THE APPLICATION OF STATISTICAL METHODS TO CIRCULAR DATA

by

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In this thesis some recent methods of analysis of circular data are applied to some illustrations from published papers: also, some new methods of estimating parameters of circular data are examined and their efficiency is verified by Monte Carlo techniques. These latter methods, also, are illustrated by examples from research papers. The three distributions used to describe circular data are:

(1) The Circular Normal Distribution (Unimodal)

$$f\,(\theta;\,K,\,\theta_o) = \frac{1}{2\pi I_o(K)}\,\exp(K\cos(\theta - \theta_o)) \qquad 0 \leq \theta < 2\pi$$

(2) Distribution B1 (Bimodal with equally weighted modes 180^o apart)

$$f(\psi;\lambda,\psi_o) = \frac{1}{2\pi I_o(\lambda)}\,\exp(\lambda\cos 2(\psi - \psi_o) \qquad 0 \leq \psi < 2\pi$$

(3) Distribution B2 (Bimodal with unequally weighted modes 180^o apart)

$$f(\theta;A,K,\theta_o) = \frac{A}{2\pi I_o(K)}\,\exp(K\cos(\theta - \theta_o)) + \frac{(1-A)}{2\pi I_o(K)}\,\exp(K\cos(\theta - \theta_o + \pi))$$

$0 \leq \theta < 2\pi$.

An appendix also contains plots, lists of most data analysed, and tables of significance points.

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CHAPTER I

INTRODUCTION

1.1 Summary

In this thesis some recent methods of analysis of circular data are applied to some illustrations from published papers; also, certain new methods of estimating parameters of circular data are examined and their efficiency is verified by Monte Carlo techniques. These latter methods, also, are illustrated by examples from research papers. Circular Data is the term used to describe observations which are recorded by an angle and by a distance from a certain point or origin. Circular data arise frequently in the biological and earth sciences where the observations are represented as points on the circumference of a unit circle, as points on a polar diagram, or as unit vectors emanating from the origin. In the past, clustered observations on the unit circle have often been analysed as though they were a sample from the normal distribution. The problem of using the linear normal distribution and other linear distributions such as the wrapped normal distribution (see Batschelet (1965), Stephens (1963a)) to describe circular data is that their analysis is difficult. The statistics used in their analysis, chiefly the mean and variance, are not invariant of the origin, that is, the computed values of these above statistics depend upon the point designated as the origin.

The object of statistical analysis is to describe the data, to estimate parameters, and to test hypotheses concerning the parameters. The three distributions used to describe circular data in this thesis are now summarised:

(a) The Circular Normal (CN) Distribution is a unimodal distribution introduced by Gumbel, Greenwood, and Durand (1953). The CN density function with mode at θ_0 and antimode at $\theta_0 + \pi$ is

$$f(\theta;\, K,\, \theta_0) = \frac{1}{2\pi I_0\,(K)}\ \exp(K \cos\,(\theta - \theta_0)) \qquad 0 \leq \theta < 2\pi \quad ,$$

where θ is the polar coordinate of a typical observation on the circumference of the unit circle and $K \geq 0$ is a measure of dispersion. $I_0(K)$ is the imaginary Bessel function of order zero and is tabulated for different values of K, see for example Jahnke and Emde (1945).

(b) Distribution B1 is an equally weighted bimodal distribution originating from Breitenberger (1963) and analysed by Stephens (1966). Its density function is

$$f(\psi\,;\lambda\,,\psi_0) = \frac{1}{2\pi\ I_0(\lambda)}\ \exp(\lambda \cos\, 2(\psi - \psi_0)) \qquad 0 \leq \psi < 2\pi,$$

where the modes occur at $\psi = \psi_0$ and $\psi = \psi_0 + \pi$ and the antimodes at $\psi = \psi_0 \pm \pi/2$. If $\lambda = K$, $2\psi = \theta$, and $2\psi_0 = \theta_0$ then distribution B1 reduces to CN(K, θ_0). Estimators of λ and ψ_0, their properties, and a complete discussion of distribution B1 are found in Stephens (1966).

(c) Distribution B2 is a bimodal distribution composed of two weighted CN distributions with modes at θ_0 and $\theta_0 - \pi$ respectively.

Its density function is

$$f(\theta;A,B,K,\theta_0) = \frac{A}{2\pi I_0(K)} \exp(K\cos(\theta-\theta_0)) + \frac{B}{2\pi I_0(K)} \exp(K\cos(\theta-\theta_0+\pi))$$

$0 \leq \theta < 2\pi$.

B2 is a modification of a distribution proposed by Gumbel (1954). $A \geq 0$ and $B \geq 0$ are the weights; $A + B = 1$, in order for B2 to integrate up to one. Note that K, the measure of dispersion, is assumed to be the same for the two weighted CN distributions.

A summary of the chapters is now given:

Chapter I indicates some areas where circular data arise and also reviews techniques that have already been applied to such data. Some basic estimation procedures and some goodness of fit statistics are also discussed.

Chapter II Some basic properties of the CN distribution are reviewed and five sets of unimodal data drawn from published papers are analysed.

Chapter III Distributions B1 and B2 are discussed in greater detail accompanied by the analysis of a number of sets of both artificial and real data.

Chapter IV is devoted to Monte Carlo checks on estimation techniques.

Chapter V reviews conclusions and presents some suggestions for further work.

Appendix contains the plots of data analysed, tables of significance points for U^2_n and $\sqrt{n}\,V_n$, an extension of a table from Stephens (1963a), and lists of most data analysed.

The CN and Bl distribution have analogues on the unit sphere (see e.g. Fisher (1953), Stephens (1967)) but these are not discussed in this thesis.

1.2 Areas in which Circular Data arise

(a) Pebble Orientation and Cross Bedding Studies. Elongated rock fragments and grains tend to align themselves parallel to the direction of river or ocean currents and therefore give an indication of the direction of flow and of a possible source area of the sediment transported. For a further discussion see Curray (1956), Krumbein (1939), Dapples and Rominger (1945), Kauranne (1960), Wadell (1936), West and Donner (1956). Inclined bedding planes also yield angular observations which are useful in determining the direction of transport. Ripple marks, oriented plant fragments, and elongated shells also provide information that is useful in making poleogeographic reconstructions -- see Tanner (1955), Crowell (1958), Chenoweth (1952), Land (1964), Looff and Hubert (1964), McKee (1940), Opdyke and Runcorn (1960).

(b) Direction of Movement of Animals. Studies of the directions taken by birds, frogs, fish etc. are important in determining the animals' orientation mechanism — see Bellrose (1958), Eaton (1934), Griffin and Goldsmith (1953), Pratt and Thouless (1955), Schmidt-Koenig (1963). One experiment consists of releasing animals singly on both sunny and cloudy days and then determining from their recorded directions whether they use the sun as a means of orientation. The directions (often bimodal) taken by turtles after being displaced inland are also useful in obtaining information on their homing mechanism, that is, to see if they do know the direction back to the sea -- see Gould (1957), Cutchis (1965). The orientation of the swimming directions of certain marine animals to polarized light has also been investigated -- see Bainbridge and Waterman (1957, 1958), Daumer, Jander, and Waterman (1963), Kalmus (1959).

(c) Other Areas. Measurements of the pleural angles of fossils give an indication of the size of the animal -- see Chronic (1952). Observations taken over time such as the time(s) of peak activity of a marine organism during a twenty-four hour period or the monthly number of auto accidents in Canada observed over one year can be treated as circular variables -- see Gumbel (1954), Batschelet (1965).

1.3 Review of Past Techniques applied to Circular Data

Reiche (1938) analysed the variability of n angles $\theta_1, \cdots, \theta_n$ by calculating the magnitude R of their resultant and forming the "consistency ratio" R/n which he said was inversely proportional to the standard error. Reiche (1938) also determines $\hat{\theta}_0$ (estimator of θ_0) and the number of observations required for the analysis by graphically computing the "flatness point" -- the point at which the cumulative curve of the resultant vector fluctuates less than 5^{o}. Krumbein (1939) computes $\hat{\theta}_o$ from the resultant but still applies the test statistics of linear normal theory. Chayes (1949) and Bainbridge and Waterman (1957) apply Pearson's X^2 test for uniformity -- often called randomness. Tukey (1954) uses a similar test and also determines $\hat{\theta}_o$ from the resultant but gives no measure of variability. Chayes (1954) determines a minimum variance origin for the data. Greenwood and Durand (1955) introduce the Rayleigh test for randomness and Curray (1956) and Schmidt-Koenig (1963) use this test. Durand and Greenwood (1958) modified the Rayleigh test when θ_o was known and produced a more powerful V-test. Gumbel, Greenwood, and Durand (1953) apply maximum likelihood techniques to the Von Mises (CN) distribution. A list of references and a summary of these and other techniques can be found in Steinmetz (1962) and Pincus (1953, 1956).

1.4 Estimation

Point estimation is a method of obtaining a scalar quantity as an estimate of a parameter. Let $f(x;\beta)$ represent a density function where β is assumed to be a vector of unknown parameters and x is a random variable. $x_1, \cdots, x_n$ is a random sample of size n from $f(x;\beta)$ if $x_1, \cdots, x_n$ are independently and indentically distributed random variables with density $f(x;\beta)$. The mathematical problem of estimation is to estimate the theoretical distribution based on a random sample from that distribution. A <u>statistic</u> is defined to be any function of the random sample and any statistic used to estimate a parameter is called an <u>estimator</u>. An observed value of an estimator is called an <u>estimate</u>. Since any statistic can be considered to be an estimator of β, some desirable properties of estimators must be determined.

(a) <u>Unbiasedness</u>. If T is an estimator of a parameter β and if $E(T) = \beta$ then T is called an unbiased estimator of β, and an observed value of T based on a fixed sample size is called an unbiased estimate of β.

(b) <u>Consistency</u>. Let T_n be an estimator of β based on a sample of size n. If for any $\epsilon > 0$ $\lim_{n \to \infty} P\left[|T_n - \beta| \geq \epsilon\right] = 0$ then T_n is called a consistent estimator of β, and an observed value of T_n is called a consistent estimate of β.

(c) <u>Efficiency</u>. Let T_1 and T_2 be two estimators of a parameter β based on a random sample of size n, then the relative efficiency of T_1 with respect to T_2 is $e = \frac{Var(T_2)}{Var(T_1)}$. If $n \to \infty$ then e is a measure of the asymptotic efficiency of T_1 with respect to T_2.

(d) <u>Minimum Variance</u>. For any unbiased estimator of the parameter β there exists a <u>minimum variance bound</u> given by

$$\sigma^2_m = \frac{1}{-n\, E\left[\frac{\partial^2}{\partial \beta^2} \log f(x;\beta)\right]} = \frac{1}{n\left[E\left(\frac{\partial}{\partial\beta}\log f(x;\beta)\right)^2\right]}.$$

Three methods of estimation are now outlined:

(1) The Method of Maximum Likelihood (MML). If $x_1, \cdots, x_n$ is an observed random sample then the likelihood function is defined as $L = f(x_1, \cdots, x_n; \beta) = \prod_{i=1}^{n} f(x_i; \beta)$, the joint probability function of the random sample. The x_i, $i = 1, \cdots, n$ are fixed, β is now regarded as the random variable and the MML involves maximizing L with respect to β. It is often convenient to maximize log L and the value of β that maximizes L (or log L) is called the ML estimate of β. Note that it is the value of β which maximizes L that is required and not the maximum value of L.

The MML does not always give unbiased estimators. Sometimes, however, upon examination of the estimators they can be made unbiased. ML estimators are consistent which implies that any bias is gradually removed as n increases, and they are asymptotically normally distributed with the theoretical minimum variance.

(2) The Method of Moments. Given $x_1, \cdots, x_n$, let the r th sample moment about the origin be $m'_r = \sum_{i=1}^{n} X_i^r / n$. β is then estimated by equating as many of the sample moments as is necessary to the corresponding population moments.

(3) The Method of Minimizing a Goodness of Fit Statistic. A good example of this method is the classical Method of Minimum X^2; however, the technique applies to any goodness of fit statistic

whose distribution is independent of the distribution being tested. The procedure involved in (3) is to first obtain the initial point estimates of the unknown parameters from the given data; secondly, to calculate the goodness of fit statistic; and thirdly, to vary these initial estimates in such a way that the calculated value of the statistic reaches a minimum. In the case of circular data an additional requirement is that the calculated value of the goodness of fit statistic in question be independent of the origin.

1.5 Some Goodness of Fit Statistics

Let $x_1, \cdots, x_n$, be a random sample drawn from the distribution $F(x)$. Assume $F_o(x)$ is any fixed distribution function, then the problem of testing the hypothesis H_o: $F(x) = F_o(x)$ is known as a <u>one sample goodness of fit problem.</u> Tests of fit are based on the <u>sample distribution function</u>

$$F_n(x) = \begin{cases} 0 & x < x_{(1)} \\ r/n & x_{(r)} \leq x < x_{(r+1)} \\ 1 & x_{(n)} \leq x \end{cases}$$

where $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ are the ordered observations.

Let $y_i = x_{(i)}$ $i = 1, \cdots, n$. Four goodness of fit statistics are now presented.

(a) Cramer, von Mises, Smirnov W^2_n. Smirnov (1936) proposed the statistic

$$W^2_n = \int_{-\infty}^{+\infty} \left[F_n(x) - F(x)\right]^2 dF(x) = \sum_{i=1}^{n} \left(F_n(y_i) - \frac{2i-1}{2n}\right)^2 + \frac{1}{12n}$$

Anderson and Darling (1952) tabulated the asymptotic distribution derived by Smirnov, and Marshall (1958) found the exact distribution of W^2_n, n=1, 2, 3. It is important to stress the fact that W^2_n does not depend on $F_o(x)$ but, on the circle, does depend upon the point at which one begins cumulating, that is, it is not independent of the origin of the polar coordinates. The Method of Minimum W^2_n can not be used, therefore, since the choice of origin is arbitrary for circular data.

Unless otherwise stated all values of W^2_n recorded in this thesis are based on the origin as the point determined by $\hat{\theta}_o$.

(b) Pearson's X^2 Pearson (1900) introduced the well known X^2 statistic

$$X^2 = \sum_{i=1}^{K} \frac{(O_i - e_i)^2}{e_i}$$

where K is the number of groups or classes into which the data is divided, O_i is the observed number of observations in the i^{th} class, and e_i is the expected number of observations in the i^{th} class (this is determined by the particular distribution considered). Under the null hypothesis that the observations come from a particular distribution $F_o(x)$, that is, H_o: $F(x) = F_o(x)$, X^2 has a chi-squared distribution with K-r-1 degrees of freedom, written $X^2 \rightarrow X^2_{K-r-1}$, where r is the number of parameters that have to be estimated. Two restrictions placed on this statistic are:

(1) K must be greater than or equal to 5.

(2) All the e_i, $i = 1, \cdots, K$ must be greater than or equal to 5.

The procedure of applying the classical X^2 goodness of fit test to determine whether a given set of data follows a particular distribution or not is outlined below.

(i) Calculate X^2

(ii) With the appropriate degrees of freedom ν look in χ^2 tables to compare X^2 with χ^2_ν. (See for example Pearson and Hartley (1966)). If X^2 is too large, reject H_0 at the required significance level.

Note that the calculated value of X^2 depends upon the position of the group boundaries.

(c) Kuiper's V_n. Kuiper (1960) proposed the statistic

$$V_n = \sup_{-\infty<x<\infty} \left[F_n(x) - F(x)\right] - \inf_{-\infty<x<\infty} \left[F_n(x) - F(x)\right]$$

$$= \max_{0\leq i\leq n} \left[\frac{i}{n} - F(y_i)\right] + \max_{0\leq i\leq n} \left[F(y_i) - \frac{(i-1)}{n}\right] \qquad \text{where } y_i = x_{(i)}$$

and showed that the distribution of V_n under H_0: $F(x)=F_0(x)$ was independent of $F(x)$, and that the calculated value of V_n was independent of the origin if the observations were points on a circle. Stephens (1965) gives the exact distribution of V_n and provides tables of significance points for both the lower and upper tails. These tables are extended in the appendix for selected values of n up to 100.

(d) Watson's U^2_n. Watson (1961, 1962) proposed the statistic

$$U^2_n = n\left(\int_{-\infty}^{+\infty} F_n(x) - F(x) - \int_{-\infty}^{+\infty}\left[F_n(y) - F(y)\right] dF(y)\right)^2 \; dF(x)$$

$$= \sum_{i=1}^{n} \left[F_n(y_i) - \frac{2i-1}{n}\right]^2 - n \left[\sum_{i=1}^{n} \frac{F_n(y_i)}{n} - \frac{1}{2}\right]^2 + \frac{1}{12n}$$

and found its asymptolic distribution. Stephens (1963, 1964) gives tables of significance points for both the upper and lower tails. The lower tail is extended in the appendix. The distribution of U^2_n, like that of V_n, is independent of the distribution being tested ($F_o(x)$), and, for circular data, the calculated value of U^2_n, again like that of V_n, does not depend on the choice of origin.

It is evident then that for circular data the Method of Minimum V_n or U^2_n can be used as criteria of good estimation -- see section 1.4 (3). It is also tobe noted that the minimum value W^2_n can take is U^2_n, that is $U^2_n = \min_{x_o} W^2_n(x_o)$ where x_o is the "origin" on the circle -- Watson (1961).

CHAPTER II

THE CIRCULAR NORMAL DISTRIBUTION

2.1 Introduction

Gumbel, Greenwood, and Durand (1953) introduced the CN distribution

$$f(\theta;K,\theta_o) = \frac{1}{2\pi I_o(K)} \exp(K \cos(\theta-\theta_o)) \qquad 0 \leq \theta < 2\pi$$

This density function has a mode at θ_o and an antimode at $\theta_o+\pi$. θ is the polar coordinate of a typical observation on the circumference of the unit circle and $K \geq 0$ is a measure of dispersion. For large $K(K > 5)$ the density is highly clustered about the mode and for small $K(K < 1)$ the density becomes more nearly uniform. At $K=0$ $f(\theta;K,\theta_o)= \frac{1}{2\pi}$, the uniform distribution. $I_o(K)$ is the imaginary Bessel function of order zero. When K is very large θ is approximately normally distributed with mean zero and variance $1/K$, written $\theta \dot{\rightarrow} N(0, \frac{1}{K})$, see Gumbel, Greenwood, and Durand (1953), Stephens (1962).

2.2 Estimation of Parameters

The techniques of analysis have been developed by Stephens (1962, 1966) and Watson and Williams (1956) and are given below. The details are not presented but are similar to those given in the discussion of distribution B2, Chapter III.

Given $\theta_1, \cdots, \theta_n$ let $C= \sum_{i=1}^{n} \cos \theta_i$, $S= \sum_{i=1}^{n} \sin \theta_i$

$X = \sum_{i=1}^{n} \cos(\theta_i - \theta_o)$. The resultant is denoted by $\overline{R}$ and has magnitude $R = \sqrt{C^2+ S^2}$.

The ML estimator of K, θ_0 known is $\frac{I_1(\hat{K})}{I_0(\hat{K})} = \frac{X}{n}$ (1)

The ML estimator of K, θ_0 unknown is $\frac{I_1(\hat{K})}{I_0(\hat{K})} = \frac{R}{n}$ (2)

The estimator of θ_0 is $\hat{\theta}_0 = \tan^{-1} \frac{S}{C}$ (3)

The procedure for describing a set of circular data by the CN distribution when θ_0 is unknown is outlined below.

<u>Procedure</u> (a) Given $\theta_1, \cdots, \theta_n$ calculate C, S and hence obtain $\hat{\theta}_0$ from (3) above.

(b) Calculate R/n and determine $\hat{K}$ from (2) -- tables for obtaining $\hat{K}$ are provided in Gumbel, Greenwood and Durand (1953), or Batschelet (1965).

2.3 Analysis of Data

The above techniques are now illustrated by seven samples drawn from published papers and the fits obtained are also illustrated by plots in the appendix. Once $\hat{\theta}_0$ and $\hat{K}$ had been obtained by the procedure set down in section 2.2 the fit was improved by using the Method of Minimum U^2_n and following the procedure outlined below. The analysis of distribution B2 in Chapter III was undertaken first and the conclusions reached there by observing the various plots comparing the Method of Minimum U^2_n and the Method of Minimum $\sqrt{n}V_n$ prompted the use of the former here. The procedure used in improving the fit once the initial estimates of θ_o and K had been obtained was:

(a) $\hat{\theta}_o$ was varied, $\hat{K}$ held fixed, to find minimum U^2_n.

(b) $\hat{K}$ was varied, holding the $\hat{\theta}_o$ obtained from (a) fixed, to find minimum U^2_n .

$\hat{\theta}_o$ was varied by single degrees and $\hat{K}$ by tenths about their respective initial values.

A typical plot consists of:

(1) The sample distribution function or step function $F_n(x)$.

(2) The estimated theoretical cumulative distribution function $F_o(x)$.

(3) The estimated theoretical density function $\frac{d}{dx} F_o(x)$

Note that for each plot the angles have been revolved by $\hat{\theta}_o$ so that the resultant or estimated directional vector pointed North ($\theta = 0$). The goodness of fit statistics $\sqrt{n}V_n$, U^2_n , and W^2_n were then calculated and Pearson's classical X^2 test was also applied to the revolved angles. Table 2.1 at the end of this chapter lists the sample sizes, the plot number, the number of cells of equal probability used for the X^2 test, and all the aforementioned values of statistics. Note that only the initial fits produced by the ML estimates are plotted. The techniques of improvement are illustrated by plots in the treatment of distribution B2 in Chapter III.

CN Sample 2.1 is from Griffin and Goldsmith (1955) and represents the initial flight directions taken by birds. Previous analysis included estimating θ_0 by the sample mean of the observations (142°). Data on page A-1.

(a) Initial Estimates. $\hat{\theta}_0$ and $\hat{K}$ obtained as in section 2.2. See table 2.1. Plot 1.

(b) Improvement. $\hat{\theta}_0$ varied, $\hat{K}$ fixed, to find minimum U^2_n. Table 2.1.

(c) Improvement. $\hat{K}$ varied, $\hat{\theta}_0$ obtained from (b) held fixed, to find minimum U^2_n. Table 2.1

Discussion. The initial estimates from (a) produce a low value of U^2_n and from plot 1 it is seen that the fit is good. The estimates obtained from (b) and (c) do not lower the value of U^2_n appreciably.

CN Sample 2.2 is from Agterberg and Briggs (1963) and represents paleocurrent directions. Previous analysis included linear normal treatment ($\hat{\theta}_0 = 14^\circ$). Data on page A-1.

(a) Initial Estimates. $\hat{\theta}_0$ and $\hat{K}$ obtained as in section 2.2 . See table 2.1. Plot 2.

(b) Improvement. $\hat{\theta}_0$ varied, $\hat{K}$ fixed, to find minimum U^2_n. See table 2.1

(c) Improvement. $\hat{K}$ varied, $\hat{\theta}_0$ obtained from (b) held fixed, to find minimum U^2_n. See table 2.1.

Discussion. The estimates from (b) and (c) lower the initial value of U^2_n obtained from (a). The change in value of U^2_n from (a) to (b) was greater than its change from (b) to (c).

CN Sample 2.3 is from Kiersch (1950) and represents directions of sloping lamination surfaces. Previous analysis followed the methods of Reiche (1938), see section 1.3, and McKee (1940) who estimated θ_o by selecting at random a portion of the total number of observations available. The direction recorded pictorially in figure 1 page 924 of Kiersch (1950) agress well with those recorded in table 2.1. Data on page A-1.

(a) Initial Estimates. $\hat{\theta}_o$ and $\hat{K}$ obtained as in section 2.2. See table 2.1. Plot 3.

(b) Improvement. $\hat{\theta}_o$ varied, $\hat{K}$ fixed, to find minimum $U^2{}_n$. See table 2.1

(c) Improvement. $\hat{K}$ varied, $\hat{\theta}_o$ obtained from (b) held fixed, to find minimum $U^2{}_n$. See table 2.1.

Discussion. The estimates of (b) and (c) have no effect on the initial value of $U^2{}_n$ obtained from (a).

CN Sample 2.4 is from Kiersch (1950). See CN Sample 2.3 for previous methods of analysis. The direction of about 200^o recorded pictorially in figure 1 page 924 Kiersch (1950) is approximately 30^o away from the $\hat{\theta}_o$ giving the best fit in table 2.1. Data on page A-1.

(a) Initial Estimates. $\hat{\theta}_o$ and $\hat{K}$ obtained as in section 2.2. See table 2.1. Plot 4

(b) Improvement. $\hat{\theta}_o$ varied, $\hat{K}$ fixed, to find minimum $U^2{}_n$. See table 2.1.

(c) Improvement. $\hat{K}$ varied, $\hat{\theta}_o$ obtained from (b) held fixed, to find minimum $U^2{}_n$. See table 2.1.

Discussion. Same as for CN Sample 2.2.

CN Sample 2.5. Source and methods of analysis as in CN Sample 2.3. The direction of about 125° recorded pictorially in figure 1 page 924 Kiersch (1950) is approximately 10° away from the $\hat{\theta}_0$ giving the best fit in table 2.1. Data on page A-1.

(a) Initial Estimates. $\hat{\theta}_0$ and $\hat{K}$ obtained as in section 2.2. See table 2.1. Plot 5.

(b) Improvement. $\hat{\theta}_0$ varied, $\hat{K}$ fixed, to find minimum U^2_n. See table 2.1

(c) Improvement. $\hat{K}$ varied, $\hat{\theta}_0$ obtained from (b) held fixed, to find minimum U^2_n. See table 2.1.

Discussion. Same as for CN Sample 2.2.

CN Sample 2.6 is from Harrison (1957-a). For previous analysis see B2 Sample 3.6, page 35. Data on page A-5.

(a) Initial Estimates. $\hat{\theta}_0$ and $\hat{K}$ obtained as in section 2.2. See table 2.1. Plot 6.

(b) Improvement. $\hat{\theta}_0$ varied, $\hat{K}$ fixed, to find minimum U^2_n. See table 2.1.

(c) Improvement. $\hat{K}$ varied, $\hat{\theta}_0$ obtained from (b) held fixed,to find minimum U^2_n . See table 2.1.

Discussion. Same as CN Sample 2.1.

CN Sample 2.7 was provided by Dr. Edwin Gould of the School of Hygiene, the Johns Hopkins University, and represents the directions taken by turtles. For another analysis see B2 Sample 3.7 on page 36

Data on page A-6.

(a) Initial Estimates. $\hat{\theta}_o$ and $\hat{K}$ obtained as in section 2.2. See table 2.1. Plot 7.

(b) Improvement. $\hat{\theta}_o$ varied, $\hat{K}$ fixed, to find minimum $U^2{}_n$. See table 2.1.

(c) Improvement. $\hat{K}$ varied, $\hat{\theta}_o$ obtained from (b) fixed, to find minimum $U^2{}_n$. See table 2.1.

Discussion. The estimates of (b) do not lower the initial value of $U^2{}_n$ obtained form (a), however the estimates of (c) lower this initial value of $U^2{}_n$ a great deal.

Conclusions. Although the above procedure for improving the fit was not checked on Monte Carlo samples from the CN distribution a similar procedure was checked on Monte Carlo samples from the more general distribution B2 in which there were three unknown parameters to be estimated. The results obtained from Monte Carlo studies in Chapter IV tables 4.4 and 4.5 indicate that the procedure used above would be satisfactory. The varying of $\hat{\theta}_o$ holding $\hat{K}$ fixed usually had the greater effect in lowing the initial value of $U^2{}_n$, but as was seen from samples 2.4 and especially 2.7, the varying of K may also be necessary to obtain an even lower value of $U^2{}_n$.

2.4 The Effect of Varying $\hat{\theta}_o$: $\hat{\theta}_o = 0$

Case 1. Assume $\hat{\theta}_o$ is varied by an amount $+\Delta\theta$.

(a) If $\theta_i \notin [\hat{\theta}_o, \hat{\theta}_o + \Delta\theta]$ for all $i = 1, \cdots, n$, then the step function is shifted to the left by an amount $\Delta\theta$.

(b) If $\theta_i \in [\hat{\theta}_o, \hat{\theta}_o + \Delta\theta]$ for j of the θ_i, $i = 1, \cdots, n$; $j = 1, \cdots, n$, then the θ_i is (are) shifted to the extreme upper portion of the step function with the end result being that the step function is shifted to the left by an amount $\Delta\theta$ and dropped down by an amount j/n.

Case 2. Assume $\hat{\theta}_o$ is varied by an amount $-\Delta\theta$.

(a) If $\theta_i \notin [\hat{\theta}_o - \Delta\theta, \hat{\theta}_o]$ for all $i = 1, \cdots, n$, then the step function is shifted to the right by an amount $\Delta\theta$.

(b) If $\theta_i \in [\hat{\theta}_o - \Delta\theta, \hat{\theta}_o]$ for j of the θ_i, $i = 1, \cdots, n$; $j = 1, \cdots, n$, then the θ_i is (are) shifted from the extreme upper portion of the step function down to the initial portion with the end result being that the step function is shifted to the right by an amount $\Delta\theta$ and raised by an amount j/n. For example in Plots 2 and 18 one could improve the fit by increasing $\hat{\theta}_o$ as in case 1(a), and in plot 27 one could improve the fit by decreasing $\hat{\theta}_o$ as in case 2(b).

2.5 The Effect of Changing $\hat{K}$: $\hat{\theta}_o = 0$

Recalling that K is a measure of dispersion one can readily observe that the effect of increasing $\hat{K}$ is to make the initial and final portions more steep and to flatten the central portion. The effect of decreasing $\hat{K}$ has the opposite effect of the above. A recommended procedure for improving the fit after plotting the distribution functions using the ML estimates is to vary $\hat{\theta}_o$ first and then, if necessary, to vary $\hat{K}$.

2.6 Discussion of the Goodness of Fit Statistics

Recalling that $U^2{}_n$ and $\sqrt{n}\, V_n$ are independent of the origin and the distribution being tested (CN distribution in this Chapter), one can use the Methods of Minimum $\sqrt{n}\, V_n$ and $U^2{}_n$ respectively to see which statistic produces the better fit. As is better exemplified in Chapter III the Method of Minimum $U^2{}_n$ produces the noticeably better fit.

