INDUCED REPRESENTATIONS
OF LIE ALGEBRAS

by

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TABLE OF CONTENTS . . . .

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B.N.

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PREFACE . . . .

The representations of Lie algebras have been considered extensively, in the bibliography are listed references most pertinent to this paper. In its treatment of the representation of Lie algebras, this paper differs from previous investigations in the particular stress it places upon the interrelation of the representation of the Lie algebra and the corresponding representation it induces on an ideal.

This point of view leads to quite explicit forms for certain representations of Lie algebras which the author believes to be original. In particular, in chapter I the concept of matrices of invariance is developed for the representation of an ideal. This concept permits us in chapter II to show that irreducible representations of a Lie algebra, in an algebraically closed field, can be expressed as a certain product whose factors are associated with the representation induced on an ideal. Conversely, if one has such factors, it is shown that they can be put together to produce an irreducible representation of the Lie algebra. A valuable guide to this work was supplied by a paper of Clifford[5)*]. In chapter III, using the Birkhoff imbedding procedure[2)], a construction is given whereby an explicit representation of a Lie algebra can be generated, in an algebraically closed field of zero characteristic, from certain representations

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* The number in the bracket refers to the bibliography.

of an ideal in the radical. The degree of the representation can be given. Furthermore, the construction is sufficiently general to give representations which include as components the indecomposable components of any representation of finite degree. The theorem of Ado is proved as an application of the construction. While this theorem has several proofs, the present one has a value in its explicitness and in the fact that the degree of the representation can be given.

All algebras, modules, and representations in this paper are to be taken over a field F, if the field is not specifically mentioned.

CHAPTER I

MATRICES OF INVARIANCE

1. Preliminary concepts. A Lie algebra L is an F-module in which there is defined a unique product aob, $(a,b,aob \in L)$, such that

(i) $k(aob) = (ka)ob = ao(kb)$, $(k \in F)$

(ii) $ao(b + c) = aob + aoc$, $(b + c)oa = boa + coa$, $(c \in L)$

(iii) $aoa = 0$,

(iv) $ao(boc) + bo(coa) + co(aob) = 0$, (the Jacobi identity).

From (ii) and (iii) we obtain the anti-commutative law,

$$aob + boa = (a + b)o(a + b) - (aoa) - (bob) = 0.$$

An F-module T, contained in a Lie algebra L, is called an ideal of L if $LoT \subseteq T$. By the anti-commutative law, $ToL \subseteq T$. Thus any ideal is two sided. Clearly, T is also a subalgebra.

If H is any associative hypercomplex algebra, then one can replace the given product ab in H by the product aob defined by $aob = ab - ba$. It is easily verified that statements (i) to (iv) above are satisfied by this 'o' multiplication. Hence H, and every module in H closed under 'o' multiplication, form a Lie algebra with respect to 'o' multiplication

A mapping $a \rightarrow a'$, where a is any element of a Lie algebra L and a' is its unique image in a Lie algebra L' is called a <u>homomorphism</u> of L in L' if

$$a + b \rightarrow a' + b',$$

$$ka \rightarrow ka', \qquad (k \in F, \text{ the field of reference})$$

$$a\circ b \rightarrow a'\circ b',$$

The images form a Lie algebra in L'. If the correspondence is one-to-one, the mapping is called an <u>isomorphism</u>.

The homomorphic mapping of L into an associative algebra H where $a\circ b \rightarrow a'\circ b' = a'b' - b'a'$ is called a <u>representation</u> of L. This representation is <u>faithful</u> if the mapping is an isomorphism.

An F-module M is called a <u>representation module</u>, or an <u>L-F-module</u>, of a Lie algebra L, if there is defined a unique product au, $(a \in L,\ u \in M)$, such that au occurs in M and

$$a(u + v) = au + av, \quad (a + b)v = av + bv, \qquad (v \in M,\ b \in L)$$

$$(ka)u = a(ku) = k(au), \qquad (k \in F)$$

$$(a\circ b)u = a(bu) - b(au).$$

Such a module assigns to each element a of L a unique linear transformation $\underline{A}(a)$ of M, defined by $au = \underline{A}(a)u$, and the correspondence $a \rightarrow \underline{A}(a)$ is a representation of L by linear transformations of M. If M has the basis $u_1, u_2, \dots, u_r$, then $\underline{A}(a)$ can be associated with the matrix $A(a) = (a_{ij})$, given by the equations

$$\underline{A}(a)u_j = \sum_{i=1}^{r} a_{ij}u_i, \quad j = 1, 2, \dots, r.$$

This association is an isomorphism, consequently the correspondence $a \rightarrow A(a)$ is a representation of L by matrices. Conversely, a representation of L determines a representation module of L by taking an F-module with a suitable number of basis elements, r say, and defining

$$au_j = \underline{A}(a)u_j = \sum_{i=1}^{r} a_{ij}u_i, \quad j = 1,2,\ldots,r.$$

where $a \rightarrow (a_{ij})$.

An F-module m, contained in a representation module M of L, is called an <u>invariant submodule</u> when $Lm \subseteq m$, i.e. when m is also a representation module. The <u>factor module</u> M/m is also a representation module for which $a(u + m) = au + m$. The usual representation properties of equivalence, irreducibility, and the various kinds of reducibility can now be expressed for Lie algebras in module terms.

2. <u>Matrices of invariance</u>. Let an ideal T, of a Lie algebra L, have the representation Q, i.e. $t \rightarrow Q(t)$ is a representation of T by matrices. If there is a matrix C(a), corresponding to an element a of L, such that

$$C(a) \circ Q(t) = Q(a \circ t),$$

for all elements t in T, then we shall call C(a) a <u>matrix of invariance</u>. If to every element of L there corresponds a matrix of invariance, then Q will be called <u>invariant under L</u>.

<u>Theorem 1.1</u> For an algebraically closed field of reference F, the matrices of invariance , of an irreducible representation Q of an ideal T, corresponding to a particular element a of the Lie algebra L, differ only by multiples of the unit matrix.

Proof. Let $C(a)$ and $C'(a)$ be two matrices of invariance for a, then we have

$$C(a)oQ(t) = Q(aot), \qquad (t \in T)$$

$$C'(a)oQ(t) = Q(aot).$$

Thus

$$C(a)oQ(t) - C'(a)oQ(t) = 0,$$

$$(C(a) - C'(a))oQ(t) = 0,$$

$$(C(a) - C'(a))Q(t) - Q(t)(C(a) - C'(a)) = 0,$$

$$(C(a) - C'(a))Q(t) = Q(t)(C(a) - C'(a)),$$

for all $t \in T$. Since $Q(t)$ is irreducible, Schur's lemma* gives

$$C(a) - C'(a) = c(a)I,$$

where $c(a) \in F$, and I is the unit matrix of the dimensions of $C(a)$.

Theorem 1.2. Let Q be an irreducible representation of an ideal T of a Lie algebra L. Let $e_1, e_2, \ldots, e_r$ be a basis of T, and $e_1, e_2, \ldots, e_r, e_{r+1}, \ldots, e_n$, be a basis of L, with respect to an algebraically closed field F. If $e_{r+1}, \ldots, e_n$ possess matrices of invariance, then each element a of L can be assigned a unique matrix of invariance $C(a)$.

Proof. From the matrices of invariance of $e_{r+1}, \ldots, e_n$, select any particular set $C(e_{r+1})$, $C(e_{r+2}), \ldots,$ $C(e_n)$. Then we define $C(e_1)$, $C(e_2)$, ..., $C(e_r)$, by the equations

$$C(e_i) = Q(e_i), \quad i = 1,2,\ldots,r.$$

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* In an algebraically closed field, the only matrices commuting with an irreducible set of matrices are scalar multiples of the unit matrix.

For $a \in L$, we have

$$a = \sum_{i=1}^{n} k_i e_i, \qquad (k_i \in F)$$

thus $C(a)$ can now be constructed by setting

$$C(a) = \sum_{i=1}^{n} k_i C(e_i),$$

and

$$\begin{aligned} C(a)oQ(t) &= \sum_{i=1}^{n} k_i C(e_i)oQ(t), \\ &= \sum_{i=1}^{n} k_i Q(e_i ot), \\ &= \sum_{i=1}^{n} Q((k_i e_i)ot), \\ &= Q((\sum_{i=1}^{n} k_i e_i)ot) = Q(aot), \text{ as required.} \end{aligned}$$

The uniqueness of $C(a)$ follows from the fact that $e_1, e_2, \ldots, e_n$, is a basis of L.

<u>Corollary</u>. The matrices of invariance of the theorem have the properties

$$C(a + b) = C(a) + C(b), \qquad (a, b \in L)$$

$$C(ka) = kC(a), \qquad (k \in F)$$

$$C(t) = Q(t).$$

<u>Proof</u>. Taking $a = \sum_{i=1}^{n} k_i e_i$, $b = \sum_{i=1}^{n} k_i' e_i$, we have

$$a + b = \sum_{i=1}^{n} k_i e_i + \sum_{i=1}^{n} k_i' e_i = \sum_{i=1}^{n} (k_i + k_i') e_i.$$

Therefore

$$\begin{aligned} C(a + b) &= \sum_{i=1}^{n} (k_i + k_i') C(e_i), \\ &= \sum_{i=1}^{n} k_i C(e_i) + \sum_{i=1}^{n} k_i' C(e_i), \\ &= C(a) + C(b). \end{aligned}$$

Also

$$ka = k \sum_{i=1}^{n} k_i e_i = \sum_{i=1}^{n} (kk_i) e_i, \text{ hence}$$

$$C(ka) = \sum_{i=1}^{n}(kk_i)C(e_i) = k\sum_{i=1}^{n}k_iC(e_i) = kC(a).$$

Finally

$$t = \sum_{i=1}^{r}k_i''e_i, \qquad (t \in T)$$

so that $$C(t) = \sum_{i=1}^{r}k_i''Q(e_i) = Q(\sum_{i=1}^{r}k_i''e_i) = Q(t).$$

3. Factor sets.

Theorem 1.3. Let T be an ideal of a Lie algebra L. Let Q be an irreducible representation of T in an algebraically closed field F, invariant under L. If matrices of invariance $C(a)$, $a \in L$, are so chosen that $C(a + b) = C(a) + C(b)$, $b \in L$, $C(ka) = kC(a)$, $k \in F$, and $C(t) = Q(t)$, $t \in T$, then

$$C(aob) = C(a)oC(b) + c(a,b)I,$$

where $c(a,b) \in F$, and I is the unit matrix of the dimensions of $C(a)$.

Proof. By the Jacobi identity,

$$(aob)ot = ao(bot) - bo(aot).$$

Since T is an ideal, $(aob)ot, aot, bot$, are elements of T; we have therefore

$$Q((aob)ot) = Q(ao(bot)) - Q(bo(aot)),$$

$$C(aob)oQ(t) = C(a)oQ(bot) - C(b)oQ(aot),$$

$$= C(a)o(C(b)oQ(t)) - C(b)o(C(a)oQ(t)),$$

$$= (C(a)oC(b))oQ(t),$$

since matrices, with respect to 'o' multiplication, satisfy the Jacobi identity. Thus

$$(C(aob) - (C(a)oC(b)))oQ(t) = 0,$$

or $$(C(aob) - (C(a)oC(b)))Q(t) = Q(t)(C(aob) - (C(a)oC(b))),$$

on replacing 'o' multiplication by ordinary multiplication.

By Schur's lemma,

$$C(aob) = C(a)oC(b) + c(a,b)I,$$

where $c(a,b) \in F$, and I is the unit matrix of the dimensions of $C(a)$.

