F I X E D P O I N T T H E O R E M S.

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FIXED POINT THEOREMS.

The problems whose history is the subject of this thesis is as follows. One wishes to obtain criteria for the existence or non-existence of points in a topological space which remain fixed under a transformation of the space into itself. Such criteria will of course be expressed in terms of the properties of the space and of the mapping. One desires further that the properties of the space be topological invariants while those of the mapping be invariant under homotopy.

Two main attacks have been made on the problem, by L. E. J. Brouwer in 1910 and by S.Lefschetz in 1925. Presentation of the results of the work of these two mathematicians forms the main part of the thesis. A less detailed account is also given of some of the lesser known researches in the subject. The thesis concludes with selected applications particularly in the field of Functional Analysis.

CONTENTS

		Page.
ACKNOWLEDGEMENT.		1
INTRODUCTION.		1
CHAPTER 1:	The Brouwer Degree.	2
CHAPTER 2:	Contimuous Vector Fields on n-dimensional spheres.	11
CHAPTER 3:	Single-Valued Continuous Transformations of n-Spheres into Themselves.	17
CHAPTER 4:	Subsequent treatments of the Brouwer Theorems.	20
CHAPTER 5:	The Lefschetz Fixed Point Formula.	25
CHAPTER 6:	Applications.	30

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INTRODUCTION.

The problem stated in its most general terms is this. Given a space S and a mapping t of S into itself, under what circumstances is there or is there not a fixed point? By a fixed point we mean, simply, a point x in S for which $tx=x$. Most investigations are concerned with single valued continuous mappings and the trend has been to search for ever larger classes of spaces for which fixed point conditions can be derived. Studies have, however, been made notably by M. H. A. Newman (15) and P. A. Smith (17) of the question of periodic transformations i.e. transformations some power of which is the identity. We shall not, however, be concerned with these. Two main attacks have been launched on the problem. Our first three chapters consist of a faithful account of the first of these due to L. E. J. Brouwer (4). This paper is regarded as the classical paper on the subject. The fourth chapter deals with subsequent discussions of the Brouwer theorems. In the fifth chapter we give an account of the second main attack begun by S. Lefschetz (10) and taken up by H. Hopf (6). The concluding chapter is devoted to selected applications. We may mention here the close relation between the question of fixed points and existence questions in analysis. For suppose one seeks a solution to an equation $\Delta f = 0$ where f is an element of some function space and Δ an operator on that space. The existence of a solution is equivalent to the existence of a fixed point under the mapping $(\Delta + I)$ where I is the identity for then $(\Delta+I)f = f$ i.e. $\Delta f = 0$. We have attempted, where possible, to lay emphasis on the methods employed and to point out crucial steps in a proof, in the hope that this paper may prove instructive rather than merely informative.

CHAPTER 1. The Brouwer Degree.

We wish to define an n-dimensional manifold in the sense of L. E. J. Brouwer and in order to do so we first introduce some preliminary notions.

A simplex star in an n-dimensional Euclidean space is a finite collection of non-overlapping simplices with a common vertex, O, the union of which is a finite neighbourhood of O and each pair of which have a common p-face ($o \leq p < n$) but no other common points.

By an n-dimensional element, E, we mean a homeomorphic image of a simplex, S, in an n-dimensional Euclidean space. The images of the vertices and faces of S are taken as the vertices and faces of E.

An n-dimensional manifold is a connected set which is made up of a set of n-dimensional elements each pair of which have no point in common or a single p-face together with its proper faces in common but no other points. The elements having a vertex in common are related in the same way as the simplices of a Euclidean simplex star defined above.

Such is the definition as given by Brouwer. We may state it briefly thus: An n-dimensional manifold is a Euclidean complex which is such that the star of each vertex is isomorphic with a set of simplices in E_n having a common vertex P and constituting a neighbourhood of P in E_n.

Concerning the properties of the manifold we observe immediately that if it is made up of a finite number of elements it is closed with respect to fundamental sequences. On the other hand if there are infinitely many elements the manifold is certainly not closed in view of the fact that the infinite covering by elements is locally finite (i.e. an arbitrary point of the manifold has a neighbourhood which intersects only finitely many elements.)

Fixing our attention now on a closed n-dimensional manifold we shall show that it is a compactum, which means that we must prove, in addition to the compactness just mentioned, that it is metric. In order to do this we will introduce a continuous, normalized, non-negative, homogeneous coordinate system from whicha metric will follow. A coordinate system will be introduced into each element but in such a way that for different elements the systems will coincide for points on their common faces. Let us see just how such a system can in fact be introduced.

Let the coordinates of a side $A_p A_q$ be denoted by U_p and U_q with the U's positive and the ratio u_p/u_q decreasing continuously from ∞ at A_p to 0 at Aq. To determine the coordinates of the points of a 2-face $A_p A_q A_r$ we map it topologically onto a plane Euclidean triangle FGH in such a way that the sides of FGH correspond to the faces ApAq, etc. of $A_p A_q A_r$. We represent the points of FGH by homogeneous barycentric coordinates with respect to F, G and H and select an arbitrary point, O, inside the triangle. Let B' be the image in FGH of a point in $A_p A_q A_r$. The straight segment OB' produced beyond B' meets some side, say GH, in C'. Let C be the inverse image of C' in $A_q A_r$; ${}_cU_q$, ${}_cU_r$ its coordinates and C" the point in GH with barycentric coordinates (0, cUq, cUr). The line through B' parallel to GH will meet OC" in some point B". We choose as the coordinates of B the barycentric coordinates of B".

In an analogous manner the coordinates of the points of the 3-faces are obtained with the aid of the barycentric coordinates of a homeomorphic Euclidean tetrahedron. The process is continued until coordinates of all the points of the manifold are assigned with the aid of those of the (n-1)-faces and the barycentric coordinates of homeomorphic n-dimensional Euclidean simplices.

We may now associate with each element a homeomorphic Euclidean simplex with edges of fixed length and whose barycentric coordinates with respect to its vertices are the same as the normal coordinates of the corresponding points in the element. We shall refer to it as the representative simplex. By a subsimplex or a plane p-dimensional subregion of an element we shall simply mean the image of a corresponding subset of its representative.simplex. Similarly, by a straight segment and its length in an element we mean the image of a straight segment and its length in the representative simplex. We can now define the distance between points of the manifold as the minimum of the lengths of the straight segmental paths joining them. Such notions as the centroid of weighted points and the volume of a subregion can be introduced for an element with the aid of its representative simplex and the homeomorphism.

The "indicatrix of an element" is defined, uniquely up to even permutations, as a sequence of its vertices. Only two indicatrixes are thereby possible. One of these is arbitrarily taken as positive, the other negative. Through the choice of the positive indicatix of an element the positive indicatix of its representative simplex is also determined. Hence also those of each subsimplex and thus also those of the faces of the element.

We consider a finite closed series of elements each consecutive pair of which have n vertices in common and choose a positive indicatrix for some element. This choice determines the negative indicatrix of the succeeding element by means of the n vertices already arranged and the new one placed in the position where it is first lacking. The positive indicatrix of each element in the sequence is thus fixed by the choice for one element. If on returning to this initial element the same positive indicatrix is induced and if this is the case for all closed sequences we call the manifold "two-sided", otherwise, "one-sided". The choice of positive indicatrix for a single element in a two-sided metric n-dimensional manifold fixes the choice for each element and each face so that we can speak of "the positive indicatrix of the manifold".