TABLE 2.1 - UNIMODAL (CN) RESULTS

CN Sample	Plot	n	$\hat{\theta}_o$	K	$\sqrt{n}\, V_n$	U^2_n	W^2_n	X^2	cells
2.1(a)	1	11	142.06	6.842	.6753	.0216	.0316		
(b)	-	11	144	6.842	.7106	.0207	.0620		
(c)	-	11	144	8.0	.6407	.0181	.0595		
2.2(a)	2	11	14.64	4.154	.7998	.0294	.0476		
(b)	-	11	22	4.154	.7283	.0221	.0249		
(c)	-	11	22	3.5	.7033	.0189	.0212		
2.3(a)	3	44	199.42	1.067	1.0553	.0474	.0474	1.0	5
(b)	-	44	200	1.067	1.0553	.0472	.0479	1.0	5
(c)	-	44	200	1.1	1.0553	.0475	.0481	.773	5
2.4(a)	4	34	216.04	1.326	1.2776	.0949	.0986	8.353	5
(b)	-	34	231	1.326	1.0945	.0747	.0859	2.176	5
(c)	-	34	231	1.2	1.0572	.0692	.0796	1.294	5
2.5(a)	5	37	110.14	1.297	.8391	.0270	.0357	1.243	5
(b)	-	37	116	1.297	.7088	.0210	.0298	1.243	5
(c)	-	37	116	1.4	.7402	.0201	.0287	1.784	5
2.6(a)	6	58	305.43	.6763	.9233	.0296	.0500	1.655	5
(b)	-	58	309	.6763	.9082	.0282	.0831	2.517	5
(c)	-	58	309	.7	.8973	.0284	.0834	1.483	5
2.7(a)	7	31	341.49	1.0679	.9801	.0543	.0544	2.065	5
(b)	-	31	339	1.0679	.9686	.0540	.0575	2.065	5
(c)	-	31	339	.9	.8630	.0469	.0503	2.065	5

CHAPTER III

TWO BIMODAL DISTRIBUTIONS

3.1 Distribution B1

$$f(\psi;\lambda,\psi_o) = \frac{1}{2\pi I_o(\lambda)} \exp(\lambda \cos 2(\psi - \psi_o)) \qquad 0 \leq \psi < 2\pi$$

This distribution has modes at $\psi = \psi_o$ and $\psi = \psi_o + \pi$ and antimodes at $\psi = \psi_o \pm \pi/2$. If $\lambda = K$, $2\psi = \theta$, and $2\psi_o = \theta_o$ then distribution B1 reduces to $CN(K,\theta_o)$. Estimators of λ and ψ_o, their properties, and a complete discussion of distribution B1 are found in Stephens (1966). The procedure for estimating λ and ψ_o is the following:

(a) Given $\psi_1, \cdots, \psi_n$ calculate

$$G = \sum_{i=1}^{n} \cos 2\psi_i, \qquad H = \sum_{i=1}^{n} \sin 2\psi_i \quad \text{then } \hat{\psi}_o = \frac{1}{2} \tan^{-1} \frac{H}{G}$$

(b) Calculate R/n and determine $\hat{\lambda}$ from $\frac{I_1(\hat{\lambda})}{I_o(\hat{\lambda})} = \frac{R}{n}$ where R is

is the size of the resultant of the doubled angles, that is, $R = \sqrt{G^2 + H^2}$. The tables mentioned in section 2.2(b) can be used to obtain $\hat{\lambda}$.

3.2 Distribution B2

$$f(\theta;A,B,K,\theta_o) = \frac{A}{2\pi I_o(K)} \exp(K \cos(\theta - \theta_o)) + \frac{B}{2\pi I_o(K)} \exp(K \cos(\theta - \theta_o + \pi))$$

$0 \leq \theta < 2\pi$

This distribution is composed of two weighted CN distributions with modes at θ_0 and $\theta_0 - \pi$ respectively. B2 is a modification of a distribution proposed by Gumbel (1954). $A \geq 0$ and $B \geq 0$ are the weights; $A + B = 1$, in order for B2 to integrate up to one. Note that K, the measure of dispersion, is assumed to be the same for the two weighted CN distributions.

Possible Unimodality of B2. It is interesting to note that distribution B2 is unimodal with the mode at $\theta_0 + \pi$ for low values of K and A, or at θ_0 for low values of K and 1-A. For convenience set $\theta_0 = 0$; then distribution B2 becomes

$$f(\theta;A,K,0) = \frac{A}{2\pi I_0(k)} \exp(K \cos \theta) + \frac{(1-A)}{2\pi I_0(K)} \exp(K \cos (\theta + \pi))$$

Differentiating with respect to θ one obtains

$$f'(\theta;A,K,0) = \frac{-A}{2\pi I_0(K)} K \sin\theta \exp(K \cos\theta) - \frac{(1-A)}{2\pi I_0(K)} K \sin(\theta+\pi) \exp(K \cos(\theta+\pi))$$

Differentiating again with respect to θ, letting $\theta = 0$, and setting the result greater than zero one imposes the condition on K and A that distribution B2 reaches a minimum at $\theta = 0$ (not a maximum). That is,

$$f''(\theta;A,K,0) = -\frac{A}{2\pi I_0(K)} K \exp(K) + \frac{(1-A)}{2\pi I_0(K)} K \exp(-K) > 0$$

$$\Longrightarrow \frac{1-A}{\exp(K)} > A \exp(K) \Longrightarrow 1-A > A \exp(2K) \Longrightarrow 1 > A(1 + \exp(2K)) \Rightarrow A < \frac{1}{1+\exp(2K)} .$$

This implies the unimodality of distribution B2 with minimum (antimode) at $\theta = 0$ and maximum (mode) at $\theta = \pi$. If the respective K and A values lie close to or within the bounds of table 3.1, that is, distribution B2 appears to be or is unimodal respectively, then the unimodal CN distribution maybe used to describe the data. This procedure was carried out with B2 Samples 3.6 and 3.7 which subsequently were called CN Samples 2.6 and 2.7. For B2 Sample 3.6 compare plots 27-31 with CN plot 6. For B2 Sample 3.7 compare plots 32-33 with the CN plot 7. For various values of K the corresponding A's are determined in table 3.1

TABLE 3.1

K	.5	.6	.7	.8	.9	1.0	1.1	1.2	1.3	1.4	1.5	2.0	3.0
A	.27	.232	.198	.168	.142	.119	.0976	.083	.069	.057	.048	.018	.0025

3.3 Estimation of Parameters : Distribution B2

The two basic estimation procedures (MML and the Method of Moments) are applied to distribution B2 in an attempt to estimate θ_o, K, and A. These procedures, it is shown, lead to expressions that become difficult to solve or difficult to interpret and in an attempt to obtain initial estimates a variation of the Method of Moments is proposed.

(a) MML. Let $\theta_1, \cdots, \theta_n$ be n independent observations from distribution B2 and let $f_i = f(\theta_i; A,K,\theta_o)$; then the likelihood function is $L = \prod_{i=1}^{n} f_i$. Considering $\log L = \sum_{i=1}^{n} \log f_i$; differentiating with respect to A,K, and θ_o respectively, and letting

$a = \exp(K\cos(\theta_i - \theta_o))$ $\qquad$ $e = \cos(\theta_i - \theta_o)$

$b = \exp(K\cos(\theta_i - \theta_o + \pi))$ $\qquad$ $f = \cos(\theta_i - \theta_o + \pi)$

$c = 2\pi I_o(K)$ $\qquad$ $g = \sin(\theta_i - \theta_o)$

$d = \dfrac{I_1(K)}{I_o(K)}$ $\qquad$ $h = \sin(\theta_i - \theta_o + \pi)$

the following equations are obtained:

$$\sum_{i=1}^{n} \frac{1}{cf_i} (a - b) = 0$$

$$\sum_{i=1}^{n} \frac{1}{cf_i} (Aea + Ada + (1-A)\, fb + (1-A)\, db) = 0$$

$$\sum_{i=1}^{n} \frac{1}{cf_i} (AKga + (1-A)\, Khb) = 0.$$

Putting "hats" on A,K, and θ_o the above equations have to be solved for $\hat{A}$, $\hat{K}$, and $\hat{\theta}_o$; however, as Gumbel (1954) points out, the estimation of the parameters A, K, and θ_o in the case of different dispersions (K_1 and K_2) and different modes (θ_{o1} and θ_{o2}) in the linear normal distribution leads to an equation of the ninth degree derived by Pearson (1894). The MML therefore does not provide a practical technique for obtaining estimators.

(b) Method of Moments. As was mentioned in section 1.1, equating the r^{th} population moment to the corresponding r^{th} sample moment for r = 1,2,3 results in the statistics, $E(\theta^r) = \frac{\sum_{i=1}^{n} \theta_i^{\,r}}{n}$,

which are not invariant (for circular observations) of the point one chooses for the origin. Also when r = 1 $E(\theta) = \frac{\sum_{i=1}^{n} \theta_i}{n}$ can give a misleading estimate of the modal direction if the observations happen to fall 180° apart in unequally weighted clusters.

(c) Method of Moments applied to $\cos(\theta - \theta_o)$ and $\sin(\theta - \theta_o)$

Following technique (c) the estimating equations are:

$$\frac{I_1(\hat{K})}{I_o(\hat{K})} (2\hat{A} - 1) = \frac{\sum_{i=1}^{n} \cos(\theta_i - \hat{\theta}_o)}{n} \qquad (1)$$

$$1 - \frac{1}{\hat{K}} \frac{I_1(\hat{K})}{I_o(\hat{K})} = \frac{\sum_{i=1}^{n} \cos^2(\theta_i - \hat{\theta}_o)}{n} \qquad (2)$$

$$0 = \frac{\sum_{i=1}^{n} \sin(\theta_i - \hat{\theta}_o)}{n} \qquad (3)$$

$$\frac{1}{\hat{K}} \frac{I_1(\hat{K})}{I_o(\hat{K})} = \frac{\sum_{i=1}^{n} \sin^2(\theta_i - \hat{\theta}_o)}{n} \qquad (4)$$

Now equations (2) and (4) provide two estimators for K. Letting $g = \frac{1}{n} \sum_{i=1}^{n} \sin^2(\theta_i - \theta_o)$ and $h = \frac{1}{n} \sum_{i=1}^{n} \cos^2(\theta_i - \theta_o)$ one obtains $\lim_{K\to o} \mathrm{Var}(g) = \frac{1}{2n} = 4 \lim_{K\to o} \mathrm{Var}(h)$

For K = 10 $\quad \mathrm{Var}(g) = \frac{111.39}{2811\,n} = \frac{.0396}{n} \qquad \mathrm{Var}(h) = \frac{34.7}{2811n} = \frac{.0123}{n}$

and for K = 2 $\quad Var(g) = \frac{.754}{2.28n} = \frac{.0331}{n} \quad Var(h) = \frac{.2804}{2.28n} = \frac{.0123}{n}$

Since the order of K is the same in both g and h it is concluded that $Var(h) < Var(g)$ and so (2) is used as an estimator of K. Since this estimator of K was derived from the Method of Moments it is denoted $\hat{K}_m$. (1) provides an estimator for A which is now denoted by $\hat{A}_m$. From (3) the Method of Moments suggests using $\hat{\theta}_o$ as the angle obtained from the resultant, however, this estimator for θ_o can give misleading information as to the direction of the modes as one can easily see by observing that the resultant of a set of observations clustered about 1^o and another set of similar size clustered about 179^o points somewhere in the vicinity of 90^o. Therefore to determine an unbiased estimator for θ_o one method suggested by Krumbein (1939), having no particular circular distribution in mind, is to double all the angles, to find the resultant of these doubled angles, and then to half the angle determined by this resultant. This above estimator of θ_o is denoted by $\hat{O}_{oD}$. The proposed set of initial estimators for distribution B2 is:

(a)For θ_o - use $\hat{\theta}_{oD}$.

(b)For K - use $\hat{K}_m$ determined from equation (2)

(c)For A - use $\hat{A}_m$ determined from equation (1)

3.4 Properties of Estimators : Distribution B2

Exact properties of the estimators are difficult to find but some approximate results can be obtained based on the following approximations:

(a) For K large (1) $\frac{I_1(K)}{I_0(K)} = 1 - \frac{1}{2K}$; when K = 5 this has accuracy within 1% and when K = 3 within 3%.

For K very large (2) $\frac{I_1(K)}{I_0(K)} = 1$ with accuracy within 5.5% for K = 10

(b) For K small (1) $\frac{I_1(K)}{I_0(K)} = \frac{K}{2}$ with accuracy just over 3% for K = .5, and just over 1% for K = .3

(2) $\frac{I_1(K)}{I_0(K)} = \frac{K}{2} - \frac{K^3}{16}$ with accuracy within 1% for K = 1/2 and within 2.5% for K = 1.

The approximate results are:

For K large (K > 10) $$\frac{1}{\hat{K}} \doteqdot \frac{n - \sum_{i=1}^{n}\cos^2(\theta_i - \hat{\theta}_o)}{n}$$

(K ≥ 3) $$\hat{K} \doteqdot \frac{n + \sqrt{n(2D - n)}}{2(n - D)} \quad \text{where } D = \sum_{i=1}^{n}\cos^2(\theta_i - \hat{\theta}_o)$$

(K ≥ 3) $$\hat{A} \doteqdot \frac{1}{2}\;\frac{2\hat{K}\sum_{i=1}^{n}\cos(\theta_i - \hat{\theta}_o) + (2\hat{K} - 1)n}{n\,(2\hat{K} - 1)}$$

For K small (K ≤ .5) $$\hat{K} \doteqdot 4\sqrt{\frac{2\sum_{i=1}^{n}\cos^2(\theta_i - \hat{\theta}_o) - n}{2n}}$$

$$(K \leqslant .5) \qquad \hat{A} \doteq \frac{1}{2} \quad \frac{2\sum_{i=1}^{n}\cos(\theta_i - \hat{\theta}_o) + n\hat{K}}{n\hat{K}} \; .$$

For results on $\hat{\theta}_o$ see Chapter IV.

3.5 Analysis of Monte Carlo Samples : Distribution B1

A search through research papers furnished no examples of recorded bimodal data that appeared at first sight to have equally weighted modes 180° apart. For $\psi_o = 0$ and $\lambda = 1,2,3$, samples of size 50, B1 samples 3.1, 3.2 and 3.3 respectively were drawn from distribution B1. The data is recorded in the appendix on page A-2 . The techniques of estimating λ and ψ_o outlined in section 3.1 were applied to each set of data. $\hat{\lambda}$, $\hat{\psi}_o$ and the relevant goodness of fit statistics are recorded in table 3.2 with plots 8, 9 and 10 respectively describing the fit obtained from the ML estimates. $\hat{\psi}_o$ was taken as the origin. In Chapter IV section 4.7 samples 3.1 and 3.2 are treated as though they had come from distribution B2.

The two aspects of this thesis are to describe the techniques with reference to practical data, and to verify the accuracy of these techniques by Monte Carlo methods. The former is now applied in the next section, and the latter in Chapter IV section 4.6.

3.6 Analysis of Data for Distribution B2

The data analysed in this section was gathered from published papers.

Initial Procedure. The initial estimates $\hat{\theta}_{oD}$, $\hat{K}_m$, and $\hat{A}_m$ were calculated as in section 3.3(c), and the goodness of fit statistics were also calculated and are recorded in table 3.3. The sample distribution function and the estimated theoretical density and distribution functions were plotted and appear in the appendix. Note that the origin in this procedure is taken to be the North Pole after all angles had been revolved by $\hat{\theta}_{oD}$ so that the estimated directional vector pointed North.

Improved Procedure. The Methods of Minimum $U^2{}_n$ and Minimum $\sqrt{n}\ V_n$ were then applied to improve the fit with the general procedure being:

(a) To vary $\hat{\theta}_{oD}$ about its initial value, holding $\hat{A}_m$ and $\hat{K}_m$ fixed.

(b) To vary $\hat{\theta}_{oD}$ and $\hat{A}_m$, holding $\hat{K}_m$ fixed.

(c) To vary $\hat{\theta}_{oD}$, $\hat{A}_m$, and $\hat{K}_m$ about their initial values.

The values of the goodness of fit statistics were recorded and the minimum values noted: Unless otherwise stated or recorded, $\hat{\theta}_{oD}$ was varied by single degrees, $\hat{A}_m$ by hundredths, and $\hat{K}_m$ by tenths about their initial values. The origin used for this analysis was the North Pole no matter where $\hat{\theta}_{oD}$ pointed, that is, the angles were not revolved. Once a certain set of estimates was decided upon, the fit was illustrated by a plot of the respective cumulative distribution functions after all the angles were revolved by $\hat{\theta}_{oD}$ so that the estimated directional vector pointed North. The goodness of fit statistics were then calculated using this above North Pole as the origin and it is these values that are recorded in table 3.3. The samples and procedure of analysis are now described. All plots are found in the appendix.

B2 Sample 3.1. This sample was provided by Dr. Edwin Gould of the School of Hygiene, The Johns Hopkins University, and represent the directions taken by turtles. Previous analysis by Cutchis (1965) consisted of CN treatment ($\hat{\theta}_o = 20.93^\circ$, $\frac{\hat{R}}{n} = .495$) and tests for randomness. For the doubled angles $\hat{\theta}_{oD}$ was also calculated (23.93^o) Data on page A-3.

(a) Initial Estimates. $\hat{\theta}_{oD}$, $\hat{A}_m$, and $\hat{K}_m$ obtained as in section 3.3. Plot 11

(b) Improvement. $\hat{\theta}_{oD}$ varied, $\hat{A}_m$ and $\hat{K}_m$ fixed, to find minimum $\sqrt{n}\,V_n$. Plot 12.

(c) Improvement. $\hat{\theta}_{oD}$, $\hat{A}_m$, and $\hat{K}_m$ are all varied to find minimum $\sqrt{n}\,V_n$. Plot 13.

Discussion. The estimates of (a) produced a value of $U^2{}_n$ that was very close to the minimum $U^2{}_n$ obtained (.0202) by varying the three initial estimates. The fit obtained from (b) appears as good as that from (c).

B2 Sample 3.2. Same source as B2 Sample 3.1. The data was previously analysed by Cutchis (1965) who applied CN treatment ($\hat{\theta}_o = 64.1^\circ$) and who also calculated $\hat{\theta}_{oD} = 62.573^\circ$ and by Stephens (1966) who applied distribution B1 techniques and X^2 tests. Both suggest that a weighted bimodal distribution be used to describe the data. Data on page A-3.

(a) Initial Estimates. $\hat{\theta}_{oD}$, $\hat{A}_m$, and $\hat{K}_m$ obtained as in section 3.3. Plot 14.

(b) Improvement. $\hat{\theta}_{oD}$ is varied, $\hat{K}_m$ and $\hat{A}_m$ fixed, to find minimum $U^2{}_n$. Plot 15.

(c) Comparison. $\hat{\theta}_{oD}$, $\hat{K}_m$, $\hat{A}_m$ are all varied to find minimum $\sqrt{n}\, V_n$. Plot 16.

(d) Comparison. $\hat{\theta}_{oD}$ is varied, $\hat{K}_m$ and $\hat{A}_m$ fixed, to seek minimum $\sqrt{n}\, V_n$. Plot 17.

Discussion. The initial estimates (a) produce a very good fit. Those of (b) produce a noticeably better fit close to the minimum value obtained by $U^2{}_n$ (.0182) from varying all three initial estimates. The fit obtained from (c) appears as good as that from (d).

B2 Sample 3.3, was read from polar diagram 30 of Harrison (1957a) and represents pebble orientations. Previous analysis includes the polar diagram and a test for uniformity. The direction of $\hat{\theta}_o$ given by Harrison (1957a) in figure 5 page 283 appears to agree with that tabulated in table 3.3 -- sample 3.3(a). Data on page A-4.

(a) Initial Estimates. $\hat{\theta}_{oD}$, $\hat{A}_m$, and $\hat{K}_m$ obtained as in section 3.3. Plot 18.

(b) Improvement. $\hat{\theta}_{oD}$ and $\hat{A}_m$ varied, $\hat{K}_m$ fixed, to find minimum $U^2{}_n$ and, as it turned out, minimum $\sqrt{n}\, V_n$. Plot 19.

(c) Comparison. $\hat{\theta}_{oD}$, $\hat{A}_m$, $\hat{K}_m$ all varied to find minimum $U^2{}_n$ and, as it turned out, minimum $\sqrt{n}\, V_n$. Plot 20.

Discussion. The estimates of (b) produce a noticeably better fit than those of (a). (b) and (c) appear to be equally good.

B2 Sample 3.4. was read from a polar diagram of Krumbein (1942) and represents pebble orientations. Previous analysis included linear normal treatment applied to the grouped data. The technique of doubling the angles etc. to estimate the orientation direction is applied to the grouped data only. The pictorially recorded direction of 90° in diagram 0-15 page 1387 in Krumbein (1942) agrees well with the results obtained for $\hat{\theta}_o$ in table 3.3. Data on page A-4.

(a) Initial Estimates. $\hat{\theta}_{oD}$, $\hat{A}_m$, and $\hat{K}_m$ obtained as in section 3.3. Plot 21.

(b) Comparison. $\hat{\theta}_{oD}$ was varied, $\hat{K}_m$ and $\hat{A}_m$ fixed, to find minimum $\sqrt{n}\, V_n$. Plot 22.

Discussion. The estimates of (a) produced a $U^2{}_n$ value very close to the minimum $U^2{}_n$ value attained (.0197) by varying all three estimates. The estimates of (b) produced the minimum $\sqrt{n}\, V_n$; also, (a) and (b) appear to produce equally good fits.

B2 Sample 3.5 was read from a polar diagram of Krumbein (1940). Previous analysis as in B2 Sample 3.4 with $\hat{\theta}_o = 53°$. Data on page A-5.

(a) Initial Estimates. $\hat{\theta}_{oD}$, $\hat{A}_m$, and $\hat{K}_m$ obtained as in section 3.3. Plot 23.

(b) Improvement. $\hat{\theta}_{oD}$ varied, $\hat{K}_m$ and $\hat{A}_m$ fixed, to seek minimum $U^2{}_n$. Plot 24.

(c) Comparison. $\hat{\theta}_{oD}$ varied, $\hat{K}_m$ and $\hat{A}_m$ fixed, to seek mimimum $\sqrt{n}\, V_n$. Plot 25.

(d) Comparison. $\hat{\theta}_{oD}$ varied by eye after observing plot 24 and following the technique of section 2.4 — case 1(a). Plot 26.

Discussion. The estimates of (a) appear to give a better fit than those of (b) even though the latter has a lower $U^2{}_n$ value. (b) and (c) estimates produce equally good fits. The minimum $U^2{}_n$ found by varying all three estimates was .0224 and the minimum $\sqrt{n}\, V_n$ was .6605.

The next two samples, B2 Samples 3.6 and 3.7, were treated at first as though they had come from distribution B2. Upon examination of the unimodal appearance of their respective density functions however, a CN fit was attempted. The CN fit to B2 Sample 3.6 appeared to be as good as the B2 fit, and the CN fit to B2 Sample 3.7 was much better than the B2 fit.

B2 Sample 3.6 was read from a polar diagram of Harrison (1957a). He records pictorially in figure 5 page 283 in Harrison (1957a) that the data have a mode at approximately 280°, ($U^2{}_n > .060$). When distribution B2 techniques were applied the best fit was obtained when $\hat{\theta}_o = 310°$ ($U^2{}_n = .0283$), and when CN techniques were applied the best fit was obtained when $\hat{\theta}_o = 309°$ ($U^2{}_n = .0282$) -- see tables 3.3 and 2.1 (Sample 2.6) respectively. Data on page A-5.

(a) Initial Estimates. $\hat{\theta}_{oD}$, $\hat{A}_m$, and $\hat{K}_m$ obtained as in section 3.3. Plot 27.

(b) Improvement. $\hat{\theta}_{oD}$ varied, $\hat{K}_m$ and $\hat{A}_m$ fixed, to find minimum U^2_n. Plot 28.

(c) Comparison. $\hat{\theta}_{oD}$ varied, $\hat{K}_m$ and $\hat{A}_m$ fixed, to find minimum $\sqrt{n}\,V_n$. Plot 29.

(d) Comparison. $\hat{\theta}_{oD}$, $\hat{K}_m$, and $\hat{A}_m$ varied to find minimum U^2_n. Plot 30.

(e) Comparison. $\hat{\theta}_{oD}$, $\hat{K}_m$, and $\hat{A}_m$ varied to seek minimum $\sqrt{n}\,V_n$. Plot 31.

Discussion. (a) estimates produce a noticeably poor fit whereas the estimates from (d) and (b) yield good fits, much better than those obtained from (e) and (c). A CN distribution was then fitted to the data after observing the unimodal appearance of the density function in plots 27 and 28, and an equally good fit was obtained with the initial estimates of K and θ_o -- see plot 6 and table 2.1 (CN Sample 2.6)

B2 Sample 3.7 Same source as B2 Sample 3.1. Previous analysis by Cutchis (1965) was unavailable. Data on page A-6.

(a) Initial Estimates. $\hat{\theta}_{oD}$, $\hat{A}_m$, and $\hat{K}_m$ obtained as in section 3.3. Plot 32.

(b) Improvement. $\hat{\theta}_{oD}$ varied in 10° intervals, $\hat{K}_m$ and $\hat{A}_m$ fixed, to find minimum U^2_n and, as it turned out, minimum $\sqrt{n}\,V_n$. Plot 33.

Discussion. (a) estimates produce a very poor fit whereas those of (b) produce a non significant U^2_n and $\sqrt{n}\,V_n$. A CN distribution was fitted to the data after observing the unimodal appearance of the estimated density function in plots 32 and 33 and a much better fit was obtained: see plot 7, table 2.1 (sample 2.7),

Conclusions. The Method of Minimum $U^2{}_n$ produces a much tighter fit than the Method of Minimum $\sqrt{n}\, V_n$ but one should always plot the theoretical and empirical distributions to confirm that a low $U^2{}_n$ value has not resulted from a shift in location. Very low values of $U^2{}_n$ and $\sqrt{n}\, V_n$ may be obtained by varying only the estimate of θ_o obtained from section 3.3(holding $\hat{A}_m$ and $\hat{K}_m$ fixed). $\hat{\theta}_{oD}$ is varied by observing the plots drawn produced by the initial estimates and following the procedure outlined in section 2.4.

It is to be emphasized that the initial estimates usually provide quite a good fit. The above procedure of varying only $\hat{\theta}_{oD}$ to obtain a good fit is tested on some Monte Carlo samples in Chapter IV section 4.6.

3.7 The Effect of Changing $\hat{A}$: $\hat{\theta}_o = 0$

A is a measure of the weight attached to each CN distribution in B2. The varying of $\hat{A}$ affects the cumulative distribution function in much the same way as the changing of $\hat{K}$ does in section 2.5. $\hat{A}$ and $\hat{K}$ should not be varied until a good estimate of θ_o is obtained, and then, if necessary, $\hat{A}$ could be varied to obtain an even better fit. Following the above procedure of improving the fit $\hat{K}$ appears to have a fairly wide range that will still produce excellent fits. One could change both $\hat{K}$ and $\hat{A}$ so as to complement each other (increasing or decreasing both) or to counteract each other (increasing one and decreasing the other) but such a procedure would require lengthy and multiple calculations with the end result in most cases being a fit which is not really too much better than that obtained by varying $\hat{\theta}_0$ alone.

3.8 Another Estimator for $K(\hat{K}_s)$: K large - distribution B2

Stephens (unpublished) has suggested the following method for obtaining a rough estimate of K from independently distributed angles $\theta_1, \cdots, \theta_n$.

(a) Double all the angles and obtain the size R of the resultant of these doubled angles, and hence obtain $\hat{K}_D$ from R/n.

(b) Find V_2 from $\hat{K}_D$ using table A-1 in the appendix. Calculate $V_1 = V_2/4$.

(c) Interpolate with $V_1 = V_2/4$ in table A-1 to get $\hat{K}_s$.

Theory. If K is large then $CN(0,K) \doteq N(0,\frac{1}{K})$ -- Gumbel, Greenwood, and Durand (1953). From Stephens (1963) -- let a point move in successive independent steps on the circumference of a unit circle, and let $f_t(\theta)$ be the density function of its polar co-ordinate after t steps. At each step assume θ increases by an amount α having the distribution function $p(\alpha)$; $-\pi < \alpha \leq \pi$. Now if

(a) The density function of α becomes concentrated at zero for each step

and (b) The number of steps increases so that the final density function $f_V(\theta)$ has finite variance

then the density function of a point which starts at the North Pole and moves with Brownian diffusion is given by

$f_V(\theta) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{m=1}^{\infty} \cos m\theta \exp(-\frac{1}{2} m^2 V)$ where V is a measure of dispersion.

If the point starts at $\theta = 0$ and has the CN(0,K) then.

$f_1(\theta) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{I_m(K)}{I_o(K)} \cos m\theta$. Putting $m = 1$ in these two Fourier expansions, assuming K is large, one gets $e^{-\frac{1}{2}V} = \frac{I_1(K)}{I_o(K)}$ and therefore $1 - \frac{1}{2}V \doteq 1 - \frac{1}{2K} \Rightarrow V \doteq \frac{1}{K}$. So $CN(0,K_1) \doteq B(V_1)$ where B represents a Brownian distribution with $V_1 = \frac{1}{K_1}$, as a parameter of dispersion. Since $CN(0,K_1) \doteq N(0,\frac{1}{K_1})$, doubling the observations in a normal distribution produces another normal distribution with the same mean and four times the original variance, and therefore $CN(0,K_2) \doteq N(0,\frac{1}{K_2}) \doteq B(V_2)$ where $V_2 = 4V_1$, so $e^{-\frac{1}{2}V_2} = e^{-2V_1}$

$= \frac{I_1(K_2)}{I_o(K_2)} \Rightarrow 1 - 2V_1 = 1 - \frac{1}{2K_2} \Rightarrow K_1 = 4K_2$

3.9 Another Estimator for A ($\hat{A}_p$)

To obtain a quick estimate of A find $\hat{\theta}_o$ as in section 3.3 and then set

$$\hat{A}_p = \frac{\text{no. of } \theta_i \text{ in } \left[\hat{\theta}_o - \frac{\pi}{2}, \hat{\theta}_o + \frac{\pi}{2}\right]}{n} \qquad i = 1, \cdots, n .$$

One would expect $\hat{A}_p$ to be a reliable estimator for large K. For a further discussion of $\hat{A}_p$ see Chapter IV.

