Lucas' Counter Example Revisited

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Abstract

We revisit Lucas' (1968) counter example for the existence of von Neumann and Morgenstern (1944) stable set (solution) for coalitional games. We show that when we endow the agents with foresight, particularly, when we replace von Neumann and Morgenstern's (1944) dominance relation with the indirect dominance relations introduced by Harsanyi (1974), Lucas' example admits a stable set.

1 Introduction

Most solution concepts for coalitional games typically assign to every game a subset of payoff allocations (for the grand coalition). Consider for example the core. It is the set of payoff allocations that no coalition can improve upon or has a feasible and beneficial objection to. von Neumann and Morgenstern (1944) solution or stable set requires that an objection made by any coalition be credible in that the objection itself must be in the solution. A stable set is a set of efficient and individually rational payoff allocations (imputations) with the properties that no coalition has a credible objection to any stable imputation (internal stability) and for every unstable imputation, some

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coalition has a credible objection (external stability). Put differently, no imputation in a stable set dominates another imputation in the same stable set via any coalition and every imputation outside of the stable set is dominated by some imputation in the stable set via some coalition. The introduction of stable set sparked the quest for results concerning its general existence. The long-standing question was answered in the negative by Lucas who provided a ten-person game without a stable set (Lucas, 1968) and a formal proof of the negative result (Lucas, 1969).

Harsanyi (1974) criticizes von Neumann and Morgenstern stable set for not taking foresight into consideration: Consider an imputation x in the stable set. It is possible that a coalition S has a feasible and beneficial objection to it. Say that this objection is y. By internal stability, y cannot be stable. Then by external stability, some coalition T has a credible objection to y using some z that is stable. However, it may well be the case that members of S prefer z to x. If members of S are forward looking, the (internal) stability of x is no longer warranted. Put differently, while z does not dominate x "directly" according to von Neumann and Morgenstern's (1944) original definition, z dominates x "indirectly" and a stable set may be susceptible to the destabilizing effect of such indirect dominance. Harsanyi (1974) also considers another version of indirect dominance, whereby each deviating coalition compares only the final payoff allocation to the one it replaces, ignoring the immediate impact of its deviation.

In this paper, we consider the stable set with Harsanyi's (1974) indirect dominance relations. We reexamine the counter example of Lucas (1968) and show that his example in fact admits a stable set. In particular, if the first dominance relation is employed the stable set is unique, while the second dominance relation gives rise to multiple stable sets. This leads to a new open question whether a stable set with indirect dominance always exists.

2 Main Result

We will adopt the notations introduced in Lucas (1969). Consider a coalitional game with transferable utility (TU), (N, v), where $N = \{1, ..., n\}$ is the finite set of players, a coalition S is a non-empty subset of N, and v is the characteristic function which assigns to every coalition S a nonnegative scalar. For the empty set we have $v(\varnothing) = 0$. Let $x = (x_1, ...x_n)$ denote the payoff allocation of the n players and $x(S) = \sum_{i \in S} x_i$. Moreover, we will use $x(ijk)$ as a shorthand for $x(\{i,j,k\})$, where $i,j,k \in N$. The set of imputations is

$$A = \left\{ x \in \mathcal{R}^n \,\middle|\, \sum_{i \in N} x_i = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for all } i \in N \right\}.$$

The following dominance relation was introduced by von Neumann and Morgenstern (1944):

Direct dominance Let $x, y \in A$. y (directly) dominates x via coalition $S \subset N$, written $y \succ_S x$, if $\sum_{i \in S} y_i \leq v(S)$ and $y_i > x_i$ for all $i \in S$. y (directly) dominates x, written $y \succ x$, if $y \succ_S x$ for some $S \subset N$.

von Neumann and Morgenstern (vN-M) stable set or solution for a TU game (N, v) is defined as follows:

Definition 1 *$K \subset A$ is a vN-M stable set for (N, v) if*

- *for all $x, y \in K$, $y \not\succ x$ (internal stability), and*
- *for all $x \in A \backslash K$, there exists $y \in K$ such that $y \succ x$ (external stability).*

The core of (N, v) is the set of undominated elements in A.

Definition 2 *The core of (N, v) is*

$$\begin{aligned} C(N,v) &= \{x \in A \mid \nexists y \in A \text{ such that } y \succ x\} \\ &= \{x \in A \mid x(S) \geq v(S) \text{ for all } S \subset N\}. \end{aligned}$$

Alternatively, vN-M stable set can be defined as an abstract stable set for the abstract system $(A, \succ)$. Recall that an abstract system is $(X, \rhd)$, where X is the set of alternatives and $\rhd$ is a dominance relation on X with the interpretation that for $x, y \in X$, $x \rhd y$ implies x dominates y. Given $B \subset X$, the dominion of B, denoted $\text{dom}(B, \rhd)$, is the set of alternatives dominated by elements of B. Formally,

$$\text{dom}(B, \rhd) = \{x \in X \mid \exists y \in B \text{ such that } y \rhd x\}.$$

Now we are ready to introduce the notion of abstract stable set.

