THE BINDING OF HYPERONS IN LIGHT HYPERNUCLEI

BY

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TABLE OF CONTENTS

SECTION I

INTRODUCTION

A. STRANGE PARTICLES

Strange particles are members of a class of unstable elementary particles whose masses lie somewhere between those of the neutron and the deuteron (exclusive of both) and which have the property that no member of this class is ever produced alone. The production of one particle is always accompanied by the simultaneous production of another particle of the class (usually a K meson unless the energy of the initial particle is sufficiently high to produce anti-particles). This property will be elaborated upon shortly.

A question which comes to mind immediately is: what makes strange particles "strange"? These particles were first observed in cosmic radiation and Fig. 1.1 illustrates the appearance of an event which indicated the existence of the new (strange) particles.

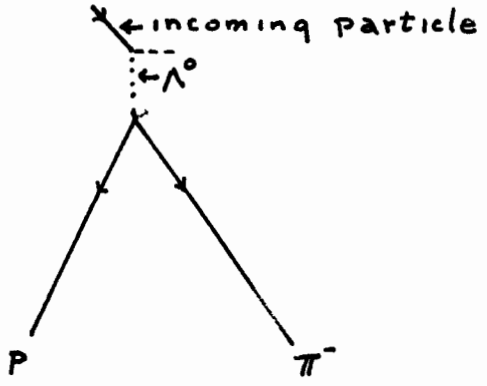


Fig. 1.1

The first observation of a "V" event by Rochester and Butler (1) indicated that a neutral particle (Λ^0 in Fig. 1.1) was produced in the high energy interaction of the incoming cosmic ray and a particle in the cloud chamber and after travelling several centimeters decayed into two oppositely charged particles (p and π^-). The rate at which these V events occurred indicate a production cross section of the order of 10^{-28} cm^2 which is the order of magnitude associated with strong interactions. Interactions of this type take place in about 10^{-23} second i.e. the proton radius divided by the speed of light. It was thought originally that the Λ^0 particles were produced by a reaction such as

$$\pi^- + p \longrightarrow \Lambda^0 + \pi^0 \qquad (1.1)$$

The half-life of the Λ^0 (measured by the distance of travel before decay) turned out to be about 10^{-10} second. One of the observed decay channels was

$$\Lambda^0 \longrightarrow p + \pi^- \qquad (1.2)$$

If (1.1) is a fast process, then the virtual process

$$\Lambda^0 \longrightarrow p + \pi^0 + \pi^- \qquad (1.3)$$

is also fast. The virtual process (1.3) can then be followed by the absorption of the π^0 by the p in which case we get the decay

$$\Lambda^0 \rightarrow p + \pi^- \qquad (1.4)$$

which should, according to the above arguments, occur in a time of the order of 10^{-23} second. The fact that the decay takes 10^{-10} second is quite an anomaly. It is this which gave these particles the qualification "strange".

There are ways of accounting for the prolonged life of the strange particles. One of these ways is to assign a high intrinsic angular momentum to the particles. The angular momentum barrier would then inhibit the decay. Pais (2) suggested an alternative approach which attributed some internal degree of freedom specified by some quantum number (call it strangeness) and that selection rules based on conservation or non-conservation of strangeness were operating in the production and decay processes. It would appear that the production processes conserve strangeness and decay processes do not. Interactions which conserve strangeness would require at least two particles of opposite strangeness to be produced simultaneously. This hypothesis of associated production is confirmed by the observation of two particles in the same cloud chamber photographs more often than the accidental coincidence rate.

Baryons are all those fermions which have a mass equal to or greater than the proton. The hyperons are the baryons which are heavier than the nucleons and are therefore strange particles. The law of conservation of baryons has never been observed to be broken: it states that the number of baryons less the number of anti-baryons is a conserved quantity. This suggests that we can assign a baryon number 1 to every baryon and a baryon number -1 to every anti-baryon and require that the total baryon number be conserved. This is the same sort of conservation law as exists for the leptons.

By assuming conservation of charge and baryon number as well as charge independence of the strong interactions, Nishijima (3) and Gell-Mann and Pais (4) were able to assign values to the isotopic spin t, its third component t_3, and strangeness S to all of the new particles. These quantities are related to the charge of the particle by

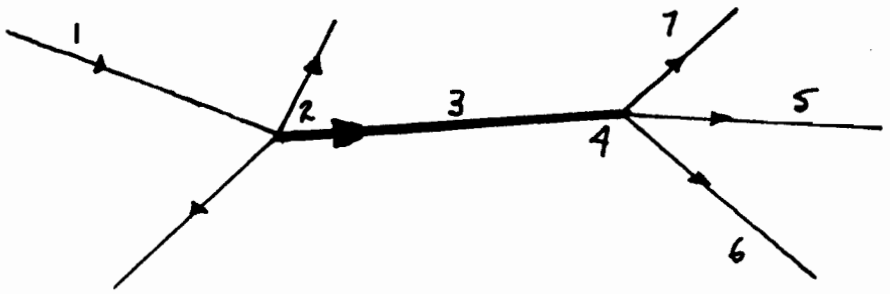
$$Q = t_3 + \frac{1}{2}(N+S) \tag{1.5}$$

where Q is the charge and N is the baryon number described above. Table 1.1 lists the strange particles (anti-particles not shown) and their measured properties.

Aside from a mass difference of approximately 77 Mev. the Λ particle seems to have properties quite similar to the Σ particles.

B. HYPERNUCLEI

Before 1953, hyperons had only been observed in cosmic rays. In 1953 Danysz and Pniewski in Warsaw observed a nuclear fragment from a cosmic ray star which stopped in their emulsion and subsequently disintegrated with an energy release of 140 to 180 Mev. The nuclear fragment was correctly interpreted as a hypernucleus containing a bound Λ° particle as well as nucleons. When the particle decayed into a nucleon and a pion, the hyperfragment disintegrated. When a Λ° particle which is bound in a nucleus decays, the pion can either be emitted as a free particle or it can be absorbed by one of the nucleons. If the pion is absorbed by a nucleon, its rest mass ($\sim$ 140 Mev.) contributes to the energy released in the hyperfragment disintergration. This was the case in the event just mentioned. An event in which a pion escapes is shown in Fig. 1.2



TYPICAL HYPERFRAGMENT EVENT

Fig. 1.2

TABLE 1.1

THE STRANGE PARTICLES

	Mass Mev	Half-life sec.	Spin	Strangeness	Isotopic spin t	t_3
Mesons						
K^+	494 ±0.2	1.22×10^{-8}	0	+1	1/2	+1/2
K^0	479.1±0.8	—	0	+1	1/2	-1/2
K^-	494 ±0.2	1.22×10^{-8}	0	-1	1/2	-1/2
$\bar{K}^0$	479.1±0.8	—	0	-1	1/2	+1/2
Hyperons						
Λ^0	1115.2±0.14	2.77×10^{-10}	1/2	-1	0	0
Σ^+	1189.4±0.25	$.83\times10^{-10}$	1/2		1	+1
Σ^0	1190.1±1.0	$<10^{-11}$	1/2		1	0
Σ^-	1196.5±0.5	1.72×10^{-10}	1/2		1	-1
Ξ^0	1311 ±8	$\sim10^{-10}$	?	-2	1/2	+1/2
Ξ^-	1319.7±5.5	$\sim2\times10^{-10}$	?		1/2	-1/2

Taken from W.S.C. Williams Introduction to Elementary Particles

Academic Press (1961)

In Fig. 1.2 a fast primary particle enters at 1 and collides with a nucleus at 2 giving rise to a shower of mesons and other objects. One of these objects is a hypernucleus 3 recognized by its dense track and by its characteristic decay at 4 into a negative pion 5 and a proton 6 and a residual nucleus 7. By analysing the decay products in such an event a determination of the binding energy of the Λ particle in the hypernucleus can be made; for the energy released in the decay is just the 37 Mev. of free decay minus the binding of the Λ particle and nucleon in the hypernucleus. This is not an entirely satisfactory method as the uncertainty is about 1/2 Mev. which unfortunately turns out to be of the same order as the Λ binding energy.

At present, however, it is the only method available. Table 2 gives the binding energy of the Λ particle in the light hypernuclei which have been identified. In the symbol for a hypernucleus, the atomic symbol gives the charge and the superscript gives the number of nuclear particles. Thus ${}_{\Lambda}He^{5}$ contains 2 protons, 2 neutrons and a Λ particle. The simplest hypernuclei ${}_{\Lambda}H^{2}$ and ${}_{\Lambda}n^{2}$ are not listed because they have never been observed although they would be easily recognized. The inference is that they do not exist. The table is not complete, however, the identification of the heavier hypernuclei is uncertain and the experimental uncertainty in the binding energy is very large. The Table is due to Dalitz and Downs. The two papers of Dalitz and Downs on hypernuclear binding energies will be referred to as DD.

TABLE 1.2

BINDING ENERGY OF A Λ - PARTICLE IN HYPERNUCLEI

${}_{\Lambda}H^{3}$	0.6 ± 0.4 Mev.
${}_{\Lambda}H^{4}$	1.8 ± 0.3
${}_{\Lambda}He^{4}$	2.0 ± 0.3
${}_{\Lambda}He^{5}$	2.9 ± 0.3
${}_{\Lambda}Li^{7}$	4.5 ± 0.4
${}_{\Lambda}Li^{8}$	5.4 ± 0.8
${}_{\Lambda}Be^{8}$	6.2 ± 0.6
${}_{\Lambda}Be^{9}$	6.4 ± 0.4

Dalitz and Downs have made a phenomenological investigation of the binding of Λ particles in the light hypernuclei (DD). The analysis is simplified by the fact the Λ binding energy is a small fraction of the nucleon binding energy per particle; the hypernucleus consists of a tightly bound nucleus and a loosely bound Λ particle. This approximation is good for ${}_{\Lambda}He^{5}$ in which the α - particle core is rather rigid. It is not bad for a ${}_{\Lambda}H^{4}$ and ${}_{\Lambda}He^{4}$ which have triton and He^{3} cores respectively. In the case of ${}_{\Lambda}H^{3}$ the approximation of a tightly bound core is not valid at all, however this problem is simple enough to treat more exactly.

Dalitz and Downs assume a reasonable range and shape for the Λ -nucleon potential and leave its depth as an adjustable parameter with which to fit the binding energy.

They use as the depth parameter, the volume integral of the Λ - nucleon potential. These turn out to be roughly 310, 280, 280, 230 Mev-Cm^3 x 10^{-39} per nucleon for the hypernuclei ${}_{\Lambda}H^3$, ${}_{\Lambda}He^4$, ${}_{\Lambda}H^4$ and ${}_{\Lambda}He^5$ respectively. To explain this marked variation in the Λ- nucleon potential DD assume that the force is spin-dependent, i.e., depends on whether the spins of the particles are aligned or anti-aligned. In ${}_{\Lambda}H^3$ where the deuteron core has a spin of 1 (both nucleon spins aligned) the Λ spin can align favourably with both of them and hence the potential per nucleon is large. In ${}_{\Lambda}H^4$ and ${}_{\Lambda}He^4$ there is one uncoupled nucleon spin with which the Λ spin can align and so the Λ - nucleon potential is correspondingly reduced. The spins in the α particle core of ${}_{\Lambda}He^5$ are all paired off so that there is no spin with which the Λ - particle can couple which results in a further reduction in the Λ - nucleon potential. The calculations of DD are not precise enough to determine whether the favoured spin alignment is parallel or anti-parallel, however, there is direct experimental evidence on this point. From the ratio of the decays of ${}_{\Lambda}He^4$ in which a free pion is emitted to those in which the pion is absorbed, it can be inferred that ${}_{\Lambda}He^4$ has spin zero. The Λ - spin therefore cancels the spin of the odd nucleon and so anti-parallel alignment is favoured. The Λ - nucleon potential found in this way is too weak to bind a Λ particle to a single nucleon in agreement with the fact that

such hypernuclei have never been observed.

The phenomenological derivation of DD does not afford a complete understanding of the Λ - nucleon system. As is the case with nucleon systems the force between two or more particles is brought about by an exchange of mesons. A complete picture would therefore require a detailed calculation involving the exchange of mesons.

At present, a quantitatively correct potential is not available. The difficulty is apparently a calculational one of finding an approximation which is both accurate and tractable. Meson theory has, however, given qualitatively correct results for the nucleon-nucleon potential, predicting its range and order of magnitude correctly as well as implying that it depends on the parity of the relative motion of the nucleons, the angle between their spins, and in the case of the spins being parallel, on the angle between the spins and the line joining the nucleons (tensor force).

A Λ particle always changes into a nucleon or another hyperon when it emits or absorbs a K meson or a pion (conservation of strangeness). A Λ particle, therefore, cannot exchange a single pion or K meson with a nucleon in the ordinary way and still remain a Λ particle. However, we can have a Λ and a nucleon interact and still be left with a Λ and a nucleon after the interaction has taken place. Consider Fig. 1.3 for the case

of the K meson type of interaction. The nucleon indicated by N emits a K meson and becomes a Λ particle. At the Λ particle the inverse process takes place, i.e., the Λ absorbs the K meson and becomes a nucleon. Wentzel (5) calculated the Λ - nucleon force due to this exchange interaction and found that it had all the wrong properties to account for the observed interactions. Another way that a Λ particle and a nucleon can interact is by an exchange of two pions. A Λ particle cannot exchange a single pion with a nucleon because the Λ particle would be left as a Σ particle. Fig. 1.4 is an illustration of the state of affairs. After the Λ particle has emitted a pion and become a Σ particle, the Σ particle can become a Λ particle again by emitting or absorbing another pion. So in all at least two pions must be involved in the interaction.

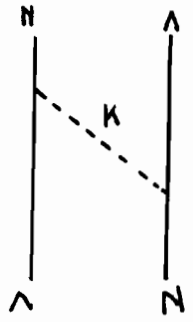


Fig. 1.3

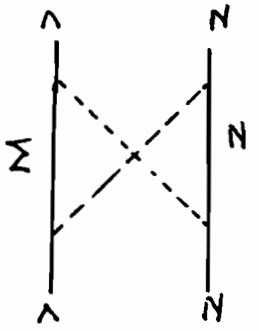


Fig. 1.4

Several authors (6, 7) have calculated the force due to this interaction. The calculations are incapable of quantitative accuracy, but the results agree qualitatively with experiment. The range of the force and its magnitude are about right. Also, there is a

spin dependence.

There are two considerations concerning the two pion interaction which come to mind immediately. The first is that there is no necessity for the Λ particle to exchange the two pions with a single nucleon. If there is a second nucleon in the vicinity the Λ can exchange one pion with each as is illustrated in Fig. 1.5.

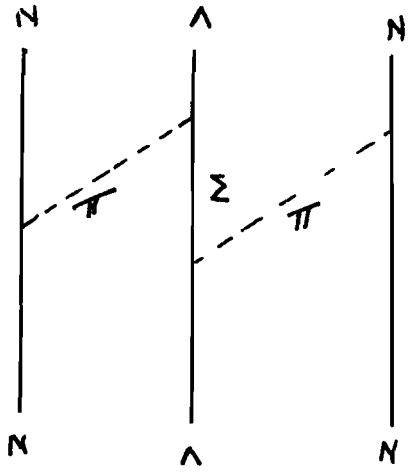


Fig. 1.5

Whenever there is more than one nucleon present, the 3-body force will be present as well as the ordinary 2-body force. For various theoretical reasons (8) the 3-body contribution is believed to be small although the arguments are not conclusive. Experimental evidence on this point is difficult to obtain. The second consideration forms the basis of this thesis. A Λ particle in a hypernucleus actually spends part of its time as a Σ particle.

This is most easily explained by saying that there is a term

in the Λ - nucleon potential which mixes Λ and Σ states of the hyperon. The effect of the Σ mixing, when it is effective is always to increase the binding energy of the Λ particle.

The mixing term is not equally effective in all hypernuclei. While the hyperon is in a Σ state, one of the nucleons must change its charge state, e.g., a neutron must become a proton or vice-versa. In ${}_{\Lambda}He^{5}$, with an α particle core, a nucleon is forbidden by the Pauli exclusion principle to change its charge state and so Σ - mixing is ineffective. In ${}_{\Lambda}H^{4}$ and ${}_{\Lambda}He^{4}$, however, the odd nucleon is quite free to change its charge state and so the potential per nucleon seen by the Λ particle is greater. In the hypertriton (${}_{\Lambda}H^{3}$), the nucleons are once again prohibited by the Pauli principle from changing their charge states; however, to the extent that the hyperon can distort the deuteron core (and the deuteron is a very loosely bound and hence easily distortable system) Σ mixing can still be effective. The object of this work is to determine whether or not potential required to bind the ${}_{\Lambda}He^{5}$ and (${}_{\Lambda}He^{4}$, ${}_{\Lambda}H^{4}$) systems is strong enough to bind the hypertriton, yet too weak to bind the hyperdeuteron. The hyperdeuteron has not been observed and is presumed not to exist.