We now turn to consideration of single-valued continuous mappings of one two-sided manifold into another. The method of studying such a mapping will be by means of approximating simplicial mappings and will lead eventually to an important property of the mapping called by Brouwer its degree and by whose name it is now designated.

Let μ, μ', be two n-dimensional closed metric two-sided manifolds and let $\alpha: \mu \to \mu'$ be single-valued and continuous. We decompose each element of μ into a finite number of subsimplices each two of which either have no common point or have a common p-face $(0 \leq p < n)$ and then also the lower dimensional faces in it but no other common points. Let ε be the least upper bound of the diameters of the subsimplices that is to say the mesh of the simplicial decomposition described. Let us denote the decomposition by ζ and refer to the subsimplices, their faces, and vertices as the base simplices etc. of ζ. We choose a positive indicatrix for μ and therewith for the base simplices of ζ also: Among the subsimplices we distinguish those which aremapped completely by α into one element of μ' and refer to them as general base simplices.

Let us now define what is meant by a simplicial mapping β belonging to ζ which approximates α. β is applied only to the general base simplices and in the following way. Let $\pi = A_1 \ldots A_{n+1}$ be a general base simplex of μ and $\pi' = \alpha(\pi) = B_1 \ldots B_{n+1}$ be the image of π under α lying of course entirely in a single element of μ'. Then $\beta(A_i) = B_i$ and $\beta(\Sigma \lambda_i A_i) = \Sigma \lambda_i B_i$ i.e. β is a linear mapping of the points of π with respect to their barycentric coordinates relative to the A_i. If the B_i do not line in a plane (n-1) dimensional region we shall take the volume of π' to be positive or negative according as its indicatrix is positive or negative while if π' is singular its volume is 0 . For $P \in \pi$, the distance $\overline{\alpha(P)\beta(P)}$ in π' has an upper bound e which along with ε decreases on each boundary.

We consider further a modified simplicial mapping γ belonging to $\mathfrak{k}$ and approximating α. γ is applicable to those base simplices of μ whose images have vertices not more than ε apart. γ is determined for these simplices in the same way as β was determined on the general base simplices. These latter are not necessarily admissible for γ.

We choose now in each element of μ' an inner simplex i.e. a subsimplex whose edges do not meet the edges of the element. ε can now be so chosen that each point of μ mapped by one of β and γ into an inner simplex belongs to a suitable subsimplex of μ with respect to both β and γ. This last subsimplex must then of course be mapped into the same element of μ'. We consider now a mapping γ which yields no singular immage simplex. Applying γ to an innersimplex J of a particular element E, let σ denote the set of points of the interior $J^\circ \subset J$ which do not lie in the image of an $(n-2)$ base face. Clearly, any two points of σ can be joined by a path of finitely many segments lying wholly in σ. The set η of points of σ which do not lie in the image of an $(n-1)$-face we call general points of J. Let P_1 and P_2 be arbitrarily fixed points and P a variable point in η. Each point of η is covered by a number of image simplices, some positive, some negative. Let the numbers of positive and negative simplices covering P be denoted by p and p' respectively and let p_1, p_1'; p_2, p_2' be these numbers corresponding to P_1 and P_2 .

Confining P to a segmental path s lying in σ which joins P_1 to P_2 , the numbers p and p' can change only when the image, τ, of an $(n-1)$-face is crossed. In fact, when the two image simplices meeting in τ lie on different sides of τ they have the same sign and the crossing of τ leaves p, p' and hence also $p-p'$ unaltered. If, however, the image simplices lie on the same side of τ they have opposite sign so that ρ and ρ' are each either increased or decreased by 1 and again p-p$'$ is unaltered. Thus $\rho_1-\rho_1'=\rho_2-\rho_2'$ and, for the general points of J under the mapping γ, $\rho-\rho'$ is a constant.

Similar numbers p and p$'$ may be defined with respect to the mapping, β. Let P_1 and P_2 in J, be general points with respect to β and suppose that $\rho_1-\rho_1'\neq\rho_2-\rho_2'$. β may be approximated so closely by a map γ_ν of type γ that P_1 and P_2 are general points of J under γ_ν while ρ_i, ρ_i' have the same values relative to γ_ν as to β. Then p-p$'$ would not be constant for γ_ν either which is a contradiction. Hence p-p$'$ is constant for the general points of β also.

We shall prove that this constant, call it C, is the same for all simplicial maps approximating α. Let β_2' be a simplicial map with underlying decomposition ξ_2' of μ and let β_1' be one whose underlying decomposition ξ_1' is a refinement of ξ_2'; i.e. the base simplices of ξ_1' are obtained by simplicial decomposition of those of ξ_2'. The images under β_1' of the base points of ξ_1' can be translated continuously along the shortest connecting path into the corresponding images under β_2' if the latter exist so that at least where J is concerned the mapping β_2' can be transformed continuously into β_1'. The total volume of the parts of the image simplices contained in J can have no discontinuities under such a continuous transformation, therefore, insofar as this is C times the volume of J and C is an integer, it must remain fixed. That is to say, the integer C for a given simplicial map is invariant under a simplicial refinement of that map. This fact will enable us to prove the assertion made at the beginning of this paragraph.

Let β_1 and β_2 be two arbitrary simplicial maps approximating α with underlying decompositions ξ_1 and ξ_2 of μ. We consider a decomposition, ξ_3, of ξ_1 of such small mesh that under the corresponding mapping β_3, it is possible to choose a general point P in J which is a general point relative to β_2 also. The vertices of each image simplex under β_3 covering P are contained in each such image under β_2. But them there will belong to each P-covering image simplex under β_2 one and only one P-covering simplex for β_3 and indeed each pair of image simplices corresponding in this way have the indicatrix in the same sense. We therefore conclude that C which was just shown the same for β_1 and β_3 is also the same for β_2 and β_3 and hence also for β_1 and β_2.

Our next step is to prove that the values C_1 and C_2 which C takes on in the inner simplices J_1 and J_2 of two elements, E_1 and E_2 with a common $(n-1)$-face, S are equal. As Euclidean antecedents of E_1 and E_2 we choose regular simplices T_1 and T_2 which are mirror images of each other with respect to a common $(n-1)$-face Σ and such that Σ corresponds to S. The definitions of subsimplex p-dimensional region, barycentre and volume of T_1+T_2 are carried over to E_1+E_2 in precisely the same way as was done for a single element. We determine in E_1 and E_2 subsimplices U_1 and U_2 containing J_1 and J_2 and having a common $(n-1)$-face lying in S but whose boundaries do not meet the remaining faces of E_1 and E_2. Those simplices whose vertex images under α all lie in the interior of E_1+E_2 we count in with the general base simplices. We determine a simplicial mapping β' approximating α which may be applied to these additional simplices also. To β' there again belong modified simplicial mappings γ'. We may now choose ε so that relative to U_1+U_2 and the general simplices in the new sense the same property, earlier imposed for the inner simplices and general simplices again remains valid. By the method used above we can deduce from this that for β' the number $p-p'$ is constant on the general points of U_1+U_2. But this constant may be identified with C_1 belonging to β' and J_1 and with C_2 belonging to β' and J_2 from which it follows that C has the same value in J_1 and J_2. By a simple extension we could show that C is constant in all the elements of μ'.