Definition 3 *$K \subset X$ is an abstract stable set for $(X, \triangleright)$ if*

- *$x \notin \mathrm{dom}(K)$ for all $x \in K$ (internal stability), and*
- *$x \in \mathrm{dom}(K)$ for all $x \in X \setminus K$ (external stability).*

A vN-M stable set for (N, v) is an abstract stable set for $(A, \succ)$. Lucas (1969) proved that the counter example he described in Lucas (1968) admits no vN-M stable set. The 10-person counter example is as follows.

Example 1 *Consider the game (N, v) such that $N = \{1,2,3,4,5,6,7,8,9,10\}$ and*

$v(N) = 5$
$v(\{1,3,5,7,9\}) = 4$
$v(\{1,2\}) = v(\{3,4\}) = v(\{5,6\}) = v(\{7,8\}) = v(\{9,10\}) = 1$
$v(\{3,5,7,9\}) = v(\{1,5,7,9\}) = v(\{1,3,7,9\}) = 3$
$v(\{3,5,7\}) = v(\{1,5,7\}) = v(\{1,3,7\}) = 2$
$v(\{3,5,9\}) = v(\{1,5,9\}) = v(\{1,3,9\}) = 2$
$v(\{1,4,7,9\}) = v(\{3,6,7,9\}) = v(\{5,2,7,9\}) = 2$
$v(\{S\}) = 0$ *for all other* $S \subset N$.

Note that the set of all the imputations is

$$A = \left\{ x \in R^{10} \,\middle|\, \sum_{i \in N} x_i = 5 \text{ and } x_i \geq 0 \text{ for all } i \in N \right\}.$$

Following Lucas (1969), we first define set B.

$$B = \{x \in A \mid x(12) = x(34) = x(56) = x(78) = x(9\ 10) = 1\}.$$

The core is given by

$$C = \{x \in B \mid x(13579) \geq 4\}.$$

Lucas (1969) proves that the game given above does not have a vN-M stable set. We shall show that if we replace the direct dominance with each of the indirect dominance relations introduced by Harsanyi (1974), then existence of a stable set is restored. Recall the following indirect dominance relation:

Indirect Dominance α Let $x, y \in A$. y indirectly dominates x, written $y \gg^{\alpha} x$, if there exist $x^0, x^1, \ldots, x^k$, where $x^0 = x$ and $x^k = y$, and $S^1, S^2, \ldots, S^k \subset N$ such that for all $j = 1, 2, \ldots, k, x^j \succ_{S^j} x^{j-1}$ and $x_i^{j-1} < y_i$ for all $i \in S^j$.

Thus, given imputation x^{j-1}, deviating coalition S^j does not only compare the current imputation with the one it brings about but also looks forward and compares the current imputation with the final imputation y induced by a sequence of coalitional deviations. Note that $y \succ x$ implies $y \gg^{\alpha} x$.

We shall show that Lucas' (1968) example admits a unique stable set with respect to indirect dominance $\gg^{\alpha}$.

Proposition 1 *For Example 1, $(A, \gg^{\alpha})$ admits a unique abstract stable set.*

Proof. We first partition A into subsets as in Lucas (1969). We start with the following variable indices following Lucas (1969):

$$\begin{aligned} (i, j, r, k) &= (1,3,4,5)\,, (3,5,6,1)\,, \text{or } \ (5,1,2,3)\,, \\ (p, q) &= (7,9) \ \text{ or } \ (9,7)\,. \end{aligned}$$

Using these indices, define the following twelve subsets of B :

$E_i = \{x \in B \mid x_j = x_{k,}\ x_i < 1,\ x(79) < 1\}$,

$E = E_1 \cup E_3 \cup E_5$,

$F_{jk} = \{x \in B \mid x_j = x_k = 1,\ 1 \leq x(79)\} - C$,

$F_p = \{x \in B \mid x_p = 1,\ x_q < 1,\ x(35q) \geq 2,\ x(51q) \geq 2,\ x(13q) \geq 2\} - C$,

$F_{79} = \{x \in B \mid x_7 = x_9 = 1\} - C$,

$F_{135} = \{x \in B \mid x_1 = x_3 = x_5 = 1\} - C$,

$F = F_{35} \cup F_{51} \cup F_{13} \cup F_7 \cup F_{79} \cup F_{135}$.