TABLE 3.2

DISTRIBUTION B1 SAMPLES

B1 SAMPLE	Plot	n	ψ_o	λ	$\hat{\psi}_o$	$\hat{\lambda}$	$\sqrt{n}\, V_n$	U^2_n	W^2_n	X^2	cells
3.1	8	50	0	1	7.63	.9267	.6263	.0182	.0473	1.0	5
3.2	9	50	0	2	177.22	2.1549	1.0778	.0546	.0615	4.2	5
3.3	10	50	0	3	1.29	3.0273	.8249	.0244	.0333	1.0	5

TABLE 3.3

DISTRIBUTION B2 SAMPLES

B2 SAMPLE	Plot	n	$\hat{\theta}_o$	$\hat{K}$	$\hat{A}$	$\sqrt{n}\,V_n$	U^2_n	W^2_n	X^2	cells
3.1(a)	11	41	23.94	4.1032	.785	.7238	.0205	.0856	2.05	5
(b)	12	41	27.5	4.1032	.785	.6574	.0269	.0776	1.07	5
(c)	13	41	27.1	3.7	.805	.6352	.0270	.0847	1.81	5
3.2(a)	14	76	62.57	3.1669	.8031	.7842	.0196	.0373	4.79	10
(b)	15	76	64.0	3.1669	.8031	.8137	.0184	.0261	4.79	10
(c)	16	76	61.1	3.0	.795	.6953	.0247	.0289	5.05	10
(d)	17	76	61.5	3.1669	.8031	.7604	.0223	.0289	3.21	10
3.3(a)	18	101	137.54	3.0843	.5609	1.5374	.1434	.2745	14.94	10
(b)	19	101	142.0	3.0843	.5609	1.4777	.1366	.1391	13.75	10
(c)	20	101	142.0	3.0843	.57	1.4640	.1362	.1382	15.14	10
3.4(a)	21	100	85.73	1.7832	.288	.6644	.0199	.0261	4.0	10
(b)	22	100	84.0	1.7832	.288	.6413	.0203	.0206	2.6	10
3.5(a)	23	100	57.34	1.6409	.3055	.8409	.0362	.1141	2.2	10
(b)	24	100	65.5	1.6409	.3055	.6812	.0233	.1141	6.6	10
(c)	25	100	64.0	1.6409	.3055	.6685	.0238	.1478	5.4	10
(d)	26	100	73.5	1.6409	.3055	.9515	.0356	.0365		10
3.6(a)	27	58	151.84	.9492	.1651	1.2205	.0777	.2476	4.24	5
(b)	28	58	130.0	.9492	.1651	.9049	.0300	.0450	.621	5
(c)	29	58	136.0	.9492	.1651	.8777	.0336	.1091	1.483	5
(d)	30	58	130.0	1.0	.15	.8761	.0283	.0435	.621	5
(e)	31	58	136.0	1.1	.17	.8278	.0323	.1091	1.483	5
3.7(a)	32	31	107.92	.8457	.1413	1.823	.2315	.8018	9.16	5
(b)	33	31	160.0	.8457	.1413	1.2854	.0839	.0992	5.94	5

CHAPTER IV

MONTE CARLO RESULTS : DISTRIBUTION B2

4.1 Introduction

As was mentioned in Chapter III not only are properties of the estimators difficult to find, but also standard estimation techniques may sometimes lead to non-invariant estimators as was exemplified when the Method of Moments suggested using $\hat{\theta}_{oR}$, the resultant direction, as an estimator of θ_o. $\hat{\theta}_{oD}$, was then proposed as another estimator of θ_o. In Chapter III, also, two estimators of K ($\hat{K}_m$ and $\hat{K}_s$) and two estimators of A ($\hat{A}_m$ and $\hat{A}_p$) were introduced. Still another estimator of K denoted $\hat{K}_n$ is introduced in section 4.3. In order to verify the efficiency and to examine the accuracy of these three sets of estimators Monte Carlo studies were undertaken. Sections 4.2 to 4.5 deal with studies concerning $\hat{\theta}_o$, $\hat{K}$, and $\hat{A}$. It should be noted that in theory the same Monte Carlo samples could have been used thoughout this chapter, however, section 4.3 was undertaken long before the others and subsequently new samples were drawn for sections 4.2 and 4.4. The same samples are used in sections 4.4 and 4.5. The general procedure of drawing and examining Monte Carlo Samples from distribution B2 is now outlined. For a fixed K, A, and θ_o a set of 15 samples of size 25 drawn from distribution B2. The appropriate statistics ($\hat{\theta}_{oR}$, $\hat{\theta}_{oD}$, $\hat{K}_s$, $\hat{K}_m$, $\hat{K}_n$, $\hat{A}_p$, $\hat{A}_m$) were calculated for each of the 15 samples within the set and then the sample mean and the standard error were calculated for each set of 15. Again, the whole of the above procedure was repeated with samples of size 50. The results are recorded in tables 4.1 to 4.3.

Up to this point in the thesis a procedure for describing bimodal data with modes of unequal strengths lying 180° apart has been presented. The procedure has been as follows:

(a) Initial estimates obtained from $\hat{\theta}_{oD}$, $\hat{K}_m$, and $\hat{A}_m$ in section 3.3.

(b) Fit improved by varying $\hat{\theta}_{oD}$, holding $\hat{K}_m$ and $\hat{A}_m$ fixed, to find minimum $U^2{}_n$.

So far the data analysed by this procedure has been drawn from published papers. The accuracy of the above procedure is now verified by testing it on Monte Carlo samples drawn from distribution B2 and on those previously drawn from distribution B1 — see sections 4.6 and 4.7 respectively.

4.2 Results concerning $\hat{\theta}_o$

For a fixed K, A, and $\theta_o = 180^{\circ}$ a set of 15 samples of size 25 was drawn from distribution B2. $\hat{\theta}_{oR}$, the angle determined by the resultant of each sample, was calculated and the sample mean and standard error were also calculated for each set of 15. A was then varied, for fixed K = 1,2,3,4,5, from .5(.1).9 and the above procedure was again applied with the results being recorded in table 4.1(a). The above procedure was repeated for $\hat{\theta}_{oD}$, the angle determined by halving the angle obtained from the resultant of the doubled angles of each sample.

Again the whole of the above procedure was repeated with samples of size 50, and these results are recorded in table 4.1(b). From tables 4.1(a) and (b) it is evident that $\hat{\theta}_{oD}$ is an unbiased estimator of θ_o with much smaller standard deviation than $\hat{\theta}_{oR}$ for all combinations of the A and K values above, except for very high values of A. It is likely in practice that A will not be extreme, but if it is then the CN distribution may be used to describe the data as was the case with B2 Samples 3.6 and 3.7 which were subsequently treated as unimodal data and analysed again as CN Samples 2.6 and 2.7. As is to be expected for values of A close to .5 and $K \leq 1$ $\hat{\theta}_{oD}$ and $\hat{\theta}_{oR}$ are subject to great variation. Also the standard errors are consistenly less for the larger sample size.

4.3 Nomogram for K

For fixed K, A, and $\theta_o = 0$ 15 samples of size 30 and 10 samples of size 50 were drawn from distribution B2 where K varied from 1(1)9 and A from .1(.1).5. The angles obtained in each sample were doubled and the corresponding $\hat{K}_D$ value was found by treating the doubled angles as though they were from a CN distribution. For each individual K value the mean of each set of $\hat{K}_D$'s generated was calculated and subsequently 10 points were plotted at each of the above values of K. A cubic was fitted by orthogonal polynomials to the 90 points thus obtained. The coefficients of the linear and quadratic terms were significant whereas that of the cubic term was not. In order to improve the fit for small values of K two more sets of 10 means were plotted against K = .5 and K = 1.5 along with the other 9 sets of 10 means corresponding to K = 1(1)9 respectively. A straight line and a quadratic, both restricted to pass through the origin, were then fitted to the now 110 points by least squares and the latter was plotted ($\hat{K}_D$ versus K) in figure 4.1. The equation of the straight line through the origin is $\hat{K}_D = .3430\ K$, and that of the quadratic is $\hat{K}_D = .3909\ K - .0068\ K^2$. The procedure for using the nomogram (figure 4.1) is now outlined:

(a) Given $\theta_1, \cdots, \theta_n$: double all these angles and compute R/n where R is the size of the resultant of the doubled angles.

(b) $\hat{K}_D$ is obtained from tables by interpolation using R/n above. See Gumbel, Greenwood, and Durand (1953) or Batschelet (1965) for the tables.

(c) With this value of $\hat{K}_D$ one enters the left hand side of figure 4.1 and reads off the K value from the curve. This K value is now denoted by $\hat{K}_n$ -- the value of K read from the nomogram.

4.4 Results concerning $\hat{K}$

Samples were drawn as in section 4.1 with $\theta_o = 0$. $\hat{K}_m$, determined by the Method of Moments in section 3.3; $\hat{K}_s$, determined as in section 3.8; and $\hat{K}_n$ determined as in section 4.3, were calculated for each sample and the sample mean and the standard error were computed for each set as in section 4.2 and were recorded in tables 4.2(a) and (b). For the smaller sample size of $N = 25$ $\hat{K}_n$, the value of K read from the nomogram, is the best of the three estimators of K. $\hat{K}_s$ tends to overestimate K more than $\hat{K}_m$ does, and $\hat{K}_m$ also has the smallest standard error of the three estimators for $K \leq 3$ and all values of A. For the larger sample size of $N = 50$ $\hat{K}_n$ again appears to be the best of the three estimators of K with $\hat{K}_m$ almost as good. $\hat{K}_s$ again tends to overestimate K but does have the smallest standard error of the three estimators for $K \leq 3$ and all values of A. Note the relatively large standard errors of $\hat{K}_m$, $\hat{K}_n$ and $\hat{K}_s$ suggest a large variation in the estimated values of K in each set of 15 samples. One concludes then that $\hat{K}_n$ and $\hat{K}_m$ are fairly reliable estimators of K with $\hat{K}_n$ being a little better than $\hat{K}_m$.

4.5 Results concerning $\hat{A}$

Samples were drawn from distribution B2 as outlined in section 4.1. $\hat{A}_m$, the estimate of A obtained from the Method of Moments in section 3.3 and $\hat{A}_p$, the estimate of A described in section 3.9 were calculated for each sample and the sample mean and standard error were calculated for each set of 15 and recorded in tables 4.3(a) and (b). For low values of K($K \leq 3$) and for all values of A (especially high values of A) $\hat{A}_m$ provides the better estimate of A. For larger values of K ($K > 3$) and all values of A, $\hat{A}_p$ and $\hat{A}_m$ appear to be equally good as estimators of A, with the standard error of $\hat{A}_p$ slightly lower than that of $\hat{A}_m$. $\hat{A}_p$, therefore, is a fairly reliable estimator of A that can be calculated easily after $\hat{\theta}_{oD}$ has been obtained. Tables 4.5(a) and (b) also verify that the standard errors are consistently less for the larger sample size.

4.6 Results from the Analysis of Distribution B2 Samples

For all combinations of the following values of parameters -- $\theta_o = 0$, K = 2,3,4,5 and A = .9,.8,.7,.6,.5 -- a sample of size 50 was drawn from distribution B2. These 20 samples were then analysed using the following procedure:

(a) The initial estimates $\hat{\theta}_{oD}$, $\hat{K}_m$, and $\hat{A}_m$ were calculated as in section 3.3.

(b) $\hat{\theta}_{oD}$ was varied, holding $\hat{K}_m$ and $\hat{A}_m$ fixed, to find minimum $U^2{}_n$.

(c) $\hat{\theta}_{oD}$, $\hat{K}_m$, and $\hat{A}_m$ were all varied to find minimum $U^2{}_n$.

For each of (a), (b), and (c) the relevant goodness of fit statistics were calculated and all the results are recorded in table 4.4. Note that the origin for all the above calculations was taken to be the North Pole (the observations were not revolved in any way).

Table 4.4 shows that the initial estimates $\hat{A}_m$ and $\hat{\theta}_{oD}$ estimate A and θ_o quite well in almost all the 20 samples generated. The initial estimates of K, however, are sometimes quite a distance from the true value of K, see samples 1, 2, 10, 13, and 17. The parameters θ_o and

K are analogous to the mean (μ) and variance (σ^2) respectively in the linear normal distribution, and the confidence intervals for μ are far smaller than those for σ^2. This fact helps to interpret the sometimes distant values of $\hat{K}$ from K in table 4.4. Table 4.4 also shows that the estimates obtained from (b) and (c) by improving the fit by the Method of Minimum U^2_n are usually quite close to the initial estimates obtained from (a).

4.7 B2 Estimation Techniques applied to B1 Samples

B1 Samples 1 and 2, section 3.5, are now treated as though they had come from distribution B2 and are analysed using the following procedure:

(a) Initial Estimates. $\hat{\theta}_{oD}$, $\hat{K}_m$, and $\hat{A}_m$ were obtained as in section 3.3.

(b) Improvement. $\hat{\theta}_{oD}$ is varied, $\hat{K}_m$ and $\hat{A}_m$ held fixed, to find minimum U^2_n.

(c) Improvement. $\hat{\theta}_{oD}$, $\hat{K}_m$, and $\hat{A}_m$ are all varied to find minimum U^2_n.

Only the results from (a) are plotted, plots 34 and 35, and all statistics are summarised in table 4.5. For (a) the origin is taken to be $\hat{\theta}_{oD}$ but for (b) and (c) the origin was the North Pole.

From table 4.5 it is seen that as expected $\hat{A}$ is very close to .5 for both the samples. Also K in the B2 distribution is seen to be about 3 times the λ in the B1 distribution.

TABLE 4.1(a)

B2 MONTE CARLO RESULTS FOR $\hat{\theta}_o$: SAMPLE SIZE = 25

			$\hat{\theta}_{oR}$		$\hat{\theta}_{oD}$	
No. of Samples	K	A	Mean	Standard Error	Mean	Standard Error
15	1	.9	178.86	23.11	183.06	50.82
15	1	.8	192.15	23.79	178.87	36.39
15	1	.7	181.64	39.04	177.05	36.36
15	1	.6	208.68	68.34	180.97	32.31
15	1	.5	169.20	112.96	177.35	27.34
15	2	.9	175.83	11.89	170.24	15.14
15	2	.8	182.94	19.84	178.78	14.84
15	2	.7	189.39	42.92	181.01	15.69
15	2	.6	183.35	35.76	173.69	12.66
15	2	.5	183.18	117.77	179.41	19.34
15	3	.9	178.78	7.87	181.51	6.70
15	3	.8	183.89	10.76	180.37	9.03
15	3	.7	180.09	22.47	182.89	7.65
15	3	.6	155.82	63.34	180.19	9.39
15	3	.5	194.52	105.13	178.58	10.02
15	4	.9	180.97	6.94	180.39	6.81
15	4	.8	177.97	11.57	181.35	7.09
15	4	.7	183.53	16.53	179.10	5.44
15	4	.6	181.80	43.13	179.31	4.18
15	4	.5	121.98	125.73	179.72	4.62
15	5	.9	178.28	7.85	179.14	5.91
15	5	.8	181.90	7.16	180.85	4.85
15	5	.7	173.27	15.45	179.92	5.11
15	5	.6	180.51	58.28	177.52	6.14
15	5	.5	188.34	126.31	180.32	3.24

TABLE 4.1(b)

B2 MONTE CARLO RESULTS FOR $\hat{\theta}_o$: SAMPLE SIZE = 50

No. of Sample	K	A	$\hat{\theta}_{oR}$ Mean	$\hat{\theta}_{oR}$ Standard Error	$\hat{\theta}_{oD}$ Mean	$\hat{\theta}_{oD}$ Standard Error
15	1	.9	179.16	14.22	172.11	20.12
15	1	.8	178.64	-16.20	184.68	20.13
15	1	.7	181.92	37.61	180.68	35.72
15	1	.6	164.82	64.68	174.32	30.92
15	1	.5	211.48	101.22	174.79	34.52
15	2	.9	181.04	10.99	179.03	9.97
15	2	.8	180.67	13.07	178.90	8.77
15	2	.7	176.88	30.35	183.17	7.13
15	2	.6	175.85	29.36	177.03	10.28
15	2	.5	211.70	89.37	180.62	15.55
15	3	.9	177.84	7.54	178.99	3.89
15	3	.8	180.74	7.69	178.99	6.23
15	3	.7	176.62	12.29	179.62	6.27
15	3	.6	184.45	58.45	178.64	5.29
15	3	.5	200.33	115.02	176.87	3.83
15	4	.9	180.95	4.64	178.98	5.37
15	4	.8	182.25	7.72	179.30	4.12
15	4	.7	180.44	13.32	181.32	3.60
15	4	.6	179.13	20.69	178.60	5.74
15	4	.5	192.07	123.19	179.36	7.68
15	5	.9	180.50	3.54	179.30	2.09
15	5	.8	182.00	10.75	178.82	4.66
15	5	.7	179.00	16.34	179.42	3.97
15	5	.6	175.34	19.04	179.21	3.77
15	5	.5	249.10	96.85	180.28	4.72

TABLE 4.2(a)

B2 MONTE CARLO RESULTS FOR $\hat{K}$: SAMPLE SIZE = 25

No. of Sample	K	A	$\hat{K}_s$ Mean	$\hat{K}_s$ Standard Error	$\hat{K}_m$ Mean	$\hat{K}_m$ Standard Error	$\hat{K}_n$ Mean	$\hat{K}_n$ Standard Error
15	1	.9	1.788	.437	1.427	.486	1.028	.543
15	1	.8	1.598	.575	1.266	.529	.868	.559
15	1	.7	1.995	.501	1.659	.554	1.288	.614
15	1	.6	1.818	.640	1.472	.704	1.099	.771
15	1	.5	1.682	.515	1.303	.582	.908	.631
15	2	.9	2.431	.773	2.140	.859	1.818	.952
15	2	.8	2.654	.849	2.388	.933	2.091	1.012
15	2	.7	2.612	.711	2.344	.791	2.047	.882
15	2	.6	2.512	.614	2.234	.686	1.929	.770
15	2	.5	2.301	.800	2.061	.745	1.759	.809
15	3	.9	3.574	.981	3.393	1.052	3.160	1.067
15	3	.8	3.230	.907	3.019	.982	2.770	1.022
15	3	.7	3.400	1.335	3.195	1.399	2.934	1.348
15	3	.6	3.412	1.150	3.213	1.214	2.962	1.187
15	3	.5	3.460	1.047	3.267	1.125	3.021	1.147
15	4	.9	4.571	1.191	4.450	1.253	4.196	1.208
15	4	.8	4.512	1.127	4.389	1.180	4.144	1.121
15	4	.7	4.628	1.048	4.513	1.096	4.267	1.039
15	4	.6	4.166	1.301	4.022	1.356	3.782	1.274
15	4	.5	4.188	.894	4.054	.937	3.837	.889
15	5	.9	5.681	1.322	5.602	1.361	5.261	1.225
15	5	.8	5.929	1.632	5.852	1.660	5.485	1.466
15	5	.7	5.134	1.295	5.035	1.343	4.743	1.246
15	5	.6	5.162	1.186	5.069	1.222	4.786	1.100
15	5	.5	5.204	1.839	5.096	1.888	4.778	1.706

TABLE 4.2(b)

B2 MONTE CARLO RESULTS FOR $\hat{K}$: SAMPLE SIZE = 50

No. of Samples	K	A	$\hat{K}_s$ Mean	$\hat{K}_s$ Standard Error	$\hat{K}_m$ Mean	$\hat{K}_m$ Standard Error	$\hat{K}_n$ Mean	$\hat{K}_n$ Standard Error
15	1	.9	1.302	.704	1.078	.468	.678	.469
15	1	.8	1.679	.394	1.310	.429	.902	.467
15	1	.7	1.577	.319	1.195	.345	.775	.365
15	1	.6	1.589	.406	1.243	.341	.829	.346
15	1	.5	1.733	.391	1.370	.425	.969	.457
15	2	.9	2.346	.662	2.084	.644	1.775	.700
15	2	.8	2.330	.315	2.032	.355	1.708	.408
15	2	.7	2.548	.527	2.275	.591	1.981	.668
15	2	.6	2.513	.436	2.236	.487	1.935	.549
15	2	.5	2.247	.459	1.938	.517	1.599	.589
15	3	.9	3.277	.675	3.078	.737	2.851	.786
15	3	.8	3.708	.697	3.545	.745	3.338	.746
15	3	.7	3.257	.473	3.060	.515	2.845	.544
15	3	.6	3.492	.663	3.313	.715	3.100	.737
15	3	.5	3.495	.704	3.315	.759	3.099	.781
15	4	.9	4.790	.705	4.690	.729	4.455	.664
15	4	.8	4.366	.746	4.244	.785	4.027	.755
15	4	.7	4.415	.717	4.298	.751	4.082	.711
15	4	.6	4.219	.626	4.092	.661	3.887	.642
15	4	.5	4.574	.692	4.465	.719	4.243	.669
15	5	.9	5.400	1.113	5.318	1.137	5.018	1.014
15	5	.8	5.653	.793	5.579	.812	5.260	.724
15	5	.7	5.338	.982	5.252	1.008	4.961	.905
15	5	.6	5.260	1.231	5.173	1.266	4.880	1.137
15	5	.5	5.525	.818	5.451	.842	5.143	.762

TABLE 4.3(a)

B2 MONTE CARLO RESULTS FOR $\hat{A}$: SAMPLE SIZE = 25

No. of Samples	K	A	$\hat{A}_m$ Mean	$\hat{A}_m$ Standard Error	$\hat{A}_p$ Mean	$\hat{A}_p$ Standard Error
15	1	.9	.803	.167	.699	.116
15	1	.8	.779	.237	.656	.110
15	1	.7	.677	.160	.616	.099
15	1	.6	.575	.140	.547	.081
15	1	.5	.497	.136	.515	.091
15	2	.9	.924	.074	.840	.081
15	2	.8	.817	.115	.765	.083
15	2	.7	.698	.080	.672	.074
15	2	.6	.597	.094	.579	.069
15	2	.5	.496	.090	.517	.073
15	3	.9	.895	.050	.885	.048
15	3	.8	.783	.083	.757	.070
15	3	.7	.707	.063	.691	.058
15	3	.6	.600	.109	.597	.099
15	3	.5	.531	.096	.520	.094
15	4	.9	.900	.054	.888	.057
15	4	.8	.793	.079	.787	.074
15	4	.7	.702	.090	.704	.083
15	4	.6	.605	.127	.595	.122
15	4	.5	.458	.102	.461	.104
15	5	.9	.893	.071	.893	.070
15	5	.8	.843	.071	.835	.074
15	5	.7	.648	.092	.645	.091
15	5	.6	.578	.099	.584	.097
15	5	.5	.475	.108	.475	.109

TABLE 4.3(b)

B2 MONTE CARLO RESULTS FOR $\hat{A}$: SAMPLE SIZE = 50

No. of Sample	K	A	$\hat{A}_m$ Mean	$\hat{A}_m$ Standard Error	$\hat{A}_p$ Mean	$\hat{A}_p$ Standard Error
15	1	.9	.842	.215	.679	.120
15	1	.8	.713	.083	.629	.050
15	1	.7	.707	.100	.627	.065
15	1	.6	.632	.131	.557	.068
15	1	.5	.519	.154	.520	.089
15	2	.9	.925	.125	.824	.059
15	2	.8	.828	.070	.773	.056
15	2	.7	.692	.077	.665	.063
15	2	.6	.576	.088	.571	.068
15	2	.5	.502	.063	.513	.055
15	3	.9	.911	.041	.887	.049
15	3	.8	.798	.063	.784	.058
15	3	.7	.687	.080	.683	.074
15	3	.6	.596	.079	.584	.069
15	3	.5	.491	.075	.497	.066
15	4	.9	.875	.035	.872	.034
15	4	.8	.810	.048	.803	.051
15	4	.7	.699	.067	.695	.069
15	4	.6	.636	.073	.632	.074
15	4	.5	.501	.051	.504	.050
15	5	.9	.887	.048	.885	.048
15	5	.8	.809	.049	.807	.045
15	5	.7	.711	.065	.708	.062
15	5	.6	.595	.083	.592	.079
15	5	.5	.461	.074	.461	.075

TABLE 4.4

20 MONTE CARLO SAMPLES FROM DISTRIBUTION B2

(a) denotes the results from the initial estimates.
(b) denotes the results from varying only $\hat{\theta}_o$.
(c) denotes the results from varying all 3 initial estimates.

Sample	n	θ_o	K	A	$\hat{\theta}_o$	$\hat{K}$	$\hat{A}$	$\sqrt{n}\,V_n$	U^2_n	W^2_n	X^2	cells
1(a)	50	0	2	.9	17.17	3.146	.9246	1.1067	.0579	.3300	22	10
(b)					16	3.146	.9246	1.0865	.0574	.3063	26.8	10
(c)					16	2.9	.92	1.0864	.0495	.2459	20	10
2(a)	50	0	2	.8	178.4	2.650	.2432	.5835	.0105	.0141	2.4	10
(b)					179	2.650	.2432	.5818	.0104	.0125	2.4	10
(c)					179	2.8	.24	.5677	.0097	.0120	2.0	10
3(a)	50	0	2	.7	1.44	2.471	.7657	1.2061	.0678	.2039	4.4	10
(b)					350	2.471	.7657	.8029	.0250	.0368	5.6	10
(c)					349	2.5	.77	.7797	.0250	.0313	7.2	10
4(a)	50	0	2	.6	0.237	1.553	.6966	1.0916	.0423	.0591	6.0	10
(b)					8	1.553	.6966	1.0382	.0341	.0343	6.8	10
(c)					8	1.8	.69	.9591	.0327	.0327	5.6	10
5(a)	50	0	2	.5	173.61	1.484	.48	.7454	.0212	.0351	4.0	10
(b)					175	1.484	.48	.7446	.0212	.0328	3.2	10
(c)					174	1.4	.49	.7548	.0205	.0323	3.2	10
6(a)	50	0	3	.9	2.039	2.904	.9158	1.2746	.1061	.1061	9.6	10
(b)					4	2.904	.9158	1.2453	.1043	.1075	14.0	10
(c)					2	2.6	.89	1.1631	.0955	.0955	14.4	10
7(a)	50	0	3	.8	5.055	3.5072	.8072	1.1019	.0457	.0482	6.4	10
(b)					2	3.5072	.8072	1.0750	.0428	.0624	10.0	10
(c)					2	2.9	.83	.9252	.0339	.0590	11.6	10

TABLE 4.4 (continued)

Sample	n	θ_0	K	A	$\hat{\theta}_0$	$\hat{K}$	$\hat{A}$	$\sqrt{n}\,V_n$	U^2_n	W^2_n	X^2	cells
8(a)	50	0	3	.7	2.92	3.0814	.7807	.6634	.0124	.0127	2.4	10
(b)					4	3.0814	.7807	.6293	.0123	.0128	2.4	10
(c)					4	2.9	.78	.6736	.0121	.0123	2.4	10
9(a)	50	0	3	.6	2.23	3.629	.5567	1.3340	.0798	.2177	5.6	10
(b)					354	3.629	.5567	1.2530	.0732	.1202	10.4	10
(c)					352	2.9	.6	1.3187	.0637	.0926	8.4	10
10(a)	50	0	3	.5	176.70	3.64	.5479	.9405	.0353	.0776	6.4	10
(b)					178	3.64	.5479	.9541	.0350	.0821	4.0	10
(c)					178	3.8	.54	.9157	.0345	.0790	5.2	10
11(a)	50	0	4	.9	177.11	3.024	.1071	.9998	.0480	.0661	5.6	10
(b)					175	3.024	.1071	1.0123	.0459	.0522	6.4	10
(c)					175	3.7	.13	.8646	.0366	.0386	10.0	10
12(a)	50	0	4	.8	170.28	4.139	.2378	1.2331	.0565	.0680	4.4	10
(b)					164	4.139	.2378	.9486	.0304	.0319	3.6	10
(c)					164	4.3	.24	.9305	.0299	.0328	4.4	10
13(a)	50	0	4	.7	171.48	2.887	.2905	1.2081	.0591	.1296	5.6	10
(b)					181	2.887	.2905	1.0846	.0408	.0420	4.4	10
(c)					181	2.7	.3	.9816	.0376	.0390	2.8	10
14(a)	50	0	4	.6	2.425	3.905	.498	.7604	.0175	.0353	.8	10
(b)					4	3.905	.498	.7192	.0173	.0298	.4	10
(c)					4	4.0	.51	.7278	.0170	.0277	.8	10
15(a)	50	0	4	.5	.989	3.567	.5391	1.0864	.0515	.0530	6.4	10
(b)					1	3.567	.5391	1.0864	.0515	.0530	6.4	10
(c)					3	3.7	.52	1.0812	.0505	.0561	6.4	10

TABLE 4.4 (continued)

Sample	n	θ_0	K	A	$\hat{\theta}_0$	$\hat{K}$	$\hat{A}$	$\sqrt{n}\,V_n$	U^2_n	W^2_n	X^2	cells
16(a)	50	0	5	.9	178.61	5.761	.1452	1.2397	.0678	.3165	11.6	10
(b)					173	5.761	.1452	1.0097	.0400	.1233	9.2	10
(c)					173	5.4	.15	1.0364	.0387	.1361	7.6	10
17(a)	50	0	5	.8	173.15	6.958	.1483	1.0150	.0426	.0742	8.0	10
(b)					176	6.958	.1483	.8790	.0311	.0362	6.4	10
(c)					176	7	.13	.82	.0301	.0361	9.2	10
18(a)	50	0	5	.7	1.213	3.588	.6905	.8773	.0290	.0355	7.2	10
(b)					5	3.588	.6905	.7872	.0255	.0582	5.2	10
(c)					5	3.3	.69	.76	.0235	.0520	4.4	10
19(a)	50	0	5	.6	4.43	5.942	.4608	1.0263	.0479	.1093	1.6	10
(b)					3	5.942	.4608	.9934	.0473	.1173	2.4	10
(c)					3	5.7	.47	1.0122	.0466	.1155	2.0	10
20(a)	50	0	5	.5	1.878	5.3254	.4536	.7792	.0209	.0310	4.4	10
(b)					359	5.3254	.4536	.7345	.0189	.0359	4.0	10
(c)					359	5.6	.44	.6997	.0175	.0347	4.0	10

TABLE 4.5

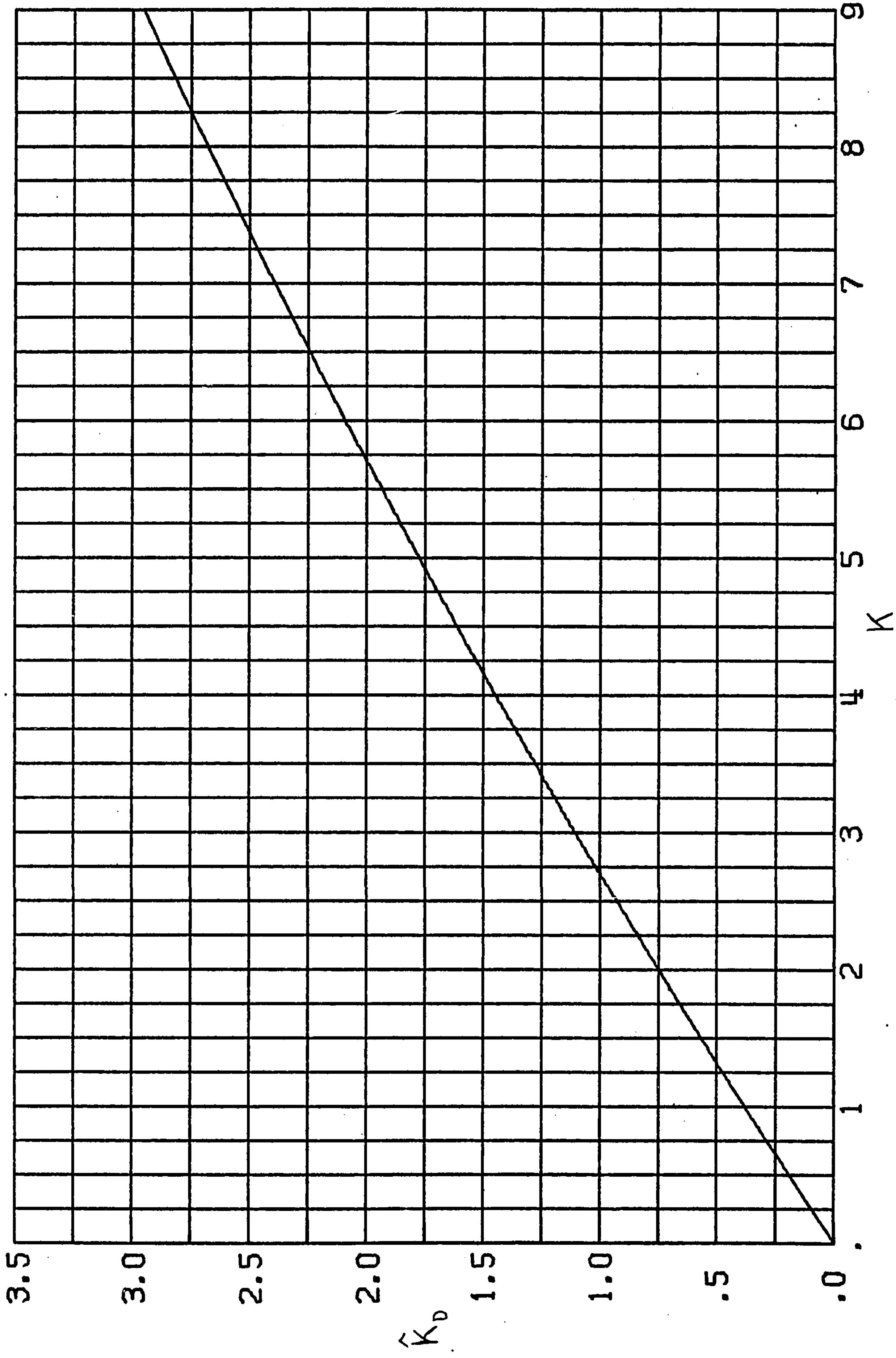
B1 SAMPLES 1 and 2 TREATED WITH B2 TECHNIQUES

B1 Sample	n	Plot	0	λ	$\hat{\theta}_0$	$\hat{K}$	$\hat{A}$	$\sqrt{n}\, V_n$	$U^2{}_n$	$W^2{}_n$
1(a)	50	34	0	1	7.633	2.7094	.5249	.7157	.0202	.0502
(b)		-			5	2.7094	.5249	.7437	.0196	.0200
(c)		-			6	2.8	.51	.6735	.0183	.0192
2(a)	50	35	0	2	177.22	6.6174	.5392	1.0678	.0442	.0490
(b)		-			180	6.6174	.5392	1.0636	.0426	.2667
(c)		-			180	6.0	.53	1.0079	.0401	.2612

Nomogram - $\hat{K}_D$ versus K

Figure 4.1

See pages 44, 45



CHAPTER V

CONCLUSIONS

5.1 Recommended Technique for Fitting Data to Distribution B2

Given $\theta_1, \cdots, \theta_n$, suspected to have unequally or equally weighted modes 180^o apart, the procedure of describing the data by distribution B2 and for improving the fit is as follows:

(a) Obtain $\hat{\theta}_{oD}$, $\hat{K}_m$, and $\hat{A}_m$ as in section 3.3

(b) Calculate $U^2{}_n$ and compare these values with lower tail significance points at the appropriate level for the given sample size. Small values of $U^2{}_n$ mean a good fit.

