bar{v}) > 0$ on δ. Since $\delta \cap \partial\omega_1$ is an open set in $\partial\omega_1$, it has positive $d\rho_{x_1}^{\omega_1}$-measure, and hence $(\omega_1, x_1)(v-\bar{v}) = \int (v-\bar{v}) \, d\rho_{x_1}^{\omega_1} > 0$. However, $(\omega_2, x_2)(v - \bar{v}) = \int (v-\bar{v}) \, d\rho_{x_2}^{\omega_2} = 0$ because $v = \bar{v}$ on $\partial\omega_2$. Therefore if $(\omega_1, x_1), (\omega_2, x_2) \in \Sigma$ are geometrically distinct, then they are distinct linear functionals on S.

Now, let $z \in \Sigma$, $z = \sum_{i=1}^{m} \alpha_i(\omega_i, x_i)$. It is possible that for some i, j, $i \neq j$, $\omega_i = \omega_j$. Let us rewrite z, collecting together those ω_i's which are identical. Then z can be written as follows:

$$(1) \qquad z = \sum_{i=1}^{m_1} \alpha_i^{\,1}(\omega_1, x_i^1) + \ldots + \sum_{i=1}^{m_p} \alpha_i^{\,p}(\omega_p, x_i^{\,p}),$$

where for any fixed k, $x_i^{\,k} \neq x_j^{\,k}$ if $i \neq j$.

We recall here that if $(\omega, x) \in \Sigma$ and $v \in S$, then $\int v\, d\rho_x^{\omega}$ is the Poisson integral of v evaluated at x, i.e., if ω is a sphere with center at y and radius r, then

$$\int v\, d\rho_x^{\omega} = 1/S(\omega) \int_{\partial\omega} [(r^2 - \|x-y\|^2)/\|x-z\|^n] v(z)\, d\mu(z),$$

where $S(\omega)$ is the surface area of ω and μ is the surface measure on $\partial\omega$.

Proposition 4.1. Let $z \in \Sigma$ such that $z(v) = 0$ for all $v \in S$. Then if z is expressed as in (1) above, for any j, $1 \leq j \leq p$,

$$\sum_{i=1}^{m_j} \alpha_i^{\,j}(\omega_j, x_i^{\,j})(v) = 0$$

for all $v \in S$.

Proof: Suppose $j = 1$ and let $\nu = \sum_{i=1}^{m_1} \alpha_i^{\,1}(\omega_1, x_i^{\,1})$. Then ν is the difference of two positive Radon measures on $\partial\omega_1$, say $\nu = \nu_1 - \nu_2$. We shall show that if $\exists\, v \in S$ such that $\sum_{i=1}^{m_1} \alpha_i^{\,1}(\omega_1, x_i^{\,1})(v) \neq 0$, then $z(v) \neq 0$ for all $v \in S$.

Let $v \in S$ such that $\sum_{i=1}^{m_1} \alpha_i^1(\omega_1, x_1)(v) \neq 0$. Then as measures on $\partial\omega_1$, $\nu_1 \neq \nu_2$, and $\exists$ some open set $V_1 \subset \partial\omega_1$ such that $\nu_1(V_1) \neq \nu_2(V_1)$. Suppose ω_1 is a sphere with center y, radius r, then

$$\nu_1(V_1)-\nu_2(V_1) = 1/S(\omega_1) \int_{V_1} [\sum_{i=1}^{m_1} \alpha_i^1(r^2 - \|x_i^1 - y\|^2)/\|x_i^1 - z\|^n] \cdot 1 \, d\mu(z),$$

