McGill University

Relative Hyperbolicity of Graphs of Free Groups with Cyclic Edge Groups

by

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SUMMARY

We prove that any finitely generated group which splits as a graph of free groups with cyclic edge groups is hyperbolic relative to certain finitely generated subgroups, known as the peripheral subgroups. Each peripheral subgroup splits as a graph of cyclic groups. Any graph of free groups with cyclic edge groups is the fundamental group of a graph of spaces X where vertex spaces are graphs, edge spaces are cylinders and attaching maps are immersions. We approach our theorem geometrically using this graph of spaces.

We apply a "coning-off" process to peripheral subgroups of the universal cover $\widetilde{X} \to X$ obtaining a space $Cone(\widetilde{X})$ in order to prove that $Cone(\widetilde{X})$ has a linear isoperimetric function and hence satisfies *weak relative hyperbolicity* with respect to peripheral subgroups.

We then use a recent characterisation of relative hyperbolicity presented by D.V. Osin to serve as a bridge between our linear isoperimetric function for $Cone(\widetilde{X})$ and a complete proof of relative hyperbolicity. This characterisation allows us to utilise geometric properties of X in order to show that $\pi_1(X)$ has a *linear relative isoperimetric function.* This property is known to be equivalent to relative hyperbolicity.

KEYWORDS: Relative hyperbolicity, Graphs of free groups with cyclic edge groups, Relative isoperimetric function, Weak relative hyperbolicity.

SOMMAIRE

On prouve qu'un groupe finiement engendré qui est décomposable en graphe de groupes libres avec groupes d'arêtes cycliques est hyperbolique relatif à un ensemble de sous-groupes finiement engendrés appelés les *sous-groupes péripheriques.* Un graphe de groupes libres avec groupes d'arêtes cycliques est le groupe fondamentale d'un graphe d'espaces X où les espaces de sommets sont des graphes et les espaces d'arêtes sont des cylindres. On utilise une approche géometrique pour la démonstration de l'hyperbolicité relative en utilisant l'espace X.

On applique aux sous-groupes péripheriques du recouvrement universel $\widetilde{X}$ un processus qui transforme les sous-groupes péripheriques en "cones" péripheriques. On obtient un espace $Cone(\widetilde{X})$ et l'on démontre que $Cone(\widetilde{X})$ a une fonction isopérimetrique linéaire. Ceci implique que l'espace X satisfait *hyperbolicité relative faible* par rapport aux sous-groupes péripheriques.

On utilise ensuite une caractérisation plus récente de l'hyperbolicité relative présentée par D.V. Osin. Cette caractérisation est le lien entre la fonction isopérimetrique linéaire de $Cone(\widetilde{X})$ et une preuve complète de l'hyperbolicité relative. On se sert de les propriétés géométriques particulières de l'espace X pour démontrer que $\pi_1(X)$ a une *fonction isopérimetrique relative* linéaire. Cette propriété est équivalente à la propriété d'hyperbolicité relative.

MOTS CLÉS: Hyperbolicité relative, Graphe de groupes libres avec groupes dárêtes cycliques, Fonction isopérimetrique relative, Hyperbolicité relative faible.

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Chapter 1

INTRODUCTION

1.1. Summary

This thesis focuses on a graph of spaces X whose vertex spaces are graphs whose edge spaces are cylinders and whose attaching maps are immersions. The goal of the thesis is to prove the *relative hyperbolicity* of the fundamental group of such a space.

We had originally planned to apply Farb's approach to relative hyperbolicity. This involves showing that $\pi_1(X)$ satisfies *weak relative hyperbolicity* with respect to certain subgroups and that it satisfies the *Bounded Coset Penetration property* (BCP property). In Chapter 2 we vertified a linear isoperimetric function for a simplicial complex corresponding to coned-off space of X. The results in Chapter 2 of this thesis prove the first of Farb's conditions, weak relative hyperbolicity. More elaborate combinatorial techniques would have led to the proof that the BCP property was also satisfied.

In Chapter 3 a more recent characterisation of relative hyperbolicity by Osin serves as a bridge between our results in Chapter 2 and a complete proof of relative hyperbolicity. It allows us to utilise the geometric properties of X. Indeed, the linear isoperimetric function we used to prove weak relative hyperbolicity is actually strong enough to prove that $\pi_1(X)$ has a *linear relative isoperimetric function.* This property is known to be equivalent to relative hyperbolicity.

1.2. Guide to Thesis

We now give a brief guide to the thesis and its main results.

In Chapter 2 we consider the universal cover $\widetilde{X} \to X$. We examine the structure of $\widetilde{X}$ and present the notion of a *peripheral subspace in* $\widetilde{X}$ as illustrated in Figure 1.1. A peripheral subspace is the union of a maximal collection of parallel edge spaces.

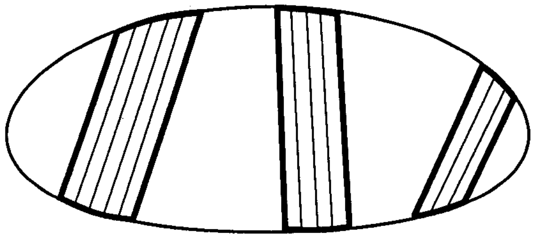


FIG. 1.1. Three peripheral subspaces in $\widetilde{X}$.

We then impose a new structure on $\widetilde{X}$ by "coning-off" each peripheral subspace to form peripheral cones. Coning-off consists of adding a 0-cell, called a conepoint, to each peripheral subspace. Then each cell in a peripheral subspace is joined to its corresponding conepoint by a higher dimensional cell. The resulting space is the *coned space* and is denoted $Cone(\widetilde{X})$. Figure 1.2 illustrates a coned space with exactly three peripheral cones.

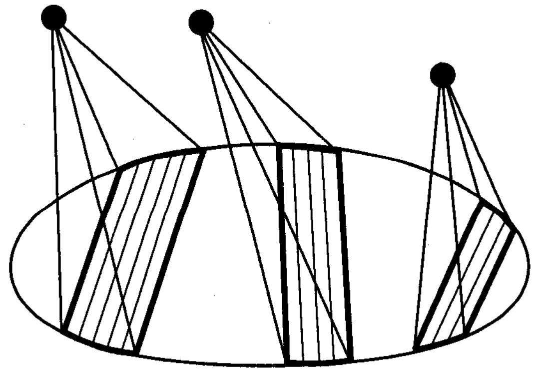


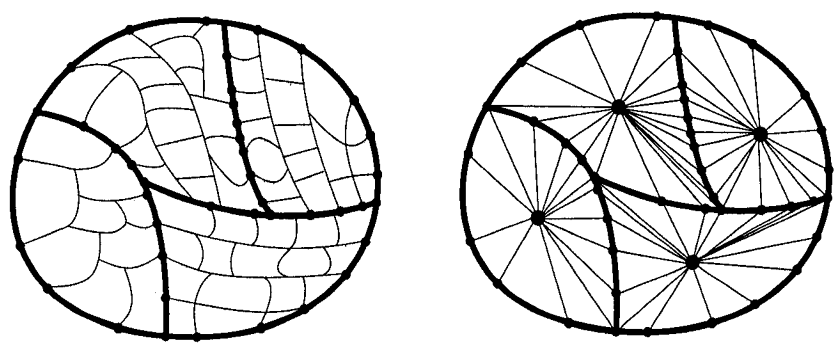
FIG. 1.2. Coning-off peripheral subspaces in $\widetilde{X}$.

We then study the non-singular disc diagrams $D \to Cone(\widetilde{X})$ in the coned space. We divide the disc diagrams into *regions* corresponding to peripheral cones in $Cone(\widetilde{X})$. We build a canoncical disc diagram $D_C \to Cone(\widetilde{X})$ from D called the *coned disc diagram* whose regions are more efficient. We do this by using a sequence of disc diagrams modified from the original $D \to Cone(\widetilde{X})$. A resulting coned diagram is illustrated in Figure 1.3. Paths in the coned space satisfy the following important property:

Lemma 1.2.1. *Any combinatorial closed simple path $P \to Cone(\widetilde{X})$ is the boundary path of a disc diagram $D_C \to Cone(\widetilde{X})$ such that $Area(D_C) \leq K|P|$ where $K = K(\widetilde{X})$ is a constant.*

This easily yields the main theorem of Chapter 2:

Theorem 1.2.2. *The coned space $Cone(\widetilde{X})$ has a linear isoperimetric function.*



1.3(a) Regions in a disc diagram.

1.3(b) Corresponding coned regions.

FIG. 1.3. Coned disc diagram in $Cone(\widetilde{X})$.

Combined with the fact that there is a bound on the number of sides of 2-cells of X this result proves that $Cone(\widetilde{X})$ is δ-hyperbolic, and hence verifies weak relative hyperbolicity of $\pi_1(X)$. According to the work of Osin, instead of verifying BCP we use our strong linear isoperimetric function property to verify a different criterion and to show relative hyperbolicity of $\pi_1(X)$. We note however, that it appears that more recent results of Manning-Groves are even more closely aligned with the method followed in this thesis.

In Chapter 3, we use Osin's presentation of this alternative characterisation of relative hyperbolicity. We first introduce a similar structure to the coned space $Cone(\widetilde{X})$ called the *capped space* and denoted $Cap(X)$, illustrated in Figure 1.4. The space $Cap(X)$ is built from X by the addition of a mapping cylinder for each *cylindrical subspace* of X (a cylindrical subspace in X is the quotient of a peripheral subspace in $\widetilde{X}$ by its stabilizer, this stabilizer is referred to as a *peripheral subgroup*).

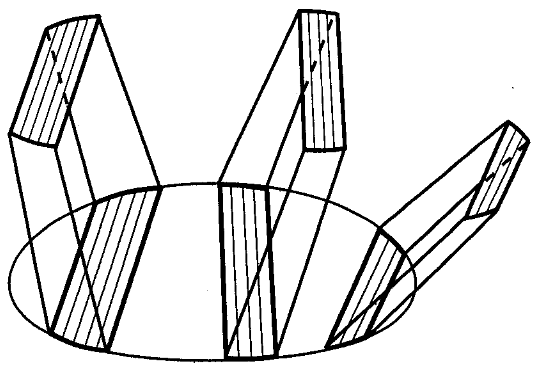


FIG. 1.4. Capped space $Cap(X)$.

We proceed to modify $Cap(X)$ by collapsing cells along free faces and contracting trees and intervals crossed with trees always preserving the fundamental group while giving the new space a desired structure. The resulting space $R(X)$ is called the *modified capped space* because it will eventually yield a *relative presentation for* $\pi_1(Cap(X))$ with respect to the peripheral subgroups.

In this chapter we also give the definition of a relative presentation for a group and we present the definition of the *2-complex of a relative presentation* to draw the link between the space $R(X)$ and a relative presentation for $\pi_1(Cap(X))$.

The mapping cylinder structure of $Cap(X)$ allows us to define three types of cells of $Cap(X)$. An open cell in the base X of $Cap(X)$ is called a *base cell.* A cap is a cylindrical subspace attached to X in the process of forming $Cap(X)$ and an open cell in a cap is called a *cap cell.* Any other open cell is called a *vertical cell.*

Given a disc diagram $D_R \to R(X)$ we define the *relative area of* D_R to be the number of 2-cells of D_R that are mapped to vertical 2-cells of $R(X)$. We define a new measure of length in $R(X)$ called *peripheral length* that is analogous to length in a relative presentation. Then, using the linear isoperimetric function for a coned disc diagram $D_C \to Cone(\widetilde{X})$ we build a disc diagram $D_R \to R(X)$ whose relative area is bounded above by a linear function of its *peripheral length.*

Proposition 1.2.3. *(Osin) Let G be a finitely generated group, generated by the set X and let $\{H_1, ..., H_n\}$ be a collection of subgroups of G. Then the following are equivalent:*

(1) G has finite relative presentation with respect to $\{H_1, ..., H_n\}$ and its corresponding isoperimetric function is linear.

(2) G is relatively hyperbolic with respect to $\{H_1, ..., H_n\}$.

The relative area of a disc diagram $D_R \to R(X)$ is used in combination with Osin's definition of the *relative isoperimetric function.* This characterisation is the property that allows us to prove the principal results of the thesis, stated below.

Theorem 1.2.4. *The group $\pi_1(Cap(X))$ has a relative presentation with linear relative isoperimetric function.*

Combining our relative isoperimetric function with Osin's criterion for relative hyperbolicity we obtain our main result:

Main Theorem 1.2.5. *The group $\pi_1(X)$ is relatively hyperbolic with respect to the peripheral subgroups.*

The thesis ends in Chapter 4 with some possible generalizations of our results as well as a contextualisation of the problem within current research in geometric group theory.

Chapter 2

WEAK RELATIVE HYPERBOLICITY

2.1. Peripheral Subspaces in Graphs of Spaces

Definition 2.1.1. We build a topological space X called a *graph of spaces* in the following manner. We begin with an *underlying graph* Γ_X with vertex set $V(\Gamma_X)$ and edge set $E(\Gamma_X)$, for every vertex v in $V(\Gamma_X)$ there is an *associated vertex space* X_v and for every edge e in $E(\Gamma_X)$, an *associated edge space* $X_e \times I$. An edge e in $E(\Gamma_X)$ attached at the vertices $i(e)$ and $\tau(e)$ gives way to corresponding attaching maps $\phi_{i(e)} : X_e \times \{0\} \to X_{i(e)}$ and $\phi_{\tau(e)} : X_e \times \{1\} \to X_{\tau(e)}$. We define X as the quotient of $(\bigcup_{v \in V(\Gamma_X)} X_v) \bigcup (\bigcup_{e \in E(\Gamma_X)} X_e \times I)$ by the above identifications $\phi_{i(e)}$ and $\phi_{\tau(e)}$.

We remark that there is an obvious map $X \to \Gamma_X$ mapping each X_v to v and each $X_e \times I$ to e.

Remark 2.1.2. Let X be a graph of spaces, and $\hat{X} \to X$ be a covering space, $\hat{X}$ is a graph of spaces in the following sense: vertex spaces of $\hat{X}$ are components of preimages of vertex spaces of X and open edge spaces of $\hat{X}$ are preimages of open edge spaces of X.

Lemma 2.1.3. *Let X be a graph of spaces and and $\widetilde{X} \to X$ be its universal cover, then the underlying graph $\Gamma_{\widetilde{X}}$ of $\widetilde{X}$ is a tree.*

Proof. We pick a closed based path $P \to \widetilde{X}$, passing through vertex spaces and edge spaces alternately. There is a continuous map from $\widetilde{X}$ to $\Gamma_{\widetilde{X}}$ mapping vertex spaces to points and edge spaces to intervals. If $\Gamma_{\widetilde{X}}$ is not a tree then P is mapped by this continuous map to an essential closed based path, contradicting $\pi_1(\widetilde{X}) = 0$

$\square$

Throughout this chapter we will be considering the following particular type of graph of spaces X. We let X be a graph of spaces where each of its vertex spaces X_v is a graph, and its edge spaces $X_e \times I$ are cylinders, that is, each X_e is a circle, we also require that the attaching maps be immersions and that X consists of a finite number of vertex and edge spaces. We can suppose that the spaces are cell complexes and the maps are combinatorial. The vertex spaces X_v inherit the cell structure of a graph. We consider each X_e circle to consist of one 0-cell and one 1-cell, then $X_e \times I$ has the cell structure of the product of these two cell complexes. A simple example of such a graph is illustrated in Figure 2.1

Definition 2.1.4. A *cylindrical space* X, is a graph of spaces whose edge spaces are cylinders, vertex spaces circles and attaching maps immersions.

Remark 2.1.5. Any graph of free groups with cyclic edge groups is the fundamental group of a space X as defined above, that is its vertex spaces are graphs and its edge spaces are cylinders. Figure 2.1 illustrates the canonical construction of such a graph of spaces when a presentation for a group. We consider the group $F_2 *_Z F_2$ given by the presentation $\langle a, b, c, d \mid aba^{-1}b^{-1} = cdcd \rangle$.

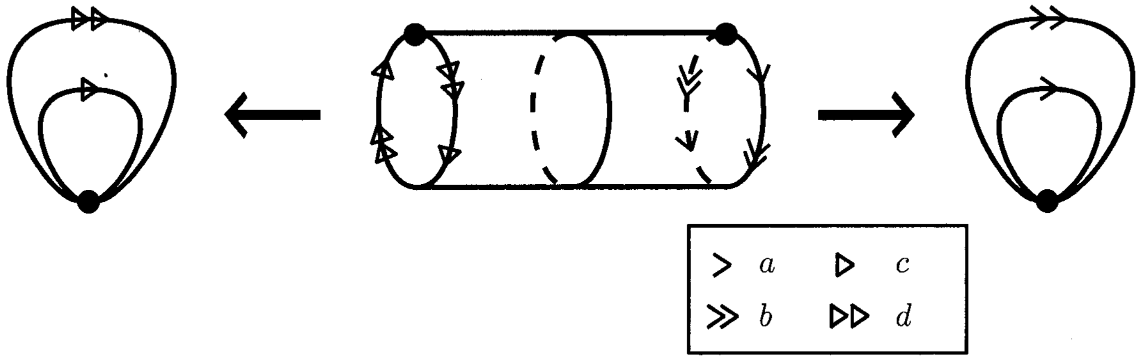


FIG. 2.1. Construction of a graph of spaces.
The construction above produces a graph of spaces whose fundamental group is $F_2 *_Z F_2$ given by the presentation $\langle a, b, c, d \mid aba^{-1}b^{-1} = cdcd \rangle$.

