

[PROPERTIES OF THE
COMPLEX GAUSSIAN DISTRIBUTION]

by

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CHAPTER 1 : Introduction

To date, the history of the Complex Normal, or Complex Gaussian Distribution belongs to one man, namely N.R. Goodman. The motivation for the postulation of the Complex Gaussian Distribution occurred in 1957 in [2] . However it was not until 1963 in [1] , that he actually defined the Complex Normal Distribution.

In this dissertation, a slightly more general definition will be given for the Complex Gaussian Distribution. The two definitions coincide wherever the Goodman definition applies; and the revised definition lends itself nicely to closed form results.

The aim of this thesis is to study the properties of the Complex Normal Distribution, while at the same time, to investigate likenesses and differences between the classical Normal Distribution and the Complex Normal Distribution in their respective probability spaces.

Chapter 2 introduces the reader to the basic concepts of the Complex Gaussian Distribution, and acquaints the reader with the notation used throughout the paper.

Chapter 3 lists the important Lemmas which will be employed to prove Theorems about the Complex Normal Distribution.

Chapter 4 contains the basic Theorems on the Complex Gaussian Distribution. Although these are interesting in their own right, their prime importance is as tools to prove results in subsequent chapters.

Chapter 5 gives additional properties of the distribution and furthermore, it gives the reader some interesting comparisons with the Gaussian Distribution.

Chapter 6 is devoted to quadratic forms of the Complex Normal, an essential ingredient to Analysis of Variance.

Chapter 7 deals with the Central Limit Criteria of the Complex Gaussian Distribution. This is perhaps one of the most striking contrasts with the Normal Distribution.

Chapter 8 is concerned with a special regression problem, leading to a characterization of the Complex Gaussian Distribution.

In Appendix I, the author leaves suggestions for further studies. These would make excellent research projects.

In Appendix II, the author indicates some applications of the Complex Gaussian Distribution.

At the conclusion of the paper the author will acknowledge the generous help he received, without which, completion of this paper would not have been possible.

CHAPTER 2 : Basic Notations and Definitions

<u>Notation</u>	<u>Meaning</u>
a) A, B, D, V	Matrices
b) A_{jk}	Sub-block of A
c) a_{jk}	element of A
d) Z, W, M	Vectors of complex numbers
e) z_j	j -th component of Z
f) I	Identity matrix
g) ϕ	Null matrix; null vector
h) $R(A)$	Matrix formed by taking only real part of each a_{jk}
i) $I(A)$	Matrix formed by taking only imaginary part of each a_{jk}
j) Z/W	" Z given W "
k) iff	If and only if
l) $\bar{A}$	Matrix of complex conjugates of each a_{jk}

<u>Notation</u>	<u>Meaning</u>
m) A'	Transpose of A
n) $ A $	$\det(A)$
o) $ Z $	Modulus of Z
p) $\bar{z}$	Arithmetic mean of $z_1, \dots, z_n$
q) s^2	$\sum_{j=1}^n \frac{(z_j - \bar{z})(\overline{z_j - \bar{z}})}{n-1}$ Sample Variance
r) $\rho(A)$	Rank of A
s) $\text{Tr}(A)$	Trace of A
t) $E(\cdot)$	Mathematical expectation
u) $M_Z(t)$	$E(\exp(t'Z))$
v) $\Phi_Z(t)$	$E(\exp(it'Z))$
w) $N(m, \sigma^2)$	Normal distribution with mean $\underline{m}$ and variance $\underline{\sigma}^2$.
x) $MVN(M, V)$	Multivariate normal distribution
y) $\sim$	Is distributed as

Definitions

D-1 Complex Random Variable

If Ω is a sample space with a probability measure, and if z is a complex valued function defined over the points of Ω , then z is a complex random variable.

Example 1: (discrete case)

Ω = outcomes of 3 flips of a balanced coin

z = (number of heads) - i(number of tails)

Then: $z = 3$ $p(z) = 1/8$

$= 2 - i$ $= 3/8$

$= 1 - 2i$ $= 3/8$

$= -3i$ $= 1/8$

$p(z) = 0$ otherwise

Example 2: (continuous case)

Ω = complex plane

z is such that:

$f(z) = \frac{1}{\pi}$ $|z| < 1$
 $= 0$ otherwise

D-2 Complex Random Vector

A complex random vector is a vector of complex random variables.

D-3 Complex Variance

If z is a complex random variable, such that $E(z)=m$, then the complex variance of z is given by:

$$\sigma^2 = E(\overline{z-m})(z-m)$$

D-4 Complex covariance Matrix

If Z is a complex random vector, such that $E(Z)=M$. Then the complex covariance matrix of Z is given by:

$$V = E(Z-M)(\overline{Z-M})'$$

D-5 Goodman Random Vector

A Goodman random vector is a complex random vector having the following property:

if $Z = X + iY$ X, Y real vectors

and if $E(x_j) = m_j$; $E(y_j) = r_j$ $j=1,2,\dots,p$

Then:

$$E \begin{bmatrix} (x_j - m_j)(x_k - m_k) & (x_j - m_j)(y_k - r_k) \\ (y_j - r_j)(x_k - m_k) & (y_j - r_j)(y_k - r_k) \end{bmatrix}$$

$$= \begin{pmatrix} \sigma_j^2 & 0 \\ 0 & \sigma_j^2 \end{pmatrix} \text{ if } j=k, \text{ and } = \begin{pmatrix} a_{jk} & -b_{jk} \\ b_{jk} & a_{jk} \end{pmatrix} \text{ if } j \neq k$$

D-6 The Complex Multivariate Normal Distribution (CMVN)

If Z is a complex random vector such that :

$E(Z)=M$, and the complex covariance matrix of Z is V .

Then : For V , non-singular:

$Z \sim \text{CMVN} [M, V]$ iff

$$f(Z) = \frac{1}{\pi^p} \frac{1}{|V|} \exp \left[-(\overline{Z-M})' V^{-1} (Z-M) \right]$$

CHAPTER 3 : Useful Lemmas

In this chapter, the tools which are needed will be listed. Many well known results will be stated without proof, while the others will be proved in detail. The first twelve lemmas are theorems in matrix analysis; the remaining lemmas are theorems in statistics and functional equations.

Lemma 3.1 (matrix lemma)

If A is a real symmetric matrix, and if B is a real skew-symmetric matrix, then a necessary and sufficient condition that $(A+iB)$ be positive definite is that

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \text{ be positive definite.}$$

Proof:

Let $Z = X + iY$ be an arbitrary vector.

$$\begin{aligned} \bar{Z}'(A + iB)Z &= (X - iY)'(A + iB)(X + iY) \\ &= X'AX + Y'AY + Y'BX - X'BY - iY'AX + iX'AY + iX'BX + iY'BY \end{aligned}$$

But A , being symmetric, implies $Y'AX = X'AY$.

And B , being skew-symmetric, implies $X'BX = X'B'X = -X'BX = 0$

Thus:

$$\bar{Z}'(A + iB)Z = X'AX + Y'AY + Y'BX - X'BY.$$

i.e.

$$\bar{Z}'(A + iB)Z = \begin{pmatrix} X' \\ Y \end{pmatrix} \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

Thus for all $Z \neq 0$ -vector (i.e. $\begin{pmatrix} X \\ Y \end{pmatrix} \neq 0$ -vector),

$A + iB$ is positive definite iff $\bar{Z}'(A + iB)Z > 0$;

that is iff right hand side of above equation is strictly positive; and that is iff the following matrix is positive definite:

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

This completes proof.

Lemma 3.2 (matrix lemma)

If V is a positive definite Hermitian matrix, then:

$$\begin{pmatrix} R(V) & -I(V) \\ I(V) & R(V) \end{pmatrix}^{-1} = \begin{pmatrix} R(V^{-1}) & -I(V^{-1}) \\ I(V^{-1}) & R(V^{-1}) \end{pmatrix}$$

Proof:

$$R(VV^{-1}) = I = R(V)R(V^{-1}) - I(V)I(V^{-1})$$

$$I(VV^{-1}) = \phi = R(V)I(V^{-1}) + I(V)R(V^{-1}) \quad (\text{see notation})$$

$$\begin{aligned} \text{But:} \quad & \begin{pmatrix} R(V) & -I(V) \\ I(V) & R(V) \end{pmatrix} \begin{pmatrix} R(V^{-1}) & -I(V^{-1}) \\ I(V^{-1}) & R(V^{-1}) \end{pmatrix} \\ & = \begin{pmatrix} R(VV^{-1}) & -I(VV^{-1}) \\ I(VV^{-1}) & R(VV^{-1}) \end{pmatrix} = I \end{aligned}$$

This completes proof.

Lemma 3.3 (matrix lemma)

Let $\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$ be a positive definite symmetric matrix

Then (1) $\begin{pmatrix} A & -B \\ B & A \end{pmatrix}^{-1}$ is of the form $\begin{pmatrix} C & -D \\ D & C \end{pmatrix}$

and (2) Given matrices C, D such that (1) holds,

Then: $(A + iB)^{-1} = (C + iD)$

Proof of (1):

$$\text{Let } \begin{pmatrix} A & -B \\ B & A \end{pmatrix}^{-1} = \begin{pmatrix} C & D' \\ D & E \end{pmatrix}$$

$$\text{Thus: } \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} C & D' \\ D & E \end{pmatrix} = I$$

Equating corresponding blocks, we obtain:

$$AC - BD = I \quad \dots(a)$$

$$AD' - BE = 0 \quad \dots(c)$$

$$BD' + AE = I \quad \dots(b)$$

$$BC + AD = 0 \quad \dots(d)$$

$$\text{From (a), } C = A^{-1}(I + BD)$$

$$\text{From (d), } D = -A^{-1}BC$$

$$\text{Therefore, } C = A^{-1}(I - BA^{-1}BC)$$

$$\text{that is: } C^{-1} = A(I + A^{-1}BA^{-1}B) \quad (*)$$

$$\text{From (b), } E = A^{-1}(I - BD')$$

$$\text{From (c) } D' = A^{-1}BE$$

Solving exactly as for C^{-1} gives:

$$E^{-1} = A(I + A^{-1}BA^{-1}B) = C^{-1} \quad (\text{by } *)$$

$$\text{Thus } E = C \quad \text{and hence } D' = A^{-1}BC = -D$$

Proof of (2)

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} C & -D \\ D & C \end{pmatrix} = I$$

implies $AC - BD = I$

and $BC + AD = \emptyset$

But $(A + iB)(C + iD) = (AC - BD) + i(BC + AD) = I$

That is : $C + iD$ is the inverse of $A + iB$.

