MACHINES IN CLOSED CATEGORIES IN GENERAL AND IN CATEGORIES OF HEYTING ALGEBRA VALUED SETS IN PARTICULAR

by

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ABSTRACT

A survey of the minimal realization theory of Arbib and Manes for "state-behavior" machines in a category is given, and how the closed category machines of Goguen are included in the above machines is discussed in detail. A survey of the non-deterministic treatment due to Arbib and Manes is given. A study of C-machines in a closed category for a monoid C is given in both the deterministic and the non-deterministic cases. A notion of u-machine in a topos for a morphism of monoids u is introduced and studied. A discussion of the category of H-valued sets as a topos is given and finally some of the concepts of automata theory for the deterministic case are investigated in this context, regarding the category of H-valued sets as a closed category and especially when H is a finite chain.

RESUME

On donne une révision de la théorie de réalisation minimale d'Arbib et Manes des machines "state-behavior" dans une catégorie. Un traitement approfondi de comment les machines de Goguen sont comprises dans les antérieures a été effectué. On donne une révision de l´approche non-déterministe d´Arbib et Manes. On a étudié des C-machines dans une catégorie fermée pour un monoïde C dans les cadres déterministe et non-déterministe. Une notion de u-machine a été introduite et étudiée dans un topos pour un morphisme de monoïdes u. On donne une description de la catégorie des ensembles H-valorisés en tant que topos; enfin, une investigation, dans ce contexte, de divers concepts de la théorie des automates dans le cadre déterministe est donnée, en considérant la catégorie des ensembles H-valorisés comme une catégorie fermée et plus spécialement lorsque H est une chaîne finie.

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INTRODUCTION

The aim of this thesis is to discuss in some detail certain aspects of automata theory (or the theory of machines) in a category and in a closed category, following the pioneering work of Arbib & Manes [2, 4, 5, 6], Bainbridge [8], Ehrig [18], and Goguen [22], to mention a few.

A topos is a special kind of category [27], and a particularly illuminating example of a topos in connexion with logic is a topos of H-valued sets for a complete Heyting algebra H, introduced by Higgs [25]. After discussing the concepts in general, they are then studied in this type of category, for the deterministic case, in order to gain some intuition for the type of generality afforded by the categorical approach.

Automata theory pre-dates digital computers and is today a basic tool in the newest theories of concurrent computation. The main point of the categorical machine theory was to give a unified treatment of important concepts arising from system theory and automata theory. The present thesis uses mainly as a framework of discussion the Arbib-Manes formulation of the categorical machine theory and the origins of that formulation lie in the elements traced in what follows .

The linear systems had a module-theoretic approach in the work due to Kalman [28] in 1969, however its origins lie in the theory of continuous-time linear dynamical systems. Arbib and Zeiger showed [7] that a number of Kalman's concepts were special cases of notions developed in automata theory and they suggested the use of category theory in the study of this new perspective; Goguen's machines in a closed category [22] formalized this much. The works done by Arbib & Manes [2] and Bainbridge [8] followed shortly. Before $Dyn(X)$ (see 1.1.1) was introduced to study categorical automata, it was very closely studied by the Prague school [1] involving Adámek, Trnková, Reiterman and others in the years 1970-74. Another element of influence was the development of triples (or monads) in the sixties.

The non-deterministic case was subsequently treated by Arbib and Manes in [4] and [32] and elements that played an important role on that development were the triples, the work done by Beck on distributive laws

[11], the Kleisli categories [29], and the notion of "scoop" introduced by Ehrig [18] and modified by Arbib and Manes.

As mentioned before a special kind of category used in this thesis is that of a topos [27]. Topos theory has its origins in two different areas of mathematical research in the sixties. Firstly the word "topos" was used by Grothendieck in the context of algebraic geometry to denote categories satisfying certain conditions; these kind of categories are called at present Grothendieck toposes. Secondly Lawvere attempted to axiomatise the category of sets and later, in 1969 Lawvere and Tierney began to investigate the consequences of taking the existence of a subobject classifier as an axiom and the outcome was the more general notion of (elementary) topos. In words of M. Bunge [16], "A topos may be viewed as a universe of variable sets, as a generalized topological space, and as a semantical universe for higher order intuitionistic logic" A generalised concept of "set" as consisting of a collection of (partial) elements with a measure of the degree of equality of these elements admits an axiomatic treatment in particular toposes, namely the toposes of H-valued sets for a complete Heyting algebra H [25].

The preliminaries of this thesis contain some basic notions of automata theory that are generalized in the categorical approach

Chapter 1 contains, in section 1.1, a survey of the minimal realization theory of Arbib and Manes for "state-behavior" machines in a category [2, 5, 32]. Section 1.2 deals with the closed category machines of Goguen [22] and contains a detailed discussion of the known fact of their inclusion in the "state-behavior" machines. The chapter ends with a survey of the treatment given by Arbib and Manes of the non-deterministic case [4, 32].

Chapter 2 includes in its section 2.1 a detailed discussion of C-machines in a closed category for a monoid C, following section 1.1 as a guide and making use of the notion of right action of a monoid on an object. Section 2.2 extends the approach of the C-machines in a closed category of section 2.1 to the non-deterministic case following section 1.3 as a framework Section 2.3 deals with a notion of u-machine in a topos for a morphism of monoids u, this notion has been obtained from the ideas related to the machines in the hyperdoctrine (cat, Set) of

Bainbridge [8]; this approach extends the one given in section 2.1 when the closed category is also a topos.

Chapter 3 contains in section 3.1 a description, without proofs, of the category of H-valued sets as a topos following the work due to Higgs [25] and then, a detailed description of the exponentiation, the coproduct and the "free monoid" on an object in that category are given. In section 3.2 the concepts of reachability, response map, run map, and observability are investigated for a machine in the category of H-valued sets, regarding it as a closed category, and especially when H is a finite chain.

In addition to a detailed and unified description of developments which are given in the literature quoted in the references, the present thesis contains some original work, as follows. In 2.2.8, the notion of a "d-machine" is introduced and this, in turn, is employed in order to transport the non-deterministic case to the context of the machines in a closed category for a monoid C. In 2.3 the ideas related to the concept of machine in the hyperdoctrine (cat, Set) of [8] are used to introduce and study the notion of u-machine in a topos relative to a morphism of monoids u, giving in this context a corresponding "minimal realization theorem". For H-valued sets, a detailed description of the notions involved in the theory of machines for the deterministic case is given, regarding the latter as a closed category. Finally, the meaning of these concepts is made precise in the case of a topos of H-valued sets for a finite chain H.

PRELIMINARIES

These preliminaries contain some basic concepts of automata theory that are generalized in the categorical approach.

Sequential machine. One may imagine a machine that can be in any one of a finite number of internal configurations or states, receive any one of a finite number of inputs, and emit any one of a finite number of outputs. One may think of it as receiving inputs, changing states, and emitting outputs once every "cycle" of some clock which times its activity. Such a system may be represented as follows [6, p. 93] :

A *sequential machine* is a sextuple $M = (X_0, Q, \delta, q_0, Y, \beta)$ where X_0, Q, and Y are sets (called the set of *inputs*, *states*, and *outputs* respectively), $q_0 \in Q$ is the *initial state*, and $\delta: Q \times X_0 \to Q$ and $\beta : Q \times X_0 \to Q$ are functions (called the *next-state function* or *dynamics* and *output map* respectively). M is said to be *finite* if X_0, Q and Y are all finite sets.

Taking the "cycle time" of M's "clock" as the unit on the time scale, one can imagine M as representing a system which starts in state q_0 at time 0, and which is such that if it is in state $q(t) \in Q$ at time t, and then receives input $x(t)$, it will emit output $y(t) = \beta(q(t)) \in Y$ and then settle into state $q(t+1) = \delta(q(t), x(t)) \in Q$ by time $t+1$

Run map. Given a sequential machine M, a single input $x \in X_0$ sends M from state q to state $\delta(q, x)$, from which a second input x' sends M to state $\delta(\delta(q, x), x')$, from which a third input x'' sends M to state $\delta(\delta(\delta(q, x), x'), x'')$, and so on In general, let $X_0{}^*$ be the free monoid generated by X_0, one defines inductively $\delta^*: Q \times X_0{}^* \to Q$ by :

$\delta^*(q, \Lambda) = q$ for all $q \in Q$ where Λ denotes the unit of $X_0{}^*$,

$\delta^*(q, wx) = \delta(\delta^*(q, w), x)$ for all $q \in Q$, $w \in X_0{}^*$, $x \in X_0$.

δ^* is called the *run map* of the dynamics $\delta: Q \times X_0 \to Q$ of M ([6, p. 97] and [2, p. 165]).

Reachability and observability. Two important questions in automata theory ([6, pp. 97-100] and [32, p. 293]) are the following .

Reachability : Given a state q of a sequential machine M does there exist a sequence of inputs which will drive M from its initial state q_0 to that state q ?

Observability . Given a sequential machine M which may be in either state q or q', does there exist a sequence of inputs, M's response to which will enable someone to tell the two states apart ?

The *reachability map* of M is the map $r : X_0^* \to Q$ that sends w to $\delta^*(q_0,w)$. M is said to be *reachable* if r is a surjective map, i.e. if every state of Q is reachable from q_0. The input/output response of M is the map $f : X_0^* \to Y$ defined by $f = \beta \cdot r$

The *observability map* of M is the map $\sigma : Q \to Y^{X_0^*}$ that sends the state q to M_q where $M_q(w) = \beta(\delta^*(q, w))$. M_q is the input/output response which would result if q (rather than q_0) were the initial state. M is said to be *observable* if σ is an injective map, i.e. if distinct states have distinct responses.

Realization A basic concept of automata theory is that of *realization:* An arbitrary function $f : X_0^* \to Y$ is called a *response*, and a sequential machine $M = (X_0, Q, \delta, q_0, Y, \beta)$ is called a *realization* of f if its input/output response coincides with f. It is intuitively clear that optimal realizations of f must be at least reachable and observable. A reachable and observable realization of f always exists [32, p. 293].

1. MACHINES IN A CATEGORY : A SURVEY.

1.1. Machines in an arbitrary category.

(1.1.1) Let C be a category. A *process* in C is an endofunctor $X : C \to C$. The category $\mathrm{Dyn}(X)$ of *X-dynamics* has as objects all pairs (Q, δ) where Q is an object of C and $\delta : XQ \to Q$ is a C-morphism, and as morphisms, *X-dynamorphisms*, $f : (Q, \delta) \to (Q', \delta')$ all C-morphisms $f : Q \to Q'$ such that :

$$\begin{array}{ccc} XQ & \xrightarrow{\delta} & Q \\ {\scriptstyle Xf}\downarrow & & \downarrow{\scriptstyle f} \\ XQ' & \xrightarrow[\delta']{} & Q' \end{array}$$

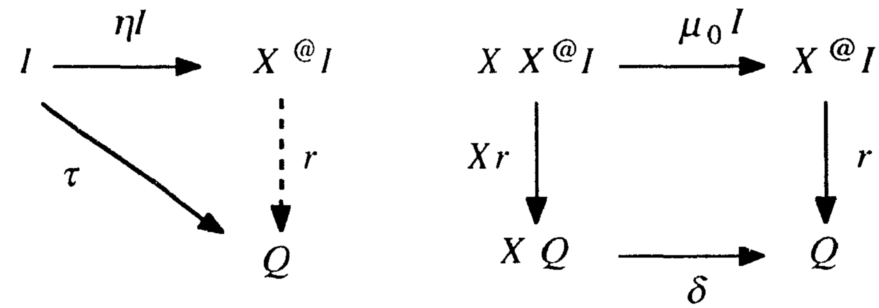
commutes. Composition and identities are defined as in C. [2, p. 177].

For example, considering the category ***Set*** of sets and maps and a fixed set X_0, one has the functor $-\times X_0 : \mathbf{Set} \to \mathbf{Set}$ that sends $f : Q \to Q'$ to $f \times 1 : Q \times X_0 \to Q' \times X_0$, and an $(-\times X_0)$-dynamics is just a map $\delta : Q \times X_0 \to Q$, the next-state function of a sequential machine.[5, pp. 315-316].

(1.1.2) Let (Q, δ) be an object of $\mathrm{Dyn}(X)$, I, Y objects of C; $\tau : I \to Q$, $\beta : Q \to Y$ morphisms in C. A *machine* in C for the process X is a 6-tuple $M = (Q, \delta, I, \tau, Y, \beta)$. Q, I, Y are called respectively the *state object*, *initial object* and *output object*; τ is the *initial state*, and β is the *output morphism*. ([2, p. 177] and [32, pp. 294-295]).

For example, considering the above process $-\times X_0 : \mathbf{Set} \to \mathbf{Set}$, $\delta : Q \times X_0 \to Q$, $q_0 : 1 \to Q$ (viewing the arrow as an element $q_0 \in Q$) and a map $\beta : Q \to Y$, the 6-tuple $M = (X_0, Q, \delta, q_0, Y, \beta)$ is a sequential machine . [5, pp. 313-314].

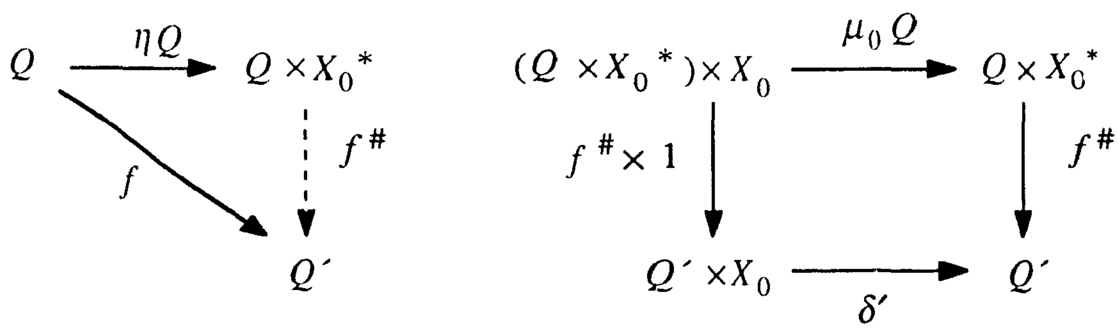
(1.1.3) X is called an *input process* if the forgetful functor $U : \mathrm{Dyn}(X) \to \boldsymbol{C}$ has a left adjoint. The free dynamics over Q with respect to U will be denoted by $((X^{@}Q\ , \mu_0 Q\ ), \eta Q)$. The unique dynamorphic extension of the initial state $\tau : I \to Q$ is called the *reachability map* $r : (X^{@}I, \mu_0 I\ ) \to (Q, \delta\ )$ of M :



The *response map* of M is defined to be the $\boldsymbol{C}$-morphism $\beta{\cdot}r : X^{@}I \to Q \to Y$. A map $f : X^{@}I \to Y$ is called a *response* and if a machine M has as response map f, one says that M is a *realization* of f. ([32, p. 295] and [2, p. 179]).

The forgetful functor $U : \mathrm{Dyn}(\text{-}{\times}X_0) \to \boldsymbol{Set}$ has a left adjoint : Given a set Q, the free dynamics over Q with respect to U is $((Q{\times}X_0{}^{*}, \mu_0 Q), \eta Q)$ where $X_0{}^{*}$ is the free monoid generated by X_0, $\mu_0 Q : (Q{\times}X_0{}^{*}){\times}X_0 \to Q{\times}X_0{}^{*}$ sends $((q, w), x)$ to (q, wx) and $\eta Q : Q \to Q{\times}X_0{}^{*}$ sends q to (q, Λ), where Λ denotes the unit of $X_0{}^{*}$.

Given $\delta' : Q'{\times}X_0 \to Q'$ and $f : Q \to Q'$, the two diagrams :



give, by recursion,

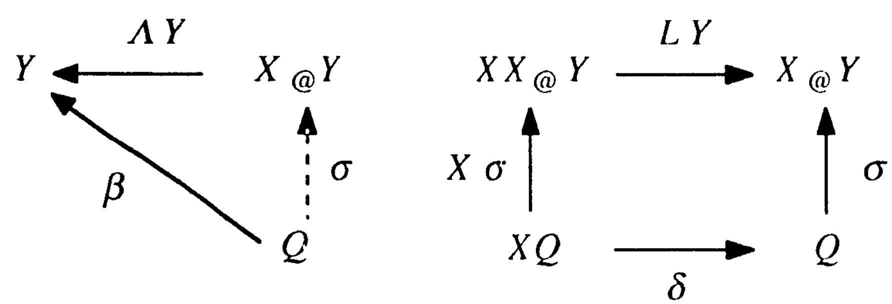
$$f^{\#}(q, \Lambda) = f(q)$$
$$f^{\#}(q, wx) = \delta'(f^{\#}(q, w), x)$$

The reachability map of $M = (X_0, Q, \delta, q_0, Y, \beta)$ is the unique dynamorphic extension of the initial state $q_0 : 1 \to Q$ and it is given by :

$$r(1, \Lambda) = q_0$$
$$r(1, wx) = \delta(r(1, w), x)$$

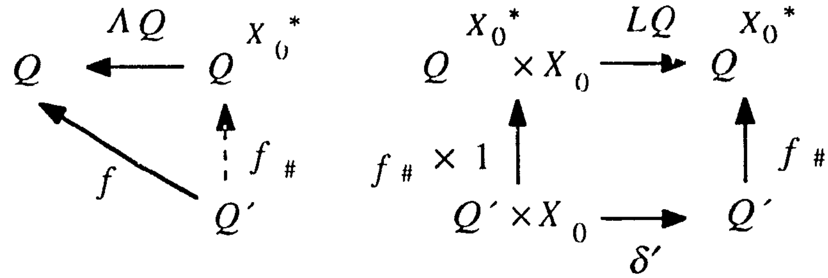
And their response map is the composition $\beta \cdot r : 1 \times X_0{}^* \to Q \to Y$. ([2, p. 176] and [32, pp. 293-296]).

(1.1.4) X is called an *output process* if the forgetful functor $U : \mathrm{Dyn}(X) \to \boldsymbol{C}$ has a right adjoint. The cofree dynamics over Q with respect to U will be denoted by $((X_@Q, LQ), \Lambda Q)$. The unique dynamorphic coextension of the output morphism $\beta : Q \to Y$ is called the *observability map* $\sigma : (Q, \delta) \to (X_@Y, LY)$ of M :



[32, p. 295].

The forgetful functor $U : \mathrm{Dyn}(\text{-}\times X_0) \to \boldsymbol{Set}$ has a right adjoint : Given a set Q , the cofree dynamics over Q with respect to U is $((Q^{X_0{}^*}, LQ), \Lambda Q)$ where $Q^{X_0{}^*}$ is the set of all maps from $X_0{}^*$ to Q, $LQ : Q^{X_0{}^*} \times X_0{}^* \to Q^{X_0{}^*}$ sends (g, x) to $g \cdot L_x$ where $L_x : X_0{}^* \to X_0{}^*$ sends w to xw and $\Lambda Q : Q^{X_0{}^*} \to Q$ sends g to $g(\Lambda)$. Given $\delta' : Q' \times X_0 \to Q'$ and $f : Q' \to Q$, the two diagrams :



give, by recursion,

$$[f\#(q')](\Lambda) = f(q')$$
$$[f\#(q')](xw) = [f\#(\delta'(q', x))](w)$$

[32, pp. 295-296].

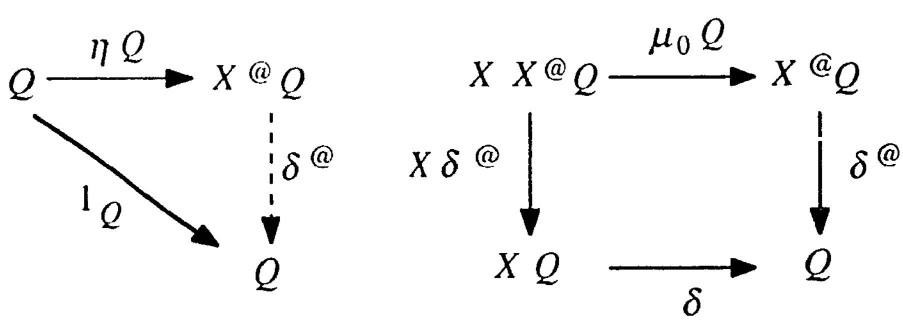
The observability map of $M = (X_0, Q, \delta, q_0, Y, \beta)$ is the unique dynamorphic coextension of the output map $\beta : Q \to Y$ and it is given by :

$$\sigma(q) = M_q \in Y^{X_0^*}$$

where $M_q(\Lambda) = \beta(q)$ and $M_q(xw) = M_{\delta(q,x)}(w)$. [6, pp.99-100].

(1.1.5) X is called a *state-behavior process* if it is both an input and an output process. [5, p. 318]. Hence the process $-\times X_0 : \mathbf{Set} \to \mathbf{Set}$ is state-behavior.

(1.1.6) Let X be an input process in C, for each object (Q, δ) of Dyn(X), the *run map* of (Q, δ) is the unique dynamorphic extension $\delta^@ : (X^@Q, \mu_0 Q) \to (Q, \delta)$ of 1_Q :



[2, p. 179].

The run map of (Q, δ) for the process $-\times X_0 : \mathbf{Set} \to \mathbf{Set}$ is $\delta^* : (Q \times X_0^*, \mu_0 Q) \to (Q, \delta)$ where :

$$\delta^*(q, \Lambda) = q$$
$$\delta^*(q, wx) = \delta(\delta^*(q, w), x)$$

[6, p. 97].

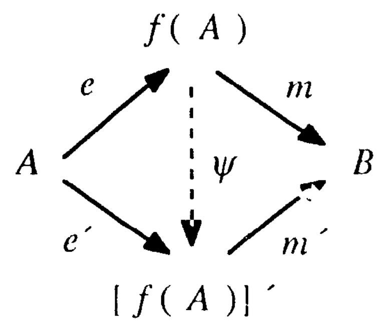
(1.1.7) An *image factorization system* for a category C is a pair (E, M), where E and M are classes of morphisms in C satisfying the following axioms :

(i) E and M are subcategories of C.

(ii) E is included in the class of epimorphisms of C and M is included in the class of monomorphisms of C.

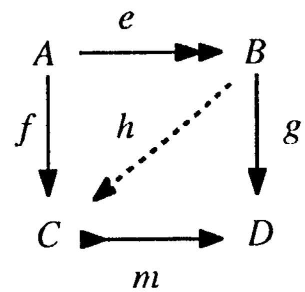
(iii) If f is an isomorphism in C, then $f \in$ E and $f \in$ M.

(iv) Every morphism $f : A \to B$ in C has an E-M factorization which is unique up to isomorphism. More precisely, there exist a pair (e, m) and an object (denoted by $f(A)$) such that $e : A \to f(A)$, $m : f(A) \to B$, $e \in$ E, $m \in$ M and $f = m \cdot e$; moreover, if (e', m') and $[f(A)]'$ are such that $e' : A \to [f(A)]'$, $m' : [f(A)]' \to B$, $e' \in$ E, $m' \in$ M and $f = m' \cdot e'$ then there exists an isomorphism $\psi : f(A) \to [f(A)]'$ such that makes the following diagram commute:



In **Set**, the pair (E, M) where E ={ surjective maps } and M = { injective maps} is an image factorization system. [5, p. 322].

(1.1.8) (Diagonal fill-in Lemma [5, p. 323]). Let (E, M) be an image factorization system for a category $\boldsymbol{C}$. Given a commutative square $g \cdot e = m \cdot f$



with $e \in$ E and $m \in$ M , there exists a unique h with $h \cdot e = f$ and $m \cdot h = g$.

(1.1.9) (Dynamorphic image Lemma [5, pp. 325-327]). Let $h : (Q, \delta) \rightarrow (Q', \delta')$ be a dynamorphism and let $e : Q \rightarrow Q''$, $m : Q'' \rightarrow Q'$ be an E-M factorization of h. Then if either X preserves E or $X^{@}$ preserves E, there exists a unique dynamics δ'' on Q'' such that $e :(Q, \delta) \rightarrow (Q'', \delta'')$ and $m :(Q'', \delta'') \rightarrow (Q', \delta')$ are dynamorphisms.

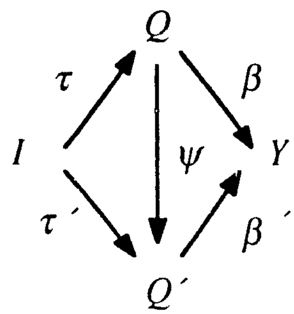
In the proof, for the case in which $X^{@}$ preserves E, the following facts are used :

- $\eta_1 : X \rightarrow X^{@}$, defined by $\eta_1 A = \mu_0 A \cdot X \eta A : XA \rightarrow XX^{@}A \rightarrow X^{@}A$ for each object A of $\boldsymbol{C}$, is a natural transformation.
- If $\delta^{@}$ is the run map of (Q, δ) then $\delta^{@} \cdot \eta_1 Q = \delta$.
- If $h : (Q, \delta) \rightarrow (Q', \delta')$ is a dynamorphism then $h \cdot \delta^{@} = \delta'^{@} \cdot X^{@} h$.

(1.1.10) (Cancellation Lemma [5, pp. 327-328]). Let $e : (Q, \delta) \rightarrow (Q', \delta')$ be a dynamorphism with $e \in$ E , let (Q'', δ'') be a dynamics and let $f : Q' \rightarrow Q''$ be an arrow in $\boldsymbol{C}$ such that $f \cdot e : (Q, \delta) \rightarrow (Q'', \delta'')$ is a dynamorphism. Then if either X preserves E or $X^{@}$ preserves E , $f : (Q', \delta') \rightarrow (Q'', \delta'')$ is a dynamorphism.

(1.1.11) Let (E, M) be an image factorization system in $\boldsymbol{C}$ and let $M = (Q, \delta, I, \tau, Y, \beta)$ be a machine; if X is an input process, M is said to be *reachable* if $r : X^{@} I \rightarrow Q$ is in E ; if X is an output process, M is said to be *observable* if $\sigma : Q \to X_{@}Y$ is in M. [32, p. 303].

(1.1.12) Fixing I and Y, but letting Q vary, let X-*mach* be the category whose objects are machines $M = (Q, \delta, I, \tau, Y, \beta)$: and whose morphisms are *simulations* $\psi : M \to M'$ i.e. dynamorphisms $\psi : (Q, \delta) \to (Q', \delta')$ such that :



Let X be an input process in C, if $\psi : M \to M'$ is a simulation of X- machines then M and M' have the same response. [5, pp. 323-324].

(1.1.13) (Minimal realization theorem [32, pp. 303-304]). Let $X : C \to C$ be a state-behavior process and let (E, M) be an image factorization system in C such that either X preserves E or $X^{@}$ preserves E then :

For any response $f : X^{@}I \to Y$ there exists a reachable and observable realization $M_f = (Q_f, \delta_f, I, \tau_f, Y, \beta_f)$ of f. Any such M_f is a terminal object in the category of reachable realizations of f and simulations and any such M_f is an initial object in the category of observable realizations of f and simulations; thus M_f is unique up to isomorphism.

Diagonal fill-in lemma, dynamorphic image lemma and cancellation lemma are crucial points in the proof of the above theorem.

1.2. Machines in a closed category with countable coproducts, relative to an object X.

(1.2.1) A *monoidal category* $\mathbf{C} = (\mathbf{C}, \otimes, E, \alpha, \lambda, \rho)$ consists of a category $\mathbf{C}$, a bifunctor $\otimes : \mathbf{C}\times\mathbf{C} \to \mathbf{C}$, an object E of $\mathbf{C}$, and three natural isomorphisms α, λ, ρ. Explicitly,

$$\alpha_{A,B,C} : A\otimes(B\otimes C) \cong (A\otimes B)\otimes C$$

is natural for all objects A, B, C of $\mathbf{C}$, and the diagram

$$\begin{array}{ccccc} A\otimes(B\otimes(C\otimes D)) & \xrightarrow{\alpha} & (A\otimes B)\otimes(C\otimes D) & \xrightarrow{\alpha} & ((A\otimes B)\otimes C)\otimes D \\ {\scriptstyle 1\otimes\alpha}\downarrow & & & & \uparrow{\scriptstyle \alpha\otimes 1} \\ A\otimes((B\otimes C)\otimes D) & & \xrightarrow[\alpha]{\qquad\qquad} & & (A\otimes(B\otimes C))\otimes D \end{array}$$

commutes for all objects A, B, C, D of $\mathbf{C}$. $\lambda_A : E\otimes A \cong A$ and $\rho_A : A\otimes E \cong A$ are natural for all objects A of $\mathbf{C}$, the diagram

$$\begin{array}{ccc} A\otimes(E\otimes C) & \xrightarrow{\alpha} & (A\otimes E)\otimes C \\ & {\scriptstyle 1\otimes\lambda}\searrow \quad \swarrow{\scriptstyle \rho\otimes 1} & \\ & A\otimes C & \end{array}$$

commutes for all objects A, C of $\mathbf{C}$, and $\lambda_E = \rho_E : E\otimes E \to E$.

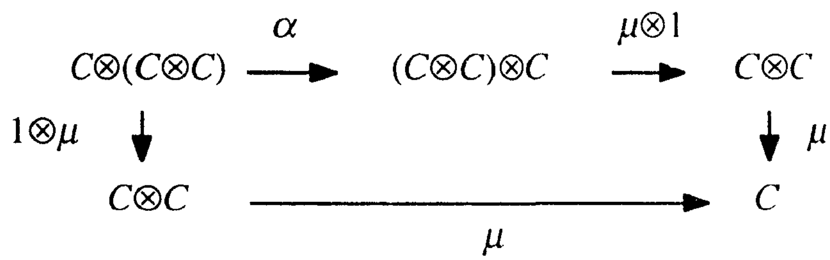
The above data imply also the commutativity of the diagrams :

$$\begin{array}{ccc} E\otimes(B\otimes C) & \xrightarrow{\alpha} & (E\otimes B)\otimes C \\ & {\scriptstyle \lambda}\searrow \quad \swarrow{\scriptstyle \lambda\otimes 1} & \\ & B\otimes C & \end{array} \qquad \begin{array}{ccc} A\otimes(B\otimes E) & \xrightarrow{\alpha} & (A\otimes B)\otimes E \\ & {\scriptstyle 1\otimes\rho}\searrow \quad \swarrow{\scriptstyle \rho} & \\ & A\otimes B & \end{array}$$

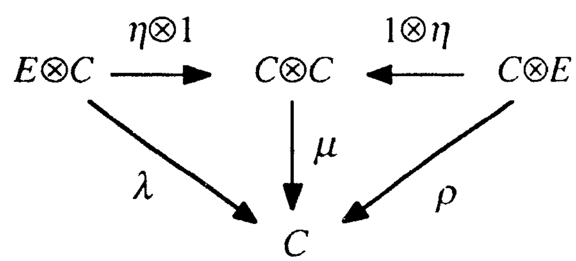
The monoidal category $\mathbf{C}$ is called a *strict monoidal category* if, further, α, λ, ρ are all equalities.

Any category with finite products is monoidal if one takes $A \otimes B$ to be (any chosen) product of the objects A, B and E to be a terminal object, while α, λ and ρ are the unique isomorphisms which commute with the respective projections. [31, pp. 157-159].

(1.2.2) A *monoid* in the monoidal category $(\mathbf{C}, \otimes, E, \alpha, \lambda, \rho)$ is an object C of $\mathbf{C}$ together with two arrows $\mu : C \otimes C \to C$, $\eta : E \to C$ such that :

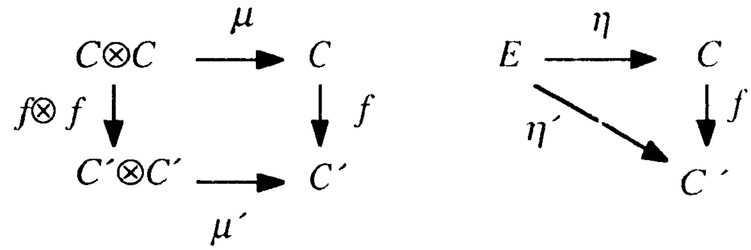


(associative law)



(unitary laws)

A morphism $f : (C, \mu, \eta) \to (C', \mu', \eta')$ of monoids is an arrow in $\mathbf{C}$ $f : C \to C'$ such that :



The category of monoids in C is denoted by Mon_C .([31, pp. 166-167]).

