Uniton Bundles

Parametrizing harmonic two-spheres in a unitary group by holomorphic vector bundles.

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Abstract

We show that a twistor construction of Hitchin and Ward can be adapted to study unitons (harmonic spheres in a unitary group). Specifically, we show that unitons are equivalent to holomorphic bundles with extra structure over a rational ruled surface. This equivalence allows us to confirm the conjecture of Wood that unitons are rational. These bundles are in turn representable by monads. By interpreting the uniton construction of Ward in this setting, we are able to give an expression for unitons of 'simplest type' in terms of the monad data (three matrices) using only matrix operations. This expression yields a proof that the components of the moduli and energy levels are one and the same for unitons of 'simplest type'.

Résumé

Nous démontrons qu'une construction twistorielle de Hitchin et de Ward peut être adaptée à l'étude des unitons (des sphères harmoniques dans un groupe unitaire). En particulier, nous démontrons que les unitons sont équivalents à des fibrés holomorphes avec une structure supplémentaire sur une surface réglée rationnelle. Cette équivalence nous permet de confirmer la conjecture de Wood stipulant que les unitons sont rationnels. Ces fibrés peuvent être représentés par des monades. L'interprétation de la construction de Ward dans ce contexte nous permet d'exprimer les unitons du type "le plus simple" en termes des données monadiennes (trois matrices) en n'utilisant que du calcul matriciel. Cette expression démontre que les composantes des modules et niveaux d'énergie sont identifiées pour les unitons du type "le plus simple".

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PREFACE

Harmonic two-spheres in a unitary group were called *unitons* by Uhlenbeck [Uhl], to suggest parallels with self-dual Yang-Mills instantons. Both are solutions to equations from mathematical physics. They are attempts to generalise the theory of electro-magnetism, their solutions representing new particles in the classical/non-quantum sense. In Yang-Mills (or gauge) theory, physical states are measured by fields taking values in a Lie algebra (u(1) for an electro-magnetic field) and the Maxwell equations are replaced by a more general curvature condition. The generalised field theory is complicated by nonlinearity/noncommutativity of the group. In physics, harmonic maps are called chiral fields, or sigma models, and the possible field 'strengths' are points in the target manifold (which may also be a group, but is more commonly a homogeneous space such as a Grassmannian) and the allowed classical states are given by critical values of the energy functional.

More than their shared physical background [Mi], however, one would hope that the harmonic maps have the same beautiful (and much-studied) structure as instantons. One word of caution: while unitons are of interest to mathematicians and mathematical physics for other reasons, they are not particles as their name may suggest, since they are defined on two dimensions and not four-dimensional Minkowski space.

If this constitutes unitons' 'physical' parentage, there is also a mathematical side of the family. Harmonic maps are closely related to minimal submanifolds, and if we allow branch points, the two are equivalent in two dimensions. The study of minimal surfaces and harmonic functions goes back to the nineteenth century. The last decades have seen a lot of work on the existence and regularity of harmonic maps taking advantage of modern tools of analysis such as Sobolev spaces of maps. Work

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on the general problem of finding harmonic maps between Riemannian manifolds has been done, but the bulk of the work (as attested to by [Rep1] and [Rep2]) has considered special cases, such as maps into manifolds of negative curvature or with restricted homotopy type, or maps of Kähler manifolds, spheres or homogeneous spaces.

The relationship between minimal surfaces and holomorphic geometry first appears in the work of Weierstraß ([Wei1], [Wei2], see also [FoTu], [Ei, pp250-264], and [Hi, §9]) who noticed that Enneper's closed form solution of the minimal surface equations in $\mathbb{R}^3$ given in terms of analytic functions and quadratures could be reformulated in terms of two holomorphic functions and their derivatives, without quadratures. The next appearance is in the work of Calabi ([Ca1], [Ca2]) who was studying minimal embeddings of Riemann surfaces in $\mathbb{R}^n$ and S^n from the point of view of determining which metrics on Riemann surfaces are realised as induced metrics from such a minimal embedding. It is in this work that what is now known as the twistor space of a target space appears implicitly. This brand of twistor theory attempts to associate to a target space an (almost) complex manifold, with a fixed projection onto the target space, such that harmonic maps into the target lift uniquely to holomorphic maps from a Riemann surface into the twistor manifold with an extra horizontality property. Many people (see [Rep2]) worked on extending this result to classify harmonic spheres in other symmetric spaces. Much of this work exploited particular properties of the target space, and thus obscured the general nature of the problem. Incrementally, however, this work led to the development of a general theory, best laid out by Burstall and Rawnsley [BuRa] for the case of homogeneous spaces with special properties. For an exposition of the mathematical development of these and related ideas see [Rep2]; for the physical point of view see [Pe].

Applying holomorphic geometry to unitons and the global topology of the moduli motivates this thesis. The main construction of this thesis takes this form, converting the question of studying the space of solutions of the harmonic map equations into that of studying a space of holomorphic bundles.

As an important subtheme of the work on harmonic spheres in Grassmannians, various people discovered a feature of the space of harmonic spheres in Grassmannians of particular importance to physics, namely, a method of constructing new solutions from known solutions. Uhlenbeck calls this a Bäcklund transform, others call it a flag transform, ∂-transform [ChWo], positive/negative transformation [Gu1], or a dressing pseudo-action[Gu2]. It was first laid out in [ZaSha,ZaMi] as the Riemann problem with zeros, which is actually an example of the dressing method developed by the same school in the seventies. The point is that every solution can be constructed beginning with constant solutions by iterating this procedure. Begun by Din and Zakrzewski [DZ] for maps into projective spaces, this line of thought ultimately led to a unique factorisation theorem for unitons in [Uhl]. By interpreting this transform/factorisation in terms of twistor-type constructions, Valli showed that unitons have a discrete energy spectrum, and gave another proof of the factorisation theorem based on energy ([Va], see also [Wo]). In this case, the twistor space turns out to be the loops on the unitary group $\Omega U(N)$ (see [PrSe])–an infinite-dimensional space. Fortunately, the image of any twistor map lies in some finite dimensional subspace determined by the uniton number. Yet another (even shorter) proof of the factorisation theorem is given by Segal ([Se2]), who makes stronger use of the loop group.

We already mentioned the perceived similarities between harmonic maps and self-dual Yang-Mills fields. From the physicists' point of view, this means that both types of solutions display particle-like behaviour. We hope that it also means that the methods which resolved the important problems for instantons will also apply to harmonic maps. In particular, in this thesis we will attempt to apply another type of twistor construction due to Ward [Wd3] (originally used by himself and Hitchin for monopoles), and get results about the uniton moduli via an equivalence with the space of special holomorphic bundles over a complex manifold. (The name twistor theory is actually borrowed from the twistor programme of Penrose in physics which has somewhat broader goals, and is concerned with one particular space, Minkowski space-time. Our twistor space, being smaller has been called mini-twistor space.) In [Wd1], Ward observed that self-dual Yang-Mills (SDYM) fields could be encoded as holomorphic bundles over $\mathbb{P}^3$ (complex projective space). Thus solutions could be constructed from holomorphic bundles. Fortunately for the mathematical physicists, algebraic geometers had already been studying holomorphic bundles over projective space (see [OSS]), and in [ADHM] these questions were further reduced to a matter of linear algebra via the monad construction of Horrocks [Ho] for bundles over $\mathbb{P}^3$. While this gave a method of constructing SDYM fields, it did not give much information about the moduli spaces. Donaldson was later able to show that the bundles are determined by their restriction to a $\mathbb{P}^2$ [Do]. Hurtubise [Hu] was then able to use this description to get topological information by describing the bundles in terms of jumping lines rather than monads. With the addition of heavy topological machinery, this approach yielded a proof of the Atiyah-Jones conjecture [AtJo], see [BHMM].

In the case of unitons Ward ([Wd3]) observed that harmonic maps $\mathbb{R}^2 \to SU(N)$ could be encoded as vector bundles by a twistor construction originally used by Hitchin for monopoles, and that for $N = 2$, finite-energy maps (unitons) correspond to bundles on the compactified base space. Oriented lines in $\mathbb{R}^3$ are parametrised by direction ($\in S^2$) and intercept with the tangent plane orthogonal to the direction ($\in T_pS^2$). The set of lines through a fixed point correspond to a section of TS^2 which turns out to be holomorphic as a section of $T\mathbb{P}^1 \cong TS^2$. (Lines through the origin are the zero section which is holomorphic, but the choice of origin was arbitrary, so the same holds for all sections.) We define a double (Penrose) twistor fibration which allows one to identify via pull-back and push-forward objects in/on $\mathbb{R}^3$ with objects in/on $T\mathbb{P}^1$ and vice versa

$$S^2 \times \mathbb{R}^3 \cong T\mathbb{P}^1 \oplus_{\mathbb{P}^1} \underline{\mathbb{R}}$$

$$\swarrow \qquad\qquad\qquad \searrow$$

$$\begin{array}{ccc}
\mathbb{R}^3 & & T\mathbb{P}^1 \\
 & & \\
\text{point} & \longleftrightarrow & \text{real section} \\
\text{oriented line} & \longleftrightarrow & \text{point} \\
(\nabla, \Phi) & \longleftrightarrow & E \to T\mathbb{P}^1
\end{array}$$

where $(\nabla = d + A, \Phi)$ define a Yang-Mills-Higgs field (monopole) on $\mathbb{R}^3$ and E is a holomorphic bundle.

Ward also shows that 'dynamic unitons' $\mathbb{R}^{2+1} \to U(N)$ can be represented as solutions to the monopole equations on $\mathbb{R}^{2+1}$. These include the traditional static, finite-energy unitons, for which he conjectures that the corresponding bundles extend to the fibrewise compactification, $\widetilde{T\mathbb{P}}^1$, of $T\mathbb{P}^1$. For $U(2)$ he shows this. We won't consider dynamic unitons in this thesis, however, because we lose the correspondence of finite energy and extendability to the compact base space which opens the door to algebro-geometric methods.

The first part of this thesis is devoted to showing that, as Ward conjectured, finite-energy, based $U(N)$ unitons, *i.e.* $\{S : S^2 \to U(N) | S(0) = \mathbb{I}\}$, correspond to bundles on the compact space:

DEFINITION. *A rank N, or $U(N)$,* uniton bundle, *E, is a holomorphic rank N bundle on $\widetilde{T\mathbb{P}}^1$ which is trivial when restricted to the following curves in $\widetilde{T\mathbb{P}}^1$*

(1) *the section at infinity*
(2) *nonpolar fibres (i.e. fibres above $\lambda \in \mathbb{C}^* \subset \mathbb{P}^1$)*
(3) *real sections of $T\mathbb{P}^1$ ($\{\{$oriented lines through $p\} \subset T\mathbb{P}^1 : p \in \mathbb{R}^3\}$)*

and which is equipped with bundle lifts

$$\begin{tikzcd} E \arrow[r, "\tilde{\delta}_t"] \arrow[d] & E \arrow[d] \\ \widetilde{T\mathbb{P}}^1 \arrow[r, "\delta_t"] & \widetilde{T\mathbb{P}}^1 \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} E \arrow[r, "\tilde{\sigma}"] \arrow[d] & E^* \arrow[d] \\ \widetilde{T\mathbb{P}}^1 \arrow[r, "\sigma"] & \widetilde{T\mathbb{P}}^1 \end{tikzcd}$$

such that $\tilde{\delta}_t$ is a one-parameter family of holomorphic transformations and $\tilde{\sigma}$ is a norm-preserving, antiholomorphic lift of σ and the induced hermitian metric on E restricted to a fixed point is positive definite; and a framing, $\phi \in H^0(P_{-1}, Fr(E))$, of the bundle E restricted to the fibre $P_{-1} = \{\lambda = -1\} \subset \widetilde{T\mathbb{P}}^1$ such that $\tilde{\sigma}(\phi) = \phi$.

Here σ and δ_t are fixed maps defined in Chapter II.

THEOREM A. *The space of based unitons, $\mathcal{U}(N)^*$, is isomorphic to the space of N-uniton bundles.*

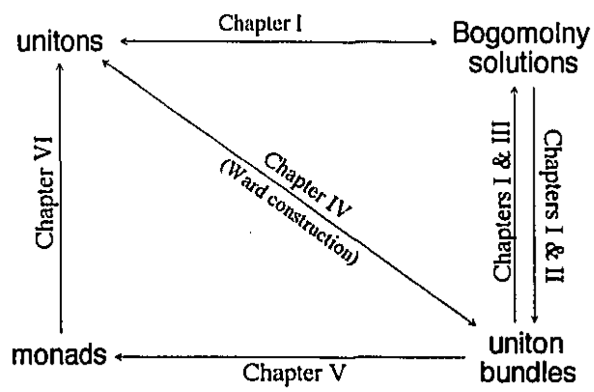
To close the first part of the thesis, we show that this construction actually generalises the construction of Ward for $U(2)$ (this not being obvious, *a priori*) and thereby extends his results on finiteness to rank N bundles. The effect of this is that although our proof is far from being constructive, we can do the computations in terms of clutching matrices as Ward did for $SU(2)$.

As a nice corollary of this, we can affirm the conjecture of Wood [Wo] that unitons are composed of rational functions of $x, y \in \mathbb{R}^2$:

COROLLARY B. *If* $S : S^2 \to U(N)$ *is a uniton, then the composition with* $U(N) \hookrightarrow \mathrm{GL}(N)$ *is rational, i.e. the functions in* x *and* y *which make up the matrix* $S \in U(N)$ *are rational.*

Plan

Much of this thesis concerns the construction of maps between spaces of unitons, bundles, monads and Bogomolny solutions. This aspect is summarised in the diagram



In the first chapter we write down the uniton equations, give the equivalence to Bogomolny solutions, and describe the twistor correspondence upon which everything else is built.

The basic construction is laid out in Chapter II. Section 1 describes coordinates adapted to $S^2 \times \mathbb{R}^3$ and $T\mathbb{P}^1 \oplus \mathbb{R}$ respectively, and time translation and the real structures in both coordinates. The extension of the resultant bundle to the compactified space $\widetilde{T\mathbb{P}}^1$ occupies section two and proceeds in two stages: Extending the bundle over nonpolar fibres and then extending to a neighbourhood of the two missing points (($\lambda = 0, \eta = \infty$) and ($\hat{\lambda} = 0, \hat{\eta} = \infty$) in the notation of II.1). The first stage uses the geometry of finiteness and time independence of the Bogomolny system. The second stage involves showing that the $\bar{\partial}$-operator defining the complex structure can be smoothed by a change of gauge without losing any topology. Finally, we show in §§3-5 that triviality over the section at infinity results from finiteness of the uniton, that $\tilde{\delta}_t$ is constructed naturally from time independence of the uniton, and how the real structure $\tilde{\sigma}$ encodes unitarity.

That the construction induces an isomorphism follows from the existence of an inverse mapping, the subject of Chapter III. Section 1 explains how $T\mathbb{P}^1$ can be embedded in $\mathbb{C}^3 \subset \mathbb{P}^3$ compactifying with the addition of a singular point to become $Q \subset \mathbb{P}^3$. Section 2 describes an algebro-geometric compact twistor fibration which includes the original fibration. Sections 3-4 describe how the connection ∇ and Higgs field Φ, constructed on $\mathbb{R}^3$ using holomorphic geometry, extend to $S^2 \times \mathbb{R}$, and how a time-independent trivialisation of the bundle can be constructed, from which the based uniton ($S^2 \to \mathrm{GL}(N)$) may be recovered by integration. Time invariance follows from the existence of a lift to the bundle of time translation on $\widetilde{T\mathbb{P}}^1$ (§5); unitarity follows from the existence of a real structure (§6).

Chapter IV shows that our construction gives the same map as Ward's construction in terms of clutching matrices, thereby giving a geometrical explanation of Ward's work and showing that finite-energy maps correspond to compact bundles in higher ranks as well. We do this by showing that Uhlenbeck's 'extended solution', $E_\lambda(z, \bar{z})$, which contains the uniton, is given by the 'monodromy' around a cycle of complex lines. By way of application, we prove that unitons are rational, *i.e.* made up of rational functions in x and y, by showing that all singularities of $E_\lambda(z, \bar{z})$ are poles.

The last part of the thesis concerns a monad construction for uniton bundles (Chapter V). A monad is a short sequence of 'homogeneous' bundles

$$0 \to F \xrightarrow{\alpha} G \xrightarrow{\beta} H \to 0$$

such that $\ker \alpha = 0$, $\operatorname{coker} \beta = 0$ and $\ker \beta / \operatorname{im} \alpha$ is a vector bundle, *i.e.* has constant rank. The advantage of this construction is that α and β can be given by matrices. For example, $\mathcal{O}^j_{\mathbb{P}^N} \to \mathcal{O}_{\mathbb{P}^N}(k)^l$, $k \geq 0$ is given by a homogeneous order-k polynomials of $j \times l$ matrices. The moduli space is then the quotient of this complex matrix space by the group of changes of frame of F, G, H. Unfortunately, this space is not always very nice topologically, but there are tools to decide when it is [MFK].

The extra structure uniton bundles carry, however, complicates matters. In §1 we sketch the properties of Hirzebruch surfaces (of which $\widetilde{T\mathbb{P}}^1$ is one) and fix notation which allow us to discuss the construction of monads. We then show how the basic theorem of Beilinson for constructing monads for stable bundles on $\mathbb{P}^N$ can be adapted to $\widetilde{T\mathbb{P}}^1$ (§2). (See also [Bu]). The resulting space of monads is large and we use the action of the group in §3 to put the monad in normal form using the special structure of the bundle: triviality over C_∞, nonpolar fibres, the lift of time translation and the real structure. We also make a seemingly arbitrary Jordan-type normalization resulting in

THEOREM C. *The space of (framed) uniton bundles is isomorphic to a space of monads (a subset of a complex linear space) quotiented by the action of a complex group. The action of the group can be used to put any monad into a unique normal form*

$$0 \to \mathcal{O}(-1,0)^k \xrightarrow[\begin{pmatrix} \mathbb{I}\eta - \omega_2^*\lambda^2 & \omega_1 \\ \rho_2\lambda^2 & \eta\mathbb{I} + \omega_2 \\ -\zeta_2^*\lambda^2 & \theta_2 \end{pmatrix}]{\begin{pmatrix} (\mathbb{I}+\gamma_1^*) + \gamma_1^*\lambda & 0 \\ 0 & \gamma_1 + \lambda(\mathbb{I}+\gamma_1) \end{pmatrix}} \begin{matrix} \mathcal{O}(-1,1)^k \\ \oplus \\ \mathcal{O}^{k+N} \end{matrix}$$

$$\xrightarrow[\begin{pmatrix} -(\mathbb{I}+\gamma_1^*) - \lambda\gamma_1^* & 0 & -(1+\lambda)\theta_2^* \\ 0 & -\gamma_1 - \lambda(\mathbb{I}+\gamma_1) & (1+\lambda)\zeta_2 \end{pmatrix}]{\begin{pmatrix} \mathbb{I}\eta + -\omega_2^*\lambda^2 & -\omega_1 \\ -\rho_2\lambda^2 & \mathbb{I}\eta + \omega_2 \end{pmatrix}} \mathcal{O}(0,1)^k \to 0,$$

where $\gamma_1 \in \mathrm{gl}(k/2)$ *is nilpotent and in Jordan Normal form,* $\omega_2 \in \mathrm{gl}(k/2)$, $\zeta_2 \in M_N^{k/2}$, $\theta_2 \in M^N_{k/2}$, *are in normal forms* $\zeta_2\theta_2 = [\gamma_1, \omega_2]$, *and* ω_1 *and* ρ_2 *are determined by* $\omega_1 + \gamma_1^*\omega_1 + \omega_1\gamma_1 = -\theta_2^*\theta_2$ *and* $\rho_2 + \rho_2\gamma_1^* + \gamma_1\rho_2 = -\zeta_2\zeta_2^*$ *and*

$$\det\begin{pmatrix} (\omega_2 + z/2) - \frac{\bar{z}}{2}((\mathbb{I}+\gamma_1)^{-1}\gamma_1)^2 & \omega_1 \\ \rho_2 & (\rho_1 - \bar{z}/2) + \frac{\bar{z}}{2}((\mathbb{I}+\gamma_0)^{-1}\gamma_0)^2 \end{pmatrix} \neq 0 \quad (2.22)$$

for all $z \in \mathbb{C}$.

(The line bundles $\mathcal{O}(p,q)$ are defined in Chapter V.)

The normal form we reach carries an intricate structure. Future work must answer what information about the moduli it carries, but we will leave the discussion of this and many other unanswered questions (notably about the moduli topology) to the conclusion.

Chapter VI describes how the interpretation of the extended solution of Uhlenbeck as the 'monodromy' of a family of cycles of complex lines shows us how to construct the uniton corresponding to monad data of 'simplest-type', using only matrix multiplication, addition and inversion.

CONSTRUCTION D. *Given monad data as in Theorem C with* $\gamma_1 = 0$, *we can construct the associated uniton as*

$$S = (\mathbb{I} + 2\Omega^* - 2\Omega - \Omega\Omega^* - \Omega^*\Omega)D^{-1}, \tag{3.7}$$

which has extended solution

$$E_\lambda = (\mathbb{I} + \lambda\Omega^* - \lambda^{-1}\Omega)(\mathbb{I} + \Omega^* - \Omega)D^{-1} \tag{3.8}$$

as an extended solution. Since all simplest-type unitons can be so constructed, all such unitons have uniton number 1 or 2.

COROLLARY E. *All simplest-type unitons can be deformed continuously into* $U(2)$ *unitons. As a result, the components of* $\mathcal{U}(N)$ *are the energy levels, i.e.*

$$\pi_0(\mathcal{U}(N)_{\text{simplest-type}}) = \mathbb{N},$$

and the energy of the uniton is given by $1/2$ *the second Chern class of the bundle in that case.*

Finally, in Chapter VII, we review the well-known construction of unitons in $U(2)$ and show how it fits into the monad picture.

Contributions to Original Knowledge

The extension of Hitchin's twistor construction to the compactified space, in particular the geometry of the C_∞ trivialisation and the smoothing of the $\bar{\partial}$-operator are novel. The compact twistor fibration and the resulting extension of Hitchin's methods are also an innovation, as is the method of encoding time-independence. The geometric explanation of Ward's construction and proof of Wood's conjecture are new. Monads for bundles over Hirzebruch surfaces were studied previously by Buchdahl, but his construction concerns stable bundles. Our monad construction encodes the extra structure of a uniton bundle via our normalisation. That $U(2)$ unitons correspond to, and can be constructed from, rational maps was known, but the construction of general simplest-type $U(N)$ unitons from monad data is new, as is the calculation of $\pi_0(\mathcal{U}(N)_{\text{simplest-type}})$.

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CONTENTS

CHAPTER I

PRELIMINARIES

1. Uniton Equations

Unitons are harmonic maps from $S^2 \xrightarrow{S} U(N)$, that is, maps satisfying

$$\frac{\partial}{\partial x}(S^{-1}\frac{\partial}{\partial x}S) + \frac{\partial}{\partial y}(S^{-1}\frac{\partial}{\partial y}S) = 0. \tag{1.1}$$

More generally, harmonic maps between Riemannian manifolds M and N are critical values of an energy functional

$$E(\phi : M \to N) = \int_M |d\phi|^2,$$

which measures the infinitesimal distortion (compression/stretching) of M in N. For example, the energy of an embedding $f : \mathbb{R}^2 \to \mathbb{R}^3$ with the standard metrics is the energy of an ideal rubber sheet, and harmonic maps $S^1 \to N$ are geodesics. In the case of maps into a matrix group, with the standard (left-invariant) metric, the energy takes the form

$$E(S) = \int_{\mathbb{R}^2} |S^{-1}\frac{\partial}{\partial x}S|^2 + |S^{-1}\frac{\partial}{\partial y}S|^2 dx \wedge dy. \tag{1.2}$$

The uniton equations are the corresponding Euler-Lagrange equations. It is worth noting that if $\frac{\partial}{\partial x}S = 0$ they reduce to the equations of a one parameter group—a geodesic!

From [SaUhl, Theorem 3.6], we know that harmonic maps from $\mathbb{R}^2 \to U(N)$ extend to S^2 *iff* they have finite energy, and that such maps are always smooth. In the following, we will use this fact and work in terms of coordinates x and y on $\mathbb{R}^2$. Unitons are determined by $A = A_x dx + A_y dy$ ($= A_z dz + A_{\bar{z}} d\bar{z}$ in characteristic coordinates)

$$A_x \stackrel{\text{def}}{=} \frac{1}{2}S^{-1}\frac{\partial}{\partial x}S, \quad A_y \stackrel{\text{def}}{=} \frac{1}{2}S^{-1}\frac{\partial}{\partial y}S, \tag{1.3}$$

and a choice of initial condition, $S_0 \in U(N)$, as we can see by thinking of $d + 2A$ as a connection, and S as a flattening gauge transformation. This gives a splitting of the space of $U(N)$-unitons $\mathcal{U}(N)$ as

$$\mathcal{U}(N) = U(N) \times \mathcal{U}^*(N), \tag{1.4}$$

where

$$\mathcal{U}^*(N) \stackrel{\text{def}}{=} \{S \in \mathcal{U}(N) : S(\infty) = 1\}$$

will be called the based unitons. Of course, the energy doesn't depend on the basing condition, and we can also write it in terms of A as

$$E = -8 \int \operatorname{tr} A_z A_{\bar{z}}.$$

Two matrices A_x, A_y come from a map $S : \mathbb{R}^2 \to U(N)$ in this way *iff* $d + 2A$ has zero curvature (S is the flat gauge) *iff*

$$0 = d(2A) + [2A, 2A] = 2\left\{\frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x + 2[A_x, A_y]\right\} dx \wedge dy. \tag{1.5 a}$$

They come from a *harmonic* map if in addition

$$0 = d^* A = \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y. \tag{1.5 b}$$

The map $S : \mathbb{R}^2 \to U(N)$ extends to a smooth map $S^2 \to U(N)$ *iff*

$$A_{\hat{z}} \stackrel{\text{def}}{=} -z^2 A_z, \text{ and } A_{\bar{\hat{z}}} \stackrel{\text{def}}{=} -\bar{z}^2 A_{\bar{z}} \tag{1.6}$$

are smooth at $z = \infty$, where we make use of complex coordinates $z = x + iy$, $\hat{z} = 1/z$. In terms of complex coordinates, the uniton equations are (any two of)

$$\begin{aligned} \frac{\partial}{\partial \bar{z}} A_z - \frac{\partial}{\partial z} A_{\bar{z}} + 2[A_{\bar{z}}, A_z] &= 0, \\ \frac{\partial}{\partial \bar{z}} A_z + [A_{\bar{z}}, A_z] &= 0, \\ \frac{\partial}{\partial z} A_{\bar{z}} + [A_z, A_{\bar{z}}] &= 0. \end{aligned} \tag{1.7}$$

1.8 Extended Solutions. We will also make use of Uhlenbeck's extended solutions E_λ (actually first employed in [Po]), which encode the unitons as follows

THEOREM 1.9 [Uhl, 2.1]. *Let $\Omega \subset S^2$ be a simply-connected neighbourhood and $A : \Omega \to T^*(\Omega) \otimes U(N)$. Then $2A = S^{-1}dS$, with S harmonic iff the curvature of the connection*

$$D_\lambda = (\frac{\partial}{\partial \bar{z}} + (1+\lambda)A_{\bar{z}}, \frac{\partial}{\partial z} + (1+\lambda^{-1})A_z) \tag{1.10}$$

vanishes for all $\lambda \in \mathbb{C}^*$.

THEOREM 1.11 [Uhl, 2.2]. *If S is harmonic and $S(\infty) = \mathbb{I}$, then there exists a unique flat frame $E_\lambda : \mathbb{P}^1 \to U(N)$ of D_λ for $\lambda \in \mathbb{C}^*$ with (a) $E_{-1} = \mathbb{I}$, (b) $E_1 = S$, (c) $E_\lambda(\infty) = \mathbb{I}$. Moreover, E is analytic and holomorphic in $\lambda \in \mathbb{C}^*$.*

THEOREM 1.12 [Uhl, 2.3]. *Suppose $E : \mathbb{C}^* \times \Omega \to G$ is analytic and holomorphic in the first variable, $E_{-1} \equiv \mathbb{I}$, and the expressions*

$$\frac{E_\lambda^{-1}\bar{\partial}E_\lambda}{1+\lambda}, \quad \frac{E_\lambda^{-1}\partial E_\lambda}{1+\lambda^{-1}}$$

are constant in λ then $S = E_1$ is harmonic.

Extended solutions are extremely useful in calculations, and we will need to refer to all these results.

2. Bogomolny Equations

The next important rewriting of the uniton equations was Ward's embedding of the harmonic map equations into the hyperbolic Bogomolny equations. The Bogomolny equations are given by

$$\nabla\Phi = *F,$$

where $\nabla = d + A$ is a connection, Φ is a Higgs' field, the curvature $F = \nabla \circ \nabla = dA + A \wedge A$ and the Hodge-star is given by $*dy \wedge dt = dx, *dt \wedge dx = dy$, and $*dx \wedge dy = \epsilon dt$, ($\epsilon = 1$ on $\mathbb{R}^3$ and $\epsilon = -1$ on $\mathbb{R}^{2,1}$). Assuming time independence of

∇ and Φ, the equations are

$$\begin{aligned}
\epsilon[A_t,\Phi] &= \frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x + [A_x,A_y],\\
\frac{\partial}{\partial x}\Phi + [A_x,\Phi] &= \frac{\partial}{\partial y}A_t + [A_y,A_t],\\
\frac{\partial}{\partial y}\Phi + [A_y,\Phi] &= -\frac{\partial}{\partial x}A_t + [A_t,A_x].
\end{aligned}\tag{2.1}$$

Write the system (1.5) as $\nabla_x A_x = -\nabla_y A_y$ and $\nabla_x A_y = \nabla_y A_x$, and (2.1) as $\nabla_x A_y - \nabla_y A_x - [A_x,A_y] = \epsilon[A_t,\Phi]$, $\nabla_x\Phi = \nabla_y A_t$, and $\nabla_y\Phi = -\nabla_x A_t$. Assuming $A_t = \alpha A_x + \beta A_y$, and $\Phi = \gamma A_x + \delta A_y$, we see that (1.5) is equivalent to (2.1) *iff* $A_t = i(\cos(\theta)A_x - \sin(\theta)A_y)$, $\Phi = i(\cos(\theta)A_y + \sin(\theta)A_x)$ in the Euclidean case and $A_t = \cos(\theta)A_x - \sin(\theta)A_y$, $\Phi = \cos(\theta)A_y + \sin(\theta)A_x$ in the hyperbolic case; $0 \le \theta < 2\pi$. In this paper, we will work on $\mathbb{R}^3$ instead of $\mathbb{R}^{2,1}$, because it is geometrically easier, although it necessitates using nonunitary real forms of $\mathrm{gl}(N)$. (Since $\mathrm{gl}(N) = \mathrm{u}(N)\otimes\mathbb{C}$, elements of $\mathrm{u}(N) \subset \mathrm{gl}(N)$ are often called real. In the following, $\mathrm{gl}(N)$ and $\mathrm{GL}(N)$ are *complex*.) Henceforth, we will assume the following choice:

$$A_t = -iA_y, \quad \Phi = iA_x, \tag{2.2}$$

corresponding to $\theta = \pi/2$. It is important to note that A_t and Φ are imaginary (*i.e.* $\in i\,\mathrm{u}(N)$), as this will determine the real structure we will use on $T\mathbb{P}^1$.

This convention allows us to associate a based uniton to time-independent solutions of $\nabla\Phi = *F$ on $\mathbb{R}^2\times\mathbb{R}$, which extend to $\{(x,y,t)\in S^2\times\mathbb{R}\}$: We can use the freedom to change gauge to put any t-independent solution into the form (2.2); the new gauge, g, is given by solving $\frac{\partial}{\partial y}g\,g^{-1} = iA_t - A_y$, $\frac{\partial}{\partial x}g\,g^{-1} = -i\Phi - A_x$, which we can do because the appropriate curvature component

$$\begin{aligned}
\left[\frac{\partial}{\partial x} + A_x + i\Phi, \frac{\partial}{\partial y} + A_y - iA_t\right] &= \left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x + [A_x,A_y] - [A_t,\Phi]\right)\\
&\quad + i\left(-(\frac{\partial}{\partial x}A_t + [A_x,A_t]) - (\frac{\partial}{\partial y}\Phi + [A_y,\Phi])\right)\\
&= (F_{xy} - \nabla_t\Phi) + i\,(F_{tx} - \nabla_y\Phi)
\end{aligned}$$

vanishes for solutions. Of course, we still have to solve $S^{-1}dS = 2A$ (*i.e.* integrate) to get a uniton.

For future reference, we extract from the previous discussion the following

THEOREM 2.3. *The space of based unitons, $\mathcal{U}(N)^*$, is isomorphic to the space of t-independent solutions $\{(\nabla, \Phi)\}$ to the Bogomolny equations with finite energy $\int_{\mathbb{R}^2} |A_x|^2 + |A_y|^2 < \infty$ (equivalently, such that $\lim_{x+iy\to\infty} A_z/z^2$ exists), which are real in the sense that in the unique t-independent gauge such that $A_t = -iAy$ and $\Phi = iA_x$, $A_x, A_y \in \mathrm{u}(N)$.*

3. Twistors

Oriented lines in $\mathbb{R}^3$ are given by a direction and a displacement from the origin perpendicular to the line's direction. Collectively, they make up the space $TS^2 \cong T\mathbb{P}^1 = \mathcal{O}_{\mathbb{P}^1}(2)$ (the exact correspondence depending on an isomorphism of S^2 and $\mathbb{P}^1$ to be specified using stereographic projection). Reversing the direction of geodesics corresponds to an antiholomorphic involution, τ, of $T\mathbb{P}^1$ which is the negative of the map on $T\mathbb{P}^1$ induced by the antipodal map, $\tau^*\lambda = -1/\bar{\lambda}$, on $\mathbb{P}^1$. The holomorphic bundle $T\mathbb{P}^1$ has sections $\eta = \frac{a}{2} - b\lambda - \frac{c}{2}\lambda^2$, where λ and η are base and fibre coordinates over $\mathbb{P}^1 \setminus \{\infty\}$. So its section space is $\mathbb{C}^3$.

Let $\underline{\mathbb{R}}$ be the trivial real line bundle. The geometry can be represented by a *twistor fibration*:

$$\begin{array}{ccc} & S^2 \times \mathbb{R}^3 \cong T\mathbb{P}^1 \oplus \underline{\mathbb{R}} & \\ \pi_{\mathbb{R}^3} \swarrow & & \searrow \pi_{T\mathbb{P}^1} \\ \mathbb{R}^3 & & T\mathbb{P}^1. \end{array} \tag{3.1}$$

The point of this construction is that there is a *twistor correspondence* between solutions (∇, Φ) to the Bogomolny equations on $\mathbb{R}^3$ and holomorphic bundles on $T\mathbb{P}^1$ which are trivial on *real sections* of $T\mathbb{P}^1$. Real sections can be defined invariantly as sections stable under an antiholomorphic involution (a *real structure*). This twistor construction is due to Hitchin [Hi], and Ward.

3.2 The Bundle. Let $\widetilde{E} = \mathbb{C}^N \times \mathbb{R}^3$ be the trivial bundle over $\mathbb{R}^3$. Define the bundle $E \to T\mathbb{P}^1$, over $\ell \in \widetilde{T\mathbb{P}^1}$ by

$$E_\ell = \left\{ s \in H^0(\ell, \widetilde{E}) : \begin{array}{l} (\nabla_u - i\Phi)s = 0, \\ \text{(where the line } \ell \subset \mathbb{R}^3 \\ \text{is parametrised by ar-} \\ \text{clength, } u) \end{array} \right\}. \tag{3.3}$$

THEOREM 3.4 [Mu2,18]. *If (∇, Φ) is a unitary solution of the Bogomolny equations $\nabla\Phi = *F$ on $\mathbb{R}^3$, then E is in a natural way a holomorphic principal bundle on the space of geodesics $T\mathbb{P}^1$ such that*

(1) *E is trivial on every real (τ-invariant) section;*
(2) *there exists a positive-definite, antiholomorphic principal bundle involution $\tilde{\tau} : E \to E$ lifting τ; i.e. if f is a local frame for E, and g a constant change of gauge ($g \in U(N)$) then $\tilde{\tau}(fg) = \tilde{\tau}(f)\bar{g}^{t-1}$ and*
(3) *$\tilde{\tau} \circ \tilde{\tau} = id$.*

Conversely, every such E defines a solution of the Bogomolny equations.

(Note that this theorem has an equivalent statement in terms of the vector bundle and its dual, and is a generalisation of [Hi, Theorem 4.2], which is stated in a way specific to $SU(2)$. In terms of vector bundles, the condition $\tau^2 = \mathrm{id}$ becomes $\tau^* = \tau$.)

We will use this theorem with the gauge group $GL(N, \mathbb{C})$, and conjugation $\bar{g} = (g^*)^{-1}$, *i.e.* the real subgroup is $U(N)$, but with an antiholomorphic involution $\tilde{\sigma}$ lifting σ which corresponds to the choice (2.2).

The bundle E comes with a natural $\bar{\partial}$ operator, *i.e.* an operator

$$\bar{\partial} : \Gamma(T\mathbb{P}^1, E \otimes T^{(p,q)}T\mathbb{P}^1)) \to \Gamma(T\mathbb{P}^1, E \otimes T^{(p,q+1)}T\mathbb{P}^1),$$

which satisfies a Leibnitz rule and $\bar{\partial}^2 = 0$. This defines a complex bundle structure (since its flat sections will give local holomorphic framings).

The $\bar{\partial}$-operator is defined as follows.

The embedding $i : \mathbb{P}^1 \hookrightarrow \mathbb{R}^3$ induces a splitting of $\mathbb{R}^3 \times \mathbb{P}^1 \to \mathbb{P}^1$ into

$$T\mathbb{P}^1 \oplus < \text{normal bundle} > .$$

On $T\mathbb{P}^1$ we put the restriction of $\pi^*_{\mathbb{R}^3}\nabla$. On the normal bundle, we put $\pi^*_{\mathbb{R}^3}(\nabla_u - i\Phi)$. This connection pulls back to a connection along the fibre directions of $\mathbb{P}^1 \times \mathbb{R}^3 \to \mathbb{P}^1$. Along the base we put the trivial connection.

Since $T\mathbb{P}^1$ has a complex structure, we can split $T(T\mathbb{P}^1)$ into holomorphic and antiholomorphic parts. The restriction of the connection to the antiholomorphic part, $\bar{\nabla}$, acts on sections of $\widetilde{E}|_\ell$ and since $\{\bar{\nabla}, \nabla_u - i\Phi\}$ are involutive (as a consequence of the Bogomolny equations) it induces a $\bar{\partial}$-operator on E.

We have encoded unitons as solutions (∇, Φ) of the Bogomolny equations with special properties (time invariance, finiteness, reality). We will show in Chapter II that these properties correspond to properties of the bundle $E \to T\mathbb{P}^1$. Namely,

(1) time translation induces a one-parameter family of automorphisms of $T\mathbb{P}^1$ which lift to bundle maps.

(2) Finiteness translates as an extension of the bundle to the fibrewise compactification of $T\mathbb{P}^1$, time translation extending as the trivial map over the fibre at infinity.

(3) Reality of the Bogomolny equations translates as a real structure (a fixed antiholomorphic principal bundle involution over an antiholomorphic involution of $T\mathbb{P}^1$, or a lift to a map $E \to E^*$) which is positive definite above a fixed point.

Our part real, part imaginary pair (∇, Φ) corresponds to a different real structure on $T\mathbb{P}^1$ than the one which fixes the real sections. We can almost do without the original structure, but will need to have it around when we set about reconstructing the uniton from the bundle.

3.5. Inverse Construction. Conversely, given such a bundle we can construct the Bogomolny solution as in [Hi].

From the sections map

$$\mathbb{P}^1 \times \mathbb{C}^3 \to \mathcal{O}(2) \cong T\mathbb{P}^1$$
$$\lambda, (a, b, c) \mapsto \eta = \frac{1}{2}a - b\lambda - \frac{1}{2}c\lambda^2$$

we get a pull-back of the bundle E to $\mathbb{P}^1 \times \mathbb{C}^3$. Over the open set Y of sections over which E is trivial, we can push this bundle down from $\mathbb{P}^1 \times Y$ to Y. Call this bundle $\widetilde{E} \to Y$.

Now put the quadratic form $(db)^2 + (da)(dc)$ on the holomorphic tangent space to $\mathbb{C}^3$. To a degenerate metric we can associate null planes on which the restricted metric is degenerate, and null lines on which the restricted metric is zero. Each null line lies in a unique null plane (its orthogonal complement). Its null planes are parametrised by $p \in T\mathbb{P}^1$ and given by the space of sections of $T\mathbb{P}^1$ through p, *i.e.* if $p = (\lambda, \eta)$ then (a, b, c) are constrained by $\eta = \frac{1}{2}(a - 2b\lambda - c\lambda^2)$. Restricted to this plane, the metric $(db)^2 + (2\lambda db + \lambda^2 dc)(dc) = (db + \lambda dc)^2$ is degenerate. Each null plane $\Pi_\lambda(y)$ inherits a flat 'null' connection: Let $p \in \widetilde{T\mathbb{P}}^1$ be the unique point of intersection of the family of sections $\Pi_\lambda(y)$. A fixed frame of $E|_p$ induces a frame of $\widetilde{E}|_{\Pi_\lambda(y)}$. Define ∇_Π to be the 'null' connection for which this frame is covariant constant.

Now fix a point $y \in Y$. Some directions (lines through y in $\mathbb{C}^3$) may lie in two different null planes (they correspond to sections intersecting in two distinct points), but null lines lie in unique null planes, so we can define a holomorphic connection on the null lines without ambiguity. The null directions form a quadric cone $Q^* = \{[a, b, c] \in \mathbb{P}^2 : b^2 + ac = 0\}$ in the $\mathbb{P}^2$ of all directions. Since a connection matrix (given by differentiating a covariant constant frame at y) is a homogeneous degree-one, matrix-valued function of $\mathbb{C}^3$ (the tangent plane to $y \in \mathbb{P}^3$), it defines a section in $H^0(Q^*, \mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathrm{gl}(N))$. The long exact sequence associated to the inclusion $Q^* \hookrightarrow \mathbb{P}^2$ and the fact that $H^0(\mathbb{P}^2, \mathcal{O}(-1)) = 0 = H^1(\mathbb{P}^2, \mathcal{O}(-1))$ tell us that such sections are uniquely extendable to $\mathbb{P}^2$. In other words, this is enough to determine uniquely a holomorphic connection at y (*i.e.* an element of $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathrm{gl}(N))$). This is the connection ∇.

