

Sharp Sobolev inequalities of higher order

Samuel Zeitler

Master of Science



McGill

Mathematics and Statistics

McGill University

Montreal, Quebec, Canada

May 30, 2024

A thesis submitted to McGill University in partial
fulfillment of the requirements of the degree of
Master of Science

©Samuel Zeitler, 2024

Contents

Abstract	iii
Abrégé	iv
Acknowledgements	v
Contribution	vi
1 Introduction	1
1.1 Concepts and Notation	1
1.2 Background and Motivation	2
1.3 Organization of the Work	4
2 Sobolev Spaces	5
2.1 Euclidean Space	5
2.2 Sobolev Spaces on Riemannian Manifolds	10
3 The Higher Order AB Program	16
3.1 Introduction	16
3.2 The Best First Constant	17
3.3 Existence Results for a Higher Order Semilinear PDE	18
4 Proof of Theorem 3.3	25
4.1 Structure of the Proof	25
4.2 Preliminary Lemmas	26
4.3 Proof of the Theorem	45
4.3.1 Asymptotic Analysis	46
4.3.2 Main Argument	57
A Appendix: The De Georgi-Nash-Moser Iterative Scheme	62
Conclusion	64

Abstract

Investigation of sharp constants in Sobolev inequalities dates back to the independent work of Aubin[3] and Talenti[43] in the 1970s on the first order Euclidean Sobolev inequality, and has been the focus of significant research in the decades since in part due to importance in geometry, physics, and PDEs (Partial Differential Equations). As a consequence, a particular collection of results on best constants for Sobolev inequalities on Riemannian manifolds has come to be known as the "AB Program" in the literature (see Druet and Hebey[10]). In recent years, natural extensions of these problems to higher order have gathered significant attention, examples of which can be found in Mazumdar[33], Mazumdar-Vétois[34], and Liu[32]. This thesis aims to cover problems involving best constants for Sobolev inequalities of higher order on Riemannian manifolds, their relationship to the sharp Sobolev inequalities on Euclidean space, and applications to existence results in PDEs. We develop the necessary background material to discuss a higher order AB Program and prove a new sharp higher order Sobolev inequality, which comprises the main result of the thesis. This result directly generalizes the previous work of Hebey-Vaugon[25] and Hebey[23] in the first and second order cases respectively, combining classical arguments from geometric analysis with results of Mazumdar[33].

Abrégé

L'étude des meilleures constantes des inégalités de Sobolev remonte aux travaux indépendants d'Aubin[3] et de Talenti[43] dans les années 1970 sur l'inégalité euclidienne de Sobolev de premier ordre, et ont fait l'objet de recherches importantes au cours des décennies qui ont suivi, en partie en raison de leur importance en géométrie, en physique et en EDP (équations aux dérivées partielles). En conséquence, une collection particulière de résultats sur les meilleures constantes pour les inégalités de Sobolev sur les variétés riemanniennes est désormais connue dans la littérature sous le nom de « programme AB » (voir Druet et Hebey[10]). Ces dernières années, les généralisations naturelles de ces problèmes aux ordres supérieurs ont retenu une attention considérable, par exemple dans les travaux de Mazumdar[33], Mazumdar-Vétois[34] et Liu[32]. Cette thèse vise à couvrir des problèmes impliquant les meilleures constantes pour les inégalités de Sobolev d'ordres supérieurs sur les variétés riemanniennes, leur relation avec les inégalités optimales de Sobolev sur l'espace euclidien et les applications à l'existence de solutions d'EDP. Nous développons le matériel nécessaire pour discuter d'un programme AB d'ordre supérieur et prouver une nouvelle inégalité de Sobolev d'ordre supérieur, ce qui constitue le résultat principal de la thèse. Ce résultat généralise directement les travaux antérieurs de Hebey-Vaugon[25] et de Hebey[23], respectivement dans les cas du premier et du second ordre, combinant des arguments issus de l'analyse géométrique avec des résultats de Mazumdar[33].

Acknowledgements

I would first like to extend the deepest gratitude to my supervisor, Professor Jérôme Vétois, for guiding me throughout the construction of this work. His kindness, precision, expertise, and unwavering patience have been remarkable, and I cannot express enough the positive influence he has had on me as a mathematician. This thesis has been an ambitious and challenging project, and without the hours he spent answering my (many) questions and reading over my work, it would never have reached its current state. The past two years have been incredibly rewarding due to his generosity and support, and I hope to continue delving into the beautiful and deep field of analysis on manifolds he has introduced me to. I would also like to thank Professor Vétois for translating the abstract into French. Finally, I would like to thank Professor Vétois and the Department of Mathematics and Statistics for their financial support over the course of my degree.

I would like to thank Professor Pengfei Guan and Doctor Joshua Flynn for organizing the Geometric Analysis seminar at McGill in conjunction with Professor Vétois. Due to their efforts, I have been able to appreciate and learn from the greater mathematical community around me and the opportunity to present at the seminar myself was a fantastic learning experience and a great privilege.

I would like to thank Doctor Axel Hundemer, whose real analysis course solidified my decision to complete my degree in mathematics, and Professor Jessica Lin, whose course introduced me to the field of PDEs and inspired me to pursue it at the graduate and research level.

I would also like to thank my friends Negar Matin, Jacob Reznikov, Shereen Elaidi, Bart Syroka, Ghilia Weldesselasie, Lakshmi Roy, William Holman-Bisseger, Maya Marsonia, Peng Tang, Samer Bishara, Cody Greyson, as well as the entire McGill mathematics graduate community for making me feel welcome and at home throughout all of my years at McGill.

Finally, I would like to thank my family for their endless and unconditional love and support.

Contribution

All writing in this thesis, except for the French translation of the abstract (by Professor Jérôme Vétois), was performed solely by the author. Some decisions regarding phrasing and presentation originate from in-person discussions with his supervisor and colleagues. Sections 1-3 and the Appendix represent development of background material interpreted from various books and papers, and Section 4 is entirely the original work of the author.

1 Introduction

1.1 Concepts and Notation

This thesis assumes general knowledge of measure theory, functional analysis, and differential and Riemannian geometry. Introductory references for these subjects can be found for example in Folland[14] and Lee[26][27].

Let (M, g) be a smooth Riemannian manifold. If $\nabla^k T$ is the k th covariant derivative of a j -degree covariant tensor T , we will occasionally use the notation

$$\nabla_{i_1} \dots \nabla_{i_k} T_{l_1 \dots l_j} := T_{l_1, \dots, l_j; i_1 \dots i_k}.$$

Einstein summation notation will be used for all tensor computations. We raise and lower indices using the metric in the usual way, for example

$$R_j^i = g^{ik} R_{kj}.$$

If S, T are two tensors of degree q , their inner product is defined by

$$\langle S, T \rangle_g = S^{i_1 \dots i_q} T_{i_1 \dots i_q}.$$

The norm of a tensor T is then defined as

$$|T| = \sqrt{\langle T, T \rangle_g}.$$

We use dv_g to denote the volume density, which is well defined regardless of whether the manifold is orientable. When working in a coordinate chart, we use the notation $\sqrt{g} := \sqrt{|\det(g)|}$. The volume density in a coordinate chart is then given by $\sqrt{g}dx$. The Laplacian operator Δ acts on a tensor T by the negative sign convention, i.e.

$$\Delta T = -\nabla^i \nabla_i T.$$

If working in a coordinate system on some open set $U \subset M$, we use ξ to represent the Euclidean metric on U . If (M, g) is compact and without boundary, then for tensors T, S of the same degree we make frequent use of the integration by parts equality

$$\int_M \langle \nabla T, \nabla S \rangle_g dv_g = \int_M \langle T, \Delta S \rangle_g dv_g.$$

Given a nonnegative integer m , we use the notational convention

$$\Delta^{\frac{m}{2}} u := \nabla \Delta^{\frac{m-1}{2}} u$$

when m is odd. We use i_g to refer to the injectivity radius of the manifold. If (M, g) is compact, we can always guarantee $i_g > 0$. If $x \in M$ and $r > 0$, we use the notation $B_x(r) = \{y \in M : d_g(y, x) < r\}$ where d_g is the Riemannian distance function. We may occasionally, when there is no confusion, instead use the alternative notation $B_r(x)$ depending on how much focus we would like to give to x and r . If $A \subset M$ and $r > 0$, we define

$$B_r(A) := \bigcup_{x \in A} B_x(r).$$

We use the notation $W^{m,p}$ for m th order Euclidean Sobolev spaces with exponent p and H_m^p for m th order Sobolev spaces on manifolds with exponent p , both of which will be defined in Section 2. If X and Y are Banach spaces, then $X \subset Y$ will always be assumed to denote a continuous embedding in the sense that there exists some C such that for all $x \in X$,

$$\|x\|_Y \leq C \|x\|_X.$$

As is typical in analysis arguments, we will use the same letter C to denote potentially different constants throughout computations where the value of C only needs to remain independent of certain parameters.

1.2 Background and Motivation

Best constants problems for Sobolev inequalities have been a topic of great development since Aubin[3] and Talenti[43] independently calculated the best constant $K(1, n, p)$ for the embedding $W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$ in the sense that

$$K(1, n, p)^{-1} = \inf_{u \in W^{1,p}(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u|^p dx}{\left(\int_{\mathbb{R}^n} u^{p^*} dx \right)^{\frac{p}{p^*}}}$$

where $1 \leq p < n$ and $p^* := \frac{pn}{n-p}$ is the critical Sobolev exponent. Because the case $p = 2$ is of particular interest, we write $K(1, n) := K(1, n, 2)$. This sharp constant is particularly notable for applications to semilinear PDEs of critical growth arising in geometry, providing a seminal role in the resolution of the Yamabe problem through the combined work of Yamabe[45], Trudinger[44], Aubin[2], and Schoen[40].

If (M, g) is a Riemannian manifold of dimension $n \geq 3$, by the Sobolev embedding $H_1^2(M) \subset L^{2^*}(M)$,

there exist constants $A, B > 0$ such that for all $u \in H_1^2(M)$,

$$\left(\int_M |u|^{2^*} dv_g \right)^{\frac{2}{2^*}} \leq A \int_M |\nabla u|^2 dv_g + B \int_M u^2 dv_g. \quad (1.1)$$

The "AB Program" (see Druet and Hebey[10] for an overview), studies sharp versions of this inequality. It was shown by Aubin[3] through an elementary partition of unity argument that the value of A in the above inequality must satisfy $A \geq K(1, n)$ and for any $A > K(1, n)$, there exists some $B(A)$ such that (1.1) is valid. Hebey-Vaugon[25] then showed the best possible value $A = K(1, n)$ is in fact obtained through a fine blow-up analysis of a sequence of positive functions u_α for $\alpha > 0$ solving the PDE

$$\Delta_g u_\alpha + \alpha u_\alpha = u_\alpha^{2^*-1}.$$

This technique of proving sharp results for inequalities like (1.1) through asymptotic analysis of a sequence of solutions to PDEs is by now classical, see for example Hebey[23], Li-Ricciardi[29], Li-Zhu[30], Druet[9], and Aubin-Li[5].

The higher order Sobolev embeddings $H_m^2(M) \subset L_m^{2^\#}(M)$, where $2_m^\# = \frac{2n}{n-2m}$ and $m < \frac{n}{2}$ give rise to a natural development of the AB Program in higher order, studying constants (A, B) such that

$$\left(\int_M |u|^{2_m^\#} dv_g \right)^{\frac{2}{2_m^\#}} \leq A \int_M (\Delta^{\frac{m}{2}} u)^2 dv_g + B \sum_{i=0}^{m-1} \int_M (\Delta^{\frac{i}{2}} u)^2 dv_g. \quad (1.2)$$

The best constants for the higher order Euclidean Sobolev embeddings with exponent $p = 2$

$$K(m, n)^{-1} = \inf_{u \in W^{m,2}(\mathbb{R}^n) \setminus \{0\}} \frac{\int (\Delta^{\frac{m}{2}} u)^2 dv_g}{\left(\int u^{2_m^\#} dv_g \right)^{\frac{2}{2_m^\#}}}$$

were calculated by Edmunds-Fortunato-Jannelli[11] in the case $m = 2$ and Swanson[42] in the case $m \geq 3$. A similar argument to that originally used by Aubin in the case $m = 1$ shows that the constant A in (1.2), must be at least $K(m, n)$, see Djadli-Hebey-Ledoux[8] for the case $m = 2$ and Mazumdar[33] for the case $m \geq 3$. It therefore is sensible to ask whether the constant A in (1.2) can be taken to be equal to the best possible value $K(m, n)$, generalizing the result of Hebey-Vaugon. A positive answer in the case $m = 2$ was shown by Hebey[23]. In this work we show this result can be extended to all $m \geq 3$.

A refined version of the first order result involving the scalar curvature was proven in Li-Ricciardi[29]. Natural next steps from our results could include estimating lower order constants in the higher order AB Program after the first constant is lowered, i.e. inequalities of the form

$$\left(\int_M u^{2^\#} \right)^{\frac{2}{2^\#}} \leq A \int_M (\Delta^{\frac{m}{2}} u)^2 dv_g + \sum_{i=0}^{m-1} B_i \int_M (\Delta^{\frac{i}{2}} u)^2 dv_g. \quad (1.3)$$

Results in this direction have been initiated in the case of hyperbolic space in Liu[32] and on the unit sphere in the case $m = 2$ in Djadli-Hebey-Ledoux[8]. One can also consider whether the higher order inequalities can be refined in the sense of [29] through local or global geometric quantities.

1.3 Organization of the Work

In Section 2, we review basic definitions regarding Sobolev spaces and prove the Sobolev inequalities in the settings of Euclidean space and compact Riemannian manifolds. In Section 3 we discuss sharp and asymptotically sharp versions of higher order Sobolev inequalities in the Riemannian setting with exponent $p = 2$. In particular we set up a higher order variant of the AB Program for best constants and discuss applications to existence results in semilinear PDEs of critical growth. Section 4 is dedicated to proving the best first constant in the higher order AB Program is attained. This is the main result and represents an original contribution to the literature. The Appendix reviews the De Giorgi-Nash-Moser iterative scheme, which is crucial in Section 4.

2 Sobolev Spaces

2.1 Euclidean Space

In this section we recall some definitions and basic results on Sobolev spaces, prove the Gagliardo-Nirenberg-Sobolev inequality, and discuss the best constants. This serves to motivate and prove the corresponding Sobolev inequalities on Riemannian manifolds in Section 2.2. All results in this section can be found in the books of Adams[1] or Evans[12].

Given a multiindex $\alpha = (\alpha_1 \dots, \alpha_n) \in \mathbb{N}^n$ we define the differential operator D^α by

$$D^\alpha u = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \dots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} u$$

for $u \in C^{|\alpha|}(\mathbb{R}^n)$.

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be an open set and $1 \leq p \leq \infty$ and let $m \geq 0$ be an integer. We define the **Sobolev Space** $W^{m,p}(\Omega)$ to be the set of functions $f \in L^p(\Omega)$ such that for all multi-indices $\alpha \in \mathbb{N}^n$ satisfying $|\alpha| \leq m$ there exists some $f_\alpha \in L^p(\Omega)$ satisfying

$$(-1)^{|\alpha|} \int_{\Omega} f_\alpha \phi dx = \int_{\Omega} f D^\alpha \phi dx$$

for all $\phi \in C_c^\infty(\Omega)$.

If $f \in W^{m,p}(\Omega)$ and $|\alpha| \leq m$, we use the notation $D^\alpha f := f_\alpha$ and call $D^\alpha f$ a **weak derivative** of f . We note if $f \in C^{|\alpha|}(\Omega)$ then the weak derivative $D^\alpha f$ coincides with the classical derivative up to a set of measure 0.

The space $W^{m,p}(\Omega)$ endowed with the Sobolev norm

$$\|u\|_{W^{m,p}(\Omega)} = \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^p(\Omega)}$$

forms a Banach space. We define $W_0^{m,p}(\Omega) \subset W^{m,p}(\Omega)$ as the completion of $C_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{m,p}}$. The following theorem of Meyers-Serrin (which we present without proof) shows when developing results in the theory of Sobolev spaces, it is often possible to prove a statement only for smooth functions and argue through density instead of working directly with the definition.

Theorem 2.1. For any $\Omega \subset \mathbb{R}^n$ open, $m \geq 0$, $1 \leq p < \infty$, the set

$$\{u \in C^\infty(\Omega) : \|u\|_{W^{m,p}(\Omega)} < \infty\}$$

forms a dense subset of $W^{m,p}(\Omega)$.

Remark 2.2. Given $R > 0$ let η_R be a nonnegative smooth function equal to 1 on $B_0(R)$ and equal to 0 on $\mathbb{R}^n \setminus B_0(R+1)$ satisfying $|\nabla^i \eta_R| \leq C$ for $0 \leq i \leq m$, where C is independent of R . Given $u \in C^\infty(\mathbb{R}^n)$ such that $\|u\|_{W^{m,p}(\mathbb{R}^n)} < \infty$, it can be easily shown that $\eta_R u \rightarrow u$ in the Sobolev norm as $R \rightarrow \infty$. Therefore it holds

$$W_0^{m,p}(\mathbb{R}^n) = W^{m,p}(\mathbb{R}^n).$$

Definition 2.2. Let $1 \leq p < n$. We define the **Sobolev conjugate** p^* by $p^* = \frac{np}{n-p}$.

It was originally proven by Sobolev[41], using a fairly difficult representation formula, that if $1 \leq p < n$, then $W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$. We present the proof in the form given by Gagliardo[15] and Nirenberg[37], which give simple and explicit constants for the embedding. This embedding is commonly referred to as the Gagliardo-Nirenberg-Sobolev inequality. Because $p^* > p$, the result allows one to in a sense "trade off" differentiability for integrability in a higher L^p space. This is especially relevant in bounded domains (and compact Riemannian manifolds, see Section 2.2) as in these spaces Hölder's inequality implies the L^p spaces are decreasing as p increases.

Theorem 2.3. For any $n > 1$ and $u \in C_c^1(\mathbb{R}^n)$,

$$\|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq \frac{1}{2} \|\nabla u\|_{L^1(\mathbb{R}^n)}.$$

Proof. We write $u(x) = u(x_1, \dots, x_n)$. We have by the fundamental theorem of calculus (and because u is assumed with compact support) that for $1 \leq i \leq n$,

$$u(x) = \int_{-\infty}^{x_i} u_{x_i} dx_i = - \int_{x_i}^{\infty} u_{x_i} dx_i.$$

Therefore we have

$$|2u(x)| \leq \int_{-\infty}^{\infty} |u_{x_i}(x)| dx_i$$

which implies

$$|u(x)|^{\frac{n}{n-1}} \leq \left(\frac{1}{2}\right)^{\frac{n}{n-1}} \prod_{i=1}^n \left(\int_{-\infty}^{\infty} |\nabla u(x)| dx_i\right)^{\frac{1}{n-1}}. \quad (2.1)$$

Note that each of the integrals on the right hand side are with respect to only one variable and the value of $\int_{-\infty}^{\infty} |\nabla u(x)| dx_i$ is a function only of the coordinates $(x_j : j \neq i)$.

We integrate (2.1) with respect to x_1 . The first term in the product on the right hand side is constant with respect to x_1 , so we pull it out of the integral and focus on the other $n - 1$ terms. We get

$$\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 \leq \left(\frac{1}{2}\right)^{\frac{n}{n-1}} \left(\int_{-\infty}^{\infty} |\nabla u(x)| dx_1\right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^n \left(\int_{-\infty}^{\infty} |\nabla u(x)| dx_i\right)^{\frac{1}{n-1}} dx_1$$

We recall the generalized Hölder's inequality, which states that if $1 \leq p_i \leq \infty$ are such that $\sum_i \frac{1}{p_i} = 1$, then $\int_{\mathbb{R}^n} |\prod_i f_i| dx \leq \prod_i \|f_i\|_{L^{p_i}}$. We apply this inequality with $p_i = n - 1$ and $f_i = \left(\int_{-\infty}^{\infty} |\nabla u(x)| dx_i\right)^{\frac{1}{n-1}}$. We achieve

$$\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 \leq \left(\frac{1}{2}\right)^{\frac{n}{n-1}} \left(\int_{-\infty}^{\infty} |\nabla u(x)| dx_1\right)^{\frac{1}{n-1}} \left(\prod_{i=2}^n \int_{D_{x_1 x_i}} |\nabla u(x)| dx_1 dx_i\right)^{\frac{1}{n-1}}$$

where $D_{x_1 x_i}$ represents the 2-dimensional plane spanned by e_1, e_i in $\mathbb{R}^n$. This notation extends naturally to other subspaces of dimension $2 \leq i \leq n - 1$.

We continue, integrating both sides with respect to x_2 . This time the first term in the iterated product on the right hand side will be constant with respect to x_2 , so it can be pulled out of the integral, and we again apply the generalized Hölder inequality to the other $n - 1$ terms, i.e. $f_1 = \left(\int_{-\infty}^{\infty} |\nabla u(x)| dx_1\right)^{\frac{1}{n-1}}$ and $f_i = \left(\int_{D_{x_1 x_{i+1}}} |\nabla u(x)| dx_1 dx_{i+1}\right)^{\frac{1}{n-1}}$ for $2 \leq i \leq n - 1$. We obtain

$$\int_{D_{x_1 x_2}} |u(x)|^{\frac{n}{n-1}} dx_1 dx_2 \leq \left(\frac{1}{2}\right)^{\frac{n}{n-1}} \left(\int_{D_{x_1 x_2}} |\nabla u(x)| dx_1 dx_2\right)^{\frac{2}{n-1}} \left(\prod_{i=3}^n \int_{D_{x_1 x_2 x_i}} |\nabla u(x)| dx_1 dx_2 dx_i\right)^{\frac{1}{n-1}}.$$

Continuing in this manner, iterating the integral of both sides k times with respect to $x_1, \dots, x_k$ with $k \leq n$ results in

$$\begin{aligned} \int_{D_{x_1 \dots x_k}} |u(x)|^{\frac{n}{n-1}} dx_1 \dots dx_k &\leq \left(\frac{1}{2}\right)^{\frac{n}{n-1}} \left(\int_{D_{x_1 \dots x_k}} |\nabla u(x)| dx_1 \dots dx_k\right)^{\frac{k}{n-1}} \\ &\quad \times \left(\prod_{i=k+1}^n \int_{D_{x_1 \dots x_k x_i}} |\nabla u(x)| dx_1 \dots dx_k dx_i\right)^{\frac{1}{n-1}}. \end{aligned}$$

Once the process has been iterated n times, we are left with exactly

$$\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}} dx \leq \left(\frac{1}{2}\right)^{\frac{n}{n-1}} \left(\int_{\mathbb{R}^n} |\nabla u(x)| dx\right)^{\frac{n}{n-1}}$$

which is what we aimed to show. □

Theorem 2.4. *Let $1 < p < n$. Then there exists a constant $C(p, n)$ such that for all $u \in C_c^1(\mathbb{R}^n)$,*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

Proof. We note the fact that $p^* \frac{n-1}{n} > 1$ implies we are able to apply the case Theorem 2.3 to $|u|^{p^* \frac{n-1}{n}} \in C_c^1(\mathbb{R}^n)$.

We obtain by Hölder's inequality

$$\|u^{p^* \frac{n-1}{n}}\|_{L^{\frac{n}{n-1}}} \leq \frac{1}{2} p^* \frac{n-1}{n} \|u^{p^* \frac{n-1}{n}-1} |\nabla u|\|_{L^1} \leq p^* \frac{n-1}{2n} \|\nabla u\|_{L^p} \|u^{p^* \frac{n-1}{n}-1}\|_{L^{p'}}$$

where p' is the Hölder conjugate for p . Then by a simple calculation $\|u^{(\frac{p^*(n-1)}{n}-1)}\|_{L^{p'}} = \|u\|_{L^{p^*}}^{p^* \frac{n-1}{n}-1}$ and $\|u^{p^*(n-1)/n}\|_{L^{n/(n-1)}} = \|u\|_{L^{p^*}}^{p^* \frac{n-1}{n}}$, so we conclude

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq \frac{p^*(n-1)}{2n} \|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

□

Corollary 2.5. *If $1 \leq p < n$, the embedding $W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$ holds.*

Remark 2.6. By considering functions of the form $u_\lambda(x) = u(\lambda x)$ and sending λ to ∞ and 0, it is easily seen that given $p \geq 1$, p^* is the only exponent for which such an estimate holds.

Corollary 2.7. *For all open sets $\Omega \subset \mathbb{R}^n$, if $1 \leq p < n$ there exists some C independent of Ω such that for all $u \in W_0^{1,p}(\Omega)$*

$$\|u\|_{L^{p^*}(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}$$

While the proof of the Gagliardo-Nirenberg-Sobolev inequality gives explicit constants for the embedding, it gives no information on the value of the optimal constant $K(1, n, p)$ for which

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq K(1, n, p)^{\frac{1}{p}} \|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

An equivalent way of phrasing the value of this constant is

$$K(1, n, p)^{-1} = \inf_{u \in W^{1,p}(\mathbb{R}^n) \setminus \{0\}} \frac{\|\nabla u\|_{L^p(\mathbb{R}^n)}^p}{\|u\|_{L^{p^*}(\mathbb{R}^n)}^p}.$$

This question was answered independently by Aubin[3] and Talenti[43]. The value is calculated to be exactly

$$K(1, n, p) = \frac{n-p}{n(p-1)} \left[\frac{p-1}{n-p} \right]^p \left[\frac{\Gamma(n+1)}{\Gamma(n/p)\Gamma(n+1-n/p)\omega_{n-1}} \right]^{\frac{p}{n}}$$

if $1 < p < n$ and

$$K(1, n, 1) = \frac{1}{n} \left[\frac{n}{\omega_{n-1}} \right]^{\frac{1}{n}}$$

where ω_{n-1} is the volume of the $n-1$ dimensional unit sphere. The extremal functions are known and are given explicitly by

$$u_{c,\mu,x_0}(x) = c \left(\frac{1}{\mu + |x - x_0|^{p/p-1}} \right)^{\frac{n-p}{p}}$$

for $c, \mu > 0$ and $x_0 \in \mathbb{R}^n$ when $1 < p < n$. When $p = 1$ the extremal functions are characteristic functions on balls.

Given a real number $p > 1$ and an integer $m \geq 1$ such that $mp < n$, we define the higher order Sobolev conjugate $p_m^\# = \frac{np}{n-mp}$. This is obtained by iterating the first order Sobolev conjugate m times. Using the well known inequality $|\nabla|\nabla^r u|| \leq |\nabla^{r+1} u|$ for $u \in C^{r+1}(\mathbb{R}^n)$ where $r \geq 0$ is an integer (see Lemma 2.11 for a proof in the more general Riemannian case), it is easy to iterate the Gagliardo-Nirenberg-Sobolev inequality to obtain

$$\|u\|_{L^{p_m^\#}(\mathbb{R}^n)} \leq C \|\nabla u\|_{L^{p_{m-1}^\#}(\mathbb{R}^n)} \leq C \|\nabla|\nabla u|\|_{L^{p_{m-2}^\#}(\mathbb{R}^n)} \leq C \|\nabla^2 u\|_{L^{p_{m-2}^\#}(\mathbb{R}^n)} \leq \dots \leq C \|\nabla^m u\|_{L^p(\mathbb{R}^n)}$$

for $u \in W^{m,2}(\mathbb{R}^n)$. In the case $p = 2$, integration by parts implies

$$\|\nabla^m u\|_{L^2(\mathbb{R}^n)} = \|\Delta^{\frac{m}{2}} u\|_{L^2(\mathbb{R}^n)}$$

for any function $u \in C_c^\infty(\mathbb{R}^n)$. Therefore we can phrase the problem of finding the higher order best constant for

$$\|u\|_{L^{2_m^\#}(\mathbb{R}^n)} \leq K(m, n, 2)^{\frac{1}{2}} \|\nabla^m u\|_{L^2(\mathbb{R}^n)}$$

as calculating

$$K(m, n, 2)^{-1} := \inf_{u \in W^{m,2}(\mathbb{R}^n) \setminus \{0\}} \frac{\|\Delta^{\frac{m}{2}} u\|_{L^2(\mathbb{R}^n)}^2}{\|u\|_{L^{2_m^\#}(\mathbb{R}^n)}^2}.$$

By the work of Edmunds-Fortunato-Jannelli[11] in the case $m = 2$ and Lions[31] and Swanson[42] in the case $m \geq 3$, the best constants and extremal functions are given by

$$K(m, n, 2) = \pi^{-m} \left(\frac{\Gamma(n)}{\Gamma(n/2)} \right)^{\frac{2m}{n}} \prod_{l=-m}^{m-1} (n+2l)^{-1}$$

and

$$u_{c,\mu,x_0}(x) = c \left(\frac{1}{\mu + |x - x_0|^2} \right)^{\frac{n-2m}{2}}$$

where $c, \mu > 0$ and $x_0 \in \mathbb{R}^n$.