This completes proof.

Lemma 3.4 (matrix lemma)

If H is any Hermitian matrix, then there exists a unitary matrix, P , such that:

$$P^*HP = D \quad \text{where } D \text{ is a real diagonal matrix.}$$

Proof: (see [8-1])

Lemma 3.5 (matrix lemma)

If $A = R(A) + iI(A)$, then a necessary and sufficient condition that A be idempotent is that

$$B = \begin{pmatrix} R(A) & -I(A) \\ I(A) & R(A) \end{pmatrix}$$

is idempotent.

Proof:

$$AA = A \text{ iff } (1) R(A)R(A) - I(A)I(A) = R(A) \\ \text{and } (2) R(A)I(A) + I(A)R(A) = I(A)$$

that is iff

$$BB = \begin{pmatrix} R(A) & -I(A) \\ I(A) & R(A) \end{pmatrix}$$

i.e. iff B is idempotent.

This completes proof.

Lemma 3.6 (matrix lemma)

If a Hermitian matrix A, is idempotent, then the rank of A is equal to the trace of A.

Proof:

Since A is Hermitian, there exists a unitary matrix P, such that $\bar{P}'AP = D$, a real diagonal matrix. (lemma 3.4)

$$\text{But: } \bar{P}'AP = \bar{P}'AAP = \bar{P}'APP\bar{P}'AP = DD$$

$$\text{Thus } D = DD \quad \text{i.e. } d_{jj} = 1 \text{ or } 0.$$

$$(*) \text{ Therefore: } \text{Tr}(D) = \rho(D) = \rho(A)$$

$$\text{But } A = P\bar{P}'AP\bar{P}' = P D \bar{P}'$$

$$\text{Hence } a_{jj} = \sum_k P_{jk} d_{kk} \bar{P}'_{kj}$$

$$\text{i.e. } \text{Tr}(A) = \sum_k d_{kk} \sum_j |P_{jk}|^2$$

$$= \text{Tr}(D) \quad (\text{since } P \text{ is unitary})$$

This completes proof. (because of equation (*))

Lemma 3.7 (matrix lemma)

Let A, B be Hermitian matrices. A necessary and sufficient condition that there exist a unitary matrix, T , with the property that

$$\bar{T}'AT = D_1 \quad \text{and} \quad \bar{T}'BT = D_2 \quad (D_i \text{ diagonal } i=1,2)$$

is that $AB = BA$.

Proof: (see [8-2])

Lemma 3.8 (matrix lemma)

If A is a positive definite Hermitian matrix, and B is a Hermitian matrix: then there exists a non-singular matrix, T such that $\bar{T}'AT = I$ and $\bar{T}'BT = D$ (D diagonal).

Proof: (see [8-3])

Lemma 3.9 (matrix lemma)

If A is a positive definite Hermitian matrix, and B is a Hermitian matrix: then a necessary and sufficient condition that A and $(A-B)$ are simultaneously positive definite is that the eigen values of $A^{-1}B$ lie in the open interval $(0,1)$.

Proof:

By Lemma 3.8, there exists non-singular T , such that:

$$\bar{T}'AT = I \quad \text{and} \quad \bar{T}'BT = D \quad (D \text{ diagonal})$$

Let λ be a characteristic root of $A^{-1}B$.

$$\text{i.e. } |A^{-1}B - \lambda I| = 0 \quad \text{i.e. iff } |\bar{T}'A(A^{-1}B)T - \lambda \bar{T}'AT| = 0$$

i.e. iff $|D - \lambda I| = 0$

Thus D and $A^{-1}B$ must have the same eigen values.

(a) If the eigen values of $A^{-1}B$ lie in $(0,1)$, then the eigen values of D must lie in $(0,1)$.

But $\bar{T}'BT=D$, T non-singular, and $d_{jj} > 0$ imply that B is positive definite.

Also, $\bar{T}'(A-B)T = (I-D)$ and $d_{jj} < 1$, imply that $(A-B)$ is also positive definite.

(b) If B is positive definite then clearly, $\bar{T}'BT$ is also positive definite. But $\bar{T}'BT = D$. Thus $d_{jj} > 0$ $j=1,2,\dots,p$

If $A-B$ is positive definite then clearly, $\bar{T}'(A-B)T = (I-D)$ is positive definite. Thus $(1-d_{jj}) > 0$ $j=1,2,\dots,p$

Hence d_{jj} lies in $(0,1)$ $j=1,2,\dots,p$ and if $|D-\lambda I| = 0$, then λ must lie in $(0,1)$

Since the eigen values of D and $A^{-1}B$ are identical, it follows that the eigen values of $A^{-1}B$ lie in $(0,1)$, the unit open interval.

This completes proof.

Lemma 3.10 (matrix lemma)

If A is a Hermitian matrix, and X and Y are vectors of the same dimension as A , then

$$\bar{X}'AY + \bar{Y}'AX = 2R(\bar{X}'AY)$$

Proof:

$$\overline{X'AY} = X' \overline{AY} = X'A'\overline{Y} = \overline{Y'AX} \quad (\text{since } \overline{Y'AX} \text{ scalar})$$

Now since: $a + \bar{a} = 2(R(a))$,

$$\overline{X'AY} + \overline{Y'AX} = 2R(\overline{X'AY})$$

This completes proof.

Lemma 3.11 (matrix lemma)

If A, B are two positive definite Hermitian matrices,

and: $C = A(A+B)^{-1}$,

then the eigen values of C lie in the open interval $(0,1)$.

Proof: (simple corollary of Lemma 3.9)

Lemma 3.12 (matrix lemma)

Let Z_1, Z_2 be independent complex, zero-mean, random vectors with complex covariance matrices V_1 and V_2 , respectively. Also: let $C = V_1(V_1+V_2)^{-1}$.

Then if: $V = E \left[(Z_1 - C(Z_1+Z_2))(\overline{Z_1 - C(Z_1+Z_2)})' \right]$,

$V^{-1}C$ is a Hermitian matrix.

Proof: (simple exercise in matrix multiplication)

Lemma 3.13 (Cramèr's Theorem, p-dimensional case)

If X, Y are independent, real, p-variate random vectors,

then $(X+Y) \sim \text{MVN}$ iff both $X \sim \text{MVN}$, and $Y \sim \text{MVN}$.

Proof: (see [7-1])

Lemma 3.14 (marginal densities of the MVN)

If $X \sim \text{MVN}[M, V]$

Then the distribution of any set of components of X is multivariate normal with means, variances, and covariances obtained by taking proper components of M , V , and V respectively.

Proof: (see [3-1])

Lemma 3.15 (conditional densities of the MVN)

If $X \sim \text{MVN}[M, V]$

and if the components of X are divided into two subgroups composing the subvectors $X^{(1)}$ and $X^{(2)}$: then

$$f(X^{(1)}/X^{(2)}) = \text{MVN}[M^{(1)} + V_{12}(V_{22})^{-1}(X^{(2)} - M^{(2)}), V_{11} - V_{12}(V_{22})^{-1}V_{21}]$$

where:

$$M = \begin{pmatrix} M^{(1)} \\ M^{(2)} \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$$

according to partition of X .

Proof: (see [3-2])

Lemma 3.16 (chi-square theorem)

If $X \sim \text{MVN}[\phi, I]$ and A is a positive semi-definite

symmetric matrix, then a necessary and sufficient condition

that $X'AX$ have the chi-square distribution with $\rho(A)$

degrees of freedom is that $AA = A$.

Proof: (see [4-1])

Lemma 3.17 (quadratic forms of normal distribution)

If $X \sim \text{MVN}[\phi, I]$ and if A, B are positive semi-definite symmetric matrices, then $X'AX$ and $X'BX$ are independent iff

$$AB = \phi$$

Proof: (see [4-2])

Lemma 3.18 (central limit theorem)

If $Y_1, Y_2, \dots$ are independent, identically distributed p -variate random vectors, such that $E(Y_j) = M$ and

$$E(Y_j - M)(Y_j - M)' = V, \text{ then}$$

$$\frac{1}{n^{1/2}} \sum_{j=1}^n (Y_j - M) \xrightarrow{L} \text{MVN}[\phi, V]$$

Proof: (see [3-3])

Lemma 3.19 (a functional equation)

If $G(Z)$ is a continuous, real valued function of Z , then

$$(a) \quad G(Z_1)G(Z_2) = G(Z_1 + Z_2) \quad \text{iff}$$

$G(Z) = f(X, Y)$ satisfies the Cauchy functional equation in X and Y . (where $Z = X + iY$, as usual)

$$(b) \quad G(Z_1)G(Z_2) = G(Z_1 + Z_2) \text{ implies that}$$

$$G(Z) = \exp R(\gamma'Z) \quad (\text{where } \gamma \text{ is a vector of complex constants})$$

Proof of (a):

$$G(Z_1)G(Z_2) = G(Z_1 + Z_2) \quad \text{iff}$$

$$f(X_1, Y_1)f(X_2, Y_2) = f(X_1 + X_2, Y_1 + Y_2)$$

Proof of (b):

(a) implies that $f(X,Y) = \exp \left[\delta' \begin{pmatrix} X \\ Y \end{pmatrix} \right]$

where δ is a $2p$ -vector of real constants. Thus:

$$G(Z) = \exp \left[R(\gamma'Z) \right] : (\gamma_j = \delta_j + i\delta_{p+j})$$

This completes proof.