Spin dependence of the Λ nucleon potential will be disregarded in order to determine whether or not Σ mixing alone can account for the observed variation in binding energies among the hypernuclei considered. In order to parallel the treatment of

Dalitz and Downs as well as to avoid too much calculational complexity, tensor forces have been neglected as well.

SECTION II

THE LAMBDA - NUCLEON POTENTIAL

It has been mentioned that there are two ways in which a Λ particle can interact with a nucleon a) by K meson exchange and b) by the exchange of two pions. On the basis of the following arguments, the K meson exchange potential has been neglected in the calculations. Wentzel (5) has investigated the possibility of the binding of Λ particles to nucleons by pure K meson exchange. He found that a scalar K meson resulted in a repulsive potential between a Λ and an α particle (or heavier nucleus) which is contrary to observations and that a pseudoscalar K meson gave an attractive potential but the coupling strength to give binding of a Λ to an α had to be at least as great as that of the pion-nucleon interaction. This would be alright were it not for the fact that such a large coupling constant would predict a cross-section for photoproduction of K's which is far greater than that which is observed experimentally. Observations therefore indicate that the Λ - nuclear force is not predominantly a K meson exchange type.

The only other simple scheme available for binding a Λ to a nucleus is by exchange of two pions. The calculations of Dallaporta and Ferrari (6) as well as those of Lichtenberg and Ross (7) indicate that proper binding is obtained if it is assumed that the pion - Λ coupling is the same as the pion-nucleon coup-

ling. There are no arguments at present which preclude this possibility and so we can accept the two pion exchanges as being primarily responsible for the Λ - nucleon interaction. Using this information as a basis, Gell-Mann (9) proposed the idea of global symmetry which we will, for the time being, assume to hold. Global symmetry implies that all baryons are coupled to the pion field with the same strength. This means that in the absence of the K meson field, all baryons are degenerate - they are simply different states of the same particle. Just as the magnetic field lifts the degeneracy between the spin states of a system, the K meson field lifts the degeneracy which exists between the baryons and results in the different masses. Gell-Mann (9) has shown that the baryons can be split up into a system of doublets (i. e. T = 1/2 particles) with no isotopic spin coupling between them. The doublets are

$$\begin{pmatrix} p \\ n \end{pmatrix} \quad \begin{pmatrix} \Sigma^+ \\ Y^0 \end{pmatrix} \quad \begin{pmatrix} Z^0 \\ \Sigma^- \end{pmatrix} \quad \begin{pmatrix} \Xi^0 \\ \Xi^- \end{pmatrix} \tag{2.1}$$

where $Y^0 = \frac{\Lambda^0 - \Sigma^0}{\sqrt{2}}$ and $Z^0 = \frac{\Lambda^0 + \Sigma^0}{\sqrt{2}}$. This implies that there is some isotopic spin operator which mixes Λ and Σ particles. Before determining just what form this operator must take, we will redefine Y^0 by choosing the phases of the different components of the Σ particle in conformity with Condon and Shortley (10) i. e., $Y^0 = \frac{\Sigma^0 - \Lambda^0}{\sqrt{2}}$

If the heavy particles are regarded as states of a single particle(the baryon) the interaction Lagrangian density is

$$g_{\pi_i} (\bar{b} \gamma_5 \underline{A}^i b) \cdot \underline{\pi} \tag{2.2}$$

where we have neglected a K meson exchange. b is the field operator that destroys a baryon; $g_{\pi_i} \underline{A}^i$ denotes the combination

$$g_{\pi N} \tau_N \pm (g_{\pi Y} \tau_Y + g_{\pi Z} \tau_Z) \pm g_{\pi \Xi} \tau_\Xi \tag{2.3}$$

The g's are coupling constants and the τ's are twice the isotopic spin operators corresponding to a particle of spin T=1/2. The symbols Y and Z have been used to denote the doublets

$$Y = \begin{pmatrix} \Sigma^+ \\ Y^0 \end{pmatrix} \qquad Z = \begin{pmatrix} Z^0 \\ \Sigma^- \end{pmatrix} \tag{2.4}$$

The operator $g_{\pi_i} \underline{A}^i$ then, couples certain states of the baryon. In writing down $g_{\pi_i} \underline{A}^i$ as the vector (2.3) we have imposed conservation of strangeness and isotopic spin.

And now if we also impose global symmetry

$$g_{\pi_i} \underline{A}^i = g_{\pi N} \left(\tau_N \pm (\tau_Y + \tau_Z) \pm \tau_\Xi \right) \tag{2.5}$$

The choice of equality of phase of $g_{\pi Y}$ and $g_{\pi Z}$ is made on the

assumption that all of the Σ particles are coupled identically with the pion field. There is still an arbitrariness in the choice of phase between a Σ particle and a Λ particle. This means that we should really define Y and Z as

$$Y = \begin{pmatrix} \Sigma^+ \\ \frac{\Sigma^0 \mp \Lambda^0}{\sqrt{2}} \end{pmatrix} \qquad Z = \begin{pmatrix} \frac{\Sigma^0 \pm \Lambda^0}{\sqrt{2}} \\ \Sigma^- \end{pmatrix} \tag{2.6}$$

When K mesons exchange is neglected, a baryon cannot change its strangeness and so the concept of all the heavy particles being different states of the same particle is largely lost. However Λ particles and Σ particles all have the same strangeness and it will still be convenient to regard Λ's and Σ's as being states of a single particle which we will refer to as a hyperon.

Although the exchange of K mesons has been neglected, they still tend to spoil the global symmetry in two ways. Through renormalization effects, they make the nucleon pion and the hyperon-pion coupling constants different. We neglect this effect. The second effect is to make the masses of the particles different. We will be concerned only with the Λ-Σ mass difference. With the neglect of K meson exchange, we cannot have any terms in the Λ-N potential which mix Λ's or Σ's with Ξ's. Therefore global symmetry is not required. We require only that the baryons can be split into doublets as in (2.1) and that all of the Σ's are coupled to the pion field with the same strength.

In order to determine the form of the operator which mixes Λ's and Σ's we must transform the isotopic spin operator in the representation with basis Y and Z to its equivalent in the representation in which the Λ and Σ particles are states of the hyperon i. e.

$$\begin{pmatrix} \Lambda^0 \\ \Sigma^+ \\ \Sigma^0 \\ \Sigma^- \end{pmatrix} \tag{2.7}$$

If Y and Z are written as states of the same particle i. e.

$$\begin{pmatrix} \Sigma^+ \\ \frac{\Sigma^0 \mp \Lambda^0}{\sqrt{2}} \\ \frac{\Sigma^0 \pm \Lambda^0}{\sqrt{2}} \\ \Sigma^- \end{pmatrix} \tag{2.8}$$

then the isotopic spin operator in this four component representation is

$$T = \begin{pmatrix} \tau & 0 \\ 0 & \tau \end{pmatrix} \tag{2.9}$$

where $\frac{1}{2}\tau$ is the isotopic spin of T=1/2 particle.

If U is the operator which transforms 2.8 into 2.7 i. e.

$$U = \begin{pmatrix} 0 & \pm\frac{1}{\sqrt{2}} & \mp\frac{1}{\sqrt{2}} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.10}$$

then the isotopic spin operator in the new representation is

$$T' = UTU^{-1} \tag{2.11}$$

If t_Σ is the isotopic spin operator for a T=1 particle (a Σ particle) the operator T' turns out to be

$$T' = t_\Sigma \pm \nu \tag{2.12}$$

Where ν is an operator which mixes Λ's and Σ's.

Defining $\nu_+ = \nu_1 + i\nu_2$ and $\nu_1 - i\nu_2$ we find that for the minus sign in (2.12)

$$\begin{aligned} &\nu_+ \Lambda^\circ = -\sqrt{2}\,\Sigma^+ \qquad \nu_+ \Sigma^- = \sqrt{2}\,\Lambda^\circ \qquad \nu_3 \Lambda^\circ = \Sigma^\circ \\ &\nu_- \Lambda^\circ = \sqrt{2}\,\Sigma^- \qquad \nu_- \Sigma^+ = -\sqrt{2}\,\Lambda^\circ \qquad \nu_3 \Sigma^\circ = \Lambda^\circ \end{aligned} \tag{2.13}$$

The minus sign in eqn. 2.12 has been chosen arbitrarily. It is not physically significant since it is determined by the relative phase of Λ and Σ states.

The operator $g_{\pi i} \underline{A}^i$ in eqn. (2.3) takes the final form

$$g_{\pi N}(\tau_N + t - \nu) \tag{2.14}$$

We have chosen the pion nucleon coupling to be equal to the pion-hyperon coupling consistent with Dallaporta and Ferrari (6) and Lichtenberg and Ross (7).

The form of 2.14 tells us how to translate the two-nucleon potential into hyperon nucleon potential; one simply replaces the isotopic spin operator τ of one of the nucleons in the two nucleon potential by $t_z - \nu$. The two nucleon potential is

$$V_{nn} = V_1 + V_2 \sigma_1 \cdot \sigma_2 + V_3 \vec{\tau}_1 \cdot \vec{\tau}_2 + V_4 \sigma_1 \cdot \sigma_2 \vec{\tau}_1 \cdot \vec{\tau}_2 \qquad (2.15)$$

By making the above substitution, we find that the hyperon nucleon potential has the form

$$V = V_1 + V_2 \sigma_\Lambda \cdot \sigma_n + V_3 (t - \nu) \cdot \vec{\tau}_n + V_4 \sigma_\Lambda \cdot \sigma_n (t - \nu) \cdot \vec{\tau}_n \qquad (2.16)$$

The Σ mixing term is the one proportioned to $\nu \cdot \vec{\tau}_n$

The object of this thesis is to determine whether or not the variation in binding energy of the Λ particles amongst the observed hyperfragments can be accounted for by Σ - mixing. In the light of this we will neglect any ordinary spin dependence of the hyperon-nucleon potential. The $\Sigma\Lambda$-nucleon interaction to be used is then

$$V(r) = V_1(r) + V_3(r)(t - \nu) \cdot \vec{\tau}_n \qquad (2.17)$$

The depth of $V_1(r)$ in equation (2.17) is determined from the work of DD. The hypernucleus ${}_\Lambda He^5$ consists of a Λ particle bound

to be an α particle core. The α particle is very tightly bound and so the problem is essentially a two-body one. The interaction between the Λ particle and α particle (from eqn. 2.17) is

$$V_1(r) + V_3(r) \sum_i (\xi - \nu) \cdot \tau_i$$

where i runs from 1-4 (4 nucleons in an α particle). In an α particle however

$$\sum_i \tau_i = 0$$

Hence we are left with $V_1(r)$; a non isotopic-spin dependent potential. Identical considerations hold for real spin. The spin of an α particle is zero and therefore the interaction of a Λ with the α particle would show no ordinary spin dependence either. We can use, therefore, the spin-independent part of the potential calculated by DD.

THE SIGMA-MIXING POTENTIAL

The Σ -mixing potential i.e. the coefficient of $(\xi - \nu) \cdot \tau_m$ in (2.17) will be written

$$V_3(r) = \lambda v(r) \qquad (2.18)$$

where the shape v(r) is an unnormalized function of the distance between the hyperon and the nucleon (or nucleus) and λ is the

potential well depth. DD have considered the influence of various shapes of the Λ-nucleon potential and found that the difference in their results was less than the errors inherent in the approximations. The potential used in the ${}_{\Lambda}H^4$ calculation will be Gaussian because this is simpler than a Yukawa or an exponential form for that system. However in treating the hypertriton i.e., ${}_{\Lambda}H^3$, a Yukawa potential will be used. In each calculation we will determine the minimum value of λ which will produce the observed binding energy of the hyperfragment concerned (in the case of the hypertriton λ will be determined for several values of the hyperon binding energy since this parameter is not accurately known). Having determined λ for each case, they will be compared by converting λ calculated for the ${}_{\Lambda}H^4$ problem to an equivalent Yukawa well depth by requiring that the volume integrals and intrinsic ranges of the Gaussian and equivalent Yukawa potentials be the same. Finally the minimum value of λ which is required to bind the hyperdeuteron will be determined and this will be compared with the other results.

At this point it is convenient to distinguish between the different states of the hyperon (2.7). When the hyperon is in the Λ state we will call it a Λ particle. When it is in a Σ state we will refer to it as a Σ particle. Reference will be made to the Λ-nucleon potential. This means the potential between a hyperon and a nucleon when the hyperon is in the Λ state. This differs from the Σ-nucleon potential by the term $\vec{t}_Y \cdot \vec{\tau}_n$ in (2.17).

Both of these interactions are modified by the term $\gamma \cdot \tau_a$ in (2.17) which mixes the Λ and the Σ states together.

SECTION III

THE $_{\Lambda}H^4$ HYPERNUCLEUS

A. THE POTENTIAL

The following assumptions are made in determining the potential between the hyperon and the H^3 core (DD):

i) the only distortion of the triton core is a uniform radial compression

ii) the relative hyperon-core motion is slow (adiabatic) compared with the internal motion of the core nucleons.

iii) all of the nucleons in the core have the same density distribution function

iiii) the hyperon nucleon potential is the same for all nucleons in the core.

Under these assumptions, the average potential provided by the interaction of the hyperon and the nucleons of the core nucleus is given by

$$U(r) = \sum_{i=1}^{A} \int V(|r-r_i|)\rho(r_i/a)\, d^3r_i \qquad (3.1)$$

where V denotes the individual hyperon-nucleon potential and $\rho(r/a)$ is the density distribution of a nucleon in the core nucleus. a is a scale factor related to the nuclear r.m.s. radius. The sum over i is over the A nucleons in the core. By assuming iiii) above, the potential can be written

$$U(r) = U_n \mu(r) = U_n \int v(|r-r'|)\rho(r'/a)\, d^3r' \qquad (3.2)$$

If v(r) is normalized to unity, then U_n denotes the total volume integral for all the hyperon-nucleon interactions in the hypernucleus. The subscript n denotes the number of nucleons in the core. For convenience in computation, the potential shape v(r) has been taken to be of Gaussian form.

$$v(r) = \frac{1}{(\pi\beta_G^2)^{3/2}} e^{-r^2/\beta_G^2} \qquad (3.3)$$

We have defined

$$U(r) = U_n \mu(r) \qquad (3.4)$$

The dependence of U(r) on "a", of course, depends on the shape of the nucleon distribution in the core nucleus. The nucleon distribution in the α particle is known to be approximated quite well by a Gaussian shape. It will be convenient to assume that a Gaussian will describe the distribution of the nucleons in the triton (DD). One can make an experimental determination of the r.m.s. radius of the charge distribution. By assumption this is the r.m.s. radius of the nucleus.

$$R^2_{r.m.s} = \frac{\int_0^\infty r^2\rho(r/a)\, d^3r}{\int_0^\infty \rho(r/a)\, d^3r} = \frac{\int_0^\infty r^4 e^{-r^2/a^2} dr}{\int_0^\infty r^2 e^{-r^2/a^2} dr} \qquad (3.5)$$

$$= \frac{3}{2}a^2$$

Therefore, the nucleon distribution function, properly normalized is

$$\rho(r/a) = \left(\frac{3}{2\pi R^2}\right)^{3/2} e^{-\frac{3r^2}{2R^2}} \tag{3.6}$$

Substituting (3.5) and (3.3) into (3.4) we get

$$U(r) = \frac{U_n}{(\pi r_0^2)^{3/2}} e^{-r^2/r_0^2} \tag{3.7a}$$

where

$$r_0^2 = \beta_0^2 + \frac{2}{3} R^2 \tag{3.7b}$$

The potential which must be determined is

$$V(r) = V_1(r) + V_3(r)\,(\gamma + t)\cdot\tau_m \tag{3.8}$$

It has already been pointed out that the depth of V_1 is calculated from the value of the potential per nucleon determined by DD in their investigation of the ${}_{\Lambda}He^5$ hypernucleus. They calculated volume integrals of potentials. The volume integral corresponding to $V_1(r)$ is (DD)

$$\frac{U_4}{4} = 227 \pm 11 \text{ Mev.-f}^3 \tag{3.9}$$

The factor 4 is introduced to reduce the volume integral to the volume integral per nucleon.

We will assume that the potentials $V_1(r)$ and $V_3(r)$ have the same shape (3.3) with different ranges. In Section I we discussed the pion exchange interactions which contribute to the hyperon-nucleon interaction. In order for the Λ particle to interact with a nucleon and remain in the Λ state, there must be an exchange of at least two pions. However an exchange of only a single pion is required for the Λ particle to become a Σ particle and vice versa. Therefore in the absence of K mesons exchange it is not unreasonable to choose as the range of the V_1 potential that corresponding to a two pion exchange and the range of the V_3 potential that corresponding to the exchange of a single pion. According to (3.4), (3.7a) and (3.9) the potential $V_1(r)$ of (3.8) is given by

$$V_1(r) = \frac{227 \pm 11}{(\pi r_0^2)^{3/2}} e^{-r^2/r_0^2} = V_1 e^{-r^2/r_0^2}$$
$$r_0^2 = \beta_G^2 + \frac{2}{3} R^2 \qquad (3.10)$$

R is the r.m.s. radius of the core nucleus and β_G is the range parameter for the Gaussian potential. β_G is the range parameter which is chosen so that the "intrinsic range" for the Gaussian potential is the same as that for a Yukawa potential with a range corresponding to a two pion exchange i.e. $\frac{1}{2m_\pi} \approx 0.7$ fermi where m_π is the pion mass.

