Copula Models and Pickands Dependence Functions

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Abstract

Copulas are multivariate cumulative distribution functions with uniform margins on the unit interval. Bivariate extreme-value copulas embody a specific form of dependence between two uniform random variables. These copulas are characterized by a function of a single variable called the Pickands dependence function. This thesis is concerned with an extension of the Pickands dependence function whose properties are studied.

Résumé

Les copules sont des fonctions de répartition multidimensionnelles à marges uniformes sur l'intervalle unité. Les copules de valeurs extrêmes bivariées traduisent une forme de dépendance spécifique entre deux variables aléatoires uniformes. Ces copules sont caractérisées par une fonction d'une seule variable appelée fonction de dépendance de Pickands. Ce mémoire porte sur une généralisation de la fonction de dépendance de Pickands, dont on étudie les propriétés.

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Introduction

In the statistical literature, copulas refer to multivariate cumulative distribution functions with uniform margins on the unit interval. This concept is used to model dependence between random variables, as justified by Sklar's Representation Theorem; see Sklar (1959). Although the use of copulas for statistical modeling grew slowly at first, as documented in Genest et al. (2009), there are now hundreds of applications of copulas in statistics and related fields, including quantitative risk management, finance, insurance, and hydrology. See, e.g., Genest and Favre (2007) for a partial survey and Genest and Nešlehová (2014) for a gentle introduction to the subject. General reference books about copulas and copula modeling are Nelsen (2006) and Joe (2014); for applications to risk management and finance, see, e.g., McNeil et al. (2015).

Bivariate extreme-value copulas are characterized by a function called the Pickands dependence function; see, e.g., Genest and Nešlehová (2012) and references therein. To be specific, a bivariate copula $C : [0,1]^2 \to [0,1]$ belongs to the extreme-value class if, and only if, there exists a convex function $A : [0,1] \to [1/2,1]$ such that, for all $u, v \in (0,1)$,

$$C(u,v) = \exp\left[\, \ln(uv) A\{\ln(v)/\ln(uv)\}\right]. \tag{1.1}$$

In order for C to be a copula, the function A, called a Pickands dependence function, must also be bounded above by the constant 1 and below by the function $t \mapsto \max(t, 1-t)$. That is, one must have, for all $t \in [0,1]$,

$$\max(t, 1-t) \leqslant A(t) \leqslant 1.$$

Many properties of a bivariate extreme-value copula can be deduced from those of the corresponding Pickands dependence function. Moreover, given the one-to-one relationship between bivariate extreme-value copulas and Pickands dependence functions, an estimate of a copula C of the form (1.1) can be derived from an estimate of A based on a random sample of size n from a bivariate distribution function whose unique underlying copula is C.

Two nonparametric estimators of A were proposed by Genest and Segers (2009), namely the Pickands and Capéraà–Fougères–Genest (CFG) estimators, respectively denoted A_n^P and A_n^{CFG}. These authors studied the large-sample behavior of these estimators of A and proved, among others, their consistency. This development naturally led Genest et al. (2011) to propose a goodness-of-fit test for bivariate extreme-value copulas based on the distance between a rank-based, parametric estimate of the Pickands dependence function and either one of the nonparametric estimators A_n^P or A_n^{CFG}. Their procedures were shown to be consistent and powerful under reasonable data generation schemes.

In the same paper, Genest et al. (2011) extended their goodness-of-fit test beyond bivariate extreme-value copulas by considering the broader class of so-called left-tail decreasing (LTD) bivariate copulas. Asymptotic properties of the CFG and Pickands estimators were derived in this context. This led the authors to generalized notions of Pickands dependence functions which arose as the weak limits A_C^P and A_C^{CFG} of the nonparametric estimators A_n^P and A_n^{CFG}, respectively, where the subscript C denotes the unique underlying LTD copula of the continuous bivariate distribution from which the data arose.

Genest et al. (2011) noted that A_C^P and A_C^{CFG} could be Pickands dependence functions, without C being an extreme-value copula, allowing the generation of bivariate extreme-value copulas from other copulas. They also remarked that when A_C^P or A_C^{CFG} is convex, the proposed goodness-of-fit test could be inconsistent. Specifically, the test might suggest that the underlying copula of the sample is extreme-value whereas in fact it is not. Therefore, it is interesting to study the functions A_C^P and A_C^{CFG}, and specifically their convexity, for an arbitrary copula C.

In Chapter 2, we will introduce formally the notions of copula and extreme-value copula, as well as the main results concerning nonparametric estimation of copulas and Pickands dependence functions. Chapter 3 concerns measures of dependence, which are used to assess the degree of dependence between two variables; also reviewed there are notions and properties of tail monotonicity that will play a role in the third part of the thesis.

In Chapter 4, which contains this thesis' original contribution, we will describe some of the properties of the function A_C^P and A_C^{CFG}. We will first extend the results of Genest et al. (2011) by proving several properties that these functions share with Pickands dependence functions. In addition, we will supply new examples of copulas C for which A_C^P is available in closed form. We will then study the derivative of A_C^P and use it to give analytical necessary and sufficient conditions for the convexity of A_C^P. At the end, we will use the notions introduced in the previous chapters to relate the convexity of A_C^P to the tail monotonicity properties of C.

2

Copulas and extreme-value copulas

2.1 Definitions and basic properties

A d-dimensional copula C is a cumulative distribution function whose support is $[0,1]^d$ and whose univariate margins are uniform on the unit interval. An equivalent definition is given below in probabilistic terms.

Definition 2.1 A function $C : [0,1]^d \to [0,1]$ is a d-variate copula if, and only if, there exist random variables $U_1, \ldots, U_d$ which are uniformly distributed on the interval $[0,1]$ and such that, for all $u_1, \ldots, u_d \in [0,1]$, $C(u_1, \ldots, u_d) = \Pr(U_1 \leqslant u_1, \ldots, U_d \leqslant u_d)$.

The following are two fundamental examples of copulas.

Example 2.1 Let $U_1, \ldots, U_d$ be mutually independent random variables which are uniformly distributed on the interval $[0,1]$. The joint cumulative distribution function of the vector $(U_1, \ldots, U_d)$ is then given, for all $u_1, \ldots, u_d \in [0,1]$, by

$$\Pi_d(u_1, \ldots, u_d) = \Pr(U_1 \leqslant u_1, \ldots, u_d \leqslant u_d) = \prod_{i=1}^{d} \Pr(U_i \leqslant u_i) = u_1 \times \cdots \times u_d.$$

This function, denoted Π_d, is called the d-dimensional independence or product copula.

Example 2.2 Let U be a uniformly distributed random variable on $[0,1]$. Consider the $d \times 1$ random vector $(U, \ldots, U)$. Let M_d denote its cumulative distribution function. Then, for all $u_1, \ldots, u_d \in [0,1]$, we have

$$\begin{aligned} M_d(u_1, \ldots, u_d) &= \Pr\left(U \leqslant u_1, \ldots, U \leqslant u_d\right) \\ &= \Pr\left\{U \leqslant \min(u_1, \ldots, u_d)\right\} = \min(u_1, \ldots, u_d). \end{aligned}$$

The copula M_d is called the d-dimensional comonotonicity copula or the Fréchet–Hoeffding upper bound. The motivation for this terminology stems from the following result, whose proof can be found in the book by Nelsen (2006).

Theorem 2.1 *Let $(U_1, \ldots, U_d)$ be a d-dimensional random vector with uniform marginals on $[0, 1]$ and let C denote its cumulative distribution function. Let $u = (u_1, \ldots, u_d) \in [0, 1]^d$, and define $W_d : [0, 1]^d \to \mathbb{R}$ by $W_d(u) = \max(0, u_1 + \cdots + u_d - d + 1)$. Then, the following statements hold:*

a) $W_d(u) \leqslant C(u) \leqslant M_d(u)$.

b) *W_d is a copula if, and only if, $d \in \{1, 2\}$.*

c) *For all $u = (u_1, \ldots, u_d)$ and $v = (v_1, \ldots, v_d) \in [0, 1]^d$,*

$$|C(u) - C(v)| \leqslant \sum_{i=1}^{d} |u_i - v_i|.$$

d) *C is almost everywhere differentiable with respect to each of its variables. If C_j denotes the first partial derivative of C with respect to the jth variable, where $j \in \{1, \ldots, d\}$, we have, for almost all $u = (u_1, \ldots, u_d) \in [0, 1]^d$,*

$$C_j(u) = \Pr\left\{ \bigcap_{k=1, k\neq j}^{d} U_k \leqslant u_k \middle| U_j = u_j \right\}.$$

The following theorem is due to Sklar (1959). It is the fundamental result which motivates copula modeling. It is often referred to as Sklar's Representation Theorem.

Theorem 2.2 *Let $X_1, \ldots, X_d$ be random variables with respective cumulative distribution functions $F_1, \ldots, F_d$. Let H denote the joint cumulative distribution function of the vector $(X_1, \ldots, X_d)$. Then, there exists a d-dimensional copula C such that, for all $x_1, \ldots, x_d \in \mathbb{R}$,*

$$H(x_1, \ldots, x_d) = C\left\{F_1(x_1), \ldots, F_d(x_d)\right\}.$$

If $X_1, \ldots, X_d$ are continuous, then C is unique. Reciprocally, if C is a d-dimensional copula, then the function $H : \mathbb{R}^d \to [0, 1]$ defined, for all $x_1, \ldots, x_d \in \mathbb{R}$, by

$$H(x_1, \ldots, x_d) = C\left\{F_1(x_1), \ldots, F_d(x_d)\right\}$$

is a cumulative distribution function on $\mathbb{R}^d$, whose univariate marginals are $F_1, \ldots, F_d$.

Example 2.3 Let $X_1, \ldots, X_d$ be mutually independent random variables with cumulative distribution functions $F_1, \ldots, F_d$, respectively. Then, the joint cumulative distribution function H of the vector $(X_1, \ldots X_d)$ is given, for all $x_1, \ldots, x_d \in \mathbb{R}$, by

$$H(x_1, \ldots x_d) = F_1(x_1) \times \cdots \times F_d(x_d) = \Pi_d \{F_1(x_1), \ldots, F_d(x_d)\}.$$

Example 2.4 Let (X, Y) be a random pair, and assume that X has distribution F, and Y has distribution G. Assume further that the functions F and G are continuous. Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be mutually independent observations from (X, Y). Let C be the unique underlying copula associated with the distribution of (X, Y) via Sklar's Representation Theorem. We seek to find the copula $C_{(n)}$ of $(X_{(n)}, Y_{(n)})$, where $X_{(n)} = \max(X_1, \ldots, X_n)$ and $Y_{(n)} = \max(Y_1, \ldots, Y_n)$. Note that, for all $x, y \in \mathbb{R}$,

$$\begin{aligned}\Pr\left\{X_{(n)} \leqslant x, Y_{(n)} \leqslant y\right\} &= \Pr\{X_1 \leqslant x, \ldots, X_n \leqslant x, Y_1 \leqslant y, \ldots, Y_n \leqslant y\} \\ &= \prod_{i=1}^{n} C\{F(x), G(y)\} = \left[C\{F(x), G(y)\}\right]^n.\end{aligned}$$

Given that the cumulative distribution functions of $X_{(n)}$ and $Y_{(n)}$ are F^n and G^n, respectively, we deduce that

$$C_{(n)}\{F(x)^n, G(y)^n\} = C\left[\{F(x), G(y)\}\right]^n.$$

Therefore, for all $u, v \in [0, 1]$, $C_{(n)}(u, v) = \{C(u^{1/n}, v^{1/n})\}^n$. This calculation can be generalized to arbitrary dimensions.

There are several well-known families of copulas. Two famous examples are the Archimedean and the elliptical class of copulas. Key references about the class of Archimedean copulas are Genest and MacKay (1986) and McNeil and Nešlehová (2009). For details about the class of elliptical copulas, see, e.g., Genest et al. (2007) or Joe (2014).

In this thesis, we will mostly be interested in the family of extreme-value copulas, and focus on the bivariate case. The definition of an extreme-value copula was already given in the Introduction; see Eq. (1.1). Details concerning this class of copulas will be given in Section 2.4 below. Before we proceed, we review in Section 2.2 some well-known results in parametric and nonparametric estimation of copulas.

2.2 Nonparametric estimation

Let $X = (X_1, \ldots, X_d)$ be a d-dimensional random vector, and assume that $X_1, \ldots, X_d$ are continuous univariate random variables with respective cumulative distribution functions $F_1, \ldots, F_d$. Let C denote the unique underlying copula associated to X via Sklar's Representation Theorem. Furthermore, let $X^{(1)}, \ldots, X^{(n)}$ be a random sample of size n from X. We write $X^{(i)} = (X_{i1}, \ldots, X_{id})$

for $i \in \{1, \ldots, n\}$.

First assume that the margins $F_1, \ldots, F_d$ are known. Then, for all $i \in \{1, \ldots, n\}$, set $U_i = (U_{i1}, \ldots, U_{id})$, where $U_{i1} = F_1(X_{i1}), \ldots, U_{id} = F_d(X_{id})$. It is easily seen that $\mathcal{U}_n = \{U_1, \ldots, U_n\}$ is a random sample from distribution C.

Definition 2.2 The empirical distribution function of the sample $\mathcal{U}_n$ is defined to be the function $C_n : [0,1]^d \to [0,1]$ defined, for all $u_1, \ldots, u_d \in [0,1]$, by

$$C_n(u_1, \ldots, u_d) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(U_{i1} \leqslant u_1, \ldots, U_{id} \leqslant u_d).$$

Remark 2.1 The empirical function C_n is an estimate of the copula C but it is actually not a copula itself because its margins are not uniform on the interval $[0,1]$. In fact, C_n takes discrete values in the set $\{0, 1/n, \ldots, n/n = 1\}$.

The following result is a special case of the standard functional Central Limit Theorem, also called Donsker's theorem; see, e.g., Donsker (1952).

Theorem 2.3 *The process* $\mathbb{C}_n = \sqrt{n}\,(C_n - C)$ *converges weakly, as* $n \to \infty$*, to a centered Gaussian process* $\mathbb{C}$ *whose covariance structure is given, for all* $u, v \in [0,1]^d$*, by*

$$\operatorname{cov}\{\mathbb{C}(u), \mathbb{C}(v)\} = C\{\min(u,v)\} - C(u)C(v),$$

where the minimum between two vectors u *and* v *in* $[0,1]^d$ *is understood component-wise.*

In practice, of course, the margins $F_1, \ldots, F_d$ are unknown and hence the above estimation method, which requires the knowledge of these margins, fails. Invoking the Glivenko–Cantelli Lemma, simple nonparametric estimators of $F_1, \ldots, F_d$ are their respective empirical distributions $F_{n1}, \ldots, F_{nd}$, where for each $j \in \{1, \ldots, d\}$, F_{nj} is defined, for all $x \in \mathbb{R}$, by

$$F_{nj}(x) = \frac{1}{n+1} \sum_{i=1}^{n} \mathbf{1}(X_{ij} \leqslant x).$$

Set, for all $i \in \{1, \ldots, n\}$, $\hat{U}_i = (\hat{U}_{i1}, \ldots, \hat{U}_{id}) = (F_{n1}(X_{i1}), \ldots, F_{nd}(X_{id}))$. A pseudo-random sample from the copula C is then given by $\hat{\mathcal{U}}_n = \{\hat{U}_1, \ldots, \hat{U}_n\}$.

