

ON MULTIVARIATE GAMMA DISTRIBUTIONS

by

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ABSTRACT

This thesis is an exposition of the various forms of Multivariate Gamma Distributions available in the literature. We begin by discussing the univariate gamma distributions and some of their most important properties. These distributions are then extended to the vector variate and matrix variate cases. Derivations, applications and properties are given for gamma distributions in these two categories. Further generalizations associated with several matrix variate gamma variables are also included.

RÉSUMÉ

Cette thèse est un exposé sur les différentes formes existantes de distributions gamma multidimensionnelles. Nous discutons d'abord la loi gamma et ses propriétés les plus importantes. Puis nous présentons ses extensions à argument vectoriel et à argument matriciel. Les dérivations, applications et propriétés des distributions gamma appartenant à ces deux catégories sont analysées. Quelques généralisations dépendant de plusieurs variables gamma à argument matriciel sont également considérées.

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CHAPTER 1

MATHEMATICAL PRELIMINARIES

INTRODUCTION.

There is a vast literature on the various forms of univariate, multivariate and matrix-variate gamma distributions. The goal of this thesis is to present various existing forms of multivariate gamma distributions, together with some of their properties and applications.

In chapter 1, we give some mathematical preliminaries that will be used throughout the whole thesis.

The univariate gamma and generalized gamma distributions are introduced in chapter 2. Functions of independent variables having the above densities, as well as, some applications are also considered. The purpose of this chapter is to provide the necessary background to understand the chapters that follow.

In chapter 3, we study the vector variate gamma distributions. In particular, we consider the multivariate exponential, the multivariate chi-square and the multivariate gamma distributions. Some of their applications are also pointed out.

Chapter 4 consists of matrix variate gamma and other densities such as matrix variate beta and matrix variate Dirichlet. Some generalizations associated with several matrix variate gamma variables are also discussed.

In the end, some results are given in the Appendix, followed by a large number of references.

1.1 GAMMA FUNCTION.

The gamma function $\Gamma(\alpha)$ can be defined by one of the following expressions:

Definition 1:

$$\begin{aligned}\Gamma(\alpha) &= \int_0^{\infty} e^{-t} t^{\alpha-1} dt \\ &= \int_0^1 (\log(1/t))^{\alpha-1} dt,\end{aligned}\tag{1.1.1}$$

for $\operatorname{Re}(\alpha) > 0$, where $\operatorname{Re}(\cdot)$ means the real part of (\cdot) .

Definition 2:

$$\begin{aligned}\Gamma(\alpha) &= \lim_{n \rightarrow \infty} \frac{n! n^{\alpha}}{\alpha(\alpha+1) \dots (\alpha+n)} \\ &= \lim_{n \rightarrow \infty} n^{\alpha} [\alpha(1+\alpha)(1+\alpha/2) \dots (1+\alpha/n)]^{-1} \\ &= \alpha^{-1} \prod_{n=1}^{\infty} [(1+1/n)^{\alpha} (1+\alpha/n)^{-1}].\end{aligned}\tag{1.1.2}$$

Definition 3:

$$1/\Gamma(\alpha) = \alpha e^{\gamma\alpha} \prod_{n=1}^{\infty} [(1+\alpha/n)e^{-\alpha/n}],\tag{1.1.3}$$

where γ is the Euler's constant,

$$\begin{aligned}\gamma &= \lim_{n \rightarrow \infty} \left(\sum_{n=1}^m (1/n) - \log m \right) \\ &= 0.5772156649 \dots\end{aligned}\tag{1.1.4}$$

The following properties hold for the gamma function, provided that the various gammas are defined.

$$\Gamma(\alpha) = \frac{1}{\alpha} \int_0^{\infty} e^{-t} t^{\alpha} dt = \frac{1}{\alpha} \Gamma(1+\alpha).\tag{1.1.5}$$

$$\Gamma(\alpha + n) = (\alpha)(\alpha + 1)(\alpha + 2) \dots (\alpha + n - 1)\Gamma(\alpha). \quad (1.1.6)$$

$$\begin{aligned} \Gamma(\alpha)/\Gamma(\alpha - n) &= (\alpha - 1)(\alpha - 2) \dots (\alpha - n) \\ &= (-1)^n \Gamma(-\alpha + n + 1)/\Gamma(-\alpha + 1). \end{aligned} \quad (1.1.7)$$

$$\begin{aligned} \Gamma(-\alpha + n)/\Gamma(-\alpha) &= (-1)^n \alpha(\alpha - 1) \dots (\alpha - n + 1) \\ &= (-1)^n \Gamma(\alpha + 1)/\Gamma(\alpha - n + 1). \end{aligned} \quad (1.1.8)$$

$$\Gamma(\alpha)\Gamma(-\alpha) = -\alpha^{-2} \prod_{n=1}^{\infty} (1 - \alpha^2/n^2)^{-1} \quad (1.1.9)$$

To prove that (1.1.1), (1.1.2) and (1.1.3) refer to the same function and for more properties of the gamma function, refer to Erdélyi et al. (1981, p. 2).

The Gauss-Legendre multiplication formula for gamma functions is given as

$$\Gamma(m\alpha) = (2\pi)^{(1-m)/2} m^{m\alpha-(1/2)} \prod_{j=0}^{m-1} \Gamma(\alpha + \frac{j}{m}) \text{ for } m = 2, 3, \dots \quad (1.1.10)$$

That is, when $m=2$ we have the duplication formula,

$$\Gamma(2\alpha) = 2^{2\alpha-1} \pi^{-1/2} \Gamma(\alpha) \Gamma(\alpha + \frac{1}{2}). \quad (1.1.11)$$

Some derivatives of the logarithm of $\Gamma(\alpha)$ are the following.

(1) The digamma function or the psi function

$$\begin{aligned} \psi(\alpha) &= \frac{d}{d\alpha} \log \Gamma(\alpha) \\ &= \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}. \end{aligned} \quad (1.1.12)$$

(2) The trigamma function

$$\begin{aligned} \psi'(\alpha) &= \frac{d}{d\alpha} [\psi(\alpha)] \\ &= \frac{d^2}{d\alpha^2} [\log \Gamma(\alpha)]. \end{aligned} \quad (1.1.13)$$

1.2 LAPLACE AND INVERSE LAPLACE TRANSFORMS.

The Laplace transform of a function $f(t)$ is defined as

$$\begin{aligned}\mathcal{L}\{f\}(s) &= F(s) \\ &= \int_0^{\infty} f(t)e^{-st} dt\end{aligned}\tag{1.2.1}$$

This integral will converge to an analytic function of $F(s)$ for a sufficiently large value of the real part of s if $f(t)$ does not become infinite too rapidly as $t \rightarrow \infty$. The boundary of the region of convergence of (1.2.1) is the line $\text{Re}(s) = \sigma$ and σ is called the abscissa of convergence of the Laplace transform.

The inverse Laplace transform of $F(s)$ is

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s) ds.\tag{1.2.2}$$

Tables for both Laplace transforms and inverse Laplace transforms can be obtained from Erdélyi et al. (1954, pp. 129-301).

1.3 THE FOURIER TRANSFORM.

If $f(x)$ is a function of a real variable X , its Fourier transform $F_t(f(x))$ is defined as

$$F_t(f(x)) = \int_{-\infty}^{\infty} e^{itx} f(x) dx,\tag{1.3.1}$$

where $i = \sqrt{-1}$ and t is a real variable. More specifically, if $f(x)$ is defined and single valued almost everywhere on the range $-\infty < x < \infty$, and is such that the integral

$$\int_{-\infty}^{\infty} |f(x)| e^{ikx} dx$$

converges for some real value k , then $F_t(f(x))$ is the Fourier transform of $f(x)$. It is usually referred to by statisticians as the characteristic function of $f(x)$, when $f(x)$ is a density function and e^{itx} is called the kernel. Conversely, if the Fourier transform is absolutely integrable over the real line $-\infty < t < \infty$, or is analytic in some horizontal strip $-\alpha < it < \beta$ of the complex plane, then $f(x)$ is uniquely determined by the inversion integral, (often referred to as the inverse Fourier transform),

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} F_t(f(x)) dt. \quad (1.3.2)$$

It is worthwhile noting that if $f(x)$ is a probability density function, its characteristic function always exists and determines the distribution function $F_X(x)$ uniquely. In particular,

$$F_X(x) = F(0) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - e^{-itx}}{it} F_t(f(x)) dt. \quad (1.3.3)$$

We may note that many of the simple properties of Laplace transforms have analogues with Fourier transforms. One may refer to Titchmarsh(1937), and Campbell and Foster(1948) for the properties and applications of Fourier transforms. Erdélyi et al. (1954, pp 117-125) have provided tables for Fourier transforms.

1.4 THE CALCULUS OF RESIDUES.

We consider some techniques of calculating the residues of integrands involving products of gamma functions

(a) By definition, at $\alpha = -\nu$ the residue of $\Gamma(\alpha)$ is

$$\begin{aligned} R_\nu &= \lim_{\alpha \rightarrow -\nu} (\alpha + \nu) \Gamma(\alpha) \\ &= \lim_{\alpha \rightarrow -\nu} \frac{\Gamma(\alpha + \nu + 1)}{(\alpha + \nu - 1)(\alpha + \nu - 2) \dots (\alpha)} \end{aligned}$$

$$= \frac{(-1)^\nu}{\nu!}. \quad (1.1.1)$$

(b) To obtain the residue of the product $\Gamma(\alpha)\Gamma(\alpha+m)$, $m = 0, 1, \dots$, we first find the poles of $\Gamma(\alpha)$ and $\Gamma(\alpha+m)$. These are given by the equations

$$\alpha = -\nu, \nu = 0, 1, 2, \dots \text{ and}$$

$$\alpha = -\nu, \nu = m, m+1, \dots \quad (1.1.2)$$

That is, for $\nu = 0, 1, 2, \dots, m-1$ the poles are of order one and for $\nu = m, m+1, \dots$ the poles are of order two. We obtain the residues,

$$\begin{aligned} R_\nu &= \lim_{\alpha \rightarrow -\nu} (\alpha + \nu) \Gamma(\alpha) \Gamma(\alpha + m) \\ &= \frac{(-1)^\nu}{\nu!} \Gamma(-\nu + m), \nu = 0, 1, \dots, m-1 \end{aligned} \quad (1.1.3)$$

and

$$\begin{aligned} R'_\nu &= \lim_{\alpha \rightarrow -\nu} \frac{\partial}{\partial \alpha} (\alpha + \nu)^2 \Gamma(\alpha) \Gamma(\alpha + m) \\ &= \lim_{\alpha \rightarrow -\nu} \frac{\partial}{\partial \alpha} \frac{\Gamma^2(\alpha + \nu + 1)}{(\alpha + \nu - 1)^2 (\alpha + \nu - 2)^2 \dots (\alpha + m)^2 (\alpha + m - 1) \dots (\alpha)} \\ &= A_0 \alpha_0 \\ &= \frac{(-1)^m}{\nu! (\nu - m)!} [2\psi(1) + 2(1 + \frac{1}{2} + \dots + \frac{1}{m - \nu}) + (\frac{1}{\nu - m - 1} + \dots + \frac{1}{\nu})], \end{aligned} \quad (1.1.4)$$

for $\nu = m, m+1, \dots$. Using the simplification of ψ function given by $\psi(n+1) = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \gamma$, where γ is given in (1.1.4), we obtain

$$R'_\nu = \frac{(-1)^m}{\nu! (\nu - m)!} [\psi(\nu + 1) + \psi(\nu - m + 1)], \nu = m, m+1, \dots, \quad (1.1.5)$$

where

$$A_0 = \frac{\partial}{\partial \alpha} \log[(\alpha + \nu)^2 \Gamma(\alpha) \Gamma(\alpha + m)], \text{ at } \alpha = -\nu \quad (1.4.6)$$

and

$$\alpha_0 = (\alpha + \nu)^2 \Gamma(\alpha) \Gamma(\alpha + m), \text{ at } \alpha = -\nu. \quad (1.4.7)$$

(c) Suppose that Δ is a gamma product with a pole of order n at $\alpha = b$, we find the residue of $\Delta x^{-\alpha}$ at $\alpha = b$. From the calculus of residues,

$$R = \frac{1}{(n-1)!} \frac{\partial^{n-1}}{\partial \alpha^{n-1}} [(\alpha - b)^n \Delta x^{-\alpha}]. \quad (1.4.8)$$

We can easily verify that

$$\begin{aligned} \frac{\partial^{n-1}}{\partial \alpha^{n-1}} [(\alpha - b)^n \Delta x^{-\alpha}] &= x^{-\alpha} \left[\frac{\partial}{\partial \alpha} + (-\log x) \right]^{n-1} (\alpha - b)^n \Delta \\ &= x^{-\alpha} \sum_{r=0}^{n-1} \binom{n-1}{r} (-\log x)^{n-1-r} \frac{\partial^r}{\partial \alpha^r} (\alpha - b)^n \Delta. \end{aligned} \quad (1.4.9)$$

Now,

$$\begin{aligned} \frac{\partial^r}{\partial \alpha^r} (\alpha - b)^n \Delta &= \frac{\partial^{r-1}}{\partial \alpha^{r-1}} \{[(\alpha - b)^n \Delta] \left[\frac{\partial}{\partial \alpha} \log(\alpha - b)^n \Delta \right]\} \\ &= \left\{ \sum_{r_1=0}^{r-1} \binom{r-1}{r_1} A^{r-1-r_1} \sum_{r_2=0}^{r_1-1} \binom{r_1-1}{r_2} A^{r_1-1-r_2} \dots \right\} B, \end{aligned} \quad (1.4.10)$$

where

$$(1) A = \frac{\partial}{\partial \alpha} \log B,$$

$$(2) B = (\alpha - b)^n \Delta,$$

(3) A^s denotes the s -th partial derivative of A with respect to α .

Evaluating (1.4.9) and (1.4.10) at $\alpha = b$, we obtain

$$R = \frac{x^{-b}}{(n-1)!} \sum_{r=0}^{n-1} \binom{n-1}{r} (-\log x)^{n-1-r} \times \left\{ \sum_{r_1=0}^{r-1} \binom{r-1}{r_1} A_0^{r-1-r_1} \sum_{r_2=0}^{r_1-1} \binom{r_1-1}{r_2} A_0^{r_1-1-r_2} \dots \right\} B_0, \quad (1.4.11)$$

where (1) $B_0 = (\alpha - b)^n \Delta$, at $\alpha = b$,

(2) $A_0^s = \frac{\partial^s}{\partial \alpha^s} [\log(\alpha - b)^n \Delta]$, at $\alpha = b$.

1.5 MELLIN AND INVERSE MELLIN TRANSFORMS.

The Mellin transform of $f(x)$ is defined as

$$\phi(s) = \int_0^\infty x^{s-1} f(x) dx. \quad (1.5.1)$$

More specifically, if $f(x)$ is a real function that is defined and single-valued almost everywhere for $x \geq 0$, and is absolutely integrable over the range $(0, \infty)$, the Mellin transform (1.5.1) exists.

If the Mellin transform exists and is an analytic function of the complex variable s for $c_1 \leq \operatorname{Re}(s) \leq c_2$, where c_1 and c_2 are real, then the inverse Mellin transform of $\phi(s)$ is

$$f(x) = \frac{1}{2\pi i} \int_{w-i\infty}^{w+i\infty} x^{-s} \phi(s) ds. \quad (1.5.2)$$

If $f(x)$ is a probability density function, the Mellin transform evaluated at $s = k$ gives the $(k-1)$ st moment of $f(x)$ about the origin, $k = 1, 2, \dots$

In order to find the inverse Mellin transform, the method of residues (sec 1.4) is particularly useful. Other recursion formulae and algorithms that are involved in evaluating the inverse Mellin transform can be obtained from Springer(1979, pp. 109-112).

CHAPTER 2

UNIVARIATE GAMMA AND GENERALIZED GAMMA DISTRIBUTIONS

2.1 INTRODUCTION

The generalized gamma distributions were discussed by Amoroso (1925) and D'addorio (1932), who fitted such a distribution to data on income rates. Further properties and statistical problems associated with the distribution have been dealt with by authors such as Stacy (1962), Bain and Weeks (1965), Roslonek (1968), Malik (1969), Jakuszenkow (1974) and Gajek (1983). Special cases of the generalized gamma distribution include the Weibull, gamma, Rayleigh, exponential and chi-square distributions. Another special case has been derived by Lienhard (1964) on a statistical-mechanical basis to describe rainfall run-off from a watershed.

In this chapter, we shall study the generalized univariate gamma including some of its special cases such as gamma, Weibull etc. Then, we shall consider the functions of independent univariate generalized gamma variables and finally we shall give some of the applications of the generalized gamma density.

2.2 GENERALIZED UNIVARIATE GAMMA

The generalized gamma distribution has various expressions. Stacy (1962) gave the expression as

$$f(x; a, d, p) = \begin{cases} \frac{p/a^d}{\Gamma(d/p)} x^{d-1} e^{-(x/a)^p}, & x > 0, d > 0, a > 0, p > 0 \\ 0, & \text{otherwise.} \end{cases} \quad (2.2.1)$$

Stacy and Mihram (1965) extended (2.2.1) to

$$f(x; a, \nu, p) = \begin{cases} \frac{|p|}{a^p \nu \Gamma(\nu)} e^{-(x/a)^p} x^{p\nu-1} & , x > 0, \nu > 0, a > 0, p \neq 0 \\ 0, & \text{otherwise.} \end{cases} \quad (2.2.2)$$

Taguchi (1980) suggested yet another expression with two shape parameters, as

$$f(x; \alpha, h) = \begin{cases} \frac{\alpha x^{\alpha-1}}{\Gamma(1/h+1)} e^{-x^{\alpha h}} & , x > 0, \alpha > 0, h > 0 \\ 0, & \text{otherwise.} \end{cases} \quad (2.2.3)$$

Other expressions can be found in Amoroso (1925) and Johnson and Kotz (1970, p.197)

Lienhard and Meyer (1967) have described a physical model generating generalized gamma distributions. Let us consider the situation where the occurrence of an event such as the failure of a component or system, depends on some variable such as the stress to which the part has been subjected to, or the time during which it has been subjected to a given level of stress or use. Let the variable be denoted by t and the number of occurrences of the event during the interval $[t_{i-1}, t_i]$ be designated by N_i where $t_i - t_{i-1} \equiv \Delta t$ and t_0 is the arbitrary origin.

The requirements that are imposed upon the N_i 's are the following:

(1) the total number of occurrences of the event is fixed,

$$\sum_{i=1}^{\infty} N_i = N. \quad (2.2.4)$$

The N_i 's and N are assumed to be large numbers.

(2) for each choice of β , the following sum is a positive constant

$$\sum_{i=1}^{\infty} \frac{N_i}{N} t_i^{\beta} = k. \quad (2.2.5)$$

(3) the number of distinguishable ways, g_i , in which the event can occur in the interval $[t_{i-1}, t_i]$ is proportional to a specified power of t_i . Thus,

$$g_i = At_i^{\alpha-1}. \quad (2.2.6)$$

Moreover, $\alpha > 0, \beta > 0$ and $k > 0$.

We derive the distribution of the number of events by determining the most probable distribution satisfying the above requirements. Let us designate as w , the number of ways in which N distinguishable occurrences of the event can take place. If N_i of these occurrences must take place in $[t_{i-1}, t_i]$ and if the number of ways the event can occur in this interval is given by g_i , it can be shown (Sommerfeld (1956)) that

$$W = N! \prod_{i=1}^{\infty} \frac{g_i^{N_i}}{N_i!}. \quad (2.2.7)$$

Suppose that \tilde{N}_i are those values of N_i 's which maximize W , subject to requirements (1), (2) and (3) above. Then it can be shown that:

(a) the explicit expression for N_i 's is given by

$$\tilde{N}_i/N = \frac{\Delta t}{\Gamma(\alpha/\beta)} \beta(\beta k/\alpha)^{-\alpha/\beta} t_i^{\alpha-1} \exp\left(-\frac{\alpha t_i^\beta}{\beta k}\right). \quad (2.2.8)$$

(b) the maximum value of w is very much larger than the value of w corresponding to N_i 's that are significantly different from the \tilde{N}_i 's, provided the N_i 's and N are large. Refer to Sommerfeld (1956) for proof.

We may suppose that the $\frac{\tilde{N}_i}{N}$'s represent the discrete probability distribution associated with the random variable T , where T is the time or stress at which the first occurrence of

the event under consideration takes place. Accordingly:

$$P(t_{i-1} \leq T < t_i) = \frac{\tilde{N}_i}{N}, \quad (2.2.9)$$

where $i = 1, 2, \dots$. We now approximate the discrete distribution in (2.2.9) with a continuous probability density function, f , as below

$$\frac{\tilde{N}_i}{N} = \int_{t_{i-1}}^{t_i} f(t) dt. \quad (2.2.10)$$

Using the mean value theorem, the integral (2.2.10) becomes $\Delta t f(\xi)$ where $(t_i - \Delta t) \leq \xi \leq t_i$.

Hence, letting $\Delta t \rightarrow 0$ and using (2.2.8), we obtain

$$f(x) = \begin{cases} \frac{\beta}{\Gamma(\alpha/\beta)} (\beta k / \alpha)^{-\alpha/\beta} t^{\alpha-1} e^{-(\alpha t^\beta)/(\beta k)} & , t \geq 0, \alpha > 0, \beta > 0, k > 0 \\ 0, & \text{otherwise.} \end{cases} \quad (2.2.11)$$

Introducing $a = \sqrt[\beta]{\beta k / \alpha}$ in equation (2.2.11) gives (2.2.1).

2.2.1 Properties.

Let $X \sim f$ denote f as the density function of a random variable X . Then (2.2.2) and the expression $X \sim f(x; a, \nu, p)$ jointly define X .

We can easily show that

$$kX \sim f(x; ka, \nu, p) \quad \text{where } k > 0, \quad (2.2.12)$$

and

$$X^m \sim f(y; a^m, \nu, p/m) \quad \text{where } m \neq 0. \quad (2.2.13)$$

The s th moment of X , ($s = 1, 2, \dots$), is

$$E(X^s) = \frac{\Gamma((\nu p + s)/p) a^\nu}{\Gamma(\nu)}, \quad s/p > -\nu. \quad (2.2.14)$$

The distribution function is

$$F(x) = \begin{cases} \Gamma_w(\nu)/\Gamma(\nu) & \text{if } p > 0 \\ 1 - (\Gamma_w(\nu)/\Gamma(\nu)) & \text{if } p < 0, \end{cases} \quad (2.2.15)$$

where $w = (x/a)^\nu$ and

$$\Gamma_w(\nu) = \int_0^w x^{\nu-1} e^{-x} dx.$$

A number of familiar distributions can be obtained as special cases of (2.2.1).

Case 1 · Setting $d = p$ in (2.2.1), we obtain the Weibull distribution.

$$f(x; a, d) = \begin{cases} \frac{d}{a^d} x^{d-1} e^{-(x/a)^d}, & x > 0, a > 0, d > 0 \\ 0, & \text{otherwise.} \end{cases} \quad (2.2.16)$$

This distribution is named after Waloddi Weibull (1939), a Swedish Physicist, who used it to represent the distribution of the breaking strength of materials. A bibliography about the distribution is given by Weibull (1951). The Weibull distribution being often suitable where the condition of "strict randomness" of the exponential does not hold, has many applications. Examples of applications can be found in papers by Berretoni (1964), Plait (1962) and so on.

Johnson and Kotz (1970, pp. 250-271) provide a good general review of the Weibull distribution, including historical development, genesis, properties, characterizations, order statistics, estimation, tables, graphs and its applications.

Setting $d = p = 2$ in (2.2.1), we obtain the Rayleigh distribution which is a special case of the Weibull distribution. It was originally derived by Lord Rayleigh in a problem in acoustics. It is a very common distribution which occurs in works on radar, the detection

of signals in the presence of noise etc. Archer (1967) and Siddiqui (1962) have given a useful summary of the properties of the Rayleigh distribution.

Case 2 : Setting $p=1$ in (2.2.1), we obtain the gamma distribution.

$$f(x; a, d, 1) = \begin{cases} \frac{x^{d-1} e^{-x/a}}{a^d \Gamma(d)} & , x > 0, a > 0, d > 0 \\ 0, & \text{otherwise.} \end{cases} \quad (2.2.17)$$

The gamma distribution is Type 3 of Pearson's system of distributions. Many of its basic properties and results are given by Johnson and Kotz (1970, pp. 166-206). A different generalization of this distribution called the Lagrangian gamma was developed by Nelson and Consul (1974), as the distribution of the time between occurrences of a generalized Poisson process.

If we let $d = n/2$ and $a = 2$, we obtain the chi-square distribution with n degrees of freedom. A general review of this distribution, including properties, characterizations and so on can be found in Johnson and Kotz (1970, pp. 166-206).

We obtain the exponential distribution when $d = p = 1$. This has widespread use in statistical procedures and its general review can also be obtained from Johnson and Kotz (1970, pp. 207-208).

2.2.2 Characterizations.

(a) Suppose that we have 3 independent positive random variables X_1, X_2, X_3 and a pair of quotients (Y_1, Y_2) , where $Y_1 = X_1/X_2$ and $Y_2 = X_2/X_3$. Malik(1969) has shown that the necessary and sufficient condition for X_k to be generalized gamma distributed with parameters d_k and p , (p and a common, $k = 1, 2, 3$) is that the joint distribution of (Y_1, Y_2)

is the bivariate distribution, given by the density

$$g(y_1, y_2) = \frac{\Gamma(d_1/p + d_2/p + d_3/p)}{\Gamma(d_1/p)\Gamma(d_2/p)\Gamma(d_3/p)} p^2 y_1^{d_1-1} y_2^{d_2-1} [1 + (y_1^p) + (y_2^p)]^{-(d_1/p + d_2/p + d_3/p)},$$

where $y_1, y_2 > 0$

$$= 0, \text{ otherwise.} \quad (2.2.18)$$

The above characterization has been generalized by Jakuszenkow (1974). She proved that a necessary and sufficient condition in order that independent random variables X_0, X_1, \dots, X_n , ($n \geq 2$), have the generalized gamma distribution (2.2.1) is that the n -dimensional random variable (Y_1, \dots, Y_n) defined as follows:

$$Y_k = \Psi_k (X_1/X_0; \dots; X_n/X_0), \quad (k = 1, \dots, n), \quad (2.2.19)$$

has a generalized Dirichlet distribution with the density below.

$$k(y_1, \dots, y_n) = \frac{\Gamma(\frac{1}{\alpha}(\sum_{j=0}^n p_j))}{\prod_{j=0}^n \Gamma(p_j/\alpha)} | \alpha | \prod_0^n \eta_k^{p_k-1}(y_1, \dots, y_n)$$

$$\times [1 + \sum_1^n \eta_k^\alpha(y_1, \dots, y_n)]^{-(\sum_0^n p_j)'\alpha} | J |, \quad (2.2.20)$$

where $p_j \alpha > 0$, $(y_1, \dots, y_n) \in \Omega$, $j = 0, 1, \dots, n$ and $k(y_1, \dots, y_n) = 0$, otherwise.