2.2 Sobolev Spaces on Riemannian Manifolds

In this section we define and prove the basic properties of Sobolev spaces on Riemannian manifolds without boundary and prove the Sobolev embeddings in the compact setting, using the results from the Euclidean case. We will always assume (M, g) is a smooth complete Riemannian manifold without boundary throughout this section. All results in this section can be found in the books of Aubin[4] or Hebey[22].

We recall that if T is a j -times covariant tensor field on M , then the covariant derivative ∇T is a $j + 1$ times covariant tensor. If u is a C^j function on M , $\nabla^k u$ is then the k -times covariant tensor of class C^{j-k} given by iterating the covariant derivative k times. In local coordinates, we compute the tensor norm

$$|\nabla^k u| = g^{i_1 j_1} \dots g^{i_k j_k} (\nabla^k u)_{i_1 \dots i_k} (\nabla^k u)_{j_1 \dots j_k} = \nabla^{i_1} \dots \nabla^{i_k} u \nabla_{i_1} \dots \nabla_{i_k} u$$

and define the L^p norm for $1 \leq p < \infty$ by

$$\|\nabla^k u\|_{L^p} = \left(\int_M |\nabla^k u|^p d\nu_g \right)^{\frac{1}{p}}.$$

Definition 2.3. Given a nonnegative integer k and $1 \leq p < \infty$, let $C_k^p(M)$ be the space of smooth functions whose first k derivatives are in $L^p(M)$, i.e.

$$C_k^p(M) = \{u \in C^\infty(M) : \|\nabla^j u\|_{L^p} < \infty \text{ for all } 1 \leq j \leq k\}.$$

The **Sobolev space** $H_k^p(M)$ is defined to be the metric completion of $C_k^p(M)$ with respect to the norm

$$\|u\|_{H_k^p(M)} = \left(\sum_{j=1}^m \|\nabla^j u\|_{L^p}^p \right)^{\frac{1}{p}}.$$

If M is compact, then for any k and p , $C_k^p(M) = C^\infty(M) = C_c^\infty(M)$. However, the Sobolev spaces for each k and p will be different due to the different imposed norms. One may notice that elements of $H_k^p(M)$ are not a priori functions (or even elements of $L^p(M)$), but rather equivalence classes of sequences of functions. However, a Cauchy sequence $\{u_i\}_i$ with respect to the norm $\|\cdot\|_{H_k^p(M)}$ is clearly Cauchy in $L^p(M)$, and therefore

converges to a limit function $u \in L^p(M)$. Then it can be shown that if two Cauchy sequences $\{u_i\}_i, \{v_i\}_i$ converge to the same limit in $L^p(M)$, then they must be equivalent (see Hebey and Robert[24], Definition 2.1 for a proof). Therefore we will from now on refer to elements of $H_k^p(M)$ by their limit in $L^p(M)$.

The first Sobolev embedding we wish to transfer to the case of compact manifold is $H_1^1(M) \subset L^{\frac{n}{n-1}}(M)$. As in the Euclidean case, the other embeddings will follow directly from this special case.

Lemma 2.8. *Let (M, g) be a compact Riemannian manifold. Then there exists a collection of finitely many charts*

$$\{(U_i, \phi_i) : 1 \leq i \leq k\}$$

covering M and constants λ, μ such that $0 < \lambda \leq \sqrt{g} \leq \mu$ on $\phi_i(U_i)$ for all $1 \leq i \leq k$.

Proof. Let

$$\{(U_i, \phi_i) : i \in \mathcal{I}\}$$

be a collection of coordinate charts covering M . For each $x \in M$, we can find some $i_x \in \mathcal{I}$ and an open set V_{i_x} containing x and compactly supported in U_{i_x} . Then $\{V_{i_x} : x \in M\}$ forms a covering of M and we take a finite subcover

$$\{(V_{i_{x_j}}, \phi_{i_{x_j}}) : 1 \leq j \leq k\}.$$

Because each $V_{i_{x_j}}$ is compactly supported in $U_{i_{x_j}}$, there exists λ_j, μ_j for $1 \leq j \leq k$ such that $0 < \lambda_j \leq \sqrt{g} \leq \mu_j$ on $\phi_{i_{x_j}}(V_{i_{x_j}})$ for all $1 \leq j \leq k$. Taking $\mu = \max_{1 \leq j \leq k} \mu_j$ and $\lambda = \min_{1 \leq j \leq k} \lambda_j$ allows us to conclude. $\square$

Theorem 2.9. *Let (M, g) be a compact Riemannian manifold. Then the embedding $H_1^1(M) \subset L^{\frac{n}{n-1}}(M)$ holds.*

Proof. We need to show that for all $u \in H_1^1(M)$, $u \in L^{\frac{n}{n-1}}(M)$ and there is some C such that

$$\|u\|_{L^{\frac{n}{n-1}}(M)} \leq C \|u\|_{H_1^1(M)}. \quad (2.2)$$

By density it suffices to restrict to $u \in C^\infty(M)$.

Let $\{(U_i, \phi_i) : 1 \leq i \leq k\}$ be a finite covering of charts of M satisfying the conclusion of Lemma 2.8. and let $\{\alpha_i : 1 \leq i \leq k\}$ be a partition of unity subordinate to this covering. Let $u \in H_1^1(M)$ be arbitrary, we first have immediately by the Minkowski inequality

$$\|u\|_{L^{\frac{n}{n-1}}(M)} = \left\| \sum_i \alpha_i u \right\|_{L^{\frac{n}{n-1}}(M)} \leq \sum_i \|\alpha_i u\|_{L^{\frac{n}{n-1}}(M)}.$$

We also have

$$\int_M |\nabla(\alpha_i u)| dv_g = \int_M |u \nabla \alpha_i + \alpha_i \nabla u| dv_g \leq \int_M |\nabla u| + |u \nabla \alpha_i| dv_g \leq C \|u\|_{H^1(M)}.$$

Therefore in order to show (2.2) it is sufficient to prove

$$\|\alpha_i u\|_{L^{\frac{n}{n-1}}} \leq C \|\nabla(\alpha_i u)\|_{L^1}.$$

We recall by the definition of integration on a manifold that

$$\int_M |\alpha_i u|^{\frac{n}{n-1}} dv_g = \int_{\mathbb{R}^n} |\alpha_i u(\phi_i^{-1}(x))|^{\frac{n}{n-1}} \sqrt{g} dx$$

and

$$\int_M |\nabla(\alpha_i u)| dv_g = \int_{\mathbb{R}^n} |\nabla(\alpha_i u(\phi_i^{-1}(x)))| \sqrt{g} dx$$

where the right hand integrands are compactly supported in $\mathbb{R}^n$. Therefore, applying Theorem 2.3 and letting λ and μ be as in the conclusion of Lemma 2.8,

$$\begin{aligned} \left(\int_M |\alpha_i u|^{\frac{n}{n-1}} dv \right)^{\frac{n-1}{n}} &\leq \mu^{\frac{n-1}{n}} \left(\int_{\mathbb{R}^n} |\alpha_i u(\phi_i^{-1}(x))| dx \right)^{\frac{n-1}{n}} \\ &\leq \frac{\mu^{\frac{n-1}{n}}}{2} \int_{\mathbb{R}^n} |\nabla(\alpha_i u(\phi_i^{-1}(x)))| dx \\ &\leq \frac{\mu^{\frac{n-1}{n}} \lambda^{-1}}{2} \int_M |\nabla(\alpha_i u)| dv_g \end{aligned}$$

and we are done. □

Theorem 2.10. *Let (M, g) be a compact Riemannian manifold. For any $1 < p < n$, the embedding $L^{p^*}(M) \subset H_1^p(M)$ holds.*

Proof. As for Theorem 2.9 we prove the statement for $u \in C^\infty(M)$, the conclusion then follows by density.

Set $\phi = |u|^{p^*(n-1)/n} \in H_1^1(M)$ and apply the inequality

$$\left(\int_M |\phi|^{\frac{n}{n-1}} dv_g \right)^{\frac{n-1}{n}} \leq C \int_M |\nabla \phi| + |\phi| dv_g.$$

We set $q = p^*(n-1)/n - 1$ and p' to be the Hölder conjugate of p . We obtain by Hölder's inequality

$$\begin{aligned} \left(\int_M |u|^{p^*} dv_g \right)^{\frac{n}{n-1}} &\leq C \frac{p^*(n-1)}{n} \int_M |u|^q |\nabla u| dv_g + C \int_M |u|^{p^*(n-1)/n} dv_g \\ &\leq C \frac{p^*(n-1)}{n} \left(\int_M |u|^{qp'} dv_g \right)^{\frac{1}{p'}} \left(\int_M |\nabla u|^p dv_g \right)^{\frac{1}{p}} \\ &\quad + C \left(\int_M |u|^{qp'} dv_g \right)^{\frac{1}{p'}} \left(\int_M |u|^p dv_g \right)^{\frac{1}{p}}. \end{aligned}$$

By a simple calculation, $qp' = p^*$ and $\frac{n-1}{n} - \frac{1}{p'} = \frac{1}{p^*}$. Therefore we obtain

$$\left(\int_M |u|^p dv_g \right)^{\frac{1}{p^*}} \leq C \frac{p(n-1)}{n} \left(\int_M |\nabla u|^p dv_g \right)^{\frac{1}{p}} + C \left(\int_M |u|^p dv_g \right)^{\frac{1}{p}}.$$

□

As the following theorem shows, we can iterate the first order Sobolev inequality to show general embeddings of higher order Sobolev spaces $H_m^p(M)$ in $H_k^q(M)$. Naturally, having higher differentiability gives stronger integrability in the lower order space. We first require a lemma.

Lemma 2.11. *Let r be a nonnegative integer and let $u \in C^{r+1}$. Then*

$$|\nabla |\nabla^r u|| \leq |\nabla^{r+1} u|. \quad (2.3)$$

Proof. Let T be the $2r+1$ degree covariant tensor defined by

$$T_{\nu\alpha_1 \dots \alpha_r \beta_1 \dots \beta_r} = \nabla_\nu \nabla_{\alpha_1} \dots \nabla_{\alpha_r} u \nabla_{\beta_1} \dots \nabla_{\beta_r} u - \nabla_\nu \nabla_{\beta_1} \dots \nabla_{\beta_r} u \nabla_{\alpha_1} \dots \nabla_{\alpha_r} u.$$

Then writing out the property $|T|^2 \geq 0$ in coordinates gives

$$\begin{aligned} &(\nabla_\nu \nabla_{\alpha_1} \dots \nabla_{\alpha_r} u \nabla_{\beta_1} \dots \nabla_{\beta_r} u - \nabla_\nu \nabla_{\beta_1} \dots \nabla_{\beta_r} u \nabla_{\alpha_1} \dots \nabla_{\alpha_r} u) \\ &\quad \times g^{\nu\mu} g^{\alpha_1\lambda_1} \dots g^{\alpha_r\lambda_r} g^{\beta_1\gamma_1} \dots g^{\beta_r\gamma_r} (\nabla_\mu \nabla_{\lambda_1} \dots \nabla_{\lambda_r} u \nabla_{\gamma_1} \dots \nabla_{\gamma_r} u - \nabla_\mu \nabla_{\gamma_1} \dots \nabla_{\gamma_r} u \nabla_{\lambda_1} \dots \nabla_{\lambda_r} u) \geq 0. \end{aligned}$$

When expanding out this product following four terms appear:

$$\begin{aligned} A_1 &= \nabla_\nu \nabla_{\alpha_1} \dots \nabla_{\alpha_r} u \nabla_{\beta_1} \dots \nabla_{\beta_r} u g^{\nu\mu} g^{\alpha_1\lambda_1} \dots g^{\alpha_r\lambda_r} g^{\beta_1\gamma_1} \dots g^{\beta_r\gamma_r} \nabla_\mu \nabla_{\lambda_1} \dots \nabla_{\lambda_r} u \nabla_{\gamma_1} \dots \nabla_{\gamma_r} u \\ A_2 &= \nabla_\nu \nabla_{\beta_1} \dots \nabla_{\beta_r} u \nabla_{\alpha_1} \dots \nabla_{\alpha_r} u g^{\nu\mu} g^{\alpha_1\lambda_1} \dots g^{\alpha_r\lambda_r} g^{\beta_1\gamma_1} \dots g^{\beta_r\gamma_r} \nabla_\mu \nabla_{\gamma_1} \dots \nabla_{\gamma_r} u \nabla_{\lambda_1} \dots \nabla_{\lambda_r} u \\ A_3 &= \nabla_\nu \nabla_{\alpha_1} \dots \nabla_{\alpha_r} u \nabla_{\beta_1} \dots \nabla_{\beta_r} u g^{\nu\mu} g^{\alpha_1\lambda_1} \dots g^{\alpha_r\lambda_r} g^{\beta_1\gamma_1} \dots g^{\beta_r\gamma_r} \nabla_\mu \nabla_{\gamma_1} \dots \nabla_{\gamma_r} u \nabla_{\lambda_1} \dots \nabla_{\lambda_r} u \\ A_4 &= \nabla_{\alpha_1} \dots \nabla_{\alpha_r} u \nabla_\nu \nabla_{\beta_1} \dots \nabla_{\beta_r} u g^{\nu\mu} g^{\alpha_1\lambda_1} \dots g^{\alpha_r\lambda_r} g^{\beta_1\gamma_1} \dots g^{\beta_r\gamma_r} \nabla_{\gamma_1} \dots \nabla_{\gamma_r} u \nabla_\mu \nabla_{\lambda_1} \dots \nabla_{\lambda_r} u. \end{aligned}$$

Substituting then gives us

$$A_1 + A_2 - A_3 - A_4 \geq 0.$$

We first recognize

$$A_1 = A_2 = |\nabla^{r+1}u|^2 |\nabla^r u|^2.$$

For the other two terms, we consider $|\nabla |\nabla^r u|^2|^2$, which in coordinates is given by

$$g^{\nu\mu} \nabla_\nu (g^{\alpha_1 \lambda_1} \dots g^{\alpha_r \lambda_r} \nabla_{\alpha_1} \dots \nabla_{\alpha_r} u \nabla_{\lambda_1} \dots \nabla_{\lambda_r} u) \nabla_\mu (g^{\beta_1 \gamma_1} \dots g^{\beta_r \gamma_r} \nabla_{\beta_1} \dots \nabla_{\beta_r} u \nabla_{\gamma_1} \dots \nabla_{\gamma_r} u).$$

When applying the product rule to in the first half of the above formula, we note by symmetry of g we have

$$g^{\alpha_1 \lambda_1} \dots g^{\alpha_r \lambda_r} \nabla_\nu \nabla_{\alpha_1} \dots \nabla_{\alpha_r} u \nabla_{\lambda_1} \dots \nabla_{\lambda_r} u = g^{\alpha_1 \lambda_1} \dots g^{\alpha_r \lambda_r} \nabla_{\alpha_1} \dots \nabla_{\alpha_r} u \nabla_\nu \nabla_{\lambda_1} \dots \nabla_{\lambda_r} u.$$

A similar equality holds when applying the product rule in the second half of the formula. We therefore obtain

$$4A_3 = 4A_4 = |\nabla |\nabla^r u|^2|^2.$$

Hence substituting in our expansion of $|T|^2 \geq 0$ gives

$$0 \leq 2|\nabla^{r+1}u|^2 |\nabla^r u|^2 - \frac{1}{2} |\nabla |\nabla^r u|^2|^2 = 2|\nabla^{r+1}u|^2 |\nabla^r u|^2 - 2|\nabla |\nabla^r u|^2|^2 |\nabla^r u|^2$$

which immediately yields our Lemma. $\square$

Theorem 2.12. *Let (M, g) be compact, $m > k \geq 0$ be integers, and $p \geq 1$ such that $n - (m - k)p > 0$. Then the embedding $H_m^p(M) \subset H_k^{p_{m-k}}(M)$ holds.*

As usual, we prove the statement for smooth functions and conclude through density. First we prove the statement in the case $k = m - 1$. By Lemma 2.11 we have $|\nabla^{m-1}u| \in H_1^p(M)$. Therefore we apply the first order Sobolev inequality and Lemma 2.11 to obtain

$$\begin{aligned} \|\nabla^{m-1}u\|_{L^{p^*}(M)} &\leq C(\|\nabla |\nabla^{m-1}u|\|_{L^p(M)} + \|\nabla^{m-1}u\|_{L^p(M)}) \\ &\leq C(\|\nabla^m u\|_{L^p(M)} + \|\nabla^{m-1}u\|_{L^p(M)}) \end{aligned}$$

Similarly, for all $0 \leq i < m - 1$ we have

$$\|\nabla^i u\|_{L^{p^*}(M)} \leq C(\|\nabla^{i+1}u\|_{L^p(M)} + \|\nabla^i u\|_{L^p(M)}).$$

Therefore, summing together the previous two inequalities gives us

$$\|u\|_{H_{m-1}^{p^*}(M)} \leq C \|u\|_{H_m^p(M)}.$$

For the general case, we note $n - (m - k)p > 0$ implies $n - p_i^\# > 0$ for all $1 \leq i \leq m - k$, and therefore iterating the previous embedding $m - k$ times allows us to obtain

$$H_m^p(M) \subset H_{m-1}^{p^*}(M) \subset H_{m-2}^{p_2^\#}(M) \subset \dots \subset H_k^{p_{m-k}^\#}(M).$$

Remark 2.13. It is easily seen by Theorem 2.12 and Hölder's inequality that if (M, g) is compact, $m > k \geq 0$ are integers, and $p, q \geq 1$ are such that $m - n/p > k - n/q$, then the embedding $H_m^p(M) \subset H_k^q(M)$ still holds. In fact, this embedding is compact (we recall an embedding $X \subset Y$ is compact if bounded subsets of X are relatively compact in Y). We refer to Chapter 2 of Hebey[22] for a proof. This is often referred to as the Rellich-Kondrachov embedding theorem. In particular, the embedding $H_m^p(M) \subset H_k^p(M)$ is compact if $m > k$.

Remark 2.14. By the well known fact that $L^p(M)$ is a reflexive space for $p > 1$, it immediately follows the Sobolev spaces $H_m^p(M)$ are reflexive for $m \geq 0$ and $p > 1$.

3 The Higher Order AB Program

3.1 Introduction

Throughout Sections 3 and 4 (except Section 4.2) we let (M, g) be a compact Riemannian manifold without boundary of dimension n and $m \geq 1$ be an integer such that $m < \frac{n}{2}$. We simplify notation by writing $2^\# := 2_m^\#$ and $K := K(m, n)$. We first note the existence of constants (A, B) such that the inequality (1.2) holds does not necessarily follow a priori from the embedding $H_m^2(M) \subset L^{2^\#}(M)$ as stated in Theorem 2.11. However, one sees as an immediate consequence of Lemma 4.4 (see also Robert[38]) the existence of a constant C such that for all $u \in H_m^2(M)$,

$$\frac{1}{C} \sum_{i=0}^m \int_M (\Delta^{\frac{i}{2}} u)^2 dv_g \leq \|u\|_{H_m^2(M)}^2 \leq C \sum_{i=0}^m \int_M (\Delta^{\frac{i}{2}} u)^2 dv_g.$$

We therefore overload notation for $\|\cdot\|_{H_m^2(M)}$ and occasionally use it to represent the equivalent norm

$$\|u\|_{H_m^2(M)}^2 = \sum_{i=0}^m \int_M (\Delta^{\frac{i}{2}} u)^2 dv_g$$

when there is no confusion.

Using this norm equivalence, we write (1.2) in the form

$$\left(\int_M |u|^{2^\#} dv_g \right)^{\frac{2}{2^\#}} \leq A \int_M (\Delta^{\frac{m}{2}} u)^2 dv_g + B \|u\|_{H_{m-1}^2(M)}^2. \quad (3.1)$$

The main focus of Sections 3 and 4 is to discuss the best first constant A in this inequality. Let $\mathcal{A}(m, n) \subset \mathbb{R}$ be defined to be the set of all values A such that there exists some $B(A)$ for which (3.1) is valid. Let $\alpha(m, n) := \inf \mathcal{A}(m, n)$. In Section 3.1 we present the result of Mazumdar[33] stating $\alpha(m, n) = K$. Results of this kind are often known as asymptotically sharp Sobolev inequalities in the literature. We also state the main result of the thesis, that it is in fact possible to take $A = K$ in (3.1). In Section 3.2 we present another result from [33], applying the asymptotically sharp inequality from Section 3.1 to prove an existence result for a higher order semilinear PDE through the concentration compactness principle of Lions[31]. We also show a sufficient condition for the solution to be smooth and positive. The use of results from the AB Program to prove the existence of solutions to PDEs originates from the work of Aubin[2] in the resolution of the Yamabe Problem (see Lee and Parker[28]). Aside from being an interesting result in its own right, this existence is necessary for proving the best constant $A = K$ in (3.1) is attained through a blow up analysis of a sequence of positive solutions u_α to the PDE

$$(\Delta + \alpha)^m u_\alpha = u_\alpha^{2^\# - 1}.$$

3.2 The Best First Constant

Directly following the proof in Mazumdar[33], we prove the result $\alpha(m, n) = K$ in two steps. We first show for any (A, B) such that (3.1) is true, $A \geq K$. We then show that for every $\epsilon > 0$, setting $A = (1 + \epsilon)K$ grants the existence of some B_ϵ such that substituting $((1 + \epsilon)K, B_\epsilon)$ in (3.1) results in a valid inequality.

Lemma 3.1. *If $A \in \mathbb{R}$ is such that there exists $B(A) \in \mathbb{R}$ satisfying inequality (3.1), then $A \geq K$.*

Proof. Suppose there exists some $A < K$ such that inequality (3.1) holds. Let $\epsilon > 0$. It can be shown by computing in geodesic coordinates (see Lemma 9.1 in Mazumdar[33]) the existence of some $\delta_0(\epsilon)$, $0 < \delta_0 < i_g$ such that for any $p \in M$, $u \in C_c^\infty(B_0(\delta))$ (on Euclidean space), and $1 \leq k \leq m$,

$$(1 - \epsilon) \left(\int_{\mathbb{R}^n} |u|^{2^\#} dx \right)^{\frac{2}{2^\#}} \leq \left(\int_M |u \circ \exp_p^{-1}|^{2^\#} dv_g \right)$$

and

$$\int_M (\Delta_g^{\frac{k}{2}}(u \circ \exp_p^{-1}))^2 dv_g \leq (1 + \epsilon) \int_{\mathbb{R}^n} (\Delta_\xi^{\frac{k}{2}} u)^2 dx.$$

Therefore if (A, B) are such that (3.1) holds then we have

$$\left(\int_{\mathbb{R}^n} |u|^{2^\#} dx \right)^{\frac{2}{2^\#}} \leq \frac{1 + \epsilon}{1 - \epsilon} A \int_{\mathbb{R}^n} (\Delta_\xi^{\frac{k}{2}} u)^2 dx + B \frac{1 + \epsilon}{1 - \epsilon} \|u\|_{H_{m-1}^2(\mathbb{R}^n)}^2.$$

Now let $v \in C_c^\infty(\mathbb{R}^n)$ and let $v_\lambda(x) := v(\lambda x)$ for $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$. For λ sufficiently large, $\text{supp}(v_\lambda) \subset B_0(\delta)$, therefore substituting in the above inequality and applying a change of variable and the chain rule gives

$$\lambda^{-(n-2m)} \left(\int_{\mathbb{R}^n} |v|^{2^\#} dx \right)^{\frac{2}{2^\#}} \leq \lambda^{-(n-2m)} \frac{1 + \epsilon}{1 - \epsilon} A \int_{\mathbb{R}^n} (\Delta_\xi^{\frac{k}{2}} v)^2 dx + C \sum_{k=0}^{m-1} \lambda^{-(n-2k)} \int_{\mathbb{R}^n} |\Delta_\xi^{\frac{k}{2}} v|^2 dx$$

and sending $\lambda \rightarrow \infty$ implies

$$\left(\int_{\mathbb{R}^n} |v|^{2^\#} dx \right)^{\frac{2}{2^\#}} \leq \frac{1 + \epsilon}{1 - \epsilon} A \int_{\mathbb{R}^n} (\Delta_\xi^{\frac{k}{2}} v)^2 dx.$$

Choosing ϵ such that $\frac{1 + \epsilon}{1 - \epsilon} A < K$ gives a contradiction.

□

Theorem 3.2. *Let $\epsilon > 0$. Then there exists B_ϵ such that*

$$\|u\|_{2^\#}^2 \leq (1 + \epsilon)K \int_M (\Delta_g^{\frac{m}{2}} u)^2 dv_g + B_\epsilon \|u\|_{H_{m-1}^2}^2$$

and K is the smallest constant for which this holds.

Proof. We prove the statement for $u \in C^\infty(M)$ and then conclude through density. Let $\epsilon, \epsilon_1 > 0$. By Lemma 9.1 in Mazumdar[33], for all $x \in M$, there exists some harmonic chart ϕ around x and some δ_x such that $i_g > \delta_x > 0$ such that for all $u \in C_c^\infty(B_{\delta_x}(x))$,

$$\left(\int_M |u|^{2^\#} dv_g \right)^{\frac{2}{2^\#}} \leq (1 + \epsilon_1) \left(\int_{\mathbb{R}^n} |u \circ \phi^{-1}|^{2^\#} dx \right)^{\frac{2}{2^\#}}$$

and

$$\int_{\mathbb{R}^n} \left(\Delta_\xi^{\frac{m}{2}} (u \circ \phi^{-1}) \right)^2 dx \leq (1 + \epsilon_1) \int_M (\Delta_g^{\frac{m}{2}} u)^2 dv_g.$$

Therefore, by the sharp higher order Euclidean Sobolev inequality as discussed at the end of Section 2.1, we can set δ_x such that for all $u \in C_c^\infty(B_{\delta_x}(x))$,

$$\left(\int_M |u|^{2^\#} dv_g \right)^{\frac{2}{2^\#}} \leq (K + \epsilon) \int_M (\Delta_g^{\frac{m}{2}} u)^2 dv_g. \quad (3.2)$$

By compactness, there exists an integer k , points $\{x_i : 1 \leq i \leq k\}$, and positive values $\{\delta_{x_i} : 1 \leq i \leq k\}$ such that M is covered by $\{B_{\delta_{x_i}}(x_i) : 1 \leq i \leq k\}$ such that for any $u \in C_c^\infty(B_{\delta_{x_i}}(x_i))$ the estimate (3.2) holds. It is easy to construct a partition of unity $\{\eta_i : 1 \leq i \leq k\}$ subordinate to this covering such that $\sqrt{\eta}$ is smooth for all $1 \leq i \leq k$. Then by the Minkowski inequality, (3.2), and Lemma 4.5 we obtain

$$\begin{aligned} \left(\int_M |u|^{2^\#} dv_g \right)^{\frac{2}{2^\#}} &\leq \sum_{i=1}^k \left(\int_M |\sqrt{\eta_i} u|^{2^\#} dv_g \right)^{\frac{2}{2^\#}} \\ &\leq \sum_{i=1}^k (K + \epsilon) \int_M (\Delta_g^{\frac{m}{2}} (\sqrt{\eta_i} u))^2 dv_g \\ &\leq (K + \epsilon) \int_M (\Delta_g^{\frac{m}{2}} u)^2 dv_g + C \|u\|_{H_{m-1}^2(M)}^2 \end{aligned}$$

which is what we intended to show. The second part of the theorem follows from Lemma 3.1. □

We delay the proof of the following theorem to Section 4.

Theorem 3.3. *There exists some B based only on (M, g) and m such that inequality (3.1) holds with $A = K$.*

3.3 Existence Results for a Higher Order Semilinear PDE

For the sake of simplicity, we consider only differential operators L of the form

$$Lu = \Delta^m u + \sum_{i=0}^{m-1} c_i \Delta^i u \quad (3.3)$$

where c_i are positive real numbers. Given $u \in H_m^2(M)$, we interpret $\int_M u L u d v_g$ in the sense of distributions, i.e.

$$\int_M u L u d v_g = \int_M (\Delta^{\frac{m}{2}} u)^2 d v_g + \sum_{i=0}^{m-1} c_i \int_M (\Delta^{\frac{i}{2}} u)^2 d v_g.$$

It is obvious L is coercive in the sense that there exists $\lambda > 0$ such that $\int_M u L u d v_g \geq \lambda \int_M u^2 d v_g$. In fact, by the Sobolev embedding theorem and the norm equivalence discussed in Section 3.1, there exists some $\lambda > 0$ such that

$$\int_M u L u d v_g \geq \lambda \|u\|_{L^{2^\#}(M)}^2 d v_g. \quad (3.4)$$

Given $f \in L^1(M)$, we say $u \in H_m^2(M)$ is a weak solution (or a solution in the sense of distributions) to the PDE $Lu = f$ if for all $\phi \in C^\infty(M)$,

$$\int_M \Delta^{\frac{m}{2}}(u) \Delta^{\frac{m}{2}}(\phi) d v_g + \sum_{i=0}^{m-1} c_i \Delta^{\frac{i}{2}}(u) \Delta^{\frac{i}{2}}(\phi) d v_g = \int_M f \phi d v_g.$$

We briefly review some classical results in the theory of linear elliptic PDEs, stated without proof and simplified in order to apply directly to the needs of this section.