Now consider our case, X is a graph of spaces whose of edge spaces are cylinders, vertex spaces graphs, attaching maps immersions and is comprised of a finite number of edge and vertex spaces. We are interested in its universal cover $\widetilde{X} \to X$. By Lemma 2.1.3 it is a tree of spaces, in the sense that its underlying graph is a tree. The vertex spaces $\widetilde{X}_v$ are universal covers of the graphs in X, and are thus trees. The edge spaces $\widetilde{X}_e \times I$ are universal covers of edge spaces in X and are thus strips made up of 2-cells whose boundary paths are homeomorphic to circles. The universal cover $\widetilde{X}$ inherits its cell structure from the base space X. Figures 2.2 and 2.3 illustrate the structure of $\widetilde{X}$. The underlying graph $\Gamma_{\widetilde{X}}$ is a tree, as mentioned previously.

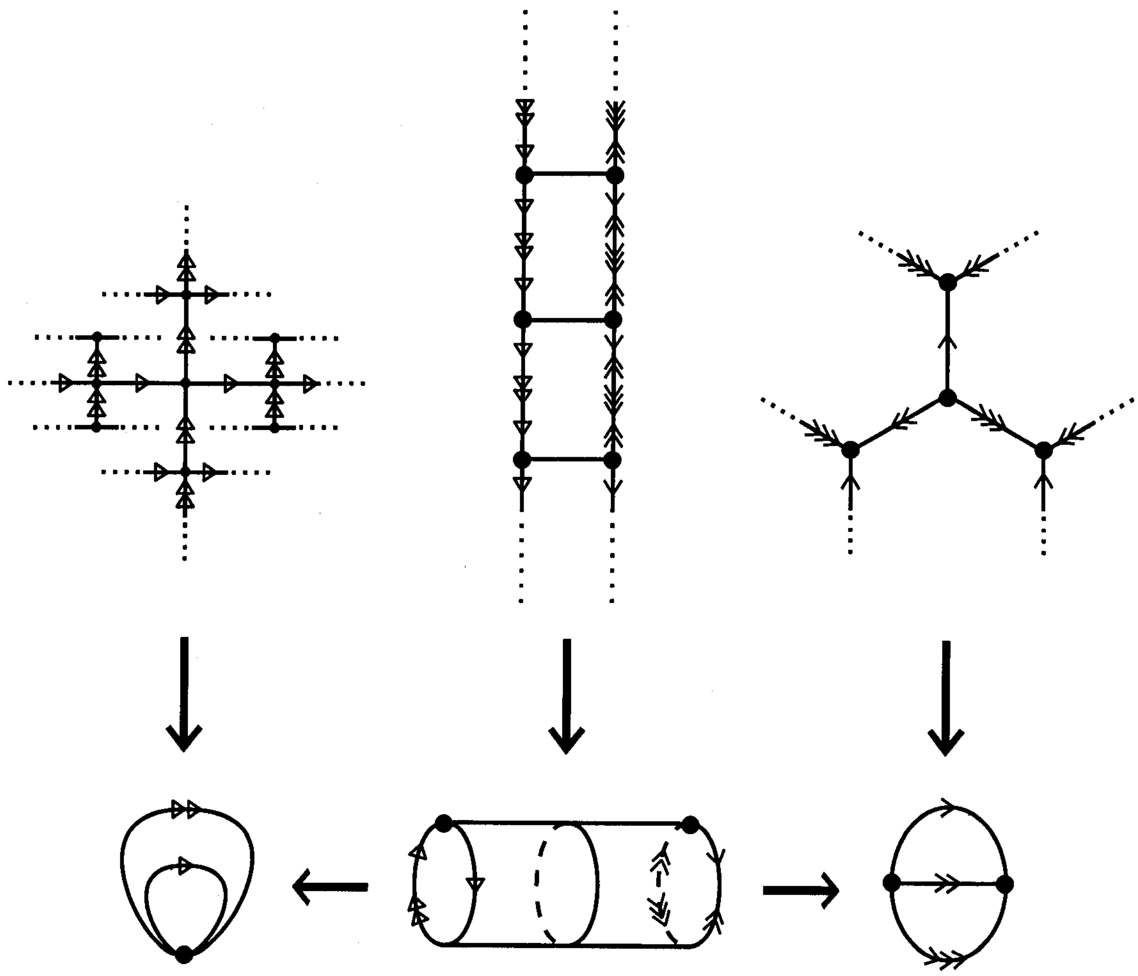
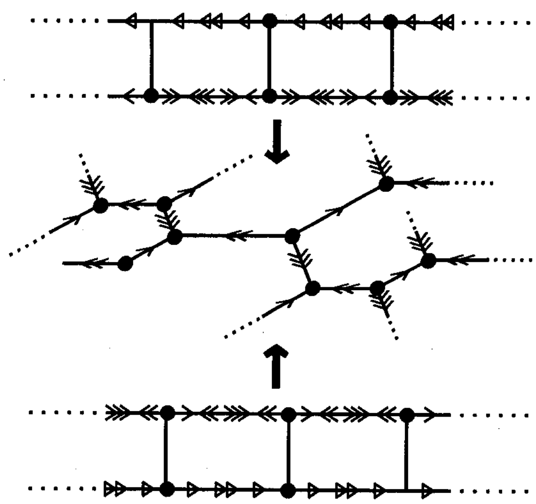
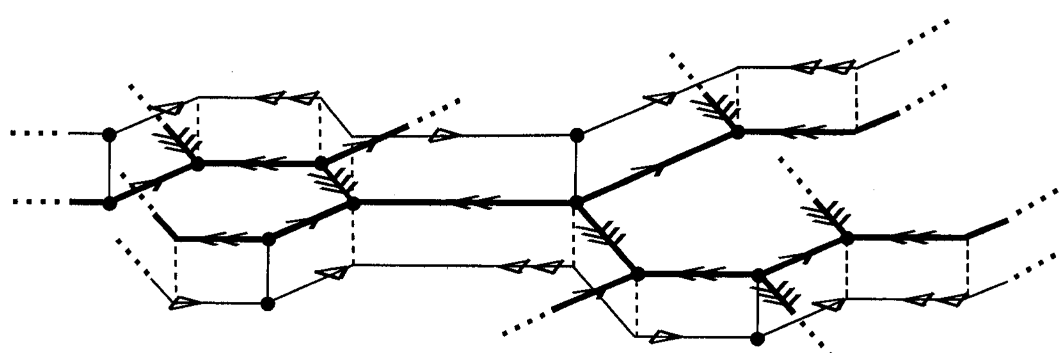


FIG. 2.2. Construction of universal cover of graph of spaces The base space is a graph of spaces X, with edge space cylinder and vertex spaces graphs. Arrows on each 1-cell represent immersed attaching maps. The universal covers of each component of X are indicated above them. The universal cover of X is constructed by gluing these universal covers together, as illustrated in Figure 2.3.



2.3(a) Gluing strips (universal cover of cylinder) to trees (universal cover of graph).



2.3(b) Resulting universal cover.

FIG. 2.3. Construction of universal cover of graph of spaces Using the example in Figure 2.2 we show how spaces are glued together to form the universal cover.

Lemma 2.1.6. *Let X be a graph of spaces whose edge spaces are cylinders, vertex spaces graphs and attaching maps immersions. The 2-cells in its universal cover $\widetilde{X}$ are homeomorphic to closed discs.*

PROOF. The proof follows from the structure of $\widetilde{X}$ as a tree of spaces. See Figure 2.3 for an illustration of such 2-cells. □

We are now interested in another way of decomposing our space $\widetilde{X}$ into subspaces. To do so, we introduce the notion of a peripheral subspaces of $\widetilde{X}$.

Definition 2.1.7. Two edge spaces in $\widetilde{X}$ are *immediately parallel* if they intersect in an infinite line.

Two edge spaces $E = \widetilde{X}_e \times I$ and $E' = \widetilde{X}_{e'} \times I$ are *parallel* if there exists a sequence of edge spaces $E = E_0, E_1, ..., E_k = E'$ such that E_i and E_{i+1} are immediately parallel for all $0 \leq i \leq k$.

Definition 2.1.8. A *peripheral subspace in* $\widetilde{X}$ is the union of edge spaces in a maximal collection of parallel edge spaces.

Definition 2.1.9. The stabilizer of a peripheral suspace in $\widetilde{X}$ is a *peripheral subgroup* and the quotient of a peripheral subspace in $\widetilde{X}$ by its peripheral subgroup is an *associated cylindrical subspace.*

Remark 2.1.10. The quotient of a peripheral subspace in $\widetilde{X}$ by its stabilizer is a cylindrical space as in Definition 2.1.4. Peripheral subspaces in $\widetilde{X}$ cover cylindrical spaces of X.

Theorem 2.1.11. *Peripheral subspaces in $\widetilde{X}$ intersect in finite line segments. Moreover the length of the intersection of any two peripheral subspaces in $\widetilde{X}$ is bounded by a constant $k = k(\widetilde{X})$.*

PROOF. We recall that the underlying graph $\Gamma_{\widetilde{X}}$ of $\widetilde{X}$ is a tree. We then note that a peripheral subspace in $\widetilde{X}$ is made up of parallel edge spaces so a peripheral subspace corresponds to a subtree in $\Gamma_{\widetilde{X}}$. Since the intersection of two subtrees of a tree is a tree, then the intersection of two distinct peripheral subspaces corresponds to a subtree in $\Gamma_{\widetilde{X}}$. Since each edge space lies in a single peripheral subspace in $\widetilde{X}$, any two peripheral subspaces cannot intersect in an edge of $\Gamma_{\widetilde{X}}$ so they must intersect in a single vertex in $\Gamma_{\widetilde{X}}$, or have empty intersection. A vertex in $\Gamma_{\widetilde{X}}$ corresponds to a vertex space in $\widetilde{X}$, a tree. So the intersection of two peripheral subspaces is the intersection of two periodic lines in a tree. This intersection must be either finite or the two periodic lines are equal. If the two lines were equal then the two peripheral subspaces could not have been distinct. So peripheral subspaces in $\widetilde{X}$ intersect in finite line segments. Since the underlying graph of X is finite, that is, there are finitely many possible periodic lines and thus finitely many intersections of periodic lines, then there is therefore a bound $K = K(\widetilde{X})$ on the length of the intersection of any two peripheral subspaces in $\widetilde{X}$. □

2.2. Alternate Viewpoint on Peripheral Subgroups

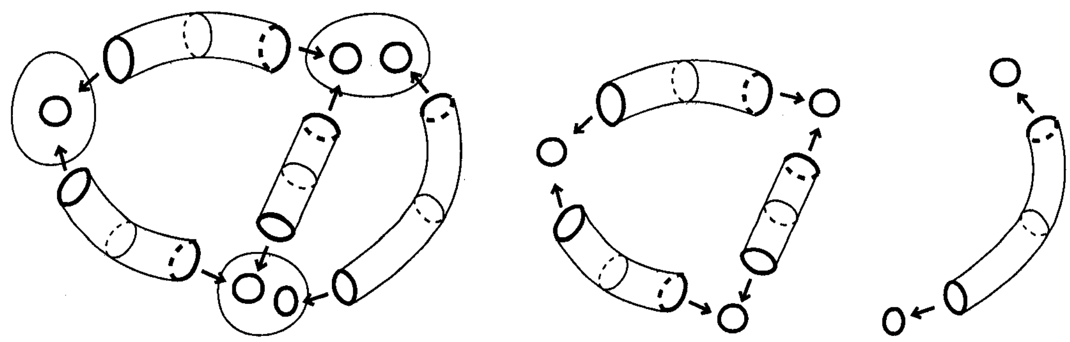
In this section we take a different approach to describing peripheral subgroups. This discussion aims to enrich the reader's understanding of peripheral subgroups however we shall not require the results of this section in subsequent sections.

Let X be a graph of spaces where each vertex space is a graph, each edge space is a cylinder and all attaching maps are immersions. Using X, we build the *induced graph of spaces* X' whose connected components are the cylindrical subspaces of X as in Definition 2.1.9. The induced graph of spaces X' has the following properties:

(1) The edge spaces of X' are the edge spaces of X

(2) The vertex spaces of X' are circles

Each attaching map of X factors through an immersed circle. Simple (non-periodic) immersed circles are in one-to-one correspondance with conjugacy classes of maximal cyclic subgroups of the fundamental groups of vertex spaces of X. The vertex spaces of X' correspond to these conjugacy classes. The attaching maps of X' will also be immersions since attaching maps of X are. An example of a graph of spaces X and its induced graph of spaces X' is sketched below in Figure 2.4.



2.4(a) Graph of spaces X.

2.4(b) Induced graph of spaces X'

FIG. 2.4. Induced graph of spaces.

The space X' is comprised of disjoint graphs of cyclic groups. The connected components X_i of X' are the *cylindrical subspaces* of X and the *peripheral subgroups* are the fundamental groups of the cylindrical subspaces. In Chapter 3 we will prove that $\pi_1 X$ is hyperbolic relative to the collection of peripheral subgroups $\{\pi_1(X_i)\}$.

We will now give an example of how peripheral subgroups can be "read off" from graphs of groups.

Example 2.2.1. Let $G = \langle a, b, s, t, u \mid ([a,b]^2)^s = [a,b], a^t = a^2, b^u = a^3 \rangle$. We consider the graph of spaces X with fundamental group G and the induced graph of spaces X'. Both X and X' are illustrated illustrated in Figures 2.5 and 2.6 below. We can see that X' has two connected components, the cylindrical subspaces, thus G has two peripheral subgroups.

The peripheral subgroups are easily "read-off" from the induced graph of spaces X', they are given by $P_1 = \langle a, b, s \mid ([a,b]^2)^s = [a,b] \rangle$ and $P_2 = \langle a, b, t, u \mid a^t = a^2, b^u = a^3 \rangle$.

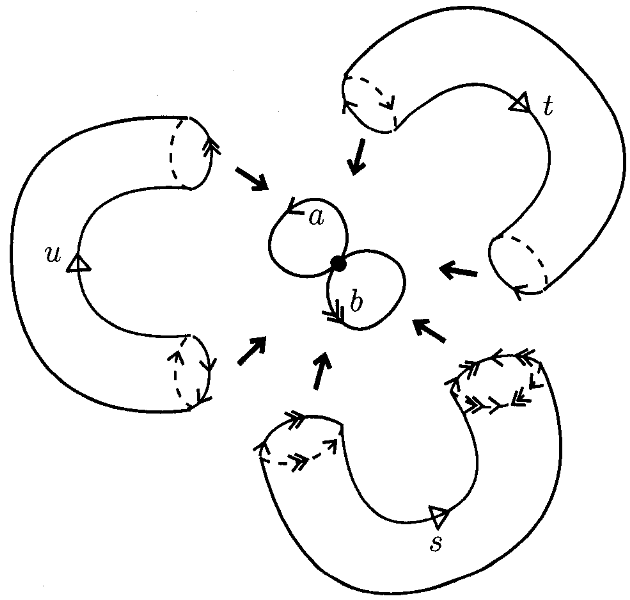


FIG. 2.5. Graph of spaces X whose fundamental group is $G = \langle a, b, s, t, u \mid ([a,b]^2)^s = [a,b], a^t = a^2, b^u = a^3 \rangle$.

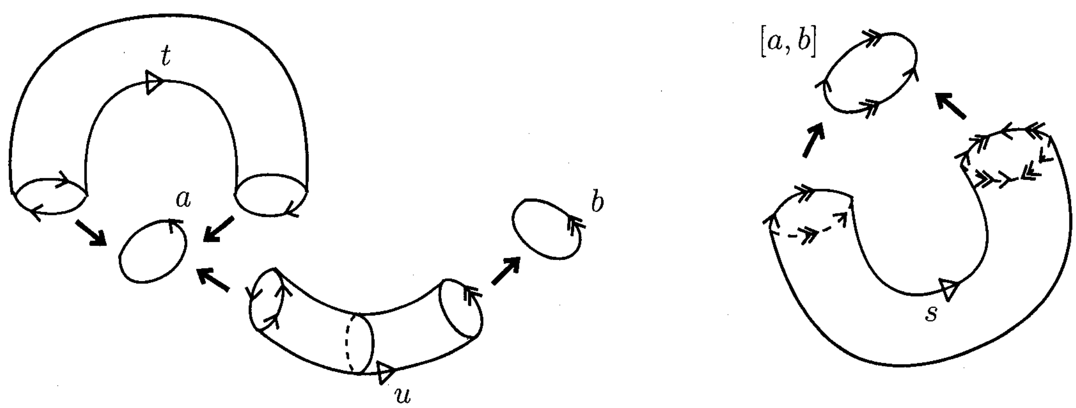
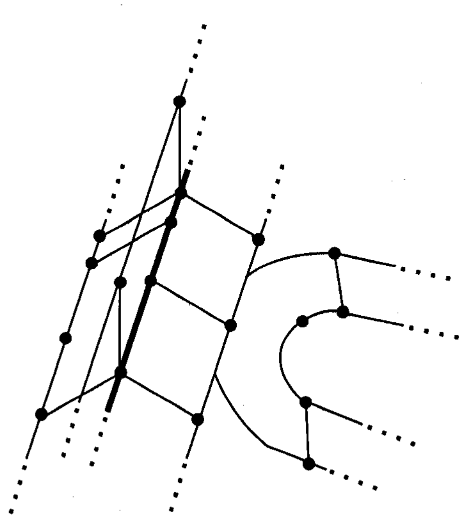


FIG. 2.6. The induced graph of spaces X' with two cylindrical subspaces.