There is a second way of defining the connection. On any line in $\mathbb{C}^3$, pick two points y_1, y_2. They correspond to two sections of $T\mathbb{P}^1$. Since $T\mathbb{P}^1$ is the total space of $\mathcal{O}_{\mathbb{P}^1}(2)$, they intersect in two points (with multiplicity). We get a double point *iff* the sections are tangent at a point *iff* the line was null. To each point y_i is associated a connection on the corresponding hyperplane $\Pi_\lambda(y_i) \subset \mathbb{C}^3$. The original line will be the intersection of the two null planes unless it was a null line, in which case it is only contained in the intersection. Taking the average of the two null connections, we get a connection on the original line, *i.e.*

$$\nabla_{\text{line containing } y_1, y_2} = \frac{1}{2}(\nabla_{\Pi(y_1)} + \nabla_{\Pi(y_2)}) \tag{3.6}$$

In the case of a null line, we get back the null connection, because any two sections will intersect in a double point.

In the case of a real line (a line in $\mathbb{R}^3$ complexified) the two null planes are conjugate, *i.e.* τ images of each other. Since this construction also gives a holomorphic connection, which agrees with the first on null lines, it follows from the preceding discussion that they are identical.

As mentioned above, we get a connection on $\mathbb{R}^3$ by making the desired choice of coordinates on $\mathbb{C}^3$ and forgetting they are complex.

3.7 Higgs' Field. Again there are two definitions for the Higgs' field on $\mathbb{C}^3$. (See [Hi].)

Given a fixed real structure, the null planes have adapted coordinates, one cutting out the null lines, the other parametrising them. Call them χ and π, and let them be coordinates on a null plane Π. They are not canonical, but any choice such that $d\chi = 0$ on null lines and $|d\chi| = 1$ on null planes will do.

By definition, the connection ∇ agrees with the flat (null) connection ∇_Π in the null direction, so they differ by a form which annihilates π. This defines a $\mathrm{gl}(N)$-valued function $\Phi(y, \Pi)$:

$$\nabla - \nabla_\Pi = i\Phi d\chi,$$

where the i is added for convenience. The null planes through a fixed point y are parametrised by $\lambda \in \mathbb{P}^1$. Since Φ is a holomorphic ($\mathrm{gl}(N)$-valued) function of λ, it is constant and therefore independent of the null plane chosen. Recalling the other definition of ∇ along a line as $1/2(\nabla_\Pi + \nabla_{\tau\Pi})$, where the line is the intersection of the two null planes, we can also write

$$\Phi d\chi = i/2(\nabla_\Pi - \nabla_{\tau\Pi}), \tag{3.8}$$

where the null plane $\tau\Pi$ also contains the line through y tangent to $\frac{\partial}{\partial\chi}$, and can be chosen uniquely.

As it turns out, we can parametrise null planes through y by $\lambda \in \mathbb{P}^1$:

$$\Pi_\lambda(y) = \left\{(a, b, c) | 2\eta = a - 2b\lambda - c\lambda^2\right\}, \tag{3.9}$$

where $\eta = \frac{1}{2}(a_y - 2b_y\lambda - c_y\lambda^2)$, and $\tau\Pi$ corresponds to taking the antipodal point to λ, *i.e.* $\Pi_{\tau(\lambda)}(y) = \Pi_{-\bar{\lambda}^{-1}}(y)$. Put this way, we can calculate $\frac{\partial}{\partial\chi}$ on $\mathbb{C}^3$. By taking derivatives, we get (after normalising $\frac{\partial}{\partial\chi}$)

$$\begin{aligned}
\frac{\partial}{\partial\chi} &= -\frac{1}{1+\lambda\bar{\lambda}}(\lambda\frac{\partial}{\partial a} + \frac{1}{2}(1-\lambda\bar{\lambda})\frac{\partial}{\partial b} + \bar{\lambda}\frac{\partial}{\partial c}) \\
&= -\frac{1}{1+\lambda\bar{\lambda}}((\lambda+\bar{\lambda})\frac{\partial}{\partial x} + (1-\lambda\bar{\lambda})\frac{\partial}{\partial t} + i(\bar{\lambda}-\lambda)\frac{\partial}{\partial y}), \text{ and} \\
\frac{\partial}{\partial\pi} &= \lambda^2\frac{\partial}{\partial z} + \lambda\frac{\partial}{\partial t} - \frac{\partial}{\partial\bar{z}} \\
&= \frac{1}{2}(\lambda^2-1)\frac{\partial}{\partial x} + \lambda\frac{\partial}{\partial t} - \frac{i}{2}(1+\lambda^2)\frac{\partial}{\partial y}.
\end{aligned} \tag{3.10}$$

Since null connections are by definition flat,

$$\begin{aligned}
0 &= [\nabla^{\Pi}_{\chi}, \nabla^{\Pi}_{\pi}] = [\nabla_{\chi} + i\Phi d\chi, \nabla_{\pi}] \\
&= F_{\chi\pi} - i\nabla_{\pi}\Phi \\
&= -\frac{i}{2}(\lambda^2-1)(\nabla_x\Phi - F_{yt}) - i\lambda(\nabla_t\Phi - F_{xy}) - (1+\lambda^2)(\nabla_y\Phi - F_{tx})
\end{aligned}$$

Since Φ does not depend on the null plane, we get a $\mathbb{P}^1$ of equations parametrised by λ. Taken together, they show that ∇, Φ satisfy the Bogomolny equations on $\mathbb{C}^3$.

In Chapter III, we will use complex algebraic methods to understand what happens in the finite (energy) case. The space of lines, $T\mathbb{P}^1$, can be embedded into $\mathbb{P}^3$ as a quasi-projective variety. It can be compactified by adding a singular point. Since E is trivial over the section at infinity anyway, we get a bundle over this variety (which is a degenerate conic). Sections of $T\mathbb{P}^1$ correspond to certain hyperplane sections. In fact, the hyperplane sections of $\mathbb{P}^3$ are given by $(\mathbb{P}^3)^*$, and the sections of $T\mathbb{P}^1$ correspond to $\mathbb{C}^3 \subset \mathbb{P}^3$. We work out a 'complex version' of $S^2 \times \mathbb{R}$, and find it sitting in $(\mathbb{P}^3)^*$. The points we need to add from $\mathbb{P}^2 = \mathbb{P}^3 \setminus \mathbb{C}^3$, are just the set of hyperplane sections restricted to which the bundle is trivial, so we do in fact get back a (finite) uniton.

4. $\widetilde{T\mathbb{P}^1}$

We will be working with coordinates (λ, η) and $(\hat{\lambda} = 1/\lambda, \hat{\eta} = \eta/\lambda^2)$ on $T\mathbb{P}^1 \cong \mathcal{O}_{\mathbb{P}^1}(2)$, where λ is the usual coordinate on $\mathbb{P}^1$ and η is the coordinate associated to

$d/d\lambda$. The bundle $T\mathbb{P}^1$ can be compactified by adding a section at infinity, which means considering the injection of $T\mathbb{P}^1$ into the projectivization $\mathbb{P}(T\mathbb{P}^1 \oplus \mathcal{O})$ (a holomorphic $\mathbb{P}^1$-fibre bundle over $\mathbb{P}^1$).

LEMMA 4.1. $\mathbb{P}(\mathcal{O}(2) \oplus \mathcal{O}(0)) \to \mathbb{P}^1$ *is the only fibrewise compactification of* $T\mathbb{P}^1$.

PROOF. A fibrewise compactification of $\mathcal{O}_{\mathbb{P}^1}(2)$ is a rational ruled surface. Since all rational ruled surfaces (strictly speaking, relatively minimal models) are of the form $S_j = \mathbb{P}(\mathcal{O}(j) \oplus \mathcal{O}(0)) \to \mathbb{P}^1$ for $j \geq 0$ [GriHa,p.514], it suffices to rule out the other possibilities. If $j \geq 3$, then S_j has no section with self-intersection less than j except the section at infinity which has self-intersection $-j$, so $\mathcal{O}(2)$, whose sections have self-intersection two cannot be embedded. For $j = 0, 1$, assume that $T\mathbb{P}^1$ could be embedded into S_j, then $S_j \setminus T\mathbb{P}^1$ would be a section which does not intersect the images of the sections of $T\mathbb{P}^1$, which would have self-intersection two, but the only sections with self-intersection two are sections with two ($j = 0$) or one ($j = 1$) pole(s) which intersect all other sections (including the infinity section), so there can be no such embedding. □

Meromorphic sections (s) of $T\mathbb{P}^1$ give all the holomorphic sections of $\widetilde{T\mathbb{P}}^1$ ([s,1] in projective coordinates), except the section at infinity ([1,0]). We will use the following notation for curves on $\widetilde{T\mathbb{P}}^1$:

$$\begin{aligned} P_\lambda &= \pi^{-1}(\lambda \in \mathbb{P}^1) = \text{a pfibre (silent p)} \\ C_0 &= \{(\lambda, [0,1])\} = \text{zero section of } T\mathbb{P}^1 \\ C_\infty &= \{(\lambda, [1,0])\} = \text{infinity section of } T\mathbb{P}^1 \\ C_{\eta=s} &= \{(\lambda, [s(\lambda), 1])\}\,. \end{aligned} \tag{4.2}$$

If $y = (a, b, c) \in \mathbb{C}^3$, we will also write C_y for $C_{\eta=\frac{1}{2}(a-2b\lambda-c\lambda^2)}$.

CHAPTER II

THE BUNDLE

In this chapter we assume (∇, Φ) comes from a uniton in the way specified in Chapter I.

1. Adapted Coordinates

Dual to stereographic projection is the embedding $\mathbb{P}^1 \hookrightarrow \mathbb{R}^3$:

$$\lambda \overset{i}{\mapsto} \left(\frac{\lambda + \bar{\lambda}}{1 + \lambda\bar{\lambda}}, -i\frac{\lambda - \bar{\lambda}}{1 + \lambda\bar{\lambda}}, \frac{1 - \lambda\bar{\lambda}}{1 + \lambda\bar{\lambda}} \right) \tag{1.1}$$

alternatively, $\mathbb{P}^1 \hookrightarrow \mathbb{C} \times \mathbb{R}$:

$$\lambda \overset{i}{\mapsto} \left(\frac{2\lambda}{1 + \lambda\bar{\lambda}}, \frac{1 - \lambda\bar{\lambda}}{1 + \lambda\bar{\lambda}} \right).$$

Using this inclusion we get an exact sequence of (real) bundles over $\mathbb{P}^1$:

$$0 \to T\mathbb{P}^1 \xrightarrow{i_*} T\mathbb{R}^3\big|_{\mathbb{P}^1} \cong \mathbb{P}^1 \times \mathbb{R}^3 \to N_{\mathbb{P}^1} \to 0,$$

where $N_{\mathbb{P}^1}$ is the normal bundle of the embedding, and

$$\begin{aligned} i_*\left(\frac{\partial}{\partial\lambda}\right) &= \frac{2}{(1+\lambda\bar{\lambda})^2}\frac{\partial}{\partial z} - 2\frac{\bar{\lambda}^2}{(1+\lambda\bar{\lambda})^2}\frac{\partial}{\partial \bar{z}} - 2\frac{\bar{\lambda}}{(1+\lambda\bar{\lambda})^2}\frac{\partial}{\partial t} \\ i_*\left(\frac{\partial}{\partial\bar{\lambda}}\right) &= -2\frac{\bar{\lambda}^2}{(1+\lambda\bar{\lambda})^2}\frac{\partial}{\partial z} + \frac{2}{(1+\lambda\bar{\lambda})^2}\frac{\partial}{\partial \bar{z}} - 2\frac{\lambda}{(1+\lambda\bar{\lambda})^2}\frac{\partial}{\partial t}. \end{aligned}$$

The isomorphism $\mathbb{R}^3 \times \mathbb{P}^1 \cong T\mathbb{P}^1 \oplus N_{\mathbb{P}^1}$ suggests that we find adapted coordinates to replace z, $\bar{z}$, t on $\mathbb{R}^3$. The fibre coordinate of $T\mathbb{P}^1 \oplus N_{\mathbb{P}^1} \to T\mathbb{P}^1$ (a trivial real line bundle) we choose will be the one associated with the canonical section

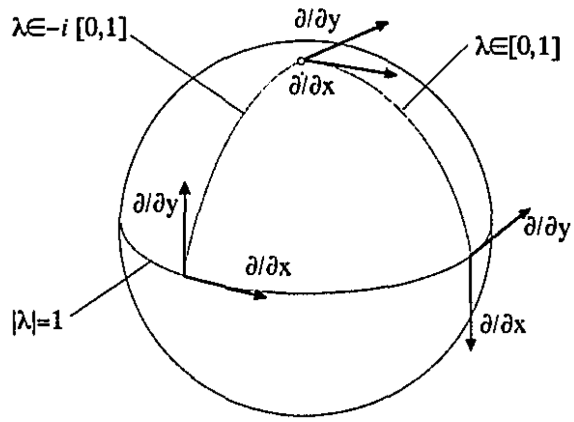

FIG. 1. The embedding of $\{\lambda \in S^2\}$ in $\mathbb{R}^3$ induces, for all λ, a splitting of $T_p\mathbb{R}^3$ into the tangent plane and the normal line. The standard real tangent vectors to $\lambda \in \mathbb{C}$ push forward to basis vectors on the plane to which we associate the coordinate η; the coordinate u is uniquely associated to the unit outward normal, and parametrises lines in the direction u. The η coordinate is not well defined at the south pole, because the bundle $T\mathbb{P}^1 \cong \mathcal{O}(2)$ has a double twist, whereas the coordinate u extends to all values of λ.

$(\lambda, i(\lambda)) \in \mathbb{P}^1 \times \mathbb{R}^3$, where i is the embedding. We will call this coordinate u, and $\eta i_*(\frac{\partial}{\partial \lambda})$ gives us the fibre coordinate of $T\mathbb{P}^1 \to \mathbb{P}^1$. For convenience, we write out:

$$\begin{pmatrix} z \\ \bar{z} \\ t \end{pmatrix} = \begin{pmatrix} \frac{2}{(1+\lambda\bar{\lambda})^2} & \frac{-2\lambda^2}{(1+\lambda\bar{\lambda})^2} & \frac{2\lambda}{1+\lambda\bar{\lambda}} \\ \frac{-2\bar{\lambda}^2}{(1+\lambda\bar{\lambda})^2} & \frac{2}{(1+\lambda\bar{\lambda})^2} & \frac{2\bar{\lambda}}{1+\lambda\bar{\lambda}} \\ \frac{-2\bar{\lambda}}{(1+\lambda\bar{\lambda})^2} & \frac{-2\lambda}{(1+\lambda\bar{\lambda})^2} & \frac{1-\lambda\bar{\lambda}}{1+\lambda\bar{\lambda}} \end{pmatrix} \begin{pmatrix} \eta \\ \bar{\eta} \\ u \end{pmatrix}, \text{ and} \tag{1.2}$$

$$\begin{pmatrix} \eta \\ \bar{\eta} \\ u \end{pmatrix} = \begin{pmatrix} 1/2 & -\lambda^2/2 & -\lambda \\ -\bar{\lambda}^2/2 & 1/2 & -\bar{\lambda} \\ \bar{\lambda}/(1+\lambda\bar{\lambda}) & \lambda/(1+\lambda\bar{\lambda}) & (1-\lambda\bar{\lambda})/(1+\lambda\bar{\lambda}) \end{pmatrix} \begin{pmatrix} z \\ \bar{z} \\ t \end{pmatrix}.$$

Remark that $\eta = \frac{1}{2}(z - 2\lambda t - \lambda^2 \bar{z})$, $\bar{\eta} = \frac{1}{2}(\bar{z} - 2\bar{\lambda} t - \bar{\lambda}^2 z)$, and restricted to the plane $\{u = 0\}$, $z = \frac{2\eta - 2\lambda^2\bar{\eta}}{(1+\lambda\bar{\lambda})^2}$, $\bar{z} = \frac{-2\bar{\lambda}^2\eta + 2\bar{\eta}}{(1+\lambda\bar{\lambda})^2}$, $\eta = \frac{1}{2}\frac{(1+\lambda\bar{\lambda})(z+\lambda^2\bar{z})}{1-\lambda\bar{\lambda}}$, $\bar{\eta} = \frac{1}{2}\frac{(1+\lambda\bar{\lambda})(\bar{\lambda}^2 z + \bar{z})}{1-\lambda\bar{\lambda}}$ (away from $\{|\lambda| = 1\}$).

Recall that two antiholomorphic involutions are involved in our construction, τ

and σ. They both come from involutions of S^2; τ restricting to the antipodal map, and σ restricting to the reflection through the equator $|\lambda| = 1$. On $T\mathbb{P}^1$ they are given by

$$\tau^*(\lambda, \eta) = (-1/\bar{\lambda}, -\bar{\lambda}^{-2}\bar{\eta}), \quad \sigma^*(\lambda, \eta) = (1/\bar{\lambda}, -\bar{\lambda}^{-2}\bar{\eta}). \tag{1.3}$$

On $\mathbb{C}^3$ they are given by

$$\tau^*(a, b, c) = (\bar{c}, \bar{b}, \bar{a}), \quad \sigma^*(a, b, c) = (\bar{c}, -\bar{b}, \bar{a}).$$

We see from these expressions that the two real structures are related to the time invariance. Real time for one is imaginary time for the other, and real sections of one are (complex) time translates of real sections of the other. From this fact and time invariance it follows that a bundle is trivial on σ-real sections *iff* it is trivial on τ-real sections. This is enough to rewrite much of the following in terms of one involution alone, but the relevance of two involutions is somehow a characteristic of this problem.

The explicit twistor correspondence, associating points of $T\mathbb{P}^1$ to lines in $\mathbb{C}^3$, and points of $\mathbb{C}^3$ to sections of $T\mathbb{P}^1$, is

$$\begin{aligned} (\lambda, \eta) \in T\mathbb{P}^1 &\mapsto \left\{ (a, b, c) \in \mathbb{C}^3 : \eta = \frac{1}{2}(a - 2b\lambda - c\lambda^2) \right\}, \\ (a, b, c) \in \mathbb{C}^3 &\mapsto \left\{ (\eta, \lambda) \in T\mathbb{P}^1 : \eta = \frac{1}{2}(a - 2b\lambda - c\lambda^2) \right\}, \end{aligned} \tag{1.4}$$

where $\mathbb{R}^3 \hookrightarrow \mathbb{C}^3$ as

$$(x, y, t) \mapsto (x + iy, 2t, x - iy).$$

One must check that the definitions of the two involutions are compatible with the twistor correspondence.

From the change of coordinates calculated above, we can relate the connection on $\mathbb{R}^3$ in the two coordinate systems:

$$\begin{pmatrix} \nabla_\eta \\ \nabla_{\bar{\eta}} \\ \nabla_u \end{pmatrix} = \begin{pmatrix} \frac{2}{(1+\lambda\bar{\lambda})^2} & \frac{-2\bar{\lambda}^2}{(1+\lambda\bar{\lambda})^2} & \frac{-2\bar{\lambda}}{(1+\lambda\bar{\lambda})^2} \\ \frac{-2\lambda^2}{(1+\lambda\bar{\lambda})^2} & \frac{2}{(1+\lambda\bar{\lambda})^2} & \frac{-2\lambda}{(1+\lambda\bar{\lambda})^2} \\ \frac{2\lambda}{1+\lambda\bar{\lambda}} & \frac{2\bar{\lambda}}{1+\lambda\bar{\lambda}} & \frac{1-\lambda\bar{\lambda}}{1+\lambda\bar{\lambda}} \end{pmatrix} \begin{pmatrix} \nabla_z \\ \nabla_{\bar{z}} \\ \nabla_t \end{pmatrix}.$$

Although we have not made the distinction, changing the $\mathbb{R}^3$ coordinates in a λ dependent way also affects $\frac{\partial}{\partial \lambda}$. Geometrically, this is because vectors on $T\mathbb{P}^1$ are Jacobi fields in $\mathbb{R}^3$ which are not linear. To be precise, we should have used coordinates $\lambda, \bar{\lambda}, z, \bar{z}, t$ and $\lambda', \bar{\lambda}', \eta, \bar{\eta}, u$, with $\lambda' = \lambda$. The distinction will be important when we want to show that the $\bar{\partial}$-operator extends to $\eta \in \mathbb{P}^1$ in some neighbourhood of $\lambda = 0$, because we will need to work with $(\eta, \bar{\eta}, u)$-coordinates.

We make use of the fact that if ζ and χ are two choices of coordinates, $d\zeta = \Lambda d\chi \iff \frac{\partial}{\partial \chi} = \Lambda^t \frac{\partial}{\partial \zeta}$. We have calculated $\begin{pmatrix} \eta \\ \bar{\eta} \\ u \end{pmatrix} = B \begin{pmatrix} z \\ \bar{z} \\ t \end{pmatrix}$ above. It follows that

$$d \begin{pmatrix} \lambda' \\ \bar{\lambda}' \\ \eta \\ \bar{\eta} \\ u \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \quad 0 \quad 0 \\ 0 & 1 & 0 \quad 0 \quad 0 \\ \frac{\partial B}{\partial \lambda} \begin{pmatrix} z \\ \bar{z} \\ t \end{pmatrix} & \frac{\partial B}{\partial \bar{\lambda}} \begin{pmatrix} z \\ \bar{z} \\ t \end{pmatrix} & B \end{pmatrix} d \begin{pmatrix} \lambda \\ \bar{\lambda} \\ z \\ \bar{z} \\ t \end{pmatrix},$$

so

$$\begin{aligned}
\frac{\partial}{\partial \bar{\lambda}} &= \frac{\partial}{\partial \bar{\lambda}'} + \left(\frac{\partial B}{\partial \bar{\lambda}} \begin{pmatrix} z \\ \bar{z} \\ t \end{pmatrix} \right)^t \begin{pmatrix} \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \bar{\eta}} \\ \frac{\partial}{\partial u} \end{pmatrix} \\
&= \frac{\partial}{\partial \bar{\lambda}'} + (\eta,\ \bar{\eta},\ u)\,(B^{-1})^t \left(\frac{\partial B}{\partial \bar{\lambda}} \right)^t \begin{pmatrix} \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \bar{\eta}} \\ \frac{\partial}{\partial u} \end{pmatrix} \\
&= \frac{\partial}{\partial \bar{\lambda}'} + \left(\frac{2\lambda}{(1+\lambda\bar{\lambda})} \bar{\eta} - u \right) \frac{\partial}{\partial \bar{\eta}} + \frac{2}{(1+\lambda\bar{\lambda})^2} \eta \frac{\partial}{\partial u}.
\end{aligned}$$

So an element of the kernel of the three operators $\nabla_u - i\Phi$, $\nabla_{\bar{\eta}}$, $\frac{\partial}{\partial \bar{\lambda}}$ is also in the kernel of

$$\nabla_{\bar{\lambda}'} \stackrel{\text{def}}{=} \frac{\partial}{\partial \bar{\lambda}'} - \left(\frac{2\lambda}{(1+\lambda\bar{\lambda})} \bar{\eta} - u \right) A_{\bar{\eta}} - \frac{2}{(1+\lambda\bar{\lambda})^2} \eta (A_u - i\Phi).$$

In these coordinates, recalling (I.2.2)

$$
\begin{aligned}
\nabla_u - i\Phi &= \frac{1}{1+\lambda\bar\lambda}\left[2\lambda\frac{\partial}{\partial z} + 2\bar\lambda\frac{\partial}{\partial\bar z} + (1-\lambda\bar\lambda)\frac{\partial}{\partial t} + 2(1+\lambda)(A_z + \bar\lambda A_{\bar z})\right]\\
\nabla_{\bar\eta} &= \frac{2}{(1+\lambda\bar\lambda)^2}\left[-\lambda^2\frac{\partial}{\partial z} + \frac{\partial}{\partial\bar z} - \lambda\frac{\partial}{\partial t} + (1+\lambda)(-\lambda A_z + A_{\bar z})\right]\\
\frac{\partial}{\partial\bar\lambda} &= \frac{\partial}{\partial\bar\lambda} \qquad\qquad (1.5)\\
\nabla_{\bar\lambda'} &= \frac{\partial}{\partial\bar\lambda'} - \frac{2(1+\lambda)}{(1+\lambda\bar\lambda)^3}\big((2\eta - 2\lambda^2\bar\eta + \lambda(1+\lambda\bar\lambda)u)A_z\\
&\qquad +(2\bar\lambda\eta + 2\lambda\bar\eta - (1+\lambda\bar\lambda)u)A_{\bar z}\big)\,.
\end{aligned}
$$

Sections of the bundle E correspond to simultaneous solutions to these operators. Roughly speaking, the system has enough solutions if it is involutive (see [Wr]).

LEMMA 1.6. *The system*

$$\left\{\nabla_u - i\Phi, \nabla_{\bar\eta}, \frac{\partial}{\partial\bar\lambda}\right\} = \{\nabla_u - i\Phi, \nabla_{\bar\eta}, \nabla_{\bar\lambda'}\}$$

is involutive iff S is harmonic.

PROOF. A system of differential operators given by generators is involutive *iff* all Lie brackets of pairs of generating operators lie in the system, *i.e.* in the span of the generators. Of course any other set of generators is as good, and finding alternative generators which have vanishing Lie brackets makes things simpler. That said, remark that

$$
\begin{aligned}
\frac{1}{2}\left\{(1+\lambda\bar\lambda)\nabla_{\bar\eta} + \lambda(\nabla_u - i\Phi)\right\} &= \frac{\partial}{\partial\bar z} - \frac{1}{2}\lambda\frac{\partial}{\partial t} + (1+\lambda)A_{\bar z}, \text{ and}\\
\frac{1}{2\lambda}\left\{-\bar\lambda(1+\lambda\bar\lambda)\nabla_{\bar\eta} + (\nabla_u - i\Phi)\right\} &= \frac{\partial}{\partial z} + \frac{1}{2\lambda}\frac{\partial}{\partial t} + (1+\lambda^{-1})A_z.
\end{aligned}
\qquad (1.7)
$$

It follows that $[\nabla_{\bar\eta}, \nabla_u - i\Phi] = 0$ *iff* $[\frac{\partial}{\partial\bar z} + (1+\lambda)A_{\bar z}, \frac{\partial}{\partial z} + (1+\lambda^{-1})A_z] = 0$, but since A_z and $A_{\bar z}$ do not depend on t, this is the case *iff* S is harmonic. That the entire system is involutive follows from the fact that (1.7) does not depend on $\bar\lambda$. □

In fact, the parametrised system of connections $(\frac{\partial}{\partial\bar z} + (1+\lambda)A_{\bar z}, \frac{\partial}{\partial z} + (1+\lambda^{-1})A_z)$ having curvature zero can be trivialised over $\{(\lambda, z) \in \mathbb{C}^* \times S^2\}$. Uhlenbeck calls these trivialisations extended solutions.

2. Compactness

We are interested in extending the bundles from $T\mathbb{P}^1$ to $\widetilde{T\mathbb{P}}^1$ ($T\mathbb{P}^1$ compactified by adding a section at ∞). The problem is that $\nabla = d + A$, which depends on $(\lambda, \eta, u) \in T\mathbb{P}^1 \times \mathbb{R}$, does not have a limit as $\eta \to \infty$. In the introduction we stated that finiteness translates as the extension of the bundle to the compactification of $T\mathbb{P}^1$. Unfortunately, it is not an easy translation.

We extend the bundle in two stages, first for $\{0 \neq \lambda \neq \infty\}$, the 'nonpolar' fibres of $\widetilde{T\mathbb{P}}^1 \to \mathbb{P}^1$, then in neighbourhoods of the poles (*i.e.* $\lambda \in \{0, \infty\}$). The first step is motivated by the geometry of the problem, the second relies on Sobolev methods to give the existence of a continuous gauge in which the $\bar{\partial}$-operator is smooth.

If $S : \mathbb{P}^1 \to G$ is a uniton, both $S(x, y) = S(z, \bar{z})$ and $S(\hat{z}, \hat{\bar{z}}) = S(\hat{x}, \hat{y})$ are continuous, where $\hat{z} = 1/z$ etc. In terms of A_z this means

$$A_z = -\frac{1}{z^2} A_{\hat{z}}, \quad A_{\bar{z}} = -\frac{1}{\bar{z}^2} A_{\hat{\bar{z}}}, \tag{2.1}$$

i.e. A_z has a strong vanishing property as $z \to \infty$. Writing this out in terms of x and y,

$$\begin{aligned} A_x &= \frac{y^2 - x^2}{(x^2 + y^2)^2} A_{\hat{x}} - \frac{2xy}{(x^2 + y^2)^2} A_{\hat{y}}, \text{ and} \\ A_y &= \frac{-2xy}{(x^2 + y^2)^2} A_{\hat{x}} + \frac{x^2 - y^2}{(x^2 + y^2)^2} A_{\hat{y}}, \end{aligned}$$

we see that A_z vanishes to order $2 - \epsilon$ at infinity. In geometric terms, it means that the 'energy' of the connection is concentrated around the t-axis in $\mathbb{R}^3$ (see Fig. 2), so that when the u-axis and the t-axis are not colinear, solutions to $\nabla_u - i\Phi$ should have limits as $u \to \infty$. The limit as $u \to \infty$ gives us a natural holomorphic framing over nonpolar fibres which extends to $\eta = \infty$, giving us the compactification there.

Because $\{\nabla_{\bar{\eta}}, \nabla_{\bar{\lambda}'}, \nabla_u - i\Phi\}$ is involutive, solutions to this system locally correspond to solutions of $\{\nabla_{\bar{\eta}}, \nabla_{\bar{\lambda}'}\}$ restricted to a plane $\{u = u_0\}$ for some u_0. As the figure suggests, away from the poles this makes sense for $u_0 = \infty$. Near the poles, however, this doesn't work, and we choose $u_0 = 0$. The transition between the resulting frames amounts to integrating $\nabla_u - i\Phi$ from $u = 0$ to $u = \infty$.

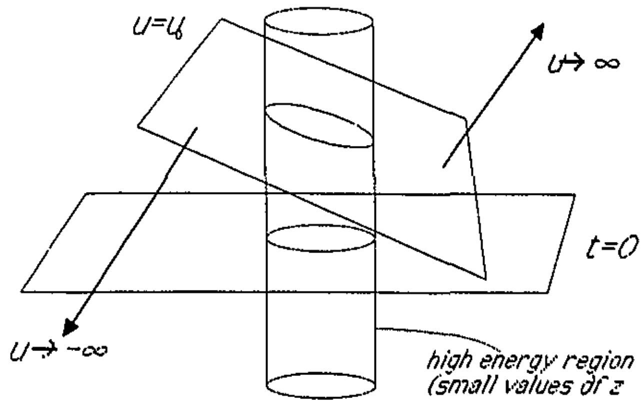

FIG. 2. A picture of $\mathbb{R}^3$ showing the high energy cylinder, and a plane $\{u = u_0\}$ for some direction $\lambda \in \mathbb{C}^*, |\lambda| \neq 1$. From the picture, we would expect solutions to $\nabla_{\bar\eta'}$ to extend to $\eta = \infty$, and solutions of $\nabla_u - i\Phi$ to extend to $u = \infty$, since in these directions the connection coefficients decrease rapidly.

2.2 C_∞ trivialisation. We will work over compact sets

$$U_k = \{(\lambda, \eta) | 1/k \leq |\lambda| \leq k, |\eta| \leq k\}, \tag{2.3}$$

with $k > 1$, and $u \in S^1$ which we obtain by by compactifying the bundle $T\mathbb{P}^1 \oplus \underline{\mathbb{R}} \to T\mathbb{P}^1$ fibrewise by adding a section at infinity–this time we are compactifying the $\underline{\mathbb{R}}$ summand into an S^1-bundle summand. Near this section, we will use the coordinate $\hat{u} = 1/u$.

LEMMA 2.4. *The three operators*

$$\widetilde{\nabla_u} = \nabla_u - i\Phi,\ \nabla_{\bar\eta},\ \textit{and}\ \nabla_{\bar\lambda'}$$

define a smooth operator

$$\widetilde{\nabla} : \Gamma(\widetilde{E}) \to \Gamma(\widetilde{E} \otimes (T''U_k \oplus TS^1))$$

on $U = U_k \times \{u \in S^1\}$.

PROOF. We must show that the operators are C^∞ on some neighbourhood of $\hat{u} = 0$:

$$\begin{aligned}
\tilde{\nabla}_{\hat{u}} &= -u^2(\nabla_u - i\Phi) \\
&= \frac{\partial}{\partial \hat{u}} - u^2 \frac{2(1+\lambda)}{1+\lambda\bar{\lambda}} \left[A_z + \bar{\lambda} A_{\bar{z}}\right] \\
&= \frac{\partial}{\partial \hat{u}} - u^2 \frac{2(1+\lambda)}{1+\lambda\bar{\lambda}} \left[-\frac{1}{z^2} A_{\hat{z}} - \frac{\bar{\lambda}}{\bar{z}^2} A_{\hat{\bar{z}}}\right] \\
&= \frac{\partial}{\partial \hat{u}} - \frac{1+\lambda}{2(1+\lambda\bar{\lambda})} \left[-\frac{(1+\lambda\bar{\lambda})^4}{((\eta - \lambda^2\bar{\eta})\hat{u} + \lambda(1+\lambda\bar{\lambda}))^2} A_{\hat{z}} \right. \\
&\qquad \left. - \frac{(1+\lambda\bar{\lambda})^4}{((\bar{\eta} - \bar{\lambda}^2\eta)\hat{u} + \bar{\lambda}(1+\lambda\bar{\lambda}))^2} A_{\hat{\bar{z}}}\right],
\end{aligned}$$

where we have replaced z and $\bar{z}$ with their expressions in η, u. The reader may verify that all the terms are smooth on some neighbourhood of $\hat{u} = 0$. That the other operators are smooth is easier to see, since

$$\begin{aligned}
A_z &= -\frac{1}{z^2} A_{\hat{z}} \\
&= -\hat{u}^2 \frac{(1+\lambda\bar{\lambda})^4}{((\eta - \lambda^2\bar{\eta})\hat{u} + \lambda(1+\lambda\bar{\lambda}))^2} A_{\bar{z}},
\end{aligned}$$

vanishes to order two, as well as being smooth at $\hat{u} = 0$, and the coefficients of $\nabla_{\bar{\eta}}$ and $\nabla_{\bar{\lambda}}$ are linear in A_z and $A_{\bar{z}}$. □

By the existence and uniqueness of solutions to linear ODEs, a solution $\tilde{s}$ on $U_k \times \{\hat{u} > 0\}$ extends to $U_k \times \{|\hat{u}| < \epsilon\}$, and the solution is differentiable. Alternatively, if, for $\tilde{s}$ a solution,

$$s_\infty(\lambda, \eta) \overset{\text{def}}{=} \lim_{u \to \infty} \tilde{s}(\lambda, \eta, u)$$

exists, then over U_k, $\tilde{s} \to s_\infty$ uniformly along with its first derivatives. Furthermore, since $\tilde{s}$ satisfies $\nabla^{(0,1)}\tilde{s} = 0$ on $U_k \times \{\hat{u} \neq 0\}$ and $||A^{(0,1)}||_{C^1} \to 0$ as $u \to \infty$ on U_k, $\nabla^{(0,1)}|_{\hat{u}=0} = \bar{\partial}$, and consequently $s_\infty(\lambda, \eta) = \tilde{s}(\lambda, \eta, \hat{u} = 0)$ is holomorphic in these variables.

We have shown that a solution $\tilde{s}$ on $\{(\lambda, \eta, u) \in \mathbb{C}^* \times \mathbb{C} \times \mathbb{R}\}$ has limits as $u \to \pm\infty$.

2.5 To eternity and back. Since $\nabla_u - i\Phi$ is a connection on $\{u \in S^1\}$ for each choice of $(\lambda, \eta) \in T\mathbb{C}^*$ with coordinates (λ, η), we may ask what is its monodromy. (Parallel translating a frame around a closed curve is a nonsingular linear transformation, called the monodromy.)

LEMMA. *$\nabla_u - i\Phi$ has trivial monodromy around $\{u \in S^1\}$.*

GEOMETRIC PROOF. We saw that $\nabla_{\bar\eta}$, $\nabla_u - i\Phi$ is equivalent to (I.1.7):

$$\frac{\partial}{\partial \bar z} - \frac{1}{2}\lambda\frac{\partial}{\partial t} + (1+\lambda)A_{\bar z}$$
$$\frac{\partial}{\partial z} + \frac{1}{2\lambda}\frac{\partial}{\partial t} + (1+\lambda^{-1})A_z$$

away from the points $(\lambda = 0, \lambda = \infty)$ where (I.1.7) has singularities. Both systems are 'underdetermined', since any solution can be multiplied by any holomorphic function in η to give another solution. The exact function, however, may be fixed by adding another differential operator to the system. Over $\{\lambda \in \mathbb{C}^*\}$, we can add $\frac{\partial}{\partial t}$, and get a completely integrable system on any fibre $\{\lambda = \lambda_0\} \subset \widetilde{T\mathbb{P}}^1 \times \mathbb{R}$ ($\lambda_0 \in \mathbb{C}^*$), *i.e.* a (full) smooth connection on E restricted to a fibre of $\widetilde{T\mathbb{P}}^1|_{\mathbb{C}^*} \times \mathbb{R} \to \mathbb{C}^*$, with zero curvature.

This connection is "*real*" in the sense that it is a connection on the real tangent space, whereas the operator $\nabla_{\bar\eta}$ restricted to a fibre of $T\mathbb{P}^1$ acts on the antiholomorphic tangent space of $T\mathbb{P}^1$, and not along one real direction in the fibre as a *real* connection would. We hesitate to call connections on the real tangent space *real*, however, because that word is usually used to describe a connection whose coefficients lie in some real Lie algebra ($\mathrm{u}(N)$), *i.e.* which lie in a real reduction of a complex principal bundle.

Since solutions of the augmented system are independent of t, we can push the system down to $\{(z, \bar z) \in \mathbb{R}^2\}$; *i.e.* all solutions to the augmented system are obtained by pulling back solutions of

$$\left(\frac{\partial}{\partial z} + (1+\lambda^{-1})A_z, \frac{\partial}{\partial \bar z} + (1+\lambda)A_{\bar z}\right) \tag{2.6}$$

on $\mathbb{R}^2$. We know from Theorem I.1.9 that (2.6) extends to a curvature-free connection on z, $\bar z \in S^2$. Since S^2 is simply-connected the connection (2.6) has trivial monodromy around any circle, in particular circles through the point at infinity which are the projections of the circles $\{u \in S^1\}$ in a fibre of $\widetilde{T\mathbb{P}}^1 \times \mathbb{R} \to \mathbb{C}^*$. (See Fig. 3.) $\square$

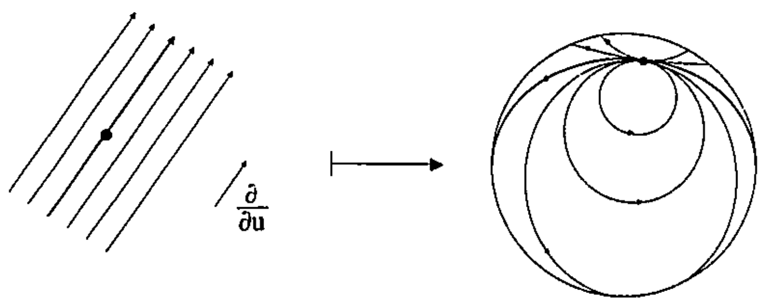

FIG. 3. For a fixed value of $\lambda \in \mathbb{C}^*$, lines parallel to $\frac{\partial}{\partial u}$ in $\mathbb{R}^3$ are projected onto lines on $\mathbb{R}^2$ which are completed as circles on S^2 tangent at the point at infinity.

ANALYTIC PROOF. Alternatively, let $\tilde{s}$ be a solution to $\nabla_u - i\Phi, \nabla^{(0,1)}$ over $T\mathbb{C}^*$, such that $\lim_{u\to\infty} \tilde{s} = \mathbb{I}$. The monodromy around S^1 is then given by

$$s_\infty(\lambda, \eta) \stackrel{\text{def}}{=} \lim_{u\to-\infty} \tilde{s}(\lambda, \eta, u),$$

and is holomorphic. In other words, $\nabla_u - i\Phi$ is a connection on S^1 for each fixed $(\lambda, \eta) \in T\mathbb{C}^*$, so calculate its monodromy by parallel transporting $\mathbb{I}$ at $\hat{u} = 0$ around S^1. It is holomorphic because $\nabla_u - i\Phi$ and $\nabla^{(0,1)}$ commute and $\nabla^{(0,1)}|_{\{\hat{u}=0\}} = \bar{\partial}$.

Since $\nabla_u - i\Phi$ is independent of t, however, it follows that $\tilde{s}$ is also independent of t; *i.e.*

$$\tilde{s}(\lambda, \eta, u) = \tilde{s}\left(\lambda, \eta + \lambda t, u - \frac{1 - \lambda\bar{\lambda}}{1 + \lambda\bar{\lambda}} t\right);$$

hence s_∞ is also independent of t, which means it is constant along one real direction in the η-plane (*i.e.* complex line). Since s_∞ is holomorphic, it is constant. Referring to Fig. 2, we see that that constant is arbitrarily small by considering the integral along lines arbitrarily distant from the high-energy $z = 0$ axis. The next lemma can be used to make this precise. $\square$

2.7 Triviality over nonpolar fibres. Reconsider $\tilde{s}_0$, a (parametrised) solution to the (parametrised) system (2.6). Such solutions are unique up to a choice of framing at some point in $\mathbb{P}^1$. Let that point be $z = \infty$ and choose the fixed framing $\tilde{s}_0 = \mathbb{I}$ there. Now let λ vary in $\{1/k < |\lambda| < k\}$. In terms of the coordinates λ, η, u, this framing is equivalent to the framing $\tilde{s} = \mathbb{I}$ over $u = \infty$. Since the family of connections on S^2 is uniformly continuous in λ, the resulting solution $\tilde{s}_0$ (and hence $\tilde{s}$) is continuous.

At $u = \infty$, this section extends to $\eta = \infty$, which defines a (trivial) bundle structure for $\tilde{E}$ over $\widetilde{T\mathbb{P}^1}|_{\mathbb{C}^*}$.

The previous lemma tells us that if we trivialise by taking the frame $\mathbb{I}$ at $u = \infty$ we can calculate the frame at $u - u_0$ by integrating $\nabla_u - i\Phi$ down from ∞ *or* up from $-\infty$. Since we will have to relate this bundle structure to one over the poles which will be defined over the $u = 0$ slice, we will need to know that for $\lambda \in \mathbb{C}^*, |\lambda| \neq 1$, $\tilde{s}$ also extends to $\eta = \infty$ when u is finite. (See Fig. 2.)

LEMMA 2.8. *For any such λ, and u_0 finite,*

$$\lim_{\eta \to \infty} \tilde{s}(\lambda, \eta, u_0) = \tilde{s}_\infty(\lambda, \eta_0) = \mathbb{I}$$

where η_0 is arbitrary.