The intrinsic range for these two potentials is related to the

range parameter as follows (11)

TABLE 3.1

RANGE PARAMETERS

	Potential Well	Intrinsic Range
Gaussian	$-V_G e^{-r^2/\beta_G^2}$	$b = 1.4354\,\beta_G$
Yukawa	$-V_Y \frac{e^{-r/\beta_Y}}{r/\beta_Y}$	$b = 2.1196\,\beta_Y$

If we equate the intrinsic ranges for the two wells in Table 3.1, we get the relation between the corresponding range parameters

$$\beta_G = \frac{2.1196}{1.4354}\,\beta_Y = 1.4767\,\beta_Y \qquad (3.11)$$

and for a two pion exchange $\beta_Y = \frac{1}{2m_\pi} \approx 0.7$ therefore $\beta_G = 1.034$. The range of the Σ mixing potential is chosen to be twice β_G because a single pion is exchanged with a nucleon in converting a Λ to a Σ particle. We will distinguish the two range parameters by subscripts 1 and 3 corresponding to the subscripts in (3.8)

$$\begin{aligned} \beta_{G_1} &= 1.034 \text{ fermi} \\ \beta_{G_3} &= 2.068 \text{ fermi} \end{aligned} \qquad (3.12)$$

B. THE ISOTOPIC SPIN WAVE FUNCTIONS

Consider the system ${}_\Lambda H^4$ i.e. a Λ particle bound to a triton. We know that the triton is one member of an isotopic spin doublet; the other member being He^3. The isotopic spin

of the triton (and He^3) is therefore 1/2. While the hyperon is in the Λ state, we can write the isotopic spin wave function for the system Λ + triton as simply ΛH^3 (this is not to be confused with ${}_{\Lambda}H^3$ which is the symbol for the hypertriton). If we define the proton to be the $T_3 = + 1/2$ component of the (p, n) doublet, then He^3 is the $T_3 = + 1/2$ component of the doublet (He^3, H^3). The hypernucleus ${}_{\Lambda}H^4$ therefore has $T_3 = - 1/2$.

In transforming eqn. (2.8) to (2.7) we found that the isotopic spin operator (2.9) becomes (2.12) which shows that in the representation in which the basis vectors are

$$\Lambda \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} ; \quad \Sigma^+ \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} ; \quad \Sigma^0 \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} ; \quad \Sigma^- \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{3.13}$$

The Σ particles behave like an isotopic spin triplet and the Λ particle as an isotopic singlet. In the untransformed representation i. e. the representation with bases

$$\Sigma^+ \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} ; \quad \frac{\Sigma^0 + \Lambda^0}{\sqrt{2}} \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} ; \quad \frac{\Sigma^0 - \Lambda^0}{\sqrt{2}} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} ; \quad \Sigma^- \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{3.14}$$

The Λ and Σ particles are mixed together and behave like components of isotopic spin doublets.

Since we are working in a representation with bases (3.13) we treat the Σ particles as T=1 particles. When the hyperon is in a Σ state, one of the nucleons in the core must change its τ_3 component of isotopic spin in order that the total T_3 will be conserved. If a neutron in H^3 becomes a proton (T_3=-1/2 $\longrightarrow$ T_3=+1/2) the nucleus becomes He^3. The proper combination of T=1 and T=1/2 isotopic spin wave functions which will give a wave function with isotopic spin T=1/2 and T_3=-1/2 is

$$\sqrt{\frac{2}{3}}\ \Sigma^- He^3 - \frac{1}{\sqrt{3}}\Sigma^0 H^3 \tag{3.15}$$

C. THE SPATIAL WAVE FUNCTIONS

We can now describe the ${}_{\Lambda}H^4$ hypernucleus by a two component wave function with bases

$$\Lambda H^3 \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\sqrt{\frac{2}{3}}\,\Sigma^- He^3 - \frac{1}{\sqrt{3}}\Sigma^0 H^3 \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.16}$$

The wave function for ${}_{\Lambda}H^4$ may be written

$$\Psi(r) = \frac{1}{\sqrt{p^2+q^2}} \begin{bmatrix} p\,\psi_\Lambda(r) \\ q\,\psi_\Sigma(r) \end{bmatrix} \tag{3.17}$$

where ψ_Λ and ψ_Σ are normalized functions which describe the position of the hyperon relative to the core nucleus when the hyperon is in a Λ and a Σ state respectively.

We will assume that the hyperon and the core are in a relative S state. Since the mass of the Σ is about 77 Mev. greater than the Λ mass, it follows that in order for the particle to exist in a hypernucleus, it must spend most of its time in the deep part of the potential in order to overcome the 77 Mev. mass difference. Suitable functions to describe the Λ and Σ components of (3.17) are

$$\psi_\Lambda(r) = \frac{1}{N_\Lambda} \frac{e^{-\frac{a}{2}r} - e^{-\frac{b}{2}r}}{r} \tag{3.18a}$$

$$\psi_\Sigma(r) = \frac{1}{N_\Sigma} e^{-c^2 r^2} \tag{3.18b}$$

The parameter c in (3.18b) must be large compared with the parameter in (3.18a) which describes the tail of the wave function (say a/2). If we assume that b > a we can determine "a" from the known binding energy. The Schrodinger equation in the region where no potential exists is

$$\frac{-\hbar^2}{2\mu} \nabla^2 \frac{e^{-\frac{a}{2}r}}{r} = -B \frac{e^{-\frac{a}{2}r}}{r}$$

Hence

$$a = \frac{\sqrt{8\mu B}}{\hbar} \tag{3.19}$$

where μ is the reduced mass of the Λ particle and the H^3 core and B is the observed binding energy of the Λ particle in ${}_{\Lambda}H^4$ (=1.85 Mev. - see Table 1.2).

The total wave function is a product of three parts, a) spatial, b) isotopic spin and c) ordinary spin. Since we are not considering any spin dependence of the Λ -nucleon force, there can be no effect of the potential on the spin wave function and so we can forget about it and consider only the spatial and isotopic spin parts.

The Hamiltonian for the ${}_{\Lambda}H^4$ hypernucleus can be written

$$\begin{pmatrix} H_{11} & \lambda H_{12}\langle\Lambda|V\cdot\tau|\Sigma\rangle \\ \lambda H_{12}\langle\Lambda|V\cdot\tau|\Sigma\rangle & H_{22} \end{pmatrix} \tag{3.20}$$

where the matrix elements are

$$\begin{aligned} H_{11} &= \langle\psi_\Lambda|-\frac{\hbar^2}{2\mu}\nabla^2+3V_1(r)|\psi_\Lambda\rangle \\ H_{22} &= \langle\psi_\Sigma|-\frac{\hbar^2}{2\mu}\nabla^2+3V_1(r)|\psi_\Sigma\rangle+\lambda\langle\psi_\Sigma|v(r)|\psi_\Sigma\rangle\langle\Sigma|t\cdot\tau|\Sigma\rangle \\ H_{12} &= \langle\psi_\Lambda|v(r)|\psi_\Sigma\rangle \end{aligned} \tag{3.21}$$

$+\Delta$

We have used Dirac's (12) notation to describe the matrix elements and

$$\begin{aligned} \langle\Lambda| &\equiv \langle\Lambda H^3| \\ |\Sigma\rangle &\equiv |\sqrt{\tfrac{2}{3}}\,\Sigma^- He^3 - \tfrac{1}{\sqrt{3}}\Sigma^0 H^3\rangle \end{aligned} \tag{3.22}$$

Δ is the Λ - Σ mass difference and λ is the depth of the V_3 potential (3.1). H_{11} is the total energy of the system when the hyperon is in a Λ state. The 3 in front of the $V_1(r)$ term is there because (3.8) refers to the potential between a hyperon and a single nucleon. There are three nucleons in H^3 and in He^3. H_{22} is the total energy when the hyperon is in a Σ state. We will neglect the kinetic plus potential energies in H_{22} with respect to Δ (the Λ - Σ mass difference) since the neglected terms are expected to be small (of the order of the binding energy) and Δ is about 77 Mev. We will discuss the approximations in Section V.

The Schrodinger equation for this system is then

$$\begin{pmatrix} H_{11} & \lambda H_{12}\langle\Lambda|V\cdot\tau|\Sigma\rangle \\ \lambda H_{12}\langle\Sigma|V\cdot\tau|\Lambda\rangle & \Delta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -B\begin{pmatrix} a \\ b \end{pmatrix} \qquad (3.23)$$

where B is the binding energy. The secular equation

$$(H_{11}+B)(\Delta+B) - \lambda^2 H_{12}^2\langle\Lambda|V\cdot\tau|\Sigma\rangle^2 = 0 \qquad (3.24)$$

minimizes the energy (maximizes the binding energy) with respect to the parameter a ($b=\sqrt{1-a^2}$). The smallest value of which will give the observed binding energy of Λ in ${}_{\Lambda}H^4$ is found

by minimizing the expression

$$\lambda^2 = \frac{(H_{11}+B)(A+B)}{H_{12}^2 \langle \Lambda | \vec{r}\cdot\vec{\tau} | \Sigma \rangle^2} \tag{3.25}$$

where H_{11} and H_{12} are given in (3.21). In Appendix A we discuss the variational principle for the potential given the energy of a system.

We now proceed to evaluate λ^2 in terms of the parameters a, b and c of (3.18a) and (3.18b). The total wave function (3.17) is normalized to unity over all space if

$$\int \psi_\Lambda^2 d^3r = 1$$
$$\int \psi_\Sigma^2 d^3r = 1 \tag{3.26}$$

where

$$\psi_\Lambda = \frac{1}{N_\Lambda} \frac{e^{-\frac{a}{2}r} - e^{-\frac{b}{2}r}}{r}$$
$$\psi_\Sigma = \frac{1}{N_\Sigma} e^{-c^2r^2}$$

Therefore

$$N_\Lambda^2 = 4\pi \int (e^{-ar} + e^{-br} - 2e^{-\frac{a+b}{2}r}) dr$$
$$= \frac{4\pi (a-b)^2}{ab(a+b)} \tag{3.27}$$

and so

$$N_\Lambda = \sqrt{\frac{4\pi (a-b)^2}{ab(a+b)}} \tag{3.28}$$

For the Σ part

$$N_\Sigma^2 = 4\pi \int_0^\infty r^2 e^{-2c^2 r^2} dr$$

$$= \left(\frac{\pi}{2c^2}\right)^{3/2}$$

Therefore

$$N_\Sigma = \left(\sqrt{\frac{\pi}{2}}\ \frac{1}{c}\right)^{3/2} \tag{3.29}$$

From (3.21) the H_{11} matrix element is

$$\int \psi_n(r) \left[-\frac{\hbar^2}{2\mu} \nabla^2 + 3V_1(r) \right] \psi_n(r) d^3r \tag{3.30}$$

The part of the Hamiltonian involved here does not have any isotopic spin dependence and so the isotopic spin part of the wage function gives a factor of 1.

First we will do the kinetic energy term

$$\text{K.E.} = \frac{1}{N_n^2} \cdot \left(-\frac{\hbar^2}{2\mu}\right) \cdot 4\pi \int \frac{\left(e^{-\frac{a}{2}r} - e^{-\frac{b}{2}r}\right)}{r} \frac{1}{r} \frac{\partial^2}{\partial r^2} \left(e^{-\frac{a}{2}r} - e^{-\frac{b}{2}r}\right) r^2 dr$$

$$= -\frac{4\pi\hbar^2}{N_n^2 \cdot 2\mu} \int \left(e^{-\frac{a}{2}r} - e^{-\frac{b}{2}r}\right)\left(\frac{a^2}{4} e^{-\frac{a}{2}r} - \frac{b^2}{4} e^{-\frac{b}{2}r}\right) dr$$

$$= -\frac{\pi}{N_A^2}\frac{\hbar^2}{2\mu}\left\{a^2\int_0^\infty e^{-ar}dr + b^2\int_0^\infty e^{-br}dr - (a^2+b^2)\int_0^\infty e^{-\frac{a+b}{2}r}dr\right\}$$

$$= -\frac{\pi}{N_A^2}\frac{\hbar^2}{2\mu}\left\{a+b-2\,\frac{a^2+b^2}{a+b}\right\}$$

$$= \frac{\pi}{N_A^2}\frac{\hbar^2}{2\mu}\,\frac{(a-b)^2}{a+b}$$

$$= \frac{\hbar^2}{8\mu}ab \qquad (3.31)$$

Using (3.10) the potential energy term becomes

$$\frac{3V_1}{N_A^2}\int_v\left(\frac{e^{-\frac{a}{2}r}-e^{-\frac{b}{2}r}}{r}\right)^2 e^{-r^2/r_0^2}d^3r$$

$$= \frac{12\pi V_1}{N_A^2}\int_0^\infty\left(e^{-ar}+e^{-br}-2e^{-\left(\frac{a+b}{2}\right)r}\right)e^{-r^2/r_0^2}dr$$

$$= \frac{12\pi V_1}{N_A^2}\left\{\frac{\sqrt{\pi}}{2}r_0\left[F\left(\frac{ar_0}{2}\right)+F\left(\frac{br_0}{2}\right)-2F\left(\frac{a+b}{4}r_0\right)\right]\right\} \qquad (3.32)$$

where

$$F(x) = e^{x2}\ (1-\text{erf } x) \qquad (3.33)$$

So the contribution of the potential term to H_{11} is

$$\frac{3\sqrt{\pi}\,r_0ab(a+b)}{2(a-b)^2}V_1\left[F\left(\frac{ar_0}{2}\right)+F\left(\frac{br_0}{2}\right)-2F\left(\frac{a+b}{4}r_0\right)\right]$$

Now putting $V_1 = -|V_1|$ we get

$$H_{11} = ab\left\{\frac{\hbar^2}{8\mu} - \frac{3\sqrt{\pi}\, r_0(a+b)}{2(a-b)^2}|V_0|\left[F(a\tfrac{r_0}{2}) + F(b\tfrac{r_0}{2}) - 2F(\tfrac{a+b}{4}r_0)\right]\right\} \quad (3.34)$$

In calculating the mixing term

$$\langle \Lambda | \nu\cdot\tau | \Sigma \rangle H_{12} = \langle \Lambda | \nu\cdot\tau | \Sigma \rangle \langle \psi_\Lambda | w(r) | \psi_\Sigma \rangle$$

we must evaluate the matrix elements of $\nu\tau$ between the states (3.16). If we define τ_+, τ_-, and ν_+, ν_- as

$$\begin{aligned} \tau_+ &= \tau_1 + i\tau_2 & \tau_- &= \tau_1 - i\tau_2 \\ \nu_+ &= \nu_1 + i\nu_2 & \nu_- &= \nu_1 - i\nu_2 \end{aligned} \quad (3.35)$$

then

$$\nu\cdot\tau = \tfrac{1}{2}(\nu_+\tau_- + \nu_-\tau_+) + \nu_3\tau_3 \quad (3.36)$$

The matrix elements of the operators ν are (see (2.13))

$$\begin{aligned} \langle \Sigma^+ \nu_+ \Lambda^\circ \rangle &= -\sqrt{2} = \langle \Lambda^\circ \nu_- \Sigma^+ \rangle \\ \langle \Sigma^- \nu_- \Lambda^\circ \rangle &= \sqrt{2} = \langle \Lambda^\circ \nu_+ \Sigma^- \rangle \\ \langle \Sigma^\circ \nu_3 \Lambda^\circ \rangle &= 1 = \langle \Lambda^\circ \nu_3 \Sigma^\circ \rangle \end{aligned} \quad (3.37)$$

All other matrix elements of ν are zero.