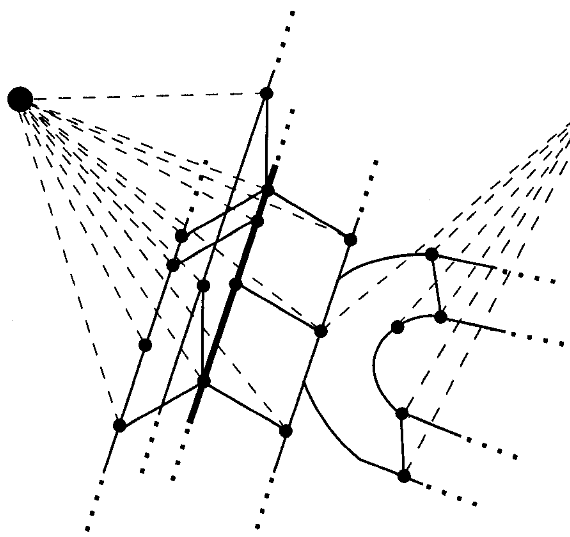
2.3. CONING OFF $\widetilde{X}$

In this section we introduce the notion of a coned space.

Definition 2.3.1. Let $P_j \subset \widetilde{X}$ be a peripheral subspace. We define the *peripheral cone associated to* P_j by $Cone_j = (P_j \times [0,1])/(P_j \times \{1\})$. We work combinatorially viewing $Cone_j$ as a cell complex. Every 0-cell in P_j is joined to a new common 0-cell by a 1-cell, we call this common 0-cell the *conepoint associated to* P_j. Attached to every 1-cell in P_j is a triangular 2-cell whose boundary is composed of the 1-cell in P_j as well as two 1-cells that meet the conepoint associated to P_j. The process of building $Cone_j$ is called *coning off P*. By coning off every peripheral subspace P_j in $\widetilde{X}$ we form $Cone(\widetilde{X}) = \widetilde{X} \bigcup_{P_j = P_j \times \{0\}} Cone_j$ which we call the *coned space.* We repeat that the 0-cells added to $\widetilde{X}$ to form in $Cone(\widetilde{X})$ are the conepoints, the new 1-cells are in one-to-one correspondance with the 0-cells of the peripheral subspaces of $\widetilde{X}$ and the new 2-cells are in one-to-one correspondance with the 1-cells of the peripheral subspaces of $\widetilde{X}$.



2.7(a) Peripheral subspaces in X.



2.7(b) Coning off the peripheral subspaces

FIG. 2.7. Building peripheral cones.
In Figure (a) two peripheral subspaces are illustrated, the bold line indicates the infinite intersection of three edge spaces belonging to the same peripheral subspace. In Figure (b) we cone off peripheral subspaces.

Definition 2.3.2. We distinguish between three types of 1-cells in $Cone(\widetilde{X})$. A *horizontal 1-cell* in $Cone(\widetilde{X})$ is a 1-cell whose interior lies entirely in an edge space of $\widetilde{X}$, a *vertical 1-cell* in $Cone(\widetilde{X})$ is a 1-cell whose interior lies entirely in a vertex space of $\widetilde{X}$ and a *cone 1-cell* in $Cone(\widetilde{X})$ is a 1-cell with an endpoint that is a conepoint.

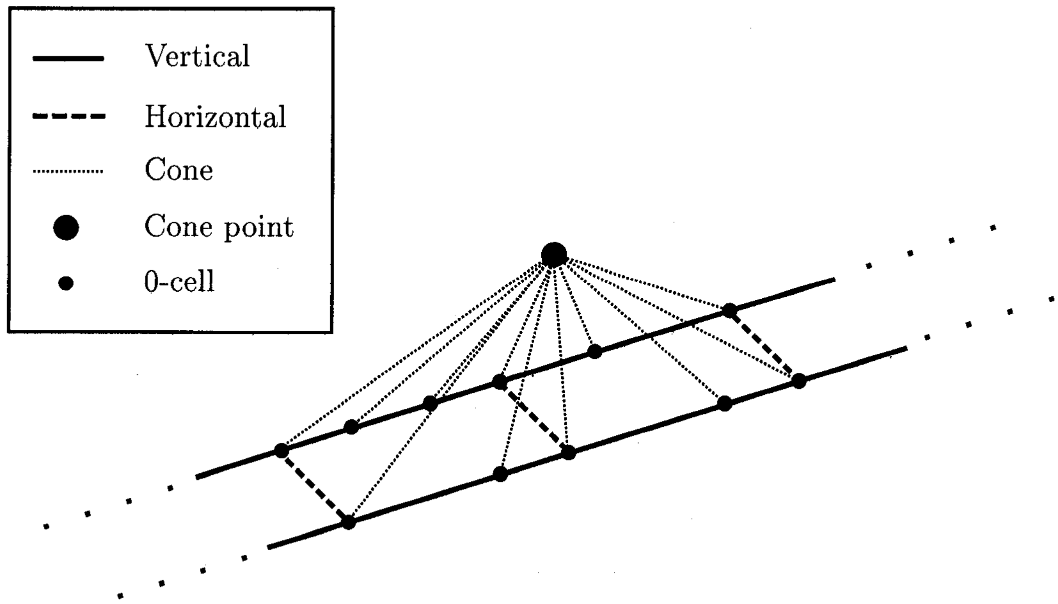


FIG. 2.8. Vertical, horizontal and cone 1-cells in a peripheral cone.

Theorem 2.3.3. *Peripheral cones in $Cone(\widetilde{X})$ intersect in finite line segments. The length of the intersection between any two peripheral cones is bounded by a constant $k = k(\widetilde{X})$.*

PROOF. Since cone 1-cells cannot lie on the boundary of two peripheral cones, then the proof follows directly from Theorem 2.1.11. The bound on the length of the intersection between any two peripheral cones is the same as the bound on the length of the intersection between any two peripheral subspaces. □

2.4. DISC DIAGRAMS AND REGIONS

In this section we recall the definition of a disc diagram. We will also define a new structure inside a disc diagram in the coned space $Cone(\widetilde{X})$ called a region.

Definition 2.4.1. A *disc diagram D* is a finite, simply-connected 2-complex embedded in R^2. A disc diagram is *non-singular* if it is homeomorphic to the unit disc. We will denote the set of 1-cells of D by I_D and the set of 0-cells of D by O_D.

Definition 2.4.2. A *disc diagram D in X* is a combinatorial map $D \to X$ where D is a disc diagram and X is a cell complex. By *combinatorial* we mean that open i-cells of D map homeomorphically to open i-cells of X.

Definition 2.4.3. A disc diagram is a finite 2-complex in R^2 and has a topological boundary denoted ∂D. The *boundary path of a disc diagram D* is a combinatorial path in the topological boundary of D, starting at a 1-cell, that travels either once or twice through each 1-cell on the topological boundary. A 1-cell on the topological boundary is traversed once if it lies on the topological boundary of a 2-cell, otherwise it is traversed twice. The order in which 1-cells incident to the same 0-cell are traversed is chosen to be clockwise from the first of the incident 1-cells in question traversed. We denote the boundary path of a disc diagram D by $\partial_p D$. The *length of a boundary path $\partial_p D$* is the total number of 1-cells that appear in $\partial_p D$, some 1-cells appear more than once in this count. We denote the length of the boundary path $\partial_p D$ by $|\partial_p D|$. (See Figure 2.9)

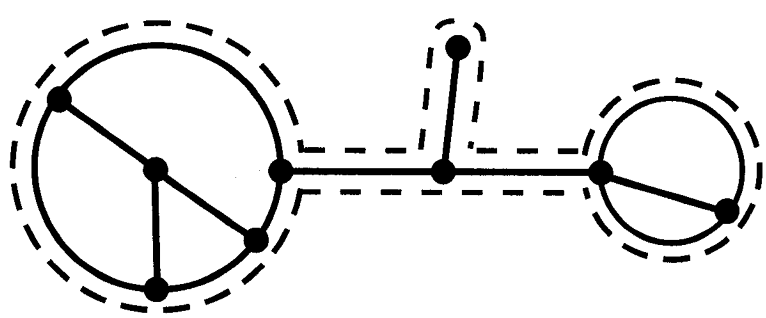


FIG. 2.9. Boundary path of a disc diagram.
The boundary path is indicated by the dashed line, $|(\partial_p D)|$=12.

Definition 2.4.4. A 1-cell in a disc diagram D is *internal* if it on the common boundary of two 2-cells of D, otherwise it is a *boundary 1-cell.* The union of all boundary 1-cells forms the topological boundary of D. A boundary 1-cell is a *singular 1-cell* if it does not belong to the boundary of a 2-cell of D. We note that a non-singular disc diagram does not contain any singular 1-cells. In a disc diagram D we denote the set of boundary 1-cells of D by $I_{\partial D}$ and the set of internal 1-cells by I_{intD}. Similarly we call a 0-cell lying on the topological boundary of D a *boundary 0-cell* otherwise we call it an *internal 0-cell* and denote the set of boundary 0-cells by $O_{\partial D}$ and the set of internal 0-cells by O_{intD}.

Remark 2.4.5. If a disc diagram D is non-singular then $|\partial_p D| = |I_{\partial D}|$.

The following lemma is the disc diagram version of the Lyndon-Van Kampen Lemma, a proof can be found in [**McC**].

Lemma 2.4.6. *Let $P \longrightarrow X$ be a combinatorial closed path in a space X. Then $P \longrightarrow X$ is nullhomotopic iff P is the boundary path of some disc diagram $D \longrightarrow X$.*

From now on we will only consider disc diagrams in the coned space $D \longrightarrow Cone(\widetilde{X})$ that are non-singular. Thus the length of the boundary path of D will coincide with the total number of boundary 1-cells of D as mentioned in Remark 2.4.5. This assumption will simplify some computations and will not affect eventual results about isoperimetric functions as Theorem 2.8.4 will later demonstrate. We now partition the open 2-cells of D by defining a new structure in a disc diagram called a region.

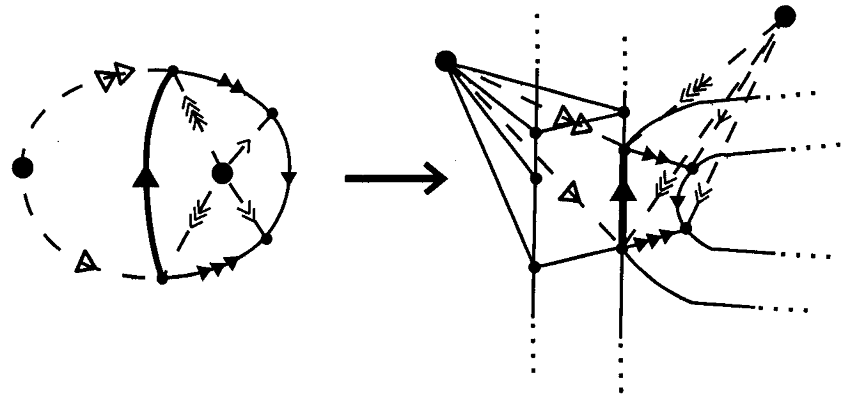
Definition 2.4.7. Two open 2-cells in a disc diagram $D \longrightarrow Cone(\widetilde{X})$ are *locally region equivalent* if their boundaries share a 1-cell and both 2-cells map to the same peripheral cone in $Cone(\widetilde{X})$. Two open 2-cells C and C' are *region equivalent* if there exists a sequence of open two cells $C = C_0, C_1, ..., C_k = C'$ such that C_i and C_{i+1} are locally region equivalent for all $0 \leq i \leq k$. Region equivalence gives rise to an equivalence relation on open 2-cells of D. We call its equivalence classes *region classes.*

Definition 2.4.8. We describe a subcomplex of a disc diagram $D \longrightarrow Cone(\widetilde{X})$ in the following way: begin with a region class of open 2-cells. Include in the subcomplex any open 1-cell of D lying on the boundary of two locally region equivalent open 2-cells in the chosen region class. Then include all 0-cells of D

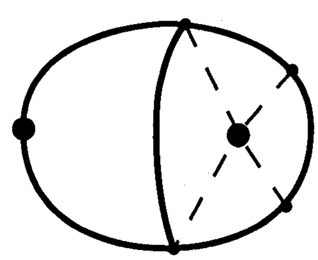
with the property that all its incident 1-cells lie on the boundary of two locally region equivalent open 2-cells in the chosen region class. The subcomplex described is called an *open region of D*.

Definition 2.4.9. The closure of an open region of D is called a *region of D*.

Remark 2.4.10. By construction, regions of D are connected along 1-cells and are thus planar surfaces.



2.10(a) Disc diagram $D \to Cone(\widetilde{X})$.



2.10(b) Regions of D.

FIG. 2.10. Regions of D given by map $D \to Cone(\widetilde{X})$

In part (a) the labeled 1-cells describe the combinatorial map to $Cone(\widetilde{X})$. Two distinct peripheral cones of $Cone(\widetilde{X})$ are indicated, they intersect in a 1-cell. In part (b) the boundaries of regions are indicated using bold lines, they are determined by which peripheral cones the 2-cells are mapped to.

Remark 2.4.11. Given a disc diagram $D \to Cone(\widetilde{X})$, by Lemma 2.1.6 and by the construction of $Cone(\widetilde{X})$, boundaries of open 2-cells in $Cone(\widetilde{X})$ are embedded circles. Therefore the boundaries of open 2-cells of D have the same property.

2.5. Justified Disc Diagrams

Given a combinatorial closed path $P \to Cone(\widetilde{X})$, our goal in this section is to establish the existence of a disc diagram $D \to Cone(\widetilde{X})$ with the following properties :

(1) $\partial_p D = P$

(2) Regions of D intersect ∂D in at least one 1-cell.

(3) Regions of D are simply-connected.

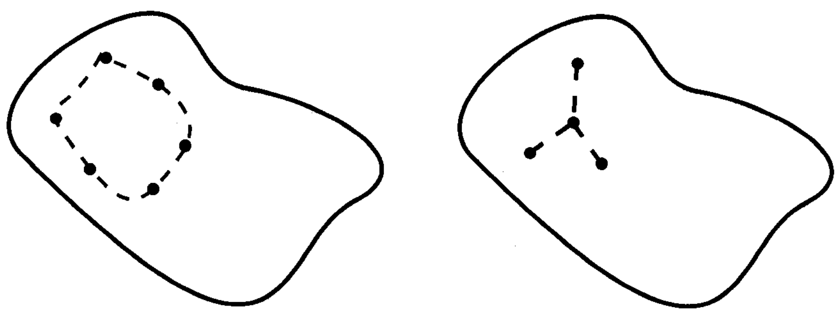
(4) The intersection of any two regions of D is a possibly empty arc whose length is bounded by a uniform constant.

Definition 2.5.1. In Definition 2.3.2 we introduced vertical, horizontal and cone 1-cells in $Cone(\widetilde{X})$. Similarly, for a disc diagram $D \to Cone(\widetilde{X})$ a *vertical/horizontal/cone 1-cell of* D as a 1-cell of D that maps to a vertical/horizontal/cone 1-cell of $Cone(\widetilde{X})$.

Remark 2.5.2. Only vertical 1-cells in $Cone(\widetilde{X})$ can sit on the boundary of two 2-cells belonging to different peripheral subspaces. Thus by the definition of a region we can assert that only vertical 1-cells of D can lie on the intersection of two distinct regions of D.

Lemma 2.5.3. *Given a combinatorial closed path $P \to Cone(\widetilde{X})$ there exists a disc diagram $D \to Cone(\widetilde{X})$ such that $\partial_p D = P$ and each region of D intersects ∂D in at least one 1-cell.*

Proof. Since $Cone(\widetilde{X})$ is simply-connected then by the Van Kampen Lemma there exists a disc diagram $D \to Cone(\widetilde{X})$ such that $\partial_p D = P$. If D has the property that regions intersect ∂D in at least one 1-cell then we are done. Otherwise we pick a region R with boundary ∂R that does not intersect ∂D in at least one 1-cell. By Remark 2.5.2 only vertical 1-cells can lie on the intersection of two distinct regions. This implies that 1-cells in ∂R are either vertical 1-cells, or 1-cells belonging to ∂D. By hypothesis R does not intersect ∂D in any 1-cell, so ∂R must consist entirely of vertical 1-cells each mapping to the same peripheral cone in $Cone(\widetilde{X})$. The boundary ∂R therefore maps to a tree T in $\widetilde{X}$. Let S be a path along ∂R which is the boundary path of a disc diagram $E \subset D$ such that $R \subset E$. We remove the interior of E and consider quotient $(D - Int(E))/S \to T$, where the map $S \to T$ comes from the map from ∂R to the tree T, see Figure 2.11. We note that $\partial R \neq \emptyset$ since E^2 contains no closed surfaces.



2.11(a) Boundary of a region in D.

2.11(b) Region replaced by a tree.

FIG. 2.11. Removal of a region that does not intersect ∂D.

The remaining complex is still a disc diagram in $Cone(\widetilde{X})$. The region without the desired property has been removed without affecting the boundary path of the disc diagram. Since the number of regions has decreased we can repeat this process and we eventually will have a disc diagram with boundary path P whose regions all intersect the boundary in at least one 1-cell. □

Lemma 2.5.4. *Let $D \to Cone(\widetilde{X})$ be a disc diagram such that every region of D intersects ∂D in at least one 1-cell, then each region of D is simply-connected.*

PROOF. By Remark 2.4.10, regions in D are planar surfaces. We consider a region R in D, its topological boundary ∂R is a graph in the plane. We define the outerboundary $\partial_o R$ of R as the union of connected components of ∂R that have access to the point at infinity. We first claim that $\partial R = \partial_o R$. This follows from the fact that every region of D must intersect ∂D in at least a 1-cell, therefore $\partial_o R \cap \partial D$ contains at least one 1-cell. In addition all 2-cells in a region intersect another 2-cell in the same region along a 1-cell. Therefore any pair of 2-cells of R can be joined to each other by a sequence of 2-cells in R each intersecting each other in a 1-cell of R, the pair of 2-cells are said to be *gallery-connected.* If we pick a 1-cell d in ∂R that is not in $\partial_o R$ then it lies on the boundary of a 2-cell c in a region R' distinct from R, and it cannot lie in ∂D. If c is gallery-connected to a 2-cell intersecting ∂D in a 1-cell then d lies in $\partial_o R$ because it has access to the point at infinity. Therefore the 2-cell c in R' cannot be gallery-connected to ∂D contradicting the fact that all regions intersect the boundary of D in at least a 1-cell. We have thus established that $\partial R = \partial_o R$. We now claim that $\partial_o R \cong S^1$, this follows from the fact that 2-cells in the same region are gallery-connected. We can thus assert that regions in D are simply-connected. □

Remark 2.5.5. Since regions in D are simply-connected, their boundary paths are defined identically to boundary paths of disc diagrams. See Definition 2.4.3. The boundary path of a region R is denoted $\partial_p R$.