This number, C, constant in all the elements of μ' and furthermore having the same value for all simplicial mappings approximating α thus characterizes a property of α. We refer to it as "the degree of the single-valued continuous mapping α." A fundamental property of the degree is that it is the same for two single-valued continuous mappings of μ into μ' which can be continuously transformed into each other. That is, <u>the degree depends only on the homotopyclass!</u>

Let us approximate two homotopic continuous mappings by simplicial mappings each pair of which have the same underlying decomposition. We can then introduce between each pair of these mappings a finite sequence of simplicial mappings based on the same decomposition for which each is a map of the type β above; each consecutive pair differ only in the images of one of the base points and then by as little as one pleases. Then there will certainly exist an inner simplex of μ' in which each consecutive pair determine the same image set and hence the same value of C. This value of C will be unaltered along the sequence of mappings considered which means that the original mappings have the same degree.

Remark: In order to see that any positive or negative integer can arise as the degree of a mapping one has only to consider the mappings of one sphere onto another as represented by the rational functions of a complex variable. In fact the degree of such a mapping is identical with the degree of the function representing it.

Finally we notice that when $\alpha(\mu)$ is not everwhere dense in μ' the degree of α must be zero. For in such a case there will exist an approximating simplicial mapping β for which we can select an inner simplex J in μ'

in which the image set is not everywhere dense. There will then exist a subregion J^* of J for which $J^* \frown \beta(\mu) = \Lambda$. Hence p , p′ and consequently C are zero there.

Hitherto we have considered only the case when μ' is two-sided and closed. However when μ' is one-sided and closed the above results still hold both with respect to a single inner simplex and with respect to a pair contained in elements with a common $(n-1)$-face. Let us consider a closed sequence of elements in which the indicatrix is reversed. Since on the one hand C must remain constant and on the other hand its sign changes in traversing the complete cycle we can only conclude that it must be zero. Again, if μ' is open we can, since $\alpha(\mu)$ is closed in μ', specify a finite collection μ'' of elements which both for α and the various β and γ contains completely the image of μ which is however not everywhere dense in μ''. This last fact together with the fact that the previous considerations are again valid for a single inner simplex or a pair contained in adjacent elements leads again to the conclusion that the degree of α must be zero.

To sum up: When a closed, two-sided, metric n-dimensional manifold μ is mapped single-valuedly and continuously into a metric n-dimensional manifold μ' there exists an integer C called the degree of the mapping which is the same for all homotopic mappings and which designates the number of times the image of μ covers positively each subregion of μ'. In particular, if μ' is one-sided or open, C is equal to zero.

CHAPTER 11. Continuous Vector Fields on n-dimensional spheres.

In what follows the term "sphere" will be reserved for spherical surfaces while solid spheres we shall call "disks".

We consider a n-dimensional sphere K in a Euclidean space R_{n+1} of dimension $(n+1)$. This may be represented in a right angled Cartesian coordinate system by the equation: $\sum_{h=1}^{n+1} x_h^2 = 1$. K may be regarded as a metric n-domensional manifold as defined above by taking, as its elements the 2^{n+1} simplices into which it is decomposed by the hyerplanes $x_h = 0$, and as its p-dimensional regions the subregions of the p-disks lying in it. As the unit point of normal coordinates in each element we choose the points with coordinates $\pm\sqrt{n^{-1}}$. A positive indicatrix is chosen for one of the elements and hence also for each spherical simplex. (A spherical simplex is made up of $n+1$ different $(n-1)$-disks and lies entirely in one hemisphere of K.)

Through each point, P, of K passes an $(n-1)$ dimensional direction sphere λ_P which can be handled precisely as K itself. The positive indicatrix of λ_P is made to depend on that of K in the following way. Let s be an $(n-1)$ dimensional spherical simplex lying in λ_P. We determine in K an n-dimensional simplex S having P as a vertex and its $(n-1)$-faces through P determined by the $(n-2)$ faces of s with the remaining one arbitrary. We then write the positive indicatrix of S with P in the last place. The others in this arrangements determine a specific order of the edges of S meeting in P hence also of the vertices of s . This last arrangement is chosen as the positive indicatrix of s .

We consider a continuous vector field in K in which we are concerned only with the directions but not the magnitudes of the vectors.

Let this field posses only a finite number of singularities, i.e. points on the sphere at which the continuity of the vector direction is disturbed. Let K be decomposed by means of an $(n-1)$ dimensional great sphere (generalization of great circle) x which contains no singular points into hemispheres H_1 and H_2 with poles π_1 and π_2. We consider x as belonging to both H_1 and H_2. The hemispheres may themselves be decomposed into finitely many spherical simplices S_{11}, S_{12}, ... in H_1 and S_{21}, S_{22},... in H_2 by means of singularityfree $(n-1)$ great spheres. In this way the decomposition of H_1 will be the mirror image in x of that of H_2.

Let σ be an $(n-1)$ -face of some particular $S_{\alpha\beta}$. From the positive indicatrix of $S_{\alpha\beta}$ we introduce that for σ when considered as a face of $S_{\alpha\beta}$. Writing the vertices in a positive indicatrixof $S_{\alpha\beta}$ in such a way that the one not in σ comes last the others determine the positive indicatrix of σ considered as a face of $S_{\alpha\beta}$. Simultaneously a positive indicatrix is fixed for the whole boundary of $S_{\alpha\beta}$in that $S_{\alpha\beta}$may be regarded as a two-sided closed $(n-1)$ dimensional manifold whose elements are the $(n-1)$-faces of $S_{\alpha\beta}$ and whose p-dimensional regions are the subregions of the p-disks lying in it.

We now project the circumference $U_{\alpha\beta}$of $S_{\alpha\beta}$together with the vectors on it through a point Q of K outside $S_{\alpha\beta}$ onto the n-dimensional hyperplane θ tangent to K at the point O diametrically opposite Q. This determines a single-valued continuous map of $U_{\alpha\beta}$ onto the sphere of directions in θ whose positive indicatrix is determined by that of λ_0. We shall establish that the degree of this mapping does not depend on the choice of Q. For let P be a point in $U_{\alpha\beta}$ then under this stereographic projection λ_P is placed in a congruence relation with the direction sphere of θ and hence also with any other sphere λ_R, $R \in U_{\alpha\beta}$. This congruence relation $\mathfrak{b}_{PR}$ between λ_P and λ_R; $P, R \in U_{\alpha\beta}$ arises in the following manner. Let V be some direction belonging to λ_P. Together with Q and R it determines a 2 -sphere ℓ lying in K and on which P, Q and R determine a circle (1-sphere) k. There will exist a direction through R belonging to ℓ which determines makes the same angle with k as V. This direction corresponds to V for the relation b_{PR} .

By continuous translation of Q through a finite distance from P and R this relation can only change continuously. Thus by a continuous translation of Q through a finite distance from $S_{\alpha\beta}$ the whole system of congruence relations between the $(n-1)$ dimensional direction spheres of the points of $U_{\alpha\beta}$ can only change continuously. Hence the degree of the mapping of $U_{\alpha\beta}$ onto the direction sphere of θ (i.e. the mapping determined by the vector field) can admit no discontinuities. It must therefore be a constant, $C_{\alpha\beta}$ say, which we may call the degree of $S_{\alpha\beta}$. We wish to evaluate the sum $\sum_{\alpha,\beta} C_{\alpha\beta}$.