Also

$$\begin{aligned}
&\langle He^3\, \tau_+\, H^3\rangle = 2\\
&\langle H^3\, \tau_-\, He^3\rangle = 2\\
&\langle H^3\, \tau_3\, H^3\rangle = -1\\
&\langle He^3\, \tau_3\, He^3\rangle = 1
\end{aligned} \tag{3.38}$$

All other matrix elements of τ are zero. The matrix element of $\gamma\cdot\tau$ between the states (3.20) is therefore

$$\begin{aligned}
\langle \Lambda | \gamma\cdot\tau | \Sigma\rangle &= \langle \Lambda H^3 | \gamma\cdot\tau | \sqrt{\tfrac{2}{3}}\,\Sigma^- He^3 - \tfrac{1}{\sqrt{3}}\,\Sigma^0 H^3\rangle\\
&= \sqrt{3}\,\langle \Lambda H^3 | \Lambda H^3\rangle = \sqrt{3}
\end{aligned} \tag{3.39}$$

We have then

$$H_{12} = \frac{1}{N_\Lambda N_\Sigma}\int \frac{e^{-\frac{a}{2}r} - e^{-\frac{b}{2}r}}{r}\, e^{-\frac{r^2}{r_m^2} - c^2 r^2}\, d^3r \tag{3.40}$$

Let

$$r_m'^2 = \frac{r_m^2}{1 + r_m^2 c^2}$$

Then

$$\begin{aligned}
H_{12} &= \frac{1}{N_\Lambda N_\Sigma}\int \frac{e^{-\frac{a}{2}r} - e^{-\frac{b}{2}r}}{r}\, e^{-r^2/r_m'^2}\, d^3r\\
&= \frac{4\pi}{N_\Lambda N_\Sigma}\int r\left(e^{-\frac{a}{2}r} - e^{-\frac{b}{2}r}\right) e^{-r^2/r_m'^2}\, dr
\end{aligned} \tag{3.41}$$

Evaluating the integral and substituting (3.28) and (3.29) we get

$$H_{12}=\left(\frac{2}{\pi}c\right)^{\frac{3}{2}}\sqrt{\frac{ab(a+b)}{4\pi(a-b)^2}}\;2\pi^{3/2}r_m'^{2}\left\{\frac{b\,r_m'}{4}F\left(b\frac{r_m'}{4}\right)-\frac{a\,r_m'}{4}F\left(a\frac{r_m'}{4}\right)\right\} \tag{3.42}$$

where F(x) is given by (3.34). It will be convenient to introduce the following change of variables:

$$\frac{b}{a}=P \qquad r_m c=\delta$$

$$\frac{r_0 a}{2}=\xi_1 \qquad \frac{r_m a}{2}=\xi_2$$

and

$$\frac{\xi_2/2}{(1+\delta^2)^{1/2}}=x$$

H_{12}^2 becomes

$$H_{12}^2=8\sqrt{2\pi}\,\frac{P(P+1)}{(P-1)^2}\,x\,\frac{(\xi_2^2-4x^2)^{3/2}}{\xi_2^3}\left[xF(x)-Px\,F(Px)\right]^2 \tag{3.43}$$

Substituting (3.34) and (3.43) into (3.25) we get

$$\lambda^2=\frac{(\Delta+B)\left[B+P\left\{\frac{t^2a^2}{8\mu}-\sqrt{\pi}\,\frac{\xi_1(P+1)}{(P-1)^2}V_1\left[F(P\xi_1)+F(\xi_1)-2F\left(\frac{P\pm1}{2}\xi_1\right)\right]\right\}\right]}{24\sqrt{2\pi}\,\frac{P(P+1)}{(P-1)^2}x\,\frac{(\xi_2^2-4x^2)^{3/2}}{\xi_2^3}\left[xF(x)-Px\,F(Px)\right]^2} \tag{3.44}$$

Note that $B=\frac{t^2a^2}{8\mu}$ is the expression which was used to evaluate a (eqn. (3.19)). The computer program which was used to determine the minimum value of (3.44) will be discussed in Section V.

SECTION IV

THE HYPERTRITON

A. THE DEUTERON PART

The hypertriton consists of a neutron, a proton and a hyperon bound together. It is well known that the free deuteron (neutron plus proton) is bound in a total spin S=1 state with an energy of 2.226 Mev. and that the spatial wave function in the ground state of the system is symmetric. The Pauli Exclusion Principle then requires that the isotopic spin state of the deuteron be T=0. This would seem to preclude the possibility of introducing a Λ -nucleon potential which has any isotopic spin dependence in it. The deuteron however is easily distorted by the hyperon since the particles in the deuteron spend a good deal of their time outside the range of the nuclear forces. This then allows us to couple the isotopic spin of the particle to the nucleon to which it happens to be closest and then to couple the resulting hyperon-nucleon system to the remaining nucleon.

In order to see the conclusions to which we are led through the use of the model just proposed for the hypertriton we must examine the total Hamiltonian and the behaviour of the wave function under the requirements of the Pauli Principle.
The total Hamiltonian for the hypertriton is

$$H = \sum_i \text{(kinetic energy)} + \text{2 nucleon potential} + \text{hyperon-nucleon potentials}$$

The 2 nucleon potential has the form

$$W_1(r_{12}) + W_2(r_{12})\sigma_1\cdot\sigma_2 + W_3(r_{12})\tau_1\cdot\tau_2 + W_4(r_{12})\sigma_1\cdot\sigma_2\,\tau_1\cdot\tau_2 \qquad (4.1)$$

The hyperon-nucleon potential is

$$V_1(r_{1h}) + V_3(r_{1h})(\tfrac{1}{2} - \nu)\cdot\tau_1 + V_1(r_{2h}) + V_3(r_{2h})(\tfrac{1}{2} - \nu)\cdot\tau_2 \qquad (4.2)$$

The expression (4.2) commutes with $\sigma_1\cdot\sigma_2$ but does not commute with $\tau_1\cdot\tau_2$. This means that the two nucleons are in a pure ordinary spin (either S=0 or S=1) but their isotopic spin state is a mixture of T=0 and T=1.

The total wave function must be antisymmetric under the operation of interchanging the two nucleons (Pauli Principle). One would expect that the ground state of the hypertriton would be described by a spatial function which is symmetric between the two nucleons. This is because a symmetric function has less rapid fluctuations than an antisymmetric one. Since the hyperon is expected to spend most of its time as a Λ particle, we will assume that the spatial function which describes the hypertriton is symmetric between the nucleons. With a symmetric spatial part, the possible values of total isotopic spin and total ordinary spin (called simply spin) allowed by the Pauli Principle are shown in Table 4.1.

TABLE 4.1

ISOTOPIC SPIN AND ORDINARY SPIN EIGENVALUES FOR Λ HYPERTRITON WITH SYMMETRIC SPATIAL FUNCTION

Isotopic Spin of Λ	Isotopic Spin of Deuteron	Total Isotopic Spin	Spin of Deuteron
0	1	1	0
0	0	0	1

When the hyperon is a Σ particle there are a few more possibilities regarding combinations of isotopic spin. We will relax the requirement that the spatial part of the Σ wave function be symmetric and consider both symmetric and antisymmetric cases. Table 4.2 lists the combinations of isotopic spin and ordinary spin of the Σ hypertriton for both symmetric and antisymmetric spatial functions.

If the spin of the deuteron is S=0, Table 4.1 shows that the isotopic spin must be T=1. For the Σ part we see from lines 2 and 4 of Table 4.2 that there are two ways in which we can have the total isotopic spin of the hypertriton equal to 1 and the spin of the deuteron equal to 0. In one case the deuteron has T=1 and the other it has T=0. This is to be expected since the operator $\vec{\gamma}\cdot\vec{\tau}$ does not commute with $\vec{\tau}_1, \vec{\tau}_2$ of the nucleons and so the total isotopic spin of the nucleons is not conserved.

TABLE 4.2

ISOTOPIC SPIN AND ORDINARY SPIN EIGENVALUES

FOR THE Σ HYPERTRITON

Isotopic Spin of Σ	Isotopic Spin of Deuteron T	Total Isotopic Spin T_{total}	Spin of Deuteron Spatial Function	
			Sym.	Antisym.
1	1	2	0	1
1	1	1	0	1
1	1	0	0	1
1	0	1	1	0

If the spin of the deuteron is S=1, Table 4.1 shows that its isotopic spin must be T=0. Due to non-conservation of T for the nucleons, we expect that the deuteron would have a component of T=1 mixed in when the hyperon becomes a Σ. From Table 4.2 we see that the only way to have $T_{total}=0$ is to have the deuteron isotopic spin T=1 which requires the spin of the deuteron to be S=1 and the spatial wave function antisymmetric.

Once again, the hyperon spends most of its time as a Λ particle. We will refer to T=0 hypertriton or the T=1 hypertriton depending on whether the nucleons are in a T=0 or a T=1 state when the hyperon is a Λ. The T=0 hypertriton has S=1 while the T=1 hypertriton has S=0. Although the free deuteron exists only in the S=1 state, the S=0 state is not far from being bound. The presence of the third particle (the hyperon) could

result in sufficient additional binding to hold the three particles together. Therefore there is no a priori reason to exclude the possibility of a bound T=1 hypertriton. The calculations will be carried out for both cases. It is expected that the T=0 hypertriton will require less Σ -mixing due to greater attraction between the nucleons in S=1 state.

B. THE T=0 HYPERTRITON

In the T=0 hypertriton, the two nucleons are in the singlet ordinary spin state which is the favoured spin state of the free deuteron. We have seen that for this spin configuration of the nucleons, the isotopic spin of the Λ hypertriton is T=0. Therefore the isotopic spin wave function is

$$\Lambda \frac{n_1 p_2 - p_1 n_2}{\sqrt{2}} \tag{4.1}$$

The proper linear combination of two T=1 isotopic spin function which has T=0, T_3=0 (since total isotopic spin must be conserved) is

$$\frac{1}{\sqrt{3}} p_1 p_2 \Sigma^- + \frac{1}{\sqrt{3}} n_1 n_2 \Sigma^+ - \frac{1}{\sqrt{6}} (n_1 p_2 + p_1 n_2) \Sigma^\circ \tag{4.2}$$

This is the isotopic spin function for the hypertriton when the hyperon is a Σ . The spatial function which goes with (4.1) is symmetric and the one which goes with (4.2) is antisymmetric (as discussed in A of this section). If we consider (4.1) and (4.2) as basis vectors in isotopic spin space and define

$$\frac{\Lambda n_1 p_2 - p_1 n_2}{\sqrt{2}} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.3}$$

$$\frac{1}{\sqrt{3}} p_1 p_2 \Sigma^- + \frac{1}{\sqrt{3}} n_1 n_2 \Sigma^+ - \frac{1}{\sqrt{6}} (n_1 p_2 + p_1 n_2) \Sigma^0 \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.4}$$

our total wave function (neglecting the spin function because it is the same for both components) is

$$\Psi(r_1, r_2, r_3) = \frac{1}{\sqrt{\ell^2 + m^2}} \begin{pmatrix} \ell \psi_1(r_1, r_2, r_3) \\ m \psi_2(r_1, r_2, r_3) \end{pmatrix} \tag{4.5}$$

where r_1, r_2 and r_3 are defined in Fig. 4.1

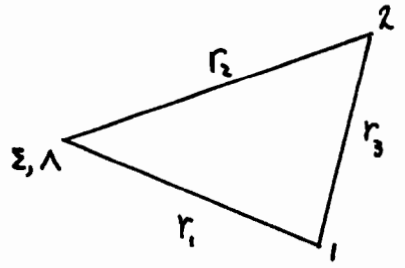


Fig. 4.1

The numbers 1 and 2 in Fig. 4.1 label the two nucleons.

The trial functions which we will use are

$$\psi_\Lambda(r_1,r_2,r_3)=\frac{1}{N_\Lambda}\left[\left(e^{-ar_1}+xe^{-br_1}\right)+\left(e^{-ar_2}+xe^{-br_2}\right)\right]\left[e^{-a_3r_3}+ye^{-b_3r_3}\right] \tag{4.6}$$

$$\psi_\Sigma(r_1,r_2,r_3)=\frac{1}{N_\Sigma}\left[e^{-cr_1}-e^{-cr_2}\right]\left[e^{-a_3r_3}+ye^{-b_3r_3}\right] \tag{4.7}$$

A trial function similar to (4.6) was used by DD. The part which describes the relative positions of the two nucleons i.e.

$$e^{-a_3r_3}+ye^{-b_3r_3} \tag{4.8}$$

was found by them to give a binding energy of 2.221 Mev. The observed binding is 2.226 Mev. so we will assume that the function (4.8) can describe the nucleons in the hypertriton. Because of a need to keep the number of parameters to a minimum for computational ease, the nucleon parts of (4.6) and (4.7) have been made the same. This does limit the flexibility of the function somewhat and it means that the nucleon function will have to adjust itself to compromise between the Λ state and the Σ state. The functions (4.6) and (4.7) do not have the proper asymptotic form to describe the function in the region where the Λ -nucleon potential vanishes. The true function would have the form

$$\frac{e^{-dr}}{r}\left(e^{-a_3r_3}+ye^{-b_3r_3}\right) \tag{4.9}$$

where r is the separation of the Λ from the deuteron.

We need not worry about the Σ part because the Σ particle is expected to spend most of its time in the deep region of the potential in order to overcome the 77 Mev. $\Lambda - \Sigma$ mass difference. In this calculation we are interested in calculating the minimum potential required to produce a certain binding. It stands to reason that that part of the function which describes the particles when they are within the region of their mutual interactions is more important than the asymptotic region where the potential is ineffective.

The Schrodinger equation for the T=0 hypertriton is the same as (3.23) i.e.,

$$\begin{pmatrix} H_{11} & \lambda H_{12}\langle \Lambda | V\cdot\tau | \Sigma \rangle \\ \lambda H_{12}\langle \Lambda | V\cdot\tau | \Sigma \rangle & H_{22} \end{pmatrix} \begin{pmatrix} \ell \\ m \end{pmatrix} = -B \begin{pmatrix} \ell \\ m \end{pmatrix} \tag{4.10}$$

where

$$H_{11} \equiv \langle \psi_\Lambda \left| -\frac{\hbar^2}{2M_\Lambda}\nabla_\Lambda^2 - \frac{\hbar^2}{2M_m}(\nabla_1^2+\nabla_2^2) + V(\Lambda-N_1) + V(\Lambda-N_2) + V(N_1-N_2) \right| \psi_\Lambda \rangle \tag{4.11}$$

$H_{22} \equiv \Delta \equiv \Lambda - \Sigma$ mass difference. We have neglected all other energy terms in H_{22} with respect to Δ .

$$H_{12}\langle \Lambda | V\cdot\tau | \Sigma \rangle \equiv \langle \psi_\Lambda | V(r_1) | \psi_\Sigma \rangle \langle \Lambda | V\cdot\tau_1 | \Sigma \rangle + \langle \psi_\Lambda | V(r_2) | \psi_\Sigma \rangle \langle \Lambda | V\cdot\tau_2 | \Sigma \rangle$$

$$K.E. = 8\pi^2 \hbar^2 \int \left\{ \frac{1}{M_n} \left(\frac{\partial \psi}{\partial r_3}\right)^2 + \left(\frac{1}{2M_n} + \frac{1}{2M_\Lambda}\right)\left[\left(\frac{\partial \psi}{\partial r_1}\right)^2 + \left(\frac{\partial \psi}{\partial r_2}\right)^2\right] \right.$$

$$\left. + \frac{1}{M_n}\left[t(231) + t(312)\right] + \frac{1}{M_\Lambda} t(123) \right\} r_1 r_2 r_3 \, dr_1 dr_2 dr_3 \qquad (4.13)$$

where $$t(i\,j\,k) = \frac{r_i^2 + r_j^2 - r_k^2}{2 r_i r_j} \frac{\partial \psi}{\partial r_i} \frac{\partial \psi}{\partial r_j}$$

and the integrals are to be carried out under the conditions that $r_1 + r_2 \geq r_3$; $r_2 + r_3 \geq r_1$ and $r_3 + r_1 \geq r_2$. M_n and M_Λ denote the nucleon and Λ particle masses respectively.

All of the potentials will have Yukawa shapes. The proton-neutron potential effective in the triplet ordinary spin state was taken to be that Yukawa potential whose range is consistent with the low energy proton-proton scattering data (13) and whose depth is determined by the known binding energy of the deuteron. The intrinsic range of this potential is 2.4995 fermi and its depth is 68.104 Mev. The part of the hyperon-nucleon potential which does not depend on isotopic spin has a volume integral of approximately 227 Mev-fermi^3. The range has been discussed in Section III and corresponds to an exchange of two pions. This means that the depth of $V_1(r)$ (3.8) for a Yukawa potential of range 1.484 fermi is 52.7 Mev.