Lemma 2.5.6. *Given a combinatorial closed path $P \to Cone(\widetilde{X})$ there exists a disc diagram $D \to Cone(\widetilde{X})$ with boundary path $\partial_p D = P$ such that every region intersects ∂D in at least one 1-cell and each region is simply-connected.*

PROOF. The proof of this Lemma follows directly from Lemma 2.5.3 and Lemma 2.5.4. □

Definition 2.5.7. We recall that each 1-cell in a disc diagram $D \to Cone(\widetilde{X})$ can be thought of as having a label and orientation, determined by how it is mapped to $Cone(\widetilde{X})$. A *backtrack on the boundary ∂R of a region R of D* is a sequence of two adjacent 1-cells on ∂R of the form ee^{-1} where e is and e^{-1} have the same label but opposite orientations.

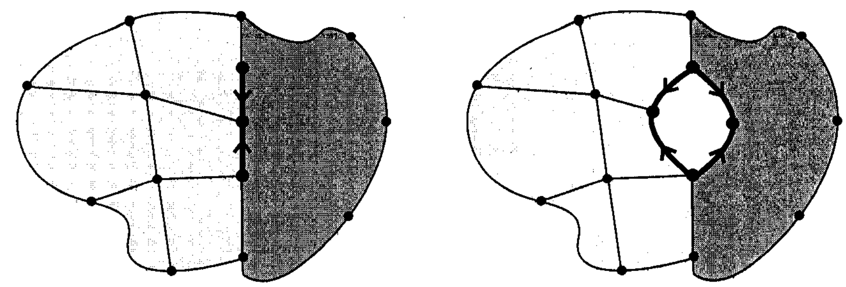
Theorem 2.5.8. *Given a combinatorial closed path $P \to Cone(\widetilde{X})$ there exists a disc diagram $D \to Cone(\widetilde{X})$ with $\partial_p D = P$ such that each region intersects ∂D in at least one 1-cell, each region is simply-connected and D has the property that there are no backtracks on the boundary of any region.*

PROOF. By Lemma 2.5.6 given a combinatorial closed path $P \to Cone(\widetilde{X})$ there exists a disc diagram $D \to Cone(\widetilde{X})$ with boundary path $\partial_p D = P$ such that every region intersects ∂D in at least one 1-cell and each region is simply-connected. We will define a process that will modify D by removing backtracks on the boundary of a region. The resulting structure D_1 will remain a disc diagram in $Cone(\widetilde{X})$ with $\partial_p D_1 = P$.

If the backtrack lies on ∂D then fold the pair of 1-cells forming the backtrack together and push them out of the disc diagram so that they form a spur but preserve the boundary path of D. Otherwise for any pair of 1-cells that form a backtrack on the boundary of R and an adjacent region R' we apply the following steps:

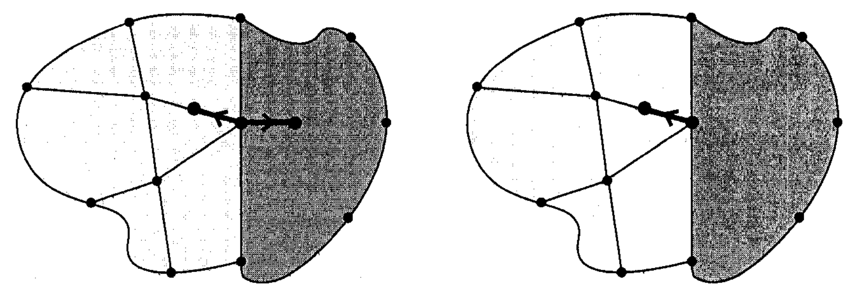
(1) Replace the pair of 1-cells, by the closed path formed by the concatenation of two copies of the pair of 1-cells. See Figure 2.12 (b).
(2) Fold the original pair of 1-cells together to form a single 1-cell, fold the new copy of the 1-cells together to form a second 1-cell.
(3) Push the first 1-cell into the first region R, and push the second 1-cell into the adjacent region R'. The backtrack is no longer on the boundary path of either of the regions. See Figure 2.12 (c).
(4) If the process creates a 1-cell that does not lie on the boundary of a 2-cell in D, it is ignored. See Figure 2.12 (d).

By this process we remove a backtrack from the intersection of two regions. The boundary path of the disc diagram is preserved by this method of removal of backtracks. In addition D_1 preserves from D the properties that its regions are simply-connected and they each intersect ∂D_1 in at least a 1-cell. The resulting disc diagram D_1 is a disc diagram in $Cone(\widetilde{X})$ because the boundary paths of



2.12(a) A backtrack in D.

2.12(b) Copying backtrack.



2.12(c) Folding and pushing-in backtrack.

2.12(d) Ignoring a 1-cell.

FIG. 2.12. Removal of backtrack in D.

each 2-cell in D have not been changed in the process. Since the disc diagram D is finite, by applying this process to every backtrack on the boundary of a region we obtain the disc diagram $D_1 \to Cone(\widetilde{X})$ with $\partial_p D_1 = P$ satisfying all required conditions. $\square$

Remark 2.5.9. It can be shown using a similar argument to the proof of the simple-connectedness of regions that the intersection of any two regions of a disc diagram D is a line segment.

Proposition 2.5.10. *If a disc diagram $D \to Cone(\widetilde{X})$ has the property that there is no backtrack on the boundary ∂R of any region R of D then there exists a constant $K = K(\widetilde{X})$ such that for any two regions of D the length of their intersection is less than or equal to K.*

PROOF. We consider a disc diagram $D \to Cone(\widetilde{X})$ such that there is no backtrack on the boundary of any region of D. We want to show that the length of the intersection of any two regions is bounded above. By Theorem 2.3.3, we know that there is a bound $K = K(\widetilde{X})$ on the length of the intersection of any two peripheral cones in $Cone(\widetilde{X})$. The intersection of two regions in D must map to the intersection of two peripheral cones in $Cone(\widetilde{X})$ by the definition of a region. Therefore if the length of the intersection of two regions in D is larger than K this implies the presence of a backtrack on the intersection of the two regions, a contradiction. □

Definition 2.5.11. The constant $K = K(\widetilde{X})$ in the previous proposition is called an *overlap constant for the regions of D.*

Definition 2.5.12. A disc diagram $D \to Cone(\widetilde{X})$ with the following properties:

(1) Each region intersects ∂D in at least one 1-cell.

(2) Each region is simply-connected.

(3) There exists an overlap constant for the regions of$D$.

is called a *justified disc diagram* in $Cone(\widetilde{X})$. It is denoted $D_J \to Cone(\widetilde{X})$.

Corollary 2.5.13. *Given a combinatorial closed path $P \to Cone(\widetilde{X})$ there exists a justified disc diagram $D_J \to Cone(\widetilde{X})$ such that $\partial_p D_J = P$.*

PROOF. The proof of this Corollary follows directly from Proposition 2.5.10 and Theorem 2.5.8. □

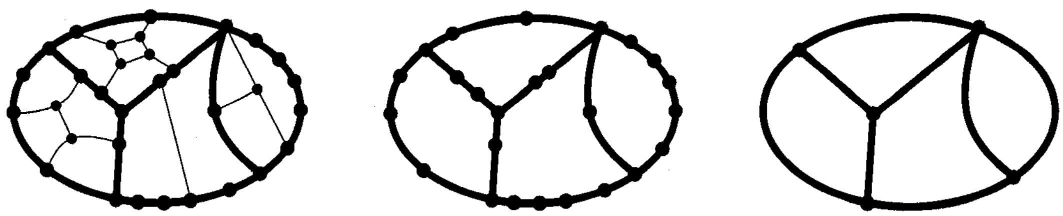
2.6. HOLLOWED DIAGRAMS

Definition 2.6.1. Given a justified disc diagram $D_J \to Cone(\widetilde{X})$. We build a *hollowed diagram*, denoted D_H, as follows:

(1) Each region in D_J is regarded as a single 2-cell.

(2) All 0-cells of valence 2 that remain once each region has been regarded as a 2-cell are ignored.

This two step process is illustrated in Figure 2.13. We note that each 2-cell of D_H corresponds to a single region in D_J and there is at most one 1-cell lying on the intersection of two 2-cells in D_H.



2.13(a) $D_J \to Cone(\widetilde{X})$ with four regions.

2.13(b) Each region regarded as 2-cell.

2.13(c) Ignoring 0-cells of valence 2.

FIG. 2.13. Construction of D_H from D_J

Our next goal is to bound the total number of 1-cells of D_H by the number of boundary 1-cells of D_H. To establish this bound we first prove some basic results about finite trees.

Lemma 2.6.2. *Let T be a finite tree with edge set E and vertex set V and let L denote the set of leaves in T. If all non-leaf vertices have valence 3 then $|E| \leq 2|L|$.*

PROOF. This proof is a simple induction on the number of vertices in the tree, based on the notion of rooted trees. In a tree with 1 vertex, there are no edges and no leaves, thus the base case is easily established. We now suppose that for any tree T with edge set E, leaf set L and vertex set V, with non-leaf vertices having valence 3 and $|V| \leq n-1$ that $|E| \leq 2|L|$. We take a new tree T with edge set E, leaf set L and vertex set V, with non-leaf vertices having valence 3 and $|V| = n$. We root this tree at a leaf r and recall the notion of *parents* and *siblings*. For any vertex $v \in T$ we define its *parent* to be a vertex to which it is connected by an edge and whose distance to the root r is smaller than that of v. Similarly we define a *child* of v to be a vertex to which v is connected by an edge and whose distance to the root r is greater than that of v. We now pick a vertex v_0 of maximal distance from the root r, v_0 is necessarily a leaf. Consider the parent v_p of v_0, it is a vertex of valence 3 since it cannot be a leaf, and it must have a child v_1 different than v_0 that is itself a leaf. If v_1 is not a leaf than this contradicts the maximal distance of v_0 from r. We consider the tree obtained by the removal of the leaves v_0 and v_1 and their incident edges, it is a tree with $n-2$ vertices, 2 fewer edges and one fewer leaf (v_p is now a leaf). By induction hypothesis $|E|-2 \leq 2(|L|-1)$ leading to the desired result $|E| \leq 2|L|$. □

Corollary 2.6.3. *Let T be a finite tree with edge set E, vertex set V and let L denote the set of leaves in T. If all non-leaf vertices have valence greater than 2 then $|E| \leq 2|L|$.*

PROOF. We transform a tree satisfying the property that all its non-leaf vertices have valence greater than 2 into one whose non-leaf vertices all have valence exactly three. We expand the tree by replacing any vertex of valence greater than 3 with an edge. We do this in such a way that one adjacent vertex of the new edge has valence three, and the other adjacent vertex has valence three or greater (see Figure 2.14). We continue this process until every vertex has valence three. In doing so, we only augment the number of edges in the tree and we dont affect the number of leaves thus Lemma 2.6.2 will give the desired result. □

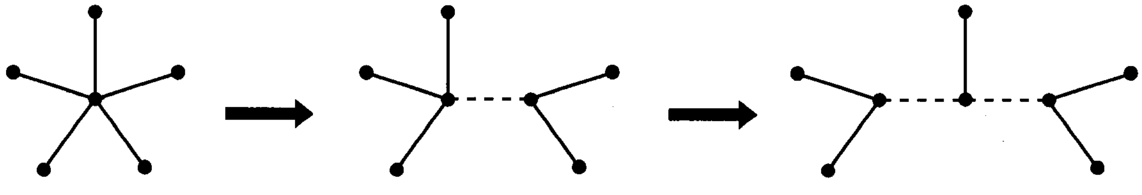
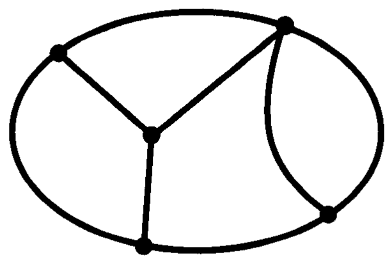


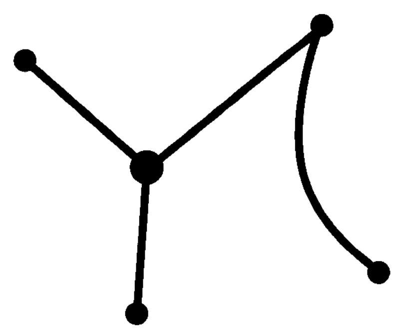
FIG. 2.14. Expansion of a vertex of valence 5.
All non-leaf vertices of the resulting graph have valence 3. The number of leaves remains unchanged while the number of edges is increased.

Definition 2.6.4. Given the 1-skeleton of a hollowed diagram D_H. By removing every open 1-cell on the boundary of D_H we obtain a planar graph denoted G_H. This graph is called the *internal graph of* D_H.

Definition 2.6.5. Given an internal graph G_H of a hollowed diagram D_H. Vertices in G_H that correspond to boundary 0-cells of D_H are called *boundary vertices*, the set of boundary vertices of G_H is denoted $V_{\partial D_H}$. The number of boundary vertices $|V_{\partial D_H}|$ is equal to $|O_{\partial D_H}|$, the number of boundary 0-cells of D_H. Vertices in G_H that are not boundary vertices are *internal vertices*, the set of internal vertices of G_H is denoted V_{intD_H}. See Figure 2.15.



2.15(a) Hollowed diagram D_H.



2.15(b) Graph G_H with one internal vertex.

FIG. 2.15. Construction of graph G_H

Lemma 2.6.6. *Given a justified disc diagram $D_J \to Cone(\widetilde{X})$ and associated hollowed diagram D_H, the internal graph G_H of D_H is a forest.*

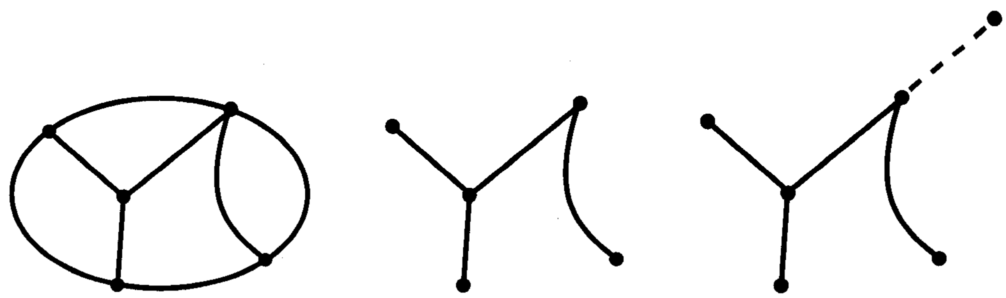
PROOF. Each 2-cell in D_H corresponds to a region in D_J. Since each region in D_J intersects ∂D_J in at least one 1-cell, then each 2-cell of D_H intersects ∂D_H in at least one 1-cell. The internal graph G_H has no cycles, since at least one edge has been removed from any cycle in D_H while building G_H. The removal of certain 1-cells from D_H could result in G_H not being a connected graph, thus G_H is a forest. □

Theorem 2.6.7. *Let G_H be the internal graph of the hollowed diagram D_H associated to the justified disc diagram $D_J \to Cone(\widetilde{X})$. We denote the edge set of G_H by E_G, then $|E_G| \leq 2|O_{\partial D_H}|$.*

PROOF. By Lemma 2.6.6, G_H is a forest, without loss of generality we assume G_H is a tree. We know that the internal vertices of G_H (those that do not correpond to boundary 0-cells of D_H) must have valence greater than 2, since in the process of building D_H all 0-cells of valence 2 are removed. Thus any leaf in G_H is a boundary vertex. However some boundary vertices of G_H may not be leaves, in fact they might have valence 2, as illustrated in Figure 2.15. We will modify the graph G_H so that the number of boundary vertices of G_H is equal to the number of leaves of the new graph (see Figure 2.16). We modify G_H in the following way : if v is a boundary vertex with degree greater than 1 then we add an edge and a corresponding leaf, if v is already a leaf than it remains in the graph unchanged. This process leads to a graph $\widetilde{G}_H$ whose number of leaves is equal to the number of boundary vertices of G_H and whose non-leaf vertices all have valence greater than 2. We recall that $|V_{\partial D_H}| = |O_{\partial D_H}|$ and we note that the number of edges $|\widetilde{E_G}|$ of $\widetilde{G}_H$ is larger or equal to $|E_G|$. In addition, $\widetilde{G}_H$ has the property that each non-leaf vertex has degree greater than 2 so by Corollary 2.6.3 $|E_G| \leq |\widetilde{E_G}| \leq 2|V_{\partial D_H}| = 2|O_{\partial D_H}|$. □

Corollary 2.6.8. *Given a justified disc diagram $D_J \to Cone(\widetilde{X})$ and an associated hollowed diagram D_H, $|I_{intD_H}| \leq 2|I_{\partial D_H}|$.*

PROOF. The internal graph G_H of D_H has the property that its number of edges $|E_G|$ is equal to $|I_{intD_H}|$, since G_H is created by removing the boundary 1-cells of D_H. The number $|O_{\partial D_H}|$ of boundary 0-cells of D_H is less than or equal to the number $|I_{\partial D_H}|$ of 1-cells on the boundary of D_H. Theorem 2.6.7 established that $|I_{intD_H}| \leq 2|O_{\partial D_H}|$, combining with the fact that $|O_{\partial D_H}| \leq |I_{\partial D_H}|$ we get the desired result that $|I_{intD_H}| \leq 2|I_{\partial D_H}|$. □



2.16(a) D_H. 2.16(b) Graph G_H. 2.16(c) $\widetilde{G}_H$, edge added at non-leaf external vertex.