Theorem 3.4 (Strong Maximum Principle). *Let $a > 0$. If $u \in C^2(M)$ and $\Delta u + au \geq 0$, then either $u \equiv 0$ or $u > 0$ everywhere.*

Proof. Robert[39], Proposition 4.3. □

Theorem 3.5 (Schauder Theory). *Let L be as in (3.3) and $f \in C_0^\beta(M)$ for $\beta \in (0, 1)$. If $u \in H_m^2$ is a weak solution to $Lu = f$, then $u \in C^{2m, \beta}(M)$ and there exists C based on (M, g) and $\sum_{i=0}^{m-1} c_i$ such that*

$$\|u\|_{C^{2m, \beta}(M)} \leq C \|f\|_{C^{0, \beta}(M)} + \|u\|_{C^0(M)}.$$

Proof. Nicolaescu[36], Theorem 10.3.11 □

Theorem 3.6 (Existence and Uniqueness). *Let L be as in (3.3). Let $f \in L^p(M)$ for $1 < p < \infty$. Then there exists a unique weak solution $u \in H_{2m}^p(M)$ to $Lu = f$.*

Proof. Mazumdar[33], Proposition 8.2. Note the uniqueness part of the theorem is trivial by coercivity. □

Corollary 3.7. *Let L be as in (3.3). Suppose L can be decomposed as $Lu = \prod_{i=1}^m (\Delta + a_i)u$ where $a_i > 0$ for all $1 \leq i \leq m$. Then for $u \in C^{2m}(M)$, if $Lu \geq 0$, then either $u \equiv 0$ or $u > 0$ everywhere.*

We aim to find weak solutions to the PDE

$$Lu = \mu_{L,f} f |u|^{2^\#-2} u \quad (3.5)$$

for positive functions $f \in C^{0,\alpha}(M)$ where

$$\mu_{L,f} := \inf_{u \in H_m^2(M)} I(u)$$

by minimizing the functional

$$I_{L,f}(u) = \frac{\int_M u L u d v_g}{\left(\int_M f |u|^{2^\#} d v_g \right)^{\frac{2}{2^\#}}}$$

over the set

$$\mathcal{N}_f := \left\{ u \in H_m^2(M) : \int_M f |u|^{2^\#} d v_g = 1. \right\}.$$

By (3.4), we have $\mu_{L,f} > 0$. By homogeneity, minimizing over $\mathcal{N}_f$ is equivalent to minimizing over all of $H_m^2(M)$. Therefore if $u_0 \in H_m^2(M)$ is a minimizer for $I_{L,f}(u)$ over $\mathcal{N}_f$, then for $\phi \in C^\infty(M)$, the property

$$\left. \frac{d}{dt} \right|_{t=0} I(u_0 + t\phi) = 0$$

implies u_0 satisfies the definition of a weak solution for (3.5).

Ideally one would like to take a minimizing sequence $\{u_i\}_i \subset \mathcal{N}_f$ such that $I_{L,f}(u_i) \rightarrow \mu_{L,f}$ and show it converges to a minimizer $u_0 \in H_m^2(M)$. Unfortunately the lack of compactness of the embedding $H_m^2(M) \subset L^{2^\#}(M)$ prevents us from achieving this in general. However, by the work of Lions[31], we can describe the sense in which this loss of compactness occurs. The following concentration compactness lemma states that if a bounded sequence in $H_m^2(M)$ does not converge in $L^{2^\#}(M)$, then it must concentrate at countably many points. Additionally, there is a sense in which this concentration satisfies a sort of Sobolev inequality. Our argument follows the presentation in Mazumdar[33] and Neumayer[35].

Lemma 3.8. *Let (M, g) be a smooth manifold of dimension n and let $2m \leq n$. Let $\{u_i\}_i$ be a bounded sequence in $H_m^2(M)$. By reflexivity of $H_m^2(M)$, the Riesz Representation Theorem for Radon measures, weak compactness for Radon measures, and the Sobolev embedding theorem, there exists $u \in H_m^2(M)$ and Radon measures ν, μ such that up to a subsequence,*

- $u_k \rightharpoonup u$ weakly in $H_m^2(M)$.
- $\mu_k := (\Delta^{\frac{m}{2}} u_k)^2 d v_g \rightharpoonup \mu$ weakly in the sense of measures.

- $\nu_k := |u_i|^{2^\#} dv_g \rightharpoonup \nu$ weakly in the sense of measures.

Then there exists an at most countable index set $\mathcal{I}$ and a set of points $\{x_i \in M : i \in \mathcal{I}\}$ and weights $\{\alpha_i \in \mathbb{R} : i \in \mathcal{I}\}$ such that

- $\nu = |u|^{2^\#} dv_g + \sum_{i \in \mathcal{I}} \alpha_i \delta_{x_i}$
- $\mu \geq (\Delta^{\frac{m}{2}} u)^2 dv_g + \frac{1}{K} \sum_{i \in \mathcal{I}} \alpha_i^{\frac{2}{2^\#}} \delta_{x_i}$

In particular $\sum_{i \in \mathcal{I}} \alpha_i^{\frac{2}{2^\#}} < \infty$.

Proof. First we consider the case $u \equiv 0$. By Theorem 3.2, for all $\epsilon > 0$ there exists B_ϵ such that for all $\phi \in C^\infty(M)$ and for all $k \in \mathbb{N}$,

$$\left(\int_M |\phi u_k|^{2^\#} dv_g \right)^{\frac{2}{2^\#}} \leq (K + \epsilon) \int_M (\Delta^{\frac{m}{2}}(\phi u_k))^2 dv_g + B_\epsilon \|\phi u_k\|_{H_{m-1}^2(M)}^2.$$

Let ϵ be fixed for now. Because $H_{m-1}^2(M)$ is compactly embedded in $H_m^2(M)$ (see Remark 2.13), the second term will converge to 0 as $k \rightarrow \infty$. Expanding the first term (or applying Lemma 4.5) and using the weak convergence of u_k to 0 implies

$$(K + \epsilon) \int_M (\Delta^{\frac{m}{2}}(\phi u_k))^2 dv_g \rightarrow (K + \epsilon) \int_M \phi^2 d\mu$$

as $k \rightarrow \infty$. After sending $\epsilon \rightarrow 0$ we therefore have for all $\phi \in C^\infty(M)$

$$\left(\int |\phi|^{2^\#} dv \right)^{\frac{2}{2^\#}} \leq K \int \phi^2 d\mu.$$

This can be thought of as a "reverse Holder's inequality" (contrasting with the use of the usual Holder's inequality to bound lower L^p norms with higher ones). Because $C^\infty(M)$ is dense in $L^p(M)$ for all Radon measures, it follows by approximating characteristic functions that for any Borel $A \subset M$

$$(\nu(A))^{\frac{2}{2^\#}} \leq K\mu(A) \tag{3.6}$$

Because μ is a finite measure, it has at most countably many atoms $\{x_i\}_{i \in I} \subset M$ where I is a countable index set. Therefore, for any $x \in M \setminus \{x_i\}_{i \in I}$, we can let A be an open set containing x with measure $\mu(A) \leq \frac{1}{K}$ and we will have

$$1 \geq K\mu(A) \geq (\nu(A))^{\frac{2}{2^\#}} \geq \nu(A)$$

i.e. ν is absolutely continuous with respect to μ on $M \setminus \{x_i\}_{i \in I}$. We can therefore calculate the Radon-Nikodym derivative f such that $d\nu = f d\mu$, and by a version of the Lebesgue Differentiation Theorem applied to manifolds (see Federer[13], Chapter 2), we have for all $x \in M \setminus \{x_i\}_{i \in I}$,

$$f(x) = \lim_{r \rightarrow 0} \frac{\nu(B_r(x))}{\mu(B_r(x))} \leq \lim_{r \rightarrow 0} K^{\frac{2^\#}{2}} \mu(B_r(x))^{\frac{2^\#}{2}-1} = 0$$

Therefore ν is supported on $\{x_i\}_{i \in I}$ and we write

$$\nu = \sum_{i \in I} \alpha_i \delta_{x_i}.$$

Then considering any x_i for $i \in I$, applying (3.6) to $B_r(x_i)$, and letting $r \rightarrow 0$ then gives us

$$\mu(\{x_i\}) \geq \frac{1}{K} (\nu(\{x_i\}))^{\frac{2}{2^\#}}.$$

Now in the case $u \neq 0$, we set $v_k = u_k - u \rightharpoonup 0$. Up to a subsequence we assume $u_k \rightarrow u$ almost everywhere, which is possible by strong convergence in $L^2(M)$. We recall the Brezis-Lieb Lemma (see Brezis-Lieb[6]) states for any $p > 1$ and any complete measure μ on M ,

$$\lim_{k \rightarrow \infty} \int_M ||u|^p - |u_k|^p + |u - u_k|^p| d\mu = 0.$$

As a consequence we have $\tilde{v}_k := |v_k|^{2^\#} dv_g \rightharpoonup \nu - |u|^{2^\#} dv_g$ weakly in the sense of measures. It is also easily seen that $\tilde{\mu}_k := (\Delta^{\frac{m}{2}}(v_k))^2 dv_g \rightharpoonup \mu - (\Delta^{\frac{m}{2}}u)^2 dv_g$ weakly in the sense of measures. Therefore the proof is completed by applying the case $u \equiv 0$ to the sequences v_k , $\tilde{\mu}_k$, and $\tilde{v}_k$.

□

The following theorem states that there exists a "minimal energy", below which we can guarantee a minimizing sequence converges strongly to a $C^{2m}(M)$ solution to (3.5).

Theorem 3.9. *Suppose L is as in (3.3) and $f \in C^{0,\alpha}(M)$ is positive. Then if*

$$\mu_{L,f} < \frac{1}{K(\max_M f)^{\frac{2}{2^\#}}},$$

then there exists a C^{2m} minimizing solution u to

$$Lu = f|u|^{2^\#-2}u.$$

Proof. Assume $\inf_{u \in \mathcal{N}_f} I(u) < \frac{1}{K(\max_M f)^{\frac{2}{2^\#}}}$. Let $\{u_k\}_k \subset \mathcal{N}_f$ be a minimizing sequence for $I(u)$. Then u_k is bounded in H_m^2 and therefore if we define v_k as in Theorem 3.8 then

$$v_k \rightharpoonup |u_0|^{2^\#} dv_g + \sum_{i \in I} \alpha_i \delta_{x_i}$$

where u_0 is the weak limit of u_k in $L^{2^\#}(M)$. An immediate consequence of the definition of weak convergence in the sense of measures and the fact that M is compact is if we define $\tilde{v}_k = f|u_k|^{2^\#} dv_g$ then

$$\tilde{v}_k \rightharpoonup f|u_0|^{2^\#} dv_g + \sum_{i \in I} f(x_i) \alpha_i \delta_{x_i}.$$

Applying again the definition of weak convergence we obtain

$$1 = \lim_{k \rightarrow \infty} \int_M d\tilde{v}_k = \int_M f|u_0|^{2^\#} dv_g + \sum_{i \in I} f(x_i) \alpha_i.$$

If we define the measure λ such that $d\lambda = f dv_g$, then $u_k \rightharpoonup u_0$ in $L^{2^\#}(M, \lambda)$. Therefore by the well known fact norms are lower semicontinuous with respect to weak convergence

$$\int f|u_0|^{2^\#} dv_g \leq \liminf_k \int f|u_k|^{2^\#} = 1.$$

Let $t \in [0, 1] = \int_M f|u_0|^{2^\#} dv_g$. If $t = 1$, then we have strong convergence in $L^{2^\#}$ and will be able to conclude u is a weak solution to the PDE. Therefore we assume $t \in [0, 1)$. Then by Lemma 3.8, the strong convergence of $u_k \rightarrow u_0$ in H_{m-1}^2 , and the fact that $\sum_{i \in I} f(x_i) \alpha_i = 1 - t$ we obtain

$$\begin{aligned} \mu_{L,f} &\geq \int_M u_0 L u_0 dv_g + \frac{1}{K} \sum_{i \in I} \alpha_i^{\frac{2}{2^\#}} \\ &\geq t^{\frac{2}{2^\#}} \mu_{L,f} + \frac{1}{K} \sum_{i \in I} \alpha_i^{\frac{2}{2^\#}} \\ &\geq t^{\frac{2}{2^\#}} \mu_{L,f} + \frac{1}{K(\max_M f)^{\frac{2}{2^\#}}} \sum_{i \in I} (f(x_i) \alpha_i)^{\frac{2}{2^\#}} \\ &= t^{\frac{2}{2^\#}} \mu_{L,f} + \frac{(1-t)^{\frac{2}{2^\#}}}{K(\max_M f)^{\frac{2}{2^\#}}} \sum_{i \in I} \left(\frac{f(x_i) \alpha_i}{1-t} \right)^{\frac{2}{2^\#}} \\ &\geq t^{\frac{2}{2^\#}} \mu_{L,f} + \frac{(1-t)^{\frac{2}{2^\#}}}{K(\max_M f)^{\frac{2}{2^\#}}} \\ &> t^{\frac{2}{2^\#}} \mu_{L,f} + (1-t)^{\frac{2}{2^\#}} \mu_{L,f} \\ &\geq \mu_{L,f} \end{aligned}$$

and we have a contradiction, therefore we must have $t = 1$. Again applying lower semicontinuity of norms under weak convergence, we have

$$\int_M u_0 L u_0 d v_g = I(u_0) \leq \lim_{k \rightarrow \infty} I(u_k)$$

and we conclude $\{u_k\}_k$ converges to a minimizer $u_0 \in H_m^2(M)$. Therefore u_0 is a weak solution to (3.5). For a proof of the $C^{2m}(M)$ regularity, we refer to the appendix (Section 8) of Mazumdar[33].

□

Now, in the case L decomposes as a product of second order operators, we follow the proof of Proposition 4.1 of Robert[39] to obtain a positivity result. We will then have the necessary background to prove Theorem 3.3.

Theorem 3.10. *Suppose L as in (3.3) decomposes as a product of second order operators $\prod_{i=1}^m (\Delta + a_i)u$. Let $f \in C^\infty(M)$ be positive. Suppose the assumptions of Theorem 3.9 hold. Then the $C^{2m}(M)$ solution to (3.5) can be chosen to be smooth and positive.*

Proof. Let u be the $C^{2m}(M)$ solution solution to (3.5) guaranteed by the conclusion of Theorem 3.9. Because $|Lu| \in C^{0,1}$, by Theorem 3.5 and 3.6 there exists $v \in C^{2m}(M)$ such that $Lv = |Lu|$. Then because $L(v \pm u) \geq 0$, by Corollary 3.7 we have $v \pm u \geq 0$. Because $u \neq 0$, we must have $v > 0$ in the conclusion of Corollary 3.7.

We calculate, applying Hölder's inequality

$$\begin{aligned} I_{L,f}(v) &= \frac{\int_M v L v d v_g}{\left(\int_M f |v|^{2^\#} d v_g\right)^{\frac{2}{2^\#}}} \\ &= \frac{\int_M \mu_{L,f} f v |u|^{2^\#-1} d v_g}{\left(\int_M f |v|^{2^\#} d v_g\right)^{\frac{2}{2^\#}}} \\ &= \frac{\int_M \mu_{L,f} (f^{\frac{1}{2^\#}} v) (|u|^{2^\#} f)^{\frac{2^\#-1}{2^\#}} d v_g}{\left(\int_M f |v|^{2^\#} d v_g\right)^{\frac{2}{2^\#}}} \\ &\leq \mu_{L,f} \frac{\left(\int_M f v^{2^\#}\right)^{\frac{1}{2^\#}} \left(\int_M |u|^{2^\#} f d v_g\right)^{\frac{2^\#-1}{2^\#}}}{\left(\int_M f |v|^{2^\#} d v_g\right)^{\frac{2}{2^\#}}} \\ &= \mu_{L,f} \frac{\left(\int_M |u|^{2^\#} f d v_g\right)^{\frac{2^\#-1}{2^\#}}}{\left(\int_M f |v|^{2^\#} d v_g\right)^{\frac{1}{2^\#}}} \end{aligned}$$

Then because $v \geq |u|$ and $u \in \mathcal{N}_f$, we continue the calculation

$$\mu_{L,f} \frac{\left(\int_M |u|^{2^\#} f dv_g\right)^{\frac{2^\#-1}{2^\#}}}{\left(\int_M f |v|^{2^\#} dv_g\right)^{\frac{1}{2^\#}}} \leq \mu_{L,f} \left(\int_M |u|^{2^\#} f dv_g\right)^{\frac{2^\#-2}{2}} = \mu_{L,f}.$$

By definition we also have $\mu_{L,f} \leq I_{L,f}(v)$. Thus, all of the inequalities in the calculation are equalities, and v is a minimizer for $I_{L,f}$. In particular we have $v = |u| > 0$. By continuity of u we must have $u < 0$ everywhere or $u > 0$ everywhere. If $u < 0$, we replace u with $-u$ to obtain a positive solution. Then by bootstrapping the PDE

$$Lu = fu^{2^\#-1}$$

we obtain u is smooth. □

4 Proof of Theorem 3.3

4.1 Structure of the Proof

In Section 4.2 we provide several lemmas which pertain to the Sobolev spaces $H_k^2(M)$ in general. While they are all necessary in order to prove Theorem 3.3 and written with the sole purpose of proving the theorem, they can easily be applied to other higher order problems (see Section 3 for examples). The main usage of the lemmas is the ability to integrate by parts around a smooth function in a way which only affects lower order error terms, for example

$$\int \eta u \Delta^k u dv_g = \int \eta \Delta u \Delta^{k-1} u dv_g + \text{lot}$$

where η is some fixed smooth function.

The theorem is then proven in Section 4.3 in two parts, following the structure of the second order case in Hebey[23]. The proof is by contradiction. Assuming the inequality (3.1) is false, we first perform a concentration point analysis of a sequence of positive functions u_α solving

$$(\Delta + \alpha)^m u_\alpha = u_\alpha^{2^\#-1}$$

and satisfying $\int u_\alpha^{2^\#} dv_g = 1$ for all $\alpha > 0$. The functions will "blow up" as $\alpha \rightarrow \infty$ and the energy (in an $L^{2^\#}(M)$ sense) will concentrate around a single point up to a subsequence. Section 4.3.1 is dedicated to

obtaining both local and global information about this concentration. Then in Section 4.3.2 we show the estimates from Section 4.3.1 imply a contradiction to the sharp higher order Euclidean Sobolev inequality, completing the proof of Theorem 3.3.

All constants C in Section 4.3 will be independent of the parameter α . All unmarked Laplacians and integrals without volume elements will be assumed to be with respect to the Riemannian metric g . We occasionally choose to state them anyway to reduce confusion when other metrics are involved (for example, Steps 1 and 2 in Section 4.3.2).

4.2 Preliminary Lemmas

In this section (M, g) will represent some smooth complete n -dimensional Riemannian manifold without boundary with bounded curvature.

In the following, given two tensors A, B we adopt the notation $A \star B$ to denote a linear combination of contractions of $A \otimes B$, possibly after raising and lowering indices using the metric and including the trivial linear combination $0 \cdot A \otimes B$. We write $A \star_k B$ in the case each element of the linear combination is covariant of degree k . Given a permutation σ of $\{1, \dots, k\}$ and a covariant k -tensor A we define $\sigma \cdot A$ by $(\sigma \cdot A)_{i_1 \dots i_k} = A_{i_{\sigma(1)} \dots i_{\sigma(k)}}$. We use the notation $A \star_k B$ to denote a sum of the form

$$\sum_{i=0}^m \sigma_i \cdot (A \star_k B)$$

for some $\{\sigma_i : 1 \leq i \leq m\}$ permutations on $\{1, \dots, k\}$. We have the relation $\nabla(A \star_k B) = \nabla A \star_{k+1} B + A \star_{k+1} \nabla B$.

All integrals will be assumed to be with respect to the Riemannian volume element dv_g unless explicitly stated otherwise. All Laplacians will be assumed to be with respect to the metric unless explicitly stated otherwise. We let R represent the Riemann curvature tensor.

Lemma 4.1. *Let $k \geq 0$ be an integer and $u \in H_k^2(M)$. Let σ be a permutation of $\{1, \dots, k\}$. Then*

$$\nabla^k u - \sigma \cdot \nabla^k u = \sum_{0 \leq l \leq k-3} \nabla^l R \star_k \nabla^{k-2-l} u.$$

Proof. Because the statement is immediate for $k = 0, 1, 2$, we operate under the assumption $k \geq 3$. We first consider what happens when σ is a transposition of two consecutive elements $(j, j+1)$ where $j \leq k-1$. In

this case, we have

$$\nabla_{i_1} \dots \nabla_{i_{j-1}} (\nabla_{i_j} \nabla_{i_{j+1}} - \nabla_{i_{j+1}} \nabla_{i_j}) \nabla_{i_{j+2}} \dots \nabla_{i_k} u = \nabla_{i_1} \dots \nabla_{i_{j-1}} \left(\sum_{l=j+2}^k R_{i_j i_{j+1} i_l}^\alpha \nabla_{i_{j+2}} \dots \nabla_{i_{l-1}} \nabla_\alpha \nabla_{i_{l+1}} \dots \nabla_{i_k} u \right)$$

This shows in this case by a Leibniz rule applied to covariant differentiation (see Gavrilov[16]) that

$$\nabla^k u - \sigma \cdot \nabla^k u = \sum_{0 \leq l \leq k-3} \nabla^l R \star_k \nabla^{k-2-l} u.$$

Now for general σ , we write $\sigma = \tau_q \dots \tau_1$ where τ_i is a transposition of consecutive elements. Let us also write $\sigma_p = \tau_p \dots \tau_1$ for $1 \leq p \leq q$. Then we write $\nabla^k u - \sigma \cdot \nabla^k u$ as a telescoping sum

$$\nabla^k u - \sigma_1 \nabla^k u + \sigma_1 \nabla^k u - \sigma_2 \nabla^k u \dots - \sigma_{q-1} \nabla^k u + \sigma_{q-1} \cdot \nabla^k u - \sigma \cdot \nabla^k u$$

and therefore by applying the case of a transposition to each difference we obtain

$$\nabla^k u - \sigma \cdot \nabla^k u = \sum_{0 \leq l \leq k-3} \nabla^l R \star_k \nabla^{k-2-l} u.$$

□

While Lemmas 4.2-4.9 are stated for functions in $u \in H_k^2(M)$, by density it will suffice to prove them for $u \in C_c^\infty(M)$, therefore all functions from now on will be assumed smooth.

Lemma 4.2. *Let $k \geq 0$ be an integer. Let*

$$(k_1, k_2) \in \{(k, k), (k-1, k+1), (k-1, k)\}.$$

For each $i \in \{1, 2\}$, let $j_i \leq \frac{k_i}{2}$, let T_{j_i} be the operator taking a k_i degree tensor and contracting on the last j_i pairs of indices, i.e. $T_{j_i}(\nabla^{k_i} u) = \nabla^{k_i-2j_i} \Delta^{j_i} u$, and let σ_i, σ'_i be permutations on $\{1, \dots, k_i\}$. Let S be an arbitrary compactly supported tensor of degree q such that $q \geq 2|(j_2 - \frac{k_2-k_1}{2}) - j_1|$. For each $k_2 - 2j_2$ degree covariant tensor A , let $A \star S$ denote some fixed contraction (possibly after raising and lowering indices using the metric) of $A \otimes S$ of degree $k_1 - 2j_1$ where all contractions either occur within S or take one index from each of A and S i.e. no contractions occur within A . Then there exists some C based on $n, k, \max |S|, \max |\nabla S|, \max |\nabla^2 S|$ and bounds for R and finitely many of its derivatives such that for all $u \in H_{k_2}^2(M)$,

$$\left| \int_M \langle T_{j_1}(\sigma_1 \cdot \nabla^{k_1} u), T_{j_2}(\sigma_2 \cdot \nabla^{k_2} u) \star S \rangle - \int_M \langle T_{j_1}(\sigma'_1 \cdot \nabla^{k_1} u), T_{j_2}(\sigma'_2 \cdot \nabla^{k_2} u) \star S \rangle \right| \leq C \sum_{i=0}^{k-1} \int_{\text{supp}(S)} |\nabla^i u|^2 \quad (4.1)$$

Proof. We note $\star$ immediately grants the existence of some operator $\star'$ such that for each $k_1 - 2j_1$ degree covariant tensor B , $B\star'S$ is some contraction of $B \otimes S$ of degree $k_2 - 2j_2$ (where all contractions are within S or take one index from B and one index from S) such that for all $k_2 - 2j_2$ degree covariant tensors A ,

$$\langle B, A\star S \rangle = \langle B\star' S, A \rangle.$$

We prove the statement in the case $\sigma'_1 = \sigma'_2 = \text{Id}$, the full statement then follows by a simple application of the triangle inequality.