where 1 is the function identical to one on $\partial\omega_1$. Let

$$f(z) = \sum_{i=1}^{m_1} \alpha_i^1(r^2 - \|x_i^1 - y\|^2)/\|x_i^1 - z\|^n.$$

Since $\nu_1(V_1)-\nu_2(V_1) \neq 0$, $\exists$ some $z_1 \in V_1$ such that $f(z_1) \neq 0$, say $f(z_1) > 0$. The function f is continuous on $\partial\omega_1$, hence $\exists$ an open set $V_2 \subset \partial\omega_1$, $z_1 \in V_2 \subset V_1$, such that $f(z) > 0 \;\; \forall z \in V_2$. Now let $X = \partial\omega_1 \cap \bigcup_{i=2}^{p} \partial\omega_i$. Then the set X is a set of ν-measure zero in $\partial\omega_1$, hence $\exists z_2 \in V_2$ such that $z_2 \notin X$. Let $\delta \in \mathcal{B}$ such that $\delta \cap \bigcup_{i=2}^{p} \partial\omega_i = \phi$ and $z_2 \in \delta \cap \omega_1 \subset V_2$. Choose a function $v \in S^+$ such that v is not harmonic on δ and let $\bar{v}$ be the function identical to v on $C(\delta)$ and equal to $\int v \, d\rho_x^{\delta}$ for $x \in \delta$. Since $(v-\bar{v}) > 0$ on δ and $(v-\bar{v}) \equiv 0$ on $C(\delta)$, $\sum_{i=1}^{m_1} \alpha_i^1(\omega_1, x_i^1)(v-\bar{v}) =$

$$1/S(\omega_1) \int_{\delta \cap \partial\omega_1} [\sum_{i=1}^{m_1} \alpha_i^1(r^2 - \|x_i^1 - y\|^2)/\|x_i^1 - z\|^n](v-\bar{v})(z) d\mu(z) =$$

$$1/S(\omega_1) \int_{\delta \cap \partial\omega_1} f(z) \cdot (v-\bar{v})(z) \, d\mu(z).$$

For all $z \in \delta \cap \partial\omega_1 \subset V_2$, $f(z)>0$ and $(v-\bar{v})(z)>0$, hence $\sum_{i=1}^{m_1} \alpha_i^1(\omega_1, x_i^1)(v-\bar{v})>0$. However, for j, $2 \le j \le p$, $\sum_{i=1}^{m_j} \alpha_i^j(\omega_j, x_i^j)(v-\bar{v}) = 0$ since $v = \bar{v}$ on $\partial\omega_j$. Therefore $z(v-\bar{v}) \neq 0$ which is the desired contradiction.

Proposition 4.2. Let $z \in \Sigma$, $z = \sum_{i=1}^{m} \alpha_i(\omega, x_i)$, $\omega \in \mathcal{B}$, $x_i \in \omega$ $\forall i$, and $x_i \neq x_j$ for $i \neq j$. Then $z(v) = 0$ for all $v \in S$ implies $\alpha_i = 0$ for each i.

Proof: Let us consider $\Omega \subset R^2$. Let $|x_i| = r_i$, $i=1,\ldots,m$, and $r = \max_{1 \le i \le m} \{r_i\}$. If C is the circle of radius r with center at the origin, then for some i, $1 \le i \le m$, x_i is on the circumference of C. Suppose $i=1$, and let y be the point on the circumference of C diametrically opposite x_1. If we relocate the origin at the point y, then $|x_1| = 2r$, and for $i=2,\ldots,m$, $|x_i| \le \gamma < 2r$. For each positive integer n, the function $f(x) = x^n$ for $x \in \Omega$ is a holomorphic function on Ω which can be written in the form

$$f(x) = h_1(x)+ih_2(x),$$

where h_i is a harmonic function, $i=1,2$. Since $z(v) = 0$ $\forall v \in S$ and $h_i \in S$, $i=1,2$, $\sum_{i=1}^{m} \alpha_i x_i^n = 0$ for each positive integer n. Then

$$\sum_{i=1}^{m} \alpha_i x_i^n / |x_1|^n = \alpha_1 \cdot x_1^n / |x_1|^n + \sum_{i=2}^{m} \alpha_i x_i^n / |x_1|^n = 0.$$

However, for each positive integer n,

$$\left|\sum_{i=2}^{m} \alpha_i x_i^{\,n}/|x_1|^n\right| \le \sum_{i=2}^{m} |\alpha_i|(|x_i|/|x_1|)^n \le (\gamma/2v)^n \sum_{i=2}^{m} |\alpha_i|,$$

and as $n\to\infty$, $(\gamma/2v)^n \to 0$. Therefore $\lim_{n\to\infty} |\alpha_1\ x_1^{\,n}/|x_1|^n| = \lim_{n\to\infty} |\alpha_1| = 0$ and $\alpha_1 = 0$. It is clear that in this way, one can show $\alpha_i = 0$ for each i.

The case where $\Omega \subset R^n$, $n > 2$ can easily be reduced to the above situation.

As a result of Lemmas 4.1, 4.2, in the case where $\Omega \subset R^n$, $n \ge 2$, Ω an open domain with positive potential, if $z \in \Sigma$, and $z(v) = 0$, for all $v \in S$ then z is the zero linear functional on S. We have already shown in Chapter I that if $v \in S$, $v \ne 0$, then $\exists\ z \in \Sigma$ such that $z(v) \ne 0$. Hence in this particular case $\langle S, \Sigma \rangle$ is a duality.