FIG. 2.16. Construction of graph $\widetilde{G}_H$.

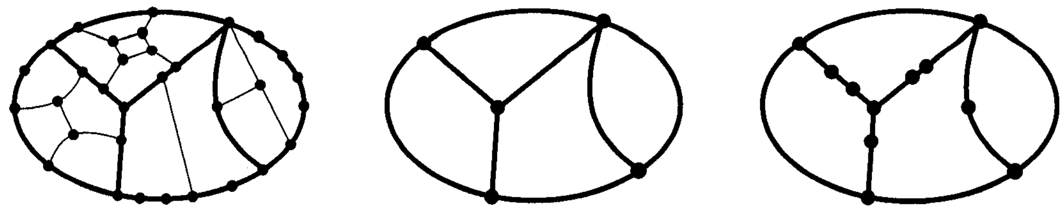
2.7. CONSTRUCTION OF THE CONED DISC DIAGRAM

In this section we will introduce a series of disc diagrams all built from the original justified disc diagram $D_J \to Cone(\widetilde{X})$ and associated hollowed diagram D_H. They are used to construct a final disc diagram $D_C \to Cone(\widetilde{X})$.

Definition 2.7.1. Given a hollowed diagram D_H associated to the justified disc diagram $D_J \to Cone(\widetilde{X})$, we create a new disc diagram called D_U by subdividing the internal 1-cells of D_H while preserving its boundary. We do this by adding the 0-cells of valence 2 that were ignored in the process of building D_H from D_J, we only add 0-cells on internal 1-cells so that $\partial_p D_H = \partial_p D_U$ (see Figure 2.17). Since we assume disc diagrams are non-singular then $|I_{\partial D_U}| = |I_{\partial D_H}|$ however $|I_{D_H}| \leq |I_{D_U}|$.

Lemma 2.7.2. *Let D_U be the disc diagram associated to hollowed diagram D_H and justified disc diagram $D_J \to Cone(\widetilde{X})$ with overlap constant k, then $|I_{D_U}| \leq 3k|I_{\partial D_U}|$.*

PROOF. The boundary 1-cells of D_H are not subdivided when building D_U so $|I_{\partial D_U}| = |I_{\partial D_H}|$. The internal 1-cells of D_U come from the subdivision of internal 1-cells of D_H in accordance to the boundaries of regions of D_J. The length of the



2.17(a) Justified disc diagram D_J.

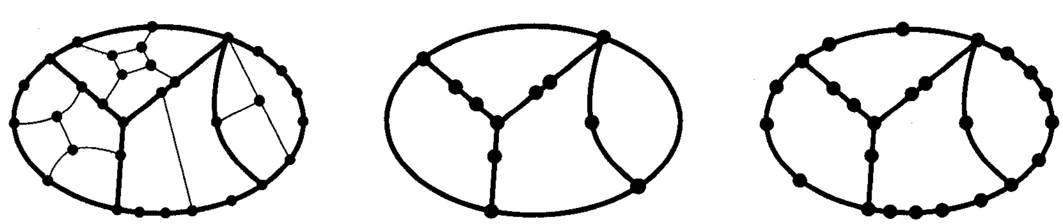
2.17(b) Associated hollowed diagram D_H.

2.17(c) Subdivide internal 1-cells of D_H to form D_U

FIG. 2.17. Construction of D_U from D_H and D_J.

intersection of any two regions in D_J is at most k, since k is the overlap constant for D_J. Thus each 1-cell of D_H will be subdivided into at most k new 1-cells to form D_U so that $|I_{D_U}| \leq k|I_{D_H}|$. Combining with Corollary 2.6.8 we get $|I_{D_U}| \leq k|I_{D_H}| = k(|I_{intD_H}| + |I_{\partial D_H}|) \leq k(2|I_{\partial D_H}| + |I_{\partial D_H}|) = 3k|I_{\partial D_H}| = 3k|I_{\partial D_U}|$ □

Definition 2.7.3. Let D_U be the disc diagram associated to hollowed diagram D_H and justified disc diagram $D_J \to Cone(\widetilde{X})$, we build a new disc diagram called D_V by taking D_U and subdividing boundary 1-cells so that the boundary of D_V is the same as the boundary of D_J. This process is illustrated in Figure 2.18.



2.18(a) Justified disc diagram D_J.

2.18(b) Associated disc diagram D_U.

2.18(c) Subdivide boundary 1-cells of D_U to form D_V

FIG. 2.18. Construction of D_V from D_U and D_J.

Lemma 2.7.4. *Let D_V be the disc diagram associated to D_U, the hollowed diagram D_H, and the justified disc diagram $D_J \to Cone(\widetilde{X})$ with overlap constant k, then $|I_{D_V}| \leq 3k|I_{\partial D_V}|$.*

PROOF. The disc diagram D_V is built by subdividing boundary 1-cells of D_U. We denote the total number of 1-cells added to the boundary of D_U to form D_V by $|A|$ so that $|I_{D_V}| = |I_{D_U}| + |A|$ and $|I_{\partial D_V}| = |I_{\partial D_U}| + |A|$. By Lemma 2.7.2 we get $|I_{D_V}| = |I_{D_U}| + |A| \leq 3k|I_{\partial D_U}| + |A| \leq 3k(|I_{\partial D_U}| + |A|) = 3k|I_{\partial D_V}|$. □

Construction 2.7.5. Given a disc diagram D_V associated to justified disc diagram $D_J \to Cone(\widetilde{X})$ we describe the construction of a new disc diagram D_C. Each 0-cell in D_V corresponds to a single 0-cell in D_J and thus corresponds to a 0-cell in $Cone(\widetilde{X})$. The new disc diagram is formed by modifying every 2-cell of D_V using a process described below. There are 3 possible modifications of a two-cell α of D_V depending on the following cases:

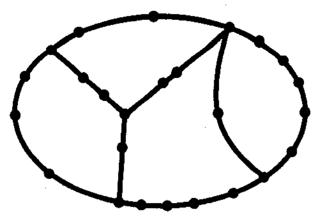
(1) No 0-cell of α corresponds to a conepoint of $Cone(\widetilde{X})$.

(2) Exactly one 0-cell of α corresponds to a conepoint of $Cone(\widetilde{X})$.

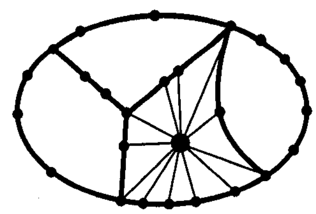
(3) More than one 0-cell of α corresponds to a conepoint of $Cone(\widetilde{X})$.

In case (1) we modify the 2-cell α by first adding one new 1-cell for every 0-cell in α, one endpoint of each newly added 1-cell is identified to its corresponding 0-cell in α. We also add a 0-cell α_c to α and identify to it the remaining endpoint of each new 1-cell. Triangular 2-cells are added in one-to-one correspondance with the original 1-cells of α. This process is illustrated in Figure 2.19.

In case (2) there is one 0-cell p of α that corresponds to a conepoint of $Cone(\widetilde{X})$. We begin by joining every 0-cell of α to p by adding new 1-cells. Some 0-cells of α will already by joined to p by a 1-cell, these are left unchanged. The new triangular 2-cells are added in one-to-one correspondance with the original 1-cells of α that were not already incident to p (1-cells that did not correspond to cone 1-cells in $Cone(\widetilde{X})$). The process is illustrated in Figure 2.20.

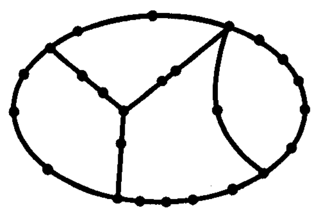


2.19(a) Disc diagram D_V.

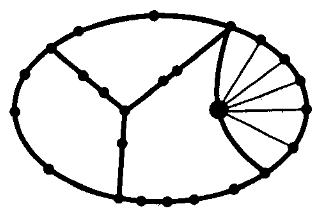


2.19(b) Modifying 2-cell of D_V to form D_C.

FIG. 2.19. Building D_C case (1).



2.20(a) Disc diagram D_V.

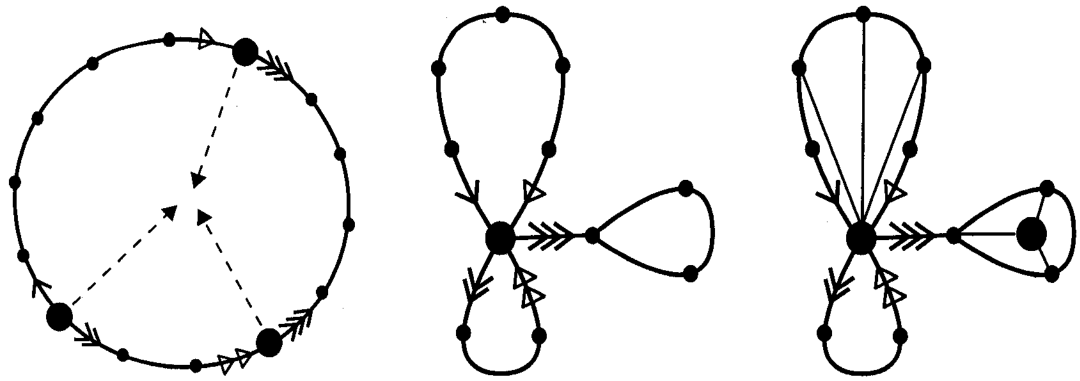


2.20(b) Modifying a 2-cell of D_V to form D_C.

FIG. 2.20. Building D_C case (2).

In case (3) there are two or more 0-cells of α that correspond to a conepoint of $Cone(\widetilde{X})$. We proceed by identifying each of the 0-cells of α corresponding to conepoints to each other. Consequently some incident 1-cells are identified if they correspond to the same cone 1-cell of $Cone(\widetilde{X})$. The 2-cell α is thus possibly divided into multiple 2-cells, each with at most one 0-cell corresponding to a conepoint of $Cone(\widetilde{X})$. Thus case (1) or (2) can be applied to each new 2-cell created during the subdivision of α. This process is illustrated in Figure 2.21.

The final disc diagram D_C is obtained by modifying every 2-cell of D_V using the appropriate case of the three cases described above.



2.21(a) A 2-cell of D_V with 0-cells corresponding to conepoints.

2.21(b) Identifying 0-cells corresponding to conepoints and some incident 1-cells.

2.21(c) Adding cells to each new 2-cell according to cases (1) and (2).

FIG. 2.21. Building D_C case (3).

Definition 2.7.6. The disc diagram D_C described in the previous construction is called the *coned disc diagram* and each subcomplex of D_C that is the result of the modification of a 2-cell of D_V is called a *coned region.*

Theorem 2.7.7. *Let D_C be a coned disc diagram corresponding to justified disc diagram $D_J \to Cone(\widetilde{X})$ then D_C is a disc diagram in $Cone(\widetilde{X})$. In addition, $\partial_p D_C = \partial_p D_J$.*

PROOF. We first give the map $D_C \to Cone(\widetilde{X})$. The coned disc diagram D_C is built from D_V by taking each 2-cell of D_V and building coned regions. The 2-cells in D_V are in one-to-one correspondance with regions in D_J and the boundaries of 2-cells in D_V are identical to those of their corresponding regions in D_J. Since D_J maps combinatorially to $Cone(\widetilde{X})$, we have an obvious way of mapping combinatorially the boundaries of coned regions of D_C into $Cone(\widetilde{X})$. In $Cone(\widetilde{X})$ each 0-cell in the base of a peripheral cone is attached to a conepoint by a 1-cell and triangular 2-cells are in one-to-one correspondance with 1-cells in the base of the peripheral cone. Each coned region of D_C agrees with this structure by construction giving a combinatorial map from $D_C \to Cone(\widetilde{X})$ mapping each coned region of D_C into a peripheral cone of $Cone(\widetilde{X})$. Finally, $\partial_p D_C = \partial_p D_J$ follows from the construction of D_C from the disc diagrams D_V, D_U, D_H and D_J. □

2.8. Linear Isoperimetric Function for $Cone(\widetilde{X})$

Definition 2.8.1. The *area of a disc diagram* D is the number of 2-cells in D, it is denoted *Area(D)*. Similarly the area of a region R in D is the number of 2-cells of R and it is denoted *Area(R)*.

Definition 2.8.2. The area, $Area(P)$, of a combinatorial closed path $P \to X$ is the minimum area of a disc diagram whose boundary path is P.

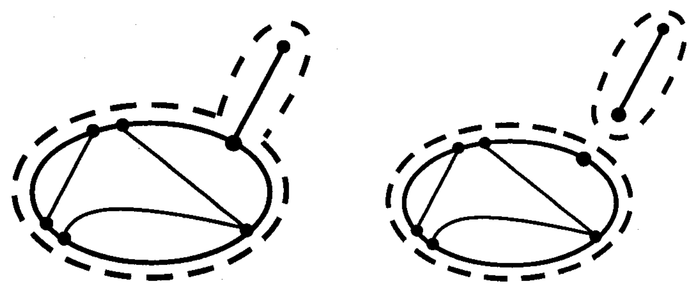
Definition 2.8.3. An *isoperimetric function* of a space X is a function $f : N \to N$ defined by

$$f(n) = max\{Area(P) \mid P \to X, |P| \leq n\}$$

where P is a nullhomotopic, combinatorial, closed path in X.

Lemma 2.8.4. *Suppose that for any combinatorial, closed, nullhomotopic, simple path $P \to Cone(\widetilde{X})$ we have $Area(P) \leq k|P|$ where $k = k(\widetilde{X})$ is a constant. Then for any combinatorial, closed, nullhomotopic path $P \to Cone(\widetilde{X})$ we have $Area(P) \leq k|P|$.*

Proof. We take an arbitrary combinatorial, closed, nullhomotopic path $P \to Cone(\widetilde{X})$. We will proceed by induction on the length of the path $|P|$. For the base case, if $|P| = 0$, then $Area(P) = 0$. We now suppose that for combinatorial closed path P, with $|P| \leq n-1$, we have $Area(P) \leq k|P|$. Now take combinatorial closed path P with $|P| = n$. If it is a simple path, then $Area(P) \leq k|P|$ by assumption. Suppose it is not a simple path, consider a disc diagram D with boundary path $\partial_p D = P$, since P is not simple then $\partial_p D$ passes through at least one singular boundary edge in ∂_D. We change the basepoint of our boundary path if necessary so that it begins at this singular boundary edge. Then the boundary path is the concatenation of two paths, $\partial_p D = \partial_1\partial_2$, where ∂_1 passes through the singular boundary edge and ∂_2 does not. We note that ∂_1 and ∂_2 are chosen so that they are both non-empty closed paths with length smaller than that of $\partial_p D$. See Figure 2.22.



2.22(a) $\partial_p D$. 2.22(b) Separating $\partial_p D$ into ∂_1 and ∂_2.

FIG. 2.22. Decomposing singular boundary path.

By the definition of the area of a path, $Area(P) \leq Area(\partial_1) + Area(\partial_2)$. By induction hypothesis, since $|\partial_1| < |P|$ and $|\partial_2| < |P|$ then $Area(\partial_1) \leq k|\partial_1|$ and $Area(\partial_2) \leq k|\partial_2|$. So combining these inequalities we get $Area(P) \leq k|\partial_1| + k|\partial_2| = k(|\partial_1\partial_2|) = k|P|$. We have thus shown that $Area(P) \leq k|P|$ for arbitrary combinatorial, closed, nullhomotopic path $P \to Cone(\widetilde{X})$. □

Lemma 2.8.5. *Let D_C be the coned disc diagram associated to the justified disc diagram $D_J \to Cone(\widetilde{X})$ with overlap constant k. Then $Area(D_C) \leq 6k|I_{\partial D_C}|$.*

PROOF. We denote the total number of 1-cells lying on the boundary of a coned region of D_C by $|R_{D_C}|$. Each 1-cell lying on the boundary of a coned region of D_C lies on the common boundary of at most two 2-cells so $Area(D_C) \leq 2|R_{D_C}|$. We remark that $|R_{D_C}| = |I_{D_V}|$ by the construction of D_C giving us $Area(D_C) \leq 2|R_{D_C}| = 2|I_{D_V}|$. Since D_J is assumed to be non-singular and $\partial_p D_J = \partial_p D_V = \partial_p D_C$ then D_V and D_C are also non-singular, so $|I_{\partial D_V}| = |I_{\partial D_C}|$. Combining with Lemma 2.7.4 we have $Area(D_C) \leq 2|I_{D_V}| \leq 6k|I_{\partial D_V} = 6k|I_{\partial D_C}|$. □

Theorem 2.8.6. *The coned space $Cone(\widetilde{X})$ has a linear isoperimetric function.*

PROOF. We consider an arbitrary combinatorial, closed, nullhomotopic simple path $P \to Cone(\widetilde{X})$. By Corollary 2.5.13 there exists a justified disc diagram $D_J \to Cone(\widetilde{X})$ with overlap constant k such that $P = \partial_p D_J$. We then consider the corresponding coned disc diagram D_C whose boundary path $\partial_p D_C = \partial_p D_J = P$. By Lemma 2.8.5, $Area(D_C) \leq 6k|I_{\partial D_C}|$, but by assuming P is simple and thus that disc diagrams are non-singular, $|I_{\partial D_C}| = |\partial_p D_C| = |P|$. By definition of the area of a path $Area(\partial_p D_C) \leq Area(D_C)$, so $Area(\partial_p D_C) \leq 6k|\partial_p D_C|$. We have shown that $Area(P) \leq 6k|P|$ for arbitrary combinatorial, closed, nullhomotopic simple path $P \to Cone(\widetilde{X})$. We can therefore apply Lemma 2.8.4 and conclude that for any combinatorial, closed, nullhomotopic path $P \to Cone(\widetilde{X})$ we have $Area(P) \leq 6k|P|$. Thus the function $f(n) = 6kn$ is a linear isoperimetric function of $Cone(\widetilde{X})$. □

Chapter 3

RELATIVE HYPERBOLICITY OF $\pi_1 X$

3.1. The Capped Space

In this chapter we again consider a graph of spaces X where each of the vertex spaces is a graph, each edge space a cylinder, the attaching maps are immersions and the number of vertex and edge spaces is finite. We will mostly be working with the universal cover $\widetilde{X} \to X$.