We have

$$\begin{aligned} & \int_M \langle T_{j_1}(\sigma_1 \cdot \nabla^{k_1} u), T_{j_2}(\sigma_2 \cdot \nabla^{k_2} u) \star S \rangle - \int_M \langle T_{j_1}(\nabla^{k_1} u), T_{j_2}(\nabla^{k_2} u) \star S \rangle \\ &= \int_M \langle T_{j_1}(\sigma_1 \cdot \nabla^{k_1} u), T_{j_2}(\sigma_2 \cdot \nabla^{k_2} u) \star S \rangle - \int_M \langle T_{j_1}(\sigma_1 \cdot \nabla^{k_1} u), T_{j_2}(\nabla^{k_2} u) \star S \rangle \\ & \quad + \int_M \langle T_{j_1}(\sigma_1 \cdot \nabla^{k_1} u), T_{j_2}(\nabla^{k_2} u) \star S \rangle - \int_M \langle T_{j_1}(\nabla^{k_1} u), T_{j_2}(\nabla^{k_2} u) \star S \rangle. \end{aligned}$$

Then considering the first difference we have by applying Lemma 4.1 and extending our $\star$ notation such that instances of $A\star_k^{(i)} B$ represent fixed choices for $A\star_k B$,

$$\begin{aligned} & \int_M \langle T_{j_1}(\sigma_1 \cdot \nabla^{k_1} u), T_{j_2}(\sigma_2 \cdot \nabla^{k_2} u) \star S \rangle - \int_M \langle T_{j_1}(\sigma_1 \cdot \nabla^{k_1} u), T_{j_2}(\nabla^{k_2} u) \star S \rangle \\ &= \int_M \langle T_{j_1}(\sigma_1 \cdot \nabla^{k_1} u), T_{j_2}(\sigma_2 \cdot \nabla^{k_2} u - \nabla^{k_2} u) \star S \rangle \\ &= \int_M \langle T_{j_1}(\sigma_1 \cdot \nabla^{k_1} u), T_{j_2} \left(\sum_{0 \leq l \leq k_2-3} \nabla^l R \star_{k_2}^{(1)} \nabla^{k_2-2-l} u \right) \star S \rangle \\ &= \int_M \langle T_{j_1}(\sigma_1 \cdot \nabla^{k_1} u), T_{j_2} \left(\sum_{0 \leq l \leq k_2-3} \nabla^l R \star_{k_2}^{(1)} \nabla^{k_2-2-l} u \right) \star S \rangle - \int_M \langle T_{j_1} \nabla^{k_1} u, T_{j_2} \left(\sum_{0 \leq l \leq k_2-3} \nabla^l R \star_{k_2}^{(1)} \nabla^{k_2-2-l} u \right) \star S \rangle \\ & \quad + \int_M \langle T_{j_1} \nabla^{k_1} u, T_{j_2} \left(\sum_{0 \leq l \leq k_2-3} \nabla^l R \star_{k_2}^{(1)} \nabla^{k_2-2-l} u \right) \star S \rangle \\ &= \int_M \langle T_{j_1} \left(\sum_{0 \leq l \leq k_1-3} \nabla^l R \star_{k_1}^{(2)} \nabla^{k_1-2-l} u \right), T_{j_2} \left(\sum_{0 \leq l \leq k_2-3} \nabla^l R \star_{k_2}^{(1)} \nabla^{k_2-2-l} u \right) \star S \rangle \\ & \quad + \int_M \langle \nabla^{k_1-2j_1} \Delta^{j_1} u, T_{j_2} \left(\sum_{0 \leq l \leq k_2-3} \nabla^l R \star_{k_2}^{(1)} \nabla^{k_2-2-l} u \right) \star S \rangle. \end{aligned}$$

We clearly have

$$\left| \int_M \langle T_{j_1} \left(\sum_{0 \leq l \leq k_1-3} \nabla^l R \star_{k_1}^{(2)} \nabla^{k_1-2-l} u \right), T_{j_2} \left(\sum_{0 \leq l \leq k_2-3} \nabla^l R \star_{k_2}^{(1)} \nabla^{k_2-2-l} u \right) \star S \rangle \right| \leq C \sum_{i=0}^{k-1} \int_{\text{supp}(S)} |\nabla^i u|^2.$$

If $(k_1, k_2) \in \{(k-1, k+1), (k-1, k)\}$, then we also clearly have

$$\left| \int_M \langle \nabla^{k_1-2j_1} \Delta^{j_1} u, T_{j_2} \left(\sum_{0 \leq l \leq k_2-3} \nabla^l R \star_{k_2}^{(1)} \nabla^{k_2-2-l} u \right) \star S \rangle \right| \leq C \sum_{i=0}^{k-1} \int_{\text{supp}(S)} |\nabla^i u|^2 dv_g.$$

In the case $(k_1, k_2) = (k, k)$, we need to integrate by parts. If $2j_1 = k$ then after integrating by parts and applying the Leibniz rule, we obtain

$$\begin{aligned} & \left| \int_M \langle \nabla^{k_1-2j_1} \Delta^{j_1} u, T_{j_2} \left(\sum_{0 \leq l \leq k_2-3} \nabla^l R \star_{k_2}^{(1)} \nabla^{k_2-2-l} u \right) \star S \rangle \right| \\ &= \left| \int_M \langle \Delta^k u, \left(\sum_{0 \leq l \leq k-3} \nabla^l R \star_{k-j_2} \nabla^{k-2-l} u \right) \star S \rangle \right| \\ &\leq \left| \int_M \langle \nabla \Delta^{k-1} u, \left(\sum_{0 \leq l \leq k-2} \nabla^l R \star_{k-j_2+1} \nabla^{k-1-l} u \right) \star_1 S \rangle \right| \\ &\quad + \left| \int_M \langle \nabla \Delta^{k-1} u, \left(\sum_{0 \leq l \leq k-3} \nabla^l R \star_{k-j_2} \nabla^{k-2-l} u \right) \star_1 \nabla S \rangle \right| \\ &\leq C \sum_{i=0}^{k-1} \int_{\text{supp}(S)} |\nabla^i u|^2. \end{aligned}$$

If $2j_1 < k$ then a similar computation gives

$$\begin{aligned} & \left| \int_M \langle \nabla^{k-2j_1} \Delta^{j_1} u, T_{j_2} \left(\sum_{0 \leq l \leq k-3} \nabla^l R \star_k^{(1)} \nabla^{k-2-l} u \right) \star S \rangle \right| \\ &= \left| \int_M \langle \nabla^{k-2j_1-1} \Delta^{j_1} u, \text{div} \left(\left(\sum_{0 \leq l \leq k-3} \nabla^l R \star_{k-2j_2} \nabla^{k-2-l} u \right) \star S \right) \rangle \right| \\ &\leq C \sum_{i=0}^{k-1} \int_{\text{supp}(S)} |\nabla^i u|^2 \end{aligned}$$

We also have, applying Lemma 4.1,

$$\begin{aligned}
& \left| \int_M \langle T_{j_1}(\sigma_1 \cdot \nabla^k u), T_{j_2}(\nabla^k u) \star S \rangle - \int_M \langle T_{j_1}(\nabla^k u), T_{j_2}(\nabla^k u) \star S \rangle \right| \\
&= \left| \int_M \langle T_{j_1} \left(\sum_{0 \leq l \leq k-3} \nabla^l R \star_k \nabla^{k-2-l} u \right), T_{j_2}(\nabla^k u) \star S \rangle \right| \\
&= \left| \int_M \langle T_{j_1} \left(\sum_{0 \leq l \leq k-3} \nabla^l R \star_k \nabla^{k-2-l} u \right) \star' S, \nabla^{k-2j_2} \Delta^{j_2} u \rangle \right| \\
&\leq C \sum_{i=0}^{k-1} \int_{\text{supp}(S)} |\nabla^i u|^2
\end{aligned}$$

after integrating by parts similarly to above. Therefore we have obtained (4.1)

□

Lemma 4.3. *Let $k \geq 0$ be an integer and η be an arbitrary compactly supported smooth function. Then there exists C , based on n, k , and bounds for R and finitely many of its derivatives, bounds for η and finitely many of its derivatives such that for all $u \in H_k^2(M)$,*

$$\left| \int_M \eta |\nabla^k u|^2 - \int_M \eta (\Delta^{\frac{k}{2}} u)^2 \right| \leq C \sum_{i=0}^{k-1} \int_{\text{supp}(\eta)} |\nabla^i u|^2$$

Proof. Throughout the proof we use the notation $A \equiv B$ to mean

$$|A - B| \leq C \sum_{i=0}^{k-1} \int_{\text{supp}(\eta)} |\nabla^i u|^2$$

. This clearly satisfies the assumptions of an equivalence relation. We therefore perform the proof of Lemma 4.3 by finding expressions $A_1 \dots A_p$ such that

$$\int_M \eta |\nabla^k u|^2 \equiv A_1 \dots \equiv A_p \equiv \int_M \eta (\Delta^{\frac{k}{2}} u)^2.$$

We prove by induction. The statement is immediate for $k = 0$ and $k = 1$. Now let $k \geq 2$ and assume the statement holds true for $k - 2$. We apply Lemma 4.2 to obtain

$$\int_M \eta |\nabla^k u|^2 = \int_M \eta \nabla^{i_1} \dots \nabla^{i_k} u \nabla_{i_1} \dots \nabla_{i_k} u \equiv \int_M \eta \nabla^{i_k} \nabla^{i_1} \dots \nabla^{i_{k-1}} u \nabla_{i_1} \dots \nabla_{i_k} u \quad (4.2)$$

We then integrate by parts on the right hand side of (4.2) to obtain

$$\int_M \eta \nabla^{i_k} \nabla^{i_1} \dots \nabla^{i_{k-1}} u \nabla_{i_1} \dots \nabla_{i_k} u = - \int_M \nabla^{i_k} \eta \nabla^{i_1} \dots \nabla^{i_{k-1}} u \nabla_{i_1} \dots \nabla_{i_k} u - \int_M \eta \nabla^{i_1} \dots \nabla^{i_{k-1}} u \nabla^{i_k} \nabla_{i_1} \dots \nabla_{i_k} u \quad (4.3)$$

Changing in order of indices on the first term on the right hand side of (4.3) by Lemma 4.2 and integrating by parts we obtain

$$\begin{aligned} - \int_M \nabla^{i_k} \eta \nabla^{i_1} \dots \nabla^{i_{k-1}} u \nabla_{i_1} \dots \nabla_{i_k} u &\equiv - \int_M \nabla^{i_k} \eta \nabla^{i_1} \dots \nabla^{i_{k-1}} u \nabla_{i_k} \nabla_{i_1} \dots \nabla_{i_{k-1}} u \\ &= - \int_M \Delta \eta \nabla^{i_1} \dots \nabla^{i_{k-1}} u \nabla_{i_1} \dots \nabla_{i_{k-1}} u + \int_M \nabla^{i_k} \eta \nabla_{i_k} \nabla^{i_1} \dots \nabla^{i_{k-1}} u \nabla_{i_1} \dots \nabla_{i_{k-1}} u \end{aligned}$$

and therefore after subtracting the second term over, we obtain

$$\left| \int_M \nabla^{i_k} \eta \nabla^{i_1} \dots \nabla^{i_{k-1}} u \nabla_{i_1} \dots \nabla_{i_k} u \right| \leq C \sum_{i=0}^{k-1} \int_{\text{supp}(\eta)} |\nabla^i u|^2. \quad (4.4)$$

Therefore

$$\int_M \eta \nabla^{i_k} \nabla^{i_1} \dots \nabla^{i_{k-1}} u \nabla_{i_1} \dots \nabla_{i_k} u \equiv - \int_M \eta \nabla^{i_1} \dots \nabla^{i_{k-1}} u \nabla^{i_k} \nabla_{i_1} \dots \nabla_{i_k} u. \quad (4.5)$$

For the right hand side of (4.5), we once again change the order of the indices with Lemma 4.2 and integrate by parts to obtain

$$\begin{aligned} - \int_M \eta \nabla^{i_1} \dots \nabla^{i_{k-1}} u \nabla^{i_k} \nabla_{i_1} \dots \nabla_{i_k} u &\equiv - \int_M \eta \nabla^{i_1} \dots \nabla^{i_{k-1}} u \nabla_{i_1} \dots \nabla_{i_{k-1}} \Delta u \\ &\equiv - \int_M \eta \nabla^{i_1} \dots \nabla^{i_{k-1}} u \nabla_{i_{k-1}} \nabla_{i_1} \dots \nabla_{i_{k-2}} \Delta u \\ &= \int_M \nabla_{i_{k-1}} \eta \nabla^{i_1} \dots \nabla^{i_{k-1}} u \nabla_{i_1} \dots \nabla_{i_{k-2}} \Delta u + \int_M \eta \nabla_{i_{k-1}} \nabla^{i_1} \dots \nabla^{i_{k-1}} u \nabla_{i_1} \dots \nabla_{i_{k-2}} \Delta u \\ &\equiv \int_M \nabla_{i_{k-1}} \eta \nabla^{i_1} \dots \nabla^{i_{k-1}} u \nabla_{i_1} \dots \nabla_{i_{k-2}} \Delta u + \int_M \eta \nabla^{i_1} \dots \nabla^{i_{k-2}} \Delta u \nabla_{i_1} \dots \nabla_{i_{k-2}} \Delta u \end{aligned} \quad (4.6)$$

Then for the first term of (4.6) we again reorganize the indices to obtain

$$\int_M \nabla_{i_{k-1}} \eta \nabla^{i_1} \dots \nabla^{i_{k-1}} u \nabla_{i_1} \dots \nabla_{i_{k-2}} \Delta u \equiv \int_M \nabla_{i_{k-1}} \eta \nabla^{i_1} \dots \nabla^{i_{k-1}} u \Delta \nabla_{i_1} \dots \nabla_{i_{k-2}} u$$

Integrating by parts gives

$$\begin{aligned} \int_M \nabla_{i_{k-1}} \eta \nabla^{i_1} \dots \nabla^{i_{k-1}} u \Delta \nabla_{i_1} \dots \nabla_{i_{k-2}} u &= \int_M \nabla^{i_k} \nabla_{i_{k-1}} \eta \nabla^{i_1} \dots \nabla^{i_{k-1}} u \nabla_{i_k} \nabla_{i_1} \dots \nabla_{i_{k-2}} u \\ &\quad + \int_M \nabla_{i_{k-1}} \eta \nabla^{i_k} \nabla^{i_1} \dots \nabla^{i_{k-1}} u \nabla_{i_k} \nabla_{i_1} \dots \nabla_{i_{k-2}} u \end{aligned}$$

We clearly have

$$\int_M \nabla^{i_k} \nabla_{i_{k-1}} \eta \nabla^{i_1} \dots \nabla^{i_{k-1}} u \nabla_{i_k} \nabla_{i_1} \dots \nabla_{i_{k-2}} u \leq C \sum_{i=0}^{k-1} \int_{\text{supp}(\eta)} |\nabla^i u|^2$$

and by reorganizing indices with Lemma 4.2 and then relabeling the indices we obtain

$$\begin{aligned} \int_M \nabla_{i_{k-1}} \eta \nabla^{i_k} \nabla^{i_1} \dots \nabla^{i_{k-1}} u \nabla_{i_k} \nabla_{i_1} \dots \nabla_{i_{k-2}} u &\equiv \int_M \nabla_{i_{k-1}} \eta \nabla^{i_1} \dots \nabla^{i_{k-2}} \nabla^{i_k} \nabla^{i_{k-1}} u \nabla_{i_1} \dots \nabla_{i_{k-2}} \nabla_{i_k} u \\ &= \int_M \nabla^{i_k} \eta \nabla^{i_1} \dots \nabla^{i_{k-1}} u \nabla_{i_1} \dots \nabla_{i_k} u \\ &\leq C \sum_{i=0}^{k-1} \int_{\text{supp}(\eta)} |\nabla^i u|^2 \end{aligned}$$

by (4.4). This shows that

$$- \int_M \eta \nabla^{i_1} \dots \nabla^{i_{k-1}} u \nabla^{i_k} \nabla_{i_1} \dots \nabla_{i_k} u \equiv \int_M \eta \nabla^{i_1} \dots \nabla^{i_{k-2}} \Delta u \nabla_{i_1} \dots \nabla_{i_{k-2}} \Delta u = \int_M \eta |\nabla^{k-2}(\Delta u)|^2 \quad (4.7)$$

By the induction hypothesis, we have

$$\left| \int_M \eta |\nabla^{k-2}(\Delta u)|^2 - \int_M \eta (\Delta^{\frac{k}{2}} u)^2 \right| \leq C \sum_{i=0}^{k-3} \int_{\text{supp}(\eta)} |\nabla^i(\Delta u)|^2 \leq C \sum_{i=0}^{k-1} \int_{\text{supp}(\eta)} |\nabla^i u|^2$$

which implies

$$\int_M \eta |\nabla^{k-2}(\Delta u)|^2 \equiv \int_M \eta (\Delta^{\frac{k}{2}} u)^2. \quad (4.8)$$

Therefore by (4.2), (4.5), (4.7) and (4.8) we have proven Lemma 4.3. □

Lemma 4.4. *Let η be a compactly supported smooth function, $k \geq 0$ be an integer, and $r > 0$ be a real number. Then there exists a constant C based on n , k , bounds for R and finitely many of its derivatives, r , and bounds for η and finitely many of its derivatives such that for all $u \in H_k^2(M)$,*

$$\left| \int_M \eta |\nabla^k u|^2 dv_g - \int_M \eta (\Delta^{\frac{k}{2}} u)^2 \right| \leq C \sum_{i=0}^{k-1} \int_{B_r(\text{supp}(\eta))} (\Delta^{\frac{i}{2}} u)^2$$

Proof. We prove by strong induction. The statement is immediate for $k = 0$ and $k = 1$ as the left hand side vanishes. Now fix $k \geq 2$ and assume the statement holds for all $j < k$. Let η_1 be a smooth nonnegative function such that $\eta_1 = 1$ on $\text{supp}(\eta)$ and $\eta_1 = 0$ on $M \setminus B_{\frac{r}{2}}(\text{supp}(\eta))$.

Then we apply Lemma 4.3 and invoke the (strong) induction hypothesis with η_1 taking the place of η and $\frac{r}{2}$ taking the place of r to obtain

$$\begin{aligned}
\left| \int_M \eta |\nabla^k u|^2 - \int_M \eta (\Delta^{\frac{k}{2}} u)^2 \right| &\leq C \sum_{i=0}^{k-1} \int_{\text{supp}(\eta)} |\nabla^i u|^2 \\
&\leq C \sum_{i=0}^{k-1} \int_M \eta_1 |\nabla^i u|^2 \\
&\leq C \sum_{i=0}^{k-1} \left(\int_M \eta_1 (\Delta^{\frac{i}{2}} u)^2 + \sum_{j=0}^{i-1} \int_{B_{\frac{r}{2}}(\text{supp}(\eta_1))} (\Delta^{\frac{j}{2}} u)^2 \right) \\
&\leq C \sum_{i=0}^{k-1} \int_{B_r(\text{supp}(\eta))} (\Delta^{\frac{i}{2}} u)^2
\end{aligned}$$

□

Lemma 4.5. *Let $k \geq 0$ be an integer. Let η be a compactly supported smooth function and $r > 0$ be a real number. Then there exists a constant C based on n , k , bounds for R and finitely many of its derivatives, r , and bounds for η and finitely many of its derivatives such that*

$$\left| \int_M (\Delta^{\frac{k}{2}}(\eta u))^2 - \int_M \eta^2 (\Delta^{\frac{k}{2}} u)^2 \right| \leq C \sum_{i=0}^{k-1} \int_{B_r(\text{supp}(\eta))} (\Delta^{\frac{i}{2}} u)^2$$

Proof. First suppose k is even. We write $k = 2l$. We recall for any tensors S_1, S_2 of the same type we have

$$\Delta \langle S_1, S_2 \rangle = \langle S_1 \Delta S_2 \rangle - 2 \langle \nabla S_1, \nabla S_2 \rangle + \langle S_2 \Delta S_1 \rangle$$

Let us define operators T_i on tensors for $i \in \{0, 1, 2\}$ by $T_0 = \text{Id}$, $T_1 = \nabla$, $T_2 = \Delta$. Then for any multi-index $\beta = (\beta_1, \dots, \beta_l) \in \{0, 1, 2\}^l$, we define $T_\beta = T_{\beta_l} \dots T_{\beta_1}$. Given β , we define $\beta' = (2 - \beta_1, \dots, 2 - \beta_l)$. Finally we define j_β to be the number of 1's appearing in β . We note $j_\beta = j_{\beta'}$, therefore if S_1, S_2 are tensors of the same type, so are $T_\beta S_1$ and $T_{\beta'} S_2$. Then we write the l th Laplacian of the product of two functions as

$$\Delta^l(\eta u) = \sum_{\beta \in \{0,1,2\}^l} (-2)^{j_\beta} \langle T_\beta u, T_{\beta'} \eta \rangle_g. \tag{4.9}$$

This formula can easily be proven by induction.

We then have

$$\int_M (\Delta^l(\eta u))^2 = \int_M \sum_{\alpha, \beta \in \{0,1,2\}^l} (-2)^{j_\beta + j_\alpha} \langle T_\beta u, T_{\beta'} \eta \rangle_g \langle T_\alpha u, T_{\alpha'} \eta \rangle_g \tag{4.10}$$

It follows that to estimate $|\int_M (\Delta^l(\eta u))^2 - \int_M \eta^2 (\Delta^l u)^2|$, it suffices to estimate the terms on the right hand side of (4.10) apart from the principal term of $\int_M \eta^2 (\Delta^l u)^2$ corresponding to the case $|\alpha| = |\beta| = 2l$. We separate this sum into three cases modulo symmetry in α and β .

Case 1: $|\alpha| \leq 2l - 1$ and $|\beta| \leq 2l - 1$. In this case, on $\text{supp}(\eta)$, we have

$$\begin{aligned} \left| (-2)^{j_\beta + j_\alpha} \langle T_\beta u, T_{\beta'} \eta \rangle_g \langle T_\alpha u, T_{\alpha'} \eta \rangle_g \right| &\leq C |T_\alpha u| |T_\beta u| \\ &\leq C (|T_\alpha u|^2 + |T_\beta u|^2) \\ &\leq C (|\nabla^{|\alpha|} u|^2 + |\nabla^{|\beta|} u|^2) \end{aligned}$$

Letting η_1 be a smooth nonnegative function such that $\eta_1 = 1$ on $\text{supp}(\eta)$ and $\eta_1 = 0$ on $M \setminus B_{\frac{r}{2}}(\text{supp}(\eta))$ and applying Lemma 4.4 with $\frac{r}{2}$ as our value of r results in

$$\left| \int_M (-2)^{j_\beta + j_\alpha} \langle T_\beta u, T_{\beta'} \eta \rangle_g \langle T_\alpha u, T_{\alpha'} \eta \rangle_g \right| \leq C \int_M \eta_1 (|\nabla^{|\alpha|} u|^2 + |\nabla^{|\beta|} u|^2) \leq C \sum_{i=0}^{2l-1} \int_{B_r(\text{supp}(\eta))} (\Delta^{\frac{i}{2}} u)^2$$

Case 2: $|\alpha| \leq 2l - 2$ and $|\beta| = 2l$. In this case we integrate by parts and argue as in Case 1 to obtain

$$\begin{aligned} \left| \int_M (-2)^{j_\alpha} \eta \Delta^l u \langle T_\alpha u, T_{\alpha'} \eta \rangle_g \right| &= \left| (-2)^{j_\alpha} \int_M \langle \nabla \Delta^{l-1} u, \nabla \eta \rangle \langle T_\alpha u, T_{\alpha'} \eta \rangle + (-2)^{j_\alpha} \int_M \eta \langle \nabla^i \Delta^{l-1} u \nabla_i T_\alpha u, T_{\alpha'} \eta \rangle_g \right. \\ &\quad \left. + (-2)^{j_\alpha} \int_M \eta \langle \nabla^i \Delta^{l-1} u \nabla_i T_{\alpha'} \eta, T_\alpha u \rangle_g \right| \\ &\leq C \sum_{i=0}^{2l-1} \int_{B_r(\text{supp}(\eta))} (\Delta^{\frac{i}{2}} u)^2 \end{aligned}$$

Case 3: $|\alpha| = 2l - 1$ and $|\beta| = 2l$. In this case we have

$$\int_M (-2)^{j_\beta + j_\alpha} \langle T_\beta u, T_{\beta'} \eta \rangle_g \langle T_\alpha u, T_{\alpha'} \eta \rangle_g = -2 \int_M \eta \Delta^l u \langle \Delta^{\gamma_1} \nabla \Delta^{\gamma_2} u, \nabla \eta \rangle$$

for some $\gamma_1 + \gamma_2 = l - 1$. In the following computation (and the rest of the proof), we extend our notation $A \equiv B$ from the proof of Lemma 4.3 to mean

$$|A - B| \leq C \sum_{i=0}^{2l-1} \int_{\text{supp}(\eta)} |\nabla^i u|^2 \leq C \sum_{i=0}^{2l-1} \int_{B_r(\text{supp}(\eta))} |\Delta^{\frac{i}{2}} u|^2$$

where the last inequality is by an application of Lemma 4.4. Integrating by parts and reorganizing the indices

by Lemma 4.2 we obtain

$$\begin{aligned}
\int_M \eta \Delta^l u \langle \Delta^{\gamma_1} \nabla \Delta^{\gamma_2} u, \nabla \eta \rangle &= \frac{1}{2} \int_M \Delta^l u \langle \Delta^{\gamma_1} \nabla \Delta^{\gamma_2} u, \nabla(\eta^2) \rangle \\
&\equiv \frac{1}{2} \int_M \Delta^l u \langle \nabla \Delta^{l-1} u \nabla(\eta^2) \rangle \\
&= \frac{1}{2} \int_M \langle \nabla^2 \Delta^{l-1} u, \nabla \Delta^{l-1} u \otimes \nabla(\eta^2) \rangle + \frac{1}{2} \int_M \langle \nabla^2(\eta^2), \nabla \Delta^{l-1} u \otimes \nabla \Delta^{l-1} u \rangle \\
&= \frac{1}{4} \int_M (\Delta^{\frac{2l-1}{2}} u)^2 \Delta(\eta^2) + \frac{1}{2} \int_M \langle \nabla \Delta^{l-1} u \otimes \nabla \Delta^{l-1} u, \nabla^2(\eta^2) \rangle.
\end{aligned}$$

Arguing as in Case 1 then shows

$$\left| \int_M (-2)^{j_\beta + j_\alpha} \langle T_\beta u, T_\beta \eta \rangle_g \langle T_\alpha u, T_\alpha \eta \rangle_g \right| \leq C \sum_{i=0}^{2l-1} \int_{B_r(\text{supp}(\eta))} (\Delta^{\frac{i}{2}} u)^2$$

and therefore Lemma 4.5 holds.

Now suppose k is odd. We write $k = 2l + 1$. Using our computations for $\Delta^l(\eta u)$, we obtain

$$\begin{aligned}
\int_M |\nabla \Delta^l(\eta u)|^2 &= \int_M \sum_{\alpha, \beta \in \{0,1,2\}^l} (-2)^{j_\beta + j_\alpha} \left(\langle \nabla^i T_\beta u, T_\beta \eta \rangle_g \langle \nabla_i T_\alpha u, T_\alpha \eta \rangle_g + \langle T_\beta u, \nabla^i T_\beta \eta \rangle_g \langle \nabla_i T_\alpha u, T_\alpha \eta \rangle_g \right) dv_g \\
&\quad + \int_M \sum_{\alpha, \beta \in \{0,1,2\}^l} (-2)^{j_\beta + j_\alpha} \left(\langle \nabla^i T_\beta u, T_\beta \eta \rangle_g \langle T_\alpha u, \nabla_i T_\alpha \eta \rangle_g + \langle T_\beta u, \nabla^i T_\beta \eta \rangle_g \langle T_\alpha u, \nabla_i T_\alpha \eta \rangle_g \right) dv_g.
\end{aligned}$$

We once again have a principal term of $\int_M \eta^2 |\nabla \Delta^l u|^2$ as the first of the four terms when $|\alpha| = |\beta| = 2l$. Therefore to prove Lemma 4.5 we once again split the other terms into cases modulo symmetry and estimate.

Case 1: $|\alpha| \leq 2l - 1$ and $|\beta| \leq 2l - 1$. The details are virtually identical to the proof of Case 1 when k is even.

Case 2: $|\alpha| \leq 2l - 2$ and $|\beta| = 2l$. In this case, we can still apply the argument from Case 1 to the second term in each row of the above formula. We perform the argument from Case 2 when k is even on the first term in the first row, the first term in the second row is handled similarly. Integrating by parts and estimating as in Case 1 results in

$$\begin{aligned}
\left| (-2)^{j_\alpha} \int_M \eta \nabla^i \Delta^l u \langle \nabla_i T_\alpha u, T_\alpha \eta \rangle_g \right| &= \left| (-2)^{j_\alpha} \int_M \Delta^l u \nabla^i \eta \langle \nabla_i T_\alpha u, T_\alpha \eta \rangle_g + (-2)^{j_\alpha} \int_M \eta \Delta^l u \langle \Delta T_\alpha u, T_\alpha \eta \rangle_g \right. \\
&\quad \left. + (-2)^{j_\alpha} \int_M \eta \Delta^l u \langle \nabla T_\alpha u, \nabla T_\alpha \eta \rangle_g \right| \\
&\leq C \sum_{i=0}^{2l} \int_{B_r(\text{supp}(\eta))} (\Delta^{\frac{i}{2}} u)^2
\end{aligned}$$

Case 3: $|\alpha| = 2l - 1$ and $|\beta| = 2l$. Once again the second term in each row of the sum can be estimated using the methods from Case 1. The first term in the second row can be estimated using the method from Case 2, leaving us with only one term to consider. We calculate, writing $T_\alpha = \Delta^{\gamma_1} \nabla \Delta^{\gamma_2}$ where $\gamma_1 + \gamma_2 = l - 1$ as in Case 3 of (i), and integrating by parts and changing the order of indices with Lemma 4.2,

$$\begin{aligned}
\int_M \eta \nabla^i \Delta^l u \nabla_i \Delta^{\gamma_1} \nabla^j \Delta^{\gamma_2} u \nabla_j \eta &= \frac{1}{2} \int_M \nabla^i \Delta^l u \nabla_i \Delta^{\gamma_1} \nabla^j \Delta^{\gamma_2} u \nabla_j (\eta^2) \\
&\equiv \frac{1}{2} \int_M \nabla^i \Delta^l u \nabla_i \nabla^j \Delta^{l-1} u \nabla_j (\eta^2) \\
&= \frac{1}{2} \int_M \Delta^l u \langle \Delta \nabla \Delta^{l-1} u \nabla (\eta^2) \rangle - \frac{1}{2} \int_M \Delta^l u \langle \nabla^2 \Delta^{l-1} u, \nabla^2 (\eta^2) \rangle \\
&\equiv \frac{1}{2} \int_M \Delta^l u \langle \nabla \Delta^l u \nabla (\eta^2) \rangle - \frac{1}{2} \int_M \Delta^l u \langle \nabla^2 \Delta^{l-1} u, \nabla^2 (\eta^2) \rangle \\
&= \frac{1}{4} \int_M (\Delta^l u)^2 \Delta (\eta^2) - \frac{1}{2} \int_M \Delta^l u \langle \nabla^2 \Delta^{l-1} u, \nabla^2 (\eta^2) \rangle \\
&\leq C \sum_{i=0}^{2l} \int_{B_r(\text{supp}(\eta))} (\Delta^{\frac{i}{2}} u)^2
\end{aligned}$$

Case 4: $|\alpha| = 2l$ and $|\beta| = 2l$. The first term is the previously mentioned principal term. The second term in the first row and first term in the second row can be estimated using the method from Case 3. The second term in the second row can be estimated using the method from Case 1. This completes the proof of Lemma 4.5. □

Lemma 4.6. *Let η be a compactly supported smooth function and $r > 0$ be a real number. Let $i_1, i_2, j_1, j_2 \in \frac{1}{2}\mathbb{N}$ be such that $i_1 + i_2 = j_1 + j_2 = k$. Then there exists a constant C based on n, k , bounds for R and finitely many of its derivatives, r , and bounds for η and finitely many of its derivatives such that*

$$\left| \int_M \eta \Delta^{i_1} u \Delta^{i_2} u - \int_M \eta \Delta^{j_1} u \Delta^{j_2} u \right| \leq C \sum_{i=0}^{k-1} \int_{B_r(\text{supp}(\eta))} (\Delta^{\frac{i}{2}} u)^2.$$

Proof. We prove

$$\left| \int_M \eta u \Delta^k u - \int_M \eta \Delta u \Delta^{k-1} u \right| \leq C \sum_{i=0}^{k-1} \int_{B_r(\text{supp}(\eta))} (\Delta^{\frac{i}{2}} u)^2 \tag{4.11}$$

and

$$\left| \int_M \eta u \Delta^k u - \int_M \eta \langle \nabla u, \nabla \Delta^{k-1} u \rangle \right| \leq C \sum_{i=0}^{k-1} \int_{B_r(\text{supp}(\eta))} (\Delta^{\frac{i}{2}} u)^2. \tag{4.12}$$

The full statement then follows by a simple induction.