Corollary. The scalars $c(a,b)$ of the theorem have the properties

(i) $\quad c(a,t) = 0, \qquad (a \in L,\ t \in T)$

(ii) $\quad c(a,a) = 0,$

(iii) $\quad c(a,b) = - c(b,a), \qquad (b \in L)$

(iv) $\quad c(a,bod) + c(b,doa) + c(d,aob) = 0, \qquad (d \in L)$

(v) $\quad c(a + s,b + t) = c(a,b) \qquad (s \in T)$

(vi) $\quad c(a + b,d) = c(a,d) + c(b,d),$

(vii) $\quad c(a,b + d) = c(a,b) + c(a,d),$

(viii) $\quad c(ka,b) = c(a,kb) = kc(a,b).$

Proof. $C(aot) = C(a)oQ(t) + c(a,t)I,$

or $\quad Q(aot) = Q(aot) + c(a,t)I,$

$$0 = c(a,t)I,$$

and so $\quad c(a,t) = 0$, giving (i).

$$C(aoa) = C(a)oC(a) + c(a,a)I,$$

$$Q(0) = 0 + c(a,a)I,$$

thus $\quad c(a,a) = 0$, giving (ii).

$$C(aob) = C(a)oC(b) + c(a,b)I$$

$$C(boa) = C(b)oC(a) + c(b,a)I.$$

Adding

$$C(a \circ b + b \circ a) = C(a) \circ C(b) + C(b) \circ C(a) + (c(a,b) + c(b,a))I.$$

Applying the anti-commutative law,

$$Q(0) = 0 + (c(a,b) + c(b,a))I,$$

and so $\quad c(a,b) = - c(b,a)$, giving (iii).

$$\begin{aligned} C(a \circ (b \circ d)) &= C(a) \circ C(b \circ d) + c(a, b \circ d)I, \\ &= C(a) \circ (C(b) \circ C(d)) + c(b,d)(C(a) \circ I) + c(a, b \circ d)I, \\ &= C(a) \circ (C(b) \circ C(d)) + c(a, b \circ d)I. \end{aligned}$$

Permuting a, b, and d, cyclicly, adding the corresponding equations, then applying the Jacobi identity, we have

$$Q(0) = 0 + (c(a, b \circ d) + c(b, d \circ a) + c(d, a \circ b))I,$$

and so $\quad 0 = c(a, b \circ d) + c(b, d \circ a) + c(d, a \circ b)$, giving (iv).

$$C((a + s) \circ (b + t)) = C(a + s) \circ C(b + t) + c(a + s, b + t)I,$$

or $\quad = C(a \circ b) + C(a \circ t) + C(s \circ b) + C(s \circ t).$

Expanding each of the expressions and comparing gives

$$c(a + s, b + t) = c(a,b), \text{ property (v).}$$

$$C((a + b) \circ d) = C(a + b) \circ C(d) + c(a + b, d)I,$$

or $\quad = C(a \circ d) + C(b \circ d).$

Expanding each expression and comparing gives

$$c(a + b, d) = c(a,d) + c(b,d), \text{ property (vi).}$$

Similarly

$$c(a, b + d) = c(a,b) + c(a,d), \text{ property (vii).}$$

$$C(a \circ (kb)) = C(k(a \circ b)),$$

$$C(a)oC(kb) + c(a,kb)I = kC(aob),$$

$$C(a)okC(b) + c(a,kb)I = k(C(a)oC(b)) + kc(a,b)I,$$

$$k(C(a)oC(b)) + c(a,kb)I = k(C(a)oC(b)) + kc(a,b)I,$$

giving $\qquad c(a,kb) = kc(a,b).$

Similarly $\qquad c(ka,b) = kc(a,b)$, giving (viii).

The elements $c(a,b)$ of F, satisfying the properties (i) to (viii), we shall call a <u>factor set</u>.

By theorem 1.3 it is shown that if matrices of invariance are so chosen that $C(a + b) = C(a) + C(b)$, $C(ka) = kC(a)$, and $C(t) = Q(t)$, then $C(aob) = C(a)oC(b) + c(a,b)I$, where $c(a,b)$ is a factor set. It is apparent, therefore, that the correspondence

$$a \rightarrow C(a)$$

is almost a representation of L. Let us call such a correspondence an <u>L-projective representation</u>, (L for Lie, and projective because of the analogy with group theory). Theorem 1.2 shows that we can construct such a representation whenever matrices of invariance exist. Furthermore, if we have an L-projective representation of L, given by

$$a \rightarrow C(a) = (c_{ij}(a)), \quad (i,j = 1,2,..,n)$$

where $c_{ij}(a) \in F$, the field of reference, we can define

$$au_j = \underline{C}(a)u_j = \sum_{i=1}^{n} c_{ij}(a)u_i, \quad (j = 1,2,..,n)$$

for an F-module with the basis elements $u_1, u_2, \ldots, u_n$ to form an L-projective representation module. It is easily verified that

(i) $\quad a(u + v) = au + av, \qquad (a \in L)$

(ii) $\quad (a + b)u = au + bu, \qquad (b \in L)$

$$\text{(iii)}\ (ka)u = a(ku) = k(au), \qquad (k \in F)$$

$$\text{(iv)}\ (a \circ b)u = a(bu) - b(au) + c(a,b)u.$$

Conversely, if there is an F-module M for which there is defined a unique product au in M for $a \in L$, a Lie algebra, $u \in M$, such that properties (i) to (iv) are satisfied, then M assigns an L-projective representation to L. We can define irreducibility and reducibility in the usual way; namely, if M properly contains an F-submodule invariant under L, then M and its representation are reducible, otherwise M and its representation are irreducible.

If the matrices $C(e_{r+1})$, $C(e_{r+2})$, ..., $C(e_n)$, of theorem 1.2, are replaced by a second set $C'(e_{r+1})$, $C'(e_{r+2})$, ..., $C'(e_n)$, then we can construct a second matrix of invariance $C'(a)$ for each $a \in L$. By theorem 1.1, $C(a) - C'(a) = c(a)I$, $c(a) \in F$. Then we can prove the following theorem.

Theorem 1.4. The set of elements $c(a) \in F$, $a \in L$, has the properties

$$c(a + b) = c(a) + c(b), \qquad (b \in L)$$

$$c(ka) = kc(a), \qquad (k \in F)$$

$$c(t) = 0.$$

Proof. By the corollary of theorem 1.2,

$$C(a + b) = C(a) + C(b), \quad C(ka) = kC(a), \quad C(t) = Q(t).$$

Replacing C in each expression by the corresponding value in C', we have

$$C'(a + b) + c(a + b)I = C'(a) + c(a)I + C'(b) + c(b)I,$$

giving
$$c(a + b) = c(a) + c(b);$$

$$C'(ka) + c(ka)I = k(C'(a) + c(a)I),$$

therefore
$$c(ka) = kc(a);$$

finally,
$$C'(t) + c(t)I = Q(t),$$

$$Q(t) + c(t)I = Q(t),$$

and
$$c(t) = 0.$$

<u>Theorem 1.5</u>. Let Q be an irreducible representation of an ideal T of a Lie algebra L in an algebraically closed field F. Let $C(a)$ be a unique matrix of invariance for each $a \in L$, such that $C(a + b) = C(a) + C(b)$, $C(ka) = kC(a)$, $C(t) = Q(t)$, $(b \in L,\ k \in F,\ t \in T)$. Then by theorem 1.3, there is a factor set $c(a,b)$ such that $C(a \circ b) = C(a) \circ C(b) + c(a,b)I$. If there is a second set of matrices of invariance $C'(a)$ such that

$$C'(a) = C(a) - c(a)I,$$

where $c(a + b) = c(a) + c(b)$, $c(ka) = kc(a)$, $c(t) = 0$, then there is an associate factor set $c'(a,b)$ such that

$$C'(a \circ b) = C'(a) \circ C'(b) + c'(a,b)I$$

and
$$c'(a,b) = c(a,b) - c(a \circ b).$$

<u>Proof</u>. $C'(a + b) = C(a + b) - c(a + b)I,$

$$= C(a) + C(b) - c(a)I - c(b)I,$$

$$= C'(a) + C'(b).$$

Similarly
$$C'(ka) = kC'(a)\ ,$$

$$C'(t) = Q(t).$$

Therefore, by theorem 1.3, there is a factor set $c'(a,b)$ such that

$$C'(aob) = C'(a)oC'(b) + c'(a,b)I.$$

Replacing C' in each expression by its equivalent in C, we have

$$\begin{aligned} C(aob) - c(aob) &= (C(a) - c(a)I)o(C(b) - c(b)I) + c'(a,b)I, \\ &= C(a)oC(b) - 0 - 0 + c'(a,b)I, \\ &= C(aob) - c(a,b)I + c'(a,b)I, \end{aligned}$$

giving $$c'(a,b) = c(a,b) - c(aob).$$

4. <u>A sufficient condition for the existence of matrices of invariance</u>.

<u>Theorem 1.6</u>. Let T be an ideal of a Lie algebra . Let the field of reference F be arbitrary. Let M be an L-F-module and m and m' T-F-submodules of M. If $M = m \dotplus m'$, then each $a \in L$ can be assigned a matrix of invariance $C(a)$, such that

$$C(a)oQ(t) = Q(aot), \qquad (t \in T)$$

where Q is the representation of T assigned by m.

<u>Proof</u>. For any $u \in M$, we have $u = u_1 + u_2$, where $u_1 \in m$ and $u_2 \in m'$. The components u_1 and u_2 are unique since the sum of m and m' is direct. Thus the correspondences

$$H_1: \quad u \rightarrow u_1 = H_1 u,$$

$$H_2: \quad u \rightarrow u_2 = H_2 u,$$

are homomorphisms of M onto m and m' respectively. We can then write

$$u = H_1 u + H_2 u.$$

In particular, for $v \in m$,

$$av = H_1av + H_2av, \qquad (a \in L)$$

then the operator H_1a is clearly a linear transformation of $\mathfrak{m}$. For $t \in T$,

$$(aot)v = a(tv) - t(av),$$
$$= H_1a(tv) + H_2a(tv) - t(H_1av + H_2av).$$

Equating components,

$$(aot)v = H_1a(tv) - t(H_1av).$$

Setting the linear transformation $H_1a = \underline{C}(a)$ and replacing t by its linear transformation $\underline{Q}(t)$ of $\mathfrak{m}$, we have

$$\underline{Q}(aot)v = \underline{C}(a)(\underline{Q}(t)v) - \underline{Q}(t)(\underline{C}(a)v),$$

or for the corresponding matrices of these linear transformations

$$Q(aot) = C(a)Q(t) - Q(t)C(a) = C(a)oQ(t).$$

CHAPTER II

INDUCED REPRESENTATIONS

1. Induced representations. Let M be a representation module of a Lie algebra L leading to the representation A of L by matrices with coefficients in the field of reference F. M will then serve as a representation module for any ideal T of L. For, if $u_1, u_2, \ldots, u_s$, is a basis of M over F and $t \in T$, then , since $t \in L$, we have

$$tu_j = \sum_{i=1}^{s} k_{ij}u_i, \quad j = 1,2,\ldots,s. \qquad (k_{ij} \in F)$$

Since M is a representation module for L, the correspondence

$$t \rightarrow (k_{ij}) = Q(t)$$

has the properties

$$Q(t + t') = Q(t) + Q(t'), \qquad (t' \in T)$$

$$Q(kt) = kQ(t), \qquad (k \in F)$$

$$Q(t \circ t') = Q(t) \circ Q(t'),$$

thus Q is a representation of T. We call it the representation induced by A in T.

Theorem 2.1. Let A be an irreducible representation of a Lie algebra L. Then A induces in any ideal T of L a representation Q, which is irreducible, or is fully reducible into equivalent irreducible components if these components are invariant under L, and conversely.

