Cyclic Orders and Graphs of Groups

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Abstract

We construct a cyclic order on the edge set of a directed tree whose vertices have cyclically ordered links. We use this construction to show that a graph of groups with left-cyclically ordered vertex groups and convex left-ordered edge groups is left-cyclically orderable.

Abrégé

Nous construisons un ordre cyclique sur l'ensemble de sommets d'un arbre orienté dont les nœuds ont des liens ordonnés cycliquement. Avec cette construction, nous montrons que lorsqu'un graphe de groupes dont les groupes de nœuds sont ordonnés cycliquement à gauche et dont les groupes de sommets sont ordonnés à gauche et convexes, le group fondamental est ordonné cycliquement à gauche.

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1 Introduction

W.Dicks and Z.Sunic gave an elegant way of totally ordering the vertex set of a directed tree [Dv20]. They applied this to give a simple proof of Vinogradov's 1949 result that free groups, and more generally, free products of left-orderable groups are left-orderable. The purpose of this text is to describe a cyclically ordered counterpart.

Our basic observation is that:

Lemma 1.1. *Let $T = (V, E)$ be a tree. Suppose there is a cyclic order on link(v) for each $v \in V$. Then there is an induced cyclic order on the directed edges of T.*

Using this natural cyclic order, we examine graphs of groups and obtain:

Theorem 1.2. *Let G split as a graph of groups with left-cyclically ordered vertex groups and convex left-ordered edge groups. Then G is left-cyclically ordered in a manner compatible with its vertex and edge groups.*

This generalizes the result of H.Baik and E.Samperton that free products of left-cyclically ordered groups are left-cyclically ordered [BS18]. A recent study in this direction, probing more deeply than our own, was given by A.Clay and T.Ghaswala who characterized when an amalgam of cyclically ordered groups is cyclically ordered [CG19].

Our work is most closely related to Dicks-Sunic in spirit and Baik-Samperton/Clay-Ghaswala in practice. Recently, there has been increased activity in the area of cyclically ordered groups. An amusing characterization declares that G is left-ordered if and only if $G \times \mathbb{Z}_n$ is left cyclically-ordered for each n [BCG20]. Finally, we refer to [Ghy01] and [Cal04] for surveys on cyclically ordered groups.

2 Cyclic Orders

Definition 2.1. (Cyclic Order) A *cyclic order* on a set A is a function $\Gamma : A \times A \times A \to \{-1, 0, 1\}$ with the following conditions:

- **Nondegeneracy**: $\Gamma(x, y, z) = \pm 1$ if x, y, z are pairwise distinct
- **Cyclicity**: if $\Gamma(x, y, z) = 1$, then $\Gamma(z, x, y) = 1$
- **Asymmetry**: $\Gamma(x, y, z) = -\Gamma(y, x, z)$
- **Transitivity**: if $\Gamma(x, y, z) = 1$ and $\Gamma(x, z, w) = 1$, then $\Gamma(x, y, w) = 1$

We write $[x, y, z]$ whenever $\Gamma(x, y, z) = 1$.

Remark 2.2. For a totally ordered set $(A, \preceq)$, there is an associated strict total order $(A, \prec)$. It is often convenient to consider strict total orders while moving between cyclic orders and total orders. Where notation is clear, we will omit the term "strict".

Definition 2.3. A *strict total order* is a binary relation $\prec$ on a set X which satisfies the following conditions for all $x, y, z \in X$:

- **Irreflexivity**: not $x \prec x$ for all $x \in X$
- **Comparability**: if $x \neq y$ then $x \prec y$ or $y \prec x$
- **Transitivity**: if $x \prec y$ and $y \prec z$ then $x \prec z$.

Remark 2.4. For a totally ordered set $(A, \prec)$, define an associated cyclic order on A, via $[a, b, c]$ provided $a \prec b \prec c$ or $b \prec c \prec a$ or $c \prec a \prec b$.

Example 2.5. Consider $[0, 2\pi)$ with the usual total order. Identifying $[0, 2\pi)$ with S^1 using $\theta \mapsto e^{\theta i}$, and applying Remark 2.4 provides a cyclic order on S^1.