(c) Plot $F_o(x)$ and $F_n(x)$

(d) Improve the fit by observing the plots in (c) and varying $\hat{\theta}_{oD}$ according to section 2.4 to find minimum $U^2{}_n$.

(e) Plot the results obtained from (d) to see if another shift of location is required, as in B2 Sample 3.5 (compare plots 24 and 26).

5.2 Recommended "Quick" Technique for Fitting Data to Distribution B2

Given $\theta_1, \cdots, \theta_n$ as above a quick procedure for obtaining initial estimates of A, K, and θ_o is:

(a) Obtain $\hat{\theta}_{oD}$ as in section 3.3

(b) Obtain $\hat{K}$ either from (1) $\hat{K} = \dfrac{\hat{K}_D}{.3430}$ where $\hat{K}_D$ is defined in section 4.3 or from (2) Figure 4.1.

(c) Obtain $\hat{A}_p$ from section 3.9.

5.3 Best Goodness of Fit Statistic

Small values of U^2_n produced a much tighter fit than did the corresponding small values of $\sqrt{n}\,V_n$. The Method of Minimum U^2_n was therefore preferred to the Method of Minimum $\sqrt{n}\,V_n$. Low values of U^2_n almost always mean a good fit but the sample distribution $F_n(x)$ function and the estimated theoretical cumulative distribution $F_n(x)$ should be plotted as a confirmation. For example the U^2_n value of plot 24 is less than that of plot 26 yet plot 26 appears to produce the better fit.

5.4 Suggestions for Further Work

(a) For distribution B2 -- to develop better estimators or to improve those suggested.

(b) For distribution B2 -- to derive statistical tests concerning the parameters A, K, and θ_o.

(c) To develop a technique of analysing unequally weighted bimodal data with modes $\psi \neq 180^\circ$ apart.

(d) To develop a technique of treating data with more than two modes.

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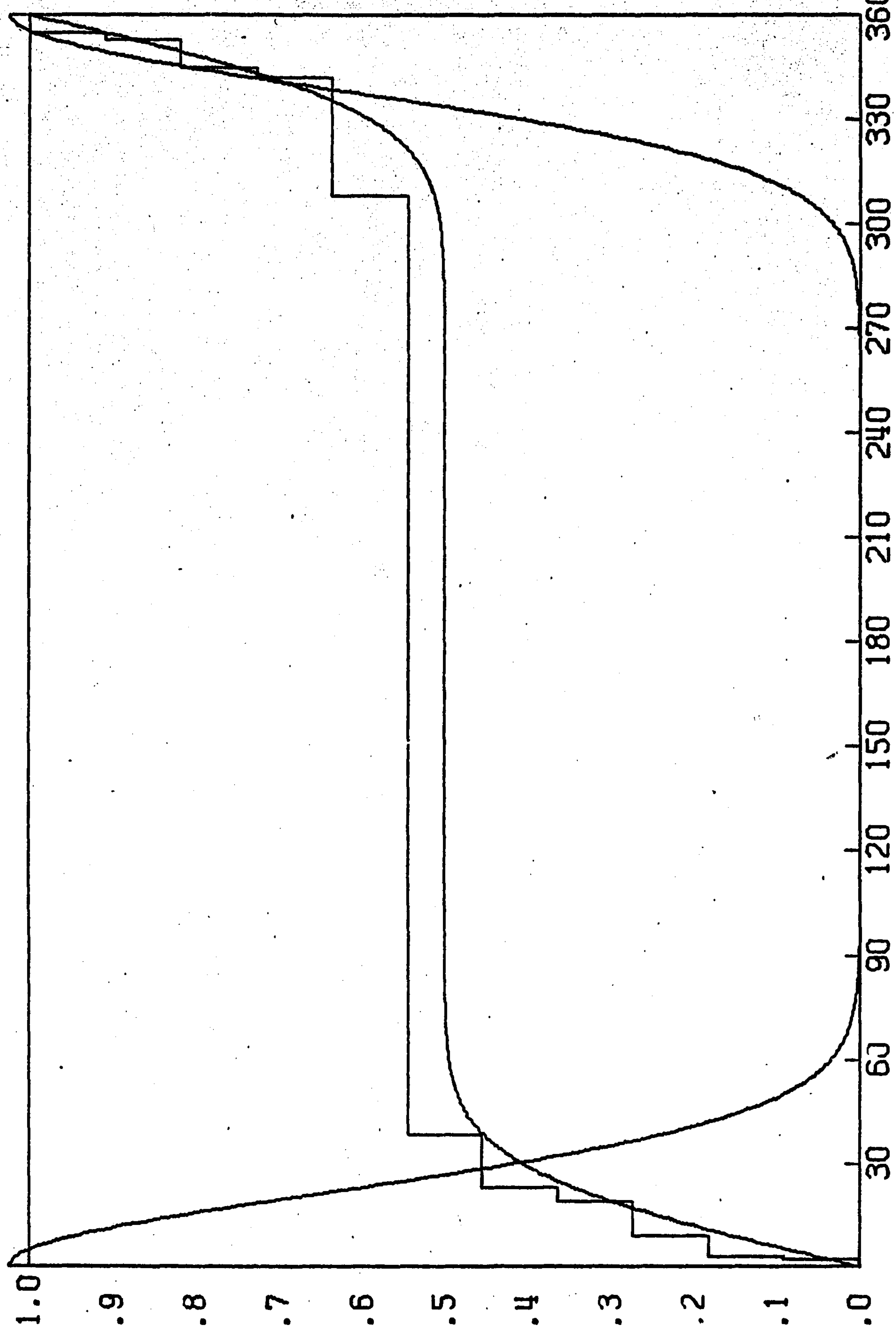
A P P E N D I X