We calculate

$$\int_M \eta u \Delta^k u = \int_M \Delta(\eta u) \Delta^{k-1} u = \int_M \eta \Delta u \Delta^{k-1} u - 2 \int_M \langle \nabla \eta, \nabla u \rangle \Delta^{k-1} u + \int_M u \Delta \eta \Delta^{k-1} u. \quad (4.13)$$

We integrate by parts on the final term of (4.13) and we have

$$\int_M u \Delta \eta \Delta^{k-1} u = \int_M \Delta^{\frac{k-1}{2}} (u \Delta \eta) \Delta^{\frac{k-1}{2}} u \leq C \sum_{i=0}^{k-1} \int_{B_r(\text{supp}(\eta))} (\Delta^{\frac{i}{2}} u)^2$$

after expanding with (4.9) and applying Lemma 4.4. For the second term of (4.13), we integrate by parts and apply (4.9) to obtain

$$\begin{aligned} 2 \int_M \langle \nabla \eta, \nabla u \rangle \Delta^{k-1} u &= 2 \int_M \Delta^{\frac{k-1}{2}} \langle \nabla \eta, \nabla u \rangle \Delta^{\frac{k-1}{2}} u \\ &= 2 \int_M \langle \nabla \eta, \Delta^{\frac{k-1}{2}} \nabla u \rangle \Delta^{\frac{k-1}{2}} u + \int_M \sum_{\beta \in \{0,1,2\}^k \setminus \{(2,\dots,2)\}} (-2)^{j_\beta} \langle T_\beta u, T_{\beta'} \eta \rangle_g \Delta^{\frac{k-1}{2}} u \end{aligned}$$

and computations as in the proof of Case 1 in Lemma 4.5 show

$$\int_M \sum_{\beta \in \{0,1,2\}^k \setminus \{(2,\dots,2)\}} (-2)^{j_\beta} \langle T_\beta u, T_{\beta'} \eta \rangle_g \Delta^{\frac{k-1}{2}} u \leq C \sum_{i=0}^{k-1} \int_{B_r(\text{supp}(\eta))} (\Delta^{\frac{i}{2}} u)^2.$$

For the remaining term, we reorganize the indices with Lemma 4.2 to obtain

$$\int_M \langle \nabla \eta, \Delta^{\frac{k-1}{2}} \nabla u \rangle \Delta^{\frac{k-1}{2}} u \equiv \int_M \langle \nabla \eta, \nabla \Delta^{\frac{k-1}{2}} u \rangle \Delta^{\frac{k-1}{2}} u = \frac{1}{2} \int_M \Delta \eta (\Delta^{\frac{k-1}{2}} u)^2 \leq C \int_{\text{supp}(\eta)} (\Delta^{\frac{k-1}{2}} u)^2$$

and we have shown (4.11).

For (4.12), we integrate by parts twice to obtain

$$\begin{aligned} \int_M \eta u \Delta^k u &= \int_M \langle \nabla(\eta u), \nabla \Delta^{k-1} u \rangle \\ &= \int_M \eta \langle \nabla u, \nabla \Delta^{k-1} u \rangle + \int_M u \langle \nabla \eta, \nabla \Delta^{k-1} u \rangle \\ &= \int_M \eta \langle \nabla u, \nabla \Delta^{k-1} u \rangle + \int_M \langle \nabla u, \nabla \eta \rangle \Delta^{k-1} u + \int_M u \Delta \eta \Delta^{k-1} u. \end{aligned}$$

The latter two terms are identical to the latter two terms of (4.13) and we conclude by applying the same computations. $\square$

Lemma 4.7. *Let $k \geq 0$ be an integer and i_1, j_1, i_2, j_2 be integers such such that $i_1 + i_2 = j_1 + j_2 = k$. Let $r > 0$ be a real number. Let η be a compactly supported smooth function. Then there exists C based on n, k, r ,*

bounds for R and finitely many of its derivatives, and bounds for η and finitely many of its derivatives such that for all $u \in H_k^2(M)$ and $\beta \geq 0$,

$$\int_M \eta(\Delta + \beta)^{i_1} u (\Delta + \beta)^{i_2} u \leq \int_M \eta(\Delta + \beta)^{j_1} u (\Delta + \beta)^{j_2} u + C \int_{B_r(\text{supp}(\eta))} \sum_{i=0}^{k-1} \sum_{j=0}^{k-1-i} \beta^{k-1-i-j} (\Delta^{\frac{j}{2}} u)^2$$

where C is independent of u and β .

Proof. We begin by stating a simple but useful reindexing identity. Let a_i, b_i be real numbers for $0 \leq i \leq k$.

Then

$$\sum_{i=0}^k \sum_{j=0}^{k-i} a_{k-i-j} b_j = \sum_{i=0}^k a_{k-i} \sum_{j=0}^i b_j \quad (4.14)$$

To prove this we use the substitution $i' = i - j$ to obtain

$$\sum_{i=0}^k a_{k-i} \sum_{j=0}^i b_j = \sum_{i=0}^k \sum_{j=0}^k a_{k-i} b_j \mathbb{1}_{j \leq i} = \sum_{j=0}^k \sum_{i=0}^k a_{k-i} b_j \mathbb{1}_{j \leq i} = \sum_{j=0}^k \sum_{i'=0}^{k-j} a_{k-i'-j} b_j = \sum_{i'=0}^k \sum_{j=0}^{k-i'} a_{k-i'-j} b_j$$

Now to prove Lemma 4.7, we will show for all k_1, k_2 such that $k_1 + k_2 = k$ and $k_1 < k_2$

$$\left| \int_M \eta(\Delta + \beta)^{k_1} u (\Delta + \beta)^{k_2} - \int_M \eta(\Delta + \beta)^{k_1+1} u (\Delta + \beta)^{k_2-1} \right| \leq C \int_{B_r(\text{supp}(\eta))} \sum_{i=0}^{k-1} \sum_{j=0}^{k-1-i} \beta^{k-1-i-j} (\Delta^{\frac{j}{2}} u)^2 \quad (4.15)$$

and full statement immediately follows by a simple induction. First we prove the case of (4.15) where $k_1 = 0, k_2 = k$. We expand, defining $c_{i,k}$ such that $(\Delta + \beta)^k = \sum_{i=0}^k c_{i,k} \beta^{k-i} \Delta^i u$ to get

$$\begin{aligned} \int_M \eta u (\Delta + \beta)^k u &= \int_M \eta u \Delta ((\Delta + \beta)^{k-1} u) + \int_M \beta \eta u (\Delta + \beta)^{k-1} u \\ &= \int_M \eta u \Delta \left(\sum_{i=0}^{k-1} c_{i,k-1} \beta^{k-1-i} \Delta^i u \right) + \int_M \beta \eta u (\Delta + \beta)^{k-1} u \\ &= \int_M \sum_{i=0}^{k-1} c_{i,k-1} \beta^{k-1-i} \eta u \Delta^{i+1} u + \int_M \beta \eta u (\Delta + \beta)^{k-1} u \end{aligned}$$

We then apply Lemma 4.6 to each term and reindex with (4.14) to obtain

$$\begin{aligned}
\int_M \sum_{i=0}^{k-1} c_{i,k-1} \beta^{k-1-i} \eta u \Delta^{i+1} u + \int_M \beta \eta u (\Delta + \beta)^{k-1} u &\leq \int_M \eta \Delta u \sum_{i=0}^{k-1} c_{i,k-1} \beta^{k-1-i} \Delta^i u + \beta \eta u (\Delta + \beta)^{k-1} u \\
&\quad + C \int_{B_r(\text{supp}(\eta))} \sum_{i=0}^{k-1} \beta^{k-1-i} \sum_{j=0}^i (\Delta^{\frac{j}{2}})^2 u \\
&= \int_M \eta (\Delta + \beta) u (\Delta + \beta)^{k-1} u \\
&\quad + C \int_{B_r(\text{supp}(\eta))} \sum_{i=0}^{k-1} \sum_{j=0}^{k-1-i} \beta^{k-1-i-j} (\Delta^{\frac{j}{2}} u)^2
\end{aligned}$$

Now let k_1, k_2 be such that $k_1 + k_2 = k$ and assume $k_1 < k_2$. We write

$$\int_M \eta (\Delta + \beta)^{k_1} u (\Delta + \beta)^{k_2} u = \int_M \eta (\Delta + \beta)^{k_1} u (\Delta + \beta)^{k_2-k_1} (\Delta + \beta)^{k_1} u$$

and apply the previous special case replacing u with $(\Delta + \beta)^{k_1} u$ and r with $\frac{r}{3}$. We obtain

$$\begin{aligned}
\int_M \eta (\Delta + \beta)^{k_1} u (\Delta + \beta)^{k_2} u &\leq \int_M \eta (\Delta + \beta)^{k_1+1} u (\Delta + \beta)^{k_2-1} u \\
&\quad + C \sum_{i=0}^{k_2-k_1-1} \sum_{j=0}^{k_2-k_1-1-i} \beta^{k_2-k_1-1-i-j} \int_{B_{\frac{r}{3}}(\text{supp}(\eta))} (\Delta^{\frac{j}{2}} (\Delta + \beta)^{k_1} u)^2 \quad (4.16)
\end{aligned}$$

Let η_1 be a smooth function equal to 1 on $B_{\frac{r}{3}}(\text{supp}(\eta))$ and equal to 0 on $M \setminus B_{\frac{2r}{3}}(\text{supp}(\eta))$. We fix an arbitrary i and j and consider $\beta^{k_2-k_1-1-i-j} \int_M \eta_1 (\Delta^{\frac{j}{2}} (\Delta + \beta)^{k_1} u)^2$. We expand

$$\begin{aligned}
\int_M \eta_1 (\Delta^{\frac{j}{2}} (\Delta + \beta)^{k_1} u)^2 &= \int_M \eta_1 \left(\sum_{l=0}^{k_1} c_{l,k_1} \beta^{k_1-l} \Delta^{l+\frac{j}{2}} u \right)^2 \\
&= \sum_{0 \leq l_1, l_2 \leq k_1} \int_M \eta_1 c_{l_1, k_1} c_{l_2, k_1} \beta^{2k_1-l_1-l_2} \Delta^{l_1+\frac{j}{2}} u \Delta^{l_2+\frac{j}{2}} u \quad (4.17)
\end{aligned}$$

We rewrite this sum as

$$\sum_{0 \leq l_1, l_2 \leq k_1} \int_M \eta_1 c_{l_1, k_1} c_{l_2, k_1} \beta^{2k_1-l_1-l_2} \Delta^{l_1+\frac{j}{2}} u \Delta^{l_2+\frac{j}{2}} u = \sum_{l=0}^{2k_1} \sum_{\substack{0 \leq l_1, l_2 \leq k_1 \\ l_1+l_2=l}} \int_M \eta_1 c_{l_1, k_1} c_{l_2, k_1} \beta^{2k_1-l} \Delta^{l_1+\frac{j}{2}} u \Delta^{l_2+\frac{j}{2}} u \quad (4.18)$$

Fixing some l, l_1, l_2 such that $l_1 + l_2 = l$ and applying Lemma 4.6 with η_1 replacing η , and $\frac{r}{3}$ replacing r

we have

$$\begin{aligned} \int_M \eta_1 c_{l_1, k_1} c_{l_2, k_1} \beta^{2k_1-l} \Delta^{l_1+\frac{j}{2}} u \Delta^{l_2+\frac{j}{2}} u &\leq c_{l_1, k_1} c_{l_2, k_1} \beta^{2k_1-l} \left(\int_M \eta_1 \Delta^{\frac{j+l}{2}} u \Delta^{\frac{j+l}{2}} u + C \sum_{p=0}^{l+j-1} \int_{B_r(\text{supp}(\eta))} (\Delta^{\frac{p}{2}} u)^2 \right) \\ &\leq C \beta^{2k_1-l} \sum_{p=0}^{l+j} \int_{B_r(\text{supp}(\eta))} (\Delta^{\frac{p}{2}} u)^2 \end{aligned}$$

Applying this inequality in the right hand side of (4.18) and using the fact that $k_2 - k_1 + 2k_1 = k$ we obtain

$$\beta^{k_2-k_1-1-i-j} \sum_{l=0}^{2k_1} \sum_{\substack{0 \leq l_1, l_2 \leq k_1 \\ l_1+l_2=l}} \int_M \eta_1 c_{l_1, k_1} c_{l_2, k_1} \beta^{2k_1-l} \Delta^{l_1+\frac{j}{2}} u \Delta^{l_2+\frac{j}{2}} u \leq C \sum_{l=0}^{2k_1} \beta^{k-1-i-j-l} \sum_{p=0}^{l+j} \int_{B_r(\text{supp}(\eta))} (\Delta^{\frac{p}{2}} u)^2 \quad (4.19)$$

Then substituting $l+j$ with l' , using the fact that $2k_1 + j \leq k-1-i$, and applying (4.14) we obtain

$$\begin{aligned} \sum_{l=0}^{2k_1} \beta^{k-1-i-j-l} \sum_{p=0}^{l+j} \int_{B_r(\text{supp}(\eta))} (\Delta^{\frac{p}{2}} u)^2 &= \sum_{l'=j}^{2k_1+j} \beta^{k-1-i-l'} \sum_{p=0}^{l'} \int_{B_r(\text{supp}(\eta))} (\Delta^{\frac{p}{2}} u)^2 \\ &\leq \sum_{l'=0}^{k-1-i} \beta^{k-1-i-l'} \sum_{p=0}^{l'} \int_{B_r(\text{supp}(\eta))} (\Delta^{\frac{p}{2}} u)^2 \\ &= \sum_{l'=0}^{k-1-i} \sum_{p=0}^{k-1-i-l'} \beta^{k-1-i-l'-p} \int_{B_r(\text{supp}(\eta))} (\Delta^{\frac{p}{2}} u)^2. \end{aligned} \quad (4.20)$$

Then by substituting $l' + i$ with l'' we obtain

$$\begin{aligned} \sum_{l'=0}^{k-1-i} \sum_{p=0}^{k-1-i-l'} \beta^{k-1-i-l'-p} \int_{B_r(\text{supp}(\eta))} (\Delta^{\frac{p}{2}} u)^2 &= \sum_{l''=i}^{k-1} \sum_{p=0}^{k-1-l''} \beta^{k-1-l''-p} \int_{B_r(\text{supp}(\eta))} (\Delta^{\frac{p}{2}} u)^2 \\ &\leq \sum_{l''=0}^{k-1} \sum_{p=0}^{k-1-l''} \beta^{k-1-l''-p} \int_{B_r(\text{supp}(\eta))} (\Delta^{\frac{p}{2}} u)^2. \end{aligned} \quad (4.21)$$

Therefore, putting together (4.17), (4.19), (4.20), (4.21) we obtain

$$\beta^{k_2-k_1-1-i-j} \int_M \eta_1 (\Delta^{\frac{j}{2}} (\Delta + \beta)^{k_1} u)^2 \leq \sum_{l''=0}^{k-1} \sum_{p=0}^{k-1-l''} \beta^{k-1-l''-p} \int_{B_r(\text{supp}(\eta))} (\Delta^{\frac{p}{2}} u)^2$$

Beuase i, j were arbitrary, we apply this bound to each term of (4.16) to conclude

$$\sum_{i=0}^{k_2-k_1-1} \sum_{j=0}^{k_2-k_1-1-i} \beta^{k_2-k_1-1-i-j} \int_{B_{\frac{r}{3}}(\text{supp}(\eta))} (\Delta^{\frac{j}{2}} (\Delta + \beta)^{k_1} u)^2 \leq C \int_{B_r(\text{supp}(\eta))} \sum_{i=0}^{k-1} \sum_{j=0}^{k-1-i} \beta^{k-1-i-j} (\Delta^{\frac{j}{2}} u)^2$$

and we are done. $\square$

Lemma 4.8. *Let $k \geq 1$ and $r > 0$ be arbitrary and let η be a smooth compactly supported function. Let i_1, i_2 and j_1, j_2 be such that $i_1 + i_2 = j_1 + j_2 = k$. Additionally, for each $0 \leq l \leq k-1$ let p_l, q_l be an arbitrary pair of nonnegative integers satisfying $p_l + q_l = l$. Then there exists C based on n, k, r , bounds for R and finitely many of its derivatives, and bounds for η and finitely many of its derivatives such that for all $u \in H_k^2(M)$ and $\beta \geq 0$ such that $(\Delta + \beta)^l u \geq 0$ for all $0 \leq l \leq k$,*

$$\int_M \eta(\Delta + \beta)^{i_1} u (\Delta + \beta)^{i_2} u \leq \int_M \eta(\Delta + \beta)^{j_1} u (\Delta + \beta)^{j_2} u + C \sum_{l=0}^{k-1} \int_{B_r(\text{supp}(\eta))} (\Delta + \beta)^{p_l} u (\Delta + \beta)^{q_l} u$$

Proof. Let $\eta \in C_c^\infty(M)$, $k \geq 0$, p_l, q_l such that $p_l + q_l = l$ for $0 \leq l \leq k$, $r > 0$ be arbitrary. Let η_1 be a function such that $\eta_1 = 1$ on $B_{\frac{r}{3}}(\text{supp}(\eta))$ and $\eta_1 = 0$ on $M \setminus B_{\frac{2}{3}r}(\text{supp}(\eta))$. By applying Lemma 4.7 with $\frac{r}{3}$ replacing r we obtain

$$\begin{aligned} \left| \int_M \eta(\Delta + \beta)^{i_1} u (\Delta + \beta)^{i_2} u - \int_M \eta(\Delta + \beta)^{j_1} u (\Delta + \beta)^{j_2} u \right| &\leq C \int_{B_{\frac{r}{3}}(\text{supp}(\eta))} \sum_{i=0}^{k-1} \sum_{j=0}^{k-1-i} \beta^{k-1-i-j} (\Delta^{\frac{j}{2}} u)^2 \\ &\leq C \int_M \sum_{i=0}^{k-1} \sum_{j=0}^{k-1-i} \eta_1 \beta^{k-1-i-j} (\Delta^{\frac{j}{2}} u)^2 \end{aligned}$$

It therefore suffices to prove for $k \geq 0$, $\eta_1 \in C_c^\infty(M)$, $s > 0$ there exists C such that

$$\int_M \sum_{i=0}^k \sum_{j=0}^{k-i} \eta_1 \beta^{k-i-j} (\Delta^{\frac{j}{2}} u)^2 \leq C \sum_{l=0}^k \int_{B_s(\text{supp}(\eta_1))} (\Delta + \beta)^{p_l} u (\Delta + \beta)^{q_l} u$$

and letting $s = \frac{r}{3}$ will allow us to conclude.

The base case $k = 0$ is immediate. Assume the statement is true for some k and let p_l, q_l satisfy $p_l + q_l = l$ for $1 \leq l \leq k+1$. Then we have after substituting $i' = i - 1$

$$\begin{aligned} \int_M \sum_{i=0}^{k+1} \sum_{j=0}^{k+1-i} \eta_1 \beta^{k+1-i-j} (\Delta^{\frac{j}{2}} u)^2 &= \int_M \sum_{j=0}^{k+1} \eta_1 \beta^{k+1-j} (\Delta^{\frac{j}{2}} u)^2 + \sum_{i=1}^{k+1} \sum_{j=0}^{k+1-i} \eta_1 \beta^{k+1-i-j} (\Delta^{\frac{j}{2}} u)^2 \\ &= \int_M \sum_{j=0}^{k+1} \eta_1 \beta^{k+1-j} (\Delta^{\frac{j}{2}} u)^2 + \sum_{i'=0}^k \sum_{j=0}^{k-i'} \eta_1 \beta^{k-i'-j} (\Delta^{\frac{j}{2}} u)^2 \end{aligned} \quad (4.22)$$

By the induction hypothesis

$$\sum_{i'=0}^k \sum_{j=0}^{k-i'} \int_M \eta_1 \beta^{k-i'-j} (\Delta^{\frac{j}{2}} u)^2 \leq C \sum_{l=0}^k \int_{B_s(\text{supp}(\eta_1))} (\Delta + \beta)^{p_l} u (\Delta + \beta)^{q_l} u.$$

Now to bound the the first term in (4.22), using the positivity assumption on $(\Delta + \beta)^l u$ we expand

$$\begin{aligned}
\int_{B_{\frac{s}{3}}(\text{supp}(\eta_1))} (\Delta + \beta)^{p_{k+1}} u (\Delta + \beta)^{q_{k+1}} u &\geq \int_M \eta_1 (\Delta + \beta)^{p_{k+1}} u (\Delta + \beta)^{q_{k+1}} u \\
&= \int_M \eta_1 \left(\sum_{l=0}^{p_{k+1}} c_{l,p_{k+1}} \beta^{p_{k+1}-l} \Delta^l u \right) \left(\sum_{l=0}^{q_{k+1}} c_{l,q_{k+1}} \beta^{q_{k+1}-l} \Delta^l u \right) \\
&= \sum_{l=0}^{k+1} \sum_{\substack{0 \leq l_1 \leq p_{k+1} \\ 0 \leq l_2 \leq q_{k+1} \\ l_1 + l_2 = l}} \int_M \eta_1 c_{l_1,p_{k+1}} c_{l_2,q_{k+1}} \beta^{k+1-l} \Delta^{l_1} u \Delta^{l_2} u dv_g
\end{aligned}$$

Then applying Lemma 4.6 to each term we obtain

$$\sum_{l=0}^{k+1} \sum_{\substack{0 \leq l_1 \leq p_{k+1} \\ 0 \leq l_2 \leq q_{k+1} \\ l_1 + l_2 = l}} \int_M \eta_1 c_{l_1,p_{k+1}} c_{l_2,q_{k+1}} \beta^{k+1-l} \Delta^{l_1} u \Delta^{l_2} u \geq \sum_{l=0}^{k+1} d_{l,k+1} \beta^{k+1-l} \left(\int_M \eta_1 (\Delta^{\frac{l}{2}} u)^2 - C \sum_{p=0}^{l-1} \int_{B_{\frac{s}{3}}(\text{supp}(\eta_1))} (\Delta^{\frac{p}{2}} u)^2 \right)$$

where

$$d_{l,k+1} = \sum_{\substack{0 \leq l_1 \leq p_{k+1} \\ 0 \leq l_2 \leq q_{k+1} \\ l_1 + l_2 = l}} c_{l_1,p_{k+1}} c_{l_2,q_{k+1}} \geq 1.$$

We therefore obtain

$$\int_{B_{\frac{s}{3}}(\text{supp}(\eta_1))} (\Delta + \beta)^{p_{k+1}} u (\Delta + \beta)^{q_{k+1}} u \geq \sum_{l=0}^{k+1} \beta^{k+1-l} \left(\int_M \eta_1 (\Delta^{\frac{l}{2}} u)^2 - C \sum_{p=0}^{l-1} \int_{B_{\frac{s}{3}}(\text{supp}(\eta_1))} (\Delta^{\frac{p}{2}} u)^2 \right).$$

Adding the remainder terms onto the left hand side and substituting $l' = l - 1$ results in

$$\sum_{l=0}^{k+1} \beta^{k+1-l} \int_M \eta_1 (\Delta^{\frac{l}{2}} u)^2 \leq \int_{B_{\frac{s}{3}}(\text{supp}(\eta_1))} (\Delta + \beta)^{p_{k+1}} u (\Delta + \beta)^{q_{k+1}} u + C \sum_{l'=0}^k \beta^{k-l'} \sum_{p=0}^{l'} \int_{B_{\frac{s}{3}}(\text{supp}(\eta_1))} (\Delta^{\frac{p}{2}} u)^2$$

Let η_2 be such that $\eta_2 = 1$ on $B_{\frac{s}{3}}(\text{supp}(\eta_1))$ and $\eta_2 = 0$ on $M \setminus B_{\frac{2}{3}s}(\text{supp}(\eta_1))$. By (4.14) and the induction hypothesis (applied to η_2 and $\frac{s}{3}$) we continue

$$\begin{aligned}
&\int_{B_{\frac{s}{3}}(\text{supp}(\eta_1))} (\Delta + \beta)^{p_{k+1}} u (\Delta + \beta)^{q_{k+1}} u + C \sum_{l'=0}^k \beta^{k-l'} \sum_{p=0}^{l'} \int_M \eta_2 (\Delta^{\frac{p}{2}} u)^2 \\
&= \int_{B_{\frac{s}{3}}(\text{supp}(\eta_1))} (\Delta + \beta)^{p_{k+1}} u (\Delta + \beta)^{q_{k+1}} u + C \sum_{l'=0}^k \sum_{p=0}^{k-l'} \int_M \beta^{k-l'-p} \eta_2 (\Delta^{\frac{p}{2}} u)^2 \\
&\leq \int_{B_s(\text{supp}(\eta_1))} (\Delta + \beta)^{p_{k+1}} u (\Delta + \beta)^{q_{k+1}} u + C \sum_{l=0}^k \int_{B_s(\text{supp}(\eta_1))} (\Delta + \beta)^{p_l} u (\Delta + \beta)^{q_l} u
\end{aligned}$$

and the claim is proven. □

Lemma 4.9. *Let $k \geq 0$ be an integer and $u \in H_k^2(M)$ be supported inside a geodesic ball $B \subset M$ centered around $x_0 \in M$. Let ξ represent the Euclidean metric on B defined through a geodesic normal coordinate chart and dx the corresponding Euclidean volume element. Let r_g be the geodesic distance function to x_0 . Then there exists C independent of u such that*

$$\int_M |\nabla_\xi^k u|^2 dx \leq \int_M (1 + Cr_g^2) \left(|\nabla_g^k u|^2 + C \sum_{i=0}^{k-1} |\nabla_g^i u|^2 \right) dv_g$$

Proof. The estimate $dx \leq (1 + Cr_g^2)dv_g$ implies it is sufficient to prove

$$\int_M |\nabla_\xi^k u|^2 dx \leq \int_M \left((1 + Cr_g^2) |\nabla_g^k u|^2 + C \sum_{i=0}^{k-1} |\nabla_g^i u|^2 \right) dx$$

We prove by strong induction on k . The base case $k = 0$ is immediate. Assume the statement holds for all values strictly smaller than same $k \geq 1$. We recall the formula for the components of the covariant derivative of a p covariant tensor T is

$$\nabla_i T_{i_1 \dots i_p} = \frac{\partial T_{i_1 \dots i_p}}{\partial x_i} - \sum_{k=1}^p \Gamma_{ii_k}^\alpha T_{i_1 \dots i_{k-1} \alpha i_{k+1} \dots i_p}.$$

Therefore, applying this to the k th covariant derivative we obtain

$$\partial_{i_1 \dots i_k} u = \nabla_{i_1 \dots i_k}^{(g)} u + \sum_{\alpha \in \{1, \dots, n\}^{k-1}} f_\alpha \partial_\alpha u + \sum_{j=0}^{k-2} \sum_{\beta \in \{1, \dots, n\}^j} g_\beta \partial_\beta u \quad (4.23)$$

where f_α represents a sum of positive and negative Christoffel symbols and g_β is a sum of positive and negative derivatives of Christoffel symbols. We have $|f_\alpha| \leq Cr_g$ and $|g_\alpha| \leq C$ in our chart. We take the equation

$$\int_M (\partial_{i_1 \dots i_k} u)^2 dx = \int_M \left(\nabla_{i_1 \dots i_k}^{(g)} u + \sum_{\alpha \in \{1, \dots, n\}^{k-1}} f_\alpha \partial_\alpha u + \sum_{j=0}^{k-2} \sum_{\beta \in \{1, \dots, n\}^j} g_\beta \partial_\beta u \right)^2 dx$$

and consider the resulting terms from the right hand side.

Terms of the form $\int_M f_{\alpha_1} f_{\alpha_2} \partial_{\alpha_1} u \partial_{\alpha_2} u dx$ where $|\alpha_1| = |\alpha_2| = k - 1$ can be bounded by taking

$$\begin{aligned} \int_M f_{\alpha_1} f_{\alpha_2} \partial_{\alpha_1} u \partial_{\alpha_2} u dx &\leq C \int_M \partial_{\alpha_1} u \partial_{\alpha_2} u dx \\ &\leq C \int_M |\partial_{\alpha_1} u|^2 + |\partial_{\alpha_2} u|^2 dx \\ &\leq C \int_M |\nabla_{\xi}^{k-1} u|^2 dx \\ &\leq C \int_M \sum_{i=0}^{k-1} |\nabla_g^i U_{\alpha}|^2 dx \end{aligned}$$

where the final inequality is by the induction hypothesis (although the full strength of the estimate is not used).