Let us project H_1 and H_2 respectively through Π_2 and Π_1 onto the n-dimensional hyperplanes Θ_1 and Θ_2 tangent to K at Π_1 and Π_2 and evaluate the sum of the degrees of the resulting mappings of $U_{1\beta}$ onto the direction sphere of Θ_1. In this sum each such $(n-1)$-face of an $S_{1\beta}$ which does not lie in x occurs twice but with opposite indicatrixes thus cancelling each other's contributions. We need therefore consider ohly those faces σ_x which lie in x.

We can regard x as a two-sided manifold with the σ_x as elements. Its positive indicatrix is given by that of a σ_x considered as a side of an $S_{1\beta}$. The projection of x onto Θ_1, has degree C_1 to which the contributions of σ_x are the same as they make to $C_{1\beta}$. Hence $C_1 = \sum C_{1\beta}$. But take the indicatrix of x now to be concordant with that of the σ_x considered as sides of $S_{2\beta}$ and project x from Π_1 onto Θ_2. As before the degree C_2 of this map is $\sum C_{2\beta}$.

Let ρ be the n-dimensional hyperplane in R_{n+1} which contains x . By reflection in ρ we have the following correspondences. The projection x_2

of x in θ_2 corresponds to x_1 the projection of x in θ_1 but with opposite indicatrix. The direction sphere of θ_2 corresponds to that of θ_1 again with opposite indicatrix. Finally, the vector distribution in X_2 corresponds to the reflected distribution in X_1, i.e. that distribution over the image points in θ_1. The problem of evaluating $\sum C_{\kappa\beta}$. i.e. C_1+C_2 is thus reduced to the question: What is the sum of the degrees of the mappings., δ and ρ of an $(n-1)$-sphere $\mathfrak{z}$ onto the direction sphere λ of an n-dimensional hyperplane θ as determined by a continuous vector distribution over the points of $\mathfrak{z}$ and by the same distribution reflected in the plane of $\mathfrak{z}$?

By means of an $(n-1)$-hyperplane we determine in $\mathfrak{z}$ an $(n-2)$-disk and poles q_1 and q_2. We consider a sequence $\Sigma_1', \Sigma_2' \ldots \Sigma_m' \ldots$ of simplicial decompositions of $\mathfrak{z}$ for which q_1 is not contained in any proper face and whose mesh decreases with increasing m. Denote by ζ_m that simplex of Σ_m' which contains q_1 and by u_m the boundary of ζ_m.

For each m we construct from δ a new single valued continuous mapping δ_m: $\mathfrak{z}$ onto λ as follows. Let j be an arbitrary great semicircle in $\mathfrak{z}$ joining q_1 to q_2 and h its point of intersection with u_m. The arc q_1 h is mapped by magnification onto the whole of j. Suppose a point $F \in q_1 h$ is thereby mapped into a point F' then δ_m is defined thus:- For $F \in q_1 h$, $\delta_m F = \delta F'$ but if $F \in j$ but $F \notin q_1 h$, $\delta_m F = \delta q_2 = g$, say.

The reflection mapping (i.e. the one belonging to the reflected distribution) of δ_m we denote by ρ_m. Since δ can be deformed continuously into δ_m and ρ into ρ_m the basic sums of δ and ρ are the same as those of δ_m, ρ_m . Through further simplicial decomposition of the base simplices of Σ_m' we arrive at a simplicial decomposition Σ_m of $\mathfrak{z}$ whose associated simplicial maps δ_m' and ρ_m' approximating δ_m and ρ_m have the properties of β in Chapter 1.

We next week the contribution to the sum of the degrees of δ_m and ρ_m made by the image sets under δ_m' and ρ_m'. Let us denote by g' the reflexion vector of the direction g at the point $q_1 \in \mathfrak{z}$. The number $\rho - \rho'$ is constant outside a neighbourhood of g' whose diameter decreases with increasing m both for $\delta_m' \zeta_m$ and $\rho_m' \zeta_m$. The latter of these two constants is unaltered if we modify $\rho_m' \zeta_m$ in such a way that as the image of each vertex P_1 of ξ_m lying in ζ_m to which the direction e_p corresponds under δ_m the reflected vector of e_p in q, takes the place of the reflected vector of e_p in P. But the result of this modification is that $\rho_m' \zeta_m$ becomes the mirror image in ζ of $\delta_m' \zeta_m$. In particular, two such mirror images have opposite indicatrixes so that the two image sets covering two points of λ which are mirror images of each other determine opposite values of $\rho-\rho'$. Thus the images of ζ_m under δ_m' and ρ_m' have opposite values everywhere outside a neighbourhood of q' of decreasing diameter and so destroy the two-part contribution to the sum of the degrees of δ_m, ρ_m

There remains for us yet to determine the contribution which the images under δ_m' and ρ_m' of a certain residual set t_m of ζ_m in $\mathfrak{J}$ makes to the degree-sum of ρ_m and δ_m. If the image $\delta_m' \zeta_m$ reduces to the single point q, there is no contribution made. Thus, for the image under ρ_m' outside a neighbourhood of q' which decreases with increasing m, the degree of the mapping of $\mathfrak{J}$ onto λ determined by the reflected distribution of a constant vector through the points of $\mathfrak{J}$ is constant.

To ascertain this degree we denote: by x_1 that point of $\mathfrak{J}$ whose radius is opposite the constant vector; by x_2 the point in $\mathfrak{J}$ diametrically opposite x_1; by ω the $(n-1)$-dimensional great-sphere having x_1 and x_2 as poles; and by a_1 and a_2 the halves of $\mathfrak{J}$ determined by ω containing x_1 and x_2 respectively. We can decompose a_1 simplicially so that the images of the simplices with positive indicatrix cover λ once only and positively. Furthermore let us decompose a_2 into simplices diametrically opposite those of a_1 and let us equip these with indicatrixes diametrically opposite those of the corresponding simplices in a_1. The resulting indicatrix is positive for even n and negative for odd n. Thus, two corresponding simplices in a_1 and a_2 determine the same image simplex in λ with positive indicatrix. Equipping the simplices of a_2 with a positive indicatrix their images cover λ again just once and positively for even n, negatively for odd n.

Hence the degree we seek is equal to 2 for even n and to 0 for odd n. Thus, also, the sum of the degrees for δ_m and ρ_m, for δ and ρ and finally for $C_{\alpha\beta}$, is 2 for even n and 0 for odd n.

When the vector field on K has no singularities its continuity is uniform. We can then choose $S_{\alpha\beta}$ so small that the variation in the directions of the stereographically projected vectors of the same $U_{\alpha\beta}$ onto Θ_1 and Θ_2 does not exceed a number chosen arbitrarily small. This would mean that $C_{\alpha\beta}$ is zero, which cannot occur if n is even. We have thus shown that a continuous vector field on a sphere of even dimension contains at least one singular point.

CHAPTER 111. Single-Valued Continuous Transformation of n-Spheres into Themselves.

