How elementary students learn to mathematically analyze word problems: The case of addition and subtraction

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The complexity "is a hallmark of educational settings."

(Cobb, Confrey, Lehrer, & Schauble, 2003, p. 9)

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Abstract

Mathematical problem solving, and more specifically the ability to mathematically analyze and model a situation, is one of the most important aspects of teaching and learning mathematics in school. Today, researchers agree that the problem-solving and mathematizing phenomena are extremely complex and that research is needed to better understand the cognitive processes involved at a phenomenological level. The lack of nuanced understanding of the ways of reasoning students might employ to analyze and model a problem prevents teachers from effectively meeting their needs.

Within the context of a larger study on the development of mathematical reasoning in early grades of elementary school, I studied how grade two elementary school students solve additive problems to answer the following questions:

1. What kind of mathematizing do students use to solving additive word problems?
2. What are the relationships between the instruction implemented and students' development of mathematizing processes?

Applying the grounded theory methodology, I analyzed multiple observations of students solving additive problems throughout one school year. I suggest models for six strategies of mathematizing, which I describe in detail. I describe the dynamics of change in the learners' ways of reasoning and the relationships between this change and the teaching implemented.

Résumé

La résolution de problèmes mathématiques, et plus particulièrement la capacité d'analyser et de modéliser mathématiquement une situation, est l'un des aspects les plus importants de l'enseignement et l'apprentissage des mathématiques à l'école. De nos jours, les chercheurs s'accordent à dire que les phénomènes de résolution de problèmes et de la mathématique d'une situation sont extrêmement complexes et que la recherche est nécessaire pour mieux comprendre au niveau phénoménologique des processus cognitifs impliqués. Le manque de compréhension nuancée du raisonnement que des apprenants

pourraient employer pour analyser et modéliser un problème empêche les enseignants de répondre à leurs besoins de façon efficace.

Dans le contexte d'une plus grande étude concernant le développement de raisonnement mathématique dans les premières années de l'école primaire, j'ai étudié les élèves de deuxième année du primaire en train de résoudre des problèmes additifs pour répondre aux questions suivantes:

1. Quels sont les moyens de mathématisation utilisent les élèves pour résoudre des problèmes écrits ayant des structures additives?
2. Quel est le rapport entre l'enseignement mis en œuvre et le développement des processus de mathématisation des élèves?

En adoptant la méthodologie de la théorisation ancrée, j'ai observé et analysé des élèves à résoudre des problèmes additifs au cours d'une année scolaire. J'ai modélisé six stratégies de mathématisation, que j'ai décrites en détail. J'ai décrit la dynamique du changement des modes de raisonnement chez les apprenants, ainsi que les relations entre ce changement et l'enseignement mis en œuvre.

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Preface

This study investigates the development of mathematical reasoning in students within the context of solving additive word problems. I analyzed 12 Grade 2 students solving problems throughout one school year. The study falls within the scope of Relational Paradigm, contributing to the originality of the work.

In this thesis, I provide detailed descriptions of five different ways in which students mathematize problems. To my knowledge, at least three of these ways have never been discussed in literature: uncoordinated use of different strategies, use of a graphic schema as a template and use of a graphic schema to understand the problem. The use of graphic schemas as template described in this study helps explain the phenomenon of inverse mathematizing in students' reasoning. Understanding this phenomenon is critical for using schemas and diagrams to help young students solve problems.

I propose two theoretical models to understand how students mathematize word problems. First, I propose a model to explain factors that affect how students reason in the process of problem solving. Second, I adapt the model of mathematical modeling proposed by Annie Savard (2008) to explain various means students use to mathematize problems.

In this study, I thoroughly analyze the relationship between changes in learners' mathematizing processes and the teaching implemented. I discuss various challenges students might have developing their knowledge. In line with the Vygotsky theory (1991), I distinguish between challenges that might be compensated for by adapted teaching and those allowing learners to develop their mathematical reasoning.

The use of a particular type of graphic representation (Arrange All Diagrams) produced through a meta-cognitive process described by the above-mentioned Savard's model (2008) might be of great interest to researchers and practitioners. In this thesis, I describe how this type of graphic representation can become a powerful didactic tool, providing students and teachers with the effective communication media to discuss the mathematical relationships and mathematical structure of problems.

Chapter 1 Introduction and Problem

1.1 Societal and practical need

Mathematical problem solving, and more specifically the ability to mathematically analyze and model a situation, is one of the most important aspects of teaching and learning mathematics in school (Lesh, Doerr, Carmona, & Hjalmarson, 2003). According to Mukhopadhyay and Greer (2001), mathematics is a “tool for describing and analyzing aspects of real world phenomena” (p. 296). The ability to mathematically analyze a situation helps students to be critical thinkers and affront various social and political issues (ibid.).

The latest educational reform in Quebec (MELS[1], 2004) prioritizes problem solving and modeling in mathematics curriculum. The teaching and learning of mathematics is implemented through the development of three transversal competences: reasoning using mathematical concepts, solving situational problems, and communicating using mathematical language. In early grades, one of the important mathematical contexts in which these transversal competencies should be developed is that of knowledge related to addition, subtraction and various additive structures (MELS, 2009). However, the curriculum is not explicit in explaining how additive structures should contribute to reasoning using mathematical concepts, nor does it elaborate on what kind of mathematical communication should be associated with or could support this knowledge development.[2]

There is a core of research (Carpenter, Fennema, & Franke, 1996; Fagnant, 2005; Julo, 2002; Ng & Lee, 2009; Novotná, 1998; Stacey & Macgregor, 2000) that examines various ways of supporting students in problem-solving activity. Authors (Baruk, 2003; Lesh et al., 2003; Lesh & Zawojewski, 2007; Poirier, 2000; Small & Cousineau, 2008; Van de Walle & Lovin, 2008) propose that problems be modeled with students to help

[1] Ministère de l'Éducation, du Loisir et du Sport du Québec

[2] As for situational problems, the curriculum does not see these tasks as a means to work on additive structures. Therefore in this study, I will concentrate my attention on the activities directly related to the development of an understanding of additive structures, such as modeling and solving simple additive word problems.

students better understand their structure. Nonetheless, as school board consultants have noticed (Gervais, Savard, & Polotskaia, 2013b), elementary teachers often complain that, in a problem-solving situation, regardless of their teaching efforts, many students choose the arithmetic operation by chance without profound reasoning. The authors argue that teachers need more explicit suggestions on how to interpret the existing theoretical knowledge and connect it to their practical experience. The lack of nuanced understanding of ways of reasoning that students might employ to analyze and model a problem prevents teachers from efficiently meeting students' needs. More work is needed to connect the knowledge issued from research and the knowledge issued from practice (Kieran, Krainer, & Shaughnessy, 2013).

1.2 Need for research

The research in education and cognitive science has accumulated a significant amount of knowledge about mathematical reasoning and problem solving in general, and specifically about simple additive word problems in early grades. Researchers (Davydov, 1990; Geary, 2006; Sierpinska, Nnadozie, & Oktaç, 2002; Sierpinska, 1994; Steen, 1999) have studied mathematical reasoning and the development of theoretical reasoning in mathematics. More specifically, the comprehension process of a text (problem) has been modeled (Cummins, Kintsch, Reusser, & Weimer, 1988; Kintsch & Greeno, 1985; Kintsch, 1994, 2005; Nesher, Greeno, & Riley, 1982; Pape, 2003, 2004). Schmidt and Bednarz (1997) studied algebraic reasoning versus arithmetic reasoning in problem solving.

The typology of simple additive word problems and their semantic structure have been studied (Nesher et al., 1982; Riley, Greeno, & Heller, 1984; Vergnaud, 1982a)[3]. Voyer (2009) studied relationships between the word formulation of a problem task and the mathematical abilities of the student.