Terms of the form $\int_M g_{\beta_1} g_{\beta_2} \partial_{\beta_1} u \partial_{\beta_2} u dx$ and $\int_M f_{\alpha} g_{\beta} \partial_{\alpha} u \partial_{\beta} u$ can be bounded through the same argument.

For terms of the form $\int_M g_{\beta} \nabla_{i_1 \dots i_k}^{(g)} u \partial_{\beta} u$, we first substitute

$$\nabla_{i_1 \dots i_k}^{(g)} u = \partial_{i_1 \dots i_k} u - \sum_{\alpha \in \{1, \dots, n\}^{k-1}} f_{\alpha} \partial_{\alpha} u - \sum_{j=0}^{k-2} \sum_{\beta \in \{1, \dots, n\}^j} g_{\beta} \partial_{\beta} u$$

After making this substitution and expanding, we only need to bound the term $\int_M g_{\beta} \partial_{i_1 \dots i_k} u \partial_{\beta} u dx$, the others can be bounded by the above computations. Euclidean integration by parts gives

$$\begin{aligned} \int_M g_{\beta} \partial_{i_1 \dots i_k} u \partial_{\beta} u dx &= - \int_M \partial_{i_1} g_{\beta} \partial_{i_2 \dots i_k} u \partial_{\beta} u dx - \int_M g_{\beta} \partial_{i_2 \dots i_k} u \partial_{i_1} \partial_{\beta} u dx \\ &\leq C \int_M |\nabla_{\xi}^{k-1} u|^2 dx + C \int_M |\nabla_{\xi}^{|\beta|} u|^2 dx + C \int_M |\nabla_{\xi}^{|\beta|+1} u|^2 dx. \end{aligned}$$

Because $|\beta| + 1 \leq k - 1$, by the induction hypothesis

$$\int_M |\nabla_{\xi}^{k-1} u|^2 dx + \int_M |\nabla_{\xi}^{|\beta|} u|^2 dx + \int_M |\nabla_{\xi}^{|\beta|+1} u|^2 dx \leq C \int_M \sum_{i=0}^{k-1} |\nabla_g^i U_{\alpha}|^2 dx$$

For terms of the form $\int_M f_{\alpha} \nabla_{i_1 \dots i_k}^{(g)} u \partial_{\alpha} u dx$, we calculate

$$\begin{aligned} \int_M f_{\alpha} \nabla_{i_1 \dots i_k}^{(g)} u \partial_{\alpha} u &\leq C \int_M r_g \nabla_{i_1 \dots i_k}^{(g)} u \partial_{\alpha} u \\ &\leq C \int_M r_g^2 (\nabla_{i_1 \dots i_k}^{(g)} u)^2 dx + C \int_M (\partial_{\alpha} u)^2 dx \end{aligned}$$

We have

$$\int_M (\partial_{\alpha} u)^2 dx \leq C \int_M \sum_{i=0}^{k-1} |\nabla_g^i u|^2 dx$$

by the induction hypothesis. Combining all of our previous computations we obtain

$$\int_M (\partial_{i_1 \dots i_k} u)^2 dx \leq \int_M (1 + Cr_g^2) (\nabla_{i_1 \dots i_k}^{(g)} u)^2 + C \int_M \sum_{i=0}^{k-1} |\nabla_g^i u|^2$$

To complete the proof of the lemma, we first recall $\delta^{ij} \leq (1 + r_g^2)g^{ij}$ where δ^{ij} represents the Kronecker delta and calculate

$$\begin{aligned} \int_M |\nabla_\xi^k U_\alpha|^2 dx &\leq \int_M \sum_{1 \leq i_1, \dots, i_k \leq n} (1 + Cr_g^2) (\nabla_{i_1 \dots i_k}^{(g)} u)^2 dx + C \int_M \sum_{i=0}^{k-1} |\nabla_g^i u|^2 \\ &= \int_M (1 + Cr_g^2) \delta^{i_1 j_1} \dots \delta^{i_k j_k} \nabla_{j_1 \dots j_k}^{(g)} u \nabla_{i_1 \dots i_k}^{(g)} u dx + C \int_M \sum_{i=0}^{k-1} |\nabla_g^i u|^2 \\ &\leq \int_M (1 + Cr_g^2) |\nabla_g^k U_\alpha|^2 + C \sum_{i=0}^{k-1} \int_M |\nabla_g^i U_\alpha|^2 dx \end{aligned}$$

□

4.3 Proof of the Theorem

Suppose for all $B > 0$, inequality (3.1) is false. Then, using notions from Section 3.3 with $f \equiv 1$, for all $\alpha > 0$ we have

$$\lambda_\alpha := \inf_{u \in \mathcal{N}_1} \int_M u(\Delta + \alpha)^m u dv_g < \frac{1}{K}. \quad (4.24)$$

where we recall we must interpret the integral in the sense of distributions. The sequence λ_α is clearly non-decreasing and so converges to some $\lambda \leq \frac{1}{K}$. By Theorem 3.10, for all $\alpha > 0$ there exists a smooth positive solution $u_\alpha \in \mathcal{N}_1$ to the PDE

$$(\Delta + \alpha)^m u_\alpha = \lambda_\alpha u_\alpha^{2^\#-1}. \quad (4.25)$$

Independently, we recall Theorem 3.2 states for all $\epsilon > 0$ there exists B_ϵ such that for all $u \in H_m^2(M)$,

$$\left(\int_M |u|^{2^\#} \right)^{\frac{2}{2^\#}} dv_g \leq (K + \epsilon) \int_M (\Delta^{\frac{m}{2}} u)^2 dv_g + B_\epsilon \|u\|_{H_{m-1}^2}^2. \quad (4.26)$$

Therefore, for all ϵ there exists α_ϵ large such that

$$\frac{1}{(1 + \epsilon)K} \leq \inf_{u \in \mathcal{N}_1} \int_M u(\Delta + \alpha_\epsilon)^m u dv_g.$$

Combining this with (4.24) shows $\lambda = \frac{1}{K}$. We note that $\|u_\alpha\|_{H_{m-1}^2} \rightarrow 0$ as $\alpha \rightarrow \infty$ while $\|u_\alpha\|_{2^\#} = 1$. Then because

$$\int_M u_\alpha^{2^\#} dv_g \leq (\sup_M u_\alpha)^{2^\#-2} \int_M u_\alpha^2 dv_g$$

we must have $\sup_M u_\alpha \rightarrow \infty$ as $\alpha \rightarrow \infty$. Let $x_\alpha \in M$ be such that $u_\alpha(x_\alpha)$ is maximum. Taking a subsequence of α we assume $x_\alpha \rightarrow x_0 \in M$ and $u_\alpha(x_\alpha)$ increases to infinity.

4.3.1 Asymptotic Analysis

Step 1: Up to a subsequence, u_α has one unique concentration point.

Let us define $\mu_\alpha = u_\alpha(x_\alpha)^{-\frac{2}{n-2m}}$. Let i_g be the injectivity radius for (M, g) . We define sequences of functions $\tilde{u}_\alpha$ and metrics $\tilde{g}_\alpha$ on $B_0(i_g/\mu_\alpha)$ by

$$\tilde{u}_\alpha(x) = \mu_\alpha^{\frac{n-2m}{2}} u_\alpha(\exp_{x_\alpha}(\mu_\alpha x)) \text{ and } \tilde{g}_\alpha(x) = (\exp_{x_\alpha}^* g)(\mu_\alpha x).$$

We note $\bigcup_\alpha B_0(i_g/\mu_\alpha) = \mathbb{R}^n$. The functions $\tilde{u}_\alpha$ are bounded in $C^0(\mathbb{R}^n)$ with $\sup \tilde{u}_\alpha = \tilde{u}(0) = 1$ and they satisfy the PDE

$$(\Delta_{\tilde{g}_\alpha} + \alpha \mu_\alpha^2)^m \tilde{u}_\alpha = \lambda_\alpha \tilde{u}_\alpha^{2^\#-1}.$$

We would like to show these functions are uniformly bounded in $C^{2m,\beta}(K)$ for K compact and therefore converge to a limit function $\tilde{u}$. To show boundedness in $C^{2m,\beta}(K)$, it suffices to show $\alpha \mu_\alpha^2$ is bounded as $\alpha \rightarrow \infty$ and apply standard regularity theory as seen in Gilbarg-Trudinger[17]. Suppose up to a subsequence $\alpha \mu_\alpha^2 \rightarrow \infty$ as $\alpha \rightarrow \infty$. Let us define for $0 \leq k \leq m$ functions $\tilde{v}_\alpha^{(k)}$ and $v_\alpha^{(k)}$ on $B_0(i_g/\mu_\alpha)$ and M respectively by

$$\tilde{v}_\alpha^{(k)} = (\Delta_{\tilde{g}_\alpha} + \alpha \mu_\alpha^2)^k \tilde{u}_\alpha = \sum_{i=0}^k c_{i,k} \alpha^{k-i} \mu_\alpha^{2(k-i)} \Delta_{\tilde{g}_\alpha}^i \tilde{u}_\alpha$$

and

$$v_\alpha^{(k)} = (\Delta_g + \alpha)^k u_\alpha = \sum_{i=0}^k c_{i,k} \alpha^{k-i} \Delta_g^i u_\alpha$$

where $c_{i,k}$ are defined the same as in Lemma 4.7.

We obtain systems of PDEs

$$\begin{cases} \Delta_{\tilde{g}_\alpha} \tilde{v}_\alpha^{(k)} + \alpha \mu_\alpha^2 \tilde{v}_\alpha^{(k)} = \tilde{v}_\alpha^{(k+1)} & \text{if } 0 \leq k \leq m-2 \\ \Delta_{\tilde{g}_\alpha} \tilde{v}_\alpha^{(m-1)} + \alpha \mu_\alpha^2 \tilde{v}_\alpha^{(m-1)} = \tilde{u}_\alpha^{2^\#-1} \end{cases} \quad (4.27)$$

and

$$\begin{cases} \Delta_g v_\alpha^{(k)} + \alpha v_\alpha^{(k)} = v_\alpha^{(k+1)} & \text{if } 0 \leq k \leq m-2 \\ \Delta_g v_\alpha^{(m-1)} + \alpha v_\alpha^{(m-1)} = u_\alpha^{2^\#-1}. \end{cases} \quad (4.28)$$

Because $u_\alpha^{2^\#-1} \geq 0$, iterating the maximum principle we obtain $v_\alpha^{(k)} \geq 0$ for all $1 \leq k \leq m$. This then shows for $x \in B_0(i_g/\mu_\alpha)$

$$\tilde{v}_\alpha^{(k)}(x) = \mu_\alpha^{\frac{n}{2}-(m-2k)} v_\alpha^{(k)}(\exp_{x_\alpha}(\mu_\alpha x)) \geq 0$$

for all $1 \leq k \leq m$.

We now show $\tilde{v}_\alpha^{(k)}$ is bounded in $L^2(B_0(R))$ for fixed $R > 0$ as $\alpha \rightarrow \infty$ for any $1 \leq k \leq m-1$. We perform the proof of this claim through the following (strong) induction:

1. If $k \leq \lfloor \frac{m}{2} \rfloor$ then $\int_{B_0(R)} (v_\alpha^{(k)})^2 dv_{\tilde{g}_\alpha} \leq C$.
2. If $\lfloor \frac{m}{2} \rfloor \leq k \leq m-2$ and $\int_{B_0(R)} (v_\alpha^{(i)})^2 dv_{\tilde{g}_\alpha} \leq C$ for $i \leq k$, then $\int_{B_0(R-2)} (v_\alpha^{(k+1)})^2 dv_{\tilde{g}_\alpha} \leq C$.

Let $k \leq \lfloor \frac{m}{2} \rfloor$. Integrating by parts gives

$$\int_M (v_\alpha^{(k)})^2 dv_g = \int_M \sum_{i=0}^{2k} d_{i,k} \alpha^{2k-i} u_\alpha \Delta_g^i u_\alpha dv_g \leq C \int_M \sum_{k=0}^{2k} \alpha^{2k-i} u_\alpha \Delta_g^i u_\alpha dv_g$$

where $d_{i,k} = \sum_{j_1+j_2=i} c_{j_1,k} c_{j_2,k}$ and we use the fact that for all integers $k \geq 0$ and $u \in H_{2k}^2$,

$$\int_M u \Delta_g^k u dv_g = \int_M (\Delta_g^{\frac{k}{2}} u)^2 dv_g \geq 0.$$

Therefore, using the PDE we obtain

$$\alpha^{m-2k} \int_M (v_\alpha^{(k)})^2 dv_g \leq C \int_M \sum_{k=0}^{2k} \alpha^{m-i} u_\alpha (\Delta_g^i u_\alpha) dv_g \leq C \int_M u_\alpha (\Delta_g + \alpha)^m u_\alpha dv_g = C \int_M u_\alpha^{2^\#} dv_g \leq C$$

We therefore have by a change of variable

$$\begin{aligned} \int_{B_0(R)} (\tilde{v}_\alpha^{(k)})^2 dv_{\tilde{g}_\alpha} &= \int_{B_0(R)} \mu_\alpha^{n-2(m-2k)} (v_\alpha^{(k)}(\exp_{x_\alpha}(\mu_\alpha x)))^2 dv_{\tilde{g}_\alpha} \\ &= \int_{B_{x_\alpha}(R\mu_\alpha)} \mu_\alpha^{-2(m-2k)} (v_\alpha^{(k)})^2 dv_g \\ &\leq C \mu_\alpha^{-2(m-2k)} \alpha^{-(m-2k)} \\ &\leq C \end{aligned}$$

and we have proven (i).

Now let us define for $1 \leq i \leq 2(m-1)$ functions $w_\alpha^{(i)}$ by

$$\begin{cases} w_\alpha^{(i)} = \left(\tilde{v}_\alpha^{\left(\frac{i}{2}\right)} \right)^2 & \text{if } i \text{ is even} \\ w_\alpha^{(i)} = \tilde{v}_\alpha^{\left(\frac{i}{2}-\frac{1}{2}\right)} \tilde{v}_\alpha^{\left(\frac{i}{2}+\frac{1}{2}\right)} & \text{if } i \text{ is odd.} \end{cases}$$

Let $\lfloor \frac{m}{2} \rfloor \leq k \leq m-2$ and let η be a smooth nonnegative function such that $\eta = 1$ on $B_0(R-1)$ and $\eta = 0$ on $\mathbb{R}^n \setminus B_0(R - \frac{1}{2})$. Assume $\int_{B_0(R)} (v_\alpha^{(l)})^2 dv_{\tilde{g}_\alpha} \leq C$ for $l \leq k$. This is equivalent to $\int_{B_0(R)} w_\alpha^{(i)} dv_{\tilde{g}_\alpha} \leq C$ for $i \leq 2k$. Because $k \geq \lfloor \frac{m}{2} \rfloor$ there exists a nonnegative integer s such that $m+s = 2k+1$. Additionally because $k \leq m-2$ we must have $s+1 \leq k$. Then applying Lemma 4.8 with $r = \frac{1}{2}$ gives

$$\int_{B_0(R-1)} \tilde{v}_\alpha^{(k+1)} \tilde{v}_\alpha^{(k)} dv_{\tilde{g}_\alpha} \leq \int_{B_0(R-\frac{1}{2})} \eta \tilde{v}_\alpha^{(k+1)} \tilde{v}_\alpha^{(k)} dv_{\tilde{g}_\alpha} \leq \int_{B_0(R-\frac{1}{2})} \eta \tilde{v}_\alpha^{(m)} \tilde{v}_\alpha^{(s)} dv_{\tilde{g}_\alpha} + C \int_{B_0(R)} \sum_{i=0}^{2k} w_\alpha^{(i)} dv_{\tilde{g}_\alpha}.$$

We note because $\tilde{g}_\alpha$ converges to the Euclidean metric uniformly on compact subsets (and its derivatives up converge to 0 uniformly on compact subsets), the constant C above is indeed independent of α . Substituting definitions of $w_\alpha^{(i)}$, applying Cauchy-Schwarz, recognizing $\tilde{v}_\alpha^{(m)} = \tilde{u}_\alpha^{2^\#-1} \leq 1$, and using the fact that $s \leq k$ gives

$$\int_{B_0(R-1)} w_\alpha^{(2k+1)} dv_{\tilde{g}_\alpha} \leq \left(\int_{B_0(R)} (\tilde{v}_\alpha^{(m)})^2 dv_{\tilde{g}_\alpha} \right)^{\frac{1}{2}} \left(\int_{B_0(R)} (w_\alpha^{(2s)}) dv_{\tilde{g}_\alpha} \right)^{\frac{1}{2}} + C \int_{B_0(R)} \sum_{i=0}^{2k} w_\alpha^{(i)} dv_{\tilde{g}_\alpha} \leq C.$$

Therefore $\int_{B_0(R-1)} w_\alpha^{(i)} dv_{\tilde{g}_\alpha} \leq C$ for $i \leq 2k+1$. Now let η be a smooth function such that $\eta = 1$ on $B_0(R-2)$ and $\eta = 0$ on $\mathbb{R}^n \setminus B_0(R - \frac{3}{2})$. Again applying Lemma 4.8 with $r = \frac{1}{2}$,

$$\int_{B_0(R-2)} (\tilde{v}_\alpha^{(k+1)})^2 dv_{\tilde{g}_\alpha} \leq \int_{B_0(R-\frac{3}{2})} \eta (\tilde{v}_\alpha^{(k+1)})^2 dv_{\tilde{g}_\alpha} \leq \int_{B_0(R-\frac{3}{2})} \eta \tilde{v}_\alpha^{(m)} \tilde{v}_\alpha^{(s+1)} dv_{\tilde{g}_\alpha} + C \int_{B_0(R-1)} \sum_{i=0}^{2k+1} w_\alpha^{(i)} dv_{\tilde{g}_\alpha}.$$

Then we once again apply Cauchy-Schwarz, $\tilde{v}_\alpha^{(m)} \leq 1$ and the fact that $s+1 \leq k$ to obtain

$$\int_{B_0(R-2)} (\tilde{v}_\alpha^{(k+1)})^2 dv_{\tilde{g}_\alpha} \leq C$$

and we have shown (ii), therefore for all $1 \leq k \leq m-1$, $\int_{B_0(R)} (\tilde{v}_\alpha^{(k)})^2 dv_{\tilde{g}_\alpha} \leq C$.

We now consider $\tilde{v}_\alpha^{(m-1)}$ which satisfies the last PDE in the system (4.27). Because $\tilde{v}_\alpha^{(m-1)}$ is positive and bounded in $L^2(B_0(R))$ and $u_\alpha^{2^\#-1}$ is bounded in $C^0(B_0(R))$, we get $\tilde{v}_\alpha^{(m-1)}$ is bounded in $C^0(B_0(R-1))$ by the De Giorgi-Nash-Moser iteration scheme. We note because $\tilde{g}_\alpha$ converges to the Euclidean metric uniformly on compact sets, the constant C in the De Giorgi-Nash-Moser iterative scheme will indeed be independent of α (specifically, in the application of Lemma A.2, we can guarantee bounds for the constants λ and Λ independent of α). Then for $1 \leq k \leq m-2$, we consider $v_\alpha^{(k)}$ in the system (4.27) and apply the De Giorgi-Nash-Moser iteration scheme again. If $\tilde{v}_\alpha^{(k+1)}$ is bounded in $C^0(B_0(R))$ and $\tilde{v}_\alpha^{(k)}$ is bounded in $L^2(B_0(R))$ then $\tilde{v}_\alpha^{(k)}$ is bounded

in $C^0(B_0(R-1))$. Therefore by induction $\tilde{v}_\alpha^{(k)}$ is C^0 bounded on compact sets for $1 \leq k \leq m$. We then take the particular equation from (4.27)

$$\Delta_{\tilde{g}_\alpha} \tilde{u}_\alpha + \alpha \mu_\alpha^2 \tilde{u}_\alpha = \tilde{v}_\alpha^{(1)}.$$

Because $\Delta_{\tilde{g}_\alpha} \tilde{u}_\alpha(0) \geq 0$, $\tilde{u}_\alpha(0) = 1$, and $\tilde{v}_\alpha^{(1)}(0) \leq C$ we obtain that $\alpha \mu_\alpha^2$ is in fact bounded. We then apply standard regularity theory to see $\tilde{u}_\alpha$ are uniformly bounded in $C_{2m}^\beta(K)$ for K compact, and therefore $\tilde{u}_\alpha$ will converge in $C_{2m}(\mathbb{R}^n)$ to a limit function $\tilde{u}$ satisfying $0 \leq \tilde{u} \leq 1$ and $\tilde{u}(0) = 1$.

Independently, by integrating by parts, substituting our PDE, and using the fact that $\int_M u_\alpha^{2^\#} dv_g = 1$ we have

$$\sum_{i=0}^m \int_M c_{i,m} \alpha^{m-i} (\Delta^{\frac{i}{2}} u_\alpha)^2 dv_g = \int_M u_\alpha (\Delta + \alpha)^m u_\alpha dv_g \leq \frac{1}{K} \int_M u_\alpha^{2^\#} dv_g = \frac{1}{K} \left(\int_M u_\alpha^{2^\#} dv_g \right)^{\frac{2}{2^\#}}. \quad (4.29)$$

We note by the inequalities above and the fact that $c_{m,m} = 1$ we have

$$\|\Delta^{\frac{m}{2}} u_\alpha\|_2^2 \leq \frac{1}{K} \|u_\alpha\|_{2^\#}^2 = \frac{1}{K}. \quad (4.30)$$

Given $\epsilon > 0$, we apply (4.29), the asymptotically sharp Sobolev inequality (4.26), and (4.30) to obtain

$$\sum_{i=0}^m \int_M c_{i,m} \alpha^{m-i} (\Delta^{\frac{i}{2}} u_\alpha)^2 dv_g \leq \|\Delta^{\frac{m}{2}} u_\alpha\|_2^2 + \frac{\epsilon}{K} \|\Delta^{\frac{m}{2}} u_\alpha\|_2^2 + \frac{B_\epsilon}{K} \|u_\alpha\|_{H_{m-1}^2} \leq \|\Delta^{\frac{m}{2}} u_\alpha\|_2^2 + \frac{\epsilon}{K^2} + \frac{B_\epsilon}{K} \|u_\alpha\|_{H_{m-1}^2}.$$

Then because $c_{0,m} = 1 = c_{m,m}$ we have for sufficiently large α

$$\alpha^m \int_M u_\alpha^2 dv_g \leq \alpha^m \int_M u_\alpha^2 dv_g + \sum_{i=1}^{m-1} \int_M (c_{i,m} \alpha^{m-i} - \frac{B_\epsilon}{K}) (\Delta^{\frac{i}{2}} u_\alpha)^2 dv_g \leq \frac{\epsilon}{K^2} + \frac{B_\epsilon}{K} \int_M u_\alpha^2 dv_g = \frac{\epsilon}{K^2} + o(1)$$

where $o(1) \rightarrow 0$ as $\alpha \rightarrow \infty$. Therefore by letting $\epsilon \rightarrow 0$ we see

$$\alpha^m \int_M u_\alpha^2 dv_g \rightarrow 0 \text{ as } \alpha \rightarrow \infty.$$

Because $\tilde{u}$ is continuous and $\tilde{u}(0) = 1$, $\int_{B_0(1)} \tilde{u}^2 dx > 0$. Thus by uniform convergence $\tilde{u}_\alpha \rightarrow \tilde{u}$ as $\alpha \rightarrow \infty$ on compact sets, there exists some $c > 0$ independent of α such that $\int_{B_0(1)} \tilde{u}_\alpha^2 \geq c$. Therefore given $\delta > 0$ small and fixed, for sufficiently large α

$$\alpha^m \int_{B_{x_\alpha(\delta)}} u_\alpha^2 dv_g = \alpha^m \mu_\alpha^{2m} \int_{B_0(\frac{\delta}{\mu_\alpha})} \tilde{u}_\alpha^2 \geq \alpha^m \mu_\alpha^{2m} \int_{B_0(1)} \tilde{u}_\alpha^2 \geq c \alpha^m \mu_\alpha^{2m}$$

which implies $\alpha\mu_\alpha^2 \rightarrow 0$ as $\alpha \rightarrow \infty$. Hence, passing to the limit, $\tilde{u}$ is a $C^{2m}(\mathbb{R}^n)$ nonnegative solution to the PDE

$$\Delta_\xi^m \tilde{u} = \frac{1}{K} \tilde{u}^{2^\#-1}.$$

in the Euclidean metric. By the Euclidean sharp Sobolev inequality (as stated in Lions[31]) we have

$$\left(\int_{\mathbb{R}^n} \tilde{u}^{2^\#} dx \right)^{\frac{2}{2^\#}} \leq K \int_M (\Delta_\xi^{\frac{m}{2}} \tilde{u})^2 dx$$

and so we obtain (after integrating by parts)

$$\left(\int_{\mathbb{R}^n} \tilde{u}^{2^\#} dx \right)^{\frac{2}{2^\#}} \leq \int_{\mathbb{R}^n} \tilde{u}^{2^\#} dx. \quad (4.31)$$

By a change of variable, for all $R > 0$ and sufficiently large α we have

$$\int_{B_0(R)} \tilde{u}_\alpha^{2^\#} = \int_{B_{x_\alpha}(\mu_\alpha R)} u_\alpha^{2^\#} \leq 1$$

and taking the limit as $\alpha \rightarrow \infty$ and then as $R \rightarrow \infty$ implies

$$\int_{B_0(\mathbb{R}^n)} \tilde{u}^{2^\#} \leq 1.$$

Hence by (4.31) we have

$$\int_{B_0(\mathbb{R}^n)} \tilde{u}^{2^\#} dx = 1$$

Fixing $R > 0$ large and $\delta > 0$ small, we have

$$1 \geq \limsup_{\alpha \rightarrow \infty} \int_{B_{x_0}(\delta)} u_\alpha^{2^\#} dv_g \geq \liminf_{\alpha \rightarrow \infty} \int_{B_{x_0}(\delta)} u_\alpha^{2^\#} dv_g \geq \liminf_{\alpha \rightarrow \infty} \int_{B_{x_\alpha}(\mu_\alpha R)} u_\alpha^{2^\#} dv_g = \int_{B_0(R)} \tilde{u}^{2^\#} dx = 1 - \epsilon_R$$

where $\epsilon_R \rightarrow 0$ as $R \rightarrow \infty$. We thus obtain

$$\lim_{\alpha \rightarrow \infty} \int_{B_{x_0}(\delta)} u_\alpha^{2^\#} dv_g = 1.$$

Since δ is arbitrary, if $y_0 \neq x_0$ for any $\delta < d(x_0, y_0)$ we must have

$$\lim_{\alpha \rightarrow \infty} \int_{B_{y_0}(\delta)} u_\alpha^{2^\#} dv_g = 0$$

otherwise we would contradict $\int_M u_\alpha^{2^\#} dv_g = 1$. This completes the proof there is only one unique concentration point x_0 along our subsequence.

Step 2: For all $x \in M$ we have the inequality

$$d_g(x_\alpha, x)^{\frac{n-2m}{2}} u_\alpha(x) \leq C. \quad (4.32)$$

We directly follow Hebey's proof in the second order case, see also Druet and Robert.

Suppose there is some sequence y_α such that

$$\sup_{x \in M} d_g(x_\alpha, x)^{\frac{n-2m}{2}} u_\alpha(x) = d_g(x_\alpha, y_\alpha)^{\frac{n-2m}{2}} u_\alpha(y_\alpha) \rightarrow \infty.$$

Similarly to what is done in step one, we define $\hat{\mu}_\alpha = u_\alpha(y_\alpha)^{-\frac{2}{n-2m}}$ and define the rescalings $\hat{u}_\alpha$ of u_α on $\mathbb{R}^n$ with metric $\hat{g}_\alpha$ as follows.

$$\hat{u}_\alpha = \hat{\mu}_\alpha^{\frac{n-2m}{2}} u_\alpha(\exp_{y_\alpha}(\hat{\mu}_\alpha x)) \quad \hat{g}_\alpha = (\exp_{y_\alpha}^* g)(\hat{\mu}_\alpha x).$$

Then $\hat{u}$ is a solution to the PDE

$$(\Delta_{\hat{g}_\alpha} + \hat{\mu}_\alpha^2 \alpha)^m \hat{u}_\alpha = \lambda_\alpha \hat{u}_\alpha^{2^\#-1}.$$

By our definition of y_α we additionally have

$$\hat{u}_\alpha(x) \leq \left(\frac{d_g(x_\alpha, y_\alpha)}{d_g(x_\alpha, \exp_{y_\alpha}(\hat{\mu}_\alpha x))} \right)^{\frac{n-2m}{2}}. \quad (4.33)$$

Fix $R > 0$. If $|x| \leq R$ the triangle inequality implies

$$d_g(x_\alpha, \exp_{y_\alpha}(\hat{\mu}_\alpha x)) \geq d_g(x_\alpha, y_\alpha) - R\hat{\mu}_\alpha.$$

Because $d_g(x_\alpha, y_\alpha)^{\frac{n-2m}{2}} u_\alpha(y_\alpha) \rightarrow \infty$, we additionally have

$$\frac{\hat{\mu}_\alpha}{d_g(x_\alpha, y_\alpha)} \rightarrow 0.$$

Therefore by (4.33),

$$\hat{u}_\alpha(x) \leq \left(1 - \frac{R\hat{\mu}_\alpha}{d_g(x_\alpha, y_\alpha)} \right)^{-\frac{n-2m}{2}}$$

which implies $\hat{u}_\alpha$ are C^0 bounded on compact sets.