Let us consider a single-valued continuous mapping, τ, of an n-sphere K into itself which has no fixed points. K may be covered by simplices, such as $S_{\alpha\beta}$ in Chapter 11, with arbitrarily small mesh. Let us consider the corresponding simplicial transformations which approximates τ. There certainly exists such a simplical transformation, t, which has no fixed point. But there can also be found a sequence of approximating simplicial maps which converge to τ and which leave the points $F_1, F_2, \dots$ say, fixed. Then, however, every limit point of this set would be a fixed point of τ.

We choose a point, O, of K, which is general with respect to t and which does not lie on any proper face. We join each point, P, of K, to its image tP by the arc of a circle through O and affix to P the vector determined by the directed arc $\overset{\frown}{PtP}$ not containing O. The result is a vector field on K whose only singularities occur at the finite number of points which t takes into O and at O itself.

We select a positive indicatrix in K and denote by $S_1, \dots S_p$ those simplices whose images $_bS_1, \dots, {}_bS_p$ cover O positively and by $S_1', \dots S_{p'}'$ those whose images $_bS_1', \dots, {}_bS_{p'}'$ cover O negatively. Let S'' be that simplex in which O is situated. We may assume the decomposition of K underlying t so dense that $S \cap tS = \Lambda$ for all S, and in particular $S'' \cap S_\alpha = \Lambda$ and $S'' \cap S_\alpha' = \Lambda$.
To determine the degrees of S_α and S_α' we project K steriographically from O onto the hyperplane Θ tangent to K at the point O' diametrically opposite O. Denote the projection of S_α by C_α, the projection of its boundary by $\mathcal{U}_\alpha$ and the projections of the boundaries of $_bS_\alpha$ by $_b\mathcal{U}_\alpha$. Now the image indicatrix

of bU_α in Θ belongs to a negative indicatrix of a simplex bC'_α bounded by bU_α. The vectors through the points $\beta \in U_\alpha$ to which the points $\beta' \in bU_\alpha$ correspond are determined through the rectilinear connecting segments $\beta\beta'$. Through a uniformly continuous transformation these are carried over into those vectors which through each point β are parallel to the straight segments $O'\beta'$. But the latter vectors cover $\lambda_{O'}$ exactly once and negatively so that the degree of S_α is -1. Through similar argument the degree of each S'_α is $+1$. The degree of S'' is found as follows. We continuously deform the vector distribution on the boundary T'' of S'' into that in which the vector through $P \in T''$ is determined by the great-circular arc OP which does not contain O'. Let us then project T'' and its vector field through O' onto the n-dimensional hyperplane Θ' tangent to K at O. Let U'' be the image of T'' in Θ'. Thereby, at each point $\beta \in U''$ the vector is determined by the straight segment $O\beta$. Thus the vector distribution on U'' covers λ_O just once and with positive indicatrix so that the degree of S'' is $+1$. As regards the remaining simplices they may be so decomposed that under stereographic projection the variation of the vector direction inside each is arbitrarily small. Hence each such simplex has degree zero and this is also true for such simplices joined together.

Adding up, the sum of the degrees of the simplices is $-p+p'+1$ which for even n must be 2 and for odd n, 0. Hence the degree p-p$'$ for r or t is -1 for even n and 1 for odd n. We have thus proved the following:

Theorem 1.

A single-valued continuous transformation of an n-sphere into itself which has no fixed point has degree -1 for even n and 1 for odd n. From this we can formulate the following special cases:

Corollary 1.

When under a single-valued continuous transformation of an n-sphere into itself the image set is not everywhere dense in the whole sphere there exists a fixed point.

Corollary 2.

Every homeomorphic mapping of an even dimensional sphere onto itself which can be deformed into the identity has a fixed point.

Corollary 3.

Every homeomorphism of a sphere of odd dimension into itself which can be deformed into a reflection must have a fixed point.

That transformations of degree $(-1)^{n+1}$ need not have fixed points may be illustrated by the rotations and reflections of R_{n+1} about a particular point.

We turn now to consider a single-valued continuous transformation $\mathcal{L}$ of an n-dimensional element E into itself. E can be regarded as the homeomorphic image of a hemisphere H_1 determined in an n-sphere K by an $(n-1)$ dimensional great-sphere x . A single-valued continuous transformation of E into itself thus corresponds to a like transformation $\mathcal{L}_1$ of H_1 into itself. Let us now extend $\mathcal{L}_1$ to the other half $H_2 \subset K$ in such a way that each pair of points which are mirror images in x are taken by $\mathcal{L}_1$ into the same point of H_1. Then this will be a single valued continuous transformation of K into itself for which the image set is not everywhere dense in K and which must therefore have a fixed point, lying of course in H_1 . But this fixed point must correspond to a fixed point of E under $\mathcal{L}$.
We have thus proved:

Theorem 2.

A single-valued continuous transformation of an n-dimensional element into itself must possess a fixed point.

It is this theorem which is to-day known as "The Brouwer Fixed Point Theorem".

Chapter IV. Subsequent treatments of the Brouwer Theorems.

The next contributions to the subject appear in two papers by B. von Kérékjarto (8,9) in December 1918. In the first of these considerations are restricted to the 2-disk and the 2-sphere. As regards the 2-disk, it is proved that a homeomorphism τ into itself has at least one fixed point. Two classical theorems are used for the proof, namely Brouwer's theorem on the "invariance of domain" (5) and the Jordan-Brouwer "separation theorem". Applied to the closed set of points, H, for which the polar angles $\varphi(P) = \varphi(\tau P)$, the first theorem demands that a point on the perimeter is mapped into the perimeter so that if H has points on the perimeter they must be fixed points. The second theorem is used to establish, in the event H is entirely inside the disk, the existance of a subcontinuum $K \subset H$ containing the centre O and the point P_0 mapped into O. One then considers the single-valued function $s(P) = r(\tau P) - r(P)$ where $r(P)$ is the length of the radius vector of P. We have $s(P) = -r(P_0)$ for $P = P_0$ and $s(P) = r(\tau O)$ for $P = O$, i.e. a negative and positive value of $s(P)$. Hence there must be a point, P_1 , for which $s(P_1) = O$ i.e. $r(\tau P_1) = r(P_1)$ but P_1 being in H_1, $\varphi(\tau P_1) = \varphi(P_1)$ so that P_1 must be a fixed point. Similar considerations are used to prove that a sense-preserving homeomorphism of the 2-sphere has a fixed point and that a sense-preserving mapping of a 2-disk onto a subregion has a fixed point.

The second paper establishes criteria for the existence under homeomorphism of fixed points in a non-simply connected closed region, namely that bounded by and including two concentric circles in the plane. Of significance is the fact that the behaviour of the bounding circles already provides the criteria. It is in fact proved that, a) a sense-preserving homeomorphism leaving the bounding circles invariant has in general no fixed point but one in which the circles are interchanged has at least two; b) a sense-reversing homeomorphism with invariant boundary circles has at least four fixed points while one which interchanges them has, in general, none.

There is an interesting contribution to the subject made by J. W. Alexander (1) in 1922; interesting in that results from classical analysis are utilized to obtain original proofs of the Brouwer theorems. The detailed discussion is restricted to 3 dimensions but can readily be extended to n.