Researchers (Barrouillet & Poirier, 1997; Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993; De Corte & Verschaffel, 1980; Fuson, Carroll, & Landis, 1996)

[3] This topology will be analyzed in more detail in the section Epistemology of additive problems, p. XXX

observed and analyzed strategies used by students to solve word problems. The difference between “expert students’ solutions” and “novice students’ solutions” has been discussed (Krutetskii, 1976; Lesh & Zawojewski, 2007). The effects of the didactic contract on the solution strategies students employ have been explored (Brousseau, 1988; Chevallard, 1988; Schubauer-Leoni & Ntamakiliro, 1994).

The role of curriculum and textbook content has also been questioned (Sarrazy, 2002; Xin, 2007). Lajoie and Bednarz (2012) used a historic approach to analyze the teaching of problem solving in Quebec.

New data has been obtained from cutting-edge research in neuro-education that looks at the problem solving phenomena from a different perspective—brain mechanisms (Stavy &Babai, 2009).

All above-mentioned researchers agree that the problem solving phenomena is extremely complex. It is widely accepted by researchers and practitioners that the ability to mathematically analyze and solve a problem is one of the most difficult to develop. Improvement is urgently needed in this area of educational research (Checkley, 2006). More specifically, for the act of understanding of a word problem and mathematizing, research is needed to better describe the cognitive processes involved at phenomenological level (Hestenes, 2010).

Researchers (DeBlois, 1997; Fagnant & Vlassis, 2013; Gamo, Sander, & Richard, 2009; Neef, Nelles, Iwata, & Page, 2003; Ng & Lee, 2009; Xin, Wiles, & Lin, 2008) propose various teaching methods and techniques to help students develop their problem-solving abilities. Some authors suggest modeling problems with students (Baruk, 2003; Poirier, 2000; Small & Cousineau, 2008; Van de Walle & Lovin, 2008), while others promote model eliciting problem solving (Lesh et al., 2003; Lesh, Galbraith, & Haines, 2010). However, research on mathematical modeling in the classroom is mainly concerned with more complex mathematical or scientific concepts, such as functions or velocity. Arithmetic problems with simple additive structures, which can be solved using one addition or one subtraction operation, are not usually considered to be requiring any mathematical modeling. There is no empirical evidence in research of how and why learning to mathematically model such problems might affect the development of students' mathematical reasoning (Lingefjärd, 2011).

The method of modeling used in Singapore, Russia and some other countries has attracted the attention of researchers (Beckmann, 2004; Hoven & Garelick, 2007; Lee, Ng, & Ng, 2009; Ng & Lee, 2009; Nunes, 2012). Although the researchers recognize the importance of this particular method to the development of mathematical reasoning, our knowledge is limited with regard to how this way of modeling affects student's learning (Nunes, 2012).

1.3 Context of the study

Usually, students start to solve word problems in their first years of regular schooling. The first problems proposed to them are ones involving addition and subtraction. In an effort to answer the theoretical and practical needs in this area formulated above, I decided to study elementary students at the very beginning of their learning to mathematically analyze simple additive word problems.

I had the privilege of participating in a collaborative study led by Dr. Annie Savard[4], called "Problem solving in first cycle elementary school: research, development, implementation." This large project was conducted in the early grades of elementary school and was funded by the Quebec Ministry of Education, Leisure, and Sport. The main goals of the project were to:

- develop and test new teaching strategies, concrete teaching/learning activities, and didactic scenarios which could support the efficient development of the problem solving ability in students
- develop and test training activities to support teachers in their working strategies shift

Within the scope of this project, the team proposed to see the problem-solving activity from a new perspective: explicit mathematical modeling of word problems. We

[4] Further in this study, I will refer to Dr. Savard's study as "collaborative project."

named this teaching approach the Equilibrated Development. My role in the project consisted in: designing learning activities, working with teachers, collecting and analyzing data.

This collaborative project provided an excellent context for my research on how students mathematize additive problems, giving me the opportunity to observe how students got from the text of a problem to the mathematics in it.

1.4 Research objectives

The Equilibrated Development teaching approach implemented in the collaborative project includes the attempt to develop in students the ability to mathematically analyze the structure of a word problem. For my PhD project, I chose to explore and try to understand how and why students develop their ways of mathematically analyzing additive word problems while being exposed to this particular teaching approach. Thus, my objective is:

to study how elementary school students develop their mathematical reasoning in the context of additive word problem solving while being exposed to Equilibrated Development teaching.

This investigation will allow a deeper understanding of how early grade learners develop their ways of mathematizing simple additive word problems. It will also help explain the relationships between the teaching employed and the development of mathematical thinking in learners. The latter will help to formulate suggestions about new learning opportunities and teaching principles.

Many terms used in the field of mathematics education are interpreted differently by researchers. Below, I explain the use of several important terms and expressions within this study. A thorough analysis of concepts behind each term is not in the scope of this study.

1.5 The terminology used in the study

Simple additive word problems

Researchers define word problems as follows (Verschaffel, Greer, & De Corte, 2000):

> Word problems can be defined as verbal descriptions of problem situations wherein one or more questions are raised, the answer to which can be obtained by the application of mathematical operations to numerical data available in the problem statement. In their most typical form, word problems take the form of brief texts describing the essentials of some situation wherein some quantities are explicitly given and others are not, and wherein the solver – typically a student who is confronted with the problem in the context of a mathematics lesson or a mathematics test – is required to give a numerical answer to a specific question by making explicit and exclusive use of the quantities given in the text and mathematical relationships between those quantities inferred from the text. (p. ix)

This study will focus on word problems with simple additive structures. In order to simplify the expression, I will henceforth use the term “simple additive word problems.” A simple additive word problem is a word problem with the following key characteristics:

- It is presented to students in the form of a written text.
- It describes a real-life, tangible context, not a purely mathematical context.
- It has one question requiring one numerical answer.
- The answer can be calculated using one addition or subtraction operation.

Example: *I have 3 apples. Someone gives me 2 more apples. How many apples do I have now?*

Mathematical expression

In the context of this study, I will use the term mathematical expression to refer to phrases composed of numbers, mathematical operations, and one equal sign (if any) that students can construct to express their understanding of a word problem or their solution to the problem. A mathematical expression in a standard form has an unknown isolated element, usually on the right side of the equal sign. It can be directly used to calculate the unknown using a calculator.

17 + 34 = ? or just 17 + 34

An open mathematical expression (non-standard form) can contain an unknown element in any other position. A mathematical expression in a non-standard form cannot be directly put into a calculator.

17 + ? = 34

Expressions like 17 + 34 will be considered mathematical expressions in standard form (abbreviated).

Mathematical structure of the problem

I will also use the concept of mathematical structure of the problem, which is usually understood as the system of mathematical relationships between quantities explicitly or implicitly described in the problem (Christou & Philippou, 1999; Elia, Gagatsis, & Demetriou, 2007; Lemoyne & Tremblay, 1986; Meron & Peled, 2004; Nunes, Bryant, Hallett, Bell, & Evans, 2009; Verschaffel, Corte, & Vierstraete, 1999). Sometimes, a distinction is made between relations which can be easily established and those difficult to see, or between known and unknown quantities participating in these relationships. Researchers (Bednarz & Janvier, 1996; Schmidt & Bednarz, 2002) speak about connected and disconnected problems. In this paper, the term *mathematical structure of the problem* will refer to all relationships and quantities, evident or not, known or unknown.

Example: In the problem mentioned earlier, 3 apples are the part of the apples I have now, the 2 apples given to me are the other part of my apples and the 5 apples I have now are composed of the 3 apples I had before and the 2 apples given to me. There are no other quantities in the situation than those described. There are no other relationships between these three quantities.

Mathematical analysis

In this study, the mathematical analysis of a problem refers to finding the mathematical structure of the problem.

Mathematizing a problem

In this study, this expression will signify any process of transforming the text of a problem into mathematical ideas and/or processes.

Modeling

In this study, modeling will signify transforming the text of a problem into an object from which a calculation plan can be derived (to solve the problem).