(1) Plots . . . . 1 - 35

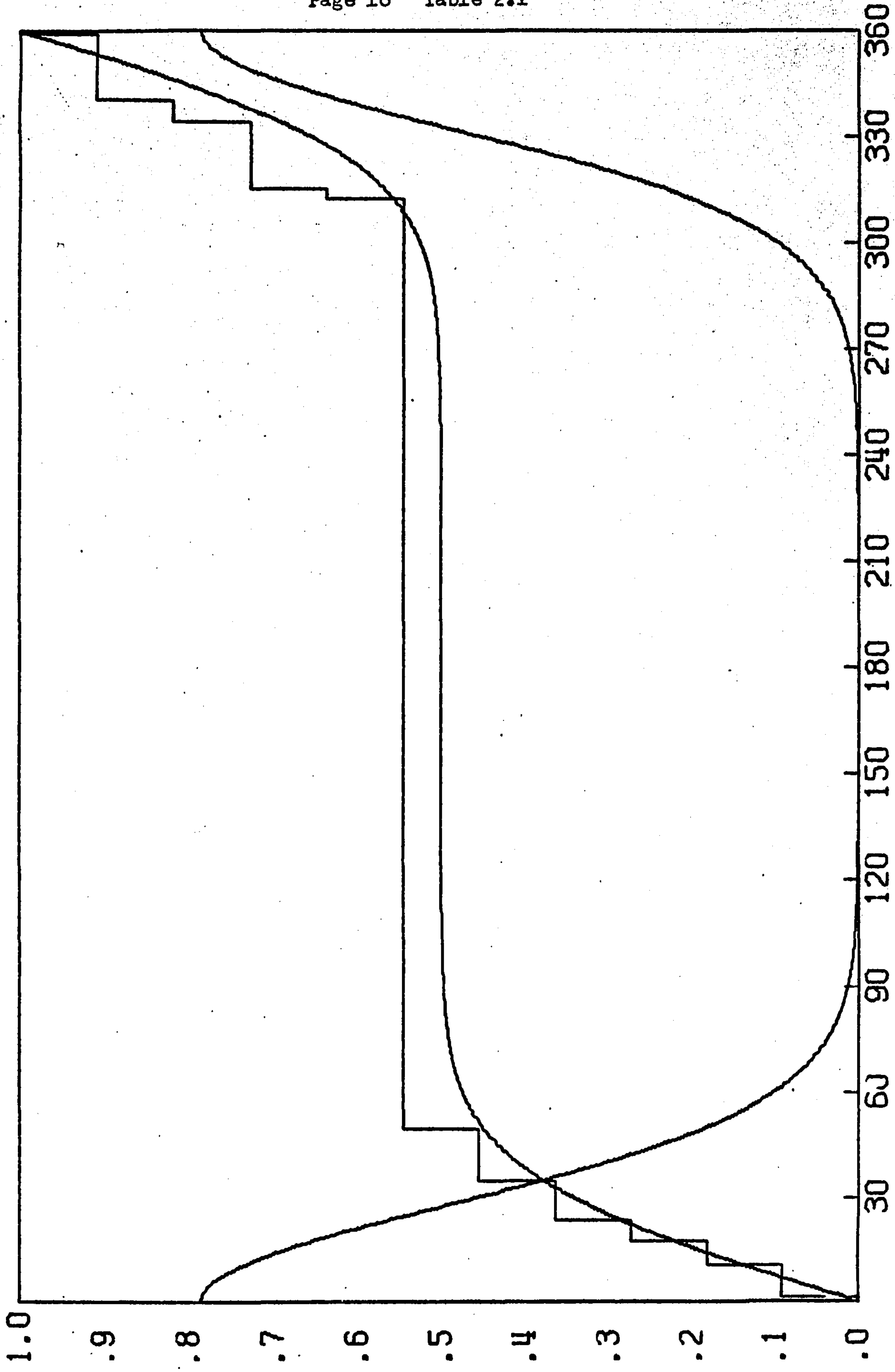
(2) Data . . . . Pages A-1 to A-6

(3) Tables . . . . Pages A-7 to A-10

Page 16 Table 2.1

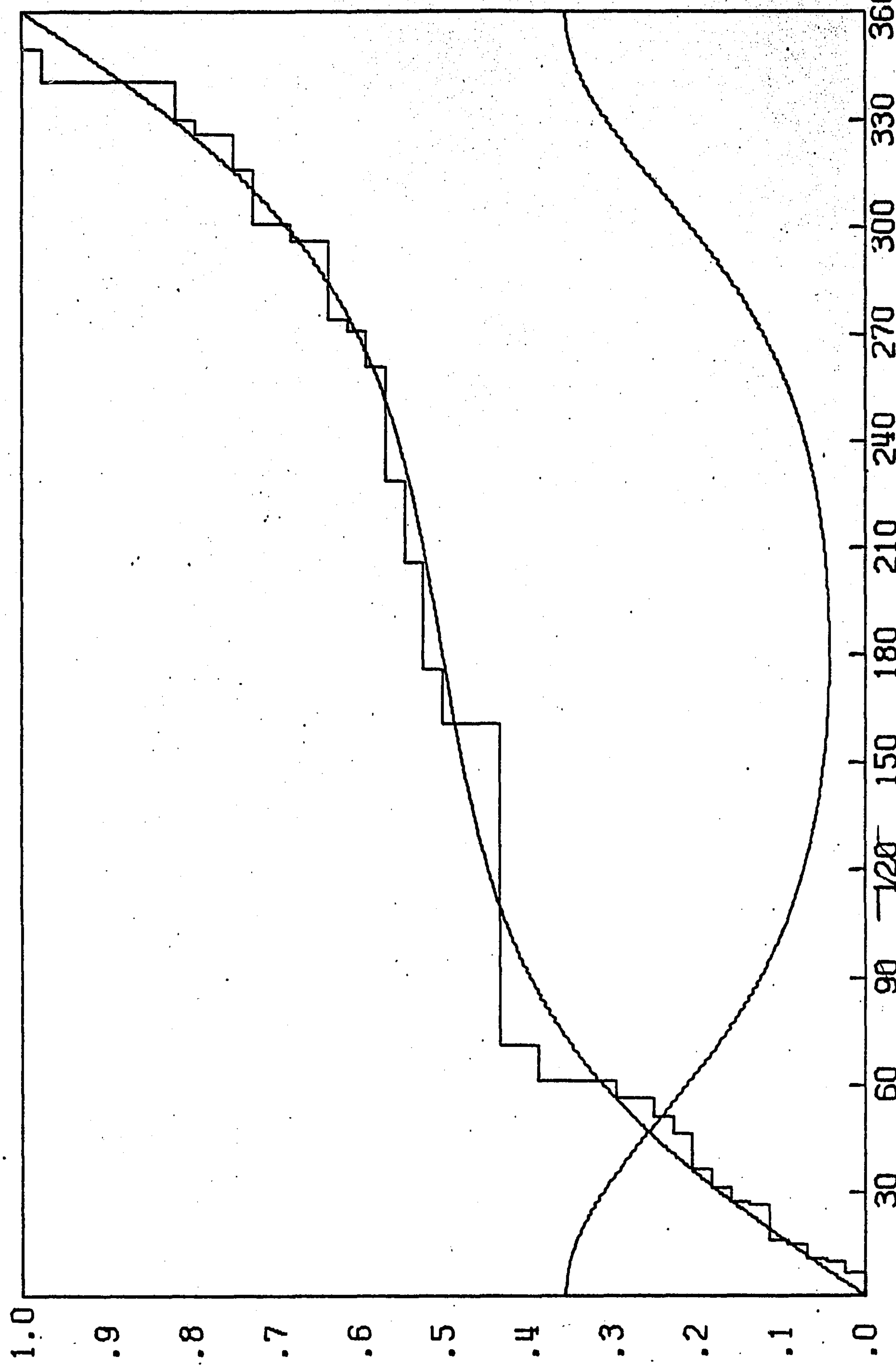


Page 16 Table 2.1



CN SAMPLE 2.3 Plot 3

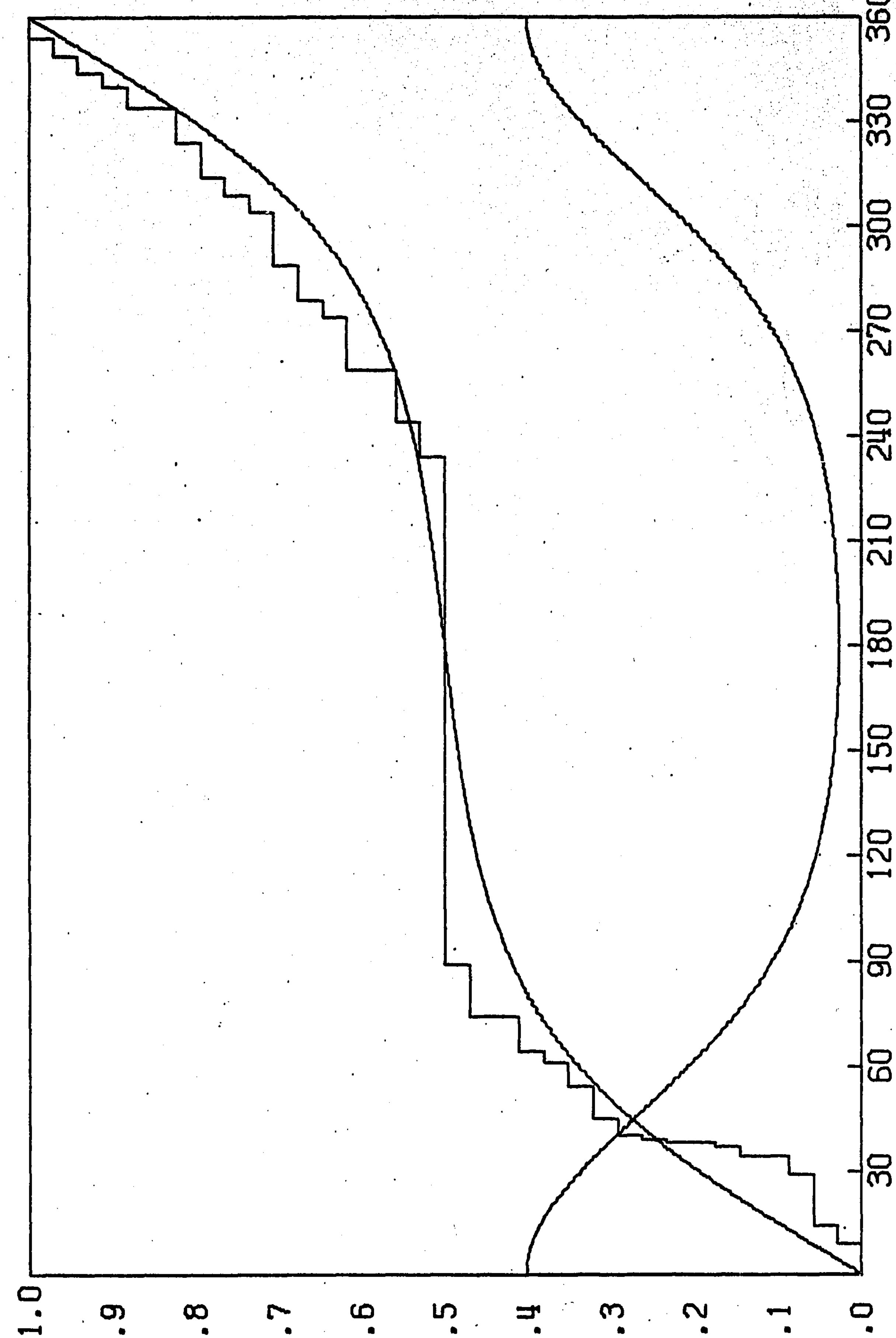
Page 17 Table 2.1



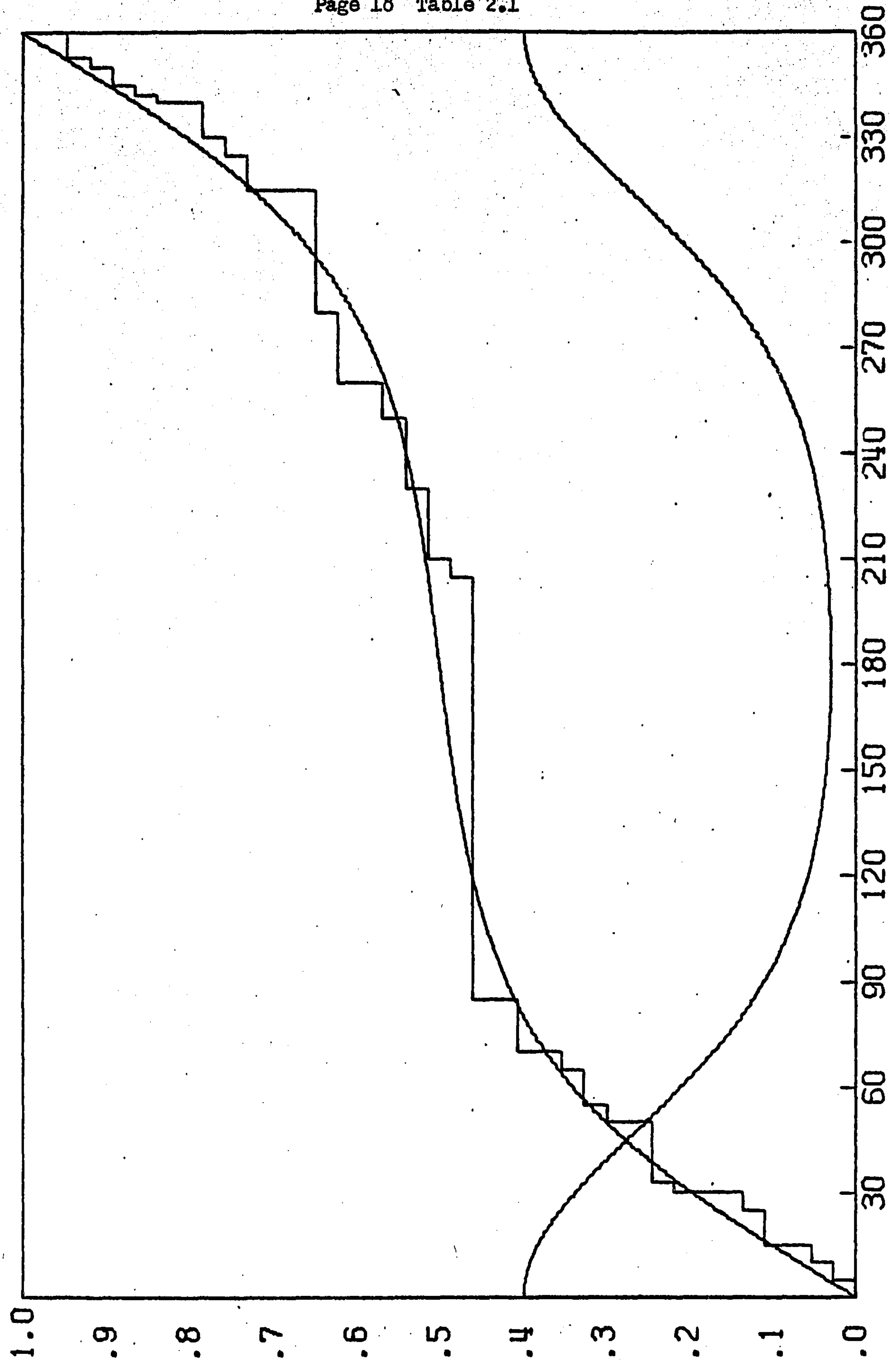
CN SAMPLE 2.4

Plot 4

Page 17 Table 2.1

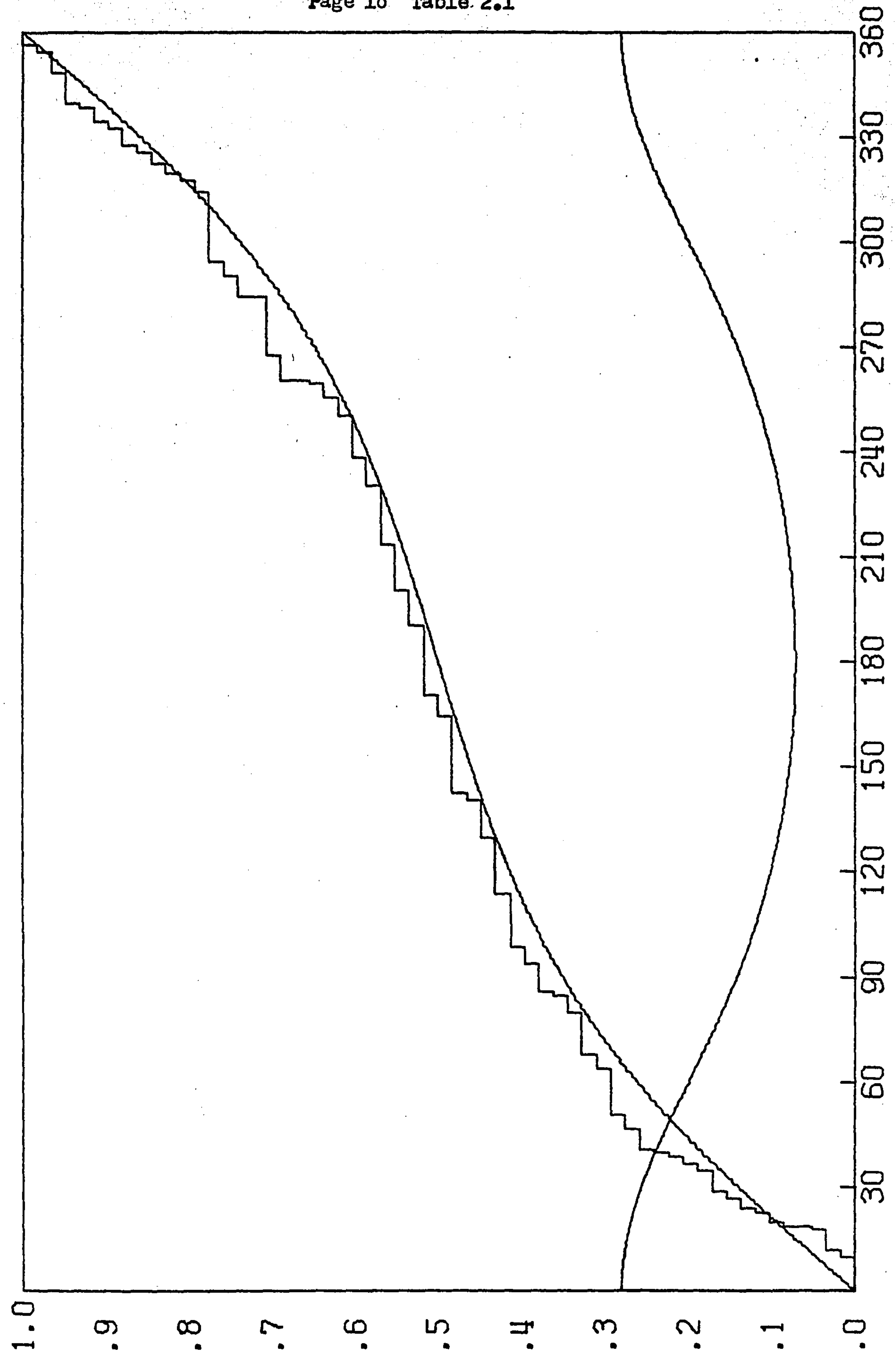


Page 18 Table 2.1



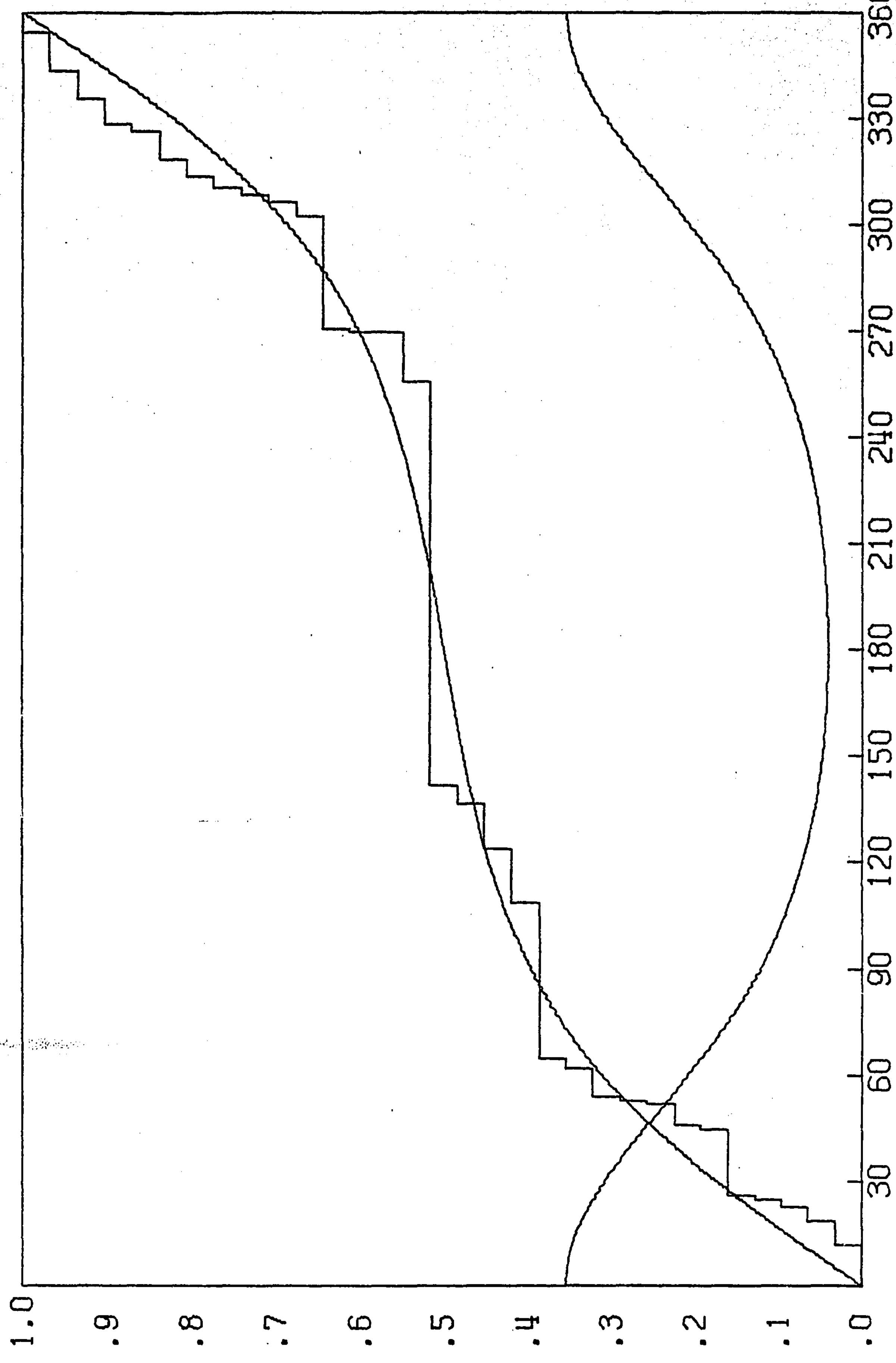
CN SAMPLE 2.6 Plot 6

Page 18 Table 2.1



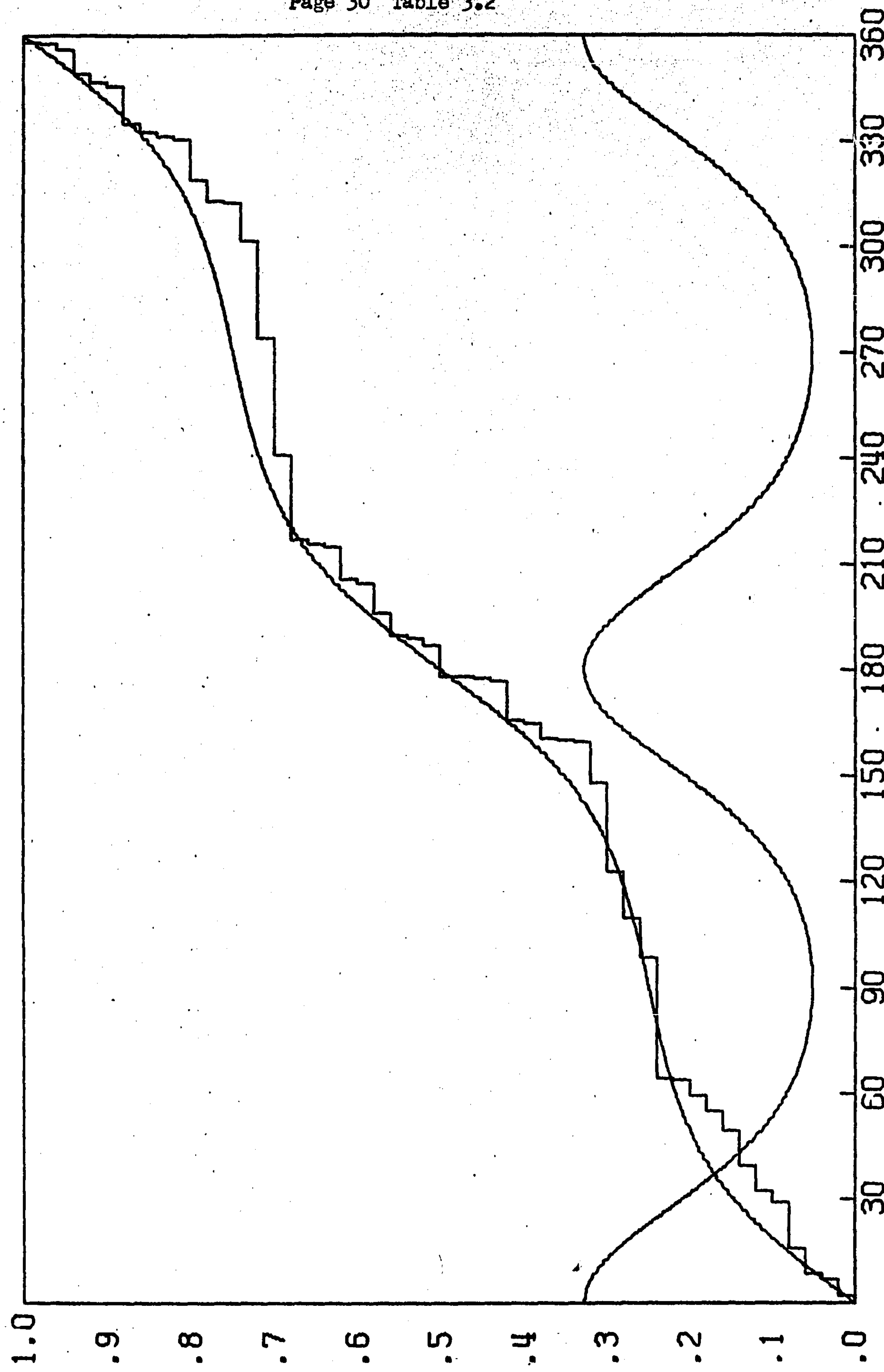
CN SAMPLE 2.7 Plot 7

Pages 18,19 Table 2.1



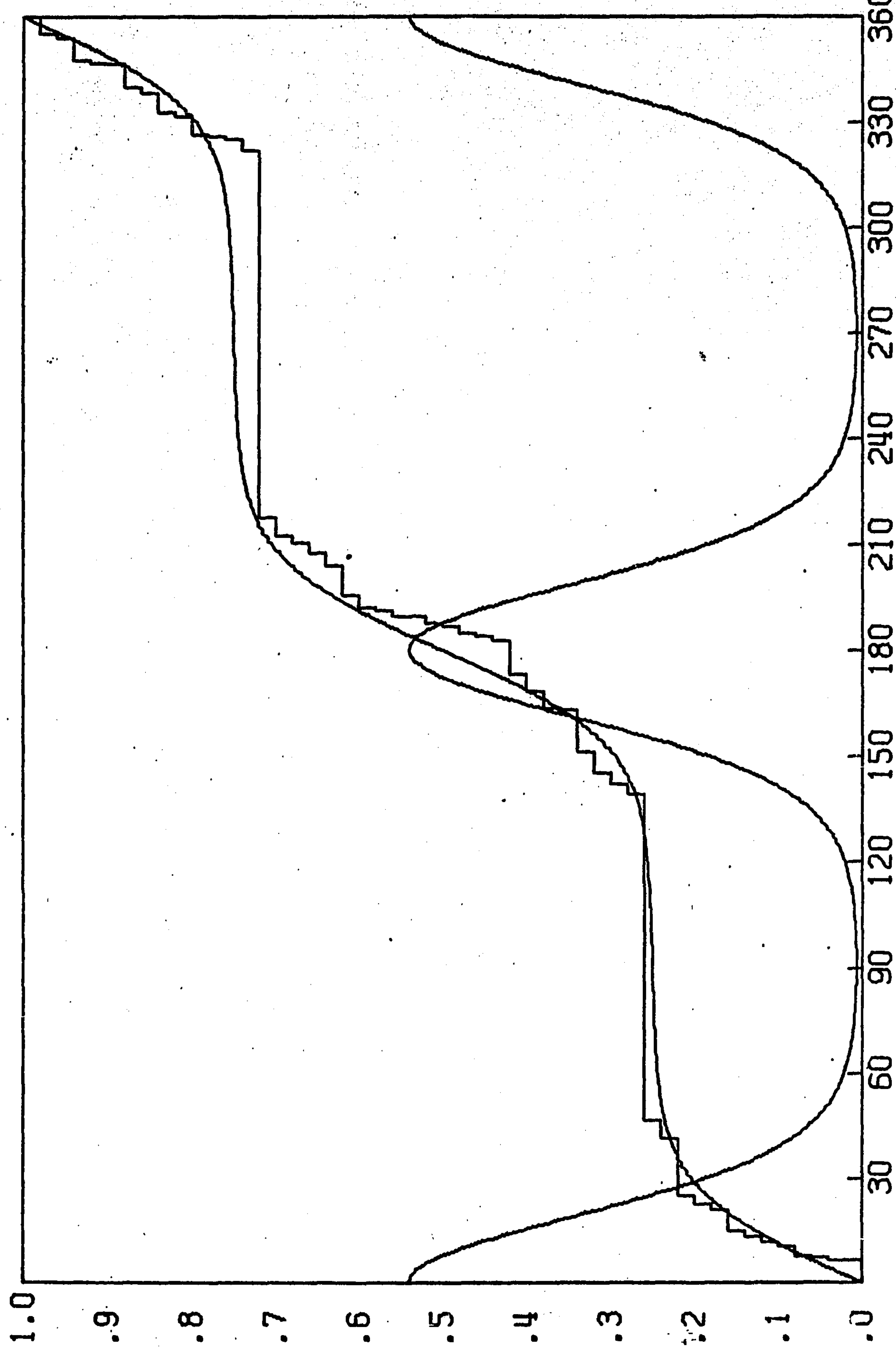
B1 SAMPLE 3.1 Plot 8

Page 30 Table 3.2



B1 SAMPLE 3.2 Plot 9

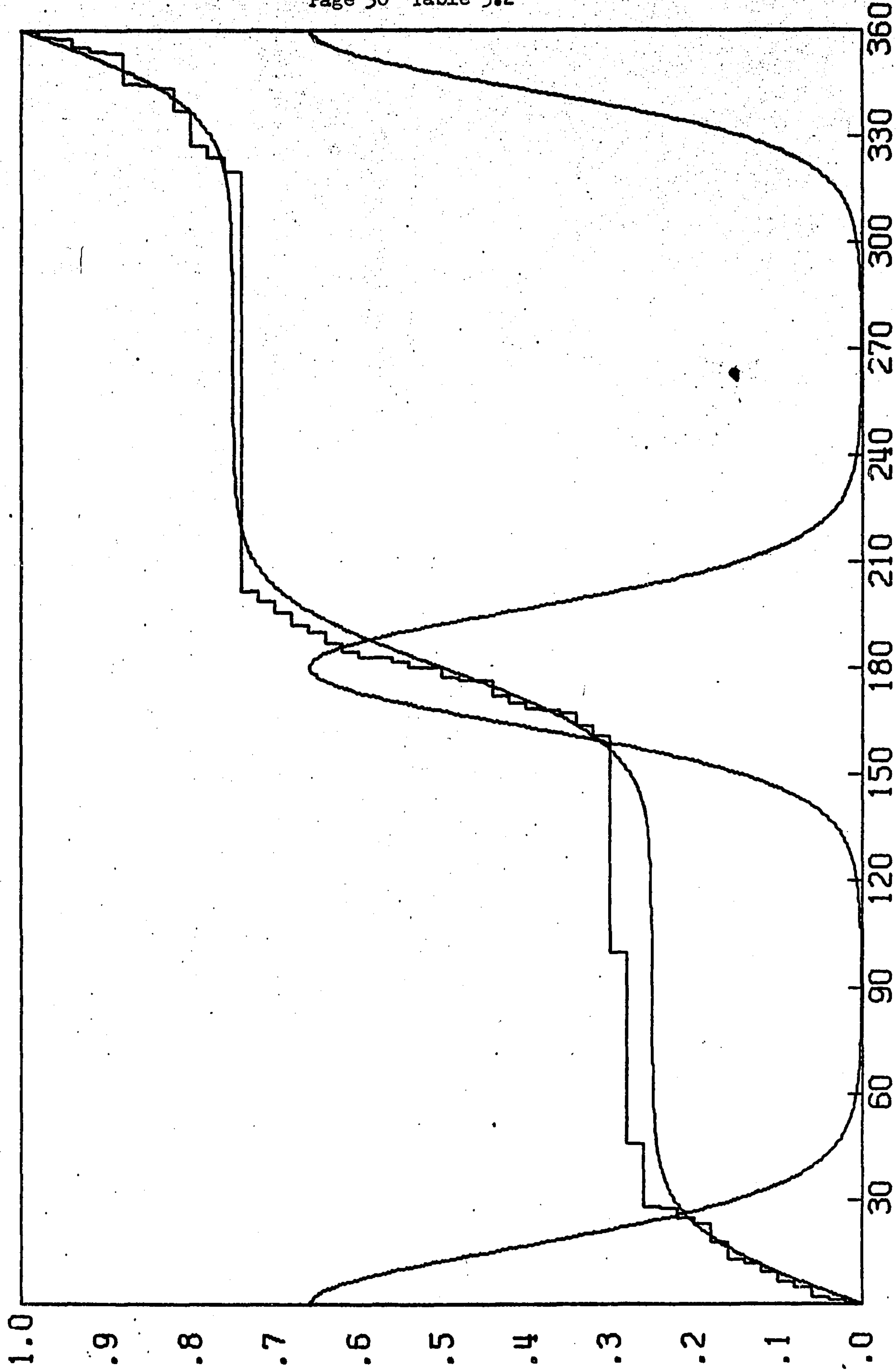
Page 30 Table 3.2



B1 SAMPLE 3.3

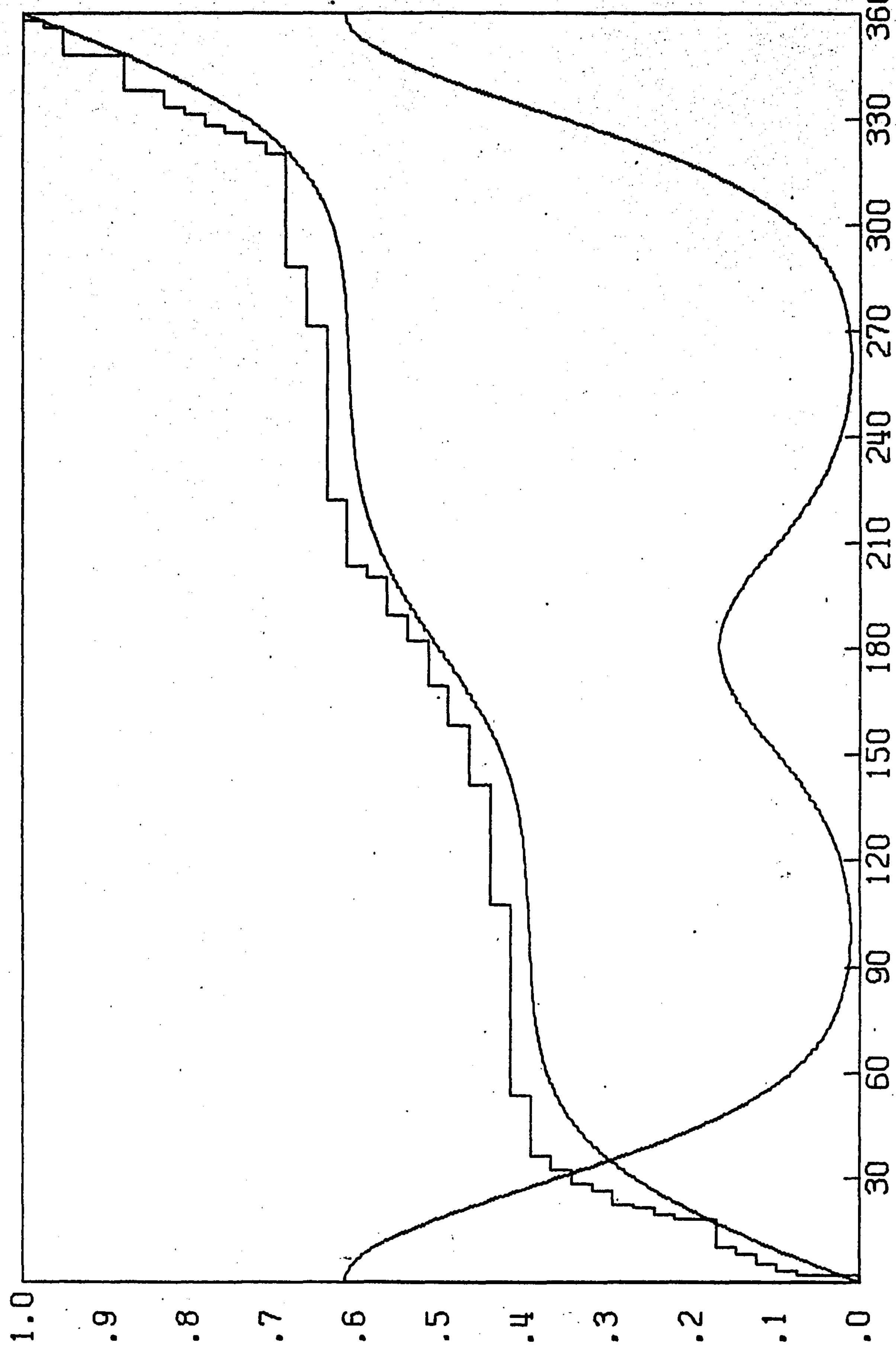
Plot 10

Page 30 Table 3.2



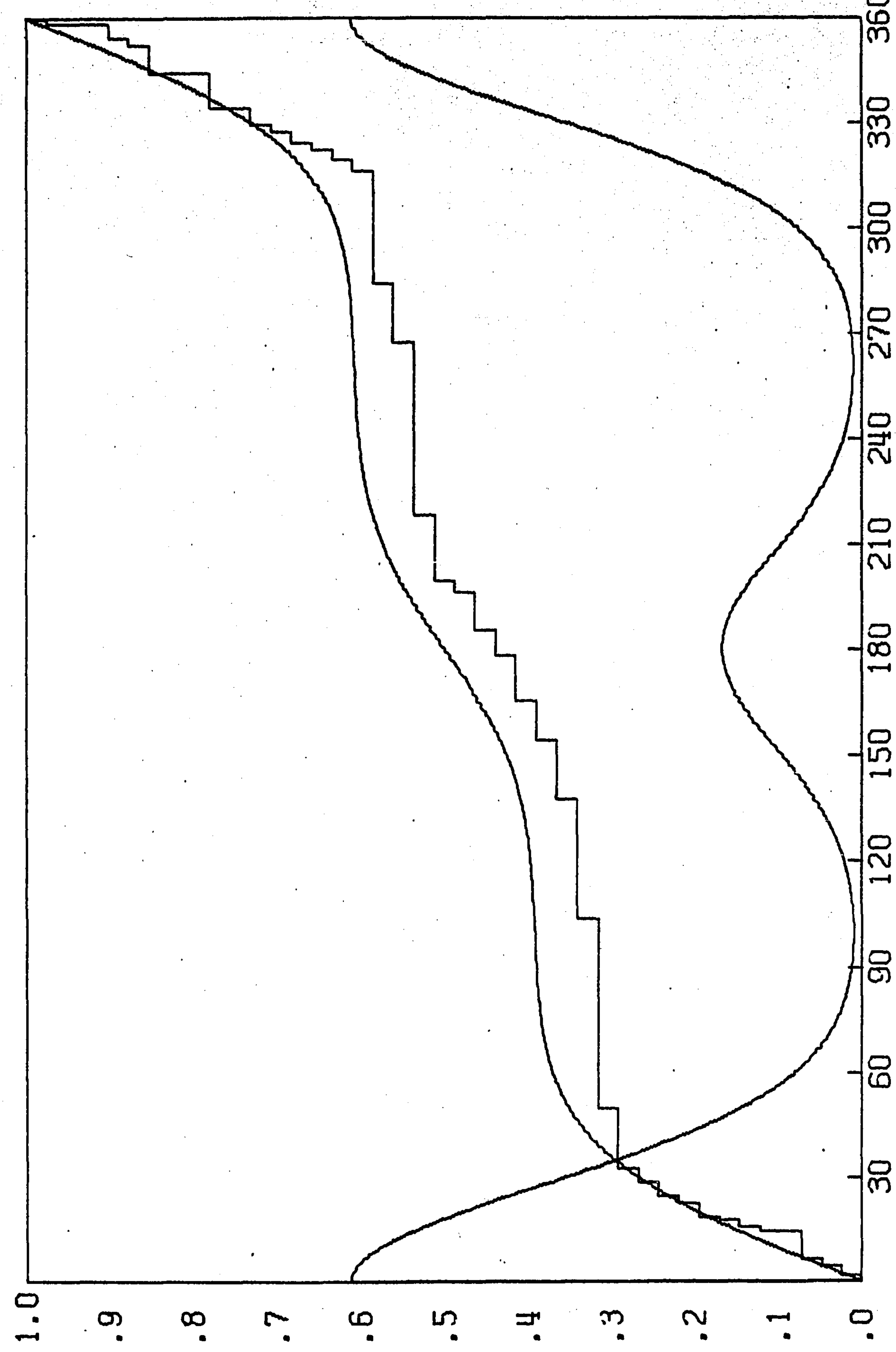
B2 SAMPLE 3.1(a)

Page 32 Table 3.3

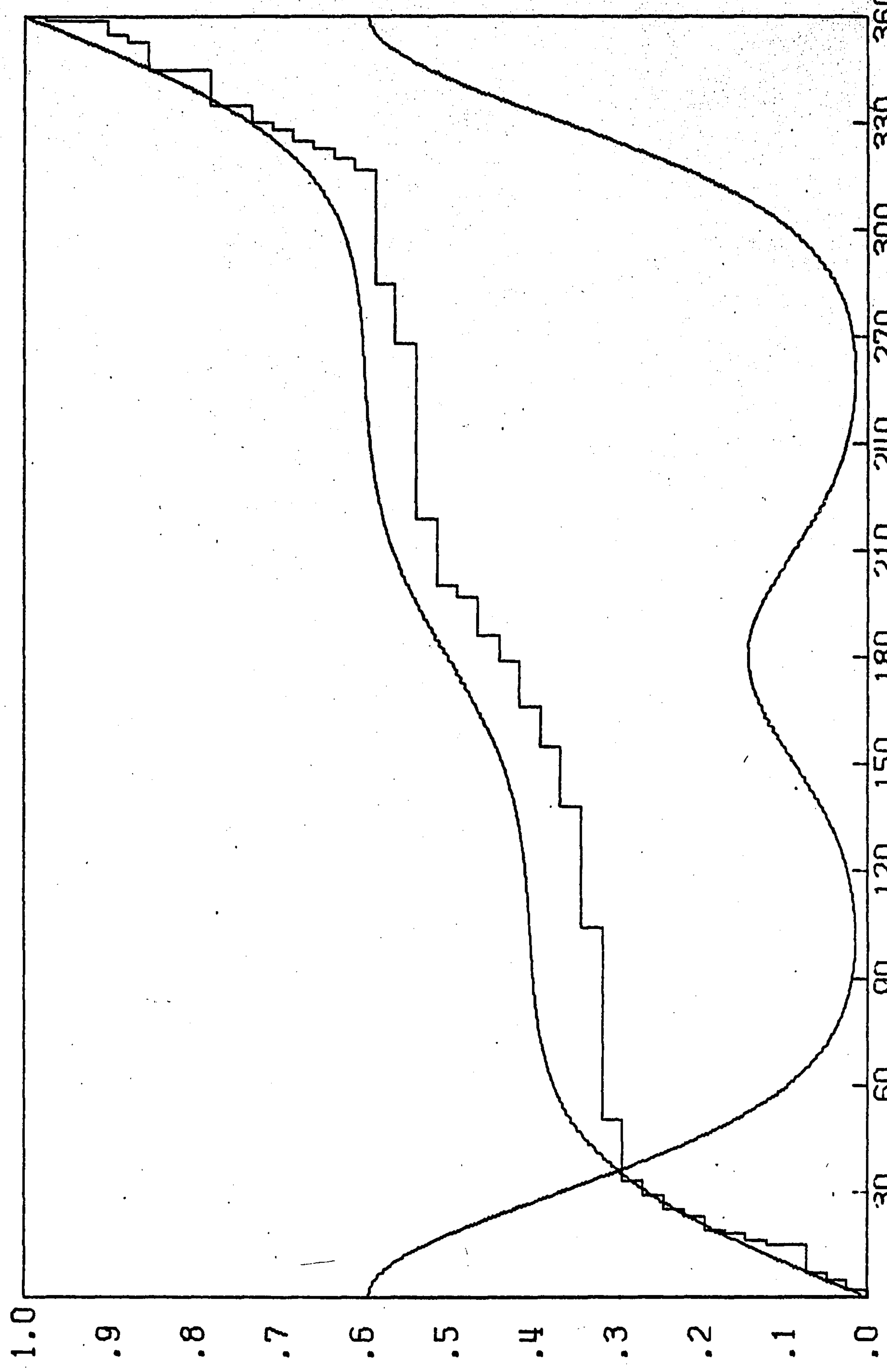


B2 SAMPLE 3.1(b) Plot 12

Page 32, 33 Table 3.3

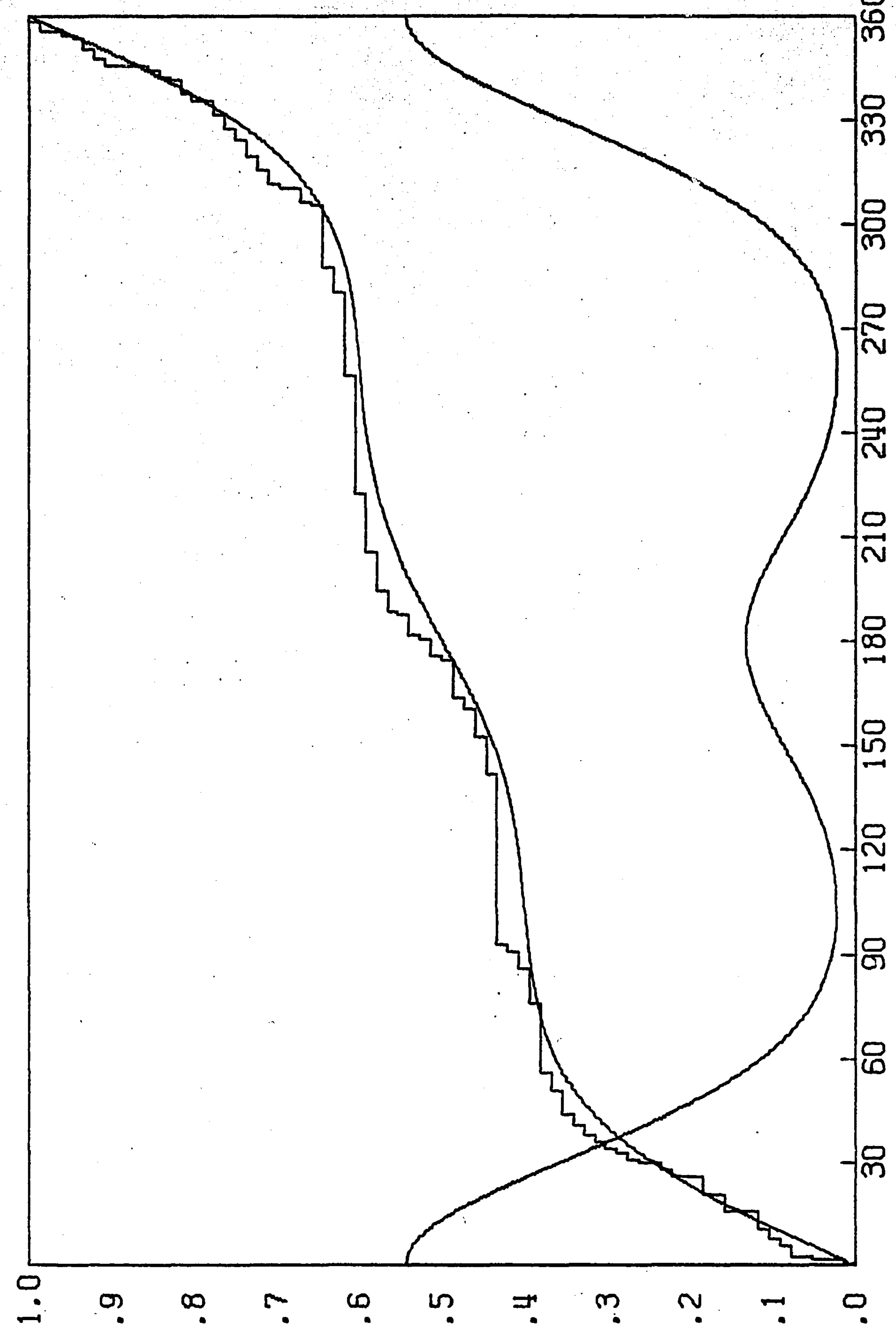


Page 32 Table 3.3



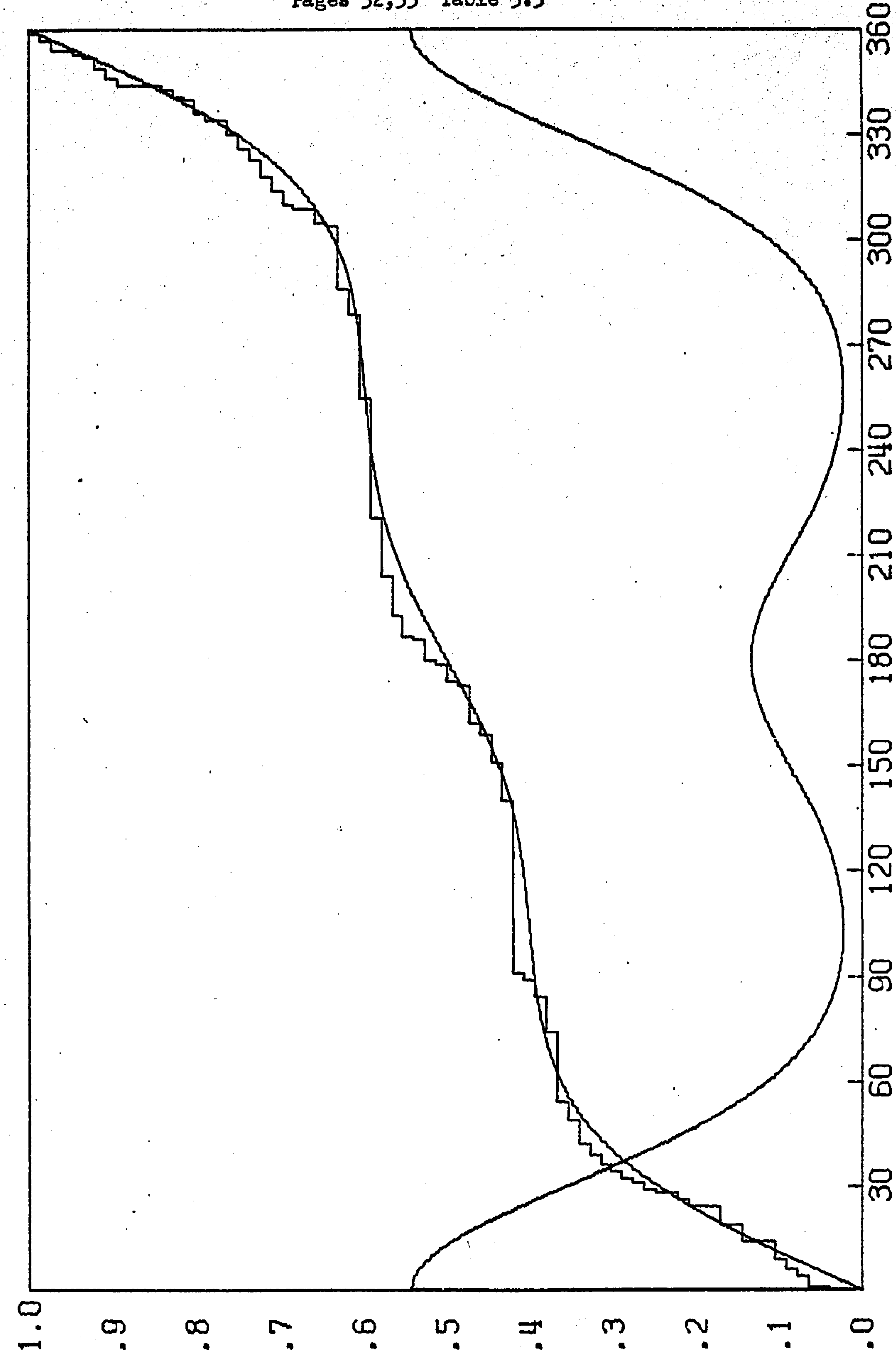
B2 SAMPLE 3.2(a)