PROOF. One has only to integrate $\nabla_u - i\Phi$ to infinity in the right direction (avoiding the high-energy cylinder around the t axis, see Fig. 2):

Let $\tilde{s}$ be the solution with limit $\mathbb{I}$ as $u \to \infty$. We are given λ fixed and without loss of generality take $u_0 = 0$. Substituting (2.1) into (1.5) and taking pointwise matrix norms we get

$$||A_u - i\Phi|| \leq \frac{K}{|z|^2},$$

for some constant K depending only on λ and $\max\left\{||A_{\dot{z}}||, ||A_{\dot{\bar{z}}}|| : z \in \mathbb{P}^1\right\}$. We would like to integrate $\nabla_u - i\Phi$ either from $-\infty$ to u_0 or from u_0 to ∞ whichever way avoids small values of z. (See Fig. 2) We can bound $|z|$ from below by

$$\begin{aligned}\frac{(1+\lambda\bar\lambda)^4}{4}|z|^2 &= \left|\eta - \lambda^2\bar\eta + \lambda(1+\lambda\bar\lambda)u\right|^2 \\ &= |\eta-\lambda^2\bar\eta|^2 + |\lambda(1+\lambda\bar\lambda)u|^2 + 2\lambda\bar\lambda(1-(\lambda\bar\lambda)^2)\rho u \\ &> |\eta|^2\min\left\{|\lambda|^4,1\right\} + |\lambda u|^2 + 2|\lambda|^2(1-|\lambda|^4)\rho u,\end{aligned}$$

where $\eta = (\rho + i\rho')\lambda$, for some real ρ, ρ'. So the right direction is $\operatorname{sign}((1-(\lambda\bar\lambda)^2)\rho)$, and for $|\eta|$ sufficiently large

$$||A_u - i\Phi|| \le \frac{K}{(|\eta|^2\min\{|\lambda|^4,1\} + |\lambda u|^2 + 2|\lambda|^2(1-|\lambda|^4)\rho u)^2}. \tag{2.9}$$

For convenience, we assume $\operatorname{sign}(1-(\lambda\bar\lambda)^2)\rho = 1$ and drop reference to it from now on. The set $\{0 \le u \le \infty\}$ is compact, but inconveniently parametrised, so we will work with

$$v = \arctan u \in [0,\pi],$$

and

$$B_v \overset{\mathrm{def}}{=} \frac{du}{dv}(A_u - i\Phi).$$

Then

$$||B_v|| \le \frac{K(1+u^2)}{(|\eta|^2\min\{|\lambda|^4,1\} + |\lambda u|^2 + 2|\lambda|^2(1-|\lambda|^4)\rho u)^2}$$

follows from (2.9). Look again at the condition $(\nabla_u - i\Phi)s = 0$ *iff* $\frac{\partial}{\partial u}ss^{-1} = A_u - i\Phi$ *iff* $\frac{\partial}{\partial v}ss^{-1} = B_v$. By definition,

$$\begin{aligned}B_v(a) = \frac{\partial}{\partial v}ss^{-1}(a) &= \lim_{b\to a}\frac{(s(b)-s(a))s(a)^{-1}}{b-a} \\ &= \lim_{b\to a}\frac{s(b)s(a)^{-1}-\mathbb{I}}{b-a}.\end{aligned}$$

The function

$$f(a,b) = \begin{cases}\frac{s(b)s(a)^{-1}-\mathbb{I}}{b-a} & (a,b)\in[0,\pi]^2\setminus\Delta \\ s'(b)s(a)^{-1} & a=b\end{cases}$$

is continuous on the diagonal $\Delta = \{b-a=0\}$ since if (a_i,b_i) is a sequence in $[0,\pi]\times[0,\pi]$ converging to a point on the diagonal, then either $a_i \ne b_i$ and we can find $c_{i,kl}$ $(1 \le k,l \le N)$ between a_i and b_i such that

$$f_{kl}(a_i,b_i) = \frac{s(b_i)s(a_i)^{-1}-\mathbb{I}}{b_i-a_i} = \frac{s(b_i)-s(a_i)}{b_i-a_i}s(a_i)^{-1} = s'(c_{i,kl})s(a_i)^{-1},$$

by the mean value theorem; or $f(a_i, b_i) = s'(b_i)s(a_i)^{-1}$. In either case they have the same limit.

For $|\eta|$ sufficiently large, $||B_v|| = ||f|_\Delta|| < \epsilon$. Since f is continuous on a compact domain, we can find a δ such that $||f|| < 2\epsilon$ on a $\sqrt{2}\,\delta$ neighbourhood of $\Delta \subset [0,\pi]\times[0,\pi]$, where $\delta = \pi/n$ for some integer. Choosing the sequence $\{v_i\} = \{0, \delta, 2\delta, ..., \pi\}$, we see that

$$s(v_i) - s(v_{i-1}) = s(v_{i-1})f(v_{i-1}, v_i)(v_i - v_{i-1})$$

(with $||f(v_{i-1}, v_i)|| < 2\epsilon$) and by induction that

$$s(v_i) - s(0) = s(0) \sum_{i_1 < i_2 < \cdots < i_k < i} \pi_{i_1}\pi_{i_2}\ldots\pi_{i_k}$$

where $\pi_j = f(v_{j-1}, v_j)(v_j - v_{j-1})$. So

$$\begin{aligned}
||s(v_i) - s(0)|| &< ||s(0)|| \cdot ||\sum_{j=1}^{i} \pi_j + \left(\sum_{j=1}^{i} \pi_j\right)^2 + \left(\sum_{j=1}^{i} \pi_j\right)^3 + \ldots|| \\
&\leq ||s(0)|| \left(\frac{\sum_{j=1}^{i} ||\pi_j||}{1 - \sum_{j=1}^{i} ||\pi_j||}\right) \\
&< ||s(0)|| \frac{2\epsilon\pi}{1 - 2\epsilon\pi} \to 0
\end{aligned}$$

as $\epsilon \to 0$. So as $|\eta| \to \infty$, $||s(\infty) - s(0)|| \to 0$ as required. □

Over the fibres, $\lambda \in \mathbb{C}^*$, the constant frame over $u = \infty$ extends in a natural way to $\eta = \infty$. Over $|\lambda| = 1$ we define the bundle $\tilde{E}$ over $\eta = \infty$ by decreeing that the constant section extends, although for u finite, $\tilde{s}$ may have bounded discontinuities approaching $\eta = \infty$ in one direction.

2.10 The bundle extends. Since $\nabla_u - i\Phi$ defines an analytic operator on $\mathbb{R}^3$ for all $\lambda \in S^2$, which in (λ, η, u) coordinates has a (locally uniform) limit as $\eta \to \infty$, the subspace of local frames, given by the kernel of this operator, has the structure of a locally free sheaf over $\mathbb{C}$—a complex vector bundle. There are several ways to define a holomorphic structure on a complex bundle. One is to specify a fixed trivialisation over open sets as a holomorphic one. The constant trivialisation at $u = \infty$ defines the holomorphic structure away from $\lambda \in \{0, \infty\}$. Over a general set we can also specify a holomorphic structure via a $\bar{\partial}$-operator. Restricted to P_0, $\nabla_{\bar{\eta}}$ and $\nabla_{\bar{\eta}'} = \frac{\partial}{\partial \bar{\eta}'} + \frac{1}{2} A_{\bar{z}} d\bar{\eta}'$ is an analytic $\bar{\partial}$-operator, which defines a holomorphic structure on $E|_{P_0}$. We need to do this on an open set.

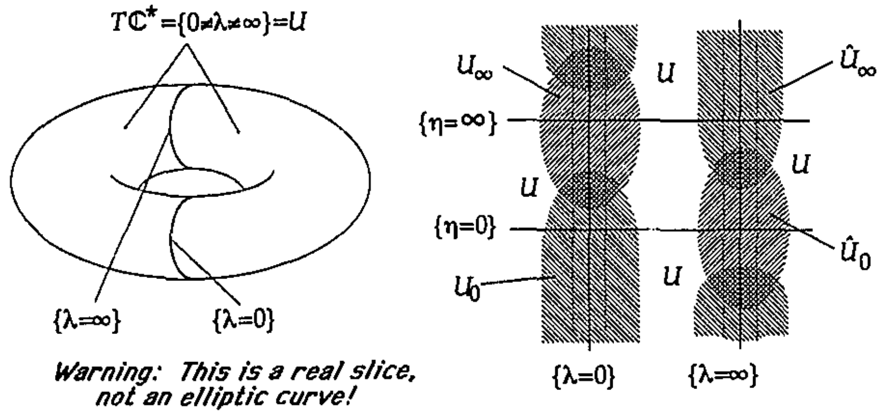

FIG. 4. Two pictures of real slices of $\widetilde{T\mathbb{P}}^1$ showing the covering.

To put these structures together, we make use of coordinate patches on $\widetilde{T\mathbb{P}}^1$

$$\begin{aligned}
U_0 &= \{(\lambda, \eta) : \lambda \neq \infty, |\eta| \leq \infty\} \\
U_\infty &= \{(\lambda, \eta) : |\lambda| < 1/2, \eta \neq 0\} \\
\hat{U}_0 &= \left\{(\hat{\lambda}, \hat{\eta}) : \hat{\lambda} \neq \infty, |\hat{\eta}| \leq \infty\right\} \\
\hat{U}_\infty &= \left\{(\hat{\lambda}, \hat{\eta}) : |\hat{\lambda}| < 1/2, \hat{\eta} \neq 0\right\} \\
U &= \{(\lambda, \eta) : 0 \neq \lambda \neq \infty\} = \mathbb{C}^* \times \mathbb{P}^1,
\end{aligned} \tag{2.11}$$

and the following subsets of $\mathbb{C}$:

$$\Xi = \{|\bar{\eta}'| \leq 1\}, \quad \Lambda = \{|\lambda| \leq 1\}. \tag{2.12}$$

On U we take the constant frame $\mathbb{I}$ $(= s_\infty)$ over $\mathbb{C}^* \times \mathbb{P}^1$, which gives a frame $\tilde{s}$ over $\{(\lambda, \eta, u) \in \mathbb{C}^* \times \mathbb{C} \times \mathbb{R}\}$ by parallel translation.

On $T\mathbb{P}^1$, $(\nabla_{\bar{\eta}}, \nabla_{\bar{\lambda}})$ is an analytic $\bar{\partial}$-operator: a (local) holomorphic frame g is a solution to $g^{-1}A_{\bar{\eta}}g + g^{-1}\frac{\partial}{\partial \bar{\eta}}g = 0 = g^{-1}A_{\bar{\lambda}}g + g^{-1}\frac{\partial}{\partial \bar{\lambda}}g$, equivalently $\frac{\partial}{\partial \bar{\eta}}gg^{-1} + A_{\bar{\eta}} = 0 = \frac{\partial}{\partial \bar{\lambda}}gg^{-1} + A_{\bar{\lambda}}$. Such a solution exists locally *iff* $[\nabla_{\bar{\eta}}, \nabla_{\bar{\lambda}}] = 0$, *i.e. iff* $(\nabla_{\bar{\eta}}, \nabla_{\bar{\lambda}})$ defines an operator

$$\nabla^{(0,1)} : C^\infty(M, E) \to C^\infty(M, E \otimes T^{(0,1)}(M)).$$

Let s_0 be such a frame of solutions over U_0 and $\hat{s}_0$ over $\hat{U}_0$.

Near the infinity section, however, we must work with coordinates $(\lambda, \eta' = 1/\eta)$ and $(\hat{\lambda}, \hat{\eta}' = 1/\hat{\eta})$. On the first patch, the $\bar{\partial}$-operator is given by

$$\begin{aligned}
\nabla_{\bar{\eta}'} = -\bar{\eta}^2\nabla_{\bar{\eta}} &= \frac{\partial}{\partial\bar{\eta}'} - \frac{(1+\lambda)(1+\lambda\bar{\lambda})^2}{2\bar{\eta}'^2}\left\{\frac{\lambda}{(\eta-\lambda^2\bar{\eta})^2}A_{\hat{z}} - \frac{1}{(\bar{\eta}-\bar{\lambda}^2\eta)^2}A_{\hat{\bar{z}}}\right\} \\
&= \frac{\partial}{\partial\bar{\eta}'} - \frac{(1+\lambda)(1+\lambda\bar{\lambda})^2}{2}\left\{\frac{\lambda}{(\lambda^2-\bar{\eta}'/\eta')^2}A_{\hat{z}} - \frac{1}{(1-\bar{\lambda}^2\bar{\eta}'/\eta')^2}A_{\hat{\bar{z}}}\right\} \\
&\stackrel{\text{def}}{=} \frac{\partial}{\partial\bar{\eta}'} + B_0A_{\hat{z}} + B_1A_{\hat{\bar{z}}}, \qquad (2.13a)\\
\nabla_{\bar{\lambda}'} &= \frac{\partial}{\partial\bar{\lambda}'} - \frac{4(1+\lambda)}{(1+\lambda\bar{\lambda})^3}\left[(\eta-\lambda^2\bar{\eta})A_z + (\bar{\lambda}\eta+\lambda\bar{\eta})A_{\bar{z}}\right] \\
&= \frac{\partial}{\partial\bar{\lambda}'} - \frac{2(1+\lambda)}{(1+\lambda\bar{\lambda})^3}\left[\frac{\bar{\eta}'}{1-\lambda^2\eta'/\bar{\eta}'}A_{\hat{z}} + \frac{\bar{\lambda}\bar{\eta}'^2/\eta'+\lambda\bar{\eta}'}{(1-\bar{\lambda}^2\bar{\eta}'/\eta')^2}A_{\hat{\bar{z}}}\right], \qquad (2.13b)
\end{aligned}$$

where for simplicity we restrict to the hypersurface $u = 0$.

Both operators are smooth away from $\eta' = 0$, and near $\eta' = 0$ are smooth functions of $\eta', \bar{\eta}'$ and $\bar{\eta}'/\eta'$. Since $A_{\hat{z}}$ and $A_{\hat{\bar{z}}}$ are smooth near $\eta' = 0$ (for $\lambda \in \Lambda$), the failure of the coefficients of $A_{\hat{z}}$ and $A_{\hat{\bar{z}}}$ to be continuous or integrable depends on the relative powers of the η' and $\eta'/\bar{\eta}'$ factors in a series expansion. The factor $\bar{\eta}'/\eta'$ is integrable but not continuous and differentiation introduces a factor of $1/\eta'$ which is not even integrable. The operator $\nabla_{\bar{\eta}'}$ has this type of discontinuity; $\nabla_{\bar{\lambda}'}$ is continuous but its derivative also has an $\bar{\eta}'/\eta'$ bounded discontinuity.

Since the discontinuity of the $\bar{\partial}$-operator is mild, it is not surprising that we will be able to find a continuous change to a holomorphic gauge, *i.e.* a gauge change g such that $0 = A^g = g^{-1}Ag + g^{-1}\bar{\partial}g$. integrability of the operator, however, is not enough to assert the existence of such a gauge, since in two complex dimensions

$$P' : g \mapsto \frac{\partial}{\partial\bar{\eta}'}gg^{-1}d\bar{\eta}' + \frac{\partial}{\partial\bar{\lambda}'}gg^{-1}d\bar{\lambda}'$$

is not invertible as a map of the appropriate Sobolev spaces, and we will need to make use of the structure of the singularity.

On $\mathbb{C}^2$ the $\bar{\partial}$-operator is not elliptic, but on $\mathbb{C}$ it is, and on $\mathbb{P}^1$ it is even surjective because its co-kernel is the kernel of its adjoint, $\bar{\partial}^* = *\bar{\partial}*$, *i.e.* holomorphic (0,1) forms, of which there are none. Local invertibility follows from a bump function argument: given a function on a neighbourhood of a point in $\mathbb{C}$, we can always multiply by a bump function supported on a smaller neighbourhood of the point resulting in a function which extends to $\mathbb{P}^1$. On $\mathbb{P}^1$, since $\bar{\partial}$ is surjective, we can find a $\bar{\partial}$ primitive, which is also a primitive of the original function on some smaller neighbourhood of the point. For a complete discussion see [AtBo, 5.1,§14]. By approaching the smoothing as a parametrised one-dimensional problem, and taking advantage of the special form of the singularity we will be able to find a continuous holomorphic gauge. Such gauges do not exist in general for $\bar{\partial}$-operators with the same integrability but without this particular type of singularity.

Because all the objects we will be dealing with, *e.g.* B_i, are smooth away from $\eta' = 0$, integrability (L^p_k, *i.e.* L^p integrability of partial derivatives up to order k) on $\Xi \times \Lambda$, and on fibres of $\Xi \times \Lambda \to \Lambda$ are equivalent. In fact $B_i \in L^2_0(\Lambda \times \Xi, \mathrm{gl}(N))$ can also be seen as a smooth map valued in a function space:

$$B_i \in \mathbb{C}^\infty(\Lambda, L^2_0(\Xi, \mathrm{gl}(N))). \tag{2.14}$$

By taking the second view of B_i, as a smooth function valued in a function space, we reinterpret the search for a smooth gauge as a parametrised one-complex-dimensional problem.

The basic tool for proving smoothness is the

SOBOLEV LEMMA. *There are inclusions*

$$L^2_2(\Xi, \mathrm{gl}(N)) \subset C^0(\Xi, \mathrm{gl}(N)) \text{ and}$$
$$L^2_3(\Lambda \times \Xi, \mathrm{gl}(N)) \subset C^0(\Lambda \times \Xi, \mathrm{gl}(N))$$

which are continuous with respect the Sobolev and supremum norms respectively.

For our purposes, continuity of the inclusions will be very important. See [GriHa, p86] for a proof.

LEMMA 2.15. *The operator* $P : g \mapsto \frac{\partial}{\partial \bar{\eta}'} g g^{-1}$ *can be extended to a smooth invertible map*

$$P : L^2_k(\Xi, GL(N, \mathbb{C}))_0 \to L^2_{k-1}(\Xi, \mathrm{gl}(N, \mathbb{C}))$$

for $k > 2$, where $L^2_k(\ )_0$, indicates the space of based maps, $g(0) = \mathbb{I}$.

PROOF. P extends to a map of Sobolev spaces because

(1) since Ξ is compact and $L^2_k(\Xi) \subset C^1(\Xi)$, we can find a constant such that $\|g'^{-1}\| < \mathrm{const}\|g\|'$ for g' in some neighbourhood of g

(2) $L^p_k(\mathbb{R}^n)$ is a Banach algebra for $k > n/p$ and L^p_j is a topological L^p_k- module for $0 \le j \le k$ [AtBo,14.5].

(3) $\frac{\partial}{\partial \bar{\eta}'}$ gives a map $L^2_k(\mathbb{C}^n, GL(N,\mathbb{C})) \to L^2_{k-1}(\mathbb{C}^n, \mathrm{gl}(N,\mathbb{C}))$ for all k, n.

We can calculate the derivative, DP, of P by expanding

$$P\left(g_0(\mathbb{I} + g_1)\right) = \frac{\partial}{\partial \bar{\eta}'} g_0 g_0^{-1} + \frac{\partial}{\partial \bar{\eta}'} g_1 g_0^{-1} - g_1 g_0^{-1} \frac{\partial}{\partial \bar{\eta}'} g_0 g_0^{-1} + \phi_{g_0}(g_1),$$

where $\phi_{g_0}(g_1)$ is tangent to the zero map (*i.e.* $\lim_{|g_1| \to 0} \frac{\phi_{g_0}(g_1)}{|g_1|} = 0$),

$$= P(g_0) + DP(g_0)(g_1) + \phi_{g_0}(g_1).$$

In particular $DP(\mathbb{I}) = \frac{\partial}{\partial \bar{\eta}'}$, so we can apply the inverse function theorem for Banach spaces [La, I.5.1] to get an inverse to P in a neighbourhood of $P(\mathbb{I}) = 0$. In fact we can get a smooth inverse because P is smooth:

We can verify the existence of higher derivatives for P either by iteratively differentiating P, or by using the chain rule and remarking (a) that $\frac{\partial}{\partial \bar{\eta}'}$ and $m(a,b) = ab$ are linear and multilinear respectively and hence both smooth [La, I.3.12]; and (b) that $g \mapsto g^{-1}$ is smooth because it's k^{th} derivative at $g = g_0$ is the k-linear map $\bigoplus^k L^2_k(\Xi, \mathrm{gl}(N,\mathbb{C})) \to L^2_k(\Xi, \mathrm{gl}(N,\mathbb{C}))$, given by

$$(g_1, \ldots, g_k) \mapsto \sum_{\sigma \in S_k} g_0^{-1} g_{\sigma(1)} g_0^{-1} g_{\sigma(2)} g_0^{-1} \cdots g_{\sigma(k)} g_0^{-1}.$$

Note that we have made repeated use of (1) and (2) above. $\square$

Unfortunately, $A_{\bar{\eta}'} \notin L^2_{3-1}(\mathbb{C}^2, \mathrm{gl}(N,\mathbb{C}))$, so we may not conclude immediately that there is a continuous change of gauge which smooths $A_{\bar{\eta}'}$. So we need to look closer at the singularity.

Looking for a better gauge involves integrating $A_{\bar{\eta}'}$. In theory, we could integrate it term by term removing the singularity one order at a time, but there is no guarantee that such can be done in a generic way for all harmonic maps, since we do not know the explicit form of $A_{\hat{z}}, A_{\hat{\bar{z}}}$. We are left to integrate their coefficients B_0, B_1 in $\nabla_{\bar{\eta}}$. The coefficients B_i are smooth away from $\eta' = 0$ where they have a bounded discontinuity of the type $\eta'/\bar{\eta}'$. We can find integrals (*i.e.* C_0 and C_1 such that $\frac{\partial}{\partial\bar{\eta}'}C_i = -B_i$),

$$\begin{aligned} C_0 &= \frac{(1+\lambda)(1+\lambda\bar{\lambda})^2}{2}\frac{-\lambda\eta'}{\lambda^2 - \bar{\eta}'/\eta'}, \\ C_1 &= \frac{(1+\lambda)(1+\lambda\bar{\lambda})^2}{2}\left(\frac{\eta'}{\bar{\lambda}^2(1-\bar{\lambda}^2\bar{\eta}'/\eta')} - \frac{\eta'}{\bar{\lambda}^2}\right) \\ &= -\frac{(1+\lambda)(1+\lambda\bar{\lambda})^2}{2}\frac{\bar{\eta}'}{1-\bar{\lambda}^2\bar{\eta}'/\eta'}, \end{aligned} \tag{2.16}$$

which are not only continuous at $\eta' = 0$, but vanish there, because they look like $\eta'\phi(\eta'/\bar{\eta}')$, where ϕ is continuous on a neighbourhood of $\{z \in \mathbb{C} : |z| = 1\}$. We can use

$$g = 1 + C_0 A_{\hat{z}} + C_1 A_{\hat{\bar{z}}},$$

to give $A^g_{\bar{\eta}'} \stackrel{\text{def}}{=} g^{-1}A_{\bar{\eta}'}g + g^{-1}\frac{\partial}{\partial\bar{\eta}'}g$ the same continuity properties (check that the four terms without a C_i factor cancel):

$$\begin{aligned} (B_0A_{\hat{z}} + B_1A_{\hat{\bar{z}}})^g &= g^{-1}(B_0A_{\hat{z}} + B_1A_{\hat{\bar{z}}})g + g^{-1}\frac{\partial}{\partial\bar{\eta}'}g \\ &= B_0A_{\hat{z}} + B_1A_{\hat{\bar{z}}} + C_1B_0[A_{\hat{z}}, A_{\hat{\bar{z}}}] + C_0B_1[A_{\hat{\bar{z}}}, A_{\hat{z}}] \\ &\quad - (C_0A_{\hat{z}} + C_1A_{\hat{\bar{z}}})(B_0A_{\hat{z}} + B_1A_{\hat{\bar{z}}})(C_0A_{\hat{z}} + C_1A_{\hat{\bar{z}}}) \\ &\quad - B_0A_{\hat{z}} - B_1A_{\hat{\bar{z}}} + C_0\left(B_2\frac{\partial}{\partial\hat{z}}A_{\hat{z}} + B_3\frac{\partial}{\partial\hat{\bar{z}}}A_{\hat{z}}\right) \\ &\quad + C_1\left(B_2\frac{\partial}{\partial\hat{z}}A_{\hat{\bar{z}}} + B_3\frac{\partial}{\partial\hat{\bar{z}}}A_{\hat{\bar{z}}}\right) - (C_0A_{\hat{z}} + C_1A_{\hat{\bar{z}}}) \\ &\quad \Bigg[-B_0A_{\hat{z}} - B_1A_{\hat{\bar{z}}} + C_0\left(B_2\frac{\partial}{\partial\hat{z}}A_{\hat{z}} + B_3\frac{\partial}{\partial\hat{\bar{z}}}A_{\hat{z}}\right) \\ &\quad + C_1\left(B_2\frac{\partial}{\partial\hat{z}}A_{\hat{\bar{z}}} + B_3\frac{\partial}{\partial\hat{\bar{z}}}A_{\hat{\bar{z}}}\right)\Bigg], \end{aligned}$$

where the terms B_2, B_3 arise because $\frac{\partial}{\partial\bar{\eta}'} \neq \frac{\partial}{\partial\hat{\bar{z}}}$: (on $u = 0$)

$$\begin{aligned} \frac{\partial}{\partial\bar{\eta}'} &= -\bar{\eta}^2\frac{\partial}{\partial\bar{\eta}} \equiv -\frac{\lambda^2(1+\lambda\bar{\lambda})^2}{2(\lambda^2 - \bar{\eta}'/\eta')^2}\frac{\partial}{\partial\hat{z}} + \frac{(1+\lambda\bar{\lambda})^2}{2(1-\bar{\lambda}^2\bar{\eta}'/\eta')^2}\frac{\partial}{\partial\hat{\bar{z}}} \pmod{\frac{\partial}{\partial t}} \\ &\stackrel{\text{def}}{=} B_2\frac{\partial}{\partial\hat{z}} + B_3\frac{\partial}{\partial\hat{\bar{z}}} \pmod{\frac{\partial}{\partial t}}. \end{aligned}$$

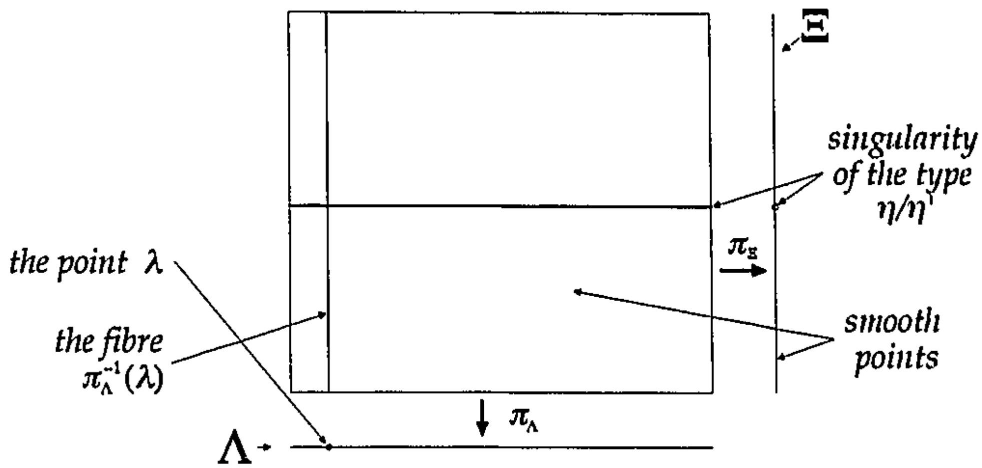

FIG. 5. A function, f, on $\Xi \times \Lambda$ defines a map from Λ to the space of functions on Ξ by assigning to λ_0 the restriction to the fibre $f_{\{(\lambda,\eta):\lambda=\lambda_0\}}$. L^p_k-integrability on $\Xi \times \Lambda$ is governed by L^p_k-integrability on fibres.

We can ignore the $\frac{\partial}{\partial t}$ terms because the connection coefficients are independent of t.

Since each (persistant) term contains a factor C_j, $A^g_{\bar\eta'}$ vanishes on $\eta' = 0$, and is continuous. Because $\frac{\partial}{\partial\bar\eta'}C_j = B_j$, $\frac{\partial}{\partial\bar\eta'}A^g_{\bar\eta'}$ is bounded but discontinuous, we see that $A^g_{\bar\eta'} \in L^2_1(U_\infty, \mathrm{gl}(N,\mathbb{C}))$, in fact

LEMMA 2.17. *The map*

$$\mathcal{A}^g_{\bar\eta'} : \Lambda \to L^2_1(\Xi, \mathrm{gl}(N,\mathbb{C}))$$

such that

$$\mathcal{A}^g_{\bar\eta'}(\lambda)({\eta'}_0) = A^g_{\bar\eta'}(\lambda, {\eta'}_0)$$

is smooth.

PROOF. We have to show (1) that the function $A^g_{\bar\eta'}|_{\lambda=\lambda_0}$ and its first $\eta', \bar\eta'$ derivatives, are square integrable, for all $\lambda_0 \in \Lambda$, and (2) that $\frac{\partial}{\partial\lambda}^p \frac{\partial}{\partial\bar\lambda}^q \mathcal{A}^g_{\bar\eta'}$ exist and are in L^2_1 for all p and q.

(1) Since

$$A^g_{\bar\eta'} = g^{-1}A_{\bar\eta'}g + g^{-1}\frac{\partial}{\partial\bar\eta'}g$$

is smooth away from $\eta' = 0$, and has a singularity of type $\eta'\phi(\eta'/\bar\eta')$ there, its first derivatives in η' or $\bar\eta'$ may have a bounded discontinuity, which doesn't effect the finiteness of the L^2 norm. In fact, we can multiply $A^g_{\bar\eta'}$ by the complement of a bump function of arbitrarily small mass concentrated at $\eta' = 0$, and find that the map $A^g_{\bar\eta'} : \Lambda \to L^2_1(\Xi, \mathrm{gl}(N, \mathbb{C}))$ is continuous.

(2) Since $A^g_{\bar\eta'}$ is smooth on $\{\eta' \neq 0\}$ we can take the λ derivatives of $\mathcal{A}^g_{\bar\eta'}$ pointwise, *i.e.* when $\eta' \neq 0$

$$\frac{\partial}{\partial\lambda'}^{p}\frac{\partial}{\partial\bar\lambda'}^{q}\mathcal{A}^g_{\bar\eta'}(\lambda_0)(\eta'_0) = \frac{\partial}{\partial\lambda'}^{p}\frac{\partial}{\partial\bar\lambda'}^{q}A^g_{\bar\eta'}|_{(\lambda_0,\eta'_0)}.$$

Examining the terms of $A^g_{\bar\eta'}$ one at a time, we find that all partials are continuous. For example,

$$\begin{aligned}\frac{\partial}{\partial\lambda'}C_0 &= \frac{(1+\lambda\bar\lambda)^2 + 2\bar\lambda(1+\lambda)(1+\lambda\bar\lambda)}{2}\frac{-\lambda\eta'}{\lambda^2 - \bar\eta'/\eta'} \\ &\quad + \frac{(1+\lambda)(1+\lambda\bar\lambda)^2}{2}\frac{-\eta'(\lambda^2 - \bar\eta'/\eta') + 2\lambda^2\eta'}{(\lambda^2 - \bar\eta'/\eta')^2}, \\ \frac{\partial}{\partial\lambda'}C_1 &= \frac{(1+\lambda\bar\lambda)^2 + 2\bar\lambda(1+\lambda)(1+\lambda\bar\lambda)}{2}\frac{\eta'}{\bar\lambda^2(1 - \bar\lambda^2\bar\eta'/\eta')}.\end{aligned}$$

Since all its partials exist and are continuous, $\mathcal{A}^g_{\bar\eta'} : \Lambda \to L^2_1(\Xi, \mathrm{gl}(N, \mathbb{C}))$, is smooth. □

Now we can exploit the fact that P has a *smooth* inverse. As a result,

$$\tilde g = P^{-1} \circ \mathcal{A}^g_{\bar\eta'} : \Lambda \to L^2_2(\Xi, GL(N, \mathbb{C}))$$

is a smooth map, and $P(\tilde g) = A^g_{\bar\eta'}$. Composing with the continuous Sobolev embedding $L^2_2 \hookrightarrow C^0$, we see that $\tilde g$ is a *continuous* change of gauge over U_∞, such that the $\bar\eta'$-operator in this gauge is trivial, *i.e.* $A^{g\tilde g}_{\bar\eta'} = 0$, and $A^{g\tilde g}_{\bar\lambda} = \tilde g^{-1}A^g_{\bar\lambda}\tilde g + \tilde g^{-1}\frac{\partial}{\partial\bar\lambda}\tilde g$ is continuous (all its ingredients are).

Since $0 = [\nabla_{\bar\lambda}, \nabla_{\bar\eta'}] = [\frac{\partial}{\partial\bar\lambda} + A^{g\tilde g}_{\bar\lambda}, \frac{\partial}{\partial\bar\eta'}]$, it follows that $A^{g\tilde g}_{\bar\lambda}$ is meromorphic in η'—holomorphic, as it is continuous. Using the fact that $A^{g\tilde g}_{\bar\lambda}$ is smooth near $\{|\eta'| = 1\}$, and differentiating the Cauchy integral:

$$\frac{\partial}{\partial\eta'}^{j} \frac{\partial}{\partial\lambda}^{k} \frac{\partial}{\partial\bar\lambda}^{l} A^{g\tilde g}_{\bar\lambda}(\lambda_0, \eta'_0) = \int_{|\eta'|=1} \frac{\left(\frac{\partial}{\partial\lambda}^{k} \frac{\partial}{\partial\bar\lambda}^{l} A^{g\tilde g}_{\bar\lambda}(\lambda, \eta')\Big|_{\lambda=\lambda_0}\right.}{(\eta' - \eta'_0)^j}.$$

we see that $A^{g\tilde g}_{\bar\lambda}$ is smooth on $\{|\eta'| < 1\}$. We can then find a smooth change of gauge $\hat g$ such that $A^{g\tilde g\hat g}_{\bar\lambda} = 0 = A^{g\tilde g\hat g}_{\bar\eta'}$. Then, $s_\infty \stackrel{\mathrm{def}}{=} g\tilde g\hat g$ is the required holomorphic trivialisation over U_∞.

Transition functions are now given by $T_0 = s_0^{-1}\tilde s$, $T_\infty = s_\infty^{-1}\tilde s$, $T_{0\infty} = s_\infty^{-1}s_0$. By construction, $\frac{\partial}{\partial\bar\eta}T_* = 0 = \frac{\partial}{\partial\bar\lambda}T_*$, so they are holomorphic transition matrices. They are nonsingular because the corresponding frames were constructed to be nonsingular.

A similar construction works over the south pole.

3. Triviality over the ∞-section

Over nonpolar fibres, $\cup\{P_\lambda : \lambda \in \mathbb{C}^*\}$, we defined a holomorphic framing of E associated to the framing $\tilde f$ of $\widetilde E$, such that

$$\lim_{u\to\infty} \tilde f = \mathbb{I}.$$

The smoothability of the $\bar\partial$-operator away from the equator $|\lambda| = 1$ tells us that we can find holomorphic framings of E in neighbourhoods of $(\lambda = 0, \eta = \infty)$ and $(\hat\lambda = 0, \hat\eta = \infty)$ which correspond to continuous framings $\tilde f_0$ and $\tilde f_\infty$ of $\widetilde E$ over the appropriate regions of $\mathbb{P}^1 \times \mathbb{R}^3$. Since away from the equator $\lim_{\eta\to\infty} \nabla_{\bar\lambda'} = \frac{\partial}{\partial\bar\lambda'}$ on every plane u =constant, $\lim_{\eta\to\infty} \tilde f_0$ is holomorphic in λ in the usual sense, so we can assume that $\lim_{\eta\to\infty} \tilde f_0 = \mathbb{I}$. We make a similar assumption about $\tilde f_\infty$.

The transition matrices are just $\tilde f^{-1}\tilde f_0$ and $\tilde f_\infty^{-1}\tilde f$, which don't depend on u because $\tilde f$, $\tilde f_0$ and $\tilde f_\infty$ all solve $\nabla_u - i\Phi$. Since

$$\lim_{\eta\to\infty} \tilde f_0 = \mathbb{I},\ \lim_{\eta\to\infty} \tilde f_\infty = \mathbb{I}, \text{ and}$$

$$\lim_{\eta\to\infty} \tilde f \stackrel{\text{Lemma 2.8}}{=} \lim_{u\to\infty} = \mathbb{I},$$

the bundle E is trivial when restricted to C_∞.

4. Time Invariance

Time translation $(z,t)\mapsto(z,t_0+t)$ induces a one-parameter group of transformations of $T\mathbb{P}^1$. In coordinates, $(\lambda,\eta)\mapsto(\lambda,\eta-t\lambda)$. The coefficients $A_z, A_{\bar{z}}$ and hence (∇,Φ) are independent of t. Another way of saying this is that they are invariant under the group of translations of t. So the space of solutions to $\nabla_u - i\Phi, \nabla_{\bar{\eta}}, \nabla_{\bar{\lambda}'}$ is invariant under time translation.

On $T\mathbb{P}^1$, the space of oriented geodesics in $\mathbb{R}^3$, time translation acts by $(\lambda,\eta)\overset{\delta_t}{\mapsto}(\lambda,\eta-t\lambda)$. The geodesic itself is shifted with respect to the geodesic parameter u,

$$u\overset{\delta_t}{\mapsto}u+\frac{1-\lambda\bar{\lambda}}{1+\lambda\bar{\lambda}}t.$$

So a solution $\tilde{s}(\lambda,\eta,u)$ such that $(\nabla_u - i\Phi)\tilde{s} = 0 = \nabla_{\bar{\eta}}\tilde{s} = \nabla_{\bar{\lambda}'}\tilde{s}$ generates a family of solutions

$$\tilde{s}_t(\lambda,\eta,u)=\tilde{s}(\lambda,\eta+\lambda t,u-\frac{1-\lambda\bar{\lambda}}{1+\lambda\bar{\lambda}}t).$$

And the map $\tilde{\delta}_t:\tilde{s}\mapsto\tilde{s}_t$, is a bundle isomorphism lifting δ_t. Since

$$\lim_{z\to\infty}A=0=\lim_{z\to\infty}\Phi,$$

the bundle map is just the identity over the section at infinity. One can also see this by remarking that the section which comes from the constant frame at infinity is preserved by the time-translation map. We will see that the specification of this map encodes the time-independence of the uniton.

5. The Real Structure

As remarked in the introduction, in adapting Hitchin's construction, there is some ambiguity as to the real structure. On $\mathbb{C}^3$ one thinks of the real structure literally as a real slice: a three dimensional subspace of the *real* six dimensional $\mathbb{C}^3$, which as a set spans the *complex* three dimensional $\mathbb{C}^3$. Any such set is the fixed set of an antilinear involution—the real structure. The $\mathbb{R}^3$ of Hitchin's original construction is the standard real slice of standard $\mathbb{C}^3$ (with conjugation as the real structure). Conjugation, however, is not the appropriate real structure for our purposes. But $\mathbb{R}^3$ is still an *invariant* set of the appropriate real structure:

$$\sigma^*(x,y,t)=(\bar{x},\bar{y},-\bar{t}=i^{-1}\bar{i}t).$$

So when our real structure acts on $E \to \mathbb{R}^3$ it not only conjugates the fibres, but reflects $\mathbb{R}^3$ in the $x-y$ plane.

5.1 On the principal bundle. One way to understand how the real structure on E arises is to work with frames rather than sections. This is because the real structure on E comes from the real structure on the complex group ($\mathrm{GL}(N)$ in our case), which induces a real structure on the trivial principal bundle of frames of $\tilde{E}$ over $\mathbb{R}^3$. The real structure fixes a real subgroup, and is $X \mapsto (X^*)^{-1}$ in the case of $U(N)$, which is both an involution and antiholomorphic with respect to the natural complex structure of $\mathrm{GL}(N)$. A frame of E, either locally, or at a point, is an invertible solution, $\tilde{f}$, to

$$(\nabla_{a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y}+c\frac{\partial}{\partial t}} - i\Phi)\tilde{f} = 0. \tag{5.2}$$

Everything here lives in $\mathrm{gl}(N)$, so we can apply the transformation

$$X \mapsto -\overline{\tilde{f}^{-1}X\tilde{f}^{-1}}^{\,t}$$

to (5.2) to get a new equation, which since $iA_t, i\Phi, A_x, A_y \in \mathrm{u}(N)$, gives

$$(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial t} + aA_x + bA_y - cA_t - i\Phi)(\tilde{f}^*)^{-1} = 0.$$

Pulling back by σ, and using the fact that A_x, A_y, A_t, Φ are independent of t, we get

$$(\nabla_{a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y}-c\frac{\partial}{\partial t}} - i\Phi)(\sigma^*\tilde{f})^{*-1} = 0.$$

So the real structure on the principal bundle $\mathbb{R}^3 \times \mathrm{GL}(N)$ induces an antiholomorphic involution of the principal bundle of frames of $E \to T\mathbb{P}^1$ which covers σ and which conjugates the natural frames above real sections (*i.e.* (frame)$\mapsto$(frame)$^{*-1}$ in a unitary frame). The specific form of this conjugation is important—other conjugations correspond to different real groups, *i.e.* $\mathrm{Gl}(N,\mathbb{R}), \mathrm{U}(n, N-n)$ *etc.*

5.3 On the vector bundle. Alternatively, we can express the real structure in terms of E and E^*. To avoid any confusion we will use $\dagger$ for conjugate transpose in this paragraph. A section of E is the same as a section, $\tilde{s}$, of $\widetilde{E}$ which satisfies

$(\nabla_u - i\Phi)\tilde{s} = 0$. Similarly, we can identify a section of E^*, as a section $\tilde{r}$ of $\widetilde{E}^*$ such that $\tilde{r}(\tilde{s})$ is constant on λ lines, *i.e.* constant in u. In terms of the standard frames of $\widetilde{E}$ and $\widetilde{E}^*$, $\tilde{r}(\tilde{s})$ is just matrix multiplication. It follows from

$$\begin{aligned} 0 = \frac{\partial}{\partial u}\tilde{r}(\tilde{s}) &= \frac{\partial}{\partial u}\tilde{r}\cdot\tilde{s} + \tilde{r}\cdot\frac{\partial}{\partial u}\tilde{s} \\ &= \frac{\partial}{\partial u}\tilde{r}\cdot\tilde{s} + \tilde{r}\cdot(-(A_u - i\Phi)\tilde{s}) \\ &= (\frac{\partial}{\partial u}\tilde{r} - \tilde{r}(A_u - i\Phi))\cdot\tilde{s} \end{aligned} \tag{5.4}$$

that $\tilde{r}$ represents a section of E^* *iff* it satisfies

$$\frac{\partial}{\partial u}\tilde{r} - \tilde{r}(A_u - i\Phi) = 0.$$

Since $A_x, A_y, iA_t, i\Phi \in \mathrm{u}(N)$, this is true *iff*

$$0 = \frac{\partial}{\partial u}\tilde{r}^\dagger - (A_{-\sigma_* u} + i\Phi)\tilde{r}^\dagger = \frac{\partial}{\partial u}\tilde{r}^\dagger + (A_{\sigma_* u} - i\Phi)\tilde{r}^\dagger.$$

Since A_x, A_y, A_t, Φ are t-independent, this is true *iff*

$$\frac{\partial}{\partial\sigma_* u}\sigma^*\tilde{r}^\dagger + (A_{\sigma_* u} - i\Phi)\sigma^*\tilde{r}^\dagger = 0,$$

i.e. iff $\sigma^*\tilde{r}^\dagger$ represents a section of E. This defines an antilinear lift $\tilde{\sigma}$:

$$\begin{array}{ccc} E & \overset{\tilde{\sigma}}{\longleftrightarrow} & E^* \\ \downarrow & & \downarrow \\ \widetilde{T\mathbb{P}^1} & \overset{\sigma}{\longleftrightarrow} & \widetilde{T\mathbb{P}^1} \end{array}.$$

To invert this second construction in terms of the the dual bundle, we need to fix coordinates. Fixing a choice of $\lambda_0 \in \mathbb{C}^*$ allows us to do this, because we can take the standard framing of $\mathbb{C}^N$ at $\lambda = \lambda_0$ and $u = \infty$, to define a fixed frame of E over P_{λ_0}, and hence over C_∞.