Example of modeling for the above-mentioned problem: $3 + 2 = X$

Representation

There are many meanings for the word “representation,” and it is used differently by researchers in the field (Savard, 2008). Sometimes the word “representation” is associated with mental activity (DeBlois, 2003; Kintsch, 1994; Lee et al., 2009), sometimes with physical objects, icons, or symbols drawn on paper (Bebout, 1990; Britt & Irwin, 2008; Bruner, 1966). Researchers (G. A. Goldin & Shteingold, 2001; Pirie & Kieren, 1994) discuss possible relationship between mental or internal representation a person can have and the ways the person can communicate it externally: graphically, schematically etc.

In this study, the words “represent” and “representation” will be used to refer to the graphic representation of something on a paper or in a document. In the context of mental activity, the expression “mental representation” will be used.

Example: The problem formulated above can be represented graphically as follows:

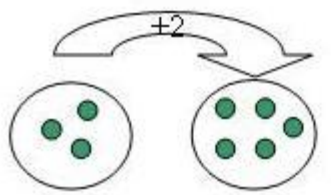

Figure 1. Representation of a problem

Schema

As well remarked by Nesher, Hershkovitz and Novotna (1997), there is some vagueness with regard to the terminology in the field of mathematical cognition theories. Vergnaud (2009) uses the word “scheme” to denote an organization of thought or behaviour. This organization appears in a similar way as a reaction to similar (problem) situations. Thus, a scheme encloses and organizes thought invariants or concepts, such as addition, subtraction, and natural number.

Fischbein (1999) uses the expression “structural schemata,” which for him denotes “behavioral-mental devices which make possible the assimilation and interpretation of information and the adequate reactions to various stimuli” (p. 11).

For Mayer (1992), a schema is “an organized structure consisting of certain elements and relations which are related to a situation and it can be used for understanding incoming information” (p. 228).

It appears that while using the words *scheme* or *schema*, some researchers speak more about mental mechanisms, and others put emphasis on the situation or the problem itself or its communicated representation (paper drawing). In this study, I will use the word *schema* in the sense of mental mechanism, similarly to the ideas of Vergnaud (2009) and Fischbein (1999). I will use the expression graphic schema in the context of paper drawing.

Chapter 2 Theoretical exploration

2.1 Two paradigms of additive problem solving

Additive problem solving has been studied for over a hundred years. Researchers from different countries and different schools of knowledge have largely addressed the epistemology of additive problems, students' solving strategies, teaching approaches and many other aspects of this subject. In order to better navigate through the sea of research in this domain, I will start by distinguishing two paradigms in which problem solving can be seen (Savard, Polotskaia, Freiman, & Gervais, 2013).

In the Operational Paradigm, addition and subtraction as arithmetic operations prevail. The knowledge of how to carry these operations is recognized as a starting point of the development of the problem solving ability. The problems themselves serve as exercises where the knowledge of arithmetic operations can be further developed. Students should interpret a word problem as referring to an operation. For example, the problem of apples will be interpreted as some apples **added** to the initial amount. Accordingly, many curriculums propose first learning about operations (how to add and subtract) and then solving various word problems to practise this knowledge and develop a conceptual understanding of the operations.

Furthermore with regard to the subtraction operation, Brissiaud (2010) explained that to have a conceptual knowledge of subtraction means having different senses (meanings) of the subtraction operation, such as finding the difference, finding the complement and taking away. However, knowing what it means to add or subtract two quantities or how to add or subtract two numbers is not enough to mathematically understand and solve additive word problems (Vergnaud, 1982b). Research has shown (Carpenter, Fennema, Franke, Linda, & Empson, 1999; Gerofsky, 2004; Nesher, Greeno, & Riley, 1982; Riley, Greeno, & Heller, 1984; Vergnaud, 1982) that the senses of subtraction-as-finding-the-difference and subtraction-as-finding-the-complement are not easily constructed directly from the intuitive sense of subtraction-as- taking-away. Researchers agree that the mathematical structure of the problem plays an important role in the problem solving. Yet in the Operational Paradigm, the knowledge of operations is seen as a means to understand mathematics in the problem.

In the Relational Paradigm, which appears in works by Davydov (1982) and more recent studies (Christou & Philippou, 1999; Iannece, Mellone, & Tortora, 2009; Xin et al., 2008), the idea of relationship prevails. Davydov (1982) describes the concept of additive relationship as "the law of composition by which the relation between two elements determines a unique third element as a function" (p. 229). Davydov (1982) argues that the additive relationship is the basis for learning addition and subtraction and should be a part of mathematics curriculum. In the Relational Paradigm, addition and subtraction operations are not the means to understand a situation (problem), but serve as tools to modify the situation, once understood as additive relationships. In the problem solving context, it means that one should first grasp the additive relationship described in the problem and then derive from this relationship the arithmetic operation needed to calculate the unknown element. For example, the apples problem should be first interpreted as final amount of apples is **composed of** initial amount of apples and the apples given.

The two paradigms are not antagonists. All researchers recognize the importance of relationships as well as operations. The difference lies primarily in the center of gravity shifting more towards operations or towards relations.

In this study, I will use the Relational Paradigm to first investigate problems as mathematical phenomena, their epistemology. I will then explore the possible cognitive challenge that various additive problems can present for learners. I will also discuss additive word problems as natural language phenomena, texts that should be read and interpreted. I will then present some models of how the learner can develop knowledge of solving additive word problems. Finally, I will discuss models that can help me interpret students' production in solving problems.

In this chapter, I will examine the state of the research related to the mentioned areas and build upon this theoretical knowledge by going deeper into the subject.

2.2 Epistemology of additive problems

Some problems which involve using one addition or subtraction operation can be difficult for students until they are between ages 12 and 14 (Vergnaud, 2009). This difficulty often relates to the nature of problems as mathematical constructs. In order to better understand possible sources of students' difficulties in solving additive word problems, it is crucial to first look at the mathematical nature of this type of task. Most of the research addressing this subject stems from the Operational Paradigm because, traditionally, solving a problem was seen as the application of an appropriate operation. However, the discussion of semantic structures of problems and their general mathematical structures were an important part of studies. This means that I can apply Relational Paradigm to analyze these studies. In the Relational Paradigm, to solve a problem means to first grasp its mathematical structure. Therefore, reviewing the classifications of additive word problems I will use available knowledge to answer the following question:

Which characteristics of additive word problems can present challenges for students when they try to grasp the mathematical structure of the problem as an additive relationship?

2.2.1 Classifications of word problems

Many researchers have studied addition and subtraction problems. Among them, Nesher, Vergnaud, Riley, Carpenter (Carpenter et al., 1999; Nesher et al., 1982; Riley et al., 1984; Vergnaud, 1982a) have proven that different types of problems can present different challenges for students. The work they did to understand students' difficulties was based on a profound analysis of the semantics of the problem's text, the mathematical structure of the problem, the concepts involved and the students' strategies of solving problems. The analysis resulted in classifications of word problems that have been used by the research community ever since. I will give a short overview of four of the most cited classifications in the literature and then try to highlight the characteristics of problems that, according to some authors, could present a challenge for learner. Table **7** of Appendix 1 presents the short description of all four classifications. In this table, I tried to coordinate all four classifications so as to preserve the correspondence between similar types of problems. There are 13 classes of problems. I named each class based on their main characteristics. Some categories of problems described by Vergnaud are not presented in this table. I will explain this decision later in the chapter. At the end of this section, I will explore what emerges from looking at problems' classifications through the lens of the Relational Paradigm.

Classification 1

I will start with the classification created by Vergnaud (1982) because in it, the relational character of related mathematical reasoning is made explicit. The classification proposed by Vergnaud (1982) was based on the nature of the quantities involved: measure, transformation and static relation. In this context, measure means quantity expressed in number, transformation means that the quantity changed with time, and static relation means that the relation between quantities is stable in time. Vergnaud (1982) distinguished six categories of relationships that can be described in a problem. Categories I, II, and III include problems where quantities are measures (5 marbles). Categories IV, V, and VI include problems in which quantities represent transformations (win 5 dollars), or relationships (5 marbles less). Each of these six categories is divided in three classes, which differ according to the place of the unknown in the corresponding mathematical equation.