Remark 2.2 The following remarks are small observations:

a) The classical definition of F_{nj} involves division by n rather than $n+1$. Replacing n by $n+1$ in the denominator of F_{nj} is to ensure that $F_{nj} < 1$ on the entire interval $[0,1]$.

b) For each $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, d\}$, let R_{ij} denote the rank of X_{ij} within the set $\{X_{1j}, \ldots, X_{nj}\}$. Then,

$$\hat{U}_i = \left(\frac{R_{i1}}{n+1}, \ldots, \frac{R_{id}}{n+1} \right).$$

In other words, $\hat{U}_i$ is a rank-based statistic.

c) The set $\hat{\mathcal{U}}_n$ does not form a random sample from C because as transpires from the above relation, knowledge of all but one of $\hat{U}_1, \ldots, \hat{U}_n$ implies complete knowledge of the remaining one. For this reason, $\hat{\mathcal{U}}_n$ is called a pseudo-sample.

Definition 2.3 The empirical copula $\hat{C}_n$ is defined to be the empirical distribution function of the set $\hat{\mathcal{U}}_n$.

The following result is the rank-based version of Donsker's result. It has a long history in the copula literature and goes back at least to Rüschendorf (1976).

Theorem 2.4 *Under suitable regularity conditions on C, the empirical copula process $\hat{\mathbb{C}}_n = \sqrt{n}\,(\hat{C}_n - C)$ defined on $[0,1]^d$ converges weakly to a process $\hat{\mathbb{C}}$, defined on $[0,1]^d$, by*

$$\hat{\mathbb{C}}(u) = \mathbb{C}(u) - \sum_{i=1}^{d} C_i(u)\mathbb{C}(1, \ldots, 1, u_i, 1, \ldots, 1).$$

where $\mathbb{C}$ is a Gaussian process on $[0,1]^d$ whose covariance structure is given, for all $u, v \in [0,1]^d$, by

$$\mathrm{cov}\left\{\mathbb{C}(u), \mathbb{C}(v)\right\} = C\left\{\min(u, v)\right\} - C(u)C(v).$$

The "suitable regularity conditions" mentioned above were strong in the work of Rüschendorf (1976) and were gradually relaxed in a series of papers. See, e.g., Genest and Nešlehová (2012) for additional references. The following condition is the weakest to date; for more details, see, e.g., Segers (2012) or Genest et al. (2017).

Condition 2.3 In order for the above convergence theorem to hold, one assumes that for each $j \in \{1, \ldots, d\}$, the derivative C_j of C with respect to the jth variable exists and is continuous on the set $\{(u_1, \ldots, u_d) \in [0,1]^d : 0 < u_j < 1\}$. This allows for a continuous extension of C_j to the boundary of $[0,1]^d$.

2.4 Extreme-value copulas

Definition 2.4 *Let C be a d-dimensional copula. We say that C is max-stable if, and only if, for all $n \in \mathbb{N}$, and $u \in [0,1]^d$, we have $\{C(u^{1/n})\}^n = C(u)$.*

Example 2.5 The independence copula Π_d is clearly max-stable. Likewise, the comonotonicity copula M_d is max-stable given that, for all $n \in \mathbb{N}$ and $u = (u_1, \ldots, u_d) \in [0,1]^d$,

$$\{M_d(u^{1/n})\}^n = \{\min(u_1^{1/n}, \ldots, u_d^{1/n})\}^n = M_d(u).$$

In contrast, the copula W_2 is not max-stable.

Definition 2.5 A d-dimensional copula C is said to be an extreme-value copula if, and only if, there exists a d-dimensional copula D such that, for all $u \in [0,1]^d$, $\lim_{n\to\infty} \{D(u^{1/n})\}^n = C(u)$. We say that D is in the domain of attraction of C.

It is obvious from the above that a max-stable copula is necessarily an extreme-value copula. The reverse is also true and stated below as a theorem. See, e.g., Joe (2014).

Theorem 2.5 *A copula C is an extreme-value copula if, and only if, it is a max-stable copula.*

Definition 2.6 The following notations will be used in the characterization of extreme-value copulas.

a) Let S_d denote the d-dimensional simplex $\{(s_1, \ldots, s_d) \in [0,1]^d : s_1 + \cdots + s_d = 1\}$.

b) Let Δ_d denote the set $\{(s_1, \ldots, s_d) \in [0,1]^d : s_1 + \cdots + s_d \leqslant 1\}$. Note that S_d and Δ_{d-1} are in one-to-one correspondence.

c) A finite measure μ on the Borel sigma-field generated by S_d is called a spectral measure if, and only if, for all $j \in \{1, \ldots, d\}$,

$$\int_{S_d} s_j d\mu(s_1, \ldots, s_d) = 1.$$

The following characterization theorem and its corollary are due to Pickands (1981).

Theorem 2.6 *Let C be a d-dimensional extreme-value copula. Then, for all $u = (u_1, \ldots, u_d) \in (0,1)^d$, one has*

$$C(u) = \exp\left[\left\{\sum_{i=1}^{d} \ln(u_i)\right\} \times B\left\{\frac{\ln(u_1)}{\sum_{i=1}^{d} \ln(u_i)}, \ldots, \frac{\ln(u_{d-1})}{\sum_{i=1}^{d} \ln(u_i)}\right\}\right],$$

where $B : \Delta_{d-1} \to [0, \infty)$ *is given, for all* $x_1, \ldots, x_{d-1} \in \Delta_{d-1}$*, by*

$$B(x_1, \ldots, x_{d-1}) = \int_{S_d} \max\{\omega_1 x_1, \ldots, \omega_{d-1} x_{d-1}, \omega_d(1 - x_1 - \cdots - x_{d-1})\} d\mu(\omega),$$

for some spectral measure μ *on* S_d*.*

Corollary 2.1 *The following are properties satisfied by the function* B*:*

a) $B(0, \ldots, 0) = 1$, $B(1, 0, \ldots, 0) = 1, \ldots, B(0, 1, 0 \ldots, 0) = 1, \ldots, B(0, \ldots 0, 1) = 1$.

b) For all $(x_1, \ldots, x_{d-1}) \in \Delta_{d-1}$,

$$\max\left(x_1, \ldots, x_{d-1}, 1 - \sum_{i=1}^{d-1} x_i\right) \leqslant B(x_1, \ldots, x_{d-1}) \leqslant 1.$$

c) B *is a convex function.*

d) Suppose that $(U_1, \ldots, U_d)$ *has distribution* C*, as defined in Theorem 2.6. When* B *is identically equal to the lower bound given in part b), then* $U_1 = \cdots = U_d$ *almost surely, whereas if* B *is identically equal to the upper bound, then* $U_1, \ldots, U_d$ *are mutually independent.*

Remark 2.3 When $d = 2$, it is common to define a function A on $[0, 1]$ by $A(t) = B(1 - t)$ for all $t \in [0, 1]$. One can then write, for all $u, v \in (0, 1)$

$$C(u, v) = \exp\left[\ln(uv) A \left\{\frac{\ln(v)}{\ln(uv)}\right\}\right].$$

From now on, we will restrict our attention to the two-dimensional case. The following result is a specialization of Corollary 2.1 to that case.

Theorem 2.7 *Let* A *be a continuous function on* $[0, 1]$*. Define a function* C *on* $(0, 1)^2$ *by setting, for all* $u, v \in (0, 1)$,

$$C(u, v) = \exp\left[\ln(uv) A \left\{\frac{\ln(v)}{\ln(uv)}\right\}\right],$$

which we extend to the boundary of $(0, 1)^2$ *so that for all* $u \in [0, 1]$*, we have* $C(u, 1) = C(1, u) = u$ *and* $C(0, u) = C(u, 0) = 0$*. Then* C *is a copula if, and only if,* A *satisfies the following properties:*

a) $\max(t, 1 - t) \leqslant A(t) \leqslant 1$ *for all* $t \in [0, 1]$*;*

b) A *is convex on* $[0, 1]$.

As already mentioned in the Introduction, the function A in Theorem 2.7 is called a Pickands dependence function. Our next result summarizes some of the well-known properties of such functions. These facts are all easily derived using basic calculus; many of them are reported, e.g., by Deheuvels (1991).

Theorem 2.8 *The following properties relate A and C:*

a) The mapping $t \mapsto A(t)/t$ is non-increasing on $(0,1]$.

b) The mapping $t \mapsto A(t)/(1-t)$ is non-decreasing on $[0,1)$.

c) A is the upper bound if, and only if, C is the independence copula, and A is the lower bound if, and only if, C is the comonotonicity copula.

d) $A(1/2) = 1/2$ if, and only if, $A(t) = \max(t, 1-t)$ for all $t \in [0,1]$.

d) For all $u, v \in [0,1]$, $uv \leqslant C(u,v) \leqslant \min(u,v)$.

e) C is symmetric if, and only if, for all $t \in [0,1]$, $A(t) = A(1-t)$.

f) C is absolutely continuous if, and only if, A is twice differentiable.

The following definition is taken from Genest and Rivest (2001).

Definition 2.7 Let C be a bivariate copula, and let (U,V) have distribution C. The distribution of $C(U,V)$ is called the Kendall distribution.

As mentioned by the authors, and as will be seen later, this distribution relates to Kendall's tau, a standard measure of the dependence between the variables U and V with joint distribution C, whence the name. The following result is due to Ghoudi et al. (1998).

Theorem 2.9 *Let C be a bivariate extreme-value copula with Pickands dependence function A. Let $W = C(U,V)$ and $Z = \ln(V)/\ln(UV)$. The joint distribution of W and Z is given, for all $w, z \in (0,1)$, by*

$$\begin{aligned}&\Pr(W \leqslant w, Z \leqslant z)\\&\qquad = (w - w\ln w)\left\{ z + z(1-z)\frac{A'(z)}{A(z)} - \int_0^z \frac{t(1-t)}{A(t)}\, dA'(t) \right\} + w \int_0^z \frac{t(1-t)}{A(t)}\, dA'(t).\end{aligned}$$

Corollary 2.2 *The following are immediate consequences of the previous proposition:*

a) For all $w \in (0,1)$,

$$\Pr(W \leqslant w) = w - \left\{ 1 - \int_0^1 \frac{t(1-t)}{A(t)}\, dA'(t) \right\} w \ln w.$$

b) For all $z \in (0,1)$,

$$\Pr(Z \leqslant z) = z + z(1-z)A'(z)/A(z).$$

c) Assuming that A is twice differentiable, we have that, for all $w, z \in (0,1)$,

$$\Pr(W \leqslant w|Z=z) = p(z)w + \{1-p(z)\}(w - w\ln w),$$

where

$$p(z) = \frac{z(1-z)}{A(z)}A''(z)/Q'(z),$$

and

$$Q(z) = \Pr(Z \leqslant z).$$

Table 2.1 lists some parametric families of Pickands dependence functions that induce bivariate extreme-value copulas.

Table 2.1: Pickands Dependence Functions of Some Parametric Extreme-Value Copulas.

Family	Dependence Function	Parameter and Range
Cuadras–Augé	$\max(1-\theta t, 1-(1-t)\theta)$	$\theta \in [0,1]$
Galambos	$1-\{t^{-\theta}+(1-t)^{-\theta}\}^{-1/\theta}$	$\theta \in (0,\infty)$
Gumbel–Hougaard	$\{t^{\theta}+(1-t)^{\theta}\}^{1/\theta}$	$\theta \in (1,\infty)$
Hüsler–Reiß	$(1-t)\Phi(\lambda+\frac{1}{2\lambda}\ln\frac{1-t}{t})+t\Phi(\lambda+\frac{1}{2\lambda}\ln\frac{t}{1-t})$	$\lambda \in (0,\infty)$

2.5 Estimation of the Pickands dependence function

Let $(X_1,Y_1),\ldots,(X_n,Y_n)$ be a random sample from a bivariate cumulative distribution function H, with continuous margins F (for X) and G (for Y) assumed to be at first, known and extreme-value copula C with dependence function A. Then $(F(X_1),G(Y_1)),\ldots,(F(X_n),G(Y_n))$ form a random sample from C. Define, for all $i \in \{1,\ldots,n\}$ and $t \in (0,1)$,

$$\zeta_i(t) = \min\left\{\frac{-\ln F(X_i)}{1-t}, \frac{-\ln G(Y_i)}{t}\right\}$$

It can then be shown that for all $i \in \{1,\ldots,n\}$, $\zeta_i(t)$ is exponentially distributed with parameter $A(t)$. Indeed, for all $x \geqslant 0$,

$$\Pr\{\zeta_i(t) > x\} = \Pr\{F(X_i) \leqslant e^{-(1-t)x}, G(Y_i) \leqslant e^{-tx}\} = C(e^{-(1-t)x}, e^{-tx}) = \exp\{-xA(t)\}.$$

Note that for all $t \in [0,1]$, $\mathbb{E}\{\zeta_i(t)\} = A(t)$ and $\mathbb{E}\{\ln\zeta_i(t)\} = -\gamma - A(t)$, where γ is the Euler–Mascheroni constant. We therefore can get a moment estimate of $A(t)$ by considering the