Finally, an antiholomorphic isomorphism $V \to V^*$ is equivalent to a nondegenerate sesquilinear form on V. (A holomorphic map would give a holomorphic metric.) As such, we have a well-defined notion of signature. In our case, σ fixes $\{(\lambda, \eta) : |\lambda| = 1, \eta \in \{0, \infty\}\}$ and other points as well. At a fixed point (λ, eta), $\tilde{\sigma} : E|_{(\lambda,\eta)} \to E|^*_{(\lambda,\eta)}$ reduces to $r \mapsto r^\dagger \cdot \_$, in terms of dual frames, and $\tilde{\sigma}(r)r = r^\dagger r \geq 0$ implies $\tilde{\sigma}$ is positive definite.

Since we will show that given either of these real structures on E, we can get back the reality of ∇ and Φ, the two real structures are equivalent.

6. Framing

Finally, let p be any fixed point of σ contained in P_{-1}. We take as the framing above this point the solution to $(\nabla_u - i\Phi)\phi = 0$ along the line $p \subset \mathbb{R}^3$ with

$$\lim_{u \to \infty} \phi = \mathbb{I}.$$

Since p is fixed by σ, the real structure $X \mapsto X^{*-1}$ takes solutions of $\nabla_u - i\Phi$ along p to solutions, and maps our particular solution to itself. Since $E|_{P_{-1}}$ is trivial,

$$H^0(P_{-1}, Fr(E)) \xrightarrow{\text{eval}} Fr(E_p)$$

is an isomorphism. This defines the 'unitary' framing

$$\phi \in H^0(P_{-1}, Fr(E))$$

of the definition.

CHAPTER III

GETTING BACK THE UNITON

Given a holomorphic bundle on $\widetilde{T\mathbb{P}}^1$, trivial over real sections, the section at infinity, and fibres $P_\lambda : 0 \neq \lambda \neq \infty$ with a (fixed) bundle isomorphism lifting time translation, which is the identity over the section at infinity and one of the two real structures, we would now like to construct a uniton. From Chapter I we know it suffices to construct a solution to the Bogomolny equations, independent of t, which extends to $z = \infty$. Hitchin has already explained how bundles on $T\mathbb{P}^1$ give solutions to the Bogomolny equations over $\mathbb{R}^3$. For independence of t and extendability to $z = \infty$ we need to use the additional structure. Extendability to $z = \infty$, not surprisingly, results from the extendability to a trivial bundle over the section at infinity. Time independence results from the lifting of δ_t, time translation.

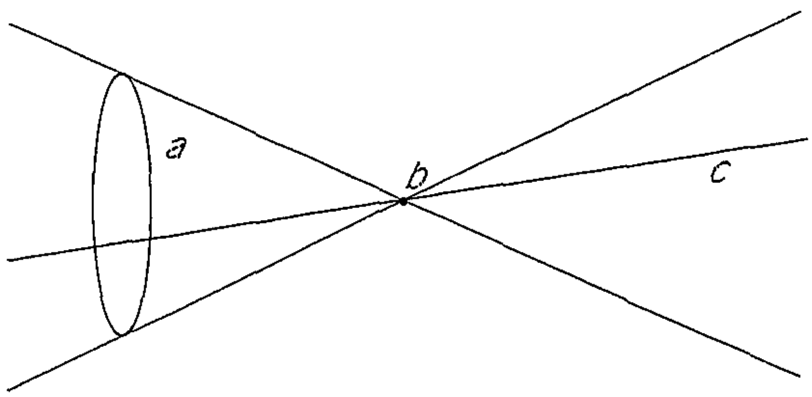

FIG. 6. The conic Q showing (a) a section of $T\mathbb{P}^1$, (b) the singular point at infinity and (c) a fibre.

1. $\widetilde{T\mathbb{P}}^1$ as a conic

As a preliminary step, we show that $T\mathbb{P}^1$ can be embedded into $\mathbb{P}^3$ as the nonsingular subset of a conic. Consider the conic Q given in homogenous coordinates $\alpha, \beta, \gamma, \delta$ on $\mathbb{P}^3$ by $\beta^2 = -4\alpha\gamma$. This conic has a singular point at $[0,0,0,1]$. Now consider the map $f : T\mathbb{P}^1 \to Q$ given by

$$\begin{aligned}(\lambda, \eta) &\mapsto [1, -2\lambda, -\lambda^2, -2\eta] = [\alpha, \beta, \gamma, \delta] \\ (\hat{\lambda}, \hat{\eta}) &\mapsto [-\hat{\lambda}^2, 2\hat{\lambda}, 1, 2\hat{\eta}].\end{aligned} \tag{1.1}$$

The map f extends to a rational map on $\widetilde{T\mathbb{P}}^1$ mapping the section at infinity to the singular point. Since the bundle is trivial over the section at infinity, when we collapse this section the bundle descends to another bundle f_*E on Q. More precisely,

LEMMA 1.2. *Pull-back of bundles ($E' \mapsto f^*E'$) from Q back to $\widetilde{T\mathbb{P}}^1$ is an isomorphism onto the set of bundles on $\widetilde{T\mathbb{P}}^1$, trivial over the section at infinity, C_∞.*

PROOF. Pullback of bundles is injective. (Push forward is a left inverse of pull back.) We only have to show it is surjective—that every bundle trivial on C_∞ is the pullback of a bundle on Q. Let $E'' \to \widetilde{T\mathbb{P}}^1$ be trivial on C_∞. Away from C_∞, f is bijective, so E'' pushes forward to a bundle on Q away from the singular point. We shall use the Theorem on Formal Functions to push forward a trivialisation of E'' in a neighbourhood of C_∞ to a trivialisation of f_*E'' in a neighbourhood of the singular point $f_*(C_\infty)$. So f_*E'' is a bundle (a *locally trivial* sheaf) whose image is E'', proving surjectivity.

Locally, the section at infinity, C_∞, looks like the zero section of $\mathcal{O}_{\mathbb{P}^1}(-2)$. Given local coordinates (λ, η'), and $(\hat{\lambda} = 1/\lambda, \hat{\eta}' = \lambda^2\eta')$ on $\mathcal{O}(-2)$, a transition matrix for E'' is given over the intersection, $\{\lambda \in \mathbb{C}^*\}$, as

$$\mathbb{I} + \eta'(\phi(\lambda, \hat{\lambda}, \eta', \hat{\eta}')),$$

where ϕ is a polynomial matrix. Since $\hat{\eta}' = \lambda^2\eta'$, we can express this in terms of two polynomials as

$$\begin{aligned}\mathbb{I}+&\eta'(\phi'(\lambda, \eta') + \phi''(\hat{\lambda}, \hat{\eta}')) \\ &= \mathbb{I} + \eta'\phi'(\lambda, \eta') + \hat{\eta}'\hat{\lambda}^2\phi''(\hat{\lambda}, \hat{\eta}'),\end{aligned}$$

but not uniquely, as $\eta' = \hat{\lambda}^2\hat{\eta}'$ *etc.* We can use this property to show inductively that the bundle must be trivial on all formal neighbourhoods of C_∞, by showing that such a transition matrix in $C^1(C_\infty^{(k)}, \mathrm{GL}(N))$, for any $k > 0$, is actually a coboundary, splitting as a product of holomorphic changes of gauge, *i.e.* it is the image of something in $C^0(C_\infty^{(k)}, \mathrm{GL}(N))$.

A bundle is trivial on the $(k-1)$st formal neighbourhood, $C_\infty^{(k-1)}$, *iff* its transition matrix has the form

$$\mathbb{I} + \eta'^k(\phi)$$

in some gauge. Using the fact that ϕ can be split as $\phi = \phi'(\lambda, \eta') + \phi''(\hat{\lambda}, \hat{\eta}')$, we can make a change of gauge:

$$\begin{aligned}(\mathbb{I} - \eta'^k\phi')(\mathbb{I} + \eta'^k(\phi' + \phi''))(\mathbb{I} - \hat{\eta}'^k\hat{\lambda}^{2k}\phi'')\\ = \mathbb{I} + \eta'^{2k}(\phi'\phi'' - \phi'(\phi' + \phi'') - \phi''(\phi' + \phi'')),\end{aligned}$$

showing that it is trivial on $C_\infty^{(2k-1)}$. Inductively, we get a trivialisation of $E''^{(k)}$, which is the same as a maximal rank section of $\mathcal{H}om(\mathcal{E}^{\oplus N}, E'')$ over the k^{th} formal neighbourhood ($C_\infty^{(k)}$), for k arbitrarily large. Now the Theorem on Formal Functions [Ha, III.11.1] says that

$$f_*\mathcal{H}om(\mathcal{E}^{\oplus N}, E'')^\wedge_{[0,0,0,1]} \xrightarrow{\sim} \varinjlim H^0(C_\infty^{(k)}, \mathcal{H}om(\mathcal{E}^{\oplus N}, E'')).$$

We have shown that the RHS has a maximal rank element. The LHS is the set of all sections of $\mathcal{H}om(\mathcal{E}^{\oplus N}, E'')$ on a neighbourhood of $f^{-1}([0,0,0,1]) = C_\infty$, up to formal equivalence, *i.e.* germs of sections. It must contain an element corresponding to the maximal-rank element of the RHS. That element is a section on some neighbourhood of the section at infinity which is nondegenerate on the infinity section. Since the determinant function is continuous, it must be nondegenerate on some neighbourhood where it gives a trivialisation. This trivialisation pushes forward to give a trivialisation of f_*E'' in a neighbourhood of the singular point, so in particular, f_*E'' is a bundle. □

The same construction would work with any bundle trivial over a rational curve of negative self-intersection embedded in a surface, because the splitting of ϕ would go through. A theorem of Castelnuovo tells us that the new surface will be smooth *iff* the curve has self-intersection -1, in which case we are just blowing down (see [Ha, Theorem V.5.7]).

The point of this construction, is that it tells us what happens to deformed real sections in the limit (as $t \to \infty$, for example). A hyperplane in $\mathbb{P}^3$, $\alpha a + \beta b + \gamma c + \delta d = 0$, restricted to Q can be written

$$a\alpha - 2\lambda b\alpha - \lambda^2 \alpha c - 2\eta\alpha d = 0 \quad \text{or} \quad -\hat{\lambda}^2\gamma a + 2\hat{\lambda}\gamma b + \gamma cz\hat{\eta}\gamma d = 0.$$

When $d \neq 0$, we can use affine coordinates $a/d, b/d, c/d$ or just restrict to the plane $d = 1$ and to pull back these sections to $T\mathbb{P}^1$ we restrict to the affine plane $\{\alpha = 1\}$ (see (1.1)). In $T\mathbb{P}^1$ coordinates, the hyperplane section $(a, b, c, 1)$ is

$$\eta = \frac{1}{2}(a - 2b\lambda - c\lambda^2),$$

a holomorphic section of $T\mathbb{P}^1$.

But what about the $\mathbb{P}^2$ of hyperplanes with $d = 0$? From Fig. 6, we can see that these are just the hyperplanes which include the singular point. Such intersections solve $a\alpha + b\beta + c\gamma = 0$ and $-4\alpha\gamma = \beta^2$, so they solve $a^2\alpha^2 + (2ac + 4b^2)\alpha\gamma + c^2\gamma^2 = 0$. When $b^2 + ac = 0$ or $b = 0$ the solution is a double line. In general we get two lines intersecting in the pinch point:

$$a^2\alpha + \left(ac + 2b^2 \pm 2b\sqrt{ac + b^2}\right)\gamma = 0,$$
$$a^2 b\beta = \left(-a^2 c + ac + 2b^2 \pm 2b\sqrt{ac + b^2}\right)\gamma.$$

So the correct way to complete the set of holomorphic sections of $T\mathbb{P}^1$ is not by adding a section at infinity (which, it turns out, is not a holomorphic section of $\widetilde{T\mathbb{P}}^1 \to \mathbb{P}^1$) but by adding a $\mathbb{P}^2$ worth of closed subvarieties of $\widetilde{T\mathbb{P}}^1$, given by the union of the section at infinity and two fibres with multiplicity.

2. Compact Twistor Fibration

Let $X \subset \mathbb{P}^3 \times \mathbb{P}^{3*}$ be the variety cut out by $\beta^2 + 4\alpha\gamma = 0$ and $a\alpha + b\beta + c\gamma + d\delta = 0$, where a, b, c, d are homogeneous coordinates on the space of hyperplane sections of

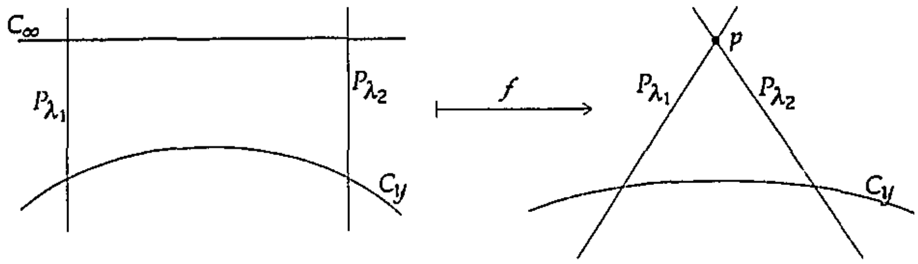

FIG. 7. The embedding of $\widetilde{T\mathbb{P}}^1 \to Q \subset \mathbb{P}^3$ maps the section at infinity to a singular point. Hyperplane sections of Q pull back to sections, C_y, of $T\mathbb{P}^1$, or to unions $P_{\lambda_1} \cup C_\infty \cup P_{\lambda_2}$ if they contain the singular point.

$\mathbb{P}^3$, $\mathbb{P}^{3*} \cong \mathbb{P}^3$. The double (twistor) fibration

$$\begin{array}{ccc} & X & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ T\mathbb{P}^1 \hookrightarrow Q & & \mathbb{P}^{3*} \hookleftarrow \mathbb{C}^3 \end{array} \tag{2.1}$$

allows us to define a bundle over

$$Y = \left\{ y \in \mathbb{P}^{3*} : E|_{X_y} \text{ is trivial} \right\}. \tag{2.2}$$

Pull back the bundle E to X, and push it forward to a sheaf over Y with stalks $\widetilde{E}|_y = H^0(X_y, E)$. By definition, $H^0(X_y, E)$ has constant dimension over Y. This is *not* true for $X \setminus Y$. In the following y will be assumed to be in Y. Since the bundle E restricted to real sections of $T\mathbb{P}^1$, given by $\{y = (a, b, c, 1) : b = \bar{b}, a = -\bar{c}\}$; the section at infinity; and fibres over $0 \neq \lambda \neq \infty$, is trivial, Y contains a neighbourhood of this set.

That $\widetilde{E}$ is in fact a bundle, *i.e.* locally trivial, follows from a

THEOREM (GRAUERT) [Ha, III.12.9]. *Let $f : X \to Y$ be a projective morphism of Noetherian schemes with Y integral, and let E be a coherent sheaf on X, flat over Y. Then if $h^i(y, E)$ is constant on Y, for some i, $R^i f_*(E)$ is locally free on Y, and for every y the natural map*

$$R^i f_*(E) \otimes k(y) \to H^i(X_y, E_y)$$

is an isomorphism.

Both X and Y are varieties over $\mathbb{C}$, and hence integral Noetherian schemes [Ha, II.3.2.1]. A map $f : X \to Y$ of schemes is projective if it factors through the projection $\mathbb{P}^n{}_Y \to Y$. Over $\mathbb{C}$, $\mathbb{P}^n{}_Y = \mathbb{P}^n \times Y$, so f is projective by definition. Since E is a bundle, it is locally trivial, not just coherent. If we can show that E is flat over Y, then we will have shown $\widetilde{E}$ is locally trivial (a vector bundle), since Y is included in the set on which $h^0(y, E)$ is constant.

Flatness is a transitive property, and $(X, E) \to (X, \mathcal{O}_X)$ is flat [Ha, III.9.2]. Theorem [Ha, III.9.9] says that $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is flat *iff*

$$P_y(m) = \dim_{\mathbb{C}} H^0(X_y, \mathcal{O}_{X_y}(m))$$

is independent of $y \in Y$ for $m \gg 0$. We can compute $P_y(m)$ from the long exact homology sequence associated to the embedding $X_y \subset \mathbb{P}^2$:

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-X_y) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{X_y} \to 0. \tag{2.3}$$

Since Q is cut out by a quadric, $X_y = Q \cap y$ is also a quadric in $\mathbb{P}^2$ (of degree two) and $\mathcal{O}_{\mathbb{P}^2}(-X_y) = \mathcal{O}_{\mathbb{P}^2}(-2)$. Plug this into the long exact sequence associated to (2.3)

$$\begin{aligned} 0 &\to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m-2)) \to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m)) \to H^0(X_y, \mathcal{O}_{X_y}(m)) \\ &\to H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m-2)) \to H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m)) \to H^1(X_y, \mathcal{O}_{X_y}(m)). \end{aligned}$$

According to Theorem B, for some suitably large μ_0, $H^q(M, \mathcal{O}(H^\mu \otimes E)) = 0$, for all $q > 0$, and $\mu > \mu_0$. In particular, $H^1(\mathcal{O}_{\mathbb{P}^2}(m-2)) = 0$ for $m > m_0$, hence

$$h^0(\mathcal{O}_{X_y}(m)) = h^0(\mathcal{O}_{\mathbb{P}^2}(m)) - h^0(\mathcal{O}_{\mathbb{P}^2}(m-2))$$

which is independent of y.

We now have a way of getting back from a bundle on $\widetilde{T\mathbb{P}^1}$ to a holomorphic bundle on Y. One may well wonder how constructing a holomorphic bundle over a complex domain of unknown shape can be seen as 'getting back' to a uniton, which we saw was equivalent to a real connection on a trivialised bundle over $S^2 \times \mathbb{R}$. In fact,

$Y_{\mathbb{R}}$ projects onto $S^2 \times \mathbb{R}$, and the bundle over Y will turn out to be trivial there. To define the connection, however, we will employ algebro-geometric machinery, hence the complexification. Real structures (antiholomorphic involutions) on $\mathbb{C}^3$ can be encoded as a choice of holomorphic coordinates, for example on $\mathbb{C}^3 \cap Y$ take $x = a+c, y = i(a-c), t = b$, with respect to which the involution is just conjugation. A holomorphic bundle is given by holomorphic sections/transition matrices in these coordinates; forgetting that they are complex-valued, we get a real-analytic bundle on $\mathbb{R}^3$. The same is true for metrics, connections etc.

3. The Connection and Higgs' Field

The construction of the connection ∇ from Chapter I also defines a connection on Y. The trouble is that we want a connection on $S^2 \times \mathbb{R}$, or S^2, but $Y_{\mathbb{R}}$ cannot contain any such set since S^2 is not the real slice of any complex variety. We must show that we can still push ∇ down by a map $Y_{\mathbb{R}} \to S^2 \times R$.

3.1 The set Y. To summarise what we know about Y:

(1) Finite real points are in Y: $\{[a, b, \bar{a}, 1] \in \mathbb{C}^3 \subset \mathbb{P}^3 : a \in \mathbb{C},\ b \in \mathbb{R}\} = \mathrm{fix}(\tau) \cap \mathbb{C}^3 \subset Y$ since E is trivial on τ-real sections of $T\mathbb{P}^1$. (In fact t-invariance of E implies $\mathrm{fix}(\sigma) \subset \{[a, b, \bar{a}, 1] : a, b \in \mathbb{C}\} \subset Y$, as well.)

(2) For infinite points, we know precisely that

$$
\begin{aligned}
Y \cap \mathbb{P}^{2*}{}_{\text{at } \infty} &= \{\text{hyperplane sections which contain neither } P_0 \text{ nor } P_\infty\} \\
&\left(= \frac{\{(\lambda_1, \lambda_2) \in \mathbb{C}^* \times \mathbb{C}^*\}}{(\lambda_1, \lambda_2) \sim (\lambda_2, \lambda_1)}\right) \\
&= \{[a, b, c, 0] \in \mathbb{P}^{2*}{}_{\text{at } \infty} : ac \neq 0\} \\
&\cong \mathbb{P}^2 \setminus (\mathbb{P}^1 \vee \mathbb{P}^1).
\end{aligned}
$$

By virtue of our choice of $\mathbb{C}^3 \subset \mathbb{P}^{3*}$, we know that the $\mathbb{P}^2$ at infinity is the set of hyperplane sections of Q through the singular point. Furthermore, any two such hyperplane sections either have a line in common or meet only at the singular point. From either definition of the connection, it's clear that the evaluation at the singular point gives a covariant constant frame of $\widetilde{E}$ over $Y \cap \mathbb{P}^{2*}{}_{\text{at } \infty}$. This is exactly the property which allows us to push the connection down to S^2.

3.2 Real Points. We know that finite real points are in Y. We must calculate the infinite ones. The involution τ acts on $\mathbb{P}^{3*}$ by $\tau(a,b,c,d) = (\bar{c},\bar{b},\bar{a},\bar{d})$, so the real points of $Y \cap \mathbb{P}^{2*}{}_{\text{at}\,\infty}$ are $\{[a,t,\bar{a},0]\}$, which we can also think of as $\{(\lambda, -\bar{\lambda}^{-1}) : \lambda \in \mathbb{P}^1\} / \sim$. Either way, we see that

$$Y_{\mathbb{R}} \cap \mathbb{R}P^{2*}_{\text{at}\,\infty} = \mathbb{R}P^2 \setminus \{[\text{pt}]\},$$

the Moebius strip. So

$$Y_{\mathbb{R}} \cong \mathbb{R}P^3 \setminus \{[\text{pt}]\}. \tag{3.3}$$

We can similarly calculate the level set

$$\begin{array}{rcccc} Y_{\mathbb{R}} \cap \{t=0\} & = & \mathbb{R}^3 \cap \{t=0\} & \cup & (\mathbb{R}P^2 \setminus \{[\text{pt}]\}) \cap \{t=0\} \\ & = & \mathbb{R}^2 & \cup & \{[a,0,\bar{a},0] : a \neq 0\} \\ & = & \mathbb{R}^2 & \cup & S^1 \\ & = & \mathbb{R}P^2 & & \end{array} . \tag{3.4}$$

3.5 Pushing down the connection. Let

$$\pi : \mathbb{R}P^2 \to S^2$$

be the real blow-down of the circle at infinity in $\mathbb{R}P^2$ to the infinite point in S^2.

We push the connection down by pushing down covariant constant frames along lines. This can be done if the inverse images of points are trivialised by covariant constant frames, which is true in our case because evaluation at the pinch point of Q gives such a frame over the circle at infinity.

It will follow from the proof of t-invariance of (∇, Φ) on $\mathbb{R}^3$ that the extension of the connection from $\mathbb{R}^2$ to S^2 implies the extension of (∇, Φ) from $\mathbb{R}^3$ to $S^2 \times \mathbb{R}$. Specifically, we know from the discussion of the Bogomolny normalisation following (I.2.2) that we can put any t-invariant pair (∇, Φ) satisfying the Bogomolny equations on $\mathbb{R}^3$ into the form $A_t = -iA_y$, $\Phi = iA_x$, so that the finiteness of A_x and A_y at infinity certainly imply the finiteness of A_t and Φ.

REMARK 3.6. It is interesting to note the parallel with the proof of compactification in the last chapter. The point $(z, \bar{z}, t)$ on $\mathbb{R}^3$ pulls back to $[z, it, \bar{z}, 1] \in \mathbb{P}^3$. The former can be thought of as a coordinate patch on $S^2 \times \mathbb{R}$, which is covered by two patches, the other one in terms of $\hat{z}$ rather than z. Thinking of them as complex numbers, $(z, \bar{z}, t)$ also give coordinates on a patch of $\mathbb{P}^3$, but if we try to compute the transition to the other patch (in terms of homogenous coordinates) we get

$$[z, it, \bar{z}, 1] = [1/\hat{z}, it, \hat{\bar{z}}, 1] = [1, it\hat{z}, \hat{z}/\hat{\bar{z}}, \hat{z}].$$

The 'coordinate' $\hat{z}/\hat{\bar{z}}$ corresponds to the real S^1 at infinity which appeared as the singularity type on the $\bar{\partial}$ operator when we tried to extend it to $\widetilde{T\mathbb{P}}^1$.

4. Choosing a trivialisation of $\tilde{E} \to Y$

Let $\lambda_0 = -1$. The bundle E is trivial over the fibre $P_{\lambda_0} \subset \widetilde{T\mathbb{P}}^1$, so evaluation at any point gives an isomorphism of $H^0(P_{\lambda_0}, E)$ with $\mathbb{C}^N$. Fix the isomorphism coming from evaluation at (λ_0, ∞). We get a map $\tilde{E} \to \mathbb{C}^N$, defined on fibres by

$$\begin{aligned} \widetilde{E}_y = H^0(C_y, E) &\xrightarrow[\cong]{\text{restr.}} H^0(C_y \cup P_{\lambda_0}, E) \\ &\xrightarrow[\cong]{\text{restr.}} H^0(P_{\lambda_0}, E) \xrightarrow[\cong]{\text{fixed framing } \phi} \mathbb{C}^N, \end{aligned} \tag{4.1}$$

where we use the canonical isomorphisms coming from restriction. This is well defined because, y is either a finite point (a section of $T\mathbb{P}^1$) and intersects P_{λ_0} in a point, or it is infinite in which case it meets the fibre at one point or on the whole fibre, in which case evaluation at any point of the fibre gives the same answer, again because E is trivial there. This map gives a trivialisation

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{\Psi_{\lambda_0}} & \mathbb{C}^N \times Y \\ \downarrow & & \downarrow \\ Y & \xrightarrow[\text{id}]{} & Y \end{array}$$

The value of this trivialisation is that in this framing the translation action δ_t lifts to $\text{id} \times \delta_t$:

$$\begin{array}{ccccc} \tilde{E}_{\delta_t y} = H^0(\delta_t C_y, E) & \xrightarrow[\cong]{\text{eval}} & E_{\delta_t C_y \cap P_{\lambda_0}} & \xleftarrow[\cong]{\text{eval}} & H^0(P_{\lambda_0}, E) = \mathbb{C}^N \\ {\scriptstyle \delta_t}\downarrow & & {\scriptstyle \delta_t}\downarrow & & \downarrow{\scriptstyle \delta_t = \text{id}} \\ \tilde{E}_y = H^0(C_y, E) & \xrightarrow[\cong]{\text{eval}} & E_{C_y \cap P_{\lambda_0}} & \xleftarrow[\cong]{\text{eval}} & H^0(P_{\lambda_0}, E) = \mathbb{C}^N \end{array} .$$

The fact that the last $\tilde{\delta}_t$ is the identity comes from the fact that $\tilde{\delta}_t$ fixes the bundle over the section at infinity, and hence must fix the bundle over non-polar fibres of $\widetilde{T\mathbb{P}}^1$ (over which the bundle is trivial). Note that Y is a time-translation independent set, because isomorphic bundles over $\mathbb{P}^1$ have the same splitting type.

It's not hard to see that if $E \to \widetilde{T\mathbb{P}}^1$ was constructed from a uniton as in the last chapter, then we have just reconstructed the original framing of the trivial bundle $\underline{\mathbb{C}^N}$.

5. Time Invariance of ∇ and Φ

The connection ∇ was constructed by considering flat frames along null sections. Consider again the flat frame given by evaluation at (λ_0, η_0), which defines a connection on a null plane through y. Translating by t, we get a null plane through $y+t$: the sections of $T\mathbb{P}^1$ through $(\lambda_0, \eta_0 + 2\lambda_0 t)$ (plus the special sections which include P_{λ_0}). By definition, the flat frame is carried by $\tilde{\delta}_t$ into another flat frame.

Specifically, a flat frame over Π_{λ_0} is given as the inverse image of a frame f,

$$f \in E(\lambda_0, \eta_0) \xleftarrow{\text{eval}} H^0(C_y, E) = \tilde{E}_y.$$

Since

$$\begin{array}{ccc} E(\lambda_0,\eta_0) & \xleftarrow{\text{eval}} & H^0(C_y,E)=\tilde{E}_y \\ {\scriptstyle \tilde{\delta}_t}\big\downarrow & & {\scriptstyle \tilde{\delta}_t}\big\downarrow \\ E(\lambda_0,\eta_0 - t\lambda_0) & \xleftarrow{\text{eval}} & H^0(C_{y+t},E)=\tilde{E}_{y+t} \end{array}$$

commutes, flat frames are sent to flat frames. Hence the null connections are invariant under $\tilde{\delta}_t$. Now ∇ and Φ are defined in terms of these connections, so they must perforce also be invariant. In terms of the special trivialisation, Ψ_λ, the connection matrices and the matrix representing Φ are independent of t.

6. Reality

It is sufficient to show that the constructed connection and Higgs field satisfy our reality condition on a dense subset of Y. For simplicity we choose to work on $Y \cap \mathbb{C}^3$. We can also assume that E comes from $(\widetilde{E}, \nabla, \Phi)$ which are independent of time. It remains to show that they are real given either real structure on E.

6.1 Principal bundle reality. We first assume that the principal bundle of frames of E comes with a fixed antiholomorphic involution, $\tilde{\sigma}$, lifting σ which is given in a unitary frame as $X \mapsto \bar{X}^{t-1}$ on the fibres of fix(σ). As was true for $\tilde{\delta}_t$ and $\widetilde{E}$, $\tilde{\sigma}$ induces a map on the bundle of frames of $\tilde{E}$, $Fr(\widetilde{E}) \cong \mathrm{GL}(N)$:

$$Fr(\tilde{E}_y) = H^0(C_y, Fr(E)) \xrightarrow{\tilde{\sigma}} H^0(C_{\sigma(y)}, Fr(E)) = Fr(\tilde{E}_{\sigma(y)}),$$

which acts on a moving frame, $f(y) \in H^0(C_y, Fr(E))$, by $f \mapsto \tilde{\sigma} \circ f \circ \sigma$, giving another moving frame $f' \in H^0(C_{\sigma(y)}, Fr(E))$. If f is holomorphic, so is its image, which is the composition of one holomorphic and two antiholomorphic maps. The same argument as for $\tilde{\delta}_t$ holds, and shows that $\tilde{\sigma}$ pulls back constant frames of the null connection on Π to constant frames of the null connection on $\sigma\Pi$, and hence the null connections back to corresponding null connections.

In particular, if $\sigma(y) = y$, then a frame f gets sent to $\tilde{\sigma}(f)$ where $\tilde{\sigma}(f)|_y = \overline{(f_y)}^{t-1}$ in any unitary basis. So if f is a covariant constant frame for $\nabla = d + A$ in direction X, then $\tilde{\sigma}(f)$ is covariant constant for ∇ in direction $\sigma(X)$ so

$$\begin{aligned}
A_X|_y &= \frac{\partial}{\partial X} f \cdot f^{-1}\Big|_y \\
A_{\sigma(X)}|_y &= \frac{\partial}{\partial \sigma(X)} \tilde{\sigma}(f) \cdot \tilde{\sigma}(f)^{-1}\Big|_y \\
&= \frac{\partial}{\partial \sigma(X)} (\bar{f})^{t-1} \cdot \bar{f}^t\Big|_y \\
&= -\, \bar{f}^{t-1} \overline{\frac{\partial}{\overline{\partial \sigma(X)}} f}^{\,t}\Big|_y \\
&= -\, \overline{\frac{\partial}{\overline{\partial \sigma(X)}} f \cdot f^{-1}}\Big|_y \\
&= -\, \overline{A_{\overline{\sigma(X}}}^{\,t}\Big|_y
\end{aligned}$$

Since $\sigma(\frac{\partial}{\partial x}) = \frac{\partial}{\partial x}$, $\sigma(\frac{\partial}{\partial y}) = \frac{\partial}{\partial y}$ and $\sigma(\frac{\partial}{\partial t}) = -\frac{\partial}{\partial t}$ (see (1.3)) we have

$$A_x = -\bar{A}_x^t, \quad A_y = -\bar{A}_y^t, \quad A_t = \bar{A}_t^t.$$

Reality for Φ either follows from t invariance (as it implied the extension of Φ to S^2) or directly:

Since $2\Phi\frac{\partial}{\partial\chi} = i(\nabla_\Pi - \nabla_{\tau\Pi})$, and Φ is independent of the null plane Π, we might as well take $\sigma(\Pi) = \Pi$, *i.e.* Π_λ for $\lambda = 1$ in the notation of section 3.5, then $\frac{\partial}{\partial\chi} = -\frac{\partial}{\partial x}$, and

$$\begin{aligned}
-2i\Phi(y)\frac{\partial}{\partial x} &= (\nabla_1 - \nabla_{-1}|_y \\
&= -\left(\overline{\nabla_1 - \nabla_{-1}}^t\right|_y \\
&= -2i\bar{\Phi}(y)^t\frac{\partial}{\partial x},
\end{aligned}$$

i.e. $\bar{\Phi}^t = \Phi$ as required. To sum up,

$$A_x, A_y, iA_t, i\Phi \in \mathrm{u}(N).$$

6.2 Vector bundle reality. If, however, we choose to work with the real structure which maps the bundle to its dual, we need to do a bit more work. In the same way that we constructed ∇ and Φ, we can construct a connection and Higgs field on the dual bundle $\widetilde{E}^*$ (both bundles are the trivial $\mathbb{C}^N$ bundle, but the connections are different). We know how to trivialise $\widetilde{E}$, and since E and E^* are trivial on the same sets, taking a dual frame to the fixed frame of E at $\lambda = -1$, and $\eta = \infty$, and using it to define a frame for $\widetilde{E}^*$ in the same way that we defined the framing of $\widetilde{E}$, we get dual frames over an open subset of $\mathbb{C}^3$ containing $\mathbb{R}^3$.

By their very definition, the null-connections ∇_Π and $\nabla_\Pi^{\mathrm{dual}}$ are dual to each other: the flat sections of $\widetilde{E}$ and $\widetilde{E}^{\mathrm{dual}}$ given by evaluation at a point (of $\widetilde{T\mathbb{P}^1}$) are dual to each other *iff* the frames of E and E^{dual} at the point are. If r and s are coordinates of flat frames along X, given in terms of dual frames of E^{dual} and E ,

$$\begin{aligned}
0 = \frac{\partial}{\partial X}(r^t s) &= \frac{\partial}{\partial X}(r(s)) \\
&= \frac{\partial}{\partial X}rs + r\frac{\partial}{\partial X}s \\
&= (-rA_X^{\mathrm{dual}})s + r(-A_X s)
\end{aligned}$$

$$\implies A^{\mathrm{dual}} = -A^t.$$

Recall that $\tilde{\sigma}$ was constructed in the original frame of $\widetilde{E}$ as conjugate transpose followed by pull-back by σ. Above the point $p = (\lambda = -1, \eta = \infty)$, the constructed

$\tilde{\sigma}$ also was given by conjugate transpose (with respect to a framing of $E|_p$ and its dual on $E^{\mathrm{dual}}|_{C_\infty}$). Conversely, a $\tilde{\sigma}$ with this property induces $\tilde{\sigma} : \widetilde{E} \leftrightarrow \widetilde{E}^{\mathrm{dual}}$ which is conjugate transpose with respect to the coordinates associated to the fixed frame at p. Finally, the form $(v, w) \mapsto \tilde{\sigma}(v)(w)$ is positive definite on $\widetilde{E}$ *iff* it is so on $E|_p$ since rank, signature and nullity are continuous.

We defined null connections in terms of flat frames coming from evaluation of $H^0(C_y, E)$ at a point—the null planes consisting of those sections which intersected the chosen point. The involution σ acts on those points, which induces an action on the corresponding null-planes and flat frames. The involution $\tilde{\sigma}$ also acts on E and hence on frames of E over a point, sending them to frames of E^{dual} over the conjugate point. This in turn induces an action on flat frames over conjugate null planes, so that if $\nabla_\Pi r = 0$, then $\nabla^{\mathrm{dual}}_\Pi(\tilde{\sigma} \circ r \circ \sigma) = 0$, which holds as well for $\nabla_\Pi + \nabla_{\tau\Pi}$ and $\nabla^{\mathrm{dual}}_\Pi + \nabla^{\mathrm{dual}}_{\tau\Pi}$. This gives a convenient way of calculating A and A^{dual}:

$$\begin{aligned}
A_X &= \frac{\partial}{\partial X} r r^{-1} \\
A_X^{\mathrm{dual}} &= (\tilde{\sigma} \circ r \circ \sigma)^{-1} \frac{\partial}{\partial X}(\tilde{\sigma} \circ r \circ \sigma) \\
&= \sigma^* \left((r^\dagger)^{-1} \frac{\partial}{\partial \sigma_* X} r^\dagger \right) \\
&= \sigma^* \left(\frac{\partial}{\partial \sigma_* X} r r^{-1} \right)^\dagger \\
&= (A_{\sigma_* X})^\dagger,
\end{aligned}$$

since A, A^{dual} are t-independent. (Note that at the second line, we use the fact that an involution $\tilde{\sigma}$ is positive definite *iff* $\tilde{\sigma}(r)s = r^\dagger s$ in terms of dual bases.) In particular, $A_x, A_y, iA_t \in \mathrm{u}(N)$. To show $i\Phi \in \mathrm{u}(N)$, recall that $2i\Phi d\chi = \nabla_\Pi - \nabla_{\tau\Pi}$, but that Φ is independent of the null-plane Π. Choose Π such that $\frac{\partial}{\partial \chi}$ is in the $x-y$-plane. Then as above we find $\Phi^{\mathrm{dual}} = -\Phi$, and $2i\Phi^{\mathrm{dual}} = (2i\Phi)^\dagger$.

To sum up, we have proven

THEOREM A. *The space of based unitons, $\mathcal{U}(N)^*$, is isomorphic to the space of rank N uniton bundles.*

CHAPTER IV

WARD'S CONSTRUCTION AND WOOD'S CONJECTURE

We are now ready to describe the link with the construction of Ward. The value of this is that Ward's construction involves only the factoring of a transition matrix (*i.e.* solving the Riemann-Hilbert problem) and no differential equations. In addition to its metaphysical significance, this result allows us to affirm the conjecture of Wood that unitons have rational functions in x and y as entries.

1. Ward's Construction

Let $E \to \widetilde{T\mathbb{P}}^1$ be a uniton bundle. Theorem A allows us to assume that E was constructed from a uniton, S, via (∇, Φ), a solution to the Bogomolny equations. We are trying to find some intrinsic definition for S on $\mathbb{R}^3 \times \mathbb{P}^1$ which can be pushed down to $\widetilde{T\mathbb{P}}^1$ and interpreted as a construction for S. For this we will need the extended solution of Uhlenbeck:

Recall that, in Chapter II, to show that the bundle E extended to the compactified fibres of $T\mathbb{C}^*$, we made use of a solution $\tilde{s}$ on $\{(z, \bar{z}, t, \lambda) \in \mathbb{R}^3 \times \mathbb{C}^*\}$, pulled back from a solution (E_λ) of the system D_λ (see (II.2.5)) on $\{(z, \lambda) \in S^2 \times \mathbb{C}^*\}$. This extended solution directly encodes the uniton:

THEOREM [Uhl,2.2]. *If S is harmonic and $S(\infty) = \mathbb{I}$, then there exists a unique flat frame $E_\lambda : \mathbb{P}^1 \to U(N)$ for D_λ with (a) $E_{-1} = \mathbb{I}$, (b) $E_1 = S$, (c) $E_\lambda(\infty) = \mathbb{I}$. Moreover, E is analytic and holomorphic in $\lambda \in \mathbb{C}^*$.*

In Lemma I.2.8 we showed that the solution $\tilde{s}$ on $\mathbb{R}^3 \times \mathbb{C}^*$ pushed down to a trivialisation of $E|_{T\tilde{\mathbb{C}}^*}$ which we called the C_∞ trivialisation. Now we will think of the solution as the expression of the pull back of the trivialisation in terms of the 'constant' $\underline{\mathbb{C}}^N$ trivialisation.

$$\pi_1^*\underline{\mathbb{C}}^N\text{-triv.} \xleftrightarrow[\text{compare}]{E_\lambda} \pi_2^* C_\infty\text{-triv.}$$

$$\begin{tikzcd} & T\mathbb{C}^* \times \mathbb{R} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \underline{\mathbb{C}}^N\text{-triv.}\quad \mathbb{R}^3 & & T\mathbb{C}^* \quad C_\infty\text{-triv.} \end{tikzcd}$$

A point $y \in \mathbb{R}^3$ corresponds to a real section of $T\mathbb{P}^1$ and we can push down the $\underline{\mathbb{C}}^N$-frame over y to a trivialisation of $E|_{C_y}$. Since $D_{-1} = \left(\frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial z}\right)$, the pulled-back C_∞ and $\underline{\mathbb{C}}^N$ trivialisations differ by a constant on $\{\lambda = -1\}$, and if the C_∞-trivialisation is chosen to agree with the framing $\phi \in H^0(P_{-1}, Fr(E))$ then the constant is $\mathbb{I}$. So the trivialisations of $E|_{C_y}$ all agree with the C_∞ trivialisation along P_{-1}. From this we see that the comparison $E_\lambda(z, \bar{z})$ can be pushed down by π_2 to the comparison at $(\lambda, \eta = 1/2(z - \lambda^2\bar{z}))$ of the C_∞ trivialisation and the $\underline{\mathbb{C}}^N$ trivialisation along $E|_{C_{(z,\bar{z},0)}}$.