Classification 2

Nesher, Greeno and Riley (1982) created a similar classification based on the semantics of the problem text. They divided problems into three categories—Combine, Change and Compare—based on the everyday meaning of the main action described in the story. This classification does not take into account the nature of the quantities involved as opposed to the classification (1), where the nature of these quantities plays an important role. Even if classification (2) was inspired by the operational vision, the semantic structures identified by researchers can be seen in Relational Paradigm as well.

Classification 3

Riley, Greeno, and Heller (1983) proposed a classification of word problems similar to the one described by Nesher et al. (1982) and based on the semantics of the text as well as the distinction between situations describing an action and situations describing a state. In this classification, all categories were grouped into two classes: change and equalizing problems are considered *action* problems, while Combine and Compare are seen as *static state* problems. It seems that this characteristic can play an important role in students' reasoning about the problem. I address this aspect later in the chapter.

Classification 4

In the classification created by Carpenter et al. (1999), a great deal of attention is paid to how the main action described in a problem corresponds to the strategy young children use to solve the problem when tokens are available. The authors stressed that to solve problems with a "join" or "separate" transformation, young students use different strategies: adding tokens or removing tokens. Therefore, the authors put Change problems into two different categories: Join and Separate. The attention authors pay to how the action described in the story corresponds to the calculation or manipulation of tokens students perform leads me to characterize their approach as belonging to the Operational Paradigm. The authors do not separate the students' possible analysis of the problem from the calculation process they use. However, it is possible that learning to solve Join and Separate problems can be difficult in different ways for the student.

Classifications of word problems through the lens of the Relational Paradigm

All the classifications of additive word problems discussed mainly rely on the semantic meaning of the verb used (join, separate, to have more). In the Operational Paradigm, this verb can be directly associated with the arithmetic operation and an equation can be formulated. Thus, the difference between problems lies in the meaning of the verb, the operation used, and the place of unknown in the equation.

In the Relational Paradigm, it is still necessary to understand the meaning of the verb describing the situation. However, the next step is to transform this understanding into an additive relationship between the three quantities involved. Grasping this relationship is not really related to whether the quantities are known or unknown. The important thing is whether the quantity is a part or the whole.

2.2.2 Position of the unknown and structure inversion

Many researchers (Nesher et al., 1982; Pape, 2004; Riley et al., 1984; Vergnaud, 1982a) pointed out that problems in which the change or relationship has a sign that is opposite to the arithmetic operation to be used to solve the problem present the most challenge for learners. Pape (2004) distinguishes consistent and inconsistent language that problems can be expressed in. In a consistent problem, the action or relationship described corresponds to the arithmetic operation to be used in the solution. In an inconsistent problem, the final arithmetic operation is opposite to the action or the relation described in the text. In the following example, the verb "lost" means "remove" and the required operation is subtraction.

Consistent problem: Pierre had 13 marbles. He lost some marbles. He now has 8 marbles. How many marbles did he lose?

In the next example, the verb "won" means "get more," but the required operation is subtraction.

Inconsistent problem: Pierre had 8 marbles. He won some marbles. He now has 13 marbles. How many marbles did he win?

Many researchers (Hershkovitz et al., 1997; Nesher, Hershkovitz, & Novotná, 2003; Pape, 2003; Valentin & Sam, 2004) have shown that some students develop a strategy of direct sequential translation of the text—data and keywords—into an arithmetic operation. As can be seen in the example above, the direct translation strategy (lost = subtract, won = add) would be successful for consistent problems, but not for inconsistent ones. Thus, learners who develop this strategy may experience significant difficulty when solving inconsistent problems as they are required to inverse the semantic structure of the problem—something that their strategy does not allow. Vergnaud (2009) stresses that for these classes of problems, an analysis of quantitative relationships[5] is needed to successfully transform the semantic structure into a standard mathematical expression. Some researchers (Bisanz, Watchorn, Piatt, & Sherman, 2009; Schliemann, Araujo, Cassundé, Macedo, & Nicéas, 1998) especially highlighted the difficulty learners have in developing an understanding of the inversion of mathematical operations. Thus, the language consistency of word problems is their essential characteristic for teaching and learning.

[5] Vergnaud (2009) uses the expression "relational calculus," which means that the student needs to deal with the relationships to figure out how to "calculate" the operation.

In the Relational Paradigm, the "operation inversion" looks different. The additive relationship is a system of three elements where each element can be found using the two others. The total is the sum of two parts, and a part is the difference between the total and the other part. We therefore do not speak about the inversion of an operation, but about deriving an operation from the additive relationship.

2.2.3 Mathematical structure of a problem and cognitive load

Many researchers (Berends & van Lieshout, 2009; Kalyuga, 2008; Lee et al., 2009; Sweller, 1988; Zahner & Corter, 2010) relate problem solving difficulties to the task's cognitive demand. In other words, different amounts of information and interactions must be processed simultaneously by a learner to solve problems of different classes. In as early as the 1940s, Van Engen (1949, mentioned in English, 2007) proposed that, to successfully solve a problem, the student should first perceive the mathematical structure of the problem correctly. In this section, I will use the Relational Paradigm to analyze the mathematical structures of three categories of problems: Simple Combine, Simple Compare,[6] and Composition-of-two-transformations.[7] I will aim to answer the following questions:

- How does the mathematical structure itself contribute to students' capacity to perceive a problem correctly?
- How do various structures differ in terms of cognitive demand?

I will represent the relationships between the quantities involved in each problem to clarify the complexity of each structure. Each quantity will be represented as a line segment.[8]

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[6] Here, I am using the same categories listed in Table 1 of Appendix 1.
[7] Category IV from Vergnaud (1982a) classification.
[8] This method of representation, called the Arrange All Diagrams method, is similar to the "distance diagram" used by Vergnaud (1982) or the now popular "Singaporean" method of representation (Fan & Zhu, 2007). The Arrange All Diagrams method consists mainly in representing each physical quantity described in the problem as a line segment. Segments can then be arranged in different ways in order to represent the relationship between quantities described in the problem or to reflect the problem solving strategy. As suggested by Vergnaud (1982), diagrammatic representation is sometimes better than the equation representation usually used by researchers in the mathematical analysis of a problem. The

Simple combine problems

I will start with the Simple Combine category[9] (composition-of-two-measures, combine or part-part-whole in other classifications). Here is a typical problem of this kind:

> *"Peter has 6 marbles in his right pocket and 8 marbles in his left pocket. He has 14 marbles all together."*(Vergnaud, 1982a, p. 43)

In order to represent the situation's mathematical structure we can visualize the 6 marbles in the right pocket as being arranged in a line. Then, imagine that the 8 marbles in the left pocket are also arranged in a line. The problem mentions that all marbles should be considered together, so we can mentally place the second line just before or after the first line.

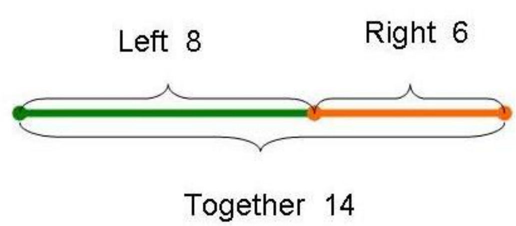

Figure 2. Mathematical structure of a Simple Combine problem

In order to correctly perceive a problem in this category, it is important to think about three quantities connected by an additive relationship—one structure made up of three elements.

Researchers (Carpenter et al., 1993, 1999; Nesher et al., 1982) have shown that children can have diverse approaches and strategies in solving Simple Combine problems depending on how they are formulated. However, a holistic analysis similar to the one presented here is essential to solve the most difficult of such problems. Nesher et al. (1982) attribute this analytical ability to a higher level of development in problem solving. Riley et al. (1983) also mention that a holistic representation is needed for

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Arrange All Diagrams method can be successfully used to represent and solve problems. In this chapter, it will be used as a powerful method of problem analysis for the purpose of research.