(1.2.3) If the monoidal category C has countable coproducts and for each object B of C, the functors $B\otimes$- and -$\otimes B : C \to C$ preserve them, then for each object A of C one has a "free monoid" (A^*, μ, i_0). In fact [31, pp. 168-169] the forgetful functor $U : \mathrm{Mon}_C \to C$ has a left adjoint. Here is a construction of (A^*, μ, i_0) :

Let A be an object of C, for each natural number n one defines the object A^n recursively as follows :

$$A^0 = E, \quad A^1 = A, \quad A^{n+1} = A\otimes A^n \quad \text{for} \quad n \geq 1.$$

Now let $(i_m : A^m \to \oplus_m A^m = A^*)_{m \in N}$ be a coproduct in C, since -$\otimes A^n$ preserves it one has that $(i_m \otimes 1 : A^m \otimes A^n \to A^*\otimes A^n)_m$ is also a coproduct. Moreover, since $(i_n : A^n \to A^*)_n$ is a coproduct in C, one has that :

$$((1\otimes i_n)\cdot(i_m \otimes 1) : A^m \otimes A^n \to A^*\otimes A^n \to A^* \otimes A^*)_{m,n} =$$
$$= (i_m \otimes i_n : A^m \otimes A^n \to A^*\otimes A^*)_{m,n}$$

is a coproduct in C.

Now, for any natural numbers m, n one defines the isomorphisms $c_{m,n} : A^m \otimes A^n \to A^{m+n}$ as follows :

For $n = 0$:

$$c_{m,0} : A^m \otimes E \to A^m = \rho$$

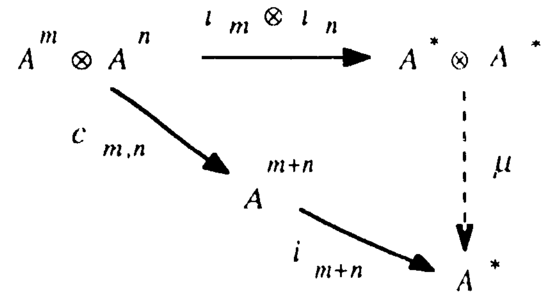
For $n \geq 1$:

$$c_{0,n} : E\otimes A^n \to A^n = \lambda$$

$$c_{1,n} : A\otimes A^n \to A^{n+1} = 1$$

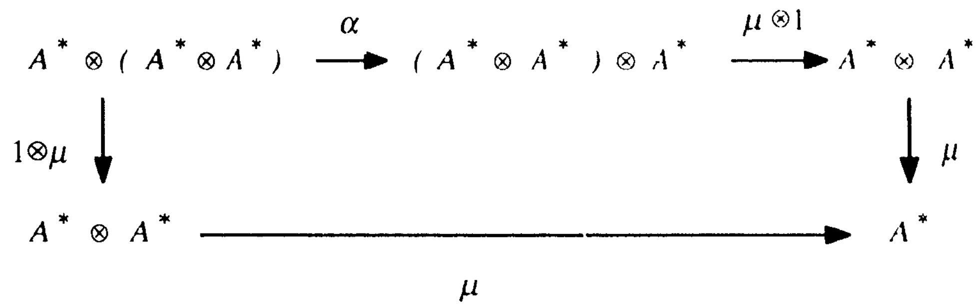
$c_{m+1,n} = (1\otimes c_{m,n})\cdot\alpha^{-1}: (A\otimes A^m)\otimes A^n \to A\otimes(A^m \otimes A^n) \to A\otimes A^{m+n}$ for $m \geq 1$.

Now, since $(i_m \otimes i_n : A^m \otimes A^n \to A^*\otimes A^*)_{m,n}$ is a coproduct, defining $\mu : A^*\otimes A^* \to A^*$ by :



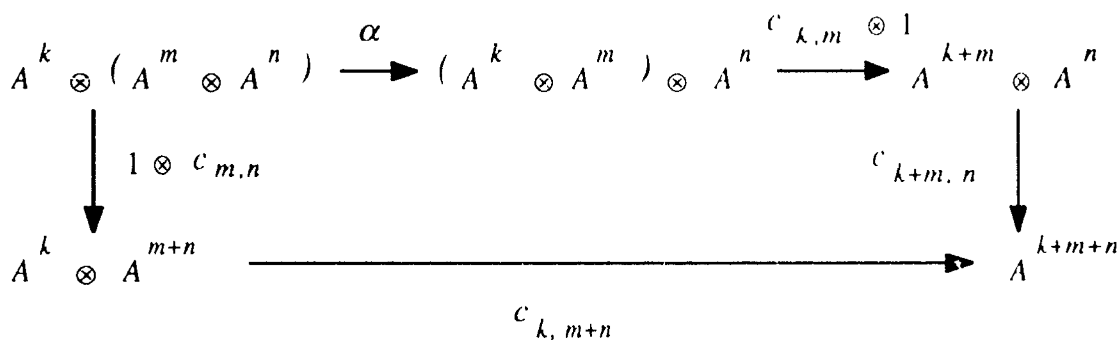
one has that (A^*, μ, i_0) is a monoid :

- The diagram



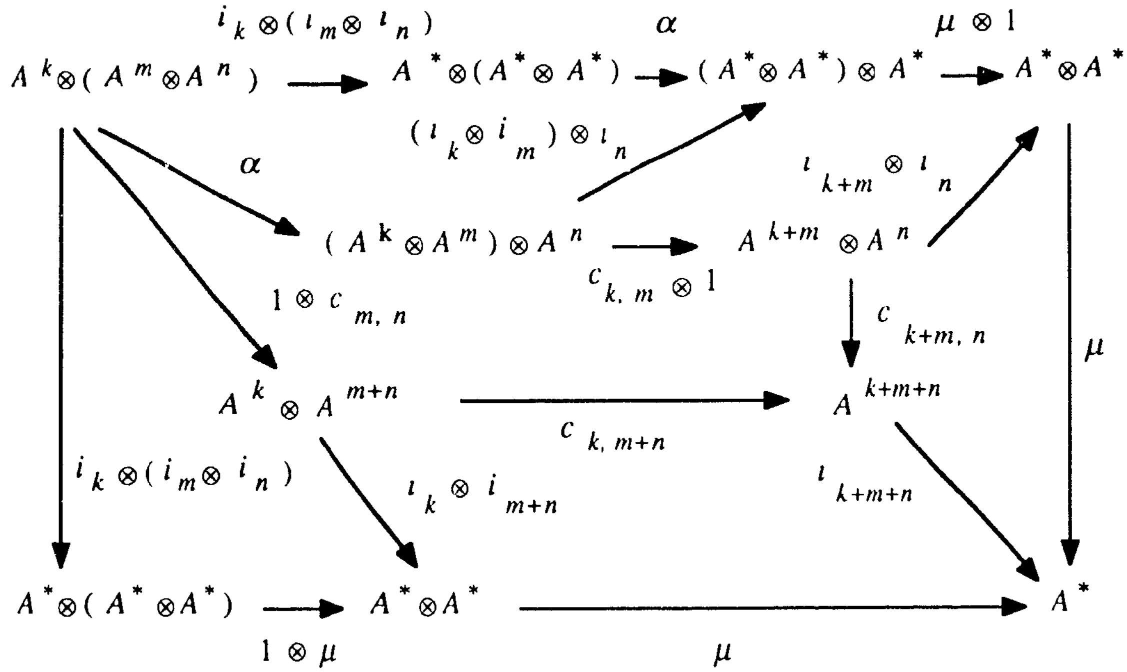
commutes :

First, from the definitions of $c_{m,n}$ one has that for any natural numbers k, m, n the diagrams

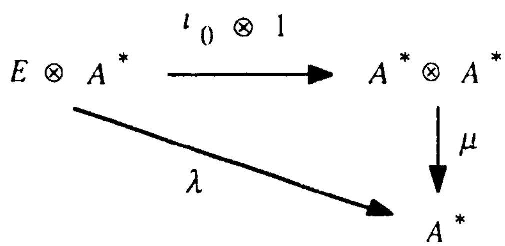


commute as can be shown by induction on k.

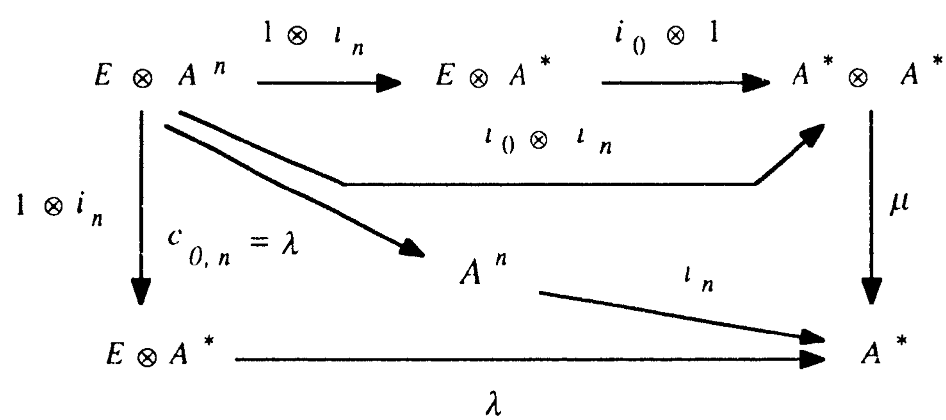
Now, since $(i_k \otimes (i_m \otimes i_n) : A^k \otimes (A^m \otimes A^n) \to A^* \otimes (A^* \otimes A^*))_{k,m,n}$ is also a coproduct in $\boldsymbol{C}$, the fact follows from the commutative diagrams :



- The diagram

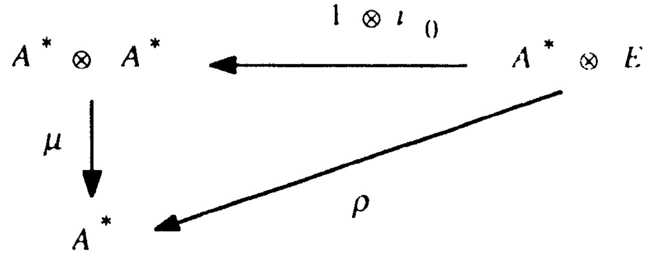


commutes because the commutativity of the following ones :

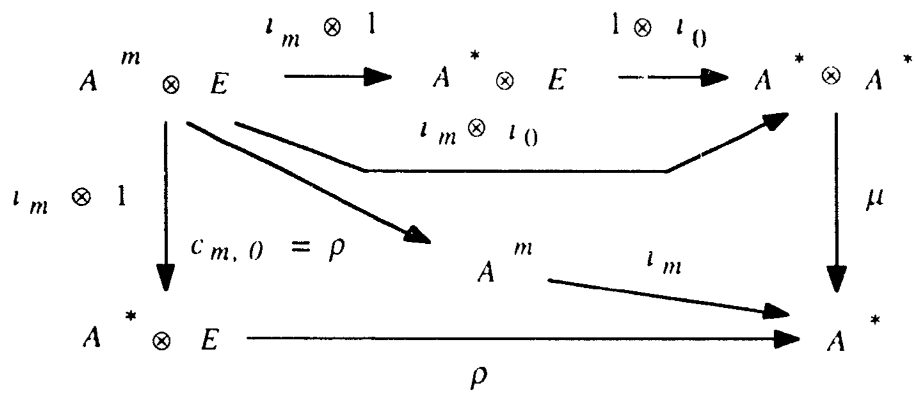


(1.2)

- The diagram



commutes because the commutativity of the following ones :

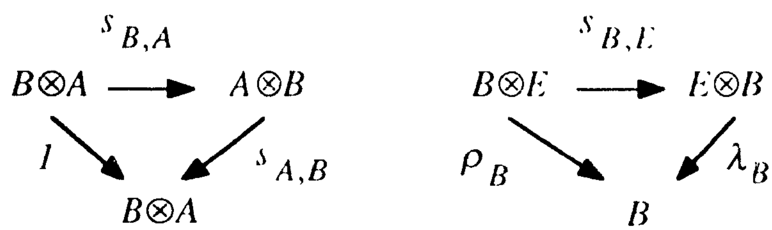


Therefore, given any object A of $\boldsymbol{C}$, the triple (A^*, μ, i_0) is a monoid.●

(1.2.4) A monoidal category $(\boldsymbol{C}, \otimes, E, \alpha, \lambda, \rho)$ is called *symmetric* if there are isomorphisms

$$s_{A,B} : A \otimes B \cong B \otimes A$$

natural in A and B for each pair of objects A, B of $\boldsymbol{C}$, such that the diagrams



$$\begin{array}{ccccc}
A\otimes(B\otimes C) & \xrightarrow{\alpha} & (A\otimes B)\otimes C & \xrightarrow{s} & C\otimes(A\otimes B) \\
{\scriptstyle 1\otimes s}\downarrow & & & & \downarrow{\scriptstyle \alpha} \\
A\otimes(C\otimes B) & \xrightarrow[\alpha]{} & (A\otimes C)\otimes B & \xrightarrow[s\otimes 1]{} & (C\otimes A)\otimes B
\end{array}$$

commute.

A monoidal category $(C, \otimes, E, \alpha, \lambda, \rho)$ where $\otimes$ is the categorical product is symmetric when $s : A\times B \cong B\times A$ is taken to be the (canonical) isomorphism which commutes with the projections.

A *closed category* C is a symmetric monoidal category in which, for any object A of C, the functor $-\otimes A : C \to C$ has a right adjoint $(-)^A : C \to C$; i. e. for each object B of C there is a morphism $\mathrm{ev} : B^A \to B$, called *evaluation*, such that for any object C of C and any morphism $f : C\otimes A \to B$ there is a unique morphism $\psi : C \to B^A$ such that makes the following diagram commute :

$$\begin{array}{ccc}
B^A\otimes A & & \\
{\scriptstyle \psi\otimes 1}\uparrow & \searrow{\scriptstyle \mathrm{ev}} & \\
C\otimes A & \xrightarrow[f]{} & B
\end{array}$$

([31, p. 180] and [18, p. 63]).

(1.2.5) Let $(C, \otimes, E, \alpha, \lambda, \rho, s)$ be a closed category with countable coproducts, an *X-monadic algebra* [22, p. 777] in C is an arrow $\delta : Q\otimes X \to Q$ in C ; and a morphism $f : \delta \to \delta'$ of such algebras is a morphism $f : Q \to Q'$ such that :

$$\begin{array}{ccc} Q\otimes X & \xrightarrow{\delta} & Q \\ {\scriptstyle f\otimes 1}\downarrow & & \downarrow{\scriptstyle f} \\ Q'\otimes X & \xrightarrow[\delta']{} & Q' \end{array}$$

Goguen denotes by $\mathrm{Mon\ddot{a}}^X$ the resulting category. In the terminology of definition (1.1.1) this is in fact the category $\mathrm{Dyn}(-\otimes X)$.

(1.2.6) Let C be a closed category with countable coproducts, then the forgetful functor $U : \mathrm{Dyn}(-\otimes X) \to C$ h..s a left adjoint. (A more general result is given in [5, p. 319]). A proof follows :

The terminology of (1.2.3) is used through this proof. Let Q be an object of C, if one takes the dynamics

$$(Q\otimes X^*, (1\otimes\mu)\cdot(1\otimes(1\otimes\iota_1))\cdot\alpha^{-1} : (Q\otimes X^*)\otimes X \to Q\otimes X^*)$$

and the C-morphism $(1\otimes\iota_0)\cdot\rho^{-1} : Q \to Q\otimes E \to Q\otimes X^*$ the only thing remains to show is that given a dynamics (Q', δ') and a C-morphism $f : Q \to Q'$ then there exists a unique dynamorphism

$$f^{\#} : (Q\otimes X^*, (1\otimes\mu)\cdot(1\otimes(1\otimes i_1))\cdot\alpha^{-1}) \to (Q', \delta')$$

such that

$$\begin{array}{ccccc} Q & \xrightarrow{\rho^{-1}} & Q\otimes E & \xrightarrow{1\otimes\iota_0} & Q\otimes X^* \\ & \searrow^{f} & & & \downarrow{\scriptstyle f^{\#}} \\ & & & & Q' \end{array}$$

commutes.

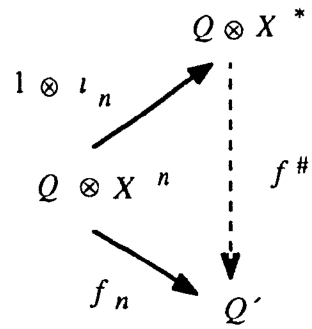
For each natural number n one defines $f_n : Q\otimes X^n \to Q'$ as follows :

$$f_0 = f\cdot\rho : Q\otimes E \to Q \to Q'$$
$$f_1 = \delta'\cdot(f\otimes 1) : Q\otimes X \to Q'\otimes X \to Q'$$

and

$f_{n+1} : Q\otimes X^{n+1} \to Q\otimes(X^n\otimes X) \to (Q\otimes X^n)\otimes X \to Q'\otimes X \to Q'$ for $n \geq 1$ is the arrow given by $\delta'\cdot(f_n\otimes 1)\cdot\alpha\cdot(1\otimes c^{-1}_{n,1})$.

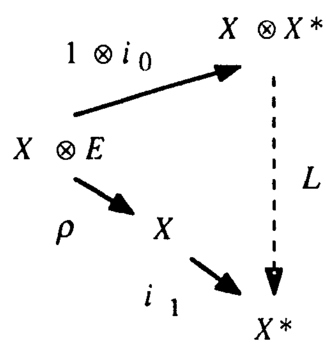
Now, since $(1\otimes i_n : Q\otimes X^n \to Q\otimes X^*)_n$ is a coproduct in $\boldsymbol{C}$, one can define an arrow $f^\#$ by :



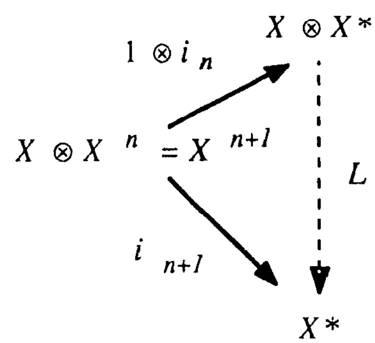
It is routine to check that this $f^\#$ works. ●

(1.2.7) Let $\boldsymbol{C}$ be a closed category with countable coproducts, then the forgetful functor U : Dyn(-$\otimes X$) $\to \boldsymbol{C}$ has a right adjoint. [18, p. 68]. A proof follows :

Let $L : X\otimes X^* \to X^*$ be defined by :

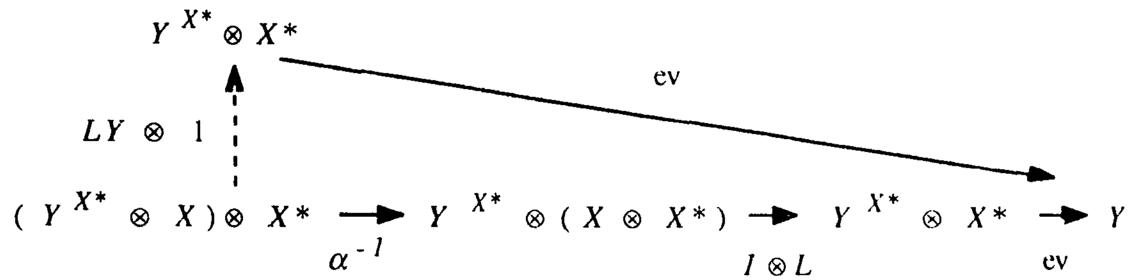


(n = 0)



(n > 0)

Let Y be an object of $\boldsymbol{C}$, define $LY : Y^{X^*}\otimes X \to Y^{X^*}$ by :

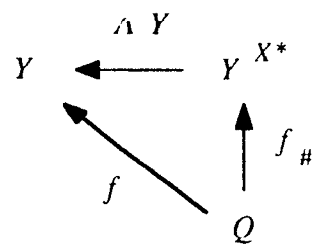


then (Y^{X^*}, LY) is an object of Dyn$(-\otimes X)$.

Consider also the C-morphism $\Lambda Y : Y^{X^*} \to Y$ given by :

$$\Lambda Y = \text{ev}\cdot(1\otimes i_0)\cdot\rho^{-1} : Y^{X^*} \to Y^{X^*}\otimes E \to Y^{X^*}\otimes X^* \to Y$$

The only thing remains to show is that given a dynamics (Q, δ) and a C-morphism $f : Q \to Y$ then there exists a unique dynamorphism $f_\# : (Q, \delta) \to (Y^{X^*}, LY)$ such that



commutes.

For each natural number n one defines $\delta_n : Q\otimes X^n \to Q$ as follows :

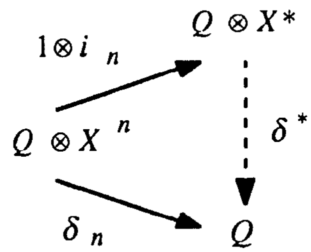
$$\delta_0 = \rho : Q\otimes E \to Q$$
$$\delta_1 = \delta : Q\otimes X \to Q$$

and

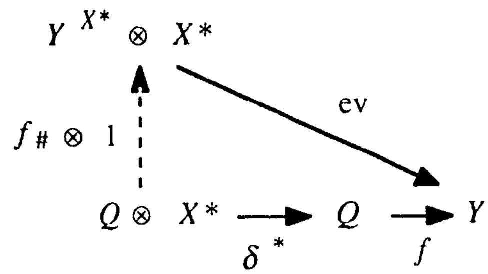
$$\delta_{n+1} = \delta_n\cdot(\delta\otimes 1)\cdot\alpha : Q\otimes(X\otimes X^n) \to (Q\otimes X)\otimes X^n \to Q\otimes X^n \to Q$$

for $n \geq 1$.

Now, one defines $\delta^* : Q\otimes X^* \to Q$ by



and $f_\#$ by



It is routine to check that this $f_\#$ works. ●

As a remark, given (Q, δ) the δ_n defined in the above proof for each natural number n have another expression :

$$\delta_0 = \rho : Q \otimes E \to Q$$

$$\delta_1 = \delta : Q \otimes X \to Q$$

$\delta_{n+1} = \delta \cdot (\delta_n \otimes 1) \cdot \alpha \cdot (1 \otimes c^{-1}{}_{n,\,1}) : Q \otimes (X \otimes X^n) \to Q$ for $n \geq 1$, as can be shown by induction, and hence $\delta^* : Q \otimes X^* \to Q$ can be considered as the unique dynamorphic extension of 1_Q, the run map.

The above points (1.2.6) and (1.2.7) show that if $\boldsymbol{C}$ is a closed category with countable coproducts then the process $-\otimes X : \boldsymbol{C} \to \boldsymbol{C}$ is state-behavior for any object X of $\boldsymbol{C}$.

(1.2.8) Let $(\boldsymbol{C}, \otimes, E, \alpha, \lambda, \rho, s)$ be a closed category with countable coproducts and let X be a fixed object in $\boldsymbol{C}$, a *machine* (relative to the object X) is a 6-tuple $M = (Q, \delta, I, \tau, Y, \beta)$ where $\delta : Q \otimes X \to Q$ is an X-monadic algebra in $\boldsymbol{C}$; I, Y, are objects of $\boldsymbol{C}$ and $\tau : I \to Q$, $\beta : Q \to Y$ are $\boldsymbol{C}$-morphisms [22, p.778]. In the terminology of definition (1.1.2) this is in fact an *$(-\otimes X)$-machine* and the minimal realization theorem (1.1.13) applied to this case is as follows:

Let $(\boldsymbol{C}, \otimes, E, \alpha, \lambda, \rho, s)$ be a closed category with countable coproducts and let (E, M) be an image factorization system in $\boldsymbol{C}$, if either $-\otimes X$ or $-\otimes X^*$ preserves E then :

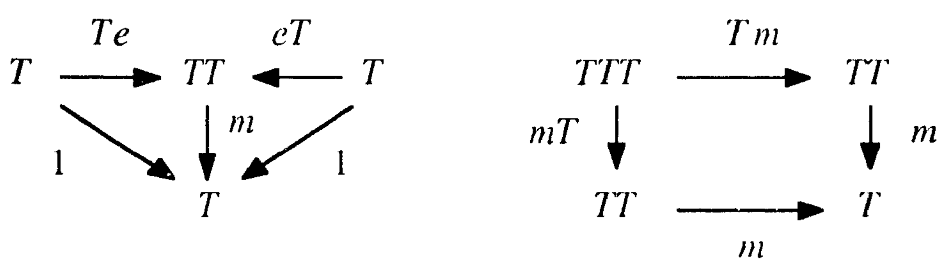
For every response $f : I \otimes X^* \to Y$ there exists a reachable and observable realization $Mf = (Qf, \delta f, I, \tau f, Y, \beta f)$ of f. Any such Mf is a terminal object in the category of reachable realizations of f and simulations and any such Mf is an initial object in the category of observable realizations of f and simulations; thus Mf is unique up to isomorphism.

1.3. Non-deterministic machines in an arbitrary category.

(1.3.1) A *triple* (*or monad*) $\mathcal{T} = (T, e, m)$ in a category $\boldsymbol{C}$ consists of a functor $T : \boldsymbol{C} \to \boldsymbol{C}$ and two natural transformations

$$e : 1_C \to T \quad , \quad m : TT \to T$$

which make the following diagrams commute



e is called the *unit* and m the *multiplication* of the triple.

Every adjunction $(F, G, \eta, \varepsilon) : \boldsymbol{C} \to \boldsymbol{D}$ gives rise to a triple $(GF, \eta, G\varepsilon F)$ in $\boldsymbol{C}$, called the triple defined by the adjunction $(F, G, \eta, \varepsilon)$. [31,pp. 133-135].

(1.3.2) Let $\mathcal{T} = (T, e, m)$ be a triple in $\boldsymbol{C}$. A $\mathcal{T}$*-algebra* (Q, ξ) is a pair consisting of an object Q of $\boldsymbol{C}$ and an arrow $\xi : TQ \to Q$ satisfying :

$$
\begin{array}{ccc}
Q & \xrightarrow{eQ} & TQ \\
 & \searrow^{1_Q} & \downarrow \xi \\
 & & Q
\end{array}
\qquad\qquad
\begin{array}{ccc}
TTQ & \xrightarrow{T\xi} & TQ \\
{\scriptstyle mQ}\downarrow & & \downarrow \xi \\
TQ & \xrightarrow[\xi]{} & Q
\end{array}
$$

A morphism $f:(Q, \xi) \to (R, \theta)$ of $\mathbb{T}$-*algebras* ($\mathbb{T}$-*homomorphism*) is a morphism $f: Q \to R$ such that :

$$
\begin{array}{ccc}
TQ & \xrightarrow{\xi} & Q \\
{\scriptstyle Tf}\downarrow & & \downarrow f \\
TR & \xrightarrow[\theta]{} & R
\end{array}
$$

The category of $\mathbb{T}$- algebras and $\mathbb{T}$- homomorphisms is denoted by $\boldsymbol{C}^{\mathbb{T}}$. [31, p. 136].

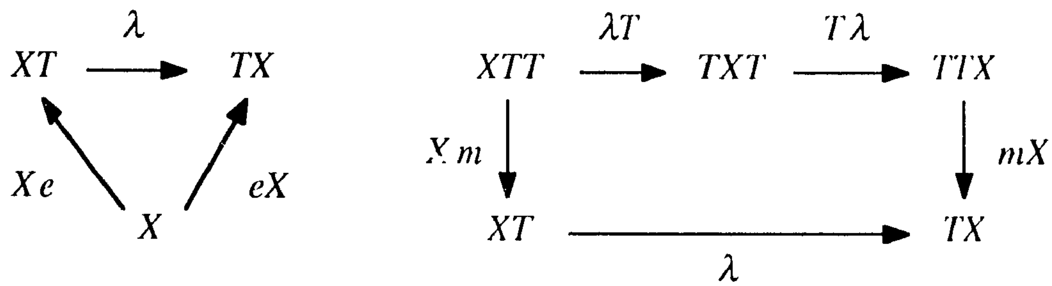
(1.3.3) The forgetful functor $U : \boldsymbol{C}^{\mathbb{T}} \to \boldsymbol{C}$ has a left adjoint $F \dashv U$ which defines the triple $\mathbb{T}$; $F : \boldsymbol{C} \to \boldsymbol{C}^{\mathbb{T}}$ is given by the rule that sends $f: Q \to R$ to $Tf: (TQ, mQ) \to (TR, mR)$, the unit of the adjunction is e and the counit assigns to each $\mathbb{T}$- algebra (Q, ξ) the $\mathbb{T}$- homomorphism $\xi : (TQ, mQ) \to (Q, \xi)$. [31, p. 136-137].

As a remark and because the adjunction, given any $\mathbb{T}$- algebra (R, ξ) and any arrow $f: Q \to R$ in $\boldsymbol{C}$, the unique $\mathbb{T}$- homomorphic extension $f^{\#} : (TQ, mQ) \to (R, \xi)$ of f

$$
\begin{array}{ccc}
Q & \xrightarrow{eQ} & TQ \\
 & \searrow_{f} & \vdots\, f^{\#} \\
 & & R
\end{array}
$$

is given by $f^{\#} = \xi \cdot Tf : TQ \to TR \to R$.

(1.3.4) Let $X : \boldsymbol{C} \to \boldsymbol{C}$ be a functor and let $\mathbb{T} = (T, e, m)$ be a triple in $\boldsymbol{C}$. A *distributive law of* X *over* $\mathbb{T}$ is a natural transformation $\lambda : XT \to TX$ such that makes the following diagrams commute :



[32, pp. 311-312].

As an example let $\mathbb{P} = (P, e, m)$ be the union triple in ***Set*** ; i.e. $P : \boldsymbol{Set} \to \boldsymbol{Set}$ sends the set X to the power set PX and $f \cdot X \to Y$ to the map $Pf : PX \to PY$ such that $(Pf)(S) = \{ f(x) : x \in S \}$ for each subset S of X.

$e : 1_{\boldsymbol{Set}} \to P$ assigns to a set X the map $eX : X \to PX$ which sends x to $\{x\}$.

$m : PP \to P$ assigns to a set X the map $mX : PPX \to PX$ which sends T to $\cup T = \{ x : x \in y \in T$ for some $y \in T \}$.

Then for any set X_0 , $\lambda : (-\times X_0) P \to P(-\times X_0)$ defined for each Q by $\lambda Q : PQ \times X_0 \to P(Q \times X_0)$ which sends (S, x) to $\{ (q, x) : q \in S \}$ is a distributive law of $-\times X_0$ over $\mathbb{P}$. [4, pp. 178 and 182].

From now on and till the end of section 1.3 a process $X : \boldsymbol{C} \to \boldsymbol{C}$, a triple $\mathbb{T} = (T, e, m)$ and a distributive law $\lambda : XT \to TX$ of X over $\mathbb{T}$ are fixed.

(1.3.5) A λ- *algebra* is a triple (Q, δ, ξ) where (Q, δ) is an X- dynamics and (Q, ξ) is a $\mathbb{T}$-algebra such that $\xi : (TQ, T\delta \cdot \lambda Q) \to (Q, \delta)$ is a dynamorphism, i.e. the following diagram commutes :

$$\begin{array}{ccccc} XTQ & \xrightarrow{\lambda Q} & TXQ & \xrightarrow{T\delta} & TQ \\ {\scriptstyle X\xi}\downarrow & & & & \downarrow{\scriptstyle \xi} \\ XQ & & \xrightarrow[\delta]{\qquad\qquad\qquad} & & Q \end{array}$$

A *λ-homomorphism* $f : (Q, \delta, \xi) \to (Q', \delta', \xi')$ between λ- algebras is a simultaneous dynamorphism and $\mathbb{T}$-homomorphism. The category of λ-algebras is denoted by $\boldsymbol{C}^{\lambda}$. [32, p. 315].

Let $\mathbb{T} = (T, e, m)$ be the identity triple in C (i.e., $TQ = Q$, $eQ = 1_Q$, $mQ = 1_Q$) and let $\lambda : XT \to TX$ be the identity distributive law (i.e., $\lambda = 1_X$), since (Q, ξ) is a $\mathbb{T}$-algebra if and only if $\xi = 1_Q$, $\boldsymbol{C}^{\lambda}$ is isomorphic to Dyn(X). Therefore the theory of λ-algebras generalizes Dyn(X). [32, p. 316].