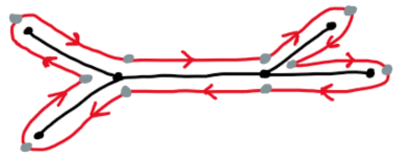

Figure 1: A clockwise boundary path induces a cyclic order on the directed edges.

Remark 2.6. Suppose $A \subset S$ is such that $(A, \prec)$ is a total order and $x \in S$ with $x \notin A$. We can cyclically order $A \cup \{x\}$ through the rule $[a_1, a_2, x]$ for all $a_1 \prec a_2$.

3 Cyclic Orders on Trees

The purpose of this section is to create a cyclic order on the set of directed edges of a tree. Note that an edge with vertices u, v is associated to two directed edges: (u, v) and (v, u).

Lemma 3.1. *Let T be a finite tree embedded in the plane. The directed edges of T are cyclically ordered.*

Proof. Regarding T as a disc diagram, the boundary path $\partial_{\mathsf{p}}(T)$ provides an embedding of the directed edges into S^1, hence inducing a cyclic order by Remark 2.4. $\square$

Definition 3.2. A tree $T = (V, E)$ is a *c-tree* if there is a cyclic order on $\text{link}(v)$ for each vertex $v \in V$. Equivalently, there is a cyclic order on the edges adjacent to each vertex.

Definition 3.3. An embedding $T \to \mathbb{R}^2$ of a locally finite c-tree is *coordinated* if for each vertex v with adjacent edges $e_1 \prec e_2 \prec \cdots \prec e_n \prec e_1$, their images $\bar{e}_1 \prec \bar{e}_2 \prec \cdots \prec \bar{e}_n \prec \bar{e}_1$ are in the same clockwise order about $\bar{v} \in \mathbb{R}^2$.

Lemma 3.4. *Let T be a locally finite c-tree. There is a coordinated embedding $T \to \mathbb{R}^2$.*

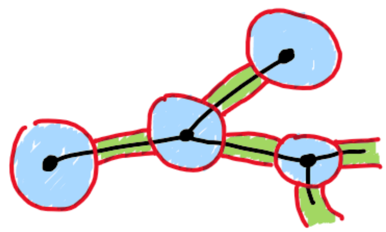

Figure 2: Handlebody decomposition of a tree in $\mathbb{R}^2$

For edges $e_1, \ldots, e_n$ about v, we regard them as cyclically ordered using the clockwise orientation of $\mathbb{R}^2$.

Proof. We produce a "thickening" of T into 0-handles and 1-handles to obtain a disk as follows. Embed a valence n-vertex v with cyclically ordered edges $e_1, \ldots, e_n$ in a unit disk, by identifying v with 0 and identifying each edge with the segment joining 0 and $e^{\frac{2\pi}{n}i}$. Join disks for adjacent vertices along neighborhoods (so that orientations cancel) to form a surface S homeomorphic to the unit disk. See Figure 2. □

Remark 3.5. The embedding of Lemma 3.4 is essentially unique (i.e. up to ambient isotopy).

Hence for any finite subtrees $T_a \subset T_b$, a coordinated embedding of T_a is essentially the same as an embedding of T_a induced by a coordinated embedding of T_b. Indeed, the way Lemma 3.4 embeds T_b induces the way it embeds T_a simply by "forgetting" $T_b - T_a$.

For any two finite subtrees, their embeddings agree with a coordinated embedding of a larger finite tree containing them.

Theorem 3.6. *Let T be a c-tree. There is an induced cyclic order on the set of directed edges of T. It is uniquely determined by the cyclic order on links of T.*

Proof. For a c-tree, take a coordinated embedding of a finite subtree T'. The clockwise boundary path of T' yields a cyclic order of the directed edges of T'. This cyclic order is consistent for $T' \subset T''$, and hence induces a cyclic order on all directed edges of T.

Uniqueness holds since the cyclic order on links is determined by the cyclic ordering on the outgoing edges at a vertex. $\square$

Remark 3.7. The condition of a finite tree T being a c-tree is equivalent to having a boundary path $\partial_{\mathsf{p}}(T)$. Note that via Lemma 3.4 we obtain a coordinated embedding of T into the plane, and we can recover a boundary path of T. Similarly, for any $\partial_{\mathsf{p}}(T)$ we could define an embedding $T \to \mathbb{R}^2$ such that the image of $\partial_{\mathsf{p}}(T)$ is homeomorphic to a circle. Tracing the boundary path will induce a cyclic order on the link of each vertex, and we recover local cyclic orders that make T a c-tree.