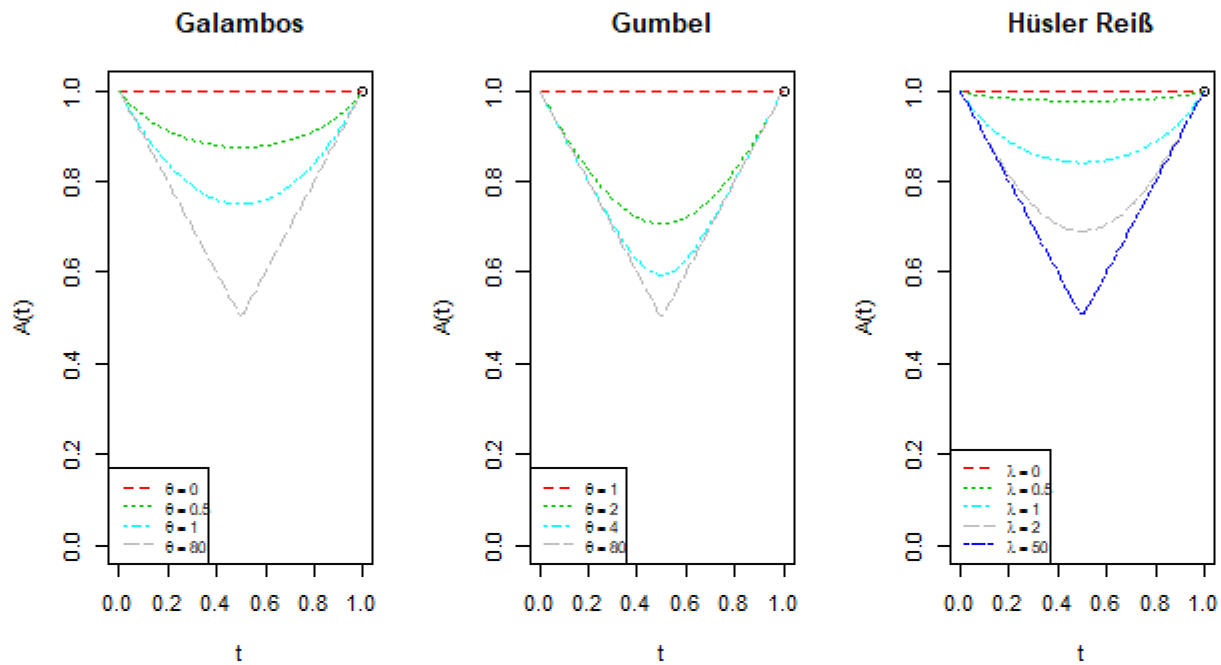

Figure 2.1: Pickands Dependence Functions for Several Parametric Families of Extreme-Value Copulas

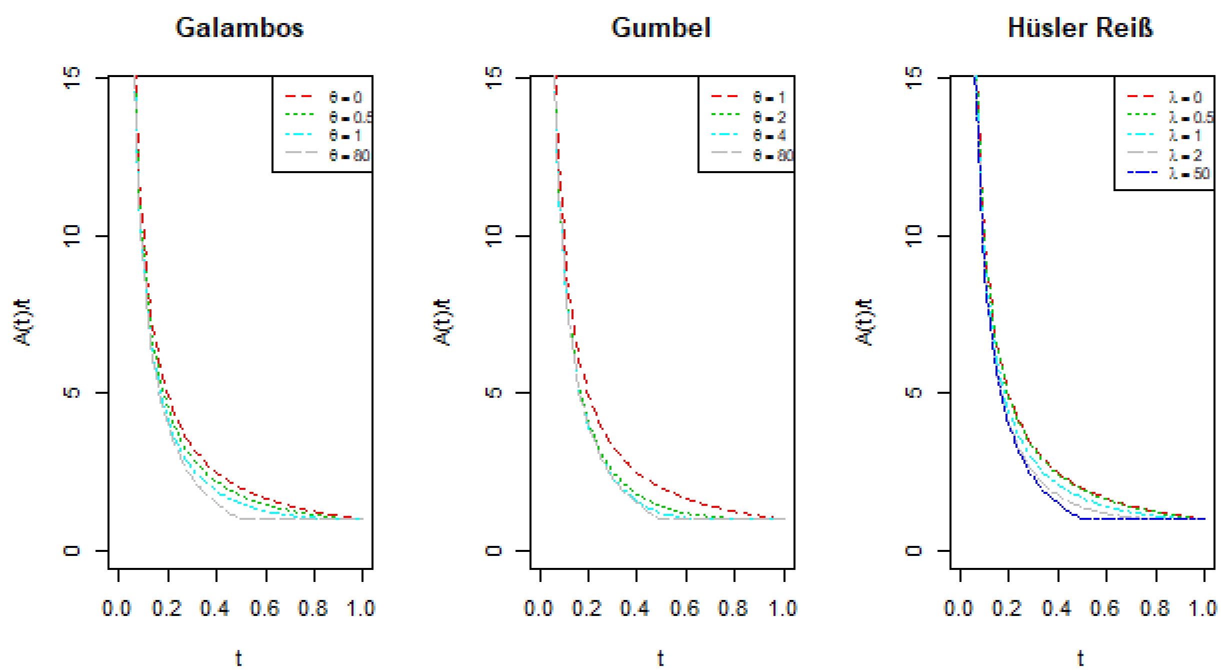

Figure 2.2: Plots of $A(t)/t$ for Several Parametric Families of Extreme-Value Copulas.

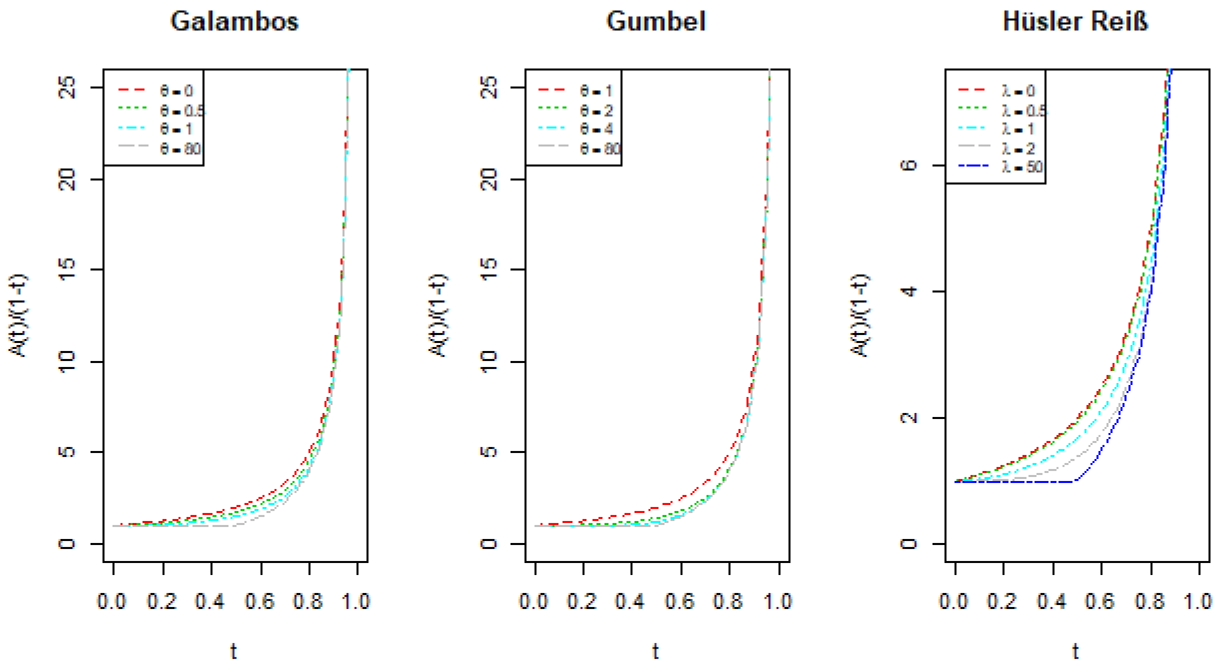

Figure 2.3: Plots of $A(t)/(1-t)$ for Several Parametric Families of Extreme-Value Copulas.

following estimators A_n^P and A_n^{CFG}, respectively defined, for all $t \in (0,1)$, by

$$A_n^P(t) = \left\{ \frac{1}{n} \sum_{i=1}^{n} \zeta_i(t) \right\}^{-1},$$

and

$$A_n^{CFG}(t) = \exp\left\{ -\gamma - \frac{1}{n} \sum_{i=1}^{n} \ln \zeta_i(t) \right\}.$$

If the margins F and G are unknown, as in most practical situations, we estimate them through their (rescaled) empirical nonparametric counterparts defined, for all $x, y \in \mathbb{R}$, by

$$F_n(x) = \frac{1}{n+1} \sum_{i=1}^{n} \mathbf{1}(X_i \leqslant x), \quad G_n(y) = \frac{1}{n+1} \sum_{i=1}^{n} \mathbf{1}(Y_i \leqslant y).$$

For each $i \in \{1, \ldots, n\}$, we then set $\hat{U}_i = F_n(X_i)$ and $\hat{V}_i = G_n(Y_i)$. Consider the empirical copula $\hat{C}_n$ of C based on the pseudo-sample $(\hat{U}_1, \hat{V}_1), \ldots, (\hat{U}_n, \hat{V}_n)$ explicitly defined, for all $u, v \in [0,1]$, by

$$\hat{C}_n(u,v) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(\hat{U}_i \leqslant u, \hat{V}_i \leqslant v).$$

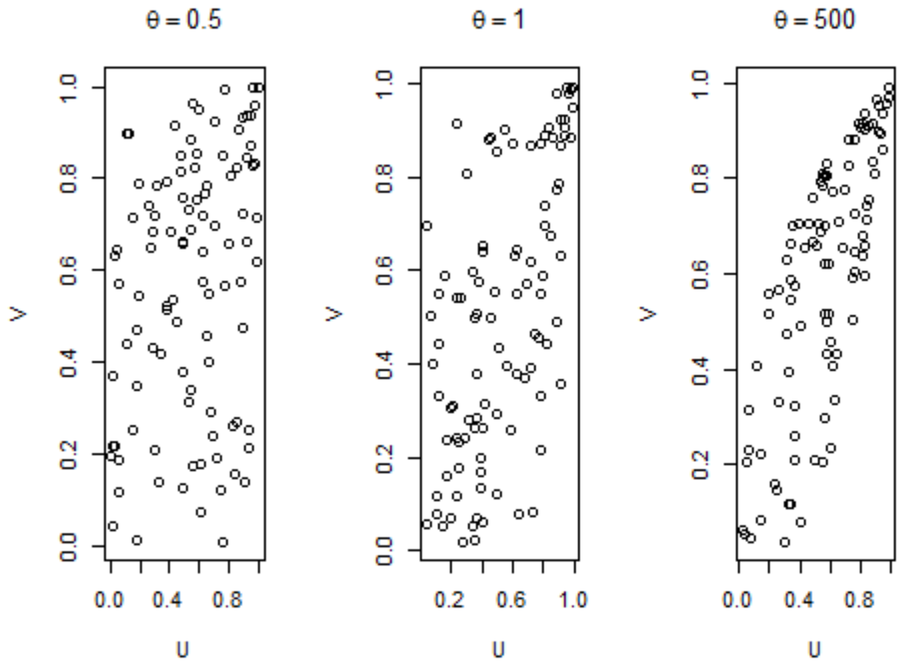

Figure 2.4: 100 Realizations from the Galambos Copula for Different Values of θ.

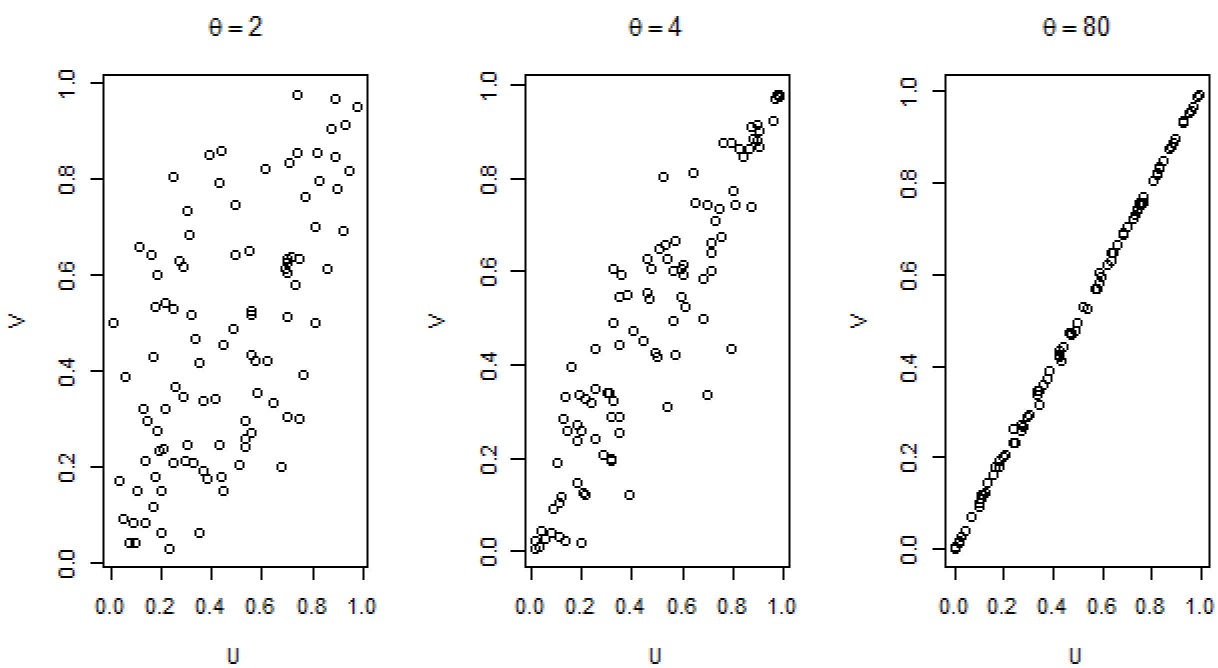

Figure 2.5: 100 Realizations from the Gumbel Copula for Different Values of θ.

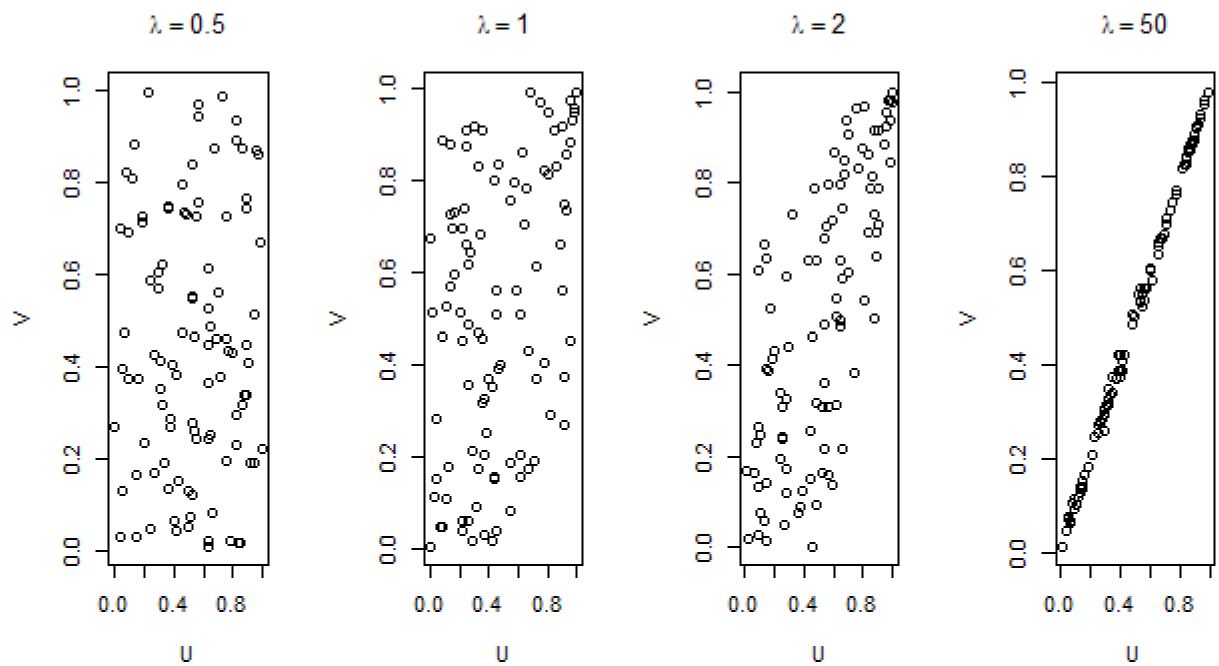

Figure 2.6: 100 Realizations from the Hüsler–Reiß Copula for Different Values of λ.