CHAPTER 4 : Basic Theorems on the Complex Normal
Distribution

THEOREM 4.1 : CMVN Integrates to Unity

If V is a positive definite Hermitian matrix, then

$$Q = \int \frac{1}{\pi^p} \frac{1}{|V|} \exp(-\bar{Z}' V^{-1} Z) d(Z) = 1$$

Where p is the dimensionality of Z , and the integration is taken over the entire $2p$ -dimensional Euclidean space.

Proof:

By Lemma 3.4, there exists a unitary matrix P , such that

$$\bar{P}' V P = D \quad (\text{a diagonal matrix})$$

This implies: $\bar{P}' V^{-1} P = D^{-1}$

Let $W = PZ$ (the Jacobian of the transformation has modulus unity) Thus:

$$Q = \int \frac{1}{\pi^p} \frac{1}{|V|} \exp(-\bar{W}' D^{-1} W) d(W)$$

But: $|V| = |D| = d_{11} \dots d_{pp}$; and if $W = U + iV$, it follows that:

$$Q = \int \frac{1}{\pi^p} \frac{1}{|D|} \exp \left[- \begin{pmatrix} U \\ V \end{pmatrix}' \begin{pmatrix} D^{-1} & \phi \\ \phi & D^{-1} \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} \right] d \begin{pmatrix} U \\ V \end{pmatrix}$$

Noting that (1): $\pi^p = (\pi^{\frac{1}{2}})^{2p}$

(2): V being positive definite implies:

$$\begin{pmatrix} D & \Phi \\ \Phi & D \end{pmatrix}$$

is also positive definite,

Then $Q = 1$ (Q has been reduced to an integral of a $2p$ -variate real, MVN distributed, random vector, over its entire range of values.)

This completes proof.

THEOREM 4.2 (transformations)

If $Z \sim \text{CMVN}[M, V]$, and P is a non-singular matrix:

Then $W = PZ \sim \text{CMVN}[PM, PVP']$

Proof:

It is clear that $E(W) = PM$, and $E(W-PM)(\overline{W-PM})' = PVP'$.

$$\begin{aligned} f(Z) &= \frac{1}{\pi^p} \frac{1}{|V|} \exp(-(\overline{Z-M})' V^{-1} (Z-M)) \\ &= \frac{1}{\pi^p} \frac{1}{|V|} \exp(-(\overline{PZ-PM})' (PVP')^{-1} (PZ-PM)) \end{aligned}$$

Letting $W = PZ$ (the desired transformation) in above, and using the fact that $g(W)$, the density function of W integrates to unity, it is clear: $W \sim \text{CMVN}[PM, PVP']$

This completes proof.

THEOREM 4.3 : Characterization of the CMVN

A necessary and sufficient condition for $Z = X + iY$ to be distributed as CMVN $[M, V]$ is that:

$$(*) \quad \begin{pmatrix} X \\ Y \end{pmatrix} \sim \text{MVN} \left[\begin{pmatrix} R(M) \\ I(M) \end{pmatrix}, \frac{1}{2} \begin{pmatrix} R(V) & -I(V) \\ I(V) & R(V) \end{pmatrix} \right]$$

Proof:

(1) Assume $Z \sim \text{CMVN} [M, V]$

$$\begin{aligned} \text{Therefore } f(Z) &= K_1 \exp \left[-(\overline{Z-M})' V^{-1} (Z-M) \right] \\ &= K_1 \exp \left\{ - \begin{pmatrix} X-R(M) \\ Y-I(M) \end{pmatrix}' \begin{pmatrix} R(V^{-1}) & -I(V^{-1}) \\ I(V^{-1}) & R(V^{-1}) \end{pmatrix} \begin{pmatrix} X-R(M) \\ Y-I(M) \end{pmatrix} \right\} \\ &= K_1 \exp \left\{ -\frac{1}{2} \begin{pmatrix} X-R(M) \\ Y-I(M) \end{pmatrix}' \left[\frac{1}{2} \begin{pmatrix} R(V) & -I(V) \\ I(V) & R(V) \end{pmatrix} \right]^{-1} \begin{pmatrix} X-R(M) \\ Y-I(M) \end{pmatrix} \right\} \quad (\text{by Lem 3.2}) \end{aligned}$$

Since V is positive definite Hermitian, then by Lemma

3.1:

$$\frac{1}{2} \begin{pmatrix} R(V) & -I(V) \\ I(V) & R(V) \end{pmatrix} = B$$

is positive definite symmetric. Integrating over the $2p$ -dimensional Euclidean space, it is clear that:

$$K_1 = \frac{1}{(2\pi)^p} \frac{1}{|B|^{\frac{1}{2}}}$$

which implies that $(*)$ is true.

(2) Now let us assume $(*)$ is true, and prove that

$$Z \sim \text{CMVN} [M, V] .$$

$(*)$ implies that:

$$f(Z) = K_2 \exp \left\{ -\frac{1}{2} \begin{pmatrix} X-R(M) \\ Y-I(M) \end{pmatrix}' \left[\frac{1}{2} \begin{pmatrix} R(V) & -I(V) \\ I(V) & R(V) \end{pmatrix} \right]^{-1} \begin{pmatrix} X-R(M) \\ Y-I(M) \end{pmatrix} \right\}$$

$$= K_2 \exp \left[- \begin{pmatrix} X-R(M) \\ Y-I(M) \end{pmatrix}' \begin{bmatrix} R(V) & -I(V) \\ I(V) & R(V) \end{bmatrix}^{-1} \begin{pmatrix} X-R(M) \\ Y-I(M) \end{pmatrix} \right]$$

By Lemma 3.3, part (1):

$$\begin{pmatrix} R(V) & -I(V) \\ I(V) & R(V) \end{pmatrix}^{-1} = \begin{pmatrix} C & -D \\ D & C \end{pmatrix}$$

Therefore:

$$\begin{aligned} f(Z) &= K_2 \exp \left[-(\overline{Z-M})' (C+iD)(Z-M) \right] \quad \text{And by Lemma 3.3 (2):} \\ &= K_2 \exp \left[-(\overline{Z-M})' (R(V)+iI(V))^{-1} (Z-M) \right] \\ &= K_2 \exp \left[-(\overline{Z-M})' V^{-1} (Z-M) \right] \end{aligned}$$

Using the notation of the first part of the proof, B , being positive definite symmetric, implies that V is positive definite Hermitian. (by Lemma 3.1)

Thus, by Theorem 4.1:

$$K_2 = \frac{1}{\pi^p} \frac{1}{|V|}$$

and this establishes the fact that $Z \sim \text{CMVN} [M, V]$.

This completes proof.

Corollary: (connection to Goodman random vectors)

$\begin{pmatrix} X \\ Y \end{pmatrix}$ has covariance matrix of the form $\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$ iff

$Z = X + iY$ is a Goodman random vector.

Proof:

The covariance matrix of $\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$ is equivalent to:

$$E \begin{bmatrix} (x_j - E(x_j))(x_k - E(x_k)) & (x_j - E(x_j))(y_k - E(y_k)) \\ (y_j - E(y_j))(x_k - E(x_k)) & (y_j - E(y_j))(y_k - E(y_k)) \end{bmatrix}$$

This reduces to:

$$\begin{pmatrix} a_{jk} & -b_{jk} \\ b_{jk} & a_{jk} \end{pmatrix} \quad \text{if } j \neq k$$

and to:

$$\begin{pmatrix} \sigma_j^2 & 0 \\ 0 & \sigma_j^2 \end{pmatrix} \quad \text{if } j = k$$

which is exactly D-5, the definition of a Goodman random vector.

This completes proof..

THEOREM 4.4 (zero-mean case)

Let $Z \sim \text{CMVN}[\phi, V]$; and P be a non-singular, unitary matrix such that: $PV = VP$

Then: $PZ \sim \text{CMVN}[\phi, V]$

Proof:

By THEOREM 4.2,

$$\begin{aligned} PZ &\sim \text{CMVN}[P\phi, PVP'] \\ &\sim \text{CMVN}[\phi, VP\bar{P}'] \end{aligned}$$

and since $PV = VP$ with $P\bar{P}' = I$

$$PZ \sim \text{CMVN}[\phi, V]$$

This completes proof.

Corollary I:

$$\exp(i\theta)Z \sim \text{CMVN}[\phi, v] \quad \text{iff} \quad Z \sim \text{CMVN}[\phi, v] .$$

Proof:

Put $P = \exp(i\theta)I$ in THEOREM 4.4 .

Corollary II: (Moment Generating Functions and
Characteristic Functions)

Let $Z \sim \text{CMVN}[\phi, v]$:

$$\text{Then } M_Z(t) \equiv E(\exp(t'Z)) = \bigoplus_Z(t) \equiv E(\exp(it'Z))$$

Proof:

Let $W = iZ$. It is clear that W has the same distribution
as Z . ($\theta = \frac{\pi}{2}$ in Corollary I)

$$M_W(t) = E(\exp(t'W)) = E(\exp(it'Z)) = \bigoplus_Z(t)$$

But $W \sim Z$ implies that $M_W(t) = M_Z(t)$

This completes proof.

THEOREM 4.5 (uncorrelated random vectors)

If $Z \sim \text{CMVN}[\phi, \sigma^2 I]$

Then $z_1, \dots, z_p$ form a mutually independent set of complex
random variables.

Proof:

$$Z \sim \text{CMVN}[\phi, \sigma^2 I] \quad \text{iff} \quad \begin{pmatrix} X \\ Y \end{pmatrix} \sim \text{MVN}[\phi, \frac{1}{2}\sigma^2 I] \quad (\text{by THM 4.3})$$

This implies that $x_1, \dots, x_p, y_1, \dots, y_p$ form a mutually
independent set of random variables.

Hence: $(x_1, y_1), \dots, (x_p, y_p)$ form a mutually independent set of random vectors in two-space. Therefore $z_1, \dots, z_p$ are a mutually independent set of complex random variables.