If we choose the two trivialisations to agree with the framing along P_{-1}, then E_λ is just the 'monodromy' around the cycle of $\mathbb{P}^1$'s:

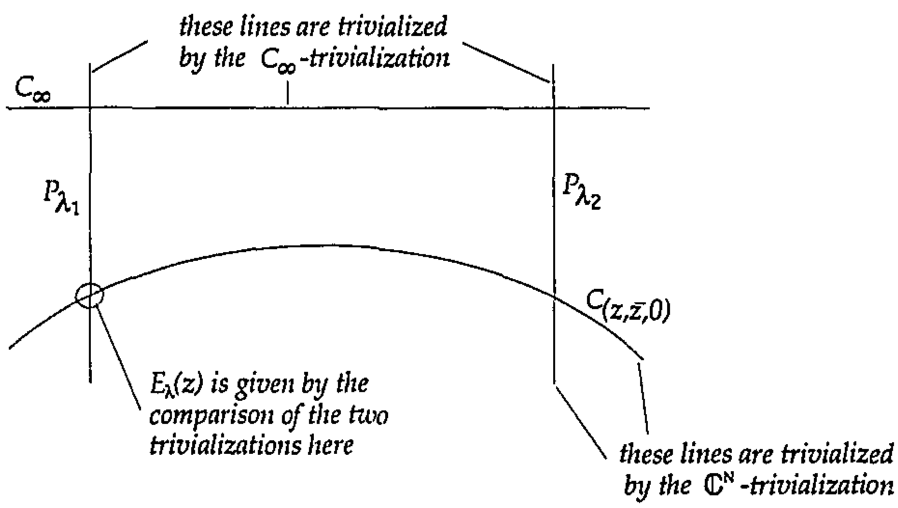

FIG. 8. Compare the trivialisations at their point of intersection.

To make this precise, what we are calling a 'monodromy' is actually the failure to commute of a cycle of homomorphisms given by the restriction map:

$$\begin{array}{ccccc} E_{\lambda,\infty} & \xleftarrow{\text{restr}} & H^0(C_\infty, E) & \xrightarrow{\text{restr}} & E_{-1,\infty} \\ \text{restr}\big\uparrow & & & & \big\uparrow\text{restr} \\ H^0(P_\lambda, E) & & & & H^0(P_{-1}, E) \\ \text{restr}\big\downarrow & & & & \big\downarrow\text{restr} \\ E_{(\lambda, z/2-\lambda t-\lambda^2\bar{z}/2)} & \xleftarrow{\text{restr}} & H^0(C_{(z,\bar{z},t)}, E) & \xrightarrow{\text{restr}} & E_{(-1, z/2+t-\bar{z}/2)}, \end{array} \tag{1.1}$$

beginning at $H^0(P_{-1}, E)$ and going clockwise. The 'monodromy' is independent of the choice of initial value, up to conjugation, as one would expect, since a change of framing of the bundle acts by conjugation on the uniton. We fix it by computing the 'monodromy' of the fixed frame $\phi \in H^0(P_{-1}, Fr(E))$.

1.2 Transition Functions. Ward's construction assumes the bundle is given by a transition matrix, so consider the covering of $\widetilde{T\mathbb{P}^1}$ given by

$$\begin{aligned} U &= \{\lambda \in \mathbb{C}, \eta \in \mathbb{C}\}, \\ \hat{U} &= \left\{\hat{\lambda} \in \mathbb{C}, \hat{\eta} \in \mathbb{C}\right\}, \\ U' &= \{\lambda \in \mathbb{C}, \eta' \in \mathbb{C}\}, \\ \hat{U}' &= \left\{\hat{\lambda} \in \mathbb{C}, \hat{\eta}' \in \mathbb{C}\right\}. \end{aligned} \tag{1.3}$$

The bundle E is determined by transition matrices $T, \hat{T}, T'$ which map fixed frames of E over U to $\hat{U}$, over $\hat{U}$ to $\hat{U}'$, and over U to U' respectively. Because E has certain triviality properties, we can choose the fixed frames such that

$$\begin{aligned} \hat{T}T{T'}^{-1}|_{C_\infty} &= \mathbb{I}, \\ T'|_{P_1} &= \mathbb{I}, \text{ and} \\ \hat{T}|_{P_{-1}} &= \mathbb{I}. \end{aligned} \tag{1.4}$$

If the bundle is trivial when restricted to a complex line ($\mathbb{P}^1$), then a framing above a point of the line extends uniquely to a nonvanishing frame on the line, because in this case evaluation

$$H^0(\mathbb{P}^1, \underline{\mathbb{C}}^N) \xrightarrow{\text{eval}} \underline{\mathbb{C}}^N_{\text{point}} = \mathbb{C}^N$$

is an isomorphism. We will think of this as defining a parallel translation within the line.

In terms of these frames parallel translation from a point on P_1 (in terms of the U frame) to a point on P_{-1} (in terms of the $\hat{U}$ frame) along $P_1 \cup C_\infty \cup P_{-1}$ is given by $\mathbb{I}$. Since the bundle is trivial above real sections, we can get a splitting of T, *i.e.* analytic functions $H : \{(z, \bar{z}, t) \in \mathbb{R}^3, \lambda \in \mathbb{C}\} \to \mathrm{GL}(N)$, and $\hat{H} : \{(z, \bar{z}, t) \in \mathbb{R}^3, \lambda \in (\mathbb{P}^1 \setminus \{0\})\} \to \mathrm{GL}(N)$, such that

$$TH_\lambda(z, \bar{z}, t) = \hat{H}_\lambda(z, \bar{z}, t).$$

Parallel translation from $P_1 \cap C_y$ (in terms of the U frame) to $P_{-1} \cap C_y$ ($\hat{U}$ frame) along C_y is given by

$$\hat{H}_{-1}(y) H_1(y)^{-1} \ \ (= E_1(y) = S(y)), \tag{1.5}$$

which gives the same formula for the uniton as in [Wd3,18]. One must verify that this doesn't depend on the choice of splitting.

Finally, we note that Ward actually takes *two* framings, along P_1 and P_{-1}. One framing would be equivalent to the restrictions (1.4). By taking two framings he does away with the basing condition. This is an important point if one wants to choose a different type of basing condition (other than $S(\infty) = \mathbb{I}$), to encode Grassmannian solutions, for example.

2. Wood's Conjecture

Noting that all known examples of unitons were matrices of functions rational in x and y (equivalently z and $\bar{z}$), Wood conjectured that this is always the case ([Wo]). While it is true that the Bogomolny solution (∇, Φ) constructed from a uniton bundle are algebraic objects there is no reason to believe that the integration

$$S^{-1} dS = 2(A_z dz + A_{\bar{z}} d\bar{z})$$

preserves rationality. Continuing S analytically, or equivalently, integrating A, we can't even rule out multivaluedness if A is holomorphic on nonsimply-connected domains.

The concrete expression (1.5), however, shows that S extends to $Y \cap \mathbb{C}^3$ (a Zariski open set), and using (1.5) and the jumping-line normal form for transition matrices (see [Hu] and [New] for proofs) we can prove

COROLLARY B. *If $S : S^2 \to U(N)$ is a uniton, then the composition with $U(N) \hookrightarrow \mathrm{GL}(N)$ is rational, i.e. the functions in x and y which make up the matrix $S \in U(N)$ are rational.*

For unitons of 'simplest type', we will be able to give an explicit formula for S in Chapter VI which is obviously rational.

PROOF. Since the solution is t-invariant, we can ignore the third dimension.

We want to show $E_\lambda(z,\bar{z}) = H_{-1}(z,\bar{z})H_\lambda(z,\bar{z})$ is a rational $\mathrm{gl}(N)$-valued function on $\{(z,\bar{z}) \in \mathbb{P}^1 \times \mathbb{P}^1\}$. A *function* is rational *iff* it is meromorphic *iff* it is meromorphic when restricted to the sets of a covering of $\mathbb{P}^1 \times \mathbb{P}^1$, and a function is meromorphic *iff* its only singularities are poles. Thus we can answer a global question with a local answer.

Consider the family of open sets

$$\left\{U_{z_0} = \left\{(z,w) \in \mathbb{P}^1 \times \mathbb{P}^1 : z \neq z_0, w \neq \bar{z}_0\right\}_{z_0 \in \mathbb{P}^1}\right\}.$$

Any three sets cover $\mathbb{P}^1 \times \mathbb{P}^1$. Symmetry allows us to consider any one set:

We will prove that S is meromorphic on U_∞, which corresponds to our choice of coordinates on S^2. Working with new coordinates $(z,\bar{z}) \mapsto (1/(z-a), 1/(\bar{z}-\bar{a}))$ amounts to working on the set U_a. The functions S agree on the overlap because analytic continuation is unique, and both the change of coordinates and the continuation defined by (1.5) are analytic.

The expression (1.5) defines S on $\mathbb{R}^2$, but extends just as well to $\mathbb{C}^2$ with potential singularities at the jumping lines. To see that they are poles, pull back the transition matrix T by

$$\varphi : \mathbb{C}^2 \times \mathbb{P}^1 \to T\mathbb{P}^1$$

$$(z,\bar{z}) \times (\lambda) \to (\lambda, \eta = z - \bar{z}\lambda^2).$$

If $(z,\bar{z},0)$ represents a jumping line of type $(k_1 \leq k_2 \leq \cdots \leq k_N)$ (*i.e.* $E|_{C_{(z,\bar{z},t)}} \cong \mathcal{O}(k_1) \oplus \cdots \mathcal{O}(k_N)$), then we can make a holomorphic change of frame on some neighbourhood of the point so that T has the form

$$T = \begin{pmatrix} z^{-k_1} & & & & \\ & \ddots & & p^i_j & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & z^{-k_N} \end{pmatrix}, \quad p^i_j = \sum_{a=-k_j+1}^{-k_i-1} p^i_{ja}(z,\bar{z})\lambda^a$$

where p^i_{ja} are holomorphic functions. A section of $E|_y$ is given by $(u^1, \ldots, u^N)^t$, $u^j = \sum_{a=0}^{\infty} u^j_a \lambda^a$ such that

$$T(y,\lambda) \begin{pmatrix} u^1 \\ \vdots \\ u^N \end{pmatrix} \text{ is holomorphic in } 1/\lambda,$$

which puts conditions on $\{u^j_a\}$. Expanding the columns of $T \cdot u$ in Laurent series in λ, the conditions come from the coefficients of positive terms in λ (λ^i, $i > 0$), which we can see are linear in u^j_a.

In particular, from the last row: $z^{-k_N} u^N$ must be holomorphic in $1/\lambda$, so u^N must be polynomial in λ of degree at most k_N. By induction on $i - N$ we see that u^j is polynomial for all j, although the degree of u^j is not bounded by k_j in general.

Since the remaining coefficient conditions are linear we can write them as a matrix. We can make a further reduction of that matrix by solving for some of the coefficients: Remark that the terms $u^j_a \lambda^a$, $a > k_j$ figure only in coefficients of $T \cdot u$ of positive order in λ, hence are determined by $u^{j'}_a$, with $j' > j$, and $a \leq k_j$. We can solve for them, getting something polynomial in the coefficients of p and linear in the coefficients of u. For each such coefficient, we can reduce the matrix of conditions by one row and one column. We call the resulting matrix, $\Gamma(y)$, the section matrix. It has $\sum_{i=1}^{N}(k_i+1)^+$ (summing only positive terms) columns, corresponding to the $\sum(k_i+1)^+$ coefficients:

$$(u^1_0, u^1_1, \ldots, u^1_{k_1}, u^2_0, \ldots, u^N_0, u^N_1, \ldots, u^N_{k_N})^t,$$

and $\sum_{i=1}^{N}(-k_i-1)^+$ rows corresponding to coefficients of $(T \cdot u)_i$ of positive order less than $-k_i$. The difference

$$\#\text{columns - }\#\text{rows} = \sum_{i=1}^{N}(k_i+1) = c_1(E) + N.$$

In particular, our section matrix has N more columns than rows. The number of sections of $E|_{C_y}$ = corank $\Gamma(y)$. Since $h^0(\mathcal{O}^N) = N$, we see that $\Gamma(y)$ has maximal rank at non-jumping lines, C_y. After possible shuffling of the u_a^j's, assume that Γ has the form

$$\Gamma = \begin{array}{c} \begin{array}{cc} \overleftrightarrow{\scriptstyle\sum_{i=1}^{N}(-k_i-1)^+} & \overleftrightarrow{\scriptstyle N} \end{array} \\ \left(\begin{array}{c|c} \boxed{\quad\Gamma'\quad} & \boxed{\Gamma''} \end{array} \right) \end{array},$$

where Γ' is invertible (and square). Then a moving frame H is given by

$$H_l = \begin{pmatrix} u^1 \\ \vdots \\ u^N \end{pmatrix}_l, \text{ where } (u_a^j) = \begin{pmatrix} \Gamma'^{-1}\Gamma_l'' \\ \Gamma_l'' \end{pmatrix},$$

which is meromorphic in p_{ja}^i, which are in turn holomorphic in z, $\bar{z}$ and t.

$$\hat{H} = TH$$

is then also meromorphic in z, $\bar{z}$ and t, so $\hat{H}^{-1}$ is meromorphic on $\mathbb{C}^2$. It follows that in a neighbourhood of the point, $S = \hat{H}^{-1}H$ is meromorphic, hence has only poles. $\square$

CHAPTER V

MONADS

Now that we have shown that unitons are equivalent to holomorphic bundles over $\widetilde{T\mathbb{P}}^1$, we are in a position to investigate the moduli space. To do this we will exploit monads as in [Do]. For a more general account of monads, see [OSS], where, in particular, they prove Beilinson's theorem, the main tool for showing the existence of a monad representation for holomorphic bundles on $\mathbb{P}^N$.

A monad is a complex of uniform bundles, whose cohomology is the desired bundle. For example, semistable two-bundles $E \to \mathbb{P}^2$ trivial on a line are expressable as the cohomology of a linear complex

$$0 \to \mathcal{O}(-1)^k \xrightarrow{\alpha} \mathcal{O}^{2k+N} \xrightarrow{\beta} \mathcal{O}(1)^k \to 0,$$

where α is injective and β is surjective, *i.e.*

$$E \cong \ker\beta / \operatorname{im}\alpha.$$

They can then be represented by three $k \times (k+N)$ and three $(k+N) \times k$ matrices, uniquely, up to the action of $\mathrm{GL}(k) \times \mathrm{GL}(2k+N) \times \mathrm{GL}(k)$ (another theorem from [OSS]). We will obtain a similar but more complicated result because we are working on a bundle of projective spaces rather than a simple projective space.

0.1 Hirzebruch Surfaces. In the introduction, we remarked that $\widetilde{T\mathbb{P}}^1$ can be obtained by projectivising $\mathcal{O}(2) \oplus \mathcal{O}$ over $\mathbb{P}^1$. Buchdahl studies such projectivisations in [Bu], where he gives a monad description for stable bundles, of a kind due originally to Beilinson. We use some of his notation.

For any sheaf F on $\widetilde{T\mathbb{P}}^1$, define the family of sheaves

$$F(p,q) = F \otimes \mathcal{O}_{\widetilde{T\mathbb{P}}^1}(pC_0 + qP_{\lambda_0}),$$

where C_0 is the zero section and P_{λ_0} is some fixed fibre (all of which are rationally equivalent). By $\mathcal{O}_{\widetilde{T\mathbb{P}^1}}(pC_0 + qP_{\lambda_0})$ we mean the line bundle given by the divisor $pC_0 + qP_{\lambda_0}$. Define $\mathcal{O}_{\widetilde{T\mathbb{P}^1}\times\widetilde{T\mathbb{P}^1}}(p,q)(p',q') = p_1^*\mathcal{O}_{\widetilde{T\mathbb{P}^1}}(p,q) \otimes p_2^*\mathcal{O}_{\widetilde{T\mathbb{P}^1}}(p',q')$, where p_1, p_2 are the projections $\widetilde{T\mathbb{P}^1} \times \widetilde{T\mathbb{P}^1} \to \widetilde{T\mathbb{P}^1}$.

This gives a complete description of line bundles on $\widetilde{T\mathbb{P}^1}$ since all subvarieties of $\widetilde{T\mathbb{P}^1}$ are linearly equivalent to combinations of fibres and the zero section. In particular $\mathcal{O}(C_\infty) = \mathcal{O}(1,-2)$ and $\mathcal{O}(0,1) = \pi^*(\mathcal{O}_{\mathbb{P}^1}(1))$. From the Leray spectral sequence, we see that complex analytic cycles generate the singular homology of $\widetilde{T\mathbb{P}^1}$. (This is not the case in general since complex analytic cycles occur only in even dimensions.) In fact, cup product is given by intersection of generic subvarieties, and since $H^4(\widetilde{T\mathbb{P}^1}, \mathbb{Z})$ is generated by a single element, or alternatively, since all points on $\widetilde{T\mathbb{P}^1}$ are rationally equivalent, we will think of $H^4(\widetilde{T\mathbb{P}^1}, \mathbb{Z}) = \mathbb{Z}[\mathrm{pt}]$, where [pt] is any point. Similarly we will take $[\widetilde{T\mathbb{P}^1}]$ to be a generator of $H^0(\widetilde{T\mathbb{P}^1}, \mathbb{Z})$, and C_0 and P_{λ_0} to generate $H^2(\widetilde{T\mathbb{P}^1}, \mathbb{Z})$.

In the construction that follows, we will encounter many cohomology groups with values in E twisted by the sheaves $\mathcal{O}(p,q)$, some appearing as obstruction groups, others in the monad itself. We will now compute some needed cohomology groups.

Since $\widetilde{T\mathbb{P}^1}$ is two complex dimensional, bundles above $\widetilde{T\mathbb{P}^1}$ have only two Chern classes, c_1 and c_2. Since we can think of c_2 as an integer (the number of points). Let

$$c_2(E) = k([\mathrm{pt}]).$$

The first class can be written $c_1(E) = pC_0 + qP_{\lambda_0}$. Since E is trivial on nonpolar fibres (P_{λ_0}, $\lambda_0 \in \mathbb{C}^*$), $0 = c_1(E|_{P_{\lambda_0}}) = p$. Since E is trivial on real sections, $0 = c_1(E|_{C_0}) = q$. So

$$c_1(E) = 0.$$

LEMMA 0.2.

$$\begin{array}{lll} H^0(E(p,q)) = 0 & \textit{when} & p < 0 \textit{ or } 2p+q < 0 \\ H^2(E(p,q)) = 0 & \textit{when} & p+2 > 0 \textit{ or } 2p+q+4 > 0, \end{array}$$

and similarly for E^* in place of E.

PROOF. Let $s \in H^0(E(p,q))$. If $p < 0$, then

$$s|_{\text{nonpolar fibre}} \in H^0(\mathcal{O}_{\mathbb{P}^1}(p)^N) = 0.$$

Since such fibres span an open subset of $\widetilde{T\mathbb{P}}^1$, $s \equiv 0$.

If $2p + q < 0$, then since a real section has intersection 2 with the zero section, and intersection one with any fibre

$$s|_{\text{real section}} \in H^0(\mathcal{O}_{\mathbb{P}^1}(2p+q)^N) = 0.$$

Again, real sections trace out an open subset of $\widetilde{T\mathbb{P}}^1$, so $s \equiv 0$.

The same is true of sections of E^*, since E^* is trivial on any subset on which E is trivial.

We use the Adjunction Formula [Ha, V.1.5] for curves in surfaces $2g_C - 2 = C \cdot (C + K_S)$ to determine $K_{\widetilde{T\mathbb{P}}^1}$, the canonical bundle. Let $K_{\widetilde{T\mathbb{P}}^1} = aP_{\lambda_0} + bC_0$. Then $g_{P_{\lambda_0}} = 0 = g_{C_0}$ and $P_{\lambda_0} \cdot P_{\lambda_0} = 0$ and $C_0 \cdot C_0 = 2$ implies $b = P_{\lambda_0} \cdot K_{\widetilde{T\mathbb{P}}^1} = -2$, $a + 2b = C_0 \cdot K_{\widetilde{T\mathbb{P}}^1} = -2 - 2 = -4$. So

$$K_{\widetilde{T\mathbb{P}}^1} = \mathcal{O}(-2, 0).$$

By Serre Duality

$$\begin{aligned} H^2(E(p,q)) &\cong H^0(E(p,q)^* \otimes K)^* \\ &= H^0(E^*(-p-2, -q)) \\ &= 0 \quad \text{if } -p-2 < 0 \text{ or } 2(-p-2) + (-q) < 0. \quad \square \end{aligned}$$

We get information about $H^1(E(p,q))$ using the Hirzebruch-Riemann-Roch Theorem [Ha, A.4.1]:

$$h^0(F) - h^1(F) + h^2(F) \equiv \chi(F) = \deg(\mathrm{ch}(F) \cdot \mathrm{td}(TX^n))_n,$$

i.e. the number of points in the intersection of the characteristic classes given (on two$_{\mathbb{C}}$-dimensional manifolds) by

$$\begin{aligned} \mathrm{ch}(F) &= \operatorname{rank} F[\widetilde{T\mathbb{P}}^1] + c_1(F) + \frac{1}{2}(c_1 \cdot c_1 - 2c_2)[\mathrm{pt}] \\ \mathrm{td}(F) &= [\widetilde{T\mathbb{P}}^1] + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2)[\mathrm{pt}] \end{aligned}$$

in particular

$$\operatorname{td}(TX) = [\widetilde{T\mathbb{P}}^1] - \frac{1}{2}K_X + \frac{1}{12}(K^2 + c_2) = [\widetilde{T\mathbb{P}}^1] - \frac{1}{2}K_X + \chi(\mathcal{O}_X).$$

But, according to [Ha, V.2.4] if $D \cdot P_{\lambda_0} \geq 0$,

$$H^i(\widetilde{T\mathbb{P}}^1, \mathcal{O}(D)) \cong H^i(\mathbb{P}^1, \pi_* \mathcal{O}(D)).$$

In particular, $H^i(\mathcal{O}_{\widetilde{T\mathbb{P}}^1}) \cong H^i(\mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}$ if $i = 0$ and 0 otherwise. So $\chi(\mathcal{O}_{\widetilde{T\mathbb{P}}^1}) = 1$, and

$$\operatorname{td}(T\widetilde{T\mathbb{P}}^1) = [\widetilde{T\mathbb{P}}^1] + C_0 + 1[\mathrm{pt}].$$

Since $\mathcal{O}(p,q)$ is a line bundle $c_2 = 0$, so

$$\operatorname{ch}(\mathcal{O}(p,q)) = [\widetilde{T\mathbb{P}}^1] + (pC_0 + qP_{\lambda_0}) + \frac{1}{2}(2p^2 + 2pq).$$

Hence

$$\begin{aligned}\chi(\mathcal{O}(p,q)) &= \deg(([\widetilde{T\mathbb{P}}^1] + pC_0 + qP_{\lambda_0} + (p+pq)[\mathrm{pt}]) \cdot ([\widetilde{T\mathbb{P}}^1] + C_0 + [\mathrm{pt}]))_2 \\ &= 1 + 2p + q + pq + p^2,\end{aligned}$$

and

$$\begin{aligned}\chi(E(p,q)) &= \deg(\operatorname{ch}(E)\operatorname{ch}(\mathcal{O}(p,q))\operatorname{td}(T\widetilde{T\mathbb{P}}^1))_2 \\ &= \deg(N[\widetilde{T\mathbb{P}}^1] - k[\mathrm{pt}]) \cdot ([\widetilde{T\mathbb{P}}^1] + (p+1)C_0 + qP_{\lambda_0} + \chi(\mathcal{O}(p,q))[\mathrm{pt}])_2 \\ &= -k + N(p^2 + pq + 2p + q + 1).\end{aligned}$$

So, in particular

$$\begin{aligned}h^1(E(0,-1)) &= h^0(E(0,-1)) + h^2(E(0,-1)) - \chi(E(0,-1)) \\ &= 0 + 0 - (-k) \\ &= k \\ h^1(E(0,-2)) &= k + N \qquad\qquad (0.3) \\ h^1(E(-1,-2)) &= k \\ h^1(E(-1,-1)) &= k.\end{aligned}$$

COROLLARY 0.4. *If $E \to \widetilde{T\mathbb{P}^1}$ is trivial on generic fibres and sections, then* $c_2(E) \geq 0$.

PROOF. This is clearly implied by our calculation of $c_2(E) = h^1(E(-1,-2))$ which as the dimension of a module must be nonnegative. The only facts about E we used to do this computation were the triviality over generic fibres and sections. $\square$

1. Beilinson's Theorem

The basic idea behind Beilinson's theorem is to construct a Koszul resolution, that is, an exact sequence of coherent sheaves

$$0 \to C_{-j} \to C_{-j+1} \to \cdots \to C_0 = \mathcal{O}_{X \times X} \to \mathcal{O}_\Delta \to 0$$

over the product of a space with itself, where Δ is the corresponding diagonal. Given a bundle E on X, pull it back by one of the projections to $X \times X$, and tensor it by this sequence. Then, the direct image (under either projection) of the last term is the original bundle, while the direct and higher direct images of the other terms are constant under the projection under which we pulled back E, but under the other projection we get sheaves with values in twisted cohomology groups of E.

The idea is to apply the standard resolution of the diagonal in $\mathbb{P}^N \times \mathbb{P}^N$ to both the fibre and base of the bundle

$$\begin{array}{ccc} \mathbb{P}^1 & \longrightarrow & \widetilde{T\mathbb{P}^1} \\ & & \downarrow{\scriptstyle \pi} \\ & & \mathbb{P}^1 \,. \end{array}$$

The resulting Koszul resolution is

$$0 \to \mathcal{O}(-1,-1)(-1,-1) \xrightarrow[s_1]{s_0,} \begin{array}{c} \mathcal{O}(0,-1)(0,-1) \\ \oplus \\ \mathcal{O}(-1,0)(-1,0) \end{array} \xrightarrow[s_0]{s_1(\oplus)} \mathcal{O} \to \mathcal{O}_{\Delta_{\widetilde{T\mathbb{P}^1}}} \to 0, \qquad (1.1)$$

where

$$s_0 = \lambda - \lambda' \quad (= \lambda \otimes 1 - 1 \otimes \lambda') \text{ and}$$
$$s_1 = \eta - \eta' \quad (= \eta \otimes 1 - 1 \otimes \eta'),$$

can be thought of as sections of $p_1^*\mathcal{O}(0,1)\otimes p_2^*\mathcal{O}(0,1)$ and $p_1^*\mathcal{O}(1,0)\otimes p_2^*\mathcal{O}(1,0)$ respectively. One must check that it is exact. (See also [GriHa, p688].) Truncating, and tensoring with p_1^*E, we get a complex

$$\begin{aligned}
C^{-2} &= p_1^*E\otimes p_1^*\mathcal{O}(-1,-1)\otimes p_2^*\mathcal{O}(-1,-1),\\
C^{-1} &= p_1^*E\otimes\left(p_1^*\mathcal{O}(0,-1)\otimes p_2^*\mathcal{O}(0,-1)\oplus p_1^*\mathcal{O}(-1,0)\otimes p_2^*\mathcal{O}(-1,0)\right),\\
C^{0} &= p_1^*E.
\end{aligned}$$

To get a monad, we use the spectral sequences in hypercohomology of this complex associated to the projection p_2, as in [EGA III,0.12.4]. Namely, let $\mathcal{H}^p(p_2,C^*)$ be the complex of sheaves associated to the presheaves

$$U\subset\widetilde{T\mathbb{P}}^1\mapsto R^q_{p_2*}(C^p),$$

where $R^q_{p_2*}(F)$ can be defined as the sheaf associated to the presheaf $U\subset\widetilde{T\mathbb{P}}^1\mapsto \check{\mathrm{H}}^q(p_2^{-1}(U),F)$;

$$H^p\left(C^*\right)=\frac{\ker(C^p\to C^{p+1})}{\operatorname{im}(C^{p-1}\to C^p)},$$

i.e. the p^{th} cohomology of the complex C^*. The hypercohomology of p_2 and C^* is then the limit of two spectral sequences with the following E_2 terms

$$\begin{aligned}
{}'E_2^{pq} &= H^q(\mathcal{H}^p(p_2,C^*)),\\
{}''E_2^{pq} &= \mathcal{H}^p(p_2,H^q(C^*))(=R^p_{p_2}(H^q_{(\text{of the complex})}(C^*))).
\end{aligned}$$

It is easy to compute ${}''E$, since the complex C^* is a truncated exact sequence. We get

$$H^q(C^*)=\begin{cases}\mathcal{O}_{\Delta_{\widetilde{T\mathbb{P}}^1}}\otimes p_1^*E & q=0\\ 0 & \text{otherwise.}\end{cases}$$

Using the fact $\mathcal{O}_{\Delta_{\widetilde{T\mathbb{P}}^1}}\otimes p_1^*E=\mathcal{O}_{\Delta_{\widetilde{T\mathbb{P}}^1}}\otimes p_2^*E$, we can conclude

$$\begin{aligned}
{}''E_2^{p0} &= R^p_{p_2}(\mathcal{O}_{\Delta_{\widetilde{T\mathbb{P}}^1}}\otimes p_2^*E)\\
&= R^p_{p_2}(\mathcal{O}_{\Delta_{\widetilde{T\mathbb{P}}^1}})\otimes E=\begin{cases}E & p=0\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

We know from [EGA III] that ${}'E_2$ also converges to the bundle E, since ${}''E$ does.

Using $R^q_{p_2*}(p_1^*L \otimes p_2^*M) = H^q(L) \otimes M$ we compute

$$'E_2^{pq} = H^q \begin{pmatrix} 0 \to H^p(E(-1,-1)) \otimes \mathcal{O}(-1,-1) \\ \xrightarrow{\mu} H^p(E(0,-1)) \otimes \mathcal{O}(0,-1) \oplus H^p(E(-1,0)) \otimes \mathcal{O}(-1,0) \\ \xrightarrow{\nu} H^p(E) \otimes \mathcal{O} \to 0 \end{pmatrix},$$

i.e.

$$'E_2^{pq} = \begin{matrix} \ker \mu_2 & \frac{\ker \nu_2}{\operatorname{im} \mu_2} & \operatorname{coim} \nu_2 \\ \ker \mu_1 & \frac{\ker \nu_1}{\operatorname{im} \mu_1} & \operatorname{coim} \nu_1 \\ \ker \mu_0 & \frac{\ker \nu_0}{\operatorname{im} \mu_0} & \operatorname{coim} \nu_0. \end{matrix}$$

Using this machinery, then, one can show that classes of bundles have monad descriptions by showing that enough cohomology groups appearing in the spectral sequence vanish to reduce it to a short complex. See [OSS, p246] for examples of this over $\mathbb{P}^n$.

In our case, $H^0(E(p,q)) = 0$ when $p < 0$ or $2p+q < 0$, and $H^2(E(p,q)) = 0$ when $p+2 > 0$ or $2p+q+4 > 0$. And after some experimentation, we discover that putting $E(0,-1)$ in place of E, the complexes with $q = 0,2$ vanish, and the spectral sequence reduces to

$$E(0,-1) = H(0 \to \mathcal{O}(-1,-1)^k \to \mathcal{O}(-1,0)^k \oplus \mathcal{O}(0,-1)^{k+N} \to \mathcal{O}^k \to 0).$$

We have completed the first step in proving the

THEOREM 1.2. *Any N-bundle $E \to \widetilde{T\mathbb{P}}^1$, trivial on nonpolar fibres, real sections and the section at infinity has a monad representation*

$$0 \to \mathcal{O}(-1,0)^k \xrightarrow[\alpha_2]{\alpha_1} \begin{matrix} \mathcal{O}(-1,1)^k \\ \oplus \\ \mathcal{O}^{k+N} \end{matrix} \xrightarrow[\beta_2]{\beta_1} \mathcal{O}(0,1)^k \to 0,$$

$$\alpha = (\alpha_1^0 + \alpha_1^1\lambda, \alpha_2^0\eta + \alpha_2^1 + \alpha_2^2\lambda + \alpha_2^3\lambda^2)^t,$$

$$\beta = (\beta_1^0\eta + \beta_1^1 + \beta_1^2\lambda + \beta_1^3\lambda^2, \beta_2^0 + \beta_2^1\lambda), \tag{1.3}$$

where $\alpha_1^i \in \mathrm{gl}(k), \alpha_2^i \in M^k_{k+N}, \beta_1^i \in \mathrm{gl}(k), \beta_2^i \in M_k^{k+N}$, α is injective, β is surjective and $0 = \beta \circ \alpha = \beta_1 \circ \alpha_1 + \beta_2 \circ \alpha_2$. A homomorphism of such bundles is represented by a monad homomorphism:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}(-1,0)^k \arrow[r] \arrow[d, "A"] & \mathcal{O}(-1,1)^k \oplus \mathcal{O}^{k+N} \arrow[r] \arrow[d, "B\,\mid\, C\searrow\downarrow D"] & \mathcal{O}(0,1)^k \arrow[r] \arrow[d, "E"] & 0 \\ 0 \arrow[r] & \mathcal{O}(-1,0)^k \arrow[r] & \mathcal{O}(-1,1)^k \oplus \mathcal{O}^{k+N} \arrow[r] & \mathcal{O}(0,1)^k \arrow[r] & 0; \end{tikzcd} \tag{1.4}$$

in the case of an isomorphism,

$$\left(A, B, C = C_0 + C_1\lambda, D, E\right) \in \mathrm{GL}(k)^2 \times (M^k_{k+N})^2 \times \mathrm{GL}(k+N) \times \mathrm{GL}(k)\}\,.$$

Monad isomorphisms are a group with multiplication given by

$$(A, B, C, D, E) \circ (A', B', C', D', E') = (AA', BB', CB' + DC', DD', EE'),$$

whose action on the monad (1.3) *is given by*

$$\begin{array}{c} \alpha_1^0 \\ \alpha_1^1 \\ \alpha_2^0 \\ \alpha_2^1 \\ \alpha_2^2 \\ \alpha_2^3 \end{array} \mapsto \begin{array}{c} B\alpha_1^0 A^{-1} \\ B\alpha_1^1 A^{-1} \\ D\alpha_2^0 A^{-1} \\ D\alpha_2^1 A^{-1} + C_0\alpha_1^0 A^{-1} \\ D\alpha_2^2 A^{-1} + C_0\alpha_1^1 A^{-1} + C_1\alpha_1^0 A^{-1} \\ D\alpha_2^3 A^{-1} + C_1\alpha_1^1 A^{-1} \end{array},$$

$$\begin{array}{c} \beta_1^0 \\ \beta_1^1 \\ \beta_1^2 \\ \beta_1^3 \\ \beta_2^0 \\ \beta_2^1 \end{array} \mapsto \begin{array}{c} E\beta_1^0 B^{-1} \\ E\beta_1^1 B^{-1} - E\beta_2^0 C_0 B^{-1} \\ E\beta_1^2 B^{-1} - E\beta_2^0 C_1 B^{-1} - E\beta_2^1 C_0 B^{-1} \\ E\beta_1^3 B^{-1} - E\beta_2^1 C_1 B^{-1} \\ E\beta_2^0 D^{-1} \\ E\beta_2^1 D^{-1}. \end{array} \tag{1.5}$$

PROOF. First we show that α and β have the specified form. Think of the maps α_i, β_i as matrices of sections of line bundles $\mathcal{O}(p,q)$.

Using the coordinates λ, η, etc., as above, $\mathcal{O}(p,q)$ has transition functions

$$\begin{tikzcd} C^0(\{\lambda \neq \infty, \eta \neq 0\}, \mathcal{O}(p,q)) \arrow[d, "\lambda'^{-q}"'] & C^0(\{\lambda \neq \infty, \eta \neq \infty\}, \mathcal{O}(p,q)) \arrow[l, "\eta^{-p}"'] \arrow[d, "\lambda^{-2p-q}"] \\ C^0(\{\lambda \neq 0, \eta \neq 0\}, \mathcal{O}(p,q)) & C^0(\{\lambda \neq 0, \eta \neq \infty\}, \mathcal{O}(p,q)) \arrow[l, "\hat{\eta}^{-p}"] \end{tikzcd}$$

sending

$$\begin{tikzcd} \sum a_{lm}\lambda'^l \eta'^{p-m} \arrow[d, "\lambda'^{-q}"'] & \sum a_{lm}\lambda^l \eta^m \arrow[l, "\eta^{-p}"'] \arrow[d, "\lambda^{-2p-q}"] \\ \sum a_{lm}\hat{\lambda}'^{2p-l-2m}\hat{\eta}'^{p-m} & \sum a_{lm}\hat{\lambda}^{2p+q-l-2m}\hat{\eta}^m \arrow[l, "\hat{\eta}^{-p}"] \end{tikzcd}$$

from which we derive bounds on l and m. In fact,

$$h^0(\mathcal{O}(p,q)) = \#\left\{(l,m) : 0 \leq m, 0 \leq l \leq 2(p-m), l - q \leq 2(p-m)\right\}.$$

To see the second part consider two bundles over $\widetilde{T\mathbb{P}}^1$. If the bundles F, F' are represented by monads, M, M', then a homomorphism of monads

$$\begin{tikzcd}
M:0 \arrow[r] & K \arrow[r, "a"] \arrow[d] & L \arrow[r, "b"] \arrow[d, "f"] & N \arrow[r] \arrow[d] & 0 \\
M':0 \arrow[r] & K' \arrow[r, "a'"] & L' \arrow[r, "b'"] & N' \arrow[r] & 0,
\end{tikzcd}$$

induces a well-defined map

$$F = \ker b / \operatorname{im} a \to \ker b' / \operatorname{im} a' = F'$$

of bundles. Every map comes from a monad map in this way if

$$\begin{aligned} &H^0(L^* \otimes K'), \quad H^0(N^* \otimes L'), \quad H^1(N^* \otimes K'), \\ &\quad H^1(L^* \otimes K'), \quad H^1(N^* \otimes L'), \quad H^2(N^* \otimes K') \end{aligned} \tag{1.6}$$

vanish, see [OSS, Lemma II.4.1.3], which may be verified by applying the vanishing lemma and the explicit calculation of $\chi(\mathcal{O}(p,q))$.

The formula for an isomorphism is argued as were the forms of α and β. □

As we are interested in bundles up to isomorphism, we will use the group of monad isomorphisms to put a general monad M into normal form, thus choosing a special representative of each isomorphism class of monads. From this normal form we can hope to read off information about the bundle just as putting a linear transformation into Jordan normal form allows one to read off the eigenvalues and identify the irreducible invariant spaces.

Finally, we are interested in bundles with additional structure: fixed bundle isomorphisms lifting the real structure σ, a framing above some fixed point and time translation δ_t. A monad (α, β) is pulled-back by $\delta_t : (\lambda, \eta) \mapsto (\lambda, \eta - t\lambda)$ to a monad

$$\begin{aligned} &(\alpha_1^0 + \alpha_1^1\lambda, \alpha_2^0\eta + \alpha_2^1 + (\alpha_2^2 + t\alpha_2^0)\lambda + \alpha_2^3\lambda^2)^t, \\ &\quad (\beta_1^0\eta + \beta_1^1 + (\beta_1^2 + t\beta_1^0)\lambda + \beta_1^3\lambda^2, \beta_2^0 + \beta_2^1\lambda). \end{aligned} \tag{1.7}$$

And the pull-back by $\sigma : (\lambda, \eta) \mapsto (1/\bar{\lambda}, -\bar{\lambda}^2\bar{\eta})$ is $\sigma^* M^{\mathrm{op}} =$

$$\begin{aligned}\big((\bar{\alpha}_1^1 + \bar{\alpha}_1^0\lambda, -\bar{\alpha}_2^0\eta + \bar{\alpha}_2^3 + \bar{\alpha}_2^2\lambda + \bar{\alpha}_2^1\lambda^2)^t,&\\ (-\bar{\beta}_1^0\eta + \bar{\beta}_1^3 + \bar{\beta}_1^2\lambda + \bar{\beta}_1^1\lambda^2, \bar{\beta}_2^1 + \bar{\beta}_2^0\lambda)\big)\,.&\end{aligned} \tag{1.8}$$

We can represent the additional structure of the bundle as a fixed monad isomorphism $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E})$ sending $\sigma^* M^{\mathrm{op}}$ to $(\alpha^\sigma, \beta^\sigma)$; $(A_t, B_t, C_t, D_t, E_t)$ sending (α, β) to $\delta_t^*(\alpha, \beta)$; and a choice of frame over some point. Remark that δ_t and σ send fibres to fibres, sections to sections, and hence preserve $\mathcal{O}(i,j)$. As to the frame, we know from the many triviality properties of the bundle that picking any two points we can specify a 'path' along curves above which the bundle is trivial and get an equivalence of the two frames, so we might as well choose a point which simplifies the process of normalising the bundle. Again, since the bundle is trivial on nonpolar fibres, we can also choose a frame of $E|_{P_{\lambda_0}}$. The monad restricted to this fibre is

$$0 \to \mathcal{O}(-1)^k \xrightarrow[\alpha_2(\lambda_0)]{\alpha_1} \begin{matrix}\mathcal{O}(-1)^k\\ \oplus\\ \mathcal{O}^{k+N}\end{matrix} \xrightarrow[\beta_2]{\beta_1} \mathcal{O}^k \to 0,$$

to which is associated the sequence

$$\begin{array}{ccccccc} H^0(\mathcal{O}(-1)^k) & \xrightarrow[\alpha_2(\lambda_0)]{\alpha_1} & H^0\begin{pmatrix}\mathcal{O}(-1)^k\\ \oplus\\ \mathcal{O}^{k+N}\end{pmatrix} & \xrightarrow[\beta_2]{\beta_1} & H^0(\mathcal{O}^k) & \longrightarrow & H^1(\mathcal{O}(-1)^k)\\ \| & & \| & & & & \|\\ 0 & & H^0(\mathcal{O}^{k+N}) & & & & 0 \end{array}$$

So a framing is given by an injection

$$\mathbb{C}^N \xrightarrow{\phi} \ker\left(\beta_2^0 + \beta_2^1\lambda_0\right) \subset \mathcal{O}^{k+N}, \tag{1.9}$$

i.e. $\phi \in M_{k+N}^N$ such that $(\beta_2^0 + \beta_2^1\lambda_0)\cdot\phi = 0$. For notational simplicity, we choose $\lambda_0 = -1$. Finally, we note that the group of monad isomorphisms acts on the representation of the framing by

$$\phi \mapsto D^{-1}\phi.$$

2. Normalisation

Normalising such a monad is best done in a sugar shack, where it can be boiled down until it gets sticky. This is carried out incrementally, using the special structure we know E has. Although the computations are tedious, the idea is very simple. If a group acts on a space, the quotient by the action is the space of orbits. The action is free *iff* all orbits are isomorphic to the group. If a special representative can be chosen from each orbit uniquely in some smooth fashion (*i.e.* so that the representatives are a continuous section of the projection of the total space onto the quotient), then the quotient is isomorphic to the space of special forms. In our case the action is only free if we include the framing. The action of $\mathrm{GL}(N)$ via conjugation on $\mathrm{gl}(N)$ is a simple example of an action, where the quotient is isomorphic to the set of matrices in Jordan normal form, but in this case the action is not free. The complication for us is that we need to work by stages, reducing at each stage to a proper set of 'special' monads acted on by a proper subgroup (the stabiliser) of the original group, and it is easy to lose sight of the purpose of the reductions.