Define, for all $t \in (0, 1)$,

$$\hat{\zeta}_i(t) = \min\left(\frac{-\ln \hat{U}_i}{1-t}, \frac{-\ln \hat{V}_i}{t}\right),$$

and define a new version of A_n^P and A_n^{CFG}, based on the pseudo-sample $(\hat{U}_1, \hat{V}_1), \ldots, (\hat{U}_n, \hat{V}_n)$, by

$$A_n^P(t) = \left\{\frac{1}{n}\sum_{i=1}^{n} \hat{\zeta}_i(t)\right\}^{-1},$$

and

$$A_n^{CFG}(t) = \exp\left\{-\gamma - \frac{1}{n}\sum_{i=1}^{n} \ln \hat{\zeta}_i(t)\right\}.$$

The following result, due to Genest and Segers (2009), gives a representation of the rank-based estimators A_n^P and A_n^{CFG} in terms of the empirical copula.

Theorem 2.10 *We have, for all $t \in (0, 1)$,*

$$\hat{A}_n^P(t) = \left\{\int_0^1 \hat{C}_n(x^{1-t}, x^t)\,\frac{\mathrm{d}x}{x}\right\}^{-1},$$

and

$$\hat{A}_n^{CFG}(t) = \exp\left[-\gamma - \int_0^1 \{\hat{C}_n(x^{1-t}, x^t) - \mathbf{1}(x > e^{-1})\} \frac{\mathrm{d}x}{x\ln(x)}\right].$$

It is interesting to look at the asymptotic properties of the above estimators. The theory was developed by Genest and Segers (2009). The following theorems, quoted from their paper, summarize the asymptotic properties of both estimators, when the margins are known and when they are not. The first result concerns the case where the margins are known.

Theorem 2.11 *Assume that A is twice continuously differentiable. Define $\mathbb{A}_n^P = \sqrt{n}\,(A_n^P - A)$ and $\mathbb{A}_n^{CFG} = \sqrt{n}\,(A_n^{CFG} - A)$. As $n \to \infty$, $\mathbb{A}_n^P \rightsquigarrow \mathbb{A}^P$ and $\mathbb{A}_n^{CFG} \rightsquigarrow \mathbb{A}^{CFG}$ in $C[0,1]$, where the weak limits are defined, for all $t \in [0,1]$, by*

$$\mathbb{A}^P(t) = -A(t)^2 \int_0^1 \mathbb{C}(x^{1-t}, x^t)\,\frac{\mathrm{d}x}{x},$$

and

$$\mathbb{A}^{CFG}(t) = A(t) \int_0^1 \mathbb{C}(x^{1-t}, x^t)\,\frac{\mathrm{d}x}{x\ln x},$$

where $\mathbb{C}$ is a Gaussian random field with covariance structure given, for all $u, v, u', v' \in [0,1]$, by

$$\operatorname{cov}\{\mathbb{C}(u,v), \mathbb{C}(u',v')\} = C\{\min(u,u'), \min(v,v')\} - C(u,v)C(u',v').$$

The following theorem is an analog of the previous one when the margins are unknown.

Theorem 2.12 *Define $\hat{\mathbb{A}}_n^P = \sqrt{n}\,(\hat{A}_n^P - A)$ and $\hat{\mathbb{A}}_n^{CFG} = \sqrt{n}\,(\hat{A}_n^{CFG} - A)$. Under Condition 2.3, the process $\hat{\mathbb{C}}_n = \sqrt{n}\,(\hat{C}_n - C)$ converges weakly to a process $\hat{\mathbb{C}}$ on $[0,1]^2$ defined, for all $u, v \in [0,1]$, by*

$$\hat{\mathbb{C}}(u,v) = \mathbb{C}(u,v) - C_1(u,v)\mathbb{C}(u,1) - C_2(u,v)\mathbb{C}(1,v),$$

where $\mathbb{C}$ is a Gaussian random field on $[0,1]^2$ with covariance structure given, for all $u, v, u', v' \in [0,1]$, by

$$\operatorname{cov}\{\mathbb{C}(u,v), \mathbb{C}(u',v')\} = C\{\min(u,u'), \min(v,v')\} - C(u,v)C(u',v').$$

If A is twice continuously differentiable, then $\hat{\mathbb{A}}_n^P$ and $\hat{\mathbb{A}}_n^{CFG}$ converge weakly, as $n \to \infty$, in $C[0,1]$

to $\hat{\mathbb{A}}^P$ *and* $\hat{\mathbb{A}}^{CFG}$ *respectively defined, for all* $t \in [0,1]$, *by*

$$\hat{\mathbb{A}}^P(t) = -A(t)^2 \int_0^1 \hat{\mathbb{C}}(x^{1-t}, x^t)\, \frac{\mathrm{d}x}{x}$$

and

$$\hat{\mathbb{A}}^{CFG}(t) = A(t) \int_0^1 \hat{\mathbb{C}}(x^{1-t}, x^t)\, \frac{\mathrm{d}x}{x \ln x}.$$

A proof of this theorem is given in the paper by Genest and Segers (2009). As will be seen later, a generalized version of the previous theorem was established by Genest et al. (2011).

3

Measures of association

The notion of independence of random variables is well known and specifically defined in probability theory. However, there are several ways in which dependence can be measured.

3.1 Measures of association

The oldest and best known measure of association between two random variables is Pearson's correlation coefficient. Let X and Y be two random variables with finite second moment and strictly positive variance. The Pearson correlation coefficient of the pair (X, Y) is given by

$$\rho_{X,Y} = \frac{\mathrm{cov}(X, Y)}{\sqrt{\mathrm{var}(X)\mathrm{var}(Y)}} .$$

The basic properties of the Pearson coefficient are the following:

a) $|\rho_{X,Y}| \leqslant 1$.
b) $|\rho_{X,Y}| = 1$ if and only if $Y = aX + b$ for some $a, b \in \mathbb{R}$ with $a \neq 0$. The sign of a is the same as the sign of ρ.
c) $\rho_{aX+b,cY+d} = \mathrm{sign}(ac)\rho_{X,Y}$ for all $a, b, c, d \in \mathbb{R}$.
d) $\rho_{X,Y} = \rho_{Y,X}$.

Further, if X and Y are independent, $\rho_{X,Y} = 0$. The converse is, however, false.

To define the sample counterpart of $\rho_{X,Y}$, let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be mutually independent observations and set $\bar{X} = (X_1 + \cdots + X_n)/n$ and $\bar{Y} = (Y_1 + \cdots + Y_n)/n$. Then $\rho_{X,Y}$ can be estimated by

$$r_{xy} = \frac{\sum\limits_{i=1}^{n} X_i Y_i - n\bar{X}\bar{Y}}{\sqrt{\sum\limits_{i=1}^{n} X_i^2 - n\bar{X}^2}\sqrt{\sum\limits_{i=1}^{n} Y_i^2 - n\bar{Y}^2}} .$$

The following concept leads to a different way of measuring the association between two random variables X and Y.

Definition 3.1 Let (X, Y) be a continuous random pair. Let (X_1, Y_1) and (X_2, Y_2) be two independent copies of (X, Y). The pairs (X_1, Y_1) and (X_2, Y_2) are said to be concordant if, and only if, $(X_1 - X_2)(Y_1 - Y_2) > 0$; they are said to be discordant if, and only if, $(X_1 - X_2)(Y_1 - Y_2) < 0$.

Note that because X and Y are assumed to be continuous, one does not need to worry about the cases $X_1 = X_2$ and $Y_1 = Y_2$, which occur with probability zero. Therefore,

$$\Pr\{(X_1 - X_2)(Y_1 - Y_2) > 0\} + \Pr\{(X_1 - X_2)(Y_1 - Y_2) < 0\} = 1.$$

The notions of concordance and discordance can be exploited to define another traditional measure of association called Kendall's tau.

Definition 3.2 The Kendall's tau for the pair (X, Y) is defined as the difference of the probability of concordance and the probability of discordance, viz.

$$\begin{aligned}\tau_{X,Y} &= \Pr\{(X_1 - X_2)(Y_1 - Y_2) > 0\} - \Pr\{(X_1 - X_2)(Y_1 - Y_2) < 0\} \\ &= 2\Pr\{(X_1 - X_2)(Y_1 - Y_2) > 0\} - 1 \\ &= 4\Pr(X_1 - X_2 > 0, Y_1 - Y_2 > 0) - 1.\end{aligned}$$

The following result is obvious from the above, upon conditioning on the pair (X_1, Y_1).

Theorem 3.1 *Let (X, Y) be a continuous pair of random variables, with joint cumulative distribution function H and unique underlying copula C. Then*

$$\tau_{X,Y} = -1 + 4\int_0^1\int_0^1 C(u, v)dC(u, v).$$

Because $\tau_{X,Y}$ depends only on the copula C, it is common to refer to it as τ_C.

An alternative proof of this proposition can be found in the book by Nelsen (2006). Note that if (U, V) is a random pair with distribution C, then $\tau_C = -1 + 4\mathbb{E}(W)$.

Example 3.2 Here are the numerical values of Kendall's tau for some specific copulas.

a) For the independence copula Π, we have

$$\tau_\pi = -1 + 4\int_0^1\int_0^1 uvdudv = 0.$$

b) Consider the Fréchet–Hoeffding lower bound W, given by $W(u,v) = \max(u+v-1,0)$ for all $u,v \in [0,1]$. Then

$$\tau_W = -1 + 4\int_0^1\int_0^1 \max(u+v-1,0)dW(u,v) = -1 + 4\int_0^1 \max(u+1-u-1,0)du = -1.$$

c) Consider the Fréchet–Hoeffding upper bound M, given by $M(u,v) = \min(u,v)$ for all $u,v \in [0,1]$. Then

$$\tau_M = -1 + 4\int_0^1\int_0^1 \min(u,v)dM(u,v) = -1 + 4\int_0^1 \min(u,u)du = 1.$$

d) Consider the Farlie–Gumbel–Morgenstern copula with parameter $\theta \in [-1,1]$, defined for all $u,v \in [0,1]$ by

$$C_\theta(u,v) = uv + \theta uv(1-u)(1-v).$$

Note that the corresponding density is given, for all $u,v \in [0,1]$, by

$$dC_\theta(u,v) = \frac{\partial^2}{\partial u \partial v} C_\theta(u,v) = 1 + \theta(1-2u)(1-2v).$$

Therefore, it can be shown easily that

$$\tau_{C_\theta} = -1 + 4\int_0^1\int_0^1 C_\theta(u,v)dC_\theta(u,v) = -1 + 4\,\frac{9+2\theta}{36} = \frac{2\theta}{9}.$$

In order to simplify calculations, a simpler formula for Kendall's tau can be found in Nelsen (2006) for the case where the copula is absolutely continuous. A more general result can be found in Li et al. (2002).

Theorem 3.2 *Let C be a copula. Then*

$$\tau_C = 1 - 4\int_0^1\int_0^1 \frac{\partial C(u,v)}{\partial u}\frac{\partial C(u,v)}{\partial v}\,dudv.$$

Next, we introduce the margin-free analog of Pearson's correlation coefficient. Elimination of the marginal effect is achieved by applying the probability integral transformation and leads to the so-called "grade correlation coefficient" better known as Spearman's rho.

Definition 3.3 Let (X, Y) and (X_1, Y_1) be two continuous independent random vectors such that $X \sim X_1$ with distribution F and $Y \sim Y_1$ with distribution G. Assume that (X, Y) has cumulative distribution function H and (X_1, Y_1) is a pair of independent random variables. Spearman's rho of the pair (X, Y) is then defined to be

$$\rho_{X,Y} = 3 \left[\Pr \{(X - X_1)(Y - Y_1) > 0\} - \Pr \{(X - X_1)(Y - Y_1) < 0\} \right].$$

Theorem 3.3 *In the same setting as above, if C denotes the copula of (X, Y) and by noticing that the independence copula is the copula of (X_1, Y_1), we have*

$$\rho_{X,Y} = 12 \int_0^1 \int_0^1 \{C(u, v) - uv\} \, dudv.$$

It is hence convenient to write $\rho_{X,Y} = \rho_C$. One can also show that

$$\int_0^1 \int_0^1 C(u, v) dudv = \int_0^1 \int_0^1 uvdC(u, v).$$

A proof of this identity, generally attributed to Hoeffding, can be found in the book by Nelsen (2006). Based on this result, it is immediate that if (X, Y) is a continuous random pair with margins F and G, then $(U, V) = (F(X), G(Y))$ has cumulative distribution function C and hence

$$\mathbb{E}(UV) = \int_0^1 \int_0^1 uvdC(u, v),$$

while $\mathbb{E}(U) = \mathbb{E}(V) = 1/2$ and $\text{var}(U) = \text{var}(V) = 1/12$. Accordingly,

$$\rho_{X,Y} = \text{corr}(U, V).$$

Example 3.3 Here are the numerical values of Spearman's rho of some families of copulas.

a) Consider the independence copula Π. It is immediate that

$$\rho_\pi = 12 \int_0^1 \int_0^1 (uv - uv) dudv = 0.$$

b) Consider the Fréchet–Hoeffding lower bound W, given by $W(u, v) = \max(u + v - 1, 0)$ for all $u, v \in [0, 1]$. Then

$$\rho_W = 12 \int_0^1 \int_0^1 uv dW(u, v) - 3 = 12 \int_0^1 u(1-u) du - 3 = -1.$$

c) Consider the Fréchet–Hoeffding upper bound M, given by $M(u, v) = \min(u, v)$ for all $u, v \in [0, 1]$. Then

$$\rho_M = 12 \int_0^1 \int_0^1 uv dM(u, v) - 3 = 12 \int_0^1 u^2 du - 3 = 1.$$

d) Consider the Farlie–Gumbel–Morgenstern copula with parameter $\theta \in [-1, 1]$ defined, for all $u, v \in [0, 1]$, by

$$C_\theta(u, v) = uv + \theta uv(1-u)(1-v).$$

Recall that the corresponding density satisfies, for all $u, v \in [0, 1]$,

$$dC_\theta(u, v) = \frac{\partial^2}{\partial u \partial v} C_\theta(u, v) = 1 + \theta(1-2u)(1-2v).$$

Therefore, one can easily deduce that

$$\rho_C = 12 \int_0^1 \int_0^1 \{uv + uv\theta(1-2u)(1-2v)\} \, du dv = \frac{\theta}{3}.$$

This result was first published by Schucany et al. (1978).