As in (3.25) the depth of the Σ -mixing potential which is required to produce the observed binding of the hypertriton

is given by

$$\lambda^2 = \frac{(H_{11}+B)(\Delta+B)}{H_{12}^2 \langle \Lambda | \gamma \cdot \tau | \Sigma \rangle^2}$$

The Σ -mixing potential terms in (4.11) are the matrix elements of

$$\frac{e^{-k_m r_1}}{k_m r_1} \gamma \cdot \tau_1 + \frac{e^{-k_m r_2}}{k_m r_2} \gamma \cdot \tau_2 \tag{4.14}$$

taken between the isotopic spin functions (4.3) and (4.4). Using (2.13) and reading n for H^3 and p for He^3 in (3.38) we find that the matrix element of $\gamma \cdot \tau_1$ and $\gamma \cdot \tau_2$ are

$$\left\langle \frac{\Lambda\, n_1 p_2 - p_1 n_2}{\sqrt{2}} \middle| \gamma \cdot \tau_1 \middle| \frac{1}{\sqrt{3}} p_1 p_2 \Sigma^- + \frac{1}{\sqrt{3}} n_1 n_2 \Sigma^+ - \frac{1}{\sqrt{6}} (n_1 p_2 + p_1 n_2) \Sigma^0 \right\rangle = \sqrt{3} \tag{4.15}$$

$$\left\langle \frac{\Lambda\, n_1 p_2 - p_1 n_2}{\sqrt{2}} \middle| \gamma \cdot \tau_2 \middle| \frac{1}{\sqrt{3}} p_1 p_2 \Sigma^- + \frac{1}{\sqrt{3}} n_1 n_2 \Sigma^+ - \frac{1}{\sqrt{6}} (n_1 p_2 + p_1 n_2) \Sigma^0 \right\rangle = -\sqrt{3}$$

Now since $\langle \Lambda | \gamma \cdot \tau_1 | \Sigma \rangle = -\langle \Lambda | \gamma \cdot \tau_2 | \Sigma \rangle$ from (4.15) and since the function ψ_Σ (4.7) is antisymmetric with respect to r_1 and r_2 we can, for calculational purposes, replace the potential (4.14) by

$$\frac{2e^{-k_m r_1}}{k_m r_1} \gamma \cdot \tau_1 \tag{4.16}$$

The main parts of the calculations for H_{11} and H_{12} are carried out in Appendix B. Only the results will be presented here. Since the expressions are rather long, names have been assigned the various parts.

$$H_{11} = \left\{ \frac{K(a,b,a_3,a_3) + 2y\,K(a,b,a_3,b_3) + y^2 K(a,b,b_3,b_3)}{N_+^2(2a_3) + 2y\,N_+^2(a_3+b_3) + y^2 N_+^2(2b_3)} \right\} \tag{4.17a}$$

where

$$K(a,b,l,m) = \frac{\hbar^2}{2m_n}\left\{ 4lm\,F_1(a,b,l+m) + \left(1+\frac{m_n}{M_\Lambda}\right) F_2(a,b,l+m) \right.$$
$$\left. + \left(1+\frac{m_n}{m_\Lambda}\right)(\epsilon+1)(l+m)\,F_3(a,b,l+m) \right\} - V_1 F_4(a,b,l+m) \tag{4.17b}$$
$$- V_{n-n} F_5(a,b,l+m)$$

The function ϵ is introduced to avoid too much repetition. It has the following property

$$\epsilon = 0 \quad \text{for } l = m$$
$$\epsilon = 1 \quad \text{for } l \neq m$$

In other words there is a factor of two in the coefficient of F_3 in the middle term of (4.17a) which does not appear in the other two terms. V_1 and V_{n-n} are the depths of the hyperon-nucleon (isotopic spin independent part) and nucleon-nucleon potentials respectively.

$$F_i(a,b,l+m) = f_i(a,a,l+m) + 2x f_i(a,b,l+m) + x^2 f_i(b,b,l+m) \tag{4.17c}$$

where i runs from 1 to 5. $N_+^2(\ell+m) = F_i(a,b,\ell+m)$

The f_i's are

$$f_1(p,q,m) = \frac{8}{(p+q)^3}\left\{\frac{1}{m^3} + \frac{(p+q+m)\left[(p+q)(p+m)+(p+q)(q+m)+(p+m)(q+m)\right]-(p+m)(q+m)(p+q)}{(p+m)^3(q+m)^3}\right\} \quad (4.17d)$$

$$f_2(p,q,m) = \frac{16pq}{m^3(p+q)^3} \quad (4.17e)$$

$$f_3(p,q,m) = pq\,\frac{(p+q+m)^2+pq+pm+qm}{(p+q)^3(p+m)^3(q+m)^3} \quad (4.17f)$$

$$f_4(p,q,m) = \frac{8(p+q+m+k)^2}{k(p+q+k)^2}\left\{\frac{1}{(p+m)^3(q+m+k)^2}+\frac{1}{(q+m)^3(p+m+k)^2}\right.$$
$$\left.+\frac{1}{m^3(p+q+m+k)^2}+\frac{1}{(k+m)^2(p+q+m)^3}\right\}+\frac{8}{k(p+q+k)^2}\left\{ \quad (4.17g)$$

$$+\frac{pm}{(p+m)^3(q+m+k)^2}+\frac{qm}{(q+m)^3(a+m+k)^2}+\frac{(p+q)m}{(m+k)^2(p+m+q)^3}\right\}$$

$$f_5(p,q,m) = \frac{8(p+q+m+k_3)^2}{k_3(p+q)^3}\left\{\frac{1}{(p+m+k_3)^2(q+m+k_3)^2}+\frac{1}{(m+k_3)^2(p+q+m+k_3)^2}\right\} \quad (4.17h)$$
$$+\frac{8pq}{k_3(p+q)^3(p+m+k_3)^2(q+m+k_3)^2}$$

The matrix element H_{12} is

$$\langle\Lambda|V_2|\Sigma\rangle I_{12} = \frac{\sqrt{3}\left\{F_6(a,b,c,2a)+2yF_6(a,b,c,a+b)+y^2F_6(a,b,c,2b)\right\}}{N_\Lambda N_\Sigma} \quad (4.18a)$$

where

$$F_6(a,b,c,m) = f_6(a,c,m)+x\,f_6(b,c,m) \quad (4.18b)$$

and

$$f_6(p,c,n) = \frac{8(p+c+n+k_m)^2}{k_m(p+c+k_m)^2}\left\{\frac{1}{n^3(p+c+n+k_m)^2}\right.$$

$$\left. - \frac{1}{(c+n)^3(p+n+k_m)^3} + \frac{1}{(p+n)^3(c+n+k_m)^2} - \frac{1}{(p+c+n)^3(n+k_m)^2}\right\} \quad (4.18c)$$

The factor $\sqrt{3}$ in (4.18a) comes from the matrix element of (4.15).

The expressions (4.17a) and (4.18a) are to be substituted into (3.25). (3.25) is then minimized with respect to the seven parameters a, b, x, a_3, b_3, y and c. The quantities k, k_3 and k_m which appear in (4.17) and (4.18) are the inverse range parameters of the 2 pion hyperon-nucleon (V_1), the nucleon-nucleon and the Σ -mixing (V_3) potentials respectively. The values of all of the parameters used in the calculations are listed in Table 4.3.

TABLE 4.3

PARAMETERS USED IN T=0 HYPERTRITON CALCULATION

$\frac{\hbar^2}{2M_n}$	$\frac{M_n}{M_\Lambda}$	V_{n-n}	k_3	V_1	k	k_m
Mev-fermi^2		Mev	(fermi)$^{-1}$	Mev	(fermi)$^{-1}$	(fermi)$^{-1}$
20.88	.842	68.104	.848	52.7	1.4286	.7143

The computations and results are shown in Section V.

C. THE T=1 HYPERTRITON

In the T=1 hypertriton, the two nucleons are in the singlet ordinary spin state. In this state, the nucleon interaction is weaker than in the triplet state. If the T=1 hypertriton is to be bound then the contribution to the potential from the hyperon-nucleon interactions must be stronger when the two nucleons are in the isotopic spin triplet state. In Table 4.2 we see that there are two ways of forming a T=1, T_z=0 isotopic spin wave function. In one case the isotopic spin of the deuteron part is 1 and in the other case it is 0. The two total isotopic spin eigenfunctions are

$$\frac{(n_1 p_2 - p_1 n_2)\,\Sigma^0}{\sqrt{2}} \tag{4.19}$$

and

$$\frac{p_1 p_2\,\Sigma^- - n_1 n_2\,\Sigma^+}{\sqrt{2}} \tag{4.20}$$

Table 4.2 also shows that if the total spin of the nucleons is to be 0 then the spatial function which goes with (4.19) must be anti-symmetric with respect to the two nucleons whereas the spatial function which goes with (4.20) must be symmetric. Recall that (3.8) has in it a term $V_3(r)\,t\cdot\tau_n$ where t represents the isotopic spin operator of the Σ particle. $t\cdot\tau_n$ does not commute with $\tau_1\cdot\tau_2$ of the nucleons therefore it will mix T=0 and T=1 states of the

nucleons together i. e., $\underline{t}\cdot\underline{\tau}_1$ and $\underline{t}\cdot\underline{\tau}_2$ have a non-zero matrix element between states (4. 19) and (4. 20).

The T=1 isotopic spin function for the Λ-hypertriton is

$$\frac{\Lambda(n_1 p_2 + p_1 n_2)}{\sqrt{2}} \tag{4.21}$$

The Σ functions are given by (4. 19) and (4. 20). If we let these three functions be represented by basis vectors in isotopic spin space i. e.

$$|\Lambda\rangle \equiv \frac{\Lambda(n_1 p_2 + p_1 n_2)}{\sqrt{2}} \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{4.22a}$$

$$|\Sigma\rangle \equiv \frac{(n_1 p_2 - p_1 n_2)\Sigma^0}{\sqrt{2}} \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{4.22b}$$

$$|\Sigma'\rangle \equiv \frac{p_1 p_2 \Sigma^- - n_1 n_2 \Sigma^+}{\sqrt{2}} \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{4.22c}$$

the total wave function can be written

$$\Psi(r_1, r_2, r_3) = \frac{1}{\sqrt{p^2+q^2+r^2}} \begin{pmatrix} p\,\psi_\Lambda(r_1, r_2, r_3) \\ q\,\psi_\Sigma(r_1, r_2, r_3) \\ r\,\psi_{\Sigma'}(r_1, r_2, r_3) \end{pmatrix} \tag{4.23}$$

For the functions ψ_Λ and ψ_Σ we can use (4.6) and (4.7). We require a function $\psi_{\Sigma'}$ to describe the Σ particle relative to the two nucleons which is symmetric with respect to them. The function

$$\psi_{\Sigma'} = \frac{1}{N_{\Sigma'}} \left(e^{-cr_1} + e^{-cr_2}\right)\left(e^{-a_3 r_3} + y e^{-b_3 r_3}\right) \qquad (4.24)$$

is chosen because the integrals involved in using it are identical with those already calculated for the T=0 case. The three spatial functions, once again, are

$$\psi_\Lambda = \frac{1}{N_\Lambda}\left[\left(e^{-ar_1} + x e^{-br_1}\right) + \left(e^{-ar_2} + x e^{-br_2}\right)\right]\left[e^{-a_3 r_3} + y e^{-b_3 r_3}\right] \qquad (4.25a)$$

$$\psi_\Sigma = \frac{1}{N_\Sigma}\left[e^{-cr_1} - e^{-cr_2}\right]\left[e^{-a_3 r_3} + y e^{-b_3 r_3}\right] \qquad (4.25b)$$

$$\psi_{\Sigma'} = \frac{1}{N_{\Sigma'}}\left[e^{-cr_1} + e^{-cr_2}\right]\left[e^{-a_3 r_3} + y e^{-b_3 r_3}\right] \qquad (4.25c)$$

The Hamiltonian for the T=1 hypertriton is

$$\begin{pmatrix} H_{11} & \lambda H_{12}\langle\Lambda|\nu\cdot\tau|\Sigma\rangle & \lambda H_{13}\langle\Lambda|\nu\cdot\tau|\Sigma'\rangle \\ \lambda H_{12}\langle\Lambda|\nu\cdot\tau|\Sigma\rangle & H_{22} & \lambda H_{23}\langle\Sigma|t\cdot\tau|\Sigma'\rangle \\ \lambda H_{13}\langle\Lambda|\nu\cdot\tau|\Sigma'\rangle & \lambda H_{23}\langle\Sigma|t\cdot\tau|\Sigma'\rangle & H_{33} \end{pmatrix} \qquad (4.26)$$

The matrix elements $H_{12}\langle\Lambda|V\cdot\tau|\Sigma\rangle$, $H_{13}\langle\Lambda|V\cdot\tau|\Sigma'\rangle$ etc. represent the following:

$$H_{12}\langle\Lambda|V\cdot\tau|\Sigma\rangle \equiv \langle\psi_\Lambda|v(r_1)|\psi_\Sigma\rangle\langle\Lambda|V\cdot\tau_1|\Sigma\rangle + \langle\psi_\Lambda|v(r_2)|\psi_\Sigma\rangle\langle\Lambda|V\cdot\tau_2|\Sigma\rangle \quad (4.27a)$$

$$H_{13}\langle\Lambda|V\cdot\tau|\Sigma'\rangle \equiv \langle\psi_\Lambda|v(r_1)|\psi_{\Sigma'}\rangle\langle\Lambda|V\cdot\tau_1|\Sigma'\rangle + \langle\psi_\Lambda|v(r_2)|\psi_{\Sigma'}\rangle\langle\Lambda|V\cdot\tau_2|\Sigma\rangle \quad (4.27b)$$

$$H_{23}\langle\Sigma|t\cdot\tau|\Sigma'\rangle \equiv \langle\psi_\Sigma|v(r_1)|\psi_{\Sigma'}\rangle\langle\Sigma|t\cdot\tau_1|\Sigma'\rangle + \langle\psi_\Sigma|v(r_2)|\psi_{\Sigma'}\rangle\langle\Sigma|t\cdot\tau_2|\Sigma'\rangle \quad (4.27c)$$

The values of the isotopic spin matrix elements are, using (2.13)

$$\begin{aligned}
\langle\Lambda|V\cdot\tau_1|\Sigma\rangle &= -\langle\Lambda|V\cdot\tau_2|\Sigma\rangle \\
&= -\left\langle\Lambda\,\frac{(n_1p_2+p_1n_2)}{\sqrt{2}}\middle|V\cdot\tau_2\middle|\frac{(n_1p_2-p_1n_2)}{\sqrt{2}}\Sigma^0\right\rangle = -1 \\
\langle\Lambda|V\cdot\tau_1|\Sigma'\rangle &= +\langle\Lambda|V\cdot\tau_2|\Sigma'\rangle \\
&= \left\langle\Lambda\,\frac{(n_1p_2+p_1n_2)}{\sqrt{2}}\middle|V\cdot\tau_2\middle|\frac{p_1p_2\Sigma^- - n_1n_2\Sigma^+}{\sqrt{2}}\right\rangle = \sqrt{2}
\end{aligned} \quad (4.28)$$

$$\begin{aligned}
\langle\Sigma|t\cdot\tau_1|\Sigma'\rangle &= -\langle\Sigma|t\cdot\tau_2|\Sigma'\rangle \\
&= -\left\langle\frac{n_1p_2-p_1n_2}{\sqrt{2}}\Sigma^0\middle|t\cdot\tau_2\middle|\frac{p_1p_2\Sigma^- - n_1n_2\Sigma^+}{\sqrt{2}}\right\rangle = \sqrt{2}
\end{aligned}$$

In (4.27a) the coefficients of the isotopic spin factors are equal and opposite in sign since ψ_Σ (4.25b) is antisymmetric in r_1 and r_2 while ψ_Λ (4.25a) is symmetric. For the same reason the coefficients of the isotopic spin factors in (4.27c) are equal

and opposite in sign. With these facts and the matrix elements (4.28) we can rewrite (4.27).

$$
\begin{aligned}
H_{12}\langle \Lambda | V\cdot\tau | \Sigma \rangle &= -2\langle \psi_\Lambda | w(r_1) | \psi_\Sigma \rangle \\
H_{13}\langle \Lambda | V\cdot\tau | \Sigma' \rangle &= 2\sqrt{2}\,\langle \psi_\Lambda | w(r_1) | \psi_{\Sigma'} \rangle \\
H_{23}\langle \Sigma | t\cdot\tau | \Sigma' \rangle &= 2\sqrt{2}\,\langle \psi_\Sigma | w(r_1) | \psi_{\Sigma'} \rangle
\end{aligned}
\tag{4.29}
$$

If we let the H's represent the spatial matrix elements, including the factors of 2, then (4.26) can be written

$$
\begin{pmatrix}
H_{11} & -\lambda H_{12} & \sqrt{2}\lambda H_{13} \\
-\lambda H_{12} & H_{22} & \sqrt{2}\lambda H_{23} \\
\sqrt{2}\lambda H_{13} & \sqrt{2}\lambda H_{23} & H_{33}
\end{pmatrix}
\tag{4.30}
$$

Now we will be consistent with the approximation made in the T=0 hypertriton case if we neglect all of the energy terms in the Σ parts of the Hamiltonian with respect to Δ the Λ-Σ mass difference. The Hamiltonian (4.30) becomes

$$
\begin{pmatrix}
H_{11} & -\lambda H_{12} & \sqrt{2}\lambda H_{13} \\
-\lambda H_{12} & \Delta & 0 \\
\sqrt{2}\lambda H_{13} & 0 & \Delta
\end{pmatrix}
\tag{4.31}
$$

The secular equation the roots of which determine the energy (or the potential - given the energy) (Appendix A) is

$$(\Delta - E)(H_{11} - E) - \lambda^2 (H_{12}^2 + 2 H_{13}^2) = 0$$

or

$$\lambda^2 = \frac{(\Delta - E)(H_{11} - E)}{H_{12}^2 + 2 H_{13}^2} \tag{4.32}$$

H_{11} and H_{12} are given by (4.17) and (4.18) respectively. H_{13} is given by (Appendix A).

$$H_{13} = \left\{ \frac{F_7(a, b, c, 2a_3) + 2y F_7(a, b, c, a_3 + b_3) + y^2 F_7(a, b, c, 2b_3)}{N_\Lambda N_{\Sigma'}} \right\} \tag{4.33a}$$

where

$$F_7(a, b, c, n) = f_7(a, c, n) + x f_7(b, c, n) \tag{4.33b}$$

and

$$f_7(p, c, n) = \frac{8(p+c+n+k_m)^2}{k_m (p+c+k_m)^2} \left\{ \frac{1}{n^3 (p+c+n+k_m)^2} + \frac{1}{(c+n)^3 (p+n+k_m)^2} + \frac{1}{(p+n)^3 (c+n+k_m)^2} + \frac{1}{(p+c+n)^3 (n+k_m)^2} \right\} \tag{4.33c}$$

The results for this system are shown and discussed in Section V along with the results for all of the other systems considered.