This completes proof.

CHAPTER 5 : Distributional Properties of the Complex
Gaussian Distribution

THEOREM 5.1 (Linear combinations)

If $Z_1, \dots, Z_n \sim \text{CMVN} [M_j, V_j]$, respectively, Then:

$$\sum_j a_j Z_j \sim \text{CMVN} \left[\sum_j a_j M_j, \sum_j a_j^2 V_j \right]$$

where $Z_1, \dots, Z_n$ form a mutually independent set of p -variate random vectors, and a_j are real numbers for $j = 1, \dots, n$.

Proof:.

Let $U_j = \begin{pmatrix} X_j \\ Y_j \end{pmatrix}$, $N_j = \begin{pmatrix} R(M_j) \\ I(M_j) \end{pmatrix}$, and

$$B_j = \frac{1}{2} \begin{bmatrix} R(V_j) & -I(V_j) \\ I(V_j) & R(V_j) \end{bmatrix}$$

It is clear that $\{U_j\}$ are mutually independent, and by

THEOREM 4.3, $U_j \sim \text{MVN} [N_j, B_j]$ $j = 1, \dots, n$

$$M_{U_j}(t) = \exp [t' N_j + \frac{1}{2} t' B_j t] \quad j = 1, \dots, n$$

where $M_{U_j}(t)$ is the Moment Generating Function of U_j .

Let $W = \sum_j a_j U_j$

$$\begin{aligned} M_W(t) &= M_{\sum_j a_j U_j}(t) = \prod_j \exp [t' a_j N_j + \frac{1}{2} t' a_j^2 B_j t] \\ &= \exp \left[t' \sum_j a_j N_j + \frac{1}{2} t' \left(\sum_j a_j^2 B_j \right) t \right] \end{aligned}$$

Therefore: $W \sim \text{MVN} \left[\sum_j a_j N_j, \sum_j a_j^2 B_j \right]$

But:

$$a_j^2 B_j = \frac{1}{2} \begin{bmatrix} a_j^2 R(v_j) & -a_j^2 I(v_j) \\ a_j^2 I(v_j) & a_j^2 R(v_j) \end{bmatrix}$$

And: $a_j N_j = a_j \begin{pmatrix} R(M_j) \\ I(M_j) \end{pmatrix}$

Thus by THEOREM 4.3, it follows that:

$$\sum_j a_j Z_j \sim \text{CMVN} \left[\sum_j a_j M_j, \sum_j a_j^2 V_j \right]$$

This completes proof.

THEOREM 5.2 (Marginals)

Let $Z \sim \text{CMVN} [M, V]$ and partition Z as:

$$Z = \begin{pmatrix} Z^{(1)} \\ Z^{(2)} \end{pmatrix} \sim \text{CMVN} \left[\begin{pmatrix} M^{(1)} \\ M^{(2)} \end{pmatrix}, \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \right]$$

Then:

$$Z^{(1)} \sim \text{CMVN} [M^{(1)}, v_{11}] \quad \text{and:}$$

$$Z^{(2)} \sim \text{CMVN} [M^{(2)}, v_{22}]$$

Where

$$Z^{(1)} = X^{(1)} + iY^{(1)}$$

and

$$Z^{(2)} = X^{(2)} + iY^{(2)}$$

PROOF:

THEOREM 4.3 implies that:

$$\begin{pmatrix} X^{(1)} \\ X^{(2)} \\ Y^{(1)} \\ Y^{(2)} \end{pmatrix} \sim \text{MVN} \left[\begin{pmatrix} R(M^{(1)}) \\ R(M^{(2)}) \\ I(M^{(1)}) \\ I(M^{(2)}) \end{pmatrix}, \frac{1}{2} \begin{pmatrix} R(V_{11}) & R(V_{12}) & -I(V_{11}) & -I(V_{12}) \\ R(V_{21}) & R(V_{22}) & -I(V_{21}) & -I(V_{22}) \\ I(V_{11}) & I(V_{12}) & R(V_{11}) & R(V_{12}) \\ I(V_{21}) & I(V_{22}) & R(V_{21}) & R(V_{22}) \end{pmatrix} \right]$$

And thus by Lemma 3.14:

$$\begin{pmatrix} X^{(1)} \\ Y^{(1)} \end{pmatrix} \sim \text{MVN} \left[\begin{pmatrix} R(M^{(1)}) \\ I(M^{(1)}) \end{pmatrix}, \frac{1}{2} \begin{pmatrix} R(V_{11}) & -I(V_{11}) \\ I(V_{11}) & R(V_{11}) \end{pmatrix} \right]$$

Hence by THEOREM 4.3,

$$Z^{(1)} \sim \text{CMVN} [M^{(1)}, V_{11}]$$

Similarly for $Z^{(2)}$

This completes proof.

THEOREM 5.3 (Conditional Densities)

Let $Z \sim \text{CMVN} [M, V]$ and partition Z as in THEOREM 5.2

Then:

$$f(Z^{(1)} / Z^{(2)}) = \text{CMVN} [M^{(1)} + V_{12}(V_{22})^{-1}(Z^{(2)} - M^{(2)}), V_{11} - V_{12}(V_{22})^{-1}V_{21}]$$

(Where $Z^{(1)} = X^{(1)} + iY^{(1)}$, and $Z^{(2)} = X^{(2)} + iY^{(2)}$)

Proof:

By THEOREM 4.3 and the commutativity of the components of the Normal distribution, it is evident that:

$$\begin{pmatrix} X^{(1)} \\ Y^{(1)} \\ X^{(2)} \\ Y^{(2)} \end{pmatrix} \sim \text{MVN} \left[\begin{pmatrix} R(M^{(1)}) \\ I(M^{(1)}) \\ R(M^{(2)}) \\ I(M^{(2)}) \end{pmatrix}, \frac{1}{2} \begin{bmatrix} R(V_{11}) & -I(V_{11}) & R(V_{12}) & -I(V_{12}) \\ I(V_{11}) & R(V_{11}) & I(V_{12}) & R(V_{12}) \\ R(V_{21}) & -I(V_{21}) & R(V_{22}) & -I(V_{22}) \\ I(V_{21}) & R(V_{21}) & I(V_{22}) & R(V_{22}) \end{bmatrix} \right]$$

Lemma 3.2 implies that:

$$\begin{pmatrix} R(V_{22}) & -I(V_{22}) \\ I(V_{22}) & R(V_{22}) \end{pmatrix}^{-1} = \begin{pmatrix} R((V_{22})^{-1}) & -I((V_{22})^{-1}) \\ I((V_{22})^{-1}) & R((V_{22})^{-1}) \end{pmatrix}$$

And hence by Lemma 3.15 :

$$f \left(\begin{pmatrix} X^{(1)} \\ Y^{(1)} \end{pmatrix} / \begin{pmatrix} X^{(2)} \\ Y^{(2)} \end{pmatrix} \right) = \text{MVN} [M^*, V^*] \quad \text{Where:}$$

$$M^* = \begin{pmatrix} R(M^{(1)}) \\ I(M^{(1)}) \end{pmatrix} +$$

$$\begin{aligned} & \begin{bmatrix} R(V_{12}) & -I(V_{12}) \\ I(V_{12}) & R(V_{12}) \end{bmatrix} \begin{bmatrix} R((V_{22})^{-1}) & -I((V_{22})^{-1}) \\ I((V_{22})^{-1}) & R((V_{22})^{-1}) \end{bmatrix} \begin{pmatrix} X^{(2)} - R(M^{(2)}) \\ Y^{(2)} - I(M^{(2)}) \end{pmatrix} \\ & V^* = 1/2 \left\{ \begin{bmatrix} R(V_{11}) & -I(V_{11}) \\ I(V_{11}) & R(V_{11}) \end{bmatrix} + \right. \\ & \left. \begin{bmatrix} R(V_{12}) & -I(V_{12}) \\ I(V_{12}) & R(V_{12}) \end{bmatrix} \begin{bmatrix} R((V_{22})^{-1}) & -I((V_{22})^{-1}) \\ I((V_{22})^{-1}) & R((V_{22})^{-1}) \end{bmatrix} \begin{bmatrix} R(V_{21}) & -I(V_{21}) \\ I(V_{21}) & R(V_{21}) \end{bmatrix} \right\} \end{aligned}$$

Multiplying out:

$$V^* = \frac{1}{2} \begin{pmatrix} R(V_{11}) & -I(V_{11}) \\ I(V_{11}) & R(V_{11}) \end{pmatrix} - \frac{1}{2} \begin{pmatrix} Q & -S \\ S & Q \end{pmatrix} \quad \dots(1)$$

Where:

$$Q = \begin{bmatrix} R(V_{12})R((V_{22})^{-1})R(V_{21}) & -I(V_{12})I((V_{22})^{-1})R(V_{21}) \\ -R(V_{12})I((V_{22})^{-1})I(V_{21}) & -I(V_{12})R((V_{22})^{-1})I(V_{21}) \end{bmatrix}$$

And:

$$S = \begin{bmatrix} I(V_{12})R((V_{22})^{-1})R(V_{21}) & + R(V_{12})I((V_{22})^{-1})R(V_{21}) \\ -I(V_{12})I((V_{22})^{-1})I(V_{21}) & + R(V_{12})R((V_{22})^{-1})I(V_{21}) \end{bmatrix}$$

This is equivalent to:

$$\begin{aligned} (Q + iS) &= \\ &= \begin{bmatrix} R(V_{12}) + iI(V_{12}) \end{bmatrix} \begin{bmatrix} R((V_{22})^{-1}) + iI((V_{22})^{-1}) \end{bmatrix} \begin{bmatrix} R(V_{21}) + iI(V_{21}) \end{bmatrix} \\ &= V_{12}(V_{22})^{-1}V_{21} \end{aligned}$$

Partition M^* as:

$$\begin{pmatrix} M^{*(1)} \\ M^{*(2)} \end{pmatrix} \text{ with same dimensions as } \begin{pmatrix} X^{(1)} \\ Y^{(1)} \end{pmatrix}$$