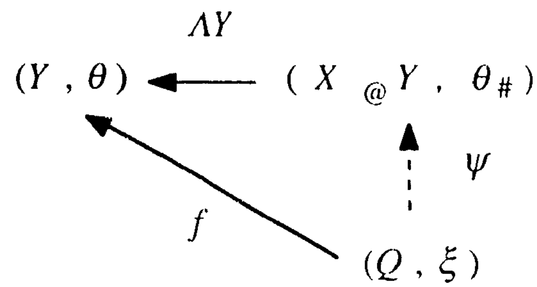
If $\boldsymbol{C}$ has an initial object 0 then for arbitrary $\mathbb{T} = (T, e, m)$ and $X : \boldsymbol{C} \to \boldsymbol{C}$ defined by $XA = 0$, $Xf = 1_0$, the $\lambda : XT \to TX$ defined by $\lambda A = e0$ is a distributive law of X over $\mathbb{T}$ and the category of λ- algebras may be identified with $\boldsymbol{C}^{\mathbb{T}}$. [4, p. 189].

(1.3.6) The forgetful functor $U : \boldsymbol{C}^{\lambda} \to$ Dyn(X) has a left adjoint, a free λ-algebra over (Q, δ) is given by $((TQ, T\delta{\cdot}\lambda Q, mQ), eQ)$, and given any λ-algebra (Y, γ, θ) and any dynamorphism $\beta : (Q, \delta) \to (Y, \gamma)$ the unique λ-homomorphic extension $\beta^{\#} : (TQ, T\delta{\cdot}\lambda Q, mQ) \to (Y, \gamma, \theta)$ of β

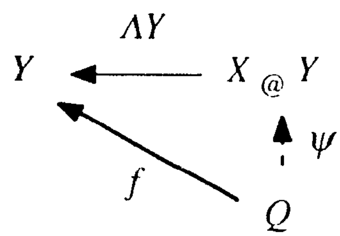
$$\begin{array}{ccc} (Q, \delta) & \xrightarrow{eQ} & (TQ, T\delta \cdot \lambda Q) \\ & {\scriptstyle \beta}\searrow & \downarrow{\scriptstyle \beta^{\#}} \\ & & (Y, \gamma) \end{array}$$

is given by $\beta^{\#} = \theta{\cdot}T\beta : TQ \to TY \to Y$. [4, pp. 191-193].

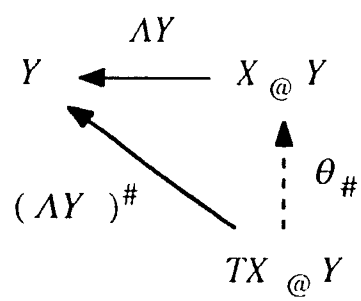
(1.3.7) If $X : \boldsymbol{C} \to \boldsymbol{C}$ is an output process, then the forgetful functor $U : \boldsymbol{C}^{\lambda} \to \boldsymbol{C}^{\mathbb{T}}$ has a right adjoint; a cofree λ-algebra over (Y, θ) is given by $((X_{@}Y, LY, \theta_{\#}), \Lambda Y)$ and given any λ-algebra (Q, δ, ξ) and any $\mathbb{T}$- homomorphism $f : (Q, \xi) \to (Y, \theta)$ the unique λ-homomorphic coextension $\psi : (Q, \delta, \xi) \to (X_{@}Y, LY, \theta_{\#})$ of f



is given by the unique dynamorphic coextension of f:



Here $\theta_{\#} : (TX_{@}Y,\ TLY{\cdot}\lambda X_{@}Y\ ) \to (\ X_{@}Y,\ LY)$ is the unique dynamorphic coextension of $(\Lambda Y)^{\#} = \theta{\cdot}T\Lambda Y : TX_{@}Y \to TY \to Y$



[4, pp. 195-196].

(1.3.8) A *λ-machine* (or *λ-automaton*) is a 7-tuple $M = (Q, \delta, I, \tau, Y, \theta, \beta)$ where (Y, θ) is a $\mathbb{T}$-algebra and $\tau : I \to TQ$, $\delta : XQ \to TQ$, $\beta : Q \to Y$ are C-morphisms. The definition of a

λ-machine is independent of λ but its response is not. [32, p. 313].

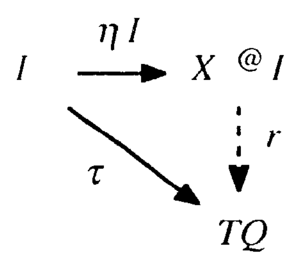
As an example, if one takes the process $-\times X_0 : \textbf{\textit{Set}} \to \textbf{\textit{Set}}$, the union triple $\mathbb{P} = (P, e, m)$, the distributive law $\lambda : (-\times X_0)P \to P(-\times X_0)$ given by $\lambda Q : PQ \times X_0 \to P(Q \times X_0)$ that sends (S, x) to $\{(q, x) : q \in S\}$, the $\mathbb{P}$-algebra $(\{0, 1\}, \max)$ the singleton $I = 1$, and the maps $\tau : 1 \to PQ$, $\delta : Q \times X_0 \to PQ$, $\beta : Q \to \{0, 1\}$, one has that $M = (Q, \delta, I, \tau, \{0, 1\}, \max, \beta)$ is a usual non-deterministic sequential machine. It is usual in automata theory to simulate such a non-deterministic sequential machine by the (deterministic) sequential machine $(PQ, \delta°, 1, \tau, \{0, 1\}, \beta^{\#})$ where $\delta° : PQ \times X_0 \to PQ$ is defined by $\delta°(S, x) = \bigcup_{q \in S} \{\delta(q, x) : q \in S\} = (\delta^{\#} \cdot \lambda Q)(S, x)$. ([4, pp. 173-175] and [32, p. 311]).

Generalizing the above fact to the general theory, one has :

(1.3.9) The λ-machine $M = (Q, \delta, I, \tau, Y, \theta, \beta)$ has an associated X-machine $M° = (TQ, \delta°, I, \tau, Y, \beta^{\#})$ where :

$$\delta° = \delta^{\#} \cdot \lambda Q \ : XTQ \to TXQ \to TQ$$

If X is an input process, the *response* of the λ-machine $M = (Q, \delta, I, \tau, Y, \theta, \beta)$ is the response of the X-machine $M° = (TQ, \delta°, I, \tau, Y, \beta^{\#})$, i.e. is the C-morphism $\beta^{\#} \cdot r : X@I \to TQ \to Y$ where $r : (X@I, \mu_0 I) \to (TQ, \delta°)$ is the unique dynamorphic extension of τ :



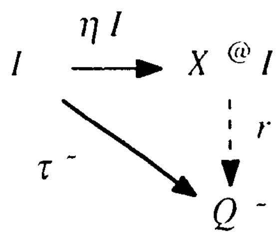
[32, p. 313].

The following concept generalizes the $M°$ of (1.3.9) (see (1.3.12)) and permits to give a generalization of the "minimal realization theorem" stated in (1.1.13) (see (1.3.15)).

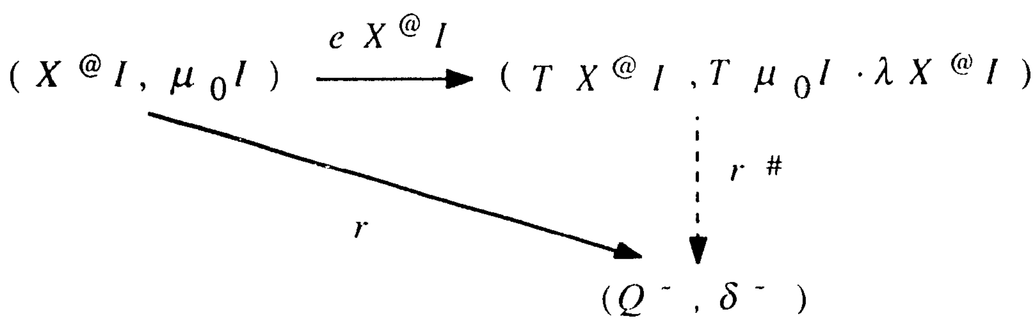
(1.3.10) An *implicit λ-machine* (or *implicit λ-automaton*) is a 8-tuple $M^\sim = (Q^\sim, \delta^\sim, \xi^\sim, I, \tau^\sim, Y, \theta, \beta^\sim)$ where $(Q^\sim, \delta^\sim, \xi^\sim)$ is a λ-algebra, $\tau^\sim : I \to Q^\sim$ is a C-morphism, and $\beta^\sim : (Q^\sim, \xi^\sim) \to (Y, \theta)$ is a $\mathbb{T}$-homomorphism. [32, p. 323].

When $\mathbb{T}$ is the identity triple and λ = id, "implicit λ -machine", and "λ -machine" coincide with "X -machine".

(1.3.11) If X is an input process, the *X-reachability map* of the implicit λ - machine $M^\sim$ is the unique dynamorphic extension $r : (X@I, \mu_0 I) \to (Q^\sim, \delta^\sim)$ of $\tau^\sim$:

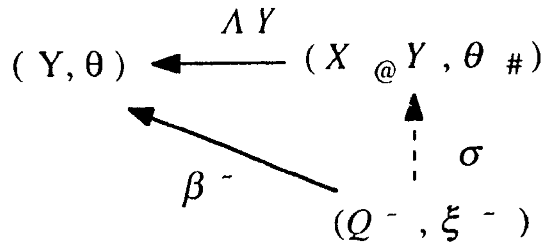


The *reachability map* of $M^\sim$ is the unique λ-homomorphic extension $r^\# : (TX@I, T\mu_0 I \cdot \lambda X@I, mX@I) \to (Q^\sim, \delta^\sim, \xi^\sim)$ of r :



The *response* of $M^\sim$ is the composition $\beta^\sim \cdot r : X@I \to Q^\sim \to Y$, i.e. it is the response of the X-machine $(Q^\sim, \delta^\sim, I, \tau^\sim, Y, \beta^\sim)$.

If X is an output process, the *observability map of* $M^\sim$ is the unique λ-homomorphic coextension $\sigma : (Q^\sim, \delta^\sim, \xi^\sim) \to (X_@ Y, LY, \theta_\#)$ of $\beta^\sim$:



[32, pp. 323-324].

The next two results say that λ-machines and implicit λ-machines compute the same responses.

(1.3.12) Let $M = (Q, \delta, I, \tau, Y, \theta, \beta)$ be a λ-machine. Then $M^\circ = (TQ, \delta^\circ, mQ, I, \tau, Y, \theta, \beta^\#)$ where $\delta^\circ = \delta^\# \cdot \lambda Q$ is an implicit λ-machine and the response of M is the response of M°. [32, p. 324].

(1.3.13) Let $(Q^\sim, \xi^\sim)$ be a T-algebra. A *scoop* of $(Q^\sim, \xi^\sim)$ is a triple (Q, i, c) where $i : Q \to Q^\sim$ and $c : Q^\sim \to TQ$ are morphisms such that $i^\# \cdot c = 1_{Q^\sim} \cdot (Q^\sim, 1_{Q^\sim}, eQ^\sim)$ is always a scoop of $(Q^\sim, \xi^\sim)$. [32, p. 325]. (This notion of scoop is a modification of the one introduced by Ehrig [18]).

Let $M^\sim = (Q^\sim, \delta^\sim, \xi^\sim, I, \tau^\sim, Y, \theta, \beta^\sim)$ be an implicit λ-machine and let (Q, i, c) be any scoop of $(Q^\sim, \xi^\sim)$. Then the λ-machine $M = (Q, \delta, I, \tau, Y, \theta, \beta)$ where :

$$\delta = c \cdot \delta^\sim \cdot Xi : XQ \to XQ^\sim \to Q^\sim \to TQ$$
$$\tau = c \cdot \tau^\sim : I \to Q^\sim \to TQ$$
$$\beta = \beta^\sim \cdot i : Q \to Q^\sim \to Y$$

has the same response as $M^\sim$. [32, pp. 325-326].

The following is a generalization of the dynamorphic image lemma stated in (1.1.9).

(1.3.14) Let h : $(Q, \delta, \xi) \to (Q', \delta', \xi')$ be a λ-homomorphism and let $i \cdot p : Q \longrightarrow\!\!> Q'' \rightarrowtail Q'$ be an E-M factorization of h in

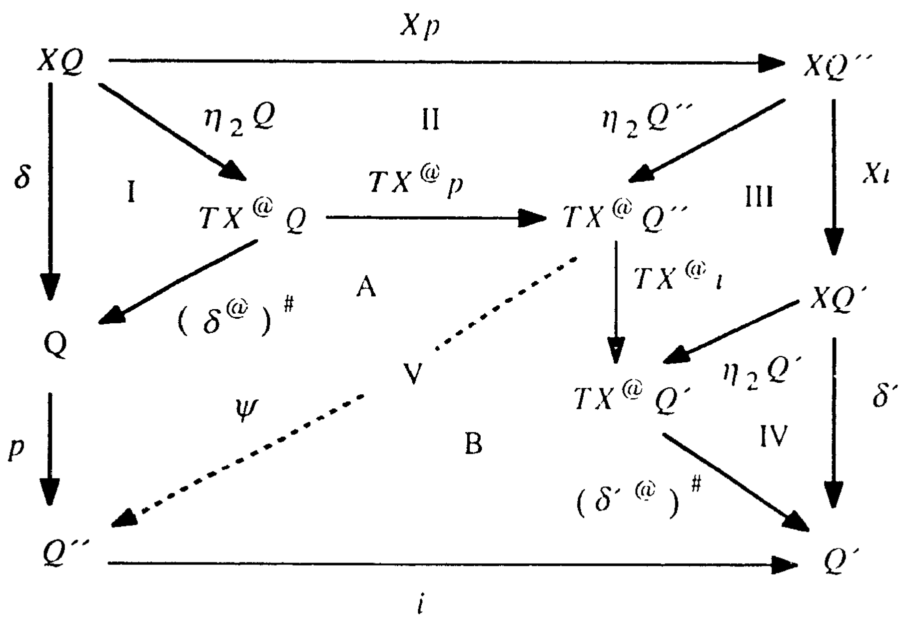
$\boldsymbol{C}$ ($\boldsymbol{C}$ is supposed to have an image factorization system (E, M)). Then, if $TX@$ preserves E, there exist unique $\delta'' : XQ'' \to Q''$ and $\xi'' : TQ'' \to Q''$ such that $p : (Q, \delta, \xi) \to (Q'', \delta'', \xi'')$ and $i : (Q'', \delta'', \xi'') \to (Q', \delta', \xi')$ are λ-homomorphisms. [32]. A proof follows :

For each A define $\eta_2 A$ to be the arrow :

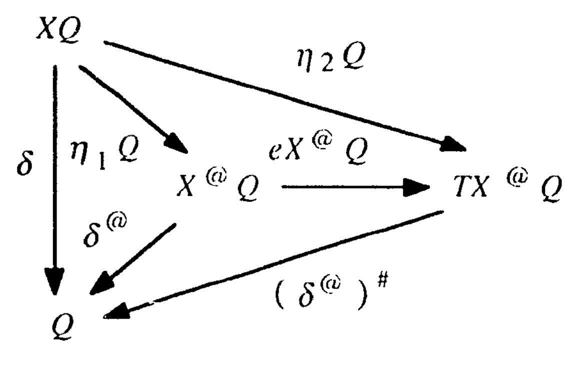
$$\eta_2 A = eX@A \cdot \mu_0 A \cdot X\eta A : XA \to XX@A \to X@A \to TX@A =$$
$$= eX@A \cdot \eta_1 A : XA \to X@A \to TX@A$$

where η_1 is given in (1.1.9). Then $\eta_2 : X \to TX@$ is a natural transformation.

Now, consider the following diagram :



From the facts stated in (1.1.9), one has the commutativity of :



and hence, I commutes. II and III commute since η_2 is a natural transformation, IV commutes by the same reason as I.

Commutativity of V :

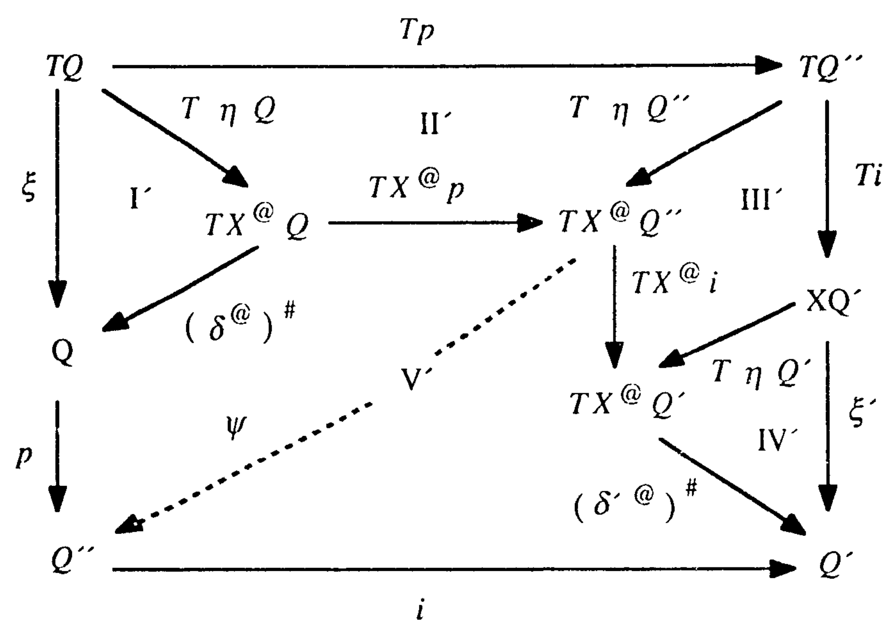
Also, from the facts stated in (1.1.9) one has that $h\cdot\delta^@ = \delta'^@\cdot X^@h$. To show that $h\cdot(\delta^@)^\# = (\delta'^@)^\#\cdot TX^@h$ but since

$$TX^@h : (TX^@Q, mX^@Q) \rightarrow (TX^@Q', mX^@Q')$$

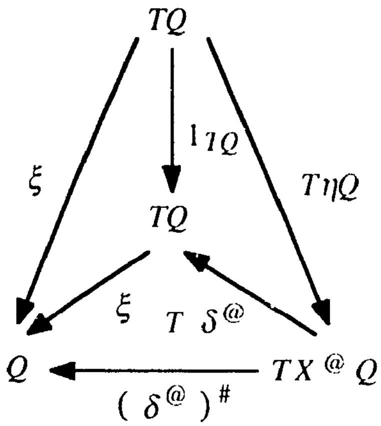
is a $\mathbb{T}$-homomorphism, suffices to show that $h\cdot(\delta^@)^\#\cdot eX^@Q = (\delta'^@)^\#\cdot TX^@h\cdot eX^@Q$, but $h\cdot(\delta^@)^\#\cdot eX^@Q = h\cdot\delta^@$ and $(\delta'^@)^\#\cdot TX^@h\cdot eX^@Q = (\delta'^@)^\#\cdot eX^@Q'\cdot X^@h = \delta'^@\cdot X^@h = h\cdot\delta$. Therefore V commutes.

Now, by diagonal fill in there exists a unique C-morphism $\psi : TX^@Q'' \rightarrow Q''$ such that makes A and B commute. Defining $\delta'' = \psi\cdot\eta_2 Q''$ one has that $p : (Q, \delta) \rightarrow (Q'', \delta'')$ and $i : (Q'', \delta'') \rightarrow (Q', \delta')$ are dynamorphisms.

Now consider the following diagram :

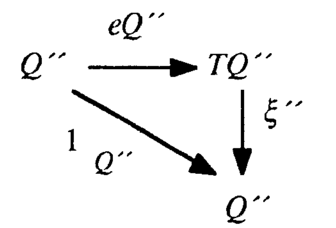


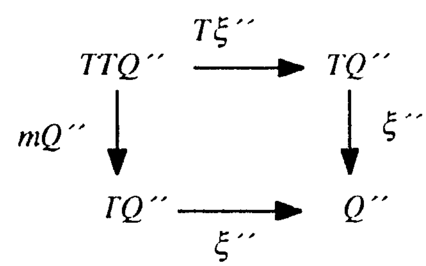
I' commutes since the following diagram



II´ and III´ commute since η is a natural transformation, IV´ by the same reason as I´, and V´ is the same as V. Define $\xi'' = \psi \cdot T\eta Q''$.

(Q'', ξ'') is a $\mathbb{T}$-algebra , i.e. the following diagrams (i) and (ii) have to commute :

(i) 

(ii) 

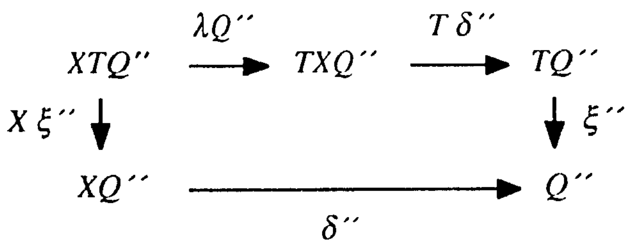
(i) : Since i is mono suffices to show that $i \cdot \xi'' \cdot eQ'' = i$; but $i \cdot \xi'' \cdot eQ'' = \xi' \cdot Ti \cdot eQ'' = i^{\#} \cdot eQ'' = i$.

(ii) : Since i is mono suffices to show that $i \cdot \xi'' \cdot T\xi'' = i \cdot \xi'' \cdot mQ''$; but since m is a natural transformation and (Q', ξ') is a $\mathbb{T}$-algebra one has respectively that $mQ' \cdot TTi = Ti \cdot mQ''$ and $\xi' \cdot T\xi' = \xi' \cdot mQ'$ hence : $i \cdot \xi'' \cdot T\xi'' = \xi' \cdot Ti \cdot T\xi'' = \xi' \cdot T\xi' \cdot TTi = \xi' \cdot mQ' \cdot TTi = \xi' \cdot Ti \cdot mQ'' =$
$= i \cdot \xi'' \cdot mQ''$.

Therefore, $p : (Q, \xi) \to (Q'', \xi'')$ and $i : (Q'', \xi'') \to (Q', \xi')$ are $\mathbb{T}$-homomorphisms.

(Q'', δ'', ξ'') is a λ-algebra :

Only remains to show that the diagram



commutes; and since i is mono suffices to show that $i{\cdot}\xi''{\cdot}T\delta''{\cdot}\lambda Q'' =$ $= i{\cdot}\delta''{\cdot}X\xi''$. But since (Q', δ', ξ') is a λ -algebra one has that $\xi'{\cdot}T\delta'{\cdot}\lambda Q' = \delta'{\cdot}X\xi'$. Hence :

$$i{\cdot}\xi''{\cdot}T\delta''{\cdot}\lambda Q'' = \xi'{\cdot}Ti{\cdot}T\delta''{\cdot}\lambda Q'' = \xi'{\cdot}T\delta'{\cdot}TXi{\cdot}\lambda Q'' =$$
$$= \xi'{\cdot}T\delta'{\cdot}\lambda Q'{\cdot}XTi = \delta'{\cdot}X\xi'{\cdot}XTi = \delta'{\cdot}Xi{\cdot}X\xi'' = i{\cdot}\delta''{\cdot}X\xi''.$$

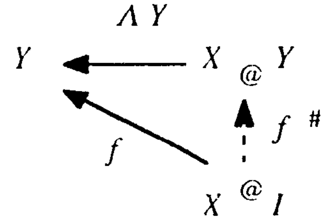
Then,

$p : (Q,\delta,\xi) \to (Q'',\delta'',\xi'')$ and $i : (Q'',\delta'',\xi'') \to (Q',\delta',\xi')$ are λ -homomorphisms.

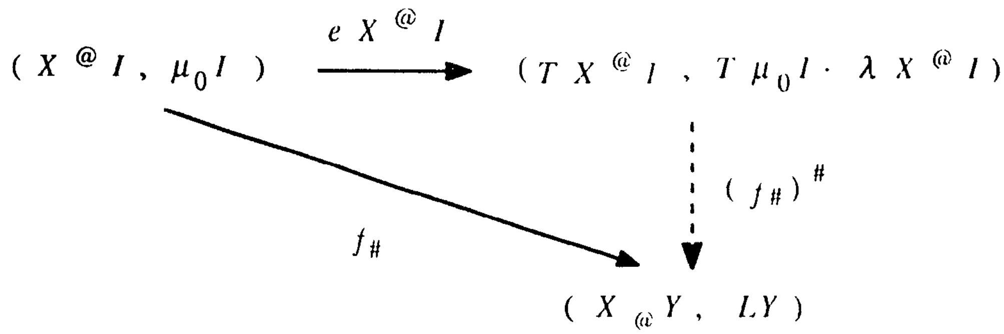
Uniqueness of δ'' and ξ'' follow since i is mono. ●

(1.3.15) Let (E, M) be an image factorization system in ***C*** , and X a state-behavior process in C such that $TX@$ preserves E. Let I and (Y, θ) be fixed. Then for every $f : X@I \to Y$ there exists an implicit λ-machine $M^{\sim} = (Q^{\sim}, \delta^{\sim}, \xi^{\sim}, I, \tau^{\sim}, Y, \theta, \beta^{\sim})$ such that the response of $M^{\sim}$ is f, the reachability map $r^{\#} : TX@I \to Q^{\sim}$ is in E, and the observability map $\sigma : Q^{\sim} \to X_{@}Y$ is in M . If $M^{\sim\prime}$ also satisfies these three conditions then $M^{\sim}$ and $M^{\sim\prime}$ are isomorphic, i.e. there exists an isomorphism $\psi : (Q^{\sim}, \delta^{\sim}, \xi^{\sim}) \to (Q^{\sim\prime}, \delta^{\sim\prime}, \xi^{\sim\prime})$ of λ-algebras such that $\psi{\cdot}\tau^{\sim} = \tau^{\sim\prime}$ and $\beta^{\sim\prime}{\cdot}\psi = \beta^{\sim}$. [32, p. 326-327]. Here is a proof that is a development of the condensed one given in [32] :

Let $f_{\#} : (X@I, \mu_0 I) \to (X_{@}Y, LY)$ be the unique dynamorphic coextension of f:



and let $f^{\sim} = (f_{\#})^{\#} : (TX@I, T\mu_0 I \cdot \lambda X@I, mX@I) \to (X_{@}Y, LY, \theta_{\#})$ be the unique λ-homomorphic extension of $f_{\#}$:

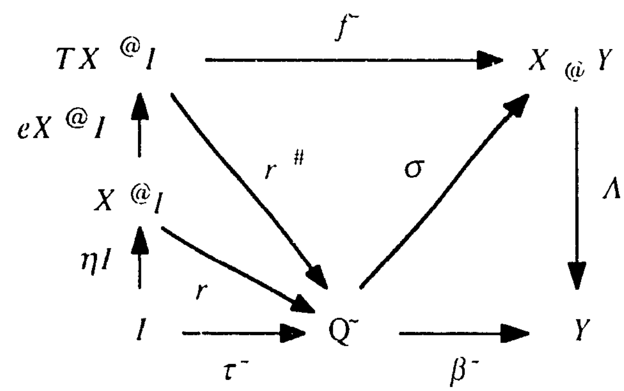


Now consider an E-M factorization of $f^{\sim}$:

$$\sigma \cdot r^{\#} : TX@I \longrightarrow\!\!> Q^{\sim} \rightarrowtail X_{@}Y$$

By (1.3.14) there exist unique $\delta^{\sim} : XQ^{\sim} \to Q^{\sim}$ and $\xi^{\sim} : TQ \to Q^{\sim}$ such that $r^{\#} : (TX@I, T\mu_0 I \cdot \lambda X@I, mX@I) \to (Q^{\sim}, \delta^{\sim}, \xi^{\sim})$ and $\sigma : (Q^{\sim}, \delta^{\sim}, \xi^{\sim}) \to (X_{@}Y, LY, \theta_{\#})$ are λ -homomorphisms.

Now, define $r : (X@I, \mu_0 I) \to (Q^{\sim}, \delta^{\sim})$ to be the dynamorphism $r = r^{\#} \cdot eX@I$, $\tau^{\sim} : I \to Q^{\sim}$ to be the C-morphism $\tau^{\sim} = r \cdot \eta I$ and $\beta^{\sim} : (Q^{\sim}, \xi^{\sim}) \to (Y, \theta)$ to be the T-homomorphism $\beta^{\sim} = \Lambda Y \cdot \sigma$.

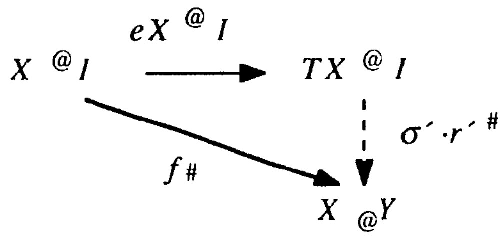


Therefore, $M^{\sim} = (Q^{\sim}, \delta^{\sim}, \xi^{\sim}, I, \tau^{\sim}, Y, \theta, \beta^{\sim})$ is an implicit λ-machine with response equal to $\beta^{\sim} \cdot r = \Lambda Y \cdot \sigma \cdot r^{\#} \cdot eX@I = \Lambda Y \cdot f^{\sim} \cdot eX@I = \Lambda Y \cdot f_{\#} = f$, reachability map equal to $r^{\#}$ and observability map equal to σ .

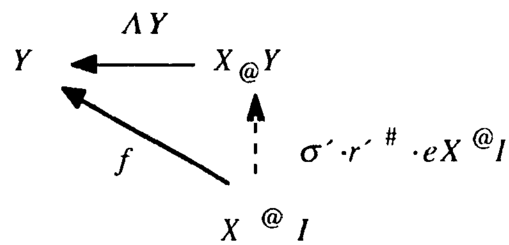
Now suppose that $M^{\sim\prime} = (Q^{\sim\prime}, \delta^{\sim\prime}, \xi^{\sim\prime}, I, \tau^{\sim\prime}, Y, \theta, \beta^{\sim\prime})$ is another implicit λ -machine such that its response is f, its reachability map $r'^{\#}$ is in E and its observability map σ' is in M.

First one has that $\sigma'\cdot r'^{\#} = f^{\sim}$:

The diagram

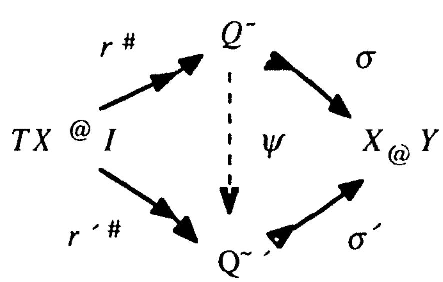


commutes if the following one does

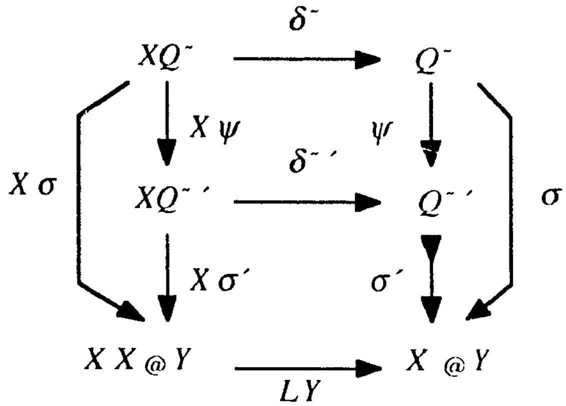


and this is true since $\Lambda Y\cdot\sigma'\cdot r'^{\#}\cdot eX@I = \beta^{\sim\prime}\cdot r' = f$.

Now, there exists a unique isomorphism $\psi : Q^{\sim} \rightarrow Q^{\sim\prime}$ with $\psi\cdot r^{\#} = r'$ and $\sigma'\cdot\psi = \sigma$:



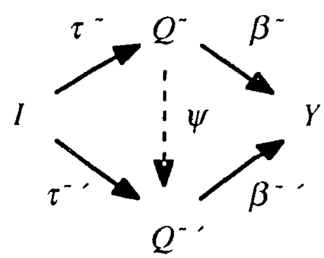
$\psi : (Q^{\sim}, \delta^{\sim}) \rightarrow (Q^{\sim\prime}, \delta^{\sim\prime})$ is a dynamorphism because in the following the exterior and bottom diagrams commute and, since σ' is mono, also does the top one :



In a similar form it can be shown that $\psi : (Q^{\sim}, \xi^{\sim}) \rightarrow (Q^{\sim\prime}, \xi^{\sim\prime})$ is a $\mathbb{T}$-homomorphism.

Therefore, $\psi : (Q^{\sim}, \delta^{\sim}, \xi^{\sim}) \rightarrow (Q^{\sim\prime}, \delta^{\sim\prime}, \xi^{\sim\prime})$ is an isomorphism of λ-algebras.