[9] Here, I am using the categories listed in Table 1 of Appendix 1.

difficult problems. According to the Relational Paradigm, this holistic analysis ensures solving any problem of this category. Therefore, to analyze the problem efficiently, the solver should treat three mental elements as the three mentioned quantities and the part-part-whole relationship that reunites them in the working memory.

Simple Compare problems

A slightly different graphic representation is needed in the case of Simple Compare problems (static-relationship-links-two-measures or compare).

> *"Peter has 8 marbles. He has 5 more marbles than John. John has 3 marbles."*(Vergnaud, 1982a, p. 43)

We can represent the first quantity, then the second quantity, and then place two line segments conveniently to compare them.

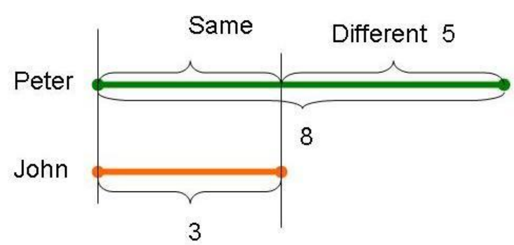


Figure 3. Mathematical structure of a Simple Compare problem

Correctly analyzing the situation requires finding a part of the larger quantity that is equivalent to the smaller quantity. By doing this, we will also see the part in which the bigger quantity differs from the smaller one. After having completed this step, we can see the same part-part-whole structure appear as in the previous examples. Peter's marbles are now represented in two parts: marbles "same as John's" and marbles "different from John's." However, one extra step was used here (as opposed to the previous category) in order to construct the representation and complete the analysis. In this analysis, four mental elements representing quantities (John, Peter, same, different) and three relations between them (bigger/smaller, equivalent, part-whole) were used. Thus, Simple Compare category problems are often more complex and more difficult than Simple Combine problems because they require using more working memory or more powerful control mechanisms for their mental representation and solution.

Composition-of-two-transformations

This is a typical Composition-of-two-transformations category situation described by Vergnaud (1982a).

> *"Peter won 6 marbles in the morning. He lost 9 marbles in the afternoon. Altogether he lost 3 marbles."*(Vergnaud, 1982a, p. 44)

Here, "won 6 marbles" can be considered a quantity with a sign or as a difference between two quantities: before the game(s) and after the game(s) in the morning. As argued by Vergnaud (1982), the first choice requires a well-developed concept of relational quantity (integer number) to keep "won 6 marbles" in mind and operate with it as one mental element. The notion of integer numbers should not only be developed, but also be profoundly and directly linked to the understanding concepts of won, lost, greater than, less than, and difference as relationships between two quantities or a relational quantity. If this knowledge is not yet developed, then the second possibility comes into play, representing "won 6 marbles" as the difference between two hypothetical quantities: before the first game and after the first game. For the same reason, the nine marbles lost in the second game would be represented as a difference between the state after the first game and the state after the second game.

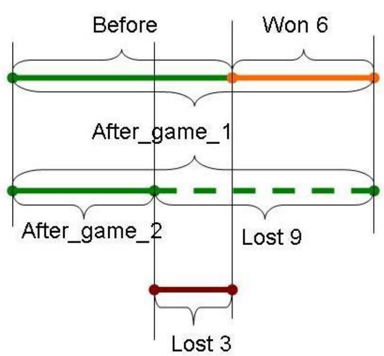


Figure 4. Mathematical structure of a composition-of-two-transformations problem

In the diagram the after_game_1 quantity is represented twice and lost_3 is represented three times because otherwise, the picture (and the reasoning) would be difficult to work with. The resulting difference lost_3 can be seen now as the difference between before and after_game_2 and at the same time as the difference between won_6 and lost_9. Even though the situation can be described by only one arithmetic equation, 9 - 6 = 3, the underlying reasoning can be very complex and involve multiple elements and relationships.

The strategy used to construct this representation is similar to the hypothesis strategy observed by Vergnaud (1982) in students solving problems in this category and others. Students started their reasoning with a concrete, arbitrarily chosen number for the initial state, then applied the described transformations and adjusted the initial number chosen if needed. This strategy can help in cases where the resulting transformation is unknown, but not in other cases. It is clear that without integer numbers and relational quantities, the situations in this category require an enormous working memory or a sophisticated procedure to represent, analyze and solve related problems. Although some researchers put this category of problems in parallel with the Simple Combine, Simple Change and Simple Compare categories (Barrouillet & Camos, 2002), the difference in the complexity of the problems in these categories is remarkable.

I analyzed three different categories of additive problems described in different sources. I represented each example situation using the Arrange All Diagram technique to illustrate the analysis of relationships involved in solving difficult problems from each category. This analysis shows that, even though each of the three examples can be described using only one arithmetic operation, situations based on relational quantities (the last example) are cognitively much more complex than situations based on physical quantities (the first example). Simple Compare category problems occupy an intermediate position. It is clear now that problems involving relational quantities should not be considered simple additive problems, and I will therefore not discuss them any further in this study.

In each analyzed category, the difficulty of the concrete problem also depends on the position of the unknown. Empirical studies (Barrouillet & Camos, 2002; Nesher et al., 1982; Nunes, Bryant, Evans, Bell, & Barros, 2011; Vergnaud, 1982a) show that problems with an unknown final state are the easiest for students. These problems can be solved without involving relational analysis (Vergnaud, 1982a), which, according to some researchers (Nunes et al., 2011), reduces the cognitive demand.

Looking from the Relational Paradigm perspective, we can say that the three examples discussed above differ from each other in the number of relationships involved. In cases of Simple Change or Simple Combine problems, there is only one additive relationship: two parts and the whole. In cases of Simple Compare problems, there is an additional relationship: one part of the bigger quantity is equal to the smaller quantity. In cases of Composition-of-two-transformations (and other problems involving two transformations or two relationships), multiple additive and equality relationships can be present. The idea that all of these problems should be considered *simple*—because they require only one operation for their solution—is not approved in the Relational Paradigm.

From the point of view of the Relational Paradigm, the easiness of the problems with an unknown final state comes from the possibility of dealing with these problems without having to grasp the additive relationship involved. At the same time, difficulties with other problems may arise if this relationship is not grasped because it makes the "inversion" difficult.

2.2.4 Summary of problems' classifications

Regardless of the differences present in the four classifications, the method of analysis used by all authors is based mainly on the mathematical structure of the problem and semantic meaning of the action or relation described in the text. Empirical evidence has confirmed that students' difficulties do relate to the problem's category (Carpenter et al., 1993; Durand & Vergnaud, 1976; Nesher et al., 1982). More specifically, they are related to the nature of the quantities involved, the place of the unknown in the mathematical structure of the problem and the type of the relation between the quantities. Problems in which the mathematical relationship described and the mathematical operation to be used have opposite signs are identified as most challenging for learners.

From the Relational Paradigm perspective, Simple Compare problems are more difficult than others because grasping the additive relationship involved requires more mental effort. Simple Change and Simple Combine problems are similar in these terms. However, the detailed analysis of the epistemology of word problems provided above is a valuable tool for interpreting students' ways of mathematizing and solving these problems. In order to create tasks in which students' different ways of problem solving can be observed and interpreted, we need to understand which of the task's characteristics could potentially affect their reasoning.

The nature of the quantities involved is the characteristic that distinguishes the categories IV, V and VI of Vergnaud's classification (1) from all other categories in all other classifications. The problems in these three categories are essentially different from others because of the number of relationships involved. Therefore, although they can be solved with one addition or subtraction operation, they should not be considered simple. It is for this reason that I did not include them in my summary of problem classifications. In Table 1, I summarize all the classes of problems described, except categories IV, V and VI. I also show the problems' characteristics in relation to the challenge they present.