D. THE HYPERDEUTERON

The hyperdeuteron consists of a Λ particle bound to a nucleon. Such a system has never been observed and the inference is that it does not exist. It is worthwhile to see what the strength of the Σ -mixing potential must be in order to allow the hyperdeuteron to be just bound i.e., binding energy B=0.

The isotopic spin states for the Λ and Σ parts of the hyperdeuteron wave function are

$$\Lambda p \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.34a}$$

$$\frac{1}{\sqrt{3}}\Sigma^{\circ}p - \sqrt{\frac{2}{3}}\Sigma^{+}n \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.34b}$$

As in the previous cases we can write the total wave function

$$\Psi(r) = \frac{1}{\sqrt{\ell^2+m^2}} \begin{pmatrix} \ell\,\psi_{\Lambda}(r) \\ m\,\psi_{\Sigma}(r) \end{pmatrix} \tag{4.35}$$

The trial wave functions which we will use to describe the system are

$$\psi_{\Lambda}(r) = \frac{1}{N_{\Lambda}} \frac{e^{-\frac{a}{2}r} - e^{-\frac{b}{2}r}}{r} \tag{4.36a}$$

where

$$N_\Lambda = \sqrt{\frac{4\pi(a-b)^2}{ab(a+b)}} \tag{4.36b}$$

$$\psi_\Sigma(r) = \frac{1}{N_\Sigma} e^{-cr} \tag{4.37a}$$

where

$$N_\Sigma = \sqrt{\frac{\pi}{c^3}} \tag{4.37b}$$

Once again we will use Yukawa potentials to describe the hyperon nucleon interaction (3.8). The depth of the potential V_1 is 52.7 Mev. (Table 4.3) and its inverse range parameter is 1.4286 (fermi)$^{-1}$. The inverse range of the Σ -mixing potential is given by .7143 (fermi)$^{-1}$ while its depth λ is to be determined by minimizing an expression like (3.25) with respect to a, b and c of (4.36) and (4.37).

The matrix element of the operator $\nu \cdot \tau$ (3.8) between the states (4.34a) and (4.34b) is determined using (2.13)

$$\left\langle \frac{1}{\sqrt{3}} \Sigma^0 p - \sqrt{\frac{2}{3}} \Sigma^+ n \middle| \nu \cdot \tau \middle| \Lambda p \right\rangle = \sqrt{3} \tag{4.38}$$

The Hamiltonian is

$$\begin{pmatrix} H_{11} & \sqrt{3}\,\lambda\, H_{12} \\ \sqrt{3}\,\lambda H_{12} & \Delta \end{pmatrix} \tag{4.39}$$

Again we have neglected the configurational energy of the Σ - nucleon system with respect to Δ - the Λ-Σ mass difference. By obtaining the secular equation corresponding to (4.37) and solving it for λ^2 we obtain

$$\lambda^2 = \frac{(H_{11}+B)(\Delta+B)}{3\,H_{12}^2} \tag{4.40}$$

where

$$\begin{aligned} H_{11} &= \langle \psi_\Lambda | -\frac{\hbar^2}{2\mu}\nabla^2 - V_1 \frac{e^{-kr}}{kr} | \psi_\Lambda \rangle \\ H_{12} &= \langle \psi_\Lambda | \frac{e^{-k_m r}}{k_m r} | \psi_\Sigma \rangle \end{aligned} \tag{4.41}$$

and μ is the reduced mass of a Λ and a nucleon (i. e. $\mu = \frac{M_n M_\Lambda}{M_n + M_\Lambda}$). The kinetic energy integral has already been done (3.31). The result is

$$K.E. = \frac{\hbar^2}{8\mu} ab \tag{4.42}$$

The potential energy integral is

$$P.E. = -\frac{V_1}{k}\frac{1}{N_1^2}\int\left(\frac{e^{-\frac{a}{2}r}-e^{-\frac{b}{2}r}}{r}\right)^2\frac{e^{-kr}}{r}d^3r$$

$$= -\frac{4\pi V_1}{k}\frac{1}{N_1^2}\int\frac{\left(e^{-ar}+e^{-br}-2e^{-\frac{a+b}{2}r}\right)e^{-kr}}{r}dr \qquad (4.43)$$

By differentiating the integral in (4.43) with respect to k, performing the resulting integral over r and then integrating with respect to k, (4.43) becomes

$$P.E. = -\frac{4\pi V_1}{k}\frac{1}{N_1^2}\ln\frac{\left(\frac{a+b}{2}+k\right)^2}{(a+k)(b+k)} \qquad (4.44)$$

Put (3.27) in (4.42) we get

$$P.E. = -\frac{V_1}{k}\frac{ab(a+b)}{(a-b)^2}\ln\frac{\left(\frac{a+b}{2}+k\right)^2}{(a+k)(b+k)} \qquad (4.45)$$

Hence

$$H_{11} = \frac{\hbar^2 ab}{8\mu} - \frac{V_1}{k}\frac{ab(a+b)}{(a-b)^2}\ln\frac{\left(\frac{a+b}{2}+k\right)^2}{(a+k)(b+k)} \qquad (4.46)$$

The term H_{12} of (4.39) written out is

$$H_{12} = \frac{4\pi}{N_1 N_2}\int\left(e^{-\frac{a}{2}r}-e^{-\frac{b}{2}r}\right)\frac{e^{-k_m r}}{k_m r}e^{-cr}r^2dr$$

$$= 2c^{3/2}\sqrt{\frac{ab(a+b)}{(a-b)^2}}\int\left(e^{-(\frac{a}{2}r+c+k_m)r} - e^{-(\frac{b}{2}r+c+k_m)r}\right)dr$$

$$= \frac{2c^{3/2}}{k_m}\sqrt{\frac{ab(a+b)}{(a-b)^2}}\left\{\frac{\frac{b-a}{2}}{(\frac{a}{2}+c+k_m)(\frac{b}{2}+c+k_m)}\right\}$$

$$= \frac{c^{3/2}\sqrt{ab(a+b)}}{k_m(\frac{a}{2}+c+k_m)(\frac{b}{2}+c+k_m)} \qquad (4.47)$$

Equation (4.40) becomes

$$\lambda^2 = \frac{\left[\left\{\frac{k^2ab}{8\mu} - \frac{V_1}{k}\frac{ab(a+b)}{(a-b)^2}\ln\frac{(\frac{a+b}{2}+k)^2}{(a+k)(b+k)}\right\}+B\right]\left[\Delta+B\right]}{3\,\frac{ab(a+b)c^3}{k_m^2(\frac{a}{2}+c+k_m)^2(\frac{b}{2}+c+k_m)^2}}$$

It was anticipated that the approximation of neglecting the energy with respect to the Λ - Σ mass difference might not be so good in this case. If the approximation is bad, the correct value of λ must be larger than the value given by (4.40). Since the hyperdeuteron is not expected to be bound, we will use the approximation above and discuss it in the results.

SECTION V

COMPUTATIONS AND RESULTS

A. MINIMIZING ROUTINE

We have calculated expressions for the strength of the Σ-mixing potential required to produce a certain binding energy for 4 different systems 1) the ${}_{\Lambda}H^4$ (or ${}_{\Lambda}He^3$) system, 2) the T=0 hypertriton, 3) the T=1 hypertriton and 4) the hyperdeuteron. In order to be able to refer to these well depths individually we will call them λ_1, λ_2, λ_3 and λ_4 respectively.

In Appendix A we have shown that the minimum potential necessary to produce a given binding will, when used in a variational calculation to determine the maximum binding energy, produce the desired binding energy. The problem at hand therefore is to minimize the expressions which we have derived in Sections III and IV for λ^2.

The computations were performed on the IBM 7070 Digital Computer at the U.S. Naval Ordnance Laboratory, Corona, California. A minimization routine was written and tested on different functions of varying degrees of complexity and was found to give results the accuracy of which were determined by some of the input data i.e., the size of the final grid.

The routine is quite general. It can accommodate any

reasonably well behaved function (provided it is known to have a minimum) of an arbitrary number of variables.

The procedure is to evaluate the function for some initial set of values of the parameters. Having done this, the first derivatives of the function for all of the variables are calculated at the starting point. On the first pass, the derivatives which are positive are reversed in sign so that all derivatives are negative. We assume that the minimum which we seek is in the direction in which all the derivatives are negative hence the routine which is being described can find only the closest minimum to the starting point. Once the derivatives are found, each variable is incremented (or decremented) by an amount proportioned to the partial derivative divided by the gradient i.e.,

$$X(N) = X(N) - \frac{\frac{\partial \lambda}{\partial X(N)}}{|GRAD|} \delta \, SIGN(N) \qquad (5.1)$$

where X(N) is the value of the Nth variable where the partial derivative was calculated, δ is the grid size and SIGN(N) determines the direction towards the minimum for each variable X(N). When a partial derivative, which has been running negative, goes positive the corresponding variable is held fixed and remains so until all of the remaining partial derivatives go positive. When this happens, all of the SIGN(N)'s are reversed and the grid size is diminished. The process repeats until the minimum

grid size is reached whereupon, the minimum value of the function along with the values of all of the variables are punched out or written on tape whatever the case may be.

B. DISCUSSION OF THE CALCULATIONS

In the ${}_{\Lambda}H^4$ problem there is considerable uncertainty in the size of the H^3 core. Since we do not have a value for the r.m.s. radius of H^3 we will consider two values which (it is hoped) will bracket the actual value. Dalitz and Downs (DD), based on arguments concerning the coulomb energy of He^3 and the known r.m.s. radius of the α particle (=1.44 fermi), choose 1.38 and 1.58 as the lower and upper limits respectively. We will use the same values.

It is expected that the presence of the hyperon in ${}_{\Lambda}H^4$ will distort the H^3 core. The only distortion which we consider is a uniform radial compression. This means that the range we should use for the r.m.s. radius of H^3 should be somewhat lower than 1.38 to 1.58 however it is felt that these values will serve to indicate the dependence of the well depth on this parameter and allow us to draw some conclusions.

The energy with which the hyperon is bound in the hypertriton is not well known since the uncertainty in the measurement is just about equal to the quantity measured. Because of this the well depth for the T=0 hypertriton has been calculated for two values of binding energy, $B_{\Lambda}=0$ and $B_{\Lambda}=.5$. The calculation

for the T=1 hypertriton was carried out to determine the strength of the Σ mixing potential which will result in zero binding energy for the whole system. The total binding energy of the hypertriton is $B = B_\Lambda$ + B deut. where B deut. is the binding energy of the deuteron. In the T=1 hypertriton, the deuteron part is in a singlet ordinary spin state. In order to bind the system, a larger Σ -mixing potential is required in order to make up for the reduced attraction of the nucleons.

As in the T=1 hypertriton, the calculation for the hyperdeuteron was calculated with 0 binding energy. This calculation was performed after the results of the T=1 hypertriton were known in order to check for consistency in the results i.e., the hyperdeuteron should not be bound by a potential which will bind the T=0 hypertriton and the ${}_\Lambda H^4$ system.

In the tables to follow, the potential parameters refer to the depth and range of the two parts of (3.8) and (when applicable) to the nucleon-nucleon potential. The hyperon-nucleon potential (3.8) is

$$V(r) = V_1(r) + \lambda v(r)\,(t-\gamma)\cdot\tau_m$$

In Table 5.1, the potential parameters refer to Gaussian potentials. In all cases, the $V_1(r)$ potential has a range corresponding to an exchange of 2 pions while the v(r) potential has a one pion range. The effective potential $V_1(r)$ for the ${}_\Lambda H^4$ case is

given by (3.10) i.e.,

$$V_1(r)_{\wedge H^4} = \frac{227}{(\pi r_0^2)^{3/2}} e^{-r^2/r_0^2} = V_{1G} \frac{\beta_G^3}{r_0^3} e^{-r^2/r_0^2} \qquad (5.2)$$

where $V_{1G} = \frac{227}{(\pi\beta_G^2)^{3/2}}$, $r_0^2 = \beta_G^2 + \frac{2}{3}R^2$. β_G is the range parameter for a Gaussian well which has the same intrinsic range as a Yukawa well with a two pion range. R is the r.m.s. radius of the H^3 core. The Σ-mixing potential $\lambda v(r)$ has the same shape but has a β_G corresponding to a single pion exchange that is, twice the β_G for $V_1(r)$. The quantity corresponding to r_o for this potential is called r_m.

In Tables 5.2 and 5.4 the potentials used are all Yukawa types i.e., $\frac{e^{-kr}}{kr}$. In the case of the two hypertriton problems we also have a nucleon-nucleon potential. The range parameter for the $V_1(r)$ potential is 1.4286(fermi)$^{-1}$; for the v(r) (Σ - mixing) potential, it is .7143(fermi)$^{-1}$. The nucleon-nucleon potentials are shown following the hypertriton tables.

The trial wave functions used in the calculations are reproduced after the corresponding table for easy reference.

TABLE 5.1

Σ -MIXING POTENTIAL WELL DEPTH IN

$_{\Lambda}H^4$ (AND $_{\Lambda}He^4$)

R	r_o	$3V_1 \frac{\beta_s^3}{r_o^3}$	r_m	a	b	c	λ_{G_1}	λ_{Y_1}
f	f	Mev.	f	f^{-1}	f^{-1}	f^{-1}	Mev.	Mev.
1.38	1.53	34.19	2.36	.55	1.95	.66	7.6	10.8
1.58	1.65	27.05	2.43	.55	1.89	.62	9.2	13.1

B_{Λ} = 1.85 Mev.

The components of the wave function are

$$\psi_1 = \frac{1}{N_1} \frac{e^{-\frac{a}{2}r} - e^{-\frac{b}{2}r}}{r}$$

$$\psi_\Sigma = \frac{1}{N_\Sigma} e^{-c^2 r^2}$$

In the above table, the column marked λ_{Y_1} is the equivalent Yukawa potential well depth per nucleon. λ_{Y_1} was obtained from λ_{G_1} by equating the volume integrals of the Gaussian and Yukawa wells and by requiring that the effective ranges be the same. Under these requirements we get

$$\lambda_{Y_1} = 1.426 \ \lambda_{G_1}$$

TABLE 5.2

Σ -MIXING POTENTIAL WELL DEPTH IN

THE T=0 HYPERTRITON

B_Λ	V_1	a	b	x	a_3	b_3	y	c	λ_2
Mev.	Mev.	f^{-1}	f^{-1}		f^{-1}	f^{-1}		f^{-1}	Mev.
0	52.7	.181	1.79	1.33	.575	2.56	.995	1.38	12.8
.5	52.7	.402	2.68	1.29	.593	2.61	.963	1.62	16.4
.5	56.0	.424	2.92	1.3	.594	2.61	.966	1.60	13.8

The triplet nucleon-nucleon potential:

$$V_{nn}(r) = -\frac{Ve^{-kr}}{kr}$$

where V = 68.104 Mev. and k = .848 (13)

The wave function components are:

$$\Psi_\Lambda = \frac{1}{N_\Lambda}\left[(e^{-ar_1} + xe^{-br_1}) + (e^{-ar_2} + xe^{-br_2})\right]\left[e^{-a_3 r_3} + ye^{-b_3 r_3}\right]$$

$$\Psi_\Sigma = \frac{1}{N_\Sigma}\left[e^{-cr_1} - e^{-cr_2}\right]\left[e^{-a_3 r_3} + ye^{-b_3 r_3}\right]$$

TABLE 5.3

Σ-MIXING POTENTIAL WELL DEPTH IN THE T=1 HYPERTRITON

B_Λ+B deut.	V_1	a	b	x	a_3	b_3	y	c	λ_3
Mev.	Mev.	f^{-1}	f^{-1}		f^{-1}	f^{-1}		f^{-1}	Mev.
0	52.7	.29	2.68	1.54	.203	1.8	1.04	1.7	32.2*

* See discussion of Results.

The singlet nucleon-nucleon potential:

$$V_{nn}(r) = \frac{Ve^{-kr}}{kr}$$

where V = 46.17 Mev. and k = .848f(13).