Let us return to the general case where Ω is simply a harmonic space of Brelot with positive potential. We define an equivalence relation R on the elements of Σ whereby, $z_1 \sim z_2$ if $z_1(s) = z_2(s)$ for all $s \in S$, $z_i \in \Sigma$, $i = 1, 2$. Let Σ' be the set of equivalence classes thus formed and define addition and scalar multiplication in the usual way, that is

$$[z_1] + [z_2] = [z_1 + z_2],$$

$$\lambda[z_1] = [\lambda z_1],$$

where $[z_i]$ is the equivalence class containing $z_i \in \Sigma$, λ any real number. Then the real vector space Σ' can be considered as a set of linear functionals on S if we define $[z](s) = z(s)$ for $z \in \Sigma$ and $s \in S$. The value of $[z](s)$ is uniquely determined for if $[z_1]=[z_2]$, then $z_1(s) = z_2(s)$ for all $s \in S$ by the equivalence relation R. The following proposition is more or less obvious.

Proposition 4.3. $\langle S, \Sigma' \rangle$ is a duality and $\tau = \sigma(S, \Sigma')$, the weak topology on S with respect to the duality.

Proof: If $s_i \in S^+$, $i=1,2$, $x \in \Omega$ such that $s_1(x) \neq s_2(x)$, then by Property 4, page 4, $\exists\ z = (\omega, x) \in \Sigma$ such that $z(s_1) \neq z(s_2)$. Therefore $[z](s_1) \neq [z](s_2)$ for $[z] \in \Sigma'$. If $[z] \in \Sigma'$ such that $[z](s) = 0$ for all $s \in S$, then $z(s) = 0$ for all $s \in S$, hence z is equivalent to the zero linear functional on S, and $[z]$ is the zero element of Σ'. Hence $\langle S, \Sigma' \rangle$ is a duality.

Let V be an arbitrary element in the base of zero neighborhoods for the $\sigma(S, \Sigma')$ topology. Then

$$V = \{ s \in S \mid |[z_i](s)| \leq 1,\ [z_i] \in \Sigma',\ i=1, \ldots, n \}.$$

Trivially, $V = V' = \{ s \in S \mid |z_i(s)| \leq 1,\ z_i \in \Sigma,\ i=1, \ldots, n \}$. However, any element in the base of zero neighborhoods for the τ-topology is of the form V', hence it is clear that $\tau = \sigma(S, \Sigma')$.

Now, let us consider $\Omega = \prod_{i=1}^{n} \Omega_i$, where for each i, Ω_i is a harmonic space of Brelot with positive potential, $M = M^+ - M^+$, where M^+ is the cone of positive multiply superharmonic functions on Ω. Let $T' = \Sigma_1' \otimes \dots \otimes \Sigma_n'$ and $\mathcal{S} = S_1 \otimes \dots \otimes S_n$. As a result of a well known theorem in topological vector spaces [14, p. 132], since $\langle S_i, \Sigma_i' \rangle$ is a duality for each i by the above proposition, $\langle \mathcal{S}, T' \rangle$ is also a duality. Now, as mentioned in Chapter III, page 42, the fact that $S_i = S_i^+ - S_i^+$ for each i, implies that the vector space $\mathcal{S} = S_1 \otimes \dots \otimes S_n$ is generated by the convex cone

$$Q = \{ \sum_{j=1}^{m} s_1^j \otimes \dots \otimes s_n^j \mid s_i^j \in S_i^+ \text{ for all } i, j \}.$$

Proposition 4.4. $\mathcal{S}$ can be embedded in M in such a way that $Q \hookrightarrow M^+$.

Proof. If $s_i \in S_i^+$, $i = 1, \dots, n$, then for $x = (x_1, \dots, x_n) \in \Omega$, the function $s_1 \otimes \dots \otimes s_n(x) = s_1(x_1) \cdot \dots \cdot s_n(x_n)$ is clearly nonnegative, hyperharmonic in each variable separately, hence it is lower semicontinuous [6, page 34]. Since s_i is superharmonic for each i, $s_1 \otimes \dots \otimes s_n$ cannot be $\equiv +\infty$ on any connected component. Therefore $s_1 \otimes \dots \otimes s_n \in M^+$, and $\mathcal{S}$ can be embedded in M so that $Q \hookrightarrow M^+$.