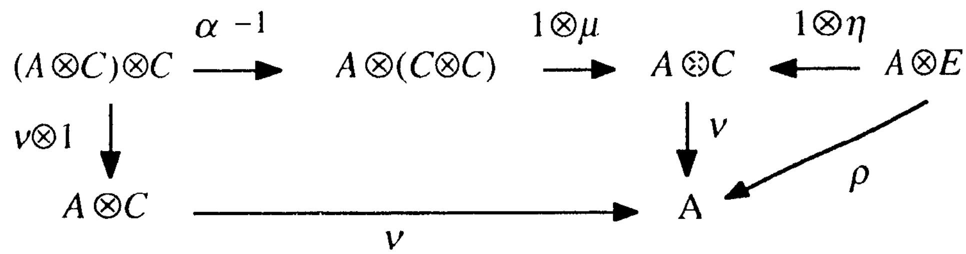
Moreover since $\tau^{\sim\prime} = r'\cdot\eta I = r'^{\#}\cdot eX@I\cdot\eta I = \psi\cdot r^{\#}\cdot eX@I\cdot\eta I =$ $= \psi\cdot r\cdot\eta I = \psi\cdot\tau^{\sim}$ and $\beta^{\sim} = \Lambda Y\cdot\sigma = \Lambda Y\cdot\sigma'\cdot\psi = \beta^{\sim\prime}\cdot\psi$ the following diagram commutes :



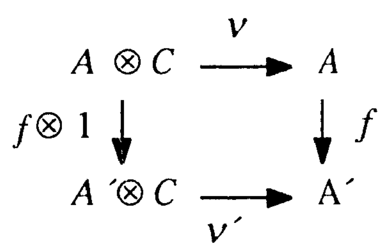
2. MACHINES IN CLOSED CATEGORIES AND IN TOPOSES.

2.1. Machines in a closed category for a monoid C.

(2.1.1) Let $(\mathcal{C}, \otimes, E, \alpha, \lambda, \rho)$ be a monoidal category, a *right action* of a monoid (C, μ, η) on an object A is an arrow $v : A\otimes C \to A$ of $\mathcal{C}$ such that the following diagram commutes :



A morphism $f : v \to v'$ of right actions of C is an arrow $f : A \to A'$ in $\mathcal{C}$ such that the following diagram commutes :



The category of right actions of (C, μ, η) is denoted by $_C$Ract. ([31, p. 170] for the similar left actions).

(2.1.2) The forgetful functor $U : {_C}\text{Ract} \to \mathcal{C}$ has a left adjoint $F : \mathcal{C} \to {_C}\text{Ract}$ which sends each object A of $\mathcal{C}$ to the right action $(1\otimes\mu)\cdot\alpha^{-1} : (A\otimes C)\otimes C \to A\otimes(C\otimes C) \to A\otimes C$. ([31, p. 170] for left actions). A proof follows :

It is straightforward to show that

(2.1)

$$(1\otimes\mu)\cdot\alpha^{-1} : (A\otimes C)\otimes C \;\rightarrow\; A\otimes(C\otimes C) \;\rightarrow\; A\otimes C$$

is a right action.

F on arrows is defined by

$$F\,(f : A \rightarrow A') = (\,f\otimes 1 : A\otimes C \rightarrow A'\otimes C\,)\,;$$

the unit $\eta^{\sim}$ of the adjunction is

$$\eta^{\sim}A = (1\otimes\eta)\cdot\rho^{-1} : A \rightarrow A\otimes E \rightarrow A\otimes C$$

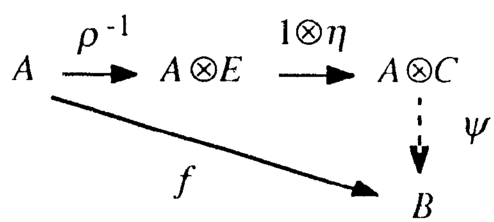
for each object A of $\boldsymbol{C}$ and the counit $\varepsilon^{\sim}$ is

$$\varepsilon^{\sim}v \;=\; v : B\otimes C \;\rightarrow\; B$$

for each right action v .

It is routine to check that the above data satisfy the conditions of an adjunction. ●

The free right action over an object A of $\boldsymbol{C}$ is given by : $((1\otimes\mu)\cdot\alpha^{-1}\!: (A\otimes C)\otimes C \rightarrow A\otimes C,\; (1\otimes\eta)\cdot\rho^{-1}\!:\!A \rightarrow A\otimes C\;)$ and given any right action $v : B\otimes C \rightarrow B$ and any arrow $f : A \rightarrow B$ the unique morphism of right actions $\psi : ((1\otimes\mu)\cdot\alpha^{-1}\!:\!(A\otimes C)\otimes C \rightarrow A\otimes C) \rightarrow (v : B\otimes C \rightarrow B)$ such that makes the following diagram commute

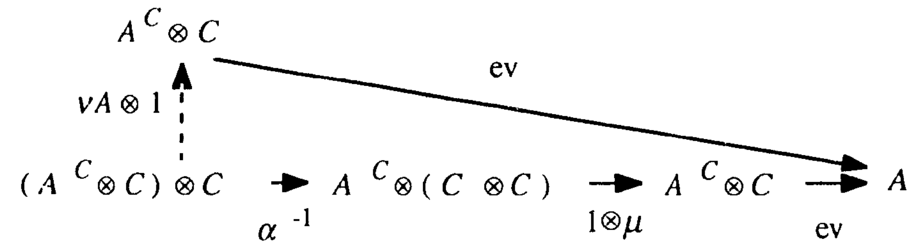


is given by $\quad \psi \;=\; \varepsilon^{\sim}v\cdot Ff \;=\; v\cdot(f\otimes 1) :\; A\otimes C \rightarrow B\otimes C \rightarrow B$.

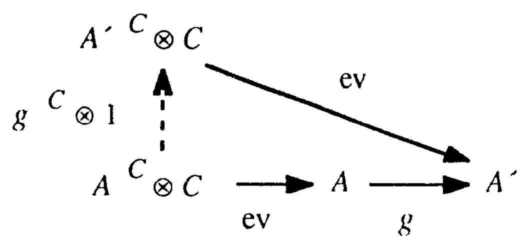
Following the remark given in [18, p 94] one has the following :

(2.1.3) Let $\boldsymbol{C}$ be a closed category and let $(C,\,\mu,\,\eta)$ be a monoid in $\boldsymbol{C}$, then the forgetful functor $U : {}_{C}\text{Ract} \rightarrow \boldsymbol{C}$ has a right adjoint $G : \boldsymbol{C} \rightarrow {}_{C}\text{Ract}$. A proof follows :

Given A an object of $\boldsymbol{C}$, define $GA = vA : A^{C}\otimes C \rightarrow A^{C}$ where vA is given by :

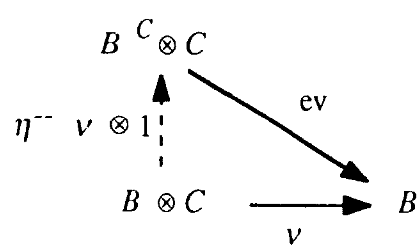


Now, G is defined on arrows as follows : let $g : A \to A'$ be an arrow in $\mathbf{C}$,
$G(g : A \to A') = g^C : (vA : A^C \otimes C \to A^C) \to (vA' : A'^C \otimes C \to A'^C)$
where g^C is given by :



The unit and couinit of the adjunction are defined respectively as follows :

For each right action $v : B \otimes C \to B$ define the arrow $\eta^{--}v : B \to B^C$ by :



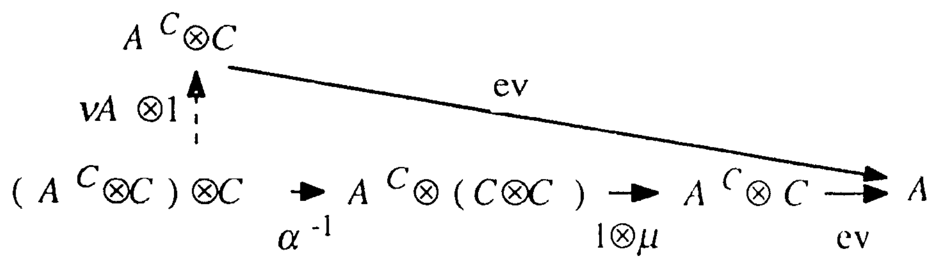
$\eta^{--}v : (v : B \otimes C \to B) \to (vB : B^C \otimes C \to B^C)$ is an arrow in ${}_C$Ract.

For each object B in $\mathbf{C}$ define the arrow $\varepsilon^{--}B : B^C \to B$ by :

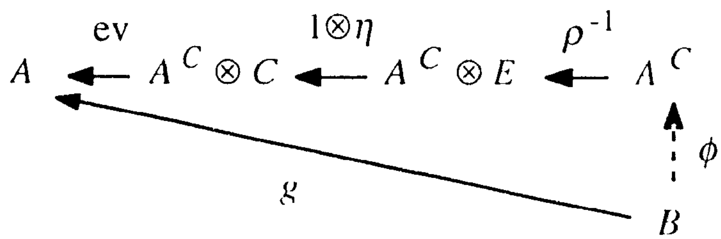
$$\varepsilon^{--}B = \text{ev}\cdot(1 \otimes \eta)\cdot \rho^{-1} : B^C \to B^C \otimes E \to B^C \otimes C \to B$$

It is routine to check that the above data satisfy the conditions of an adjunction. ●

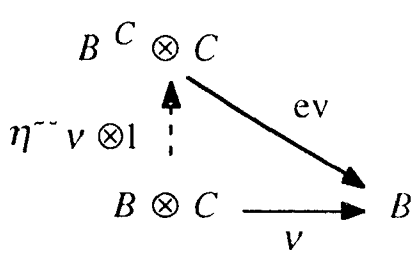
The cofree right action over an object A of C is given by :
$(\nu A : A^C \otimes C \to A^C,\ \varepsilon^{\sim\sim}A = \mathrm{ev}\cdot(1\otimes\eta)\cdot\rho^{-1} : A^C \to A^C \otimes E \to A^C \otimes C \to A)$
where νA is defined by :

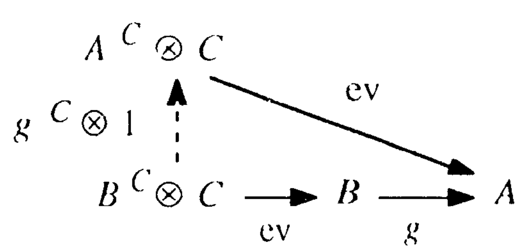


Moreover, given any right action $\nu : B\otimes C \to B$ and any arrow $g : B \to A$, the unique morphism of right actions $\phi : (\nu : B\otimes C \to B) \to (\nu A : A^C \otimes C \to A^C)$ such that makes the following diagram commute



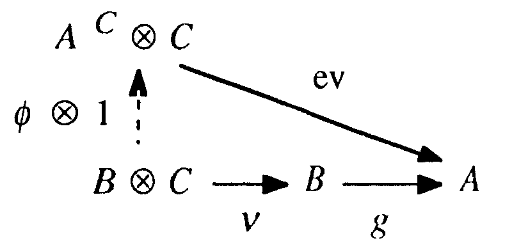
is given by $\phi = Gg\cdot\eta^{\sim\sim}\nu = g^C\cdot\eta^{\sim\sim}\nu : B \to B^C \to A^C$ where $\eta^{\sim\sim}\nu$ and g^C are defined respectively by :





or,

(2.1)

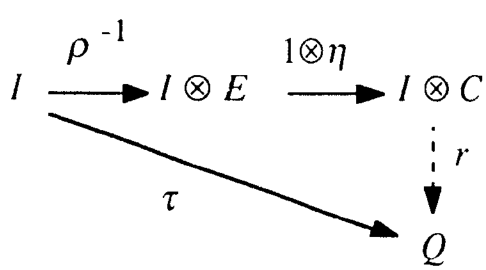


From now on and till the end of section 2.1 a closed category $(C, \otimes, E, \alpha, \lambda, \rho, s)$ and a monoid (C, μ, η) in it are fixed.

Following the remark given in [18, p. 94] and imitating definitions from (1.1.2) to (1.1.4) one has :

(2.1.4) A (C,μ,η)-*machine* is a 6-tuple $M = (Q, v, I, \tau, Y, \beta)$ where $v : Q\otimes C \to Q$ is a right action and $\tau : I \to Q$, $\beta : Q \to Y$ are morphisms in $\mathbf{C}$. Q, I, Y are called respectively the *state object* , *initial object* and *output object* ; τ is the *initial state* and β is the *output morphism.*

The *reachability map* of M is the unique morphism of right actions $r : ((1\otimes\mu)\cdot\alpha^{-1} : (I \otimes C)\otimes C \to I\otimes(C\otimes C) \to I\otimes C) \to (v : Q\otimes C\to Q)$ such that makes the following diagram commute :

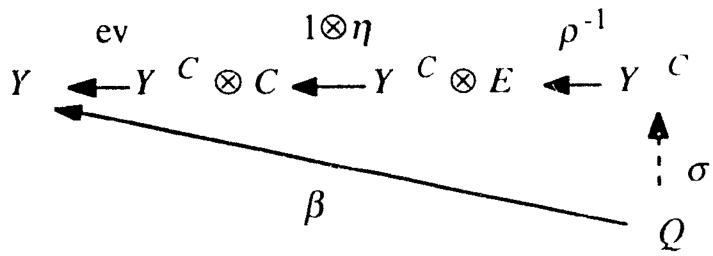


i.e. $r = v\cdot(\tau \otimes 1) : I\otimes C \to Q\otimes C \to Q$

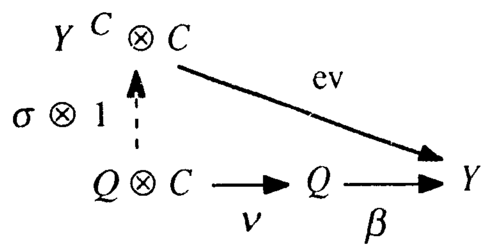
The *response map* is the $\mathbf{C}$-morphism $\beta\cdot r : I\otimes C \to Q \to Y.$

A map $f : I\otimes C\to Y$ is called a *response* and if a (C,μ,η)-*machine* M has as response map f, one says that M is a *realization* of f.

The *observability map* of M is the unique morphism of right actions $\sigma : (v : Q\otimes C \to Q) \to (vQ : Q^C\otimes C \to Q^C)$ such that makes the following diagram commute :



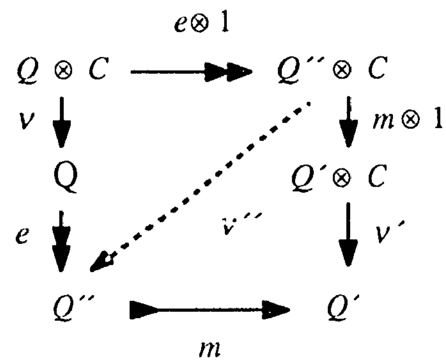
i.e. σ is also defined by the following diagram :



Now, imitating the points (1.1.9) to (1.1.13) one has the following (2.1.5) to (2.1.9) :

(2.1.5) Suppose that $\boldsymbol{C}$ has an image factorization system (E,M) such that $-\otimes C$ preserves E. Let $h:(v:Q\otimes C \to Q) \to (v':Q'\otimes C \to Q')$ be a morphism of right actions and let $m\cdot e : Q \twoheadrightarrow Q'' \rightarrowtail Q'$ be an E-M factorization of h. Then there exists a unique right action $v'': Q''\otimes C \to Q''$ on Q'' such that $e:(v:Q\otimes C \to Q) \to (v'':Q''\otimes C \to Q'')$ and $m:(v'':Q''\otimes C \to Q'') \to (v':Q'\otimes C \to Q')$ are morphisms of right actions.

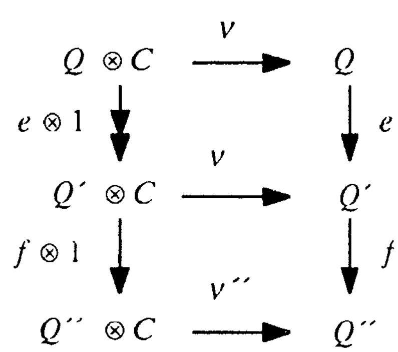
Proof : Since $h: v \to v'$ is a morphism of right actions one has that $h\cdot v = v'\cdot(h\otimes 1)$. Now, since $e\otimes 1$ is in E, by diagonal fill in define v'':



The only thing remains to show is that v'' is a right action but since m is mono this fact follows from a routine diagram. ●

(2.1.6) Suppose that $\boldsymbol{C}$ has an image factorization system (E,M) such that $-\otimes C$ preserves E. Let $e : (v : Q \otimes C \to Q) \to (v' : Q' \otimes C \to Q')$ be a morphism of right actions with $e \in$ E, let $v'' : Q'' \otimes \mathrm{C} \to Q''$ be a right action and let $f : Q' \to Q''$ be an arrow in $\boldsymbol{C}$ such that $f{\cdot}e : (\mathrm{v} : Q \otimes C \to Q) \to (v'' : Q'' \otimes C \to Q'')$ is a morphism of right actions. Then $f : (v' : Q' \otimes C \to Q') \to (v'' : Q'' \otimes C \to Q'')$ is a morphism of right actions.

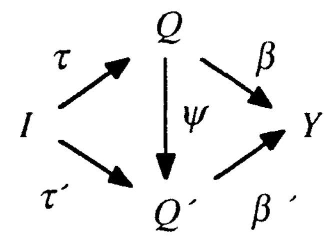
Proof : Consider the following diagram :



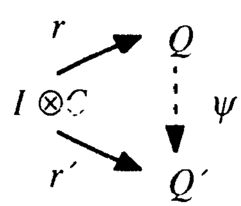
The perimeter and the top square commute, and since $e \otimes 1$ is epi, the bottom square also commutes. ●

(2.1.7) Let (E, M) be an image factorization system in C and $M = (Q, \nu, I, \tau, Y, \beta)$ a (C,μ,η)-machine, M is said to be *reachable* if $r : I \otimes C \to Q$ is in E, and M is said to be *observable* if $\sigma : Q \to Y^C$ is in M.

(2.1.8) Fixing I and Y, but letting Q vary, let (C,μ,η)*-mach* be the category whose objects are (C,μ,η)-machines $M = (Q, \nu, I, \tau, Y, \beta)$; and whose morphisms are *simulations* $\psi : M \to M'$ i.e. morphisms of right actions $\psi : (\nu : Q \otimes C \to Q) \to (\nu' : Q' \otimes C \to Q')$ which make the following diagram commute :



If $\psi : M \to M'$ is a simulation of (C,μ,η)-machines, then since the following diagram commutes :

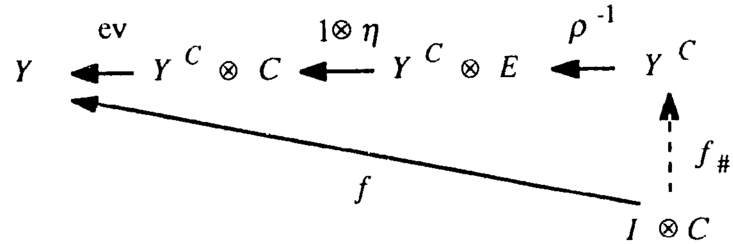


M and M' have the same responses.

Now one has in this context of (C,μ,η) - machines the analogous to the " minimal realization theorem" stated in (1.1.13) :

(2.1.9) Let (E, M) be an image factorization system in C such that $-\otimes C$ preserves E. Then, for every response $f : I \otimes C \to Y$ there exists a reachable and observable realization $M_f = (Q_f, \nu_f, I, \tau_f, Y, \beta_f)$ of f. Any such M_f is a terminal object in the category of reachable realizations of f and simulations and any such M_f is an initial object in the category of observable realizations of f and simulations; thus M_f is unique up to isomorphism. Here is a proof that is similar to the one given by Manes of the fact stated in (1.1.13) :

Given $f: I\otimes C \to Y$ let $f_\#$ be the arrow $((1\otimes\mu)\cdot\alpha^{-1}: (I\otimes C)\otimes C \to I\otimes(C\otimes C) \to I\otimes C)) \to (\nu Y : Y^C\otimes C \to Y^C)$ i.e. the unique coextension of f:

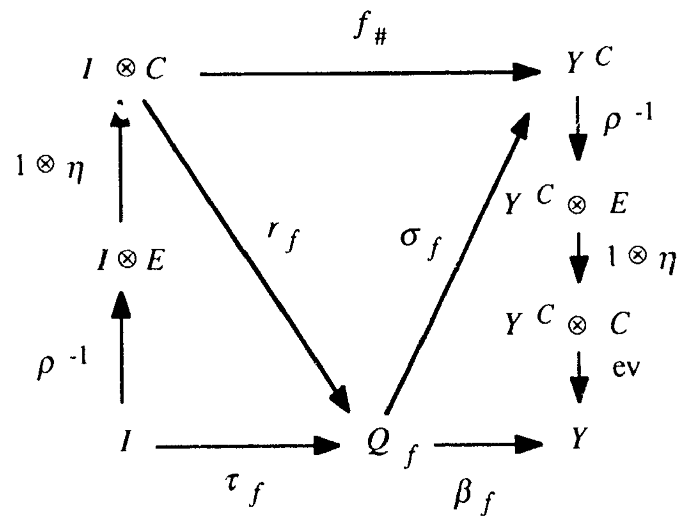


Consider an E-M factorization of $f_\#$:

$$\sigma_f \cdot r_f : I\otimes C \longrightarrow\!\!> Q_f \rightarrowtail Y^C$$

By (2.1.5) there exists a unique right action ν_f on Q_f such that : $r_f : ((1\otimes\mu)\cdot\alpha^{-1}: (I\otimes C)\otimes C \to I\otimes(C\otimes C) \to I\otimes C)) \to (\nu_f : Q_f\otimes C \to Q_f)$ and $\sigma_f : (\nu_f : Q_f\otimes C \to Q_f) \to (\nu Y : Y^C\otimes C \to Y^C)$ are morphisms of right actions.

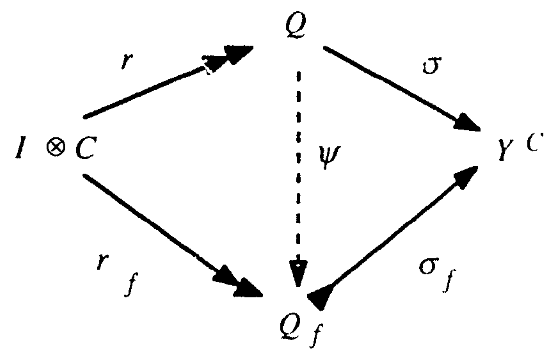
Now define $\tau_f = r_f\cdot(1\otimes\eta)\cdot\rho^{-1}$ and $\beta_f = \mathrm{ev}\cdot(1\otimes\eta)\cdot\rho^{-1}\cdot\sigma_f$.



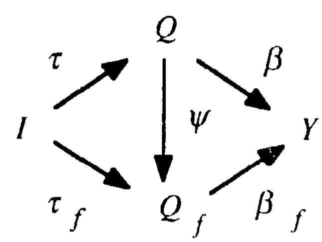
(2.1)

Therefore $M_f = (Q_f, v_f, I, \tau_f, Y, \beta_f)$ is a reachable and observable realization of f.

Now suppose that $M = (Q, v, I, \tau, Y, \beta)$ is a reachable realization of f. First one has that $\sigma \cdot r = f_\#$ by uniqueness of coextensions : $(\mathrm{ev}\cdot(1\otimes\eta)\cdot\rho^{-1})\cdot\sigma\cdot r = \beta\cdot r = f$. By diagonal fill in there exists unique $\psi : Q \to Q_f$ with

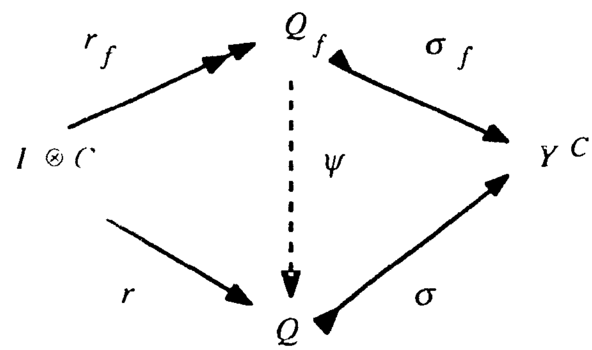


$\psi : (v : Q\otimes C \to Q) \to (v_f : Q_f\otimes C \to Q_f)$ is a morphism of right actions because (2.1.6); also since $\tau_f = r_f \cdot(1\otimes\eta)\cdot\rho^{-1} = \psi\cdot r\cdot(1\otimes\eta)\cdot\rho^{-1} = \psi\cdot\tau$ and $\beta = \mathrm{ev}\cdot(1\otimes\eta)\cdot\rho^{-1}\cdot\sigma = \mathrm{ev}\cdot(1\otimes\eta)\cdot\rho^{-1}\cdot\sigma_f\cdot\psi = \beta_f\cdot\psi$ the following diagram commutes :

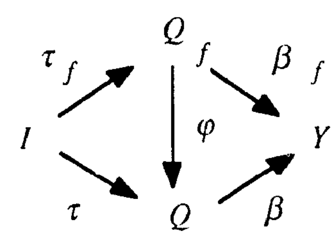


Hence $\psi : M \to M_f$ is a simulation.

Analogously suppose that $M = (Q, v, I, \tau, Y, \beta)$ is an observable realization of f, as before $\sigma\cdot r = f_\#$. By diagonal fill in there exists unique $\varphi : Q_f \to Q$ with



$\varphi : (v_f \ : Q_f \otimes C \to Q_f) \ \to \ (v : Q \otimes C \to Q\ )$ is a morphism of right actions because (2.1.6); also since $\tau = r \cdot (1 \otimes \eta) \cdot \rho^{-1} = = \varphi \cdot r_f \cdot (1 \otimes \eta) \cdot \rho^{-1} = \varphi \cdot \tau_f$ and $\beta_f = \mathrm{ev} \cdot (1 \otimes \eta) \cdot \rho^{-1} \cdot \sigma_f = = \mathrm{ev} \cdot (1 \otimes \eta) \cdot \rho^{-1} \cdot \sigma \cdot \varphi = \beta \cdot \varphi$ the following diagram commutes :



Hence $\varphi : M_f \to M$ is a simulation. ●

The next point shows a relationship between the approach given in this section and the one given in section 1.2 :

(2.1.10) Let $\boldsymbol{C}$ be a closed category with countable coproducts and let X be an object of $\boldsymbol{C}$ then one has the monoid (X^*, μ, i_0) given in (1.2.3). If by one hand one considers the process $-\otimes X \ : \boldsymbol{C} \to \boldsymbol{C}$ and the category Dyn($-\otimes X$) and by the other hand the category $_{(X^*)}$Ract then both categories are isomorphic [22, p. 778] ; and then also, fixing I and Y objects of $\boldsymbol{C}$, the categories $(-\otimes X)$-*mach* and (X^*, μ, i_0)-*mach* are isomorphic. A proof follows.

- The functor Φ : Dyn(-$\otimes X$) $\rightarrow$ ${}_{(X^*)}$Ract

Let (Q, δ) be an object of Dyn(-$\otimes X$) and let $\delta^* : Q\otimes X^* \rightarrow Q$ be defined as in the proof of (1.2.7), then δ^* is a right action.

Now, define $\Phi((Q,\delta)) = (\delta^* : Q\otimes X^* \rightarrow Q)$ and if $f : (Q, \delta) \rightarrow (Q', \delta')$ is a dynamorphism, $\Phi(f) = f$. f is a morphism of right actions and Φ is a functor.

- The functor Ψ : ${}_{(X^*)}$Ract $\rightarrow$ Dyn(-$\otimes X$)

Let $\nu : Q\otimes X^* \rightarrow Q$ be a right action, define $\delta_\nu : Q\otimes X \rightarrow Q$ to be the arrow $\nu\cdot(1\otimes i_1) : Q\otimes X \rightarrow Q\otimes X^* \rightarrow Q$ and $\Psi(\nu : Q\otimes X^* \rightarrow Q) = (Q , \delta_\nu)$.

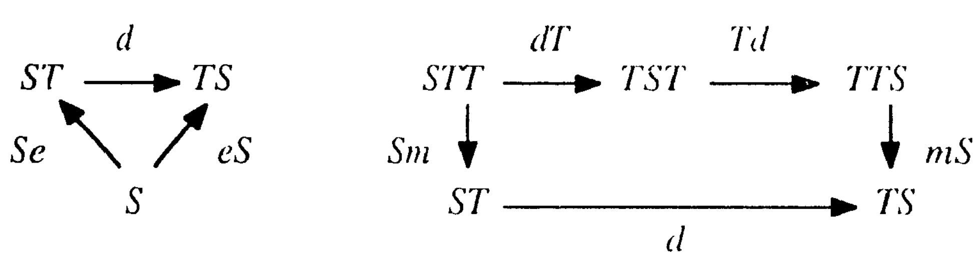
Given $f : (\nu : Q\otimes X^* \rightarrow Q) \rightarrow (\nu' : Q'\otimes X^* \rightarrow Q')$ a morphism of right actions define $\Psi(f) = f$. f is a dynamorphism and Ψ is a functor.

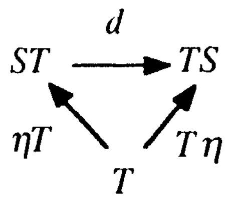
Finally one has that $\Psi\cdot\Phi$ is the identity on Dyn(-$\otimes X$) and $\Phi\cdot\Psi$ is the identity on ${}_{(X^*)}$Ract. ●

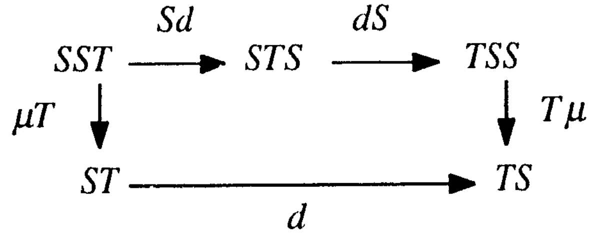
2.2. Non-deterministic C-machines.

In the following points (2.2.1) to (2.2.4) C represents an arbitrary category.

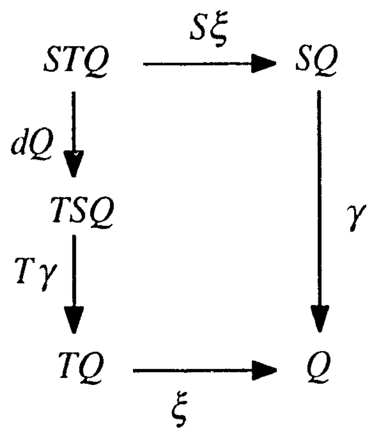
(2.2.1) Let $\mathbb{S} = (S, \eta, \mu)$ and $\mathbb{T} = (T, e, m)$ be triples in the category C; a *distributive law* of $\mathbb{S}$ over $\mathbb{T}$ (Beck, [11]) is a natural transformation $d : ST \rightarrow TS$ such that :







If $d : ST \to TS$ is a distributive law of $\mathbb{S}$ over $\mathbb{T}$ a *d-algebra* is (Q, γ, ξ) where (Q, γ) is an $\mathbb{S}$-algebra and (Q, ξ) is a $\mathbb{T}$-algebra subject to the d-law :



[32, p. 334-335].

(2.2.2) Let $X : \mathbf{C} \to \mathbf{C}$ be an input process then $\mathbb{X}@ = (X@, \eta, \mu_0@)$ where $\mu_0@Q$ is the run map of $(X@Q, \mu_0 Q)$ defines a triple in $\mathbf{C}$. [32, p. 299].

(2.2.3) Let $\mathbb{T} = (T, e, m)$ be a triple in $\mathbf{C}$, $X : \mathbf{C} \to \mathbf{C}$ an input process and $\lambda : XT \to TX$ a distributive law of X over $\mathbb{T}$. Then defining for each Q in $\mathbf{C}$, dQ as the unique morphism such that :

$$\begin{array}{ccc} TQ & \xrightarrow{\eta TQ} & X^{@}TQ \\ & \searrow^{T\eta Q} & \downarrow dQ \\ & & TX^{@}Q \end{array} \qquad \begin{array}{ccc} XX^{@}TQ & \xrightarrow{\mu_0 TQ} & X^{@}TQ \\ \downarrow XdQ & & \downarrow dQ \\ XTX^{@}Q & \xrightarrow[T\mu_0 Q \cdot \lambda X^{@}Q]{} & TX^{@}Q \end{array}$$

$d : X@T \to TX@$ is a distributive law of $\mathbb{X}@$ over $\mathbb{T}$.(From exercise 6(a) of [32, p. 335]).