Here Ω is the image of $\{X_k \mid x_k > 0, k = 0, 1, \dots, n\}$ under transformation (2.2.19). This transformation is assumed to be one-to-one with respect to the variables $Z_k = X_k/X_0$, $k = 1, \dots, n$ for $X_k > 0$. The functions $Z_k = \eta_k(y_1, \dots, y_n)$ are inverse to the functions (2.2.19) and are of class C_1 . J is the jacobian of the transformation

$$Z_k = X_k/X_0$$

$$= \eta_k(y_1, \dots, y_n), \quad (2.2.21)$$

where $k = 1, 2, \dots, n$ and $J \neq 0$ for $(y_1, \dots, y_n) \in \Omega$.

(b) Roy (1984) proved another characterization which is as follows:

If X, Y are two independent nonnegative random variables, then X and Y have the generalized gamma distribution $f(x; \alpha_1, \beta_1, \gamma)$ and $f(y; \alpha_2, \beta_2, \gamma)$ respectively, if and only if the conditional distribution of $X | (Z = X/Y)$ is the generalized gamma distribution. In this case, the distribution has parameters $\alpha = \alpha_1 + \alpha_2$, $\beta = (\beta_1 \beta_2 z^\gamma) / (\beta_1 + \beta_2 z^\gamma)$ and γ .

2.3 FUNCTIONS OF INDEPENDENT UNIVARIATE GENERALIZED GAMMA VARIABLES.

Let X_1, X_2, \dots, X_n be an independent set of random variables, X_i , having the generalized gamma distribution,

$$\begin{aligned} f(x_i; a_i, \alpha_i, \beta_i) &= \frac{\beta_i}{\Gamma(\alpha_i/\beta_i)} a_i^{\alpha_i/\beta_i} x_i^{\alpha_i-1} \exp[-a_i x_i^{\beta_i}], \\ &\text{where } x_i, a, \alpha, \beta > 0, \text{ for } i = 1, \dots, n \\ &= 0, \text{ elsewhere.} \end{aligned} \tag{2.3.1}$$

Stacy (1962) gave an expression for the distribution of the sum of X_1, X_2, \dots, X_n and he pointed out that his results are more general than the results obtained by Robbins(1948), who studied the distribution of a definite quadratic form. Malik (1967) gave the exact distribution of the quotient of independent generalized gamma variates. In this section, we will find the distribution of a linear combination of independent generalized gamma random variables using the techniques given by Mathai and Saxena (1973). We will also derive the distribution for the products and ratios of independent generalized univariate gamma variates using the method given by Mathai (1972).

2.3.1 Distribution of the sum of independent univariate generalized gamma variates

Let $Y = \sum_{i=1}^n X_i$, where the X_i 's are defined as above. Now, the density (2.3.1) can be written as an H-function, (Appendix A1),

$$\begin{aligned} f(x_i; a_i, \alpha_i, \beta_i) &= c_i x_i^{-1} \mathbf{H}_{0,1}^{1,0}[a_i x_i^{\beta_i} \mid \alpha_i/\beta_i, 1], \\ &\text{when } x_i > 0, \alpha_i > 0, a_i > 0, \beta_i > 0, i = 1, \dots, n \\ &= 0, \text{elsewhere.} \end{aligned} \quad (2.3.2)$$

Here

$$c_i = \frac{\beta_i}{\Gamma(\alpha_i/\beta_i)} a_i^{(\alpha_i - \beta_i^2)/\beta_i}. \quad (2.3.3)$$

The moment generating function of X_i can be obtained as,

$$\begin{aligned} M(t) &= \int_0^\infty c_i x_i^{-1} e^{x_i t} \mathbf{H}_{0,1}^{1,0}[a_i x_i^{\beta_i} \mid \frac{\alpha_i}{\beta_i}, 1] dx_i \\ &= c_i \mathbf{H}_{1,1}^{1,1}[\frac{a_i}{t^{\beta_i}} \mid (1, \beta_i), (\frac{\alpha_i}{\beta_i}, 1)]. \end{aligned} \quad (2.3.4)$$

Since X_1, \dots, X_n are assumed to be independent, we obtain a moment generating function $\Phi(t)$ for Y as,

$$\begin{aligned} \Phi(t) &= \prod_{i=1}^n M(t) \\ &= \prod_{i=1}^n c_i \mathbf{H}_{1,1}^{1,1}[\frac{a_i}{t^{\beta_i}} \mid (1, \beta_i), (\frac{\alpha_i}{\beta_i}, 1)]. \end{aligned} \quad (2.3.5)$$

Using the series expansion for the H-function given by Braaksma(1964), we obtain (2.3.5)

in a series form,

$$\Phi(t) = \prod_{i=1}^n c_i \prod_{i=1}^n \sum_{\nu_i=0}^{\infty} \frac{\Gamma(-\nu_i) \Gamma(1 - \alpha_i/\beta_i + (1 + \nu_i)) \Gamma(\beta_i(1 + \nu_i))}{\Gamma(1 + \nu_i) \Gamma(\alpha_i/\beta_i - 1 - \nu_i)}$$

$$\times \frac{(-1)^{\nu_i}}{\nu_i!} \frac{a_i^{(1+\nu_i)}}{t^{\beta_i(1+\nu_i)}}, \quad (2.3.6)$$

for the case when the poles of the integrand are simple. Collecting the terms containing the powers of t , we obtain the quantity

$$t^{-\sum_{i=1}^n [\beta_i(1+\nu_i)]}. \quad (2.3.7)$$

The density of Y can now be obtained by taking the inverse Laplace Transform of $\Phi(t)$ in (2.3.6). The inverse Laplace transform of (2.3.6) can be seen to be

$$\frac{y^{\sum_{i=1}^n [\beta_i(1+\nu_i)]-1}}{\Gamma(\sum_{i=1}^n [\beta_i(1+\nu_i)])}. \quad (2.3.8)$$

After multiplying and summing up, we get the density $h(y)$ of Y as

$$\begin{aligned} h(y) &= \left\{ \prod_{i=1}^n c_i \right\} \\ &\quad \prod_{i=1}^n \sum_{\nu_i=0}^{\infty} \frac{\Gamma(-\nu_i) \Gamma(1 - \alpha_i/\beta_i + (1 + \nu_i)) \Gamma(\beta_i(1 + \nu_i))}{\Gamma(1 + \nu_i) \Gamma(\alpha_i/\beta_i - 1 - \nu_i)} \times a_i^{(1+\nu_i)} \\ &\quad \times \frac{(-1)^{\nu_i}}{\nu_i!} \times \frac{y^{\sum_{i=1}^n [\beta_i(1+\nu_i)]-1}}{\Gamma(\sum_{i=1}^n [\beta_i(1+\nu_i)])}, \\ &\quad \text{for } y > 0, a_i > 0, \beta_i > 0, \alpha_i > 0, \\ &= 0, \text{ otherwise,} \end{aligned} \quad (2.3.9)$$

where c_i is defined in (2.3.3). In order to find the distribution of general linear combinations of variables X_1, \dots, X_n , one may refer to Mathai and Saxena(1973).

2.3.2 Distribution of products and ratios of independent generalized gamma variates.

Let random variables X_1, \dots, X_k be independently distributed according to (2.3.1) and let

$$Y = \frac{X_1 \dots X_m}{X_{m+1} \dots X_k}. \quad (2.3.10)$$

If the density $g(y)$ of Y exists, then the h th moment of Y about the origin, is given by

$$\begin{aligned} E(Y^h) &= \prod_{j=1}^m E(x_j^h) \prod_{j=m+1}^k E(x_j^{-h}) \\ &= \prod_{j=1}^m \left\{ \frac{\Gamma[(\alpha_j + h)/\beta_j]}{\Gamma(\alpha_j/\beta_j)} a_j^{-h/\beta_j} \right\} \prod_{j=m+1}^k \left\{ \frac{\Gamma[(\alpha_j - h)/\beta_j]}{\Gamma(\alpha_j/\beta_j)} a_j^{h/\beta_j} \right\} \\ &= C \prod_{j=1}^m \frac{\Gamma(\alpha_j/\beta_j + h/\beta_j)}{a_j^{h/\beta_j}} \prod_{j=m+1}^k \frac{\Gamma(\alpha_j/\beta_j - h/\beta_j)}{a_j^{-h/\beta_j}}, \end{aligned} \quad (2.3.11)$$

where

$$C = \frac{1}{\prod_{j=1}^k \Gamma(\alpha_j/\beta_j)}. \quad (2.3.12)$$

Whenever the various gammas exist, from the theory of Mellin transforms (2.3.11) uniquely determines $g(y)$. Using the inverse Mellin transform of (2.3.11), we find that

$$\begin{aligned} g(y) &= \frac{y^{-1}}{2\pi i} \int_{c-i\infty}^{c+i\infty} E(y^h) y^{-h} dh \\ &= \frac{C y^{-1}}{2\pi i} \int_{c-i\infty}^{c+i\infty} \prod_{j=1}^m \frac{\Gamma(\alpha_j/\beta_j + h/\beta_j)}{a_j^{h/\beta_j}} \prod_{j=m+1}^k \frac{\Gamma(\alpha_j/\beta_j - h/\beta_j)}{a_j^{-h/\beta_j}} y^{-h} dh, \end{aligned} \quad (2.3.13)$$

where

$$(a) \quad i = \sqrt{-1},$$

$$(b) \quad C \text{ is given by (2.3.12).}$$

From Braaksma (1964), (2.3.13) can be written as an H-function. So, we obtain

$$g(y) = \begin{cases} C y^{-1} H_{n,m}^{m,n} \left[(a/a') y \mid \begin{smallmatrix} (1-\alpha_{m+1}/\beta_{m+1}, 1/\beta_{m+1}), \dots, (1-\alpha_k/\beta_k, 1/\beta_k) \\ (\alpha_1/\beta_1, 1/\beta_1), \dots, (\alpha_m/\beta_m, 1/\beta_m) \end{smallmatrix} \right], & 0 < y < \infty \\ 0, & \text{otherwise.} \end{cases} \quad (2.3.14)$$

where

$$(a) \quad n = k - m,$$

$$(b) \quad a = \prod_{j=1}^m a_j^{1/\beta_j},$$

$$(c) \quad a' = \prod_{j=m+1}^k a_j^{1/\beta_j} \text{ and}$$

$g(y)$ in (2.3.14) exists if

$$(1) \quad \sum_{j=1}^m 1/\beta_j - \sum_{j=m+1}^k 1/\beta_j > 0,$$

$$(2) \quad \beta_j(\alpha_j/\beta_j + \nu) \neq \beta_h(-\alpha_h/\beta_h - r),$$

where $\nu, r = 0, 1, \dots$; $j = 1, \dots, m$; $h = m + 1, \dots, k$.

A representation of (2.3.14) in computable form is available from Mathai (1972) or Mathai and Saxena (1978).

Case 1 :

$$\alpha_1 = \alpha_2 = \dots = \alpha_k = \alpha.$$

$$\beta_1 = \beta_2 = \dots = \beta_k = \beta.$$

$$a_1 = a_2 = \dots = a_k = a.$$

The integral in (2.3.13) can be simplified using a simple transformation, to obtain

$$\begin{aligned} g(y) &= C \frac{\beta}{2\pi i} y^{-1} \int_{c-i\infty}^{c'+i\infty} \Gamma^m(\alpha/\beta + h) \Gamma^n(\alpha/\beta - h) [a^{m-n} y^\beta]^{-h} dh \\ &= C \beta a^{(m-n)/\beta} y^{-1} G_{n,m}^{m,n} [a^{m-n} y^\beta \mid_{(\alpha/\beta), \dots, (\alpha/\beta)}^{(1-\alpha/\beta), \dots, (1-\alpha/\beta)}] \end{aligned} \quad (2.3.15)$$

where $y > 0, a > 0, \alpha > 0, \beta > 0$. $G(\cdot)$ is a G-function which is a special case of an H-function (Appendix A1). For a definition of the Meijer's G-function, see Braaksma(1964).

The function (2.3.15) is available as the sum of the residues at the poles of $\Gamma^m(b+h)$ where

$b = \alpha/\beta$ The poles of $\Gamma^m(b+h)$ can be obtained from the equation,

$$b+h+\nu=0, \nu=0,1,\dots \quad (2.3.16)$$

and each one is of order m . Hence density $g(y)$ can be written as,

$$g(y) = c\beta a^{-\frac{m-n}{\beta}} y^{-1} \sum_{\nu=0}^{\infty} R_{\nu}, y > 0, m > n. \quad (2.3.17)$$

where R_{ν} is the residue of $\Gamma^m(b+h)\Gamma^n(b-h)[a^{m-n}y^{\beta}]^{-h}$ at $h=-(b+\nu)$. Using the calculus of residues (sec 1.4), we obtain

$$\begin{aligned} R_{\nu} &= \lim_{h \rightarrow -(b+\nu)} \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial h^{m-1}} (b+h+\nu)^m \Gamma^m(b+h) \\ &\quad \times \Gamma^n(b-h) (a^{m-n}y^{\beta})^{-h} \\ &= \frac{z^{b+\nu}}{(m-1)!} \left(\frac{\partial}{\partial h} - \log(z) \right)^{m-1} (b+h+\nu)^m \\ &\quad \times \Gamma^m(b+h) \Gamma^n(b-h), \text{ at } h=-(b+\nu), \text{ where } z = a^{m-n}y^{\beta}. \end{aligned} \quad (2.3.18)$$

Simplifying (2.3.18), we can write

$$\begin{aligned} R_{\nu} &= \frac{z^{b+\nu}}{(m-1)!} \sum_{r=0}^{m-1} \binom{m-1}{r} (-\log z)^{m-1-\nu} \times \\ &\quad \left[\sum_{r_1=0}^{r-1} \binom{r-1}{r_1} A_{\nu}(r-1-r_1) \sum_{r_2=0}^{r_1-1} \binom{r_1-1}{r_2} A_{\nu}(r_1-1-r_2) \dots \right] B_{\nu} \end{aligned} \quad (2.3.19)$$

where (i)

$$\begin{aligned} B_{\nu} &= (b+h+\nu)^m \Gamma^m(b+h) \Gamma^n(b-h), \text{ at } h=-(b+\nu) \\ &= \frac{(-1)^{m\nu}}{(\nu!)^m} \Gamma^n(2b+\nu). \end{aligned} \quad (2.3.20)$$

(ii)

$$A_{\nu}(r) = \frac{\partial^r}{\partial h^r} \log[(b+h+\nu)^m \Gamma^m(b+h) \Gamma^n(b-h)], \text{ at } h=-(b+\nu). \quad (2.3.21)$$

(a) When $m = 2, n = 0$, we obtain the density of $X_1 X_2$ as

$$\begin{aligned} g(y) &= C \beta z^{-1/\beta} \sum_{\nu=0}^{\infty} \frac{z^{b+\nu}}{(\nu!)^2} [-\log z + 2\psi(\nu+1)] \\ &= C \beta z^{-1/\beta} 2z^b K_0(2\sqrt{z}) \end{aligned} \quad (2.3.22)$$

[From Erdélyi et al. (1981, p. 9).] where,

(i) $K_n(z)$ is a modified Bessel function of integer order (See 0.3.4),

(ii) $z = a^{m-n} y^\beta$,

(iii) $b = \alpha/\beta$,

(iv) $\psi(z) = \frac{d}{dz} \log \Gamma(z) = -\gamma + (z-1) \sum_{n=0}^{\infty} [(n+1)n+z]^{-1}$.

(b) When $m = 2, n = 1$

$$g(y) = \begin{cases} C \beta z^{-1/\beta} \sum_{\nu=0}^{\infty} \frac{\Gamma(2b+\nu)}{(\nu!)^2} z^{b+\nu} [-\log z + 2\psi(\nu+1) + \psi(2b+\nu)] & , y > 0 \\ 0, & \text{otherwise.} \end{cases} \quad (2.3.23)$$

Case 2:

$$\beta_1 = \beta_2 = \dots = \beta_k = \beta.$$

In this case (2.3.14) can be written as,

$$g(y) = \begin{cases} C \beta y^{-1} G_{n,m}^{m,n} \left[y^\beta \frac{a_1 \dots a_m}{a_{m+1} \dots a_k} \mid \begin{matrix} (1-\alpha_j/\beta), j=m+1, \dots, k \\ (\alpha_j/\beta), j=1, \dots, m \end{matrix} \right], & 0 < y < \infty, m > n \\ 0, & \text{otherwise} \end{cases} \quad (2.3.24)$$

(2.3.24) can be put into a computable form by using the results from Mathai (1970), after

identifying the poles in $\prod_{j=1}^m \Gamma(\alpha_j/\beta + h)$.

Case 3:

$\beta_1 = \beta_2 = \dots = \beta_k$ are rational.

In this case, there exists a β such that

$$1/\beta_j = m_j/\beta, \quad j = 1, \dots, k \quad (2.3.25)$$

where m_1, \dots, m_k are positive integers since β_1, \dots, β_k are positive. Then,

$$\Gamma(\alpha_j/\beta_j + h/\beta_j) = \Gamma(m_j\alpha_j/\beta + m_j h/\beta). \quad (2.3.26)$$

Expanding (2.3.26) by using Gauss-Legendre multiplication formula (1.1.10), we obtain

$$\begin{aligned} \prod_{j=1}^m \Gamma(\alpha_j/\beta_j + h/\beta_j) \prod_{j=m+1}^k \Gamma(\alpha_j/\beta_j - h/\beta_j) = \\ \prod_{j=1}^m \prod_{r=0}^{m_j-1} \Gamma(\alpha_j/\beta + r/m_j + h/\beta) \prod_{j=m+1}^k \prod_{r=0}^{m_j-1} \Gamma(\alpha_j/\beta + r/m_j - h/\beta). \end{aligned} \quad (2.3.27)$$

So, (2.3.14) reduces to the form

$$g(y) = \begin{cases} C\beta y^{-1} G_{n', m'}^{m', n'} [y^\beta t \mid \begin{smallmatrix} (1-\alpha_j/\beta-r/m_j), j=m+1, \dots, k, r=0, 1, \dots, m_j-1 \\ (\alpha_j/\beta+r/m_j), j=1, \dots, m, r=0, 1, \dots, m_j-1 \end{smallmatrix}], & 0 < y < \infty \\ 0, & \text{otherwise,} \end{cases} \quad (2.3.28)$$

where

- (i) $t = \prod_{j=1}^m a_j^{m_j} / \prod_{j=m+1}^k a_j^{m_j}$,
- (ii) $m' = \sum_{j=1}^m m_j$,
- (iii) $n' = \sum_{j=m+1}^k m_j m', \quad n' > 0$.

Again, we can put (2.3.28) into computable forms by using the results in Mathai(1970).

2.4 APPLICATIONS.

(a) Generalized gamma distribution in reliability.

Special cases of the generalized gamma distribution such as gamma, Weibull, exponential and so on, are of great interest in reliability theory.

Let X be the age to first failure of a piece of equipment. We assume that X has a density function $f(x)$ and a distribution function $F(x)$. We define the reliability of the equipment at age x as

$$\begin{aligned} R(x) &= \text{prob. } (X > x) \\ &= 1 - F(x) \end{aligned} \tag{2.4.1}$$

and the failure rate, $Z(x)$ at age x as

$$Z(x) = f(x)/R(x). \tag{2.4.2}$$

The exponential failure density function is characterized by a constant failure rate. That is, the reliability for a given operating interval is the same, no matter from what portion of the useful life of a device the interval is taken.

The Rayleigh failure density function is characterized by a linearly increasing failure rate. That is, we have an intense aging of the equipment taking place and failures do not satisfy the conditions of stationary random process. So, as time increases the probability of failure-free operation decreases at a much higher rate than in the case of the exponential law.

The gamma distribution represents satisfactorily the distribution of the times of the occurrence of failures of redundant systems when the redundancies are connected according to the method of replacement and under the condition that the flows of failures of the main

system and of all the redundancies are simple.(Refer to Polovko (1968).) This distribution can also be a characteristic of the times of occurrence of failures of complex electromechanical systems if the components fail instantaneously during the initial stage of operation and during the wearout period of the system.

The Weibull distribution, just like the gamma distribution, can be used as a characteristic of the reliability of equipment during its burn-in period. This distribution is observed in the case of some mechanical parts and, in particular, it is used in the study of the reliability of ball bearings. Polovko(1968, pp. 73-95) has discussed more about the above failure density functions.

(b) The generalized gamma distribution is employed as a radial distribution in Engineering. It is associated with two dimensional kill probability, target evacuation and distribution of populations. McNolty(1968) assumed that the distribution of the distance from a typical impact point to a point target, is given by

$$g(s)ds = \frac{(\lambda s)^Q}{\gamma^{Q-1}} e^{-1/2[(\gamma^2/\lambda) + \lambda s^2]} I_{Q-1}(\gamma s) ds, \quad (2.4.3)$$

where $I_{Q-1}(x)$ is the modified Bessel function of order $Q - 1$, $\lambda, Q > 0$ and here Q is not necessarily an integer.

As a radial density function, expression (2.4.3) includes many useful special cases - the Rayleigh distribution, the non-central chi-square distribution with two degrees of freedom etc. Using the above distribution, the probability density functions for the random phase angle and the x, y components can be derived.

(c) Generalized gamma distribution in demography.

The growth of the urban component of a nation and the shift in population from rural to urban areas in the course of industrial development are processes that show both regularity and predictability. Consequently, it is possible to describe the concentration of population in terms of population density and area by comparatively simple expressions. Sherratt(1960) has shown that the distribution of urban populations can be a generalized gamma distribution. This is very useful to the complex of urban areas in the initial planning for services such as protection of population and industry against possible attack, as well as the logistic problems of supplying food, communications, housing, transportation and so on.

CHAPTER 3

VECTOR VARIATE GAMMA DISTRIBUTIONS

INTRODUCTION

The vector variate gamma distributions can be considered to be generalizations of the univariate gamma distributions to the vector variable cases such that all the marginals are again gamma. They are commonly known as the multivariate gamma distributions in the literature. In this chapter, we shall review some of the most popular generalizations such as multivariate exponential distributions which involve the exponential, multivariate chi-square distributions which involve the chi-square, and multivariate gamma distributions.

3.1 MULTIVARIATE EXPONENTIAL DISTRIBUTIONS.

These are distributions for which all the marginal distributions are exponential. There are a number of different bivariate exponential distributions and their extensions available in the literature.

Gumbel(1960) gives several bivariate exponential distributions which he derives from different types of bivariate distribution functions. By studying the joint distribution function,

$$F(x, y) = 1 - e^{-x} - e^{-y} + e^{-x-y-\delta xy}, \quad x \geq 0, y \geq 0, \quad (3.1.1)$$

where the parameter δ satisfies the inequalities

$$0 \leq \delta \leq 1, \quad (3.1.2)$$

he obtains the joint density function of X and Y as

$$f(x, y) = \begin{cases} e^{-x(1+\delta y-y)}[(1+\delta x)(1+\delta y) - \delta] & , x > 0, y > 0 \\ 0, & \text{otherwise.} \end{cases} \quad (3.1.3)$$

The marginal distributions of X and Y can be shown to be standard exponential and when $\delta = 0$, X and Y are independent.

The second system of Gumbel's bivariate exponential is obtained by applying the general formula for a bivariate distribution function,

$$F(x, y) = F_1(x)F_2(y)[1 + \alpha(1 - F_1(x))(1 - F_2(y))] \text{ where } -1 \leq \alpha \leq 1, \quad (3.1.4)$$

and the bivariate density function is given by

$$f(x, y) = f_1(x)f_2(y)[1 + \alpha(2F_1(x) - 1)(2F_2(y) - 1)]. \quad (3.1.5)$$

Letting $F_1(x)$ and $F_2(y)$ be exponential functions, we obtain

$$F(x, y) = (1 - e^{-x})(1 - e^{-y})[1 + \alpha e^{-x-y}], x \geq 0; y \geq 0, \quad (3.1.6)$$

and

$$f(x, y) = \begin{cases} e^{-x-y}[(1 + \alpha(2e^{-x} - 1)(2e^{-y} - 1))] & , x > 0, y > 0 \\ 0, & \text{otherwise.} \end{cases} \quad (3.1.7)$$

Once again the marginal distributions of X and Y are exponentials. If $\alpha = 0$, then the two variables are independent.

The above models satisfy all the criteria for bivariate exponential distributions. Some authors remark that they do not appear to be appropriate models to particular physical situations.

Moran (1967) gives a bivariate distribution as the joint distribution of

$$X = X_1^2 + X_2^2$$

and

$$Y = X_3^2 + X_4^2,$$

where X_1, X_3 are jointly normally distributed with zero means and variances $1/2$ and correlation a ($0 \leq a \leq 1$) and X_2 and X_4 are independent of (X_1, X_3) but have the same distribution. By considering the joint characteristic function of X and Y which is given by

$$\{(1 - it_1)(1 - it_2) + a^2 t_1 t_2\}^{-1} = \sum_{n=0}^{\infty} \frac{(-a^2 t_1 t_2)^n}{\{(1 - it_1)(1 - it_2)\}^{n+1}}, \quad (3.1.8)$$

he shows that the joint density can be expanded in the form

$$f(x, y) = \sum_{n=0}^{\infty} a^{2n} f_n(x, y). \quad (3.1.9)$$

Here $f_n(x, y)$ has the Fourier transform

$$\begin{aligned} \frac{(-t_1 t_2)^n}{\{(1 - it_1)(1 - it_2)\}^{n+1}} &= \frac{1}{(1 - it_1)(1 - it_2)} \left\{ \frac{1 - (1 - it_1)}{(1 - it_1)} \right\}^n \left\{ \frac{1 - (1 - it_2)}{(1 - it_2)} \right\}^n \\ &= \sum_{j=0}^n \binom{n}{j} \frac{(-1)^j}{(1 - it_1)^{j+1}} \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{(1 - it_2)^{k+1}}. \end{aligned} \quad (3.1.10)$$

It then follows that

$$f_n(x, y) = \sum_{j=0}^n \binom{n}{j} \frac{(-1)^j}{j!} x^j e^{-x} \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{k!} y^k e^{-y}. \quad (3.1.11)$$

Downton (1970) derives a bivariate exponential distribution by using a simple failure model. He shows that the joint density function of the component lifetimes is given by

$$f(t_1, t_2) = \frac{\mu_1 \mu_2}{1 - \rho} \exp\left\{-\frac{(\mu_1 t_1 + \mu_2 t_2)}{1 - \rho}\right\} \\ \times I_0\left\{\frac{2(\rho \mu_1 t_1 \mu_2 t_2)^{1/2}}{1 - \rho}\right\}, t_1, t_2 > 0, 0 \leq \rho \leq 1, \mu_1, \mu_2 > 0. \quad (3.112)$$

μ_i is the mean interval between shocks of component $i, i = 1, 2$. I_0 is the modified Bessel function of the first kind of order zero, (see Appendix A3). This model is a special case of the bivariate gamma distribution given by Kibble (1941), see section 3.3.