Now that we have local C^0 bounds, following the De Giorgi-Nash-Moser iteration scheme argument seen in Step 1 shows for all $1 \leq k \leq m$, the functions $\hat{v}^{(k)} = (\Delta_{\hat{g}_\alpha} + \hat{\mu}_\alpha^2 \alpha)^k(\hat{u}_\alpha)$ are bounded on compact sets.

Let us first consider the case (after possibly taking a subsequence) $y_\alpha \rightarrow y_0 \neq x_0$. Then for all $\epsilon > 0$, because $\Delta_{\hat{g}_\alpha} \hat{u}_\alpha \leq \hat{v}_\alpha^{(1)} \leq C$,

$$\Delta_{\hat{g}_\alpha} (\hat{u}_\alpha)^{1+\epsilon} = (1+\epsilon) \hat{u}_\alpha^\epsilon \Delta_{\hat{g}_\alpha} \hat{u}_\alpha - \epsilon(1+\epsilon) \hat{u}_\alpha^{\epsilon-1} |\nabla_{\hat{g}_\alpha} \hat{u}_\alpha|^2 \leq C \hat{u}_\alpha^\epsilon$$

Now fixing ϵ small such that $\frac{2}{\epsilon} > \frac{n}{2}$ and applying the De Giorgi-Nash-Moser iteration scheme gives for all $p > 0$ there exists C_p such that

$$\sup_{B_0(R-1)} (\hat{u}_\alpha)^{1+\epsilon} \leq C_p \left(\|(\hat{u}_\alpha)^{1+\epsilon}\|_{L^p(B_0(R))} + \|\hat{u}_\alpha^\epsilon\|_{L^{\frac{2}{\epsilon}}(B_0(R))} \right).$$

Setting $p = \frac{2}{1+\epsilon}$ gives us

$$\sup_{B_0(R-1)} (\hat{u}_\alpha)^{1+\epsilon} \leq C \|\hat{u}_\alpha\|_{L^2(B_0(R))}^\epsilon (1 + \|\hat{u}_\alpha\|_{L^2(B_0(R))}).$$

Because $\hat{u}_\alpha(0) = 1$, this implies $\liminf_{\alpha \rightarrow \infty} \|\hat{u}_\alpha\|_{L^2(B_0(R))} > 0$. However, because x_0 is the only concentration point we additionally have

$$\|\hat{u}_\alpha\|_{L^2(B_0(R))} \leq C \left(\int_{B_0(R)} \hat{u}_\alpha^{2^\#} \right)^{\frac{1}{2^\#}} = C \left(\int_{B_{y_\alpha}(R\hat{\mu}_\alpha)} u_\alpha^{2^\#} \right)^{\frac{1}{2^\#}} \rightarrow 0.$$

and we have a contradiction.

Now let us consider the case (after potentially taking a subsequence) $y_\alpha \rightarrow x_0$. In order to apply the same logic as Step 1 in showing $\hat{u}_\alpha$ converges to a limit function in $C^{2m}(\mathbb{R}^n)$, we need to show $(\Delta_{\hat{g}_\alpha} \hat{u}_\alpha)(0)$ is bounded from below. This corresponds to showing $\hat{\mu}_\alpha^2(\Delta_g u)(y_\alpha)$ is bounded from below. We let r_α correspond to the function $d(x_\alpha, \cdot)$. Because $y_\alpha \rightarrow x_0$, y_α will be contained in an exponential chart around x_α for α sufficiently large. In particular, $d_g(x_\alpha, x)^{\frac{n-2m}{2}} u_\alpha(x)$ is twice differentiable at y_α for α sufficiently large, so we have $\Delta_g(r_\alpha^{\frac{n-2m}{2}} u_\alpha)(y_\alpha) \geq 0$ and $\nabla_g(r_\alpha^{\frac{n-2m}{2}} u_\alpha)(y_\alpha) = 0$. The latter equation implies

$$r_\alpha^{\frac{n-2m}{2}} \nabla_g u_\alpha(y_\alpha) = (-\nabla_g(r_\alpha^{\frac{n-2m}{2}}) u_\alpha)(y_\alpha).$$

Then, using the well known formulas in an exponential chart $|\nabla_g f(r)| = |f'(r)|$ and $\Delta_g f(r) = \Delta_\xi f(r) - f'(r) \partial_r(\ln \sqrt{g})$ for a radial function f , we compute, with all expressions are evaluated at y_α ,

$$\begin{aligned} 0 \leq \Delta_g(r_\alpha^{\frac{n-2m}{2}} u_\alpha) &= r_\alpha^{\frac{n-2m}{2}} \Delta_g u_\alpha + 2 \left(\frac{n-2m}{2} \right)^2 r_\alpha^{\frac{n-2m-4}{2}} u_\alpha \\ &\quad - \left(\frac{n-2m}{2} \right) \left(\frac{3n-2m-4}{2} \right) r_\alpha^{\frac{n-2m-4}{2}} u_\alpha - \frac{n-2m}{2} r_\alpha^{\frac{n-2m-2}{2}} \partial_r(\ln \sqrt{g}) u_\alpha \end{aligned}$$

which implies

$$\frac{n-2m}{2} r_\alpha^{-1} \partial_r (\ln \sqrt{g}) u_\alpha + \left(\frac{n-2m}{2} \right) \left(\frac{n+2m-4}{2} \right) r_\alpha^{-2} u_\alpha \leq \Delta_g u_\alpha.$$

The Cartan expansion of the metric shows $\partial_r (\ln \sqrt{g}) \geq -Cr$ where C is based only on the curvature of the metric, therefore we have

$$\frac{n-2m}{2} \left[-C + \frac{n+2m-4}{2} r_\alpha^{-2} \right] u_\alpha \leq \Delta_g u_\alpha$$

The left hand side is bounded from below (positive for sufficiently large α because $y_\alpha \rightarrow x_0$), so our claim holds and we let $\hat{u} : \mathbb{R}^n \rightarrow \mathbb{R}$ be the limit function of $\hat{u}_\alpha$.

Let $\delta > 0$ be small and R be fixed. We recall notions from Step 1, using the same $\tilde{u}$ and $\tilde{u}_\alpha$,

$$\int_{B_0(R)} \tilde{u}_\alpha^{2^\#} d\nu_{\tilde{g}_\alpha} = \int_{B_{x_\alpha}(R\mu_\alpha)} u_\alpha^{2^\#} d\nu_g$$

and

$$\int_{\mathbb{R}^n} \tilde{u}^{2^\#} dx = 1.$$

We therefore have

$$\lim_{\alpha \rightarrow \infty} \int_{B_{x_\alpha}(R\mu_\alpha)} u_\alpha^{2^\#} d\nu_g = \int_{B_0(R)} \tilde{u}^{2^\#} dx = 1 - \epsilon_R$$

where $\epsilon_R \rightarrow 0$ as $R \rightarrow \infty$.

Combining this with the fact that

$$\lim_{\alpha \rightarrow \infty} \int_{B_{x_0}(\delta)} u_\alpha^{2^\#} d\nu_g = 1$$

immediately implies

$$\int_{B_{x_0}(\delta) \setminus B_{x_\alpha}(R\mu_\alpha)} u_\alpha^{2^\#} \leq \epsilon_R + o(1).$$

Because $\hat{\mu}_\alpha \rightarrow 0$ and $y_\alpha \rightarrow x_0$, for sufficiently large α we obtain

$$\int_{B_{y_\alpha}(\hat{\mu}_\alpha)} u_\alpha^{2^\#} \leq \int_{B_{y_\alpha}(\hat{\mu}_\alpha) \cap B_{x_\alpha}(R\mu_\alpha)} u_\alpha^{2^\#} + \epsilon_R + o(1).$$

As stated before $\hat{\mu} = o(d_g(x_\alpha, y_\alpha))$. Combining this fact with $\mu_\alpha \leq \hat{\mu}_\alpha$ implies $B_{y_\alpha}(\hat{\mu}_\alpha) \cap B_{x_\alpha}(R\mu_\alpha) = \emptyset$ for sufficiently large α .

Therefore

$$\int_{B_{y_\alpha}(\hat{\mu}_\alpha)} u_\alpha^{2^\#} \leq \epsilon_R + o(1).$$

Taking the limit as $\alpha \rightarrow \infty$ shows

$$\int_{B_0(1)} \hat{u}^{2^\#} dx \leq \epsilon_R.$$

However, the left hand side is a positive constant independent of R or α and the right hand side converges to 0 as we send $R \rightarrow \infty$ and we have a contradiction, therefore (4.32) holds.

As a consequence of Step 2, we prove for all $0 \leq k \leq m-1$, $v_\alpha^{(k)} \rightarrow 0$ in $C_{loc}^0(M \setminus \{x_0\})$. By Step 2, u_α is bounded in $C_{loc}^0(M \setminus \{x_0\})$ and $v_\alpha^{(m-1)}$ satisfies

$$\Delta v_\alpha^{(m-1)} \leq \frac{1}{K} u_\alpha^{2^\#-1}.$$

Integrating the PDE (4.25) implies $\int \alpha^m u_\alpha \leq C$ by the divergence theorem, so for $0 \leq k \leq m-1$,

$$\int_M v_\alpha^{(k)} = \int_M \alpha^k u_\alpha \rightarrow 0$$

as $\alpha \rightarrow \infty$. Additionally, because $\int_M u_\alpha \rightarrow 0$ and u_α is bounded in $C_{loc}^0(M \setminus \{x_0\})$, for any $p > 0$ and for any compact subset E of $M \setminus \{x_0\}$

$$\int_E u_\alpha^p \rightarrow 0$$

as $\alpha \rightarrow \infty$. Therefore by the De Giorgi-Nash-Moser iteration scheme $v_\alpha^{(m-1)} \rightarrow 0$ in $C_{loc}^0(M \setminus \{x_0\})$. Now, assuming for some $1 \leq k \leq m-1$ $v_\alpha^{(k)} \rightarrow 0$ in $C_{loc}^0(M \setminus \{x_0\})$, then we consider

$$\Delta_g v_\alpha^{(k-1)} \leq v_\alpha^{(k)}$$

and applying the De Giorgi-Nash-Moser iteration scheme again shows $v_\alpha^{(k-1)} \rightarrow 0$ in $C_{loc}^0(M \setminus \{x_0\})$ and the claim follows by induction.

Step 3: L^2 concentration holds, i.e. for each $\delta > 0$ and for all $0 \leq i \leq m-1$

$$\lim_{\alpha \rightarrow \infty} \frac{\int_{M \setminus B_{x_0}(\delta)} |\Delta^{\frac{i}{2}} u_\alpha|^2}{\int_M u_\alpha^2} = 0$$

Our first claim is that for all $0 \leq k \leq m-1$,

$$\int_{M \setminus B_{x_0}(\delta)} (\Delta^{\frac{k}{2}} u_\alpha)^2 \leq C \alpha^k \int_{M \setminus B_{x_0}(2^{-(2k+1)}\delta)} u_\alpha^2 \quad (4.34)$$

Let $\delta > 0$ be arbitrary. Let us as before write $v_\alpha^{(m-1)} = (\Delta + \alpha)^{m-1} u_\alpha$. By the last PDE in the system (4.28) we have

$$\Delta v_\alpha^{(m-1)} + \alpha v_\alpha^{(m-1)} - \frac{1}{\lambda_\alpha} u_\alpha^{2^\#-1} = 0.$$

We write this as

$$\sum_{i=1}^m c_{i,m} \alpha^{m-i} \Delta^i u_\alpha + \alpha^m u_\alpha - \frac{1}{\lambda_\alpha} u_\alpha^{2^\#-1} = 0.$$

Because $u_\alpha \rightarrow 0$ in $C_{loc}^0(M \setminus \{x_0\})$ we have $\alpha^m u_\alpha - \frac{1}{\lambda_\alpha} u_\alpha^{2^\#-1} \geq 0$ for sufficiently large α on compact subsets of $M \setminus \{x_0\}$. Additionally, by the simple identity $(a+b)^k - b^k = a \sum_{i=0}^{k-1} b^i (a+b)^{k-1-i}$ we have

$$\sum_{i=1}^m c_{i,m} \alpha^{m-i} \Delta^i u_\alpha = \Delta \left(\sum_{i=0}^{m-1} \alpha^i v_\alpha^{(m-1-i)} \right).$$

Therefore, applying the De Giorgi-Nash-Moser scheme we obtain

$$\sup_{M \setminus B_{x_0}(\delta)} \sum_{i=0}^{m-1} \alpha^i v_\alpha^{(m-1-i)} \leq C \int_{M \setminus B_{x_0}(\delta/2)} \sum_{i=0}^{m-1} \alpha^i v_\alpha^{(m-1-i)}$$

Let η be a smooth nonnegative function equal to 1 on $M \setminus B_{\frac{\delta}{2}}(x_0)$ and equal to 0 on $B_{\frac{\delta}{4}}(x_0)$. Then we continue the calculation, integrating by parts the expression $\int_M \eta \sum_{i=0}^{m-1} \alpha^i v_\alpha^{(m-1-i)}$ to obtain

$$\int_{M \setminus B_{x_0}(\delta/2)} \sum_{i=0}^{m-1} \alpha^i v_\alpha^{(m-1-i)} \leq \int_M \eta \sum_{i=0}^{m-1} \alpha^i v_\alpha^{(m-1-i)} \leq C \alpha^{m-1} \int_{M \setminus B_{x_0}(\delta/4)} u_\alpha. \quad (4.35)$$

Let η now be a smooth nonnegative function equal to 1 on $M \setminus B_{2\delta}(x_0)$ and equal to 0 on $B_\delta(x_0)$. Now we prove (4.34) by induction. The base case $k = 0$ holds by immediately. Now suppose for $1 \leq k \leq m-1$ (4.34) is true for $i < k$. Considering the $m-1-k$ th term of 4.35 individually and applying Cauchy-Schwarz gives us

$$\alpha^{m-1-k} \int_M \eta v_\alpha^{(k)} u_\alpha \leq C \alpha^{m-1} \left(\int_{M \setminus B_{x_0}(\delta/4)} u_\alpha \right)^2 \leq C \alpha^{m-1} \int_{M \setminus B_{x_0}(\delta/4)} u_\alpha^2$$

which we can write as

$$\int_M \eta u_\alpha \sum_{l=0}^k \alpha^{k-l} c_{l,k} \Delta^l u_\alpha \leq C \alpha^k \int_{M \setminus B_{x_0}(\frac{\delta}{4})} u_\alpha^2.$$

Applying Lemma 4.6 to each term with $r = \frac{\delta}{2}$ we obtain

$$\int_M \eta \sum_{l=0}^k c_{l,k} \alpha^{k-l} (\Delta^{\frac{l}{2}} u_\alpha)^2 \leq C \alpha^k \int_{M \setminus B_{x_0}(\frac{\delta}{4})} u_\alpha^2 + C \int_{M \setminus B_{x_0}(\delta/2)} \sum_{l=0}^k \alpha^{k-l} \sum_{p=0}^{l-1} (\Delta^{\frac{p}{2}} u_\alpha)^2. \quad (4.36)$$

Given a fixed $l \leq k$ for all $0 \leq p \leq l-1$ we have by the induction hypothesis

$$\int_{M \setminus B_{x_0}(\delta/2)} \alpha^{k-l} (\Delta^{\frac{p}{2}} u_\alpha)^2 \leq C \int_{M \setminus B_{x_0}(2^{-(2k)}\delta)} \alpha^{k-l} \alpha^p u_\alpha^2 \leq C \alpha^k \int_{M \setminus B_{x_0}(2^{-(2k)}\delta)} u_\alpha^2. \quad (4.37)$$

We therefore obtain by (4.36) and 4.37)

$$\int_{M \setminus B_{x_0}(\delta/2)} (\Delta^{\frac{k}{2}} u_\alpha)^2 \leq C \alpha^k \int_{M \setminus B_{x_0}(2^{-(2k)} \delta)} u^2$$

which is equivalent to (4.34). For all $1 \leq k \leq m-1$ we obtain the bound

$$\int_{M \setminus B_{x_0}(\delta)} (\Delta^{\frac{k}{2}} u_\alpha)^2 \leq C \alpha^k \int_{M \setminus B_{x_0}(2^{-(2m+1)} \delta)} u^2 \quad (4.38)$$

Now multiplying the PDE (4.25) by η and using the fact that $u_\alpha \rightarrow 0$ in $C_{\text{loc}}^0(M \setminus \{x_0\})$ we obtain

$$\sum_{i=0}^m \int_M \eta c_{i,m} \alpha^{m-i} u_\alpha \Delta^i u_\alpha = \int_M \eta u_\alpha^{2^\#} \leq \int_{M \setminus B_{x_0}(2^{-(2m+1)} \delta)} u_\alpha^{2^\#} \leq C \int_{M \setminus B_{x_0}(2^{-(2m+1)} \delta)} u_\alpha^2$$

We apply Lemma 4.6 with $r = \frac{\delta}{2}$ to each term and substitute $i' = i - 1$ to obtain

$$\begin{aligned} \sum_{i=0}^m \int_M \eta c_{i,m} \alpha^{m-i} (\Delta^{\frac{i}{2}} u_\alpha)^2 &\leq C \alpha^{m-1} \int_{M \setminus B_{x_0}(2^{-(2m+2)} \delta)} u_\alpha^2 + C \sum_{i=1}^m \alpha^{m-i} \sum_{j=0}^{i-1} \int_{M \setminus B_{x_0}(\frac{\delta}{2})} (\Delta^j u_\alpha)^2 \\ &= C \alpha^{m-1} \int_{M \setminus B_{x_0}(2^{-(2m+2)} \delta)} u_\alpha^2 + C \sum_{i'=0}^{m-1} \alpha^{m-1-i'} \sum_{j=0}^{i'} \int_{M \setminus B_{x_0}(\frac{\delta}{2})} (\Delta^j u_\alpha)^2 \end{aligned}$$

Given some $0 \leq i' \leq m-1$ and $0 \leq j \leq i'$ we apply (4.34) to obtain

$$\alpha^{m-1-i'} \int_{M \setminus B_{x_0}(\frac{\delta}{2})} (\Delta^j u_\alpha)^2 \leq C \alpha^{m-1-i'+j} \int_{M \setminus B_{x_0}(2^{-(2m+2)} \delta)} u_\alpha^2 \leq C \alpha^{m-1} \int_{M \setminus B_{x_0}(2^{-(2m+2)} \delta)} u_\alpha^2.$$

Hence we have

$$\sum_{i=0}^m \int_M \eta c_{i,m} \alpha^{m-i} (\Delta^{\frac{i}{2}} u_\alpha)^2 \leq C \alpha^{m-1} \int_{M \setminus B_{x_0}(2^{-(2m+2)} \delta)} u_\alpha^2. \quad (4.39)$$

After dividing both sides by α^{m-1} we have

$$\alpha \int_{M \setminus B_{x_0}(2\delta)} u_\alpha^2 \leq C \int_{M \setminus B_{x_0}(2^{-(2m+2)} \delta)} u_\alpha^2 \quad (4.40)$$

and we have

$$0 \leq \lim_{\alpha \rightarrow \infty} \frac{\int_{M \setminus B_{x_0}(\delta)} u_\alpha^2}{\int_M u_\alpha^2} \leq \lim_{\alpha \rightarrow \infty} \frac{\int_{M \setminus B_{x_0}(\delta)} u_\alpha^2}{\int_{M \setminus B_{x_0}(2^{-(2m+3)} \delta)} u_\alpha^2} \leq \lim_{\alpha \rightarrow \infty} \frac{C}{\alpha} = 0$$

proving Step 3 for $i = 0$. For $1 \leq i \leq m - 1$ we divide both sides of (4.39) by α^{m-i} and apply (4.40) $i - 1$ times to obtain

$$\int_{M \setminus B_{x_0}(\delta)} (\Delta^{\frac{i}{2}} u_\alpha)^2 \leq C \alpha^{m-1-i} \int_{M \setminus B_{x_0}(2^{-(2m+3)}\delta)} u_\alpha^2 \leq C \int_{M \setminus B_{x_0}(2^{-(i-1)(2m+3)}\delta)} u_\alpha^2$$

and we have

$$0 \leq \lim_{\alpha \rightarrow \infty} \frac{\int_{M \setminus B_{x_0}(\delta)} (\Delta^{\frac{i}{2}} u_\alpha)^2}{\int_M u_\alpha^2} \leq C \lim_{\alpha \rightarrow \infty} \frac{\int_{M \setminus B_{x_0}(2^{-(i-1)(2m+3)}\delta)} u_\alpha^2}{\int_M u_\alpha^2} = 0$$

and the proof of Step 3 is complete.

4.3.2 Main Argument

We first note an immediate consequence of Step 3 is for all $0 \leq k \leq m - 1$

$$\|u_\alpha\|_{H_k^2(M)} \leq C \sum_{i=0}^k \int_{B_{x_0}(\delta)} (\Delta^{\frac{i}{2}} u_\alpha)^2. \quad (4.41)$$

Now we let $\frac{i_g}{2} > \delta > 0$ and let η be a nonnegative smooth function defined on $\mathbb{R}^n$ such that $\eta = 1$ on $B_0(\delta)$ and equal to 0 on $\mathbb{R}^n \setminus B_0(2\delta)$. We define η_α on M by $\eta_\alpha(x) = \eta(\exp_{x_\alpha}^{-1}(x))$. We then define U_α by $U_\alpha = \eta_\alpha u_\alpha$. We have uniform bounds for all derivatives of η_α .

Because U_α is only nonzero inside a geodesic chart, we consider the Euclidean metric ξ on $B_{x_\alpha}(2\delta)$ defined through the pullback of $\exp_{x_\alpha}^{-1}$. We write $dx = dv_\xi$ and the Euclidean Sobolev inequality gives

$$\left(\int_M U_\alpha^{2^\#} dx \right)^{\frac{2}{2^\#}} \leq K \int_M (\Delta_\xi^{\frac{m}{2}} U_\alpha)^2 dx \quad (4.42)$$

Euclidean integration by parts shows

$$\int_M (\Delta_\xi^{\frac{m}{2}} U_\alpha)^2 dx = \int_M |\nabla_\xi^m U_\alpha|^2 dx.$$

Then Lemma 4.9 gives

$$\int_M |\nabla_\xi^m U_\alpha|^2 dx \leq \int_M (1 + Cr_\alpha^2) \left(|\nabla_g^m U_\alpha|^2 + C \sum_{i=0}^{m-1} |\nabla_g^i U_\alpha|^2 \right) dv_g$$

where $r_\alpha = d_g(x_\alpha, \cdot)$. Then by Lemma 4.3

$$\int_M (\Delta_\xi^{\frac{m}{2}} U_\alpha)^2 dx \leq \int_M (\Delta_g^{\frac{m}{2}} U_\alpha)^2 dv_g + C \int_M r_\alpha^2 |\nabla_g^m U_\alpha|^2 dv_g + C \sum_{i=0}^{m-1} \int_M |\nabla_g^i U_\alpha|^2 dv_g \quad (4.43)$$

We consider the third term in (4.43). Applying norm equivalence for $H_m^2(M)$ (see Robert[38]), Lemma 4.5 and (4.41) results in

$$\sum_{i=0}^{m-1} \int_M |\nabla_g^i U_\alpha|^2 dv \leq C \sum_{i=0}^{m-1} \int_M |\Delta_g^{\frac{i}{2}} U_\alpha|^2 dv \leq C \sum_{i=0}^{m-1} \int_M |\Delta_g^{\frac{i}{2}} u_\alpha|^2 dv \leq C \sum_{i=0}^{m-1} \int_{B_{x_0}(\delta)} (\Delta^{\frac{i}{2}} u_\alpha)^2 \quad (4.44)$$

where Lemma 4.5 is applied to each individual term of the second sum. For the first term of (4.43), we use Lemma 4.5, 4.6 and (4.41) to obtain

$$\begin{aligned} \int_M (\Delta_g^{\frac{m}{2}} U_\alpha)^2 &\leq \int_M \eta_\alpha^2 (\Delta_g^{\frac{m}{2}} u_\alpha)^2 + C \sum_{i=0}^{m-1} \int_{B_{x_0}(\delta)} (\Delta^{\frac{i}{2}} u_\alpha)^2 \\ &\leq \int_M \eta_\alpha^2 u_\alpha \Delta^m u_\alpha + C \sum_{i=0}^{m-1} \int_{B_{x_0}(\delta)} (\Delta^{\frac{i}{2}} u_\alpha)^2 \end{aligned}$$

We then substitute $\Delta^m u_\alpha = u_\alpha^{2^\#-1} - \sum_{i=0}^{m-1} c_{i,m} \alpha^{m-i} \Delta^i u_\alpha$, apply Lemma 4.6 to each term, and apply (4.41) to obtain

$$\int_M (\Delta_g^{\frac{m}{2}} U_\alpha)^2 \leq \lambda_\alpha \int_M \eta_\alpha^2 u_\alpha^{2^\#} - \left(\sum_{i=0}^{m-1} c_{i,m} \alpha^{m-i} \int_M \eta_\alpha^2 u_\alpha \Delta^i u_\alpha \right) + C \sum_{i=0}^{m-1} \int_{B_{x_0}(\delta)} (\Delta^{\frac{i}{2}} u_\alpha)^2 \quad (4.45)$$

$$\begin{aligned} &\leq \frac{1}{K} \int_M \eta_\alpha^2 u_\alpha^{2^\#} - \left(\sum_{i=0}^{m-1} c_{i,m} \alpha^{m-i} \int_M \eta_\alpha^2 (\Delta^{\frac{i}{2}} u_\alpha)^2 \right) \\ &\quad + C \sum_{i=1}^{m-1} \alpha^{m-i} \sum_{j=0}^{i-1} \int_M (\Delta^{\frac{j}{2}} u)^2 + C \sum_{i=0}^{m-1} \int_{B_{x_0}(\delta)} (\Delta^{\frac{i}{2}} u_\alpha)^2 \end{aligned} \quad (4.46)$$

$$\begin{aligned} &\leq \frac{1}{K} \int_M \eta_\alpha^2 u_\alpha^{2^\#} - \left(\sum_{i=0}^{m-1} c_{i,m} \alpha^{m-i} \int_M \eta_\alpha^2 (\Delta^{\frac{i}{2}} u_\alpha)^2 \right) \\ &\quad + C \sum_{i=1}^{m-1} \alpha^{m-i} \sum_{j=0}^{i-1} \int_{B_{x_0}(\delta)} (\Delta^{\frac{j}{2}} u)^2 + C \sum_{i=0}^{m-1} \int_{B_{x_0}(\delta)} (\Delta^{\frac{i}{2}} u_\alpha)^2. \end{aligned} \quad (4.47)$$

Considering the remainder terms, substituting $i' = i - 1$ we have

$$\sum_{i=1}^{m-1} \alpha^{m-i} \sum_{j=0}^{i-1} \int_{B_{x_0}(\delta)} (\Delta^{\frac{j}{2}} u_\alpha)^2 + \sum_{i=0}^{m-1} \int_{B_{x_0}(\delta)} (\Delta^{\frac{i}{2}} u_\alpha)^2 = \sum_{i'=0}^{m-2} \alpha^{m-1-i'} \sum_{j=0}^{i'} \int_{B_{x_0}(\delta)} (\Delta^{\frac{j}{2}} u_\alpha)^2 + \sum_{i=0}^{m-1} \int_{B_{x_0}(\delta)} (\Delta^{\frac{i}{2}} u_\alpha)^2 \quad (4.48)$$

$$= \sum_{i'=0}^{m-1} \alpha^{m-1-i'} \sum_{j=0}^{i'} \int_{B_{x_0}(\delta)} (\Delta^{\frac{j}{2}} u_\alpha)^2 \quad (4.49)$$

Then reindexing with (4.14) gives

$$\sum_{i'=0}^{m-1} \alpha^{m-1-i'} \sum_{j=0}^{i'} \int_{B_{x_0}(\delta)} (\Delta^{\frac{j}{2}} u_\alpha)^2 = C \sum_{i=0}^{m-1} \sum_{j=0}^{m-1-i} \int_{B_{x_0}(\delta)} \alpha^{m-1-i-j} (\Delta^{\frac{j}{2}} u_\alpha)^2 \quad (4.50)$$

$$= C \sum_{j=0}^{m-1} \sum_{i=0}^{m-1-j} \int_{B_{x_0}(\delta)} \alpha^{m-1-j-i} (\Delta^{\frac{j}{2}} u_\alpha)^2 \quad (4.51)$$

$$= C \sum_{j=0}^{m-1} \int_{B_{x_0}(\delta)} (\Delta^j u_\alpha)^2 \sum_{i=0}^{m-1-j} \alpha^i. \quad (4.52)$$

Therefore we obtain

$$\int_M (\Delta_g^{\frac{m}{2}} U_\alpha)^2 \leq \frac{1}{K} \int_M u_\alpha^{2^\#} - \sum_{i=0}^{m-1} \left(c_{i,m} \alpha^{m-i} - \sum_{j=0}^{m-1-i} C \alpha^j \right) \int_{B_{x_0}(\delta)} (\Delta^{\frac{j}{2}} u)^2. \quad (4.53)$$

We now estimate the second term of (4.43). Applying Lemmas 4.3, 4.5, 4.6, substituting the PDE (4.25), and applying (4.44) gives us

$$\begin{aligned} \int_M r_\alpha^2 |\nabla_g^m U_\alpha|^2 dv &\leq \int_M r_\alpha^2 (\Delta_g^{\frac{m}{2}} U_\alpha)^2 + C \|U_\alpha\|_{H_{m-1}^2} \\ &\leq \int_M \eta_\alpha^2 r_\alpha^2 (\Delta_g^{\frac{m}{2}} u_\alpha)^2 + C \|U_\alpha\|_{H_{m-1}^2} \\ &\leq \int_M \eta_\alpha^2 r_\alpha^2 u_\alpha \Delta^m u_\alpha + C \|U_\alpha\|_{H_{m-1}^2} \\ &\leq \frac{1}{K} \int_M \eta_\alpha^2 r_\alpha^2 u_\alpha^{2^\#} dv_g - \left(\sum_{i=0}^{m-1} c_{i,m} \alpha^{m-i} \int_M \eta_\alpha^2 r_\alpha^2 u_\alpha \Delta^i u_\alpha \right) + C \sum_{i=0}^{m-1} \int_{B_{x_0}(\delta)} (\Delta^{\frac{j}{2}} u_\alpha)^2 \\ &\leq \frac{1}{K} \int_M \eta_\alpha^2 r_\alpha^2 u_\alpha^{2^\#} dv_g + C \sum_{j=0}^{m-1} \int_{B_{x_0}(\delta)} (\Delta^j u_\alpha)^2 \sum_{i=0}^{m-1-j} \alpha^i \end{aligned} \quad (4.54)$$

where to obtain (4.54) we have applied the computations from (4.45) – (4.52).