Lemma 3.8 (G-invariance)**.** *Suppose G acts on a c-tree T so that cyclic orders on vertex links are G-invariant. Then the induced cyclic order on directed edges of T is G-invariant.*

Proof. This holds by Theorem 3.6 since the induced cyclic order on directed edges of T is determined by the cyclic orderings on vertex links. $\square$

4 Cyclic Orders and Tree Augmentation

We provide an alternate method to obtain that the directed edges of a c-tree can be cyclically ordered. This approach constructs a correspondence between directed edges and spurs.

Definition 4.1. For vertices of a tree $x, y, z \in V$, the *median* $m(x, y, z)$ is the vertex equal to the intersection of geodesics $xy \cap yz \cap zx$.

Lemma 4.2. *Let $T = (V, E)$ be a c-tree, there is a cyclic order on the set $L \subseteq V$ of leaves of T.*

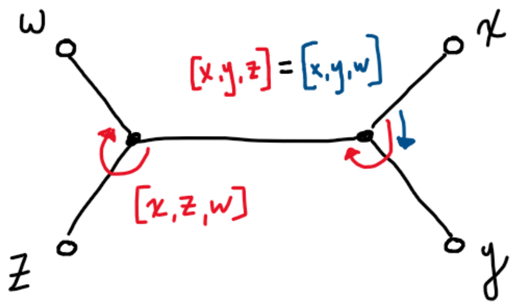

Figure 3: This explains transitivity for Lemma 4.2

Proof. When $x, y, z \in L$ are distinct, the median $m = m(x, y, z)$ has three distinct edges adjacent to m pointing to x, y, and z. These edges e_x, e_y, and e_z are cyclically ordered around m. Declare a cyclic order on L via:

$$[x, y, z] \text{ in } L \iff [e_x, e_y, e_z] \text{ in } \mathrm{link}(m).$$

Nondegeneracy, cyclicity, and asymmetry all follow immediately as the link of the median is cyclically ordered. For leaves $x, y, z, w \in L$, transitivity follows if $m(x, y, z) = m(x, z, w)$. Otherwise, let S be the smallest subtree containing $\{x, y, z, w\}$. S takes the form of an "H" with two leaves at $m_1 = m(x, y, z)$ and two leaves at $m_2 = m(x, z, w)$. Via Lemma 3.4, we can embed S into the plane so that links of vertices are cyclically ordered clockwise. If $[x, y, z]$ and $[x, z, w]$ hold, then $[x, y, w]$ also holds, see Figure 3. $\square$

Definition 4.3. (Augmented Tree) Let T be a directed c-tree, the *augmented tree* $\overline{T}$ is obtained by adding an *augmented edge* e_{aug} at the barycenter of each directed edge e, see Figure 4.

More precisely, for each edge $e \in E$, let b_e be its barycenter and cut e into two half edges, e_{out} and e_{in}. Orient the half edges so that e_{in} and e_{out} are incoming and outgoing at

Figure 4: The direction of e determines the position of the spur.

b_e. Under this construction links of vertices in the original tree T are unchanged, and the link of each barycenter vertex b_e is $\{e_{in}, e_{out}, e_{aug}\}$. Cyclically order the link of each b_e using the rule $[e_{in}, e_{out}, e_{aug}]$. Direct augmented edges away from barycenters, and note that the augmented tree $\overline{T}$ is now a directed c-tree.

Theorem 4.4. *There is an induced cyclic order on the set of directed edges of a c-tree T.*

Proof. Construct the augmented tree $\overline{T}$ and note that each directed edge of T is associated to a spur of $\overline{T}$. Apply Lemma 4.2 to cyclically order these spurs. $\square$

Remark 4.5. The cyclic orders from Theorem 3.6 and Theorem 4.4 are identical. For any finite directed c-tree, a coordinated embedding gives rise to a boundary path inducing a cyclic order on the edges. In Theorem 4.4 we merely "augment" each edge to acquire a cyclic order on the leaves. The augmented tree $\overline{T}$ has a boundary path induced by the coordinated embedding of T, and the cyclic order corresponding to this boundary path restricts to the cyclic order on T.