2.1 Triviality above the Infinity section. Let $E' = E(0,-1)|_{C_\infty}$. Since $E|_{C_\infty}$ is trivial, $E' \cong \mathcal{O}(-1)^N$ has no sections. We use this to get information about α_2^0 and β_1^0. Restricting the monad to C_∞ and twisting by $\mathcal{O}(-1)$ (equivalently, twisting by $\mathcal{O}(0,-1)$ first) we get

$$0 \to \mathcal{O}(-1)^k \xrightarrow[\alpha_2^0]{\alpha_1^0+\alpha_1^1\lambda} \begin{matrix}\mathcal{O}^k\\ \oplus\\ \mathcal{O}(-1)^{k+N}\end{matrix} \xrightarrow[\beta_2^0+\beta_2^1\lambda]{\beta_1^0} \mathcal{O}^k \to 0. \tag{2.2}$$

(Note: twisting doesn't effect α, β, so we won't rename them.) To any monad is associated a *display*,

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}(-1)^k \arrow[r, "\alpha"] \arrow[d, equal] & \ker\beta \arrow[r] \arrow[d] & E' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}(-1)^k \arrow[r, "\alpha"] & \mathcal{O}^k \oplus \mathcal{O}(-1)^{k+N} \arrow[r] \arrow[d, "\beta"] & \operatorname{coker}\alpha \arrow[r] \arrow[d, "\beta"] & 0 \\
 & & \mathcal{O}^k \arrow[r, equal] \arrow[d] & \mathcal{O}^k \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \tag{2.3}$$

which has exact rows and columns.

Since $E' \cong \mathcal{O}(-1)^N$, $H^0(E') = 0 = H^1(E')$. From the long exact sequences associated to the first row and last column, then, we see that $H^0(\ker\beta) = 0 = H^1(\ker\beta)$ and $H^0(\operatorname{coker}\alpha) \overset{\beta}{\cong} H^0(\mathcal{O}^k)$. Looking at the last column: since nonzero sections of $\mathcal{O}^k$ have no zeros, neither can sections of $\operatorname{coker}\alpha$. From the second row, we see that sections of $\operatorname{coker}\alpha$ come from sections of $\mathcal{O}^k \oplus \mathcal{O}(-1)^{k+N}$, which are parametrised by $(u, 0)$, $u \in \mathbb{C}^k$. If $(u, 0)$ represents a section of $\operatorname{coker}\alpha$, then it has a zero as a section of $\operatorname{coker}\alpha$ *iff* it is in the image of α. If this never happens, and given that α is injective, α_2 must be injective (*iff* α_2^0 has rank k). Meanwhile, the second column tells us that $H^0(\mathcal{O}^k \oplus \mathcal{O}(-1)^{k+N}) \overset{\beta}{\cong} H^0(\mathcal{O}^k)$, *i.e.* β_1^0 is invertible. We can use the action of the group to put the monad in the form

$$\alpha_2^0 = \begin{pmatrix} \mathbb{I}_{k\times k} \\ 0_{N\times k} \end{pmatrix} \quad \beta_1^0 = (\,\mathbb{I}\,),$$

which has as stabiliser, the subgroup with

$$D = \begin{pmatrix} A & D_1 \\ 0 & D_2 \end{pmatrix}, \quad E = B, \tag{2.4}$$

where $D_1 \in M_k^N$ and $D_2 \in \mathrm{GL}(N)$, and A, B and C are unrestricted as before.

2.5 $\beta \circ \alpha = 0$. We can break down the condition $\beta \circ \alpha = 0$ by considering coefficients of $\lambda^i\eta^j$ separately; we obtain

$$
\begin{aligned}
0 &= \beta_1^0\alpha_1^0 + \beta_2^0\alpha_2^0\\
0 &= \beta_1^0\alpha_1^1 + \beta_2^1\alpha_2^0\\
0 &= \beta_1^1\alpha_1^0 + \beta_2^0\alpha_2^1\\
0 &= \beta_1^1\alpha_1^1 + \beta_1^2\alpha_1^0 + \beta_2^0\alpha_2^2 + \beta_2^1\alpha_2^1\\
0 &= \beta_1^2\alpha_1^1 + \beta_1^3\alpha_1^0 + \beta_2^0\alpha_2^3 + \beta_2^1\alpha_2^2\\
0 &= \beta_1^3\alpha_1^1 + \beta_2^1\alpha_2^3.
\end{aligned}
$$

With the normalisations we have already made we see that

$$\beta_2^0\begin{pmatrix}\mathbb{I}\\ 0\end{pmatrix} = \alpha_1^0 \quad \beta_2^1\begin{pmatrix}\mathbb{I}\\ 0\end{pmatrix} = \alpha_1^1.$$

We have

PROPOSITION 2.6. *The bundle $E \to \widetilde{T\mathbb{P}^1}$ is trivial over C_∞ iff it has a monad representation of the form*

$$0 \to \mathcal{O}(-1)^k \xrightarrow[\begin{pmatrix}\mathbb{I}\\ 0\end{pmatrix}\eta+\alpha_2^1+\alpha_2^2\lambda+\alpha_2^3\lambda^2]{\beta_2^0\begin{pmatrix}\mathbb{I}\\ 0\end{pmatrix}+\beta_2^1\begin{pmatrix}\mathbb{I}\\ 0\end{pmatrix}\lambda_0} \begin{matrix}\mathcal{O}(-1)^k\\ \oplus\\ \mathcal{O}^{k+N}\end{matrix} \xrightarrow[\beta_2^0+\beta_2^1\lambda]{(\,\mathbb{I}\,)\eta+\beta_1^1+\beta_1^2\lambda+\beta_1^3\lambda^2} \mathcal{O}^k \to 0.$$

It is unique up to an isomorphism of the form (1.4) *restricted to the subgroup* (2.4).

2.7 Nonpolar fibres. The restriction of E to a fibre P_{λ_0}, for some $\lambda_0 \in \mathbb{C}^*$, is given by a monad over $P_{\lambda_0} \cong \mathbb{P}^1$:

$$0 \to \mathcal{O}(-1)^k \xrightarrow[\alpha_2(\lambda_0)]{\alpha_1^0+\alpha_1^1\lambda_0} \begin{matrix}\mathcal{O}(-1)^k\\ \oplus\\ \mathcal{O}^{k+N}\end{matrix} \xrightarrow[\beta_2^0+\beta_2^1\lambda_0]{\beta_1(\lambda_0)} \mathcal{O}^k \to 0.$$

Since E is trivial over nonpolar fibres, this monad has N nonvanishing sections. Using the same long exact sequences as in the last section, since $H^1(\mathcal{O}(-1)) = 0$, we see that these sections are isomorphic to sections of $\ker\beta$, which are naturally contained in the sections of $\mathcal{O}(-1)^k \oplus \mathcal{O}^{k+N}$. Since the first summand has no sections, we see that $H^0(\ker\beta) \cong \ker(\beta_2^0+\beta_2^1\lambda_0)$. In particular, we see that $\beta_2^0+\beta_2^1\lambda_0$

is surjective for $\lambda_0 \in \mathbb{C}^*$, otherwise $\ker\beta$ would have more than N sections. And $\alpha_1^0 + \alpha_1^1\lambda_0$ must also be surjective, otherwise (a) the injectivity of α, (b) $\beta \circ \alpha = 0$ and (c) $\dim\ker\beta_2(\lambda_0) = N$ would force some section of $\ker\beta/\operatorname{im}\alpha$ to have a zero, but sections of trivial bundles have no zeros. Since $\alpha_1(\lambda_0)$ is invertible and B acts on it by multiplication, we can use the action of B to set it equal to any invertible matrix. We take $\lambda_0 = -1$ and set

$$\alpha_1^0 - \alpha_1^1 = \mathbb{I} \quad \Longrightarrow \beta_2^0 - \beta_2^1 = (\mathbb{I},\ X),$$

where $X \in M_N^k$, resulting in a restriction to the subgroup with

$$B = A.$$

Since the remaining group acts on $\beta_2(\lambda_0)$ by

$$(\,\alpha_1(\lambda_0), \quad X\,) \mapsto (\,A\alpha_1(\lambda_0)A^{-1}, \quad A\alpha_1(\lambda_0)(-D_1)A^{-1} + AX\mathbb{I}\,),$$

we can make the normalisation $X = 0$, again for $\lambda_0 = -1$, resulting in the reduction to

$$D_1 = 0.$$

From the condition $(\beta_2^0 - \beta_2^1)\phi = 0$ on ϕ, the representation of the framing (1.9), we see that

$$\phi = \begin{pmatrix} 0 \\ Y \end{pmatrix}, \quad Y \in \mathrm{GL}(N).$$

We can use the action of D_2 to set $Y = \mathbb{I}$. This corresponds to the reduction from the group of bundle isomorphisms to the subgroup of isomorphisms preserving a fixed framing.

We could use the action of A to put α_1^0 into Jordan normal form, then $\alpha_1^1 = \alpha_1^0 - \mathbb{I}$ would be as well. Since $\alpha_1^0 + \alpha_1^1\lambda$ is invertible for $\lambda \in \mathbb{C}^*$, we see their only possible eigenvalues are 0 and 1. Assume without loss of generality that

$$\alpha_1^0 = \begin{pmatrix} \mathbb{I} + \gamma_0 & \\ & \gamma_1 \end{pmatrix}, \quad \alpha_1^1 = \begin{pmatrix} \gamma_0 & \\ & \mathbb{I} + \gamma_1 \end{pmatrix},$$

where γ_i is nilpotent and therefore $\mathbb{I}+\gamma_i$ is invertible. Let

$$\begin{aligned} k_0 &= \operatorname{rank}(\mathbb{I}+\gamma_0), \\ k_1 &= \operatorname{rank}(\mathbb{I}+\gamma_1). \end{aligned}$$

The stabiliser of this block form is $\mathrm{GL}(k_1)\oplus\mathrm{GL}(k_2)$. We can now use $\mathrm{GL}(k_2)$ to put $\mathbb{I}+\gamma_1$ into Jordan normal form and $\mathrm{GL}(k_1)$ to put $\mathbb{I}+\gamma_0$ into transpose Jordan normal form, *i.e.* γ_0 (γ_1) is zero except for possible 1's on the sub(super)diagonal. The reason for this choice will be apparent when we consider reality of the monad.

Since the stabiliser of a matrix in Jordan normal form is not trivial, this does not reduce the action of A completely, but to some subgroup

$$\left\{ A = \begin{pmatrix} F & \\ & G \end{pmatrix} \right\} < \mathrm{GL}(k_1)\oplus\mathrm{GL}(k_2).$$

In this thesis we will give special attention to the case when $\gamma_i = 0$, in which case the stabiliser is $\mathrm{GL}(k_1)\oplus\mathrm{GL}(k_2)$. We will call these monads of 'simplest type'.

We are left with the unrestricted action of the subgroup corresponding to C, which acts on α_2^1 by

$$\alpha_2^1 \mapsto \begin{pmatrix} F & & \\ & G & \\ & & \mathbb{I} \end{pmatrix} \alpha_2^1 \begin{pmatrix} F^{-1} & \\ & G^{-1} \end{pmatrix} + C_0 \begin{pmatrix} \mathbb{I}+\gamma_0 & \\ & \gamma_1 \end{pmatrix} \begin{pmatrix} F^{-1} & \\ & G^{-1} \end{pmatrix}.$$

Since γ_0 is invertible, we can use this action to put

$$\alpha_2^1 \text{ into the form } \begin{array}{cc} \begin{array}{cc} k_1 & k_2 \\ \leftrightarrow & \leftrightarrow \end{array} & \\ \begin{pmatrix} & \omega_1 \\ 0 & \omega_2 \\ & \theta_2 \end{pmatrix} & \begin{array}{cc} \updownarrow & k_1 \\ \updownarrow & k_2 \\ \updownarrow & N \end{array} \end{array}$$

which is fixed by the action of $C_0 = (0\ C_0')$, $C_0' \in M_{k_2}^{k+N}$. Similarly, C acts on α_2^3 and we can use C_1 to put

$$\alpha_2^3 \text{ into the form } \begin{pmatrix} \rho_1 & \\ \rho_2 & 0 \\ \theta_1 & \end{pmatrix},$$

which is stabilised by $C_1 = (C_1', 0)$, $C_1' \in M_{k_1}^{k+N}$. The subgroup with $C = (C_1', \lambda C_0')$ continues to act on α_2^2 via

$$\alpha_2^2 \mapsto \begin{pmatrix} F & & \\ & G & \\ & & \mathbb{I} \end{pmatrix} \alpha_2^2 \begin{pmatrix} F^{-1} & \\ & G^{-1} \end{pmatrix} + \left(C_1'(\mathbb{I}+\gamma_0)F^{-1},\, C_0'(\mathbb{I}+\gamma_1)G^{-1}\right)$$

which can be used to set

$$\alpha_2^2 \text{ to zero,}$$

which fixes the remaining action of C_0 and C_1. As the stabiliser of the normal form

$$\begin{aligned} \alpha_1 &= \begin{pmatrix} \mathbb{I}+(1+\lambda)\gamma_0 & 0 \\ 0 & \lambda\mathbb{I}+(\mathbb{I}+\lambda)\gamma_1 \end{pmatrix} & \beta_1 &= \mathbb{I}\eta + \beta_1^1 + \beta_1^2\lambda + \beta_1^3\lambda^2 \\ \alpha_2 &= \begin{pmatrix} \mathbb{I}\eta+\rho_1\lambda^2 & \omega_1 \\ \rho_2\lambda^2 & \mathbb{I}\eta+\omega_2 \\ \theta_1\lambda^2 & \theta_2 \end{pmatrix} & \beta_2 &= \begin{pmatrix} \alpha_1 & \zeta(1+\lambda) \end{pmatrix} \end{aligned} \tag{2.8}$$

we are left with $(A, B, C, D, E) =$

$$\left(\begin{pmatrix} F & \\ & G \end{pmatrix}, \begin{pmatrix} F & \\ & G \end{pmatrix}, (0), \begin{pmatrix} F & & \\ & G & \\ & & \mathbb{I} \end{pmatrix}, \begin{pmatrix} F & \\ & G \end{pmatrix} \right), \tag{2.9}$$

F in some subgroup of $\mathrm{GL}(k_1)$, G in some subgroup of $\mathrm{GL}(k_2)$.

PROPOSITION 2.10. *A bundle E given by the monad* (1.3) *can be put into the form* (2.8) *iff it is trivial on the section at infinity and the nonpolar fibres. This form is stabilised by the subgroup of monad isomorphisms given by* (2.9).

2.11 Monad Condition. Consider the restriction of E to the section C_0:

$$0 \to \mathcal{O}(-1)^k \xrightarrow[\begin{pmatrix} \rho_1\lambda^2 & \omega_1 \\ \rho_2\lambda^2 & \omega_2 \\ \theta_1\lambda^2 & \theta_2 \end{pmatrix}]{\begin{pmatrix} (\mathbb{I}+\gamma_0)+\gamma_0\lambda & \\ & \gamma_1+(\mathbb{I}+\gamma_1)\lambda \end{pmatrix}} \begin{matrix} \mathcal{O}(-1)^k \\ \oplus \\ \mathcal{O}^{k+N} \end{matrix}$$

$$\xrightarrow[\begin{pmatrix} -(\mathbb{I}+\gamma_0)-\gamma_0\lambda & & (1+\lambda)\zeta_1 \\ & -\cdot\; -(\mathbb{I}+\gamma_1)\lambda & (1+\lambda)\zeta_2 \end{pmatrix}]{(\beta_1^1+\beta_1^2\lambda+\beta_1^3\lambda^2)} \mathcal{O}^k \to 0.$$

From the fact $\beta \circ \alpha = 0$, we extract:

$$(\beta_1^1+\beta_1^2\lambda+\beta_1^3\lambda^2) \begin{pmatrix} (\mathbb{I}+\gamma_0)+\gamma_0\lambda & \\ & \gamma_1+(\mathbb{I}+\gamma_1)\lambda \end{pmatrix}$$
$$+ \begin{pmatrix} (\rho_1+\gamma_0\rho_1-\zeta_1\theta_1)\lambda^2+(\gamma_0\rho_1-\zeta_1\theta_1)\lambda^3 & (\omega_1+\gamma_0\omega_1-\zeta_1\theta_2)+(\gamma_0\omega_1-\zeta_1\theta_2)\lambda \\ (\gamma_1\rho_2-\zeta_2\theta_1)\lambda^2+(\rho_2+\gamma_1\rho_2-\zeta_2\theta_1)\lambda^3 & (\gamma_1\omega_2-\zeta_2\theta_2)+(\omega_2+\gamma_1\omega_2-\zeta_2\theta_2)\lambda \end{pmatrix} = 0.$$

Comparing coefficients of λ, we derive the following forms for

$$\beta_1^1 = \begin{pmatrix} 0 & -\omega_1 \\ 0 & \omega_2 \end{pmatrix}, \quad \beta_1^2 = 0, \quad \beta_1^3 = \begin{pmatrix} \rho_1 & 0 \\ -\rho_2 & 0 \end{pmatrix} \tag{2.12}$$

and the relations

$$\begin{aligned} \zeta_1\theta_1 &= [\gamma_0, \rho_1] \\ \zeta_2\theta_1 &= \rho_2 + \rho_2\gamma_0 + \gamma_1\rho_2 \\ \zeta_1\theta_2 &= \omega_1 + \gamma_0\omega_1 + \omega_1\gamma_1 \\ \zeta_2\theta_2 &= [\gamma_1, \omega_2]. \end{aligned} \tag{2.13}$$

To a bundle E, then, we have associated a monad

$$0 \to \mathcal{O}(-1,0)^k \xrightarrow[\begin{pmatrix} \mathbb{I}\eta + \rho_1\lambda^2 & \omega_1 \\ \rho_2\lambda^2 & \eta\mathbb{I} + \omega_2 \\ \theta_1\lambda^2 & \theta_2 \end{pmatrix}]{\begin{pmatrix} (\mathbb{I}+\gamma_0)+\gamma_0\lambda & \\ & \gamma_1 + (\mathbb{I}+\gamma_1)\lambda \end{pmatrix}} \begin{matrix} \mathcal{O}(-1,1)^k \\ \oplus \\ \mathcal{O}^{k+N} \end{matrix}$$

$$\xrightarrow[\begin{pmatrix} -(\mathbb{I}+\gamma_0)-\gamma_0\lambda & & (1+\lambda)\zeta_1 \\ & -\gamma_1-(\mathbb{I}+\gamma_1)\lambda & (1+\lambda)\zeta_2 \end{pmatrix}]{\begin{pmatrix} \mathbb{I}\eta + \rho_1\lambda^2 & -\omega_1 \\ -\rho_2\lambda^2 & \mathbb{I}\eta + \omega_2 \end{pmatrix}} \mathcal{O}(0,1)^k \to 0 \tag{2.14}$$

where $\gamma_i \in \mathrm{gl}(k_i)$ are nilpotent and in (transpose) Jordan normal form, $\rho_1, \omega_2 \in \mathrm{gl}(k_i)$, $\zeta_2, \zeta_1 \in M_N^{k_i}$ and $\theta_1, \theta_2 \in M_{k_i}^N$, and ρ_2, ω_1 are determined by γ, ζ, θ. This representation is unique up to the action of $F \in \mathrm{Stab}_{\gamma_0} \mathrm{GL}(k_1)$, $G \in \mathrm{Stab}_{\gamma_1} \mathrm{GL}(k_2)$ whose action on (2.14) is given by

$$\begin{gathered} \rho_1 \mapsto F\rho_1F^{-1}, \ \rho_2 \mapsto G\rho_2F^{-1}, \\ \omega_1 \mapsto F\omega_1G^{-1}, \ \omega_2 \mapsto G\omega_2G^{-1}, \\ \theta_1 \mapsto \theta_1F^{-1}, \ \theta_2 \mapsto \theta_2G^{-1}, \\ \zeta_1 \mapsto F\zeta_1, \ \zeta_2 \mapsto G\zeta_2. \end{gathered}$$

2.15 Jordan Normal Form. We now turn our attention to ω_2, θ_2, and ζ_2, which describe the bundle behaviour at the north pole ($\lambda = 0$). The behaviour near the south pole will similarly depend on $\rho_1, \theta_1, \zeta_1$, and in the real case it will mirror the behaviour at the other pole.

We will need the following

LEMMA. *Let*

$$J_j(a) = \begin{pmatrix} a+\eta & 1 & & \\ & a+\eta & \ddots & \\ & & \ddots & 1 \\ & & & a+\eta \end{pmatrix} \in \mathrm{gl}(j), \tag{2.16}$$

then

$$\mathrm{Stab}_{J_j(a)} = \left\{ \sum_{i=0}^{j-1} b_i N^i \in \mathrm{GL}(j) : b_i \in \mathbb{C} \right\} = \left\{ \begin{pmatrix} b_0 & b_1 & & b_{j-1} \\ & b_0 & \ddots & \vdots \\ & & \ddots & b_1 \\ & & & b_0 \end{pmatrix} \right\},$$

where

$$N = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}.$$

PROOF.

$$J_j(a) = a\mathbb{I} + N,$$

and $a\mathbb{I}$ is in the centre of $\mathrm{GL}(j)$, so

$$\mathrm{Stab}_{J_j(a)} = \mathrm{Stab}_N .$$

Let $X \in \mathrm{Stab}_N$, then XN (NX) is just X shifted to the right (up) with the first column (last row) replaced with zeros.

$$(XN - NX)_{lm} = X_{l-1,m} - X_{l,m-1},$$

so X is Toeplitz (has single valued (sub/super) diagonals).

$$0 = X_{0,m} = X_{1,m-1} = X_{2,m-2} = \cdots = X_{m-1,1}$$

for $m > 0$. So X is upper diagonal. If b_i is the value of the i^{th} super diagonal, $X = \sum b_i N^i$, as required. $\square$

If Y is made up of l similar Jordan blocks, then elements of the stabiliser will be blocks of the form

$$X = \begin{pmatrix} \sum b_i^{(11)} N^i & \sum b_i^{(12)} N^i & \dots \\ \sum b_i^{(21)} N^i & \ddots & \\ \vdots & & \end{pmatrix}.$$

LEMMA. *Since*

$$\det X = \left(\det \check{X}\right)^j,$$

where $\check{X}_{lm} = b_0^{(lm)}$, *$X$ is invertible iff $\check{X}$ is.*

PROOF. Examining the formula

$$\det X = \sum_{\nu \in S_{k_2}} \prod_{i=1}^{k_2} X_{\nu(i),i}$$

we see that the ν term vanishes unless

$$\nu(1), \nu(j+1), \ldots, \nu((l-1)j+1) \in \{1, j+1, \ldots, (l-1)j+1\}$$

because X has block upper-triangular form. A ν of this form corresponds to a vanishing term *iff* $\nu(i), \nu(j+i), \ldots, \nu((A-1)j+i) \in \{i, j+i, \ldots, (A-1)j+i\}$, for $i = 1, \ldots, j$. We see that ν corresponds to a vanishing term unless $\nu \in (S_A)^j$. Omitting the vanishing terms, we are left with

$$\det X = \sum_{\nu=(\nu_1,\ldots,\nu_j)\in(S_A)^j} \prod_{m=1}^{A} \prod_{i=1}^{j} b_0^{\nu_i(m)m} = \left(\det b_0^{nm}\right)^j. \quad \square$$

The stabiliser of a matrix in block Jordan normal form splits into $\check{X} \in \mathrm{GL}(l)$ and the remaining nilpotent part. After a possible reordering of blocks the stabiliser of a general matrix in Jordan normal form can be put into block diagonal form with blocks of the form X corresponding to the set of Jordan blocks with the same size and eigenvalue.

Applying this to Stab_{α_1}, we can use the $\mathrm{GL}(l)$ part of each block of Stab_{α_1} to put the corresponding submatrix of ω_2 into Jordan normal form. If $\gamma_1 = 0$, we can of course put all of ω_2 into Jordan normal form. As the stabiliser of this normal form, we are left with the subgroup of Stab_{α_1} given by restricting the $\mathrm{GL}(l)$ parts to be block Jordan stabilisers. The injectivity of $\alpha|_{\lambda=0,\,\eta=a}$ now implies that for each set of similar Jordan blocks with eigenvalue a of the submatrix of ω_2 the first columns of

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \theta_2 \end{pmatrix}$$

of each Jordan block must be linearly independent, and therefore represent a point of a Grassmannian. Choosing an appropriate representative, say the standard representatives of the Schubert cycles, we can use the $GL(l)$ action of the stabiliser to put those columns in that form, further reducing the stabiliser. Then we can use the nilpotent part of the embedded stabiliser to make the other columns under the Jordan blocks of γ_1 orthogonal to the first columns, using either the hermitian or Euclidean metric on $\mathbb{C}^{k+N}$.

Repeating this process, we are left with the columns of

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \theta_2 \end{pmatrix}$$

which start Jordan blocks of γ_1 in normal form, with stabiliser the nilpotent part of the stabiliser of γ_1. Since the resulting columns of

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \theta_2 \end{pmatrix}$$

which start Jordan blocks of γ_1 are linearly independent, we can use the action of this nilpotent part to make the remaining columns orthogonal to the first columns, resulting in a normal form stabilised by $\{\mathbb{I}\}$.

The same procedure can be applied to ρ_1, and θ_1, but with transpose Jordan normal forms for reasons which will be apparent.

Special Case ($\gamma_i = 0$). We will now give more explicit details of the above normalisation in the case when $\gamma_i = 0$, both to make the above clearer and to provide the coordinate descriptions which we will need in the next two chapters.

Since G acts on ω_2 by conjugation, we can put ω_2 into Jordan normal form. If we agree to a lexicographical ordering of $\mathbb{C}$, we can fix the order of the Jordan blocks up to a permutation of blocks with the same rank and eigenvalue. Unless ω_2 is diagonalisable with distinct eigenvalues, its Jordan form has a nonzero stabiliser, which continues to act on θ_2 and ζ_2. Once we divide ω_2 into groups of Jordan blocks, each group with a distinct eigenvalue and rank, we get a corresponding decomposition of $\mathbb{C}^{k_2}$ into subspaces. They are the invariant subspaces of the stabiliser (under the standard action of $\mathrm{GL}(k_2)$ on $\mathbb{C}^{k_2}$), so we may consider them one at a time.

Assume for the moment that ω_2 has A Jordan blocks of one type, *i.e.*

$$\omega_2 = \begin{pmatrix} J_j(a) & & & \\ & J_j(a) & & \\ & & \ddots & \\ & & & J_j(a) \end{pmatrix}.$$

The injectivity of α imposes an independence condition on θ. At $\lambda = 0, \eta = -a$, $\mathbb{I}\eta + \omega_2$ is singular, in particular the first, $j+1$st, $2j+1$st $\ldots$ columns are zero, so

$$\begin{aligned} &\alpha|_{\lambda=0,\eta=-1}(b_1 e_{k_1+1} + b_2 e_{k_1+j+1} + \cdots + b_A e_{k_1+j(A-1)+1}) \\ &\quad = \left(\begin{matrix} \zeta_1(b_1(\theta_2)_1 + b_2(\theta_2)_{j+1} + \ldots) \\ \begin{pmatrix} 0 \\ b_1(\theta_2)_1 + b_2(\theta_2)_{j+1} + \ldots \end{pmatrix} \end{matrix} \right) \begin{matrix} \updownarrow k \\ \updownarrow k \\ \updownarrow N \end{matrix} \in \left. \begin{matrix} \mathcal{O}(-1,1)^k \\ \oplus \\ \mathcal{O}^{k+N} \end{matrix} \right|_{\substack{\lambda=0 \\ \eta=-a}} . \end{aligned}$$

Since α is injective, we see that $(\theta_2)_1 \wedge (\theta_2)_{j+1} \wedge \cdots \wedge (\theta_2)_{(A-1)j+1} \neq 0$. Similarly, the surjectivity of β at the same point implies that $(\zeta_2^t)^1 \wedge (\zeta_2^t)^{j+1} \wedge \cdots \wedge (\zeta_2^t)^{(A-1)j+1} \neq 0$, where $\zeta = (\zeta_1,\ \zeta_2)^t$.

An element $X \in \mathrm{Stab}_{\omega_2}$ (as computed in the Lemma) acts on

$$\begin{gathered} ((\theta_2)_1, (\theta_2)_{j+1}, \ldots, (\theta_2)_{j(l-1)+1}) \in M_l^N, \text{ and} \\ ((\zeta_2^t)_{j-1}, (\zeta_2^t)_{2j-1}, \ldots, (\zeta_2^t)_{jl-1})^t \in M_N^l \end{gathered}$$

via the standard action of $b_0^{lm} \in \mathrm{GL}(A)$, *i.e.* by taking linear recombinations. So we should think of θ_2 and ζ_2 as defining points on a Grassmannian:

$$M_A^N / \mathrm{GL}(A) \cong \mathrm{Gr}_{A,N}.$$

If ω_2 has Jordan blocks of different rank but the same eigenvalue, then we get an element of an enlarged flag manifold (actually the space of linearly independent subspaces with prescribed ranks).

In any case, within each invariant block, we can use the action of b_0^{lm} to put

$$((\zeta_2^t)_1, (\zeta_2^t)_{j+1}, \ldots, (\zeta_2^t)_{j(A-1)+1})$$

into some normal form, say the one which gives coordinates on the Schubert cycles [GriHa,1.5]. This reduces the stabiliser to the subgroup with $b_0^{lm} = \mathbb{I}$.

There are two 'standard' metrics on $\mathbb{C}^N$ each of which can be used to put ζ_2 into a normal form. If we put the hermitian metric on $\mathbb{C}^N$, we can use the remaining action of the stabiliser to put

$$\{(\zeta_2^t)_2, (\zeta_2^t)_3, \dots, (\zeta_2^t)_j, (\zeta_2^t)_{j+2}, \dots\} \subset \{(\zeta_2^t)_1, (\zeta_2^t)_{j+1}, \dots, (\zeta_2^t)_{j(A-1)+1}\}^{\perp_{\text{hermitian}}}.$$

This shows that the data ζ_2 describe a point in the orthogonal bundle to the universal bundle contained in $\underline{\mathbb{C}}^N_{Gr_{A,N}}$.

Alternatively, we can put the holomorphic or Euclidean metric on $\mathbb{C}^N$. In this case we have to worry about null vectors, so the same procedure doesn't work. Instead we have to consider the usual coordinate patches of $\mathrm{Gr}_{A,N}$. If $((\zeta_2^t), (\zeta_2^t), \dots)^t$ is in Schubert cycle form, *i.e.*

$$\begin{array}{c}
\begin{array}{cccccccccccccc}
 & & & i_1 & & & & i_2 & & & & i_3 & & \dots \\
 & & & \downarrow & & & & \downarrow & & & & \downarrow & &
\end{array} \\
\begin{pmatrix}
0 & \dots & 0 & 1 & * & \dots & * & 0 & * & \dots & * & 0 & * & \dots \\
0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 & * & \dots & * & 0 & * & \dots \\
0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 & * & \dots \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\end{array}$$

Then we can put

$$\{(\zeta_2^t)_2, (\zeta_2^t)_3, \dots, (\zeta_2^t)_j, (\zeta_2^t)_{j+2}, \dots\} \subset \{e_{i_1}, e_{i_2}, \dots, e_{i_a}\}^{\perp_{\text{holomorphic}}}. \tag{2.17}$$

Apply this procedure to each invariant subspace of Stab_{ω_2}, and we are left with a normalised monad which uniquely represents the bundle E, *i.e.* its stabiliser in the group of monad isomorphisms is the trivial subgroup. We can apply the same procedure to ρ_1 using the action of F, but for real reasons, it will be better to put it into an equivalent normalisation whose form will be dictated in the next section.

PROPOSITION 2.18. *Bundles $E \to \widetilde{T\mathbb{P}}^1$ trivial on real sections, the section at infinity and nonpolar fibres, with a fixed framing and a lift, $\tilde{\delta}_t$, of time translation are uniquely represented by monads* (2.14) *in Jordan normal form as described above.*

PROOF. We have shown that these bundle properties determine the monad normal form, it remains to show that the monad form completely determines the lift of time translation: Time translation is a one parameter group of monad isomorphisms mapping the pulled-back monad into this normal form. Since $\alpha_1, \beta_1^0, \beta_2, \alpha_2^0$ are pulled back to themselves, we see the group is restricted to

$$A_t = B_t = E_t = D_t = \mathbb{I}.$$

And since α_2^1 and α_2^3 are pulled back to themselves,

$$C_{0,t}\alpha_1^0 = 0 \iff C_{0,t}\alpha_1^1 = C_{0,t}$$
$$C_{1,t}\alpha_1^1 = 0 \iff C_{1,t}\alpha_1^0 = C_{1,t}.$$

Finally, $(\delta_t^*\alpha)_2^2 = \alpha_2^2 + t\alpha_2^0$ implies $(C_{1,t}\alpha_1^0 + C_{0,t}\alpha_1^1)A_t^{-1} = t\alpha_2^0$ implies

$$C_{1,t} = \begin{pmatrix} t(\mathbb{I}+\gamma_0)^{-1} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad C_{0,t} = \begin{pmatrix} 0 & 0 \\ 0 & t(\mathbb{I}+\gamma_1)^{-1} \\ 0 & 0 \end{pmatrix}. \quad \square$$

So, in this normal form, time translation is completely determined, as we would expect since there is only one isomorphism of E preserving the framing.

2.19 Triviality over the real sections (part I). While we encoded triviality on generic sections into the monad, we did not explicitly encode triviality above real sections. In this section we will uncover the condition for a monad to represent a bundle trivial over the real sections, and in section (2.28) we will show that when $\gamma_i = 0$ this condition is implied by the other monad conditions.

The basic idea is that a bundle over $\mathbb{P}^1$ with $c_1 = 0$ is trivial *iff* it has no sections after twisting with $\mathcal{O}(-1)$. Time invariance of the bundle means we only need to consider real sections with $t = 0$. Restricting the monad to such a real section, and twisting by $\mathcal{O}(-1)$ gives

$$0 \to \mathcal{O}(-3)^k \xrightarrow[\alpha_2]{\alpha_1} \begin{matrix} \mathcal{O}(-2)^k \\ \oplus \\ \mathcal{O}(-1)^{k+N} \end{matrix} \xrightarrow[\beta_2]{\beta_1} \mathcal{O}^k \to 0.$$

We can derive the long exact sequences

$$0 \to H^0(\ker\beta) \to H^0(\mathcal{O}(-2)^k \oplus \mathcal{O}(-1)^{k+N}) = 0 \to H^0(\mathcal{O}^k) \to H^1(\ker\beta) \to \dots$$

which implies $H^0(\ker\beta)=0$, and

$$0=H^0(\ker\beta)\to H^0(E(-1))\xrightarrow{\delta^*}H^1(\mathcal{O}(-3)^k)\xrightarrow{\alpha}H^1(\ker\beta)\to\dots$$

which implies $H^0(E(-1))=0$ *iff* $\alpha: H^1(\mathcal{O}(-3)^k)\to H^1(\ker\beta)$ is injective.

For the space $H^1(\mathcal{O}(-3)^k)$ we will take

$$\left\{\begin{pmatrix}(a_1'-(\mathbb{I}+\gamma_0)^{-1}\gamma_0 a_2')\lambda^{-1}+a_2'\lambda^{-2}\\ a_1''\lambda^{-1}+(a_2''-(\mathbb{I}+\gamma_1)^{-1}\gamma_1 a_1'')\lambda^{-2}\end{pmatrix}: a_i=\begin{pmatrix}a_i'\\ a_i''\end{pmatrix}\in\mathbb{C}^k\right\}$$

as representative cocycles. In terms of power series, $C^1(\ker\beta)$ is the set of Laurent power series of (k+k+N) vectors which converge on $\lambda\in\mathbb{C}^*$, lying pointwise in the kernel of β. To get H^1 we have to factor out by series convergent on $\lambda\in\mathbb{C}$ (positive power series) and $\lambda\in\mathbb{C}^*\cup\infty$ (negative power series), *i.e.* taking into account the twisting of the vector bundles, by the sets

$$\left\{\begin{pmatrix}\sum_{i\geq 0}\begin{pmatrix}c\lambda^i\\ d\lambda^i\end{pmatrix}\\ \sum_{i\geq 0}\begin{pmatrix}e\lambda^i\\ f\lambda^i\\ g\lambda^i\end{pmatrix}\end{pmatrix}\in\ker\beta\right\},\text{ and}$$
$$\left\{\begin{pmatrix}\sum_{i\leq -2}\begin{pmatrix}c\lambda^i\\ d\lambda^i\end{pmatrix}\\ \sum_{i\leq -1}\begin{pmatrix}e\lambda^i\\ f\lambda^i\\ g\lambda^i\end{pmatrix}\end{pmatrix}\in\ker\beta\right\}. \tag{2.20}$$

We see immediately that any coefficients of λ^{-1} in the first (c,d) component persist under this quotienting, but the other coefficients may or may not, their fate being tangled in the structure of $\ker\beta$, *i.e.* after composing with $\ker\beta\hookrightarrow\mathcal{O}(-2)^k\oplus\mathcal{O}(-1)^{k+N}$, the images $\beta(c_1\lambda^{-1},d_1\lambda^{-2})\in H^1(\mathcal{O}(-2)^k\oplus\mathcal{O}(-1)^{k+N})$ are nonzero, while the other images are zero meaning that their images in $C^1(\mathcal{O}(-2)^k\oplus\mathcal{O}(-1)^{k+N})$ split, so we still have to determine whether they split in $\ker\beta$. Since their images in $C^1(\ker\beta)$ are linearly independent of the images of cocycles of the form $\begin{pmatrix}-(\mathbb{I}+\gamma_0)^{-1}\gamma_0 a_2'\lambda^{-1}+a_2'\lambda^{-2}\\ a_1''\lambda^{-1}-(\mathbb{I}+\gamma_1)^{-1}\gamma_1 a_1''\lambda^{-2}\end{pmatrix}$ (whose c and d coefficients of $\lambda^{-1},\lambda^{-2}$ respectively are zero) the kernel of α is contained in the span of these second cocycles. The image of these cocycles under α is

$$\alpha\begin{pmatrix} -(\mathbb{I}+\gamma_0)^{-1}\gamma_0 a_2'\lambda^{-1} + a_2'\lambda^{-2} \\ a_1''\lambda^{-1} - (\mathbb{I}+\gamma_1)^{-1}\gamma_1 a_1''\lambda^{-2} \end{pmatrix}$$
$$= \begin{pmatrix} \begin{pmatrix} (\lambda^{-2}(\mathbb{I}+\gamma_0) - \gamma_0(\mathbb{I}+\gamma_0)^{-1}\gamma_0)a_2' \\ (-\lambda^{-2}\gamma_1(\mathbb{I}+\gamma_1)^{-1}\gamma_1 + (\mathbb{I}+\gamma_1))a_1'' \end{pmatrix} \\ (W_0 a_2' + W_1 a_1'') \end{pmatrix}, \quad (2.21)$$

where

$$W_0 = \begin{pmatrix} \lambda^{-2}\frac{z}{2} - \lambda^{-1}\frac{z}{2}(\mathbb{I}+\gamma_0)^{-1}\gamma_0 + (\rho_1 - \frac{\bar{z}}{2}) - \lambda(\rho_1 - \frac{\bar{z}}{2})(\mathbb{I}+\gamma_0)^{-1}\gamma_0 \\ \rho_2 - \lambda\rho_2(\mathbb{I}+\gamma_0)^{-1}\gamma_0 \\ \theta_1 - \lambda\theta_1(\mathbb{I}+\gamma_0)^{-1}\gamma_0 \end{pmatrix},$$
$$W_1 = \begin{pmatrix} -\lambda^{-2}\omega_1(\mathbb{I}+\gamma_1)^{-1}\gamma_1 + \lambda^{-1}\omega_1 \\ -\lambda^{-2}(\omega_2 + \frac{z}{2})(\mathbb{I}+\gamma_1)^{-1}\gamma_1 + \lambda^{-1}(\omega_2 + \frac{z}{2}) + \frac{\bar{z}}{2}(\mathbb{I}+\gamma_1)^{-1}\gamma_1 - \frac{\bar{z}}{2} \\ -\lambda^{-2}\theta_2(\mathbb{I}+\gamma_1)^{-1}\gamma_1 + \lambda^{-1}\theta_2 \end{pmatrix}.$$

The kernel of this map on H^1 is the inverse image of the coboundaries $d(C^0(\ker\beta))$. So our cocycle is mapped to zero *iff* its α-image is of the form $X + \hat{X}$, where X and $\hat{X}$ are in the sets (2.20) respectively. As a result of twisting by $\mathcal{O}(-1)$, these two sets are independent so there is no ambiguity as to the splitting:

$$X = \begin{pmatrix} \begin{pmatrix} -\gamma_0(\mathbb{I}+\gamma_0)^{-1}\gamma_0 a_2' \\ (\mathbb{I}+\gamma_1)a_1'' \end{pmatrix} \\ \begin{pmatrix} (\rho_1 - \bar{z}/2)\chi_0 a_2' \\ \rho_2\chi_0 a_2' - \lambda\frac{\bar{z}}{2}\chi_1 a_1'' \\ \theta_1\chi_0 a_2' \end{pmatrix} \end{pmatrix}, \quad \hat{X} = \begin{pmatrix} \begin{pmatrix} (\mathbb{I}+\gamma_0)a_2'\lambda^{-2} \\ -\gamma_1(\mathbb{I}+\gamma_1)^{-1}\gamma_1 a_1''\lambda^{-2} \end{pmatrix} \\ \begin{pmatrix} \lambda^{-2}\frac{z}{2}\chi_0 a_2' + \lambda^{-1}\omega_1\chi_1 a_1'' \\ \lambda^{-1}(\omega_2 + z/2)\chi_1 a_1'' \\ \lambda^{-1}\theta_2\chi_1 a_1'' \end{pmatrix} \end{pmatrix},$$

where $\chi_0 = \mathbb{I} - \lambda(\mathbb{I}+\gamma_0)^{-1}\gamma_0$ and $\chi_1 = \mathbb{I} - \lambda^{-1}(\mathbb{I}+\gamma_1)^{-1}\gamma_1$.

The existence of this splitting tells us that (2.21) represents the zero class in $H^1(\mathcal{O}(-2)^k \oplus \mathcal{O}(-1)^{k+N})$. It is zero in $H^1(\ker\beta)$ *iff* $0 = \beta(X) = -\beta(\hat{X})$. We find that α will be injective *iff*

$$\beta(\hat{X}) = \begin{pmatrix} \omega_1(\mathbb{I}+\gamma_1)a_1'' + ((\rho_1 - \bar{z}/2)(\mathbb{I}+\gamma_0) + \frac{z}{2}\gamma_0(\mathbb{I}+\gamma_0)^{-1}\gamma_0)a_2' \\ -((\omega_2 + z/2)(\mathbb{I}+\gamma_1) - \frac{\bar{z}}{2}\gamma_1(\mathbb{I}+\gamma_1)^{-1}\gamma_1)a_1'' - \rho_2\gamma_0 a_2' \end{pmatrix} \neq 0$$

for all $z, \in \mathbb{C}$, equivalently, *iff*

$$\det\begin{pmatrix} (\omega_2 + z/2) - \frac{\bar{z}}{2}((\mathbb{I}+\gamma_1)^{-1}\gamma_1)^2 & \omega_1 \\ \rho_2 & (\rho_1 - \bar{z}/2) + \frac{z}{2}((\mathbb{I}+\gamma_0)^{-1}\gamma_0)^2 \end{pmatrix} \neq 0 \quad (2.22)$$

for all $z \in \mathbb{C}$. Since we haven't used the specific form of the Jordan normalisation, we have

PROPOSITION 2.23. *A monad in the form* (2.14) *represents a bundle which is trivial over real sections iff the real determinant* (2.22) *is nonvanishing as a function of* $z \in \mathbb{C}$.