The wave function components are

$$\psi_\Lambda = \frac{1}{N_\Lambda}\left[(e^{-ar_1} + xe^{-br_1}) + (e^{-ar_2} + xe^{-br_2})\right]\left[e^{-a_3 r_3} + ye^{-b_3 r_3}\right]$$

$$\psi_\Sigma = \frac{1}{N_\Sigma}\left[e^{-cr_1} - e^{-cr_2}\right]\left[e^{-a_3 r_3} + ye^{-b_3 r_3}\right]$$

$$\psi_{\Sigma'} = \frac{1}{N_{\Sigma'}}\left[e^{-cr_1} + e^{-cr_2}\right]\left[e^{-a_3 r_3} + ye^{-b_3 r_3}\right]$$

TABLE 5.4

Σ -MIXING POTENTIAL WELL DEPTH IN THE HYPERDEUTERON

B	V_1	a	b	c	λ_4
Mev.	Mev.	f^{-1}	f^{-1}	f^{-1}	Mev.
0	52.7	0	1.27	1.54	29.2

The wave function components are:

$$\psi_\Lambda = \frac{1}{N_\Lambda} \frac{e^{-\frac{a}{2}r} - e^{-\frac{b}{2}r}}{r}$$

$$\psi_\Sigma = \frac{1}{N_\Sigma} e^{-cr}$$

C. DISCUSSION OF RESULTS.

In setting up the calculations certain approximations were made. The main approximation which requires some discussion at this point is the one in which the energy of the Σ particle is neglected relative to the Λ - Σ mass difference. The only case in which the approximation was good turned out to be the T=0 hypertriton. As a check on the approximation, the calculated parameters were used to determine the total energy of the Σ part. This check was not carried out for the T=1 hypertriton. For the ${}_{\Lambda}H^4$ system the energy of the Σ particle, using the parameter c from Table 5.1, turned out to be about 17% of Δ (the Λ - Σ mass difference). For the T=0 hypertriton, the

energy was 2% of Δ . In the hyperdeuteron, the approximation broke down completely and the part we had neglected turned out to be larger than the mass difference. Because of the simplicity of the system the calculation was redone. The only purpose which the new calculation served was to point out that the inclusion of the Σ energy only increased the value of λ_4 required to bind the system and so there would be little point in recalculating the T=1 case in hopes of improvement. The value of λ_4 shown in Table 5.4 is the new calculation (i.e., including the Σ energy).

In connection with the ${}_{\Lambda}H^4$ case, it would appear that the calculations can be improved somewhat by including the Σ particle energy in the expression for the well depth. However the uncertainty in some of the parameters causes changes in the calculated value of λ_1 which are of this order of magnitude. Furthermore it is likely that more complicated distortions of the H^3 core in ${}_{\Lambda}H^4$ would allow greater correlations of the hyperon with the individual nucleons and result in a reduction in the calculated value of λ for ${}_{\Lambda}H^4$.

The values of the parameters for the deuteron part of the T=0 hypertriton (Table 5.2) are appreciably different from those which DD found. This is not surprising since the presence of the Σ particle would tend to compress the deuteron resulting in larger values of a_3 and b_3. It would also appear that the re-

lative amounts of tail function (given by $e^{-a_3 r_3}$) and inner function ($e^{-b_3 r_3}$) are made more equal in this case. The relative amounts are given by y. The values which DD found for a_3, b_3 and y, which minimize the spin dependent Λ -nucleon potential are $a_3 = .38$, $b_3 = 1.14$ and $y = 2.14$.

SECTION VI

CONCLUSIONS

We have attempted to account for the binding of hyperons in the light hypernuclei ${}_{\Lambda}He^{5}$, ${}_{\Lambda}H^{4}$, ${}_{\Lambda}He^{4}$ and ${}_{\Lambda}H^{3}$ by assuming that the hyperon nucleon potential has a term in it which mixes Λ and Σ states of the hyperon. In order to see how strong the λ -mixing potential must be under the most adverse circumstances the potential was assumed to have no ordinary spin dependence.

The values for the minimum Σ -mixing well depth as calculated in the ${}_{\Lambda}H^{4}$ and the T=0 hypertriton agree to the extent that the range of the variations of the well depths for the two cases overlap. That the results should agree this closely may be a little fortuitous in view of the approximations made. However it is felt that the value calculated in the T=0 hypertriton problem may be a little better than that calculated for ${}_{\Lambda}H^{4}$.

From the calculations for the T=1 hypertriton and the hyperdeuteron it would appear that these systems are just about equally "unbound" in that they probably require about the same Σ -mixing well depth to become bound.

The calculations of Dalitz and Downs for the T=1 hypertriton indicated that the volume integral of the potential required to bind that system was roughly 20% larger than the potential

required to produce a binding energy of 1 Mev. for the Λ particle in the T=0 hypertriton and about 40% larger than the potential to produce $B_\Lambda = 0$.

In view of their results and the results of the present work it is unlikely that the T=1 hypertriton is bound. This means that the other members of the T=1 triplet i.e., ${}_\Lambda n^3$ and ${}_\Lambda He^3$ are not expected to exist either. Of course the coulomb repulsion of the two protons in ${}_\Lambda He^3$ would tend to break that system up anyway. The ${}_\Lambda n^3$ system has not been observed but this could have been due to the difficulty of detecting it because it carries no charge. The inference from the work done here is that the system does not exist.

If instead of well depths, we express the hyperon nucleon potential in terms of volume integrals we can compare the isotopic spin dependent potential with the ordinary spin dependent potential of DD. If V_p and V_a are the volume integrals of the triplet and singlet Λ -nucleon potentials then the spin dependent potential can be written

$$U = \frac{3V_p + V_a}{4} + \frac{V_p - V_a}{4} \sigma_\Lambda \cdot \sigma_N \qquad (6.1)$$

For the isotopic spin dependent potential we have

$$U' = U_1 + U_{mix}(t-r)\cdot\tau_N \qquad (6.2)$$

The value of U_1 is 227 Mev. f^3 which was determined by calculating $\frac{3V_p + V_a}{4}$ from the work of DD. The volume integral of the Σ mixing potential is to be compared with $\frac{V_p - V_a}{4}$. The potentials V_a and V_p which best fit the hypernuclear binding energies according to DD are

$$\begin{aligned} V_a &= 482 \pm 16 \text{ Mev. } f^3 \\ V_p &= 142 \pm 18 \text{ Mev. } f^3 \end{aligned} \tag{6.3}$$

Therefore the magnitude of the spin dependent volume integral is

$$\frac{V_p - V_a}{4} \approx -85 \text{ Mev. } f^3 \tag{6.4}$$

where we have neglected the errors. Denoting by U_{DD} the volume integral which Dalitz and Downs found, we can write it in the form

$$U_{DD} = 227 - 85\, \sigma_\Lambda \cdot \sigma_m \tag{6.5}$$

The volume integral of the Yukawa shaped Σ -mixing potential is

$$4\pi \lambda r_\gamma^3 \approx 517 \text{ Mev. } f^3$$

Whence

$$U^1 = 227 + 517\, (t \cdot \nu) \cdot \tau_m \tag{6.6}$$

The spin dependent potential is therefore considerably more effective in binding the hyperons in the light hypernuclei than is the Σ -mixing potential.

The correct hyperon-nucleon potential would contain some linear combination of ordinary spin and isotopic spin dependent parts. A measure of the amount of Σ-mixing could be found by fitting the Λ-nucleon scattering and the Σ production cross-sections with an ordinary spin-isotopic spin dependent potential.

APPENDIX A

THE VARIATIONAL PRINCIPLE FOR THE POTENTIAL

In the calculations of this thesis we want to find the minimum value of a potential which will produce a given ground state binding energy. Suppose we have a two component wave function

$$\Psi = \begin{bmatrix} a \\ b \end{bmatrix} \tag{A.1}$$

and a Hamiltonian

$$\begin{bmatrix} H_{11}(\alpha_i) & \lambda H_{12}(\alpha_i) \\ \lambda H_{12}(\alpha_i) & H_{22}(\alpha_i) \end{bmatrix} \tag{A.2}$$

λ is the strength of the potential which mixes the two states. The matrix elements have been determined as functions of all the parameters in the wave function used to describe the ground state. From the Schrodinger equation we find

$$\begin{bmatrix} H_{11}(\alpha_i) + B & \lambda H_{12}(\alpha_i) \\ \lambda H_{12}(\alpha_i) & H_{22}(\alpha_i) + B \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \tag{A.3}$$

where we have written B for -E (the ground state binding energy).

The secular equation resulting from (A. 3) is

$$(H_{11}(\alpha_i)+B)(H_{22}(\alpha_i)+B) - \lambda^2 H_{12}^2(\alpha_i) = 0 \qquad \text{(A. 4)}$$

Suppose now that we know λ and want to find the maximum binding energy. We do this by varying all of the α i's in (A. 4) for all α i.

The set of equations we will get to determine B max. is

$$\frac{\partial}{\partial\alpha_i}(H_{11}(\alpha_i)+B)(H_{22}(\alpha_i)+B) - \lambda^2\frac{\partial}{\partial\alpha_i}H_{12}^2(\alpha_i) = 0$$

$$= (H_{11}(\alpha_i)+B)\frac{\partial}{\partial\alpha_i}H_{22}(\alpha_i) + (H_{22}(\alpha_i)+B)\frac{\partial}{\partial\alpha_i}H_{11}(\alpha_i) - \lambda^2\frac{\partial}{\partial\alpha_i}H_{12}^2(\alpha_i) \qquad \text{(A. 5)}$$

If now we know the binding energy B and wish to determine the minimum value of λ which will produce this value of B, we consider B to be constant and differentiate (A. 4) once again with respect to the α i. The set of equations is

$$\frac{\partial}{\partial\alpha_i}(H_{11}(\alpha_i)+B)(H_{22}(\alpha_i)+B) - \lambda^2\frac{\partial}{\partial\alpha_i}H_{12}^2(\alpha_i) - H_{12}^2(\alpha_i)\frac{\partial\lambda^2}{\partial\alpha_i} = 0$$

$$(H_{11}(\alpha_i)+B)\frac{\partial}{\partial\alpha_i}H_{22}(\alpha_i) + \frac{\partial H_{11}(\alpha_i)}{\partial\alpha_i}(H_{22}(\alpha_i)+B) - \lambda^2\frac{\partial}{\partial\alpha_i}H_{12}^2(\alpha_i) - H_{12}^2(\alpha_i)\frac{\partial\lambda^2}{\partial\alpha_i} \qquad \text{(A. 6)}$$

at the minimum, $\dfrac{\partial\lambda^2}{\partial\alpha_i}$ =0 and therefore

$$(H_{11}(\alpha_i)+B)\frac{\partial}{\partial\alpha_i}H_{22}(\alpha_i) + \frac{\partial H_{11}(\alpha_i)}{\partial\alpha_i}(H_{22}(\alpha_i)+B) - \lambda^2\frac{\partial}{\partial\alpha_i}H_{12}^2(\alpha_i) = 0 \qquad \text{(A. 7)}$$

Note that equations (A. 5) and (A. 7) are identical. Hence the value of the potential strength λ in equation (A. 5) required to produce the observed binding B is equal to the minimum value of λ found in (A. 7) assuming that B has the observed value.

APPENDIX B

CALCULATION OF THE SPATIAL INTERGRALS FOR THE HYPERTRITON PROBLEM

In order to avoid repetition, we will define a function

$$\psi_{\pm} = \frac{1}{N_\Lambda}\left[(e^{-ar_1} \pm e^{-ar_2}) + x(e^{-br_1} \pm e^{-br_2})\right]\left[e^{-a_3 r_3} + y e^{-b_3 r_3}\right] \qquad \text{(B.1)}$$

This will be handy for doing the normalizations because the Σ parts of the wave function are

$$\frac{1}{N_\Sigma}(e^{-cr_1} - e^{-cr_2})(e^{-a_3 r_3} + y e^{-b_3 r_3}) \qquad \text{(B.2)}$$

and

$$\frac{1}{N_{\Sigma'}}(e^{-cr_1} + e^{-cr_2})(e^{-a_3 r_3} + y e^{-b_3 r_3}) \qquad \text{(B.3)}$$

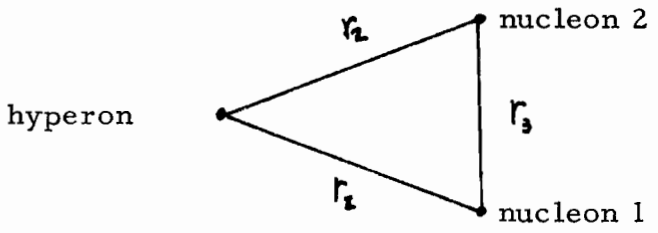
We need only put a=c and x=0 to obtain (B.2) or (B.3) from (B.1). To normalize (B.1) to unity, we will first expand the square.

$$|\psi_{\pm}|^2 = \frac{1}{N_\Lambda^2}\Big\{ e^{-2ar_1 - 2a_3 r_3} + 2y e^{-2ar_1 - (a_3+b_3)r_3} + y^2 e^{-2ar_1 - 2b_3 r_3}$$

$$\pm 2e^{-a(r_1+r_2) - 2a_3 r_3} \pm 4y e^{-a(r_1+r_2) - (a_3+b_3)r_3} \pm 2y^2 e^{-a(r_1+r_2) - 2b_3 r_3}$$

$$
\begin{aligned}
&+e^{-2ar_2-2a_3r_3} + 2ye^{-2ar_2-(a_3+b_3)r_3} + y^2e^{-2ar_2-2b_3r_3} \\
&+x^2e^{-2br_1-2a_3r_3} + 2x^2ye^{-2br_1-(a_3+b_3)r_3} + y^2x^2e^{-2br_1-2b_3r_3} \\
&\pm 2x^2e^{-b(r_1+r_2)-2a_3r_3} \pm 4x^2ye^{-b(r_1+r_2)-(a_3+b_3)r_3} \pm 2x^2y^2e^{-b(r_1+r_2)-2b_3r_3} \\
&+x^2e^{-2br_2-2a_3r_3} + 2x^2ye^{-2br_2-(a_3+b_3)r_3} + x^2y^2e^{-2br_2-2b_3r_3} \\
&+2xe^{-(a+b)r_1-2a_3r_3} + 4xye^{-(a+b)r_1-(a_3+b_3)r_3} + 2xy^2e^{-(a+b)r_1-2b_3r_3} \qquad \text{(B.4)} \\
&\pm 2xe^{-ar_1-br_2-2a_3r_3} \pm 4xye^{-ar_1-br_2-(a_3+b_3)r_3} \pm 2xy^2e^{-ar_1-br_2-2b_3r_3} \\
&\pm 2xe^{-ar_2-br_1-2a_3r_3} \pm 4xye^{-ar_2-br_1-(a_3+b_3)r_3} \pm 2xy^2e^{-ar_2-br_1-2b_3r_3} \\
&+2xe^{-(a+b)r_2-2a_3r_3} + 4xye^{-(a+b)r_2-(a_3+b_3)r_3} + 2xy^2e^{-(a+b)r_2-2b_3r_3}
\end{aligned}
$$

Each of these terms is to be integrated over the positions of the three particles r_1, r_2 and r_3 defined by



From the diagram it is obvious that $r_i + r_j \geq r_k$. Each term is of the form

$$\iiint e^{-\ell r_1 - m r_2 - n r_3} \, r_1 r_2 r_3 \, dr_1 dr_2 dr_3 \tag{B.5}$$

we have left off a factor of $8\pi^2$ which is unimportant for the time being. If we define

$$I(\ell, m, n) = \iiint e^{-\ell r_1 - m r_2 - n r_3} \, dr_1 dr_2 dr_3 \tag{B.6}$$

then (B.5) can be determined from (B.6) since

$$-\frac{\partial^3 I(\ell, m, n)}{\partial \ell \partial m \partial n} = \iiint e^{-\ell r_1 - m r_2 - n r_3} \, r_1 r_2 r_3 \, dr_1 dr_2 dr_3 \tag{B.7}$$

or in general

$$I_{pqr}(\ell, m, n) = (-1)^{p+q+r} \frac{\partial I(\ell, m, n)}{\partial \ell^p \partial m^q \partial n^r} = \iiint e^{-\ell r_1 - m r_2 - n r_3} \, r_1^p r_2^q r_3^r \, dr_1 dr_2 dr_3 \tag{B.8}$$

In evaluating I(l, m, n) we will do the r_3 integral first. The limits on r_3 are from $|r_1-r_2|$ to (r_1+r_2). The straight bars around a quantity indicate "magnitude of". The r_3 integral is

$$\int_{|r_1-r_2|}^{(r_1+r_2)} e^{-nr_3} dr_3 = -\frac{1}{n} e^{-nr_3}\Big|_{|r_1-r_2|}^{r_1+r_2} = \frac{1}{n} e^{-n|r_1-r_2|} - \frac{1}{n} e^{-n(r_1+r_2)}$$