5 Ordered and Cyclically Ordered Groups

Definition 5.1 (Left-Ordered Group)**.** A group G is *left-ordered* if there is a total order $(G, \prec)$ such that for all $x, y, g \in G$ we have:

$$x \prec y \quad \Longrightarrow \quad gx \prec gy.$$

G is left-ordered if and only if $G = P \sqcup \{1_G\} \sqcup N$ with $PP \subset P$ and $NN \subset N$ where $P = \{g \in G \ : \ 1_G \prec g\}$ and $N = \{g \in G \ : \ g \prec 1_G\}$. Then $g \prec h \iff g^{-1}h \in P$

Definition 5.2 (Left-Cyclically Ordered Group)**.** A group G is *left-cyclically ordered* if there is a cyclic order on G that is left-invariant in the sense that:

$$[a, b, c] \quad \Longrightarrow \quad [ga, gb, gc].$$

Remark 5.3. Let G act freely on a cyclically ordered set X. We cyclically order G via:

$$[a, b, c] \text{ in } G \iff [ax, bx, cx] \text{ in } X.$$

Lemma 5.4. *Let G act faithfully and order-preservingly on an ordered set $(X, <)$. Then G has an induced left-order.*

Proof. Choose a well-ordering $\prec_w$ on X. For $g \neq h \in G$, let p be $\prec_w$-minimal with $gp \neq hp$. Declare $g \prec h$ if $gp < hp$.

This relation is irreflexive as $gp \not< gp$. Since G acts faithfully on X, for $g \neq h \in G$ there exists $x \in X$ with $gx \neq hx$, so comparability holds. G-invariance holds since $kgp < khp \iff gp < hp$. Let p_1 and p_2 be $\prec_w$-minimal with $xp_1 \neq yp_1$ and $yp_2 \neq zp_2$. If $p_1 = p_2$ we are done. If $p_1 \prec_w p_2$ then $yp_1 = zp_1$ and $xp_1 < zp_1$. If $p_2 \prec_w p_1$ then $xp_2 = yp_2$ and $xp_2 < zp_2$. Thus transitivity holds for $(G, \prec)$. □

Theorem 5.5. *Let G act faithfully and order-preservingly on a cyclically ordered set X. Then G has an induced left-cyclic order.*

Proof. Let $p \in X$ and $\dot{X} = X - \{p\}$. Observe that $\dot{X}$ is totally ordered and $H = stab(p)$ acts faithfully on $\dot{X}$. Via Lemma 5.4, H is left-ordered. There is a total order $(gH, \prec)$ for each left coset, by declaring $g\alpha \prec g\beta \iff \alpha \prec \beta$. This is independent of the choice g of representative, since $(H, \prec)$ is left H-invariant.

Let $\{g_i\}$ be a choice of coset representatives for H in G. Our orderings on cosets provides a partial ordering on $\sqcup_i g_i H$. This partial ordering is G-invariant by definition. Finally, this partial ordering on G extends to a G-invariant cyclic ordering by cyclically ordering the left cosets using their bijection with Gp. Specifically $[a, b, c]$ holds if either:

1. $a \prec b \prec c$ and a, b, c lie in the same coset,
2. $a \prec b$ with $ap = bp \neq cp$, or $b \prec c$ with $bp = cp \neq ap$, or $c \prec a$ with $cp = ap \neq bp$,
3. $[ap, bp, cp]$ in X. □

6 Ordering Collections of Cosets

Definition 6.1. A subgroup H of a left-ordered group $(G, \prec)$ is *convex* if for all $h_1, h_2 \in H$ and $g \in G$, if $h_1 \prec g \prec h_2$ then $g \in H$.

Definition 6.2 (Convex Subgroup)**.** Let G be a left-cyclically ordered group and $H \subset G$ a proper subgroup. We say H is *convex* in G if for every $g \in G - H$ and $f \in G$ and $h_1, h_2 \in H$, if $[h_1, f, h_2]$ and $[h_1, h_2, g]$ then $f \in H$.

There is an alternate definition for a convex subgroup of a left-cyclically ordered group that requires the following preliminary notion. The two definitions of convexity are shown to be equivalent in [CG19, after Lem. 5.2].