The following definition is taken from Scarsini (1984).

Definition 3.4 Let (X, Y) be a continuous pair of random variables whose copula is C. A measure of association κ of X and Y (denote $\kappa_{X.Y}$ or κ_C) is called a measure of concordance if, and only if, it satisfies the following properties:

a) $\kappa_{X,Y}$ exists for every continuous pair (X, Y);

b) $\kappa_{X,Y} = \kappa_{Y,X}$;

c) $|\kappa_{X,Y}| \leqslant 1$;

d) $\kappa_{X,X} = -\kappa_{X,-X} = 1$;

e) $\kappa_{X,Y} = 0$ if X and Y are independent;

f) $\kappa_{-X,Y} = \kappa_{X,-Y} = -\kappa_{X,Y}$;

g) $\kappa_K \leqslant \kappa_C$ for all copulas C and K such that $K \leqslant C$ point-wise;

f) if $\{(X_n, Y_n)\}_{n \geqslant 1}$ is a sequence of continuous random vectors with respective copulas C_n, which converges point-wise to a copula C, then $\kappa_{C_n} \to \kappa_C$ as $n \to \infty$.

Theorem 3.4 *Kendall's tau and Spearman's rho are measures of concordance. Furthermore, if Y and X are continuous, and Y is almost surely an increasing (resp. decreasing) function of X, then $\kappa_{X,Y} = \kappa_M = 1$ (resp. $\kappa_{X,Y} = \kappa_W = -1$).*

3.2 Positive quadrant dependence

Definition 3.5 Let (X, Y) be a pair of continuous random variables with joint cumulative distribution function H, unique underlying copula C, and marginals F and G for X and Y, respectively. The variables X and Y are said to be positive quadrant dependent (PQD) if, and only if,

$$\forall_{x,y \in \mathbb{R}} \quad H(x, y) \geqslant F(x)G(y).$$

It is easy to show that X and Y are PQD, written PQD(X, Y), if, and only if,

$$\forall_{u,v \in [0,1]} \quad C(u, v) \geqslant uv.$$

It is immediate that the independence copula Π is PQD. Other examples are provided below.

Example 3.3 Consider the Fréchet–Hoeffding upper bound M. It is a PQD copula, given that, for all $u, v \in [0, 1]$, $u \geqslant uv$ and $v \geqslant uv$, and hence $M(u, v) \geqslant uv$.

Example 3.4 Consider the Farlie–Gumbel–Morgenstern copula, with parameter $\theta \in [-1, 1]$ defined, for all $u, v \in [0, 1]$, by

$$C_\theta(u, v) = uv + \theta uv(1 - u)(1 - v).$$

This copula is PQD if, and only if, $\theta \in [0, 1]$.

Example 3.5 The Fréchet–Hoeffding lower bound W is not a PQD copula. In fact we have, for all $u, v \in [0, 1]$, $W(u, v) \leqslant uv$, and for example, $W(1/2, 1/4) = 0$ but $\Pi(1/2, 1/4) = 1/8$.

Below is a collection of easily established facts mentioned, e.g., in the book by Nelsen (2006).

Theorem 3.5 *We have the following properties for a PQD pair of continuous random variables (X, Y) with copula C:*

a) PQD(X, Y) *if, and only if,* $\mathbb{E}\{f(X)g(Y)\} \geqslant 0$ *for all (almost everywhere) non-decreasing functions f and g on* Ran(X) *and* Ran(Y)*, respectively.*

b) *Any measure of concordance of C is non-negative. In particular, Kendall's tau of C is non-negative and Spearman's rho of C is non-negative.*

c) $\mathrm{PQD}(X, Y)$ *if, and only if,*

$$\forall_{x,y\in\mathbb{R}} \quad \Pr(X \leqslant x|Y \leqslant y) \Pr(Y \leqslant y) = \Pr(Y \leqslant y|X \leqslant x) \Pr(X \leqslant x) \geqslant xy.$$

3.3 Diagonal copulas

In developments to follow, we will need to call on the notion of diagonal copula. Here are a definition and some basic facts about these mathematical objects reported by Nelsen (2006).

Definition 3.6 Let C be a bivariate copula. The diagonal of C is defined to be the function $\delta_C : [0, 1] \to [0, 1]$ given, for all $z \in [0, 1]$, by $\delta_C(z) = C(z, z)$.

Theorem 3.6 *The diagonal δ_C of a copula C satisfies the following conditions:*

a) $\delta_C(0) = 0$ *and* $\delta_C(1) = 1$*;*

b) *for all $t \in [0, 1]$, $\max(2t - 1, 0) \leqslant \delta_C(t) \leqslant t$, and $\delta_C(t) = t$ for all $t \in [0, 1]$ if, and only if, $C \equiv M$;*

c) *for all $t_1, t_2 \in [0, 1]$ such that $t_1 \leqslant t_2$, $0 \leqslant \delta_C(t_2) - \delta_C(t_1) \leqslant 2(t_2 - t_1)$.*

Proof. We establish each one of these claims in turn.

a) We have $\delta_C(0) = C(0, 0) = 0$ and $\delta_C(1) = C(1, 1) = 1$.

b) We have that $\max(u + v - 1, 0) \leqslant C(u, v) \leqslant \min(u, v)$ for all $u, v \in [0, 1]$. Plugging $u = v = t$ in the previous inequality, we get $\max(2t - 1, 0) \leqslant \delta_C(t) \leqslant t$. Assume that, for all $t \in [0, 1]$, $\delta_C(t) = t$. Then,

$$\forall_{u,v\in[0,1]} \quad \min(u, v) = \delta_C\{\min(u, v)\} = C\{\min(u, v), \min(u, v)\} \leqslant C(u, v).$$

Since for all $u, v \in [0, 1]$, $C(u, v) \leqslant \min(u, v)$, then for all $u, v \in [0, 1]$, $C(u, v) = \min(u, v)$.

c) Assume that $0 \leqslant t_1 \leqslant t_2 \leqslant 1$. Then $\delta_C(t_2) - \delta_C(t_1) = C(t_2, t_2) - C(t_1, t_1) \geqslant 0$. Furthermore, $C(t_2, t_2) - C(t_1, t_1) \leqslant t_2 - t_1 + t_2 - t_1 = 2(t_2 - t_1)$.

The following result is due to Fredricks and Nelsen (1997).

Theorem 3.7 *Let δ be a function from $[0, 1]$ to $[0, 1]$ that satisfies the following three properties:*

a) *for all $t \in [0, 1]$, $\delta(t) \leqslant t$;*

b) $\delta(1) = 1$;

c) for all $t_1, t_2 \in [0, 1]$ *such that* $t_1 \leqslant t_2$, $0 \leqslant \delta_C(t_2) - \delta_C(t_1) \leqslant 2(t_2 - t_1)$.

Then, there exists a copula C whose diagonal is δ. In particular, we can take, for all $u, v \in [0, 1]$,

$$C(u, v) = \min\left[u, v, \{\delta(u) + \delta(v)\}/2\right].$$

Such a copula is called a diagonal copula with diagonal δ.

Theorem 3.8 *Let C be a diagonal copula, with diagonal δ. Then C is a PQD copula if, and only if, $\delta(t) \geqslant t^2$ for all $t \in [0, 1]$.*

Proof. Assume that C is a PQD copula. Then, for all $t \in [0, 1]$, $\delta(t) = C(t, t) \geqslant t \times t = t^2$. Reciprocally, assume that the diagonal C is PQD. Hence for all $t \in [0, 1]$, $C(t, t) = \min\{t, \delta(t)\} \geqslant t^2$ and therefore, for all $t \in [0, 1]$, $\delta(t) \geqslant t^2$. □

Theorem 3.9 *Let C be a symmetric copula with diagonal δ, and let C_δ be the diagonal copula arising from δ. Then we have that, for all $u, v \in [0, 1]$, $C(u, v) \leqslant C_\delta(u, v)$.*

Proof. We already know that for all $u, v \in [0, 1]$, $C(u, v) \leqslant \min(u, v)$. It remains to show that $C(u, v) \leqslant \{\delta(u) + \delta(v)\}/2$. Note that

$$\begin{aligned} &C\{\min(u, v), \max(u, v)\} + C\{\max(u, v), \min(u, v)\} \\ &\qquad \leqslant C\{\min(u, v), \min(u, v)\} + C\{\max(u, v), \max(u, v)\}. \end{aligned}$$

Therefore, $2C(u, v) \leqslant C(u, u) + C(v, v)$, which proves the claim. □

3.4 Tail and stochastic monotonicities

Definition 3.7 Let X and Y be random variables. The following definitions are statements about the tail monotonicity of (X, Y).

a) Y is left-tail decreasing in X, denoted LTD($Y|X$), if, and only if, $\Pr(Y \leqslant y|X \leqslant x)$ is a non-increasing function of x for all y.

b) Y is right-tail increasing in X, denoted RTI($Y|X$), if, and only if, $\Pr(Y > y|X > x)$ is a non-decreasing function of x for all y.

The concepts of $\mathrm{LTD}(X|Y)$ and $\mathrm{RTI}(X|Y)$ can be defined in a similar way.

Remark 3.1 Any of the above four properties implies positive quadrant dependence of X and Y. Furthermore, if the random variable (X, Y) is exchangeable, then LTD($Y|X$) and LTD($X|Y$) are equivalent, and so are RTI($X|Y$) and RTI($X|Y$). They are respectively denoted by LTD(X, Y) and RTI(X, Y).

The tail monotonicities of a pair (X, Y) of continuous random variables with copula C have equivalent statements in terms of C. This is recalled next. The proof of this theorem can be found in the book by Nelsen (2006).

Theorem 3.10 *Let (X, Y) be a continuous random pair with copula C. Then*

a) LTD($Y|X$) *if, and only if, for any $v \in (0, 1)$, $C(u, v)/u$ is non-increasing in u and if, and only if, for any $v \in (0, 1)$, $\partial C(u, v)/\partial u \leqslant C(u, v)/u$ for almost every $u \in (0, 1)$.*

b) RTI($Y|X$) *if, and only if, for any $v \in (0, 1)$, $\{v - C(u, v)\}/(1 - u)$ is non-increasing in u and if, and only if, for any $v \in (0, 1)$, $\partial C(u, v)/\partial u \geqslant \{v - C(u, v)\}/(1 - v)$ for almost every $v \in (0, 1)$.*

c) LTD($X|Y$) *if, and only if, for any $u \in (0, 1)$, $C(u, v)/v$ is non-increasing in v and if, and only if, for any $u \in (0, 1)$, $\partial C(u, v)/\partial v \leqslant C(u, v)/v$ for almost every $v \in (0, 1)$.*

d) RTI($X|Y$) *if, and only if, for any $u \in (0, 1)$, $\{u - C(u, v)\}/(1 - v)$ is non-increasing in v and if, and only if, for any $u \in (0, 1)$, $\partial C(u, v)/\partial v \geqslant \{u - C(u, v)\}/(1 - v)$ for almost every $v \in (0, 1)$.*

Remark 3.2 Here are some remarks about the above four properties.

a) Any of the four properties above implies positive quadrant dependence of C; see, e.g., Nelsen (2006).

b) Let C and K be two copulas and let $\theta \in (0, 1)$. Assume that C and K satisfy any one of the properties above. Then the mixture copula $C_\theta = \theta C + (1 - \theta)K$ satisfies the same property. To see this, assume for example that C and K satisfy $\mathrm{RTI}(X|Y)$. To show that C_θ satisfies it as well, note that

$$\frac{u - C_\theta(u, v)}{1 - v} = \theta \, \frac{u - C(u, v)}{1 - v} + (1 - \theta) \, \frac{u - K(u, v)}{1 - v} \,,$$

which is non-increasing in v for all u, as it is the sum of two non-increasing functions in v for all u.

c) The four properties do not in general imply one another.

Definition 3.8 Let X and Y be continuous random variables. We say that

a) Y is stochastically increasing in X, denoted SI($Y|X$), if, and only if, $\Pr(Y > y|X = x)$ is a non-decreasing function of x for all y.

b) X is stochastically increasing in Y, denoted SI$(X|Y)$, if, and only if, $\Pr(X > x|Y = y)$ is a non-decreasing function of y for all x.

The proof of the following theorem can be found in the book by Nelsen (2006).

Theorem 3.11 *Suppose that X, Y are two continuous random variables with copula C. The following results hold true:*

a) If SI*$(Y|X)$, then* $\mathrm{LTD}(Y|X)$ *and* $\mathrm{RTI}(Y|X)$.
b) If SI*$(X|Y)$, then* $\mathrm{LTD}(X|Y)$ *and* $\mathrm{RTI}(X|Y)$.
c) SI $(Y|X)$ *if, and only if, for all $v \in (0,1)$, $\partial C(u,v)/\partial u$ is non-increasing in u and if, and only if, for all $v \in (0,1)$, $C(u,v)$ is a concave function of u.*
d) SI$(X|Y)$ *if, and only if, for all $u \in (0,1)$, $\partial C(u,v)/\partial v$ is non-increasing in v and if, and only if, for all $u \in (0,1)$, $C(u,v)$ is a concave function of v.*

It is not hard to see that if C and K are two copulas satisfying, e.g., $SI(Y|X)$, then the mixture copula $C_\theta = \theta C + (1-\theta)K$, $\theta \in (0,1)$ satisfies it as well.

Definition 3.9 Let (X, Y) be a pair of continuous random variables. Assume that X has distribution F and that Y has distribution G. The upper tail dependence coefficient, denoted λ_U, is the following limit (if it exists):

$$\lambda_U = \lim_{t \uparrow 1} \Pr\{Y > G^{-1}(t) | X > F^{-1}(t)\}.$$

The lower tail dependence coefficient, denoted λ_L, is the following limit (if it exists):

$$\lambda_L = \lim_{t \downarrow 0} \Pr\{Y \leqslant G^{-1}(t) | X \leqslant F^{-1}(t)\}.$$

Theorem 3.12 *Let (X, Y) be a pair of continuous random variables with margins F and G, respectively, and let C be the unique copula of (X, Y). Then the upper and lower tail dependence coefficients (if they exist) depend only on C, viz.*

$$\lambda_U = 2 - \lim_{t \to 1^-} \frac{1 - C(t,t)}{1-t}.$$

and

$$\lambda_L = \lim_{t \to 0^+} C(t,t)/t.$$

A proof of this statement can be done by direct calculations; see, e.g., Nelsen (2006).

One notices that any two copulas with the same diagonal section have the same lower and upper tail dependence coefficient, and that these coefficients lie in the interval $[0,1]$.