The key to the method is the "Gaussian Integral":

$$\iint \begin{vmatrix} x & y & z \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} \frac{du\,dv}{r^3}$$

where x, y, z are the coordinates in real 3-space of the points on the image of the unit sphere S_1 under continuous transformation; $r = \sqrt{x^2+y^2+z^2}$ and u and v are parametric coordinates for the unit sphere (e.g. latitude and longitude. For continuous mappings

$$x = x(u,v)\ ;\quad y = y(u,v)\ ;\quad z = z(u,v)$$

are continuous functions. The integral taken over S_1 has the value $\pm 4\pi$ and in fact $+4\pi$ by suitable choice of u and v. Evaluation of the integral for a given continuous mapping $S_2 = T S_1$, leads to the definition of an index, k, of the mapping T. It is shown that the integral for S_2 is $4k\pi$ where k is an integer which is the difference between the number of times a ray from the origin cuts S_2 from the negative to the positive side and the number of times it cuts S_2 the opposite way. Although the image, S_2, is considered provisionally as made up of finitely many analytic pieces this condition is later removed in so far as S_2 may be closely approximated by a second image which is of the analytic type. The value of the integral for S_2 is then defined to be its value for a such a sufficiently well approximating analytic surface. In that a continous deformation of the integration surface which does not cross the origin (discontinuities would then arise) leaves the value of the integral unaltered the definition above of the value of the integral over an arbitrary image S_2 is sufficiently invariant.

The properties of this integral and the resulting definition of the index, k, are now used in studying a continuous transformation σ of S_1 into itself. Letting $\sigma(x_1, y_1, z_1) = (x_2, y_2, z_2)$ one considers

$$\iint \begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ \frac{\partial}{\partial u}(x_1 - x_2) & \frac{\partial}{\partial u}(y_1 - y_2) & \frac{\partial}{\partial u}(z_1 - z_2) \\ \frac{\partial}{\partial v}(x_1 - x_2) & \frac{\partial}{\partial v}(y_1 - y_2) & \frac{\partial}{\partial v}(z_1 - z_2) \end{vmatrix} \frac{du\,dv}{r^3}$$

where $r = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}$. The absence of fixed points is characterised by $r \neq 0$ everywhere. Under these circumstances the integral is well defined.

By radial contraction of σS_1 into the origin the integral is seen to be 4π while similar contraction of S_1 shows it to be $-4k\pi$ where k is the index of σ. It is thus proved that a continuous transformation of S_1 into itself must have a fixed point if the index, k, is different from -1.

We conclude this chapter with a very elegant proof of Theorem 2 as we have found it in Lefschetz "Introduction to Topology" (11). Aside from its elegance we have found it particularly instructive for the following reason. It may be recalled that the technique used by Brouwer consisted in simplicial decomposition of the manifold which led to consideration of associated simplicial maps which approximated the continuous mapping in question. Indeed, the very property of continuity of the mapping enabled the successful approximation by means of simplicial maps referred to decompositions of sufficiently small mesh. The success of this technique culminated in the definition of the degree of the mapping a characteristic of satisfactory invariance in that one could prove its invariance with respect to homotopy. In short, then, this powerful technique concerns itself not so much with the underlying triangulations but with their associated linear mappings.

In contrast with this, the proof which is to follow is based on results of a study of the decomposition per se! We refer to the result known as Sperner's Lemma (18) which states the following:

Consider a complex $K = Cl\, \sigma^n$ where $\sigma^n = A_0 \dots A_n$, i.e. the n-dimensional simplex σ^n together with all its faces. Let $K^{(s)}$ be a barycentric subdivision of K with vertices $\{a_i\}$ and t an assignement which associates with each a_i a vertex $A_{j(i)}$ which is a vertex of that face of σ^n which contains a_i. Then there exists a simplex (and in fact an odd number of them) $a_{i_0} \dots a_{i_n}$ such that the $t a_{i_n}$ are all distinct. Now to prove the theorem.

The n-dimensional element, or n-disk, may be taken to be an n-dimensional simplex $\sigma^n = A_0 \dots A_n$ with barycentric coordinates $x_0, \dots, x_n$. Then a continuous mapping $t : \sigma^n \to \sigma^n$ consists of relations

$$x_i' = f_i(x_0, \dots, x_n) \qquad i = 0, 1, 2, \dots n,$$

where $(x_0', \dots x_n') = t(x_0, \dots x_n)$ in which the f_i are continuous functions.

Consider the closed subsets F_i of σ^n given by

$$f_i \; (x_0, \dots x_n) \leq x_i$$

Intersection of the F_i implies the existence of a point for which $x_i' \leq x_i$ $(i = 0, \dots n)$ But $\sum x_i' = \sum x_i = 1$ so that $x_i' = x_i$ $(i = 0, \dots n)$ i.e. $(x_0, \dots x_n)$ is a fixed point. Thus in order to prove the theorem it suffices to show that

$$\bigcap_{i=0}^{n} F_i \neq \Lambda$$

Consider the faces $A_i \dots A_j$. Here we have $x_i + \dots + x_j = 1$ while $x_k = 0$ for $k \neq i, \dots j$. Since $x_i' + \dots + x_j' \leq 1$ for at least one index, r, among $i, \dots j$, $x'_r \leq x_r$ so that

$$A_i \dots A_j \subset F_i \cup \dots \cup F_j$$

Let us take a barycentric subdivision of $\mathcal{C}\ell\,\sigma^n$ of mesh ε with vertices $\{B_h\}$. We assign to each vertex B_h a vertex $A_{i(h)}$ in the following way. If $B_h \in A_i \dots A_j$ then $B_h \in F_r$ for some F_r among $F_i, \dots F_j$. We may take that F_r with smallest r and then assign to B_h the vertex $A_{i(h)} = A_r$. Now, by Sperner's Lemma there is a simplex $B_{h_0} \dots B_{h_n}$ such that A_i is assigned distinctly to B_{h_i} which means $B_{h_i} \in F_i$. Hence this set $B_{h_0}, \dots, B_{h_n}$ of diameter $\leq \varepsilon$ meets all the F_i. But ε may be chosen as small as we please and so even smaller than the Lebesgue number of the covering $\{F_i\}$. It follows therefore that the F_i must intersect thus proving the theorem.

CHAPTER V. "The Lefschetz Fixed Point Formula."

Prerequisite to proving the Lefschetz Fixed Point Theorem we must quote two essential results. The first of these is due to J. W. Alexander (2) and may be stated thus: Every mapping, φ , of one polyhedron $|K|$ into another $|L|$ is ε-homotopic to a simplicial mapping $\sigma: K_1 \to L_1$ where K_1 and L_1 are suitable subdivisions of the complexes K. and L. Indeed, mesh L_1 is less than ε and the homotopy paths are each contained in the closure of a simplex of .

The second result to which we refer, asserts that a mapping σ of one polyhedron, $|K|$ into another, $|L|$ induces a homomorphism σ^* of the homology groups of $|K|$ into the corresponding homology groups of $|L|$. Furthermore, σ^* is the same for all mappings homotopic to σ. We speak here of the homology groups of a polyhedron. This we may do in virtue of the fact that the homology groups of different triangulations of the same polyhedron are isomorphic.

After these preliminaries we turn now to the development of the fixed point formula. The first step consists of an investigation of the homomorphism of the homology groups induced by the mapping of a complex K into itself.