Definition 6.3. Let G be a left-cyclically ordered group. A subgroup $H \subset G$ is *left-ordered by restriction* if $[h^{-1}g^{-1}, 1_G, gh]$ holds whenever $[h^{-1}, 1_G, h]$ and $[g^{-1}, 1_G, g]$ for $g, h \in H$.

When $H \subset G$ is left-ordered by restriction, define the *left-order by restriction* on H using the positive cone:

$$P = \{h \in H \ : \ [h^{-1}, 1_G, h] \text{ in } G\}.$$

Definition 6.4 (Convex Subgroup)**.** Let G be a left-cyclically ordered group. A subgroup $H \subset G$ is *convex* if H is left-ordered by restriction and whenever $h_1, h_2 \in H$ and $g \in G$, if $[h_1, 1_G, h_2]$ and $[h_1, g, h_2]$ then $g \in H$.

For subsets U, V of an ordered set $(X, \prec)$, declare $U \ll V$ if there exists $v \in V$ with $u \prec v$ for all $u \in U$. Note that within a left-ordered group $(G, \prec)$ we have $U \ll V \iff gU \ll gV$ for all $g \in G$.

Lemma 6.5. *Let $(G, \prec)$ be an ordered group and H a convex subgroup. The relation $\ll$ restricts to a G-invariant total order on the collection G/H of left-cosets.*

Proof. Comparability of $(G/H, \ll)$ holds as cosets are disjoint and H is a convex subgroup. Transitivity follows given that $(G, \prec)$ is left-ordered. If there exists $U \in G/H$ with $U \ll U$, then there exists $v \in U$ with $u \prec v$ for all $u \in U$, so $v \prec v$ which is impossible. □

Lemma 6.6. *Let G be a left-cyclically ordered group. Let $H \subset K \subsetneq G$ be convex subgroups. $(K, \prec)$ is left-ordered by restriction. If $H \not\ll K$, then $H = K$.*

Proof. Consider a coset $kH \neq H$. If $H \not\ll K$, then $\forall k \in K$, there exists $h \in H$ such that $k \prec h$. By Lemma 6.5, $k' \prec h'$ for all $k' \in kH$ and $h' \in H$. In particular, $k \prec 1$. Left multiplying gives $1 \prec k^{-1}$. Since $H \not\ll K$, we have $k^{-1} \prec h'$ for some $h' \in H$. Finally, $1 \prec k^{-1} \prec h'$ shows that $k^{-1} \in H$ by convexity, a contradiction as $k \notin H$. Hence $H = K$. □

Lemma 6.7. *Suppose H and K are convex subgroups of the left-cyclically ordered group G. Either $H \subset K$ or $K \subset H$.*

Proof. By Lemma 6.6, if $H \cap K \not\ll H$ or $H \cap K \not\ll K$, then $H \cap K$ is equal to H or K respectively. Else, $H \cap K \ll H$ and $H \cap K \ll K$. Thus for all $\alpha \in H \cap K$ we can find $h \in H$ and $k \in K$ such that $\alpha \prec h$ and $\alpha \prec k$. Without loss of generality, assume $\alpha \prec h \prec k$ and by convexity, $h \in K$. Thus $H \cap K = H$ and $H \subset K$. □

Corollary 6.8. *Suppose K and H are convex subgroups of a left-cyclically ordered group G. Let xK and yH be left cosets of K and H respectively.*

If $xK \cap yH \neq \emptyset$ then either $xH \subset yK$ or $yK \subset xH$

Proof. This follows directly from Lemma 6.7. □

Definition 6.9. It will be convenient to consider *indexed collections* of subsets $\{H_i\}_{i\in I}$ allowing "repeats" in the sense that $H_i = H_j$ though $i \neq j$.