Now by ... (1) and by THEOREM 4.3,

$$\begin{aligned} f(Z^{(1)}/Z^{(2)}) &= \text{CMVN} \left[(M^{*(1)} + iM^{*(2)}), V_{11} - (Q + iS) \right] \\ &= \text{CMVN} \left[(M^{*(1)} + iM^{*(2)}), V_{11} - V_{12}(V_{22})^{-1}V_{21} \right] \end{aligned}$$

Thus it remains merely, to show that:

$$M^{*(1)} = R \left[M^{(1)} + V_{12}(V_{22})^{-1}(Z^{(2)} - M^{(2)}) \right] \quad \dots(2)$$

$$M^{*(2)} = I \left[M^{(1)} + V_{12}(V_{22})^{-1}(Z^{(2)} - M^{(2)}) \right] \quad \dots(3)$$

$$\begin{aligned}
\text{R.H.S. of } \dots(2) &= \\
R(M^{(1)}) + [R(V_{12})R((V_{22})^{-1}) - I(V_{12})I((V_{22})^{-1})] (X^{(2)} - R(M^{(2)})) \\
- [I(V_{12})R((V_{22})^{-1}) + R(V_{12})I((V_{22})^{-1})] (Y^{(2)} - I(M^{(2)})) \\
&= M^{*(1)} \quad (\text{by multiplying out in formula for } M^* \text{ listed earlier in proof})
\end{aligned}$$

Similarly, $\dots(3)$ is true. Thus:

$$f(Z^{(1)}) / Z^{(2)} = \text{CMVN} [M^{(1)} + V_{12}(V_{22})^{-1}(Z^{(2)} - M^{(2)}), V_{11} - V_{12}(V_{22})^{-1}V_{21}]$$

This completes proof.

Remark: THEOREMS 5.1, 5.2, and 5.3 are exact analogues of the classical multivariate normal distribution.

Next, we shall examine whether or not Cramér's Theorem (Lemma 3.13) can be extended to the Complex Gaussian distribution.

THEOREM 5.4 :

If Z_1 , and Z_2 are independent complex random vectors; and

$$f(Z_1 + Z_2) = \text{CMVN}$$

In order that Z_1 and Z_2 both have the complex Gaussian distribution, it is necessary and sufficient that both Z_1 and Z_2 be Goodman Random Vectors.

Proof:

If Z_1, Z_2 are distributed CMVN, then by Corollary of THEOREM 4.3, Z_1, Z_2 are Goodman Random Vectors.

$f(Z_1 + Z_2) = \text{CMVN}$ implies, by THEOREM 4.3, that:

$$\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} + \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} \sim \text{MVN}$$

also: Since Z_1 is independent of Z_2 , $\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}$ is independent of $\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}$.

Thus conditions of Lemma 3.13 are satisfied. That is,

$\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}$ and $\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}$ are both distributed as MVN. But if Z_1, Z_2 are Goodman random vectors, then by the Corollary of Theorem 4.3 and by Theorem 4.3 itself, Z_1 and Z_2 are distributed as CMVN.

This completes proof.

THEOREM 5.5

If Z_1, Z_2 are independent complex random vectors, then $(Z_1 + Z_2) \sim \text{CMVN}[M, V]$ iff $\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} \sim \text{MVN}[L, A]$ and $\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} \sim \text{MVN}[N, B]$;

$$\text{where } (L + N) = \begin{pmatrix} R(M) \\ I(M) \end{pmatrix} \quad \text{and} \quad (A + B) = 1/2 \begin{pmatrix} R(V) & -I(V) \\ I(V) & R(V) \end{pmatrix}$$

Proof:

By Theorem 4.3, $(Z_1 + Z_2) \sim \text{CMVN}[M, V]$ iff

$$\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} + \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} \sim \text{MVN} \left[\begin{pmatrix} R(M) \\ I(M) \end{pmatrix}, 1/2 \begin{pmatrix} R(V) & -I(V) \\ I(V) & R(V) \end{pmatrix} \right]$$

and that is, by Lemma 3.13 iff:

$\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} \sim \text{MVN}[L, A]$ and $\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} \sim \text{MVN}[N, B]$ with A, B, L, N satisfying the equations above.

This completes proof.

Corollary 1 :

If Z_1 and Z_2 are independently and identically distributed random vectors such that:

$$\begin{aligned} (Z_1 + Z_2) &\sim \text{CMVN} [M, V] & \text{Then} \\ Z_1 &\sim \text{CMVN} [\tfrac{1}{2}M, \tfrac{1}{2}V] \end{aligned}$$

Corollary 2 :

If $z_1, \dots, z_n$ are independent complex random variables having distribution $\text{CN}(m, \sigma^2)$ (univariate CMVN)

$$\text{Then } \hat{z} = \frac{1}{n} \sum_j z_j \sim \text{CN} \left[\left(m, \frac{\sigma^2}{n} \right) \right]$$

Proof:

Put $a_j = \frac{1}{n}$ in THEOREM 5.1.

CHAPTER 6 : Quadratic Forms of the Complex Gaussian
Distribution

THEOREM 6.1 (chi-square theorem)

If $Z \sim \text{CMVN}[\Phi, 2I]$, and A is a positive
semi-definite Hermitian matrix :

Then $\bar{Z}'AZ \sim \chi^2_{2\rho(A)}$ iff $AA = A$

Proof:

Let $Z = X+iY$

Therefore: $\bar{Z}'AZ = \begin{pmatrix} X \\ Y \end{pmatrix}' \begin{bmatrix} R(A) & -I(A) \\ I(A) & R(A) \end{bmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$

Since $Z \sim \text{CMVN}[\Phi, 2I]$, THEOREM 4.3 implies that:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \text{MVN}\left[\Phi, \frac{1}{2} \begin{bmatrix} R(2I) & -I(2I) \\ I(2I) & R(2I) \end{bmatrix}\right] = \text{MVN}[\Phi, I]$$

Now let $B = \begin{pmatrix} R(A) & -I(A) \\ I(A) & R(A) \end{pmatrix}$

By Lemma 3.16 :

$$\begin{pmatrix} X \\ Y \end{pmatrix}' B \begin{pmatrix} X \\ Y \end{pmatrix} \sim \chi^2_{\rho(B)} \quad \text{iff } B \text{ is idempotent.}$$

But by Lemma 3.5, A is idempotent iff B is idempotent.

Since A is Hermitian, $a_{jj} = \bar{a}_{jj}$ (i.e. $I(a_{jj}) = 0$)

Thus: $\text{Tr}(A) = \text{Tr}(R(A)) = \rho(A)$ (by Lemma 3.6)

Since a symmetric matrix is also a Hermitian matrix, then if

$BB = B$: $\text{Tr}(B) = 2\text{Tr}(R(A))$ which implies, by Lemma 3.6:

$$\rho(B) = 2\rho(A)$$

Hence: $\bar{Z}'AZ \sim \chi_{2\rho(A)}^2$ iff $AA=A$

THEOREM 6.2 (chi-square theorem)

If $Z \sim \text{CMVN}[\phi, V]$; and A is a positive semi-definite Hermitian matrix, Then $\bar{Z}'AZ \sim \chi_{2\rho(A)}^2$ iff $\frac{1}{2}AV$ is idempotent.

Proof:

Since V is positive definite Hermitian, there exists a non-singular matrix, P , such that:

$$\begin{aligned} PV\bar{P}' &= 2I & \text{That is:} \\ V &= 2P^{-1}(\bar{P}')^{-1} & (*) \end{aligned}$$

Let: $W = PZ$ Hence by THEOREM 4.2,

$$W \sim \text{CMVN}[P\phi, PV\bar{P}'] = \text{CMVN}[\phi, 2I]$$

$$\text{Also: } \bar{W}'((\bar{P}')^{-1}AP^{-1})W = \bar{Z}'AZ$$

Hence THEOREM 6.1 and the fact that P is non-singular imply that:

$$\bar{Z}'AZ \sim \chi_{2\rho(A)}^2 \text{ iff } (\bar{P}')^{-1}AP^{-1} \text{ is idempotent.}$$

$$\text{i.e. iff: } (\bar{P}')^{-1}AP^{-1}(\bar{P}')^{-1}AP^{-1} = (\bar{P}')^{-1}AP^{-1}$$

$$\text{i.e. iff: } \bar{P}'(\bar{P}')^{-1}AP^{-1}(\bar{P}')^{-1}AP^{-1}(\bar{P}')^{-1} = \bar{P}'(\bar{P}')^{-1}AP^{-1}(\bar{P}')^{-1}$$

$$\text{i.e. iff: } (\frac{1}{2}AV)(\frac{1}{2}AV) = (\frac{1}{2}AV) \quad (\text{by } (*))$$

This completes proof.

Remark: In order to simplify the proof of the following Theorem, we shall introduce two special $p \times p$ matrices:

$C^* =$ (a matrix with each $c^*_{jk} = -1/p$)
 and $H^* = I + C^* =$ (a matrix with all off-diagonal terms $= -1/p$
 and each diagonal term $= (p-1)/p$.)

Recalling definitions of $\bar{z}$, the sample mean; and $s\bar{s}$, the sample variance, from the table of notation (page 4), we shall prove the following important Theorem.

**THEOREM 6.3 : The Distribution of the Sample
 Variance, Zero Mean Case**

If $z_1, \dots, z_p$ are independent complex random variables from $CN(0, \sigma^2)$, then:

$$\frac{2(p-1)s\bar{s}}{\sigma^2} \sim \chi^2_{2p-2}$$

Proof:

$$\begin{aligned} \sum_j (z_j - \bar{z})(\overline{z_j - \bar{z}}) &= \sum_j |z_j|^2 - p\bar{z}\bar{z} \\ &= \bar{Z}'IZ + \bar{Z}'C^*Z \quad (Z' = (z_1, \dots, z_p)) \\ &= \bar{Z}'(I + C^*)Z = \bar{Z}'(H^*)Z \end{aligned}$$

It is clear that $Z \sim CMVN[\Phi, \sigma^2 I]$.