REMARK. If $\gamma_i = 0$, the condition (2.22) is equivalent to the condition

$$\beta|_{\lambda,\eta=z/2} \circ \alpha|_{\lambda=\infty,\hat{\eta}=-\bar{z}/2} \neq 0 \quad \text{for all } z \in \mathbb{C},$$

which is certainly necessary for $E|_{C_{(z,\bar{z},0)}}$ to be trivial. The analogous condition for monads describing bundles on $\mathbb{P}^N$ is easily seen to be sufficient as well. This is not as easy in our case because of the extra twist in the monad.

2.24 Reality. We will work with the real structure given as a map

$$\sigma^* E^{\mathrm{op}} \to E^*,$$

where op indicates the opposite (conjugate) complex structure. Since E^* satisfies the same triviality and additional properties as E, we can represent E^* as a monad $\tilde{M}$ with all the normalisations we made on E. We can also dualise the monad for E by taking the dual of the exact sequences in the display which reverses the direction of the maps and transforms $\mathcal{O}(p,q)$ into $\mathcal{O}(-p,-q)$, its dual. This results in a monad for E^* which is unfortunately not easy to work with. For example, one may check that it doesn't satisfy the conditions required for monad isomorphisms to be in bijection with bundle isomorphisms.

We would prefer to work with a monad $\tilde{M}$ of the same form as M because then finding a bundle map is equivalent to finding a monad map, but we will need to use the dual monad M^* to determine $\tilde{M}$:

PROPOSITION 2.25. *If E is given by a monad M in the form* (2.14) *then its dual*

E^* is given by a monad $\tilde{M}$:

$$\mathcal{O}(-1,0)^k \xrightarrow[\begin{pmatrix} \mathbb{I}\eta + \rho_1^t\lambda^2 & -\rho_2^t \\ -\omega_1^t\lambda^2 & \eta\mathbb{I} + \omega_2^t \\ -\zeta_1^t\lambda^2 & -\zeta_2^t \end{pmatrix}]{\begin{pmatrix} (\mathbb{I}+\gamma_0^t)+\gamma_0^t\lambda & 0 \\ 0 & \gamma_1^t + (\mathbb{I}+\gamma_1^t)\lambda \end{pmatrix}} \begin{matrix} \mathcal{O}(-1,1)^k \\ \oplus \\ \mathcal{O}^{k+N} \end{matrix}$$

$$\xrightarrow[\begin{pmatrix} -(\mathbb{I}+\gamma_0^t)-\gamma_0^t\lambda & & (1+\lambda)\theta_1^t \\ & -\gamma_1^t + (\mathbb{I}+\gamma_1^t)\lambda & (1+\lambda)\theta_2^t \end{pmatrix}]{\begin{pmatrix} \mathbb{I}\eta + \rho_1^t\lambda^2 & +\rho_2^t \\ \omega_1^t\lambda^2 & \mathbb{I}\eta + \omega_2^t \end{pmatrix}} \mathcal{O}(0,1)^k.$$

PROOF. The bundle E is given by a monad M to which is associated a display. By dualising the display (taking the duals of spaces and the transpose of maps) we see that E^* is given by a dual monad, M^*. We will construct a singular map of monads $\tilde{M} \to M^*$ which induces a singular map of bundles. One may check that monads of the form M^* don't have the nice properties with respect to maps that our usual monads have, *i.e.* the obstruction groups to the existence of monad representatives for bundle maps, *etc.* (1.6) fail to vanish. As a result, we must guess bundle maps and prove that they have the desired properties directly.

Consider the map of monads $f = (f_1, f_2, f_3) : \tilde{M} \to M^*$:

$$\begin{tikzcd}[column sep=huge]
\mathcal{O}(-1,0)^k \arrow[r, "W", "\begin{pmatrix} \mathbb{I}\eta+\rho_1^t\lambda^2 & -\rho_2^t \\ -\omega_1^t\lambda^2 & \eta\mathbb{I}+\omega_2^t \\ -\zeta_1^t\lambda^2 & -\zeta_2^t \end{pmatrix}"'] \arrow[d, "f_1"'] & \begin{matrix} \mathcal{O}(-1,1)^k \\ \oplus \\ \mathcal{O}^{k+N} \end{matrix} \arrow[r, "\begin{pmatrix} \mathbb{I}\eta+\rho_1^t\lambda^2 & \rho_2^t \\ \omega_1^t\lambda^2 & \mathbb{I}\eta+\omega_2^t \end{pmatrix}", "\begin{pmatrix} -W & \begin{matrix} (1+\lambda)\theta_1^t \\ (1+\lambda)\theta_2^t \end{matrix} \end{pmatrix}"'] \arrow[d, "f_2"] & \mathcal{O}(0,1)^k \arrow[d, "f_3"] \\
\mathcal{O}(0,-1)^k \arrow[r, "\begin{pmatrix} \mathbb{I}\eta+\rho_1^t\lambda^2 & -\rho_2^t \\ -\omega_1^t\lambda^2 & \mathbb{I}\eta+\omega_2^t \end{pmatrix}", "\begin{pmatrix} \multicolumn{2}{c}{-W} \\ (1+\lambda)\zeta_1^t & (1+\lambda)\zeta_2^t \end{pmatrix}"'] & \begin{matrix} \mathcal{O}(1,-1)^k \\ \oplus \\ \mathcal{O}^{k+N} \end{matrix} \arrow[r, "W", "\begin{pmatrix} \mathbb{I}\eta+\rho_1^t\lambda^2 & \rho_2^t & \theta_1^t\lambda^2 \\ \omega_1^t\lambda^2 & \eta\mathbb{I}+\omega_2^t & \theta_2^t \end{pmatrix}"'] & \mathcal{O}(1,0)^k
\end{tikzcd}$$

where

$$f_1 = -\begin{pmatrix} \lambda^2\mathbb{I} & \\ & \mathbb{I} \end{pmatrix}$$

$$f_2 = \begin{pmatrix} (0) & -\begin{pmatrix} \lambda^2\mathbb{I} & & 0 \\ & \mathbb{I} & 0 \end{pmatrix} \\ \begin{pmatrix} \lambda^2\mathbb{I} & \\ & \mathbb{I} \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (1+\lambda)\mathbb{I} \end{pmatrix} \end{pmatrix} : \begin{matrix} \mathcal{O}(-1,1)^k \\ \oplus \\ \mathcal{O}^{k+N} \end{matrix} \rightarrow \begin{matrix} \mathcal{O}(1,-1)^k \\ \oplus \\ \mathcal{O}^{k+N} \end{matrix}$$

$$f_3 = \begin{pmatrix} \lambda^2\mathbb{I} & \\ & \mathbb{I} \end{pmatrix}$$

$$W = \begin{pmatrix} (\mathbb{I}+\gamma_0^t)+\gamma_0^t\lambda & 0 \\ 0 & \gamma_1^t+(\mathbb{I}+\gamma_1^t)\lambda \end{pmatrix}.$$

This is a map of monads with singularities, but away from the bad points, $\{\lambda \in \{0,-1,\infty\}\}$, it is a valid monad map and gives an isomorphism

$$\tilde{E}|_{\widetilde{T\mathbb{P}^1}\setminus\{\lambda\in\{0,-1,\infty\}\}} \overset{f}{\cong} E^*|_{\widetilde{T\mathbb{P}^1}\setminus\{\lambda\in\{0,-1,\infty\}\}}.$$

We will show that it has a pole and zero along P_∞ and P_{-1} respectively, and hence represents a bundle map which is $(1+\lambda)$ times a bundle isomorphism, which implies our proposition.

Over nonpolar fibres, the first monad has natural representatives for a trivialisation, namely

$$\begin{pmatrix} (0) \\ \begin{pmatrix} (1+\lambda)(\mathbb{I}+(1+\lambda)\gamma_0^t)^{-1}\theta_1^t a \\ (1+\lambda)(\mathbb{I}+(1+\lambda)\gamma_1^t)^{-1}\theta_2^t a \\ a \end{pmatrix} \end{pmatrix} \quad a \in \mathbb{C}^N.$$

These sections are mapped to sections of the second monad which give equally natural trivialisations away from P_{-1} where they have a zero.

Over sections $C_{\eta=z} \setminus P_\infty$, where $-z/2$ is not an eigenvalue of ω_2, we can find equally good sections trivialising E^t:

$$\begin{pmatrix} \begin{pmatrix} 0 \\ -(1+\lambda)(\omega_2^t+z)^{-1}\theta_2^t a \end{pmatrix} \\ \begin{pmatrix} (1+\lambda)(\mathbb{I}+(1+\lambda)\gamma_1^t)^{-1}(\theta_1^t - \rho_2^t(\omega_2^t+z)^{-1}\theta_2^t)a \\ 0 \\ a \end{pmatrix} \end{pmatrix} \quad a \in \mathbb{C}^N.$$

These sections are mapped by f to sections

$$\begin{pmatrix} \begin{pmatrix} \lambda^2(1+\lambda)(\mathbb{I}+(1+\lambda)\gamma_1^t)^{-1}(\theta_1^t - \rho_2^t(\omega_2^t+z)^{-1}\theta_2^t)a \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ -(1+\lambda)(\omega_2^t+z)^{-1}\theta_2^t aa \end{pmatrix} \end{pmatrix} \quad a \in \mathbb{C}^N$$

which have no poles on $\lambda \in \mathbb{C}$ but have a zero at $\lambda = -1$. Since the set of eigenvalues of ω_2 is closed in $\mathbb{C}$, we can put these two results together and see that f is holomorphic on a neighbourhood of P_0 minus a set of codimension two. In this way, Hartog's Theorem then implies that f is holomorphic on $\widetilde{T\mathbb{P}}^1 \setminus P_\infty$, with a simple pole at P_{-1}.

Similarly, we can find sections on a dense set of sections $C_{\eta=z\lambda^2}$ which have no zeros or poles on $\hat{\lambda} \in \mathbb{C}$ but whose images under f have a zero at $\hat{\lambda} = -1$ and a pole at $\hat{\lambda} = 0$. It follows that f has a pole and zero at P_∞ and P_{-1} respectively, so $f/(1+\lambda)$ is the required bundle isomorphism. $\square$

By the correspondence theorem for monads and bundles, the real structure is then given by a monad map of $\sigma^* M^{\mathrm{op}}$ to M^*. (We use op to indicate the opposite, *i.e.* conjugate, complex structure.) Since the opposite complex structure is given by taking the conjugates of frames and of transition matrices (*i.e.* the monad construction is natural with respect to conjugation) it is given by the conjugate monad.

To determine the normalised monad for $\sigma^* E^{\mathrm{op}}$ (*i.e.* with the opposite complex structure), we pullback by σ the normalised monad M and conjugate everything, as computed in (1.8):

$$0 \to \mathcal{O}(-1,0)^k \xrightarrow[\begin{pmatrix} -\mathbb{I}\eta+\bar{\rho}_1 & \bar{\omega}_1\lambda^2 \\ \bar{\rho}_2 & -\eta\mathbb{I}+\bar{\omega}_2\lambda^2 \\ \bar{\theta}_1 & \bar{\theta}_2\lambda^2 \end{pmatrix}]{\begin{pmatrix} (\mathbb{I}+\bar{\gamma}_0)\lambda+\bar{\gamma}_0 & \\ & \bar{\gamma}_1\lambda+(\mathbb{I}+\bar{\gamma}_1) \end{pmatrix}} \begin{matrix} \mathcal{O}(-1,1)^k \\ \oplus \\ \mathcal{O}^{k+N} \end{matrix}$$

$$\xrightarrow[\begin{pmatrix} -(\mathbb{I}+\bar{\gamma}_0)\lambda+\bar{\gamma}_0 & & (1+\lambda)\bar{\zeta}_1 \\ & -\bar{\gamma}_1\lambda+(\mathbb{I}+\bar{\gamma}_1) & (1+\lambda)\bar{\zeta}_2 \end{pmatrix}]{\begin{pmatrix} -\mathbb{I}\eta+\bar{\rho}_1 & -\bar{\omega}_1\lambda^2 \\ -\bar{\rho}_2 & -\mathbb{I}\eta+\bar{\omega}_2\lambda^2 \end{pmatrix}} \mathcal{O}(0,1)^k \to 0 \qquad (2.26)$$

which one sees disturbs the normal form (2.14). For $\sigma^* M^{\mathrm{op}}$ to be isomorphic to $\tilde{M}$ we must be able to put $\sigma^* M^{\mathrm{op}}$ into the normal form (2.14) at which point if ρ_1, ω_2, θ_1 and θ_2 are all in 'Jordan normal form', they must be equal as a result of the uniqueness of the monad representative, *i.e.* Proposition 2.18. In particular, this is only possible if

$$k_1 = k_2 = k/2.$$

Given this, the group element

$$\begin{pmatrix} & -\mathbb{I}_{k/2} \\ -\mathbb{I}_{k/2} & \end{pmatrix}, \begin{pmatrix} & -\mathbb{I}_{k/2} \\ -\mathbb{I}_{k/2} & \end{pmatrix}, (0), \begin{pmatrix} & & \mathbb{I}_{k/2} \\ \mathbb{I}_{k/2} & & \\ & \mathbb{I}_N & \end{pmatrix}, \begin{pmatrix} & \mathbb{I}_{k/2} \\ \mathbb{I}_{k/2} & \end{pmatrix}$$

puts (2.26) into normal form. The result is that

$$\gamma_0 = \gamma_1^*, \quad \rho_1 = -\omega_2^*, \quad \rho_2 = \rho_2^*, \quad \omega_1 = \omega_1^*, \quad \zeta_2 = -\theta_1^*, \quad \theta_2 = -\zeta_1^*. \tag{2.27}$$

(Use the reality of θ and ζ and the reality of (2.13) to see the second and third conditions.) As a result, if γ_1, θ_2, ζ_2, ω_2 are in 'Jordan normal form' γ_0, θ_1, ζ_1, ρ_1 must be in a conjugate normal form.

2.28 Triviality over the real sections (part II). As promised, we will now prove

LEMMA 2.29. *The determinant condition* (2.22) *is implied by the other monad conditions when* $\gamma_1 = 0$.

PROOF. Imposing reality (*i.e.* (2.27)) and $\gamma_1 = 0$, condition (2.22) becomes

$$\det \begin{pmatrix} \omega_2 + z/2 & -\zeta_2\zeta_2^* \\ -\theta_2^*\theta_2 & -(\omega_2 + z/2)^* \end{pmatrix} \neq 0 \text{ for all } z \in \mathbb{C}. \tag{2.30}$$

We proceed in two steps. First we show that the determinant can be decomposed into a sum of subdeterminants each of which is real and nonnegative. When $-2z$ is not an eigenvalue of ω_2, the sum has an obvious nonzero term. In the second step, we find a nonnegative term in the expression for the determinant in the case that $-2z$ is an eigenvalue. In this step we will need to use the injectivity of α and surjectivity of β.

To calculate the determinant we will use the fact that the determinant of a product is the product of the determinants. Specifically, we will multiply the matrix (2.30) by

$$\begin{pmatrix} P^*(\omega_2 + z/2)^{-1} & \\ & P^* \end{pmatrix} \text{ on the left,}$$

and by

$$\begin{pmatrix} P & \\ & -(\omega_2 + z/2)^{*-1}P \end{pmatrix} \text{ on the right,}$$

where $P \in \mathrm{GL}(k/2)$ is such that $\theta_2 P$ has orthogonal columns, equivalently, such that $P^*\theta_2^*\theta_2 P$ is diagonal with nonnegative eigenvalues. Let the diagonal elements be $y_1, y_2, \ldots, y_{k/2}$.

The determinant of the resulting matrix,

$$\begin{pmatrix} \mathbb{I} & -XX^* \\ \mathrm{diag}(y_1, y_2, \ldots) & \mathbb{I} \end{pmatrix}, \tag{2.31}$$

will be

$$1/\left(|P|^4 \prod_{i=1}^{A} |a_i + z/2|^{2j_i}\right) \tag{2.32}$$

times the original, (2.30).

Now we calculate the determinant by expanding by minors. The first $k/2$ columns have at most two nonzero elements, and it's not hard to see that the determinant decomposes into a sum of $2^{k/2}$ subdeterminants:

$$\sum_{\mathcal{A}\subset\{1,2,\ldots,k/2\}} \left(\prod_{a\in\mathcal{A}} y_a\right) \det XX^*_{\mathcal{A}\mathcal{A}}$$

where $Y_{\mathcal{A}\mathcal{A}}$ indicates the submatrix $(X_{ij} : (i,j) \in \mathcal{A}\times\mathcal{A}$.

To see that $\det(XX^*)_{\mathcal{A}\mathcal{A}}$ is real and nonnegative, let $Q_{\mathcal{A}}$ be a nonsingular matrix such that X^*Q has orthogonal columns (*i.e.* $Q_{\mathcal{A}}$ represents column operations putting it into this form). Then Q^*XX^*Q is diagonal with nonnegative eigenvalues and hence has real nonnegative determinant, as is $\det QQ^* = \det Q \det Q^* = |\det Q|^2$.

If $-2z$ is not an eigenvalue, the summand (2.32) $\det(XX^*)_{\emptyset\emptyset} =$ (2.32) is positive, and hence so is the determinant. It remains to find a nonzero summand when $-2z$

is an eigenvalue. In this case the factor (2.32) has a zero, so we have to find a term with the appropriate pole.

If $-2z$ is a given eigenvalue of ω_2, we can assume without loss of generality that ω_2 has l blocks with this eigenvalue and that they are the first l blocks of ω_2, with sizes $j_1, \ldots, j_l$. Let

$$\nu(i) = j_1 + j_2 + \cdots + j_{i-1} + 1 \text{ and } J = j_1 + \cdots + j_l.$$

As a result of the surjectivity of β, we can assume without loss of generality that ζ_2 is in the normal form specified above, *i.e.* that the last rows beside Jordan blocks with the same eigenvalue are mutually orthonormal and that the intervening rows are orthogonal to those rows. This is equivalent to the requirement that the $j_1{}^{\text{st}}$, $j_2{}^{\text{nd}}$, ... rows and columns of $\zeta_2\zeta_2^*$ are zero except for a 1 on the diagonal.

As a result of the injectivity of α implies that the first, $\nu(2)^{\text{nd}}$, ... columns of θ_2 are independent, and we may therefore assume without loss of generality that $P_{\mathcal{A}\mathcal{A}}$ is nonsingular, and $\mathbb{C}^{\mathcal{A}} \supset P(\mathbb{C}^{\mathcal{A}})$, where $\mathcal{A} = \{1, \nu(2), \nu(3), \ldots\}$ and $\mathbb{C}^{\mathcal{A}}$ is the space of vectors spanned by the a^{th} standard basis vector of $\mathbb{C}^{k/2}$, for $a \in \mathcal{A}$.

As a result of the assumed property of P,

$$\begin{aligned}\det XX^*_{\mathcal{A}\mathcal{A}} &= \det P^*_{\mathcal{A}\mathcal{A}} \left((\omega_2 + z/2)^{-1}\zeta_2\zeta_2^*(\omega_2 + z/2)^{*-1}\right)_{\mathcal{A}\mathcal{A}} P_{\mathcal{A}\mathcal{A}} \\ &= |\det P_{\mathcal{A}\mathcal{A}}|^2 \det \left((\omega_2 + z/2)^{-1}\zeta_2\zeta_2^*(\omega_2 + z/2)^{*-1}\right)_{\mathcal{A}\mathcal{A}}.\end{aligned}$$

The assumed normalisation of ζ_2 allows us to assert that

$$\begin{aligned}\left((\omega_2 + z/2)^{-1}\zeta_2\zeta_2^*(\omega_2 + z/2)^{*-1}\right)_{\mathcal{A}\mathcal{A}} &= \operatorname{diag}(|a_1 + z/2|^{-2j_1}, |a_1 + z/2|^{-2j_2}, \ldots) \\ &\quad + \text{ terms of `lower order'}\end{aligned}$$

by which we mean that the i^{th} row contains no other power of $(a_1 + z/2)$ of degree $-j_i$ or lower, and that the i^{th} column contains no other power of $\overline{(a_1 + z/2)}$ of degree $-j_i$ or lower. As a result

$$|a_1 + z/2|^{2J} \det \left((\omega_2 + z/2)^{-1}\zeta_2\zeta_2^*(\omega_2 + z/2)^{*-1}\right)_{\mathcal{A}\mathcal{A}} = 1.$$

Since the eigenvalue a_1 was chosen arbitrarily, it follows that the determinant condition (2.22) is always satisfied for monads satisfying the reality, time translation and other triviality conditions. □

We have proven

THEOREM C. *The space of (framed) uniton bundles is isomorphic to a space of monads (a subset of a complex linear space) quotiented by the action of a complex group. The action of the group can be used to put any monad into a unique normal form*

$$0 \to \mathcal{O}(-1,0)^k \xrightarrow[\begin{pmatrix} \mathbb{I}\eta - \omega_2^*\lambda^2 & \omega_1 \\ \rho_2\lambda^2 & \eta\mathbb{I} + \omega_2 \\ -\zeta_2^*\lambda^2 & \theta_2 \end{pmatrix}]{\begin{pmatrix} (\mathbb{I}+\gamma_1^*) + \gamma_1^*\lambda & 0 \\ 0 & \gamma_1 + \lambda(\mathbb{I}+\gamma_1) \end{pmatrix}} \begin{matrix} \mathcal{O}(-1,1)^k \\ \oplus \\ \mathcal{O}^{k+N} \end{matrix}$$

$$\xrightarrow[\begin{pmatrix} -(\mathbb{I}+\gamma_1^*) - \lambda\gamma_1^* & 0 & -(1+\lambda)\theta_2^* \\ 0 & -\gamma_1 - \lambda(\mathbb{I}+\gamma_1) & (1+\lambda)\zeta_2 \end{pmatrix}]{\begin{pmatrix} \mathbb{I}\eta + -\omega_2^*\lambda^2 & -\omega_1 \\ -\rho_2\lambda^2 & \mathbb{I}\eta + \omega_2 \end{pmatrix}} \mathcal{O}(0,1)^k \to 0,$$

where $\gamma_1 \in \mathrm{gl}(k/2)$ *is nilpotent and in Jordan Normal form,* $\omega_2 \in \mathrm{gl}(k/2)$, $\zeta_2 \in M_N^{k/2}$, $\theta_2 \in M_{k/2}^N$, *are in normal forms* $\zeta_2\theta_2 = [\gamma_1, \omega_2]$, *and* ω_1 *and* ρ_2 *are determined by* $\omega_1 + \gamma_1^*\omega_1 + \omega_1\gamma_1 = -\theta_2^*\theta_2$ *and* $\rho_2 + \rho_2\gamma_1^* + \gamma_1\rho_2 = -\zeta_2\zeta_2^*$ *and*

$$\det\begin{pmatrix} (\omega_2 + z/2) - \frac{\bar{z}}{2}((\mathbb{I}+\gamma_1)^{-1}\gamma_1)^2 & \omega_1 \\ \rho_2 & (\rho_1 - \bar{z}/2) + \frac{z}{2}((\mathbb{I}+\gamma_0)^{-1}\gamma_0)^2 \end{pmatrix} \neq 0 \quad (2.22)$$

for all $z \in \mathbb{C}$.

CHAPTER VI

CONSTRUCTION OF A SIMPLEST-TYPE UNITON

Having shown that unitons can be represented by monads, we now reconstruct the unitons from a monad of simplest type, closing the circle unitons $\to$ Bogomolny solutions $\to$ uniton bundles $\to$ monads. The key is the link to the original construction of Ward (Chapter IV), which gives the extended solution as the monodromy of the bundle E around a cycle of complex lines. To construct the extended solution in this manner, we need fixed frames of E restricted to the lines, C_∞, $C_{z,\bar{z},0}$, P_λ (for z in $\mathbb{C}$ and λ in $\mathbb{C}^*$).

1. Parametrising sections over C_∞ and P_λ

Recall that a monad with $\gamma_i = 0$ restricted to C_∞ is

$$0 \to \mathcal{O}^k \xrightarrow[\begin{pmatrix} \mathbb{I} & \\ & \mathbb{I} \\ 0 & 0 \end{pmatrix}]{\begin{pmatrix} \mathbb{I} & \\ & \lambda\mathbb{I} \end{pmatrix}} \begin{matrix} \mathcal{O}(1)^k \\ \oplus \\ \mathcal{O}^{k+N} \end{matrix} \xrightarrow[\begin{pmatrix} -\mathbb{I} & & (1+\lambda)\zeta_1 \\ & -\lambda\mathbb{I} & (1+\lambda)\zeta_2 \end{pmatrix}]{\mathbb{I}} \mathcal{O}(1)^k \to 0.$$

From this we obtain the long exact sequences

$$0 \to H^0(\ker\beta) \to H^0(\mathcal{O}(1)^k \oplus \mathcal{O}^{k+N}) \xrightarrow{\beta} H^0(\mathcal{O}(1)^k) \to 0, \text{ and}$$

$$0 \to H^0(\mathcal{O}^k) \xrightarrow{\alpha} H^0(\ker\beta) \to H^0(E) \to 0.$$

The first sequence tells us that

$$\ker\beta_{H^0} = H^0(\ker\beta) = \left\{ \begin{pmatrix} (-\beta_2 a) \\ (a) \end{pmatrix} : a \in \mathbb{C}^{k+N} \right\}$$

and the second sequence tells us

$$\operatorname{im}\alpha_{H^0} = \left\{ \begin{pmatrix} \left(-\beta_2 \begin{pmatrix} a_0 \\ 0 \end{pmatrix}\right) \\ \begin{pmatrix} a_0 \\ 0 \end{pmatrix} \end{pmatrix} : a_0 \in \mathbb{C}^k \right\}.$$

(Recall that $\alpha_1(A) = -\beta_2\left((A,0)^t\right)$.) Together they tell us that

$$H^0(E|_{C_\infty}) \cong \left\{ \left(\begin{pmatrix} -(1+\lambda)\zeta_1 a \\ -(1+\lambda)\zeta_2 a \end{pmatrix} \atop \begin{pmatrix} 0^{(k)} \\ a \end{pmatrix} \right) : a \in \mathbb{C}^N \right\} \cong \mathbb{C}^N. \tag{1.1}$$

Similarly, the restriction of the monad to P_λ for $\lambda \in \mathbb{C}^*$:

$$0 \to \mathcal{O}(-1)^k \underset{\alpha_2}{\xrightarrow{\alpha_1}} \begin{matrix} \mathcal{O}(-1)^k \\ \oplus \\ \mathcal{O}^{k+N} \end{matrix} \underset{\beta_2}{\xrightarrow{\beta_1}} \mathcal{O}^k \to 0$$

leads to the isomorphisms

$$H^0(P_\lambda, E) \cong H^0(P_\lambda, \ker\beta) \cong \left\{ \left(\begin{matrix} (0) \\ \begin{pmatrix} (1+\lambda)\zeta_1 a \\ (1+\lambda^{-1})\zeta_2 a \\ a \end{pmatrix} \end{matrix} \right) : a \in \mathbb{C}^N \right\} \cong \mathbb{C}^N. \tag{1.2}$$

2. Parametrising sections over $C_{z,\bar{z},0}$

Unfortunately, this case is a little bit more complicated. Since the extended solution is time-independent, it is enough to compute it for $t = 0$, so we will only concern ourselves with such sections. In this case the monad restricts to

$$0 \to \mathcal{O}(-2)^k \xrightarrow[\begin{pmatrix} z/2 + (-\bar{z}/2 + \rho_1)\lambda^2 & \zeta_1\theta_2 \\ \zeta_2\theta_1\lambda^2 & (z/2+\omega_2) - \bar{z}/2\lambda^2 \\ \theta_1\lambda^2 & \theta_2 \end{pmatrix}]{\begin{pmatrix} \mathbb{I} & \\ & \lambda\mathbb{I} \end{pmatrix}} \begin{matrix} \mathcal{O}(-1)^k \\ \oplus \\ \mathcal{O}^{k+N} \end{matrix}$$

$$\xrightarrow[\begin{pmatrix} -\mathbb{I} & & (1+\lambda)\zeta_1 \\ & -\lambda\mathbb{I} & (1+\lambda)\zeta_2 \end{pmatrix}]{\begin{pmatrix} z/2 + (-\bar{z}/2 + \rho_1)\lambda^2 & -\zeta_1\theta_2 \\ -\zeta_2\theta_1\lambda^2 & (z/2+\omega_2) - \bar{z}/2\lambda^2 \end{pmatrix}} \mathcal{O}(1)^k \to 0.$$

The fact that $H^1(\mathcal{O}(-2)^k) \neq 0$ means that we cannot think of sections of E as sections of $\ker\beta$. Some sections will be contributed by $H^1(\mathcal{O}(-2)^k)$. To compute a basis of sections, we will have to look at the exact sequence of complexes of Čech cochains with respect to the cover

$$\left\{ U_\lambda = \{\lambda \neq \infty\}, U_{\hat{\lambda}} = \{\lambda \neq 0\} \right\},$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^0(\mathcal{O}(-2)^k) & \xrightarrow{\alpha^0} & C^0(\ker\beta) & \xrightarrow{p^0} & C^0(E) & \longrightarrow & 0 \\ & & {\scriptstyle d^\alpha}\downarrow & & {\scriptstyle d^\beta}\downarrow & & {\scriptstyle d^E}\downarrow & & \\ 0 & \longrightarrow & C^1(\mathcal{O}(-2)^k) & \xrightarrow{\alpha^1} & C^1(\ker\beta) & \xrightarrow{p^1} & C^1(E) & \longrightarrow & 0 \end{array} \tag{2.1}$$

and essentially trace through the proof of the snake lemma. Since α^0, d^α and α^1 are all injective (from the definition of a monad and the fact that $H^0(\mathcal{O}(-2)) = 0$), we can factor out the image of $C^0(\mathcal{O}(-2))$ in the first two columns resulting in the modified complex

$$\begin{array}{ccccc} 0 \quad \rightarrow & C^0(\ker\beta)/\alpha(C^0(\mathcal{O}(-2)^k)) & \underset{\cong}{\xrightarrow{p^{0\prime}}} & C^0(E) & \rightarrow 0 \\ & {\scriptstyle d^{\beta\prime}}\downarrow & & {\scriptstyle d^E}\downarrow & \\ 0 \rightarrow H^1(\mathcal{O}(-2)^k) \xrightarrow{\alpha^{1\prime}} & C^1(\ker\beta)/\alpha^1(d^\alpha(C^0(\mathcal{O}(-2)^k))) & \xrightarrow{p^{1\prime}} & C^1(E) & \rightarrow 0. \end{array} \tag{2.2}$$

Then

$$H^0(E) = \ker d^E \cong \ker\left(p^{1\prime} \circ d^{\beta\prime}\right) \subset C^0(\ker\beta)/\alpha(C^0(\mathcal{O}(-2)^k)),$$

which, since the bottom row is still exact,

$$= \left(d^{\beta\prime}\right)^{-1}\left(\alpha^{1\prime}(H^1(\mathcal{O}(-2)^k))\right).$$

It's easy to see what the image of $\alpha^{1\prime}$: looks like. If we use the standard representatives for

$$H^1(\mathcal{O}(-2)^k) \cong \left\{\begin{pmatrix} f\lambda^{-1} \\ g\lambda^{-1}\end{pmatrix} : f, g \in \mathbb{C}^{k/2}\right\},$$

the image of $\alpha^{1\prime}$ is

$$\left\{\left(\begin{array}{c}\begin{pmatrix} f\lambda^{-1} \\ g \end{pmatrix} \\ \begin{pmatrix} \lambda^{-1}(\frac{\bar z}{2}f + \zeta_1\theta_2 g) + \lambda(\rho_1 - \bar z/2)f \\ \lambda^{-1}(\omega_2 + z/2)g + \lambda(\zeta_2\theta_1 f - \frac{\bar z}{2}g) \\ \lambda\theta_1 f + \lambda^{-1}\theta_2 g \end{pmatrix}\end{array}\right)\right\} \in C^1(\ker\beta).$$

In calculational terms, we take the quotienting by $C^0(\mathcal{O}(-2))$ to mean that $X = ((a,b)(c,d,e)) \in \Gamma(U_\lambda, \ker\beta)$ satisfies $a = 0$ and $b = b_0$ (*i.e.* does not depend on λ) and $\hat{X} = \left((\hat{a},\hat{b}),(\hat{c},\hat{d},\hat{e})\right) \in \Gamma(U_{\hat{\lambda}}, \ker\beta)$ satisfies $\hat{a} = \hat{a}_0$ and $\hat{b} = 0$, which is compatible with the choice of basis for $H^1(\mathcal{O}(-2)^k)$. This normalisation allows us to see that if

$$\alpha^1 \begin{pmatrix} f\lambda^{-1} \\ g\lambda^{-1} \end{pmatrix} = d^\beta(X,\hat{X}) = X + \hat{X}$$

then $f = \hat{a}_0$ and $g = b_0$. Since X and $\hat{X}$ are local sections of $\ker\beta$, we have relations

$$0 = \beta(X) = \begin{pmatrix} -\zeta_1\theta_2 g - c + (1+\lambda)\zeta_1 e \\ (z/2+\omega_2)g - \bar{z}/2\lambda^2 g - \lambda d + (1+\lambda)\zeta_2 e \end{pmatrix}, \text{ and}$$
$$0 = \beta(\hat{X}) = \begin{pmatrix} \lambda(\rho_1 - \bar{z}/2)f + \lambda^{-1}z/2f - \hat{c} + (1+\lambda)\zeta_1\hat{e} \\ -\lambda\zeta_2\theta_1 f - \lambda\hat{d} + (1+\lambda)\zeta_2\hat{e} \end{pmatrix},$$

which yield relations

$$\begin{aligned}
c &= (1+\lambda)\zeta_1 e - \zeta_1\theta_2 g \\
\hat{d} &= (\hat{\lambda}+1)\zeta_2\hat{e} - \zeta_2\theta_1 f \\
d &= \left((1+\lambda)\zeta_2 e - \zeta_2 e_0\right)/\lambda - \bar{z}/2\ \lambda g \\
\hat{c} &= \left((\hat{\lambda}+1)\zeta_1\hat{e} - \zeta_1\hat{e}_0\right)/\hat{\lambda} + z/2\ \hat{\lambda} f \\
0 &= (\omega_2 + z/2)g + \zeta_2 e_0 \\
0 &= (\rho_1 - \bar{z}/2)f + \zeta_1\hat{e}_0.
\end{aligned} \tag{2.3}$$

Using the first four relations, we calculate

$$d^\beta(X,\hat{X}) = \begin{pmatrix} \begin{pmatrix} f\lambda^{-1} \\ g \end{pmatrix} \\ \begin{pmatrix} (1+\lambda)\zeta_1 e - \zeta_1\theta_2 g + \lambda\left((\lambda^{-1}+1)\zeta_1\hat{e} - \zeta_1\hat{e}_0\right) + \lambda^{-1}z/2f \\ \left((1+\lambda)\zeta_2 e - \zeta_2 e_0\right)/\lambda - \bar{z}/2\lambda g + (\lambda^{-1}+1)\zeta_2\hat{e} - \zeta_2\theta_1 f \\ e + \hat{e} \end{pmatrix} \end{pmatrix},$$

from whose last row we see that $d^\beta(X,\hat{X}) = \alpha^1(f\lambda^{-1},\, g\lambda^{-1})$ implies that

$$\hat{e}_0 = -e_0, \quad e_1 = \theta_1 f, \quad \hat{e}_1 = \theta_2 g,$$

and these are all the relations we can derive, *i.e.* if X and $\hat{X}$ satisfy these relations, $d^\beta(X,\hat{X}) = \alpha^1(f\lambda^{-1},\, g\lambda^{-1})$.

Putting it together, we see that on the open set where $-2z$ does not correspond to an eigenvalue of ω_2, respectively $2\bar{z}$ of ρ_1, $H^0(E)$ is parametrised by $e_0 \in \mathbb{C}^N$, *i.e.* the map $\mathbb{C}^N \to C^0(\ker\beta)$ given by $e_0 \mapsto (\phi_0(e_0), -\phi_1(e_0))$

$$\begin{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 \\ -(z/2+\omega_2)^{-1}\zeta_2 e_0 \end{pmatrix} \\ \begin{pmatrix} (1+\lambda)\zeta_1(e_0+\lambda\theta_1(\rho_1-\bar{z}/2)^{-1}\zeta_1 e_0)+\zeta_1\theta_2(z/2+\omega_2)^{-1}\zeta_2 e_0 \\ ((1+\lambda)\zeta_2(e_0+\lambda\theta_1(\rho_1-\bar{z}/2)^{-1}\zeta_1 e_0)-\zeta_1 e_0)/\lambda+\bar{z}/2\lambda(z/2+\omega_2)^{-1}\zeta_2 e_0 \\ e_0+\lambda\theta_1(\rho_1-\bar{z}/2)^{-1}\zeta_1 e_0 \end{pmatrix} \end{pmatrix} \\ \begin{pmatrix} \begin{pmatrix} \hat{\lambda}(\rho_1-\bar{z}/2)^{-1}\zeta_1 e_0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} \big((\hat{\lambda}+1)\zeta_1(-e_0-\hat{\lambda}\theta_2(z/2+\omega_2)^{-1}\zeta_2 e_0)+\zeta_1 e_0\big)/\hat{\lambda}+\hat{\lambda}z/2(\rho_1-\bar{z}/2)^{-1}\zeta_1 e_0 \\ (\hat{\lambda}+1)\zeta_2(-e_0-\hat{\lambda}\theta_2(z/2+\omega_2)^{-1}\zeta_2 e_0)-\zeta_2\theta_1(\rho_1-\bar{z}/2)^{-1}\zeta_1 e_0 \\ -e_0-\hat{\lambda}\theta_2(z/2+\omega_2)^{-1}\zeta_2 e_0 \end{pmatrix} \end{pmatrix} \end{pmatrix} \tag{2.4}$$

induces an isomorphism $\mathbb{C}^N \to H^0(C_{(z,\bar{z},0)}, E)$.

REMARK 2.5. We have actually only calculated parametrisations of E restricted to an open set of sections. This is sufficient for the construction to follow, since by the definition of the monad the monodromy is analytic, and hence determined by an open set. Restricted to the closed set of sections $C_{(z,\bar{z},0)}$ for z an eigenvalue of $-2\omega_2$, however, it doesn't follow immediately that the space of sections has dimension N, nor that those sections don't have zeros at $\lambda = 0, \infty$. In fact, if we try to derive these properties we find that they require the nonvanishing of the same determinant (V.2.22) (with $\gamma_1 = 0$) we found in the last chapter.

3. Parallel Translation

We defined the extended solution E_λ as the 'monodromy' (*i.e.* failure to commute) of the cycle of isomorphisms:

$$\begin{tikzcd}
E_{\lambda,\infty} & H^0(C_\infty, E) \arrow[l, "\text{restr}"'] \arrow[r, "\text{restr}"] & E_{-1,\infty} \\
H^0(P_\lambda, E) \arrow[u, "\text{restr}"] \arrow[d, "\text{restr}"'] & & H^0(P_{-1}, E) \arrow[u, "\text{restr}"'] \arrow[d, "\text{restr}"] \\
E_{(\lambda, z/2-\lambda t-\lambda^2\bar{z}/2)} & H^0(C_{(z,\bar{z},t)}, E) \arrow[l, "\text{restr}"'] \arrow[r, "\text{restr}"] & E_{(-1, z/2+t-\bar{z}/2)}.
\end{tikzcd}$$

As with traditional monodromies, choice of a different starting point or frame effects the extended solution by a conjugation. We can fix it by specifying a framing at a point or alternatively over C_∞ or P_{-1} (but not over $C_{(z,\bar{z},t)}$, because it moves). Of course going around the other way gives the inverse solution. We will base our monodromies at the fixed frame $\phi \in H^0(P_{-1}, Fr(E))$. Since the extended solution is independent of t, we can greatly simplify the calculation by assuming $t = 0$ in the following.

This description of the extended solution is well-suited to the monad representation since we have been able to calculate explicit parametrisations of the spaces of sections E over nonpolar fibres and over a dense set of sections of $T\mathbb{P}^1$ (including the section at infinity).

Let $\psi : \mathbb{P}^1 \to \widetilde{T\mathbb{P}}^1$ parametrise one of the above sections or fibres. The parametrisation of E restricted to such a line is a map

$$\Psi : \mathbb{C}^N \times \mathbb{P}^1 \to \ker\beta$$
$$\Psi_p : \mathbb{C}^N \to \ker\beta|_{\psi(p)}.$$

Evaluation is given by composing with the quotient map

$$\ker\beta|_{[\mathrm{pt}]} \to \ker\beta|_{[\mathrm{pt}]}/\operatorname{im}\alpha|_{[\mathrm{pt}]} = E|_{[\mathrm{pt}]}.$$

If ψ^1 and ψ^2 parametrise lines which intersect at $\psi^1(p_1) = q = \psi^2(p_2)$, then the map $H^0(\psi^1(\mathbb{P}^1), E) \to H^0(\psi^2(\mathbb{P}^1), E)$ given by evaluation at q maps $a \in \mathbb{C}^N$ to $b \in \mathbb{C}^N$ such that $\Psi^1_{p_1}(a) \equiv \Psi^2_{p_2}(b)(\mod \operatorname{im}\alpha|_q)$. Since the maps Ψ and α are analytic, the resulting evaluation map is as well, and in practice is easy to calculate.