Table 1 *Summary of Problems' Classifications*

Categories	Place of unknown	Classification (1)	Classification (2)	Classification (3)	Classification (4)	Example	Equation	Solution expression	Action / state (inconsistency)
Simple positive change	Result unknown	Transformation links two quantities[10]	Change	Change	Join	*Peter had 17 marbles before playing. Tom gave him 4 marbles. How many marbles does he have now?*	17 + 4 = ?	17 + 4 =	Action (consistent)
	Start unknown	Transformation links two quantities	Change	Change	Join	*Peter had some marbles before playing. He then won 4 marbles. He now has 13 marbles. How many marbles did he have before playing?*	? + 4 = 13	13 - 4 =	Action (inconsistent)
	Change unknown	Transformation links two quantities	Change	Change	Join	*Peter had 7 marbles before playing. He then won some*	7 + ? = 13	13 - 7 =	Action (inconsistent)

[10] Vergnaud (1982) uses the word "measures."

						marbles. He now has 13 marbles. How many marbles did he win?			
Simple negative change	Result unknown	Transformation links two quantities	Change	Change	Separate	*Peter had 17 marbles before playing. He then lost 4 marbles. How many marbles does he have now?*	17 - 4 = ?	17 - 4 =	Action (consistent)
	Start unknown	Transformation links two quantities	Change	Change	Separate	*Peter had some marbles before playing. He then lost 4 marbles. He now has 13 marbles. How many marbles did he have before playing?*	? - 4 = 13	13 + 4 =	Action (consistent)
	Change unknown	Transformation links two quantities	Change	Change	Separate	*Peter had 17 marbles before playing. He lost some marbles. He*	17 - ? = 13	17 - 13 =	Action (consistent)

						now has 13 marbles. How many marbles did he lose?			
	Positive action (difference) unknown	Static relationship links two quantities	Compare	Equalizing	Compare	*Joe has 3 marbles, Tom has 8 marbles. How many marbles does Joe need to have as many as Tom?*	3 + ? = 8	8 - 3 =	Action (inconsistent)
Simple compare	Negative action (difference) unknown	Static relationship links two quantities	Compare	Equalizing	Compare	*Joe has 8 marbles, Tom has 3 marbles. How many marbles does Joe need to have as many as Tom?*	8 - ? = 3	8 - 3 =	Action (consistent)
Simple combine	Total unknown	Composition of two quantities	Combine	Combine	Part-part-whole	*Peter has 6 marbles in his right-hand pocket and 8 marbles in his left-hand pocket. How many marbles does Peter*	6 + 8 = ?	6 + 8 =	State (consistent)

						have altogether?			
	Part unknown	Composition of two quantities	Combine	Combine	Part-part-whole	*Peter has 6 marbles in his right-hand pocket and some marbles in his left-hand pocket. He has 14 marbles altogether. How many marbles does Peter have in his left-hand pocket?*	6 + ? = 14	14 - 6 =	State (inconsistent)
Simple positive compare	Difference unknown	A static relationship links two quantities	Compare	Compare	Compare	*Tom has 6 marbles. Joe has 4 marbles. How many more marbles does Tom than Joe?*		6 - 4 =	State
	Compared unknown	A static relationship links two quantities	Compare	Compare	Compare	*Tom has 6 marbles. Joe has 4 more marbles than Tom. How many marbles*	6 + 4 = ?	6 + 4	State (consistent)

						does Joe have?			
	Referent unknown	A static relationship links two quantities	Compare	Compare	Compare	*Tom has 6 marbles. He has 4 marbles more than Joe. How many marbles does Joe have?*	6 = ? + 4	6 - 4	State (inconsistent)
Simple negative compare	Difference unknown	A static relationship links two quantities	Compare	Compare	Compare	*Tom has 4 marbles. Joe has 6 marbles. How many marbles less does Tom have than Joe?*		6 - 4 =	State
	Compared unknown	A static relationship links two quantities	Compare	Compare	Compare	*Tom has 6 marbles. Joe has 4 marbles less than Tom. How many marbles does Joe have?*	6 - 4 = ?	6 - 4	State (consistent)
	Referent unknown	A static relationship links two quantities	Compare	Compare	Compare	*Tom has 6 marbles. He has 4 marbles less than Joe. How many*	6 = ? - 4	6 + 4	State (inconsistent)

	marbles does Joe have?

The analysis provided above shows how a problem's mathematical structure can contribute to determining the cognitive demand with regard to solving the problem. The cognitive demand is caused by the necessity to analyze the mathematical structure of difficult problems in each category. However, this analysis did not take into consideration the problem text's lexical structure or the problem's context. It also ignores the availability of manipulatives while solving the problem. All of these aspects affect the level of difficulty of a particular problem and the student's success in solving it.

The relational analysis of the situation as a whole in the problem solving process depends first on the manner in which students use the information available in the text of the problem, and second, on the knowledge available to students from their previous experience. In the next section, I will discuss students' problem solving process in more detail from a linguistic point of view and in relation to previous knowledge. Later, I will provide an overview of how researchers conceptualize the process of development of the knowledge related to problem solving and what previous knowledge can be present in students at various stages of this development.

2.3 Word problem as a phenomenon related to natural language

I have described the mathematical structures that can be present in additive problems and how these structures can contribute to the cognitive challenge that students encounter while solving them (see section 2.2.3 of this chapter). I have shown that some problems, which can be solved with one addition or subtraction, should not be considered simple. Therefore, I will further be considering only the first three categories of problems: Simple Combine, Simple Change and Simple Comparison.

Recent research (Gamo et al., 2009; Voyer, 2009) has shown that difficulties in the problem solving process can also come from the context of the problem and especially from the particular wording of the problem. In this section, I will analyze how students might understand and interpret the text of a given problem. I will try to answer the following questions with regard to the Relational Paradigm:

- How might the interpretation of a problem's text support or hinder the clarification of the mathematical structure for the student?

- How can the text of the problem contribute to the difficulty learners may experience?

2.3.1 Construction/integration model

The understanding of a text is a complex multilayered process (Kintsch, 1988) that includes a linguistic level, conceptual levels, and the level "at which the text itself has lost its individuality and its information content has become integrated into some larger structure" (Kintsch, 1988, p. 163). In their works, Kintsch and Greeno (Kintsch & Greeno, 1985; Kintsch, 1988, 1994, 2005) propose a very detailed model of text understanding and shows how it can work for the understanding of textual problems. The main characteristic of this model is that it represents coexistence and constant coordination of two converging processes—construction and integration—which happen simultaneously within the reading-comprehension process. The first of these processes treats sensorial information coming from the text (letters and words). Using this initial information, the reader (student) can *construct* perceptions of the situation in his head. The second process involves finding a mental schema into which the incoming information can be *integrated* without provoking a contradiction. According to Kintsch (2005) prior knowledge plays an important role in the integration process, providing the reader with existing schemas. However, each invoked mental schema is always under scrutiny of the incoming information and the broader context of the situation.

Some researchers distinguish different levels of understanding of word problem texts: *text base* (van Dijk & Kintsch, 1983), *problem model* (Kintsch & Greeno, 1985), and *situation model* (Reusser, 1990; Voyer, 2009). Other researchers see this distinction as vague (Nesher et al., 2003). According to the construction/integration model, the passage from text to understanding that students may have of a problem is not merely a sequence of states and can be very complex even for simple additive problems. First, the mental schemas available (or not) in a student's previous knowledge can be of various natures: linguistic, semantic, situational, arithmetic, etc. Second, some students may not develop the ability to scrutinize invoked mental schemas or the process may just fail for various reasons. For example, if students are not familiar with the relational expression "more than," then the sentence "Peter has 5 marbles more than Tom" may be understood as “Peter has 5 marbles” (Okamoto, 1996). Third, appropriate schemas may not yet be available, and some schemas may be easier to invoke than others or more strongly associated with certain contexts or words than others (Nunes et al., 2011; Peltenburg, Heuvel-Panhuizen, & Robitzsch, 2012; Peters, Smedt, Torbeyns, Ghesquière, & Verschaffel, 2011; Selter, Prediger, Nührenbörger, & Hußmann, 2011; Valentin & Sam, 2004). I will now look at the reasoning schemas available to students at different stages of knowledge development.