We shall now present Freund's and Marshall and Olkin's famous bivariate exponential densities and their extensions. We will also give some of their properties.

3.1a Freund's bivariate exponential distribution and extension.

This bivariate extension of the exponential distribution, is designed, in particular, for the life testing of two-component systems, which can function even after one of the components has failed. It might, thus, apply to the study of engine failures in two-engine planes, to the wear of two pens on an executive's desk, or to the performance of a person's eyes, ears, kidneys, or other paired organs.

The model can be introduced by considering two random variables X and Y which represent the lifetimes of two components A and B in a two-component system. Let X^* represent the lifetime of component A if component B is replaced with a component of the same kind each time it fails (if necessary more than once). Let Y^* represent the lifetime of component B if component A is replaced with a component of the same kind each time it fails (if necessary more than once). X^* and Y^* are assumed to be independent random

variables having exponential distributions.

$$f(x^*) = \begin{cases} \alpha e^{-\alpha x^*} & , x^* > 0, \alpha > 0 \\ 0, & \text{otherwise.} \end{cases} \quad (3.1.13)$$

$$f(y^*) = \begin{cases} \beta e^{-\beta y^*} & , y^* > 0, \beta > 0 \\ 0, & \text{otherwise.} \end{cases} \quad (3.1.14)$$

From the above assumption, we obtain the element of probability that the first failure of an A component occurs in the neighborhood of x^* and that the B component has not yet failed as

$$(\alpha e^{-\alpha x^*} dx^*) \int_{x^*}^{\infty} \beta e^{-\beta y^*} dy^* = \alpha e^{-x^*(\alpha+\beta)} dx^*. \quad (3.1.15)$$

Similarly, the element of probability that the first failure of a B component occurs in the neighborhood of y^* and that the A component has not yet failed is

$$\beta e^{-y^*(\alpha+\beta)} dy^*. \quad (3.1.16)$$

We now consider the case when the components are not replaced. The element of probability that component A fails in the neighborhood of x and that B has not yet failed is

$$\alpha e^{-x(\alpha+\beta)} dx, \quad (3.1.17)$$

analogous to (3.1.15).

The element of probability that component B fails in the neighborhood of y and that A has not yet failed is

$$\beta e^{-y(\alpha+\beta)} dy, \quad (3.1.18)$$

analogous to (3.1.16).

We also suppose that, the probability density of X given that $Y=y$ is

$$\begin{cases} \alpha' e^{-\alpha'(x-y)} & , x > y, \alpha' > 0 \\ 0, & \text{otherwise,} \end{cases} \quad (3.1.19)$$

and that the probability density of Y given that $X=x$ is

$$\begin{cases} \beta' e^{-\beta'(y-x)} & , y > x, \beta' > 0 \\ 0, & \text{otherwise.} \end{cases} \quad (3.1.20)$$

It follows from the above assumptions that the joint density of X and Y is

$$f(x, y) = \begin{cases} \alpha\beta' \exp\{-\beta'y - (\alpha + \beta - \beta')x\} & , 0 < x < y, \\ \beta\alpha' \exp\{-\alpha'x - (\alpha + \beta - \alpha')y\}, & 0 < y < x. \end{cases} \quad (3.1.21)$$

(i) Properties.

(a) The marginal densities are

$$f(x) = \begin{cases} \frac{(\alpha - \alpha')(\alpha + \beta) \exp\{-(\beta + \alpha)x\}}{\alpha + \beta - \alpha'} + \frac{\alpha' \beta \exp\{-\alpha'x\}}{\alpha + \beta - \alpha'} & , x > 0 \\ 0, & \text{otherwise,} \end{cases} \quad (3.1.22)$$

provided $\alpha + \beta - \alpha' \neq 0$, and

$$f(y) = \begin{cases} \frac{(\beta - \beta')(\alpha + \beta) \exp\{-(\beta + \alpha)y\}}{\alpha + \beta - \beta'} + \frac{\alpha \beta' \exp\{-\beta'y\}}{\alpha + \beta - \beta'} & , y > 0 \\ 0, & \text{otherwise,} \end{cases} \quad (3.1.23)$$

provided $\alpha + \beta - \beta' \neq 0$. It can be observed that the marginal distributions are exponential

only in the special case $\alpha' = \beta' = \alpha + \beta$, and in this case

$$f(x) = \begin{cases} \alpha' e^{-\alpha'x} & , x > 0 \\ 0, & \text{otherwise,} \end{cases} \quad (3.1.24)$$

$$f(y) = \begin{cases} \beta' e^{-\beta' y} & , y > 0 \\ 0, & \text{otherwise.} \end{cases} \quad (3.1.25)$$

(b) The moment generating function of (3.1.21) is

$$E(e^{t_1 x + t_2 y}) = \frac{\alpha + \beta}{1 - [(t_1 + t_2)/(\alpha + \beta)]} \left\{ \frac{\beta}{1 - t_1/\alpha'} + \frac{\alpha}{1 - t_2/\beta'} \right\}. \quad (3.1.26)$$

Using (3.1.26), we obtain

$$\begin{aligned} E(X) &= \frac{\alpha' + \beta}{\alpha'(\alpha + \beta)}; \\ E(Y) &= \frac{\alpha + \beta'}{\beta'(\alpha + \beta)}. \end{aligned} \quad (3.1.27)$$

$$\begin{aligned} Var(X) &= \frac{\alpha'^2 + 2\alpha\beta + \beta^2}{\alpha'^2(\alpha + \beta)^2}; \\ Var(Y) &= \frac{\beta'^2 + 2\alpha\beta + \alpha^2}{\beta'^2(\alpha + \beta)^2}. \end{aligned} \quad (3.1.28)$$

$$Cov(X, Y) = \frac{\alpha'\beta' - \alpha\beta}{\alpha'\beta'(\alpha + \beta)^2}. \quad (3.1.29)$$

$$Corr(X, Y) = (\alpha'\beta' - \alpha\beta) \{(\alpha'^2 + 2\alpha\beta + \beta^2)(\beta'^2 + 2\alpha\beta + \alpha^2)\}^{-1/2}. \quad (3.1.30)$$

(c) The conditional density function of Y, given X (if $\alpha + \beta \neq \alpha'$), can be shown to be

$$f(y | x) = \begin{cases} \frac{\alpha\beta'(\alpha + \beta - \alpha') \exp\{\beta' y - (\beta + \alpha - \beta')x\}}{(\alpha - \alpha')(\alpha + \beta) \exp\{-(\beta + \alpha)x\} + \alpha'\beta \exp\{-\alpha'x\}} & , y \geq x \\ \frac{\alpha'\beta(\alpha + \beta - \alpha') \exp\{-(\alpha + \beta - \alpha')y - (\alpha')x\}}{(\alpha - \alpha')(\alpha + \beta) \exp\{-(\beta + \alpha)x\} + \alpha'\beta \exp\{-\alpha'x\}} & 0 \leq y < x. \end{cases} \quad (3.1.31)$$

Using (3.1.31), it is straightforward to show that

$$\begin{aligned} E(Y | X = x) &= \{(\alpha\beta'^{-1}(\alpha + \beta - \alpha')(1 + \beta'x) - \alpha'\beta(\alpha + \beta - \alpha')^{-1}) \\ &\quad [1 + (\alpha + \beta - \alpha')x](e^{-(\beta + \alpha)x} + \alpha'\beta(\alpha + \beta - \alpha')^{-1}e^{-\alpha'x})\} \\ &\quad \div \{(\alpha - \alpha')(\alpha + \beta)e^{-(\beta + \alpha)x} + \alpha'\beta e^{-\alpha'x}\}. \end{aligned} \quad (3.1.32)$$

Now if $\alpha' \rightarrow \infty$, that is, if A has not failed prior to B, then we have a linear regression of Y on X. That is

$$E(Y | X = x) = x + \frac{\alpha}{\beta'(\alpha + \beta)}, \quad (3.1.33)$$

and the linear correlation coefficient becomes

$$\rho = \frac{\beta'}{\sqrt{\alpha^2 + 2\alpha\beta + \beta'^2}}. \quad (3.1.34)$$

Similarly if $\beta' \rightarrow \infty$, then

$$E(X | Y = y) = y + \frac{\beta}{\alpha'(\alpha + \beta)}, \quad (3.1.35)$$

and the linear correlation coefficient becomes

$$\rho = \frac{\alpha'}{\sqrt{\alpha'^2 + 2\alpha\beta + \beta^2}}. \quad (3.1.36)$$

It can be noted that in general $-1/3 < \rho < 1$. The linear correlation coefficient approaches +1 when $\alpha' \rightarrow \infty$ and $\beta' \rightarrow \infty$, which corresponds to the case where the two-component system cannot function if either component fails. The linear correlation coefficient approaches -1/3 when $\alpha = \beta$, $\alpha' \rightarrow 0$ and $\beta' \rightarrow 0$, which corresponds to the case where either component becomes "almost infallible" as soon as the other one fails. Since this would not be a very realistic situation, the two limiting cases $\rho = +1$ and $\rho = -1/3$ are excluded under the assumptions of the model.

(ii) Extension.

Weinman (1966) has extended distribution (3.1.21) to a multivariate exponential distribution, by considering a system of m identical components with times to failure X_1, X_2, \dots ,

X_m . Each of the failure times is supposed to have the exponential distribution

$$f(x) = \begin{cases} \alpha_0^{-1} e^{-x/\alpha_0} & , x > 0, \alpha_0 > 0 \\ 0, & \text{otherwise.} \end{cases} \quad (3.1.37)$$

Moreover, it is supposed that if k components have failed (and not replaced), the conditional joint distribution of the lifetimes of the remaining $(m-k)$ components is that of independent random variables, each having the distribution,

$$f(x) = \begin{cases} \alpha_k^{-1} e^{-x/\alpha_k} & , x > 0, \alpha_k > 0 \\ 0, & \text{otherwise.} \end{cases} \quad (3.1.38)$$

It can be shown that the joint density of X_1, X_2, \dots, X_m is

$$f(x_1, x_2, \dots, x_m) = \prod_{j=0}^{m-1} [\alpha_j^{-1} e^{-(m-j)\alpha_j^{-1}(x_{j+1}-x_j)}], \quad (3.1.39)$$

where $x_0 = 0$ and $x_1 \leq x_2 \leq \dots \leq x_m$ are the x_j 's arranged in increasing order of magnitude.

The moment generating function of (3.1.39) is

$$E[\exp(\sum_{i=1}^m t_i X_i)] = (m!)^{-1} \sum_p^* \prod_{j=0}^{m-1} [1 - \alpha_j \sum_{i=j+1}^m t_{p(i)} / (m-j)]^{-1}, \quad (3.1.40)$$

where $\{t_{p(1)}, \dots, t_{p(m)}\}$ is one of the $m!$ possible permutations of $\{t_1, \dots, t_m\}$ and \sum^* denotes summation over all such permutations. The distribution is clearly symmetrical in X_1, X_2, \dots, X_m . For each j ($=1, 2, \dots, m$),

$$E(X_j) = m^{-1} \sum_{j=0}^{m-1} \alpha_j. \quad (3.1.41)$$

$$\begin{aligned} Var(X_j) = m^{-2} & \left[\sum_{j=0}^{m-1} (m+j)(m-j)^{-1} \alpha_j^2 + \right. \\ & \left. 2 \sum_{j < j'} j(m-j) \alpha_j \alpha_{j'} \right]. \end{aligned} \quad (3.1.42)$$

$$\begin{aligned} \text{Cov}(X_j, X_{j'}) &= \frac{m-2}{m-1} \left[\sum_{j=0}^{m-1} \left(m - \frac{m+j}{m-j} \right) \alpha_j^2 - \right. \\ &\quad \left. 2 \sum_{j < j'} j(m-j) \alpha_j \alpha_{j'} \right]. \end{aligned} \quad (3.1.43)$$

The joint moment generating function of the ordered variables $X'_1 \leq X'_2 \leq \dots \leq X'_m$ can be found to have the form,

$$\prod_{j=0}^{m-1} \left[1 - (\alpha_j \sum_{i=j+1}^m t_i) (m-j)^{-1} \right]^{-1}, \quad (3.1.44)$$

and the joint density of X'_1, X'_2, \dots, X'_m is $m!$ times density in (3.1.39).

3.1b Marshall and Olkin's bivariate exponential distribution and extension.

First, we suppose that the components of a two-component system die after receiving a shock which is always fatal. The occurrences of shocks are assumed to be governed by independent poisson processes with parameters $\lambda_1, \lambda_2, \lambda_{12}$ according to whether the shock applies to component 1 only, component 2 only or both components.

Thus if X_1 and X_2 denote the lifelength of the first and second components respectively, then

$$\begin{aligned} f(x_1, x_2) &= e^{[-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_{12} \max(x_1, x_2)]} \\ &\quad \text{for } x_1 \geq 0, x_2 \geq 0. \end{aligned} \quad (3.1.45)$$

Next, we consider a two component and three independent Poisson processes $Z_1(t; \delta_1)$, $Z_2(t; \delta_2)$, $Z_{12}(t; \delta_{12})$ governing the occurrence of shocks just like above, except that the shocks received need not be fatal. We describe the state of the system by the ordered pairs (0,0), (0,1), (1,0), (1,1), where a 1 in the first (second) place indicates that the first

(second) component is operating and a 0 indicates that it is not. Suppose that events in process $Z_1(t; \delta_1)$, are shocks to first component causing a transition from (1,1) to (0,1) with probability p_1 and from (1,1) to (1,1) with probability $1 - p_1$. Similarly, events in $Z_2(t; \delta_2)$ are transitions from (1,1) to (1,0) or (1,1) with probability p_2 and $1 - p_2$ respectively. Events in $Z_{12}(t; \delta_{12})$ are shocks to both components which cause a transition from state (1,1) to states (0,0), (0,1), (1,0), (1,1) with respective probabilities $p_{00}, p_{01}, p_{10}, p_{11}$. Furthermore, we assume that each shock to a component represents an independent opportunity for failure.

Let X_1 and X_2 denote the life length of the first and second components. Since $Z_1(t; \delta_1)$, $Z_2(t; \delta_2)$, $Z_{12}(t; \delta_{12})$ are independent and have independent increments, we have

$$\begin{aligned}
f(x_1, x_2) &= P[(X_1 > x_1), (X_2 > x_2)] \\
&= \left\{ \sum_{k=0}^{\infty} e^{-\delta_1 x_1} \frac{(\delta_1 x_1)^k}{k!} (1 - p_1)^k \right\} \left\{ \sum_{l=0}^{\infty} e^{-\delta_2 x_2} \frac{(\delta_2 x_2)^l}{l!} (1 - p_2)^l \right\} \\
&\quad \times \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} [e^{\delta_{12} x_1} \frac{(-\delta_{12} x_1)^m}{m!} (p_{11})^m] \right. \\
&\quad \times [e^{-\delta_{12}(x_2 - x_1)} \frac{(\delta_{12}(x_2 - x_1))^n}{n!} (p_{11} + p_{01})^n] \Big\} \\
&= \exp\{-x_1[\delta_1 p_1 + \delta_{12}(p_{10})] - x_2[\delta_2 p_2 + \delta_{12}(1 - p_{11} - p_{10})]\}, \\
&\quad \text{for } x_2 \geq x_1 \geq 0.
\end{aligned} \tag{3.1.46}$$

By symmetry, for $x_1 \geq x_2 \geq 0$

$$f(x_1, x_2) = e^{\{-x_1[\delta_1 p_1 + \delta_{12}(1 - p_{11} - p_{10})] - x_2[\delta_2 p_2 + \delta_{12} p_{10}]\}}. \tag{3.1.47}$$

Consequently, by combining (3.1.46) and (3.1.47), we obtain

$$f(x_1, x_2) = e^{[-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_{12} \max(x_1, x_2)]}, \quad x_1 > 0, x_2 > 0, \tag{3.1.48}$$

$$(ii) \lambda_2 = \delta_2 p_2 + \delta_{12} p_{10};$$

$$(iii) \lambda_{12} = \delta_{12} p_{00}.$$

It should be noted that when $p_1 = p_2 = 1$, $p_{00} = 1$, we have the specialized fatal model.

(i) Properties.

(a) The function (3.1.48) can be shown to have an absolutely continuous and a singular part.

$$f(x_1, x_2) = \frac{\lambda_1 + \lambda_2}{\lambda} f_\alpha(x_1, x_2) + \frac{\lambda_{12}}{\lambda} f_s(x_1, x_2), \quad (3.1.49)$$

where $f_s(x_1, x_2) = e^{[-\lambda \max(x_1, x_2)]}$ is the singular part and

$$\begin{aligned} f_\alpha(x_1, x_2) = & \frac{\lambda}{\lambda_1 + \lambda_2} e^{[-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_{12} \max(x_1, x_2)]} \\ & - \frac{\lambda_{12}}{\lambda_1 + \lambda_2} e^{[-\lambda \max(x_1, x_2)]}, \end{aligned}$$

is absolutely continuous. For a detailed proof, one may refer to Marshall and Olkin (1967).

The presence of a singular part in the distribution function is a reflection of the fact that if X_1 and X_2 have bivariate exponential distribution, then $X_1 = X_2$ with positive probability, whereas the line $x_1 = x_2$ has two-dimensional Lebesgue measure zero.

(b) The moment generating function for the bivariate exponential distribution is given by:

$$\begin{aligned} \Psi(t_1, t_2) &= \int_0^\infty \int_0^\infty e^{-t_1 x_1 - t_2 x_2} dF(x_1, x_2) \\ &= \int \int_{x_1 > x_2} e^{-t_1 x_1 - t_2 x_2} \lambda_2 (\lambda_1 + \lambda_{12}) f(x_1, x_2) dx_1 dx_2 \\ &\quad + \int \int_{x_1 < x_2} e^{-t_1 x_1 - t_2 x_2} \lambda_1 (\lambda_2 + \lambda_{12}) f(x_1, x_2) dx_1 dx_2 \\ &\quad + \int_0^\infty e^{-t_1 x_1 - t_2 x_2} \lambda_{12} f_s(x_1, x_1) dx_1. \end{aligned}$$

Evaluation of these integrals yield the result

$$\Psi(t_1, t_2) = \frac{(\lambda + t_1 + t_2)(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12}) + t_1 t_2 \lambda_{12}}{(\lambda + t_1 + t_2)(\lambda_1 + \lambda_{12} + t_1)(\lambda_2 + \lambda_{12} + t_2)}, \text{ where } \lambda = \lambda_1 + \lambda_2 + \lambda_{12}. \quad (3.1.50)$$

From (3.1.50), we obtain

$$\begin{aligned} E(X_1) &= \frac{1}{\lambda_1 + \lambda_{12}}. \\ E(X_2) &= \frac{1}{\lambda_2 + \lambda_{12}}. \end{aligned} \quad (3.1.51)$$

$$\begin{aligned} Var(X_1) &= \frac{1}{(\lambda_1 + \lambda_{12})^2}. \\ Var(X_2) &= \frac{1}{(\lambda_2 + \lambda_{12})^2}. \end{aligned} \quad (3.1.52)$$

$$\begin{aligned} Cov(X_1, X_2) &= E(X_1 X_2) - E(X_1)E(X_2) \\ &= \frac{1}{\lambda_1} \left(\frac{1}{\lambda_1 + \lambda_{12}} + \frac{1}{\lambda_2 + \lambda_{12}} \right) - \left(\frac{1}{\lambda_1 + \lambda_{12}} \right) \left(\frac{1}{\lambda_2 + \lambda_{12}} \right) \\ &= \frac{\lambda_{12}}{\lambda(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12})}. \end{aligned} \quad (3.1.53)$$

The correlation is

$$\rho(X_1, X_2) = \frac{\lambda_{12}}{\lambda_1}. \quad (3.1.54)$$

(c) If X is an exponential random variable, then it is known that aX is exponential for all $a > 0$. However, if (X_1, X_2) is bivariate exponential then (aX_1, bX_2) is bivariate exponential of the type we considered only if $a = b > 0$. The density function of (aX_1, bX_2) for $a, b > 0$ is easily seen to be of the form

$$g(x_1, x_2) = \exp[-\lambda_1 x_1 - \lambda_2 x_2 - \max(\lambda_3 x_1 - \lambda_4 x_2)], \quad x_1 > 0, x_2 > 0. \quad (3.1.55)$$

This distribution has exponential marginals and includes the bivariate exponential distribution as the special case $\lambda_3 = \lambda_4$.

Another change of variables in the bivariate exponential distribution which may be of interest is $(X_1^{1/\beta}, X_2^{1/\gamma})$. This has a bivariate Weibull distribution, namely,

$$f(x_1, x_2) = \exp[-\lambda x_1^\beta - \lambda^2 x_2^\gamma - \lambda^{12} \max(x_1^\beta, x_2^\gamma)], x_1 > 0, x_2 > 0 \quad (3.1.56)$$

(d) A characterization of the bivariate exponential distribution is the following:

(X_1, X_2) has a bivariate exponential distribution if and only if there exist independent exponential random variables U, V and W such that $X_1 = \min(U, W)$ and $X_2 = \min(V, W)$. The above is an immediate consequence of the fatal shock model discussed at the beginning of this section.

Marshall and Olkin (1967) have also introduced a generalized bivariate exponential distribution derived from shock models. They derive the moment generating function and give some other properties of the distribution. Saw (1969) has generalized (3.1.48) by replacing $\max(x_1, x_2)$ by an increasing function of $\max(x_1, x_2)$. This can be interpreted as arising from a situation in which the joint failure rate can vary with time. The marginal distributions are still exponential. Saw suggested the function

$$\lambda \int_0^{\max(x_1, x_2)} t(\gamma + t)^{-1} dt = \lambda [\max(x_1, x_2) - \gamma \log(\gamma + \max(x_1, x_2))], \quad (3.1.57)$$

for which

$$f(x_1, x_2) = (\gamma + \max(x_1, x_2))^{\lambda\gamma} e^{[-\lambda_1 x_1 - \lambda_2 x_2 - \lambda \max(x_1, x_2)]},$$

$$\text{for } x_1 > 0, x_2 > 0. \quad (3.1.58)$$

(ii) **Extension.**

Marshall and Olkin (1967) have generalized their bivariate exponential distribution in the following way. In a system of m components the distribution of times between "fatal shocks" to the combination $\{a_1, \dots, a_s\}$ of components is supposed to have an exponential distribution with expected value $\lambda_{\{a_1, \dots, a_s\}}^{-1}$. The $2^m - 1$ different distributions of this kind are supposed to be a mutually independent set. The joint density function of lifetimes X_1, X_2, \dots, X_m of the components is obtained as

$$h(x_1, x_2, \dots, x_m) = \exp\left[-\sum_{j=1}^m \lambda_j x_j - \sum_{j_1 < j_2} \lambda_{j_1 j_2} \max(x_{j_1}, x_{j_2}) - \sum_{j_1 < j_2 < j_3} \lambda_{j_1 j_2 j_3} \max(x_{j_1}, x_{j_2}, x_{j_3}) - \dots - \lambda_{12 \dots m} \max(x_1, \dots, x_m)\right]. \quad (3.1.59)$$

It is observed that (3.1.59) is a mixed distribution.

(a) Method of parameter estimation.

Arnold (1968) has pointed out that it is not simple to estimate the λ 's by standard maximum likelihood or method of moments. Instead, he has suggested the following method of estimation which exploits the singular nature of the distribution.

We define

$$Z_{a_1, \dots, a_s} = \begin{cases} 1 & \text{if } X_{a_1} = X_{a_2} = \dots = X_{a_s} < X_j \\ 0, & \text{otherwise,} \end{cases} \quad (3.1.60)$$

where $j \neq a_1, \dots, a_s$. Given n independent sets $X_j = (X_{1j}, \dots, X_{mj})$ each having the joint distribution (3.1.59), the estimator of $\lambda_{\{a_1, \dots, a_s\}}$ can be obtained as,

$$\left[n^{-1} \sum_{j=1}^n Z_{a_1, \dots, a_s, (j)}\right] \left[(n-1)^{-1} \sum_{j=1}^n \min(X_{1j}, \dots, X_{mj})\right]^{-1}. \quad (3.1.61)$$

The numerator and denominator of (3.1.61) are mutually independent. For each $s \in S_n$, $Z_{a_1, \dots, a_s(j)}$ can be shown to have a binomial distribution with parameters $(\lambda_{a_1, \dots, a_s}/\lambda)$, where $\lambda = \text{sum of } \lambda_{\{a_1, \dots, a_s\}} \text{'s over all possible sets } \{a_1, \dots, a_s\}$. All $U_j = \min(X_{1j}, \dots, X_{mj})$ ($j = 1, \dots, n$), are i.i.d exponential random variables with common mean $1/\lambda$. So, it follows that $\frac{1}{n-1} \sum_{j=1}^n U_j$ has a scaled gamma distribution with parameters n and $[(n-1)\lambda]^{-1}$. Using the above and the independence of the numerator and denominator of (3.1.61), it can be easily shown that the estimator is unbiased. The variance of the estimator is

$$[n(n-2)]^{-1} \lambda_{a_1, \dots, a_s} [(n-1)\lambda + \lambda_{a_1, \dots, a_s}]. \quad (3.1.62)$$

Note that if n is not large, many of the estimators (3.1.61) will have the value zero. In fact for each Y_j , only one Z (at most) will not be zero, so there must be at least $(2^n - 1 - n)$ estimators with zero values.

Before concluding this section, it is worthwhile mentioning other authors such as Block and Basu (1974), Block (1975) and Friday and Patil (1977) who have also discussed bivariate exponential distributions and bivariate exponential extensions.

3.2 MULTIVARIATE CHI-SQUARE DISTRIBUTIONS.

Multivariate distributions with chi-square marginals will be considered here. Krishnamah, Hagis and Steinberg (1963) refer to them as multivariate chi-square distributions and Miller et al. (1958) refer to them as generalized Rayleigh distributions. Various expressions have been derived for both the central and non-central multivariate chi-square distributions. We shall discuss only the central multivariate chi-square distributions but the results can be extended to the non-central cases.