The set T′ can be considered as a set of linear functionals on M as follows. If $m \in M$, $z_i = (\omega_i, x_i) \in \Sigma_i$, then let

$$\begin{aligned}[z_1] \otimes \dots \otimes [z_n](m) &= z_1 \otimes \dots \otimes z_n(m) \\ &= \int \dots \int m(\sigma_1, \dots, \sigma_n)\, d\rho_{x_1}^{\omega_1}(\sigma_1) \dots d\rho_{x_n}^{\omega_n}(\sigma_n).\end{aligned}$$

This number is uniquely determined by the way in which the equivalence relation was defined on Σ_i, $i=1, \ldots, n$. Since each element of T' is some finite linear combination of linear functionals of this form, $T' \subset M^*$, the algebraic dual of M.

Proposition 4.5. $\langle M, T' \rangle$ is a duality and the $\sigma(M, T')$ topology on M is equal to the γ topology (defined in Chapter II, p. 14).

Proof: In Chapter II, Theorem 2.1, we showed that if $m_1, m_2 \in M^+$, $m_1(x) \neq m_2(x)$ for some $x = (x_1, \ldots, x_n) \in \Omega$, then $\exists\, \omega_i$, $i=1, \ldots, n$ such that $x_i \in \omega_i$ for each i, and

$$[(\omega_1, x_1) \otimes \ldots \otimes (\omega_n, x_n)](m_1 - m_2) \neq 0.$$

If $z_i = (\omega_i, x_i)$ for each i, then $[z_i] \in \Sigma_i'$ and

$$[z_1] \otimes \ldots \otimes [z_n](m_1 - m_2) = (z_1 \otimes \ldots \otimes z_n)(m_1 - m_2) \neq 0.$$

Now, if $z \in T'$, $z(m) = 0$ for all $m \in M$, then $z(s) = 0$ for all $s \in \mathcal{S}$. Since $\langle \mathcal{S}, T' \rangle$ is a duality, $z = 0$, and therefore $\langle M, T' \rangle$ is a duality.

To show that $\gamma = \sigma(M, T')$ one simply observes the following. If $\sum_{j=1}^{m} z_1^{\,j} \otimes \ldots \otimes z_n^{\,j}$, $z_i^{\,j} \in \Sigma_i$ for all i, j, is an arbitrary element of T, let $F\left(\sum_{j=1}^{m} z_1^{\,j} \otimes \ldots \otimes z_n^{\,j}\right) = \sum_{j=1}^{m} [z_1^{\,j}] \otimes \ldots \otimes [z_n^{\,j}]$. Then F is an onto map from T to T' such that for $z \in T$, $m \in M$, $z(m) = [F(z)](m)$. Recall that $T = \Sigma_1 \otimes \ldots \otimes \Sigma_n$. Then if V is an arbitrary element of the base of zero neighborhoods for $\sigma(M, T')$,

$$V = \{m \in M \mid |[F(z_i)](m)| \leq 1, \ z_i \in T, i=1, \ldots, n\}$$

$$= \{m \in M \mid |z_i(m)| \leq 1, \ z_i \in T, \ i=1, \ldots, n\}$$

$$= V'.$$

Now V' is a γ neighborhood of zero and any element of the base of γ neighborhoods of zero is of the same form as V'.

Clearly $\sigma(M, T') = \gamma$ on M.

<u>Proposition 4.6.</u> $\mathcal{S}$ is $\sigma(M, T')$ dense in M, hence $M \subset \widehat{(\mathcal{S}, \sigma)}$, the completion of $\mathcal{S}$ with the $\sigma(M, T')$ topology.

<u>Proof</u>: Since $\langle \mathcal{S}, T' \rangle, \langle M, T' \rangle$ are dualities and $\mathcal{S} \subset M$, by Propositions 4.1, 4.2, 4.3 we have that $\mathcal{S}$ is $\sigma(M, T')$ dense in M. [10, p.237]

Let $[z_i] \in \Sigma_i'$, i=1, ..., n, and for $m \in M$, let $P_{[z_1]\otimes\ldots\otimes[z_n]}(m) = |[z_1]\otimes\ldots\otimes[z_n](m)|$. Then this defines a seminorm on M, and $\sigma(M, T')$ is generated by the family of seminorms $\mathcal{P} = \{P_{[z_1]\otimes\ldots\otimes[z_n]} / [z_i] \in \Sigma_i'$ for each i$\}$, because T' is the linear span of the set of all elements of the form $[z_1]\otimes\ldots\otimes[z_n]$, $[z_i] \in \Sigma_i'$ for each i. In addition to the $\sigma(M, T')$ topology on $\mathcal{S}$, there is another topology π, called the projective topology, which is formed by considering the spaces S_i with the $\sigma(S_i, \Sigma_i')$ topology. The topology π is defined as the finest Hausdorff locally convex topology on $\mathcal{S}$ such that the canonical map $\varphi: S_1 \times \ldots \times S_n \to \mathcal{S} = S_1 \otimes \ldots \otimes S_n$ is continuous.