2.3.2 Stages of development and knowledge schemas

The analysis provided in sections 2.2.1, 2.2.2 and 2.2.3 of this chapter helps identify the different levels of complexity present in problems of different categories. Researchers all agree that successfully solving of problems of different categories or classes should correspond to certain stages of knowledge development. The following is a short overview of these stages in an attempt to identify the reasoning schemas possibly available to the learner at different stages of knowledge development.

Nesher and colleagues’ model

To try to explain empirical data, Nesher and colleagues (Nesher et al., 1982) proposed an order of knowledge development stages in students. According to this order, the knowledge necessary to solve simple additive word problems develops as follows.

At stage 1, students are able to solve Change problems if the result is unknown and Combine problems if the total is unknown. At stage 2, students can also solve Change problems if the change is unknown and find the difference in Compare problems. Essentially, at these two stages, students are able to understand problems in some linear unidirectional way and directly represent the situation or state described in the problem using tokens or counting blocks. This method helps them to see and count the unknown quantity.

At stage 3, students acquire the reversibility of an action and are able to transform the described action or situation into another action. At stage 4, students can use a holistic mental representation of the problem in a flexible way. As stated by the authors (Nesher et al., 1982), the ultimate proof of the final stage of development is that the student “is able to read the word 'more', and yet perform a subtraction operation" (Nesher, 1982 p. 392). The authors argue that, to successfully solve a difficult problem, students need to understand the story as a whole with individual elements interconnected, forming a coherent relational network. In Vergnaud's (1982) terminology, *relational calculus* (the ability to reason in terms of relations) acquisition is an important condition for successful problem solving.

Riley and Greeno's model

Riley and Greeno(1988) also proposed three stages of knowledge development. At the first stage, children can represent and count sets as they appear in the problem's text. Stage 2 is characterized by the ability to think about unknown sets and certain relationships between sets. At stage 3, children can mentally combine relationships described in the text with the part-whole relationships between known and unknown sets. In addition, for each stage of development the authors propose a model of possible reasoning in the problem solving process.

Okamoto and Case's model

Okamoto (Case & Okamoto, 1996; Okamoto, 1996) further developed the models proposed by Riley and Greeno(1988) and linked them to students' knowledge of the concept of numbers. Okamoto (1996) proposed three stages of development. At the first stage, children are able to represent and mentally coordinate one number at a time as a line of physical objects. They are able to solve problems where the final state of an action or a total of two sets is unknown. At the second stage, children are able to mentally represent two number lines and tentatively coordinate them. This gives them the possibility of solving other problems in the transformation and composition categories. Finally, children can mentally coordinate two number lines with ease, which helps them to solve compare problems. Some empirical results (Okamoto, 1996) confirm this model. Although the number concept is at the centre of Okamoto's (1996) models, her research shows that the relationship coordination plays an important role in successful problem solving.

Fuson and colleagues' model

Fuson et al. (1996) made an interesting attempt to discover and describe stages of knowledge development with regard to comparison problem solving by first and second grade students. These stages are described in terms of what students can and cannot do at a particular stage of development. It is interesting that at the first stage, students can see which of two compared quantities is more or less, but cannot figure out how to proceed to find the difference. At the second and third stages, they usually apply a direct procedure or action suggested by the wording of the problem. Only at the third stage are students able to coordinate all available data and relationships properly to find the unknown quantity at any place in the mathematical structure of the problem.

Summary of stages and possible schemas

Looking at the models discussed above through the lens of the Relational Paradigm, the development of the ability to solve additive problems can be generalized as follows.

1. In early stages of development, students understand the problem in a sequential way and can work with it using physical objects and representing numbers and operations in the order described in the problem. Some researchers (Carpenter et al., 1999) believe that this way of solving, which they call "direct modeling," is the natural way all beginners usually use. It seems that the reasoning schema learners primarily use at these stages is sequential, and thus helps them transform the action described in a text into a mimicking process or an arithmetic operation consistent with the action.
2. Those who master additive problem solving can mentally represent known and unknown quantities and their part-whole relations, and deal with these representations as a whole system in a flexible manner (Lesh & Zawojewski, 2007). The reasoning schema that learners likely use at this stage is a holistic and relational one.

In the next section, the two types of schemas I have described will be explored in more detail. It is important to mention here that sequential schemas can be available for learners at very early stages of knowledge development. Holistic relational schemas seem to be the target of teaching and learning as they are likely not initially available to all students. Holistic relational schemas provide students with the possibility of deciding which arithmetic operation to use without any limitations on the numbers involved. This means that the calculation plan (operation or sequence of operations) can be independent from actual values of numbers—the way an algebraic solution is usually created.

2.3.3 Text and schema interplay

Sequential and holistic relational schemas

Nesher et al. (1982) stressed that at least two things affect students' understanding of a problem and their success in problem solving. Firstly, students should possess knowledge about the main action or relation described in the text. They should be able to understand the expression (take, add, give, more, less, etc.) in some way in order to connect it to their mathematical knowledge. The second important element is the arithmetic additive schemas from their previous knowledge. In Kintsch (2005) terminology, some initial semantic and arithmetic schemas should exist as the student's previous knowledge.

According to Nesher et al. (1982), the understanding students have at the beginning of problem-solving knowledge development is closely related to their everyday understanding of the physical actions described in the problem. This corresponds to model (1) proposed by Riley et al. (1983) and to the one proposed by Okamoto (1996). In other words, while analyzing the text, the child creates mental schemas to represent the quantities and the action in a sequential way—the way they understand the action. This likely happens because the other schema which helps integrate all the quantities and relations simultaneously is not available. In this situation, the student is able to solve problems where the action is easily imaginable and can be directly mimicked. The process of mimicking helps students perform the known mathematical procedure (matching one by one, adding/removing blocks, counting forward, counting backward, adding/subtracting, etc.). These linear sequential schemas are also promoted at the beginning of the math curriculum, when students learn to count objects, represent numbers and mimic addition or subtraction actions. They are thus the most recent and most “popular” for students.

It is important to stress here that it is not the mathematical structure of the problem itself that determines the accessibility of stage 1 problems for beginners. The clear indication of the action in the text provides the possibility of finding the answer using a simple sequential schema that corresponds to an everyday level of understanding of the action. No overall schema of the mathematical structure of the problem is needed in such problems. In the following examples of problems that can be solved using this kind of understanding, I underline the text which indicates a solution process:

> Peter had 7 marbles before playing. He <u>lost</u> 4 marbles. How many marbles does he have now? (Simple Change problem)
>
> Peter has 3 marbles in his right hand and 4 marbles in his left hand. How many marbles would he have if he <u>put all the marbles together</u>? (Simple Combine problem)
>
> 5 birds saw 3 worms on the ground, and <u>each bird tried to get a worm</u>. How many birds didn't get a worm? (Simple Compare problem)

The last example caught the attention of many researchers (Grishchenko, 2009; Hudson, 1983; Kintsch, 1988; Nunes & Bryant, 2009; Okamoto, 1996; Rodriguez, 2004). The fact is that, in the equalizing form given in the example above, this problem is accessible to beginners and can be classified as a Level 1 problem. In the classic Compare form—There were 5 birds in a tree and 5 worms on the ground. How many more birds are there than worms?—the problem becomes much more difficult (Hudson, 1983; Okamoto, 1996) and should be categorized as a Level 3 problem.