Proof : TQ is an object of C and $(TX@Q \, , T\mu_0 Q \cdot \lambda X@Q)$ is a dynamics , dQ is defined as to be the unique dynamorphic extension $dQ : (X@TQ \, , \mu_0 TQ) \to (TX@Q \, , T\mu_0 Q \cdot \lambda X@Q \,)$ of $T\eta Q : TQ \to TX@Q$.

- $d : X@T \to TX@$ is a natural transformation :

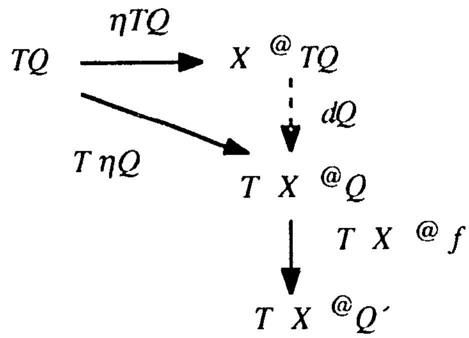
Given $f : Q \to Q'$ one has to show that the following diagram commutes :

$$\begin{array}{ccc} X^{@}TQ & \xrightarrow{dQ} & TX^{@}Q \\ X^{@}Tf \downarrow & & \downarrow TX^{@}f \\ X^{@}TQ' & \xrightarrow[dQ']{} & TX^{@}Q' \end{array}$$

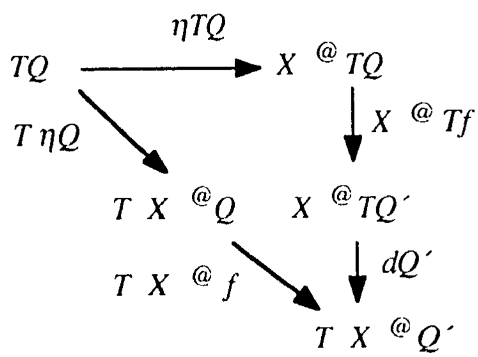
Now $TX@f : TXQ \to TX@Q'$ is a dynamorphism since :

$$\begin{array}{ccccc} XTX^{@}Q & \xrightarrow{\lambda X^{@}Q} & TXX^{@}Q & \xrightarrow{T\mu_0 Q} & TX^{@}Q \\ XTX^{@}f \downarrow & & \downarrow TXX^{@}f & & \downarrow TX^{@}f \\ XTX^{@}Q' & \xrightarrow[\lambda X^{@}Q']{} & TXX^{@}Q' & \xrightarrow[T\mu_0 Q']{} & TX^{@}Q' \end{array}$$

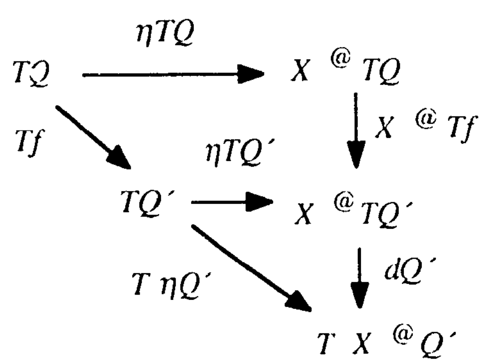
By definition of dQ one has :



Then, by uniqueness of the dynamorphic extension, it suffices to show that the following diagram commutes :



This follows from the equality $X@f \cdot \eta Q = \eta Q' \cdot f$ and from the diagram :



Therefore $d : X@T \rightarrow TX@$ is a natural transformation.

- For each Q the following diagram

$$\begin{array}{ccc} X^{@}TQ & \xrightarrow{dQ} & TX^{@}Q \\ & {\scriptstyle X^{@}eQ}\nwarrow \quad \nearrow{\scriptstyle eX^{@}Q} & \\ & X^{@}Q & \end{array}$$

commutes :

From (1.3.6) $eX@Q : (X@Q \,, \mu_0 Q) \rightarrow (TX@Q, T\mu_0 Q \cdot \lambda X@Q)$ is a dynamorphism.

By one hand one has that :

$$\begin{array}{ccc} Q & \xrightarrow{\eta Q} & X^{@}Q \\ {\scriptstyle eQ}\downarrow & & \downarrow{\scriptstyle X^{@}eQ} \\ TQ & \xrightarrow{\eta TQ} & X^{@}TQ \\ & {\scriptstyle T\eta Q}\searrow & \downarrow{\scriptstyle dQ} \\ & & TX^{@}Q \end{array}$$

and, by uniqueness of the dynamorphic extension, it suffices to show that the following diagram commutes :

$$\begin{array}{ccc} Q & \xrightarrow{\eta Q} & X^{@}Q \\ {\scriptstyle eQ}\downarrow & & \downarrow{\scriptstyle eX^{@}Q} \\ TQ & \xrightarrow[T\,\eta Q]{} & TX^{@}Q \end{array}$$

and this diagram commutes since e is a natural transformation.

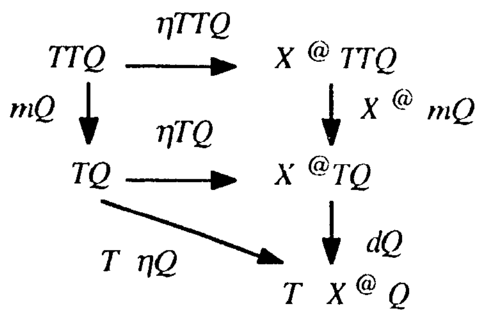
- For each Q the following diagram

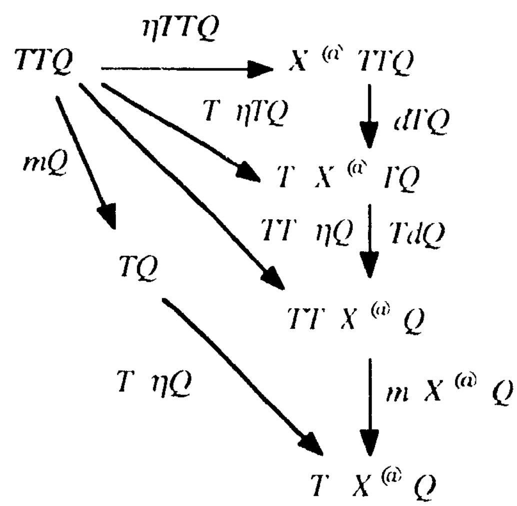
$$\begin{array}{ccccc} X^{@}TTQ & \xrightarrow{dTQ} & TX^{@}TQ & \xrightarrow{TdQ} & TTX^{@}Q \\ {\scriptstyle X^{@}mQ}\downarrow & & & & \downarrow{\scriptstyle mX^{@}Q} \\ X^{@}TQ & & \xrightarrow[dQ]{\qquad\qquad} & & TX^{@}Q \end{array}$$

commutes :
$mX@Q\cdot TdQ : (TX@TQ, T\mu_0 TQ\cdot\lambda X@TQ) \to (TX@Q, T\mu_0 Q\cdot\lambda X@Q)$ is a dynamorphism :

$$\begin{array}{ccccccc} XTX^{@}TQ & \xrightarrow{\lambda X^{@}TQ} & TXX^{@}TQ & & \xrightarrow{T\mu_0 TQ} & & TX^{@}TQ \\ {\scriptstyle XTdQ}\downarrow & & \downarrow{\scriptstyle TXdQ} & & & & \downarrow{\scriptstyle TdQ} \\ XTTX^{@}Q & \xrightarrow{\lambda TX^{@}Q} & TXTX^{@}Q & \xrightarrow{T\lambda X^{@}Q} & TTXX^{@}Q & \xrightarrow{TT\mu_0 Q} & TTX^{@}Q \\ {\scriptstyle XmX^{@}Q}\downarrow & & & & \downarrow{\scriptstyle mXX^{@}Q} & & \downarrow{\scriptstyle mX^{@}Q} \\ XTX^{@}Q & & \xrightarrow[\lambda X^{@}Q]{\qquad\qquad} & & TXX^{@}Q & \xrightarrow[T\mu_0 Q]{} & TX^{@}Q \end{array}$$

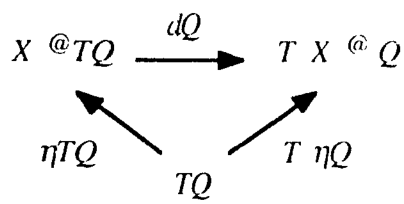
Now one has :





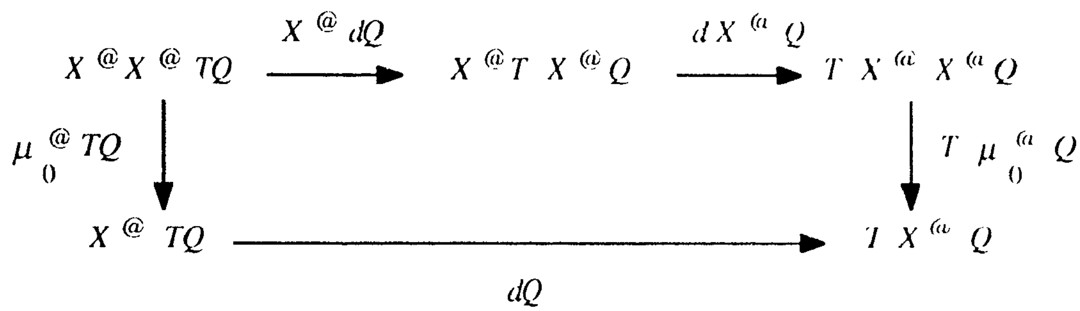
and by uniqueness of the dynamorphic extension the morphisms $dQ{\cdot}X@mQ$ and $mX@Q{\cdot}TdQ{\cdot}dTQ$ are equal.

- For each Q the following diagram



commutes. (This is by definition of dQ).

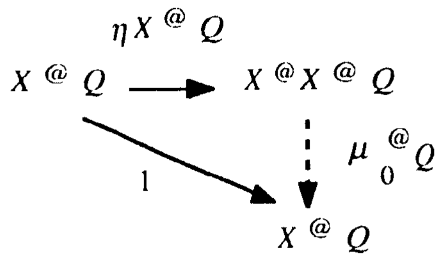
- For each Q the following diagram

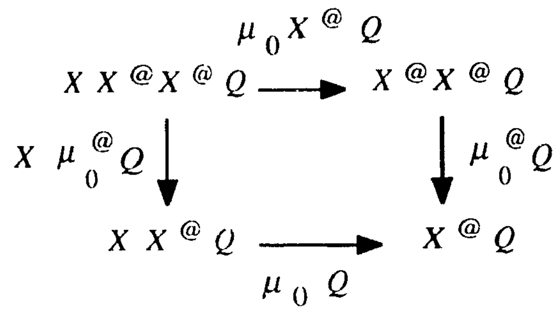


(2.2)

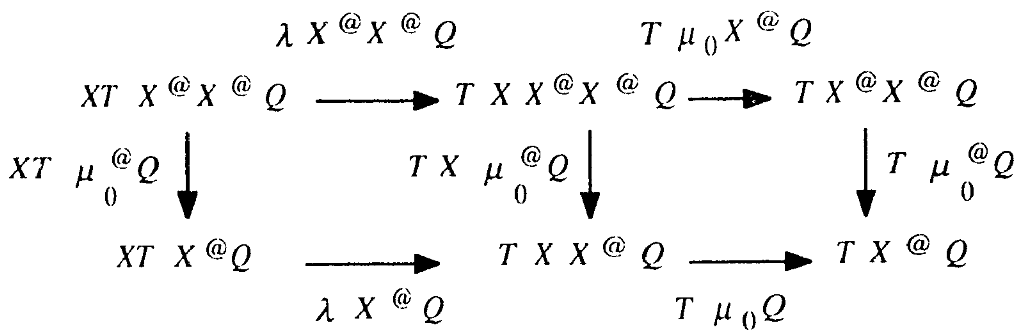
commutes :

$\mu_0 @ Q$ is the unique dynamorphic extension of $1_{X@Q}$:

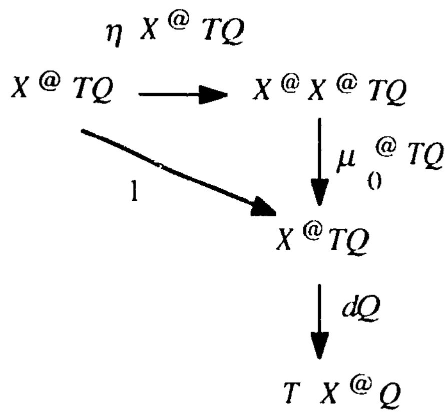


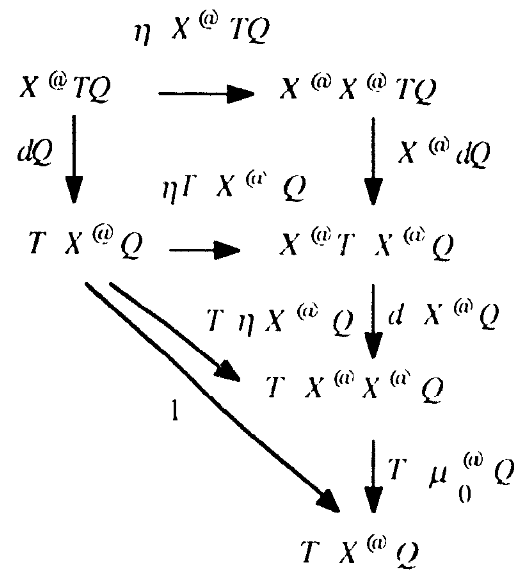


$T\mu_0 @ Q : (TX@X@Q, T\mu_0 X@Q \cdot \lambda X@X@Q) \to (TX@Q, T\mu_0 Q \cdot \lambda X@Q)$ is a dynamorphism :



Now one has :

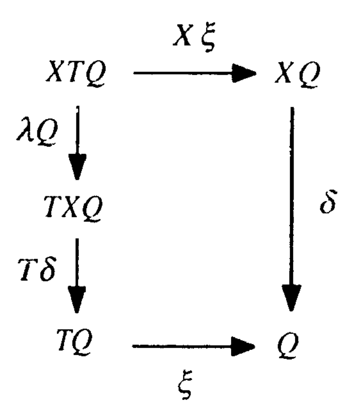




and by uniqueness of the dynamorphic extension the morphisms $dQ{\cdot}\mu_0@TQ$ and $T\mu_0@Q{\cdot}dX@Q{\cdot}X@dQ$ are equal. ●

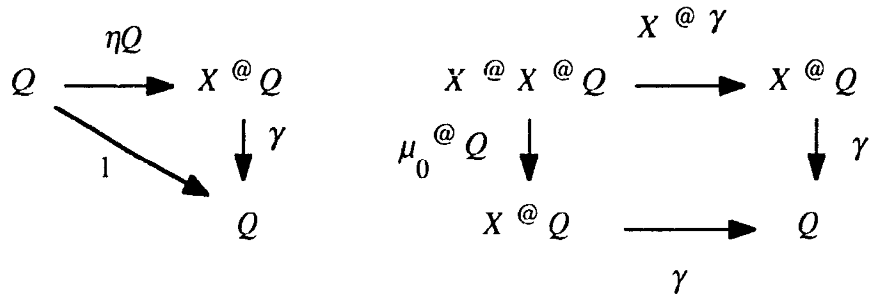
(2.2.4) In the context of (2.2.3) the category of λ-algebras is isomorphic to the category of d-algebras. (From exercise 6(b) of [32, p. 335]).

Proof: C^{λ} has as objects (Q, δ, ξ) where (Q, δ) is an X-dynamics and (Q, ξ) a $\mathbb{T}$-algebra such that $\xi : (TQ, T\delta{\cdot}\lambda Q) \to (Q, \delta)$ is a dynamorphism :

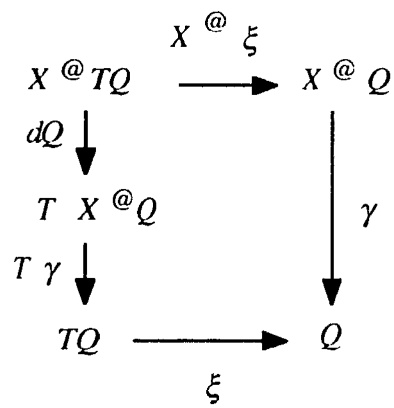


and as morphisms $f : (Q, \delta, \xi) \to (Q', \delta', \xi')$ a simultaneous dynamorphism and $\mathbb{T}$-homomorphism.

C^d has as objects (Q, γ, ξ) where (Q, γ) is such that $\gamma : X@Q \to Q$ verifies :

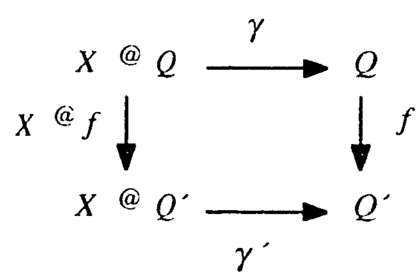


(Q, ξ) is a $\mathbb{T}$-homomorphism and the diagram



has to commute.

C^d has as morphisms $f : (Q, \gamma, \xi) \to (Q', \gamma', \xi')$ where $f : (Q, \xi) \to (Q', \xi')$ is a $\mathbb{T}$-homomorphism and also the diagram



commutes.

Now define $F : \mathbf{C}^{\lambda} \to \mathbf{C}^{d}$ and $G : \mathbf{C}^{d} \to \mathbf{C}^{\lambda}$ by the following rules :

$F\ (f : (Q, \delta, \xi) \to (Q', \delta', \xi')\ ) = (f : (Q, \delta^{@}, \xi) \to (Q', \delta'^{@}, \xi')\ )$ where $\delta^{@}$ is the run map of (Q, δ).

$G\ (g : (Q, \gamma, \xi) \to (Q', \gamma', \xi')\ ) =$

$= g : (Q\ ,\ \gamma \cdot \eta_1 Q : XQ \to Q\ , \xi) \to (Q'\ ,\ \gamma' \cdot \eta_1 Q' : XQ' \to Q'\ , \xi')$

where $\eta_1 : X \to X^{@}$, as in (1.1.9), is defined for each Q as $\mu_0 Q \cdot X\eta Q : XQ \to XX^{@}Q \to X^{@}Q$ and given any dynamics (Q, δ) is $\delta^{@} \cdot \eta_1 Q = \delta$.

The remaining details, i.e. F and G are well defined functors, $G \cdot F = 1_{\mathbf{C}^{\lambda}}$, and $F \cdot G = 1_{\mathbf{C}^{d}}$ can be checked easily. ●

Monoidal and closed categories will be treated again .

(2.2.5) If $(\mathbf{C}\ , \otimes, E, \alpha, \lambda, \rho)$ is a monoidal category and (C, μ, η) is a monoid in it one can consider [32, pp. 214-215] the triple $(\ \text{-}\otimes C\ , \eta'\ , \mu')$ on $\mathbf{C}$ given by :

$\text{-}\otimes C : \mathbf{C} \to \mathbf{C}$

$\eta' : 1_{\mathbf{C}} \to \text{-}\otimes C$ is the natural transformation that assigns to each object A the arrow $\eta' A : A \to A \otimes C$ defined by $\eta' A = (1 \otimes \eta) \cdot \rho^{-1} : A \to A \otimes E \to A \otimes C$

$\mu' : (\ \text{-}\otimes C\ ) \cdot (\ \text{-}\otimes C\ ) \to (\ \text{-}\otimes C\ )$ is the natural transformation that assigns to each object A the arrow $\mu' A : (A \otimes C) \otimes C \to A \otimes C$ defined by $\mu' A = (1 \otimes \mu) \cdot \alpha^{-1} : (A \otimes C) \otimes C \to A \otimes (C \otimes C) \to A \otimes C$.

Hence to have a distributive law of $(\ \text{-}\otimes C\ , \eta', \mu')$ over $\mathbb{T} = (T, e, m)$ (both triples on the monoidal category $\mathbf{C}$) means to have a natural transformation

$$d : (\ \text{-}\otimes C\ ) \cdot T \to T \cdot (\ \text{-}\otimes C\ )$$

such that for each object A of $\mathbf{C}$ the following diagrams commute :

$$
\begin{array}{ccc}
TA\otimes C & \xrightarrow{dA} & T(A\otimes C) \\
{\scriptstyle eA\otimes 1}\nwarrow & & \nearrow{\scriptstyle e(A\otimes C)} \\
& A\otimes C &
\end{array}
\qquad
\begin{array}{ccccc}
TTA\otimes C & \xrightarrow{dTA} & T(TA\otimes C) & \xrightarrow{TdA} & TT(A\otimes C) \\
\downarrow{\scriptstyle mA\otimes 1} & & & & \downarrow{\scriptstyle m(A\otimes C)} \\
TA\otimes C & & \xrightarrow[dA]{\qquad\qquad} & & T(A\otimes C)
\end{array}
$$

$$
\begin{array}{ccc}
TA\otimes C & \xrightarrow{dA} & T(A\otimes C) \\
{\scriptstyle \eta' TA}\nwarrow & & \nearrow{\scriptstyle T\eta' A} \\
& TA &
\end{array}
\qquad
\begin{array}{ccccc}
(TA\otimes C)\otimes C & \xrightarrow{dA\otimes 1} & (T(A\otimes C))\otimes C & \xrightarrow{d(A\otimes C)} & T((A\otimes C)\otimes C) \\
\downarrow{\scriptstyle \mu' TA} & & & & \downarrow{\scriptstyle T\mu' A} \\
TA\otimes C & & \xrightarrow[dA]{\qquad\qquad} & & T(A\otimes C)
\end{array}
$$

And a *d-algebra* will be (Q, v, ξ) where $v : Q\otimes C \to Q$ is a $(\text{-}\otimes C\ , \eta', \mu')$-algebra (that is the same as to say that v is a right action) and (Q, ξ) a $\mathbb{T}$-algebra such that the following diagram commutes :

$$
\begin{array}{ccccc}
TQ\otimes C & \xrightarrow{dQ} & T(Q\otimes C) & \xrightarrow{Tv} & TQ \\
{\scriptstyle \xi\otimes 1}\downarrow & & & & \downarrow{\scriptstyle \xi} \\
Q\otimes C & & \xrightarrow[v]{\qquad\qquad} & & Q
\end{array}
$$

A *d-homomorphism* $f : (Q, v, \xi) \to (Q', v', \xi')$ will be a simultaneous morphism of right actions and $\mathbb{T}$-homomorphism. The corresponding category will be denoted again by C^d.

Similarly to (1.3.5) if $\mathbb{T} = (T, e, m)$ is the identity triple and $d : (\text{-}\otimes C\ )\cdot T \to T\cdot(\text{-}\otimes C\ )$ is the identity distributive law, C^d is isomorphic to ${}_C\text{Ract}$. Therefore the theory of d-algebras generalizes ${}_C\text{Ract}$.

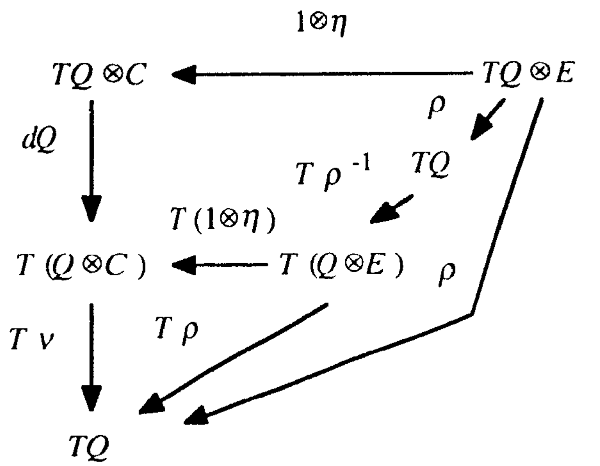
From now on and till the end of section 2.2 $\boldsymbol{C}$ will be a closed category; (C, μ, η) a monoid in it, $(\,\text{-}\otimes C\,, \eta', \mu'\,)$ the associated triple; $\mathbb{T} = (T, e, m)$ another triple in $\boldsymbol{C}$ and d a distributive law of $(\,\text{-}\otimes C\,, \eta', \mu'\,)$ over $\mathbb{T}$.

In a similar manner to (1.3.6) one has :

(2.2.6) The forgetful functor $U : \boldsymbol{C}^d \to {}_C\mathrm{Ract}$ has a left adjoint. A free d-algebra over $v : Q\otimes C \to Q$ is given by $((TQ, Tv{\cdot}dQ, mQ), eQ)$; and given any d-algebra (Y, γ, θ) and any morphism of right actions $\beta : (\,v : Q\otimes C \to Q\,) \to (\,\gamma : Y\otimes C \to Y\,)$ the unique d-homomorphic extension $\beta^{\#} : (TQ\,, Tv{\cdot}dQ\,, mQ) \to (Y, \gamma, \theta)$ of β is given by $\beta^{\#} = \theta{\cdot}T\beta : TQ \to TY \to Y$. Here is the proof:

Let $v : Q\otimes C \to Q$ be a right action then the arrow $Tv{\cdot}dQ : TQ\otimes C \to T(Q\otimes C) \to TQ$ is also a right action :

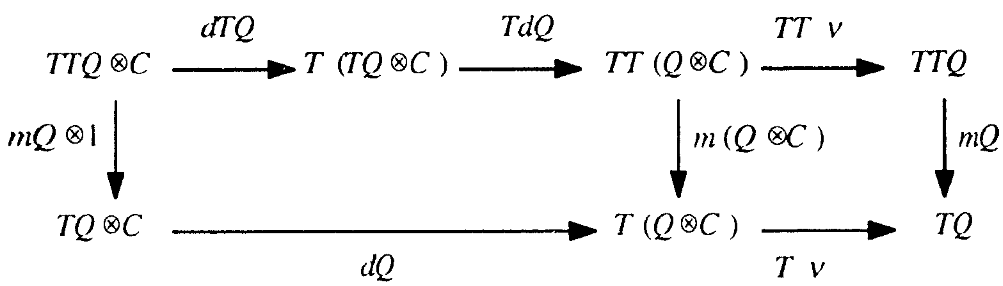
$$\begin{array}{ccccccc}
(TQ\otimes C)\otimes C & & \xrightarrow{\alpha^{-1}} & TQ\otimes(C\otimes C) & & \xrightarrow{1\otimes\mu} & TQ\otimes C \\
\downarrow{\scriptstyle dQ\otimes 1} & & & & & & \downarrow{\scriptstyle dQ} \\
(T(Q\otimes C))\otimes C & \xrightarrow{d(Q\otimes C)} & T((Q\otimes C)\otimes C) & \xrightarrow{T\alpha^{-1}} & T(Q\otimes(C\otimes C)) & \xrightarrow{T(1\otimes\mu)} & T(Q\otimes C) \\
\downarrow{\scriptstyle Tv\otimes 1} & & \downarrow{\scriptstyle T(v\otimes 1)} & & & & \downarrow{\scriptstyle Tv} \\
TQ\otimes C & \xrightarrow{dQ} & T(Q\otimes C) & & \xrightarrow{Tv} & & TQ
\end{array}$$



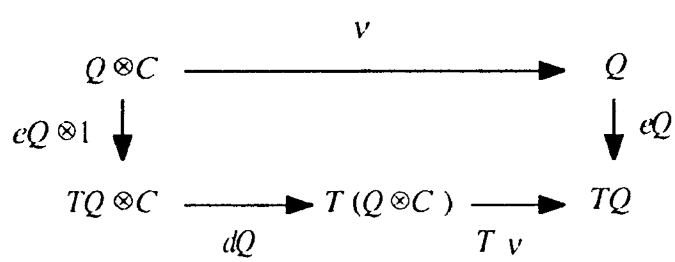
Now follows the remainder of the proof that is similar to the proof given by Arbib and Manes of the fact stated in (1.3.6) :

- $(TQ\ , Tv{\cdot}dQ\ , mQ)$ is a d-algebra :

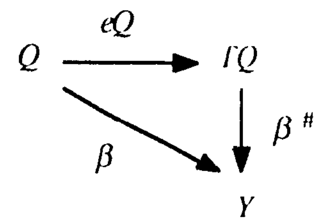
The only thing that remains to show is the commutativity of the following diagram, and this is obvious from the definitions :



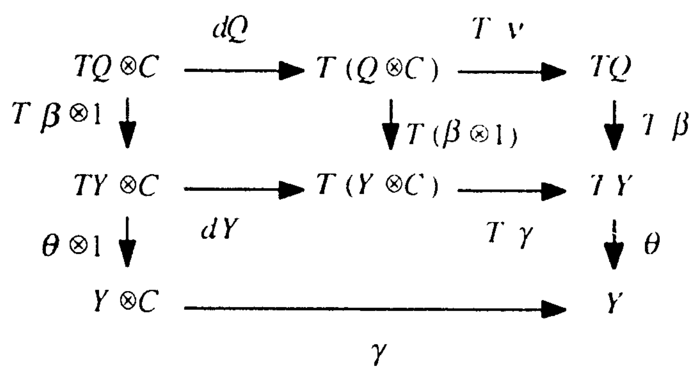
- $eQ : (\ v : Q{\otimes}C \rightarrow Q\ ) \rightarrow (\ Tv{\cdot}dQ : TQ{\otimes}C \rightarrow TQ\ )$ is a morphism of right actions :



- Suppose given (Y, γ, θ) an object of C^d and $\beta : (v : Q \otimes C \to Q) \to (\gamma : Y \otimes C \to Y)$ a morphism of right actions, since $U : C^{\mathbb{T}} \to C$ has a left adjoint there exists a unique $\mathbb{T}$-homomorphism $\beta^{\#} : (TQ\ , mQ) \to (Y\ , \theta)$ given by $\theta{\cdot}T\beta : TQ \to TY \to Y$ such that the following diagram commutes :



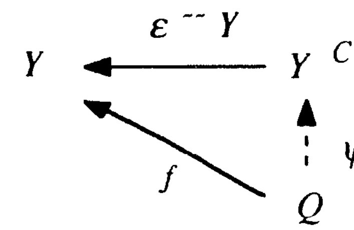
That $\beta^{\#} : (Tv{\cdot}dQ : TQ \otimes C \to TQ\ ) \to (\ \gamma : Y \otimes C \to Y\ )$ is a morphism of right actions follows from the diagram :



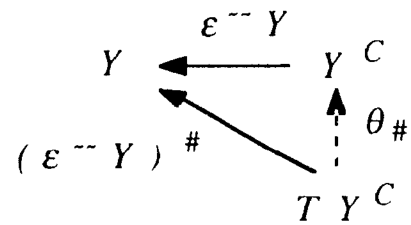
●

In a similar manner to (1.3.7) one has :

(2.2.7) The forgetful functor $U : C^d \to C^{\sim\sim}$ has a right adjoint. A cofree d-algebra over $(Y,\ \theta)$ is given by $(\ (Y^C,\ vY,\ \theta_{\#}),\ \varepsilon^{\sim\sim}Y)$, and given any d-algebra $(Q,\ v,\ \xi)$ and any $^{\sim\sim}$-homomorphism $f : (Q,\ \xi) \to (Y,\ \theta)$, the unique d-homomorphic coextension $\psi : (Q,\ v,\ \xi) \to (\ Y^C,\ vY,\ \theta_{\#})$ of f is given by the unique morphism $\psi : (Q,\ v) \to (Y^C,\ vY)$ of right actions that makes the following diagram commute :

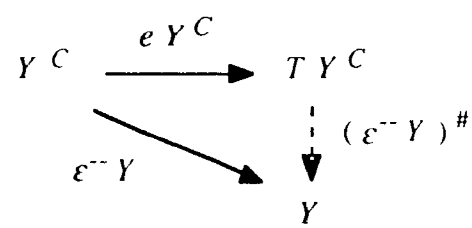


Here $\theta_{\#} : (T\nu Y \cdot dY^C : TY^C \otimes C \rightarrow TY^C) \rightarrow (\nu Y : Y^C \otimes C \rightarrow Y^C)$ is the unique morphism of right actions such that :



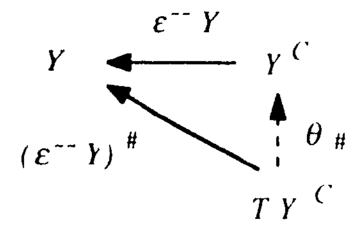
where $(\varepsilon^{--}Y)^{\#} = \theta \cdot T\varepsilon^{--}Y : TY^C \rightarrow TY \rightarrow Y$. Here is a proof that is similar to the one given by Arbib and Manes of the fact stated in (1.3.7) :