For each i, the $\sigma(S_i, \Sigma_i')$ topology on S_i is generated by the family of seminorms $\{P_{[z]} \mid [z] \in \Sigma_i'\}$ where $P_{[z]}(s) = |[z](s)|$ for $s \in S_i$, $[z] \in \Sigma_i'$. If $[z_i] \in \Sigma_i'$, i=1, . . . , n, $s \in \mathbb{S}$, let

$$P_{[z_1]} \otimes \ldots \otimes P_{[z_n]}(s) = \inf\{ \sum_{j=1}^{m} P_{[z_1]}(s_1^j) \cdot \ldots \cdot P_{[z_n]}(s_n^j) \mid s = \sum_{j=1}^{m} s_1^j \otimes \ldots \otimes s_n^j \}.$$

Then the family of seminorms $\mathfrak{G} = \{P_{[z_1]} \otimes \ldots \otimes P_{[z_n]} \mid [z_i] \in \Sigma_i'$ for each i$\}$ generates the π topology on $\mathbb{S}$ [7, p.31].

Theorem 4.1. The two families of seminorms, $\mathbb{P}$, $\mathfrak{G}$, are identical on $\mathbb{S}$.

Proof: We will prove this for the case of $\mathbb{S} = S_1 \otimes S_2$ and the method carries over directly to finitely many variables.

Let $[z_i] \in \Sigma_i'$, i=1, 2, and $s \in \mathbb{S}$. If $s = \sum_{j=1}^{m} s_1^j \otimes s_2^j$, $s_i^j \in S_i$ for all i, j, then

$$\begin{aligned} P_{[z_1] \otimes [z_2]}(s) &= \left| \sum_{j=1}^{m} [z_1](s_1^j) \cdot [z_2](s_2^j) \right| \\ &\leq \sum_{j=1}^{m} |[z_1](s_1^j)| \cdot |[z_2](s_2^j)| \\ &= \sum_{j=1}^{m} P_{[z_1]}(s_1^j) \cdot P_{[z_2]}(s_2^j). \end{aligned}$$

Hence, on $\mathbb{S}$,

(1) $$P_{[z_1] \otimes [z_2]} \leq P_{[z_1]} \otimes P_{[z_2]}.$$

Now, let $\{s^i\}_{i\in I}$, $\{t^j\}_{j\in J}$ be basis for the kernel of $[z_1]$, kernel of $[z_2]$ respectively. Assume that $1 \notin I$, $1 \notin J$, and let $s^1 \in S_1 - \text{Ker}[z_1]$, $t^1 \in S_2 - \text{Ker}[z_2]$ where $\text{Ker}[z_i]$ is the kernel of z_i. Let $\{1\} \cup I = I'$ and $\{1\} \cup J = J'$. Then $\{s^i\}_{i\in I'}$, $\{t^j\}_{j\in J'}$ are basis for S_1, S_2 respectively, and $s \in \mathcal{S}$ implies $s = \sum_{i,j} \alpha^{i,j} s^i \otimes t^j$ where this is a finite sum. Then,

$$P_{[z_1]\otimes[z_2]}(s) = \left| \sum_{i,j} \alpha^{i,j}[z_1](s^i)\cdot[z_2](t^j)\right|$$

$$= |\alpha^{1,1}[z_1](s^1)\cdot[z_2](t^1)|,$$

since $[z_1](s^i)\cdot[z_2](t^j) = 0$ if $i \neq 1$ or $j \neq 1$. Similarly,

$$P_{[z_1]}\otimes P_{[z_2]}(s) \le \sum_{i,j} |\alpha^{i,j}|\, P_{[z_1]}(s^i)\cdot P_{[z_2]}(t^j)$$

$$= \sum_{i,j} |\alpha^{i,j}|\,|[z_1](s^i)|\,|[z_2](t^j)|$$

$$= |\alpha^{1,1}[z_1](s^1)\cdot[z_2](t^1)|.$$

This together with (1) above imply that

$$P_{[z_1]\otimes[z_2]} = P_{[z_1]}\otimes P_{[z_2]} \quad \text{on } \mathcal{S}.$$

As a result of the last theorem, $(\widehat{\mathcal{S},\sigma})$, the completion of $\mathcal{S}$ with respect to the $\sigma(M, T')$ topology, is uniformly isomorphic to $(\widehat{\mathcal{S},\pi})$ the completion of $\mathcal{S}$ with respect to π. Therefore we can apply Theorem 3.2 to obtain the following result.