Then A can be partitioned into $A \setminus B, B \setminus (C \cup E \cup F)\,, C, E$, and F.

We shall prove by construction that the unique abstract stable set for $(A, \gg^{\alpha})$ is

$$K = C \cup F_7 \cup F_9 \cup F_{79} \cup F_{135}.$$

Lucas (1969) shows that

$$\text{dom}(C, \succ) \supset [A \setminus B] \cup [B \setminus (C \cup E \cup F)]\,.$$

Since $y \succ x$ implies $y \gg^{\alpha} x$, we have

$$\text{dom}(C, \gg^{\alpha}) \supset [A \setminus B] \cup [B \setminus (C \cup E \cup F)]\,.$$

Moreover, it is easy to see that $\text{dom}(A, \gg^\alpha) \cap C = \varnothing$. Thus, an abstract stable set K for $(A, \gg^\alpha)$ necessarily contains C. We shall further show that $\text{dom}(C, \gg^\alpha) \supset E \cup F_{jk}$ for all j, k defined earlier, thereby establishing that $A \setminus K$ cannot be part of any abstract stable set and that K is externally stable. To this end, first consider

$$E_1 = \{x \in B \mid x_3 = x_5 = 1, x_1 < 1, x(79) < 1\}.$$

Let $x \in E_1$. Then, $x(13579) < 4$ and $x_4 = x_6 = 0$. Consider $y \in A$ such that

$$\begin{aligned} 1 &> y_1 > x_1, \quad y_7 > x_7, \quad y_9 > x_9, \quad y(79) < 1, \\ y_4 &= 2 - (y_1 + y_7 + y_9) > x_4 = 0, \\ y_2 &= y_3 = y_5 = y_6 = y_8 = y_{10} = \frac{1}{2}. \end{aligned}$$

Obviously, $y(1479) = 2 = v(\{1,4,7,9\})$ and $y \succ_{\{1,4,7,9\}} x$. Now let z be such that

$$\begin{aligned} 1 &> z_1 > y_1, \; z_2 = 1 - z_1, \\ 1 &> z_3 > y_3, \; z_4 = 1 - z_3, \\ 1 &> z_5 > y_5, \; z_6 = 1 - z_5, \\ 1 &> z_7 > y_7, \; z_8 = 1 - y_7, \\ 1 &> z_9 > y_9, \; z_{10} = 1 - y_9, \\ z(13579) &= 4 = v(\{1,3,5,7,9\}). \end{aligned}$$

Such a z exists as $y(13579) < 3$. By construction, $z \gg^\alpha x$ and $z \in C$.

Similar arguments establish that $\text{dom}(C, \gg^\alpha) \supset E_3$ and $\text{dom}(C, \gg^\alpha) \supset E_5$.

Now consider

$$F_{35} = \{x \in B \mid x_3 = x_5 = 1, \; x(79) \geq 1\} \setminus C.$$

Let $x \in F_{35}$. Then, $x_4 = x_6 = 0$. $x \notin C$ implies that $x(13579) < 4$. In view of the fact that $x_3 = x_5 = 1$, we have $x(179) < 2$. Since $x_4 = 0$, we have $x(1479) < 2$. Also, given that $x(79) \geq 1$, $x_1 < 1$. Consider $y \in A$ such that

$$\begin{aligned} 1 &> y_1 > x_1, \; y_7 > x_7, \\ y_9 &> x_9, \; y(179) < 2, \\ y_4 &= 2 - (y_1 + y_7 + y_9) > x_4 = 0, \\ y_2 &= y_3 = y_5 = y_6 = y_8 = y_{10} = \frac{1}{2}. \end{aligned}$$

Obviously, $y(1479) = 2 = v(\{1,4,7,9\})$. Now let z be such that

$$\begin{aligned}
1 &> z_1 > y_1,\ z_2 = 1 - z_1, \\
1 &> z_3 > y_3,\ z_4 = 1 - z_3, \\
1 &> z_5 > y_3,\ z_6 = 1 - z_5, \\
1 &> z_7 > y_7,\ z_8 = 1 - y_7, \\
1 &> z_9 > y_9,\ z_{10} = 1 - y_9, \\
z(13579) &= 4 = v(\{1,3,5,7,9\}).
\end{aligned}$$

Such a z exists as $y(13579) < 4$. By construction, $z \gg^\alpha x$ and $z \in C$. Similarly, $\text{dom}(C, \gg^\alpha) \supset F_{51}$ and $\text{dom}(C, \gg^\alpha) \supset F_{13}$.