Thus:

$$\frac{2(p-1)s\bar{s}}{\sigma^2} = \bar{Z}' \left(\frac{2}{\sigma^2} \right) H^* Z \quad \text{and for:}$$

$$A = \left(\frac{2}{\sigma^2} \right) H^* ,$$

$\frac{1}{2}A(\sigma^2 I) = H^*$, which is clearly idempotent.

Hence, by THEOREM 6.2,

$$\bar{Z}'AZ \sim \chi^2_{2p(A)}$$

But: $\rho(A) = \rho(H^*) = \text{Tr}(H^*) = (p-1)$

Thus: $\frac{2(p-1)s\bar{s}}{\sigma^2} \sim \chi^2_{(2p-2)}$

This completes proof.

THEOREM 6.4 : The Distribution of the Sample Variance,
General Case

If $z_1, \dots, z_p$ is an independent random sample from

$CN(m, \sigma^2)$, then:

$$\frac{2(p-1)s\bar{s}}{\sigma^2} \sim \chi^2_{2p-2}$$

Proof:

Let $w_j = z_j - m$ $j = 1, 2, \dots, p$

It is clear that: $\hat{w} = \hat{z} - m$ Thus: $w_j - \hat{w} = z_j - \hat{z}$

It is hence evident that the set of random variables $\{z_j\}$ and the set of random variables $\{w_j\}$ have the identical sample variances. But $w_j \sim CN(0, \sigma^2)$. Hence by THEOREM 6.3,

$$\frac{2(p-1)s\bar{s}}{\sigma^2} \sim \chi^2_{2p-2}$$

This completes proof.

THEOREM 6.5 : Independence of Two Quadratic Forms

Let $Z \sim \text{CMVN}[\phi, \sigma^2 I]$

Given: $\bar{Z}'AZ$ and $\bar{Z}'BZ$, two positive semi-definite Hermitian quadratic forms, then:

$\bar{Z}'AZ$ is independent of $\bar{Z}'BZ$ iff $AB = \phi$.

Proof:

(i) We will first assume that $AB = \phi$.

Hence: $(\overline{AB})' = \phi$ which implies that
 $\bar{B}'\bar{A}' = \phi$ which implies, because

A, B Hermitian:

$$BA = \phi$$

Therefore: $AB = BA$

Lemma 3.7 implies that there exists a unitary matrix T such that:

$$\bar{T}'AT = D_1 \quad \text{and} \quad \bar{T}'BT = D_2$$

$$D_1 D_2 = \bar{T}'AT \bar{T}'BT = \bar{T}'ABT \quad (\text{since } T \text{ is unitary})$$

$$= \bar{T}'\phi T = \phi$$

Therefore: $(D_1)_{jj} \neq 0$ implies that $(D_2)_{jj} = 0$

and $(D_2)_{jj} \neq 0$ implies that $(D_1)_{jj} = 0$

Let:

$$\bar{T}'AT = \begin{pmatrix} D^*_1 & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & \phi \end{pmatrix} \quad \bar{T}'BT = \begin{pmatrix} \phi & \phi & \phi \\ \phi & D^*_2 & \phi \\ \phi & \phi & \phi \end{pmatrix}$$

Where: $\rho(A) = \rho(D^*_1) = p_1$
 $\rho(B) = \rho(D^*_2) = p_2$

and block matrix of zeros in lower right hand corner of each matrix has dimensions $p_3 \times p_3$. ($p_1 + p_2 + p_3 = p$)

Let $W = \bar{T}'Z$ that is $Z = TW$

THEOREM 4.4 implies that $W \sim \text{CMVN} [\Phi, \sigma^2 I]$.

$$\bar{Z}'AZ = \bar{W}'\bar{T}'ATW = \bar{W}'D_1W = \sum_{j=1}^{p_1+p_2} \bar{w}_j w_j (D^*_1)_{jj}$$

Similarly: $\bar{Z}'BZ = \sum_{j=p_1+1}^{p_1+p_2} \bar{w}_j w_j (D^*_2)_{jj}$

Therefore: $\bar{Z}'AZ$ depends upon only the first p_1 w_j 's.

and $\bar{Z}'BZ$ depends upon only the next p_2 w_j 's.

But by THEOREM 4.5, $\{w_j\}$ are mutually independent.

Therefore, $\bar{Z}'AZ$ is independent of $\bar{Z}'BZ$.

(ii) Now let us assume that $\bar{Z}'AZ$ and $\bar{Z}'BZ$ are independent.

$$\bar{Z}'AZ = \begin{pmatrix} X' \\ Y' \end{pmatrix} \begin{bmatrix} R(A) & -I(A) \\ I(A) & R(A) \end{bmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\bar{Z}'BZ = \begin{pmatrix} X' \\ Y' \end{pmatrix} \begin{bmatrix} R(B) & -I(B) \\ I(B) & R(B) \end{bmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

Since $Z \sim \text{CMVN} [\Phi, \sigma^2 I]$, then by THEOREM 4.3,

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \text{MVN} [\Phi, \frac{1}{2}\sigma^2 I]$$

Also by Lemma 3.1, A and B are positive semi-definite Hermitian matrices iff :

$\begin{pmatrix} R(A) & -I(A) \\ I(A) & R(A) \end{pmatrix}$ and $\begin{pmatrix} R(B) & -I(B) \\ I(B) & R(B) \end{pmatrix}$ are both positive semi-definite symmetric matrices.

Since we assumed $\bar{Z}'AZ$ and $\bar{Z}'BZ$ independent, the conditions of Lemma 3.17 are satisfied. That is:

$$\begin{pmatrix} R(A) & -I(A) \\ I(A) & R(A) \end{pmatrix} \begin{pmatrix} R(B) & -I(B) \\ I(B) & R(B) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Multiplying out, we obtain:

$$R(A)R(B) - I(A)I(B) = 0$$

$$\text{and } R(A)I(B) + I(A)R(B) = 0$$

And this in turn implies that:

$$AB = (R(A) + iI(A))(R(B) + iI(B)) = 0 + i0 = 0$$

This completes proof.

THEOREM 6.6 : Independence of Linear and Quadratic Forms

If $Z \sim \text{CMVN}[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \sigma^2 I]$; A is a $p \times p$ positive semi-definite

Hermitian matrix; and B is a $q \times p$ matrix:

Then $\bar{Z}'AZ$ is independent of BZ iff $BA = 0$.

Proof:

(i) First we shall assume $BA = 0$.

Let T be such that: $T'AT = D = \begin{pmatrix} D^* & 0 \\ 0 & 0 \end{pmatrix}$ (T unitary)

Where D is diagonal, and D^* is of full rank, p_1 .

Let $W = \bar{T}'Z$; that is since T is unitary: $Z = TW$

Also by THEOREM 4.4 : $W \sim \text{CMVN} [\phi, \sigma^2 I]$.

Now $BA = \phi$ implies that $BAT = \phi$, which in turn implies:

$$BT\bar{T}'AT = \phi , \text{ that is } BTD = \phi .$$

$$\text{Let } BT = X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

$$XD = \phi \text{ implies that } X_{11}D^* = \phi \text{ and } X_{21}D^* = \phi$$

But since diagonal matrices of full rank have no proper divisors of zero: it is clear that $X_{11} = \phi$ and $X_{21} = \phi$.

Thus X is of the form $X = (\phi, X_2)$

Let us partition W into $\begin{pmatrix} W_1 \\ W_2 \end{pmatrix}$ so that W_1 has dimensionality p_1 .

Hence $BZ = BTW = XW = X_2W_2$ depends upon only the components of W_2 .

$\bar{Z}'AZ = \bar{W}'\bar{T}'ATW = \bar{W}'DW = \bar{W}_1'D^*W_1$ depends only upon the components of W_1 .

However: THEOREM 4.5 implies that $w_1, w_2, \dots, w_p$ form a mutually independent set of complex random variables. This further implies that W_1 , and W_2 are independent. Thus any function of W_1 is independent of any function of W_2 . In particular, therefore:

$$\bar{Z}'AZ \text{ is independent of } BZ$$

(ii) We shall now assume that $\bar{Z}'AZ$ and BZ are independent. Thus $\bar{Z}'AZ$ is independent of any function of BZ . In particular, $\bar{Z}'AZ$ is independent of $\bar{Z}'\bar{B}'BZ$.

But $\bar{B}'B$ is positive semi-definite Hermitian. Thus conditions of THEOREM 6.5 (ii) are satisfied. Hence:

$$\bar{B}'BA = \phi \quad \text{This implies}$$

$$\bar{A}'\bar{B}'BA = \phi$$

Let $C = BA$. Therefore $\bar{C}'C = \phi$.

Now $(\bar{C}'C)_{jj}$, the element on the diagonal of the j -th row of $\bar{C}'C$, being ZERO implies:

$$\sum_k |c_{kj}|^2 = 0 \quad j = 1, \dots, p$$

which implies:

$$c_{jk} = 0 \quad j = 1, \dots, p \text{ and } k = 1, \dots, q$$

That is :

$$C = \phi$$

or:

$$BA = \phi$$

This completes proof.

CHAPTER 7 : Central Limit Theorems for Complex
Random Vectors

The most important reason why the Normal distribution has been so useful to statisticians is the enormous class of random variables which converge in law to the Normal distribution. Intuitively, one would expect similar results to hold for complex random variables, and the Complex Normal distribution. Unfortunately, this is not the case.