Consider a random sample of size n , represented by n independent vectors $(X_{i1}, X_{i2}, \dots, X_{im})'$, $(i = 1, \dots, n)$, drawn from a given multivariate normal distribution with covariance matrix $\Sigma > 0$ and having each diagonal element equal to 1. Let $S_j = \sum_{i=1}^n (X_{ij} - \bar{X}_{.j})^2$, $(j = 1, 2, \dots, m)$, $\bar{X}_{.j} = \sum_{i=1}^n X_{ij}/n$.

In this section, we shall consider two ways of deriving the multivariate chi-square distribution:

- (i) as the joint distribution of S_j , $j = 1, \dots, m$, which is defined above, using the conditional distribution method;
- (ii) as the joint distribution of quadratic forms and of traces of Wishart matrices, using the characteristic function method.

(3.2a) Multivariate chi-square distribution as the joint distribution of S_j 's.

Case (1): $m = 2$.

Expressions for the bivariate chi-square distribution, using the joint distribution of S_j , $j = 1, 2$ have been obtained by Bose (1935), Johnson (1962) and Vere-Jones (1967).

Let $\vec{Y}_{11} = (X_{11} \dots X_{n1})$, $\vec{Y}_{12} = (X_{12} \dots X_{n2})$, where $X_{1j} \dots X_{nj}$, $j = 1, 2$, are independent and $\vec{X}_i = (X_{i1} X_{i2})' \sim \mathcal{N}_2(0, \Sigma)$ with

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

The conditional density of X_{i1} given X_{i2} is normal with mean ρX_{i2} and variance $(1 - \rho^2)$.

Consequently, given X_{i2} we can represent X_{i1} as $\rho X_{i2} + \sqrt{1 - \rho^2} U_i$, where U_i , $i = 1, 2, \dots, n$ are independent and $U_i \sim \mathcal{N}(0, 1)$. Thus the conditional density of \vec{Y}_{11} given \vec{Y}_{12} is that of n independent random variables with expected values $\rho \vec{Y}_{12}$ and common variance $(1 - \rho^2)$.

Now, if $S_j = \sum_{i=1}^n (X_{ij} - \bar{X}_{.j})^2$ for $j = 1, 2$, then given \bar{Y}_{12}, S_1 is distributed as

$$\sum_{i=1}^n [\rho(X_{i2} - \bar{X}_2) + \sqrt{1 - \rho^2}(U_i - \bar{U})]^2, \quad (3.2.1)$$

that is, as $(1 - \rho^2) \chi^2$ (noncentral χ^2 with $(n - 1)$ degrees of freedom and noncentrality parameter $\rho^2 S_2 (1 - \rho^2)^{-1}$). This was proved by Johnson (1962).

The conditional density function of S_1 given S_2 is given by

$$\begin{aligned} f(s_1 | s_2) &= \exp\left\{-\frac{\rho^2 s_2}{2(1 - \rho^2)}\right\} \sum_{j=0}^{\infty} \frac{1}{j!} \left[\frac{\rho^2 s_2}{2(1 - \rho^2)}\right]^j 2^{\frac{(n-1+2j)}{2}} \\ &\quad \times \left[\Gamma\left(\frac{n-1+2j}{2}\right)\right]^{-1} [s_1(1 - \rho^2)^{-1}]^{\frac{(n-1+2j)}{2}-1} \exp\left\{-\frac{s_1}{2(1 - \rho^2)}\right\} \\ &\quad \text{for } s_1 > 0, \\ &= 0, \text{ otherwise.} \end{aligned} \quad (3.2.2)$$

Using $f(s_1 | s_2)$ and the density function of S_2 , which is chi-square with $n - 1$ degrees of freedom, we obtain the joint density of S_1 and S_2 as

$$\begin{aligned} g(s_1, s_2) &= \sum_{j=0}^{\infty} c_j [\Gamma(n - 1 + 2j)/2]^{-2} [s_1(1 - \rho^2)^{-1}]^{(n-1+2j)/2-1} \exp\left\{\frac{-(s_1 + s_2)}{2(1 - \rho^2)}\right\} \\ &\quad [s_2(1 - \rho^2)^{-1}]^{\frac{(n-1+2j)}{2}-1} 2^{-(n-1+2j)} \text{ for } s_1 > 0, s_2 > 0 \\ &= 0, \text{ otherwise,} \end{aligned} \quad (3.2.3)$$

where

$$c_j = \frac{\Gamma\left(\frac{n-1+2j}{2}\right) (1 - \rho^2)^{(n-1)/2} \rho^{2j}}{j!}. \quad (3.2.4)$$

The expression for a general bivariate chi-square distribution can be obtained by considering the variables $S'_j = S_j \sigma_j^2$. Moreover, we can replace $(n - 1)$ by $\nu > 0$ but not necessarily an integer. From (3.2.3), we note that the joint distribution of S_1 and S_2 is a mixture of

joint distributions, with weights c_j , $\sum_0^\infty c_j = 1$, in which S_1 and S_2 each have independent $\chi^2(n-1+2j)$.

Bose(1935) shows that the density of $G = \frac{S_1}{S_2}$ can be written as,

$$f_1(g) = \frac{(1-\rho^2)^{\nu/2}}{B(\frac{1}{2}, \frac{1}{2}\nu)} \frac{g^{\frac{\nu}{2}-1}}{(1+g)^\nu} \left(1 - \frac{4\rho^2 g}{(1+g)^2}\right)^{-\frac{1}{2}(\nu+1)}, g > 0, \quad (3.2.5)$$

where $B(\frac{1}{2}, \frac{1}{2}\nu) = \Gamma(\frac{1}{2})\Gamma(\frac{1}{2}\nu)/\Gamma(\frac{1+\nu}{2})$.

Krishnaiah et al.(1963) give the joint density of $U = \sqrt{S_1}$ and $V = \sqrt{S_2}$ as

$$f_2(u, v) = 4(1-\rho^2)^{n/2} \sum_{i=0}^{\infty} \left[\frac{\Gamma(n/2+i)}{\Gamma(n/2)i!} \right. \\ \left. \times \frac{\rho^{2i}(uv)^{n+2i-1} e^{-(u^2+v^2)/(2(1-\rho^2))}}{[2^{n/2+i}\Gamma(n/2+i)(1-\rho^2)^{n/2+i}]^2}, u > 0, v > 0. \right] \quad (3.2.6)$$

They call the joint density as the bivariate chi distribution and study its various properties.

Case (2): $m > 2$.

Derivation of the joint distribution of S_1, \dots, S_m is more difficult and we will consider special cases only. Using the above method for $m = 2$, it can be shown that the conditional distribution of S_1 given $(X_{12}, \dots, X_{n2}) (X_{13}, \dots, X_{n3}) \dots (X_{1m}, \dots, X_{nm})$ is that of $(1 - \rho_{1.23\dots m}^2)^{-1} \times$ (noncentral χ^2 with $(n-1)$ degrees of freedom and noncentrality parameter

$$(1 - \rho_{1.23\dots m}^2)^{-1} \left[\sum_{j=2}^m a_j^2 S_j + \sum_{j \leq k}^m \sum_{k=2}^m a_j a_k P_{jk} \right],$$

where $a_j = \rho_{1j.2\dots(j-1)(j+1)\dots m}$, $P_{jk} = \sum_{i=1}^n (X_{ij} - \bar{X}_j)(X_{ik} - \bar{X}_k)$ and $\rho_{1.23\dots m}^2$ is the multiple correlation of X_1 on X_2, \dots, X_m . Moreover, the joint distribution of S_2, \dots, S_m , $P_{23}, \dots, P_{m-1,m}$, can be shown to be Wishart distribution $W_{m-1}(n-1; V_{11})$, where V_{11} is the cofactor of the first diagonal element of Σ , the covariance matrix (Anderson (1984)).

Thus we can obtain the joint distribution of $S_1, S_2, \dots, S_m, P_{23}, \dots, P_{m-1,m}$, but it is very difficult to eliminate the P 's. When

$$\Sigma = \begin{pmatrix} 1 & \rho & \dots & \rho \\ \rho & 1 & \dots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \dots & 1 \end{pmatrix}, \quad (3.2.7)$$

Johnson(1962) suggests the approximate formula

$$f(s_1, s_2, \dots, s_m) \approx \sum_{j=0}^{\infty} c_j \prod_{k=1}^m \int_0^{s_k} \frac{t^{[(n-1-2j)/2]-1} e^{-t/2}}{2^{(n-1-2j)/2} \Gamma[(n-1-2j)/2]} dt, \text{ for } s_i > 0 \quad (3.2.8)$$

where $i = 1, \dots, m$ and c_j 's are as in (3.2.4) and $f(s_1, \dots, s_m) = 0$, otherwise. It seems that (3.2.8) gives reasonably accurate values for $m = 2$ or 4 , but that the accuracy decreases with increasing m .

(i) Properties.

(a) The joint characteristic function of S_1, S_2, \dots, S_m is

$$E(e^{i \sum_{j=1}^m t_j S_j}) = |I - 2i \Sigma D_t|^{-\nu/2}, \quad (3.2.9)$$

where $D_t = \text{diag}(t_1, \dots, t_m)$. For proof of (3.2.9) see for example, Krishnamoorthy and Parthasarathy (1951).

(b) If $\rho_{ij} = \rho$ for all i, j the joint distribution of S_1, S_2, \dots, S_m is infinitely divisible (that is, for any $\alpha > 0$, $|I - 2i \Sigma D_t|^{-\alpha}$ is a characteristic function), see Moran and Vere-Jones (1969).

(3.2b.1) Bivariate chi-square distribution as the joint distribution of quadratic forms, using characteristic function method.

Let $\vec{X}' = (\vec{X}'_{(1)}, \vec{X}'_{(2)})$ where $\vec{X}_{(1)}$ and $\vec{X}_{(2)}$ are $(r \times 1)$ and $(s \times 1)$ respectively with $r \leq s$. Suppose \vec{X}' has a multinormal distribution with zero mean and non-singular covariance matrix

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}, \quad (3.2.10)$$

where Σ_{11} is the covariance matrix of $\vec{X}_{(1)}$ and Σ_{22} is the covariance matrix of $\vec{X}_{(2)}$, then the quadratic forms

$$Y_j = \vec{X}'_j \Sigma_{jj}^{-1} \vec{X}_j, \quad j = 1, 2, \quad (3.2.11)$$

can be shown to have χ^2 distributions with r and s degrees of freedom. The joint distribution of Y_1 and Y_2 is thus a form of bivariate chi-square distribution. Following Jensen(1970), we obtain an expression for the joint density of Y_1 and Y_2 . He points out that the distribution is the same as that of

$$Y_1 = \sum_{j=1}^r Z_{1j}^2, \quad Y_2 = \sum_{j=1}^s Z_{2j}^2,$$

where (Z_{1j}, Z_{2j}) ($j = 1, \dots, r$) are independent and have standardized bivariate normal distributions with correlations $\rho_1, \rho_2, \dots, \rho_r$ (the canonical correlations between $\vec{X}_{(1)}$ and $\vec{X}_{(2)}$).

The joint characteristic function of Y_1 and Y_2 can be shown to be

$$(1 - 2it_1)^{-r/2} (1 - 2it_2)^{-s/2} \sum_{j=0}^{\infty} C_j(\rho_1, \rho_2, \dots, \rho_r) \left[\frac{-4t_1 t_2}{(1 - 2it_1)(1 - 2it_2)} \right]^j, \quad (3.2.12)$$

where

$$(i) \quad C_j(\rho_1, \rho_2, \dots, \rho_r) = \sum_{j_1 + \dots + j_r = j} a_{j_1}(\rho_1) \dots a_{j_r}(\rho_r),$$

$$(ii) \quad a_{jl}(\rho_l) = \rho_l^j \Gamma(j + 1/2) [\sqrt{\pi} \Gamma(j + 1)]^{-1},$$

(iii) $\rho_l = \text{corr}(Z_{l1}, Z_{l2})$.

Now Gurland (1955) has shown that the inverse Fourier exponential transform of the function $w(t; g, h) = (1 - it)^{-g} [\frac{it}{1-it}]^h$ is

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} w(t; g, h) dt = h! \Gamma(g) L_h^{g-1}(x) \frac{x^{g-1} e^{-x}}{\Gamma(g) \Gamma(g+h)}, \quad (3.2.13)$$

where $L_j^{\alpha-1}(x)$ is the Laguerre polynomial

$$L_j^{\alpha-1}(x) = \left[\frac{\Gamma(\alpha) \Gamma(\alpha+j)}{j!} \right]^{1/2} \sum_{h=0}^j \frac{(-i)^h}{\Gamma(\alpha+h)} \binom{j}{h} x^h.$$

Applying the inversion formula (3.2.13) to the terms in (3.2.12), we obtain the joint density of Y_1 and Y_2 , $k(y_1, y_2)$ as

$$\begin{aligned} k(y_1, y_2) &= \left[\frac{y_1^{r/2-1} e^{-y_1/2}}{2^{r/2} \Gamma(r/2)} \right] \left[\frac{y_2^{s/2-1} e^{-y_2/2}}{2^{s/2} \Gamma(s/2)} \right] \\ &\times \sum_{j=0}^{\infty} \left[\frac{(j!)^2 \Gamma(r/2) \Gamma(s/2)}{\Gamma(r/2+j) \Gamma(s/2+j)} \right] C_j(\rho_1, \rho_2, \dots, \rho_r) L_j^{r/2-1}(y_1/2) L_j^{s/2-1}(y_2/2) \\ &y_1 > 0, y_2 > 0, \Gamma(\alpha) > 0. \end{aligned} \quad (3.2.14)$$

We can renormalize the Laguerre polynomials and write (3.2.14) in the standard form

$$\begin{aligned} k(y_1, y_2) &= \left[\frac{y_1^{r/2-1} e^{-y_1/2}}{2^{r/2} \Gamma(r/2)} \right] \left[\frac{y_2^{s/2-1} e^{-y_2/2}}{2^{s/2} \Gamma(s/2)} \right] \\ &\times \sum_{j=0}^{\infty} M_j L_j^* L_j^{**}, y_1 > 0, y_2 > 0. \end{aligned} \quad (3.2.15)$$

Where L_j^* and L_j^{**} are Laguerre polynomials of degree j which are orthonormal with respect to the weight functions $\left[\frac{y_1^{r/2-1} e^{-y_1/2}}{2^{r/2} \Gamma(r/2)} \right]$ and $\left[\frac{y_2^{s/2-1} e^{-y_2/2}}{2^{s/2} \Gamma(s/2)} \right]$ and

$$M_j = j! \frac{[\Gamma(r/2) \Gamma(s/2)]^{\frac{1}{2}}}{[\Gamma(r/2+k) \Gamma(s/2+k)]^{\frac{1}{2}}} C_j(\rho_1, \dots, \rho_r).$$

It can be shown that the series (3.2.15) is absolutely convergent almost everywhere. Using the same procedure Jensen (1970) also obtains expressions as Laguerrian series for a general trivariate chi-square distribution and a multivariate chi-square distribution when the covariance matrices of Y_k 's are each of Jacobi form, that is, all the elements are zero except those on the principal and its two adjacent diagonals.

(3.2b.2) Multivariate chi-square distribution as the joint distribution of traces of Wishart matrices.

Let $W(p \times p)$ be a central Wishart matrix having ν degrees of freedom, positive definite parameter matrix $\Sigma(p \times p)$ and rank $\min(p, \nu)$. Block partitions of W and Σ are W_{jk} and Σ_{jk} , respectively, both $(p_j \times p_k)$, where $1 \leq j, k \leq q$ and $p_1 + \dots + p_q = p$.

Consider scalars $v_j = \text{tr} W_{jj} \Sigma_{jj}^{-1}$, $1 \leq j \leq q$. It can be easily shown that each v_j has a chi-square distribution with νp_j degrees of freedom. The characteristic function of W is

$$\Phi_W(T) = |I_p - 2iT\Sigma|^{-\nu/2}, \quad (3.2.16)$$

where T is real, symmetric and $T = (\gamma_{ij} t_{ij})$ with $\gamma_{ij} = 1$ for $i = 1, \dots, p$, $\gamma_{ij} = 1/2$ for $i \neq j$. Using (3.2.16), we obtain the characteristic function of $\vec{v} = (v_1, \dots, v_q)$, as

$$\Phi_{\vec{v}}(\vec{t}) = |I_p - 2iH^*(\vec{t})\Sigma|^{-\nu/2}, \quad (3.2.17)$$

where $H^*(\vec{t}) = \text{diag}(t_1 \Sigma_{11}^{-1}, \dots, t_q \Sigma_{qq}^{-1})$ is a block diagonal matrix. Let $u_j = \frac{1}{2} v_j$, $1 \leq j \leq q$. Since Σ_{jj} is symmetric and positive definite, we write $\Sigma_{jj} = \Sigma_{jj}^{1/2} \Sigma_{jj}^{1/2}$. Using (3.2.17), we obtain the characteristic function of $\vec{u} = (u_1, \dots, u_q)$, as

$$\Phi_{\vec{u}}(\vec{t}) = |I_p - iH(\vec{t})R|^{-\nu/2}, \quad (3.2.18)$$

where now $H(\vec{t}) = \text{diag}(t_1 I_{p_1}, \dots, t_q I_{p_q})$ and R is a block partitioned matrix with elements

$$\begin{aligned} R_{jj} &= I_{p_j} \text{ and} \\ R_{jk} &= \Sigma_{jj}^{-1/2} \Sigma_{jk} \Sigma_{kk}^{-1/2} \text{ for } 1 \leq j, k \leq q. \end{aligned} \quad (3.2.19)$$

Letting $Z_j = it_j/(1 - it_j)$, $1 \leq j \leq q$ in (3.2.18) and factoring terms out of the determinant, we write

$$\Phi_{\vec{u}(\vec{t})} = \prod_{j=1}^q (1 - it_j)^{-(\nu p_j)/2} [g(z_1, \dots, z_q)]^{-\nu/2}, \quad (3.2.20)$$

where $g(z_1, \dots, z_q) = |I_p - A(z)|$,

$$A(z) = \begin{pmatrix} 0 & z_1 R_{12} & \dots & z_1 R_{1q} \\ z_2 R_{21} & 0 & \dots & z_2 R_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ z_q R_{q1} & z_q R_{q2} & \dots & 0 \end{pmatrix} \quad (3.2.21)$$

and R_{jk} is of order $(p_j \times p_k)$, $p_1 + \dots + p_q = p$. Now,

$$\begin{aligned} [g(z_1, \dots, z_q)]^{-\nu/2} &= [1 - B(\vec{Z})]^{-\nu/2} \\ &= \sum_{m=0}^{\infty} \frac{\Gamma(\nu/2 + m)}{\Gamma(\nu/2) m!} B^m(\vec{Z}), \end{aligned} \quad (3.2.22)$$

where, by the multinomial expansion, $B^m(\vec{Z})$ is the finite sum

$$B^m(\vec{Z}) = \sum_{a_1=0}^m \dots \sum_{a_q=0}^m A_a z_1^{a_1} \dots z_q^{a_q}, \quad (3.2.23)$$

(a_1, \dots, a_q) are non-negative integers and unless at least two of them are positive, the corresponding coefficient A_a of such first order terms vanishes. We now write

$\sum_a^{(m1)} = \sum_{a_1=0}^m \dots \sum_{a_q=0}^m$ where $(m1') = (m, \dots, m)$.

Combining (3.2.20) to (3.2.23) and replacing $z_1^{a_1} \dots z_q^{a_q}$ by $\prod_{j=1}^q [it_j/(1-it_j)]^{a_j}$, we obtain the characteristic function as

$$\begin{aligned} \Phi_{\vec{u}}(\vec{t}) &= \sum_{m=0}^{\infty} \frac{\Gamma(\nu/2 + m)}{\Gamma(\nu/2)m!} \\ &\times \sum_a^{(m1)} A_a \prod_{j=1}^q [it_j/(1-it_j)]^{a_j} (1-it_j)^{-\nu p_j/2}. \end{aligned} \quad (3.2.24)$$

Using the inverse Fourier transform of (3.2.24), we obtain the joint probability density function of $u_j = \frac{1}{2} tr W_{jj} \Sigma_{jj}^{-1}$ $1 \leq j \leq q$, in the form of a series given by

$$f(u) = \sum_{m=0}^{\infty} \frac{\Gamma(\nu/2 + m)}{\Gamma(\nu/2)m!} \sum_{(a)}^{(m)} A_a f_a(\vec{u}; 1/2\nu\vec{p}), \quad (3.2.25)$$

where

$$(i) f_a(\vec{u}; 1/2\nu\vec{p}) = \prod_{j=1}^q \frac{h_j \Gamma(\theta_j)}{\Gamma(\theta_j + h_j)} \frac{u_j^{\nu p_j/2 - 1} e^{-u_j}}{\Gamma(\nu p/2)} L_h(\vec{u}; \nu\vec{p}/2).$$

$$(ii) L_h(\vec{u}; \nu\vec{p}/2) = \sum_{m=0}^h (-1)^m \binom{h+\theta-1}{h-m} x^m / m!.$$

$$(iii) \text{ The coefficients } A_a \text{ depend on the matrix } \Sigma \text{ through } R_{jk} = \Sigma_{jj}^{-1/2} \Sigma_{jk} \Sigma_{kk}^{-1/2}, 1 \leq j, k \leq q.$$

Now the joint probability density function of $\vec{V} = (v_1, \dots, v_q)$ can be obtained by a simple change of scale. Some properties and approximations of the joint distribution function can be seen from Jensen(1970). Note that special cases of (3.2.25) are given when $p=q$ by Kibble (1941) and more generally by Krishnamoorthy and Parthasarathy(1951).

We remark that the expressions for the multivariate chi-square obtained in this section are generally complicated, thus requiring considerable calculations in order to become explicit and useful. In the next section, we will consider Mathai and Moschopoulos models (1990, 1990a), which include multivariate chi-square density as a particular case and give

relatively simple expressions in computable forms.

3.3 MULTIVARIATE GAMMA DISTRIBUTIONS.

A random vector $\vec{Z} = (Z_1, \dots, Z_m)'$ is said to have a multivariate gamma distribution if it has gamma marginals. Different methods have been used by authors to construct various forms of bivariate and multivariate gamma distributions.

Kibble(1941) has introduced a bivariate gamma distribution function having linear regression under all conditions, as a series which is bilinear in Laguerre polynomials. He shows that a bivariate distribution function in which each of the variates $Z_i, i=1,2$ has the density

$$g(z_i) = \begin{cases} \frac{z_i^{p-1} \exp(-z_i)}{\Gamma(p)} & \text{if } p > 0, z_i > 0 \\ 0, & \text{otherwise,} \end{cases} \quad (3.3.1)$$

may be represented by

$$g(z_1)g(z_2)\left[1 + \frac{\rho^2}{p} L_1(z_1, p) L_1(z_2, p) + \frac{\rho^4}{2!p(p+1)} L_2(z_1, p) L_2(z_2, p) + \dots\right], \quad (3.3.2)$$

where $|\rho| < 1$ and $L_r(z, p), p > 0$ is the generalized Laguerre polynomial of degree r .

Krishnamoorthy and Parthasarathy (1951) extended Kibble's result to n variables by using the moment generating function for the distribution of the sums of squares in a sample of size m from a n -variate normal distribution.

Dussauchoy and Berland (1974) define a multivariate gamma random vector $\vec{Z} = (Z_1, \dots, Z_m)'$ as having a characteristic function

$$\Psi_Z(u_1, \dots, u_m) = \prod_{j=1}^m [\Psi_{Z_j}(u_j + \sum_{k=j+1}^n \beta_{jk} u_k)]$$

$$\div [\Psi_{Z_j}(\sum_{k=j+1}^n \beta_{jk} u_k)], \quad (3.3.3)$$

where :

(i) $\Psi_{Z_j}(u_j) = (1 - iu_j a_j)^{-e_j}$ for all $j = 1, \dots, m$, is the characteristic function of the component Z_j , e.g Z_j is a gamma random variable with parameters (a_j, e_j) .

(ii) $\beta_{jk} \geq 0$; $a_j \geq \beta_{jk}$; $a_k > 0$ for all $j < k = 1, 2, \dots, m$ and $0 < e_1 \leq e_2 \leq \dots \leq e_m$.

They obtain explicit form for the bivariate density only.

Gaver (1970) generates multivariate gamma distribution with gamma marginals through the mixture of gamma variables with negative binomial weights. He does not give an explicit form of the density but instead gives the Laplace transform of the density of a multivariate gamma vector $\vec{Z} = (Z_1, \dots, Z_m)'$ as

$$L_Z(s_1, \dots, s_m) = \left[\frac{\alpha}{(1 + \alpha) \prod_{j=1}^m (1 + s_j) - 1} \right]^k \text{ for } k > 0, \alpha > 0. \quad (3.3.4)$$

Becker and Roux (1981) have introduced a bivariate gamma model by considering the lifetimes of components of operating systems when the components are subjected to shocks. Their model appears to be the only bivariate gamma density which includes Freund's bivariate exponential distribution mentioned in section (3.1), as a special case.

Without doubt the most popular method of developing multivariate gamma densities has been from linear combinations of independent gamma variables. Various forms of bivariate gamma have thus been obtained by Moran(1967, 1969), Kibble (1941), Ghitis (1967) and Eagleson (1964). In their simplest form, the multivariate gamma distributions are constructed as follows:

Let X_0, X_1, \dots, X_m be independent random variables with X_j having a standard gamma distribution with density function

$$f_j(x_j) = \begin{cases} [\Gamma(\alpha_j)]^{-1} x_j^{\alpha_j-1} e^{-x_j} & \text{if } x_j > 0, R(\alpha_j) > 0, \\ 0, & \text{otherwise,} \end{cases} \quad (3.3.5)$$

where $j = 0, 1, 2, \dots, m$. We now consider the random variables $Z_j = X_0 + X_j$, $j = 1, 2, \dots, m$. Since the joint density of X_0, X_1, \dots, X_m is

$$\prod_{j=0}^m [\Gamma(\alpha_j)]^{-1} x_j^{\alpha_j-1} e^{-x_j},$$

it can be easily seen that the joint density of X_0, Z_1, \dots, Z_m is given by

$$\begin{aligned} g(x_0, z_1, \dots, z_m) &= \left[\prod_{j=0}^m [\Gamma(\alpha_j)]^{-1} \right] x_0^{\alpha_0-1} \prod_{j=1}^m [z_j - x_0]^{\alpha_j-1} \\ &\quad \times \exp\left\{ (m-1)x_0 - \sum_{j=1}^m z_j \right\} \\ &\quad \text{for } z_j \geq x_0 \geq 0; j = 1, \dots, m, R(\alpha_j) > 0. \end{aligned} \quad (3.3.6)$$

In order to find the marginal densities of Z_1, \dots, Z_m , we need to evaluate

$$\int_0^z x_0^{\alpha_0-1} \left[\prod_{j=1}^m [z_j - x_0]^{\alpha_j-1} \right] e^{(m-1)x_0} dx_0, \quad (3.3.7)$$

where $z = \min(z_1, \dots, z_m)$. In the general case (3.3.7) leads to very complicated expressions.