Definition 3.1.1. Given a map $f : A \to Y$, the *mapping cylinder of f* is the quotient space $M_f = ((A \times I) \cup Y)$ subject to the equivalence relation $(a, 1) \sim f(a)$.

Definition 3.1.2. We recall Definition 2.1.8 of a peripheral subspace in $\widetilde{X}$, the union of edge spaces of a maximal collection of parallel edge spaces of $\widetilde{X}$. The stabilizer of a peripheral suspace in $\widetilde{X}$ is a *peripheral subgroup* and the quotient of a peripheral subspace in $\widetilde{X}$ by its peripheral subgroup is an *associated cylindrical subspace.*

Remark 3.1.3. Given an associated cylindrical subspace C_P, it is by definition the quotient of a peripheral subspace P by its stabilizer, the peripheral subgroup $Stab(P)$, moreover the fundamental group $\pi_1(C_P) = Stab(P)$.

Definition 3.1.4. We view $\widetilde{X}$ as a cell complex and we take the quotient of each peripheral subspace by its peripheral subgroup to get the associated cylindrical subspaces $\{C_j\}_{j \in J}$, where $C_j = P_j / Stab(P_j)$ and $j \in J$ is the index set of orbits of peripheral subspaces P_j. Each associated cylindrical subspace maps into X by a map f_j: $C_j \to X$. We form the mapping cylinder $(\{C_j \times I\}_{j \in J} \cup X)$ of $f = \cup f_j$ where f is the map $f : \cup C_j \to X$ with $f\mid_{C_j} = f_j$. The map f attaches the associated cylindrical subspaces to X by the f_j maps. The resulting space is the *capped space* and is denoted $Cap(X)$. The capped space $Cap(X)$ has an obvious cell structure, in particular each $C_j \times I$ has an induced cell structure, it is made up of a 2-complex inherited from C_j which we call the *cap* crossed with an interval I. The space X is referred to as the *base.*

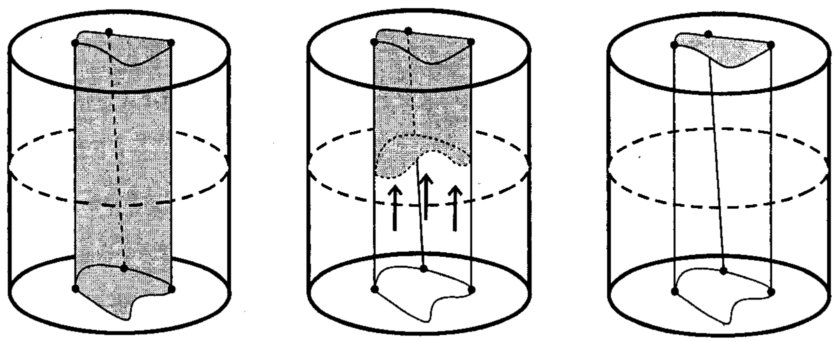
Definition 3.1.5. We distinguish between several different types of 1-cells and 2-cells in the capped space $Cap(X)$. An open 1-cell/2-cell in the base X is called a *base 1-cell/2-cell.* An open 1-cell/2-cell in a cap is called a *cap 1-cell/2-cell.* An open 1-cell/2-cell that is neither a base nor a cap 1-cell/2-cell is called a *vertical 1-cell/2-cell.*

3.2. A Modification of $Cap(X)$

In this section we will make a series of modifications to the capped space $Cap(X) = M_0$, introducing intermediate spaces M_1, M_2, M_3 as we modify, and finally obtaining the space $M_4 = R(X)$. At each stage $\pi_1 M_i \cong \pi_1 M_{i-1}$, the space $R(X)$ will thus have the same fundamental group as $Cap(X)$.

Remark 3.2.1. Each modification that is performed to obtain intermediate spaces consists of collapsing cells along free faces, contracting trees and contracting trees crossed with intervals. Therefore in each intermediate space, the cells correspond to cells in the original space $Cap(X)$. Thus without ambiguity, we continue to refer to cells in intermediate complexes as base, cap or vertical cells according to their origin in $Cap(X)$.

Modification 3.2.2. Given the capped space $Cap(X)$ we note that every 2-cell of $\widetilde{X}$ lies in a unique associated peripheral subspace and thus a unique cylindrical subspace in X, each 2-cell in the base of $Cap(X)$ thus corresponds to a unique 2-cell in a unique cap. Since every 2-cell in the base is a free face of a 3-cell corresponding to that unique 2-cell and cap, we may therefore collapse along each such free face. The 1-skeleton of the base remains. Figure 3.1 illustrates this modification. The resulting space is a 2-complex denoted M_1.



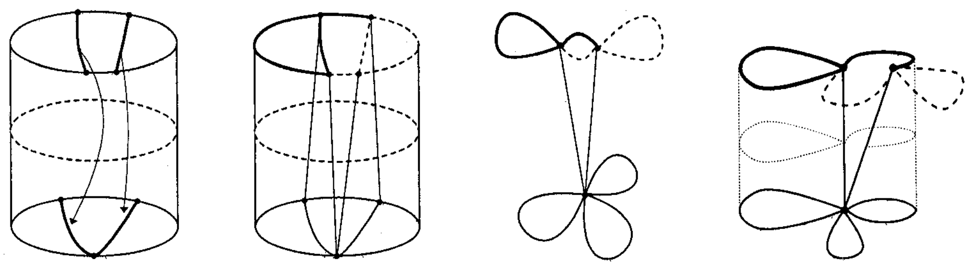
3.1(a) A 2-cell in the base of $Cap(X)$.

3.1(b) Collapsing along free face.

3.1(c) The 1-skeleton of the base of $Cap(X)$ remains.

FIG. 3.1. Modification of $Cap(X)$ to produce M_1.

Modification 3.2.3. Given the space M_1 we pick a maximal tree T_b in the base. Its pre-image in the set of caps (under the mapping cylinder attaching map) is a forest, in particular in every cap C_j the pre-image of T_b is a forest F_j. We consider every forest cross interval $F_j \times I$, then we contract $F_j \times t$ for all values of t. The tree T_b in the base is simultaneously contracted to a point. The base now consists of a unique 0-cell. Figure 3.2 illustrates this modification. The resulting complex is denoted M_2.



3.2(a) Part of maximal tree in base and its pre-image in cap.

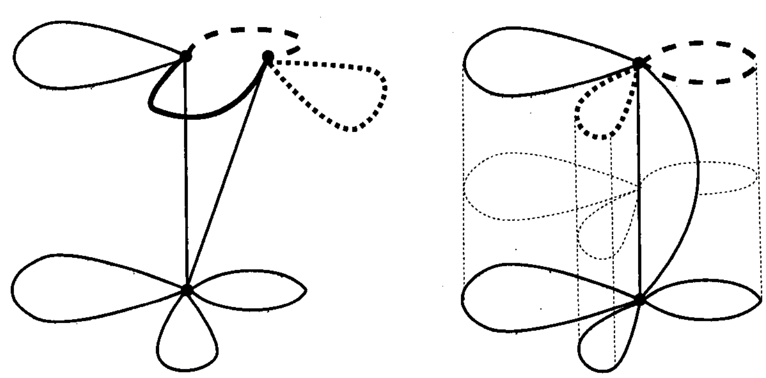
3.2(b) Three selected cap 1-cells in bold.

3.2(c) Contraction of pre-image (1-skeleton).

3.2(d) Two 2-cells corresponding to bold cells in Figure (b) added to Figure (c).

FIG. 3.2. Modification of M_1 to produce M_2.

Modification 3.2.4. Given the space M_2, we select a maximal tree in each cap. We contract each maximal tree to a point. We note that the contractions only take place in the caps C_j. This modification is illustrated in Figure 3.3. The resulting complex is denoted M_3.

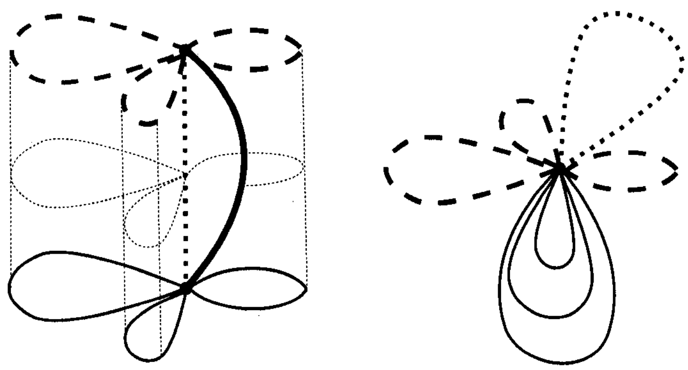


3.3(a) 1-skeleton in M_2 same complex as in Figure 3.2. Maximal tree in cap indicated in bold.

3.3(b) Contraction of maximal tree leads to 2-complex M_3.

FIG. 3.3. Modification of M_2 to produce M_3.

Modification 3.2.5. Given the space M_3, for every cap cross interval $C_j \times I$ we pick a vertical 1-cell and contract it to a point (the earlier modification processes guarantee the existence of at least one vertical 1-cell in each $C_j \times I$). This modification is illustrated in Figure 3.4. The resulting complex is called the *modified capped space* and denoted $R(X)$ or M_4.



3.4(a) Complex from Figure 3.3, vertical 1-cell to be contracted is bold.

3.4(b) 1-skeleton of resulting complex $M_4 = R(X)$.

FIG. 3.4. Modification of M_3 to produce $M_4 = R(X)$.

3.3. A RELATIVE PRESENTATION FOR $\pi_1(Cap(X))$

In this section we review the notion of the relative presentation of a group as presented in **[Os]**, and we find that the geometry of the modified capped space $R(X)$ yields an obvious relative presentation for $\pi_1(Cap(X)) = \pi_1(R(X))$ with respect to peripheral subgroups of $\widetilde{X}$.

We first review basic definitions pertaining to the relative presentation of a group. We will then utilise the geometric structure of the space $R(X)$ to derive a relative presentation for $\pi_1(Cap(X))$.

Definition 3.3.1. Let Y be a subset of a group G, and H a subgroup of G. We say that Y is a *relative generating set of G with respect to H* if G is generated by

$Y \cup H$. More generally, G is generated by Y relative to a collection of subgroups $\{H_i\}_{i\in I}$ if it is generated by $Y \cup \left(\bigcup_{i\in I} H_i\right)$.

Definition 3.3.2. Let G be a group, $\{H_i\}_{i\in I}$ a collection of subgroups of G and Y a relative generating set of G with respect to $\{H_i\}_{i\in I}$. We consider the free product $Z = F(Y) * (*_{i\in I} H_i)$, where $F(Y)$ is the free group freely generated by Y. Since Z is a free product it can be presented by the disjoint union of presentations for its factors. Namely, let H_i have presentation $\langle H_i | S_i\rangle$ for each i. Then $\langle Y \cup (\bigcup_{i\in I} H_i) \mid \bigcup_{i\in I} S_i\rangle$ is a presentation for Z. Let R be a set of elements in Z. The group G has a *relative presentation* $\left(F(Y) * (*_{i\in I} H_i)\right)/R$ *with respect to the subgroups* $\{H_i\}_{i\in I}$ if $\langle Y \cup (\bigcup_{i\in I} H_i) \mid \bigcup_{i\in I} S_i, R\rangle$ is a presentation for G. Each element in the set R is called a *super-relator.*

We can relate the relative presentation of a group to the geometric structure of a corresponding space by using a relative presentation to form a cell-complex. This cell complex is introduced in the following definition.

Definition 3.3.3. Given a group G with relative presentation $\left(F(Y) * (*_{i\in I} H_i)\right)/R$ we build a 2-complex. For each $i \in I$ let K_i be a based space with $\pi_1(K_i) \cong H_i$ and for each $y \in Y$ let L_y be a based circle. Let W be the based wedge of the K_i's and L_y's along their basepoints. Each super-relator in R can be represented by a combinatorial closed path in the 1-skeleton of W. We attach a *super 2-cell* to W for each super-relator in R along its corresponding combinatorial closed path. The resulting space $\mathcal{R}$ is called the *2-complex of the relative presentation.* By the Seifert-Van Kampen Theorem $\pi_1(W) \cong \left(\ F(Y) * (*_{i\in I} H_i)\right)$ and $\pi_1(\mathcal{R}) \cong G$.

Theorem 3.3.4. *There exists a relative presentation of* $\pi_1(Cap(X))$ *with respect to the peripheral subgroups such that the 2-complex of this relative presentation is isomorphic to the modified capped space* $R(X)$.

Proof. We first give a description of $R(X)$ to show that it is isomorphic to the 2-complex of some relative presentation. It will then be clear that this particular relative presentation is in fact a relative presentation for $\pi_1(Cap(X))$ with respect to the peripheral subgroups as required.

The modified capped space $R(X)$ is a based wedge W of cell complexes (see Figure 3.4 (b)) along with a set S of 2-cells each attached to the 1-skeleton of W along a combinatorial closed path. The space W is a wedge of based loops and based 2-complexes. The loops arise from single base 1-cells or single vertical 1-cells. The 2-complexes arise from the cap cells. The 2-cells in S arise from the vertical 2-cells of $Cap(X)$.

We recall that the modified capped space $R(X)$ has the same fundamental group as $Cap(X)$ whereas each cell complex in W consisting entirely of cap cells has the fundamental group of a corresponding associated cylindrical subspace of X. The fundamental group of an associated cylindrical subspace $C_k = (P_k)/Stab(P_k)$ is the peripheral subgroup $Stab(P_k)$.

The structure that we have just described agrees with the definition of a 2-complex of a relative presentation, namely a relative presentation for $\pi_1(Cap(X))$ with respect to the peripheral subgroups.

We can give explicitely this relative presentation. Each based 2-complex of W consisting entirely of cap cells has fundamental group $\pi_1(C_k) = Stab(P_k)$ where C_k is an associated cylindrical subspace corresponding to the cap 2-complex. The set of based loops in W that arise from base 1-cells or vertical 1-cells corresponds to a set of elements $A = \{a_i\}_{i\in I}$ of $\pi_1(Cap(X))$. The set $(\bigcup_{k\in K} Stab(P_k) \cup A)$ freely generates $\pi_1(Cap(X))$. Moreover each 2-cell in S represents a combinatorial closed path in W and hence corresponds to a relator over the alphabet $\bigcup_{k\in K} Stab(P_k) \cup A$. This set of corresponding relators R is in fact the set of super-relators making $\big(F(A) * (*_{k\in K} Stab(P_k))\big)/R$ a relative presentation of $\pi_1(Cap(X))$ with respect to the peripheral subgroups $\{Stab(P_k)\}_{k\in K}$. Moreover our earlier discussion describes how the 2-complex of the relative presentation $\big(F(A) * (*_{k\in K} Stab(P_k))\big)/R$ is in fact isomorphic to the modified capped space $R(X)$ as desired. □

3.4. Peripheral length

Definition 3.4.1. Given a combinatorial path p in $Cap(X)$ or in M_i where $1 \leq i \leq 4$, we will decompose p as the concatenation of certain combinatorial subpaths. The decomposition $p = p_1p_2......p_\ell$ where each p_i is either a maximal subpath in the 1-skeleton of a cap, a single base 1-cell or a single vertical 1-cell is the *peripheral decomposition of* p. Each p_i is a *peripheral syllable* of p and the *peripheral length* of p is the number ℓ of peripheral syllables in the peripheral decomposition of p. We denote the peripheral length of p by $|p|_{pl}$.

The *non-cap length* of a path p in $Cap(X)$ or in M_i where $1 \leq i \leq 4$ is the number of 1-cells in p that are not cap 1-cells, equivalently it is the number of peripheral syllables that are either a single base 1-cell or a single vertical 1-cell. We denote the non-cap length of p by $|p|_{nc}$.

We recall that cells in spaces M_i are classified in accordance to their origin in the space $Cap(X)$. This is why the notion of peripheral and non-cap length applies to the spaces M_i as well as the space $Cap(X)$.

Definition 3.4.2. Let P_R be a path in $R(X)$. The space $R(X)$ arises from a series of modifications to intermediate spaces beginning with the capped space $Cap(X)$. A path P_A in $Cap(X)$ who, after the four modifications described in Modifications 3.2.2 to 3.2.5 results in the path P_R is a *path in* $Cap(X)$ *leading to* P_R. Similarly if a path P_i in M_i where $0 \leq i \leq 4$ is the result of one of the Modifications 3.2.2 to 3.2.5 applied to a path P_{i-1} in M_{i-1} then P_{i-1} is a *path leading to* P_i.

Theorem 3.4.3. *Given a combinatorial closed path* $P_R \to R(X)$*, there exists a path* P_A *in* $Cap(X)$ *leading to* P_R *and a constant* $k = k(X)$ *such that* $|P_A|_{nc} \leq k|P_R|_{pl}$.