Lemma 6.10. *Let $(G, \prec)$ be a left-ordered group and $\{H_i\}_{i\in I}$ an indexed collection of convex subgroups. Choose a total order $\prec_I$ on I. There is a G-invariant total order on the indexed collection of left cosets $\{gH_i \ : \ g \in G, i \in I\}$.*

Proof. Let $\ll_*$ denote the relation defined by:

$$g_1H_i \ \ll_* \ g_2H_j \quad \Longleftrightarrow \quad \begin{cases} g_1H_i \neq g_2H_j \text{ and } g_1H_i \ll g_2H_j \\ g_1H_i = g_2H_j \text{ and } i \prec_I j \end{cases}$$

Transitivity, and comparability of $\ll_*$ follow given that $(G, \prec)$ and $(I, \prec_I)$ are total orders. It is impossible for $g_1H_i \ll_* g_1H_i$, as this would imply $i \prec_I i$. Thus $\ll_*$ is irreflexive, and therefore a total order.

Let $g_1H_i \ll_* g_2H_j$. If $g_1H_i \neq g_2H_j$, then G-invariance of $(G, \prec)$ guarantees $\alpha g_1H_i \ll_* \alpha g_2H_j$ for all $\alpha \in G$. If $g_1H_i = g_2H_j$, the order depends only on $(I, \prec_I)$, and $\alpha g_1H_i \ll_* \alpha g_2H_j$ for all $\alpha \in G$. Thus, $\ll_*$ is G-invariant. □

The following is proven in [CMR18]:

Lemma 6.11. *If G is a left-cyclically ordered group and H is a convex subgroup, the cosets G/H inherit a G-invariant cyclic order defined as follows for distinct cosets g_1H, g_2H, g_3H.*

$$[g_1H, g_2H, g_3H] \iff [g_1, g_2, g_3] \tag{1}$$

Proof. It suffices to prove that for all $h \in H$ and for all distinct triples $(g_1, g_2, g_3) \in G^3$ such that $g_iH \neq g_jH$ for $i \neq j$, if $[g_1, g_2h, g_3]$ then $[g_1, g_2, g_3]$.

Without loss of generality, let $h \in H$ and $g_1, g_2, g_3 \in G$ such that $[h^{-1}, 1_G, h]$ and $[g_1, g_2, g_3]$. Suppose that $[g_3, g_2h, g_1]$, equivalently $[g_2^{-1}g_3, h, g_2^{-1}g_1]$. Cyclically permute $[g_1, g_2, g_3]$ and left multiply by g_2^{-1} to obtain $[g_2{-}1g_3, g_2^{-1}g_1, 1_G]$. By transitivity $[g_2^{-1}g_3, h, 1_G]$, so $g_2^{-1}g_3 \prec h \prec 1_G$. Thus $g_2^{-1}g_3 \in H$ by convexity. This contradicts that $g_2H \neq g_3H$. $\square$

Theorem 6.12. *Let G be a left-cyclically ordered group and let $\{H_i\}_{i\in I}$ be an indexed collection of convex subgroups. Choose a total order $\prec_I$ on I. There is a G-invariant cyclic order on the indexed collection of left-cosets $\{gH_i : g \in G, i \in I\}$.*

Proof. For any finite set of convex subgroups $\{H_j\}_{j\in J} \subseteq \{H_i\}_{i\in I}$, by Lemma 6.7 there is a chain of inclusions

$$G = H_0 \supset H_1 \supseteq \cdots \supseteq H_n.$$

This chain of inclusions determines a graph of groups, whose underlying graph is a length-n subdivided interval. Direct all edges away from the root vertex v_0, whose vertex group is G. The edge e_i terminates at the vertex v_i, and $G_{e_i} = G_{v_i} = H_i$. As this graph of groups is telescopic its fundamental group is G.

Let $T = (V, E)$ be the Bass-Serre tree corresponding to this graph of groups. The vertex set $V = \sqcup_{i=0}^{n}\{gH_i \ : \ g \in G\}$ consists of the indexed collection of left-cosets of vertex groups, likewise the edge set.

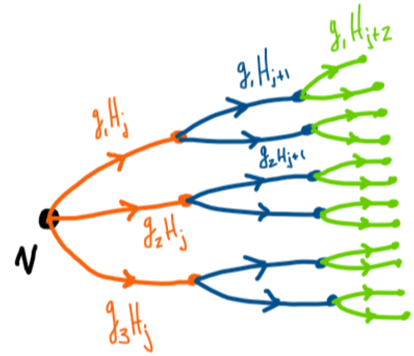

Figure 5: Part of a finite coset tree.