<u>Proof</u>. Let M be an irreducible L-F-module leading to the representation A of L by matrices. Select any irreducible T-F-module $m \subseteq M$. Let m assign to T the representation Q, invariant under L.

If $m = M$, then A induces in T the irreducible representation Q.

If $m \neq M$, then there is an $a \in L$, such that $am \not\subseteq m$, otherwise M is reducible. Since Q is invariant under L, there is a matrix of invariance $C(a)$ corresponding to a and, consequently, a corresponding linear transformation $\underline{C}(a)$ of m. From $am + m$, form the set m_2 of the elements

$$au - \underline{C}(a)u, \qquad (u \in m).$$

It is easily verified that m_2 is an F-module. Further

$$\begin{aligned} t(au - \underline{C}(a)u) &= t(au) - t(\underline{C}(a)u), \\ &= (t \circ a)u + a(tu) - t(\underline{C}(a)u), \\ &= \underline{Q}(t \circ a)u + a(tu) - t(\underline{C}(a)u), \\ &= (\underline{Q}(t) \circ \underline{C}(a))u + a(tu) - t(\underline{C}(a)u), \\ &= \underline{Q}(t)(\underline{C}(a)u) - \underline{C}(a)(\underline{Q}(t)u) + a(tu) \\ &\quad - \underline{Q}(t)(\underline{C}(a)u) \\ &= a(tu) - \underline{C}(a)(tu) \in m_2. \end{aligned}$$

Thus m_2 is a T-F-module. The correspondence

$$u \rightarrow au - \underline{C}(a)u$$

is then an operator homomorphism over F and T of m onto m_2. But m is irreducible, hence the homomorphism is an isomorphism. Since $m \neq m_2$, we have

$$m + am = m + m_2 = m \dotplus m_2.$$

If $m \dotplus m_2 = M$, the theorem is proved. If $m \dotplus m_2 \neq M$, there exists $b \in L$ such that either

$$bm + m \not\subseteq m \dotplus m_2,$$

or

$$bm_2 + m_2 \not\subseteq m \dotplus m_2.$$

Otherwise

$$L(m \dotplus m_2) = Lm + Lm_2 \subseteq Lm + m + Lm_2 + m_2 \subseteq m \dotplus m_2,$$

making M reducible, contrary to assumption. Suppose $bm_2 + m_2$ is not contained in $m \dotplus m_2$. We then form the set of elements, m_3 say,

$$bu - \underline{C}(b)u, \qquad (u \in m_2)$$

and by replacing a,m,m_2 in our previous remarks by b,m_2,m_3, respectively, it follows that

$$bm_2 + m_2 = m_2 + m_3 = m_2 \dotplus m_3,$$

and

$$m_2 \cong m_3$$

over T. If $m \dotplus m_2 \dotplus m_3 = M$, the theorem is true; otherwise we can continue the process. In fact, if

$$m_1 \dotplus m_2 \dotplus \dots\dots \dotplus m_n \neq M,$$

$$m = m_1 \cong m_2 \cong \dots\dots \cong m_n,$$

then there exists $g \in L$ such that

$$gm_i + m_i \not\subseteq m_1 \dotplus m_2 \dotplus \dots\dots \dotplus m_n, \qquad (1 \leq i \leq n)$$

otherwise, M is reducible, since

$$g\sum_{i=1}^{n} m_i = \sum_{i=1}^{n} gm_i \subseteq \sum_{i=1}^{n}(gm_i + m_i) \subseteq \sum_{i=1}^{n} m_i \ .$$

We can then form the set m_{n+1} of the elements

$$gu - \underline{C}(g)u, \qquad (u \in m_i)$$

Then

$$gm_i + m_i = m_i + m_{n+1} = m_i \dotplus m_{n+1},$$

and

$$m_{n+1} \cong m_i$$

over T. Since M is finite and each additional module is non-zero, a finite number of the above constructions will exhaust M. With the final module, m_r say, we have

$$M = m_1 \dotplus m_2 \dotplus \dots \dotplus m_r,$$
$$m_1 \cong m_2 \cong \dots \cong m_r,$$

over T. Thus M considered as a T-F-module is completely reducible into irreducible T-F-modules operator isomorphic to m; i.e. A induces on the ideal T a representation Q which is completely reducible into equivalent irreducible components.

<u>Proof of the converse</u>. Since A induces in T a representation which is fully reducible, the corresponding representation module M, considered as a T module, can be written in the form

$$M = m_i \dotplus (m_1 \dotplus \dots \dotplus m_{i-1} \dotplus m_{i+1} \dotplus \dots \dotplus m_r).$$

Theorem 1.6 then assures us of the existence of matrices of invariance for the representation assigned to T by m_i.

2. <u>Product representations</u>. In order to consider the nature of induced representations in greater detail, we define a further matrix product, the Lie-Kronecker product. This product in the representation theory of Lie algebras has properties comparable to those of the Kronecker product of matrices in the representation theory of groups. If A and B are any square matrices, not necessarily of the same dimensions, their <u>Lie - Kronecker product</u>, designated by $A \circledast B$, is defined by the equation

$$A \circledast B = A x I_B + I_A x B,$$

where 'x' is the Kronecker product of matrices, and I_A, I_B are the unit matrices with the dimensions of A and B, respectively. This product can be derived in a natural way by a consideration of product modules.

Let M and N be any F-modules. The <u>product module</u>, designated by MN, is defined as the module generated by the formal products uv, $u \in M$, $v \in N$, with the defining relations

$$\begin{aligned}
&\text{(i)} \quad u_1v_1 + u_2v_2 = u_2v_2 + u_1v_1, &&(u_1,u_2 \in M,\ v_1,v_2 \in N)\\
&\text{(ii)} \quad u(v_1 + v_2) = uv_1 + uv_2, &&\\
&\text{(iii)} \quad (u_1 + u_2)v = u_1v + u_2v, &&\\
&\text{(iv)} \quad u(kv) = (ku)v. &&(k \in F)
\end{aligned}$$

we define scalar multiplication by the equation

$$k\sum_{i=1}^{h} \pm u_iv_i = \sum_{i=1}^{h}(\pm ku_i)v_i .$$

Since this definition preserves the relations (i) to (iv), MN is an F-module.

Let M and N be L-projective representation modules. We can then define a linear transformation $\underline{A}(a)$ of the product module MN by the equations

$$\underline{A}(a)(uv) = (au)v + u(av), \qquad (a \in L)$$

$$\underline{A}(a)\sum_{i=1}^{h} \pm u_iv_i = \sum_{i=1}^{h}\pm(au_i)v_i + \sum_{i=1}^{h}\pm u_i(av_i).$$

The linear transformation $\underline{A}(a)$ is uniquely determined by a, so we define the product $a(uv)$ by

$$a(uv) = \underline{A}(a)(uv).$$

<u>Theorem 2.2</u> Let M and N assign projective representations

to the Lie algebra L, whose factor sets are c(a,b) and d(a,b), respectively. Then the product module MN, for which there is defined a left multiplication by elements of L as above, assigns an L-projective representation to L with the factor set c(a,b) + d(a,b).

<u>Proof</u>. Since the multiplication is unique and clearly distributive, we need only verify the remaining three properties for L-projective representation modules.

$$\text{(i)}\quad \begin{aligned}(a + b)(uv) &= ((a + b)u)v + u((a + b)v),\\ &= (au + bu)v + u(av + bv),\\ &= (au)v + u(av) + (bu)v + u(bv),\\ &= a(uv) + b(uv).\end{aligned}$$

$$\text{(ii)}\quad \begin{aligned}(ka)(uv) &= ((ka)u)v + u((ka)v),\\ &= (k(au))v + u(k(av)),\\ &= k(a(uv)) = a(k(uv)).\end{aligned}$$

$$\text{(iii)}\quad (a\circ b)(uv) = ((a\circ b)u)v + u((a\circ b)v),$$

$$= (a(bu) - b(au) + c(a,b)u)v + u(a(bv) - b(av) + d(a,b)v),$$

$$= (a(bu) - b(au))v + u(a(bv) - b(av)) + (c(a,b) + d(a,b))uv,$$

$$\begin{aligned}= {} & (a(bu))v + (bu)(av) + (au)(bv) + u(a(bv))\\ & - (b(au))v - (au)(bv) - (bu)(av) - u(b(av))\\ & + (c(a,b) + d(a,b))uv, \quad \text{(after insertion of suitable terms)}\end{aligned}$$

$$= a((bu)v + u(bv)) - b((au)v + u(av)) + (c(a,b) + d(a,b))uv,$$

$$= a(b(uv)) - b(a(uv)) + (c(a,b) + d(a,b))uv.$$

Thus MN is an L-projective representation module of L and assigns an L-projective representation with the factor set $c(a,b) + d(a,b)$.

In order to exhibit a matrix representation of L assigned by MN, let $u_1, u_2, \ldots, u_n$, be a basis of M, and $v_1, v_2, \ldots, v_r$, a basis of N. The pairs $u_i v_j$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,r$, in some fixed order, then form a basis for MN. Let us take

$$u_1v_1, u_1v_2, \ldots, u_1v_r, u_2v_1, \ldots, u_2v_r, \ldots, u_nv_r,$$

as a basis for MN. Let $C(a) = (c_{ig})$ be the n x n matrix assigned to $a \in L$ by M, and $U(a) = (d_{jh})$ be the r x r matrix assigned to a by N, and A(a) the nr x nr matrix assigned to a by MN, then

$$\begin{aligned} a(u_iv_j) &= (au_i)v_j + u_i(av_j), \\ &= \left(\sum_{g=1}^{n} c_{gi}u_g\right)v_j + u_i \sum_{h=1}^{r} d_{hj}v_h, \\ &= \left(\sum_{g=1}^{n} c_{gi}u_g\right)\sum_{h=1}^{r} \delta_{hj}v_h + \left(\sum_{g=1}^{n} \delta_{gi}u_g\right)\sum_{h=1}^{r} d_{hj}v_h, \\ &= \sum_{g=1}^{n}\sum_{h=1}^{r}(c_{gi}\delta_{hj} + \delta_{gi}d_{hj})u_gv_h. \qquad \left(\delta_{ij} = \left(\begin{matrix} 0 \text{ for } i \neq j \\ 1 \text{ for } i = j \end{matrix}\right)\right) \end{aligned}$$

Arranging the coefficients, $c_{gi}\delta_{hj} + \delta_{gi}d_{hj}$, in matrix form as directed by the choice of basis, we have

$$A(a) = C(a)xI_U + I_CxU(a) = C(a)\oplus U(a).$$

Let us designate $C(a)\oplus U(a)$ by $C\oplus U(a)$, then if M and N induce the L-projective representations C and U on L, MN induces the L-projective representation $C\oplus U$ on L. Let us observe that an L-projective representation becomes an ordinary representation when its factor set is zero.

<u>Theorem 2.3</u>. Let A be an irreducible representation of a

Lie algebra L in an algebraically closed field F. Let A induce in an ideal T of L a representation Q, completely reducible to r irreducible components equivalent to a representation G. A is then the Lie-Kronecker product of two irreducible L-projective representations C and U of L, where C has the degree of G, U, the degree r, and their factor sets differ only in sign. U is actually an L-projective representation of the residue class algebra L - T.

<u>Proof</u>. Let M and m_1 be the representation modules assigning the representations A and G to L and T, respectively, then

$$M = m_1 \dotplus m_2 \dotplus \dots \dotplus m_r,$$

and

$$m_1 \cong m_i, \quad i = 2,3,\dots,r,$$

over T. Let α_i be the operator isomorphism between m_1 and m_i, i.e. the isomorphism $m_1 \cong m_i$ is accomplished by the correspondence

$$u \to \alpha_i u, \qquad (u \in m_1,\ \alpha_i u \in m_i)$$

such that for $t \in T, k \in F$, $tu \to \alpha_i tu = t\alpha_i u$, $ku \to \alpha_i ku = k\alpha_i u$.