By Step 2, we have

$$r_\alpha^2 u_\alpha^{2^\#} = u_\alpha (r_\alpha u_\alpha^{\frac{n}{n-2m}}) (r_\alpha u_\alpha^{\frac{2}{n-2m}})^{\frac{2m-2}{n-2m}} \leq C u_\alpha (r_\alpha u_\alpha^{\frac{n}{n-2m}})^{\frac{2m-2}{n-2m}}$$

and therefore

$$\eta_\alpha^2 r_\alpha^2 u_\alpha^{2^\#} \leq C \eta_\alpha u_\alpha (r_\alpha \eta_\alpha u_\alpha^{\frac{n}{n-2m}})^{\frac{2m-2}{n-2m}}.$$

Because $\frac{n-2(m-1)}{2n} + \frac{1}{2} + \frac{2m-2}{2n} = 1$, we apply Hölder's inequality to the right hand side to obtain

$$\int_M \eta_\alpha^2 r_\alpha^2 u_\alpha^{2^\#} \leq C \left(\int_M (\eta_\alpha u_\alpha)^{\frac{2n}{n-2(m-1)}} dv_g \right)^{\frac{n-2(m-1)}{2n}} \left(\int_M \eta_\alpha^2 r_\alpha^2 u_\alpha^{2^\#} \right)^{\frac{1}{2}} \left(\int_M u_\alpha^{2^\#} \right)^{\frac{2m-2}{2n}}.$$

Because $\int_M u_\alpha^{2^\#} = 1$, this gives us

$$\int_M \eta_\alpha^2 r_\alpha^2 u_\alpha^{2^\#} \leq C \left(\int_M (\eta_\alpha u_\alpha)^{\frac{2n}{n-2(m-1)}} \right)^{\frac{n-2(m-1)}{n}}. \quad (4.55)$$

We then apply the Sobolev embedding theorem and (4.44)

$$\left(\int_M (\eta_\alpha u_\alpha)^{\frac{2n}{n-2(m-1)}} \right)^{\frac{n-2(m-1)}{n}} \leq C \|u_\alpha\|_{H_{m-1}^2} \leq C \sum_{i=0}^{m-1} \int_{B_{x_0}(\delta)} (\Delta^{\frac{i}{2}} u_\alpha)^2 \quad (4.56)$$

therefore, by putting together (4.54), (4.55), and (4.56) we obtain

$$\int_M r_\alpha^2 |\nabla_g^m U|^2 dv \leq C \sum_{j=0}^{m-1} \int_{B_{x_0}(\delta)} (\Delta^j u_\alpha)^2 \sum_{i=0}^{m-1-j} \alpha^i (\Delta^{\frac{i}{2}} u_\alpha)^2 \quad (4.57)$$

Therefore, by combining (4.43), (4.44), (4.53), (4.57) we obtain

$$K \int_M (\Delta_{\xi}^{\frac{m}{2}} U_\alpha)^2 dx \leq \int_M u_\alpha^{2^\#} - \sum_{i=0}^{m-1} \left(K c_{i,m} \alpha^{m-i} - \sum_{j=0}^{m-i-1} C \alpha^j \right) \int_{B_{x_0}(\delta)} (\Delta^{\frac{i}{2}} u_\alpha)^2 \quad (4.58)$$

Now we consider the left hand side of (4.42). By the Cartan expansion of the metric and the definition of U_α we have

$$\int_M U_\alpha^{2^\#} dx \geq \int_{B_{x_0}(\delta)} u_\alpha^{2^\#} dv_g - C \int_M r_\alpha^2 U_\alpha^{2^\#} dv_g.$$

Since $0 \leq \eta_\alpha \leq 1$ we must have $\eta_\alpha^{2^\#} \leq \eta_\alpha^2$ and so by (4.55) and (4.56) we have

$$\int_M \eta_\alpha^{2^\#} r_\alpha^2 u_\alpha^{2^\#} \leq \int_M \eta_\alpha^2 r_\alpha^2 u_\alpha^{2^\#} \leq C \sum_{i=0}^{m-1} \int_{B_{x_0}(\delta)} (\Delta^{\frac{i}{2}} u_\alpha)^2.$$

Therefore we obtain

$$\int_M U_\alpha^{2^\#} dx \geq \int_{B_{x_0}(\delta)} u_\alpha^{2^\#} dv_g - C \sum_{i=0}^{m-1} \int_{B_{x_0}(\delta)} (\Delta^{\frac{i}{2}} u_\alpha)^2$$

The first term converges to 1 from below and the second term converges to 0, therefore raising both sides to

$\frac{2}{2^\#} < 1$ we have

$$\left(\int_M U_\alpha^{2^\#} dx \right)^{\frac{2}{2^\#}} \geq \int_{B_{x_0}(\delta)} u_\alpha^{2^\#} - C \sum_{i=0}^{m-1} \int_{B_{x_0}(\delta)} (\Delta^{\frac{i}{2}} u_\alpha)^2$$

Independently we have by Step 3

$$\int_{M \setminus B_{x_0}(\delta)} u_\alpha^{2^\#} \leq \left(\sup_{M \setminus B_{x_0}(\delta)} u_\alpha \right)^{2^\#-2} \int_{M \setminus B_{x_0}(\delta)} u_\alpha^2 \leq C \int_{B_{x_0}(\delta)} u_\alpha^2$$

allowing us to strengthen our previous inequality to

$$\left(\int_M U_\alpha^{2^\#} dx \right)^{\frac{2}{2^\#}} \geq \int_M u_\alpha^{2^\#} - C \sum_{i=0}^{m-1} \int_{B_{x_0}(\delta)} (\Delta^{\frac{i}{2}} u_\alpha)^2 \quad (4.59)$$

Therefore combining (4.42), (4.58), and (4.59) we obtain

$$\sum_{i=0}^{m-1} \left(K c_{i,m} \alpha^{m-i} - \sum_{j=0}^{m-i-1} C \alpha^j \right) \int_{B_{x_0}(\delta)} (\Delta^{\frac{i}{2}} u_\alpha)^2 \leq 0$$

and we obtain a contradiction when α is sufficiently large. □

A Appendix: The De Giorgi-Nash-Moser Iterative Scheme

The De Giorgi-Nash-Moser Iterative Scheme, for Riemannian manifolds is a crucial tool in Section 4.3. We state the Euclidean De Giorgi-Nash-Moser Iterative Scheme and use it to show it indeed applies in a useful sense when generalized to the Riemannian case. The following is Theorem 4.14 in the book of Han and Lin[20].

Theorem A.1 (Euclidean De Giorgi-Nash-Moser Scheme). *Let $\Omega \subset \mathbb{R}^n$ be a connected open set. Let $a_{ij} \in L^\infty(\Omega)$ satisfy*

$$\lambda|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Let $f \in L^q(\Omega)$ for some $q > \frac{n}{2}$. Let $u \in W^{1,2}(\Omega)$ be a nonnegative subsolution in following sense:

$$\int_{\Omega} a_{ij} D_i u D_j \phi dx \leq \int_{\Omega} f \phi dx$$

for any $\phi \in W_0^{1,2}(\Omega)$ such that $\phi \geq 0$ in Ω . Then for any ball $B_R \subset \Omega$ and $0 < r < R$ and any $p > 0$, there exists C based on $\lambda, \Lambda, q, p, n$ such that

$$\sup_{B_r} u \leq C \left[\frac{1}{(R-r)^{\frac{n}{p}}} \|u\|_{L^p(B_R)} + R^{2-\frac{n}{q}} \|f\|_{L^q(B_R)} \right].$$

The following lemma shows we can apply the Euclidean De Giorgi-Nash-Moser iterative scheme in small geodesic balls on Riemannian manifolds.

Lemma A.2. *Let (M, g) be a smooth complete n dimensional Riemannian manifold with bounded curvature. Let $x_0 \in M$, $\delta > 0$ such that $i_{x_0, \delta} > \delta$ where $i_{x_0, \delta}$ is the injectivity radius at x_0 . Let $B_\delta(x_0)$ be the geodesic ball of radius δ at x_0 . Let $f \in L^q(B_\delta(x_0))$ for $q > \frac{n}{2}$ be a nonnegative function. Let $0 < \lambda \leq \sqrt{g} \leq \Lambda$ in normal coordinates on $B_\delta(x_0)$. It is easily seen through the Cartan expansion of the metric in normal coordinates that λ and Λ can be taken to depend on curvature bounds for the manifold. Let $u \in H_1^2(B_\delta(x_0))$ be a nonnegative function satisfying $\Delta_g u \leq f$. Then for any $p > 0$, there exists C based on $(\lambda, \Lambda, \delta, p, q, n)$ such that*

$$\sup_{B_{\delta/2}(x_0)} u \leq C \left(\|u\|_{L^p(B_\delta(x_0))} + \|f\|_{L^q(B_\delta(x_0))} \right)$$

Proof. We first recall the formula for the Laplacian operator in coordinates $\Delta u = -\frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j u)$ (see Chapter 2 of Lee[26]). Let $\phi \in W_0^{1,1}(B_\delta(0))$ be nonnegative. Computing in geodesic normal coordinates, we

have

$$\int_{B_0(\delta)} -\partial_i(\sqrt{g}g^{ij}\partial_j u(\exp_{x_0}(x)))\phi(x)dx \leq \int_{B_0(\delta)} f(\exp_{x_0}(x))\phi(x)\sqrt{g}dx \leq \int_{B_0(\delta)} \Lambda f(\exp_{x_0}(x))\phi(x)dx.$$

Integrating by parts on the left hand side shows the definition of a subsolution in Theorem A.1 is satisfied and therefore applying the Euclidean De Giorgi-Nash-Moser scheme (passing to Riemmanian integrals through the bounds λ and Λ) yields our lemma. $\square$

Theorem A.3 (De Giorgi-Nash-Moser Scheme). *Let (M, g) be a smooth complete n dimensional Riemannian manifold with bounded curvature. Let $\Omega \subset M$ be a bounded open set. Let $A \subset \Omega$ be a compact proper subset of Ω . Let $f \in L^q(\Omega)$ for $q > \frac{n}{2}$ be a nonnegative function. Let $u \in H^{1,1}(\Omega)$ be a nonnegative function satisfying $\Delta_g u \leq f$. Then for any $p > 0$, there exists C based on $((M, g), \Omega, A, p, q, n)$ such that*

$$\sup_A u \leq C \left(\|u\|_{L^p(\Omega)} + \|f\|_{L^q(\Omega)} \right).$$

Proof. By the Hopf-Rinow theorem, it is easily seen there exists $\delta_1 > 0$ such that for all $x \in \Omega$, $i_{x,g} > \delta_1$. Because A is compact in Ω , there also exists some $\delta_2 > 0$ such that for all $x \in A$, $B_{\delta_2}(x) \subset \Omega$. Taking $\delta := \min(\delta_1, \delta_2)$, considering a finite covering $\{B_{\delta/2}(x_i) : x_i \in A, 1 \leq i \leq k\}$ of A , and applying Lemma A.2 immediately yeilds the theorem. $\square$

Conclusion

The result of Theorem 3.3 represents an important development in the AB Program and is interesting in the sense that it requires methods which are not necessary in the second order proof by Hebey[23] while still maintaining the same structure. We hope to see more higher order questions of this sort being attacked in the future, as they often require more than just added technicality and viewing a problem in its most general state frequently reveals aspects which would not be shown in lower order cases. We also hope the lemmas in Section 4.2 (or sharper versions thereof) can be of use in other higher order problems.

The refined version of Theorem 3.3 (for dimension $n \geq 6$ and $m = 1$) proven in Li-Ricciardi[29] makes use of the "conformal Laplacian" $\Delta + R$ where R represents the scalar curvature. This operator has natural generalizations to higher order in the "GJMS operators" (see Graham, Janne, Mason, and Sparling[18]). It may then be reasonable to seek refinements of Theorem 3.3 in general (with possible restrictions on the dimension or geometry) involving the GJMS operators. A key obstacle which is inherent to higher order problems is that of positivity. One reason this causes difficulty is $H_m^p(M)$ does not have the property that if $u \in H_m^p(M)$ then $|u| \in H_m^p(M)$ if $m > 2$ (while this does hold for $m = 1$). Additionally, if a higher order operator is not a composition of coercive second order operators, then the maximum principle may not apply. We refer to Robert[39], Proposition 4.4 for an example of a sign changing minimizing solution to an equation like (3.5) in fourth order. It therefore should not be taken for granted that Theorem 3.3 can be proven by working solely with the operators $(\Delta + \alpha)^m$ for $\alpha > 0$, where positivity is trivial by iterating the second order maximum principle. Some recent results on positivity for fourth or higher order operators (with a focus on GJMS operators) can be found in Case-Malchiodi[7], Gursky-Malchiodi[19], and Hang-Yang[21]. This makes it a very interesting problem unique to higher order to formulate blow up analysis arguments while taking account of this obstacle.

References

- [1] Robert A. Adams and John J. F. Fournier. *Sobolev spaces*. Second. Vol. 140. Pure and Applied Mathematics (Amsterdam). Elsevier/Academic Press, Amsterdam, 2003, pp. xiv+305. ISBN: 0-12-044143-8.
- [2] Thierry Aubin. “Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire”. In: *J. Math. Pures Appl. (9)* 55.3 (1976), pp. 269–296. ISSN: 0021-7824,1776-3371.
- [3] Thierry Aubin. “Problèmes isopérimétriques et espaces de Sobolev”. In: *J. Differential Geometry* 11.4 (1976), pp. 573–598. ISSN: 0022-040X,1945-743X. URL: <http://projecteuclid.org/euclid.jdg/1214433725>.
- [4] Thierry Aubin. *Some nonlinear problems in Riemannian geometry*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998, pp. xviii+395. ISBN: 3-540-60752-8. DOI: 10.1007/978-3-662-13006-3. URL: <https://doi.org/10.1007/978-3-662-13006-3>.
- [5] Thierry Aubin and Yan Yan Li. “On the best Sobolev inequality”. In: *J. Math. Pures Appl. (9)* 78.4 (1999), pp. 353–387. ISSN: 0021-7824. DOI: 10.1016/S0021-7824(99)00012-4. URL: [https://doi.org/10.1016/S0021-7824\(99\)00012-4](https://doi.org/10.1016/S0021-7824(99)00012-4).
- [6] Haim Brézis and Elliott Lieb. “A relation between pointwise convergence of functions and convergence of functionals”. In: *Proc. Amer. Math. Soc.* 88.3 (1983), pp. 486–490. ISSN: 0002-9939,1088-6826. DOI: 10.2307/2044999. URL: <https://doi.org/10.2307/2044999>.
- [7] Jeffrey Case and Andrea Malchiodi. “A factorization of the GJMS operators of special Einstein products and applications”. In: *arXiv preprint arXiv:2007.10180* (2022).
- [8] Zindine Djadli, Emmanuel Hebey, and Michel Ledoux. “Paneitz-type operators and applications”. In: *Duke Math. J.* 104.1 (2000), pp. 129–169. ISSN: 0012-7094,1547-7398. DOI: 10.1215/S0012-7094-00-10416-4. URL: <https://doi.org/10.1215/S0012-7094-00-10416-4>.
- [9] Olivier Druet. “The best constants problem in Sobolev inequalities”. In: *Math. Ann.* 314.2 (1999), pp. 327–346. ISSN: 0025-5831,1432-1807. DOI: 10.1007/s002080050297. URL: <https://doi.org/10.1007/s002080050297>.
- [10] Olivier Druet and Emmanuel Hebey. “The AB program in geometric analysis: sharp Sobolev inequalities and related problems”. In: *Mem. Amer. Math. Soc.* 160.761 (2002), pp. viii+98. ISSN: 0065-9266,1947-6221. DOI: 10.1090/memo/0761. URL: <https://doi.org/10.1090/memo/0761>.

- [11] David Eric Edmunds, Donato Fortunato, and Enrico Jannelli. “Critical exponents, critical dimensions and the biharmonic operator”. In: *Arch. Rational Mech. Anal.* 112.3 (1990), pp. 269–289. ISSN: 0003-9527. DOI: 10.1007/BF00381236. URL: <https://doi.org/10.1007/BF00381236>.
- [12] Lawrence C. Evans. *Partial differential equations*. Second. Vol. 19. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2010, pp. xxii+749. ISBN: 978-0-8218-4974-3. DOI: 10.1090/gsm/019. URL: <https://doi.org/10.1090/gsm/019>.
- [13] Herbert Federer. *Geometric measure theory*. Vol. Band 153. Die Grundlehren der mathematischen Wissenschaften. Springer-Verlag New York, Inc., New York, 1969, pp. xiv+676.
- [14] Gerald B. Folland. *Real analysis*. Second. Pure and Applied Mathematics (New York). Modern techniques and their applications, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1999, pp. xvi+386. ISBN: 0-471-31716-0.
- [15] Emilio Gagliardo. “Proprietà di alcune classi di funzioni in più variabili”. In: *Ricerche Mat.* 7 (1958), pp. 102–137. ISSN: 0035-5038.
- [16] Aleksie Vladimirovich Gavrilo. “Leibniz’s formula for a covariant derivative and some of its applications”. In: *Mat. Tr.* 13.1 (2010), pp. 63–84. ISSN: 1560-750X. DOI: 10.3103/s1055134412020022. URL: <https://doi.org/10.3103/s1055134412020022>.
- [17] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Reprint of the 1998 edition. Springer-Verlag, Berlin, 2001, pp. xiv+517. ISBN: 3-540-41160-7.
- [18] C. Robin Graham et al. “Conformally invariant powers of the Laplacian. I. Existence”. In: *J. London Math. Soc. (2)* 46.3 (1992), pp. 557–565. ISSN: 0024-6107,1469-7750. DOI: 10.1112/jlms/s2-46.3.557. URL: <https://doi.org/10.1112/jlms/s2-46.3.557>.
- [19] Matthew J. Gursky and Andrea Malchiodi. “A strong maximum principle for the Paneitz operator and a non-local flow for the Q -curvature”. In: *J. Eur. Math. Soc. (JEMS)* 17.9 (2015), pp. 2137–2173. ISSN: 1435-9855,1435-9863. DOI: 10.4171/JEMS/553. URL: <https://doi.org/10.4171/JEMS/553>.
- [20] Qing Han and Fanghua Lin. *Elliptic partial differential equations*. Second. Vol. 1. Courant Lecture Notes in Mathematics. Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2011, pp. x+147. ISBN: 978-0-8218-5313-9.

- [21] Fengbo Hang and Paul C. Yang. “Sign of Green’s function of Paneitz operators and the Q curvature”. In: *Int. Math. Res. Not. IMRN* 19 (2015), pp. 9775–9791. ISSN: 1073-7928,1687-0247. DOI: 10.1093/imrn/rnu247. URL: <https://doi.org/10.1093/imrn/rnu247>.
- [22] Emmanuel Hebey. *Nonlinear analysis on manifolds: Sobolev spaces and inequalities*. Vol. 5. Courant Lecture Notes in Mathematics. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999, pp. x+309. ISBN: 0-9658703-4-0; 0-8218-2700-6.
- [23] Emmanuel Hebey. “Sharp Sobolev inequalities of second order”. In: *J. Geom. Anal.* 13.1 (2003), pp. 145–162. ISSN: 1050-6926,1559-002X. DOI: 10.1007/BF02931001. URL: <https://doi.org/10.1007/BF02931001>.
- [24] Emmanuel Hebey and Frédéric Robert. “Sobolev spaces on manifolds”. In: *Handbook of global analysis*. Elsevier Sci. B. V., Amsterdam, 2008, pp. 375–415, 1213. ISBN: 978-0-444-52833-9. DOI: 10.1016/B978-044452833-9.50008-5. URL: <https://doi.org/10.1016/B978-044452833-9.50008-5>.
- [25] Emmanuel Hebey and Michel Vaugon. “Meilleures constantes dans le théorème d’inclusion de Sobolev”. In: *Ann. Inst. H. Poincaré C Anal. Non Linéaire* 13.1 (1996), pp. 57–93. ISSN: 0294-1449,1873-1430. DOI: 10.1016/S0294-1449(16)30097-X. URL: [https://doi.org/10.1016/S0294-1449\(16\)30097-X](https://doi.org/10.1016/S0294-1449(16)30097-X).
- [26] John M. Lee. *Introduction to Riemannian manifolds*. Second. Vol. 176. Graduate Texts in Mathematics. Springer, Cham, 2018, pp. xiii+437.
- [27] John M. Lee. *Introduction to smooth manifolds*. Second. Vol. 218. Graduate Texts in Mathematics. Springer, New York, 2013, pp. xvi+708. ISBN: 978-1-4419-9981-8.
- [28] John M. Lee and Thomas H. Parker. “The Yamabe problem”. In: *Bull. Amer. Math. Soc. (N.S.)* 17.1 (1987), pp. 37–91. ISSN: 0273-0979,1088-9485. DOI: 10.1090/S0273-0979-1987-15514-5. URL: <https://doi.org/10.1090/S0273-0979-1987-15514-5>.
- [29] Yanyan Li and Tonia Ricciardi. “A sharp Sobolev inequality on Riemannian manifolds”. In: *Commun. Pure Appl. Anal.* 2.1 (2003), pp. 1–31. ISSN: 1534-0392,1553-5258. DOI: 10.1016/S1631-073X(02)02529-3. URL: [https://doi.org/10.1016/S1631-073X\(02\)02529-3](https://doi.org/10.1016/S1631-073X(02)02529-3).

- [30] Yanyan Li and Meijun Zhu. “Sharp Sobolev trace inequalities on Riemannian manifolds with boundaries”. In: *Comm. Pure Appl. Math.* 50.5 (1997), pp. 449–487. ISSN: 0010-3640,1097-0312. DOI: 10.1002/(SICI)1097-0312(199705)50:5<449::AID-CPA2>3.3.CO;2-5. URL: [https://doi.org/10.1002/\(SICI\)1097-0312\(199705\)50:5%3C449::AID-CPA2%3E3.3.CO;2-5](https://doi.org/10.1002/(SICI)1097-0312(199705)50:5%3C449::AID-CPA2%3E3.3.CO;2-5).
- [31] Pierre-Louis Lions. “The concentration-compactness principle in the calculus of variations. The limit case. I”. In: *Rev. Mat. Iberoamericana* 1.1 (1985), pp. 145–201. ISSN: 0213-2230. DOI: 10.4171/RMI/6. URL: <https://doi.org/10.4171/RMI/6>.
- [32] Genqian Liu. “Sharp higher-order Sobolev inequalities in the hyperbolic space $\mathbb{H}^n$ ”. In: *Calc. Var. Partial Differential Equations* 47.3-4 (2013), pp. 567–588. ISSN: 0944-2669,1432-0835. DOI: 10.1007/s00526-012-0528-x. URL: <https://doi.org/10.1007/s00526-012-0528-x>.
- [33] Saikat Mazumdar. “GJMS-type operators on a compact Riemannian manifold: best constants and Coron-type solutions”. In: *J. Differential Equations* 261.9 (2016), pp. 4997–5034. ISSN: 0022-0396,1090-2732. DOI: 10.1016/j.jde.2016.07.017. URL: <https://doi.org/10.1016/j.jde.2016.07.017>.
- [34] Saikat Mazumdar and Jérôme Vétois. “Existence results for the higher-order Q-curvature equation”. In: *arXiv preprint arXiv:2007.10180* (2022).
- [35] Robin Neumayer. “The Yamabe Problem”. In: *expository notes* (). URL: <https://www.math.cmu.edu/~rneumaye/YamabeProblem.pdf>.
- [36] Liviu Nicolaescu. “Lectures on the Geometry of Manifolds”. In: *lecture notes (unpublished)* (2022). URL: <https://www3.nd.edu/~lnicolae/Lectures.pdf>.
- [37] Louis Nirenberg. “On elliptic partial differential equations”. In: *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (3) 13 (1959), pp. 115–162. ISSN: 0391-173X.
- [38] Frédéric Robert. “Admissible Q -curvatures under isometries for the conformal GJMS operators”. In: *Nonlinear elliptic partial differential equations*. Vol. 540. Contemp. Math. Amer. Math. Soc., Providence, RI, 2011, pp. 241–259. ISBN: 978-0-8218-4907-1. DOI: 10.1090/conm/540/10668. URL: <https://doi.org/10.1090/conm/540/10668>.

- [39] Frédéric Robert. “Fourth order equations with critical growth in Riemannian geometry”. In: *lecture notes (unpublished)* (2009). URL: <https://dev-iecl.univ-lorraine.fr/wp-content/uploads/2021/04/LectRobertFourth.pdf>.
- [40] Richard Schoen. “Conformal deformation of a Riemannian metric to constant scalar curvature”. In: *J. Differential Geom.* 20.2 (1984), pp. 479–495. ISSN: 0022-040X,1945-743X.
- [41] Sergei Sobolev. “Sur un théorème d’analyse fonctionnelle”. In: *Rec. Math. [Mat. Sbornik] N.S.* 4(46).3 (1938), pp. 471–497.
- [42] Charles A. Swanson. “The best Sobolev constant”. In: *Appl. Anal.* 47.4 (1992), pp. 227–239. ISSN: 0003-6811,1563-504X. DOI: 10.1080/00036819208840142. URL: <https://doi.org/10.1080/00036819208840142>.
- [43] Giorgio Talenti. “Best constant in Sobolev inequality”. In: *Ann. Mat. Pura Appl. (4)* 110 (1976), pp. 353–372. ISSN: 0003-4622. DOI: 10.1007/BF02418013. URL: <https://doi.org/10.1007/BF02418013>.
- [44] Neil S. Trudinger. “Remarks concerning the conformal deformation of Riemannian structures on compact manifolds”. In: *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)* 22 (1968), pp. 265–274. ISSN: 0391-173X.
- [45] Hidehiko Yamabe. “On a deformation of Riemannian structures on compact manifolds”. In: *Osaka Math. J.* 12 (1960), pp. 21–37. ISSN: 0388-0699.