(i) The distribution for the bivariate case is obtained by Cheriyan(1941) and that of the trivariate case is obtained by Ramabhadran(1951).

(ii) If $\alpha_1 = \alpha_2 = \dots = \alpha_m = 1$, then

$$f(z_1 \dots z_m) = \begin{cases} [\Gamma(\alpha_0)]^{-1} \exp(-\sum_{j=1}^m z_j) h(z; \alpha_0) & \text{if } z_j > 0, R(\alpha_0) > 0, \\ 0, & \text{otherwise,} \end{cases} \quad (3.3.8)$$

where $h(z; \alpha_0) = \int_0^z x_0^{\alpha_0-1} e^{(m-1)x_0} dx_0$.

More parameters can be introduced in the multivariate gamma distribution by considering the joint distribution of $\lambda_j(X_0 + X_j)$ ($j = 1, 2, \dots, m$). The case $m = 2$ is called the double gamma distribution. Some of its characteristics and the method of fitting the parameters $\alpha_1, \alpha_2, \alpha_0, \lambda_1, \lambda_2$ by using sample moments can be found in Ghirtis(1967).

We shall now present some methods of constructions of multivariate gamma distributions with three parameter gamma marginals and we shall also discuss some of their properties.

3.3a Multivariate gamma distributions with three parameter gamma marginals.

Let V_0, V_1, \dots, V_m be mutually independent random variables with V_j having a gamma distribution with location, scale and shape parameters $\gamma_j, \beta_j, \alpha_j$ respectively. The density function of V_i is given by

$$\begin{aligned} f_j(v_j) &= \frac{(v_j - \gamma_j)^{\alpha_j-1}}{\beta_j^{\alpha_j} \Gamma(\alpha_j)} \exp\{-(v_j - \gamma_j)/\beta_j\} \\ &\quad \text{if } \alpha_j > 0, \beta_j > 0, v_j > \gamma_j, \\ &= 0, \text{ otherwise.} \end{aligned} \tag{3.3.9}$$

We denote the above statement by $V_j \sim \Gamma(\alpha_j, \beta_j, \gamma_j)$. Kowalczyk and Tyrcha (1989) define a random vector $\vec{Z} = (Z_1, \dots, Z_m)'$ to have a multivariate gamma distribution, if $Z_j = [\sigma_j(V_0 + V_j - \alpha_j)/\sqrt{\alpha_j}] + \mu_j$, where $V_0 = \Gamma(\nu_0, 1, 0)$, $V_j = \Gamma(\alpha_j - \nu_0, 1, 0)$, $0 \leq \nu_0 \leq \min(\alpha_1, \dots, \alpha_m)$, $\sigma_j > 0$ and μ_j is a real number, $j = 1, \dots, m$. They study various properties of the distribution but they do not give an explicit form of the multivariate density.

On the other hand, Mathai and Moschopoulos (1990,1990a) who start with $V_j \sim$

$\Gamma(\alpha_j, \beta_j, \gamma_j)$ provide explicit forms of two multivariate gamma densities which include several existing forms of bivariate and multivariate gamma distributions as special cases, together with the properties. Since their two multivariate gamma models belong to different categories, we will look at both of them.

3.3b Mathai-Moschopoulos multivariate gamma model-1.

(i) Mathai and Moschopoulos (1990) have introduced a multivariate gamma, as the density of the vector $\vec{Z}' = (Z_1, \dots, Z_m)$ where

$$Z_i = \frac{\beta_i}{\beta_0} V_0 + V_i, i = 1, \dots, m, \quad (3.3.10)$$

$V_i \sim \Gamma(\alpha_i, \beta_i, \gamma_i), i = 0, 1, \dots, m$ and the V_i 's are mutually independent. Considering the transformations (3.3.10) and $Z_{m+1} = V_0$, the joint density of (Z_1, \dots, Z_{m+1}) is

$$\begin{aligned} f_{m+1}(z_1, \dots, z_{m+1}) &= C(z_{m+1} - \gamma_0)^{\alpha_0 - 1} \exp\left\{-\frac{(z_{m+1} - \gamma_0)}{\beta_0}\right\} \\ &\times \prod_{j=1}^m (z_j - \frac{\beta_j}{\beta_0} z_{m+1} - \gamma_j)^{\alpha_j - 1} \\ &\times \exp\left\{-(z_j - \frac{\beta_j}{\beta_0} z_{m+1} - \gamma_j)/\beta_j\right\}, \end{aligned} \quad (3.3.11)$$

where $C = [\beta_0^{\alpha_0} \Gamma(\alpha_0) \prod_{j=1}^m \beta_j^{\alpha_j} \Gamma(\alpha_j)]^{-1}$.

Letting $u = z_{m+1} - \gamma_0, u_j = \frac{\beta_0}{\beta_j}(z_j - \gamma_j) - \gamma_0, j = 1, 2, \dots, m$ in (3.3.11), we obtain

$$\begin{aligned} f_{m+1}(u_1, \dots, u_m, u) &= C \prod_{j=1}^m \left(\frac{\beta_j}{\beta_0}\right)^{\alpha_j - 1} u^{\alpha_0 - 1} \exp\{-u/\beta_0\} \\ &\times \prod_{j=1}^m (u_j - u)^{\alpha_j - 1} \exp\left\{-\frac{(u_j - u)}{\beta_0}\right\}, \end{aligned} \quad (3.3.12)$$

where $0 < u < \min(u_1, \dots, u_m)$. Integrating out u in (3.3.12), we obtain

$$g_m(u_1, \dots, u_m) = C_1 \int_0^{\min(u_1, \dots, u_m)} u^{\alpha_0 - 1} e^{-u/\beta_0}$$

$$\times \prod_{j=1}^m (u_j - u)^{\alpha_j - 1} \exp\left\{-\frac{(u_j - u)}{\beta_0}\right\} du, \quad (3.3.13)$$

where $C_1 = C \prod_{j=1}^m (\frac{\beta_1}{\beta_0})^{\alpha_j - 1}$.

We observe that the above density is of different form for each of $m!$ orderings of u_1, \dots, u_m .

Consider $u_1 < u_2 < \dots < u_m$, the part of the density in this region is then

$$\begin{aligned} g_m(u_1, \dots, u_m) &= C_1 \int_0^{u_1} u^{\alpha_0 - 1} e^{-u/\beta_0} (u_1 - u)^{\alpha_1 - 1} \dots (u_m - u)^{\alpha_m - 1} \\ &\quad \times \exp\{ -[(u_1 - u) + \dots + (u_m - u)]/\beta_0 \} du \\ &= C_1 \left[\prod_{j=1}^m u_j^{\alpha_j - 1} \right] \int_0^{u_1} u^{\alpha_0 - 1} e^{-u/\beta_0} \left(1 - \frac{u}{u_1}\right)^{\alpha_1 - 1} \\ &\quad \dots \left(1 - \frac{u}{u_m}\right)^{\alpha_m - 1} \exp\left\{-\frac{1}{\beta_0} \left[u_1 \left(1 - \frac{u}{u_1}\right) + \dots + u_m \left(1 - \frac{u}{u_m}\right)\right]\right\} du. \end{aligned}$$

After simplifying, we obtain

$$\begin{aligned} g_m(u_1, \dots, u_m) &= C_1 \left[\prod_{j=1}^m u_j^{\alpha_j - 1} \right] u_1^{\alpha_0} \int_0^1 y^{\alpha_0 - 1} e^{-\frac{u_1 y}{\beta_0}} (1 - y)^{\alpha_1 - 1} \\ &\quad \times \left(1 - \frac{u_1}{u_2} y\right)^{\alpha_2 - 1} \dots \left(1 - \frac{u_1}{u_m} y\right)^{\alpha_m - 1} \exp\left\{-\frac{1}{\beta_0} \left[u_1 (1 - y) \right. \right. \\ &\quad \left. \left. + u_2 \left(1 - \frac{u_1}{u_2} y\right) + \dots + u_m \left(1 - \frac{u_1}{u_m} y\right)\right]\right\} dy. \end{aligned} \quad (3.3.14)$$

We expand the exponentials in series forms to get

$$\begin{aligned} g_m(u_1, \dots, u_m) &= C_1 \left[\prod_{j=1}^m u_j^{\alpha_j - 1} \right] u_1^{\alpha_0} \sum_{r_0=0}^{\infty} \sum_{r_1=0}^{\infty} \dots \sum_{r_m=0}^{\infty} \frac{(-u_1/\beta_0)^{r_0}}{r_0!} \frac{(-u_1/\beta_0)^{r_1}}{r_1!} \\ &\quad \dots \frac{(-u_m/\beta_0)^{r_m}}{r_m!} \times \int_0^1 y^{\alpha_0 + r_0 - 1} (1 - y)^{\alpha_1 + r_1 - 1} \left(1 - \frac{u_1}{u_2} y\right)^{\alpha_2 + r_2 - 1} \\ &\quad \dots \left(1 - \frac{u_1}{u_m} y\right)^{\alpha_m + r_m - 1} dy. \end{aligned}$$

Since $u < \frac{u_1}{u_j} < 1, j = 1, \dots, m$,

$$\begin{aligned}
g_m(u_1, \dots, u_m) &= \frac{[\prod_{j=1}^m (\frac{\beta_1}{\beta_0})^{\alpha_j-1}]}{[\prod_{j=0}^m \beta_j^{\alpha_j} \Gamma(\alpha_j)]} [\prod_{j=1}^m u_j^{\alpha_j-1}] \\
&\times u_1^{\alpha_0} \sum_{r_0=0}^{\infty} \sum_{r_1=0}^{\infty} \dots \sum_{r_m=0}^{\infty} \frac{(-u_1/\beta_0)^{r_0}}{r_0!} \frac{(-u_1/\beta_0)^{r_1}}{r_1!} \dots \frac{(-u_m/\beta_0)^{r_m}}{r_m!} \\
&\times \frac{\Gamma(\alpha_0 + r_0) \Gamma(\alpha_1 + r_1)}{\Gamma(\alpha_0 + r_0 + \alpha_1 + r_1)} \\
&\times F_D(\alpha_0 + r_0; \alpha_2 + r_2; \dots; \alpha_m + r_m; \alpha_0 + r_0 + \alpha_1 + r_1; u_1/u_2, \\
&\dots, u_1/u_m), \tag{3.3.15}
\end{aligned}$$

where F_D is the Lauricella function, see Mathai and Saxena (1978). Thus for each permutation (i_1, i_2, \dots, i_m) of the integers $(1, 2, \dots, m)$ such that $u_{i_1} < \dots < u_{i_m}$, that part of the density has the form

$$\begin{aligned}
g_m(u_{i_1}, \dots, u_{i_m}) &= \frac{[\prod_{j=1}^m (\frac{\beta_1}{\beta_0})^{\alpha_j-1}]}{[\prod_{j=0}^m \beta_j^{\alpha_j} \Gamma(\alpha_j)]} [\prod_{j=1}^m u_j^{\alpha_j-1}] \\
&\times u_{i_1}^{\alpha_0} \sum_{r_0=0}^{\infty} \sum_{r_1=0}^{\infty} \dots \sum_{r_m=0}^{\infty} \frac{(-u_{i_1}/\beta_0)^{r_0}}{r_0!} \frac{(-u_{i_1}/\beta_0)^{r_1}}{r_1!} \dots \frac{(-u_{i_m}/\beta_0)^{r_m}}{r_m!} \\
&\times \frac{\Gamma(\alpha_0 + r_0) \Gamma(\alpha_1 + r_1)}{\Gamma(\alpha_0 + r_0 + \alpha_1 + r_1)} \\
&\times F_D(\alpha_0 + r_0; \alpha_{i_2} + r_2; \dots; \alpha_{i_m} + r_m; \alpha_0 + r_0 + \alpha_{i_1} + r_1; u_{i_1}/u_{i_2}, \\
&\dots, u_{i_1}/u_{i_m}). \tag{3.3.16}
\end{aligned}$$

The Lauricella function F_D has a convergent series representation for $|\frac{u_{i_1}}{u_{i_j}}| < 1, j = 2, \dots, m$ and $R(\alpha_0) > 0, R(\alpha_{i_1}) > 0$. These conditions are satisfied since $u_{i_1} < u_{i_2} < \dots < u_{i_m}$ and $\alpha_0, \alpha_1, \dots, \alpha_m$ are parameters of the gamma densities.

3.3b.1 Properties.

(a) Moment generating function.

Let $\vec{t} = (t_1, \dots, t_m)'$, then we have

$$\begin{aligned} M_{\vec{Z}}(\vec{t}) &= E(\exp\{(t_1 z_1 + \dots + t_m z_m)\}) \\ &= E(\exp\{\frac{1}{\beta_0}(\beta_1 t_1 + \dots + \beta_m t_m)V_0\}) \\ &\quad \times E(\exp\{t_1 V_1\}) \dots E(\exp\{t_m V_m\}). \end{aligned} \quad (3.3.17)$$

Using

$$\begin{aligned} M_X(t) &= E(e^{tx}) \\ &= \int_{\gamma}^{\infty} \frac{(x-\gamma)^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} \exp\{-\frac{x-\gamma}{\beta}\} e^{tx} dx \\ &= \frac{e^{\gamma t}}{(1-\beta t)^{\alpha}}, \end{aligned} \quad (3.3.18)$$

we obtain,

$$\begin{aligned} M_{\vec{Z}}(\vec{t}) &= \frac{\exp\{\frac{\gamma_0}{\beta_0}(\beta_1 t_1 + \dots + \beta_m t_m)\}}{[1 - (\beta_1 t_1 + \dots + \beta_m t_m)^{\alpha_0}]^{\alpha_0}} \\ &\quad \times \frac{\exp\{(t_1 \gamma_1 + \dots + t_m \gamma_m)\}}{(1 - \beta_1 t_1)^{\alpha_1} \dots (1 - \beta_m t_m)^{\alpha_m}} \\ &= \frac{\exp\{(\vec{g} + (\gamma_0/\beta_0)\vec{b})'\vec{t}\}}{(1 - \vec{b}'\vec{t})^{\alpha_0} \prod_{i=1}^m (1 - \beta_i t_i)^{\alpha_i}}, \end{aligned} \quad (3.3.19)$$

where $\vec{b} = (\beta_1, \dots, \beta_m)'$, $\vec{g} = (\gamma_1, \dots, \gamma_m)'$, $\vec{t} = (t_1, \dots, t_m)'$, $|\beta_i t_i| < 1$, for all i and $|\vec{b}'\vec{t}| < 1$. Using the moment generating function, we see that

- (i) $Z_i \sim \Gamma(\alpha_0 + \alpha_i, \beta_i + \frac{\gamma_0}{\beta_0}\beta_i + \gamma_i)$.
- (ii) $E(Z_i) = (\alpha_0 + \alpha_i)\beta_i + \frac{\gamma_0}{\beta_0}\beta_i + \gamma_i$.
- (iii) $Var(Z_i) = (\alpha_0 + \alpha_i)\beta_i^2$.
- (iv) $Cov(Z_i, Z_j) = \alpha_0\beta_i\beta_j, i \neq j$.

From (d), we observe that Z_i and $Z_j, (i \neq j)$ are positively correlated.

(v) **Reproductive properties:** The class of multivariate gamma is closed under

(1) transformations of the form $\vec{W} = \vec{Z} + \vec{d}$, where $\vec{d} = (d_1, \dots, d_m)'$;

(2) convolutions $\vec{Z}_1 + \vec{Z}_2$ where \vec{Z}_1, \vec{Z}_2 are independent, \vec{Z}_1 is multivariate gamma with parameters $\alpha_i, \beta_i, \gamma_i$ and \vec{Z}_2 is multivariate gamma with parameters $\alpha'_i, \beta'_i, \gamma'_i$, $i = 0, 1, \dots, m$.

For proofs refer to Mathai and Moschopoulos (1990).

(b) Moments and Cumulants.

The moments of the form $E(Z_i^s)$ and product moments $E(Z_i^s Z_j^r)$ can be evaluated using the moments $E(V_i^s)$. Using (3.3.18), we obtain

$$\frac{d^s M_{V_i}(t)}{dt^s} = \sum_{k_1=0}^s \binom{s}{k_1} \left[\frac{d^{k_1}}{dt^{k_1}} (1 - \beta_i t)^{-\alpha_i} \right] \left[\frac{d^{s-k_1}}{dt^{s-k_1}} e^{\gamma_i t} \right].$$

Letting $t = 0$,

$$\begin{aligned} M_i^{(s)} &= E(V_i^s) \\ &= \sum_{k_1=0}^s \binom{s}{k_1} \alpha_i (\alpha_i + 1) \dots (\alpha_i + k_1 - 1) \beta_i^{k_1} \gamma_i^{s-k_1} \\ &= \sum_{k_1=0}^s \binom{s}{k_1} (\alpha_i)_{k_1} \beta_i^{k_1} \gamma_i^{s-k_1}, \end{aligned} \quad (3.3.20)$$

where $(\alpha)_a = \alpha(\alpha + 1) \dots (\alpha + a - 1)$, $(\alpha)_0 = 1$.

Using (3.3.20), it can be shown that

$$\begin{aligned} E(Z_i^s) &= E(\beta_i / \beta_0 V_0 + V_i)^s \\ &= E \left[\sum_{r=0}^s \binom{s}{r} (\beta_i / \beta_0)^r V_0^r V_i^{s-r} \right] \\ &= \sum_{r=0}^s \binom{s}{r} (\beta_i / \beta_0)^r \sum_{k_0=0}^r \binom{r}{k_0} (\alpha_0)_{k_0} \beta_0^{k_0} \gamma_0^{r-k_0} \end{aligned}$$

$$\times \sum_{k_i=0}^{s-r} \binom{s-r}{k_i} (\alpha_i)_{k_i} \beta_i^{k_i} \gamma_i^{s-r-k_i}. \quad (3.3.21)$$

From (3.3.20), we can also derive the product moments,

$$\begin{aligned} E(Z_i^s Z_j^n) &= E[(\beta_i/\beta_0 V_0 + V_i)^s (\beta_j/\beta_0 V_0 + V_j)^n] \\ &= \sum_{r=0}^s \sum_{t=0}^n \binom{s}{r} \binom{n}{t} (\beta_i/\beta_0)^r (\beta_j/\beta_0)^t M_0^{r+t} M_i^{s-r} M_j^{n-t}, \end{aligned} \quad (3.3.22)$$

where $M_i^{(s)}$ for $i = 1, \dots, m$ are available from (3.3.20). From (3.3.19), the logarithm of the moment generating function is

$$\begin{aligned} \ln M_{\vec{Z}}(\vec{t}) &= -\alpha_0 \ln(1 - \sum_{i=1}^m \beta_i t_i) - \sum_{i=1}^m \alpha_i \ln(1 - \beta_i t_i) + \\ &\quad \sum_{i=1}^m [(\gamma_0/\beta_0) \beta_i + \gamma_i] t_i. \end{aligned} \quad (3.3.23)$$

For $s_i \geq 2$, the cumulants of \vec{Z} are

$$\begin{aligned} K_{s_i} &= \frac{\delta^{s_i}}{\delta t_i^{s_i}} \ln(M_{\vec{Z}}(\vec{t})) \big|_{\vec{t}=0} \\ &= \left[\frac{\alpha_0 (s_i - 1)! \beta_i^{s_i}}{(1 - \beta_1 t_1 - \dots - \beta_m t_m)^{s_i}} + \frac{\alpha_i (s_i - 1)! \beta_i^{s_i}}{(1 - \beta_i t_i)^{s_i}} \right] \big|_{\vec{t}=0} \\ &= (s_i - 1)! \beta_i^{s_i} (\alpha_0 + \alpha_i). \end{aligned} \quad (3.3.24)$$

The joint cumulants are

$$\begin{aligned} K_{s_i, s_j} &= \frac{\delta^{s_i + s_j}}{\delta t_i^{s_i} \delta t_j^{s_j}} \ln(M_{\vec{Z}}(\vec{t})) \big|_{\vec{t}=0} \\ &= \alpha_0 (s_i + s_j - 1)! \beta_i^{s_i} \beta_j^{s_j}. \end{aligned} \quad (3.3.25)$$

In addition, the authors derive expressions for conditional densities and conditional moments. They also give asymptotic results and methods for parameter estimations.

3.3c Mathai-Moschopoulos multivariate gamma model-2.

Mathai and Moschopoulos (1990a) give another multivariate gamma, which has a relatively simple form, as the joint distribution of $\vec{Z} = (Z_1, \dots, Z_m)'$, where

$$Z_1 = V_1,$$

$$Z_2 = V_1 + V_2,$$

$$\vdots$$

$$Z_m = V_1 + V_2 + \dots + V_m.$$

$V_i \sim \Gamma(\alpha_i, \beta, \gamma_i), i = 1, \dots, m$ and V_1, \dots, V_m are mutually independent. The joint distribution of $\vec{Z} = (Z_1, \dots, Z_m)'$ is

$$\begin{aligned} f(z_1, \dots, z_m) &= 1/(\beta^{\alpha_m}) \frac{(z_1 - \gamma_1)^{\alpha_1 - 1}}{\prod_{i=1}^m \Gamma(\alpha_i)} (z_2 - z_1 - \gamma_2)^{\alpha_2 - 1} \dots \\ &\quad \times (z_m - z_{m-1} - \gamma_m)^{\alpha_m - 1} \times e^{1/\beta(z_m - (\gamma_1 + \dots + \gamma_m))} \\ &\quad \text{for } \alpha_1 > 0, \beta > 0, \gamma_i \text{ real}, z_m < \infty, \\ &\quad \gamma_1 < z_1, z_{i-1} + \gamma_i < z_i, i = 1, 2, \dots, m, \\ &= 0, \text{ otherwise.} \end{aligned} \tag{3.3.26}$$

3.3c.1 Properties.

(a) Moment generating function.

Using (3.3.18), the moment generating function of \vec{Z} is

$$\begin{aligned} M_{\vec{Z}}(\vec{t}) &= M_{\vec{Z}}(t_1, \dots, t_m) = E(e^{(t_1 z_1 + \dots + t_m z_m)}) \\ &= \frac{e^{\gamma_1(t_1 + \dots + t_m)}}{(1 - \beta(t_1 + \dots + t_m))^{\alpha_1}} \frac{e^{\gamma_2(t_2 + \dots + t_m)}}{(1 - \beta(t_2 + \dots + t_m))^{\alpha_2}} \\ &\quad \dots \frac{e^{\gamma_m t_m}}{(1 - \beta t_m)^{\alpha_m}}, \end{aligned} \tag{3.3.27}$$

which exists if $|t_i + t_{i+1} + \dots + t_m| < 1/\beta$ for $i = 1, 2, \dots, m$. Using the m.g.f, we obtain the properties below:

(i) The marginal distribution of Z_i is gamma,

$$Z_i \sim \Gamma(\alpha_i^*, \beta, \gamma_i^*), i = 1, \dots, m, \quad (3.3.28)$$

where $\alpha_i^* = \alpha_i + \dots + \alpha_m$, $\gamma_i^* = \gamma_1 + \dots + \gamma_m$.

(ii) The mean and variance of Z_i are respectively

$$\begin{aligned} E(Z_i) &= \beta \alpha_i^* + \gamma_i^*, \\ \text{Var}(Z_i) &= \beta^2 \alpha_i^*. \end{aligned} \quad (3.3.29)$$

(iii) Z_i and Z_j are correlated. For $i < j$,

$$\text{Cov}(Z_i, Z_j) = \text{Cov}(Z_i, Z_i + V_{i+1} + \dots + V_j) = \text{Var}(Z_i) = \beta^2 \alpha_i^*.$$

$$\rho = \text{Corr}(Z_i, Z_j) = \sqrt{\frac{\alpha_i^*}{\alpha_j^*}}. \quad (3.3.30)$$

It is obvious that the correlation is always positive. For details about the covariance matrix of \vec{Z} , see Mathai and Moschopoulos (1990a).

(iv) Reproductive property.

Suppose that \vec{Z}_1 and \vec{Z}_2 have a multivariate gamma distribution with parameters $\alpha_j, \beta, \gamma_j, j = 1, \dots, m$, and $\alpha'_j, \beta, \gamma'_j, j = 1, \dots, m$, respectively and that they are independently distributed. Using the m.g.f in (3.3.27), it is seen that $\vec{Z}_1 + \vec{Z}_2$ is also distributed as a multivariate gamma with parameters $\alpha_j + \alpha'_j, \beta, \gamma_j + \gamma'_j, j = 1, \dots, m$.

(b) Moments and Cumulants.

The logarithm of the m.g.f in (3.3.27) gives the cumulant generating function of \vec{Z} as

$$\begin{aligned} K_{\vec{Z}}(\vec{t}) = & \gamma_1 \sum_{i=1}^m t_i + \gamma_2 \sum_{i=2}^m t_i + \dots + \gamma_m t_m \\ & - \alpha_1 \ln(1 - \beta \sum_{i=1}^m t_i) - \alpha_2 \ln(1 - \beta \sum_{i=2}^m t_i) - \alpha_m \ln(1 - \beta t_m). \end{aligned} \quad (3.3.31)$$

So, the n th cumulant of Z_i is

$$\begin{aligned} K_n &= \frac{\delta^n}{\delta t_i^n} \ln(M_{\vec{Z}}(\vec{t})) \big|_{\vec{t}=0} \\ &= \gamma_i^* + \beta \alpha_i^*, \text{ if } n = 1 \\ &= (n-1)! \beta^n \alpha_i^*, \text{ if } n \geq 2. \end{aligned} \quad (3.3.32)$$

The (n_1, n_2) th product cumulant of Z_i and Z_j is given by

$$\begin{aligned} K_{n_1, n_2} &= \frac{\delta^{n_1+n_2}}{\delta t_i^{n_1} \delta t_j^{n_2}} \ln(M_{\vec{Z}}(\vec{t})) \big|_{\vec{t}=0} \\ &= (n_1 + n_2 - 1)! \beta^{n_1+n_2} \alpha_r^* \text{ where } r = \min(i, j) \end{aligned} \quad (3.3.33)$$

The moments of Z_i can be obtained from the moments of V_i by

$$\frac{d^n M_{v_i}(t)}{dt^n} = \sum_{k_1=0}^n \binom{n}{k_1} \left[\left(\frac{d^{k_1}}{dt^{k_1}} (1 - \beta t)^{-\alpha_i} \right) \left(\frac{d^{n-k_1}}{dt^{n-k_1}} e^{\gamma_i t} \right) \right].$$

Letting $t = 0$, in the above equation the n -th moment of V_i is

$$\begin{aligned} M_i^{(n)} &= E(V_i^n) \\ &= \sum_{k_1=0}^n \binom{n}{k_1} \alpha_i (\alpha_i + 1) \dots (\alpha_i + k_1 - 1) \beta^{k_1} \gamma_i^{n-k_1} \\ &= \sum_{k_1=0}^n \binom{n}{k_1} (\alpha_i)_{k_1} \beta^{k_1} \gamma_i^{n-k_1}, \end{aligned} \quad (3.3.34)$$

where $(\alpha)_a = \alpha(\alpha+1) \dots (\alpha+a-1), (\alpha)_0 = 1$. So,

$$\begin{aligned} E(Z_1^n) &= E(V_1 + \dots + V_1)^n \\ &= \sum_{k(r_1, \dots, r_1, n)} \frac{n!}{r_1! \dots r_1!} \prod_{l=1}^1 E(V_l^{r_l}) \\ &= \sum_{k(r_1, \dots, r_1, n)} \frac{n!}{r_1! \dots r_1!} \prod_{l=1}^1 (M_l^{r_l}), \end{aligned} \quad (3.3.35)$$

where $M_l^{r_l}$ is given in (3.3.34) and $k(r_1, \dots, r_j, n) = \{(r_1, \dots, r_j) \in N_+^j \mid r_1 + \dots + r_j = n\}$

and N_+ is the set of non-negative integers.