Remark 3.3 It is not hard to see that if C is a copula with diagonal δ_C, then

$$\lambda_U = 2 - \delta_C'(1^-), \quad \lambda_L = \delta_C'(0^+).$$

Example 3.5 Here are some examples:

a) For the independence copula, it is clear that $\lambda_U = \lambda_L = 0$. Any two continuous random variables whose copula is Π are independent, and hence have no dependence.

b) If M denotes the Fréchet–Hoeffding upper bound, which is the case of total dependence, one has $\lambda_L = \lambda_U = 1$.

c) If W denotes the Fréchet–Hoeffding lower bound, one has $\lambda_L = \lambda_U = 0$; hence this copula exhibits no tail dependence although any two continuous random variables whose copula is C are dependent.

3.5 Dependence properties of extreme-value copulas

Let C be an extreme-value copula with Pickands dependence function $A : [0,1] \to [1/2, 1]$. Recall that for all $u, v \in (0,1)$, we have

$$C(u,v) = \exp\left[\ln(uv) A\left\{\frac{\ln(v)}{\ln(uv)}\right\}\right].$$

The following result gives analytical expressions for Kendall's tau and Spearman's rho for bivariate extreme-value copulas. These results are reported, e.g., by Ghoudi et al. (1998).

Theorem 3.13 *Let C be an extreme-value copula with Pickands dependence function A.*

a) Kendall's tau for C is

$$\tau = \int_0^1 \frac{t(1-t)}{A(t)}\, dA'(t).$$

b) Spearman's rho for C is

$$\rho = -3 + 12 \int_0^1 \frac{1}{\{A(t)+1\}^2}\, dt.$$

Proof. Both results are proved by Ghoudi et al. (1998). The first involves computing the Kendall distribution and its expectation. For Spearman's rho, we present an alternative proof. We need to compute

$$\int_0^1 \int_0^1 C(u,v) du dv.$$

We introduce the following change of variables:

$$x = \ln uv, \quad y = \ln(v)/\ln(uv),$$

so that $u = e^{x(1-y)}$ and $v = e^{xy}$. Computing the Jacobian, and making the necessary substitutions in the integral, we get

$$\frac{\rho+3}{12} = \int_0^1 \int_{-\infty}^0 \exp\{xA(y)\}\,(-xe^x)dxdy = \int_0^1 \int_0^\infty \exp\left[-x\{\,A(y)+1\,\}\right] xdxdy,$$

which easily reduces to

$$\frac{\rho+3}{12} = \int_0^1 \frac{1}{\{A(y)+1\}^2}\,dy.$$

This concludes the argument. □

We now look at the dependence and tail properties of C. First observe that C is PQD. Indeed, for all $u, v \in (0,1)$,

$$0 \leqslant A\left(\frac{\ln v}{\ln uv}\right) \leqslant 1, \quad \ln uv \leqslant 0 \quad \Rightarrow \quad \ln uvA\left(\frac{\ln v}{\ln uv}\right) \geqslant \ln uv$$

and hence for all $u, v \in (0,1)$,

$$C(u,v) \geqslant \exp(\ln uv) = uv.$$

One can also verify that C is right-tail increasing (RTI) and left-tail decreasing (LTD). These properties can be checked directly, although it was shown by Garralda-Guillem (2000) that C is stochastically increasing, which is stronger than all of the above properties.

Finally, we compute below the lower and upper tail dependence coefficients for C. We notice that, for all $t \in (0,1)$,

$$\delta_C(t) = C(t,t) = t^{2A(1/2)},$$

and hence $\delta_C'(t) = 2A(1/2)t^{2A(1/2)-1}$. Therefore,

$$\lambda_L = 2 - 2A(1/2)$$

and $\lambda_U = 0$ if $A(1/2) > 1/2$ while $\lambda_U = 1$ if $A(1/2) = 1/2$. The latter case corresponds to the Fréchet–Hoeffding upper bound.

4

Main results

Convergence properties of the Pickands and Capéraà–Fougères–Genest estimators $\hat{A}_n^P$ and $\hat{A}_n^{CFG}$ have been proved by Genest and Segers (2009). This result was extended to a broader class of copulas than those which are extreme-value by Genest et al. (2011). This was first done by noticing that the estimators $\hat{A}_n^P$ and $\hat{A}_n^{CFG}$ could be defined for any copula, and not necessarily an extreme-value copula. Indeed, one can define, for any $t \in [0,1]$,

$$A_C^P(t) = \left\{ \int_0^1 C(x^{1-t}, x^t) \, \frac{\mathrm{d}x}{x} \right\}^{-1},$$

and

$$A_C^{CFG}(t) = \exp\left[-\gamma - \int_0^1 \left\{ C(x^{1-t}, x^t) - \mathbf{1}(x > e^{-1}) \right\} \frac{\mathrm{d}x}{x \ln(x)} \right].$$

The following result is due to Genest et al. (2011).

Theorem 4.1 *Let C be a copula that is left-tail decreasing in each variable given the other, i.e., C satisfies $\mathrm{LTD}(Y|X)$ and $\mathrm{LTD}(X|Y)$. Assume further that C has a continuous density. Then, as $n \to \infty$, $\sqrt{n}\,(\hat{A}_n^P - A_C^P) \rightsquigarrow \hat{\mathbb{A}}_C^P$ and $\sqrt{n}\,(\hat{A}_n^{CFG} - A_C^{CFG}) \rightsquigarrow \hat{\mathbb{A}}_C^{CFG}$, where for all $t \in (0,1)$,*

$$\hat{\mathbb{A}}_C^P(t) = -\{A_C^P(t)\}^2 \int_0^1 \mathbb{C}(x^{1-t}, x^t) \, \frac{\mathrm{d}x}{x},$$

$$\hat{\mathbb{A}}_C^{CFG}(t) = A_C^{CFG}(t) \int_0^1 \mathbb{C}(x^{1-t}, x^t) \, \frac{\mathrm{d}x}{x \ln x},$$

and $\mathbb{C}$ a Gaussian random field on $[0,1]^2$ with covariance structure given, for all $u, v, u', v' \in$

$[0, 1]$, *by*

$$\operatorname{cov}\{\mathbb{C}(u,v), \mathbb{C}(u',v')\} = C\{\min(u,u'), \min(v,v')\} - C(u,v)C(u',v').$$

As shown by Genest et al. (2011), the functions A_C^P and A_C^{CFG} can be used to generate extreme-value copulas. For example, if C belongs to the Farlie–Gumbel–Morgenstern family, A_C^P and A_C^{CFG} are Pickands dependence functions, given that they are both convex and are point-wise bounded by the functions $t \mapsto \max(t, 1-t)$ and $t \mapsto 1$ on $[0, 1]$. They generate extreme-value copulas, known respectively as FGM-P and FGM-CFG families. Further, they can also be used to test whether an LTD copula is an extreme-value copula.

Several properties of Pickands dependence functions are retained by A_C^P and A_C^{CFG}, as mentioned by Genest et al. (2011). We will show that A_C^P retains several additional properties of Pickands dependence functions, and we will study this function in detail. Analytical necessary and sufficient conditions will be given so that A_C^P is convex.

Let C be an arbitrary copula and suppose that C is positive quadrant dependent (PQD), i.e.,

$$\forall_{u,v\in[0,1]} \quad C(u,v) \geqslant uv.$$

As is well known, we also have

$$\forall_{u,v\in[0,1]} \quad C(u,v) \leqslant \min(u,v) \equiv M(u,v).$$

Therefore, the following integral always exists and is non-zero:

$$1 \leqslant \int_0^1 C(x^{1-t}, x^t)\,\mathrm{d}x/x \leqslant \min\{1/t, 1/(1-t)\}.$$

As we will focus exclusively on the function A_C^P from here on, we henceforth drop the superscript ^{P} and set, for all $t \in [0, 1]$,

$$A_C(t) = 1 \Big/ \int_0^1 C(x^{1-t}, x^t)\,\mathrm{d}x/x. \tag{4.1}$$

The map $A_C : [0,1] \to [1/2, 1]$ is then well defined and $A_C \geqslant A_M$ point-wise. Furthermore, $A_C(t) \leqslant 1$ for all $t \in [0, 1]$ when C is PQD.

4.1 Basic analytical properties and examples

A few elementary properties of A_C were mentioned by Genest et al. (2011). They are properties a)–c) of the following result, which lists additional analytical facts about A_C.

Theorem 4.2 *The function A_C defined in (4.1) satisfies the following properties:*

a) If $C(u,v) \geqslant uv$ for all $u,v \in [0,1]$, then $\max(t,1-t) \leqslant A_C(t) \leqslant 1$ for all $t \in [0,1]$.

b) If $C(u,v) = C(v,u)$ for all $u,v \in [0,1]$, then $A_C(t) = A_C(1-t)$ for all $t \in [0,1]$.

c) If C is an extreme-value copula with Pickands dependence function A, then $A_C = A$.

d) If C and D are two PQD copulas, and for all $u,v \in [0,1]$, $C(u,v) \leqslant D(u,v)$, then for all $t \in [0,1]$, $A_C(t) \geqslant A_D(t)$.

e) If C and D are two PQD copulas, then the function C_θ defined on $[0,1]^2$ by $C_\theta = \theta C + (1-\theta)D$, where θ is parameter in $(0,1)$, is a PQD copula, and for all $t \in [0,1]$,

$$A_{C_\theta}(t) = \frac{A_C(t)A_D(t)}{\theta A_D(t) + (1-\theta)A_C(t)}.$$

Proof. We establish each claim in turn.

a) Notice that $uv \leqslant C(u,v) \leqslant \min(u,v)$ for all $u,v \in [0,1]$ and hence, if $t \in [0,1]$, then, for any $x \in [0,1]$, we have, by plugging x^{1-t} and x^t for u and v, respectively,

$$x \leqslant C(x^{1-t}, x^t) \leqslant \min(x^{1-t}, x^t).$$

Consequently, dividing by x and integrating, we find

$$1 = \int_0^1 dx \leqslant \frac{1}{A_C(t)} \leqslant \int_0^1 \min(x^{-t}, x^{t-1})dx.$$

Therefore, $A_C(t) \leqslant 1$. Furthermore, given that

$$\int_0^1 \min(x^{-t}, x^{t-1})dx \leqslant \min\left(\int_0^1 x^{-t}dx, \int_0^1 x^{t-1}dx\right) = \min\{1/(1-t), 1/t\},$$

and hence we have, for all $t \in [0,1]$, $\max(t,1-t) \leqslant A_C(t) \leqslant 1$.

b) Assume that C is symmetric. Then, for all $t \in [0,1]$, we have

$$\frac{1}{A_C(t)} = \int_0^1 C(x^{1-t}, x^t)\,\frac{dx}{x} = \int_0^1 C(x^t, x^{1-t})\,\frac{dx}{x} = \frac{1}{A_C(1-t)}\,.$$

c) If C is an extreme-value copula with Pickands dependence function A, then, for all $u,v \in (0,1)$, we have

$$C(u,v) = \exp\left[\ln(uv)A\{\ln(v)/\ln(uv)\}\right],$$

and hence
$$\frac{1}{A_C(t)} = \int_0^1 C(x^{1-t}, x^t)\frac{dx}{x} = \int_0^1 x^{A(t)-1}dx = \frac{1}{A(t)}.$$

Therefore, $A_C(t) = A(t)$ for all $t \in [0,1]$.

d) Assume that for all $u, v \in [0,1]$, $C(u,v) \leqslant D(u,v)$. Then, for any $t \in [0,1]$ and all $x \in [0,1]$,
$$C(x^{1-t}, x^t) \leqslant D(x^{1-t}, x^t),$$
which implies that
$$\int_0^1 C(x^{1-t}, x^t)\,\frac{\mathrm{d}x}{x} \leqslant \int_0^1 D(x^{1-t}, x^t)\,\frac{\mathrm{d}x}{x}$$
and hence, for all $t \in [0,1]$, $A_C(t) \geqslant A_D(t)$.

e) It is clear that C_θ is a distribution function, as it is the mixture of two distributions. To check that it is a copula, we observe that, for all $u \in [0,1]$,
$$C_\theta(u,1) = \theta C(u,1) + (1-\theta)D(u,1) = \theta u + (1-\theta)u = u,$$
and, for all $v \in [0,1]$,
$$C_\theta(1,v) = \theta C(1,v) + (1-\theta)D(1,v) = \theta v + (1-\theta)v = v.$$
Therefore, if C and D are PQD, we have, for all $u, v \in [0,1]$,
$$C_\theta(u,v) = \theta C(u,v) + (1-\theta)D(u,v) \geqslant \theta uv + (1-\theta)uv = uv.$$
Furthermore, for all $t \in [0,1]$,
$$\begin{aligned}\frac{1}{A_{C_\theta}(t)} &= \int_0^1 C_\theta(x^{1-t}, x^t)\,\frac{\mathrm{d}x}{x} \\ &= \theta\int_0^1 C(x^{1-t}, x^t)\,\frac{\mathrm{d}x}{x} + (1-\theta)\int_0^1 D(x^{1-t}, x^t)\,\frac{\mathrm{d}x}{x} \\ &= \theta\frac{1}{A_C(t)} + (1-\theta)\frac{1}{A_D(t)},\end{aligned}$$
We deduce that, for all $t \in [0,1]$,
$$A_{C_\theta}(t) = \frac{A_C(t)A_D(t)}{\theta A_D(t) + (1-\theta)A_C(t)}.$$

This concludes the argument. □

Example 4.1 In general, the function A_C is not analytically tractable. However, there are cases where one can get a closed form for A_C. Here are two examples.

a) In Genest et al. (2011), the function A_C is computed for the Farlie–Gumbel–Morgenstern copula (Morgenstern, 1956). It is stated there that if $C_\theta(u,v) = uv + \theta uv(1-u)(1-v)$ for all $u, v \in [0,1]$ and some $\theta \in [0,1]$, then

$$A_{C_\theta}(t) = \frac{2t^2 - 2t - 4}{2t^2 - 2t - 4 + (3t^2 - 3t)\theta}\,.$$

b) We will generalize the example in the previous part. Consider the copula defined, for all $u, v \in [0,1]$, by

$$C_{\theta,\alpha}(u,v) = uv + \theta u^\alpha v^\alpha (1-u)(1-v),$$

where $\alpha \in [1,\infty)$ and $\theta \in [0,1]$. For this copula, which is a generalization of the Farlie–Gumbel–Morgenstern copula, we find, for all $x, t \in (0,1)$,

$$C_{\theta,\alpha}(x^{1-t}, x^t) = x + \theta x^\alpha (1 - x^t)(1 - x^{1-t}) = x + \theta x^\alpha (1 + x - x^t - x^{1-t}),$$

and

$$\frac{C_{\theta,\alpha}(x^{1-t}, x^t)}{x} = 1 + \theta x^{\alpha-1}(1 + x - x^t - x^{1-t}).$$

Therefore,

$$\begin{aligned}\int_0^1 \frac{C_{\theta,\alpha}(x^{1-t}, x^t)}{x}\,dx &= 1 + \theta\left(\frac{1}{\alpha} + \frac{1}{\alpha+1} - \frac{1}{\alpha+t} - \frac{1}{\alpha-t+1}\right)\\ &= \frac{\alpha(\alpha+1)\{-t^2 + t + \alpha(\alpha+1)\} + (2\alpha+1)\theta(-t^2+t)}{\alpha(\alpha+1)\{-t^2 + t + \alpha(\alpha+1)\}}\,.\end{aligned}$$

We conclude that, for all $t \in (0,1)$,

$$A_{C_{\theta,\alpha}}(t) = \frac{\alpha(\alpha+1)\{-t^2 + t + \alpha(\alpha+1)\}}{\alpha(\alpha+1)\{-t^2 + t + \alpha(\alpha+1)\} + (2\alpha+1)\theta(-t^2+t)}\,.$$

4.2 Shared properties with Pickands dependence functions

The following properties are satisfied by any Pickands dependence function. The first two properties have been used in the study of Archimax copulas and the derivation of the Kendall distribution for extreme-value copulas; see Capéraà et al. (2000) and Charpentier et al. (2014), as well as Ghoudi et al. (1998).