THEOREM 7.1 : (a central limit condition)

If $Z_1, \dots, Z_m$ are independent, identically distributed Goodman random vectors, mean M , and complex covariance matrix V , Then:

$$\frac{1}{m^2} \sum_{\alpha} (Z_{\alpha} - M) \xrightarrow{L} \text{CMVN}[\phi, V]$$

Proof:

Let $Z_{\alpha} = X_{\alpha} + iY_{\alpha}$. Since Z_{α} , Goodman, it follows:

$$S = \frac{1}{2} \begin{pmatrix} R(V) & -I(V) \\ I(V) & R(V) \end{pmatrix} \quad \text{and} \quad E \begin{pmatrix} X_{\alpha} \\ Y_{\alpha} \end{pmatrix} = \begin{pmatrix} R(M) \\ I(M) \end{pmatrix}$$

where S is covariance matrix of $\begin{pmatrix} X_{\alpha} \\ Y_{\alpha} \end{pmatrix}$

Thus $\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} \dots \begin{pmatrix} X_m \\ Y_m \end{pmatrix}$ are independent identically distributed random vectors satisfying conditions of Lemma 3.18. This implies:

$$\frac{1}{m^{\frac{1}{2}}} \sum_{\alpha=1}^m \left[\begin{pmatrix} X_{\alpha} \\ Y_{\alpha} \end{pmatrix} - \begin{pmatrix} R(M) \\ I(M) \end{pmatrix} \right] \xrightarrow{L} \text{MVN}[\phi, S]$$

which, by THEOREM 4.3, is equivalent to

$$\frac{1}{m^{\frac{1}{2}}} \sum_{\alpha=1}^m (Z_{\alpha} - M) \xrightarrow{L} \text{CMVN}[\phi, V]$$

This completes proof.

THEOREM 7.2 : (a startling counter-example)

If $Z_1, \dots, Z_m$ are independent identically distributed random vectors, which are not Goodman random vectors, having mean M , and complex variance-covariance matrix V , then:

$$T_m = \frac{1}{m^{\frac{1}{2}}} \sum_{\alpha} (Z_{\alpha} - M) \quad \text{does not converge in distribution to } \text{CMVN}[\phi, V].$$

Proof:

Let us assume T_m does converge in law to $\text{CMVN}[\phi, V]$, and obtain a contradiction.

Using the notation of THEOREM 7.1, our assumption is equivalent to (by THEOREM 4.3)

$$\frac{1}{m^{\frac{1}{2}}} \sum_{\alpha=1}^m \left[\begin{pmatrix} X_{\alpha} \\ Y_{\alpha} \end{pmatrix} - \begin{pmatrix} R(M) \\ I(M) \end{pmatrix} \right] \xrightarrow{L} \text{MVN}[\phi, S]$$

But since $\left\{ \begin{pmatrix} X_\alpha \\ Y_\alpha \end{pmatrix} \right\}$ are independent, identically distributed random vectors, S is the covariance matrix of each

$$\begin{pmatrix} X_\alpha \\ Y_\alpha \end{pmatrix} \quad \alpha = 1, 2, \dots, m$$

But form of S implies that each Z_α is a Goodman random vector. This is contrary to the given fact that each Z_α is not a Goodman random vector. This clearly indicates that T_m does not converge in law to $\text{CMVN}[\phi, v]$.

This completes proof.

THEOREM 7.3 (application to sample mean)

If $z_1, \dots, z_m$ are independent, identically distributed Goodman random variables (1-dimensional Goodman random vectors), with mean r , and complex variance σ^2 . Then:

$$\frac{\bar{z} - r}{(\sigma^2/m)^{1/2}} \xrightarrow{L} \text{CN}(0, 1), \text{ as } m \text{ becomes}$$

infinitely large.

Proof:

$$\frac{\hat{z} - r}{(\sigma^2/m)^{1/2}} = \frac{1}{m^{1/2}} \sum_{\alpha} \frac{z_{\alpha} - r}{(\sigma^2)^{1/2}}$$

Since each z_{α} is a Goodman random variable, it is obvious that each of:

$$\frac{z_{\alpha} - r}{(\sigma^2)^{1/2}} \quad \text{is also a Goodman random variable.}$$

Thus by THEOREM 7.1, and the fact that:

$$E[(z_{\alpha} - r)(\overline{z_{\alpha} - r})] = \sigma^2 \quad \alpha = 1, 2, \dots, m$$

it is clear that:

$$\frac{\hat{z} - r}{(\sigma^2/m)^{1/2}} \xrightarrow{L} \text{CN}[0, 1]$$

This completes proof.

CHAPTER 8 : A Characteristic Property of the Complex
Gaussian Distribution

THEOREM 8.1 (main result of chapter)

If Z_1 and Z_2 are independent p -variate complex random vectors, with continuous density functions $J_1(Z_1)$ and $J_2(Z_2)$, which are non-vanishing at $Z_1 = \phi$, and $Z_2 = \phi$, respectively, and if V , a positive definite $p \times p$ Hermitian matrix, and C , a $p \times p$ non-singular matrix satisfy :

1) $V^{-1}C$ is Hermitian

and 2) The eigen values of C lie in open interval, $(0,1)$

Then if $f(Z_1/Z_1 + Z_2) = \text{CMVN} [C(Z_1 + Z_2), V]$
(where / means " given that ")

$Z_1 \sim \text{CMVN}$, and $Z_2 \sim \text{CMVN}$.

Proof:

Let $M(\cdot)$ be the marginal density of (Z_1+Z_2)

$$\begin{aligned} f(Z_1/Z_1+Z_2) &= \frac{J_1(Z_1)J_2(Z_2)}{M(Z_1+Z_2)} \\ &= \frac{1}{\pi^p} \frac{1}{|V|} \exp \left[-(\overline{Z_1 - C(Z_1+Z_2)})' V^{-1} (Z_1 - C(Z_1+Z_2)) \right] \dots (A) \end{aligned}$$

Thus the following special cases of (A) are readily obtained:

$$J_1(Z_1)J_2(\phi) = \frac{M(Z_1)}{\pi^p |V|} \exp \left[-\bar{Z}_1'(\bar{I}-\bar{C})'V^{-1}(I-C)Z_1 \right] \dots(1)$$

$$J_1(\phi)J_2(Z_2) = \frac{M(Z_2)}{\pi^p |V|} \exp \left[-\bar{Z}_2'\bar{C}'V^{-1}CZ_2 \right] \dots(2)$$

$$J_1(\phi)J_2(\phi) = \frac{M(\phi)}{\pi^p |V|} \dots(3)$$

Multiplying (1) by (2) and substituting for $J_1(\phi)J_2(\phi)$ by (3), we obtain: (B):

$$\frac{J_1(Z_1)J_2(Z_2)}{M(\phi)M(Z_1+Z_2)} = \dots(B)$$

$$\frac{M(Z_1)M(Z_2)}{M(\phi)M(Z_1+Z_2)} \frac{1}{\pi^p} \frac{1}{|V|} \exp \left[-\left(\bar{Z}_1'(\bar{I}-\bar{C})'V^{-1}(I-C)Z_1 + \bar{Z}_2'\bar{C}'V^{-1}CZ_2 \right) \right]$$

Equating right hand sides of (A) and (B), we obtain (C):

$$M(Z_1)M(Z_2) =$$

$$M(\phi)M(Z_1+Z_2) \exp \left[\bar{Z}_1'(\bar{I}-\bar{C})'V^{-1}CZ_2 + \bar{Z}_2'\bar{C}'V^{-1}(I-C)Z_1 \right] \dots(C)$$

But $(\bar{I}-\bar{C})'V^{-1}C = V^{-1}C - \bar{C}'V^{-1}C$, (and is thus Hermitian)

$$= (\bar{C}'V^{-1}(\bar{I}-\bar{C}))'$$

Thus by Lemma 3.10, (C) can be reduced to: (D)

$$M(Z_1)M(Z_2) = M(\phi)M(Z_1+Z_2) \exp \left[2R \left(\bar{Z}_1'(\bar{I}-\bar{C})'V^{-1}CZ_2 \right) \right] \dots(D)$$

Let $\alpha(Z) = \frac{1}{M(\phi)} \exp \left[R \left(\bar{Z}' (\bar{I}-C)' V^{-1} CZ \right) \right]$

Thus:

$$\frac{\alpha(Z_1+Z_2)}{\alpha(Z_1)\alpha(Z_2)} = M(\phi) \exp \left[R \left(\bar{Z}_1' A Z_2 + \bar{Z}_2' A Z_1 \right) \right]$$

where $A = (\bar{I}-C)' V^{-1} C$ (Hermitian)

Thus by Lemma 3.10 and equation (D), we obtain :

$$\frac{\alpha(Z_1+Z_2)}{\alpha(Z_1)\alpha(Z_2)} = \frac{M(Z_1)M(Z_2)}{M(Z_1+Z_2)} \dots (E)$$

Let $G(Z) = \alpha(Z)M(Z)$

Equation (E) implies :

$$G(Z_1)G(Z_2) = G(Z_1+Z_2)$$

and thus by Lemma 3.19 :

$$G(Z) = \exp R(\gamma' Z)$$

thus by definitions of G , and α ,

$$M(Z_1) = M(\phi) \exp \left[-R \left(\bar{Z}_1' (\bar{I}-C)' V^{-1} CZ_1 - \gamma' Z_1 \right) \right]$$

with above in equation (1), we obtain :

$$J_1(Z_1)J_2(\phi) = J_1(\phi)J_2(\phi) \exp \left[-\bar{Z}_1' (\bar{I}-C)' V^{-1} (I-C) Z_1 - \bar{Z}_1' (\bar{I}-C)' V^{-1} CZ_1 + R(\gamma' Z_1) \right]$$

(Above since $R(\bar{Z}' B Z) = \bar{Z}' B Z$) (for B Hermitian)

Simplifying, we obtain :

$$J_1(Z_1) = J(\phi) \exp \left[-\bar{Z}_1' V^{-1} (I-C) Z_1 + R(\gamma' Z_1) \right]$$

Let $V^{-1}(I-C) = S$ and $V^{-1}C = T$

$$\begin{aligned} \text{Noting } (Z_1 - \frac{1}{2}S^{-1}\bar{Y})'S(Z_1 - \frac{1}{2}S^{-1}\bar{Y}) &= \\ \bar{Z}_1'SZ_1 - \frac{1}{2}\bar{Y}'Z_1 - \frac{1}{2}\bar{Z}_1'\bar{Y} + \frac{1}{4}\bar{Y}'S^{-1}\bar{Y} &= \\ \bar{Z}_1'SZ_1 - R(\bar{Y}'Z_1) + \frac{1}{4}\bar{Y}'S^{-1}\bar{Y} \end{aligned}$$

Hence the equation for $J_1(Z_1)$ reduces to:

$$J_1(Z_1) = J_1(\phi) \exp \left\{ -\frac{1}{2}(Z_1 - \frac{1}{2}S^{-1}\bar{Y})'S(Z_1 - \frac{1}{2}S^{-1}\bar{Y}) + \frac{1}{4}\bar{Y}'S^{-1}\bar{Y} \right\}$$

Similarly it can be shown that:

$$J_2(Z_2) = J_2(\phi) \exp \left\{ -\frac{1}{2}(Z_2 - \frac{1}{2}T^{-1}\bar{Y})'T(Z_2 - \frac{1}{2}T^{-1}\bar{Y}) + \frac{1}{4}\bar{Y}'T^{-1}\bar{Y} \right\}$$

Since $C = (V^{-1})^{-1}V^{-1}C$ has all its eigen values in the open interval $(0,1)$, and since V^{-1} is positive definite Hermitian, and $V^{-1}C$ is Hermitian, Lemma 3.9 implies both $S = V^{-1} - V^{-1}C$ and $T = V^{-1}C$ are positive definite Hermitian.