Let $A(a) = (A_{ij}(a))$, $i,j = 1,2,\dots,r$, $a \in L$, where A_{ij} are submatrices of the dimensions of $G(t)$, $t \in T$, then we have

$$a\alpha_j u = \sum_{i=1}^{r} \alpha_i \underline{A}_{ij}(a)u, \qquad (u \in m_1)$$

where $\underline{A}_{ij}(a)$ is the linear transformation of m_1 corresponding to the matrix $A_{ij}(a)$. In the following calculations, we will omit the bar under $A_{ij}(a)$; the context will indicate whether a linear transformation or its matrix is meant.

Since T is an ideal, $a \circ t \in T$, hence

$$(aot)\alpha_j u = \alpha_j((aot)u), \qquad (u \in m_1)$$

$$a(t\alpha_j u) - t(a\alpha_j u) = \alpha_j((aot)u),$$

$$a(\alpha_j(tu)) - t\sum_{i=1}^{r}\alpha_i A_{ij}(a)u = \alpha_j((aot)u),$$

$$\sum_{i=1}^{r}\alpha_i A_{ij}(a)(tu) - \sum_{i=1}^{r}\alpha_i t(A_{ij}(a)u) = \sum_{i=1}^{r}\alpha_i\delta_{ij}((aot)u),$$

where $\delta_{ij} = \left(\begin{smallmatrix}0 \text{ for } i\neq j\\ 1 \text{ for } i=j\end{smallmatrix}\right)$. Thus

$$\sum_{i=1}^{r}\alpha_i(A_{ij}(a)(tu) - t(A_{ij}(a)u)) = \sum_{i=1}^{r}\alpha_i\delta_{ij}((aot)u).$$

Replacing the element t of T by its corresponding linear transformation $G(t)$,(omitting the bar), we have

$$\sum_{i=1}^{r}\alpha_i(A_{ij}(a)oG(t))u = \sum_{i=1}^{r}\alpha_i(\delta_{ik}G(aot))u.$$

Since $M = m_1 \dotplus (m_2 \dotplus \dots \dotplus m_r)$, theorem 1.6 assures us of the invariance of G under L. Hence we can construct matrices of invariance $C(a)$, $a \in L$, according to theorem 1.2, and then the correspondence $a \rightarrow C(a)$ is an L-projective representation. Therefore we can write our last equation in the form

$$\sum_{i=1}^{r}\alpha_i(A_{ij}(a)oG(t) - \delta_{ij}C(a)oG(t))u = 0.$$

Consequently

$$(A_{ij}(a) - \delta_{ij}C(a))oG(t) = 0,$$

for all $t \in T$ and $i,j = 1,2,\dots,r$. Applying Schur's lemma, after replacing 'o' multiplication by ordinary multiplication of matrices, we have

$$A_{ij}(a) - \delta_{ij}C(a) = U_{ij}(a)I_G, \qquad (U_{ij}(a) \in F)$$

I_G being the unit matrix with the dimensions of G, thus

$$A_{ij}(a) = \delta_{ij}C(a) + U_{ij}(a)I_G,$$

and so

$$A(a) = (A_{ij}(a)) = C(a)xI_U + I_C xU(a),$$
$$= C(a)\oplus U(a),$$
$$= C\oplus U(a),$$

where $U(a) = (U_{ij}^{(a)})$ and since $I_C = I_G$.

As $C\oplus U$ is a representation and C is an L-projective representation, we have

$$C\oplus U(a + b) = C\oplus U(a) + C\oplus U(b), \qquad (a,b \in L)$$
$$C(a + b)xI_U + I_C xU(a + b) = C(a)xI_U + I_C xU(a) + C(b)xI_U + I_C xU(b),$$
$$I_C xU(a + b) = I_C x(U(a) + U(b)),$$

thus
$$U(a + b) = U(a) + U(b).$$

Similarly expanding the expressions in the equations

$$C\oplus U(ka) = kC\oplus U(a),$$

and
$$C\oplus U(aob) = C\oplus U(a)oC\oplus U(b),$$

we obtain

$$U(ka) = kU(a)$$
$$U(aob) = U(a)oU(b) - c(a,b)I,$$

where $c(a,b)$ is the factor set belonging to the L-projective representation C. Thus U is an L-projective representation of L with a factor set differing only in sign from that of C.

For $t \in T$, we have $A(t) = Q(t) = G(t)xI_r$, where I_r is the unit matrix of degree r. Also

$$A(t) = C(t)xI_U + I_C xU(t),$$
$$= G(t)xI_r + I_C xU(t),$$

thus
$$0 = I_C xU(t),$$

giving $$U(t) = 0.$$
Hence U gives a representation to the residue class algebra L - T.

C and U are irreducible L-projective representations of L, for suppose U is reducible. Let M_C and M_U be the L-projective representation modules assigning C and U to L, respectively, then the product module $M_C M_U$ assigns A to L. Since U is reducible, M_U contains an invariant submodule m_U. The product module $M_C m_U$ is then invariant under L, and is contained in $M_C M_U$. Thus A is reducible, contrary to its irreducibility. Thus U is irreducible. Similarly C is irreducible.

3. The imbedding of irreducible representations.

Theorem 2.4. Let T be an ideal of a Lie algebra L. Let Q be an irreducible representation of T invariant under L. Let c(a,b), ($a,b \in L$), be a factor set of an L-projective representation C of L. Then a necessary and sufficient condition that Q can be imbedded in an irreducible representation of L is that the factor set -c(a,b) can be realized by an L-projective representation U* of L-T.

Proof. The necessity of the condition is shown by theorem 2.3. The condition is also sufficient. For, taking an irreducible component U of the representation U*, we set $A = C \otimes U$. By theorem 2.2, A is certainly a representation. To show A is irreducible we require the following lemma.

<u>Lemma 2.1</u>. Let L have the equivalent representations $C\otimes U$ and $C'\otimes U'$, formed according to the theorem from an irreducible representation Q of T. If C and C' are equivalent, then U and U' are equivalent.

<u>Proof</u>. For $a \in L$, we have

$$XC'(a)\otimes U'(a)X^{-1} = C(a)\otimes U(a),$$
$$Y^{-1}C'(a)Y = C(a),$$

consequently,

$$X(YC(a)Y^{-1})\otimes U'(a)X^{-1} = C(a)\otimes U(a),$$
$$X(YC(a)Y^{-1}\times I_{U'} + I_C\times U'(a))X^{-1} = C(a)\otimes U(a),$$
$$X(Y\times I_{U'})(C(a)\otimes U'(a))(Y^{-1}\times I_{U'})X^{-1} = C(a)\otimes U(a). \quad \ldots.(1)$$

Setting $X(Y\times I_{U'}) = Z$, and replacing a by $t \in T$, the equation becomes

$$Z(Q(t)\times I_{U'})Z^{-1} = Q(t)\times I_U$$

or $$Z(Q(t)\times I_U) = (Q(t)\times I_U)Z \qquad \text{for all } t \in T.$$

Applying Schur's lemma gives Z the form $I_Q\times W$, where W is non-singular. Substituting this form of Z in (1) gives

$$C(a)\times I_{U'} + I_C\times WU'(a)W^{-1} = C(a)\times I_U + I_C\times U(a)$$
$$WU'(a)W^{-1} = U(a),$$

proving the lemma.

Returning to the theorem, we can now prove that the representation $A = C\otimes U$ is irreducible. Let m and n be the modules assigning the L-projective representations C and U to L. By theorem 2.2, $M = mn$ assigns the ordinary representation $C\otimes U$

to L. Further, since

$$A(t) = C(t)xI_U + I_C xU(t), \qquad (t \in T)$$
$$= Q(t)xI_U,$$

M, considered as a T-F-module is the direct sum of irreducible T-F-modules operator isomorphic to m, i.e.

$$M = m_1 \dot{+} m_2 \dot{+} \dots \dot{+} m_s, \quad m_1 = m, \quad m_1 \cong m_i \text{ over } T.$$

In this form M certainly has a Remak decomposition.[17)]

Let us assume that M is reducible, then M properly contains an L-F-module M'. With suitably chosen subscripts we then have

$$M = M' \dot{+} m_{r+1} \dot{+} \dots \dot{+} m_s.$$

Since any submodule of a module with a Remak decomposition has a Remak decomposition and, furthermore, since different Remak decompositions of the same module are equal in length, and the components are operator isomorphic in some order, we have

$$M = m_1' \dot{+} m_2' \dot{+} \dots \dot{+} m_r' \dot{+} m_{r+1} \dot{+} \dots \dot{+} m_s,$$

where $m_1' \cong m_i' \cong m_j$, $i = 1,2,..,r$, $j = 1,2,..,s$. These operator isomorphisms assure us of the irreducibility of the m_i'. Let α_i be the operator isomorphism of m_1' onto m_i', or m_i if $i \geqslant r+1$. Then for $u \in m_1'$,

$$a\alpha_g u = \sum_{i=1}^{r} \alpha_i A_{ig}'(a), \qquad (a \in L, \ g \leqslant r)$$

since M' is invariant under L. Also

$$a\alpha_h u = \sum_{i=1}^{r} \alpha_i A_{ig}'(a), \qquad (s \geqslant h > r).$$

As in theorem 2.3, these equations lead to

$$A_{ik}'(a) = C'(a)\delta_{ik} + I\ U_{ik}'(a),$$

but with the further property that

$$I_C U'_{ik}(a) = 0, \text{ for } k \leqslant r, \ i > r,$$

giving
$$U'_{ik}(a) = 0, \text{ for } k \leqslant r, \ i > r.$$

Thus $U'(a) = (U'_{ik}(a))$ is a reducible representation, making $A'(a) = C'(a) \otimes U'(a)$ also reducible. Since there is an operator isomorphism between the two Remak decompositions of M and also between m_1 and m'_1 it follows that the representation $C \otimes U$ is equivalent to $C' \otimes U'$ and C is equivalent to C'. By the lemma U is equivalent to U', contrary to U being irreducible. Thus the assumption that M is reducible is contradicted.

4. Indecomposable representations. As we have developed it, the theory of induced representations for indecomposable representations over an algebraically closed field is less manageable than in the irreducible case because the commuting matrices lie in a primary ring, rather than being simply scalar multiples of the unit matrix. However, with certain clearly indicated changes of definition consistent with the nature of the commuting matrices, the theorems of chapter I have analogies for indecomposable representations. For example, analogous to theorem 1.1 we have:

Theorem 2.5. For an algebraically closed field of reference F, the matrices of invariance of an indecomposable representation Q of an ideal T, for a particular element a of a Lie algebra L, differ only by matrices lying in a primary ring P.

<u>Proof</u>. Let $C(a)$ and $C'(a)$ be matrices of invariance corresponding to a, then

$$C(a)oQ(t) = Q(aot), \qquad (t \in T)$$

$$C'(a)oQ(t) = Q(aot).$$

Thus

$$(C(a) - C'(a))oQ(t) = 0,$$

$$(C(a) - C'(a))Q(t) = Q(t)(C(a) - C'(a)).$$

Since $Q(t)$ is indecomposable

$$C(a) - C'(a) = c^*(a),$$

where $c^*(a)$ is a matrix in a primary ring P with the dimensions of $C(a)$.