Putting this back into (B.6) we get

$$\int\int e^{-lr_1-mr_2}\left\{\frac{1}{n} e^{-n|r_1-r_2|} - \frac{1}{n} e^{-n(r_1+r_2)}\right\} dr_1 dr_2$$

$$= \frac{1}{n}\int\int e^{-lr_1-mr_2-n|r_1-r_2|} dr_1 dr_2 - \frac{1}{n}\int\int e^{-lr_1-mr_2-n(r_1+r_2)} dr_1 dr_2 \qquad \text{(B.9)}$$

If the r_2 integral is done next, (B.9) becomes

$$\frac{1}{n}\int_0^\infty e^{-(l+n)r_1}\int_0^{r_1} e^{-(m+n)r_2} dr_2 dr_1 + \frac{1}{n}\int_0^\infty e^{-(l-n)r_1}\int_{r_1}^\infty e^{-(m+n)r_2} dr_2 dr_1$$

$$-\frac{1}{n}\int_0^\infty e^{-(l+n)r_1} dr_1 \int_0^\infty e^{-(m+n)r_2} dr_2$$

If we interchange the order of integration in the first integral and perform the last integral we get

$$\frac{1}{n}\int_0^\infty e^{-(m-n)r_2}\int_{r_2}^\infty e^{-(l+n)r_1} dr_1 dr_2 + \frac{1}{n}\int_0^\infty e^{-(l-n)r_1}\int_{r_1}^\infty e^{-(m+n)r_2} dr_2 dr_1$$

$$-\frac{1}{n(l+n)(m+n)}$$

The remaining integrals are of the same form. We can get one from the other by interchanging l and m. Doing the first

$$\frac{1}{n}\int_0^\infty e^{-(m-n)r_2}\left\{\frac{e}{l+n}^{-(l+n)r_2}\right\}dr_2 = \frac{1}{n(l+n)}\int_0^\infty e^{-(l+n)r_2}\,dr_2$$

$$= \frac{1}{n(l+n)(l+m)}$$

Therefore the integral in (B.6) becomes

$$\frac{1}{n}\left\{\frac{1}{(l+n)(l+m)} + \frac{1}{(m+n)(l+m)} - \frac{1}{(l+n)(m+n)}\right\}$$

$$= \frac{2}{(l+m)(m+n)(n+l)} \qquad \text{(B.10)}$$

The integrals involved in the normalization are $I_{111}(l, m, n)$. If we take derivatives of (B.10) with respect to l, m and n, collect all of the terms and take the negative of the result we find that

$$I_{111}(l,m,n) = 8\,\frac{(l+m+n)\left[(l+m)(l+n)+(l+m)(m+n)+(l+n)(m+n)\right]-(l+m)(m+n)(l+n)}{(l+m)^3(l+n)^3(m+n)^3} \qquad \text{(B.11)}$$

In particular we have terms in (B.2) which are of the form I_{111} (l, o, n) and $I_{111}(l, l, n)$. If we put m=0 in (B.11) we find

$$I_{111}(l,0,n) = \frac{8}{l^3n^3} \qquad \text{(B.12)}$$

Putting m=1 results in

$$I_{111}(\ell,\ell,m) = \frac{8\ell^2+5\ell m+m^2}{\ell^3(\ell+m)^5} \tag{B.13}$$

We see from (B.11) or from (B.6) that

$$I_{111}(\ell,m,n) = I_{111}(m,\ell,n)$$

and therefore we can reduce the normalizing factor N^2 to

$$N_{\pm}^2(x,a_3,b_3) = J_{\pm}(x,2a_3) + 2y\,J_{\pm}(x,a_3+b_3) + y^2 J_{\pm}(x,2b_3) \tag{B.14}$$

$$J_{\pm}(x,m) = 2\Big\{\big[I_{111}(2a,0,m) \pm I_{111}(a,a,m)\big] + 2x\big[I_{111}(a+b,0,m) \pm I_{111}(a,b,n)\big] + x^2\big[I_{111}(2b,0,m) \pm I_{111}(b,b,n)\big]\Big\} \tag{B.15}$$

The normalizing factors for the three functions (4.25) are

$$N_\Lambda^2 = N_+^2(x,a_3,b_3) \tag{B.16}$$

$$N_\Sigma^2 = N_-^2(0,a_3,b_3) \tag{B.17}$$

$$N_{\Sigma'}^2 = N_+^2(0,a_3,b_3) \tag{B.18}$$

We will now evaluate the kinetic energy term (4.13). The wave function for which the kinetic energy is tobe calculated is

$$\psi_\Lambda = \frac{1}{N_\Lambda}\Big[(e^{-ar_1}+e^{-ar_2}) + x(e^{-br_1}+e^{-br_2})\Big]\Big[e^{-a_3r_3} + ye^{-b_3r_3}\Big] \tag{B.19}$$

The kinetic energy is given by

$$\hbar^2 \iiint \left\{ \frac{1}{M_n} \left(\frac{\partial \psi}{\partial r_3} \right)^2 + \left(\frac{1}{2M_n} + \frac{1}{2M_1} \right) \left[\left(\frac{\partial \psi}{\partial r_1} \right)^2 + \left(\frac{\partial \psi}{\partial r_2} \right)^2 \right] \right.$$

$$\left. + \frac{1}{M_n} \left[t(231) + t(312) \right] + \frac{1}{M_1} t(123) \right\} r_1 r_2 r_3 \, dr_1 dr_2 dr_3 \qquad \text{(B. 20)}$$

where

$$t(ijk) = \frac{r_i^2 + r_j^2 - r_k^2}{2 r_i r_j} \frac{\partial \psi}{\partial r_i} \frac{\partial \psi}{\partial r_j} \qquad \text{(B. 20a)}$$

Differentiating (B. 19) with respect to r_3 and squaring, the result is

$$\left(\frac{\partial \psi}{\partial r_3} \right)^2 = \left\{ \left[e^{-2ar_1} + e^{-2ar_2} + 2e^{-a(r_1+r_2)} \right] + 2x \left[e^{-(a+b)r_1} + e^{-(a+b)r_2} + e^{-ar_1 - br_2} \right.\right.$$

$$\left.\left. + e^{-br_1 - ar_2} \right] + x^2 \left[e^{-2br_1} + e^{-2br_2} + 2e^{-b(r_1+r_2)} \right] \right\} \left\{ a_3^2 e^{-2a_3 r_3} + b_3^2 y^2 e^{-2b_3 r_3} + 2a_3 b_3 y e^{-(a_3+b_3)r_3} \right\} \qquad \text{(B. 21)}$$

Performing the integrals indicated in (B. 20) we get

$$\iiint \left(\frac{\partial \psi}{\partial r_3} \right)^2 r_1 r_2 r_3 \, dr_1 dr_2 dr_3 = 2 \left\{ a_3^2 \left[f_1(a,a,2a_3) + 2x f_1(a,b,2a_3) + x^2 f_1(b,b,2a_3) \right] \right.$$

$$+ 2a_3 b_3 y \left[f_1(a,a,a_3+b_3) + 2x f_1(a,b,a_3+b_3) + x^2 f_1(b,b,a_3+b_3) \right]$$

$$\left. + b_3^2 y^2 \left[f_1(a,a,2b_3) + 2x f_1(a,b,2b_3) + x^2 f_1(b,b,2b_3) \right] \right\} \qquad \text{(B. 22)}$$

where

$$f_1(l,m,n) = \left[I_{111}(l+m,0,n) + I_{111}(l,m,n) \right] \qquad \text{(B. 23)}$$

If we define

$$F_1(a,b,x) = f_1(a,a,x) + 2x f_1(a,b,x) + x^2 f_1(b,b,x) \qquad \text{(B.24)}$$

(B.22) becomes

$$\iiint \left(\frac{\partial\psi}{\partial r_3}\right)^2 r_1 r_2 r_3 \, dr_1 dr_2 dr_3 = 2\left\{a_3^2 F_1(a,b,2a_3) + 2a_3 b_3 y F_1(a,b,a_3+b_3) + b_3^2 y^2 F_1(a,b,2b_3)\right\} \qquad \text{(B.25)}$$

The function (B.19) is symmetric in r_1 and r_2 therefore the contributions from $\left(\frac{\partial\psi}{\partial r_1}\right)^2$ and $\left(\frac{\partial\psi}{\partial r_2}\right)^2$ are equal.

$$\left(\frac{\partial\psi}{\partial r_1}\right)^2 = \left[a^2 e^{-2ar_1} + b^2 x^2 e^{-2br_1} + 2abx e^{-(a+b)r_1}\right]\left[e^{-a_3 r_3} + y e^{-b_3 r_2}\right]^2 \qquad \text{(B.26)}$$

Writing $f_2(p,q,x) = pq\, I_{111}(p+q,0,x)$ the contribution to the kinetic energy from the $\left[\left(\frac{\partial\psi}{\partial r_1}\right)^2 + \left(\frac{\partial\psi}{\partial r_2}\right)^2\right]$ terms

$$\iiint \left[\left(\frac{\partial\psi}{\partial r_1}\right)^2 + \left(\frac{\partial\psi}{\partial r_2}\right)^2\right] r_1 r_2 r_3 \, dr_1 dr_2 dr_3 =$$

$$2\left\{F_2(a,b,2a_3) + 2y F_2(a,b,a_3+b_3) + y^2 F_2(a,b,2b_3)\right\} \qquad \text{(B.27)}$$

where

$$F_2(a,b,x) = f_2(a,a,x) + 2x f(a,b,x) + x^2 f(b,b,x) \qquad \text{(B.28)}$$

The terms t(231) and t(312) are identical as can be seen from (B.20a) and the fact that the function ψ_A is symmetric in r_1 and r_2. We require

$$t(231)=\tfrac{1}{2}\iiint(r_2^2+r_3^2-r_1^2)\frac{\partial\psi}{\partial r_2}\frac{\partial\psi}{\partial r_3}\,r_1\,dr_1\,dr_2\,dr_3 \tag{B.29}$$

$$\frac{\partial\psi}{\partial r_2}=-(ae^{-ar_2}+bxe^{-br_2})(e^{-a_3r_3}+ye^{-b_3r_3}) \tag{B.30}$$

$$\frac{\partial\psi}{\partial r_3}=-[(e^{-ar_1}+e^{-ar_2})+x(e^{-br_1}+e^{-br_2})][a_3e^{-a_3r_3}+b_3ye^{-b_3r_3}] \tag{B.31}$$

When (B.30) and (B.31) are substituted into (B.29) we get integrals of the form $I_{120}(l,m,n)$, $I_{103}(l,m,n)$ and $I_{300}(l,m,n)$. If we define

$$\begin{aligned} g(l,m,n)&=\{I_{120}(l,m,n)+I_{102}(l,m,n)-I_{300}(l,m,n)\}\\ &=l\,G(l,m,n)\end{aligned} \tag{B.32}$$

where

$$G(l,m,n)=\frac{(l+m+n)^2+lm+ln+mn}{(l+m)^3(m+n)^3(l+n)^3}$$

then t(231) can be written

$$\begin{aligned} t(231)=\tfrac{1}{2}\{&a_3[a^2G(a,a,2a_3)+2abx\,G(a,b,2a_3)+b^2x^2G(b,b,2a_3)]\\ &+(a_3+b_3)y[a^2G(a,a,a_3+b_3)+2abx\,G(a,b,a_3+b_3)+b^2x^2G(b,b,a_3+b_3)]\\ &+b_3y^2[a^2G(a,a,2b_3)+2abx\,G(a,b,2b_3)+b^2x^2G(b,b,2b_3)]\}\end{aligned} \tag{B.33}$$

Now defining

$$f_3(l, m, n) = lmG(l, m, n)$$

and

$$F_3(a, b, n) = f_3(a, a, n) + 2x\, f_3(a, b, n) + x^2 f_3(b, b, n)$$

(B. 33) becomes

$$t(231) = \frac{1}{2}\left\{a_3 F_3(a, b, 2a_3) + (a_3 + b_3)\, y\, F_3(a, b, a_3 + b_3) + b_3 y^2 F_3(a, b, 2b_3)\right\} \quad \text{(B. 34)}$$

Adding the contribution from t(312) we get

$$t(231) + t(312) = \left\{a_3 F_3(a, b, 2a_3) + (a_3 + b_3)\, y\, F_3(a, b, a_3 + b_3) + b_3 y^2 F_3(a, b, 2b_3)\right\} \quad \text{(B. 35)}$$

The final term of the kinetic energy t(123) is

$$t(123) = \frac{1}{2}\iiint (r_1^2 + r_2^2 r_3^2)(ae^{-ar_1} + bxe^{-br_1})(ae^{-ar_2} + bxe^{-br_2})(e^{-a_3 r_3} + y e^{-b_3 r_3})^2 r_3\, dr_1 dr_2 dr_3 \quad \text{(B. 36)}$$

The integrals occur in the combinations

$$g'(l, m, n) = I_{201}(l, m, n) + I_{021}(l, m, n) - I_{003}(l, m, n)$$

$$= 16m\, \frac{(l+m+n)^2 + lm + ln + mn}{(l+m)^3 (m+n)^3 (l+n)^3} = n\, G(l, m, n) \quad \text{(B. 37)}$$

Using (B. 37) the integral (B. 36) becomes

$$t(123) = \left\{a_3 F_3(a, b, 2a_3) + y(a_3 + b_3) F_3(a, b, a_3 + b_3) + b_3 y^2 F_3(a, b, 2b_3)\right\} \quad \text{(B. 38)}$$

Comparing (B. 38) with (B. 34) we see that

$$t(1, 2, 3) = 2t(231)$$

Collecting all of the terms together the kinetic energy (B. 20) is

$$K.E = \left\{ \frac{T(a_1, b_1, a_3, a_3) + 2y\,T(a_1, b_1, a_3, b_3) + y^2 T(a_1, b_1, b_3, b_3)}{N_\Lambda^2} \right\} \quad \text{(B. 39)}$$

where we have written

$$T(a, b, l, m) = \frac{\hbar^2}{2M_m} \left\{ 4lm\, F_1(a, b, l+m) + \left(1 + \frac{M_m}{M_\Lambda}\right) F_2(a, b, l+m) \right.$$

$$\left. + \left(1 + \frac{M_m}{M_\Lambda}\right) (\epsilon + 1)(l+m)\, F_3(a, b, l+m) \right\} \quad \text{(B. 40)}$$

A factor ϵ has been introduced in front of F_3 because there is a factor of 2 in that term when $l \neq m$ and is absent when $l = m$ i. e.,

$$\epsilon = 0 \quad \text{when} \quad l = m$$

$$\epsilon = 1 \quad \text{when} \quad l \neq m$$

The integrals involved in the potential energy are all of the same type since all of the potentials are of the form $\frac{e^{-kr}}{kr}$. The potential energy is

$$P.E. = -\iiint \left[(e^{-ar_1} + e^{-ar_2}) + x(e^{-br_1} + e^{-br_2}) \right]^2 \left[e^{-a_3 r_3} + y e^{-b_3 r_3} \right]^2$$

$$\cdot \left[V_1 \frac{e^{-kr_1}}{kr_1} + \frac{V_1 e^{-kr_2}}{kr_2} + V_{mm} \frac{e^{-k_3 r_3}}{k_3 r_3} \right] r_1 r_2 r_3 \, dr_1 dr_2 dr_3 \quad \text{(B. 41)}$$

Since the same type of integrals are involved here as in the kinetic energy calculation we will not go through the details. The result is written in (4.17c), (4.17g) and (4.17h).

The only point of interest in the calculation of the matrix element H_{12} of (4.10) is the fact that although the contributions to the integrals from the two nucleons are equal and opposite in sign, H_{12} does not vanish because the matrix elements of the operator $V \cdot \mathcal{P}$ (4.10) are also equal and opposite in sign. Thus the sum of the contributions is equal to twice the contribution from one of the nucleons. Therefore

$$H_{12} = \frac{2}{N_1 N_2} \iiint [(e^{-ar_1} + e^{-ar_2}) + x(e^{-br_1} + e^{-br_2})][e^{-cr_1} - e^{-cr_2}][e^{-a_3 r_3} + y e^{-b_3 r_3}]^2 \cdot \frac{e^{-k_m r_1}}{k_m r_1} r_1 r_2 r_3 \, dr_1 dr_2 dr_3$$

Once again there is nothing more to demonstrate in these integrals and so the result is written in (4.18a) and (4.18b). The matrix element H_{13} of (4.30) is

$$H_{13} = \frac{2}{N_1 N_2'} \iiint [(e^{-ar_1} + e^{-ar_2}) + x(e^{-br_1} + e^{-br_2})][e^{-cr_1} + e^{-cr_2}][e^{-a_3 r_3} + y e^{-b_3 r_3}]^2 \cdot \frac{e^{-k_m r_1}}{k_m r_1} r_1 r_2 r_3 \, dr_1 dr_2 dr_3$$

H_{13} can be obtained from H_{12} by simply changing all of the minus signs in the result of H_{12} to plus signs.

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