Theorem 4.3 *The following statements hold true:*

a) The function $t \mapsto A_C(t)/t$ *is non-increasing on* $(0, 1]$.

b) The function $t \mapsto A_C(t)/(1-t)$ *is non-decreasing on* $[0, 1)$.

c) If $A_C(1/2) = 1/2$*, then* $A_C(t) = \max(t, 1-t)$ *for all* $t \in [0, 1]$.

Proof. To prove claim a), first observe that if $\lambda \geqslant 1$, then, for all $t \in [0, 1]$, we have $A_C(t) \leqslant \lambda A_C(t/\lambda)$. Indeed, from the definition of A_C and upon setting $u = -\ln(x)$, one finds

$$\frac{1}{A_C(t)} = \int_0^1 C(x^{1-t}, x^t) = \int_0^\infty C\{e^{-u(1-t)}, e^{-ut}\}du = \frac{1}{\lambda}\int_0^\infty C\{e^{-v(1-t)/\lambda}, e^{-vt/\lambda}\}dv,$$

where the last identity is justified by the change of variable $v = \lambda u$. Now observe that because $\lambda \geqslant 1$ and $t \in [0, 1]$, one has $1 - t/\lambda \geqslant (1-t)/\lambda \geqslant 0$ and hence

$$\frac{1}{A_C(t)} \geqslant \frac{1}{\lambda}\int_0^\infty C\{e^{-v(1-t/\lambda)}, e^{-vt/\lambda}\}dv = \frac{1}{\lambda A_C(t/\lambda)}.$$

Now to see that the function $t \mapsto A_C(t)/t$ is non-increasing on $(0, 1]$, let $0 < t_1 \leqslant t_2 \leqslant 1$. Then setting $\lambda = t_2/t_1 \geqslant 1$ and using the previous relation, we find

$$A_C(t_2) \leqslant \frac{t_2}{t_1} A_C\left(\frac{t_2}{t_2/t_1}\right)$$

or, equivalently, $A_C(t_2)/t_2 \leqslant A_C(t_1)/t_1$.

To establish claim b), let $\lambda \geqslant 1$ be fixed. Then, for all $t \in [0, 1]$, we have $A_C(1-t) \leqslant \lambda A_C(1-t/\lambda)$.The proof uses similar arguments, substitutions and inequalities as the above proof. Indeed, we have

$$\begin{aligned}\frac{1}{A_C(1-t)} &= \int_0^1 C(x^t, x^{1-t})\,\frac{dx}{x} = \int_0^\infty C(e^{-ut}, e^{-u(1-t)})du \\ &= \frac{1}{\lambda}\int_0^\infty C\{e^{-vt/\lambda}, e^{-v(1-t)/\lambda}\}dv \\ &\geqslant \frac{1}{\lambda}\int_0^\infty C\{e^{-vt/\lambda}, e^{-v(1-t/\lambda)}\}dv = \frac{1}{\lambda A_C(1-t/\lambda)}.\end{aligned}$$

Therefore, if $\lambda \geqslant 1$, then $A_C(1-t) \leqslant \lambda A_C(1-t/\lambda)$ for all $t \in [0, 1]$.

To verify that the function $t \mapsto A_C(t)/(1-t)$ is non-decreasing on $[0,1)$, let $0 \leqslant t_1 \leqslant t_2 < 1$. Then $1 \geqslant 1-t_1 \geqslant 1-t_2 > 0$. Hence setting $\lambda = (1-t_1)/(1-t_2) \geqslant 1$ and using the previous relation, we find

$$A_C\{1-(1-t_1)\} \leqslant \frac{1-t_1}{1-t_2} A_C\left(1 - \frac{1-t_1}{\frac{1-t_1}{1-t_2}}\right)$$

and hence

$$A_C(t_1) \leqslant \frac{1-t_1}{1-t_2} A_C(t_2),$$

which implies that

$$\frac{A_C(t_1)}{1-t_1} \leqslant \frac{A_C(t_2)}{1-t_2}.$$

Finally, let us consider claim c). Assume that $A_C(1/2) = 1/2$. Let $t \in [0,1/2]$. Then

$$\frac{A_C(0)}{1-0} \leqslant \frac{A_C(t)}{1-t} \leqslant \frac{A_C(1/2)}{1-1/2}$$

and hence

$$1 \leqslant \frac{A_C(t)}{1-t} \leqslant 1,$$

which implies that $A_C(t) = 1-t$ on $[0,1/2]$. Next, let $t \in [1/2,1]$. Then

$$\frac{A_C(1)}{1} \leqslant \frac{A_C(t)}{t} \leqslant \frac{A_C(1/2)}{1/2},$$

and hence $A_C(t) = t$ on $[1/2,1]$. This concludes the argument. □

The following result gives an analytical property of A_C.

Theorem 4.4 *A_C is Lipschitz on any subinterval $[c,d] \subset (0,1)$ and hence is almost everywhere differentiable on $(0,1)$.*

Proof. The following function $f_C : [0,1] \mapsto \mathbb{R}$, $f_C(t) = 1/A_C(t)$. Let $c \leqslant t_1 < t_2 \leqslant d$. As shown in Nelsen (2006) and stated in Theorem 2.1, we have

$$|C(u,v) - C(u',v')| \leqslant |u-u'| + |v-v'|$$

for all $u, u'v, v' \in [0,1]$. Therefore,

$$\begin{aligned}
|A_C(t_2) - A_C(t_2)| &= \left|\frac{1}{f_C(t_1)} - \frac{1}{f_C(t_2)}\right| = \left|\frac{f_C(t_1) - f_C(t_2)}{f_C(t_1) f_C(t_2)}\right| \\
&\leqslant |f_C(t_1) - f_C(t_2)| \\
&\leqslant \int_0^\infty |C\{e^{-u(1-t_1)}, e^{-ut_1}\} - C\{e^{-u(1-t_2)}, e^{-ut_2}\}| du \\
&\leqslant \int_0^\infty |e^{-u(1-t_1)} - e^{-u(1-t_2)}| + |e^{-ut_1} - e^{-ut_2}| du \\
&\leqslant \left\{\frac{1}{c^2} + \frac{1}{(1-d)^2}\right\} (t_2 - t_1).
\end{aligned}$$

This concludes the argument. □

Theorem 4.5 *For almost every* $t \in [0,1]$,

$$\begin{aligned}
-A_C'(t) = \int_0^\infty x e^{-x(1-t) f_C(t)} C_1\{e^{-x(1-t) f_C(t)}, e^{-x f_C(t) t}\} dx \\
- \int_0^\infty x e^{-x t f_C(t)} C_2\{e^{-x(1-t) f_C(t)}, e^{-x t f_C(t)}\} dx.
\end{aligned}$$

Proof. First observe that, for almost all $t \in (0,1)$, we have

$$f_C'(t) = \int_0^\infty u e^{-u(1-t)} C_1\{e^{-u(1-t)}, e^{-ut}\} - u e^{-ut} C_2\{e^{-u(1-t)}, e^{-ut}\} du.$$

After making the substitution $u = x f_C(t)$, we find

$$\begin{aligned}
f_C'(t) = f_C(t)^2 \Big[\int_0^\infty x e^{-x(1-t) f_C(t)} C_1\{e^{-x(1-t) f_C(t)}, e^{-x f_C(t) t}\} dx \\
- \int_0^\infty x e^{-x t f_C(t)} C_2\{e^{-x(1-t) f_C(t)}, e^{-x t f_C(t)}\} dx \Big]
\end{aligned}$$

and hence

$$-A'_C(t) = \int_0^\infty xe^{-x(1-t)f_C(t)}C_1\{e^{-x(1-t)f_C(t)}, e^{-xf_C(t)t}\}dx$$
$$- \int_0^\infty xe^{-xtf_C(t)}C_2\{e^{-x(1-t)f_C(t)}, e^{-xtf_C(t)}\}dx.$$

This concludes the argument. □

Example 4.2 Let C be an extreme-value copula with Pickands dependence function A. That is, for all $u, v \in (0,1)$, we have

$$C(u,v) = \exp\big[\{\ln(uv)A\{\ln(v)/\ln(uv)\}\big].$$

The partial derivatives of C with respect to u and v are respectively given (almost everywhere) by

$$C_1(u,v) = \frac{\partial}{\partial u}C(u,v) = \frac{1}{u}\left[A\left\{\frac{\ln(v)}{\ln(uv)}\right\} - \frac{\ln(v)}{\ln(uv)}A'\left\{\frac{\ln(v)}{\ln(uv)}\right\}\right]C(u,v)$$

and

$$C_2(u,v) = \frac{\partial}{\partial v}C(u,v) = \frac{1}{v}\left[A\left\{\frac{\ln(v)}{\ln(uv)}\right\} + \frac{\ln(u)}{\ln(uv)}A'\left\{\frac{\ln(v)}{\ln(uv)}\right\}\right]C(u,v).$$

It has been previously shown that $A_C(t) = A(t)$ for all $t \in [0,1]$. Then we must have $A'_C(t) = A'(t)$ for almost all $t \in [0,1]$. We verify this result as follows. First,

$$\int_0^\infty xe^{-x(1-t)f_C(t)}C_1\{e^{-x(1-t)f_C(t)}, e^{-xf_C(t)t}\}dx$$
$$= \int_0^\infty xe^{-x(1-t)f_C(t)}e^{x(1-t)f_C(t)}\{A(t) - tA'(t)\}\exp\{-xf_C(t)A(t)\}\,dx$$
$$= \int_0^\infty x\{A(t) - tA'(t)\}e^{-x}dx = A(t) - tA'(t).$$

Second,

$$\int_0^\infty xe^{-xtf_C(t)} C_2\{e^{-x(1-t)f_C(t)}, e^{-xf_C(t)t}\} dx$$
$$= \int_0^\infty xe^{-xtf_C(t)} e^{xtf_C(t)} \{A(t) + (1-t)A'(t)\} \exp\{-xf_C(t)A(t)\} dx$$
$$= \int_0^\infty x\{A(t) + (1-t)A'(t)\} e^{-x} dx = A(t) + (1-t)A'(t).$$

We conclude that, for all $t \in (0,1)$,

$$-A'_C(t) = A(t) - tA'(t) - \{A(t) + (1-t)A'(t)\} = -A'(t).$$

4.3 Characterizations of copulas with $A_C(1/2) \in \{1/2, 1\}$

The following proposition allows us to give a characterization of copulas C such that $A_C(1/2) = 1$ and $A_C(1/2) = 1/2$.

Theorem 4.6 *The following statements hold true:*

a) $A_C(1/2) = 1$ *if, and only if, the diagonal section* δ_C *of* C *is given by* $\delta_C(z) = z^2$ *for all* $z \in [0,1]$.

b) $A_C(1/2) = 1/2$ *if, and only if, for all* $u, v \in [0,1]$, $C(u,v) = \min(u,v)$.

Proof. First consider claim a). One direction is clear. For, if $\delta_C(z) = z^2$ for all $z \in [0,1]$, then

$$\frac{1}{A_C(1/2)} = \int_0^1 \frac{C(x^{1/2}, x^{1/2})}{x} dx = \int_0^1 dx = 1.$$

Now assume that $A_C(1/2) = 1$. Then

$$\int_0^1 \frac{C(x^{1/2}, x^{1/2})}{x} dx = 1$$

or equivalently,

$$\int_0^1 \left\{ \frac{C(x^{1/2}, x^{1/2})}{x} - 1 \right\} dx = 0.$$

The fact that C is PQD implies that $C(x^{1/2}, x^{1/2})/x \geqslant 1$ for all $x \in [0,1]$, and hence we must then have $C(x^{1/2}, x^{1/2})/x = 1$ for all $x \in [0,1]$ or equivalently $\delta_C(x^{1/2}) = x$ for all $x \in [0,1]$ and hence $\delta_C(z) = z^2$ for all $z \in [0,1]$.

Turning to claim b), we have already seen in Theorem 4.3 that $A_M(t) = \max(t, 1-t)$ for all $t \in [0,1]$. To establish the other implication, note that if $A_C(1/2) = 1/2$, then

$$\int_0^1 \frac{C(x^{1/2}, x^{1/2})}{x}\, dx = 2.$$

Making the substitution $x = s^2$, we deduce that

$$\int_0^1 \frac{C(s,s)}{s}\, ds = 1$$

or, equivalently, that

$$\int_0^1 \{C(s,s)/s - 1\} ds = 0.$$

Now, given that $C(s,s) \leqslant s$ for all $s \in [0,1]$, we must have $C(s,s) = s$ for all $s \in [0,1]$ and hence $\delta_C(s) = s$ for all $s \in [0,1]$. Therefore, we have $C(u,v) = \min(u,v)$ for all $u, v \in [0,1]$. This concludes the argument. $\square$

4.4 Analytical necessary and sufficient conditions for the convexity of A_C

Theorem 4.7 *Define the functions g_C, $k_C : (0,1) \mapsto \mathbb{R}$ by setting, for all $t \in (0,1)$,*

$$g_C(t) = \int_0^\infty x e^{-x(1-t)f_C(t)} C_1\{e^{-x(1-t)f_C(t)}, e^{-xf_C(t)t}\} dx$$

and

$$k_C(t) = \int_0^\infty x e^{-xtf_C(t)} C_2\{e^{-x(1-t)f_C(t)}, e^{-xtf_C(t)}\} dx.$$

The following statements hold true:

a) For almost all $t \in (0,1)$, $g_C(t) = A_C(t) - tA'_C(t)$ and $k_C(t) = A_C(t) + (1-t)A'_C(t)$.

b) If A_C is twice differentiable, then A_C is convex if, and only if, g_C is non-increasing and if, and only if, k_C is non-decreasing.

c) If C is symmetric, then, for almost all $t \in [0,1]$, $g_C(t) = k_C(1-t)$.