By replacing the matrices $c(a)I$ and $c(a,b)I$ of chapter I by the matrices $c^*(a)$ and $c^*(a,b)$ of P, theorems 1.2 and 1.3, and of course their corollaries, have analogous proofs for indecomposable representations. Defining a factor set as the set of matrices $c^*(a,b)$ possessing the properties of the corollary to theorem 1.3, with c replaced by c^*, the concept of L-projective representation can be extended to indecomposable representations. Theorems 1.4, 1.5, and 1.6 are then valid for indecomposable representations. With these basic theorems available, we can now consider how far the analogy extends to induced representations.

<u>Theorem 2.6</u>. Let A be any representation of a Lie algebra L. Then A induces on any ideal T of L a representation which is indecomposable, or is decomposable into indecomposable components. These components are invariant under L.

Proof. Let M be the L-F-module assigning the representation A to L. As a T-F-module, M is either a direct sum of two T-F-modules or not. If not, the theorem is true. If M is a direct sum, the summands are in turn either direct sums or not. Since M is finite th♦ process will teminate, giving

$$M = m_1 \dotplus m_2 \dotplus \dots \dotplus m_r,$$

where $m_1, m_2, \dots, m_r$, are indecomposable T-F-modules. Thus M considered as a T-F-module assigns a representation Q to T which is decomposable into indecomposable components Q_i, the representation assigned to T by m_i, $i = 1,2,\dots,r$.

Since M can be put in the form

$$M = m_i \dotplus (m_1 \dotplus \dots \dotplus m_{i-1} \dotplus m_{i+1} \dotplus \dots \dotplus m_r),$$

theorem 1.6 assures us that Q_i is invariant under L.

Theorem 2.7. Let A be any representation of a Lie algebra L in an algebraically closed field F. Let A induce in T a representation Q, decomposable into the indecomposable components $Q_1, Q_2, \dots, Q_r$, then A(a) can be partitioned so that $A(a) = (A_{ij}(a))$ with the properties

$$A_{ii}(a) = C_i(a) + c_i^*(a), \qquad (a \in L)$$

$$A_{ij}(a)Q_i(t) = Q_j(t)A_{ij}(a), \; i \neq j, \; (t \in T)$$

where C_i is the L-projective representation with $C_i(t) = Q_i(t)$, $c_i^*(a) \in P_i$, the commuting ring of Q_i.

Proof. Let M be the representation module assigning the representation A to L. Then M has the following decomposition with respect to T,

$$M = m_1 \dotplus m_2 \dotplus \dots \dotplus m_r,$$

where m_i is indecomposable and assigns the representation Q_i to T, $i = 1,2,\dots,r$. For $u_i \in m_i$, we have(omitting bars from linear transformations)

$$tu_i = Q_i(t)u_i \in m_i, \qquad (t \in T)$$
$$au_i = \sum_{j=1}^{r} A_{ij}(a)u_i,$$

where $A_{ij}(a)u_i \in m_j$. Hence

$$a(tu_i) = a(Q_i(t)u_i),$$
$$(aot)u_i + t(au_i) = \sum_{j=1}^{r} A_{ij}(a)(Q_i(t)u_i),$$
$$Q_i(aot)u_i = \sum_{j=1}^{r} A_{ij}(a)(Q_i(t)u_i) - \sum_{j=1}^{r} Q_j(t)(A_{ij}(a)u_i).$$

Since the module sum is direct, the components are unique, and a comparison of these gives

$$Q_i(aot) = A_{ii}(a)Q_i(t) - Q_i(t)A_{ii}(a)$$
$$= A_{ii}(a)oQ_i(t),$$

and

$$0 = A_{ij}(a)Q_i(t) - Q_j(t)A_{ij}(a), \qquad (j \neq i)$$

1 By theorem 2.6, Q_i is invariant under L, hence we can construct an L-projective representation C_i of L. Then we can put

$$Q_i(aot) = A_{ii}(a)oQ_i(t),$$

in the form

$$C_i(a)oQ_i(t) = A_{ii}(a)oQ_i(t),$$

giving $(C_i(a) - A_{ii}(a))Q_i(t) = Q_i(t)(C_i(a) - A_{ii}(a))$.

Hence $$C_i(a) - A_{ii}(a) = c_i^*(a),$$

i.e. an element of the primary ring P_i of matrices commuting with $Q_i(t)$ for all $t \in T$.

It is clear from this theorem that the Lie-Kronecker product does not have an application to indecomposable representations, except in the very special case when the Q_i are equivalent and the required elements of the commuting ring are multiples of the unit matrix. Since the Lie-Kronecker product was essential to the formation of representations, this aspect of indecomposable representations is left open for further study.

CHAPTER III

GENERATED REPRESENTATIONS FOR ALGEBRAS OF CHARACTERISTIC ZERO

1. <u>The Birkhoff imbedding algebra</u>. Any Lie algebra having the basis $e_1, e_2, \ldots, e_r$, over any field F, can be imbedded in a linear associative algebra H(L) according to the procedure of G. Birkhoff as follows: since the e_i form a basis of L, we have

$$e_i \circ e_j = \sum_{\alpha=1}^{r} k_{ij}^{\alpha} e_\alpha, \quad (i,j = 1,2,\ldots,r).$$

Furthermore, we can form an infinity of the expressions

$$\begin{array}{l}
e_r,\ e_{r-1},\ \ldots,\ e_1, \\
e_re_r,\ e_re_{r-1},\ \ldots,\ e_re_1, \\
\qquad e_{r-1}e_{r-1}, \ldots, e_{r-1}e_1, \\
\qquad\qquad \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\qquad\qquad\qquad e_1e_1, \\
e_re_re_r,\ e_re_re_{r-1},\ \ldots\ldots,\ e_re_re_1, \\
\qquad\qquad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\qquad\qquad\qquad\qquad e_1e_1e_1, \\
\cdot\cdot \qquad \cdot\cdot \qquad \cdot\cdot \qquad \cdot\cdot
\end{array}$$

These formal products will form the basis of a linear associative algebra with respect to a multiplication defined by the equation

$$(e_{i_1}e_{i_2}\cdots\cdots e_{i_h})(e_{j_1}e_{j_2}\cdots e_{j_s}) = \sum k_q(e_{q_1}e_{q_2}\cdots\cdots e_{q_f})$$

where $i_1 \geqq i_2 \geqq \cdots \geqq i_h$, $j_1 \geqq j_2 \geqq \cdots \geqq j_s$, $q_1 \geqq q_2 \geqq \cdots \geqq q_f$, and the sum is the result of rearanging the formal product

$$e_{i_1}e_{i_2}\cdots\cdots e_{i_h}e_{j_1}e_{j_2}\cdots\cdots e_{j_s}$$

and the consequent products, so that the subscripts are monotone decreasing, according to the rule

$$e_i e_j = e_j e_i + \sum_{\alpha=1}^{r} k_{ij}^{\alpha} e_{\alpha},$$

the k_{ij}^{α} being supplied by the corresponding product $e_i o e_j$ in L. This multiplication can be shown to be independent of the sequence of rearrangements employed, consequently it is unique and associative. Let us add a principal unit element $e = e_{i_1}^0 e_{i_2}^0 \ldots e_{i_h}^0$ to the basis such that $ea = ae = a$ for all elements a in the algebra, then we will designate the resulting algebra with unit element by the symbol H(L).

Conversely, in any associative algebra H we can define a product $[ab]$ by the equation $[ab] = ab - ba$, $(a,b \in H)$, then with respect to this new multiplication H forms a Lie algebra, L(H) say. Furthermore, every submodule of H, H' say, closed under the new multiplication forms a Lie algebra L(H'). Thus from the associative algebra H(L) we can form L(H(L)). The module H'(L) with the basis $e_1, e_2, \ldots e_r$, is closed under '$[\ ]$' multiplication for if $i \geqslant j$, $[e_i e_j] = e_i e_j - e_j e_i = e_i e_j - (e_i e_j + \sum_{\alpha=1}^{r} k_{ji}^{\alpha} e_{\alpha})$ $= \sum_{\alpha=1}^{r} k_{ji}^{\alpha} e_{\alpha}$. Hence we can form L(H'(L)). By the correspondence $e_i \leftrightarrow e_i$ between L and L(H'(L)) we have

$$e_i o e_j \longleftrightarrow [e_i e_j],$$

$$e_i + e_j \longleftrightarrow e_i + e_j,$$

$$ke_i \longleftrightarrow ke_i$$

giving $L \cong L(H'(L))$. Thus replacing the symbol $[ab]$ by aob, $(a,b \in L(H'(L)))$, we can consider any element of L as also being an element of $H(L)$. Notice that a representation A of L gives a representation to $H(L)$ by the definition

$$A(e_{i_1}e_{i_2}\dots e_{i_r}) = A(e_{i_1})A(e_{i_2})\dots A(e_{i_r}).$$

Conversely, a representation A of $H(L)$ gives a representation to L since it gives a representation to $L(H'(L))$.

2. Induced and generated representations. Let T be an ideal of a Lie algebra L over a field F. A T-F-module m generates an L-F-module M if M and m occur in the same L-F-module M^* and M is the intersection of all L-F-modules in M^* containing m. The representation assigned to L by M is said to be generated by the representation Q assigned to T by m. M' and its representation of L are said to be induced by m and Q, respectively, if for every L-F-module M generated by m, there is an operator homomorphism of M' onto M leaving each element of m invariant. To give M' an explicit form let $e_1, e_2, \dots, e_s$ be a basis of T, and $e_1, e_2, \dots, e_s, e_{s+1}, \dots, e_r$, a basis of L over a field F. Then M' is obtained as the direct sum of the formal power products

$$e_r^{n_r} e_{r-1}^{n_{r-1}} \dots e_{s+1}^{n_{s+1}} m$$

taken over the powers, with the following rules of computation:

$$\text{(i)}\ \sum e_r^{n_r}\dots e_{s+1}^{n_{s+1}} u_n + \sum e_r^{n_r}\dots e_{s+1}^{n_{s+1}} u_n' = \sum e_r^{n_r}\dots e_{s+1}^{n_{s+1}}(u_n + u_n'),$$

$$\text{(ii)}\ k\sum e_r^{n_r}\dots e_{s+1}^{n_{s+1}} u_n = \sum e_r^{n_r}\dots e_{s+1}^{n_{s+1}}(ku_n), \qquad (k \in F)$$

$$\text{(iii)}\ a\sum e_r^{n_r}\dots e_{s+1}^{n_{s+1}} u_n = \sum e_r^{n_r}\dots e_{s+1}^{n_{s+1}} \sum a_{\alpha,n}u_\alpha,$$

where u_n depends on $n_r, \dots, n_{s+1}$ and is in m and where

$ae^{\alpha_r}\dots e^{\alpha_{s+1}} = \sum e^{n_r}\dots e^{n_{s+1}} a_{\alpha,n}$ is obtained by the Birkhoff multiplication procedure and gives $a_{\alpha,n}$ as an element of the imbedding algebra of T, $a_{\alpha,n}$ depending on $\alpha_r,\ldots,\alpha_{s+1}$ and $n_r,\ldots,n_{s+1}$. If $T \subset L$, the induced representation is clearly of infinite degree. That M', so defined, is induced by m is seen by observing that, if M is any L-F-module generated by m, then M contains the F-module H(L)m whose generators are

$$e_r^{n_r} \ldots\ldots e_{s+1}^{n_{s+1}} u_i, \quad (i = 1,2,\ldots,q)$$

with the defining relations supplied by M, $u_1,u_2,\ldots,u_q$ being a basis of m. Since H(L)m is invariant under left multiplication by elements of L, it is an L-F-module. It follows that M = H(L)m. Mapping each basis element of M'onto the formally equal generator of M then gives an operator homomorphism of M' onto M, as required. Further, the representations of L generated by representations of T are components of the induced representation. In this chapter we consider the induced representation by determining a construction for its finite components, namely finite generated representations.