Theorem 4.2. The space M can be embedded in $(\widehat{\mathcal{S}, \pi})$ in such a way that,

(i) the two topologies γ, π coincide on M,

(ii) the convex cone Q has the property that Q-Q is dense in M; $\overline{\bar{Q}-\bar{Q}} = (\widehat{\mathcal{S}, \pi}) \supset M$; $M^+ \supset \bar{Q}$,

(iii) $\bar{Q}$ has a compact metrizable base C and $C = \overline{co}\, B_1 \otimes \ldots \otimes B_n$, the closed convex hull of $B_1 \otimes \ldots \otimes B_n$.

Proof: By Proposition 4.6, $M \subset (\widehat{\mathcal{S}, \sigma})$. By Proposition 4.5, the γ topology coincides on M with the $\sigma(M, T')$ topology. Since $(\widehat{\mathcal{S}, \sigma})$ is uniformly isomorphic to $(\widehat{\mathcal{S}, \pi})$ by the remarks after Theorem 4.1, $M \subset (\widehat{\mathcal{S}, \pi})$ and γ coincides on M with π. By Proposition 4.4, $\mathcal{S} = Q-Q$ is $\sigma(M, T')$ dense in M, hence Q-Q is π dense in M. Since M^+ is γ complete by Theorem 2.4, hence π complete, $Q \subset M^+$ implies $\bar{Q} \subset M^+$. The remaining part of (ii) and (iii) except for the metrizability of C follow directly from Theorem 3.2. That C is metrizable is easily seen. The base A for M^+ was $\{m \in M^+ \mid z_0(m) = 1\}$ where $z_0 = [(\omega_1^0, x_1^0) \otimes \ldots \otimes (\omega_n^0, x_n^0)]$ and for each i, (ω_i^0, x_i^0) is the linear functional which generates the base B_i for S_i^+. Then if $m = b_1 \otimes \ldots \otimes b_n$, $b_i \in B_i$ for each i, $z_0(m) = 1$. Hence $B_1 \otimes \ldots \otimes B_n \subset A$, and A being closed, convex implies $C = \overline{co}\, B_1 \otimes \ldots \otimes B_n \subset A$. Since $A \subset M^+$ is γ metrizable and γ coincides with π on M^+, A is π metrizable. Thus $C \subset A$ implies C is π metrizable. The proof is complete.

A natural question to ask at this point is whether the cone $\bar{Q}$ could be equal to M^+. We will examine this question by considering the extreme points of the base C for $\bar{Q}$ and of the base A for M^+.

Proposition 4.7. If $C = \overline{co}\, B_1 \otimes \ldots \otimes B_n$, then

$$\mathcal{E}(C) = \{b_1 \otimes \ldots \otimes b_n \mid b_i \in \mathcal{E}(B_i) \text{ for each } i\}.$$

Proof: We have already shown in Theorem 2.8 that any element of the form $b_1 \otimes \ldots \otimes b_n$, $b_i \in \mathcal{E}(B_i)$ is an extreme point of A. Since elements of this form are in C as well and $C \subset A$, we have

$$\mathcal{E}(C) \supset \{b_1 \otimes \ldots \otimes b_n \mid b_i \in \mathcal{E}(B_i) \text{ for each } i\}.$$

Since B_i is compact in S_i, $\sigma(S_i, \Sigma_i')$, for each i, $B_1 \otimes \ldots \otimes B_n$ is π compact in $\mathcal{S}$ by definition of π. By the Milman Theorem [10, p.332], $\mathcal{E}(C) \subset B_1 \otimes \ldots \otimes B_n$. Suppose $b_i \in B_i$ for each i, and that $b_1 \notin \mathcal{E}(B_1)$, say $b_1 = \lambda a_1 + (1-\lambda) a_2$, $a_1 \neq b_1$, $a_1, a_2 \in B_1$, $0 < \lambda < 1$. Then

$$\begin{aligned} b_1 \otimes \ldots \otimes b_n &= (\lambda a_1 + (1-\lambda) a_2) \otimes b_2 \otimes \ldots \otimes b_n \\ &= \lambda a_1 \otimes b_2 \ldots \otimes b_n + (1-\lambda) a_2 \otimes b_2 \otimes \ldots \otimes b_n . \end{aligned}$$

Clearly, if $b_1 \neq a_1$, $b_i \neq 0$ for any i,

$$b_1 \otimes \ldots \otimes b_n \neq a_1 \otimes b_2 \otimes \ldots \otimes b_n,$$

and $b_1 \otimes \ldots \otimes b_n \notin \mathcal{E}(C)$. The proof is complete.