Let (Y, θ) be an object of $C^{\mathcal{T}}$. From (2.1.3) one knows that $U : {}_C\text{Ract} \rightarrow \boldsymbol{C}$ has a right adjoint $G : \boldsymbol{C} \rightarrow {}_C\text{Ract}$ and that the cofree right action over an object A of $\boldsymbol{C}$ is given by $(\nu A : A^C \otimes C \rightarrow A^C , \ \varepsilon^{--}A = ev \cdot (1 \otimes \eta) \cdot \rho^{-1} : A^C \rightarrow A)$. Consider $\varepsilon^{--}Y : Y^C \rightarrow C = \text{ev} \cdot (1 \otimes \eta) \cdot \rho^{-1} : Y^C \rightarrow Y^C \otimes E \rightarrow Y^C \otimes C \rightarrow Y$ and the unique $\mathcal{T}$ - homomorphic extension $(\varepsilon^{--}Y)^{\#} : (TY^C , mY^C) \rightarrow (Y, \theta)$ of $\varepsilon^{--}Y$:



$$(\varepsilon^{--}Y)^{\#} = \theta \cdot T\varepsilon^{--}Y : TY^C \rightarrow TY \rightarrow Y =$$
$$= \theta \cdot T \, \text{ev} \cdot T(1 \otimes \eta) \cdot T\rho^{-1} : TY^C \rightarrow T(Y^C \otimes E) \rightarrow T(Y^C \otimes C) \rightarrow TY \rightarrow Y.$$

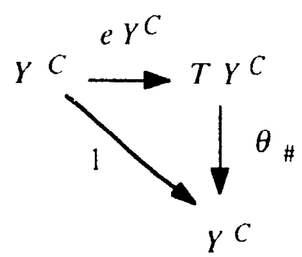
Now given the right action $\nu Y : Y^C \otimes C \to Y^C$, from (2.2.6) $T\nu Y \cdot dY^C : TY^C \otimes C \to T(Y^C \otimes C) \to TY^C$ is also a right action and considering $(\varepsilon^{--}Y)^{\#} . TY^C \to Y$ as an arrow in $\mathbf{C}$ let $\theta_{\#} : (T\nu Y \cdot dY^C : TY^C \otimes C \to TY^C) \to (\nu Y : Y^C \otimes C \to Y^C)$ be the unique morphism of right actions such that :



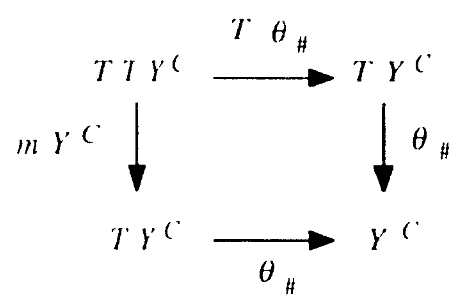
- $(Y^C, \theta_{\#})$ is a T-algebra :

One has to show that :

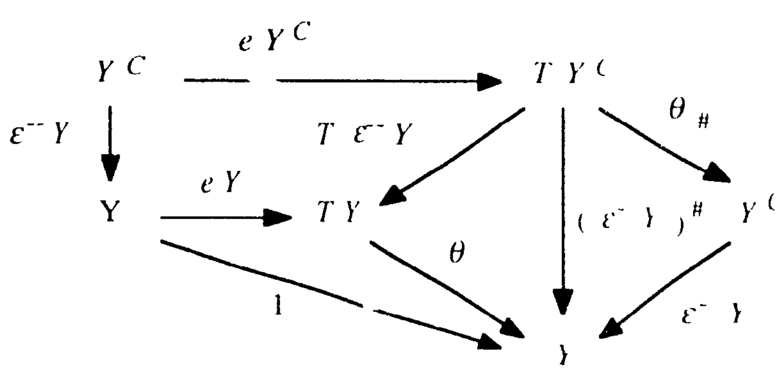
(i)



(ii)

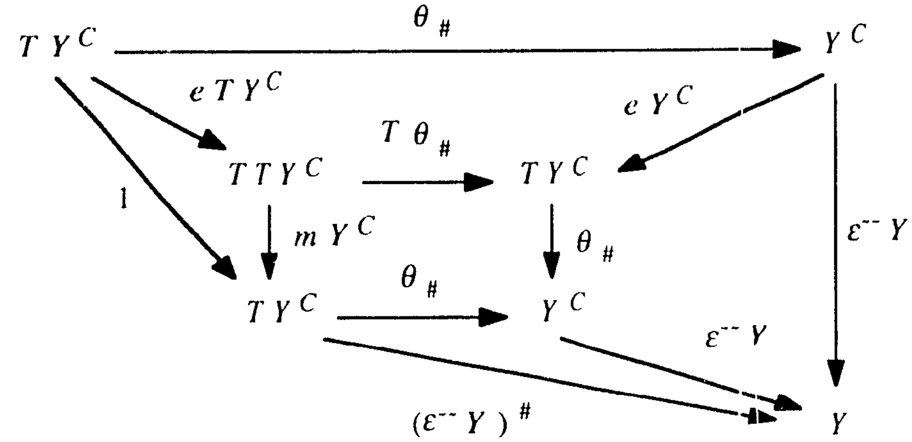


The diagram



commutes, then the morphisms of right actions $\theta_{\#} \cdot eY^C$ and 1_{Y^C} are equal followed by $\varepsilon^{--}Y$, and hence equal. Therefore (i) commutes.

Now, consider the diagram

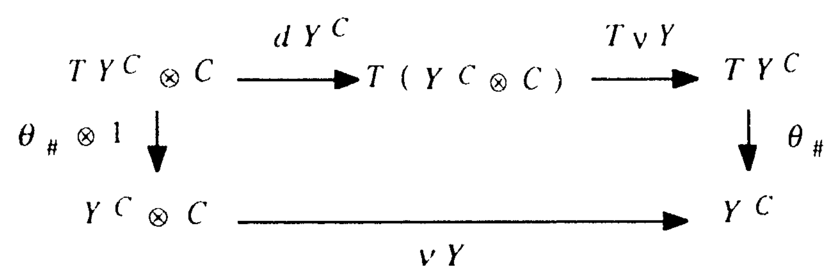


The $\mathcal{T}$-homomorphisms $(\varepsilon^{--}Y)^\#\cdot mY^C$ and $(\varepsilon^{--}Y)^\#\cdot T\theta_\#$ are equal preceded by eTY^C, and hence equal.

Now the morphisms of right actions $\theta_\#\cdot T\theta_\#$ and $\theta_\#\cdot mY^C$ are equal followed by $\varepsilon^{--}Y$ and so are equal. Therefore (ii) commutes.

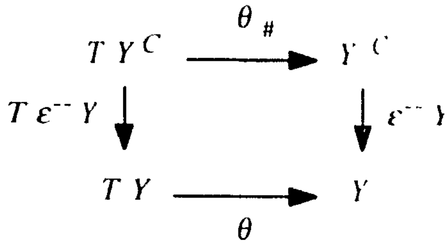
- $(Y^C, vY, \theta_\#)$ is a d-algebra :

Since $\theta_\#: (TvY\cdot dY^C : TY^C\otimes C \rightarrow TY^C) \rightarrow (vY : Y^C\otimes C \rightarrow Y^C)$ is a morphism of right actions :



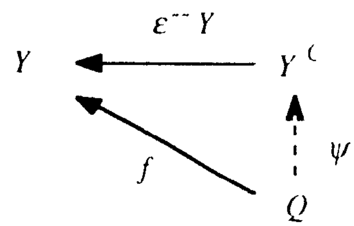
- $\varepsilon^{--}Y : (Y^C, \theta_\#) \rightarrow (Y, \theta)$ is a $\mathcal{T}$-homomorphism :

The diagram

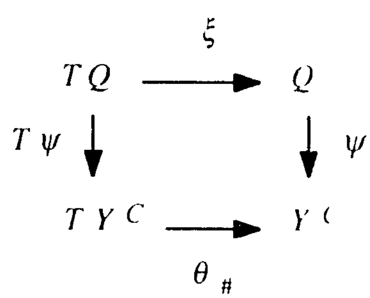


commutes since $\theta \cdot T\varepsilon^{\sim\sim}Y = (\varepsilon^{\sim\sim}Y)^{\#}$ and $\varepsilon^{\sim\sim}Y \cdot \theta_{\#} = (\varepsilon^{\sim\sim}Y)^{\#}$.

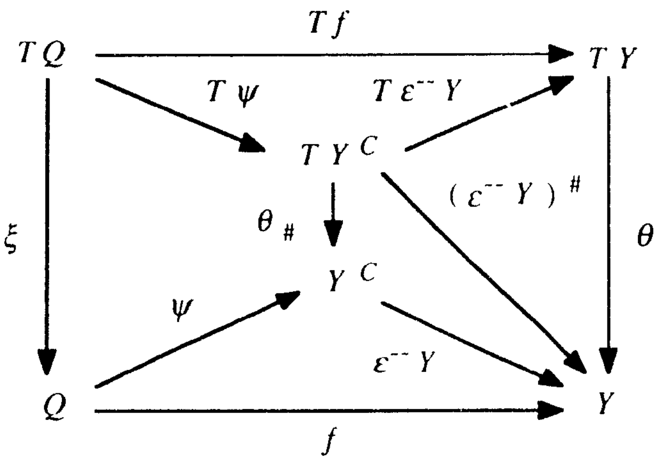
- Now suppose given a d-algebra (Q, ν, ξ) and a $\mathcal{T}$-homomorphism $f : (Q, \xi) \rightarrow (Y, \theta)$. Let $\psi : (\ \nu : Q \otimes C \rightarrow Q\ ) \rightarrow (\ Y^C, \nu Y\ )$ be the unique morphism of right actions such that



It suffices to show that $\psi \cdot (Q, \xi) \rightarrow (Y^C, \theta_{\#})$ is a $\mathcal{T}$-homomorphism, i.e. that the following diagram commutes :



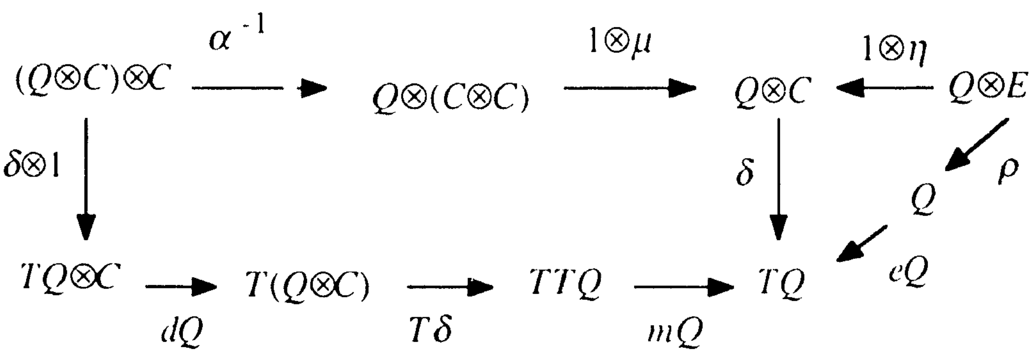
Consider the diagram :



The exterior part commutes since f is a T-homomorphism. $\theta_{\#}\cdot T\psi$ and $\psi\cdot\xi$ are morphisms of right actions that are equal followed by $\varepsilon^{--}Y$, and hence equal. ●

Now the concept of d-machine is introduced :

(2.2.8) A *d-machine* is a 7-tuple $M = (Q,\ \delta\ I,\ \tau,\ Y,\ \theta,\ \beta)$ where (Y, θ) is a T-algebra, $\tau : I \to TQ$ and $\beta : Q \to Y$ are $\boldsymbol{C}$-morphisms and $\delta : Q{\otimes}C \to TQ$ is a $\boldsymbol{C}$ -morphism such that :



If $M = (Q,\ \delta,\ I,\ \tau,\ Y,\ \theta,\ \beta)$ is a d-machine, a routine diagram shows that $\delta^{\circ} = \delta^{\#}\cdot dQ\ : TQ{\otimes}C \to T(Q{\otimes}C\,) \to TQ$ is a right action.

Now, trying to translate (1.3.9) to (1.3.15) to the present situation, the following (2.2.9) to (2.2.15) have been obtained.

(2.2.9) Associated with the d-machine $M = (Q, \delta, I, \tau, Y, \theta, \beta)$ is the (C, μ, η)-machine $M^\circ = (TQ, \delta^\circ, I, \tau, Y, \beta^\#)$ where :

$$\delta^\circ = \delta^\# \cdot dQ = mQ \cdot T\delta \cdot dQ : TQ \otimes C \rightarrow T(Q \otimes C) \rightarrow TQ \quad \text{and}$$

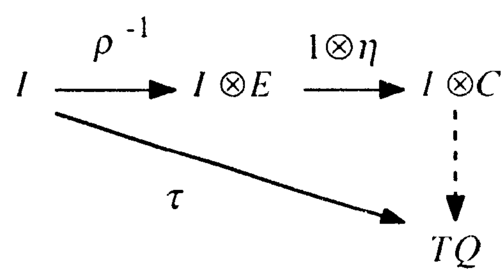
$$\beta^\# = \theta \cdot T\beta : TQ \rightarrow TY \rightarrow Y.$$

The *response* of the d-machine $M = (Q, \delta, I, \tau, Y, \theta, \beta)$ is the response of the (C, μ, η)-machine $M^\circ = (TQ, \delta^\circ, I, \tau, Y, \beta^\#)$, i.e. it is the $\boldsymbol{C}$-morphism :

$$\beta^\# \cdot r : I \otimes C \rightarrow TQ \rightarrow Y$$

where

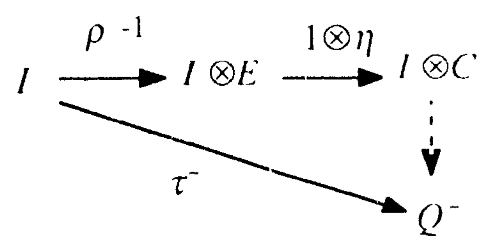
$r : ((1 \otimes \mu) \cdot \alpha^{-1} : (I \otimes C) \otimes C \rightarrow I \otimes (C \otimes C) \rightarrow I \otimes C) \rightarrow (\delta^\circ : TQ \otimes C \rightarrow TQ)$ is the unique morphism of right actions such that :



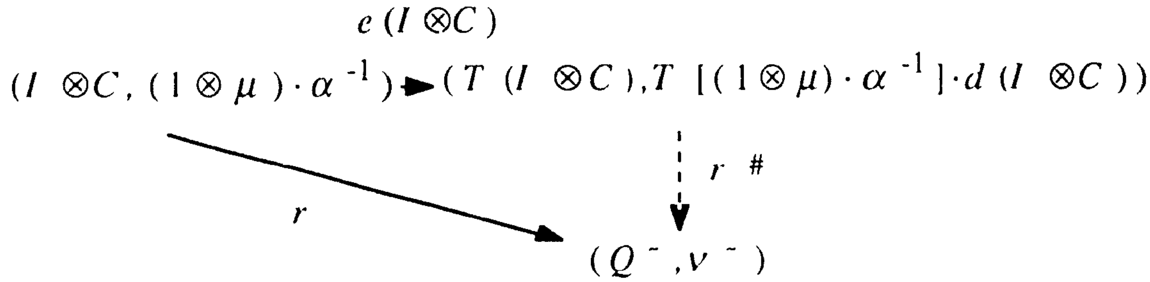
(2.2.10) An *implicit d-machine* is a 8-tuple $M^\sim = (Q^\sim, \nu^\sim, \xi^\sim, I, \tau^\sim, Y, \theta, \beta^\sim)$ where $(Q^\sim, \nu^\sim, \xi^\sim)$ is a d-algebra, $\tau^\sim : I \rightarrow Q^\sim$ is a $\boldsymbol{C}$-morphism, and $\beta^\sim : (Q^\sim, \xi^\sim) \rightarrow (Y, \theta)$ is a $\mathbb{T}$-homomorphism.

When $^{\sim\sim}$ is the identity triple and d = id, " implicit d -machine " and " d -machine " coincide with " (C, μ, η)-machine ".

(2.2.11) The (C, μ, η)-*reachability map* of the implicit d-machine $M^\sim$ is the unique morphism of right actions $r : (I \otimes C, (1 \otimes \mu) \cdot \alpha^{-1}) \rightarrow (Q^\sim, \nu^\sim)$ such that :

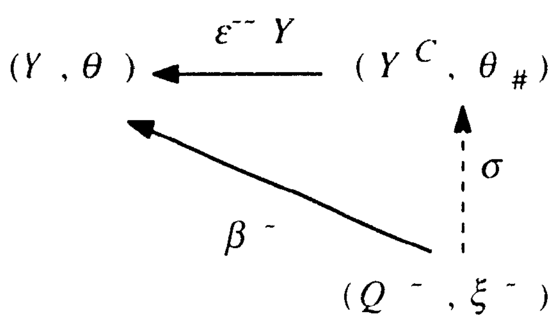


The *reachability map of* M^{-} is the unique d-homomorphic extension
$r^{\#}: (T(I\otimes C), T[(1\otimes\mu)\cdot\alpha^{-1}]\cdot d(I\otimes C),\ m(I\otimes C)) \to (Q^{-}, \nu^{-}, \xi^{-})$ of r :



The *response* of M^{-} is the composition $\beta^{-}\cdot r : I\otimes C \to Q^{-} \to Y$, i.e. is the response of the (C, μ, η)-machine $(Q^{-}, \nu^{-}, I, \tau^{-}, Y, \beta^{-})$.

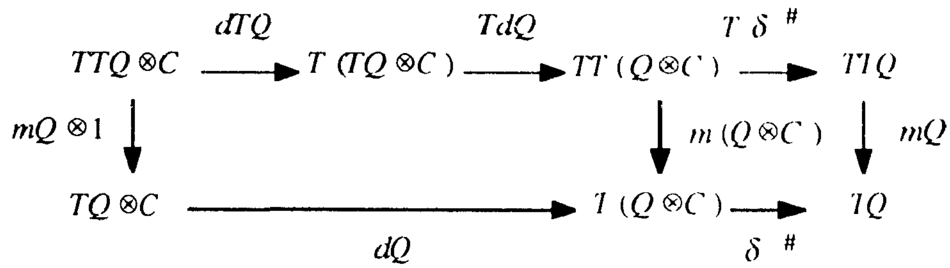
The *observability map* of M^{-} is the unique d-homomorphic coextension $\sigma : (Q^{-}, \nu^{-}, \xi^{-}) \to (Y^{C}, \nu Y, \theta_{\#})$ of β^{-} :



The next two results show that d-machines and implicit d-machines compute the same responses.

(2.2.12) Let $M = (Q, \delta, I, \tau, Y, \theta, \beta)$ be a d-machine. Then $M^{\circ} = (TQ,\ \delta^{\circ},\ mQ,\ I,\ \tau,\ Y,\ \theta,\ \beta^{\#})$, where $\delta^{\circ} = \delta^{\#}\cdot dQ$, is an implicit d-machine and the response of M is the response of M°.

Proof : (TQ, δ°, mQ) is a d -algebra :



The rest of the proof follows immediately from the definitions. ●

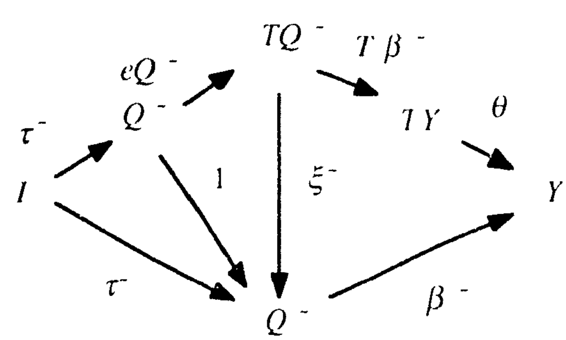
(2.2.13) Let $M^{\sim} = (Q^{\sim}, \nu^{\sim}, \xi^{\sim}, I, \tau^{\sim}, Y, \theta, \beta^{\sim})$ be an implicit d-machine. Then the d-machine $M = (Q^{\sim}, \delta, I, \tau, Y, \theta, \beta^{\sim})$ where $\delta = eQ^{\sim}\cdot\nu^{\sim} : Q^{\sim}\otimes C \to Q^{\sim} \to TQ^{\sim}$ and $\tau = eQ^{\sim}\,\tau^{\sim} : I \to Q^{\sim} \to TQ^{\sim}$ has the same response as $M^{\sim}$. Here is the proof :

- A routine diagram shows that M is a d-machine.
- M and $M^{\sim}$ have the same responses :

Since the response of the d-machine M is the response of the (C, μ, η)-machine $M^{\circ} = (TQ^{\sim}, \delta^{\circ}, I, \tau, Y, \beta^{\sim\#})$, where $\delta^{\circ} = \delta^{\#}\cdot dQ = mQ^{\sim}\cdot T\delta\cdot dQ = mQ^{\sim}\cdot TeQ^{\sim}\cdot T\nu^{\sim}\cdot dQ = T\nu^{\sim}\cdot dQ$ and $\beta^{\sim\#} = \theta\cdot T\beta^{\sim}$, and the response of the implicit d-machine $M^{\sim}$ is the response of the (C, μ, η)-machine $M^{\sim\prime} = (Q^{\sim}, \nu^{\sim}, I, \tau^{\sim}, Y, \beta^{\sim})$, it suffices to construct a simulation $\psi : M^{\circ} \to M^{\sim\prime}$.

But since $(Q^{\sim}, \nu^{\sim}, \xi^{\sim})$ is a d-algebra one has that $\xi^{\sim} : (TQ^{\sim}\ \delta^{\circ}) \to (Q^{\sim}, \nu^{\sim})$ is a morphism of right actions.

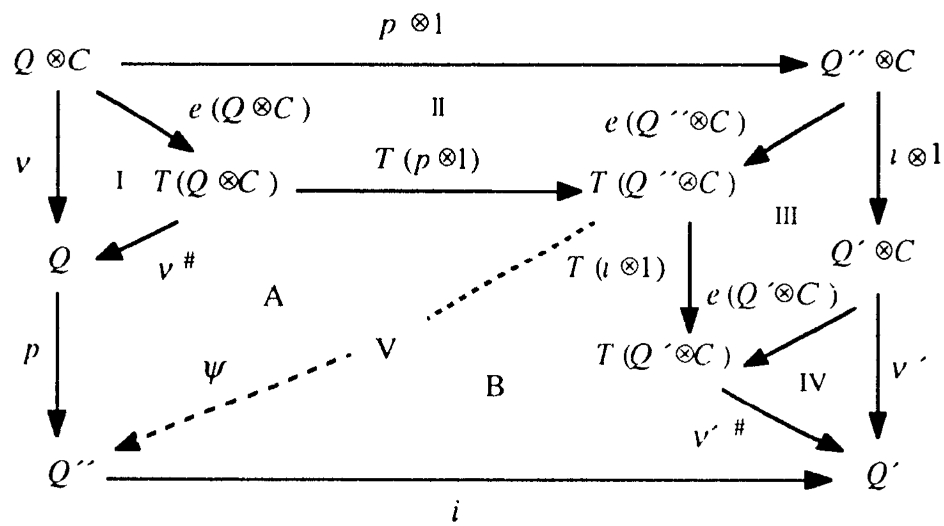
Finally, since the following diagram commutes



one has that ξ^{-} is a simulation. ●

(2.2.14) Let $h : (Q, v, \xi) \to (Q', v', \xi')$ be a d-homomorphism and let $i \cdot p : Q \twoheadrightarrow Q'' \rightarrowtail Q'$ be an E-M factorization of h in $\boldsymbol{C}$ ($\boldsymbol{C}$ is supposed to have an image factorization system (E, M)). Then, if $T \cdot (- \otimes C)$ preserves E, there exist unique $v'' : Q'' \otimes C \to Q''$ and $\xi'' : TQ'' \to Q''$ such that $p : (Q, v, \xi) \to (Q'', v'', \xi'')$ and $i : (Q'', v'', \xi'') \to (Q', v', \xi')$ are d-homomorphisms.

Proof : Consider the diagram

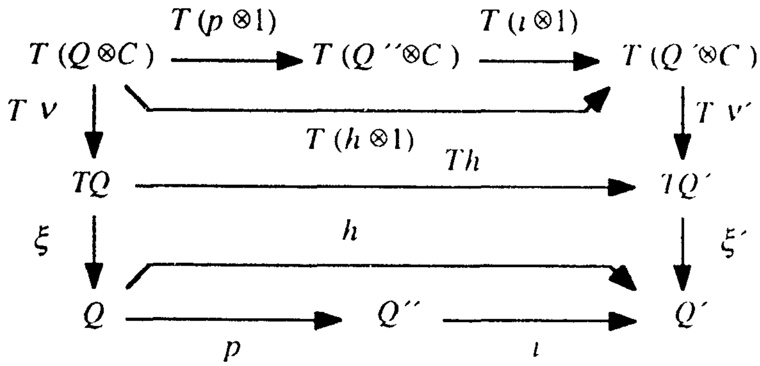


I commutes by definition of $v^{\#}$.

II and III commute since e is a natural transformation.

IV commutes by definition of $v'^{\#}$.

V commutes because the following diagram :

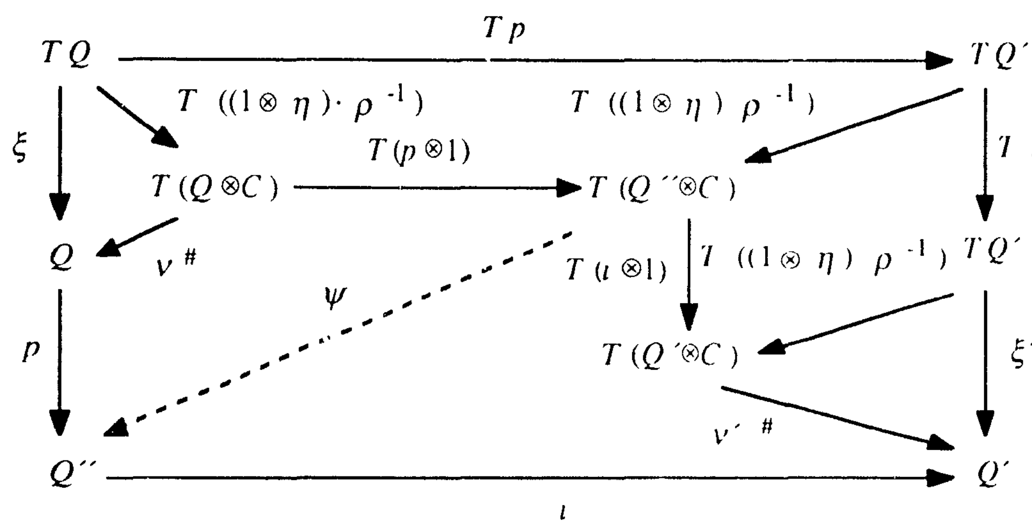


Now, by diagonal fill in, there exists a unique $\boldsymbol{C}$-morphism $\psi : T(Q''\otimes C) \to Q''$ such that makes the diagrams A and B commute.

Define $v'' = \psi\cdot e(Q''\otimes C)$ and using the fact that ι is a monomorphism a routine diagram shows that $v'' : Q''\otimes C \to Q''$ is a right action.

Then $p : (Q, v) \to (Q'', v'')$ and $i : (Q'', v'') \to (Q', v')$ are morphisms of right actions.

Consider now the diagram

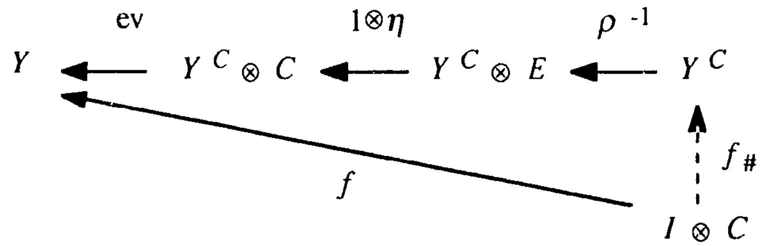


and define $\xi'' = \psi\cdot T((1\otimes\eta)\cdot\rho^{-1})$.

The rest of the proof is analogous to the proof of (1.3.14). ●

(2.2.15) Let (E, M) be an image factorization system in C and suppose that $T\cdot(-\otimes C)$ preserves E . Let I and (Y, θ) be fixed. Then for every $f : I\otimes C \to Y$ there exists an implicit d-machine $M^{\sim} = (Q^{\sim}, v^{\sim}, \xi^{\sim}, I, \tau^{\sim}, Y, \theta, \beta^{\sim})$ such that the response of $M^{\sim}$ is f, the reachability map $r^{\#} : T(I\otimes C) \to Q^{\sim}$ is in E, and the observability map $\sigma : Q^{\sim} \to Y^{C}$ is in M. If $M^{\sim\prime}$ also satisfies these three conditions then $M^{\sim}$ and $M^{\sim\prime}$ are isomorphic (i.e., there exists an isomorphism $\psi : (Q^{\sim}, v^{\sim}, \xi^{\sim}) \to (Q^{\sim\prime}, v^{\sim\prime}, \xi^{\sim\prime})$ of d-algebras such that $\psi\cdot\tau^{\sim} = \tau^{\sim\prime}$ and $\beta^{\sim\prime}\cdot\psi = \beta^{\sim}$).

Proof: Let
$f_{\#}$: $((1\otimes\mu)\cdot\alpha^{-1}: (I\otimes C)\otimes C \to I\otimes(C\otimes C) \to I\otimes C) \to (vY: Y^{C}\otimes C \to Y^{C})$
be the unique morphism of right actions coextension of f:



and let
$f^{\sim} = (f_{\#})^{\#}$: $(T(I\otimes C), T((1\otimes\mu)\cdot\alpha^{-1})\cdot d(I\otimes C), m(I\otimes C)) \to (Y^{C}, vY, \theta_{\#})$ be the unique d-homomorphic extension of $f_{\#}$.

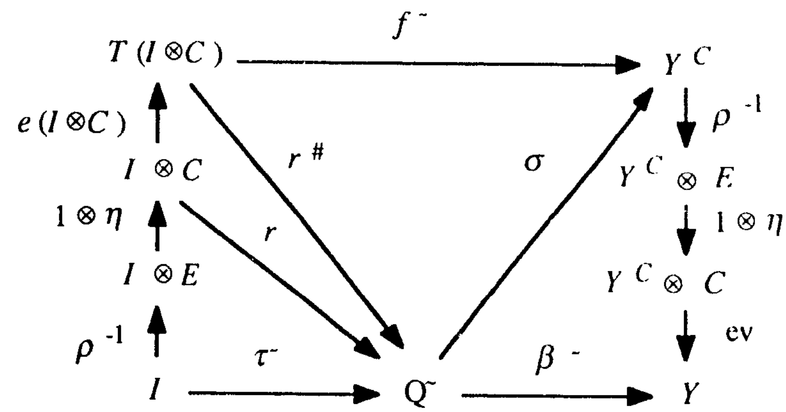
Now consider an E-M factorization of $f^{\sim}$:

$$\sigma\cdot r^{\#} : T(I\otimes C) \longrightarrow\!\!\!\!\!\!\to Q^{\sim} \rightarrowtail Y^{C}$$

By (2.2.14) there exist unique $v^{\sim} : Q^{\sim}\otimes C \to Q^{\sim}$ and $\xi^{\sim} : TQ^{\sim} \to Q^{\sim}$ such that
$r^{\#}$: $(T(I\otimes C), T((1\otimes\mu)\cdot\alpha^{-1})\cdot d(I\otimes C), m(I\otimes C)) \to (Q^{\sim}, \xi^{\sim}, v^{\sim})$ and $\sigma : (Q^{\sim}, \xi^{\sim}, v^{\sim}) \to (Y^{C}, vY, \theta_{\#})$ are d-homomorphisms.

Now define $r : (I\otimes C, (1\otimes\mu)\cdot\alpha^{-1}) \to (Q^{\sim}, v^{\sim})$ to be the morphism of right actions $r = r^{\#}\cdot e(I\otimes C)$ and $\tau^{\sim} : I \to Q^{\sim}$ to be the

C-morphism $\tau^{\sim} = r\cdot(1\otimes\eta)\cdot\rho^{-1}$ and $\beta^{\sim} : (Q^{\sim}, \beta^{\sim}) \to (Y, \theta)$ to be the $\mathbb{T}$-homomorphism $\beta^{\sim} = \mathrm{ev}\cdot(1\otimes\eta)\cdot\rho^{-1}\cdot\sigma$. Then one has the following commutative diagram :



The rest of the proof is similar to the proof of (1.3.15). ●

(2.2.16) Let $(C, \otimes, E, \alpha, \lambda, \rho, s)$ be a closed category with countable coproducts and let X be an object of C, then one can consider the monoid (X^*, μ, i_0) given in (1.2.3). The triple in C that gives rise the functor $-\otimes X$ in the sense of (2.2.2) and the triple that gives rise the monoid (X^*, μ, i_0) in the sense of (2.2.5) are the same one : $(-\otimes X^*, i_0', \mu')$ where $i_0'Q = (1\otimes i_0)\cdot\rho^{-1} : Q \to Q\otimes E \to Q\otimes X^*$ and $\mu'Q = (1\otimes\mu)\cdot\alpha^{-1} : (Q\otimes X^*)\otimes X^* \to Q\otimes(X^*\otimes X^*) \to Q\otimes X^*$ for each object Q of C.