Let us notice that the representation of L given in theorem 2.4, by M in which m is imbedded, is generated by m since any L-F-module in M generated by m must contain M itself otherwise M is reducible.

3. <u>A necessary condition for generating representations</u>.

If a representation of an ideal T of a Lie algebra L is to generate a representation of L, a necessary condition is given by

Theorem 3.1. Any representation A of L in an algebraically closed field of characteristic zero induces a nil representation on the intersection of LoL and the radical R of L

Proof. Since R is defined as the maximal solvable ideal of L and, since LoL is an ideal of L, T_1 = LoL is a solvable ideal of L.

Since A can always be reduced to a form with its irreducible components along the main diagonal and since our theorem concerns only diagonal elements, there is no loss of generality in taking A to be irreducible. Then A induces a representation on T_1 which is irreducible or is reducible to irreducible components A_i along the main diagonal, with zeros below. Since $T_1 \leqslant$ LoL, we have $trA_i(t) = 0$, $t \in T_1$. Since T_1 is solvable in an algebraically closed field, by the theorem of Lie A_i is a matrix of degree one, giving $A_i = 0$. Thus A induces a nil representation on T_1.

4. The generation of a representation (special case). By the theorem of Levi, any Lie algebra over a field of characteristic zero can be expressed in the form

$$L = V \dotplus R,$$

where V is a semi-simple subalgebra and R is the radical of L. Let T be an ideal such that

$$R \geqslant T \geqslant T_1 = \mathrm{LoL} \cap R.$$

Since R is solvable, we can form the subalgebra $L_1 = Fa \dotplus T$, where a is a basis element of R not in T, and F is the field of reference. Since $L \circ L_1 \leqslant L \circ R \leqslant T_1 \leqslant T \leqslant L_1$, L_1 is an ideal of L. We will now show that any representation Q, of finite degree, of T which induces a nil representation on T_1 generates a representation of L_1 of finite degree.

<u>Lemma 3.1</u>. Within H(L), let F(a) be the ring obtained by the adjunction of the basis element a to the field of reference F, then
$$L \circ F(a) \leqslant F(a)T_1.$$

<u>Proof</u>. Let $\underline{a}$ be the regular representation of a, i.e. the representation assigned to a by L as representation module where $\underline{a}u = a \circ u$, $(u \in L)$. Let L now be imbedded in the associative algebra H(L), then we can set
$$h \circ a = h\underline{a} = ha - ah, \qquad (h \in L).$$

By induction
$$h \circ a^n = h\underline{a}^n + \binom{n}{1} a h \underline{a}^{n-1} + \dots + \binom{n}{n-1} a^{n-1} h\underline{a} \qquad \dots (1),$$
where $h \circ a^n = h \circ (a^n)$ and $h\underline{a}^n = (\dots(h \circ a)\circ a)\dots)\circ a)$, a appearing n times. Since $L\underline{a} = L \circ a \leqslant T_1$, by induction $L\underline{a}^n \leqslant T_1$. Applying this property to equation (1), we have
$$h \circ a^n = t_1^{(n)} + \binom{n}{1} a t_1^{(n-1)} + \dots + \binom{n}{n-1} a^{n-1} t_1^{(1)},$$
where $t_1^{(i)} \in T_1$, $i = 1,2,\dots,n$. Consequently, for any polynomial p(a) over the field F,
$$h \circ p(a) \in F(a)T_1,$$
giving
$$L \circ p(a) \leqslant F(a)T_1, \text{ or } L \circ F(a) \leqslant F(a)T_1.$$

Let the characteristic equation for $\underline{a}$, the regular representation of a, be

$$f(x) = k_0 + k_1 x + \ldots\ldots + k_{n-1}x^{n-1} + x^n = 0,$$

and let its roots be

$$\alpha_1, \alpha_2, \ldots\ldots, \alpha_n.$$

In the imbedding algebra H(L), let us now define the function $f^{(s)}(a)$ recursively, as follows:

$$f^{(0)}(a) = e, \text{ the unit element,}$$
$$f^{(1)}(a) = a,$$
$$f^{(2)}(a) = f^{(1)}(a)\ f^{(1)}(a-\alpha_1)\ f^{(1)}(a-\alpha_2)\ \ldots\ f^{(1)}(a-\alpha_n),$$
$$\cdots\cdots\cdots$$
$$f^{(s)}(a) = f^{(s-1)}(a)\ f^{(s-1)}(a-\alpha_1)\ \ldots\ldots\ldots\ f^{(s-1)}(a-\alpha_n).$$

Notice that $f^{(2)}(a) = af(a)$, where $f(x)$ is the characteristic polynomial of $\underline{a}$.

<u>Theorem 3.3</u>. $\quad Lof^{(s)}(a-k) \subseteq f^{(s-1)}(a-k)F(a)T_1 \qquad (k \in F).$

<u>Proof</u>. We have

$$ho(a-k) = t_1, \qquad\qquad (h \in L,\ t_1 \in T_1)$$
$$ho(a-k)^2 = t_1\underline{a} + \binom{2}{1}(a-k)t_1, \quad \text{and by induction,}$$
$$\cdots\cdots\cdots$$
$$ho(a-k)^{n+1} = t_1\underline{a}^n + \binom{n+1}{1}(a-k)t_1\underline{a}^{n-1} + \ldots\ldots + \binom{n+1}{n}(a-k)^n t_1.$$

Multiplying each of the rows by $k_0, k_1, \ldots, k_{n-1}$, respectively, and adding

$$ho((a-k)f(a-k)) = t_1 f(\underline{a}) + (a-k)(t_1^{(n-1)} + p_1(a)t_1^{(n-2)} + \ldots + p_{n-1}(a)t_1)$$

where $t_1^{(i)} \in T_1$ and $p_i(a) \in F(a)$, $i = 1,2,\ldots,n-1$, that is

$$hof^{(2)}(a-k) = 0 + f^{(1)}(a-k)(t_1^{(n-1)} + \dots + p_{n-1}(a)t_1).$$

Thus $$Lof^{(2)}(a-k) \leqslant f^{(1)}(a-k)F(a)T_1.$$

Suppose now that the theorem is true for $s = q$, i.e.

$$Lof^{(q)}(a-k) \leqslant f^{(q-1)}(a-k)F(a)T_1,$$

then for $h \in L$

$$hof^{(q+1)}(a-k) = ho(f^{(q)}(a-k)f^{(q)}(a-k-\alpha_1)\dots f^{(q)}(a-k-\alpha_n))$$

$$= \sum_{i=0}^{n} f^{(q)}(a-k)\dots(hof^{(q)}(a-k-\alpha_i))\dots f^{(q)}(a-k-\alpha_n), \qquad (\alpha_0 = 0)$$

$$= \sum_{i=0}^{n} f^{(q)}(a-k)\dots(f^{(q-1)}(a-k-\alpha_i)g_i)\dots f^{(q)}(a-k-\alpha_n), \ (g_i \in F(a)T_1)$$

$$= f^{(q-1)}(a-k)f^{(q-1)}(a-k-\alpha_1)\dots f^{(q-1)}(a-k-\alpha_n)(\sum_{i=0}^{n} \overset{*}{f}^{(q)}(a-k)\dots$$

$$\dots g_i \dots \overset{*}{f}^{(q)}(a-k-\alpha_n)),$$

where $\overset{*}{f}^{(q)}(a-k-\alpha_j)$ is the product

$$f^{(q)}(a-k-\alpha_j) = f^{(q-1)}(a-k-\alpha_j)f^{(q-1)}(a-k-\alpha_j-\alpha_1)\dots f^{(q-1)}(a-k-\alpha_j-\alpha_n)$$

with the first factor absent. Thus

$$hof^{(q+1)}(a-k) = f^{(q)}(a-k)\sum_{i=0}^{n} \overset{*}{f}^{(q)}(a-k)\dots g_i \dots \overset{*}{f}^{(q)}(a-k-\alpha_n),$$

giving

$$hof^{(q+1)}(a-k) \in f^{(q)}(a-k)F(a)T_1,$$

and $$Lof^{(q+1)}(a-k) \leqslant f^{(q)}(a-k)F(a)T_1,$$ proving the theorem.

<u>Corollary</u>. $Lo(f^{(s)}(a-k)F(a)) \leqslant f^{(s-1)}(a-k)F(a)T_1$.

<u>Proof</u>. Let $p(a)$ be a polynomial in $F(a)$, then for $h \in L$

$$ho(f^{(s)}(a-k)p(a)) = (hof^{(s)}(a-k))p(a) + f^{(s)}(a-k)(hop(a)),$$

$$= f^{(s-1)}(a-k)g_1 + f^{(s)}(a-k)g_2$$

$$= f^{(s-1)}(a-k)(g_1 + f^{*(s)}(a-k)g_2),$$

where $g_1 \in F(a)T_1$ by the theorem, and $g_2 \in F(a)T_1$ by lemma 3.1.

Thus $$Lof^{(s)}(a-k)F(a) \leqslant f^{(s-1)}(a-k)F(a)T_1.$$

Our aim in developing the properties of $f^{(s)}(a-k)F(a)$ is to determine a certain invariant subalgebra B of $H(L_1)$ such that the difference algebra $H(L_1) - B$ has a finite basis and $B \cap L_1 = 0$. We then determine a representation module which assigns the null representation to B. This module gives an explicit representation to L_1. To further our aim we now form a two sided ideal in $H(T)$.

Since T is in the radical R of L, T is solvable. By the ordinary theorem of Lie, every irreducible representation of a solvable Lie algebra is of degree one in an algebraically closed field. Consequently Q, the given representation of T, for a suitable choice of coordinates, has the form

$$Q(t) = \begin{bmatrix} q_1(t) & & & & * \\ & q_2(t) & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & q_s(t) \end{bmatrix}$$

where $q_1(t)$, $q_2(t)$, ..., $q_s(t)$ are irreducible representations of t of degree one. Let T_1 have the basis $t_\sigma, t_{\sigma+1}, \ldots, t_h$, which can be extended by $t_1, t_2, \ldots, t_{\sigma-1}$, to give a basis of T. Let D be the two sided ideal generated in $H(T)$ by

$$h(t_1),\ h(t_2), \ldots,\ h(t_{\sigma-1}),\ t_\sigma,\ t_{\sigma+1},\ \ldots,\ t_h,$$

where $h(t_i) = (t_i - q_1(t_i))(t_i - q_2(t_i))\ldots(t_i - q_s(t_i))$.

We notice that the representation assigned to $H(T)$ by Q has the property that the representative matrix $Q(t')$, $(t' \in D)$, is triangular with zeros along the main diagonal, i.e. is properly triangular. Consequently $Q(D^s) = Q^s(D) = 0$ Further $L \circ D \subseteq D$, for let $d \in D$. It is sufficient to consider d in the form

$$d = t^{(1)}t^{(2)}\ldots t^{(w)}$$

where $t^{(i)} \in T$, $i = 1,2,\ldots,w$, and at least one of the $t^{(i)}$ is a generator. Then for $h \in L$,

$$h \circ d = \sum_{j=1}^{w} t^{(1)}..(h \circ t^{(j)})..t^{(w)},$$

$$= \sum_{j=1}^{w} t^{(1)}..t_1^{(j)}..t^{(w)},$$

where $t_1^{(j)} \in T_1 \subseteq D$. Since D is a two sided ideal, $h \circ d \in D$, giving $L \circ D \subseteq D$.