3.3c.2 Marginal Densities.

From the definition of the Z_i , it is clear that subsets of (Z_1, \dots, Z_m) should have the same form of densities. We will show that this is in fact true.

The joint density of (Z_1, \dots, Z_{m-1}) is obtained by integrating out Z_m from (3.3.26), that is

$$\begin{aligned} f(z_1, \dots, z_{m-1}) &= \int_{z_{m-1} + \gamma_m}^{\infty} (z_m - z_{m-1} - \gamma_m)^{\alpha_m - 1} \exp\left[-\frac{1}{\beta}(z_m - (\gamma_1 + \dots + \gamma_m))\right] dz_m, \\ &\times \frac{(z_1 - \gamma_1)^{\alpha_1 - 1}}{\beta^{\alpha_m} \prod_{i=1}^m \Gamma(\alpha_i)} (z_2 - z_1 - \gamma_2)^{\alpha_2 - 1} \\ &\dots (z_m - z_{m-1} - \gamma_m)^{\alpha_m - 1}. \end{aligned}$$

Letting $u = z_m - z_{m-1} - \gamma_m$ and integrating out, we obtain

$$\begin{aligned} f(z_1, \dots, z_{m-1}) &= \beta^{\alpha_m} \Gamma(\alpha_m) \exp\left[-\frac{1}{\beta}(z_{m-1} - (\gamma_1 + \dots + \gamma_{m-1}))\right] \\ &\times \frac{(z_1 - \gamma_1)^{\alpha_1 - 1}}{\beta^{\alpha_m} \prod_{i=1}^m \Gamma(\alpha_i)} (z_2 - z_1 - \gamma_2)^{\alpha_2 - 1} \\ &\dots (z_m - z_{m-1} - \gamma_m)^{\alpha_m - 1}, \end{aligned} \quad (3.3.36)$$

which is the same form as in (3.3.26).

We can also obtain the joint density of $Z_1, \dots, Z_{i-1}, Z_{i+1}, \dots, Z_m$ by integrating out Z_i , that is we have the the integral

$$\begin{aligned} & \int_{z_{i-1}+\gamma_i}^{z_{i+1}-\gamma_{i+1}} (z_i - z_{i-1} - \gamma_i)^{\alpha_i-1} (z_{i+1} - z_i - \gamma_{i+1})^{\alpha_{i+1}-1} dz_i \\ &= \int_0^{z_{i+1}-\gamma_{i+1}-z_{i-1}-\gamma_i} u^{\alpha_i-1} (z_{i+1} - \gamma_{i+1} - z_{i-1} - \gamma_i - u)^{\alpha_{i+1}-1} du \\ &= \frac{\Gamma(\alpha_i)\Gamma(\alpha_{i+1})}{\Gamma(\alpha_i + \alpha_{i+1})} (z_{i+1} - \gamma_{i+1} - z_{i-1} - \gamma_i)^{\alpha_i+\alpha_{i+1}-1} \end{aligned} \quad (3.3.37)$$

Since the location parameter of $z_{i+1} - z_{i-1}$ is $\gamma_i + \gamma_{i+1}$ and the shape parameter is $\alpha_i + \alpha_{i+1}$, the above density is also of the same form as the density of Z_1, \dots, Z_m

Several other results concerning conditional densities and densities of ratios of the Z_i 's can be obtained from Mathai and Moschopoulos (1990a).

3.4 APPLICATIONS.

The need to evaluate the different forms of the multivariate gamma distributions and to study their properties, arises in diverse areas of investigation. In this section, we shall point out several such situations.

(a) Multivariate exponential distribution in competing risks and life lengths

The analysis of human mortality for large populations frequently involves data on causes of death derived from death certificates. Sometimes, the examination of data containing information on all the conditions reported on the death certificates show that multiple conditions have been reported as causing death and that little information is available on the certificate to ascertain the underlying cause of death from among conditions listed

In order to analyse the multiple cause of death data, Marshall and Olkin's (1967) multivariate exponential distribution has been used by David (1974) to describe human mortality

In this "fatal shock model", individuals are represented as a multiple - component system and a "risk" process is defined for every possible configuration of component failure. Thus, for a k component system, one would define 2^k vectors of dimension k , each composed of a configuration of zeros and ones - a one coded if a specific component had been observed to contribute to system failure and a zero if it had not. Each of the 2^k vectors defined a "pattern" of component failures which was assumed to be generated by independently operating "fatal" shocks. This model has the attractive property that it can represent any level of interaction among the failing components.

(b) Multivariate chi-square distributions occur naturally as the joint distributions of statistics useful for simultaneous tests of hypotheses in a variety of cases.

(i) In the context of multivariate linear models, Jensen (1970) has shown that the joint distribution of the large sample form of the Lawley-Hotelling statistic in simultaneous tests involving subsets of the responses, as well as subsets of the factors, is a multivariate chi-square.

(ii) Krishnaiah (1965) has used the multivariate chi-square, in the finite intersection tests procedure for the multiple comparisons of mean vectors where the underlying distributions are multivariate normal with a common known covariance matrix.

(c) In rain-making experiments it is a common occurrence that one measures the rainfall X in a target area, and uses the rainfall Y in a control area as a controlling variate to increase the precision of the required test. In many rain-making experiments it has been shown that individual distribution of X and Y are well fitted by gamma distribution and usually such

that their probability densities are monotonically decreasing Moran (1969) gave a bivariate gamma distribution on which the tests on the rain-making experiments are based

(d) In stochastic processes, suppose that $V_i, i = 1, \dots, m$, are the times between successive occurrences of a phenomenon, for example arrivals or time delays of an airplane at several airports, and they are identically distributed. Let $Z_i = Z_{i-1} + V_i$ for $i = 1, \dots, m$ and $Z_0 = 0$, then Z_i is the total time required for i th occurrence or the total delay in the i th airport. In this case, the process $Z_i, i \in N$, can be called a renewal process and the times V_i can be called renewal times. Mathai and Moschopoulos (1990a) have shown that the joint distribution of $\vec{Z} = (Z_1, \dots, Z_m)'$ is a multivariate gamma, under some assumptions on the V_i 's.

(e) In reliability analysis, an item is installed at time $Z_0 = 0$ and when it fails, it is replaced by an identical (or different) item. Then, when the new item fails it is replaced again by another item and the process continues. In this case $Z_i = Z_{i-1} + V_i$, where V_i is the time of operation of the i th part and Z_i is the time at which the i th replacement is needed. The joint distribution of $\vec{Z} = (Z_1, \dots, Z_m)'$ has been shown to be a multivariate gamma by Mathai and Moschopoulos (1990a), under some assumptions on the V_i 's.

CHAPTER 4

MATRIX VARIATE GAMMA DISTRIBUTIONS

INTRODUCTION

In this chapter, we extend the vector variate gamma distributions dealt with in chapter 3, to the matrix variate case. The matrix variate gamma and other densities such as matrix variate beta and Dirichlet which are related to it, appear in the distributions of various test statistics in multivariate analysis and in distributions connected with the concepts of generalized variance, canonical correlation matrices and so on. We shall give a concise presentation of the various properties and distributions connected with the central matrix variate gamma distribution. In addition we shall consider some special functions associated with several matrix-variate gamma variables.

The following notations will be used throughout the chapter:

A'	transpose of matrix A .
$ A $	determinant of matrix A .
$A = A' > 0$	A is a positive definite symmetric matrix.
$\int_{A>0}(\cdot)dA$	integral over A such that A is symmetric and positive definite.
$tr A$	trace of A .
$\ (\cdot)\ $	norm of (\cdot) .
$R(\cdot)$	real part of (\cdot) .
$A \sim f(\cdot)$	A is distributed as $f(\cdot)$.

4.1 Matrix variate gamma density .

Let A be a $(p \times p)$ positive definite symmetric matrix. A is said to have a matrix variate gamma density with parameters (α, B) , if its density is of the form:

$$\begin{aligned} f_1(A) &= \frac{|A|^{\alpha-(p+1)/2}}{|B|^{\alpha} \Gamma_p(\alpha)} \exp(-\text{tr} B^{-1} A) \text{ for } A = A' > 0, \\ &\quad B = B' > 0, R(\alpha) > (p-1)/2, \\ &= 0 \quad \text{otherwise,} \end{aligned} \quad (4.1.1)$$

where $\Gamma_p(\alpha)$ is the multivariate gamma function defined by,

$$\Gamma_p(\alpha) = \pi^{\frac{p(p-1)}{4}} \prod_{i=1}^p \Gamma(\alpha - \frac{1}{2}(i-1)). \quad (4.1.2)$$

Letting $C = B^{-\frac{1}{2}} A B^{-\frac{1}{2}}$, from Deemer and Olkin (1951) we have $dC = |B|^{-\frac{p+1}{2}} dA$. The density function of C obtained from (4.1.1) is known as the standard matrix variate gamma density, and is of the form

$$\begin{aligned} f_2(C) &= \frac{|C|^{\alpha-(p+1)/2}}{\Gamma_p(\alpha)} \exp(-\text{tr} C), \quad C = C' > 0, R(\alpha) > (p-1)/2, \\ &= 0 \text{ otherwise,} \end{aligned} \quad (4.1.3)$$

(i) Properties.

Property 1: If we let $C = TT'$ in (4.1.3) where $T = (t_{ij})$ with $t_{ij} = 0$ for $i < j$, $t_{ii} > 0 \forall i$, then the t_{ij} 's are independently distributed with the t_{ii}^2 's having univariate gamma densities with parameters $(\alpha - \frac{i-1}{2}, 1)$.

Proof:

$C = TT'$ and $t_{ii} > 0 \forall i$. Then $0 < t_{ii} < \infty \forall i$ and $-\infty < t_{ij} < \infty$ for $i \neq j$ such that $TT' > 0$. Using the results from Deemer and Olkin (1951), we have

$$dC = 2^p \prod_{i=1}^p t_{ii}^{p+1-i} dT.$$

Moreover $|C| = \prod_{i=1}^p t_{ii}^2$ and

$$\begin{aligned} \text{tr}(TT') &= t_{11}^2 + (t_{21}^2 + t_{22}^2) + \dots + (t_{p1}^2 + \dots + t_{pp}^2) \\ &= \sum_{i=1}^p \sum_{j=1}^i t_{ij}^2. \end{aligned}$$

Hence from (4.1.3), we obtain

$$\begin{aligned} f_2(C) dC &= \frac{2^p}{\Gamma_p(\alpha)} \prod_{i=1}^p t_{ii}^{p+1-i} \prod_{i=1}^p (t_{ii}^2)^{\alpha - \frac{p+1}{2}} \exp\left(-\sum_{i=1}^p \sum_{j=1}^i t_{ij}^2\right) dT \\ &= \left\{ \prod_{i=1}^p 2(t_{ii}^2)^{\alpha - i/2} \frac{\exp(-t_{ii}^2)}{\Gamma(\alpha - \frac{i-1}{2})} \right\} \left\{ \prod_{i>j} \frac{\exp(-t_{ij}^2)}{\sqrt{\pi}} \right\} dT. \end{aligned} \quad (4.1.4)$$

From (4.1.4), we see that the joint density function of the t_{ij} 's factors. This implies that the t_{ij} 's are independently distributed. If we let $U_i = t_{ii}^2$, then U_i has a univariate gamma density given by

$$f_3(u_i) = \frac{u_i^{\alpha - \frac{i-1}{2} - 1} \exp(-u_i)}{\Gamma(\alpha - \frac{i-1}{2})} \quad 0 < u_i < \infty.$$

Property 2: The moment generating function of matrix A having the density (4.1.1) is defined by

$$M_A(T) = E(\exp(\text{tr}TX)),$$

where $T = (\gamma_{ij}t_{ij})$ with $\gamma_{ii} = 1$ for $i = 1, \dots, p$, $\gamma_{ij} = 1/2$ for $i \neq j$ and T is symmetric.

Hence,

$$\begin{aligned} M_A(T) &= \frac{1}{|B|^\alpha \Gamma_p(\alpha)} \int_{A>0} |A|^{\alpha - \frac{p+1}{2}} \exp(-\text{tr}(B^{-1} - T)X) dX \quad \text{for } (B^{-1} - T) > 0, \\ &= |I - BT|^{-\alpha} \quad \text{where } \|BT\| < 1. \end{aligned} \quad (4.1.5)$$

Property 3: Reproductive property.

Suppose that C and D are two independent matrices having matrix variate gamma densities with parameters (α_1, B) and (α_2, B) respectively. Then the matrix $C + D$ also has a matrix variate gamma density with parameters $(\alpha_1 + \alpha_2, B)$.

Proof:

$$\begin{aligned} M_{C+D}(T) &= |I - BT|^{-\alpha_1} |I - BT|^{-\alpha_2} \\ &= |I - BT|^{-(\alpha_1 + \alpha_2)}, \quad \|BT\| < 1, \end{aligned} \quad (4.1.6)$$

which is the m.g.f of a matrix variate gamma $((\alpha_1 + \alpha_2), B)$.

Property 4: In (4.1.1), let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ and $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ such that A_{11} and B_{11} are $(r \times r)$ matrices, A_{22} and B_{22} are $((p-r) \times (p-r))$ matrices. Then the marginal densities of A_{11} and A_{22} are matrix variate gamma with parameters (α, B_{11}) and (α, B_{22}) respectively.

Proof:

$$\text{Let } T_0 = \begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix}.$$

$$\begin{aligned} M_{A_{11}}(T_{11}) &= M_A(T_0) = |I - BT_0|^{-\alpha} \\ &= \left| \begin{pmatrix} I_r & 0 \\ 0 & I_{p-r} \end{pmatrix} - \begin{pmatrix} B_{11}T_{11} & 0 \\ B_{21}T_{11} & 0 \end{pmatrix} \right|^{-\alpha} \\ &= \{|I_r - B_{11}T_{11}| |I_{p-r}|\}^{-\alpha} \\ &= |I_r - B_{11}T_{11}|^{-\alpha}, \quad \|B_{11}T_{11}\| < 1. \end{aligned} \quad (4.1.7)$$

From the uniqueness of the moment generating function, A_{11} is distributed as matrix variate

gamma (α, B_{11}) . Similarly, we can show that A_{22} is distributed as matrix variate gamma (α, B_{22}) .

Property 5: Given that matrix A has density (4.1.1), we shall derive the density of $U = \text{tr} A$. Several methods have been used to derive the density of U and we shall give the one which expresses it in terms of a zonal polynomial.

The moment generating function of U is given by

$$\begin{aligned} M_U(t) &= E(e^{t \text{tr} A}) \\ &= \frac{1}{|B|^\alpha \Gamma_p(\alpha)} \int_{A>0} |A|^{\alpha - \frac{p+1}{2}} \exp(-\text{tr}(B^{-1} - tI)A) dA. \end{aligned}$$

Without any loss of generality we may assume that $(B^{-1} - tI) > 0$. Consequently, we have

$$\begin{aligned} M_U(t) &= |B^{-1} - tI|^{-\alpha} |B|^{-\alpha} \\ &= |I - tB|^{-\alpha} = \prod_{j=1}^p (1 - t\lambda_j)^{-\alpha}, \\ &\quad |t| < \frac{1}{\lambda_j}, \quad j = 1, \dots, p, \end{aligned} \tag{4.1.8}$$

where $\lambda_1, \dots, \lambda_p$ are the eigenvalues of matrix B and $\lambda_j > 0$, $j = 1, \dots, p$.

For any $\delta > 0$, we can write

$$\begin{aligned} |I - tB|^{-\alpha} &= |I - B/\delta + B(1 - \delta t)/\delta|^{-\alpha} \\ &= |B|^{-\alpha} \delta^{p\alpha} (1 - \delta t)^{-p\alpha} |I - (I - \delta B^{-1})/(1 - \delta t)|^{-\alpha} \\ &= \left[\prod_{j=1}^p \lambda_j^{-\alpha} \right] \delta^{p\alpha} \sum_{k=0}^{\infty} \sum_K \frac{(\alpha)_K}{k!} C_K(I - \delta B^{-1})(1 - \delta t)^{-(p\alpha + k)}, \end{aligned} \tag{4.1.9}$$

where $\mathbf{k} = (k_1, k_2, \dots, k_p)$ denotes a partition of the nonnegative integer k into not more than p parts $k_1 \geq k_2 \geq \dots \geq k_p \geq 0$, $k = k_1 + k_2 + \dots + k_p$, C_K denotes the zonal polynomial of order p and

$$(\alpha)_K = \prod_{i=1}^p (\alpha - (i-1)/2) k_i, \quad K = (k_1, k_2, \dots, k_p).$$

The above expansion is valid when $\| (I - \delta B^{-1}) / (1 - \delta t) \| < 1$. A sufficient condition might be having the absolute value of the largest eigenvalue less than unity and it can always be met by adjusting the value of the arbitrary quantity δ and choosing t . Thus the density function of U is the following.

$$\begin{aligned} g(u) &= \left[\prod_{j=1}^p \lambda_j^{-\alpha} \right] \delta^{p\alpha} \sum_{k=0}^{\infty} \sum_K \frac{(\alpha)_K}{k!} C_K (I - \delta B^{-1}) \frac{u^{p\alpha+k-1}}{\delta^{p\alpha+k} \Gamma(p\alpha+k)} e^{-u/\delta} \\ &= \left[\prod_{j=1}^p \lambda_j^{-\alpha} \right] \frac{u^{p\alpha-1}}{\Gamma(p\alpha)} \exp(-u/\delta) \sum_{k=0}^{\infty} \sum_K (\alpha)_K C_K (I - \delta B^{-1}) \left(\frac{u}{\delta} \right)^k \\ &\quad \div [k! (p\alpha)_k] \quad \text{for } U > 0, \\ &= 0, \quad \text{elsewhere.} \end{aligned} \tag{4.1.10}$$

The variable U is connected to various problems in different fields such as random division of intervals and distribution of spacings, see Dwass(1961), to test statistics and traces of Wishart matrices, see Mathai(1980), Mathai and Pillai(1982) and to time series problems, see MacNeill (1974).

Property 6: Suppose that matrix C has density (4.1.3) then the density of $V = |C|$ can be derived by first finding the $(s-1)$ -th moment of V .

$$\begin{aligned} E(V^{s-1}) &= \int_{C>0} \frac{|C|^{(s-1)+\alpha-(p+1)/2}}{\Gamma_p(\alpha)} \exp(-trC) dC \\ &= \frac{\Gamma_p(\alpha+s-1)}{\Gamma_p(\alpha)} \quad \text{where } R(\alpha+s-1) > \frac{p-1}{2} \\ &= \prod_{j=1}^p \frac{\Gamma(\alpha+s-1-\frac{j-1}{2})}{\Gamma(\alpha-\frac{j-1}{2})} \\ &= E(X_1^{s-1}) E(X_2^{s-1}) \dots E(X_p^{s-1}), \end{aligned} \tag{4.1.11}$$

where X_1, \dots, X_p are independent scalar gamma variables with parameters $(\alpha, 1), (\alpha$

$1/2, 1), \dots, (\alpha - (p - 1)/2, 1)$. Now the density of V is uniquely determined by $E(V^{s-1})$ through the inverse Mellin transform. Mathai(1971) obtains the density of V in the form:

$$f_4(v) = C G_{p,0}^{p,0}(v \mid \alpha - 1, \alpha - 3/2, \dots, \alpha - 1 - (p - 1)/2), \quad (4.1.12)$$

$$\text{where } C = \prod_{j=1}^p \frac{1}{\Gamma(\alpha - \frac{j-1}{2})} \text{ and}$$

$$G_{0,p}^{p,0}(v \mid \alpha - 1, \alpha - 3/2, \dots, \alpha - 1 - (p - 1)/2) = \frac{C}{2\pi i} \int_{c-i\infty}^{c+i\infty} \prod_{j=1}^p \Gamma(\alpha + s - 1 - \frac{j-1}{2}) v^{-s} ds$$

with $0 < v < \infty$ and $i = \sqrt{-1}$.

4.2 Densities related to matrix variate gamma density .

(4.2a) Wishart density.

A matrix $W_{p \times p}$, where $W = W' > 0$, has a central Wishart density, denoted by $W_p(N, \Sigma)$, if its density is of the form (4.1.1) with $\alpha = N/2$ and $B = 2\Sigma$. That is, the density is given by

$$\begin{aligned} g_1(W) &= \frac{|W|^{N/2-(p+1)/2}}{2^{Np/2} |\Sigma|^{N/2} \Gamma_p(N/2)} \exp(-(tr \Sigma^{-1} W)/2) \text{ for } W = W' > 0, \\ &\quad \Sigma = \Sigma' > 0, N/2 > (p - 1)/2, \\ &= 0 \quad \text{otherwise.} \end{aligned} \quad (4.2.1)$$

The Wishart density is a p -dimensional generalization of the chi-square density. Both the central and non-central Wishart densities have been discussed in detail in the literature. For the derivations of the central Wishart density, see for example Wishart(1928), Hsu(1939) and Fisher(1939). For the non-central case one can consult James(1955) and Constantine (1963).

Anderson(1984) and Muirhead(1982) give a comprehensive discussion of the properties.

(4.2b) Matrix variate beta densities.

Let A_1 and A_2 be two independent $(p \times p)$ symmetric positive definite matrices having gamma densities with parameters (α_1, I) and (α_2, I) respectively. We will show that the density of $U_1 = (A_1 + A_2)^{-1/2} A_1 (A_1 + A_2)^{-1/2}$ is a type-1 matrix variate beta density with parameters (α_1, α_2) and denoted by $\beta_1(\alpha_1, \alpha_2)$.

Since A_1 and A_2 are independently distributed, their joint density is given by

$$\begin{aligned} f(A_1, A_2) dA_1 dA_2 &= \frac{|A_1|^{\alpha_1 - (p+1)/2}}{\Gamma_p(\alpha_1) \Gamma_p(\alpha_2)} |A_2|^{\alpha_2 - (p+1)/2} \exp(-\text{tr}(A_1 + A_2)) dA_1 dA_2, \\ &\text{for } A_1 = A'_1 > 0, A_2 = A'_2 > 0, R(\alpha_1) > (p-1)/2, \\ &R(\alpha_2) > (p-1)/2. \end{aligned} \quad (4.2.2)$$

Letting $V_1 = A_1 + A_2$ for fixed A_1 , we have $dV_1 = dA_2$.

Letting $U_1 = V_1^{-1/2} A_1 V_1^{-1/2}$ for fixed V_1 , we obtain $dA_1 = |V_1|^{(p+1)/2} dU_1$.

Thus, we obtain the joint density of U_1 and V_1 as

$$\begin{aligned} f_2(U_1, V_1) dU_1 dV_1 &= \frac{|U_1|^{\alpha_1 - (p+1)/2}}{\Gamma_p(\alpha_1) \Gamma_p(\alpha_2)} |V_1|^{\alpha_1} |V_1 - V_1^{-1/2} U_1 V_1^{-1/2}|^{\alpha_2 - (p+1)/2} \exp(-\text{tr} V_1) dU_1 dV_1, \\ &= \frac{|V_1|^{\alpha_1 + \alpha_2 - (p+1)/2}}{\Gamma_p(\alpha_1) \Gamma_p(\alpha_2)} |U_1|^{\alpha_1 - (p+1)/2} |I - U_1|^{\alpha_2 - (p+1)/2} \exp(-\text{tr} V_1) dU_1 dV_1, \\ &\text{for } V_1 > 0, U_1 > 0, (I - U_1) > 0. \end{aligned} \quad (4.2.3)$$

Integrating out V_1 in (4.2.3) gives the marginal density of U_1 as

$$f_3(U_1) = \frac{\Gamma_p(\alpha_1 + \alpha_2)}{\Gamma_p(\alpha_1) \Gamma_p(\alpha_2)} |U_1|^{\alpha_1 - (p+1)/2} |I - U_1|^{\alpha_2 - (p+1)/2},$$

$$\begin{aligned}
& \text{for } 0 < U_1 < I, \quad R(\alpha_i) > (p-1)/2, \quad i = 1, 2, \\
& = 0, \quad \text{otherwise,}
\end{aligned} \tag{4.2.4}$$

which is $\beta_1(\alpha_1, \alpha_2)$. Similarly, we can obtain a type-2 matrix variate beta density with parameters (α_1, α_2) , denoted by $\beta_2(\alpha_1, \alpha_2)$, as the density of $A = A_2^{-1/2} A_1 A_2^{-1/2}$, where A_1 and A_2 are independently distributed matrix variate gammas with parameters (α_1, I) and (α_2, I)

The joint density of A_1 and A_2 is given by (4.2.2). Using the transformation $A_1 = A_2^{1/2} A A_2^{1/2}$ for fixed A_2 , $\Rightarrow dA_1 = |A_2|^{(p+1)/2} dA$. Consequently, the joint density of A and A_2 is given by

$$\begin{aligned}
q(A, A_2) dA dA_2 &= \\
& \frac{|A_2^{1/2} A A_2^{1/2}|^{\alpha_1 - (p+1)/2}}{\Gamma_p(\alpha_1) \Gamma_p(\alpha_2)} |A_2|^{\alpha_2 - (p+1)/2 + (p+1)/2} \exp(-\text{tr}(A_2^{1/2} A A_2^{1/2} + A_2)) dA dA_2 \\
&= \frac{|A_2|^{\alpha_1 + \alpha_2 - (p+1)/2}}{\Gamma_p(\alpha_1) \Gamma_p(\alpha_2)} |A|^{\alpha_1 - (p+1)/2} \exp(-\text{tr} A_2(A + I)) dA dA_2.
\end{aligned} \tag{4.2.5}$$

Integrating out A_2 in (4.2.5), we obtain the marginal density of A as

$$\begin{aligned}
g_1(A) &= \frac{\Gamma_p(\alpha_1 + \alpha_2)}{\Gamma_p(\alpha_1) \Gamma_p(\alpha_2)} |A|^{\alpha_1 - (p+1)/2} |I + A|^{-(\alpha_1 + \alpha_2)}, \\
& A = A' > 0, \quad R(\alpha_i) > (p-1)/2, \quad i = 1, 2, \\
& = 0, \quad \text{otherwise,}
\end{aligned} \tag{4.2.6}$$

Remark:

The densities of U_1 and A are matrix-variate generalizations of the following results: If X and Y are independently distributed gamma random variables, then (i) $\frac{X}{X+Y}$ has a type-1 beta density and (ii) $\frac{X}{Y}$ has a type-2 beta density.