Now let $\mathbb{T} = (T, e, m)$ be a triple in C and $\lambda : (-\otimes X)\cdot T \to T\cdot(-\otimes X)$ a distributive law of $-\otimes X$ over $\mathbb{T}$. Following (2.2.3) the distributive law λ gives rise to a distributive law $d : (-\otimes X^*)\cdot T \to T\cdot(-\otimes X^*)$ of $(-\otimes X^*, i_0', \mu')$ over $\mathbb{T}$.

Then, in that context, by one hand one can consider the category of implicit λ-machines and by the other hand the category of implicit d-machines because (2.2.4) the category of λ-algebras is isomorphic to the category of d-algebras and hence both categories of implicit machines are isomorphic.

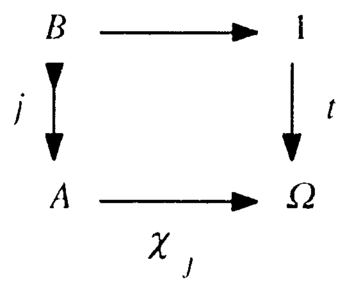
2.3. *u*-machines in a topos for a morphism of monoids *u*.

(2.3.1) A category $\boldsymbol{K}$ that verifies the following three conditions is called an (*elementary*) *topos* :

(i) $\boldsymbol{K}$ has all finite limits.

(ii) $\boldsymbol{K}$ is *cartesian closed* , i.e. for each object A of $\boldsymbol{K}$ there is an *exponential* functor $(\ -\ )^A : \boldsymbol{K} \to \boldsymbol{K}$ which is right adjoint to the functor $-\times A$.

(iii) $\boldsymbol{K}$ has a *subobject classifier* , i.e. there is an object Ω and a morphism $t : 1 \to \Omega$ such that, for each monomorphism $j : B \rightarrowtail A$ in $\boldsymbol{K}$, there is a unique $\chi_j : A \to \Omega$ making the following diagram a pullback :



In a topos a monic arrow (respectively epi) is an equalizer (respectively coequalizer). A topos is balanced; i.e. a morphism which is both mono and epi is an isomorphism. A topos has finite colimits and an image factorization system given by the epis and monos. [27, pp. 23-41].

(2.3.2) Let $\boldsymbol{C}$ be a category with finite limits. An *internal category* in $\boldsymbol{C}$ is $\mathbb{C} = (C_0, C_1, d_0, d_1, i\ , m)$ where :

(i) C_0 and C_1 are objects of $\boldsymbol{C}$, called respectively the *object of objects* and the *object of morphisms* of $\mathbb{C}$.

(ii) four morphisms $d_0 : C_1 \to C_0$, $d_1 : C_1 \to C_0$, $i : C_0 \to C_1$, and $m : C_2 \to C_1$ where $C_2 = C_1 \times_{C_0} C_1$ represents the pullback :

$$\begin{array}{ccc} C_2 & \xrightarrow{\pi_2} & C_1 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle d_0} \\ C_1 & \xrightarrow[d_1]{} & C_0 \end{array}$$

(iii) such that $d_0 \cdot i = d_1 \cdot i = 1_{C_0}$, $d_0\, m = d_0\, \pi_1$, $d_1 \cdot m = d_1 \cdot \pi_2$, $m\cdot(1\times m) = m\cdot(m\times 1) : C_3 = C_1 \times_{C_0} C_1 \times_{C_0} C_1 \to C_1$, and $m\cdot(1\times i) = m\cdot(i \times 1) = 1_{C_1}$.

(Whenever C_1 appears as one factor of a pullback over C_0, it is written on the left (respectively, on the right) of the symbol $\times_{C_0}$ if it is considered with the structure map d_1 (respectively, d_0)).

An *internal functor* (or morphism of internal categories) $f : \mathbb{C} \to \mathbb{D}$ is a pair of morphisms $f_0 : C_0 \to D_0$, $f_1 : C_1 \to D_1$ commuting with d_0, d_1, i , and m The category of internal categories and functors of $\boldsymbol{C}$ is denoted by cat($\boldsymbol{C}$).

An object $\mathbb{C}$ of cat ($\boldsymbol{C}$) such that C_0 is the terminal object 1 is called a *monoid* .

Let $\mathbb{C}$ be an object of cat($\boldsymbol{C}$) . An *internal diagram* F on $\mathbb{C}$ consists of an object $\gamma_0 : F_0 \to C_0$ of $\boldsymbol{C}/C_0$, and a morphism $e : F_1 = F_0 \times_{C_0} C_1 \to F_0$ such that $\gamma_0 \cdot e = d_1 \cdot \pi_2$, $e \cdot (1\times i) = 1_{F_0}$, and $e\cdot(e\times 1) = e\cdot(1\times m) : F_2 = F_0 \times_{C_0} C_1 \times_{C_0} C_1 \to F_0$.

A *morphism of internal diagrams* $f : F \to G$ is a morphism $F_0 \to G_0$ over C_0 , commuting with the structure morphisms e

The category of internal diagrams on $\mathbb{C}$ is denoted by $\boldsymbol{C}^{\mathbb{C}}$

The categories cat($\boldsymbol{C}$) and $\boldsymbol{C}^{\mathbb{C}}$ have finite limits and if $f : \mathbb{C} \to \mathbb{D}$ is a morphism of cat($\boldsymbol{C}$) then the pullback functor :

$$f^* : \mathrm{cat}(\boldsymbol{C}) / \mathbb{D} \to \mathrm{cat}(\boldsymbol{C}) / \mathbb{C}$$

induces a functor $f^* : \boldsymbol{C}^{\mathbb{D}} \to \boldsymbol{C}^{\mathbb{C}}$.

If $\boldsymbol{K}$ is a topos and $\mathbb{C}$ is an object of cat($\boldsymbol{K}$) then $\boldsymbol{K}^{\mathbb{C}}$ is a topos and, given $f : \mathbb{C} \to \mathbb{D}$ a morphism of cat($\boldsymbol{K}$) , the functor $f^* : \boldsymbol{C}^{\mathbb{D}} \to \boldsymbol{C}^{\mathbb{C}}$ has both left and right adjoints. Also one has that the following diagram commutes :

$$\begin{array}{ccc} K^{\mathcal{D}} & \xrightarrow{f^*} & K^{\mathcal{C}} \\ U\Big\downarrow & & \Big\downarrow U \\ K/D_0 & \xrightarrow[f_0^*]{} & K/C_0 \end{array}$$

where U denotes the forgetful functors. [27, pp. 47-56].

In [10, pp. 226-227] the concept of *left $\mathcal{C}$-object* is given, (analogous to the one of internal diagram) as a triple (A, φ, ψ) where $\varphi : A \to C_0$, and $\psi : A_1 \to A$ where A_1 denotes the fiber product $\{ (g,a) : a \in A, g \in C_1$ and $\varphi(a) = d_0(g) \}$ for which a list of conditions have to be satisfied Again $C^{\mathcal{C}}$ denotes the category of left $\mathcal{C}$ -objects (with the appropiate morphisms).

If $\boldsymbol{K}$ is a topos and if $f : \mathcal{C} \to \mathcal{C}'$ is a morphism in cat($\mathcal{C}$), the induced functor $f^* : \boldsymbol{K}^{\mathcal{C}'} \to \boldsymbol{K}^{\mathcal{C}}$, that has left and right adjoints, may be defined on objects by stipulating that $f^* (A', \varphi', \psi') = (A, \varphi, \psi)$ where $A = [(c, a') : fc = \varphi' a']$, φ is the first projection, and $\psi(g\,(c, a'))$ (where necessarily $d_0(g) = c$ and $\varphi' a' = fc$) must be $(d_1(g), \psi'(f(g), a'))$. [10, p. 230].

(2.3.3) In view of (2.3.2) one can consider the following : Let $\boldsymbol{K}$ be a topos and let (C, μ, ι) be a monoid in it in the sense of definition (1.2.2) and with respect to $\times$, that monoid may be viewed as an internal category in $\boldsymbol{K}$ with object of objects the terminal object 1, and then $\boldsymbol{K}^{(C, \mu, \iota)}$ (in the notation of (2.3.2)) is the category of right actions of (C, μ, ι). Hence, if $f : (C, \mu, \iota) \to (C', \mu', \iota')$ is a morphism of monoids, the functor :

$$f^* : {}_{C'}\text{Ract} \to {}_{C}\text{Ract}$$

that sends $g : (v'_A : A \times C' \to A) \to (v'_B : B \times C' \to B)$ to $g : (v'_A \cdot (1 \times f) : A \times C \to A \times C' \to A) \to (v'_B \cdot (1 \times f) : B \times C \to B \times C' \to B)$ has left and right adjoints :

$$\Sigma_f\ ,\ \Pi_f\ :\ {}_{C}\text{Ract} \to {}_{C'}\text{Ract}\ ;\ \ \Sigma_f \dashv f^* \dashv \Pi_f$$

The ideas contained in (2.3.4) and (2.3.5) have been obtained from the machine theory that E. S. Bainbridge [8] gives in the hyperdoctrine (***cat*** , ***Set***) .

(2.3.4) Let K be a topos and let (E, m, e) and (C, μ, ι) be monoids in it. If $u : (E, m, e) \to (C, \mu, \iota)$ is a morphism of monoids then following (2.3.3) one has a functor u^*, that will be denoted by sub u (of substitution), sub $u : {}_C\text{Ract} \to {}_E\text{Ract}$ that has a left adjoint Σ_u and a right adjoint Π_u When one takes (E, m, e) to be $(1, 1\times 1, 1 \to 1)$ and $\eta . (1, 1\times 1, 1 \to 1) \to (C, \mu, \iota)$, ${}_1\text{Ract}$ can be identified with K and sub η with $U : {}_C\text{Ract} \to K$ and one recovers the deterministic approach given in (2 1) for the case of a topos K

(2.3.5) Let K be a topos, fix two monoids (E, m, e) and (C, μ, ι) in it and a morphism $u : (E, m, e) \to (C, \mu, \iota)$ between them. Then one has the corresponding functor sub $u : {}_C\text{Ract} \to {}_E\text{Ract}$ that has a left adjoint Σ_u and a right adjoint Π_u. A *u-machine* is a 5-tuple $M = ((Q, \nu_Q), (I, \omega_I), \tau, (Y, \omega_Y), \beta)$ where (Q, ν_Q) is a C-right action, (I, ω_I) and (Y, ω_Y) are E-right actions, and $\tau : (I, \omega_I) \to \text{sub}\, u\, (Q, \nu_Q)$ and $\beta : \text{sub}\, u\, (Q, \nu_Q) \to (Y, \omega_Y)$ are morphisms in ${}_E\text{Ract}$.

The *reachability map* of M is the unique morphism $r : \Sigma_u (I, \omega_I) \to (Q, \nu_Q)$ such that :

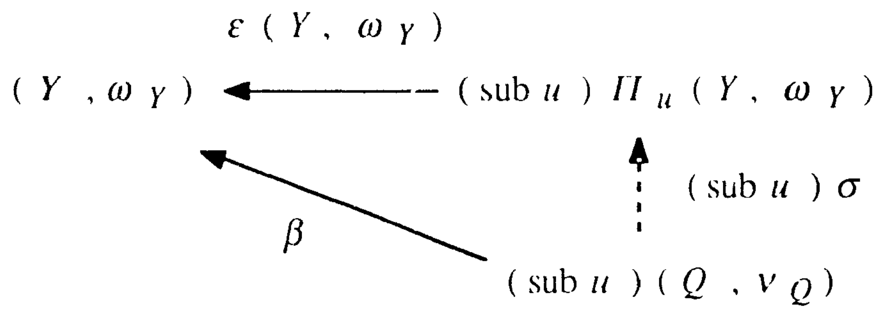
$$\begin{array}{ccc} (I, \omega_I) & \xrightarrow{\eta(I, \omega_I)} & (\text{sub}\, u)\, \Sigma_u(I, \omega_I) \\ & \searrow^{\tau} & \Big\downarrow (\text{sub}\, u)\, r \\ & & (\text{sub}\, u)\, (Q, \nu_Q) \end{array}$$

(η is the unit of $\Sigma_u \dashv \text{sub}\, u$).

The *response map* of M is the morphism in ${}_E$Ract $\beta\cdot(\text{sub } u)\, r\; : (\text{sub } u)\, \Sigma_u\, (I,\, \omega_I) \rightarrow (\text{sub } u)\, (Q,\, v_Q) \rightarrow (Y,\, \omega_Y)$.

A morphism $f: (\text{sub } u)\, \Sigma_u\, (I,\, \omega_I) \rightarrow (Y,\, \omega_Y)$ is called a *response* and if a u-machine M has as response map f, M is said to be a *realization* of f.

The *observability map* of M is the unique morphism $\sigma : (Q,\, v_Q) \rightarrow \Pi_u\, (Y,\, \omega_Y)$ such that :



(ε is the counit of sub u —| Π_u).

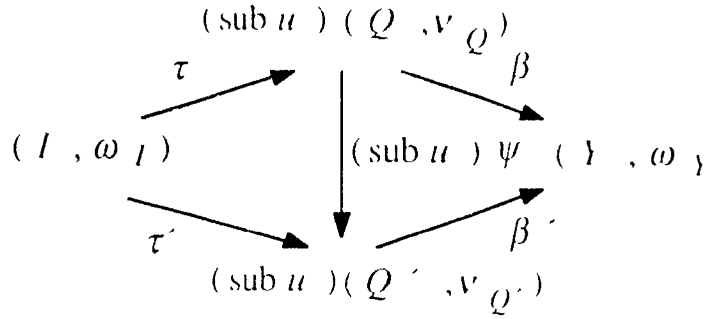
The machine M is said to be *reachable* if (sub u) r is epi and M is said to be *observable* if (sub u) σ is mono.

(2.3.6) Let $\boldsymbol{K}$ be a topos and let $(C,\, \mu,\, \iota)$ be a monoid in it. Let $\varphi : (Q,\, v) \rightarrow (Q',\, v')$ be a morphism of right actions such that $\varphi : Q \twoheadrightarrow Q'$ is epi in $\boldsymbol{K}$, let $(Q'',\, v'')$ be a right action and let $\psi : Q' \rightarrow Q''$ be a morphism in $\boldsymbol{K}$ such that $\psi\cdot\varphi : (Q,\, v) \rightarrow (Q'',\, v'')$ is a morphism of right actions. Then $\psi : (Q',\, v') \rightarrow (Q'',\, v'')$ is a morphism of right actions.

Proof: Since $\varphi : Q \twoheadrightarrow Q'$ is epi and the functor $-\times C : \boldsymbol{K} \rightarrow \boldsymbol{K}$ has a right adjoint, it preserves colimits (in particular coequalizers) then, $\varphi\times 1 : Q\times C \twoheadrightarrow Q'\times C$ is also epi.

The rest of the proof is like in (2.1.6). ●

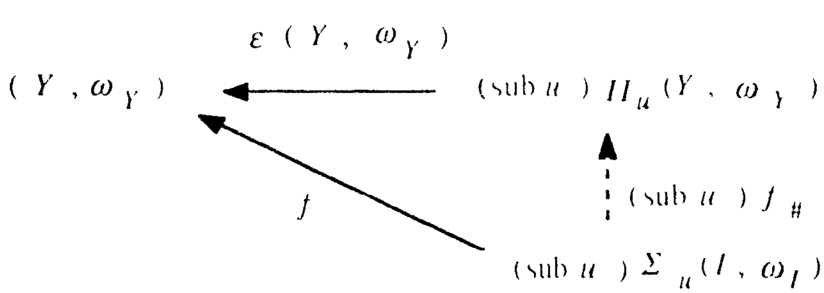
(2.3.7) Fixing (I, ω_I) and (Y, ω_Y), but letting (Q, v_Q) vary, let *u-mach* be the category whose objects are u - machines $M = ((Q, v_Q), (I, \omega_I), \tau, (Y, \omega_Y), \beta)$; and whose morphisms are *simulations* $\psi : M \to M'$ i.e. morphisms of C-right actions $\psi : (Q, v_Q) \to (Q', v_{Q'})$ such that :



The following is the analogous, in the present context, to the "minimal realization theorem" stated in (1.1.13) :

(2.3.8) With the above notation, let $f : (\text{sub } u)\ \Sigma_u (I, \omega_I) \to (Y, \omega_Y)$ be a response, then there exists a reachable and observable realization $M_f = ((Q_f, v_f), (I, \omega_I), \tau_f, (Y, \omega_Y), \beta_f)$ of f. Any such M_f is a terminal object in the category of reachable realizations of f and simulations and any such M_f is an initial object in the category of observable realizations of f and simulations; thus M_f is unique up to isomorphism.

Proof : Given $f : (\text{sub } u)\ \Sigma_u (I, \omega_I) \to (Y, \omega_Y)$, let $f\# : \Sigma_u (I, \omega_I) \to \Pi_u (Y, \omega_Y)$ be the unique morphism such that :



and now consider an epi-mono factorization of $f_{\#}$ in ${}_C$Ract (that is a topos) :

$$\sigma_f \cdot r_f \ : \Sigma_u (I, \omega_I) \longrightarrow\!\!\!\!\to (Q_f, \nu_f) \rightarrowtail H_u (Y, \omega_Y)$$

Define τ_f and β_f respectively as follows :

(sub u)$r_f \cdot \eta (I, \omega_I)$: $(I, \omega_I) \to$ (sub u)$\Sigma_u (I, \omega_I) \longrightarrow\!\!\!\!\to$(sub u)(Q_f, ν_f)

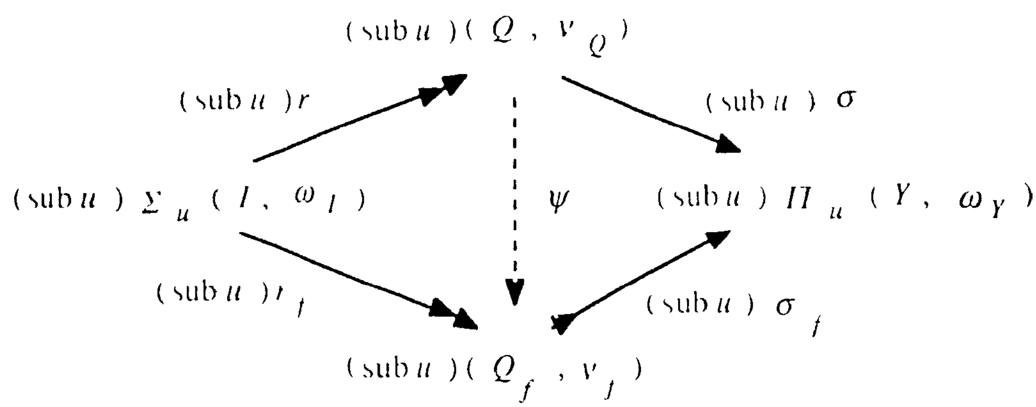
$\varepsilon (Y, \omega_Y) \cdot$(sub u)σ_f : (sub u)$(Q_f, \nu_f) \rightarrowtail$ (sub u)$\Pi_u (Y, \omega_Y) \to (Y, \omega_Y)$

(since sub u has a left and right adjoint it preserves limits and colimits and in particular monos (equalizers) and epis (coequalizers) , since ${}_C$Ract and ${}_I$Ract are toposes)

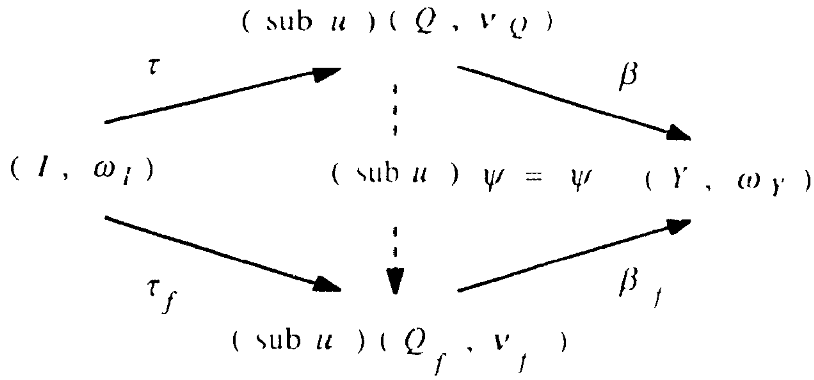
Therefore, $M_f = ((Q_f, \nu_f), (I, \omega_I), \tau_f, (Y, \omega_Y), \beta_f)$ is a reachable and observable realization of f.

Now suppose that $M = ((Q, \nu_Q), (I, \omega_I), \tau, (Y, \omega_Y), \beta)$ is a reachable realization of f. First one has that $\sigma \cdot r = f_{\#}$.

Now, by diagonal fill in in ${}_E$Ract, one has a unique ψ such that :



Now, $\psi : Q \to Q_f$ is a morphism in $\boldsymbol{K}$ such that $\psi \cdot r = r_f \ : \ \Sigma_u (I, \omega_I) \to (Q_f, \nu_f)$ is a morphism of C-right actions and r is epi in $\boldsymbol{K}$ then , by (2.3.6) , $\psi : (Q, \nu_Q) \to (Q_f, \nu_f)$ is a morphism of C-right actions; also the diagram



commutes since :

$$\tau_f = (\text{sub } u)\, r_f \cdot \eta(I, \omega_I) = \psi \cdot (\text{sub } u)\, r \cdot \eta\, (I, \omega_I) = \psi \cdot \tau$$

$$\beta = \varepsilon\, (Y, \omega_Y) \cdot (\text{sub } u)\ \sigma = \varepsilon\, (Y, \omega_Y) \cdot (\text{sub } u) \sigma_f \cdot \psi = \beta_f \cdot \psi$$

Hence, $\psi : M \rightarrow M_f$ is a simulation.

Analogously if $M = ((Q, \nu_Q), (I, \omega_I), \tau , (Y, \omega_Y), \beta)$ is an observable realization of f , there is a unique simulation $\varphi . M_f \rightarrow M$. ●

3. MACHINES IN CATEGORIES OF HEYTING ALGEBRA VALUED SETS.

3.1. A discussion of H-valued sets as a topos.

(3.1.1) An equivalent definition for a category K to be a topos is that K has to verify :

(i) K has all finite limits.

(ii) For every object X of K, there exists a *power object* $\mathbb{P}X$ and a subobject $\in_X \rightarrowtail \mathbb{P}X\times X$ such that, for every object Y and every subobject $R \rightarrowtail Y\times X$, there exists a unique $r : Y \to \mathbb{P}X$ such that

$$\begin{array}{ccc} R & \longrightarrow & \in_X \\ \downarrow & & \downarrow \\ Y\times X & \xrightarrow[r\times 1]{} & \mathbb{P}X\times X \end{array}$$

is a pullback. [27, p. 43].

(3.1.2) Let H be a complete Heyting algebra where the smallest and largest elements will be denoted respectively by 0 and 1. An *H-valued set* is a pair (X, κ) where X is a set and $\kappa : X\times X \to H$ is a map such that :

$\kappa(x, x') = \kappa(x', x)$ for all x, x' in X ; and

$\kappa(x, x') \wedge \kappa(x', x'') \leq \kappa(x, x'')$ for all x, x', x'' in X.

Given (X, κ) and (Y, κ) H-valued sets (the κ-functions are not distinguished notationally) an *H-valued mapping* $f : (X, \kappa) \to (Y, \kappa)$ is a map $f : X\times Y \to H$ such that :

(i) $\kappa(x, x') \wedge f(x, y) \leq f(x', y)$

$f(x, y) \wedge \kappa(y, y') \leq f(x, y')$

for all x, x' in X ; y, y' in Y.

(ii) $f(x, y) \wedge f(x, y') \leq \kappa(y, y')$ for all x in X ; y, y' in Y.

(iii) $\bigvee_y f(x, y) = \kappa(x, x)$

($\kappa(x, x)$ will be denoted by $\varepsilon(x)$ for all x in X).

A consequence of (ii) and (iii) is that $f(x, y) \leq \varepsilon(x) \wedge \varepsilon(y)$ for all x in X, y in Y.

Let $f : (X, \kappa) \to (Y, \kappa)$ and $g : (Y, \kappa) \to (Z, \kappa)$ be H-valued mappings, then $(g \cdot f) : (Y, \kappa) \to (Z, \kappa)$ is defined by :

$$(g \cdot f)(x, z) = \bigvee_y f(x, y) \wedge g(y, z)$$

To each H-valued set (X, κ) is associated an H-valued mapping the "identity" $1_{(X, \kappa)} : (X, \kappa) \to (X, \kappa)$ given by κ.

The above data give a category : The category ***H-Set*** of H-valued sets.

The significance is that (X, κ) may contain elements only partially, the degree of membership of x is mesured by $\varepsilon(x) = \kappa(x, x)$, and the degree of equality between x and x' in (X, κ) is mesured by $\kappa(x, x')$. For an H-valued mapping $f : (X, \kappa) \to (Y, \kappa)$ the significance of $f(x, y)$ is that it gives the degree of equality between y and the image of x by f. [25, pp. 4-5].

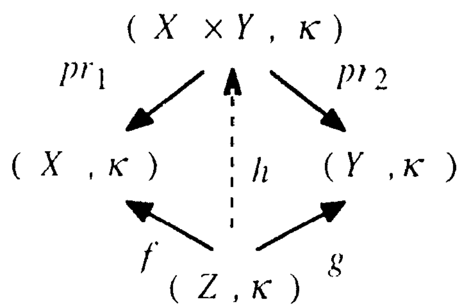
(3.1.3) Let (X, κ) and (Y, κ) be H-valued sets and $f^0 : X \to Y$ be a map, defining $f : X \times Y \to H$ by $f(x, y) = \varepsilon(x) \wedge \kappa(f(x), y)$, if f is a morphism from (X, κ) to (Y, κ) one says that f^0 *represents* f. In particular 1_X represents $1_{(X, \kappa)}$ for each H-valued set (X, κ).

A function $f^0 : X \to Y$ represents a morphism from (X, κ) to (Y, κ) iff $\kappa(x, x') \leq \kappa(f^0(x), f^0(x'))$ for all x, x' in X.

If $f : (X, \kappa) \to (Y, \kappa)$ and $g : (Y, \kappa) \to (Z, \kappa)$ are morphisms and f is represented by f^0 then $(g \cdot f)(x, z) = \varepsilon(x) \wedge g(f^0(x), z)$; in particular if g is also represented by a map g^0 then $g \cdot f$ is represented by $g^0 \cdot f^0$. [25, p. 6]

(3.1.4) The object $(\{e\}, \kappa)$ where $\{e\}$ is a singleton and $\kappa(e, e) = 1$ is a terminal object in the category *H-Set*. Given any object (Q, κ), the unique morphism $(Q, \kappa) \to (\{e\}, \kappa)$ is represented by the unique map $Q \to \{e\}$. [25, p. 7].

(3.1.5) Let (X, κ) and (Y, κ) be objects of *H-Set*, a product of them is given by $(X \times Y, \kappa)$ where $\kappa((x, y), (x', y')) = \kappa(x, x') \wedge \kappa(y, y')$, and the projections $pr_1 : (X \times Y, \kappa) \to (X, \kappa)$, $pr_2 : (X \times Y, \kappa) \to (Y, \kappa)$ that are represented by the corresponding projections $X \times Y \to X$, $X \times Y \to Y$ in ***Set***. Moreover, if $f : (Z, \kappa) \to (X, \kappa)$ and $g : (Z, \kappa) \to (X, \kappa)$ are morphisms, the unique morphism h that makes the following diagram commute



is given by $h(z, (x, y)) = f(z, x) \wedge g(z, y)$. [25, pp. 7-8].

(3.1.6) Let (X, κ) be an H-valued set, consider the set $P(X, \kappa) = \{\alpha : X \to H : \alpha(x) \le \varepsilon(x) \text{ and } \alpha(x) \wedge \kappa(x, x') \le \alpha(x') \text{ for all } x, x' \text{ in } X\}$. If $\alpha \in P(X, \kappa)$, defining $\kappa_\alpha : X \times X \to H$ by :

$$\kappa_\alpha(x, x') = \alpha(x) \wedge \kappa(x, x')$$

one has that $(X, \kappa)_\alpha = (X, \kappa_\alpha)$ is an H-valued set and $\kappa_\alpha : (X, \kappa)_\alpha \to (X, \kappa)$ is a morphism represented by 1_X. [25, p. 8].

(3.1.7) • A morphism $f : (X, \kappa) \to (Y, \kappa)$ is mono iff $f(x, y) \wedge f(x', y) \le \kappa(x, x')$ for all x, x' in X ; y in Y.

• A morphism $f : (X, \kappa) \to (Y, \kappa)$ is epi iff

$$\varepsilon(y) \le \bigvee_x f(x, y)$$

for all y in Y.

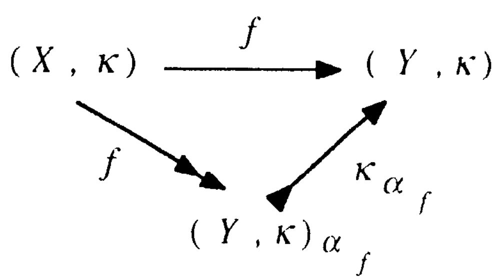
• If morphisms $f, g : (X, \kappa) \to (Y, \kappa)$ satisfy $f(x, y) \le g(x, y)$ for all x in X, y in Y, then $f = g$.

• If $f : (X, \kappa) \to (Y, \kappa)$ is both a monomorphism and an epimorphism, then it is an isomorphism. [25, pp. 8-10].

(3.1.8) Let $f : (X, \kappa) \to (Y, \kappa)$ be a morphism, defining α_f in $P(Y, \kappa)$ by

$$\alpha_f(y) = \bigvee_x f(x, y)$$

then $f : (X, \kappa) \twoheadrightarrow (Y, \kappa)_{\alpha_f}$ is an epimorphism, and



commutes.

The subobjects of an H-valued set (X, κ) are in bijective correspondence with the elements of $P(X, \kappa)$: To $g : (Z, \kappa) \rightarrowtail (X, \kappa)$ corresponds α_g and to α in $P(X, \kappa)$ corresponds $\kappa_\alpha : (X, \kappa)_\alpha \rightarrowtail (X, \kappa)$.

It is a consequence of this result that the intersection of two subobjects of an H-valued set (X, κ) always exists : if α and β are in $P(X, \kappa)$ then $\alpha \wedge \beta$, defined pointwise, is also in $P(X, \kappa)$ and obviously describes the intersection of the subobjects corresponding to α and β.

Together with the fact that ***H-Set*** has finite products, this implies that ***H-Set*** has all finite limits. [25, pp. 10-11]

(3.1.9) (Power objects). For each H-valued set (X, κ),

$$\in_{(X,\,\kappa)} \rightarrowtail \mathbb{P}(X, \kappa) \times (X, \kappa)$$

is defined as follows :

$$\mathbb{P}(X, \kappa) = (\, P(X, \kappa), \kappa) \quad \text{where } \kappa(\alpha, \beta) = \bigwedge_{x} \alpha(x) \leftrightarrow \beta(x)$$

In particular, $\varepsilon(\alpha) = 1$ for all α in $P(X, \kappa)$.

Let $\in$ in $P(\,\mathbb{P}(X, \kappa)\times(X, \kappa))$ be given by $\in(\alpha, x) = \alpha(x)$; then :

$$\in_{(X,\,\kappa)} = (\,\mathbb{P}(X, \kappa)\times(X, \kappa))_{\in} \quad \text{and}$$

$\in_{(X,\,\kappa)} \rightarrowtail \mathbb{P}(X, \kappa)\times(X, \kappa)$ is the mono $\kappa_{\in}$.