Proof. To establish claim a), consider the function $h : (0, \infty) \times [0, 1] \mapsto [0; \infty)$ defined by

$$h(\alpha, t) = \int_0^\infty C\{e^{-u(1-t)\alpha}, e^{-ut\alpha}\} du.$$

Notice that by the change of variable $v = \alpha u$ in the previous integral, we get

$$h(\alpha, t) = \frac{1}{\alpha} f_C(t).$$

Therefore, the function h is differentiable with respect to α, and

$$\frac{\partial}{\partial\alpha} h(\alpha, t) = -\frac{1}{\alpha^2} f_C(t).$$

Moreover,

$$\frac{\partial}{\partial\alpha} h(\alpha, t) = -(1-t) \int_0^\infty u e^{-u(1-t)\alpha} C_1\{e^{-u(1-t)\alpha}, e^{-ut\alpha}\} du \\ - t \int_0^\infty u e^{-ut\alpha} C_2\{e^{-u(1-t)\alpha}, e^{-ut\alpha}\} du.$$

Setting $\alpha = f_C(t)$ in the two expressions for $\partial h(\alpha, t)/\partial\alpha$, we get, for all $t \in [0, 1]$,

$$-(1-t) g_C(t) - t k_C(t) = -A_C(t).$$

We also know that, for almost all $t \in (0, 1)$, $A'_C(t) = k_C(t) - g_C(t)$. Combining the previous two equations, and solving for k_C and g_C, we get that, for almost all $t \in (0, 1)$,

$$k_C(t) = A_C(t) + (1-t) A'_C(t),$$

and

$$g_C(t) = A_C(t) - t A'_C(t).$$

Turning to claim b), assume that A_C is twice differentiable. Then A_C is convex if, and only if, for all $t \in (0, 1)$, $A''_C(t) \geqslant 0$. Notice that $g'_C(t) = -t A''_C(t)$ and that $k'_C(t) = (1-t) A''_C(t)$. The conclusion is then immediate.

Finally, to prove claim c), assume that C is symmetric. Then we have, for all $t \in [0, 1]$, $A_C(t) = A_C(1-t)$. Differentiating both sides with respect to t, we deduce that, for almost all $t \in (0, 1)$,

$A'_C(t) = -A'_C(1-t)$. Therefore, we get that, for almost all $t \in [0,1]$,

$$\begin{aligned} g_C(1-t) &= A_C(1-t) - (1-t)A'_C(1-t) \\ &= A_C(t) + (1-t)A'_C(t) = k_C(t). \end{aligned}$$

This concludes the argument. □

4.5 Probabilistic arguments concerning A_C

Theorem 4.8 *Let C be a copula and (U,V) have distribution C. For any fixed $t \in (0,1)$, define the random variable*

$$\xi(t) = \min\{-(\ln U)/(1-t), -(\ln V)/t\}.$$

Then, for all $t \in (0,1)$, $A_C(t) = 1/\mathbb{E}_C\{\xi(t)\}$, where $\mathbb{E}_C$ denotes the expectation under the assumption that $(U,V) \sim C$.

Proof. For all $x \geqslant 0$, we have

$$\begin{aligned} \Pr\{\xi(t) \geqslant x\} &= \Pr[\min\{-(\ln U)/(1-t), -(\ln V)/t\} \geqslant x] \\ &= \Pr\{U \leqslant e^{-x(1-t)}, V \leqslant e^{-xt}\} = C\{e^{-x(1-t)}, e^{-xt}\}. \end{aligned}$$

Given that, for all $t \in [0,1]$, $\xi(t)$ is almost surely non-negative, we have that

$$\mathbb{E}_C\{\xi(t)\} = \int_0^\infty \Pr\{\xi(t) \geqslant x\}\, dx = \int_0^\infty C\{e^{-x(1-t)}, e^{-xt}\} dx.$$

It follows that, for all $t \in [0,1]$, $A_C(t) = 1/\mathbb{E}_C\{\xi(t)\}$. This concludes the argument. □

We now give a probabilistic interpretation for the following result.

Theorem 4.9 *The following statements hold true:*

a) If $C(u,v) = C(v,u)$ for all $u,v \in [0,1]$, then $A_C(t) = A_C(1-t)$ for all $t \in [0,1]$.

b) If C is an extreme-value copula with Pickands dependence function A, then $A_C = A$.

Proof. To establish claim a), assume that C is symmetric. Then, for all $t \in [0,1]$,

$$\begin{aligned} \mathbb{E}_C\{\xi(t)\} &= \mathbb{E}_C\{\min\{-(\ln U)/(1-t), -(\ln V)/t\}\} \\ &= \mathbb{E}_C[\min\{-(\ln V)/(1-t), -(\ln U)/t\}] \\ &= \mathbb{E}_C\{\xi(1-t)\}. \end{aligned}$$

Hence, for all $t \in [0,1]$, $A_C(t) = A_C(1-t)$.

To establish claim b), assume that C is an extreme-value copula with Pickands dependence function A. We know that, for all $t \in (0,1)$, $\xi(t)$ follows an exponential law with mean $1/A(t)$. Therefore, $\mathbb{E}_C\{\xi(t)\} = 1/A(t)$ and hence, for all $t \in [0,1]$, $A_C(t) = A(t)$. This concludes the argument. □

A probabilistic argument can also be used to give a proof of the following result.

Theorem 4.10 *The following statements hold true:*

a) The function $t \mapsto A_C(t)/t$ is non-increasing on $(0,1]$.

b) The function $t \mapsto A_C(t)/(1-t)$ is non-decreasing on $[0,1)$.

We note that the proofs are similar, due to common properties of expectations and integrals.

Proof. To establish claim (i), let $0 < t_1 \leqslant t_2 \leqslant 1$. Note that

$$\begin{aligned}
\mathbb{E}_C\left\{\xi(t_1)\right\} &= \mathbb{E}_C\left[\min\left\{-\ln(U)/(1-t_1), -\ln(V)/t_1\right\}\right] \\
&= \frac{t_2}{t_1}\,\mathbb{E}_C\left[\min\left\{-\ln(U)\frac{t_1}{t_2-t_1t_2}, -\ln(V)/t_2\right\}\right] \\
&\leqslant \frac{t_2}{t_1}\,\mathbb{E}_C\left[\min\{-\ln(U)/(1-t_2), -\ln(V)/t_2\}\right] \\
&\leqslant \frac{t_2}{t_1}\,\mathbb{E}_C\{\xi(t_2)\}.
\end{aligned}$$

and hence $A_C(t_2)/t_2 \leqslant A_C(t_1)/t_1$. We note that $t_1/(t_2-t_1t_2) = 1/(t_2/t_1-t_2) \leqslant 1/(1-t_2)$, which has been used in the first inequality.

The proof of claim b) is similar and left to the reader. □

4.6 Counterexamples for convexity

If C is a PQD copula such that $A_C(1/2) = 1$, then the diagonal δ_C of C must satisfy $\delta_C(z) = z^2$ for all $z \in [0,1]$. Using this result, we will find a copula C such that A_C is not convex. First of all, we ensure that $\delta_C(z) = z^2$ for all $z \in [0,1]$ and hence that $A_C(1/2) = 1$. We then find a point in $t_0 \in (0,1)$ such that $A(t_0) \neq 1$. This will ensure that A_C is not convex.

Lemma 4.1 *Let $A : [0,1] \to [0,1]$ be a convex function such that for all $t \in [0,1]$, $\max(t, 1-t) \leqslant A(t) \leqslant 1$, and $A(0) = A(1) = 1$. Then $A(1/2) = 1$ if, and only if, $A(t) = 1$ for all $t \in [0,1]$.*

Proof. Assume that $A(1/2) = 1$. Let $t \in [0,1/2]$. We write $1/2$ as a convex combination of t and 1, viz.

$$\frac{1}{2} = \frac{1}{2(1-t)}\,t + \frac{1-2t}{2(1-t)}\,1.$$

Using the convexity of A, we deduce that

$$1 = A(1/2) \leqslant \frac{1}{2(1-t)}\, A(t) + \frac{1-2t}{2(1-t)}\, A(1),$$

which implies that

$$A(t) \geqslant \{2(1-t)\} \left\{ 1 - \frac{1-2t}{2(1-t)} \right\} = 1,$$

and hence $A(t) = 1$ for all $t \in [0, 1]$. A similar argument holds if $t \in [1/2, 1]$, by expressing $1/2$ as a convex combination of 0 and t. Thus we can conclude. □

Consider the copula $C_\star$ defined, for all $u, v \in [0, 1]$, by

$$C_\star(u, v) = \min\{u, v, (u^2 + v^2)/2\}.$$

Notice that C is a PQD diagonal copula. For all $x \in [0, 1]$, we have

$$C_\star(x^{1/2}, x^{1/2}) = \min(x^{1/2}, x^{1/2}, x) = x$$

and hence

$$\frac{1}{A_{C_\star}(1/2)} = \int_0^1 \frac{C_\star(x^{1/2}, x^{1/2})}{x} dx = \int_0^1 dx = 1.$$

Accordingly, $A_{C_\star}(1/2) = 1$. It can be easily seen that, for all $x \in [0, 1]$,

$$C_\star(x^{2/3}, x^{1/3}) = \min \left\{ x^{1/3}, x^{2/3}, \frac{1}{2}\,(x^{4/3} + x^{2/3}) \right\} = \frac{1}{2}\,(x^{4/3} + x^{2/3}).$$

Consequently,

$$\frac{1}{A_{C_\star}(1/3)} = \frac{1}{2} \int_0^1 (x^{1/3} + x^{-1/3}) dx = 9/8$$

and hence $A_{C_\star}(1/3) = 8/9$. Therefore, in view of the previous lemma, we can conclude that $A_{C_\star}$ is not a convex function. □

Theorem 4.11 *Let $\theta \in (0, 1)$ be a parameter and consider the mixture copula $C_\theta = \theta C_\star + (1-\theta)\Pi$. Then A_{C_θ} is non-convex for all $\theta \in (0, 1)$.*

Proof. As shown before we have, for all $t \in (0, 1)$,

$$A_{C_\theta}(t) = \frac{A_{C_\star}(t)}{\theta + (1-\theta) A_{C_\star}(t)}\,.$$

Note that $A_{C_\theta}(1/2) = 1$ but that, at the same time,

$$A_{C_\theta}(1/3) = \frac{8/9}{\theta + (1-\theta)8/9} = \frac{1}{1+\theta/8}\,.$$

Further note that, for any $y \in (8/9, 1)$, one can find $\theta \in (0,1)$ such that $A_{C_\theta}(1/3) = y$. Finally, observe that if C is a symmetric PQD copula such that $A_C(1/2) = 1$, then $A_C(t) \geqslant A_{C_\star}(t)$ for all $t \in [0,1]$. □

Theorem 4.12 *There exists a symmetric copula C which is stochastically increasing and for which A_C is not convex.*

Proof. We consider the mixture of two symmetric extreme-value copulas. As shown before, tail monotonicities and stochastic monotonicities are preserved under mixtures. Therefore, any mixture of two symmetric extreme-value copulas is stochastically increasing.

Let C_3 be the Gumbel–Hougaard copula defined, for all $u, v \in (0,1)$, by

$$C_3(u,v) = \exp\{(|\ln u|^3 + |\ln v|^3)^{1/3}\}.$$

This is an extreme-value copula with Pickands dependence function $A_3 : [0,1] \to [1/2, 1]$ defined, for all $t \in [0,1]$, by

$$A_3(t) = \sqrt[3]{(1-t)^3 + t^3}.$$

Consider the mixture $C = C_3/2 + \Pi/2$. Therefore, for all $t \in [0,1]$, we have

$$A_C(t) = \frac{2A_{C_3}(t)}{1 + A_{C_3}(t)}.$$

The derivative of A_C, defined for all $t \in (0,1)$, is given by

$$A'_C(t) = \frac{4t-2}{(3t^2-3t+1)^{2/3}\,\{(3t^2-3t+1)^{1/3}+1\}^2}\,.$$

We have $A'_C(0) = -1/2$ and $A'_C(0.2) < -0.56$, which implies that the derivative is not non-decreasing, and hence A_C is not convex. □

5
Conclusion

This thesis was concerned with bivariate extreme-value copulas. The first chapter introduced the problem that appeared in Genest et al. (2011). The convexity of a generalization of the Pickands dependence function, which we denoted by A_C^P, where C denotes the underlying copula, was the subject of our study. Genest et al. (2011) realized that the convexity of A_C^P will lead to an inconsistency in a test based on the Pickands estimator.

In Chapter 2 we introduced the notion of a copula, which is defined to be a distribution on the d-dimensional unit cube, with uniform margins. We then recalled several analytical properties of a copula, including uniform continuity properties, as well as the Fréchet–Hoeffding upper and lower bounds. Further, we related the conditional distributions of a copula to its partial derivatives. Then, we introduced the notion of an empirical copula and its asymptotic properties. Afterwards, we introduced extreme-value copulas and focused on the bivariate case. We gave examples of such copulas, and saw that they are characterized by a univariate function defined on $[0, 1]$ called the Pickands dependence function. This function is a convex function on $[0, 1]$, bounded point-wise by $t \mapsto \max(t, 1 - t)$ and $t \mapsto 1$. We also mentioned that this function dictates the behavior of the extreme-value copula. Furthermore, we introduced the Pickands and Capéraà–Fougères–Genest estimators and mentioned their asymptotic properties.

Chapter 3 consisted of studying measures of dependence and correlation between two random variables. We introduced Spearman's ρ and Kendall's τ of two random variables X and Y, and saw that they depend only on their copula. We also introduced the concepts of tail monotonicities and stochastic ordering of copulas, and proved that many of these properties are preserved by mixtures. Finally, we introduced upper and lower tail dependence coefficients and computed them for extreme-value copulas.

In Chapter 4, we presented our main results, relating to the generalized Pickands dependence function, which we denoted A_C. We sought to extend the results of Genest et al. (2011). After establishing obvious properties of A_C, we proved that the map $t \mapsto A_C(t)/t$ is non-increasing and that the function $t \mapsto A_C(t)/(1-t)$ is non-decreasing. These properties are also satisfied by a Pickands dependence functions and play an important role in establishing properties of bivariate extreme-value copulas. We further showed that A_C is differentiable almost everywhere in $(0, 1)$.

In the last part of the thesis, we tried to characterize the copulas such that $A_C(1/2) = 1/2$. This happens if, and only if, C is the Fréchet–Hoeffding upper bound, just as in the case of the Pickands dependence function. However, if $A_C(1/2) = 1$, then one can only infer about the diagonal of C, which must agree with the diagonal of the independence copula. Using this fact, we found an example of a copula C for which A_C is not convex. We also studied the derivative and deduced a closed form for the derivative of A_C by using a substitution, which allows us to circumvent the usual rules of differentiation. We then gave necessary and sufficient conditions in order for A_C to be convex. An interesting problem is to try to find families of copulas for which A_C is convex.

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