Expressions for the central matrix variate beta densities have been derived by Khatri(1959, 1970), Olkin and Rubin(1964) and Mitra(1970) and those for the non-central case by Kshirsagar(1960) and De Waal(1970). Tan (1969) shows that if $U_1 \sim \beta_1(\alpha_1, \alpha_2)$ then

$$A = U_1^{\frac{1}{2}}(I_p - U_1)^{-1}U_1^{\frac{1}{2}} \sim \beta_2(\alpha_1, \alpha_2),$$

and if $A \sim \beta_2(\alpha_1, \alpha_2)$ then

$$U_1 = A^{\frac{1}{2}}(I_p + A)^{-1}A^{\frac{1}{2}} \sim \beta_1(\alpha_1, \alpha_2).$$

Property 1: Decomposition of the matrix-variate beta density.

Tan(1969) shows that if the $(p \times p)$ matrix $A_1 \sim \beta_1(\alpha_1, \alpha_2)$, then A_1 can be decomposed as a product of $2p - 1$ independent random variables, p of which are univariate type-1 beta variables, $\beta_1(\alpha_1, \alpha_2), \beta_1(\alpha_1 - 1/2, \alpha_2), \dots, \beta_1(\alpha_1 - (p - 1)/2, \alpha_2)$, while the others are Dirichlet variables, $D(1/2, \alpha_2 - 1/2), D(1/2, 1/2, b - 1), \dots, D(1/2, \dots, 1/2, b - (p - 1)/2)$

Property 2: Mitra(1970) proved that if matrix $A_1 \sim \beta_1(\alpha_1, \alpha_2)$, then for each fixed non-null vector \vec{a} , $(\vec{a}'A_1\vec{a})/(\vec{a}'\vec{a})$ has a univariate type-1 beta density with parameters (α_1, α_2) . This implies in particular that the diagonal elements of A_1 are distributed as univariate type-1 beta random variables with parameters (α_1, α_2) .

He also showed that if the $(p \times p)$ matrix $A_1 \sim \beta_1(\alpha_1, \alpha_2)$, then for every fixed non-null vector \vec{a} , $(\vec{a}'\vec{a})/(\vec{a}'A_1^{-1}\vec{a})$ has a univariate type-1 beta density with parameters $(\alpha_1 - (p - 1)/2, \alpha_2)$. Extensions of the above results can be seen from Khatri(1970). Other properties have been discussed by Tan(1969), and Roux(1971).

(4.2c) Matrix variate Dirichlet density.

Matrix variate Dirichlet densities are generalizations of type-1 and type-2 matrix variate

beta densities. We shall use the same procedure as Olkin and Rubin (1964) to derive these generalized densities.

Let A_0, A_1, \dots, A_k be $(p \times p)$ symmetric, positive definite and independently distributed matrix variate gamma variables with parameters $(\alpha_j, I), j = 0, \dots, k$. We have to find the joint density of

$$W_j = \left(\sum_{i=0}^k A_i \right)^{-1/2} A_j \left(\sum_{i=0}^k A_i \right)^{-1/2}, \quad j = 1, \dots, k. \quad (4.2.7)$$

The joint density function of A_0, A_1, \dots, A_k is given by

$$h(A_0, A_1, \dots, A_k) = \prod_{j=0}^k \frac{|A_j|^{\alpha_j - (p+1)/2}}{\Gamma_p(\alpha_j)} \exp(-\text{tr} A_j) \\ \text{for } A_j = A'_j > 0, R(\alpha_j) > (p-1)/2, j = 0, \dots, k. \quad (4.2.8)$$

Let $A = \sum_{i=0}^k A_i$ for fixed A_0 and $W_j = A^{-1/2} A_j A^{-1/2}, j = 1, \dots, k$.

$$\Rightarrow dA_0 \dots dA_k = |A|^{(k(p+1))/2} dA dW_1 \dots dW_k.$$

Consequently, the joint density of A, W_1, \dots, W_k is

$$h_1(A, W_1, \dots, W_k) = [1 / \prod_{i=0}^k \Gamma_p(\alpha_i)] |A - A^{1/2} \sum_{j=1}^k W_j A^{1/2}|^{\alpha_0 - (p+1)/2} |A|^{(k(p+1))/2} \\ \times |A^{1/2} W_1 A^{1/2}|^{\alpha_1 - (p+1)/2} \dots |A^{1/2} W_k A^{1/2}|^{\alpha_k - (p+1)/2} e^{-\text{tr} A} \\ = \left\{ \prod_{j=1}^k |W_j|^{\alpha_j - (p+1)/2} \right\} |I - \sum_{j=1}^k W_j|^{\alpha_0 - (p+1)/2} e^{-\text{tr} A} |A|^{\alpha_0 + \dots + \alpha_k} \\ \div [\Gamma_p(\alpha_0) \dots \Gamma_p(\alpha_k)], \quad 0 < W_j < I, 0 < \sum_{j=1}^k W_j < I, \\ R(\alpha_j) > (p-1)/2, \quad j = 1, \dots, k. \quad (4.2.9)$$

Integrating out A , we obtain the joint density of W_1, \dots, W_k in the form

$$h_2(W_1, \dots, W_k) = C \left\{ \prod_{j=1}^k |W_j|^{\alpha_j - \frac{p+1}{2}} \right\} |I - \sum_{j=1}^k W_j|^{\alpha_0 - \frac{p+1}{2}},$$

$$W_j > 0, I - \sum_{j=1}^k W_j > 0, j = 1, \dots, k, \quad (4.2.10)$$

where C is defined by $\frac{\Gamma_p(\alpha_0 + \dots + \alpha_k)}{\prod_{j=0}^k \Gamma_p(\alpha_j)}$. The above density is known as type-1 matrix variate Dirichlet density. Similarly, we obtain type-2 (or inverted) matrix variate Dirichlet density as a generalization of type-2 matrix variate beta density by considering the joint density of

$$V_j = A_0^{-\frac{1}{2}} A_j A_0^{-\frac{1}{2}}, j = 1, \dots, k, \quad (4.2.11)$$

where A_0, \dots, A_k have been defined above. Making the transformation (4.2.12) in (4.2.9), with the Jacobian being $|A_0|^{\frac{k(p+1)}{2}}$, we obtain the joint density of A_0, V_1, \dots, V_k as

$$\begin{aligned} h_3(A_0, V_1, \dots, V_k) &= \frac{1}{\prod_{j=0}^k \Gamma_p(\alpha_j)} \left\{ \prod_{j=1}^k |A_0^{\frac{1}{2}} V_j A_0^{\frac{1}{2}}|^{\alpha_j - \frac{p+1}{2}} \right\} |A_0|^{\alpha_0 - \frac{p+1}{2}} \\ &\times |A_0|^{\frac{k(p+1)}{2}} \exp\{-\text{tr} A_0(I + V_1 + \dots + V_k)\} \text{ where } A_0 = A'_0 > 0, \\ V_j = V'_j > 0, R(\alpha_j) > 0, j &= 1, \dots, k. \end{aligned} \quad (4.2.12)$$

Integrating out A_0 in (4.2.13), we obtain the joint density of V_1, \dots, V_k given by

$$\begin{aligned} h_4(V_1, \dots, V_k) &= \frac{\Gamma_p(\alpha_0 + \dots + \alpha_k)}{\prod_{j=0}^k \Gamma_p(\alpha_j)} \left\{ \prod_{j=1}^k |V_j|^{\alpha_j - \frac{p+1}{2}} \right\} |I + \sum_{j=1}^k V_j|^{-(\alpha_0 + \dots + \alpha_k)}, \\ V_j = V'_j > 0, j &= 1, \dots, k, R(\alpha_j) > \frac{p-1}{2}, j = 0, \dots, k, \end{aligned} \quad (4.2.13)$$

which is a type-2 matrix variate Dirichlet density. Other derivations of matrix variate Dirichlet density can be seen from Tan(1969), Mitra(1970) and Roux(1971) who gives his expressions in terms of generalized hypergeometric functions.

(4.3) Further generalizations associated with several matrix-variate gamma variables.

In this section, we shall look at some generalizations associated with several-variate gamma variables. Most of these generalizations can be obtained by using Lauricella functions which appear in many problems connected with geometric probabilities, time series situations, queueing situations and engineering problems. Various types of scalar Lauricella functions have been studied by Exton(1976) and Mathai and Saxena (1987). In this section, we shall consider the generalizations of scalar Lauricella functions of type A , B and D , to Lauricella functions of many matrix variables, which have been given by Mathai(1989). These functions will be defined by using integral representations and some of their properties will be studied.

(4.3.1) Lauricella function F_A of many matrix variates.

The Lauricella function F_A of many matrix variates, denoted by $F_A(a, b_1, \dots, b_n; c_1, \dots, c_n; X_1, \dots, X_n)$ analogous to the corresponding scalar case (Mathai and Saxena (1978)) is defined by the following integral representation.

$$\begin{aligned}
 F_A(a, b_1, \dots, b_n; c_1, \dots, c_n; X_1, \dots, X_n) = & \\
 & \frac{[\prod_{i=1}^n \Gamma_p(c_i)]}{[\prod_{i=1}^n \Gamma_p(b_i) \Gamma_p(c_i - b_i)]} \int_0^I \dots \int_0^I |U_1|^{b_1 - \frac{p+1}{2}} \dots |U_n|^{b_n - \frac{p+1}{2}} \\
 & \times |I - U_1|^{c_1 - b_1 - \frac{p+1}{2}} \dots |I - U_n|^{c_n - b_n - \frac{p+1}{2}} |I - X_1^{\frac{1}{2}} U_1 X_1^{\frac{1}{2}} - \dots - X_n^{\frac{1}{2}} U_n X_n^{\frac{1}{2}}|^{-a} \\
 & dU_1 \dots dU_n,
 \end{aligned} \tag{4.3.1}$$

for $R(b_1), \dots, R(b_n), R(c_1 - b_1), \dots, R(c_n - b_n) > \frac{p-1}{2}$, $X_i = X_i' > 0$, $U_i = U_i'$, $i = 1, \dots, n$, $\|X_1\| + \dots + \|X_n\| < 1$.

Using definition (4.3.1), we shall give some results on F_A .

Result (1):

$$\begin{aligned}
 & F_A(a, b_1, \dots, b_n; c_1, \dots, c_n; X_1, \dots, X_n) \\
 &= \frac{1}{\Gamma_p(a)} \int_{T>0} |T|^{a-\frac{p+1}{2}} \exp(-\text{tr } T) {}_1F_1(b_1; c_1; X_1^{\frac{1}{2}} T X_1^{\frac{1}{2}}) \\
 &\dots {}_1F_1(b_n; c_n; X_n^{\frac{1}{2}} T X_n^{\frac{1}{2}}) dT, \quad (4.3.2)
 \end{aligned}$$

where T is again a $(p \times p)$ symmetric positive definite matrix and ${}_1F_1$ is a hypergeometric function of matrix argument.

Proof:

Let $X_i^{\frac{1}{2}}$ be the symmetric square root of the symmetric positive definite matrix X_i , $i = 1, \dots, n$. Now, $\sum_{i=1}^n X_i^{\frac{1}{2}} U_i X_i^{\frac{1}{2}}$ is positive definite and $0 < \sum_{i=1}^n X_i^{\frac{1}{2}} U_i X_i^{\frac{1}{2}} < I$ as $0 < U_i < I$, $U_i = U_i'$, $X_i = X_i' > 0$, $i = 1, \dots, n$, $0 < \sum_{i=1}^n X_i < I$. Since $I - \sum_{i=1}^n X_i^{\frac{1}{2}} U_i X_i^{\frac{1}{2}} > 0$, we can write its determinant as an integral (see Mathai and Saxena (1978)) and it is given by

$$\begin{aligned}
 & |I - \sum_{i=1}^n X_i^{\frac{1}{2}} U_i X_i^{\frac{1}{2}}|^{-a} \\
 &= \int_{T>0} \frac{|T|^{a-\frac{p+1}{2}}}{\Gamma_p(a)} \exp(-\text{tr}(I - \sum_{i=1}^n X_i^{\frac{1}{2}} U_i X_i^{\frac{1}{2}})T) dT, \quad (4.3.3)
 \end{aligned}$$

for $R(a) > \frac{p-1}{2}$. We replace the last factor in (4.3.1) by the equivalent integral in (4.3.3), then interchange the integrals and integrate out U_i , the steps being valid. The integral is evaluated by noting that $\text{tr} X_i^{\frac{1}{2}} U_i X_i^{\frac{1}{2}} T = \text{tr} T^{\frac{1}{2}} X_i^{\frac{1}{2}} U_i X_i^{\frac{1}{2}} T^{\frac{1}{2}}$ and then using the result (5.2.18) of Mathai and Saxena (1978) (see appendix A4). Consequently, we obtain the following

$$\int_0^I |U_i|^{b_i-\frac{p+1}{2}} |I - U_i|^{c_i-b_i-\frac{p+1}{2}} \exp(\text{tr} X_i^{\frac{1}{2}} U_i X_i^{\frac{1}{2}} T) dU_i$$

$$= \frac{\Gamma_p(b_i)\Gamma_p(c_i - b_i)}{\Gamma_p(c_i)} {}_1F_1(b_i; c_i; X_i^{\frac{1}{2}} T X_i^{\frac{1}{2}}), \quad i = 1, \dots, n. \quad (4.3.4)$$

The result is easily obtained by using (4.3.3) and (4.3.4).

Result (2):

For $\|X_1\| + \dots + \|X_n\| < 1$, $R(a)$, $R(d - a) > \frac{p-1}{2}$,

$$\begin{aligned} & F_A(a, b_1, \dots, b_n; c_1, \dots, c_n; X_1, \dots, X_n) \\ &= \frac{\Gamma_p(d)}{\Gamma_p(a)\Gamma_p(d-a)} \int_0^I |U|^{a-\frac{p+1}{2}} |I-U|^{d-a-\frac{p+1}{2}} F_A(d, b_1, \dots, b_n; \\ & \quad c_1, \dots, c_n; U^{\frac{1}{2}} X_1 U^{\frac{1}{2}} \dots U^{\frac{1}{2}} X_n U^{\frac{1}{2}}) dU. \end{aligned} \quad (4.3.5)$$

Proof:

Using (4.3.1), we write F_A on the L.H.S of (4.3.5) as a multiple integral which is given by

$$\begin{aligned} & F_A(d, b_1, \dots, b_n; c_1, \dots, c_n; U^{\frac{1}{2}} X_1 U^{\frac{1}{2}} \dots U^{\frac{1}{2}} X_n U^{\frac{1}{2}}) \\ &= \frac{[\prod_{i=1}^n \Gamma_p(c_i)]}{[\prod_{i=1}^n \Gamma_p(b_i)\Gamma_p(c_i - b_i)]} \int_0^I \dots \int_0^I \left[\prod_{i=1}^n |U_i|^{b_i-\frac{p+1}{2}} |I-U_i|^{c_i-b_i-\frac{p+1}{2}} \right] \\ & \quad \times \left| I - \sum_{i=1}^n U_i^{\frac{1}{2}} X_i^{\frac{1}{2}} U_i X_i^{\frac{1}{2}} U_i^{\frac{1}{2}} \right|^{-d} \left[\prod_{i=1}^n dU_i \right]. \end{aligned} \quad (4.3.6)$$

Substituting (4.3.6) in (4.3.5) and integrating out U by using the Euler integral (5.2.25) of

Mathai and Saxena(1978), we obtain

$$\begin{aligned} & \int_0^I |U|^{a-\frac{p+1}{2}} |I-U|^{d-a-\frac{p+1}{2}} \left| I - \sum_{i=1}^n U_i^{\frac{1}{2}} X_i^{\frac{1}{2}} U_i X_i^{\frac{1}{2}} U_i^{\frac{1}{2}} \right|^{-d} dU \\ &= \frac{\Gamma_p(a)\Gamma_p(d-a)}{\Gamma_p(d)} {}_2F_1(a, d; d; \sum_{i=1}^n X_i^{\frac{1}{2}} U_i X_i^{\frac{1}{2}}), \end{aligned} \quad (4.3.7)$$

for $\|\sum_{i=1}^n X_i^{\frac{1}{2}} U_i X_i^{\frac{1}{2}}\| < 1$. Note that the condition for convergence is satisfied because

$$\|X_1^{\frac{1}{2}} U_1 X_1^{\frac{1}{2}} + \dots + X_n^{\frac{1}{2}} U_n X_n^{\frac{1}{2}}\| \leq \|X_1\| \|U_1\| + \dots + \|X_n\| \|U_n\| \leq \|X_1\| + \dots + \|X_n\| < 1$$

since $\|U_i\| < 1$ for $i = 1, \dots, n$. Moreover we can permute the matrices in the determinant when they are symmetric positive definite. That is

$$|I - AB| = |I - A^{\frac{1}{2}}BA^{\frac{1}{2}}| = |I - B^{\frac{1}{2}}AB^{\frac{1}{2}}| = |I - BA|.$$

From (4.3.7), one upper parameter is equal to one lower parameter in ${}_2F_1$ hence it reduces to a ${}_1F_0$ which is a binomial series. That is

$$\begin{aligned} {}_2F_1(a, d; d; \sum_{i=1}^n X_i^{\frac{1}{2}} U_i X_i^{\frac{1}{2}}) &= {}_1F_0(a, -; \sum_{i=1}^n X_i^{\frac{1}{2}} U_i X_i^{\frac{1}{2}}) \\ &= |I - \sum_{i=1}^n X_i^{\frac{1}{2}} U_i X_i^{\frac{1}{2}}|^{-a} \text{ for } \|\sum_{i=1}^n X_i^{\frac{1}{2}} U_i X_i^{\frac{1}{2}}\| < 1. \end{aligned} \quad (4.3.8)$$

Using the above fact in (4.3.6) and interpreting the resulting integral as an F_A , result (2) follows.

Result (3):

$$\begin{aligned} & \frac{[\prod_{i=1}^n \Gamma_p(d_i)]}{[\prod_{i=1}^n \Gamma_p(b_i) \Gamma_p(d_i - b_i)]} \int_0^I \dots \int_0^I [\prod_{i=1}^n |U_i|^{b_i - \frac{p+1}{2}} |I - U_i|^{d_i - b_i - \frac{p+1}{2}}] \\ & \times F_A(a, d_1, \dots, d_n; c_1, \dots, c_n; X_1^{\frac{1}{2}} U_1 X_1^{\frac{1}{2}}, \dots, X_n^{\frac{1}{2}} U_n X_n^{\frac{1}{2}}) [\prod_{i=1}^n dU_i] \\ & = \frac{[\prod_{i=1}^n \Gamma_p(c_i)]}{[\prod_{i=1}^n \Gamma_p(d_i) \Gamma_p(c_i - d_i)]} \int_0^I \dots \int_0^I [\prod_{i=1}^n |U_i|^{d_i - \frac{p+1}{2}} |I - U_i|^{c_i - d_i - \frac{p+1}{2}}] \\ & F_A(a, b_1, \dots, b_n; d_1, \dots, d_n; X_1^{\frac{1}{2}} U_1 X_1^{\frac{1}{2}}, \dots, X_n^{\frac{1}{2}} U_n X_n^{\frac{1}{2}}) [\prod_{i=1}^n dU_i], \end{aligned} \quad (4.3.9)$$

for $R(d_i - b_i) > \frac{p-1}{2}, R(c_i - d_i) > \frac{p-1}{2}, R(d_i) > \frac{p-1}{2}, R(b_i) > \frac{p-1}{2}, R(c_i) > \frac{p-1}{2}, \|X_i\| < 1, i = 1, \dots, n$.

Proof:

On L.H.S of (4.3.9), we substitute F_A as a multiple integral in a set of n new variables V_1, \dots, V_n to obtain a $2n$ -fold integral in U_1, \dots, U_n and V_1, \dots, V_n . We denote this integral

by T and it is of the form

$$T = \frac{[\prod_{i=1}^n \Gamma_p(c_i)]}{[\prod_{i=1}^n \Gamma_p(b_i) \Gamma_p(d_i - b_i) \Gamma_p(c_i - d_i)]} \int_0^I \dots (2n) \dots \int_0^I [\prod_{i=1}^n |U_i|^{b_i - \frac{p+1}{2}} \\ \times |I - U_i|^{d_i - b_i - \frac{p+1}{2}} |V_i|^{d_i - \frac{p+1}{2}} |I - V_i|^{c_i - d_i - \frac{p+1}{2}}] \\ |I - \sum_{i=1}^n X_i^{\frac{1}{2}} V_i^{\frac{1}{2}} U_i V_i^{\frac{1}{2}} X_i^{\frac{1}{2}}|^{-a} [\prod_{i=1}^n dU_i dV_i]. \quad (4.3.10)$$

Result (3) follows by interpreting the U -integrals in (4.3.10) as an F_A .

Result (4):

Each side of (4.3.9) is an $F_A(a, b_1, \dots, b_n; c_1, \dots, c_n; X_1, \dots, X_n)$.

Proof:

In (4.3.10), consider a transformation of U_i going to W_i through

$$V_i^{\frac{1}{2}} U_i V_i^{\frac{1}{2}} = W_i, \forall i, \Rightarrow |V_i|^{\frac{p+1}{2}} dU_i = dW_i, \forall i,$$

$$0 < W_i < V_i \text{ and } U_i = V_i^{-\frac{1}{2}} W_i V_i^{-\frac{1}{2}}, \forall i.$$

Thus T now becomes

$$T = C \int \dots (2n) \dots \int_{\substack{0 < W_i < V_i \\ 0 < V_i < I}} [\prod_{i=1}^n |W_i|^{b_i - \frac{p+1}{2}} |V_i - W_i|^{d_i - b_i - \frac{p+1}{2}} \times \\ |I - V_i|^{c_i - d_i - \frac{p+1}{2}} |I - X_1^{\frac{1}{2}} W_1 X_1^{\frac{1}{2}} - \dots - X_n^{\frac{1}{2}} W_n X_n^{\frac{1}{2}}|^{-a} [\prod_{i=1}^n dW_i dV_i], \quad (4.3.11)$$

where C is the constant representing the gamma products. Let $I - V_i = (I - W_i)^{\frac{1}{2}} Z_i (I -$

$W_i)^{\frac{1}{2}}, \forall i$ then $dV_i = |I - W_i|^{\frac{p+1}{2}} dZ_i$ and

$$|V_i - W_i| = |(I - W_i) - (I - V_i)| = |I - W_i| |I - Z_i|, \quad 0 < Z_i < I \forall i.$$

Making the above substitutions in T , we obtain

$$T = C \int_0^I \dots \int_0^I [\prod_{i=1}^n |W_i|^{b_i - \frac{p+1}{2}} |I - W_i|^{c_i - b_i - \frac{p+1}{2}}]$$

$$\begin{aligned} & \times |I - X_1^{\frac{1}{2}} W_1 X_1^{\frac{1}{2}} - \dots - X_n^{\frac{1}{2}} W_n X_n^{\frac{1}{2}}|^{-a} \left[\prod_{i=1}^n dW_i \right] \\ & \times \int_0^I \dots \int_0^I \left[\prod_{i=1}^n |Z_i|^{c_i - d_i - \frac{p+1}{2}} |I - Z_i|^{d_i - b_i - \frac{p+1}{2}} dZ_i \right] \end{aligned} \quad (4.3.12)$$

Since

$$\int_0^I |Z_i|^{c_i - d_i - \frac{p+1}{2}} |I - Z_i|^{d_i - b_i - \frac{p+1}{2}} dZ_i = \frac{\Gamma_p(c_i - d_i) \Gamma_p(d_i - b_i)}{\Gamma_p(c_i - b_i)}, \quad \forall i \quad (4.3.13)$$

Substituting (4.3.13) in (4.3.12) we have,

$$T = F_A(a, b_1, \dots, b_n; c_1, \dots, c_n; X_1, \dots, X_n).$$

(4.3.2) Lauricella function F_B of several matrix variates.