If (Y, κ) is an H-valued set, the subobjects of $(Y, \kappa)\times(X, \kappa) = (Y\times X\,, \kappa)$ are bijective with the functions γ in $P(Y\times X\,, \kappa)$. The condition that

$$\begin{array}{ccc}
(Y\times X\,, \kappa)_{\gamma} & \xrightarrow{\quad g \quad} & \in_{(X\,,\,\kappa)} \\
\Big\downarrow{\scriptstyle \kappa_{\gamma}} & & \Big\downarrow{\scriptstyle \kappa_{\in}} \\
(Y\,, \kappa)\times(X\,, \kappa) & \xrightarrow[h\,\times\,1]{} & \mathbb{P}(X\,, \kappa)\times(X\,, \kappa)
\end{array}$$

is a pullback for some g defines a bijection, F say, from the set M of morphisms $h : (Y, \kappa) \to \mathbb{P}(X, \kappa)$ to the set $P = P(Y\times X\,, \kappa)$:

$$F(h)(x, y) = \bigvee_{\beta \in P(X,\,\kappa)} h(y, \beta) \wedge \beta(x) \quad \text{and}$$

$F^{-1}(\gamma) : (Y, \kappa) \to \mathbb{P}(X, \kappa)$ is the morphism represented by the map $y \to \gamma(y\,, -)$.

Therefore, the category *H-Set* is a topos. [25, pp. 11-12].

(3.1.10) Let (X, κ) and (Y, κ) be H-valued sets. Then every morphism $h : (Y, \kappa) \to \mathbb{P}(X, \kappa)$ is represented by a unique function

$h^0 : Y \to P(X, \kappa)$ such that $h^0(y)(x) \leq \varepsilon(y)$ for all x in X and y in Y. In fact $h^0(y) = \gamma(y, -)$ where $\gamma = F(h)$ (see 3.1.9). [25, p. 13].

(3.1.11) Let (X, κ) be an H-valued set, a *predicate* of type (X, κ) is a subobject of (X, κ). Then one can identify the predicates of type (X, κ) with the corresponding elements of $P(X, \kappa)$.

Given (X, κ), $\in : P(X, \kappa)\times X \to H$ that sends (α, x) to $\alpha(x)$ is a predicate of type $\mathcal{P}(X, \kappa)\times(X, \kappa)$.

Given (X, κ) and (Y, κ), $\Phi : P(X\times Y, \kappa) \to H$ that sends f to

$$\Phi(f) = \bigwedge_{x, y, y'} \left[f(x, y) \wedge f(x, y') \to \kappa(y, y') \right] \wedge \bigwedge_{x} \left[\varepsilon(x) \to \bigvee_{y} f(x, y) \right]$$

is a predicate of type $\mathcal{P}(X\times Y, \kappa)$.

If f is in $P(X\times Y, \kappa)$ and g in $P(Y\times Z, \kappa)$ where (X, κ), (Y, κ), (Z, κ) are H-valued sets, $g\cdot f$ defined by

$$(g\cdot f)(x, z) = \bigvee_{y} f(x, y) \wedge g(y, z)$$

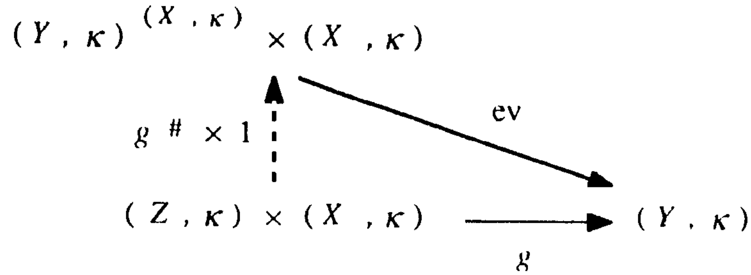
is in $P(X\times Z, \kappa)$. [25, pp. 14-17].

(3.1.12) (Exponentiation). Given (X, κ) and (Y, κ) H-valued sets, the subobject $(\mathcal{P}(X\times Y, \kappa))_{\Phi}$ of $\mathcal{P}(X\times Y, \kappa)$ determined by Φ is denoted by

$$(Y, \kappa)^{(X, \kappa)}$$

it being the thing which makes *H-Set* cartesian closed. [25, p. 18].

If one defines $\mathrm{ev} : (Y, \kappa)^{(X, \kappa)}\times(X, \kappa) \to (Y, \kappa)$ as the map $(P(X\times Y, \kappa)\times X)\times Y \to H$ that sends $((f, x), y)$ to $\Phi(f)\wedge f(x, y)$, given (Z, κ) and $g : (Z, \kappa)\times(X, \kappa) \to (Y, \kappa)$ the unique $g^{\#} : (Z, \kappa) \to (Y, \kappa)^{(X, \kappa)}$ such that makes the following diagram commute



is represented by the map $g^0 : Z \to P(X\times Y\,,\kappa)$ which sends z to $g\,((z\,,-),-)$, and $g^{\#}(z\,,f) = \varepsilon(z) \wedge \kappa_{\Phi}(\,g((z\,,-),-),f) =$
$= \varepsilon(z) \wedge \kappa(\,g((z\,,-),-),f)$.

Proof: $(\,\mathbb{P}(X\times Y,\,\kappa))_{\Phi} = (P(X\times Y,\,\kappa),\,\kappa_{\Phi})$ where $\kappa_{\Phi}(f,\,g) = \Phi(f) \wedge \kappa(f,\,g)$.

- ev is a morphism :

(i) • $\kappa_{\Phi}(f, f') \wedge \kappa(x,\,x') \wedge f(x,\,y) \wedge \Phi(f) \leq$
$\leq \kappa_{\Phi}(f, f') \wedge f(x',\,y) \wedge \Phi(f) = \Phi(f) \wedge \kappa(f, f') \wedge f(x',\,y) \leq$
$\leq \Phi(f') \wedge f'(x',\,y)$ for all $x,\,x'$ in X ; f, f' in $P(X\times Y,\,\kappa)$. (The last inequality follows since Φ and $\in$ are predicates).

• $\mathrm{ev}((f,\,x),\,y) \wedge \kappa(y,\,y') = \Phi(f) \wedge f(x,\,y) \wedge \kappa(y,\,y') \leq$
$\leq \Phi(f) \wedge f(x,\,y')$ for all x in X; $y,\,y'$ in Y; f in $P(X\times Y\,,\,\kappa)$.

(ii) $\mathrm{ev}((f,\,x),\,y) \wedge \mathrm{ev}((f,\,x),\,y') = \Phi(f\,) \wedge f(x,\,y) \wedge \Phi(f\,) \wedge f(x,\,y') =$
$= f(x,\,y) \wedge f(x,\,y') \wedge$

$$\wedge \bigwedge_{x,\,y,\,y'} [f(x,y) \wedge f(x,\,y') \to \kappa(y,\,y')] \wedge \bigwedge_{x} \left[\varepsilon\,(x\,) \to \bigvee_{y} f(x,\,y)\right] \leq$$

$\leq f(x,\,y) \wedge f(x,\,y') \wedge [f(x,y) \wedge f(x,\,y') \to \kappa(y,\,y')] \leq \kappa(y,\,y')$ for all x in X ; $y,\,y'$ in Y; f in $P(X\times Y,\,\kappa)$. (The last inequality follows since in H , $a \wedge (\,a \to b\,) \leq b$ for any $a,\,b$ in H).

(iii) $\bigvee_{y} ev((f,\,x),\,y) = \varepsilon(f,\,x)$ for all x in X; f in $P(\,X\times Y,\,\kappa)$:

$\leq$) $\bigvee_y f(x, y) \wedge \Phi(f) \leq \varepsilon(x) \wedge \Phi(f) = \varepsilon(f, x)$

$\geq$) $\varepsilon(f, x) = \Phi(f) \wedge \varepsilon(x) \leq \Phi(f)$.

One also has that $\varepsilon(f, x) = \Phi(f) \wedge \varepsilon(x) =$

$$= \varepsilon(x) \wedge \bigwedge_{x, y, y'} [f(x, y) \wedge f(x, y') \rightarrow \kappa(y, y')] \wedge \bigwedge_x \left[\varepsilon(x) \rightarrow \bigvee_y f(x, y)\right] \leq$$

$$\leq \varepsilon(x) \wedge \left[\varepsilon(x) \rightarrow \bigvee_y f(x, y)\right] \leq \bigvee_y f(x, y)$$

- $g^0 : Z \rightarrow P(X \times Y, \kappa)$ represents a morphism from (Z, κ) to $(Y, \kappa)^{(X, \kappa)}$:

First one has the following inequalities :

(1) $\varepsilon(z) \leq \bigwedge_x \left[\varepsilon(x) \rightarrow \bigvee_y g((z, x), y)\right]$ for all z in Z.

(2) $\kappa(z, z') \leq \bigwedge_{x, y} [g((z, x), y) \leftrightarrow g(z', x), y)]$ for all z, z' in Z.

(1) : Since g is a morphism one has that

$$\bigvee_y g((z, x), y) = \varepsilon(z, x) = \varepsilon(z) \wedge \varepsilon(x)$$

and the certainty of the inequality $\varepsilon(x) \wedge \varepsilon(z) \leq \varepsilon(x) \wedge \varepsilon(z)$ implies the certainty of $\varepsilon(z) \leq \varepsilon(x) \rightarrow \varepsilon(x) \wedge \varepsilon(z)$. Then one has inequality (1).

(2) : Since g is a morphism, $g((z, x), y) \wedge \kappa(z, z') \leq g((z', x), y)$ and hence $\kappa(z, z') \leq g((z, x), y) \rightarrow g((z', x), y)$. Analogously $\kappa(z, z') \leq$ $\leq g((z', x), y) \rightarrow g((z, x), y)$. And then ,

$\kappa(z, z') \leq g((z, x), y) \leftrightarrow g((z', x), y)$ for all x, y. Therefore one has (2).

Now from (1) and (2) one has :

$$\kappa(z, z') = \varepsilon(z) \wedge \kappa(z, z') \leq$$

$$\leq \bigwedge_x \left[\varepsilon(x) \to \bigvee_y g((z, x), y)\right] \wedge \bigwedge_{x, y} \left[g((z, x), y) \leftrightarrow g((z', x), y)\right] =$$

$$= \bigwedge_{x, y, y'} \left[g((z, x), y) \wedge g((z, x), y') \to \kappa(y, y')\right] \wedge$$

$$\wedge \bigwedge_x \left[\varepsilon(x) \to \bigvee_y g((z, x), y)\right] \wedge \bigwedge_{x, y} \left[g((z, x), y) \leftrightarrow g((z', x), y)\right] =$$

$$= \Phi(g((z ,-),-)) \wedge \kappa(g((z ,-),-), g((z' ,-),-)) = \kappa_\Phi(g^0(z), g^0(z')).$$

One has used the fact that :

$$\bigwedge_{x, y, y'} \left[g((z, x), y) \wedge g((z, x), y') \to \kappa(y, y')\right] = 1$$

since $g((z, x), y) \wedge g((z, x), y') \leq \kappa(y, y')$.

Now,

$$g^\#(z, f) = \varepsilon(z) \wedge \kappa_\Phi(g^0(z), f) = \varepsilon(z) \wedge \kappa_\Phi(g((z , -), -), f) =$$

$$= \varepsilon(z) \wedge \Phi(g((z ,-),-)) \wedge \kappa(g((z ,-),-), f) = \varepsilon(z) \wedge \kappa(g((z ,-), -), f)$$

(The last inequality follows since $\varepsilon(z) \leq \Phi(g((z , -), -))$).

- $\mathrm{ev}\cdot(g^\#\times 1) = g$:

$g^\#\times 1 : (Z, \kappa)\times(X, \kappa) \to (Y, \kappa)^{(X, \kappa)}\times(X, \kappa)$ is represented by the map $Z\times X \to P(X\times Y, \kappa)\times X$ that sends (z, x) to $(g^0(z), x)$, hence :

$$[\ \mathrm{ev}\cdot(g^\#\times 1)]((z, x), y) = \varepsilon(z, x) \wedge \mathrm{ev}((g^0(z), x), y) =$$

$$= \varepsilon(z) \wedge \varepsilon(x) \wedge \Phi(g^0(z)) \wedge (g^0(z))(x, y) =$$

$$= \varepsilon(z) \wedge \varepsilon(x) \wedge \Phi(g^0(z)) \wedge g((z, x), y) = g((z, x), y)$$

- Uniqueness of $g^\#$:

Suppose that $h : (Z, \kappa) \to (Y, \kappa)^{(X, \kappa)}$ is a morphism such that $\mathrm{ev}\cdot(h\times 1) = g$. Then $h = g^\#$ has to be shown .

First, one has the following formula :

(3) $$\bigvee_{f} h(z,f) \wedge f(x, y) = g((z, x), y) \quad \text{for all } x \text{ in } X; y \text{ in } Y;$$

z in Z :

$$g((z, x), y) = [\ \mathrm{ev}\cdot(h\times 1)\ ]((z, x), y) =$$

$$= \bigvee_{f, x'} (h \times 1)((z, x), (f, x')) \wedge ev((f, x'), y) =$$

$$= \bigvee_{f, x'} h(z,f) \wedge \kappa(x, x') \wedge \Phi(f) \wedge f(x', y) =$$

$$= \bigvee_{f} h(z,f) \wedge \Phi(f) \wedge f(x, y) = \bigvee_{f} h(z,f) \wedge \varepsilon_{\Phi}(f) \wedge f(x, y) =$$

$$= \bigvee_{f} h(z,f) \wedge f(x, y)$$

Hence (3) follows.

Now consider

$$(Z, \kappa) \underset{h}{\overset{g^{\#}}{\rightrightarrows}} (Y, \kappa)^{(X, \kappa)} \xrightarrow{\kappa_{\Phi}} \mathbb{P}(X \times Y, \kappa)$$

and since κ_{Φ} is mono suffices to show that $\kappa_{\Phi}\cdot h = \kappa_{\Phi}\cdot g^{\#}$.

Now, from (3.1.10), the morphism $\kappa_{\Phi}\cdot h : (Z, \kappa) \to \mathbb{P}(X\times Y, \kappa)$ is represented by a map $h^0 : Z \to P(X\times Y, \kappa)$ where $h^0(z) = \gamma(z, -)$ and

$$\gamma(z, (x, y)) = \bigvee_{f} (\kappa_{\Phi}\cdot h)(z,f) \wedge f(x, y) =$$

$$= \bigvee_{f} \left(\bigvee_{f'} h(z, f') \wedge \kappa_{\Phi}(f', f) \right) \wedge f(x, y) =$$

$$= \bigvee_{f} h(z, f) \wedge f(x, y) = g((z, x), y)$$

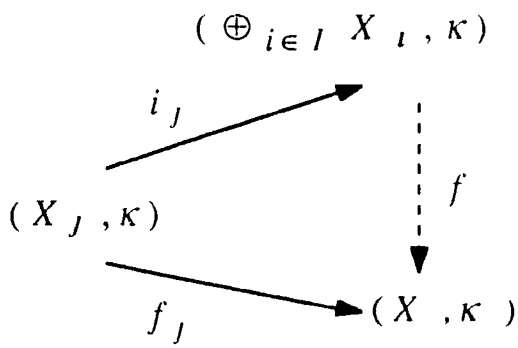
(The last equality is the formula (3)).

Hence $\kappa_{\Phi} \cdot h$ is represented by the map $z \rightarrow g((z\ ,\ \text{-}), \text{-})$ and $\kappa_{\Phi} \cdot g^{\#}$ is also represented by $z \rightarrow g((z\ ,\ \text{-}), \text{-})$, therefore one has that $\kappa_{\Phi} \cdot h = \kappa_{\Phi} \cdot g^{\#}$. ●

(3.1.13) Let $\{(X_i\ ,\ \kappa)\}_{i\ \in\ I}$ be a family of objects in H -***Set*** , then a coproduct of this family is given by :

$$\{\ i_j : (X_j\ ,\ \kappa) \rightarrow (\oplus_{i \in I} X_i\ ,\ \kappa)\ \}_{j \in I}$$

where $\oplus_{i\ \in\ I} X_i = \{\ (x_i\ , i) : i \in I\ \}$ is the disjoint union in ***Set***, $\kappa : (\oplus_{i \in I} X_i) \times (\oplus_{i\ \in I} X_i) \rightarrow H$ is the map that sends $((x_i\ , i\ ), (x_j', j))$ to $\kappa(x_i\ , x_j')$ if $j = i$ and to 0 otherwise and the injections i_j are represented by the corresponding injections $X_j \rightarrow \oplus_{i \in I} X_i$ in ***Set*** . Moreover, given a family $\{\ f_j : (X_j\ , \kappa) \rightarrow (X, \kappa)\}_{j \in I}$ of morphisms in H -***Set*** the unique morphism $f : (\oplus_{i\ \in\ I} X_i\ , \kappa) \rightarrow (X, \kappa)$ such that makes the following diagrams commute for each $j \in I$,



is given by $f((x_j\ , j), x) = f_j\ (x_j\ , x)$ for all $(x_j\ , j)$ in $\oplus_{i \in I} X_i$; x in X .

The proof is straightforward. ●

(3.1.14) Given (X, κ) an object of *H-Set*, the "free monoid on (X, κ)" given following (1.2.3) is $((X^*, \kappa), \mu, i_0)$ where :

- $X^* = \oplus_{n \in N} X^n$ (the disjoint union in *Set*).
- $\kappa : (\oplus_{n \in N} X^n) \times (\oplus_{n \in N} X^n) \to H$ is the map that sends the element $((x_1,\ldots,x_n, n), (x_1',\ldots,x_m', m))$ to $\kappa(x_1, x_1') \wedge .. \wedge \kappa(x_n, x_m')$ if $m = n$ and to 0 if $m \neq n$ (in particular if $m = n = 0$, $\kappa((e, 0), (e, 0)) = 1$).
- $\mu : (X^*, \kappa) \times (X^*, \kappa) \to (X^*, \kappa)$ is the morphism represented by the map $X^* \times X^* \to X^*$ that sends $((x_1,\ldots,x_n, n), (x_1',\ldots,x_m', m))$ to $(x_1,\ldots,x_n, x_1',\ldots,x_m', n+m)$.
- $i_0 : (\{e\}, \kappa) \to (X^*, \kappa)$ is the 0-injection.

The proof is straightforward following the construction given in (1.2.3). ●

3.2. An analysis of some concepts of automata theory in the context of H-valued sets, especially when H is a finite chain.

Given (X, κ) an object of *H-Set* one can consider the process $-\times(X, \kappa) : H\text{-}Set \to H\text{-}Set$ and the category $\mathrm{Dyn}(-\times(X, \kappa))$, then the study of the machines in a closed category of (1.2) applies to this case :

(3.2.1) An (X, κ)-machine will be a 6-tuple $M = ((Q, \kappa), \delta, (I, \kappa), \tau, (Y, \kappa), \beta)$ where (Q, κ), (I, κ) and (Y, κ) are H-valued sets (the state object, initial object and output object respectively), $\delta : (Q, \kappa) \times (X, \kappa) \to (Q, \kappa)$ an H-valued mapping (the dynamics) and $\tau : (I, \kappa) \to (Q, \kappa)$, $\beta : (Q, \kappa) \to (Y, \kappa)$ H-valued mappings (initial state and output morphism respectively).

Following the significance of the H-valued sets such a machine may be thought as to have an input object (X, κ) that may contain "inputs" only partially and a degree of equality between any two "inputs" x and x' is given.The same significance for the state object (Q, κ), initial object

(I, κ) and output object (Y, κ). For the transition map, $\delta((q, x), q')$ gives the degree of equality between the "new state" that results by application of "input" x to "state" q and any "state" q' of Q. For the output morphism, $\beta(q, y)$ gives the degree of equality between the "output emitted" by "state" q and any "output" y of Y. Analogously for the initial state.

From now on some of the concepts will be interpreted in the particular case of H to be a finite chain with

$$a \to b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise} \end{cases}$$

[23, p. 187].

(3.2.2) The *reachability map* of M, that is the unique dynamorphic extension $r : (I, \kappa)\times(X^*, \kappa) \to (Q, \kappa)$ of τ, will be given by :

$$r((i, (x_1,\ldots,x_n, n)), q) =$$

$$= \bigvee_{q_1,\ldots,q_n} \tau(i, q_1) \wedge \delta((q_1, x_1), q_2) \wedge\ldots\wedge \delta((q_{n-1}, x_{n-1}), q_n) \wedge \delta((q_n, x_n), q)$$

(in the case $n = 0$, $r((i, (e, 0)), q) = \tau(i, q)$).

Proof : Going to the proof of (1.2.6) one has :

$\tau_0 = \tau\cdot\rho : (I, \kappa)\times(\{e\}, \kappa) \to (I, \kappa) \to (Q, \kappa)$, then $\tau_0((i, e), q) =$

$$= \varepsilon(i, e) \wedge \tau(i, q) = \tau(i, q).$$

$\tau_1 = \delta\cdot(\tau\times 1) : (I, \kappa)\times(X, \kappa) \to (Q, \kappa)\times(X, \kappa) \to (Q, \kappa)$, then

$$\tau_1((i, x), q) = \bigvee_{q', x'} (\tau\times 1)((i, x), (q', x')) \wedge \delta((q', x'), q) =$$

$$= \bigvee_{q', x'} \tau(i, q') \wedge \kappa(x, x') \wedge \delta((q', x'), q) = \bigvee_{q'} \tau(i, q') \wedge \delta((q', x), q)$$

$$\tau_2 = \delta\cdot(\tau_1\times 1)\cdot\alpha : (I,\ \kappa)\times((X,\ \kappa)\times(X,\ \kappa)) \to$$
$$\to ((I,\ \kappa)\times(X,\ \kappa))\times(X,\ \kappa) \to (Q,\ \kappa)\times(X,\ \kappa) \to (Q,\ \kappa)$$

then $\tau_2((i, (x_1, x_2)), q) =$

$$= \varepsilon((i, (x_1, x_2))) \wedge (\delta\cdot(\tau_1\times 1))(((i, x_1), x_2), q) =$$
$$= (\delta\cdot(\tau_1\times 1))(((i, x_1), x_2), q) =$$

$$= \bigvee_{q', x'} (\tau_1 \times 1)(((i, x_1), x_2), (q', x')) \wedge \delta((q', x'), q) =$$

$$= \bigvee_{q', x'} \tau_1((i, x_1), q') \wedge \kappa(x_2, x') \wedge \delta((q', x'), q) =$$

$$= \bigvee_{q'} \tau_1((i, x_1), q') \wedge \delta((q', x_2), q) =$$

$$= \bigvee_{q'} \left(\bigvee_{q''} \tau(i, q'') \wedge \delta((q'', x_1), q') \right) \wedge \delta((q', x_2), q) =$$

$$= \bigvee_{q', q''} \tau(i, q'') \wedge \delta((q'', x_1), q') \wedge \delta((q', x_2), q) =$$

$$= \bigvee_{q_1, q_2} \tau(i, q_1) \wedge \delta((q_1, x_1), q_2) \wedge \delta((q_2, x_2), q)$$

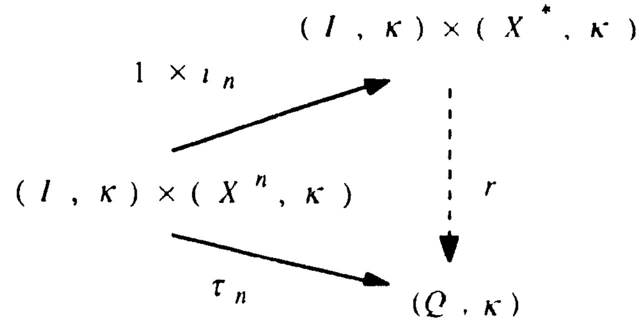
Now by induction can be shown that :

$$\tau_n((i, (x_1, \ldots, x_n)), q) =$$

$$= \bigvee_{q_1, \cdots, q_n} \tau(i, q_1) \wedge \delta((q_1, x_1), q_2) \wedge \cdot \cdot \wedge \delta((q_{n-1}, x_{n-1}), q_n) \wedge \delta((q_n, x_n), q)$$

Now, $r : (I, \kappa)\times(X^* \ \kappa) \to (Q, \kappa)$ is defined by :

(3.2)



Now, the reachability map may be thought as follows : Begining in i and applying the "sequence of inputs" $(x_1,\ldots,x_n, n)$, $r((i, (x_1,\ldots,x_n, n)), q)$ says the degree in which q is reached.

If H is the finite chain, the fact that q be reached in degree $a \in H$ means that there exists a sequence $q_1,\ldots,q_n$ such that :

$$\tau(i, q_1) \wedge \delta((q_1, x_1), q_2) \wedge \delta((q_2, x_2), q_3) \wedge \ldots \wedge \delta((q_n, x_n), q) = a$$

i.e. the degree in which τ sends i to q_1 is at least a ; the degree in which δ sends (q_j, x_j) to q_{j+1} is at least a , $1 \leq j \leq n-1$; the degree in which δ sends (q_n, x_n) to q is at least a ; and at least one of those degrees has to be exactly a .

M is *reachable* if r is epi, i.e. if for each q in Q is

$$\bigvee_{i, (x_1,\ldots,x_n, n)} r((i, (x_1,\ldots,x_n, n)), q) = \varepsilon(q)$$

that, in the case H is the finite chain, means that there exist an i and two sequences $x_1,\ldots,x_n$, $q_1,\ldots,q_n$ such that :

$$\tau(i, q_1) \wedge \delta((q_1, x_1), q_2) \wedge \delta((q_2, x_2), q_3) \wedge \ldots \wedge \delta((q_n, x_n), q) = \varepsilon(q)$$

(if $n = 0$ it means simply that $\tau(i, q) = \varepsilon(q)$)

i.e. the degree in which q is reached is the maximum possible since $\varepsilon(q)$ indicates the degree of membership of q to (Q, κ).

The *response map* of M , that will be

$\beta\cdot r : (I, \kappa)\times(X^*, \kappa) \to (Q, \kappa) \to (Y, \kappa)$ is given by

$$(\beta\cdot r)((i, (x_1,\ldots,x_n, n)), y) = \bigvee_q r((i, (x_1,\ldots,x_n, n)), q) \wedge \beta(q, y)$$

that may be thought as follows : it measures the degree in which, beginning with i and after apply the "sequence of inputs" $(x_1,\ldots,x_n, n)$, the machine will emit "output" y.

If H is the finite chain, to say that $(\beta\cdot r)((i, (x_1,\ldots,x_n, n)), y) = a \in H$ means that there exists q in Q that is reached, from $(i, (x_1,\ldots,x_n, n))$, in degree at least a and the degree in which q emits y by β is also at least a; moreover at least one of those degrees has to be exactly a.

(3.2.3) The *run map* of $((Q, \kappa), \delta)$, that is the unique dynamorphic extension $\delta^* :(Q, \kappa)\times(X^*, \kappa) \to (Q, \kappa)$ of $1_{(Q, \kappa)}$ will be given by :

$n = 0$: $\delta^*((q, (e, 0)), q') = \kappa(q, q')$

$n \geq 1$: $\delta^*((q, (x_1,\ldots,x_n, n)), q') =$

$$= \bigvee_{q_1, \ldots q_{n-1}} \left(\begin{array}{c} \delta((q, x_1), q_1) \wedge \delta((q_1, x_2), q_2) \wedge\cdots\wedge \\ \wedge\, \delta((q_{n-2}, x_{n-1}), q_{n-1}) \wedge \delta((q_{n-1}, x_n), q') \end{array} \right)$$

(when $n = 1$: $\delta^*((q, (x, 1)), q') = \delta((q, x), q')$).

The proof can also be obtained from (1.2.6) in a similar manner to the one given in (3.2.2). ●

The run map may be thought as follows : $\delta^*((q, (x_1,\ldots,x_n, n)), q')$ measures the degree in which, after applying $(x_1,\ldots,x_n, n)$ to q, the dynamics assumes "state" q'.

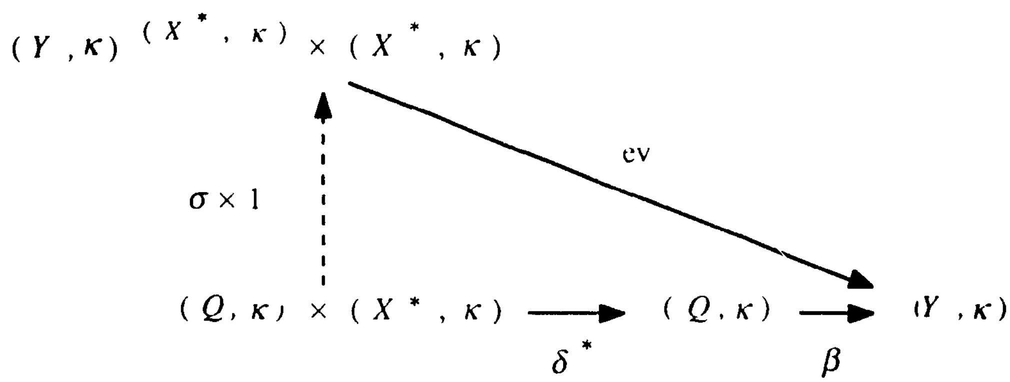
If H is the finite chain to say that $\delta^*((q, (x_1,\ldots,x_n, n), q') = a \in H$ means that there exists a sequence $q_1,\ldots,q_{n-1}$ such that :

$$\delta((q, x_1), q_1) \wedge \delta((q_1, x_2), q_2) \wedge\ldots\wedge \delta((q_{n-1}, x_n), q') = a$$

whose significance may be thought as follows : when applying the "sequence of inputs" $(x_1,\ldots,x_n, n)$ to q , the dynamics "goes from q to q_1" in degree at least a , from q_j to q_{j+1} in degree at least a for $1 \leq j \leq n-2$, from q_{n-1} to q' in degree at least a , moreover at least one of those degrees has to be exactly a .

(3.2.4) The *observability map* of M , that is the unique dynamorphic coextension $\sigma : (Q, \kappa) \to (Y, \kappa)^{(X^*, \kappa)}$ of β will be the morphism represented by the map $Q \to P(X^* \times Y, \kappa)$ which sends q to $(\beta \cdot \delta^*)((q, -), -)$ where δ^* is the run map.

Proof : Going to the proof of (1.2.7) one has that σ is defined by :



and by (3.1.12) one has that σ is represented by the map $Q \to P(X^* \times Y, \kappa)$ which sends q to $(\beta \cdot \delta^*)((q, -), -)$. ●

The observability map may be thought as follows : $(\beta \cdot \delta^*)((q, -), -)$ gives the "response emitted" by the machine beginning in "state" q, and $\sigma(q, f)$ measures the degree of equality between the above "response" and f.

If H is the finite chain, to say that $\sigma(q, f) = a \in H$ means that :

$$\varepsilon(q) \wedge \kappa\Phi(\beta \cdot \delta^*((q, -), -), f) = \varepsilon(q) \wedge \kappa(\beta \cdot \delta^*((q, -), -), f) =$$

$$= \varepsilon(q) \wedge \bigwedge_{(x^*, y) \in X^* \times Y} (\beta \cdot \delta^*)((q, x^*), y) \leftrightarrow f(x^*, y) = a$$

i.e. that for all pairs (x^*, y) one has that :

$\varepsilon(q) \wedge [(\beta\cdot\delta^*)((q, x^*), y) \to f(x^*, y)] \wedge [f(x^*, y) \to (\beta\cdot\delta^*)((q, x^*), y)] \geq a$

and the equality has to hold for at least one pair (x^*, y).

That means that $\varepsilon(q) \geq a$ and for each pair (x^*, y) $(\beta\cdot\delta^*)((q, x^*), y) = f(x^*, y)$ or $(\beta\cdot\delta^*)((q, x^*), y) \wedge f(x^*, y) \geq a$, and moreover $\varepsilon(q) = a$ or for at least one pair (x^*, y), $(\beta\cdot\delta^*)((q, x^*), y) \wedge f(x^*, y) = a$.

M is *observable* if σ is mono, i.e. if for any q, q' in Q is :

$$\varepsilon(q) \wedge \varepsilon(q') \wedge \kappa_\Phi(\ (\beta\cdot\delta^*)((q, \text{-}), \text{-})\ ,\ (\beta\cdot\delta^*)((q', \text{-}), \text{-})\ ) =$$

$$= \varepsilon(q) \wedge \varepsilon(q') \wedge \kappa(\ (\beta\cdot\delta^*)((q, \text{-}), \text{-})\ ,\ (\beta\cdot\delta^*)((q', \text{-}), \text{-})\ ) \leq \kappa(q, q')$$

For any q, q' in Q, the degree in which $(\beta\cdot\delta^*)((q, \text{-}), \text{-})$ and $(\beta\cdot\delta^*)((q', \text{-}), \text{-})$ are equal, minimum tha degree of membership of q and q', has to be less or equal than the degree of equality between q and q'.

(3.2.5) Finally, since the machines (relative to the object (X, κ)) are a special case of the ones given in (1.2) one has that the "minimal realization theorem" works.

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