Now, as a result of the above Proposition and Theorem 2.8, $\mathcal{E}(C) \subset \mathcal{E}(A)$. The example[1)] below will show that $\mathcal{E}(A)$ is not necessarily contained in $\mathcal{E}(C)$, hence one cannot conclude that $\bar{Q} = M^+$ in general.

Let $\Omega_i = (0,1)$ the unit interval in R^1, $i=1,2$, $\mathcal{B}_i = \{(a,b) \mid 0<a\leq b<1,$ a, b rational$\}$. The harmonic function on Ω_i are the linear functions. If h is a minimal positive harmonic function on Ω_i, then either $h(x) = kx$, k a constant >0, or $h(x) = \ell(1-x)$, ℓ a constant >0. If p is an extremal potential on Ω_i, then $p(x) \to 0$ as $x \to 0$, and $p(x) \to 0$ as $x \to 1$. Now, let $\Omega = \Omega_1 \times \Omega_2$, $S_i = S_i^+ - S_i^+$ where S_i^+ is the cone of positive superharmonic functions on Ω_i, M^+ be the cone of positive multiply superharmonic functions on Ω.

<u>Theorem 4.3</u>. There is a function $v \in M^+$ such that v is an extreme generator of M^+ but v is not a tensor product of extreme generators of the cones S_i^+, $i=1,2$.

<u>Proof</u>: Let v be a function defined on $\Omega_1 \times \Omega_2$ such that $v(x,y) = \inf\{x+y, 1-x, 1-y\}$. Then $v \in M^+$, and the graph of v is given in Figure (1)

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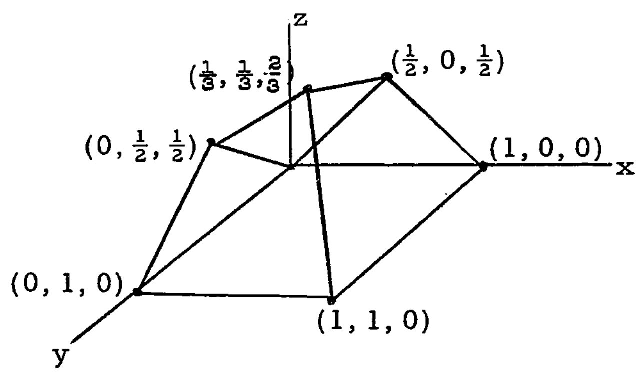
1) This example was suggested to me by Professor Carl Herz, McGill University.

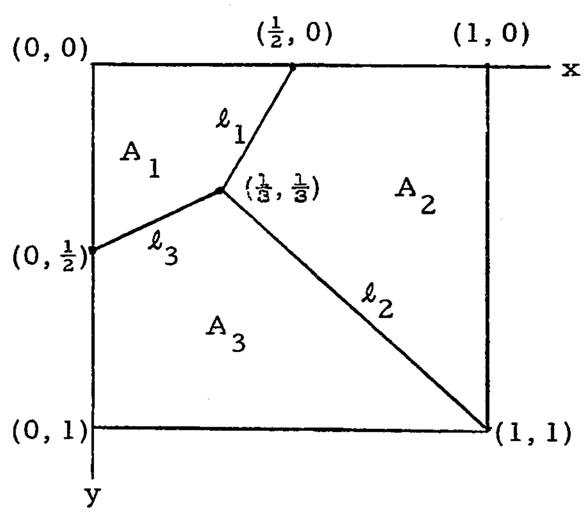
Figure (1)



If p_i is an extremal potential on Ω_i, i=1, 2, then $p_1 \otimes p_2(\frac{1}{2}, y) \to 0$ as $y \to 0$. Since $v(\frac{1}{2}, y) \to \frac{1}{2}$ as $y \to 0$, it is clear that v is not a tensor product of two extremal potentials. If h_i is a minimal positive harmonic function on Ω_i, p_i an extremal potential on Ω_i, i=1, 2, then by examining the behaviour of $h_1 \otimes h_2$, $h_1 \otimes p_2$, $p_1 \otimes h_1$, close to the boundary of the unit square, one can easily see that v cannot be a tensor product of elements in S_1^+, S_2^+.