Pages 32,33 Table 3.3



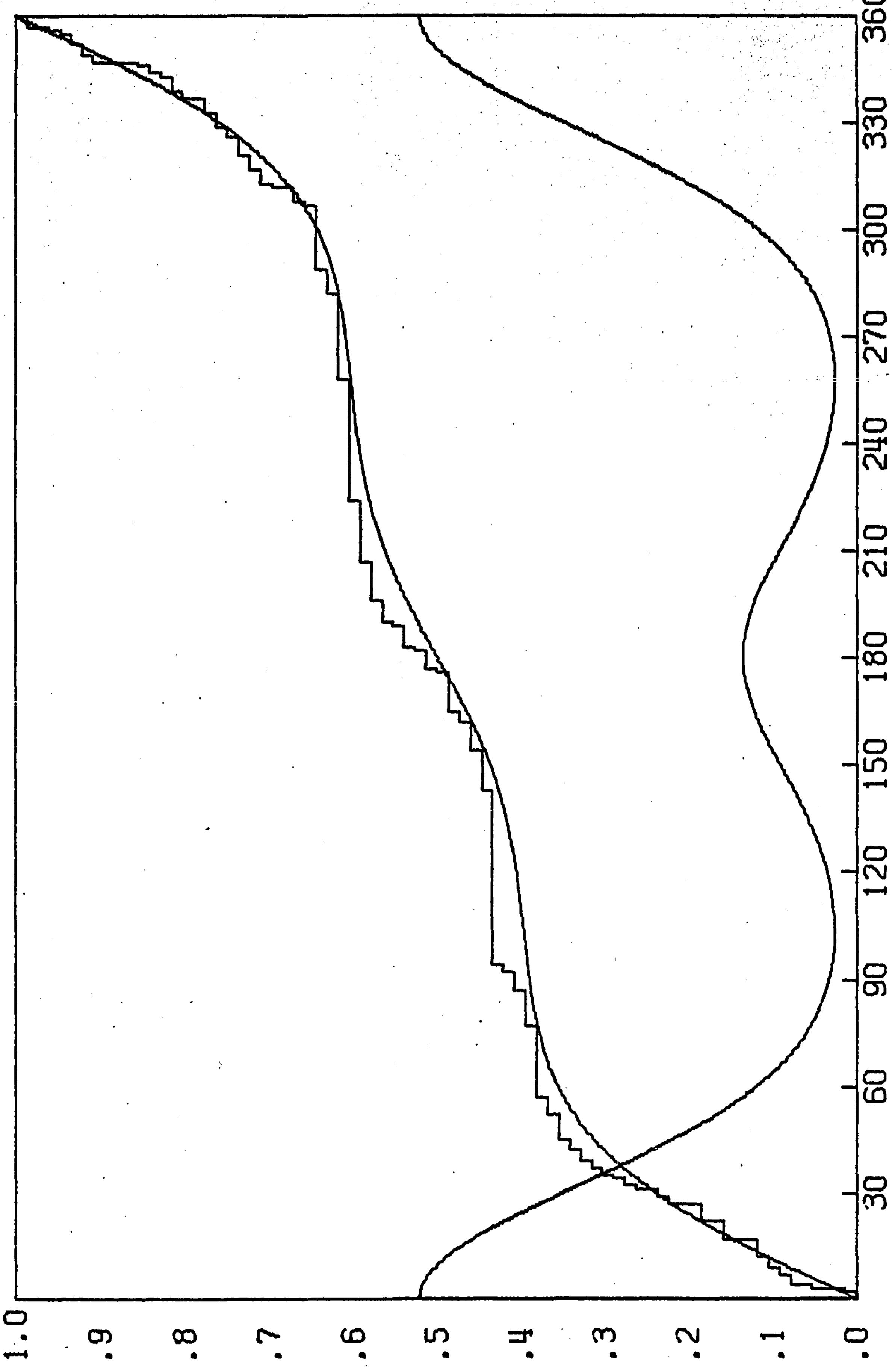
B2 SAMPLE 3.2(b) Plot 15

Pages 32,33 Table 3.3



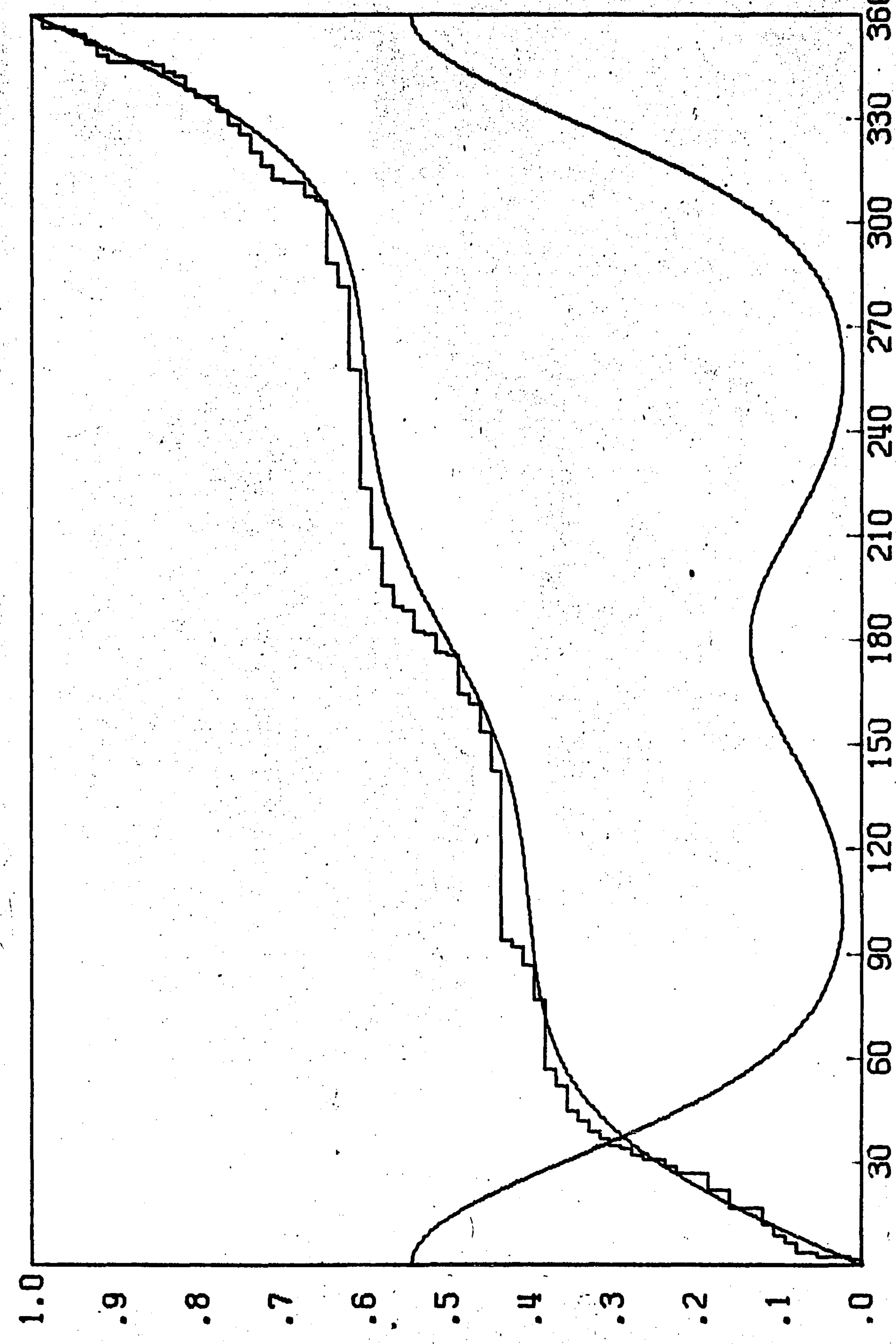
B2 SAMPLE 3.2(c) Plot 16

Pages 32,33 Table 3.3



B2 SAMPLE 3.2(d) Plot 17

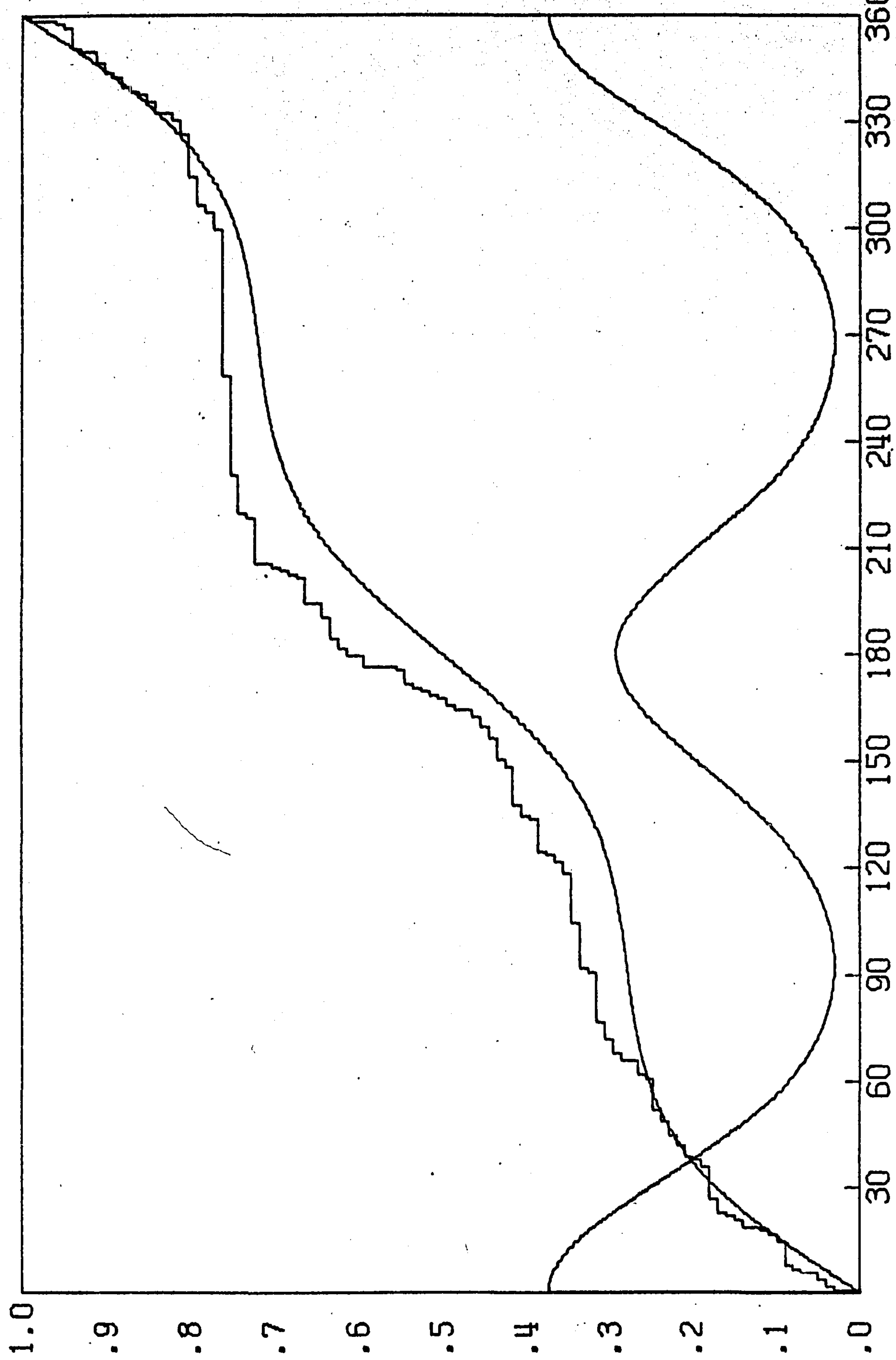
Pages 32,33 Table 3.3



B2 SAMPLE 3.3(a)