We begin with the hardest case, the map $H^0(C_{(z,\bar{z},0)}, E) \to H^0(P_\lambda, E)$. The images of $\Psi_p : H^0(P_\lambda, E) \to \mathbb{C}^k \oplus \mathbb{C}^{k+N}$ are all contained in the second summand, *i.e.* their first components are zero. To calculate the translation from $H^0(C_{(z,\bar{z},0)}, E)$ to $H^0(P_\lambda, E)$ we have to put the cocycle representative (2.4) into this form. We can take either the local section over U_λ or the local section over $U_{\hat{\lambda}}$. To the former we have to add

$$\alpha\begin{pmatrix} 0 \\ \lambda^{-1}(\omega_2 + \bar{z}/2)^{-1}\zeta_2 e_0 \end{pmatrix}$$

to get the representative

$$\left(\begin{pmatrix}0\\0\end{pmatrix}\;\begin{pmatrix}(1+\lambda)\zeta_1(e_0+\lambda\theta_1(\rho_1-\bar{z}/2)^{-1}\zeta_1 e_0)+(1+\lambda^{-1})\zeta_1\theta_2(\omega_2+z/2)^{-1}\zeta_2 e_0\\ (1+\lambda^{-1})\zeta_2(e_0+\lambda\theta_1(\rho_1-\bar{z}/2)^{-1}\zeta_1 e_0)\\ e_0+\lambda\theta_1(\rho_1-\bar{z}/2)^{-1}\zeta_1 e_0+\lambda^{-1}\theta_2(\omega_2+z/2)^{-1}\zeta_2 e_0\end{pmatrix}\right)$$

of the form (1.2). So in terms of the chosen trivialisations, the translation

$$\Psi_p^{-1}\circ\Psi_\lambda : H^0(C_{(z,\bar{z},0)})\to H^0(P_\lambda)$$

is

$$\mathbb{I}+\lambda\theta_1(\rho_1-\bar{z}/2)^{-1}\zeta_1+\lambda^{-1}\theta_2(\omega_2+z/2)^{-1}\zeta_2. \tag{3.1}$$

By the same method of translating by the image of α we calculate the parallel translation from sections above C_∞ to sections above P_λ to be the identity in the chosen bases.

Putting these together, we get

$$\begin{aligned}E_\lambda &= (\mathbb{I}+\lambda\theta_1(-\bar{z}/2+\rho_1)^{-1}\zeta_1+\lambda^{-1}\theta_2(z/2+\omega_2)^{-1}\zeta_2)^{-1}\\ &\quad(\mathbb{I}-\theta_1(-\bar{z}/2+\rho_1)^{-1}\zeta_1-\theta_2(z/2+\omega_2)^{-1}\zeta_2)\\ &=\left(\mathbb{I}-(\theta_1,\ \theta_2)\begin{pmatrix}\lambda^{-1}(\rho_1-\bar{z}/2)^{-1} & \\ & \lambda(\omega_2+z/2)^{-1}\end{pmatrix}\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix}\right)^{-1}\\ &\quad\left(\mathbb{I}-(\theta_1,\ \theta_2)\begin{pmatrix}(\rho_1-\bar{z}/2)^{-1} & \\ & (\omega_2+z/2)^{-1}\end{pmatrix}\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix}\right)\end{aligned}$$

and imposing reality, we get

$$\begin{aligned}E_\lambda &= (\mathbb{I}-\lambda\left(\theta_2(z/2+\omega_2)^{-1}\zeta_2\right)^*+\lambda^{-1}\theta_2(z/2+\omega_2)^{-1}\zeta_2)^{-1}\\ &\quad(\mathbb{I}+\left(\theta_2(z/2+\omega_2)^{-1}\zeta_2\right)^*-\theta_2(z/2+\omega_2)^{-1}\zeta_2).\end{aligned} \tag{3.2}$$

Let

$$\Omega=\theta_2(z/2+\omega_2)^{-1}\zeta_2. \tag{3.3}$$

Then

$$\theta_1(\rho_1-\bar{z}/2)^{-1}\zeta_1=-\Omega^*$$

by the reality conditions. The monad condition ($\zeta_2\theta_2 = 0$) implies

$$\Omega\Omega = 0 = \Omega^*\Omega^*.$$

The matrices $\Omega\Omega^*$ and $\Omega^*\Omega$ are hermitian and hence have real eigenvalues. Since they are normal and commute with each other, they are simultaneously diagonalisable. To summarise, we can find a unitary (but nonconstant!) frame so that

$$\Omega = \begin{pmatrix} 0 & \Omega' \\ 0 & 0 \end{pmatrix} \tag{3.4}$$

in block form, where Ω' is not necessarily square, and $\Omega'\Omega'^*$ and $\Omega'^*\Omega'$ are diagonal. From this picture we see that the eigenvalues of $\Omega\Omega^*$ and $\Omega^*\Omega$ are all nonnegative. It follows that

$$D \overset{\text{def}}{=} \mathbb{I} + \Omega\Omega^* + \Omega^*\Omega \tag{3.5}$$

has eigenvalues bounded away form zero, so we can invert it.

One remarks that Ω and Ω^* both commute with D. Since D is independent of λ, D commutes with $(\mathbb{I}+\lambda\Omega^*-\lambda^{-1}\Omega)$, as does D^{-1}. That the inverse of $\mathbb{I}-\lambda\Omega^*+\lambda^{-1}\Omega$ is

$$(\mathbb{I} - \lambda\Omega^* + \lambda^{-1}\Omega)^{-1} = (\mathbb{I} + \lambda\Omega^* - \lambda^{-1}\Omega)D^{-1} \tag{3.6}$$

follows from

$$(\mathbb{I} - \lambda\Omega^* + \lambda^{-1}\Omega)(\mathbb{I} + \lambda\Omega^* - \lambda^{-1}\Omega) = D.$$

We have shown

Construction D. *Given monad data as in Theorem C with $\gamma_1 = 0$, we can construct the associated uniton as*

$$S = (\mathbb{I} + 2\Omega^* - 2\Omega - \Omega\Omega^* - \Omega^*\Omega)D^{-1}, \tag{3.7}$$

which has extended solution

$$E_\lambda = (\mathbb{I} + \lambda\Omega^* - \lambda^{-1}\Omega)(\mathbb{I} + \Omega^* - \Omega)D^{-1} \tag{3.8}$$

as an extended solution. Since all simplest-type unitons can be so constructed, all such unitons have uniton number 1 or 2.

4. For the Sceptical

4.1 Extended solution. We can check that S satisfies the uniton equations directly, but it is much easier to check the equations for the extended solution.

Using

$$\frac{\partial}{\partial z}\Omega^* = 0, \quad \frac{\partial}{\partial z}D^{-1} = -D^{-1}(\frac{\partial}{\partial z}\Omega\Omega^* + \Omega^*\frac{\partial}{\partial z}\Omega)D^{-1}$$

and commutativity of D^{-1} and Ω, we can calculate

$$\frac{1}{1+1/\lambda}E_\lambda^{-1}\frac{\partial}{\partial z}E_\lambda = D^{-1}\left(-\frac{\partial}{\partial z}\Omega - \frac{\partial}{\partial z}\Omega\Omega^* + \Omega^*\frac{\partial}{\partial z}\Omega + \Omega^*\frac{\partial}{\partial z}\Omega\Omega^*\right)D^{-1} \quad (4.3a)$$

and

$$\frac{1}{1+\lambda}E_\lambda^{-1}\frac{\partial}{\partial \bar{z}}E_\lambda = D^{-1}\left(\frac{\partial}{\partial \bar{z}}\Omega^* - \frac{\partial}{\partial \bar{z}}\Omega^*\Omega + \Omega\frac{\partial}{\partial \bar{z}}\Omega^* + -\Omega\frac{\partial}{\partial \bar{z}}\Omega^*\Omega\right)D^{-1}, \quad (4.3b)$$

verifying that E_λ is in fact an extended solution. (See Theorem I.1.12.)

4.4 Nonsingularity. We can also check that S is nonsingular. In fact, we will show that E_λ is nonsingular on $\{(\lambda, z) \in \mathbb{C}^* \times \mathbb{P}^1\}$.

We can make a unitary (but not holomorphic) change of gauge so that $\Omega\Omega^*$ and $\Omega^*\Omega$ are diagonal and Ω is block diagonal of the form (3.4). Let j be the size of the first block of zeros, equivalently, the height of Ω. In such a frame, the (i,i)th element of $\Omega\Omega^*$ is the squared norm of the ith row of Ω', $|\Omega'^i|^2$, and the $(i+j,\ i+j)$th element of $\Omega^*\Omega$ is the squared norm of the ith column of Ω'. That the off-diagonal elements of $\Omega\Omega^*$ are zero implies that the rows of Ω' are orthogonal; the columns of Ω' are similarly orthogonal, and together, these imply that Ω' is square.

Since D has positive real eigenvalues, it has a positive square root. Since D is diagonal in this frame, we can write $D^{1/2}$ explicitly as

$$D^{1/2} = \operatorname{diag}(\sqrt{1+|\Omega'^1|^2}, \sqrt{1+|\Omega'^2|^2}, \ldots, \sqrt{1+|\Omega'_1|^2}, \sqrt{1+|\Omega'_2|^2}, \ldots).$$

Using the commutativity of D and Ω, we can write E_λ in the form

$$\begin{aligned} E_\lambda &= (\mathbb{I} + \lambda\Omega^* - 1/\lambda\Omega)D^{-1}(\mathbb{I} + \Omega^* - \Omega) \\ &= (D^{-1/2} + \lambda\Omega^* D^{-1/2} - 1/\lambda\Omega D^{-1/2})(D^{-1/2} + D^{-1/2}\Omega^* - D^{-1/2}\Omega). \end{aligned}$$

In terms of our diagonalising basis, the multiplication by $D^{-1/2}$ on the left, respectively right, acts on Ω or Ω^* by scaling each row, respectively column, X, by $(1+|X|^2)^{-1/2}$, with the result that norms of the rows and columns are bounded by 1 and therefore $|\Omega|$ and $|\Omega^*|$ are bounded by N. It follows that

$$|E_\lambda| \leq 3N(1+|\lambda|+1/|\lambda|).$$

Applying a similar argument to

$$E_\lambda^{-1} = (\mathbb{I}-\Omega^*+\Omega)(\mathbb{I}-\lambda\Omega^*+1/\lambda\,\Omega)D^{-1},$$

we see that $|E_\lambda^{-1}|$ is similarly bounded. Together, the two conditions imply that E_λ is nonsingular on $\mathbb{P}^1\times\mathbb{C}^*$.

COROLLARY E. *All simplest-type unitons can be deformed continuously into $U(2)$ unitons. As a result, the components of $\mathcal{U}(N)$ are the energy levels, i.e.*

$$\pi_0(\mathcal{U}(N)_{\text{simplest-type}}) = \mathbb{N},$$

and the energy of the uniton is given by $1/2$ the second Chern class of the bundle in that case.

PROOF. As is well known, and will be demonstrated in the next chapter, $U(2)$ unitons factor through a $\mathbb{P}^1\subset U(2)$, and hence are parametrised by rational maps $\mathbb{P}^1\to\mathbb{P}^1$ and embeddings $\mathbb{P}^1\hookrightarrow U(2)$. The space of rational maps has components given by degree. Holomorphic maps $\mathbb{P}^1\to\mathbb{P}^1$ are the same as line bundles over $\mathbb{P}^1$ and degree corresponds to the Chern class of the bundle. By a theorem of Valli [Va, Theorem 3] this is the same as the energy of the uniton.

Given a set of monad data, we will give a path in the space of monads without the condition (V.2.22) to a monad whose corresponding uniton bundle is decomposable into a trivial bundle and a uniton bundle of rank two. Since the construction of the uniton from the monad data is continuous with respect to the matrix norms, our path preserves energy. Since energy is discrete, and the energy levels of $\mathcal{U}(2)$ are its components, it will follow that the components of $\mathcal{U}(N)$ are its energy levels as well.

The path itself is simple. Let the monad be given by ω_2, θ_2 and ζ_2, and the eigenvalues of the Jordan blocks of ω_2 be given by $a_1, a_2, \ldots, a_L$.

If we perturb the eigenvalues of ω_2 so they are distinct, the injectivity of α and surjectivity of β reduce to the condition that the first columns of θ_2 under Jordan blocks and the last rows of ζ_2 beside Jordan blocks be nonzero. These can be deformed to be colinear and the other rows and columns can be made zero, then the row and column can be rotated by $\mathrm{GL}(N)$ into the forms $(0,*,0,\ldots,0)$ and $(*,0,\ldots,0)^t$ respectively.

If g_t is a path in $\mathrm{GL}(N)$ such that

$$g_0 = \mathbb{I}, \quad g_1(\theta_2)_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (\zeta_2)^N g_1^{-1} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \end{pmatrix},$$

$3N\epsilon$ is the minimum nonzero distance $|a_i - a_j|$ between eigenvalues, and $f : [0,1] \to \mathbb{C}$, is a path between 0 and 1 in $\mathbb{C}$ such that no two colinear columns of ζ_2 or two colinear rows of θ_2 have $f(t)/(1-f(t))$ as their ratio, then the deformation $(\omega_2, \theta_2, \zeta_2)(t)$ $(0 \le t \le 1)$ is given by

$$a_i(t) = a_i + \epsilon t$$

$$(\theta_2)_i(t) = \begin{cases} (1-f(t))g_t(\theta_2)_i + f(t)g_t(\theta_2)_1 & \text{if } (\theta_2)_i \text{ is the first column under its Jordan block} \\ (1-t)g_t(\theta_2)_i & \text{otherwise} \end{cases}$$

$$(\zeta_2)_i(t) = \begin{cases} (1-f(t))(\zeta_2)_i g_t^{-1} + f(t)(\zeta_2)_N g_t^{-1} & \text{if } (\zeta_2)_i \text{ is the last row beside its Jordan block} \\ (1-t)(\zeta_2)_i g_t^{-1} & \text{otherwise} \end{cases}$$

Such a monad corresponds to a uniton bundle which is decomposable into the sum of a $U(2)$ uniton bundle and a trivial bundle, and hence a $U(2)$ uniton.

Alternatively, remark that Ω is of the form

$$\begin{pmatrix} \tilde{\Omega} & 0 \\ 0 & 0 \end{pmatrix}$$

where $\tilde{\Omega} \in \mathrm{gl}(2)$, and as a result S is of the form

$$\begin{pmatrix} \tilde{S} & 0 \\ 0 & \mathbb{I} \end{pmatrix}$$

where $S \in U(2)$. $\square$

CHAPTER VII

EXAMPLE: $S^2 \to U(2)$

This section treats the simplest case: $U(2)$ unitons given by rank two uniton bundles. It is well known that such maps factor through spheres, and hence are closely linked to the rational maps. We will give an ahistorical proof that based unitons correspond to rational maps ($\mathbb{P}^1 \to \mathbb{P}^1$), and show that the action of $U(2)$ on $\mathcal{U}(2)^*$ by conjugation ($S \mapsto USU^*$) corresponds to the usual $Gl(2)$ action on rational maps, *i.e.* the correspondence is equivariant. We then show that this is the same map as given by Construction D, and we prove that the determinant condition (V.2.22) is implied by the other monad conditions in the $U(2)$ case.

1. Rational maps

From [Uhl] we know that $S : S^2 \to U(N)$ is an n-uniton for $n < N$. As a result, harmonic maps $S : S^2 \to U(2)$ are 1-unitons, which have a simple form.

THEOREM [Uhl,9.3]. *$S : \Omega \to U(N)$ is a one-uniton iff $S = Q(\pi - \pi^\perp)$, $Q \in U(N)$, $\pi^* = \pi$, $\pi^2 = \pi$,* $\operatorname{rank}\pi$ *is constant, $\pi^\perp \bar\partial \pi = 0$, i.e. π is projection onto a holomorphic subbundle of $\Omega \times \mathbb{C}^N$.*

We can see the decomposition $S = Q(\pi - \pi^\perp)$ as a composition of three maps. In the middle is the inclusion

$$\mathbb{P}^1 = \mathrm{Gr}_{2,1} \overset{I}{\hookrightarrow} U(2) : \pi \in \mathrm{Gr}_{2,1} \mapsto \pi - \pi^\perp.$$

In terms of z, a coordinate on $\mathbb{P}^1$,

$$\begin{aligned}
\pi_z &= \frac{1}{1+z\bar z}\begin{pmatrix}1\\ z\end{pmatrix}\begin{pmatrix}1\\ z\end{pmatrix}^*,\\
\pi_z^\perp &= \frac{1}{1+z\bar z}\begin{pmatrix}-\bar z\\ 1\end{pmatrix}\begin{pmatrix}-\bar z\\ 1\end{pmatrix}^*,\\
I(z) = \pi_z - \pi_z^\perp &= \frac{1}{1+z\bar z}\begin{pmatrix}1 - z\bar z & 2\bar z\\ 2z & z\bar z - 1\end{pmatrix}.
\end{aligned} \tag{1.1}$$

Now we can identify holomorphic subbundles, $\pi \subset \underline{\mathbb{C}}^2$, as π_f, for some rational function $f : \mathbb{P}^1 \to \mathbb{P}^1$. Finally, we can left-translate harmonic maps $L_Q : S \mapsto QS$. Summarising, a general one uniton is a composition

$$\mathbb{P}^1 \xrightarrow{f} \mathbb{P}^1 \xhookrightarrow{I} U(2) \xrightarrow{L_Q} U(2)$$

where $f \in \operatorname{Rat} \mathbb{P}^1$, $Q \in U(2)$.

LEMMA 1.2. *This decomposition is unique!*

Another way of looking at this decomposition is that the image of a harmonic map is an embedded sphere (S^2), and the harmonic map is just a rational map of spheres. The question is, is the association to $Q \in U(2)$ of an embedded sphere injective? Since left translation is a group action, we can ask the stronger question, is $\{Q | Q\mathrm{Gr}_{2,1} = \mathrm{Gr}_{2,1}\} = \{\mathbb{I}\}$. In fact it is $\{\pm\mathbb{I}\}$, and we can calculate the action of $L_{-\mathbb{I}}$:

$$\begin{aligned}
(-\mathbb{I})I(z) &= \frac{(-1/z)(-1/\bar{z})}{(-1/z)(-1/\bar{z})}\left(-\frac{1}{1+z\bar{z}}\begin{pmatrix} 1-z\bar{z} & 2\bar{z} \\ 2z & z\bar{z}-1 \end{pmatrix}\right) \\
&= \frac{1}{(-1/z)(-1/\bar{z})}\begin{pmatrix} -(-1/z)(-1/\bar{z})+1 & 2(-1/z) \\ 2(-1/\bar{z}) & -1+(-1/z)(-1/\bar{z}) \end{pmatrix} \\
&= I(-1/\bar{z})
\end{aligned}$$

to be I composed with the antipodal map ($z \mapsto -1/\bar{z}$), which is of degree -1. So up to orientation, left translation of spheres is not free, but acting on oriented spheres, it is, *i.e.* if we were considering both holomorphic and antiholomorphic maps the decomposition would not be unique, but for rational maps it is.

PROOF. We use the fact that $S \in \mathrm{Gr}_{2,1} \subset U(N)$ satisfies $S^2 = \mathbb{I}$ (in fact this is equivalent to S being in $\mathrm{Gr}_{2,i}$ for some i). Let $Q\mathrm{Gr}_{2,1} = \mathrm{Gr}_{2,1}$. Then for all $I(z) \in \mathrm{Gr}_{2,1}$, $(QI(z))^2 = \mathbb{I}$. This puts conditions on Q. In particular

$$\begin{aligned}
\mathbb{I} = (QI(0))^2 &= \left(\begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\right)^2 = \begin{pmatrix} q_1^2 - q_2q_3 & q_2(q_4-q_1) \\ q_3(q_1-q_4) & q_4^2 - q_2q_3 \end{pmatrix} \\
\mathbb{I} = (QI(1))^2 &= \left(Q\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\right)^2 = \begin{pmatrix} q_2^2 + q_1q_4 & q_1(q_2+q_3) \\ q_4(q_2+q_3) & q_3^2 + q_1q_4 \end{pmatrix} \\
\mathbb{I} = (QI(-i))^2 &= \left(Q\begin{pmatrix} & i \\ -i & \end{pmatrix}\right)^2 = \begin{pmatrix} -q_2^2 + q_1q_4 & q_1(q_2-q_3) \\ -q_4(q_2-q_3) & -q_3^2 + q_1q_4 \end{pmatrix}.
\end{aligned}$$

We see immediately that $q_1q_4 = 1$, $q_2 = 0$, $q_3 = 0$, $q_1^2 = 1 = q_4^2$ which implies $Q = \pm\mathbb{I}$. $\square$

So the space of $U(2)$-unitons is

$$\mathcal{U}(2) \cong U(2) \times \operatorname{Rat}\mathbb{P}^1.$$

Rational maps can be written as $p(z)/q(z)$, with $(p,q) = 1$. Their topological degree is given by $\max\{\deg p, \deg q\}$ (always positive, because holomorphic maps preserve orientation). They contain the based maps

$$\operatorname{Rat}^* = \{f \in \operatorname{Rat} : f(\infty) = 0\} = \left\{\frac{p(z)}{q(z)} : (p,q) = 1, \deg p < \deg q\right\}.$$

Various groups act on Rat via the action of $PGL(2,\mathbb{C})$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \frac{p}{q} \mapsto \frac{ap+bq}{cp+dq}.$$

This map preserves degree because GL(2) is connected and degree components are disjoint. Rat is a Rat^* bundle over $\mathbb{P}^1$, given by

$$\operatorname{rat} : \operatorname{Rat} \to \mathbb{P}^1 : f \mapsto f(\infty).$$

$PGL(2,\mathbb{C})$ acts on $\mathbb{P}^1$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : [x,y] \mapsto [ax+by, cx+dy],$$

making $\operatorname{Rat} \to \mathbb{P}^1$ an equivariant bundle.

$$\{P \in PGL(2,\mathbb{C}) : P(\operatorname{Rat}^*) = \operatorname{Rat}^*\} = \left\{\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}\right\}.$$

Conjugation acts on the $\mathcal{U}(N)^*$ component of $\mathcal{U}(N) = U(N) \times \mathcal{U}(N)^*$, the space of $U(N)$ unitons. In this case $\mathcal{U}(2)^* \cong \operatorname{Rat}\mathbb{P}^1$.

Claim. *If $p/q \in \operatorname{Rat}\mathbb{P}^1$ and $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(2)$ then*

$$UI(p/q)U^* = I\left(\frac{Cq+Dp}{Aq+Bp}\right).$$

PROOF. Write

$$I(p/q) = \frac{1}{p\bar{p}+q\bar{q}} \begin{pmatrix} q & -\bar{p} \\ p & \bar{q} \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} q & -\bar{p} \\ p & \bar{q} \end{pmatrix}^* .$$

Then

$$\begin{aligned} UI(p/q)U^* &= \frac{1}{p\bar{p}+q\bar{q}} U \begin{pmatrix} q & -\bar{p} \\ p & \bar{q} \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \left(U \begin{pmatrix} q & -\bar{p} \\ p & \bar{q} \end{pmatrix} \right)^* \\ &= \frac{1}{p\bar{p}+q\bar{q}} \begin{pmatrix} Aq+Bp & -A\bar{p}+B\bar{q} \\ Cq+Dp & -C\bar{p}+D\bar{q} \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \\ &\qquad \begin{pmatrix} Aq+Bp & -A\bar{p}+B\bar{q} \\ Cq+Dp & -C\bar{p}+D\bar{q} \end{pmatrix}^* . \end{aligned}$$

The claim follows from the facts that for $U \in U(2)$

$$\begin{aligned} &(Aq+Bp)\overline{(Aq+Bp)} + (Cq+Dp)\overline{(Cq+Dp)} \\ &\qquad = (A\bar{A}+C\bar{C})q\bar{q} + (A\bar{B}+C\bar{D})\bar{p}q + (\bar{A}B+\bar{C}D)p\bar{q} + (B\bar{B}+D\bar{D})p\bar{p} \\ &\qquad = p\bar{p}+q\bar{q}, \end{aligned}$$

and

$$\frac{Cq+Dp}{Aq+Bp} = -\overline{\frac{-A\bar{p}+B\bar{q}}{-C\bar{p}+D\bar{q}}}$$

iff

$$0 = p^2(\bar{C}D+\bar{A}B) + pq(C\bar{C}-D\bar{D}+A\bar{A}-B\bar{B}) + q^2(-C\bar{D}-A\bar{B}). \quad \square$$

This action is not free, it has stabiliser $\{e^{i\theta}\mathbb{I}\}$. An element $\begin{pmatrix} e^{i\theta_1} & \\ & e^{i\theta_2} \end{pmatrix}$ acts on $f \in \text{Rat}$ by $f \mapsto e^{i(\theta_2-\theta_1)}f$. Is this the same S^1 action induced by the freedom to choose gauges when integrating the Bogomolny solutions?

2. Meanwhile, back at the monad ranch...

We have another description of $\mathcal{U}(2)$ and we would like to relate the two. Think of $\mathcal{U}(2)^* = \mathcal{U}(2)/U(2)$ as the space of monads with $N = 2$. The second Chern class gives a stratification

$$\mathcal{U}(2)^* = \bigcup_{k/2} \mathcal{U}(2)^*_{k/2}.$$

The quantity $k/2$ is also the jumping type of $E|_{P_0}$. Degree gives a stratification

$$\mathrm{Rat} = \bigcup_j \mathrm{Rat}_j .$$

The map $\mathcal{U}(2)/U(2) \to \mathrm{Rat}$ of Construction D preserves this stratification.

Recall the Jordan block normalisation of a simplest-type monad M, given by the data $\omega_2, \theta_2, \zeta_2$. From the fact that monads are complexes, we saw that $\zeta_2\theta_2 = 0$, but both ζ_2 and θ_2 must be nonzero if α and β are to be injective and surjective respectively at $\lambda = 0$ and η =an eigenvalue of ω_2. It follows that the column space of θ_2 and row space of ζ_2 are one dimensional, and that they are perpendicular to one another with respect to the Euclidean metric on $\mathbb{C}^2$. From the discussion of the normalisation, we see that the Jordan blocks of ω_2 have distinct eigenvalues, and the monad will be given by

$$\begin{aligned}
\omega_2 &= \begin{pmatrix} J_{j_1}(a_1) & & \\ & \ddots & \\ & & J_{j_L}(a_L) \end{pmatrix}, \\
\theta_2 &= \begin{pmatrix} 1 \\ b \end{pmatrix} (1, 0, \ldots, 0, 1, 0, \ldots, 0), \qquad (2.1)\\
\zeta_2 &= \begin{pmatrix} (\zeta_2)^1_{j_1 - 1} \\ \vdots \\ (\zeta_2)^1_0 \\ (\zeta_2)^2_{j_2-1} \\ \vdots \end{pmatrix} \begin{pmatrix} -b & 1 \end{pmatrix}
\end{aligned}$$

generically, where $(\zeta_2)^i_0 \neq 0$ for all i. Putting this rank condition into the determinant (V.2.22), we can give another proof that it is always satisfied for simplest-type $U(2)$ monads of simplest type. In fact,

$$\begin{aligned}
(\mathrm{V}.2.22) &= \prod_{i=1}^{A} |(a_i + z/2)|^{2j_i} \left(1 + \left| \sum_{i=1}^{A} \sum_{l=1}^{j_i} (a_i + z/2)^{-l} \zeta'_{\nu(i)+l-1} \right|^2 \right) \\
&= |q|^2 + |p|^2
\end{aligned}$$

where $f = p/q$ as below.

In the case $b = 0$, we calculate

$$\Omega = \theta_2(\omega_2 + z/2)^{-1}\zeta_2 = \begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix} \tag{2.5}$$

where

$$f(z) = -\sum_{i=0}^{L}\sum_{j=0}^{j_i-1} \frac{(\zeta_2)^i_j (z-a_i)^{l_i-j-1}}{(z-a_i)^{l_i}} = \frac{p}{q}, \tag{2.6}$$

is a based rational map of degree $k/2$. We see that

$$\begin{aligned} S &= \begin{pmatrix} 1 & f \\ -\bar{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & f \\ -\bar{f} & 1 \end{pmatrix} \begin{pmatrix} 1+f\bar{f} & \\ & 1+f\bar{f} \end{pmatrix}^{-1} \\ &= \frac{1}{1+f\bar{f}} \begin{pmatrix} 1-f\bar{f} & 2f \\ -2\bar{f} & 1-f\bar{f} \end{pmatrix}. \end{aligned}$$

This is equivalent to (1.1) under a change of frame and of basing condition.

REMARK 2.8. We see from this calculation that *all* $U(2)$ unitons are of simplest type!

REMARK 2.9. To understand the general uniton moduli, we should exploit this bundle structure. Of course, the general moduli are not bundles over $\mathbb{P}^1$ but are made of bundles over Grassmannians and flag manifolds glued together along the real subvarieties given by the determinant condition.

CONCLUSION

In this thesis we have developed a tool for studying unitons, *i.e.* a correspondence which identifies harmonic maps $S^2 \to U(N)$ with framed holomorphic uniton bundles over $\widetilde{T\mathbb{P}}^1$ with extra structure, and with a monad description of those bundles. It is reasonable to ask how useful this tool is. Does it help answer the three types of questions concerning unitons:

(1) construction question: Is there a 'reasonable' procedure for constructing some or all unitons?

(2) local questions: Are they smooth? Are they composed of rational functions? How can we calculate their energies? When do they factor through a totally-geodesic imbedding of a Grassmannian?

(3) global questions: What energy values are possible? Find a complete set of invariants, *i.e.* quantities which determine the connected components. Calculate $\pi_i(\mathcal{U}(N))$, $H^i(\mathcal{U}(N))$. Is the inclusion $\mathcal{U}(N) \to \mathcal{U}(N+1)$ a homotopy/homology isomorphism up to some dimension depending on N? Can we measure its failure to be an isomorphism? Does $\mathcal{U}(N)$ admit a complex structure?

Of course, these questions should be posed in the context of previous results and methods. A less quantifiable question is how well we understand unitons. This encompasses all of the previous questions, but also asks whether we can integrate the various approaches into a whole. Can we interpret energy, uniton number *etc.* in terms of all the known constructions? Does our construction suggest new questions or new approaches to established problems?

Construction. In [Wo], Wood amplifies Uhlenbeck's uniton factorisation, and relates it to the flag-transform method for Grassmannian solutions, thereby giving an algorithm for constructing harmonic maps from holomorphic maps into Grassmannians using only algebraic operations, differentiation and integral transforms. Constructing the uniton from bundle data, *i.e.* by factoring a transition matrix, is an advance over integral transforms, but it still requires one to know which transition matrices are allowed. The construction from the monad data for simplest-type unitons using only matrix operations is much more straight forward, but needs to be extended to the general case.

Local Questions. Based on his parametrisation of $U(3)$ unitons, and numerical calculations for higher rank, Wood conjectured that unitons were composed of rational functions, our construction allows us to verify this for all ranks. As concerns energy, we were able to show that for simplest-type unitons it is given by the second Chern class of the bundle, something easy to read off the monad data. We also showed that these unitons can only have uniton number 1 or 2. To be able to read off the uniton number from the monad data we have to find the meromorphic section of $E \to \widetilde{T\mathbb{P}}^1$ which corresponds to the 'Uhlenbeck normalisation' of the extended solution which seems central to the determination of the uniton number in both Uhlenbeck's and Segal's work.

We know that the space of unitons contains spaces of harmonic spheres in Grassmannians. If we could tell when a uniton factors through a Grassmannian, we could construct Grassmannian solutions as we did unitons. This may be determined by a condition on the jumping type of the bundle, or on the structure of the monad.

Global Questions. These are the least known. Using Uhlenbeck's factorisation, Valli showed that the energy spectrum of unitons is discrete, and can be normalised to be positive integers, thereby linking energy and uniton number. Since the energy functional is continuous, this result implies that the moduli space has countably many components, As a result of our retraction of $\mathcal{U}(N)_{\text{simplest-type}}$ to $\mathcal{U}(2)$, energy levels are the same as components, *i.e.* $\pi_0(\mathcal{U}(N)_{\text{simplest-type}}) = \mathbb{N}$.

The other work in this direction is the work of Guest and Ohnita [GuO2] which uncovers deformations of harmonic maps from one-parameter subgroups of the loop group acting on Uhlenbeck's extended solution via a dressing action. The main obstacle to this method is the possibility of 'bubbling off' of harmonic spheres resulting in a deformation which fails to be continuous. This method was used by Guest to show that certain unitons can be deformed so that their image is contained in a unitary group of strictly smaller rank, and in particular, that unitons with images in a projective space are always homotopic in the space of such unitons to a uniton with image in $\mathbb{P}^2$. Based on this result Crawford [Cra] has shown that the components of harmonic spheres in complex projective space are given by energy and degree alone. It would be interesting to try to write these deformations in terms of the monad data. This might help answer the question of when unitons factor through Grassmannians.

This thesis suggests that the moduli components (π_0) are the same as energy levels. This would follow from an extension of the deformation of simplest-type unitons into $\mathcal{U}(2)$, or from an extension of the proof that the determinant condition (V.2.22) is, in general, implicit in the other monad conditions.

It also opens up two related methods of investigation of the higher homotopy of the moduli space. One is to investigate the space of framed jumps, the second is to study the space of monads. As mentioned in the preface, both methods were used in studying instanton moduli and in particular in proving the Atiyah-Jones conjecture. Of course, the uniton situation is somewhat different. Uniton bundles have two fixed jumping lines (the polar fibres) in a ruling of $\widetilde{T\mathbb{P}^1}$, and from the monad description of simplest-type bundles we can read off that they are jumps of length one in the language of [BHMM], *i.e.* they cease to jump on the first formal neighbourhoods of the jumping lines. It should be possible to calculate the uniton in terms of a transition matrix for one jump and a choice of framing along the infinity section. From such an expression, one would hope to read off the relationship between jumping line and uniton data. For example, one might find that restricted types of jumps correspond to Grassmannian solution, or perhaps adding a uniton could be interpreted as some sort of tensor product of bundles in this way.

This work certainly improves our understanding of unitons since it gives a new construction for general unitons in terms of uniton bundles and for simplest-type unitons in terms of monads, reducing the problem to linear algebra, it answers an open question about the rationality of the constituent functions, and it allows us to calculate $\pi_0(\mathcal{U}(N)_{\text{simplest-type}})$, but it leaves many stimulating questions unanswered.

(1) What is the link between our construction and the method of Uhlenbeck worked on by many people in the uniton case and also in the Grassmannian case?

(2) What is the link to the loop group methods of Segal [Se2], which give $\mathcal{U}(N)$ a complex structure?

(3) Can the deformation of simplest-type unitons into $U(2)$ unitons be extended to general unitons. In other words, can the isomorphism

$$\pi_0(\mathcal{U}(2)) \to \pi_0(\mathcal{U}(N)_{\text{simplest-type}}),$$

be extended to an isomorphism

$$\pi_0(\mathcal{U}(2)) \to \pi_0(\mathcal{U}(N)).$$

Given $k > 0$, is there an N_k such that the map

$$\pi_k(\mathcal{U}(N_k)) \to \pi_k(\mathcal{U}(N))$$

induced by the inclusion $U(N_k) \hookrightarrow U(N)$ is an isomorphism for all $N \geq N_k$?

(4) Are simplest-type unitons the same as one unitons? Is the uniton number the size of the largest block in the Jordan decomposition of γ_1? If this were so, we could compute the homotopy of the type-components and use the long exact sequence in homotopy to compute the homotopy of the moduli space.

(5) Can the determinant condition (V.2.22) be simplified? Eliminated? Or perhaps reduced to checking for a finite number of values of z.

(6) Can the monad description be interpreted as a sort of cell complex description of $\mathcal{U}(N)$? If so, how are the cells glued together.

(7) Which jumps are allowed and how may they be glued in to construct a uniton bundle?

(8) Is there an expression for the uniton in terms of a transition matrix for the jump and framing?

(9) Is energy given by the multiplicity of the jump at P_0/P_∞, and the uniton number given by the degree of the first formal neighbourhood on which it jumps down?

BIBLIOGRAPHY

[An] C. K. Anand, *Harmonic Maps*, McGill masters' thesis, 1990.

[AtBo] M. F. Atiyah & R. Bott, *The Yang-Mills Equations over Riemann Surfaces*, Phil. Trans. R. Soc. Lond. A **308** (1982), 323–615.

[ADHM] M. F. Atiyah, V. G. Drinfeld, N. J. Hitchin and Yu. I. Manin, *Construction of Instantons*, Phys. Lett. **65A** (1978), 185-187.

[AtJo] M. F. Atiyah and J. D. S. Jones, *Topological aspects of Yang-Mills theory*, Commun. math. Phys. **61** (1978), 97–118.

[Gu2] J. M. Bergvelt and M. A. Guest, *Actions of loop groups on harmonic maps*, Trans. Amer. Math. Soc. **326** (1991), no. 2, 861–886.

[BHMM] C. P. Boyer, J. C. Hurtubise, B. M. Mann and R. J. Milgram, *The topology of instanton moduli spaces. I: The Atiyah-Jones conjecture*, Annals of Math.(2).

[BuRa] F. E. Burstall and J. H. Rawnsley, *Twistor Theory for Riemannian Symmetric Sapces*, LNM 1424, Springer Verlag, 1990.

[Bu] N. P. Buchdahl, *Stable 2-Bundles on Hirzebruch Surfaces*, Math. Z. **194** (1984), 143–152.

[Ca1] E. Calabi, *Minimal Immersions of Surfaces in Euclidean Spheres*, J. Differential Geometry **1** (1967), 111–125.

[Ca2] ———, *Quelques Applications de l'Analyse Complexe aux Surfaces d'Aire Minima*, Séminaire de Math. Supérieures 30 (1967), 59–81.

[Cra] T. A. Crawford, *The Space of Harmonic maps from the 2-sphere to Complex Projective Space*, McGill preprint.

[Do] S. K. Donaldson, *Instantons and Geometric Invariant Theory*, Commun. Math. Phys. **93** (1984), 453–460.

[DZ] A. M. Din and W.J. Zakrzewski, *General classical solutions in the* $\mathbb{C}P^{n-1}$ *model*, Nuclear Phys. B **174** (1980), 397–406.

[EGA3] A. Grothendieck and J. Dieudonné, *EGA III: Étude cohomologique des faisceaux cohérents*, Publ. Math. IHES **4** (1960).

[Ei] L. P. Eisenhart, *A treatise on the differential geometry of curves and surfaces*, Ginn & Co., Boston, 1909.

[Rep1] J. Eells and L. Lemaire, *A report on harmonic maps*, Bull. Lond. Math. Soc. **10** (1978), 1–68.

[Rep2] ———, *Another report on harmonic maps*, Bull. Lond. Math. Soc. **20** (1988), 385–524.

[FoTu] A. T. Fomenko and A. A. Tuzhlin, *Elements of the Geometry and Topology of Minimal Surfaces in Three-Dimensional Space*, AMS translation, 1991.

[GriHa] Phillip Griffiths and Joseph Harris, *Principles of Algebraic Geometry*, John Wiley & Sons, New York, 1978.

[Gro] A. Grothendieck, *Sur la classification des fibrés holomorphes sur la sphère de Riemann*, Am. J. Math. **79** (1957), 121–138.

[Gu1] M. A. Guest, *Harmonic two-spheres in complex projective space and some open problems*, Exposition. Math. **10** (1992), no. 1, 61–87.

[GuO1] M. A. Guest and Y. Ohnita, *Group actions and deformations for harmonic maps*, J. Math. Soc. Japan **45** (1993), no. 4, 671–704.

[GuO2] ———, *Group actions and deformations for harmonic maps (preprint)*.

[Ha] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York, 1987.

[Hi] N. J. Hitchin, *Monopoles and Geodiesics*, Commun. Math. Phys. **83** (1982), 579–602.

[Ho] G. Horrocks, *Vector bundles on the punctured spectrum of a local ring*, Proc. Lond. Math. Soc. (3) **14**, 689-713.

[Hu] J. C. Hurtubise, *Instantons and Jumping Lines*, Commun. Math. Phys. **105** (1986), 107–122.

[La] Serge Lang, *Introduction to Differentiable Manifolds*, Wiley Interscience, 1962.

[Mi] C. W. Misner, *Harmonic maps as models for physical theories*, Physical Review D **18** (1978), no. 12, 4510–4524.

[MFK] D. Mumford, J. Fogarty and F. Kirwan, *Geometric Invariant Theory, 3rd ed.*, Springer-Verlag, 1994.

[New] S. J. H. New, McGill University.

[Mu1] M. K. Murray, *Monopoles and Spectral Curves*, Commun. Math. Phys. **90** (1983), 263–271.

[Mu2] ———, *Non-Abelian Magnetic Monopoles*, Commun. Math. Phys. **96** (1984), 539–565.

[OSS] C. Okonek, M. Schneider and H. Spindler, *Vector bundles on complex projective spaces*, Birkhauser, Boston, 1980.

[Os] R. Osserman, *A Survey of Minimal Surfaces*, Van Nostrand Reinhold, 1969.

[Pe] A. M. Perelomov, *Chiral Models: geometrical aspects*, Physics Reports **146** (1987), no. 3, 135–213.

[Po] K. Pohlmeyer, *Integrable Hamiltonian systems and interactions through constraints*, Comm. Math. Phys. **46** (1976), 207-221.

[PrSe] A. Pressley and G. Segal, *Loop Groups*, Oxford University Press, 1986.

[SaUhl] J. Sacks and K. Uhlenbeck, *The existence of minimal immersions of 2-spheres*, Annals of Math. **113** (1981), 1-24.

[Se1] G. Segal, *The topology of rational functions*, Acta Math. **143** (1979), 39–72.

[Se2] ———, *Loop groups and harmonic maps*, Advances in Homotopy Theory (Cortona 1988), London Math. Soc. Lecture Note 139, Cambridge Univ. Press, 1989, pp. 153–164.

[Ti] Y. Tian, *The Based $SU(n)$-Instanton Moduli Spaces*, Math. Annallen.

[Uhl] K. Uhlenbeck, *Harmonic Maps into Lie Groups (Classical Solutions of the Chiral Model)*, J. Differential Geometry **30** (1989), 1–50.

[Va] G. Valli, *On the energy spectrum of harmonic 2-spheres in unitary groups*, Topology **27** (1988), 129–136.

[Wd1] R. S. Ward, *On self-dual gauge fields*, Physics Letters **61A** (1977), no. 2, 81–82.

[Wd2] ———, *Soliton solutions in an integrable chiral model in 2+1 dimensions*, J. Math. Phys. **29**, no. 2.

[Wd3] ———, *Classical Solutions of the Chiral Model, Unitons, and Holomorphic Vector Bundles*, Commun. Math. Phys. **123** (1990), 319–332.

[Wr] F. Warner, *Foundations of differentiable manifolds and Lie groups*, Springer-Verlag, 1983.

[Wei1] K. Weierstraß, *Über die flächen, deren mittlere krümmung überall gleich null ist*, Monatsberichte der Berliner Akademie (1866), 612–625, (It's easier to find it in his collected works.).

[Wei2] ———, *Untersuchungen über die flächen, deren mittlere krümmung überall gleich null ist*, Monatsberichte der Berliner Akademie (1886), 39–52, (It's easier to find it in his collected works.).

[Wo] J. C. Wood, *Explicit construction and parametrisation of harmonic two-spheres in the unitary Group*, Proc. London Math. Soc. (3) **58**, 608–624.

[ZaMi] V. E. Zakharov and A. V. Mikhailov, *Relativistically invariant two-dimensional models of field theory which are integrable by means of the inverse scattering problem method*, Sov. Phys. JETP **47** (1978), no. 6, 1017–1027.

[ZaSha] V. E. Zakharov and A. B. Shabat, *Integration of nonlinear equations of mathematical physics by the method of inverse scattering II*, Functional Anal. Appl. **13** (1979), 166–174.