There is a directed edge from g_1H_k to g_2H_{k+1} when $g_1H_k \supset g_2H_{k+1}$. Under this construction, each left-coset gH_j for $j > 0$ is represented by a directed edge.

We turn T into a directed c-tree. For the root vertex G, note that $\text{link}(G) = G/H_1$. We can cyclically order link(G) via Lemma 6.11. For any other vertex gH_k, there is one incoming parent edge of link(gH_k), and outgoing edges representing containment of left-subcosets of H_{k+1}. By Lemma 6.5, $(H_k, \ll)$ induces a total order on included left-subcosets of H_{k+1}. Translating by g provides a total order on left-subcosets of gH_k. Cyclically order link(gH_k) via Remark 2.6, i.e. $[g_1H_{k+1}, g_2H_{k+1}, gH_k]$ if $g_1H_{k+1} \prec g_2H_{k+1}$.

As T is now a directed c-tree and each left-coset is represented by a directed edge, Theorem 3.6 provides a cyclic order on all left-cosets of $\{H_j\}_{j\in J}$. This holds for any finite collection of convex subgroups. The cyclic order is consistent for graphs of groups $\mathcal{G}' \subset \mathcal{G}''$ as defined above. Hence, this induces a cyclic order on all left-cosets in $\{gH_i \ : \ g \in G, i \in I\}$. As G is the fundamental group of this graph of groups, the cyclic order on the link of each vertex is G-invariant. Via Lemma 3.8, the cyclic order on left-cosets is G-invariant. See Figure 5. $\square$

7 Groups Acting on Trees

7.1 Action on tree

Definition 7.1. An inclusion $H \to K$ of a left-ordered group into a left-cyclically ordered group is *order-preserving* if

$$a \prec b \prec c \text{ in } H \quad \Longrightarrow \quad [a, b, c] \text{ in } K.$$

Definition 7.2. An action $G \curvearrowright S$ of a left-cyclically ordered group G on a cyclically ordered set S is *consistent* if $[\alpha, \beta, \gamma] \iff [g\alpha, g\beta, g\gamma]$ for all $g \in G$.

Theorem 7.3. *Let G act without inversions on a tree $T = (V, E)$. Suppose:*

1. *The stabilizer G_v is left-cyclically ordered for each vertex $v \in V$.*
2. *The stabilizer G_e is left-ordered for each edge $e \in E$.*
3. *The inclusion $G_e \subset G_v$ is convex whenever v is a vertex of e.*

Then there is a c-tree $\widetilde{T} = (\widetilde{V}, \widetilde{E})$ such that:

1. *For each $e \in E$, the order on G_e is induced by the action of G_e on $\widetilde{T}$.*
2. *For each $v \in V$, the cyclic order on G_v is induced by the action of G_v on $\widetilde{T}$.*
3. *There exists a spur $\tilde{e} \in \widetilde{E}$ such that $G\tilde{e}$ is a free orbit.*
4. *There is a G-invariant cyclic order on the orbit $G\tilde{e}$ that induces a cyclic order on G.*

Proof. Let $\widetilde{T}$ be obtained from T as follows. For each $v \in V$ add a spur to v for each element of the stabilizer G_v. These spurs are in correspondence with cosets of the trivial subgroup of G_v which is a convex subgroup. $G_e \subset G_v$ for every other edge is convex by hypothesis. Hence we can cyclically order link(v) via Theorem 6.12 for each $v \in V$, and $\widetilde{T}$ is a c-tree.

Let $\tilde{e}$ be an added spur. Cyclically order the leaf-edges and hence $G\tilde{e}$ by Theorem 4.4. By Lemma 3.8, the cyclic order on $G\tilde{e}$ is G-invariant. Finally, since G acts freely on $G\tilde{e}$, Remark 5.3 provides a left-cyclic order on G. □

7.2 Graph of groups statement

Corollary 7.4. *Let G split as a graph Γ of groups. Suppose each vertex group G_v is left-cyclically ordered, and each edge group G_e is left-ordered. Suppose each inclusion $G_e \hookrightarrow G_v$ of an edge group is convex. Then G has a left-cyclic order that restricts to the cyclic order of each vertex group G_v.*

Proof. Let $T = (V, E)$ be the Bass-Serre tree over Γ, which we assume to be directed. V consists of all left-cosets of vertex groups of Γ in G, and E consists of left-cosets of edge groups of Γ in G. That is, allowing for repeats (of edge or vertex groups):

$$V = \{gG_v \ : \ g \in G, \ v \in \text{ Vertices}(\Gamma)\}$$

$$E = \{gG_e \ : \ g \in G, \ e \in \text{ Edges}(\Gamma)\}.$$

Varying $g \in G$, there is an edge gG_e directed from gG_u to gG_v in T precisely when e is directed from u to v in Γ.