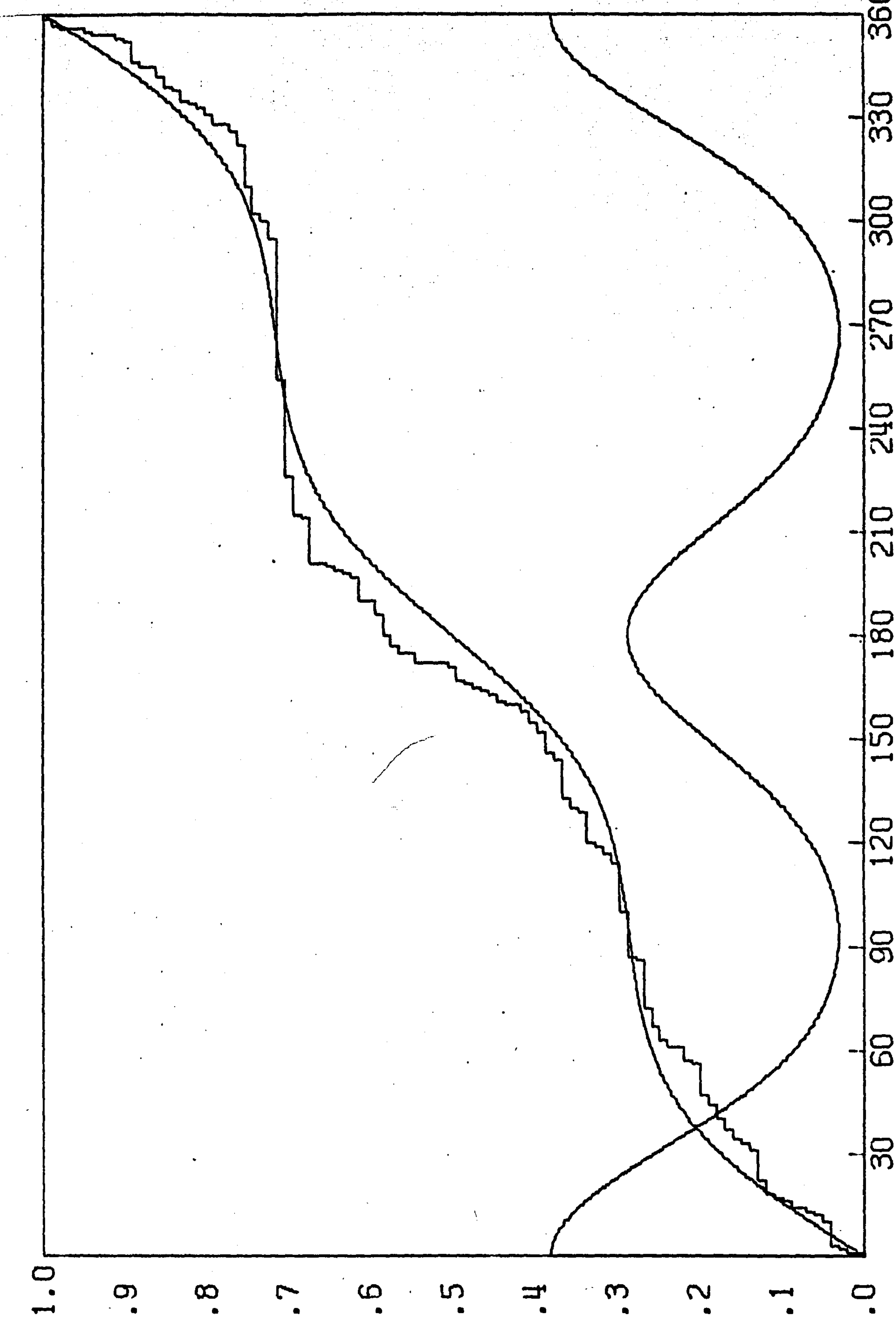
Plot 18

Pages 33,34 Table 3.3



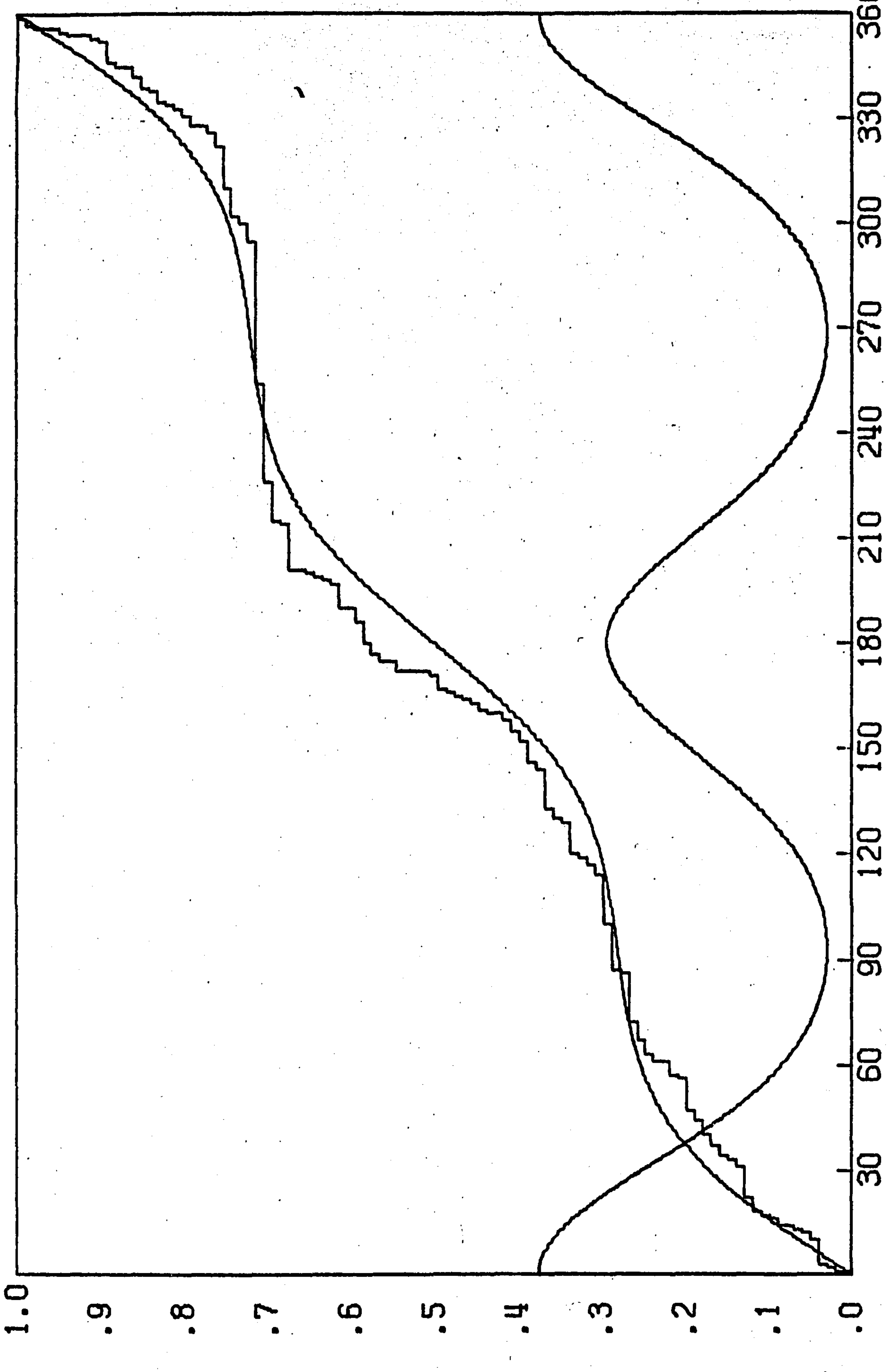
B2 SAMPLE 3.3(b) Plot 19

Pages 33,34 Table 3.3

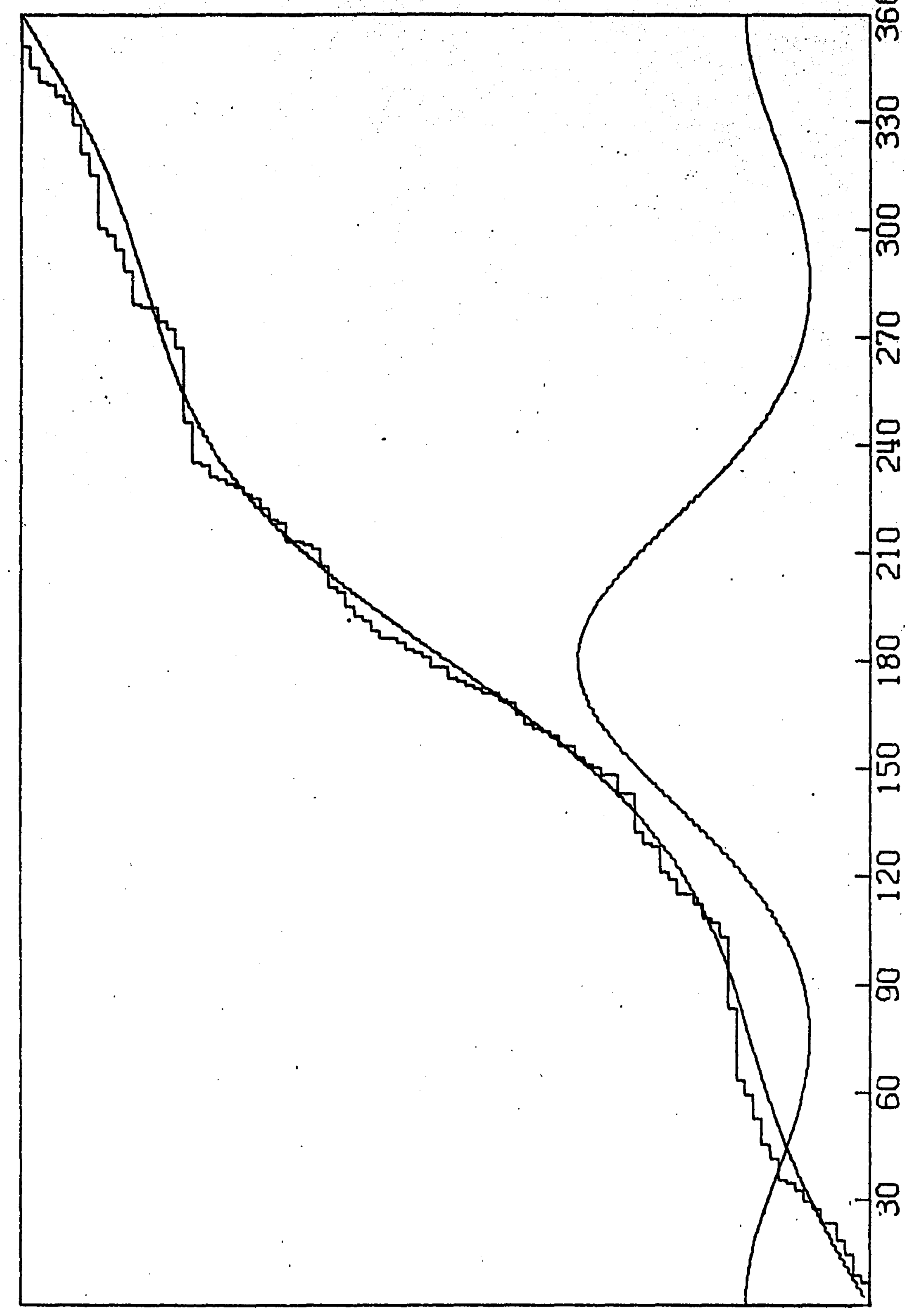


B2 SAMPLE 3.3(c) Plot 20

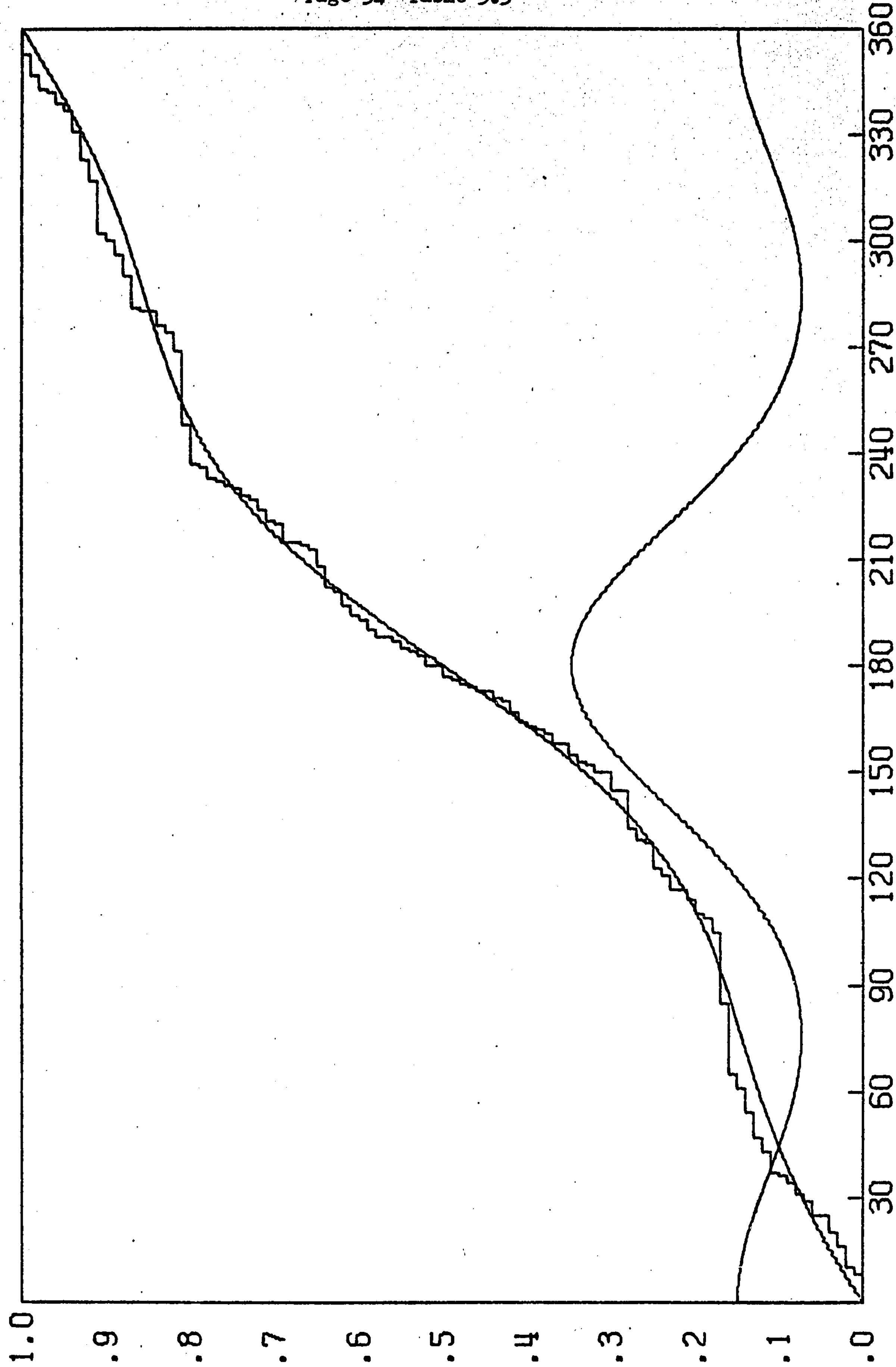
Pages 33,34 Table 3.3



Page 34 Table 3.3

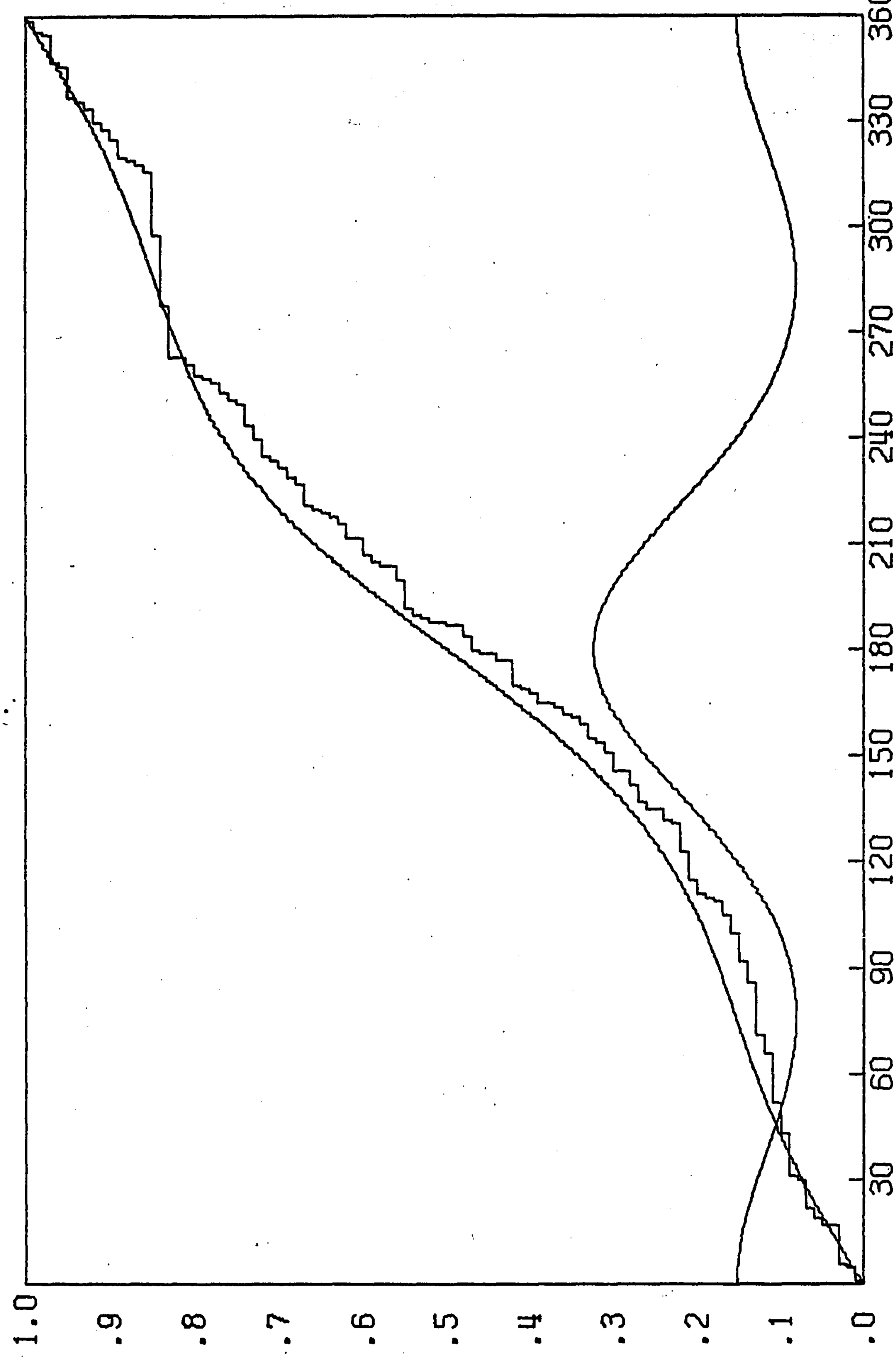


Page 34 Table 3.3



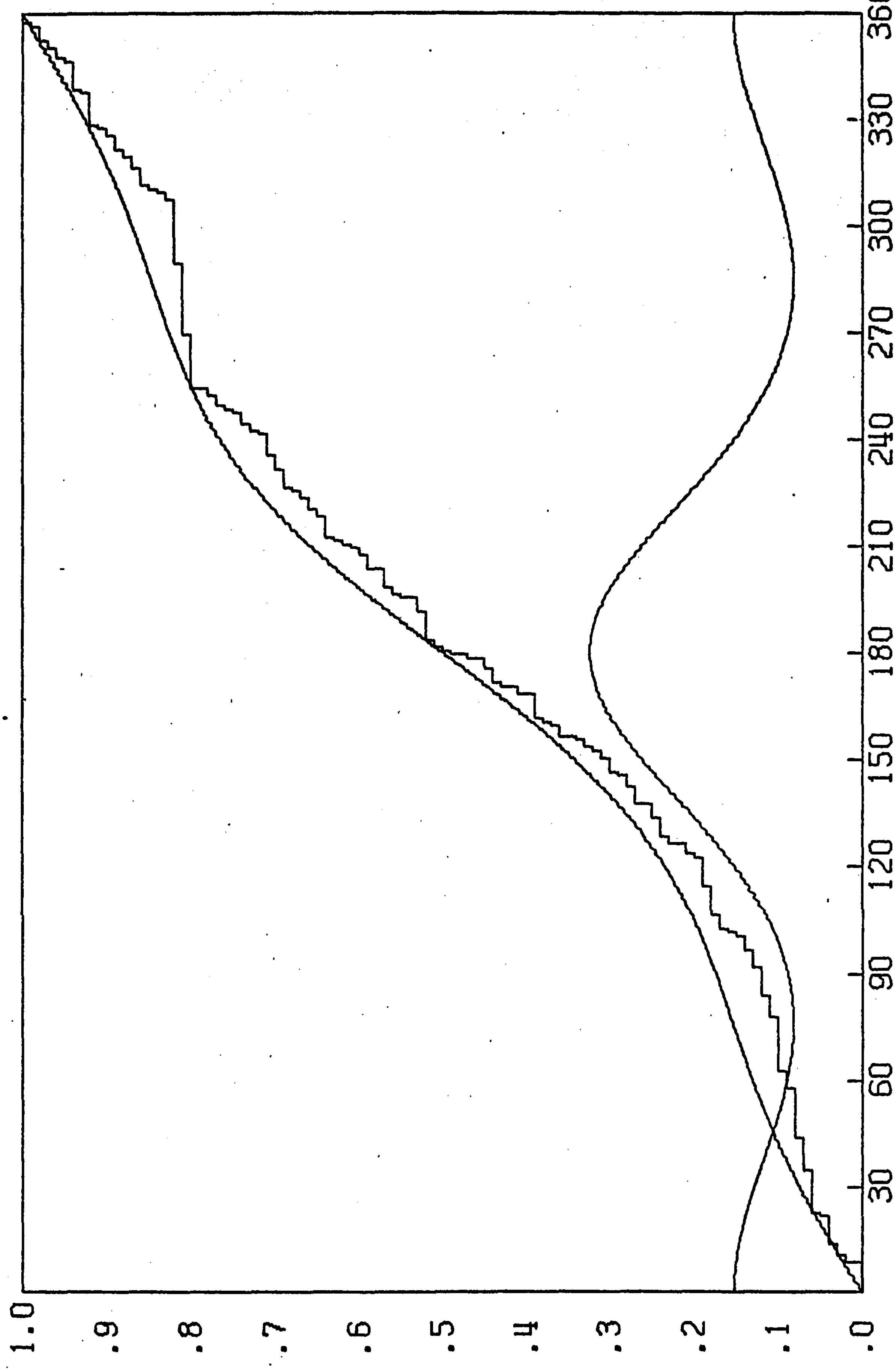
B2 SAMPLE 3.5(a)

Pages 34,35 Table 3.3



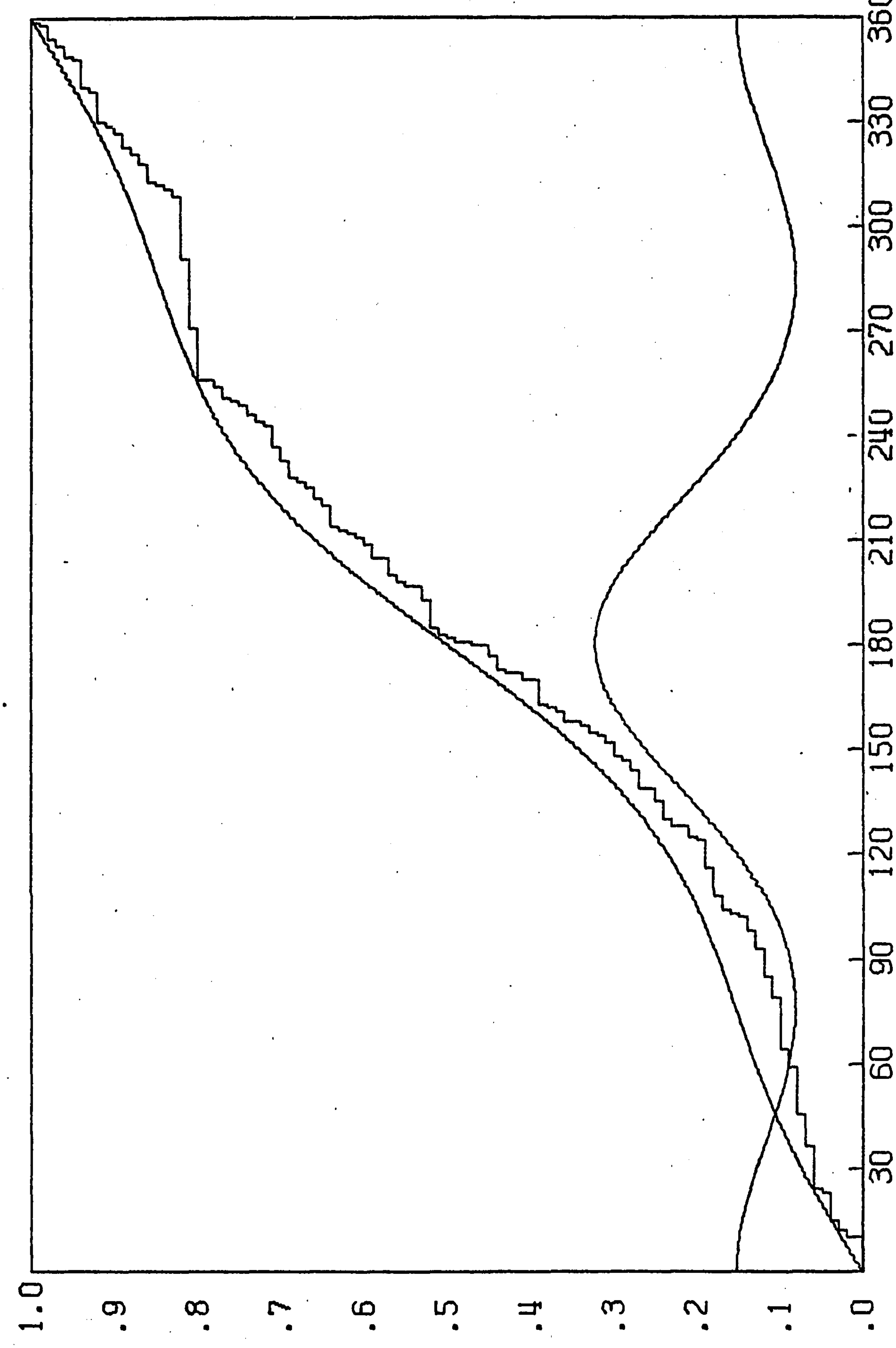
B2 SAMPLE 3.5(b)

Pages 34,35 Table 3.3



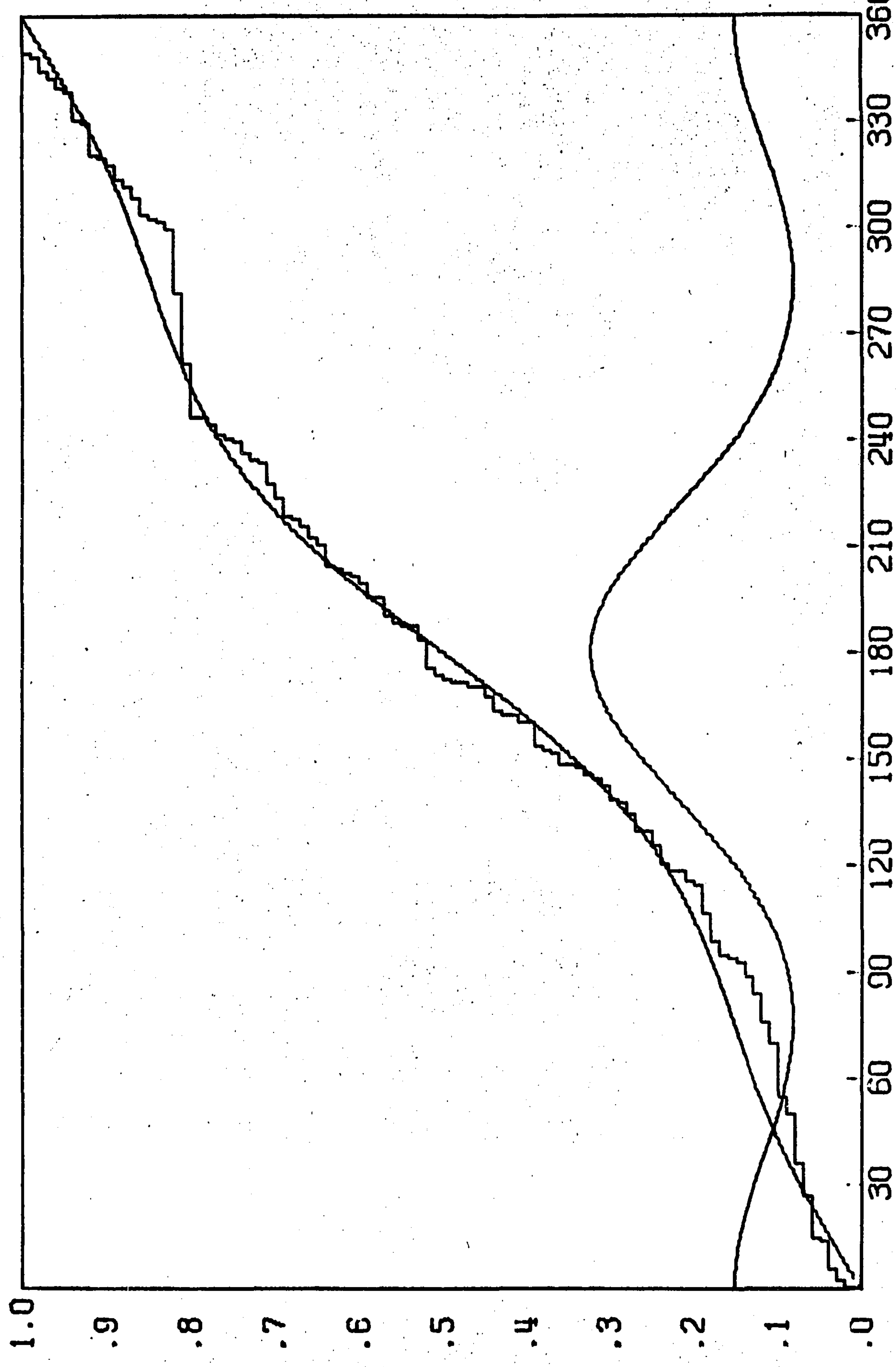
B2 SAMPLE 3.5(c) Plot 25

Pages 34,35 Table 3.3



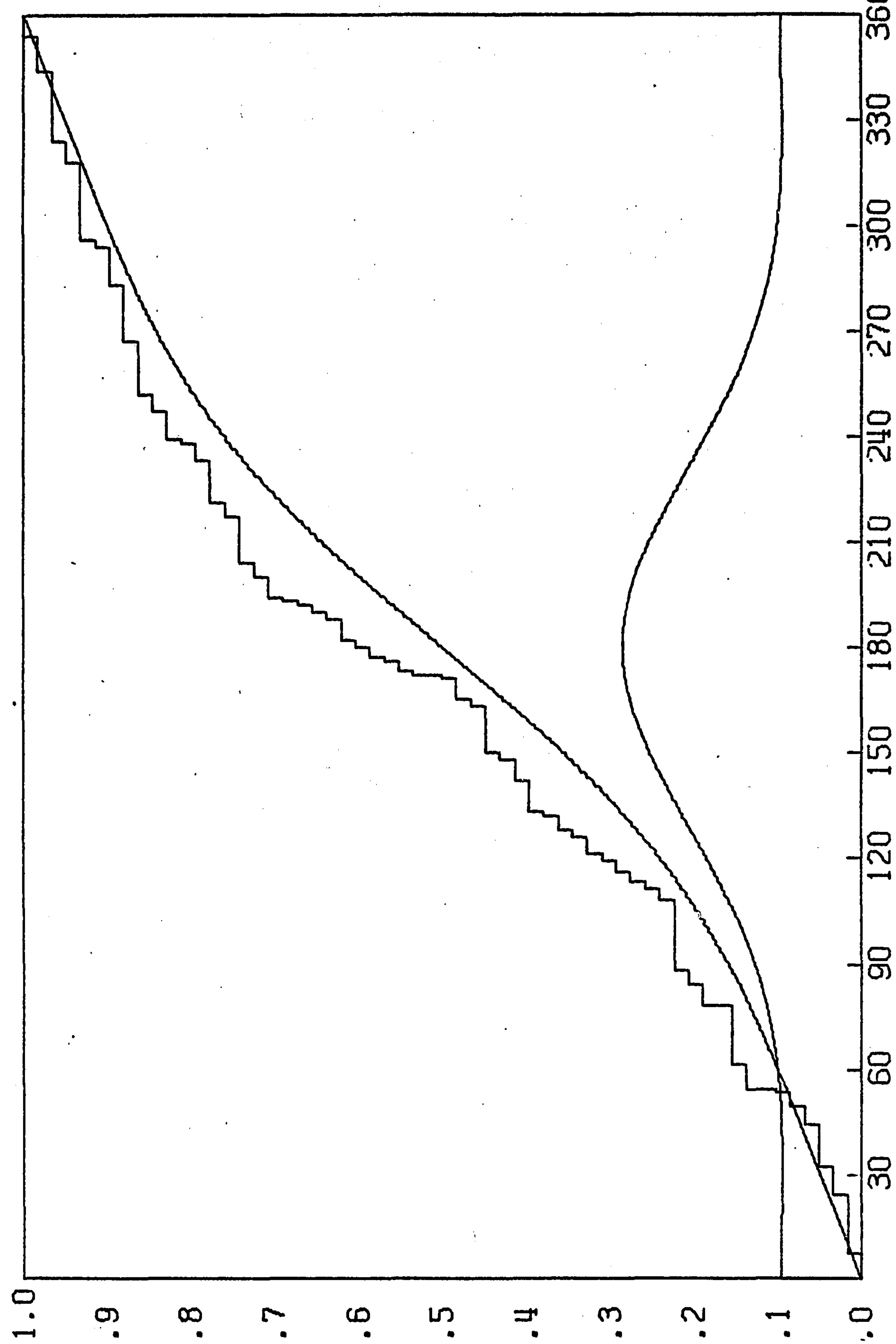
B2 SAMPLE 3.5(d)