Proof. Let the path P_R have peripheral length ℓ with peripheral decomposition $p_1p_2...p_\ell$ so each p_i is either a maximal subpath in the 1-skeleton of a cap, a single base 1-cell or a single vertical 1-cell. We show the existence of a path

P_A in $Cap(X)$ leading to P_R and establish the desired inequality by studying individually each of the Modifications 3.2.2 to 3.2.5.

We first note that given a vertex v_i in a path in M_i, if v_i is the result of a contraction of a tree in M_{i-1} then we can recover a subpath of the contracted tree in M_i that was contracted to v_i.

We build a path in $P_3 \to M_3$ by recovering from certain vertices of P_R any vertical 1-cell that was contracted to a point in Modification 3.2.5. The path P_3 is formed by adding these vertical 1-cells to the path P_R in the place of the vertices that are the result of a contraction. A new peripheral syllable corresponding to a vertical 1-cell is potentially added to the beginning and the end of each peripheral syllable in P_R, therefore $|P_3|_{nc} \leq |P_3|_{pl} \leq |P_R|_{pl} + 2|P_R|_{pl} = 3|P_R|_{pl} = 3\ell$.

We next build a path P_2 in M_2 by recovering from certain vertices of P_3 any subpath that was contracted to a point in Modification 3.2.4. This modification consists of contracting a set of maximal trees $\{T_j\}$ in caps of M_2. The path P_2 is formed by replacing the vertices that are the result of such a contraction by the corresponding subpath in a cap. Since all subpaths added consist only of cap cells, then the non-cap length of P_2 and P_3 are identical and $|P_2|_{nc} = |P_3|_{nc} \leq 3\ell$.

Modification 3.2.3 consists of contracting a maximal tree T_b in the base of M_1 as well as contracting trees crossed with intervals, where each tree is a component of the pre-image of T_b. We build a path P_1 in M_1 again by first recovering from certain vertices of P_2 any subpath in the base of M_1 or in one of its caps that was contracted to a point in P_2. We can also recover from certain vertical 1-cells of P_2 a vertical 1-cell that was identified to it when the trees crossed with intervals were contracted.

The path P_1 is then built by replacing vertices and vertical 1-cell of P_2 that were the result of contractions by the corresponding recovered subpath or vertical 1-cell. The replacing of vertical 1-cells by new ones will not affect the peripheral length or the non-cap length of P_1 and any replacing of a vertex by a subpath in a cap will not affect the non-cap length of P_1. However replacing a vertex in the base by a subpath can affect the non-cap length of P_1. A vertex can be replaced by a subpath in the base of M_1 consisting of at most D_b base 1-cells where D_b is the diameter of the tree T_b. Such a subpath in the base can be added to each non-cap peripheral syllable of P_2, since any subpath in a cap is joined to a subpath in the base by a vertical 1-cell (when in M_1).

We thus have $|P_1|_{nc} \leq |P_2|_{nc} + |P_2|_{nc} D_b \leq 3|P_3|_{nc} + 3|P_3|_{nc} D_b = 3\ell + 3\ell D_b = 3\ell(D_b + 1)$.

The first modification, consists of collapsing 2-cells along free faces. The 1-skeleton of $Cap(X)$ and M_1 are thus the same. We can pick a path P_A in $Cap(X)$ that leads to P_1 after Modification 3.2.2, by simply considering the path P_1 as a path in $Cap(X)$. Since by construction each P_i is a path leading to the path P_{i+1} then P_A is a path leading to P_R. Since Modification 3.2.2 does not affect length of paths then $|P_A|_{nc} = |P_1|_{nc} \leq 3\ell(D_b + 1)$, let $k = 3(1 + D_B)$ then $|P_A|_{nc} \leq k\ell = k|P_R|_{pl}$. □

Definition 3.4.4. The path P_A leading to P_R constructed in Theorem 3.4.3 is called the *canonical path leading to P_R*.

3.5. Disc Diagram in $Cap(\widetilde{X})$

In this section we will re-encounter the coned space $Cone(\widetilde{X})$ that was introduced in Definition 2.3.1. We will naturally call upon the main results from Chapter 2 pertaining to disc diagrams in $Cone(\widetilde{X})$, in particular, coned disc diagrams. The goal of this section is to produce a disc diagram in $R(X)$ from a coned disc diagram in $Cone(\widetilde{X})$, while preserving important properties of the coned disc diagram pertaining to area.

Definition 3.5.1. Given the universal cover $\widetilde{X} \to X$, we now define $Cap(\widetilde{X})$ in analogy to $Cap(X)$. For each peripheral subspace P of $\widetilde{X}$ let $i_P : P \to \widetilde{X}$ be the inclusion map. Let $f = \bigcup i_P$, that is, $f : \cup P \to \widetilde{X}$ where $f\mid_P = i_P$. Then we let $Cap(\widetilde{X}) = M_f$ where M_f is the mapping cylinder of the map f. The space $Cap(\widetilde{X})$ is called the *capped universal cover*.

Remark 3.5.2. The capped universal cover $Cap(\widetilde{X})$ is the universal cover of the capped space, that is, $Cap(\widetilde{X}) = \widetilde{Cap(X)}$.

Definition 3.5.3. The capped universal cover $Cap(\widetilde{X})$ is a mapping cylinder where $P \times I$ is identified to $\widetilde{X}$ for every peripheral subspace P of $\widetilde{X}$. In analogy to $Cap(X)$, the space $\widetilde{X}$ is the *base* and each peripheral subspace P in $P \times I$ is a *cap.* Any cell that is neither in the base nor a cap is a *vertical cell.*

Remark 3.5.4. There is a quotient map $\phi_{cap} : Cap(\widetilde{X}) \to Cone(\widetilde{X})$ defined by identifying each cap to a conepoint.

Definition 3.5.5. We let $P_R \to R(X)$ be a combinatorial closed path, and $P_A \to Cap(X)$ the canonical path leading to P_R. We lift the path P_A to the path $\widetilde{P_A} \to Cap(\widetilde{X})$. Using the map ϕ_{cap} from Remark 3.5.4 that identifies caps to conepoints, we obtain a combinatorial closed path P_C in $Cone(\widetilde{X})$. The path P_C is called the *cone path associated to* P_R.

Remark 3.5.6. Since caps in $Cap(\widetilde{X})$ collapse to conepoints then the length of the path P_C is equal to the non-cap length of the path $\widetilde{P_A}$.

Our next goal is to modify a disc diagram in the coned space with boundary path P_C to obtain a disc diagram in the modified capped space $R(X)$ with boundary path P_R, where $P_C \to Cone(\widetilde{X})$ is the cone path associated to $P_R \to R(X)$.

We begin with a combinatorial closed path $P_R \to R(X)$ and consider its canonical path $P_A \to Cap(X)$ leading to P_R and the cone path $P_C \to Cone(\widetilde{X})$ associated to P_R.

In Section 2.8 we saw that given a combinatorial closed path $P \to Cone(\widetilde{X})$, there exists a disc diagram $D \to Cone(\widetilde{X})$ with boundary path $\partial_p D = P$ and $Area(D) \leq k|P|$, where $k = k(\widetilde{X})$ is a constant. We called this disc diagram the coned disc diagram corresponding to the path P (see Definition 2.7.6).

We take the cone path P_C associated to P_R and form the coned disc diagram corresponding to P_C which we denote D_C. As mentioned above, the coned disc diagram has the properties that $\partial_p D_C = P_C$ and $Area(D_C) \leq k|P_C|$ where $k = k(\widetilde{X})$ is some constant.

We will eventually transform D_C into a disc diagram in $R(X)$. The first step is to produce a disc diagram in $Cap(\widetilde{X})$ with boundary path $\widetilde{P_A}$. This will be done by expanding 0-cells of D_C corresponding to conepoints into disc diagrams in $Cap(\widetilde{X})$ that correspond to caps collapsed in the process of mapping $Cap(\widetilde{X})$ to $Cone(\widetilde{X})$ by the map ϕ_{cap}. The following theorem gives the process for the modification of the coned disc diagram D_C into a disc diagram in $Cap(\widetilde{X})$.

Definition 3.5.7. Given a disc diagram in the modified capped space $D \to R(X)$, the *relative area* of D is the number of 2-cells of D that are mapped to vertical 2-cells of $R(X)$. The relative area of D is denoted $Area_{rel}(D)$.

Theorem 3.5.8. *Let $P_R \to R(X)$ be a combinatorial closed path, let $P_A \to Cap(X)$ be the canonical path leading to P_R and let $P_C \to Cone(\widetilde{X})$ be the cone path associated to P_R with corresponding coned disc diagram $D_C \to Cone(\widetilde{X})$. There exists a disc diagram $\widetilde{D_A} \to Cap(\widetilde{X})$ such that $\partial_p\widetilde{D_A} = \widetilde{P_A}$ and $Area_{rel}(\widetilde{D_A}) = Area_{rel}(D_C)$.*

Proof. We will transform the disc diagram $D_C \to Cone(\widetilde{X})$ into a disc diagram in $Cap(\widetilde{X})$ with boundary path $\widetilde{P_A}$. During this process no vertical 2-cells of D_C will be removed, nor will any vertical 2-cells be added, in this way $Area_{rel}(\widetilde{D_A}) = Area_{rel}(D_C)$.

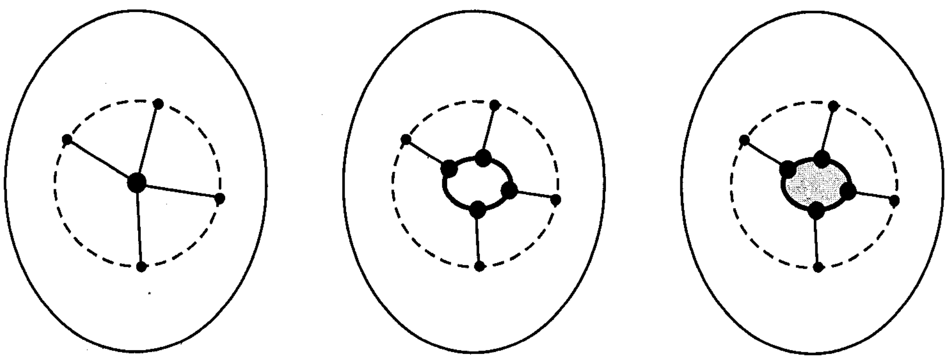
The path $\widetilde{P_A}$ is mapped to the path P_C by the map ϕ_{cap} that identifies caps in $Cap(\widetilde{X})$ to conepoints, thus we will only be concerned with 0-cells of D_C that correspond to conepoints in $Cone(\widetilde{X})$.

Given a 0-cell c in D_C corresponding to a conepoint there are two cases to consider:

(1) c is an internal 0-cell.

(2) c is a boundary 0-cell.

In case (1) (see Figure 3.5) any 1-cells adjacent to c are cone 1-cells, they describe a sequence of triangles about c whose base is a closed path in the peripheral subspace associated to c. We remove c from D_C and replace it by a copy of the closed path in the peripheral subspace. This will agree with the structure of $Cap(\widetilde{X})$. Since peripiheral subspaces are simply-connected this closed path can be filled to form a disc diagram in $Cap(\widetilde{X})$. The boundary path of the disc diagram D_C is unchanged by this operation since c was internal, in addition only cap 2-cells are used to fill a closed path in a peripheral subspace, so the relative area of D_C remains unchanged.



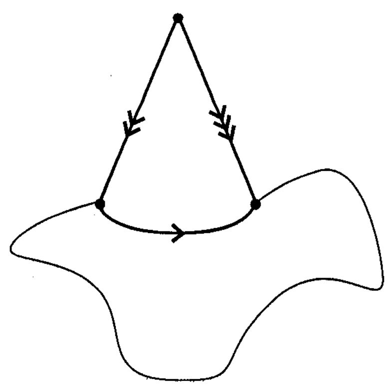
3.5(a) Conepoint in interior.

3.5(b) Closed path replacing conepoint.

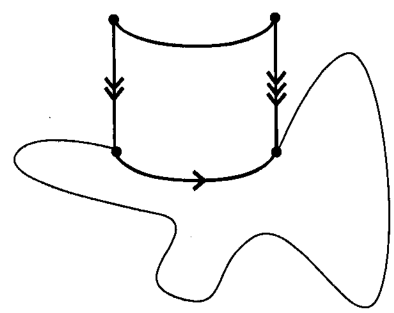
3.5(c) Filling closed path.

FIG. 3.5. Disc diagram in $Cap(\widetilde{X})$ from a disc diagram in $Cone(\widetilde{X})$, case (1).

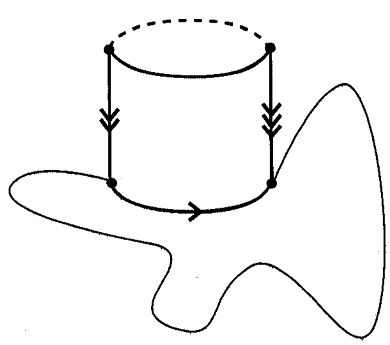
In case (2) c is on the boundary of D_C, thus the boundary path of D_C will be affected by our modification. We first assume that D_C is non-singular (see Figure 3.6). As in case (1) all 1-cells adjacent to c are cone 1-cells and they describe a sequence of triangles about c. In this case the base of the triangles form a connected path p in the peripheral subspace corresponding to c, however the path is not closed since c is not internal. We proceed by replacing c by a copy of the path p so that the path p is now on the boundary of the disc diagram. However we must assure that the new disc diagram has boundary path $\widetilde{P_A}$ and p is not necessarily a subpath of $\widetilde{P_A}$. We therefore consider the endpoints of p, that describe a subpath q of $\widetilde{P_A}$ joining them. The subpath q is concatenated to p since they have the same endpoints forming a closed path in a peripheral subspace. The closed path can be filled to form a disc diagram in $Cap(\widetilde{X})$. In this way q lies on the boundary of the disc diagram as required and no vertical 2-cells are added to the disc diagram hence preserving relative area.



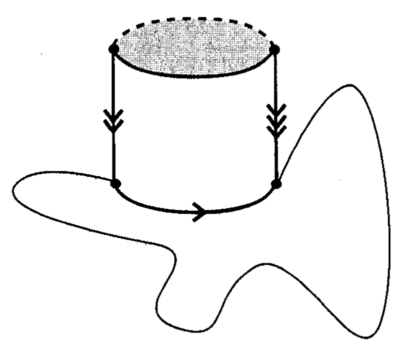
3.6(a) One 0-cell corresponding to conepoint on boundary.



3.6(b) Path p in peripheral subspace replacing 0-cell.

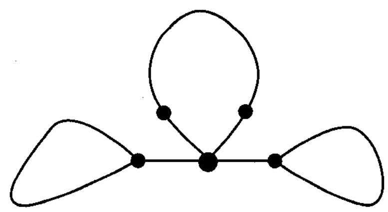


3.6(c) Adding q a corresponding subpath of $\widetilde{P_A}$.

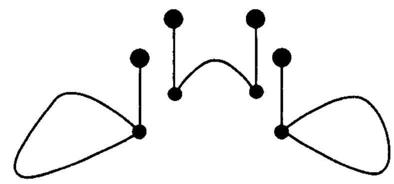


3.6(d) Filling closed path.

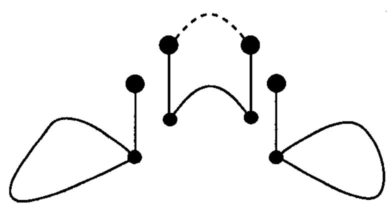
FIG. 3.6. Disc diagram in $Cap(\widetilde{X})$ from a disc diagram in $Cone(\widetilde{X})$, case (2), non-singular D_C.



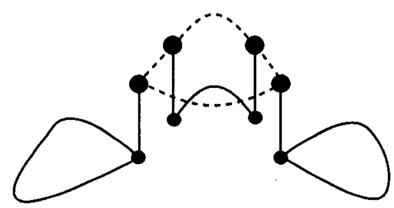
3.7(a) Singular disc diagram D_C.



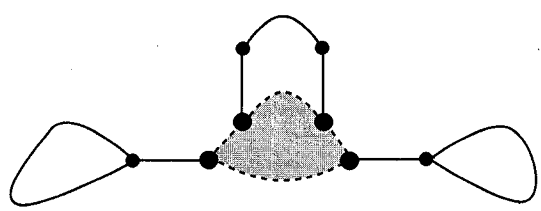
3.7(b) 0-cell pulled apart to illustrate cone 1-cells.



3.7(c) Path corresponding to path in peripheral subspace.



3.7(d) Adding subpaths of $P_{\widetilde{A}}$ to form a closed path.



3.7(e) Filling closed path to form disc diagram in $Cap(\widetilde{X})$.

FIG. 3.7. Disc diagram in $Cap(\widetilde{X})$ from a disc diagram in $Cone(\widetilde{X})$, case (2), singular D_C.

We lastly consider case (2) where D_C is singular (see Figure 3.7). Any 1-cells adjacent to c are cone 1-cells, at least one of which is singular. The cone 1-cells describe a sequence of triangles whose base is a disconnected family of paths each in the same peripheral subspace. We again want to replace c by a closed combinatorial path in a peripheral subspace. We will do this by joining each of the paths in the base of the triangles to form a single closed path. The path in the base of the triangles describe endpoints of subpaths of $\widetilde{P_A}$ that must be added. Using in addition the fact that any singular cone 1-cell figures twice in the boundary path of the disc diagram, we can find subpaths of $\widetilde{P_A}$ joining the paths in the base of the triangles, forming a closed path. We replace the conepoint by our newly constructed closed path and fill it to form a disc diagram in $Cap(\widetilde{X})$. Only the subpaths of $\widetilde{P_A}$ will be part of the boundary of the disc diagram. Again the relative area of D_C has not been affected by this process.