With the function $f^{(s)}(a-k)$ and the ideal D, we can now construct a two sided ideal B of $H(L_1)$, the imbedding algebra of the Lie algebra $L_1 = Fa \dotplus T$. Let us set

$$C_i = f^{(i)}(a-k)F(a),$$

$$D^0 = H(T),$$

$$D^1 = D$$

$$D^2 = DD, \text{ and similarly for higher powers,}$$

then

<u>Theorem 3.4</u>. $B = C_sD^0 + C_{s-1}D^1 + \ldots + C_1D^{s-1} + C_0D^s$ is a two sided ideal of $H(L_1)$

<u>Proof</u>. It is sufficient to verify the invariance of B under left and right multiplication by T and a. We have

$$TC_iD^{s-i} \leqslant (ToC_i)D^{s-i} + C_iTD^{s-i},$$
$$\leqslant C_{i-1}T_1D^{s-i} + C_iD^{s-i},$$
$$\leqslant C_{i-1}D^{s-i+1} + C_iD^{s-i},$$
$$\leqslant C_{i-1}D^{s-(i-1)} + C_iD^{s-i}.$$

$$TC_0D^s \leqslant (ToC_0)D^s + C_0TD^s,$$
$$\leqslant C_0T_1D^s + C_0D^s,$$
$$\leqslant C_0D^s + C_0D^s,$$
$$\leqslant C_0D^s.$$

Thus $\quad TB \leqslant B.$

Also $\quad C_iD^{s-i}T \leqslant C_iD^{s-i}$, giving $BT \leqslant B$.

Further

$$aC_iD^{s-i} \leqslant C_iD^{s-i}, \text{ giving } aB \leqslant B,$$

and

$$C_iD^{s-i}a \leqslant C_i(D^{s-i}oa) + C_iaD^{s-i},$$
$$\leqslant C_iD^{s-i} + C_iD^{s-i},$$
$$\leqslant C_iD^{s-i},$$

giving $\quad Ba \leqslant B.$

Since $H(L_1)$ possesses a unit element B is a subalgebra, hence we can form the residue class algebra $H(L_1) - B$. An examination of the basis of $H(L_1)$ and of B shows this difference algebra has a finite basis. Since no element of L_1 is in B, the elements of L_1 lie in different residue classes. Thus a faithful? representation of $H(L_1) - B$ will give a faithful? representation to L_1. Using the representation module m assigning the representation Q to T, we can construct a finite representation module for $H(L_1) - B$. Let us set $a = t_{h+1}$, then the basis of $H(L_1)$ can be taken in the form

$$t_{i_1} t_{i_2} \cdots t_{i_g}, \quad (i_1 \geqslant i_2 \geqslant \cdots \geqslant i_g)$$

where these have the alternative form

$$a^j t_{i_h} \cdots t_{i_g} .$$

Let the module m, assigning Q to T, have the basis $u_1, u_2, \ldots u_s$. Then any L_1-F-module generated by m will have generators of the form $a^j u_i$, $i = 1,2,\ldots,s$, $j = 0,1,\ldots$, with certain defining relations. These relations will be determined by finding relations such that $Bu_i = 0$, $i=1,2,.,s$, at the same time leaving $u_1, u_2, \ldots, u_s$ linearly independent. The linearly independent generators will then form a basis of the L_1-F-module . To determine this basis we observe that

$$Bu_i = 0 \qquad (i = 1,2,\ldots,s)$$

is satisfied if

$$f^{(s)}(a-k)u_i = 0$$

$$f^{(s-j)}(a-k)D^j u_i = 0 \qquad (j = 1,2,\ldots,s).$$

Let us first consider the nature of $D^j u_i$. Since m also serves as a representation module for H(T), we have

$$D^j u_i = Q(D^j)u_i = Q^j(D)u_i,$$

and by the nature of D, the matrix of $Q^j(D)$ will have the form

$$\begin{bmatrix} 0 & \cdots & 0 & d_{1j+1} & & & & * \\ & & & 0 & d_{2j+2} & & & \\ & & & & 0 & d_{3j+3} & & \\ & & & & & 0 & & \\ & & & & & & 0 & \\ & & & & & & & d_{s-js} \\ & & & & & & & 0 \\ & & & & & & & \vdots \\ & & & & & & & 0 \end{bmatrix}$$

Let us now determine a set of conditions sufficient for

$$f^{(s-j)}(a-k)D^j u_i = 0. \qquad (i = 1,2,\dots,s)$$

For $j = s$,

$$f^{(0)}(a-k)D^s u_i = f^{(0)}(a-k)0 = 0, \text{ identically.}$$

For $j = s-1$, the only non-trivial equation is

$$f^{(1)}(a-k)D^{s-1}u_s = f^{(1)}(a-k)d_{1s}u_1 = 0,$$

which is satisfied by setting

$$f^{(1)}(a-k)u_1 = 0. \qquad \dots\dots\dots(1)$$

For $j = s-2$, the only non-trivial equations are

$$f^{(2)}(a-k)D^{s-2}u_s = f^{(2)}(a-k)(d_{1s}u_1 + d_{2s}u_2) = 0,$$

$$f^{(2)}(a-k)D^{s-2}u_{s-1} = f^{(2)}(a-k)(d_{1s-1}u_1) = 0.$$

Since $f^{(2)}(a-k)$ contains $f^{(1)}(a-k)$ as a factor, $f^{(2)}(a-k)u_1 = 0$ now, by equation (1), thus the only new condition is

$$f^{(2)}(a-k)d_{2s}u_2 = 0,$$

which is satisfied by setting

$$f^{(2)}(a-k)u_2 = 0. \qquad \dots\dots\dots(2)$$

For $j = s-3$, since $f^{(3)}(a-k)$ contains $f^{(2)}(a-k)$ as a factor and, with equations (1) and (2), the only new condition is satisfied by setting

$$f^{(3)}(a-k)u_3 = 0.$$

Continuing thus, we have

$$f^{(1)}(a-k)u_1 = f^{(2)}(a-k)u_2 = \dots = f^{(s)}(a-k)u_s = 0,$$

as a set of sufficient conditions for

$$f^{(s-j)}(a-k)D^j u_i = 0$$

and hence for $\qquad Bu_i = 0.$

Let $f^{(j)}(a-k) = a^{w_j} - k_{jw_j-1}a^{w_j-1} - \dots - k_{j2}a^2 - k_{j1}a - k_{j0}$

then the given relations determine a representation module M of $H(L_1) - B$ with the basis

$$\begin{array}{ccccccccc}
u_1 & u_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & u_s \\
 & au_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & au_s \\
 & \cdot\cdot & & & & & & & \cdot\cdot \\
 & a^{w_2-1}u_2 & & \cdot & \cdot & \cdot & \cdot & \cdot & a^{w_2-1}u_s \\
 & & & a^{w_2}u_3 & & \cdot & \cdot & \cdot & a^{w_2}u_s \\
 & & & \cdot\cdot & & & & & \cdot\cdot \\
 & & & & & \cdot\cdot & & & \cdot\cdot \\
 & & & & & & & & a^{w_{s-1}}u_s \\
 & & & & & & & & \cdot\cdot \\
 & & & & & & & & \cdot\cdot \\
 & & & & & & & & a^{w_s-1}u_s.
\end{array}$$

To exhibit a representation assigned by M to L_1, let us fix the basis in the above order. Then applying $t \in T$ to each of the basis elemants, we have

$$tu_j = \sum_{i=1}^{s} q_{ij}(t)u_i \quad j=1,2,..,s \qquad (Q(t) = (q_{ij}(t))).$$

$$tau_j = (t\circ a)u_j + atu_j = (t\underline{a})u_j + a(tu_j), \qquad j = 2,3,..,s,$$

$$= \sum_{i=1}^{s} q_{ij}(t\underline{a})u_i + a\sum_{i=1}^{s} q_{ij}(t)u_i,$$

$$= \sum_{i=1}^{s} q_{ij}(t\underline{a})u_i + q_{1j}(t)k_{10}u_1 + \sum_{i=2}^{s} q_{ij}(t)au_i.$$

$$ta^2u_j = (t\underline{a}^2)u_j + \binom{2}{1}a(t\underline{a}u_j) + a^2(tu_j), \qquad j = 2,3,..,s,$$

$$= \sum_{i=1}^{s} q_{ij}(t\underline{a}^2)u_i + (\binom{2}{1}q_{1j}(t\underline{a})k_{10} + q_{1j}(t)k_{10}^2)u_1$$

$$+ \binom{2}{1}\sum_{i=2}^{s} q_{ij}(t\underline{a})au_i + \sum_{i=2}^{s} q_{ij}(t)a^2u_i.$$

Continuing thus the matrix representation of t can be determined and it has the following form

$$\begin{bmatrix} Q(t) & {}_1Q'(t\underline{a}) & {}_1Q'(t\underline{a}^2) & & & * \\ & {}^1_1Q(t) & \binom{2}{1}{}^1_1Q(t\underline{a}) & & & \\ & & {}^1_1Q(t) & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & {}^{s-1}_{s-1}Q(t) \end{bmatrix}$$

where ${}^i_jQ(t)$ is $Q(t)$ with i rows from the top, and j columns from the left, absent. $Q'(t)$ is $Q(t)$ with certain additions to its elements. To determine the matrix corresponding to a, multiply each basis element by a, using the relations

$$a^{w_j}u_j = k_{jw_j-1}a^{w_j-1}u_j + \dots + k_{j2}a^2u_j + k_{j1}au_j + k_{j0}u_j,$$

then a corresponds to

$$\begin{bmatrix} k_{10} & 0 & 0 & \cdots & 0 & \\ 0 & & & & & \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 0 & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & \end{bmatrix}$$

Thus , using the regular representation of a, we have generated from Q a representation, Q_2 say, of L_1. Notice that if Q is faithful then Q_2 is also faithful since $Q\ (t + k'a) = 0$ only if $t = k' = 0$. We further observe that our construction allows us to assign one arbitrary eigen value to a, namely k. The other eigen values are then $k + c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n$, where $c_1, c_2, \dots, c_n$ are zero or positive integers whose maximum values depend on the degree of the representation Q.

5. <u>The generation of a representation(general case)</u>.

If L_1 of section 4 does not equal the radical of L, we can select a basis element b in the radical not in L_1 and form the subalgebra

$$L_2 = Fb \dotplus L_1.$$

L_2 is an ideal since

$$L \circ L_2 \leqslant L \circ R \leqslant T_1 \leqslant T \leqslant L_1 \leqslant L_2 .$$

The representation Q_2 of L_1 for an element $t \in T$ is triangular for a choice of coordinates making Q triangular, since the matrices ${}^{j}_{j}Q(t)$ lie along the main diagonal and all elements below these are zero. Since Q induces a nil representation on $T_1 = L \circ L \cap R \leqslant T$, the form of Q_2 will cause it to also induce a nil representation on T_1. Hence we can repeat the procedure of section 3, replacing L_1 by L_2, T by L_1, a by b, Q by Q_2, m by M, T_1 remaining the same. With this repetition, we generate from Q_2 a representation Q_3 of L_2

Continuing, step by step, we finally exhaust R. At this stage we will have a representation, Q_r say, of the radical R. In order to extend Q_r to L itself, we assume Q is invariant under V, the semi-simple subalgebra of L, then we can prove

<u>Theorem 3.5</u>. If T has a representation Q which, in addition to inducing a nil representation on $L \circ L \cap R$, is invariant under V, where $L = V \dotplus R$, then Q_2, the representation of L_1 generated by Q, is invariant under V.

<u>Proof</u>. We recall that Q is assigned by the module m and Q_2 by M. Since Q is invariant under V, we have an L-projective representation C such that

$$C(v) \circ Q(t) = Q(v \circ t). \qquad (t \in T,\ v \in V)$$

In module terms

$$(v \circ t)u = C(v)(tu) - t(C(v)u). \qquad (u \in m)$$

To prove our theorem it is necessary to define a linear transformation $C_2(v)$ of M such that

$$(v \circ h)w = C_2(v)(hw) - h(C_2(v)w) \quad (h \in L_1,\ w \in M)$$

and furthermore the relations

$$f^{(1)}(a-k)u_1 = f^{(2)}(a-k)u_2 = \dots = f^{(s)}(a-k)u_s = 0,$$

must be preserved under $C_2(v)$.