Hence by THEOREM 4.1 :

$$J_1(\phi) \exp(\frac{1}{4}\bar{Y}'S^{-1}\bar{Y}) = \frac{1}{\pi^p} \frac{1}{|S^{-1}|}$$

Therefore: $Z_1 \sim \text{CMVN} \left[\frac{1}{2}S^{-1}\bar{Y}, S^{-1} \right]$

Similarly: $Z_2 \sim \text{CMVN} \left[\frac{1}{2}T^{-1}\bar{Y}, T^{-1} \right]$

This completes proof.

Corollary : (Characterization)

If Z_1, Z_2 are independent random vectors having respective densities $\text{CMVN}[\phi, A]$ and $\text{CMVN}[\phi, B]$, then:

$$f(Z_1/Z_1+Z_2) = \text{CMVN}[C(Z_1+Z_2), V],$$

with $V^{-1}C$ Hermitian, and the eigen values of C lie in the open interval $(0,1)$.

Proof:

By THEOREM 5.1, $(Z_1+Z_2) \sim \text{CMVN}[\phi, A+B]$

and hence by THEOREM 5.3,

$$f(Z_1/Z_1+Z_2) = \text{CMVN}[A(A+B)^{-1}(Z_1+Z_2), (I-A(A+B)^{-1})A]$$

Putting $C = A(A+B)^{-1}$ and $V = (I-C)A$

Lemmas 3.11 and 3.12 imply that the eigen values of C lie in $(0,1)$; and that $V^{-1}C$ is Hermitian.

This completes proof.

We thus have a characterization of the Complex Normal Distribution for the zero mean case.

THEOREM 8.2 : (a regression problem)

Given: (1) Z_1, Z_2 are independent p-variate complex random vectors with zero means and respective complex covariance matrices V_1 and V_2

and (2) If Linear Regression of Z_1 on $Z_1 + Z_2$ implies:

$$f(Z_1/Z_1+Z_2) = \text{CMVN} [\alpha(Z_1+Z_2), V]$$

where α is the coefficient matrix of regression;

$$\text{Then } Z_1 \sim \text{CMVN} [\phi, V_1] \quad \text{and} \quad Z_2 \sim \text{CMVN} [\phi, V_2]$$

Proof:

$$\text{Let } \alpha = V_1(V_1+V_2)^{-1} + \beta = C + \beta$$

$$\text{where } C = V_1(V_1+V_2)^{-1}$$

It is easily seen that:

$$(I-C)V_1 = CV_2 \quad \dots (*)$$

$$(\text{Since: } (I-C)V_1 - CV_2 = V_1 - C(V_1+V_2) = \phi)$$

$$\text{Also: } V_1(\overline{I-C})' = V_2\overline{C}' \quad \dots (**) \quad (\text{obtained from } (*))$$

Let H be the complex covariance matrix of $(Z_1 - \alpha(Z_1+Z_2))$

and J be the complex covariance matrix of $(Z_1 - C(Z_1+Z_2))$.

Thus

$$H =$$

$$(I-C)V_1(\overline{I-C})' + CV_2\overline{C}' - \beta V_1(\overline{I-C})' - (I-C)V_1\overline{\beta}' + CV_2\overline{\beta}' + \beta V_2\overline{C}' + \beta(V_1+V_2)\overline{\beta}'$$

and by (*) and (**) this reduces to:

$$\begin{aligned} H &= (I-C)V_1(\overline{I-C})' + CV_2\overline{C}' + \beta(V_1+V_2)\overline{\beta}' \\ &= J + \beta(V_1+V_2)\overline{\beta}' \end{aligned}$$

Since $\beta(V_1+V_2)\overline{\beta}'$ is positive semi-definite for all matrices

β , H is minimum at $\alpha = C$. (i.e. for all vectors W , $\overline{W}'HW \geq \overline{W}'JW$)

Therefore by definition of regression, $\alpha = C$.

But by Lemma 3.11, $C = V_1(V_1+V_2)^{-1}$ has all its eigen values in the open interval $(0,1)$;

and by Lemma 3.12, $V^{-1}C$ is Hermitian.

Since we assumed $f(Z_1/Z_1+Z_2) = \text{CMVN} [C(Z_1+Z_2)', V]$,

the conditions of THEOREM 8.1 are satisfied.

Therefore $Z_1 \sim \text{CMVN} [\phi, V_1]$ and $Z_2 \sim \text{CMVN} [\phi, V_2]$

This completes proof.

Remark: The results in this chapter exactly parallel those of the MVN. (see [9-1])

APPENDIX 1 : Suggestions For Further Studies

(1) Since it is well known that Laplace Transform has not necessarily a unique inverse in the complex plane, what would one use to replace the characteristic function?

Example 1.1 :

Let $z \sim \text{CN}(0, \sigma^2)$ with $z = x + iy$

$$\begin{aligned} M_z(t) &= E(\exp(tz)) = E(\exp(tx)\exp(ity)) \\ &= E(\exp(tx))E(\exp(ity)) \quad \text{since } x, y \text{ independent.} \\ &= \exp(\tfrac{1}{2}t^2\sigma^2)\exp(-\tfrac{1}{2}t^2\sigma^2) \quad \text{by THEOREM 4.3} \\ &= 1 \end{aligned}$$

Thus for all values of σ^2 the moment generating function, and hence by Corollary 2, THEOREM 4.4, the characteristic function is the same.

(2) What closed form results can be obtained for the following logical extension of the complex normal distribution:

$$Z = X + iY \sim \text{CMVN} \quad \text{iff} \quad \begin{pmatrix} X \\ Y \end{pmatrix} \sim \text{MVN} \quad ?$$

(3) What other distributions have complex counterparts, which can be systematically treated ? N.R. Goodman [1] , has treated the Complex Wishart Distribution in some detail.

APPENDIX II: Applications

N.R. Goodman in [2] , lists several applications of Complex Gaussian processes, although at that time he did not call them such. He was motivated in his research by experimenters' needs in such fields as micrometeorology, oceanography, electrical engineering, and aeronautical engineering. Goodman was asked to statistically estimate parameters characterizing "models" of physical systems.

In this dissertation, the distributional properties of some of these estimates have been developed. In addition, the tests of hypotheses concerning certain two-dimensional stationary Gaussian vector processes can be conveniently performed.

The complex Gaussian distribution is an excellent mode of treating Brownian Motion problems in which one considers displacements jointly along the X-axis and the Y-axis.

Analysis of Variance models can easily be constructed using results of Chapter 6. This would be a nice method for treating experiments with bivariate observations. For example we may be interested in the joint yields of copper and zinc at two different temperatures, using three different catalysts. The CMVN reduces this problem to one operation.

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BIBLIOGRAPHY

[1] Goodman, N. R. (1963). Statistical Analysis Based on a Certain Complex Gaussian Distribution. Annals of Mathematical Statistics, Vol. 34, No. 1 .

[2] Goodman, N. R. (1957). On the Joint Estimation of the Spectra, Cospectrum and Quadrature Spectrum of a Two-dimensional Stationary Gaussian Process. Clearinghouse, Washington, D.C. (Xerox).

[3] Anderson, T. W. (1958). An Introduction to Multivariate Statistical Analysis. John Wiley Publication.

[3-1] THEOREM 2.4.3 page 24

[3-2] THEOREM 2.5.1 page 29

[3-3] THEOREM 4.2.3 page 74

[4] Graybill, F. A. (1961). An Introduction to Linear Statistical Models, Volume 1. McGraw-Hill Publication.

[4-1] THEOREM 4.6 page 82

[4-2] THEOREM 4.10 page 84

[5] Cramèr, H. (1946). Mathematical Methods of Statistics. Princeton University Press.

[6] Stoll, R. R. (1952). Linear Algebra and Matrix Theory.
McGraw-Hill Publication.

[7] Linnik, Y. V. (1964). Decomposition of Probability
Distributions. Oliver and Boyd Publishing Company.

[7-1] THEOREM 6.3.1 page 96

[8] Bellman, R. (1960). Introduction to Matrix Analysis.
McGraw-Hill Publication.

[8-1] THEOREM 7 page 59

[8-2] Theorem 5* page 56

[8-3] THEOREM 6* page 58

* The author proves these for symmetric matrices. The
proof for Hermitian matrices is identical, except that
orthonormality is replaced by Unitary.

[9] Seshadri, V. (1966). A Characteristic Property of
the Multivariate Normal Distribution. Annals of Mathematical
Statistics, Vol.37, No.6.

[9-1] THEOREM page 1830