Suppose $\{S_i^p\}$ is a basis for the rational p-cycles, i.e. a maximum set of p-cycles independent with respect to homology; i will range from 1 to R^p where R^p is the p^{th} Betti number. A simplicial map σ of K into K induces a transformation:

$$\sigma S_i^p \sim \sum_j a_{ij}^p S_j^p \quad .$$

S_j^p we shall consider fixed it if appears in $\sum a_{ij}^p S_j^p$. Denoting the matrix $\|a_{ij}^p\|$ by a^p we shall study the number $\sum_p (-1)^p \text{trace } a^p$

which we may denote by $\varphi_1(\sigma)$.

Suppose that $\{T_i^p\}$ is a new base for the module of p-cycles then there will be relations

$$T_i^p = \sum \lambda_{ij}^p S_j^p \qquad \text{with } \lambda^p \text{ non-singular and}$$
$$\sigma T_i^p = \sum b_{ij}^p T_j^p$$

so that $b^p = (\lambda^p)^{-1} a^p \lambda^p$. Hence trace b^p trace a^p which proves that $\varphi_1(\sigma)$ is independent of the choice of bases for the cycles.

Now let $\{\beta_i^p\}$, $\{\gamma_i^p\}$, $\{\delta_i^p\}$ be successively extended bases for the p-boundaries, p-cycles and p-chains respectively; so chosen that $\partial \delta_i^{p+1} = \beta_i^p$ for all admissible p. σ will now induce the following transformations:

$$\sigma\beta_i^p = \sum b_{ij}^p \beta_j^p$$
$$\sigma\gamma_i^p = \sum b_{ij}^{p\prime} \beta_j^p + \sum c_{ij}^p \gamma_j^p$$
$$\sigma\delta_i^p = \sum b_{ij}^{p\prime\prime} \beta_j^p + \sum c_{ij}^{p\prime} \gamma_j^p + \sum d_{ij}^p \delta_j^p$$

Let $\quad \varphi(\sigma) = \sum (-1)^p(\text{trace } b^p + \text{trace } c^p + \text{trace } d^p)$.

Applying ∂ to the last of the three relations above,

$$\partial\sigma\delta_i^p = \partial\sum b_{ij}^{p\prime\prime}\beta_j^p + \partial\sum c_{ij}^{p\prime}\gamma_j^p + \partial\sum d_{ij}^p\delta_j^p$$

Since $\partial\sigma = \sigma\partial$ and $\partial\delta_i^p = \beta_i^{p-1}$ we get

$$\sigma\beta_i^{p-1} = \sum d_{ij}^p \beta_j^{p-1}$$

so that comparing this with the first relation for p-1 we see that $d_{ij}^p = b_{ij}^{p-1}$ and hence trace $d^p =$ trace b^{p-1}. The ingenuity of including the factor $(-1)^p$ now becomes opparent in that $(-1)^p$ trace d^p cancels $(-1)^{p-1}$ trace b^{p-1} and d^0 and b^n being 0 anyway $\varphi(\sigma)$ reduces to

$$\varphi(\sigma) = \sum(-1)^p \operatorname{trace} C^p$$

In other words, $\varphi(\sigma)$ is completely determined by the transformation of the rational cycles and since C_{ij}^p are integers, $\varphi(\sigma)$ is an integer.

Since no linear combination of the γ_i^p is a boundary their homology classes $\{\Gamma_i^p\}$ foma base for the p^{th} rational homology group $H^p(K)$. The homomorphism σ^* induced by σ is given precisely by

$$\sigma^*\Gamma_i^p = \sum C_{ij}^p \Gamma_j^p$$

Hence $\varphi(\sigma)$ is completely determined by σ^* . In particular a sufficient condition that σ have fixed elements, as defined above, is that $\varphi(\sigma) \neq 0$

We shall now use the consequences of the first result we quoted in order to prove:

<u>Theorem 111</u>

The 'trace invariant' $\varphi(f)$ of a continuous mapping f of a polyhedron $|K|$ into itself has the property that $\varphi(f) \neq 0$ is a sufficient condition for f to have a fixed point.

In speaking of $\varphi(f)$ when f is a continuous mapping we mean of course φ as determined by any simplicial mapping homotopic to f.

Let us assume that f has no fixed points, i.e. $d(x, f(x)) > 0$ for all $x \in |K|$. Since $|K|$ is a compactum there exists a ρ such that $d(x, f(x)) \geqslant \rho > 0$ for all $x \in |K|$. Let K_1 be a simplicial subdivision of K with the property that each simplex in K_1 has diameter less than $\rho/2$. Now, f is homotopic to a simplicial mapping $g: K_1 \to K_1$ so that for each $x \in |K|$, $f(x)$ and $g(x)$ are in the same simplex of K_1. Now suppose that for some simplex $S \in K_1$, $gS = S$. This will mean that for $x \in S$, $g(x) \in S$ so that $d(x, g(x)) < \rho/2$. But we have also $f(x), g(x) \in S_1 \in K_1$ so that $d(f(x), g(x)) < \rho/2$

Hence $\quad d(x, f(x)) \leq d(x, g(x)) + d(g(x), f(x)) \leq \rho/2 + \rho/2 = \rho$

which is a contradiction.

It follows then that under the assumption that f be free of fixed points the chain transformation induced by a simplicial map homotopic to f can have no fixed element i.e. $\varphi(f) = 0$ which proves the theorem.

We shall give one or two examples to illustrate the theorem. In theorem 11, $K = \mathcal{C}l\sigma^n$. K is therefore zero-cyclic, i.e. all the p-cycles are homologous to zero for $p > 0$ and every 0-cycle is homologous to a multiple of a given one. Hence $\varphi(f) = 1$ so that f always has a fixed point.

The rational homology groups of the projective plane are those of a point so that here again $\varphi(f) = 1$ and hence every continuous mapping of the projective plane into itself has a fixed point.

The n-sphere may be considered on the boundary of an n-simplex. The latter is cyclic at dimension 0 and dimension n. Thus, if A is the class of a vertex and Γ^n the basic nth homology class we have $fA = A$; $f\Gamma^n = d\Gamma^n$ where d is the degree of f. Hence $\varphi(f) = 1+(-1)^n d$ from which we conclude that every sense-preserving mapping $(d>0)$ of an even dimensional sphere and every sense-reversing mapping $(d<0)$ of an odd-dimensional sphere must have a fixed point. Indeed every mapping of an n-sphere into itself whose degree is different from ± 1 has a fixed point. This is the converse of Theorem 1.

The examples just cited illustrate the power of the fixed point formula. Lefschetz (12) in 1937 generalized the theorem still further to apply to LC^* spaces.

An LC space is defined in the following way. Let $K = \{\sigma\}$ be a finite Euclidean complex and L a closed subcomplex of K which contains all its vertices. An LC^* space, R, is characterized by the properties: 1) R is a compactum 2) For any $\varepsilon > 0$ there is an $\eta > 0$ such that if there is a pair (K,L) as just defined and a continuous mapping t_0 of $|L|$ into R such that mesh $\{t_0(L \cap Cl\,\sigma)\} < \eta$ then t_0 can be extended to a continuous mapping t of $|K|$ such that mesh $\{t\sigma\} < \varepsilon$.