To achieve the latter, we observe that the following change in the basis of m can be assumed to have taken place at the time of constructing D.

Let m_1 consist of all elements u of m such that $Q(d)u = 0, (d \in D)$ then $\qquad C(v)(Q(d)u) - Q(d)(C(v)u) = Q(v \circ d), \qquad (u \in m_1)$

giving $\qquad 0 \quad - Q(d)(C(v)u) = 0,$

and so $C(v)u \in m$, i.e. m_1 is an F-module invariant under $C(v)$.

Let m_2 consist of all elements u of m such that $Q(d)u \in m_1$.

Then $\quad C(v)(Q(d)u) - Q(d)(C(v)u) = Q(v\circ d)u. \qquad (u \in m_2)$

Since m_1 is invariant under $C(v)$, this equation gives $C(v)u \in m_2$, i.e. m_2 is an F-module invariant under $C(v)$.

Continuing thus, we can form an ascending series of modules invariant under $C(v)$ which, since $Q(d)$ is properly triangular, will finally exhaust m, then

$$0 = m_0 \subset m_1 \subset m_2 \ \ldots\ldots \subset m_q = m \ . \qquad (q \leqslant s)$$

Notice that $Q(d)m_j \subseteq m_{j-1}$ by definition, thus $Q(T_1)m_j \subseteq m_{j-1}$.
We now choose a basis $u_1, u_2, \ldots, u_{h_1}$, of m_1 and extend it by $u_{h_1+1}, \ldots, u_{h_2}$ to form a basis for m_2, and continue so until the final extension $\ldots, u_s$ is a basis of m. This basis is suitable for our purpose.

Let us define $C_2(v)$ by the equation

$$C_2(v)w = C_2(v)\sum_{i=1}^{s} p_i(a)u_i, \quad (w \in M,\ p_i(a) \in F(a))$$

$$= (\underline{v}\sum_{i=1}^{s} p_i(a))u_i + (\sum_{i=1}^{s} p_i(a))(C(v)u_i),$$

where $\underline{v}$ is the regular representation of v with respect to L.

To show that the defining relations of M are invariant under $C_2(v)$, we recall that $f^{(i)}(a-k)u_j = 0$ for all $j \leqslant i$. For the relations

$$f^{(i)}(a-k)u_i = 0, \qquad (i = 1,2,\ldots,s)$$

we have

$$C_2(v)f^{(i)}(a-k)u_i = (\underline{v}f^{(i)}(a-k))u_i + f^{(i)}(a-k)(C(v)u_i),$$

$$= (v\circ f^{(i)}(a-k))u_i + f^{(i)}(a-k)\sum_{j=1}^{i} k_j u_j, \qquad (k_j \in F)$$

$$\subseteq F(a)f^{(i-1)}(a-k)Q(T_1)u_i + \sum_{j=i}^{l} k_j f^{(i)}(a-k)u_j$$
$$\subseteq F(a)f^{(i-1)}(a-k)\sum_{j=i-1}^{l} Fu_j + 0$$
$$= 0, \text{ as required.}$$

To show the invariance of the representation assigned to L_1 by M, it is sufficient to verify the invariance of (1) a and (2) any $t \in T$.

(1) $C_2(v)(aw) - a(C_2(v)w)$ $\qquad (w \in M)$

$$= C_2(v)(a\sum_{i=1}^{s} p_i(a)u_i - a(C_2(v)\sum_{i=1}^{s} p_i(a)u_i) \qquad (p_i(a) \in F(a))$$

$$= \sum_{i=1}^{s}(C_2(v)(ap_i(a)u_i) - a(C_2(v)p_i(a)u_i)),$$

$$= \sum_{i=1}^{s}((\underline{v}(ap_i(a))u_i + ap_i(a)(C(v)u_i) - a((\underline{v}p_i(a))u_i) - ap_i(a)(C(v)u_i))$$

$$= \sum_{i=1}^{s}((v\circ a)(p_i(a)u_i) + a((v\circ p_i(a))u_i) - a((v\circ p_i(a))u_i))$$

$$= (v\circ a)w, \text{ as required.}$$

(2) Replacing a by $t \in T$, a similar calculation gives $C_2(v)(tw) - t(C_2(v)w) = (v\circ t)w$, as required.

By this theorem it is evident that if Q is invariant it generates, step by step, an invariant representation of the radical, the matrices of invariance being defined at each step. We can now show that an invariant representation of the radical can be extended to the whole Lie algebra without a change of degree. For convenience, let us take the invariant representation of the radical to be Q and the matrices of invariance, which are assigned, according to theorem 1.2, to each element d of L to be $C(d)$. Then the matrices $C(v)$, $v \in V$, generate a Lie algebra V^*

<u>Theorem 3.6</u>. The elements x of V^* which annihilate $Q(t)$, $t \in R$, the radical, form an ideal Z of V^*.

<u>Proof.</u> Since $x o Q(t) = 0$, we have

$$(C(v) o x) o Q(t) = C(v)(x o Q(t)) - x o (C(v) o Q(t)) = 0 - x o Q(v o t) = 0.$$

Thus $C(v) o x \in Z$ and $C(v) o Z \leqslant Z$, as required.

<u>Theorem 3.7</u>. The algebra of residue classes $V^* - Z$ is semi-simple.

<u>Proof</u>. For v', v'' in V, we have

$$(C(v') o C(v'') - C(v' o v'')) o Q(t) = (C(v') o C(v'')) o Q(t) - C(v' o v'') o Q(t),$$

$$= (C(v') o Q(t)) o C(v'') + C(v') o (C(v'') o Q(t)) - Q((v' o v'') o t),$$

$$= Q(v' o t) o C(v'') + C(v') o Q(v'' o t) - Q((v' o v'') o t),$$

$$= Q((v' o t) o v'') + Q(v' o (v'' o t)) - Q((v' o v'') o t),$$

$$= Q((v' o t) o v'' + v' o (v'' o t) - (v' o v'') o t) = Q(0) = 0.$$

Thus $$C(v') o C(v'') = C(v' o v'') \bmod Z$$

and certainly $$kC(v') = C(kv') \bmod Z,$$

$$C(v') + C(v'') = C(v' + v'') \bmod Z.$$

It follows that the mapping

$$v \longrightarrow C(v) \bmod Z$$

is a homomorphism of V onto $V^* - Z$, Since V is semi-simple, the kernel K of the mapping is semi-simple, and also $V - K$. Then

$V^* - Z$, being isomorphic to $V - K$, is semi-simple.

Theorem 3.8. If $V^* - Z$ is semi-simple, it has a representative algebra(an extension of the theorem of Levi).

Proof. By the theorem of Levi, $V^* = W \dotplus R(V^*)$, $R(V^*)$ being the radical of V^*. Since $V^* - Z$ is semi-simple, $Z \leqslant R(V^*)$. Consequently, $W = W_1 \dotplus W \cap Z$, where $W \cap Z$ is an ideal of W. Thus the difference algebra $W - W \cap Z$ has W_1 as a representative algebra. Also

W is a representative algebra of $V^* - R(V^*)$,

thus W_1 is a representative algebra of $(V^* - R(V^*)) - W \cap Z$,

and then W_1 is a representative algebra of $V^* - Z$.

As a result of this theorem, for every $v \in V$, there is a $C'(v) \in W_1$ such that $C(v) = C'(v) \bmod Z$. On replacing $C(v)$ by $C'(v)$, we have

$$C'(v' + v'') = C'(v') + C'(v''),$$

$$C'(kv') = kC'(v'),$$

$$C'(v' \circ v'') = C'(v') \circ C'(v'').$$

With this replacement and setting $C'(r) = C(r) = Q(r)$, $r \in R$, C' is an ordinary representation of L.

Remarks. We have now shown how, under certain conditions a representation Q of an ideal T, occuring in the radical, generates a representation of the radical and this representation can be extended to the whole Lie algebra without change of degree. Moreover the construction of the representation is such that it permits us to assign one arbitrary eigen value to each basis element of the radical not in T.The other eigen values differ from the assigned value by sums of integral multiples of the characteristic roots of its regular representation, the multiplicity being governed by the degree of Q.

This flexibility gives the construction the ability to reproduce any indecomposable representation , in an algebraically closed field of characteristic zero, as a component of a representation the construction assigns to the Lie algebra. For if one has an indecomposable representation in such a field, it induces on the radical a representation such that the eigen values of any basis element not in T_1 differ only by sums of multiples of the characteristic roots of its regular representation. On T_1 a nil representation must be induced which is clearly invariant under L. Taking T to be T_1 and Q as the nil representation, our construction can reproduce the given conditions but not precisely, since more eigen values will be assigned to the radical elements than the given ones. The possible extensions of the representation to the whole algebra will include the original extension, consequently our representation will include the given representation as a component.

6. <u>The theorem of Ado</u>. Birkhoff[2] has shown that every nilpotent Lie algebra has a faithful representation of finite degree, namely the regular representation of the imbedding

algebra modulo a certain invariant subalgebra. For a suitable choice of basis this representation is properly triangular, and so is a nil representation. Using this result we can prove the theorem of Ado, which states that any Lie algebra over an algebraically closed field of zero characteristic has a faithful representation by matrices of finite degree.

Let L be a Lie algebra over an algebraically closed field of characteristic zero. Let its maximal nilpotent ideal be T and the faithful nil representation by the Birkhoff procedure be Q. We have $L = V \dotplus R$, where R is the radical of L and hence contains T. Q is already a nil representation, and since $T_1 = L \circ L \cap R$ is nilpotent, T_1 is contained in T and so Q induces a nil representation on T_1. With the following theorem, all the conditions are satisfied for determining a representation of L from Q by our construction.

<u>Theorem 3.9</u>. Q is invariant under V.

<u>Proof</u>. We have that H(T) is a subalgebra of H(L). Also, to $v \in V$ there corresponds the derivation D(v) of H(L) defined by

$$D(v)(X) = vX - Xv \qquad (X \in H(L)).$$

Since
$$D(v)(t) = vt - tv = v \circ t \in T$$

for all t in T, we have $D(v)(T) \subseteq T$ and also $D(v)(H(T)) \subseteq H(T)$. Hence D(v) induces a derivation on H(T). Moreover, D(v) leaves invariant the subalgebra S of H(T) generated by T, and so leaves invariant S^W. Hence D(v) induces a derivation C(v) of the hypercomplex system $H(T) - S^w$. The representation Q is the regular representation of $H(T) - S^w$ with a suitable value of the exponent w.

Since $C(v)$ is a derivation of $H(T) - S^w$, for $t \in T$, $u \in H(T) - S^w$, we have

$$C(v)(tu) = C(v)(t)u + tC(v)(u),$$

$$= (v \circ t)u + tC(v)(u),$$

$$(v \circ t)u = C(v)(tu) - tC(v)(u)$$

or
$$Q(v \circ t) = C(v)Q(t) - Q(t)C(v) = C(v) \circ Q(t),$$

so Q is invariant under V.

The representation A of L determined from Q by our construction is faithful over the radical R of L since Q is faithful. Let P be the regular representation of L, then the representation $U = A \dotplus P$ is faithful. Consider the elements x in L for which $U(x) = A(x) \dotplus P(x) = 0$. $P(x) = 0$ places x in the centre of L, i.e. $L \circ x = 0$. The centre is nilpotent, therefore $x \in T$. Since A is faithful over T, $A(x) = 0$ gives $x = 0$. Hence U is faithful proving Ado's theorem.

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