The stabilizer of a vertex gG_v equals gG_vg^{-1}, and similarly the stabilizer of an edge gG_e equals gG_eg^{-1}. Conjugation preserves the cyclic orders on G_v for each vertex, and similarly preserves the orderings on G_e for each edge, thus vertex and edge stabilizers are cyclically ordered. The conditions of Theorem 7.3 are satisfied, and the rest of this proof collects the conclusions of the theorem.

Let $\widetilde{T}$ be the tree obtained from T by adding a spur at each vertex for each element of the stabilizer. Via Theorem 6.12, cyclically order the link of each vertex of $\widetilde{T}$, so that $\widetilde{T}$ is a c-tree by Theorem 3.6. For an added spur $\tilde{e}$, the orbit $G\tilde{e}$ is free and so Remark 5.3 provides a cyclic order on G. □

Lemma 7.5. *Let G be a left-cyclically ordered group. Let H be a subgroup with $|G : H| \geq 3$. Then $H \subset G$ is convex if and only if the left cosets G/H have an induced cyclic order as in Equation (1) of Lemma 6.11.*

Proof. The forward direction is shown in Lemma 6.11, while the converse is proven in [CG19, Lem 5.1]. □

Lemma 7.6. *Let G act faithfully on a cyclically ordered set X. Let $Y \subsetneq X$ be convex. Then $stab(Y)$ is a convex subgroup of G, with respect to the cyclic order on G induced by the action.*

Proof. Observe that $H = stab(Y)$ acts faithfully on $X - Y$. Following the proof of Theorem 5.4, H is left-ordered and each coset gH inherits a total order. Note that the set of cosets $\{gH\}_{g\in G}$ is cyclically ordered by choosing representatives for each left-coset and restricting the cyclic order of X. Thus, in the induced cyclic order on G, the left-cosets of H are cyclically ordered. Thus H is convex by Lemma 7.5. □

Remark 7.7. Every group acting faithfully without inversions on a c-tree arises as in Corollary 7.4. The edge stabilizers are convex subgroups of the vertex stabilizers. Indeed, for each edge e at a vertex v, the left-cosets of $stab(e)$ in G_v correspond G_v-equivariantly to the edges in the G_v-orbit of e. The G_v-invariant cyclic order on the edges yields a G_v-invariant cyclic ordering on the cosets. Thus they are convex via Lemma 7.5. Finally, every action on a tree arises as the Bass-Serre tree of a graph of groups.

References

[BCG20] Jason Bell, Adam Clay, and Tyrone Ghaswala. Promoting circular-orderability to left-orderability, 2020.

[BS18] Hyungryul Baik and Eric Samperton. Spaces of invariant circular orders of groups. *Groups Geom. Dyn.*, 12(2):721–763, 2018.

[Cal04] Danny Calegari. Circular groups, planar groups, and the Euler class. In *Proceedings of the Casson Fest*, volume 7 of *Geom. Topol. Monogr.*, pages 431–491. Geom. Topol. Publ., Coventry, 2004.

[CG19] Adam Clay and Tyrone Ghaswala. Free products of circularly ordered groups with amalgamated subgroup. *J. Lond. Math. Soc. (2)*, 100(3):775–803, 2019.

[CMR18] Adam Clay, Kathryn Mann, and Cristóbal Rivas. On the number of circular orders on a group. *J. Algebra*, 504:336–363, 2018.

[Dv20] Warren Dicks and Zoran Šunić. Orders on trees and free products of left-ordered groups. *Canad. Math. Bull.*, 63(2):335–347, 2020.

[Ghy01] Étienne Ghys. Groups acting on the circle. *Enseign. Math. (2)*, 47(3-4):329–407, 2001.