Let $m \in M^+$ such that $(v-m) \in M^+$. Then for $(x, y) \in \Omega_1 \times \Omega_2$, $m(x, y) < v(x, y)$. Let us consider the open sets A_1, A_2, A_3 and the lines ℓ_1, ℓ_2, ℓ_3 illustrated in Figure (2).

Figure (2)



Then on A_1, $v(x,y) = x+y$; on A_2, $v(x,y) = 1-x$; on A_3, $v(x,y) = 1-y$. Since on A_1 $v(x,y)$ is multiply harmonic, m is multiply superharmonic, we have $m-v$ is multiply superharmonic and $v-m \in M^+ \Rightarrow v-m$ is multiply harmonic and hence m is multiply harmonic on A_1. This implies here that in A_1 for a fixed y, $m(\ ,y)$ is linear and for a fixed x, $m(x,\ )$ is linear. Therefore $m(x,y) = ax+by+c$ for some constants $a, b, c \geq 0$. Note however, $m(x,y) \leq v(x,y)$ implies $c = 0$. In a similar way we can deduce that on A_2 $m(x,y) = d(1-x)$; on A_3, $m(x,y) = e(1-y)$. Suppose that along the line ℓ_1, the planes defining m on A_1 and A_2 do not meet at some point (x_0, y_0). Then for y_0 fixed we have two choices given in Figures (3), (4), for the graph of $m(\ ,y_0)$ in the x-z plane.

Figure (3)

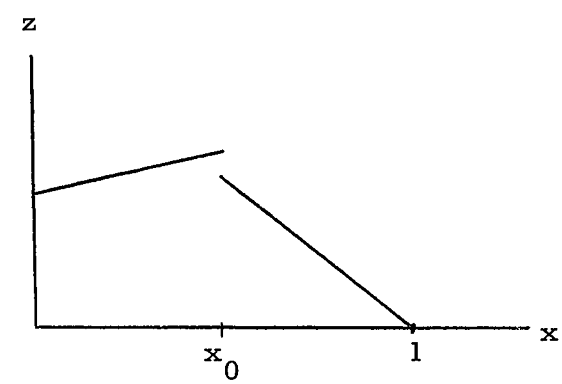
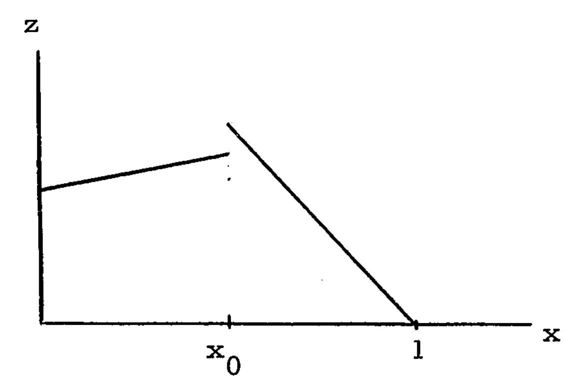


Figure (4)



Since m must be lower semicontinuous on $\Omega_1 \times \Omega_2$, $m(\ ,y_0)$ must be lower semicontinuous. Therefore in Figure (3), $m(x_0, y_0) \le d(1-x_0)$ and in Figure (4), $m(x_0, y_0) \le ax_0 + by_0$. In either case, the function $(v-m)(\ ,y_0)$ will not be lower semicontinuous at x_0. Since this contradicts the fact that $(v-m) \in M^+$, the planes defining m on A_1 and A_2 must meet along the line ℓ_1. Similarly along the lines ℓ_2 and ℓ_3. Then $m(x,y) = \inf\{ax+by, d(1-x), e(1-y)\}$. Now, the following equations must be satisfied:

(1) $d(1-1/3) = e(1-1/3)$,

(2) $a(1/2)+b(0) = d(1-1/2)$

(3) $a(0)+b(1/2) = e(1-1/2)$.

By (1), $d = e$; by (2), $a = d$; by (3) $b = e$. Hence $m(x,y) = r \inf\{x+y, (1-x), (1-y)\}$ where $0 \le r < 1$.

We conclude therefore that v is an extreme generator in the cone M^+. The proof is complete.

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