F_B of symmetric positive definite matrices X_1, \dots, X_n is defined as follows

$$\begin{aligned} & F_B(a_1, \dots, a_n; b_1, \dots, b_n; c; X_1, \dots, X_n) \\ & = \frac{\Gamma_p(c)}{\{\prod_{i=1}^n \Gamma_p(a_i)\} \Gamma_p(c - a_1 - \dots - a_n)} \int \dots \int_{0 < U_i < I, \sum_{i=1}^n U_i < I} |U_1|^{a_1 - \frac{p+1}{2}} \dots |U_n|^{a_n - \frac{p+1}{2}} \\ & \times |I - \sum_{i=1}^n U_i|^{c - a_1 - \dots - a_n - \frac{p+1}{2}} \prod_{i=1}^n |I - X_i^{\frac{1}{2}} U_i X_i^{\frac{1}{2}}|^{-b_i} dU_i, \end{aligned} \quad (4.3.14)$$

for $X_i = X_i' > 0$, $0 < X_i < I$, $R(a_i) > \frac{p-1}{2}$, $i = 1, \dots, n$, $R(c) > \frac{p-1}{2}$, $R(c - a_1 - \dots - a_n) > \frac{p-1}{2}$. Moreover

$$|I - X_i^{\frac{1}{2}} U_i X_i^{\frac{1}{2}}| = |I - X_i U_i| = |I - U_i X_i| = |I - U_i^{\frac{1}{2}} X_i U_i^{\frac{1}{2}}|$$

Result (5):

For $R(a_i) > \frac{p-1}{2}$, $i = 1, \dots, n$, $R(c) > \frac{p-1}{2}$, $R(c - a_1 - \dots - a_n) > \frac{p-1}{2}$,

$$\begin{aligned} & F_B(a_1, \dots, a_n; b_1, \dots, b_n; c; X_1, \dots, X_n) \\ & = \frac{1}{\Gamma_p(b_1) \dots \Gamma_p(b_n)} \int_{T_1 > 0} \dots \int_{T_n > 0} \exp(-\text{tr}(T_1 + \dots + T_n)) |T_1|^{b_1 - \frac{p+1}{2}} \dots |T_n|^{b_n - \frac{p+1}{2}} \\ & \times \phi_2(a_1, \dots, a_n; c; X_1^{\frac{1}{2}} T_1 X_1^{\frac{1}{2}} \dots X_n^{\frac{1}{2}} T_n X_n^{\frac{1}{2}}) dT_1 \dots dT_n, \end{aligned} \quad (4.3.15)$$

where

$$\begin{aligned}
& \phi_2(a_1, \dots, a_n; c; Y_1, \dots, Y_n) \\
&= \frac{\Gamma_p(c)}{[\prod_{i=1}^n \Gamma_p(a_i)] \Gamma_p(c - a_1 - \dots - a_n)} \int \dots \int_{\substack{0 < U_1 < I \\ 0 < \sum_{i=1}^n U_i < I}} \left[\prod_{i=1}^n |U_i|^{a_i - \frac{p+1}{2}} \right] \\
&\times |I - U_1 - \dots - U_n|^{c - a_1 - \dots - a_n - \frac{p+1}{2}} \\
&\times \exp(\text{tr}(U_1^{\frac{1}{2}} Y_1 U_1^{\frac{1}{2}} + \dots + U_n^{\frac{1}{2}} Y_n U_n^{\frac{1}{2}})) dU_1 \dots dU_n,
\end{aligned} \tag{4.3.16}$$

the $Y_i, i = 1, \dots, n$ are symmetric positive definite matrices.

Proof:

For $(I - X_i^{\frac{1}{2}} U_i X_i^{\frac{1}{2}}) > 0$, we can write

$$|I - X_i^{\frac{1}{2}} U_i X_i^{\frac{1}{2}}|^{-b_i} = \frac{1}{\Gamma_p(b_i)} \int_{T_i > 0} |T_i|^{b_i - \frac{p+1}{2}} \exp(-\text{tr}(I - X_i^{\frac{1}{2}} U_i X_i^{\frac{1}{2}}) T_i) dT_i, \tag{4.3.17}$$

for $R(b_i) > \frac{p-1}{2}$, $i = 1, \dots, n$. Substituting (4.3.17) in (4.3.15) and interchanging the integrals, the step being valid, we obtain

$$\begin{aligned}
F_B &= \frac{\Gamma_p(c)}{\Gamma_p(a_1) \dots \Gamma_p(a_n) \Gamma_p(c - a_1 - \dots - a_n) \Gamma_p(b_1) \dots \Gamma_p(b_n)} \\
&\times \int_{T_1 > 0} \dots \int_{T_n > 0} |T_1|^{b_1 - \frac{p+1}{2}} \dots |T_n|^{b_n - \frac{p+1}{2}} \exp(-\text{tr}(T_1 + \dots + T_n)) \\
&\times \left[\int \dots \int_{\substack{0 < U_1 < I \\ 0 < \sum_{i=1}^n U_i < I}} |U_1|^{a_1 - \frac{p+1}{2}} \dots |U_n|^{a_n - \frac{p+1}{2}} \right. \\
&|I - U_1 - \dots - U_n|^{c - a_1 - \dots - a_n - \frac{p+1}{2}} \exp(\text{tr}(\sum_{i=1}^n (X_i^{\frac{1}{2}} T_i X_i^{\frac{1}{2}}) U_i)) \\
&\left. \times dU_1 \dots dU_n \right] dT_1 \dots dT_n.
\end{aligned} \tag{4.3.18}$$

The result follows by interpreting the inner integral as a ϕ_2 .

(4.3.3) Lauricella function F_D of several matrix variates.

Lauricella function F_D of matrices X_i such that $X_i = X_i' > 0$, $\|X_i\| < 1$, $i = 1, \dots, n$, is defined by

$$\begin{aligned} F_D(a, b_1, \dots, b_n; c; X_1, \dots, X_n) \\ = \frac{\Gamma_p(c)}{\Gamma_p(a)\Gamma_p(c-a)} \int_0^1 |U|^{a-\frac{p+1}{2}} |I-U|^{c-a-\frac{p+1}{2}} |I-UX_1|^{-b_1} \\ |I-UX_n|^{-b_n} dU, \text{ for } R(c) > \frac{p-1}{2}, R(a) > \frac{p-1}{2}, R(c-a) > \frac{p-1}{2} \quad (4.3.19) \end{aligned}$$

Result (6):

$$\begin{aligned} F_D(a, b_1, \dots, b_n; c; X_1, \dots, X_n) \\ = \frac{\Gamma_p(d_1) \dots \Gamma_p(d_n)}{\Gamma_p(b_1) \dots \Gamma_p(b_n) \Gamma_p(d_1-b_1) \dots \Gamma_p(d_n-b_n)} \int_0^1 \dots \int_0^1 |U_1|^{b_1-\frac{p+1}{2}} \\ \times |U_n|^{b_n-\frac{p+1}{2}} |I-U_1|^{d_1-b_1-\frac{p+1}{2}} \dots |I-U_n|^{d_n-b_n-\frac{p+1}{2}} F_D(a, d_1, \dots, d_n; c, \\ U_1^{\frac{1}{2}} X_1 U_1^{\frac{1}{2}} \dots U_n^{\frac{1}{2}} X_n U_n^{\frac{1}{2}}) dU_1 \dots dU_n. \quad (4.3.20) \end{aligned}$$

Proof:

The proof is similar to the one used in result (2), that is we first replace F_D on the L.H.S of (4.3.20) using definition (4.3.19). Then we integrate out U_1, \dots, U_n . We have the following integrals,

$$\begin{aligned} \int_0^1 |U_i|^{b_i-\frac{p+1}{2}} |I-U_i|^{d_i-b_i-\frac{p+1}{2}} |I-U_i U^{\frac{1}{2}} X_i U^{\frac{1}{2}}|^{-d_i} dU_i \\ = \frac{\Gamma_p(d_i-b_i)\Gamma_p(b_i)}{\Gamma_p(d_i)} {}_2F_1(b_i, d_i; d_i; U^{\frac{1}{2}} X_i U^{\frac{1}{2}}), \quad i = 1, \dots, n \quad (4.3.21) \end{aligned}$$

The integrals are evaluated by the Euler's integrals (Mathai and Saxena (1978, (5.2.25)))

Since there is one common parameter in ${}_2F_1$, it reduces to a binomial sum. That is

$${}_2F_1(b_i, d_i; d_i; U^{\frac{1}{2}} X_i U^{\frac{1}{2}}) = {}_1F_0(b_i; U^{\frac{1}{2}} X_i U^{\frac{1}{2}})$$

$$\begin{aligned}
&= |I - U^{\frac{1}{2}} X_i U^{\frac{1}{2}}|^{-b_i} \\
&= |I - U X_i|^{-b_i}, \text{ for } \|U X_i\| < 1, i = 1, \dots, n. \quad (4.3.22)
\end{aligned}$$

The result follows by substituting back in (4.3.20).

Result (7):

For $R(b_i) > \frac{p-1}{2}$, $\|X_i\| < 1$, $i = 1, \dots, n$

$$\begin{aligned}
&\Gamma_p(b_1) \dots \Gamma_p(b_n) F_D(a, b_1, \dots, b_n; X_1, \dots, X_n) \\
&= \int_{T_1 > 0} \dots \int_{T_n > 0} \exp(-\text{tr}(T_1 + \dots + T_n)) |T_1|^{b_1 - \frac{p+1}{2}} \dots \\
&\times |T_n|^{b_n - \frac{p+1}{2}} {}_1F_1(a; c; X_1^{\frac{1}{2}} T_1 X_1^{\frac{1}{2}} + \dots + X_n^{\frac{1}{2}} T_n X_n^{\frac{1}{2}}) dT_1 \dots dT_n. \quad (4.3.23)
\end{aligned}$$

Proof:

Since for all i , $(I - U X_i) > 0$ we can replace $|I - U X_i|$ in (4.3.19) by

$$|I - U X_i|^{-b_i} = \frac{1}{\Gamma_p(b_i)} \int_{T_i > 0} |T_i|^{b_i - \frac{p+1}{2}} \exp(-\text{tr}(I - X_i^{\frac{1}{2}} U X_i^{\frac{1}{2}}) T_i) dT_i, \quad (4.3.24)$$

for $R(b_i) > \frac{p-1}{2}$, $i = 1, \dots, n$. We substitute (4.3.24) in F_D and after collecting the factors containing U we integrate it out using the result from appendix A4. Thus we have

$$\begin{aligned}
&\int_{U > 0} |U|^{a - \frac{p+1}{2}} |I - U|^{c - a - \frac{p+1}{2}} \exp(\text{tr}(\sum_{i=1}^n X_i^{\frac{1}{2}} T_i X_i^{\frac{1}{2}}) U) dU \\
&= \frac{\Gamma_p(a) \Gamma_p(c - a)}{\Gamma_p(c)} {}_1F_1(a; c; X_1^{\frac{1}{2}} T_1 X_1^{\frac{1}{2}} + \dots + X_n^{\frac{1}{2}} T_n X_n^{\frac{1}{2}}). \quad (4.3.25)
\end{aligned}$$

Substituting the above equation in the L.H.S of (4.3.23) yields the result.

The next result will be stated without proof and later it will be used to establish the remaining results.

Result (8):

$$\begin{aligned}
 F_D(a, b_1, \dots, b_n; c; X_1, \dots, X_n) \\
 = \frac{\Gamma_p(c)}{\Gamma_p(b_1) \dots \Gamma_p(b_n) \Gamma_p(c - b_1 - \dots - b_n)} \int \dots \int |U_1|^{b_1 - \frac{p+1}{2}} \\
 \times |U_n|^{b_n - \frac{p+1}{2}} |I - U_1 - \dots - U_n|^{c - b_1 - \dots - b_n - \frac{p+1}{2}} \\
 |I - X_1^{\frac{1}{2}} U_1 X_1^{\frac{1}{2}} - \dots - X_n^{\frac{1}{2}} U_n X_n^{\frac{1}{2}}|^{-a} dU_1 \dots dU_n.
 \end{aligned} \tag{4.3.26}$$

Note that the integral is over $0 < U_i < I$, $i = 1, \dots, n$ and $0 < \sum_{i=1}^n U_i < I$. The conditions are the same as the ones in (4.3.19).

Result (9):

$$\begin{aligned}
 F_D(a, b_1, \dots, b_n; c; X_1, \dots, X_n) = \frac{1}{\Gamma_p(a)} \int_{T>0} |T|^{a - \frac{p+1}{2}} \exp(-tr(T)) \phi_2(b_1, \\
 b_n; c; X_1^{\frac{1}{2}} T X_1^{\frac{1}{2}}, \dots, X_n^{\frac{1}{2}} T X_n^{\frac{1}{2}}) dT,
 \end{aligned} \tag{4.3.27}$$

for $R(a) > \frac{p-1}{2}$, the other conditions are the same as the ones in (4.3.19) and the ϕ_2 function is defined in (4.3.16).

Proof:

We substitute for ϕ_2 and integrate out T to obtain

$$\begin{aligned}
 \frac{1}{\Gamma_p(a)} \int_{T>0} |T|^{a - \frac{p+1}{2}} \exp(-tr(I - X_1^{\frac{1}{2}} U_1 X_1^{\frac{1}{2}} - \dots - X_n^{\frac{1}{2}} U_n X_n^{\frac{1}{2}})T) dT \\
 = |I - X_1^{\frac{1}{2}} U_1 X_1^{\frac{1}{2}} - \dots - X_n^{\frac{1}{2}} U_n X_n^{\frac{1}{2}}|^{-a}
 \end{aligned} \tag{4.3.28}$$

We now substitute the above equation on the R.H.S of (4.3.27) and interpret it by using (4.3.26) and the result follows.

Result (10):

$$\begin{aligned}
 & F_D(a, b_1, \dots, b_n; c; X_1, \dots, X_n) \\
 &= \frac{\Gamma_p(c)}{\Gamma_p(d)\Gamma_p(c-d)} \int_0^1 |U|^{d-\frac{p+1}{2}} |I-U|^{c-d-\frac{p+1}{2}} \\
 & \times F_D(a, b_1, \dots, b_n; d; X_1^{\frac{1}{2}} U X_1^{\frac{1}{2}}, \dots, X_n^{\frac{1}{2}} U X_n^{\frac{1}{2}}) dU, \quad (4.3.29)
 \end{aligned}$$

for $R(c) > \frac{p-1}{2}$, $R(d) > \frac{p-1}{2}$, $R(c-d) > \frac{p-1}{2}$, $\|X_i\| < 1$, $i = 1, \dots, n$.

Proof:

Let W represent the R.H.S of equation (4.3.29), thus using (4.3.19), we can rewrite W as

$$\begin{aligned}
 W &= \frac{\Gamma_p(c)}{\Gamma_p(d)\Gamma_p(c-d)} \int_0^1 |U|^{d-\frac{p+1}{2}} |I-U|^{c-d-\frac{p+1}{2}} \\
 & \times F_D(a, b_1, \dots, b_n; d; X_1^{\frac{1}{2}} U X_1^{\frac{1}{2}}, \dots, X_n^{\frac{1}{2}} U X_n^{\frac{1}{2}}) dU, \\
 &= \frac{\Gamma_p(c)}{\Gamma_p(d)\Gamma_p(c-d)} \int_0^1 |U|^{d-\frac{p+1}{2}} |I-U|^{c-d-\frac{p+1}{2}} \\
 & \times \frac{\Gamma_p(d)}{\Gamma_p(a)\Gamma_p(d-a)} \int_0^1 |T|^{a-\frac{p+1}{2}} |I-T|^{d-a-\frac{p+1}{2}} |I-TX_1^{\frac{1}{2}} U X_1^{\frac{1}{2}}|^{-b_1} \\
 & \dots |I-TX_n^{\frac{1}{2}} U X_n^{\frac{1}{2}}|^{-b_n} dT dU \\
 &= \frac{\Gamma_p(d)}{\Gamma_p(a)\Gamma_p(d-a)} \int_0^1 |T|^{a-\frac{p+1}{2}} |I-T|^{d-a-\frac{p+1}{2}} \\
 & \times F_D(d, b_1, \dots, b_n; c; X_1^{\frac{1}{2}} T X_1^{\frac{1}{2}}, \dots, X_n^{\frac{1}{2}} T X_n^{\frac{1}{2}}) dT. \quad (4.3.30)
 \end{aligned}$$

We now replace the R.H.S of (4.3.30) by the multiple integral representation in (4.3.26) and integrate out T . As it was shown earlier, the T -integral leads to a ${}_2F_1$ which reduces to a ${}_1F_0$. Thus interpreting the R.H.S of (4.3.30) by using the multiple integral representation of (4.3.26), the result follows.

(4.3.4) Mixed Results.

We shall consider a result involving F_B and F_D of many matrix variates, similar results involving F_A and F_D can be seen from Mathai(1989).

Result (11):

For $R(b_i) > \frac{p-1}{2}$, $\|X_i\| < 1$, $i = 1, \dots, n$, $R(d) > \frac{p-1}{2}$, $R(d - b_1 - \dots - b_n) > \frac{p-1}{2}$,

$$\begin{aligned} & F_B(b_1, \dots, b_n, a_1, \dots, a_n; c; X_1, \dots, X_n) \\ &= \frac{\Gamma_p(d)}{\Gamma_p(b_1) \dots \Gamma_p(b_n) \Gamma_p(d - b_1 - \dots - b_n)} \\ & \times \int \dots \int |U_1|^{b_1 - \frac{p+1}{2}} \dots |U_n|^{b_n - \frac{p+1}{2}} |I - U_1 - \dots - U_n|^{d - b_1 - \dots - b_n - \frac{p+1}{2}} \\ & \times F_D(d, a_1, \dots, a_n; c; X_1^{\frac{1}{2}} U_1 X_1^{\frac{1}{2}}, \dots, X_n^{\frac{1}{2}} U_n X_n^{\frac{1}{2}}) dU_1 \dots dU_n. \end{aligned} \quad (4.3.31)$$

Proof:

Using (4.3.19), we replace F_D by the single integral representation and substitute it in (4.3.31). Then we make the transformation of U_i going to $W_i = U^{\frac{1}{2}} U_i U^{\frac{1}{2}}$, $i = 1, \dots, n$. If we denote the R.H.S of (4.3.31) by Y , we have

$$\begin{aligned} Y &= \frac{\Gamma_p(c)}{\Gamma_p(b_1) \dots \Gamma_p(b_n) \Gamma_p(d - b_1 - \dots - b_n) \Gamma_p(c - d)} \int_{W_1} \dots \int_{W_n} \int_U |W_1|^{b_1 - \frac{p+1}{2}} \\ & \dots |W_n|^{b_n - \frac{p+1}{2}} |I - X_1^{\frac{1}{2}} W_1 X_1^{\frac{1}{2}}|^{-a_1} \dots |I - X_n^{\frac{1}{2}} W_n X_n^{\frac{1}{2}}|^{-a_n} \\ & \times |I - U|^{c - d - \frac{p-1}{2}} |U - W_1 - \dots - W_n|^{d - b_1 - \dots - b_n - \frac{p-1}{2}} \left[\prod_{i=1}^n dW_i \right] dU. \end{aligned} \quad (4.3.32)$$

We note that

$$\begin{aligned} U - W_1 - \dots - W_n &= (I - W_1 - \dots - W_n) - (I - U) \\ &= (I - W_1 - \dots - W_n)^{\frac{1}{2}} (I - Z) (I - W_1 - \dots - W_n)^{\frac{1}{2}}, \end{aligned}$$

where $(I - U) = (I - W_1 - \dots - W_n)^{\frac{1}{2}} Z (I - W_1 - \dots - W_n)^{\frac{1}{2}}$.

Using these substitutions and the matrix variate beta integral the Z -integral reduces to the following:

$$\int_0^I |Z|^{c-d-\frac{p-1}{2}} |I-Z|^{d-b_1-\dots-b_n-\frac{p-1}{2}} dZ = \frac{\Gamma_p(c-d)\Gamma_p(d-b_1-\dots-b_n)}{\Gamma_p(c-b_1-\dots-b_n)}.$$

Thus we obtain,

$$\begin{aligned} Y &= \frac{\Gamma_p(c)}{\Gamma_p(b_1)\dots\Gamma_p(b_n)\Gamma_p(c-b_1-\dots-b_n)} \int \dots \int |W_1|^{b_1-\frac{p+1}{2}} \\ &\quad \dots |W_n|^{b_n-\frac{p+1}{2}} |I-W_1-\dots-W_n|^{c-b_1-\dots-b_n-\frac{p-1}{2}} |I-X_1^{\frac{1}{2}}W_1X_1^{\frac{1}{2}}|^{-a_1} \\ &\quad \dots |I-X_n^{\frac{1}{2}}W_nX_n^{\frac{1}{2}}|^{-a_n} dW_1 \dots dW_n \\ &= F_B(b_1, \dots, b_n; a_1, \dots, a_n; c; X_1, \dots, X_n). \end{aligned} \tag{4.3.33}$$

APPENDIX

(A1) H-FUNCTION.

The H-function is applicable in many problems arising in physical sciences, engineering, and statistics. It is the most generalized special function and it is studied in some detail in Braaksma (1964). An H-function is defined in terms of a Mellin-Barnes type integral as follows:

$$\begin{aligned} H_{p,q}^{m,n}(z) &= H_{p,q}^{m,n}[z]_{(b_q, B_q)}^{(a_p, A_p)} \\ &= H_{p,q}^{m,n}[z]_{(b_1, B_1), \dots, (b_q, B_q)}^{(a_1, A_1), \dots, (a_p, A_p)} \\ &= \frac{1}{2\pi i} \int_L \chi(s) z^s ds, \end{aligned} \quad (0.1.1)$$

where

$$i = (-1)^{1/2}, \quad z \neq 0,$$

$$z^s = \exp[s \log |z| + i \arg(z)],$$

in which $\log |z|$ represents the natural logarithm of $|z|$ and $\arg z$ is not necessarily the principal value. An empty product is interpreted as unity. Here, we have

$$\chi(s) = \frac{\prod_{j=1}^m \Gamma(b_j - B_j s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + B_j s)} \frac{\prod_{j=1}^n \Gamma(1 - a_j + A_j s)}{\prod_{j=n+1}^p \Gamma(a_j - A_j s)}, \quad (0.1.2)$$

where m, n, p and q are nonnegative integers such that $0 \leq n \leq p$, $1 \leq m \leq q$; $A_j (j = 1, \dots, p)$, $B_j (j = 1, \dots, q)$ are positive numbers; $a_j (j = 1, \dots, p)$, $b_j (j = 1, \dots, q)$ are complex numbers such that

$$A_j(b_h + \nu) \neq B_h(a_j - \lambda - 1) \quad (0.1.3)$$

for $\nu, \lambda = 0, 1, \dots$; $h = 1, \dots, m$; $j = 1, \dots, n$. L is a contour separating the points

$$s = \frac{b_j + \nu}{B_j}, \quad (j = 1, \dots, m; \nu = 0, 1, \dots)$$

which are the poles of $\Gamma(b_j - B_j s)$, $(j = 1, \dots, m)$ from the points

$$s = \frac{a_j - \nu - 1}{A_j}, \quad (j = 1, \dots, n; \nu = 0, 1, \dots),$$

which are the poles of $\Gamma(1 - a_j + A_j s)$, $j = 1, \dots, n$. The contour L exists on account of (0.1.3). The H-function is an analytic function of z and makes sense if the following existence conditions are satisfied.

Case 1: For all $z \neq 0$ with $\mu > 0$.

Case 2: For $0 < |z| < \beta^{-1}$ with $\mu = 0$.

Here $\mu = \sum_{j=1}^q B_j - \sum_{j=1}^p A_j$ and

$$\beta = \prod_{j=1}^p A_j^{A_j} \prod_{j=1}^q B_j^{-B_j}.$$

Remark: When we have $A_1 = \dots = A_p = B_1 = \dots = B_q = 1$ in the H-function, we obtain the Meijer's G-function.

(A2) HYPERGEOMETRIC SERIES.

A hypergeometric function with $p + q$ parameters is defined as follows

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n}{(b_1)_n} \dots \frac{(a_p)_n}{(b_q)_n} \frac{z^n}{n!} \quad (0.2.1)$$

for $p \leq q$ or $p = q + 1$ and $|z| < 1$, where for example $(a)_0 = 1$ and $(a)_n = a(a+1) \dots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$, $n = 1, 2, \dots$

Here (0.2.1) diverges for all $z \neq 0$ if $p > q + 1$. Assume that none of the b_1, \dots, b_q is zero or a negative integer and if any of the a_1, \dots, a_p is a negative integer the series terminates. a_1, \dots, a_p are often known as the upper parameters and b_1, \dots, b_q are called the lower

parameters.

A particular case of the hypergeometric function is Gauss' hypergeometric function ${}_2F_1(\cdot)$ which is given by

$$\begin{aligned} {}_2F_1(a, b; c; z) &= F(a, b; c; z) \\ &= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}. \end{aligned} \quad (0.2.2)$$

The Euler's integral representation for the Gauss' hypergeometric function is

$$\begin{aligned} F(a, b; c; z) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} dt \\ &\text{for } R(c) > R(b) > 0. \end{aligned} \quad (0.2.3)$$

(A3) BESSEL FUNCTION.

The different Bessel functions listed below are solutions of Bessel differential equation.

$$\begin{aligned} J_v(z) &= \sum_{m=0}^{\infty} (-1)^m (z/2)^{2m+v} / [m! \Gamma(m+v+1)] \\ &= (z/2)^v {}_0F_1(v+1; -z^2/4) / \Gamma(v+1). \end{aligned} \quad (0.3.1)$$

(0.3.1) is called a Bessel function of the first kind where z is the variable and v is the order.

The Bessel function of the second kind or Neumann's function is given by

$$Y_v(z) = (\sin v\pi)^{-1} [J_v(z) \cos(v\pi) - J_{-v}(z)] \quad (0.3.2)$$

The modified Bessel function of the first kind is given by

$$\begin{aligned} I_v(z) &= \sum_{m=0}^{\infty} (z/2)^{2m+v} / [m! \Gamma(m+v+1)] \\ &= \frac{(z/2)^v}{\Gamma(v+1)} {}_0F_1(v+1; z^2/4) \\ &= \frac{(z/2)^v}{\Gamma(v+1)} e^{-z} {}_1F_1(v+1/2; 2v+1; 2z). \end{aligned} \quad (0.3.3)$$

The modified Bessel function of integer order is given by

$$K_n(z) = (-1)^{n+1} I_n(z) \log(z/2) + \sum_{m=0}^{n-1} \frac{(-1)^m}{2m!} (n-m-1)! (z/2)^{2m-n} \\ + (-1)^n / 2 \sum_{m=0}^{\infty} (z/2)^{n+2m} \frac{[\psi(n+m+1) + \psi(m+1)]}{m!(n+m)!}, \quad (0.3.4)$$

for $n = 1, 2, \dots$ where $\psi(\cdot)$ is defined in (1.1.12).

(A4) CONFLUENT HYPERGEOMETRIC FUNCTIONS OF MATRIX ARGUMENT.

$${}_1F_1(\alpha; \beta; Z) = \frac{\Gamma_m(\beta)}{(2\pi i)^{m(m+1)/2}} \int_{R(\Lambda)=X_0>0} \exp(\text{tr}(\Lambda)) \\ \times |I - Z\Lambda^{-1}|^{-\alpha} |\Lambda|^{-\beta} d\Lambda, \\ \text{for } X_0 > R(Z), R(\beta) > m, \\ = \frac{\Gamma_m(\beta)}{\Gamma_m(\alpha)\Gamma_m(\beta-\alpha)} \int_0^I \exp(\text{tr}(\Lambda Z)) |\Lambda|^{\alpha-(m+1)/2} \\ |I - \Lambda|^{\beta-\alpha-(m+1)/2} d\Lambda, \text{ for } R(\alpha) > \frac{m+1}{2} - 1, \\ R(\beta) > \frac{m+1}{2} - 1 \text{ and } R(\beta-\alpha) > \frac{m+1}{2} - 1. \quad (0.4.1)$$

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