Lastly, we establish that K is internally stable by showing that

$$\text{dom}(B, \gg) \cap (F_7 \cup F_9 \cup F_{79} \cup F_{135}) = \varnothing.$$

First, consider

$$F_7 = \{x \in B \mid x_7 = 1, x_9 < 1, x(359) \geq 2, x(519) \geq 2, x(139) \geq 2\} \setminus C.$$

We shall show that $\text{dom}(B, \gg^\alpha) \cap F_7 = \varnothing$. Let $x \in F_7$ and $y \in B$. Assume in negation that $y \gg^\alpha x$. Let S be the coalition that "deviates" from x. First, $x \in B$ implies that S cannot be any of the coalitions $\{1,2\}, \{3,4\}, \{5,6\}, \{7,8\}$, and $\{9,10\}$. Next, since $x_7 = 1$, player 7 cannot be a member of S. Also, since $x(359) \geq 2 = v(\{3,5,9\})$, $x(519) \geq 2 = v(\{5,1,9\})$, and $x(139) \geq 2 = v(\{1,3,9\})$, S cannot be any of the coalitions $\{3,5,9\}$, $\{5,1,9\}$, $\{1,3,9\}$ either. This contradicts that $y \gg^\alpha x$; hence $\text{dom}(B, \gg^\alpha) \cap F_7 = \varnothing$. A similar argument applies to F_9.

Consider

$$F_{79} = \{x \in B \mid x_7 = x_9 = 1\} \setminus C.$$

We shall show that $\text{dom}(B, \gg^\alpha) \cap F_{79} = \varnothing$. Let $x \in F_{79}$ and $y \in B$. Assume in negation that $y \gg^\alpha x$. Let S be the coalition that "deviates" from x. First, $x \in B$ implies that S cannot be any of the coalitions $\{1,2\}, \{3,4\}, \{5,6\}, \{7,8\}$, and $\{9,10\}$. Then $7 \notin S$ and $9 \notin S$ as $x_7 = x_9 = 1$. This contradicts that $y \gg^\alpha x$; hence $\text{dom}(B, \gg^\alpha) \cap F_7 = \varnothing$. A similar argument establishes that $\text{dom}(B, \gg^\alpha) \cap F_{135} = \varnothing$.

In conclusion, the unique stable set is $K = C \cup F_7 \cup F_9 \cup F_{79} \cup F_{135}$. ■

In addition to the indirect dominance defined earlier, Harsanyi (1974) also discussed the following alternative indirect dominance whereby a deviating coalition only compares the current imputation with the final one.

Indirect Dominance β Let $x, y \in A$. y indirectly dominates x, written $y \gg^{\beta} x$, if there exist $x^0, x^1, \ldots, x^k$, where $x^0 = x$ and $x^k = y$, and $S^1, S^2, \ldots, S^k \subset N$ such that for all $j = 1, 2, \ldots, k$, $x^j(S^j) \leq v(S^j)$ and $x_i^{j-1} < y_i$ for all $i \in S^j$.

Note that $y \succ x$ also implies $y \gg^{\beta} x$.

Proposition 2 *For Example 1, $(A, \gg^{\beta})$ admits an abstract stable set.*

Proof. We shall now show that $(A, \gg^{\beta})$ admits a stable set. Let $K = \{y\}$ where

$$y = \left(\frac{4}{5}, \frac{1}{5}, \frac{4}{5}, \frac{1}{5}, \frac{4}{5}, \frac{1}{5}, \frac{4}{5}, \frac{1}{5}, \frac{4}{5}, \frac{1}{5}\right).$$

We shall show that $y \gg^{\beta} x$ for all $x \in A \setminus K$. Let $x \in A \setminus K$. Then, there exists $i \in N$ such that $x_i < y_i$. Let $z \in A$ be such that $z_i = 0 = v(\{i\})$ and $z_j = \frac{5}{9}$ for all $j \neq i$. Since $z_\ell < y_\ell$ for all $\ell \in \{1, 3, 5, 7, 9\}$ and $y(13579) = v(\{1, 3, 5, 7, 9\}) = 4$. We have $y \gg^{\beta} x$. It is easy to construct other abstract stable sets by changing y appropriately. ■

3 Conclusion

We have show that Lucas' (1968) counter example for the existence of vN-M (1944) stable set nevertheless admits a stable set when we replace vN-M's (1944) direct dominance with any one of Harsanyi's (1974) indirect dominance relations that capture foresight on the part of the players. The general existence of stable set with indirect dominance remains an open question.

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