By applying the processes described in cases (1) and (2) to all 0-cells of D_C corresponding to conepoints we build a disc diagram $\widetilde{D_A} \to Cap(\widetilde{X})$ such that $\partial_p \widetilde{D_A} = \widetilde{P_A}$ and $Area_{rel}(\widetilde{D_A}) = Area_{rel}(D_C)$. □

Definition 3.5.9. The disc diagram $\widetilde{D_A}$ obtained in Theorem 3.5.8 is called the *capped disc diagram in* $Cap(\widetilde{X})$.

3.6. Disc Diagram in R(X)

In this section we will construct a disc diagram $D_R \to R(X)$ using the capped disc diagram $\widetilde{D_A} \to Cap(\widetilde{X})$ built in the previous section. We will relate the relative area of D_R to the peripheral length of its boundary path in order to establish a relative isoperimetric inequality.

Construction 3.6.1. Let P_R be a combinatorial closed path in $R(X)$, let $P_A \to Cap(X)$ be the canonical path leading to P_R and let P_C be the cone path associated to P_R with associated coned disc diagram D_C. We consider the capped disc diagram $\widetilde{D_A} \to Cap(\widetilde{X})$ with the properties $\partial_p \widetilde{D_A} = \widetilde{P_A}$ and $Area_{rel}(\widetilde{D_A}) = Area_{rel}(D_C)$. Using the capped disc diagram $\widetilde{D_A}$ we naturally obtain a disc diagram $D_A \to Cap(X)$ such that $\partial_p(D_A) = P_A$. The Modifications 3.2.2 to 3.2.5 map $Cap(X)$ to $R(X)$, we apply the modification to D_A by collapsing appropriate cells and we obtain a disc diagram $D_R \to R(X)$. Since P_A is the canonical path leading to P_R then $\partial_p D_R = P_R$.

Definition 3.6.2. The disc diagram D_R obtained in Construction 3.6.1 is called the *modified capped disc diagram in* $R(X)$.

Lemma 3.6.3. *Let* $D_R \to R(X)$ *be the modified capped disc diagram corresponding to the combinatorial closed path* $P_R \to R(X)$*. There exists a constant* $K = K(X)$ *such that* $Area_{rel}(D_R) \leq K|P_R|_{pl}$.

PROOF. We let $P_A \to Cap(X)$ be the canonical path leading to P_R and let P_C be the cone path associated to P_R with corresponding coned disc diagram D_C. We recall that the modified capped disc diagram D_R has the property that $\partial_p D_R = P_R$. We now review the construction of D_R starting with the coned disc diagram D_C.

In Theorem 3.5.8 we used D_C to build a new disc diagram $\widetilde{D_A} \to Cap(\widetilde{X})$ with $\partial_p \widetilde{D_A} = \widetilde{P_A}$, we showed that $Area_{rel}(\widetilde{D_A}) = Area_{rel}(D_C)$. In Section 2.7 we saw that the coned disc diagram D_C has the property that $Area(D_C) \leq k|P_C|$ for some constant $k = k(X)$ and that $\partial_p D_C = P_C$. Since $Area_{rel}(D_C) \leq Area(D_C)$ we obtain the inequality $Area_{rel}(\widetilde{D_A}) = Area_{rel}(D_C) \leq Area(D_C) \leq k|P_C|$.

In the construction of D_R from D_A we collapse cells and thus we lose area. This means $Area(D_R) \leq Area(\widetilde{D_A})$ and $Area_{rel}(D_R) \leq Area_{rel}(\widetilde{D_A})$. We recall that the path P_C has no cap 1-cells, since the map $\phi_{cap} : Cone(\widetilde{X}) \to Cap(\widetilde{X})$ identifies all cap cells to a point. We therefore have $|P_C| = |\widetilde{P_A}|_{nc} = |P_A|_{nc}$ and so $Area(D_C) \leq k|P_C| = k|P_A|_{nc}$. By Theorem 3.4.3 there exists a constant k'=k'(X) such that $|P_A|_{nc} \leq k'|P_R|_{pl}$.

Combining the above inequalities we have

$$\begin{aligned} Area_{rel}(D_R) &\leq Area_{rel}(\widetilde{D_A}) = Area_{rel}(D_C) \\ &\leq Area(D_C) \\ &\leq k|P_C| = k|P_A|_{nc} \\ &\leq kk'|P_R|_{pl} \end{aligned}$$

We thus obtain the desired result $Area_{rel}(D_R) \leq K|P_R|_{pl}$ where $K = kk'$. □

3.7. Relative Linear Isoperimetric Function and Relative Hyperbolicity

In this section we give the definition of a relative isoperimetric function and relate it to the relative area of a disc diagram in the modified capped space $R(X)$. We will then show that a bound on the relative area of a disc diagram in $R(X)$ will give a relative linear isoperimetric function for $\pi_1(Cap(X))$ and conclude that $\pi_1(X)$ is relatively hyperbolic.

Definition 3.7.1. [Os] Let G be a group with relative presentation $\big(F(Y) * (*_{i\in I} H_i)\big)/R$. We recall there is a natural homomorphism $\phi : Z = \big(F(Y) * (*_{i\in I} H_i)\big) \to G$. We say the function $f : N \to N$ is a *relative isoperimetric function of* $\big(F(Y) * (*_{i\in I} H_i)\big)/R$ if for every word W of length $|W| \leq n$ over the alphabet $\bigcup_{i\in I} H_i \cup Y$ mapped to the identity in G, the word W represents the same element in Z as a product of $k \leq f(n)$ conjugates of superrelators. That is, there exists an expression $W =_Z \prod_{j=1}^{k} z_j^{-1} R_j^{\pm} z_j$ where $z_j \in Z$, $R_j^{\pm} \in R$ and $k \leq f(n)$.

Theorem 3.7.2. *The group $\pi_1(Cap(X))$ has a relative presentation with linear relative isoperimetric function.*

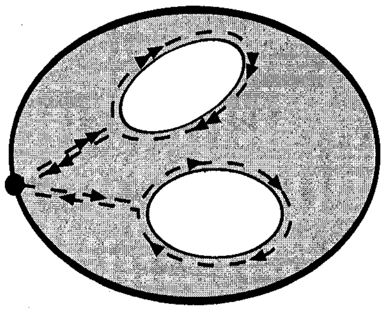
Proof. In Theorem 3.3.4 we produced a relative presentation for $\pi_1(Cap(X))$ from the space $R(X)$. Indeed a relative presentation for $\pi_1(Cap(X))$ with respect to the peripheral subgroups was given by $\big(F(A) * (*_{k\in K} Stab(P_k))\big)/R$. The space $R(X)$ is a wedge W of cell complexes, certain cell complexes H_k arising from cylindrical subspaces C_k (caps) with $\pi_1(C_k) = Stab(P_k)$ and a set of loops represented by the set A of elements of $\pi_1 Cap(X)$ arising from vertical or base 1-cells. Each super-relator in R represents an attaching map of a super 2-cell along a path in the 1-skeleton of W.

Let W be a word of length $|W| \leq n$ over the alphabet $A \bigcup (\cup_{k\in K} Stab(P_k))$ representing the identity in $\pi_1(Cap(X))$. This word is represented by a combinatorial closed path $P_R \to R(X)$. By definition the peripheral length of P_R is smaller or equal to the length of the word W, that is $|P_R|_{pl} = |W| \leq n$. We consider the modified capped disc diagram $D_R \to R(X)$ corresponding to P_R. By Theorem 3.6.3 there exists a constant $K = K(X)$ such that $Area_{rel}(D_R) \leq K|P_R|_{pl} = K|W|$.

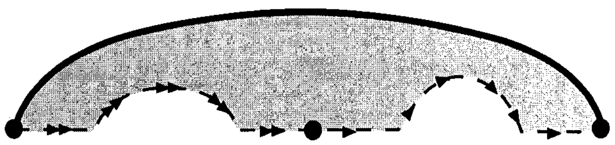
We will use D_R to show that W is the same element in $Z = (F(A)*(*_{k\in K}Stab(P_k)))$ as the product or $Area_{rel}(D_R)$ conjugates of super-relators. We first build a disc diagram by removing all 2-cells from D_R that mapped to super 2-cells of $R(X)$. We recall that every such super 2-cell corresponds to a super-relator in R. The removal of a 2-cell in D_R is done by first travelling along a path. The path is the concatenation of three subpaths p_1, p_2 and p_1^{-1}. The first subpath p_1 begins at the base point b on the boundary of D_R and travels to the boundary path of some super 2-cell. The subpath p_2 continues by travelling exactly once along the boundary path of the super 2-cell. The third subpath p_1^{-1} travels along p_1 in the opposite direction back to the basepoint. The path $p_1p_2p_1^{-1}$ represents a word over the alphabet $\bigcup_{k\in K} Stab(P_k) \cup A$ where $M = Area_{rel}(D_R)$. The path p_2 represents a super-relator R_i in R and p_1 represents some word $z_i \in Z$, we thus represent the word by $z_i R_i^{\pm 1} z_i^{-1}$. We travel along a path P that circles around each super-relator from D_R (see Figure 3.8(a)). The path P corresponds to a word $W_p = \prod_{i=1}^{M} z_i R_i z_i^{-1}$ representing an element of $\pi_1(Cap(X))$ with the index $M = Area_{rel}(D_R)$ (the number of super 2-cells removed). We cut along this path P removing each super 2-cell (see Figure 3.8(b)). The boundary path of the remaining disc diagram is the concatenation of the original path P_R representing the word W and the path P representing the word $W_p = \prod_{i=1}^{M} z_i R_i z_i^{-1}$. Since all super 2-cells have been removed from the disc diagram then the fact that the concatenation of these two paths is a combinatorial closed path representing words W and W_p translates to $W = \prod_{i=1}^{M} z_i r_i z_i^{-1}$ with equality in the group $\big(F(A) * (*_{k\in K} Stab(P_k))\big)$.

We therefore have shown that for the word W of length $|W| \leq n$ we have an expression $W = \prod_{i=1}^{M} z_i R_i z_i^{-1}$ with at most $M = Area_{rel}(D_R)$ terms. But

$Area_{rel}(D_R) \leq K|P|_{pl} \leq K|W| \leq Kn$. So the function $f(n) = Kn$ is a relative linear isoperimetric function of $\pi_1(Cap(X))$ with respect to the peripheral subgroups for the relative presentation $\big(F(A) * (*_{k \in K} Stab(P_k))\big)/R$.



3.8(a) A disc diagram in $R(X)$ with 2 super 2-cells



3.8(b) The disc diagram obtained by removal of super 2-cells

□

Proposition 3.7.3. [Os] *Let G be a finitely generated group, generated by the set Y and let $\{H_1, ..., H_n\}$ be a collection of subgroups of G. Then the following are equivalent:*

(1) G has finite relative presentation with respect to $\{H_1, ..., H_n\}$ and its corresponding relative isoperimetric function is linear.

(2) G is relatively hyperbolic with respect to $\{H_1, ..., H_n\}$.

Corollary 3.7.4. *The group $\pi_1(Cap(X))$ is relatively hyperbolic with respect to the peripheral subgroups.*

PROOF. In Theorem 3.7.2 we showed that $\pi_1(Cap(X))$ has a relative presentation with respect to the peripheral subgroups and its corresponding relative isoperimetric function is linear. Since $\pi_1(Cap(X))$ is finitely presented then by Proposition 3.7.3 it is relatively hyperbolic with respect to the peripheral subgroups. □

Main Theorem 3.7.5. *The group $\pi_1(X)$ is relatively hyperbolic with respect to the peripheral subgroups.*

PROOF. It is clear by the mapping cylinder cell structure of $Cap(X)$ that it can be deformation retracted to X. Therefore $\pi_1(X) \cong \pi_1(Cap(X))$, so $\pi_1(X)$ is relatively hyperbolic with respect to the peripheral subgroups. $\square$

Chapter 4

CONCLUSION

4.1. Possible Extensions of Results

There exist several combination theorems for word-hyperbolic groups, most notably that of Bestvina and Feighn in [**BF**]. More recently, Dahmani generalized their work to relatively hyperbolic groups. In this section we will state one of Dahmani's combination theorems for relatively hyperbolic groups and explain how it relates to our results. We will then state some conjectures which we hope could be proven using generalizations of the combinatorial arguments presented throughout this thesis.

Theorem 4.1.1. [**Dah**] *Let Γ be the fundamental group of an acylindrical finite graph of relatively hyperbolic groups, whose edge groups are fully quasi-convex subgroups of the adjacent vertex groups. Let G be the family of the images of the maximal parabolic subgroups of the vertex groups, and their conjugates in Γ. Then, (Γ, G) is a relatively hyperbolic group.*

Note that we have quoted Dahmani's theorem directly, here a group is hyperbolic relative to its parabolic subgroups, we have used the term peripheral instead of parabolic throughout this thesis. The term *acylindrical* means that there is a number k such that the stabilizer of any segment of length k in the Basse-Serre tree is finite.

Let us examine Theorem 4.1.1 in our context, namely where vertex groups are free groups and edge groups are cyclic. The acylindrical condition implies that each of our peripheral subgroups are cyclic. Thus Γ is hyperbolic relative to a set of cyclic subroups and is thus word-hyperbolic. In our context, though vertex groups and edge groups are quite limited, by not imposing the acylindrical condition we get a much richer interaction between edge groups. This leads to the definite failure of word-hyperbolicity and the need to include non-cyclic peripheral subgroups for relative hyperbolicity. In conclusion, Dahmani's very strong hypotheses totally precludes any interesting relatively hyperbolic behavior.

We now discuss two generalizations of our results. For simplicity we restrict ourselves by preserving the hypothesis that all edge groups are cyclic. We then first let vertex groups be word-hyperbolic and then consider a second case when vertex groups are relatively hyperbolic.

Let X be a graph of word-hyperbolic groups with cyclic edge groups. We let V_X denote the collection of conjugacy classes maximal cyclic subgroups of vertex groups of X. A representative of an element of V_X is called *relevant* if some edge group of X is conjugated into it.

We *build* an induced graph of groups X' with the following properties:

(1) The vertex groups of X' are the relevant representatives of elements of V_X

(2) The edge groups of X' are the edge groups of X

Each connected component X_i of X' is a *cylindrical graph of groups.* The natural map $X_i \to X'$ is an induced π_1-injection. The fundamental group of each X_i is a *peripheral subgroup* of X'.

Conjecture 4.1.2. *The fundamental group of the graph of groups X with word-hyperbolic vertex groups and cyclic edge groups is hyperbolic relative to the peripheral subgroups of X'.*

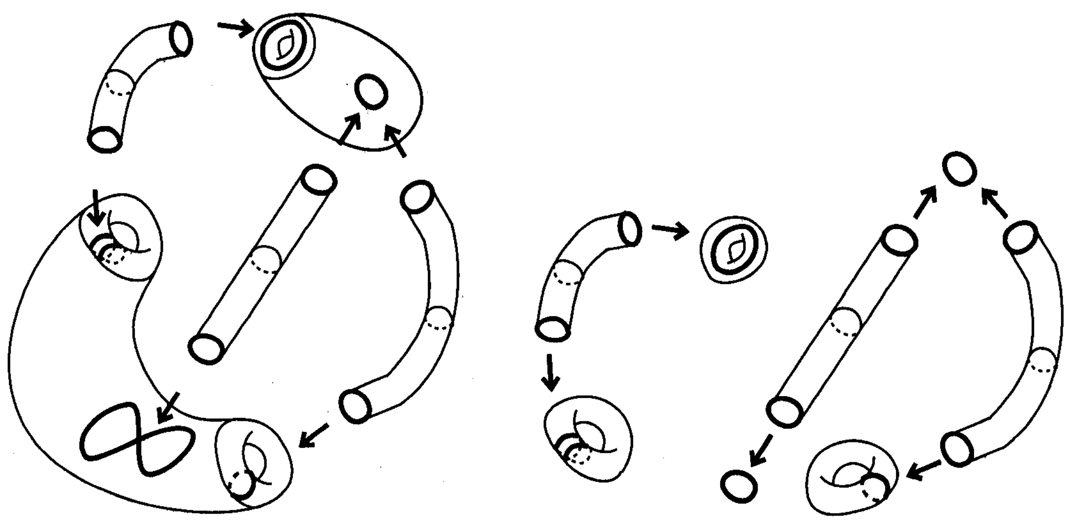
We now turn to the case where X is a graph of relatively hyperbolic groups with cyclic edge groups. Each vertex group of X is hyperbolic relative to a set of peripheral subgroups. We let V_X denote the set of peripheral subgroups of all vertex groups of X together with the set of all conjugacy classes of maximal cyclic subgroups of vertex groups of X that do not conjugate into a peripheral subgroup. A reprsentative of an element of V_X is called *relevant* if some edge group of X is conjugated into it. We note that conjugates intersect uniquely because peripheral subgroups are malnormal [**Gro**].

We *build* an induced graph of groups X' with the following properties:

(1) The vertex groups of X' are the relevant representatives of elements of V_X

(2) The edge groups of X' are the edge groups of X

Each connected component X_i of X' is called a *cylindrical graph of groups.* The natural maps $X_i \to X'$ are π_1-injections and the fundamental group of each X_i is a *peripheral subgroup.* An example of a graph of relatively hyperbolic groups with cylic edge groups and its induced graph of groups is illustrated in Figure 4.1.



4.1(c) Graph of relatively hyperbolic groups with cyclic edge groups

4.1(d) Induced graph of groups

Conjecture 4.1.3. *The fundamental group of the graph of groups X with relatively hyperbolic vertex groups and cyclic edge groups is hyperbolic relative to the peripheral subgroups of X'.*

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