It has been shown by Lefschetz (13) that the LC^* spaces may be identified with the "absolute neighbourhood retracts" as defined by K. Borsuk. The latter are defined in the following way. Given a topological space B and a subspace A, a continuous mapping t: $B \to A$ which is the identity on A is called "a retraction of B onto A." If a retraction of B onto A exists then A is known as a retract of B. More generally, A is a "neighbourhood retract of B" if there is a neighbourhood, C, of A (i.e. an open set containing A) which may be retracted into A. When A is a separable metric space it is said to be "an absolute neighbourhood retract" if every homeomorphic image A_1 of A as a closed subset of any other separable metric space, B, is a neighbourhood retract of B. A detailed presentation of the theory of retraction may be found in Lefschetz' "Topics in Topology," (14).

CHAPTER VI - Applications.

Noteworthy applications of fixed point theorems have been made in the field of functional analysis in endeavouring to obtain "existence theorems". The first major contribution is to be fouhd in a paper by G. D. Birkhoff and O.D. Kellogg (3) which appeared in 1922. The procedure followed here has essentially two steps. The first is to generalize results for two and three dimensions to spaces of dimension n and then to function space by means of a limiting process. Methods of classical analysis are used to prove that a bounded connected region of E_n has a fixed point under a mapping for which the coordinate transformations $x_i' = f_i(x_1, \cdots x_n)$ are algebraic. This is then extended to arbitrary continuous functions by means of the Weierstrass theorem on the approximation of continuous functions by polynomials.

The authors then pass to consideration of the space R_f of real functions which consists of the totality of real functions $f(s)$ defined on the closed interval $[0,1]$ which are uniformly bounded, i.e. $|f| < B < +\infty$ for all f and all $s \in [0,1]$ and equicontinuous $\eta(\varepsilon)$, $\eta(\varepsilon)$ being convex. The last property means that there exists a function defined and bounded on which approaches 0 with ε such that $|f(s+h) - f(s)| \leq \eta$ for $|h| \leq \varepsilon$ all s and $s+h$ in $[0,1]$ and all f. The convexity of $\eta(\varepsilon)$ means that for every a, b and θ in $[0,1]$

$$\eta(a + \theta(b-a)) \geq \eta(a) + \theta(\eta(b) - \eta(a))$$

It is proved that a single valued continuous mapping, S of R_f into itself has a fixed point. This is done by considering the effect of S on polygonal functions $\pi(s,x)$ at the points $x \in R_n$ where R_n is a region of n-space whose points have coordinates satisfying the relations:

$$|x_i| \leq B \quad ; \quad |x_{i+j} - x_i| \leq \eta\left(\tfrac{j}{n-1}\right) \quad \begin{pmatrix} i = 1, 2, \ldots n \\ j = 1, 2, \ldots n-i \end{pmatrix}$$

The function $\pi(s,x) = x_i$ for $s = s_i = \frac{i-1}{n-1}$ $(i = 1, 2, \ldots n)$ and is linear for intermediate values of S. The functions, $\pi(s,x)$, as thus defined are evidently in R_f .

A transformation T of R_n is now defined by means of $\pi'(s,x) = S\pi(s,x)$. $Tx = x'$ is obtained by setting $x_i' = \pi'(s_i, x)$. The authors assert that $x' \in R_n$ and in fact that R_n is a bounded connected region. T is algebraic on R_n and so R_n has a fixed point under T. Let us denote this fixed point by a

The function $\pi(s,a)$ coincides with $S\pi(s,a)$ at the n points s_i and between these points the variation of either function is not greater than $\eta\left(\frac{1}{n-1}\right)$ so that for all $s \in [0,1]$, $\left|\pi(s,a) - S\pi(s,a)\right| \leq 2\eta\left(\frac{1}{n-1}\right)$

Denoting by $\delta_f = \sqrt{\int_0^1 \left(f(s) - Sf(s)\right)^2 ds}$ the distance by which f is moved by S,

$$\delta_\pi \leq 2\eta\left(\frac{1}{n-1}\right) ;$$

clearly $\delta_\pi \to 0$ as $n \to \infty$ so that

$$\inf \delta_f = 0$$

which means that R_f has a fixed point under S.

This result is used to answer affirmatively the question as to the existence in R_f of a solution of a differential equation $y^{(n)} = F(x, y, y', \dots y^{(n-1)})$ satisfying n linear conditions on the interval $(0, a)$:

$$\int_0^a \sum_{j=0}^{n-1} P_{ij}(x)\, y^{(j)}(x)\,dx + \sum_{j=0}^{n-1}\sum_{k=1}^{m} q_{ijk}\, y^{(j)}(x_k) = c_i$$
$$(i = 1, 2, \dots n ; \quad 0 \leq x_1 \leq \dots \leq x_m \leq a)$$

where the $P_{ij}(x)$ are continuous and the conditions are such as to determine uniquely a polynomial y of degree $n-1$. The problem is reduced to proving the existence of a fixed point of the transformation S :

$$Sy = \int_0^x\int_0^x \cdots \int_0^x F(x, y, y', \dots y^{(n-1)})\,dx\,dx \dots dx$$
$$+ a_0 + a_1 x + \dots + a_{n-1} x^{n-1}.$$

Results of a more general nature which include those obtained by Birkhoff and Kellogg have been obtained by J. Schauder (16) in 1930. We shall merely state the theorems which he has proved.

1. Every single valued continuous mapping of a convex campactum in a real vector space into itself has a fixed point.

2. If H is convex and closed in a Banach space and t is a continuous mapping such that tH is conditionally compact then t has a fixed point. (The conditional compactness of a subspace A in a space R means that every sequence $\{x_n\}$ in A has a subsequence $\{x_{n'}\}$ convergent to a point x in R).

3. Let R be a strongly separable Banach space and H a strongly closed and convex subset of R which is weakly, sequentially conditionally compact. Then every weakly continuous mapping of H into itself has a fixed point. ("Strongly" refers to the 'strong' topology, namely that with which R is equipped in virtue of the metric. The 'weak' topology is induced by means of the space, R^*, conjugate to R, i.e. the space of linear funtionals on R, in the following way. If $\{I\}$ are the intervals of the real line and $R^* = \{f\}$ then the intersections of finitely many $f^{-1}I$ constitute a base for the weak topology of R.

A further application, in quite a different field may be found in a paper by H. Hopf and H. Samelson (7) which appeared in 1940. The problem considered there is that of determining topological properties which spaces must have in order to serve as "operation spaces" (Wirkungs räume) for closed Lie groups. An operation space W is a manifold related to a Lie group G in thefollowing way. To each element a of G there corresponds an analytic topological mapping f_a of W onto itself. The mappings f_a must satisfy the cinditions:

1. $f_a(f_b(\xi)) = f_{ab}(\xi)$
2. The point $f_a(\xi)$ depends continuously on the pair (a, ξ)
3. For each pair (ξ, η) of elements in W there is at least one a in G for which $f_a(\xi) = \eta$

Up to 1940 it had been known that the fundamental group of W has an abelian subgroup of finite index and that the Betti numbers p_r, $r=1,2,\cdots$ satisfy the inequalities $p_r \geqslant \binom{p_1}{r}$ and $p_r \geqslant \binom{n}{r}$. The authors provethat the Euler-Poincaré characteristic $\chi(W)$ must be either zero or positive and then proceed to restrict still further the positive numbers which are admissible as characteristics. The method utilizes the trace invariants of the f_a and the fact that $\varphi(f)$ where f can he deformed into the identity is the same as $\chi(W)$ both being in fact $\sum(-1)^r p_r$.

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