Pages 34,35 Table 3.3



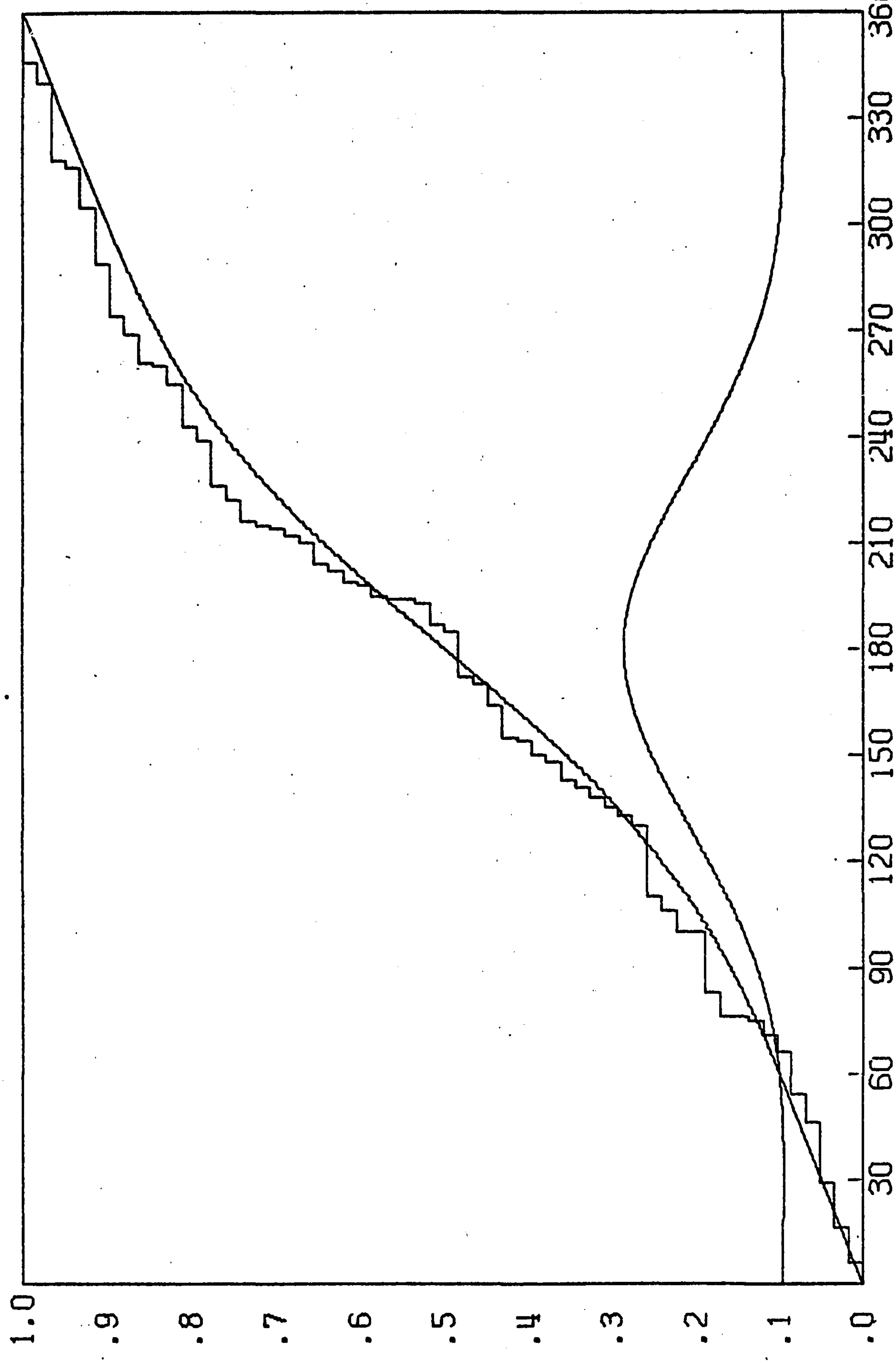
B2 SAMPLE 3.6(a) Plot 27

Pages 35,36 Table 3.3

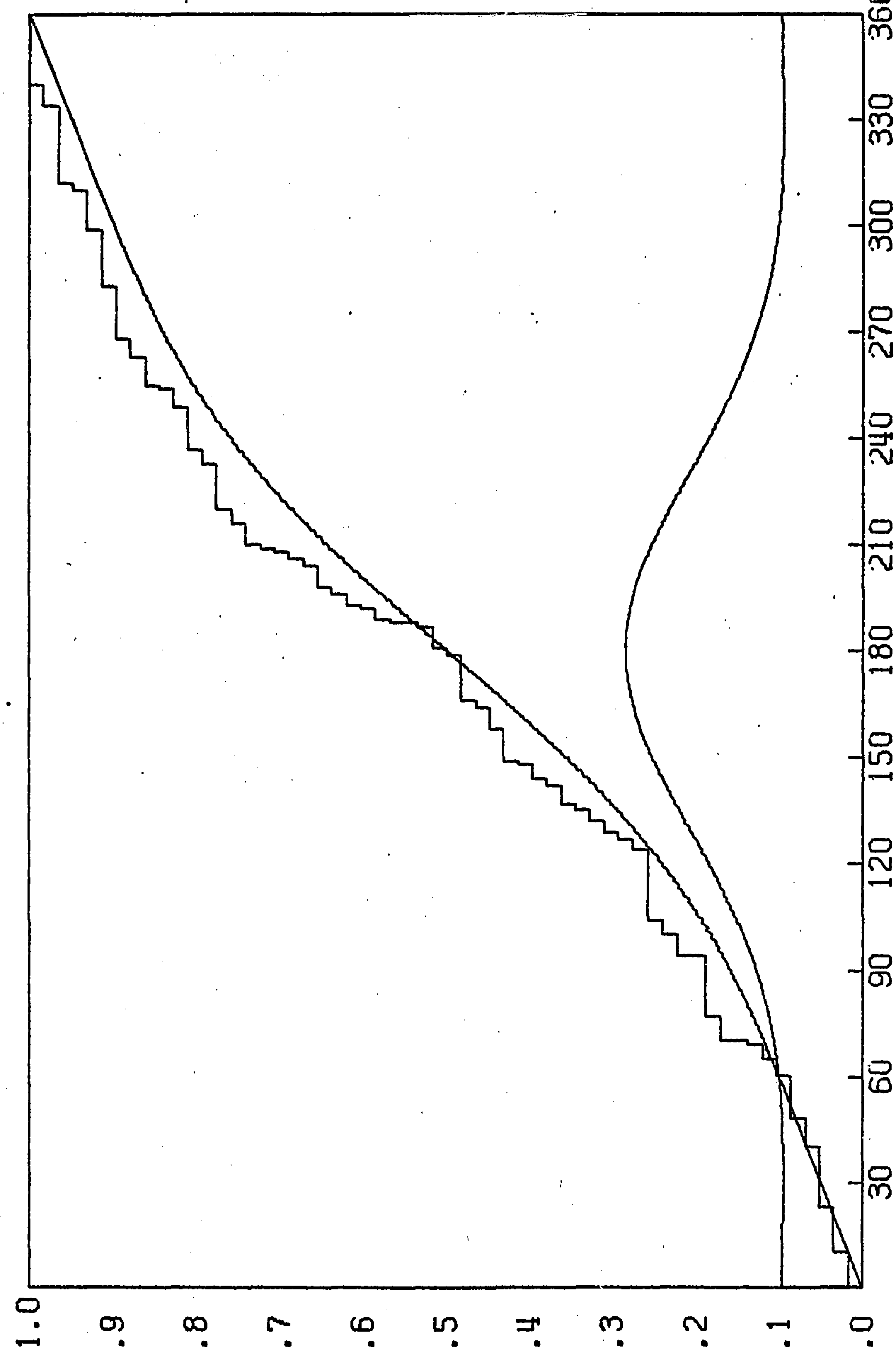


B2 SAMPLE 3.6(b) Plot 28

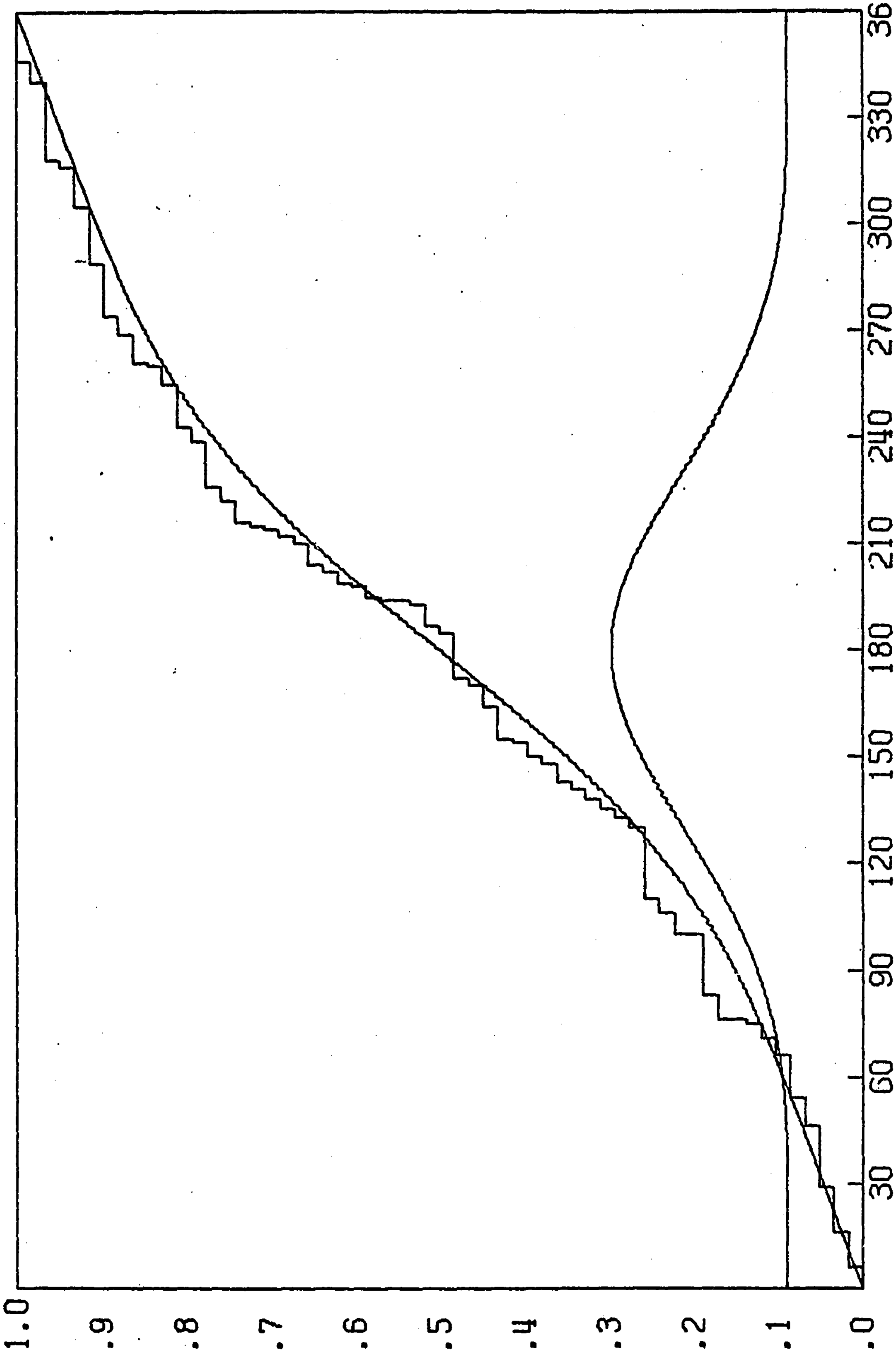
Pages 35,36 Table 3.3



Pages 35,36 Table 3.3

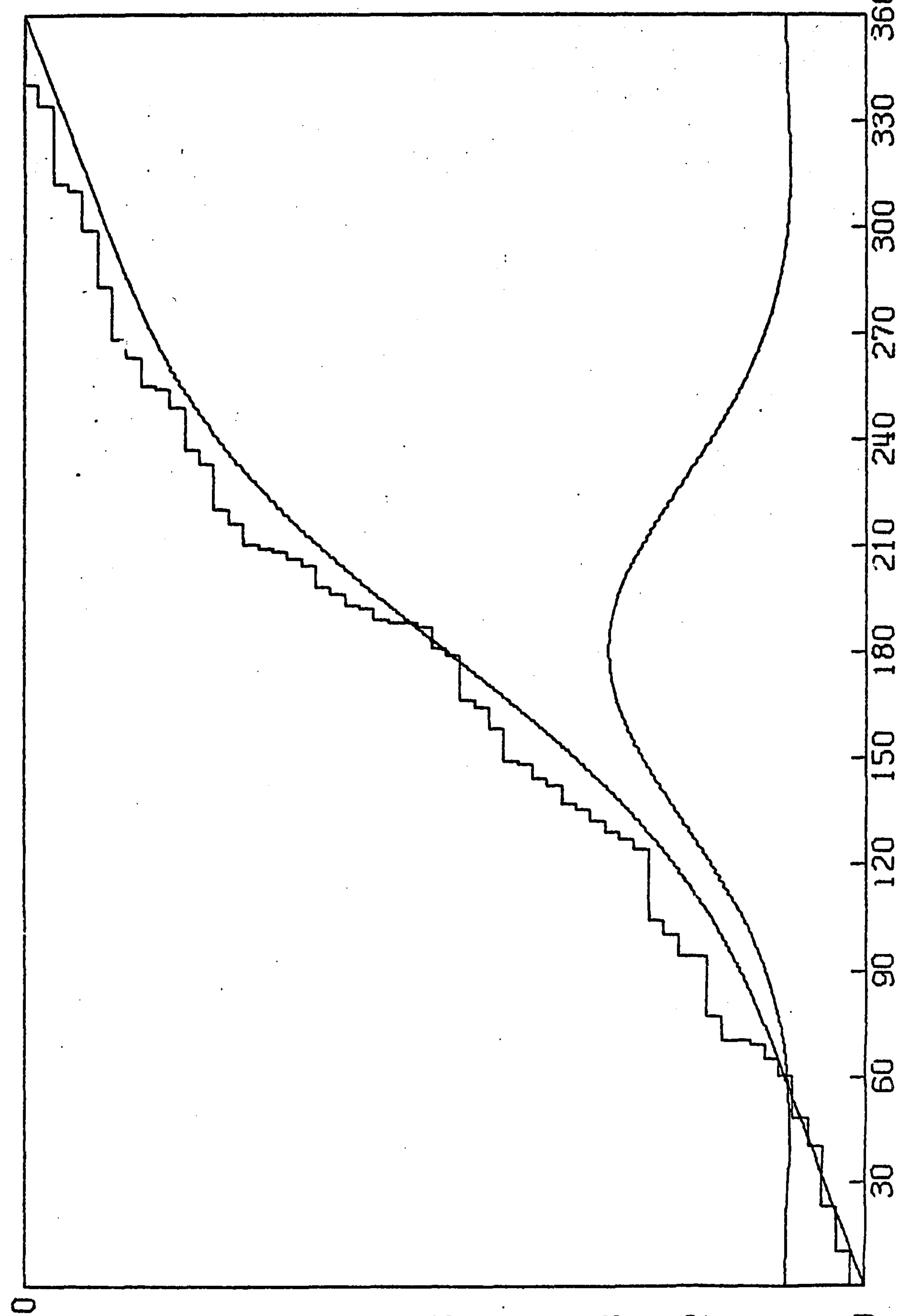


Pages 35,36 Table 3.3



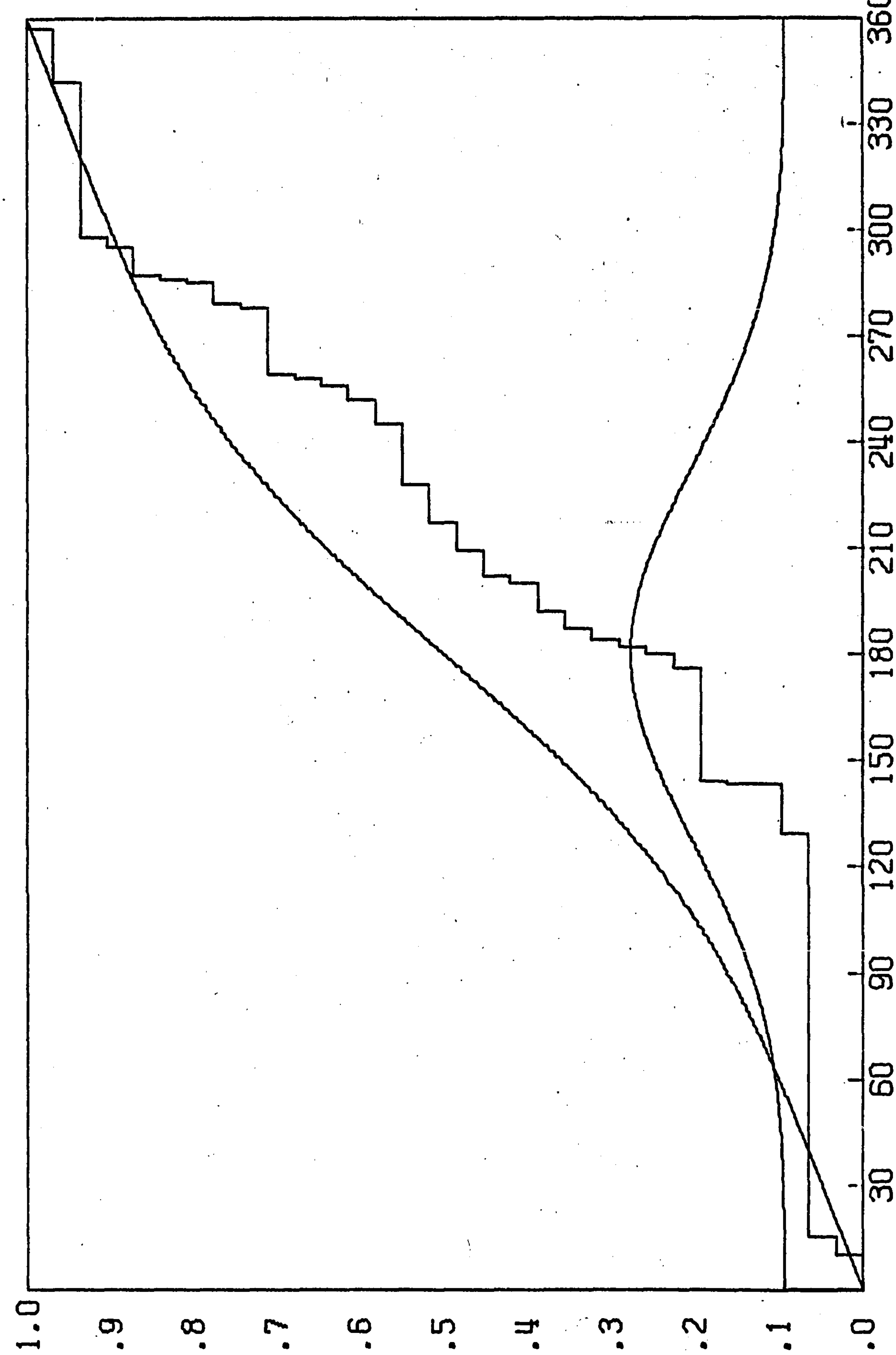
B2 SAMPLE 3.6(e) Plot 31

Pages 35,36 Table 3.3



B2 SAMPLE 3.7(a) Plot 32

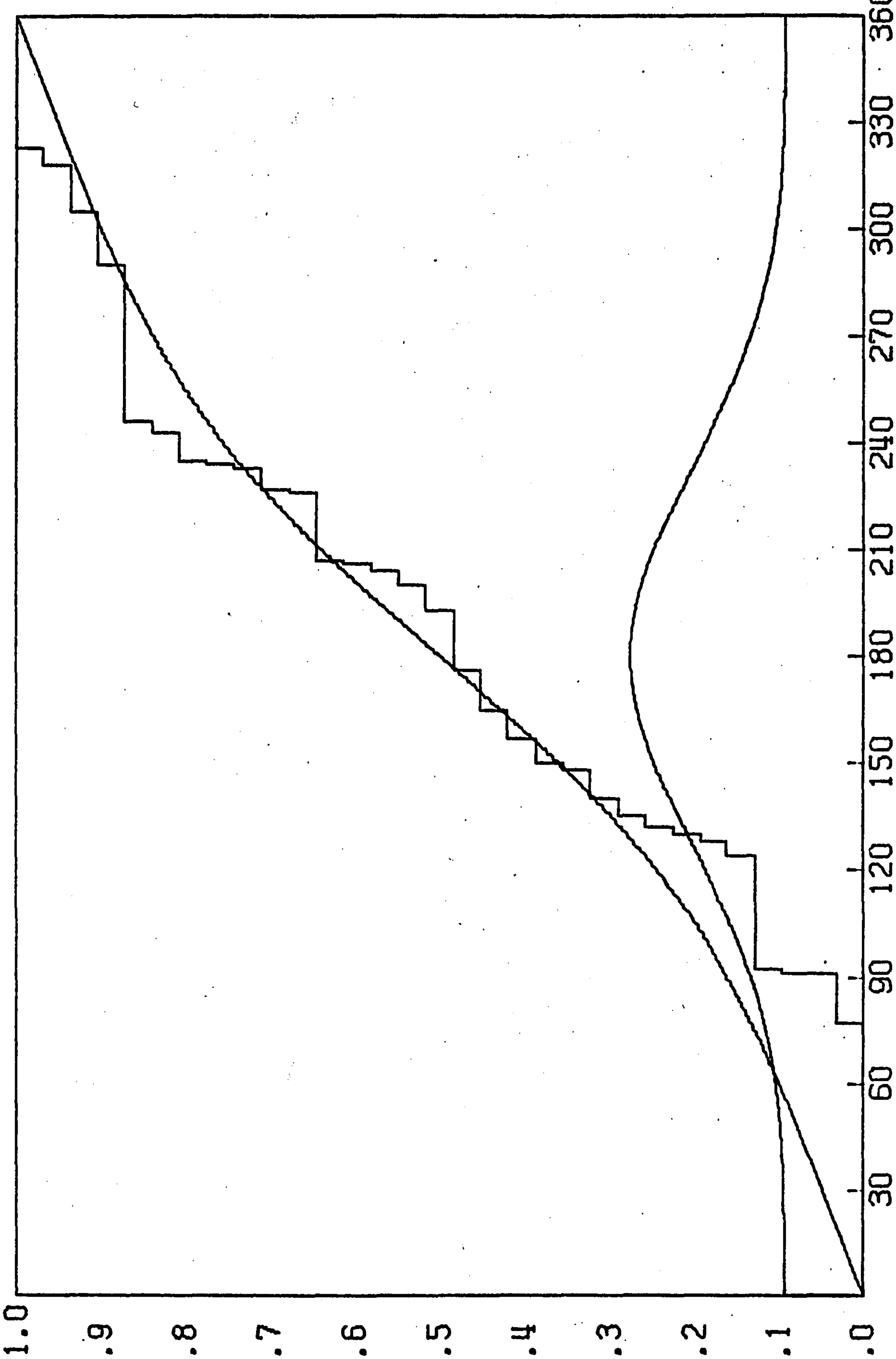
Pages 36.37 Table 3.3



B2 SAMPLE 3.7(b)

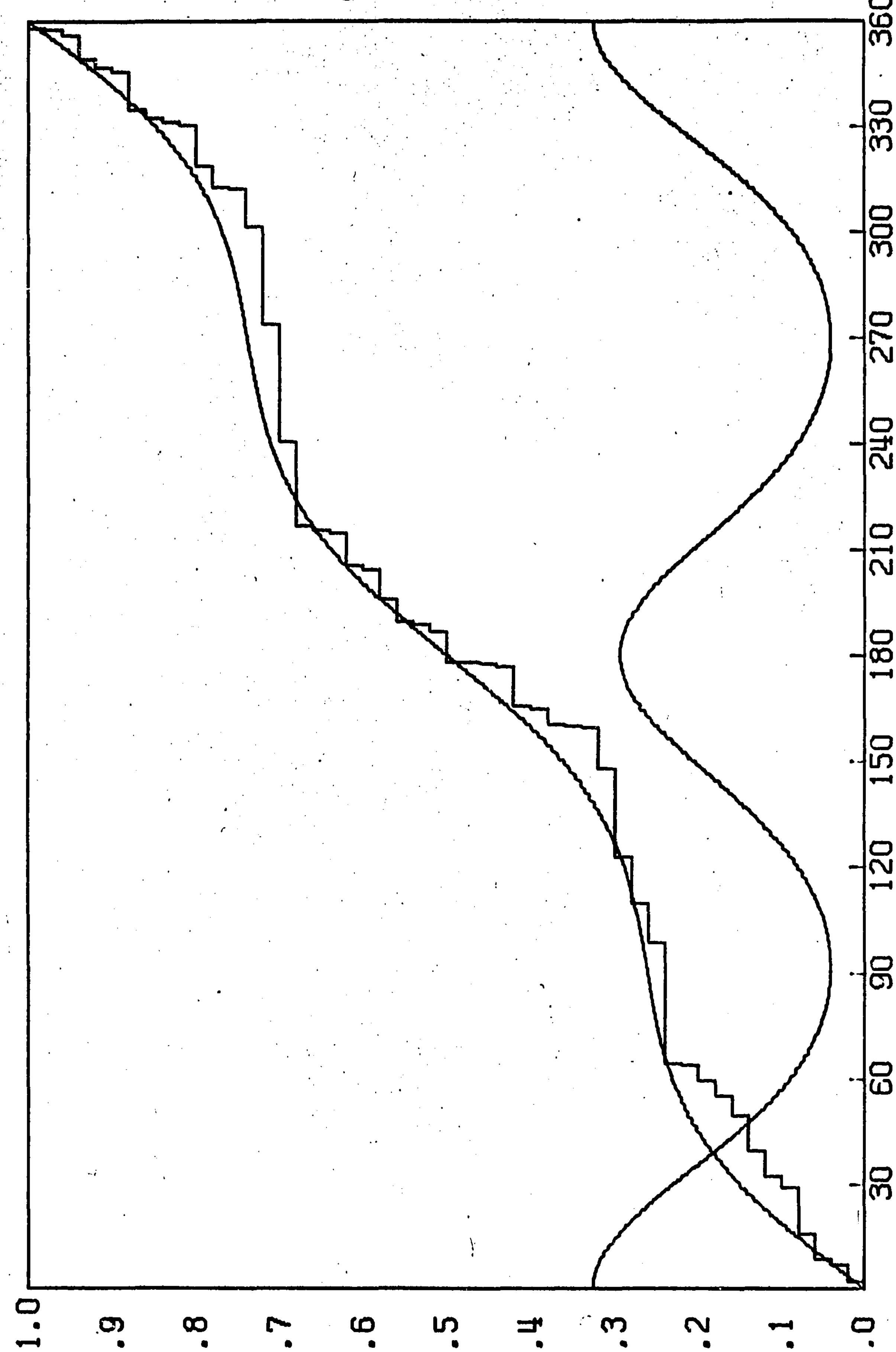
Plot 33

Pages 36,37 Table 3.3



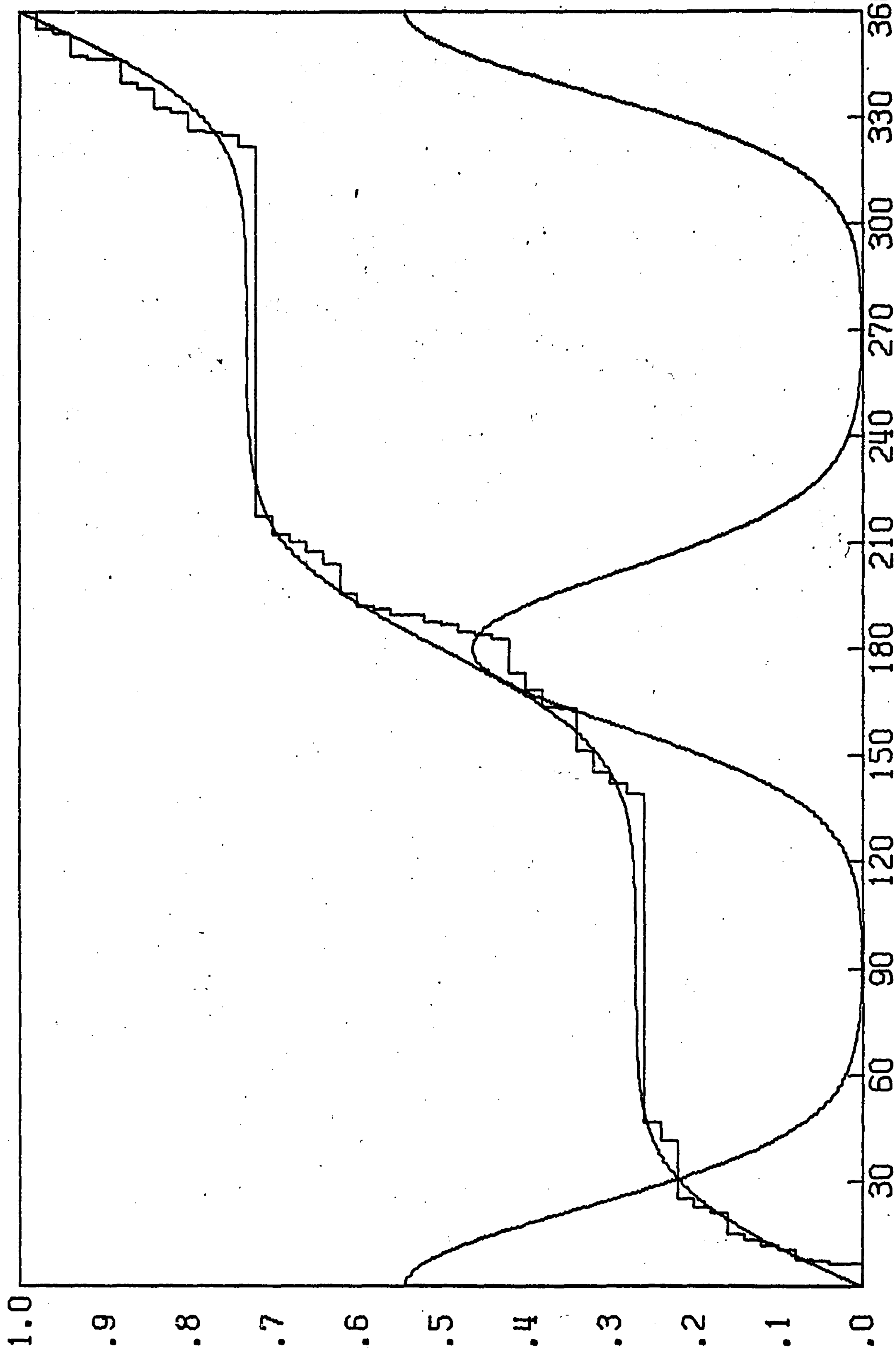
B1 SAMPLE 3.1 - B2 TECHNIQUES APPLIED Plot 34

Page 48 Table 4.5



B1 SAMPLE 3.2 - B2 TECHNIQUES APPLIED Plot 35

Page 48 Table 4.5



DATA FROM RESEARCH PAPERS

Sample 2.1 Page 16
Table 2.1

161	127	165	180
151	137	90	124
135	144	145	

Sample 2.2 Page 16
Table 2.1

-5	25	16	-11
-30	64	32	-33
14	38	49	

Sample 2.3 Page 17
Table 2.1

0	100	140	180	189	225	255	270
0	110	155	180	206	226	255	270
0	113	165	180	209	230	260	
15	135	165	180	210	235	260	
45	135	169	180	214	245	260	
68	140	180	180	215	250	260	

Sample 2.4 Page 17
Table 2.1

90	180	250	280
100	190	250	290
115	190	253	290
115	196	254	305
130	200	254	
135	205	255	
145	210	256	
160	225	261	
165	230	270	
170	245	277	

Sample 2.5 Page 18
Table 2.1

0	75	103	135	165	320
10	80	110	140	175	340
10	90	110	140	180	
30	90	115	140	180	
65	92	120	143	195	
65	95	125	160	195	
65	100	125	160	315	

B1 MONTE CARLO SAMPLES

B1 Sample 3.1 Page 30
Table 3.2

3.7182	56.9236	167.6164	197.1887	319.9494
5.4026	62.4757	168.2302	203.5842	320.6022
5.7003	66.9130	172.5338	212.1515	326.5602
9.6334	71.0545	173.1626	213.3146	338.0818
14.3539	71.5664	184.2424	222.4545	338.8897
16.0266	105.8336	185.3733	223.2531	340.0581
23.4053	117.2761	185.4736	224.3081	342.3268
36.4892	130.2958	185.6501	248.5739	353.1798
39.6699	155.5369	194.5911	281.8282	354.4481
46.7687	167.3148	196.4118	309.1716	356.9899

B1 Sample 3.2 Page 30
Table 3.2

1.1258	24.7522	155.6367	184.3212	223.7235
1.8580	27.4391	157.3156	184.88	316.4911
4.1698	29.5057	163.8848	187.6001	319.3234
4.7939	34.9770	163.9545	188.6241	322.2319
6.6450	139.2687	165.0068	190.2954	328.2415
6.8921	142.5484	171.2823	191.9801	340.5395
8.3440	143.3416	172.2442	198.0043	340.6692
9.1477	143.5618	177.0821	199.5030	345.4194
12.6692	148.7608	183.5740	202.0601	350.5906
21.1884	150.0679	183.6519	218.6103	359.8215

B1 Sample 3.3 Page 30
Table 3.2

.7572	24.2832	169.6922	184.2339	328.6209
1.5733	25.5807	171.4582	185.8403	338.4501
2.3712	28.5941	173.3347	188.0936	344.9509
3.3255	28.8413	177.8303	191.3320	345.7968
5.9681	47.0242	177.8832	193.3874	346.0253
7.6849	100.7673	178.3473	197.0109	354.8917
10.3887	162.0068	181.4334	199.9409	355.3690
13.0399	164.8599	181.4691	202.7441	356.5232
14.0063	168.5845	182.9752	321.3066	359.0003
18.9332	169.3744	184.1224	325.3334	359.2711

DATA FROM RESEARCH PAPERS

B2 Sample 3.1 Page 32	2	27	52	224	357
Table 3.3	2	29	56	227	
	12	32	60	246	
	12	34	77	295	
	12	42	131	312	
	20	42	165	344	
	22	43	182	347	
	26	45	193	350	
	26	46	206	352	
	26	50	213	355	

B2 Sample 3.2 Page 32,33	-156	8	48	65	95
Table 3.3	-145	9	48	68	96
	-137	13	48	70	98
	-134	13	48	73	100
	-123	14	50	78	103
	-122	18	53	78	106
	-117	22	56	78	113
	-116	27	57	83	118
	-110	30	58	83	138
	-109	34	58	88	148
	-103	38	61	88	153
	- 92	38	63	88	155
	- 75	40	64	90	
	- 41	44	64	92	
	- 17	45	64	92	
	- 10	47	65	93	

DATA FROM RESEARCH PAPERS

B2 Sample 3.3 Page 33
Table 3.3

8	134	158	242	309	357
36	135	159	256	313	
77	136	160	259	314	
82	136	164	261	314	
84	136	173	262	314	
92	137	175	271	314	
104	138	176	272	317	
108	138	179	275	317	
110	138	182	286	319	
110	139	186	288	322	
113	141	189	294	328	
115	143	198	297	332	
116	143	199	300	332	
117	144	203	302	339	
120	145	203	302	340	
121	152	205	303	341	
124	154	209	305	342	
127	155	214	306	343	
127	156	228	307	343	
128	156	229	308	356	

B2 Sample 3.4 Page 34
Table 3.3

0	104	205	255	286
4	109	207	257	292
4	109	214	257	297
5	113	215	258	298
14	115	218	259	299
20	118	229	260	299
24	120	229	261	304
26	121	234	264	305
41	127	234	264	308
47	131	236	267	311
55	138	237	268	312
61	145	239	269	314
63	149	242	271	315
66	169	242	272	316
67	189	245	272	317
71	193	246	274	320
77	194	247	277	321
92	198	248	278	322
94	201	251	281	353
100	201	254	285	358

DATA FROM RESEARCH PAPERS

B2 Sample 3.5 Page 34,35	13	76	192	236	277
Table 3.3	15	79	194	237	278
	16	87	199	241	284
	17	88	203	244	286
	22	100	203	244	289
	25	109	208	245	291
	27	123	211	245	292
	31	128	212	246	297
	33	143	216	247	301
	34	149	218	249	307
	43	157	219	257	308
	44	162	221	261	310
	52	166	222	261	313
	53	167	222	262	314
	56	168	225	264	315
	58	172	226	269	318
	62	180	227	269	320
	63	188	234	273	320
	74	189	234	275	335
	74	192	236	276	355

B2 Sample 3.6 Page 35,36	9	176	271	329
Table 3.3	13	184	273	332
CN Sample 2.6 Page 18	25	196	278	334
Table 2.1	30	201	280	340
	31	205	284	342
	39	206	285	344
	44	206	294	345
	59	213	300	346
	75	230	302	352
	86	230	315	356
	88	236	317	
	110	240	323	
	116	260	324	
	136	263	324	
	146	265	325	
	159	268	328	

DATA FROM RESEARCH PAPERS

B2 Sample 3.7 Page 36,37	0	43	252	317
Table 3.3	4	46	284	325
CN Sample 2.7 Page 18,19	6	90	288	336
Table 2.1	7	105	290	353
	26	118	292	
	27	123	295	
	33	237	300	
	34	251	308	
	35	251	310	

TABLE A-1 (see page 38)

K	V
.1	5.9940
.2	4.6151
.3	3.8165
.4	3.2582
.5	2.8335
.6	2.4947
.7	2.2163
.8	1.9827
.9	1.7840
1.0	1.6131
1.1	1.4650
1.2	1.3358
1.3	1.2225
1.4	1.1228
1.5	1.0346
1.6	.9564
1.7	.8869
1.8	.8249
1.9	.7694
2.0	.7197
2.1	.6751
2.2	.6348
2.3	.5985
2.4	.5656
2.5	.5358
2.6	.5086
2.7	.4839
2.8	.4612
2.9	.4405
3.0	.4215
3.1	.4040
3.2	.3878
3.3	.3728
3.4	.3590
3.5	.3461
3.6	.3341
3.7	.3229
3.8	.3125
3.9	.3027
4.0	.2935
4.1	.2848
4.2	.2767
4.3	.2690
4.4	.2618
4.5	.2549
4.6	.2484
4.7	.2422
4.8	.2364
4.9	.2308
5.0	.2255
5.1	.2204
5.2	.2156
5.3	.2109
5.4	.2065
5.5	.2023
5.6	.1982
5.7	.1943
5.8	.1905
5.9	.1869
6.0	.1834
6.1	.1801
6.2	.1769
6.3	.1738
6.4	.1708
6.5	.1679
6.6	.1651
6.7	.1624
6.8	.1598
6.9	.1572
7.0	.1548
7.1	.1524
7.2	.1501
7.3	.1479
7.4	.1457
7.5	.1436
7.6	.1415
7.7	.1396
7.8	.1376
7.9	.1357
8.0	.1339
8.1	.1321
8.2	.1304
8.3	.1287
8.4	.1271
8.5	.1255
8.6	.1239
8.7	.1224
8.8	.1209
8.9	.1195
9.0	.1180
9.1	.1167
9.2	.1153
9.3	.1140
9.4	.1127
9.5	.1114
9.6	.1102
9.7	.1090
9.8	.1078
9.9	.1067
10.0	.1055
11.0	.0954
12.0	.0871
13.0	.0801
14.0	.0742
15.0	.0690
16.0	.0646
17.0	.0606
18.0	.0572
19.0	.0541
20.0	.0513

TABLE A-2

U^2_n Lower Tail Significance Points

N	Significance levels expressed as percentages: 15	10	5	2.5	1	.5
7	.0395	.0348	.0295	.0257	.0220	.0200
8	.0392	.0344	.0291	.0254	.0218	.0198
9	.0389	.0341	.0288	.0252	.0216	.0196
10	.0387	.0339	.0286	.0250	.0215	.0195
15	.0384	.0336	.0282	.0245	.0209	.0189
20	.0382	.0335	.0280	.0242	.0206	.0185
25	.0382	.0334	.0279	.0240	.0204	.0184
30	.0381	.0334	.0278	.0239	.0203	.0182
35	.0381	.0333	.0277	.0239	.0202	.0181
40	.0380	.0333	.0277	.0238	.0201	.0181
60	.0379	.0332	.0276	.0237	.0200	.0179
80	.0379	.0332	.0275	.0236	.0199	.0178
100	.0379	.0332	.0275	.0236	.0199	.0178

TABLE A-3

$\sqrt{n}\, V_n$ Lower Tail Significance Points

	Significance levels expressed as percentages					
N	15	10	5	2.5	1	.5
15	.905	.857	.791	.742	.691	.658
20	.913	.864	.798	.749	.696	.662
25	.917	.870	.802	.754	.702	.665
30	.923	.874	.807	.757	.703	.669
35	.926	.877	.810	.760	.706	.672
40	.929	.880	.813	.763	.709	.675
45	.932	.883	.815	.765	.712	.678
50	.934	.885	.817	.766	.714	.681
55	.936	.887	.819	.768	.716	.683
60	.937	.889	.821	.769	.717	.684
65	.939	.890	.822	.771	.719	.686
70	.940	.891	.823	.772	.720	.687
75	.941	.892	.824	.773	.721	.688
80	.942	.894	.826	.773	.722	.689
85	.943	.895	.827	.774	.723	.690
90	.944	.896	.828	.775	.724	.691
95	.944	.896	.828	.776	.724	.691
100	.945	.897	.829	.777	.725	.692
∞	.973	.9275	.8613	.8095	.7550	.7212

TABLE A-4

$\sqrt{n}\, V_n$ Upper Tail Significance Points

N	Significance levels expressed as percentages 15	10	5	2.5	1	.5
15	1.448	1.532	1.650	1.760	1.887	1.978
20	1.460	1.546	1.665	1.776	1.908	1.998
25	1.469	1.556	1.673	1.789	1.922	2.010
30	1.476	1.562	1.684	1.797	1.930	2.022
35	1.481	1.567	1.690	1.803	1.936	2.029
40	1.484	1.571	1.695	1.808	1.941	2.034
45	1.487	1.574	1.698	1.812	1.945	2.038
50	1.490	1.576	1.701	1.815	1.949	2.042
55	1.492	1.579	1.703	1.818	1.952	2.045
60	1.494	1.582	1.705	1.820	1.955	2.047
65	1.496	1.584	1.706	1.822	1.957	2.049
70	1.497	1.585	1.707	1.824	1.959	2.051
75	1.499	1.587	1.709	1.825	1.961	2.053
80	1.500	1.588	1.711	1.826	1.962	2.055
85	1.501	1.589	1.712	1.827	1.963	2.056
90	1.503	1.589	1.714	1.829	1.965	2.058
95	1.504	1.590	1.715	1.830	1.966	2.059
100	1.505	1.590	1.716	1.831	1.967	2.060
∞	1.537	1.620	1.747	1.862	2.001	2.098