Researchers explain this phenomenon in different ways. Cummins et al. (1988) point out the lack of knowledge of keywords (more than, less than). Nunes and Bryant (2009) argue that in the “matching” version, students do not need to apply relational thinking. Okamoto (1996) stresses that in two versions, different mathematical knowledge is needed (one-to-one correspondence and other higher knowledge). It is possible to use the one-to-one procedure in both cases. However, the absence of the direct indication of this action in the classic version obscures this choice of sequential schema for students and creates a need for the analysis of relationships as an intermediate step (Grishchenko, 2009; Nunes & Bryant, 2009; Okamoto, 1996). This relational reasoning should be supported by a relational schema (see previous section) that is likely not available at early stages of knowledge development. In addition, the lack of knowledge of the expressions *less than* and *more than* can prevent students from associating the text with the relational schema even when the schema is available.

If the manipulation of physical objects is available for students, the sequential schema can be used to solve more difficult problems, which should theoretically be supported by a relational schema. For example, the *Simple Negative Change* problem can be solved correctly, "even though this problem also involves an unknown change set. This is because the effect of [physically] decreasing an initial set by some amount to get a specified final amount is that the change set and final set are now physically separated and both appear in the child's actual display" (Nesher et al., 1982, p. 386).

At level 2 (Nesher et al., 1982), students can solve *Simple Change* problems (change is unknown), *equalizing* problems and *Simple Compare* problems that can be treated via equalizing procedure. Examples:

> Joe has 8 marbles. Tom has 3 marbles. How many marbles does Joe need to have as many as Tom?
>
> Joe has 8 marbles. Tom has 3 more marbles than Joe. How many marbles does Tom have?

As argued by Nesher et al. (1982), in case of *equalizing* problems and sometimes with *Simple Compare* problems, the student is not really concerned with the abstract comparison relationship. "He regards them as 'make this smaller one large'" (Nesher et al., 1982, p. 387). Once again, the sequential schema can be used by the learner. The holistic relational schema is simply not needed.

The second model described by Riley et al. (1984) differs from Nesher et al.'s (1982) level 2 in that it involves better memorization of the change set in *Simple Change* problems where the change set is unknown. It is important to stress that in both cases, situations with physical manipulations were analyzed. So, the child could mimic the change using physical blocks and see or remember the change amount. Therefore, the same sequential schema could be used.

Level 3 (Nesher et al., 1982) includes the understanding of reversibility of a change and the capacity to relate two quantities with the inclusion relationship. According to the authors, this helps students solve *Simple Change* problems, where the initial set is unknown. However, if the numbers used in the problems are small, students can also use the same sequential understanding (as in the levels 1 and 2). For example, children can understand that they need to start with 2 and add 3 to obtain 5, but will not be able to say that in order to find the starting number, they need to subtract 3 from 5. Another possibility is that children, mentally or by using blocks, start with a number (for example, 3), add 3, and compare the result with what is required. They then adjust the initial number (hypothesis strategy described in (Vergnaud, 1982a)). The students end up using the same sequential schema several times. Thus, problems of this type with small numbers mastered by students do not really require a holistic understanding of the problem and relational schema (Carpenter, Moser, & Bebout, 1988).

At level 4 (Nesher et al., 1982), students are capable of seeing the data in the problems in a holistic and flexible fashion. This corresponds to model 3 proposed by Riley et al. (1984) and model 3 proposed by Okamoto (1996). According to Vergnaud (1982), students at this stage are capable of thinking about relations as well as numbers. In other words, while reading the text of the problem, students are able to invoke a relational additive mental schema and thus organize their thinking about the quantities described in the problem in a holistic and flexible way. For example, students can easily construct an appropriate arithmetic operation to find the unknown in any position in the mathematical structure of the problem.

Consistent and inconsistent problems from the reading point of view

The distinction between the sequential schema and holistic relational schema can also explain the phenomenon of "consistent" and "inconsistent" problems (Pape, 2004) discussed in section 2.2.2. In a consistent problem, the action or relationship described corresponds to the arithmetic operation required for the solution. In an inconsistent problem, the final arithmetic operation is opposite to the action or relationship described. Researchers (Hershkovitz et al., 1997; Nesher et al., 2003; Pape, 2003; Valentin & Sam, 2004) have shown that some students develop a strategy of direct sequential translation of the text—data and the keywords—into an arithmetic sentence.

Example of consistent problem translation:

Pierre had <u>13 marbles</u>. He <u>lost</u> some marbles. He now has <u>8 marbles</u>. How many marbles did he lose?

Translation: 13 - 8 (correct)

This strategy can be seen as another example of a sequential schema of reasoning that, in some cases, can help students solve a problem, but may also prevent them from looking for and constructing the holistic relational schema needed for other problems.

The correct solution to the consistent problem in the example above can be explained by the student's use of the holistic flexible schema. It can also be explained using the construction-integration model of text comprehension (Kintsch, 1988) in two different ways. If the reading process is dominated by the sequential arithmetically sound schema *number-keyword-number*, it can rapidly lead to the direct translation of the key elements into the arithmetic operation: "*13…lost…8"* -> (13 - 8).

If the sequential unidirectional semantic schema *known-remove-known* dominates the reading process, this process can produce an incorrect perception of the problem's text that will then be translated into the arithmetic operation. This is how the latter scenario could take place for the same problem.

> Pierre had 13 marbles. He then lost some marbles. Now, he has 8 marbles. How many marbles did he lose?

When reading the problem, getting to the point "Pierre had 13 marbles. Then he lost," students can unconsciously employ a well-known sequential schema: *initial state known–*

negative change known–result unknown. This schema then takes over the whole process of reading and understanding, so the student finally perceives the problem as follows:

> Pierre had 13 marbles. He then **lost 8 marbles**. How many marbles **does he have now**?

In this reading, the *lost marbles* and *current marbles* have changed their roles. However, this erroneous perception gives students the chance to construct a correct mathematical expression: 13 - 8 = 5. DeBlois (1997) observed similar behaviour working with students with difficulties. In her experimentation, while discussing the solved problem with the researcher, the student attributed an incorrect role to a number correctly used in a correct arithmetic expression. Giroux and Sainte-Marie (2001) also described similar behaviour in students solving Simple Compare problems.

The example of consistent and inconsistent problems highlights the hidden power of the sequential schema of reasoning and the role this schema can play in providing success in solving various problems (but not all of them), thus preventing students from holistic schema development. In the Relational Paradigm, the distinction between sequential and holistic schemas of reasoning might be the key element in understanding the development of mathematical reasoning by students. This is in line with the results of Stern (1993), which show that the flexibility of relational reasoning represents the main difficulty young children have in solving inconsistent comparison problems.

2.4 Problems, their wording and knowledge development

I have discussed the epistemology of additive word problems and the role that natural language can play in the problem solving process, giving learners more access or hindering this access to the mathematics of the problem. From this analysis the following propositions can be formulated:

1. Sequential reasoning and holistic relational reasoning about a problem constitute the main distinction between successfully and unsuccessfully solving difficult problems.

2. In some specific conditions (using physical objects, consistent problems), students can successfully solve problems of different types without evoking the mathematical structure of the problem or using relational reasoning.
3. Not all problems require relational reasoning. This depends on the mathematical structure of the problem, the place of unknown and the specific wording of the problem. Only difficult and inconsistent problems require relational reasoning.

From these propositions, it follows that teaching methods and curriculum content can potentially contribute significantly to promoting or inhibiting specific cognitive schemas in students. This is in line with the view of many researchers, who argue that using challenging situations (in our context, difficult problems) in math teaching better promotes knowledge development in students (Freiman, 2006; Jackson & Cobb, 2010; Savard, 2008; Stein & Lane, 1996). At the same time, the presence of holistic relational schemas can likely be better observed in students while solving difficult and inconsistent problems.

In the following section, I will briefly discuss how the knowledge of additive problem solving can grow.

2.5 Models of knowledge development in relation to additive problems

2.5.1 Current model

All the existing models of students' reasoning that I have discussed above were created to explain how students develop their knowledge about additive word problem solving. These views on knowledge development in additive problem solving differ in how much attention researchers pay to the number concept and to representational tools available for students. Within the Relational Paradigm, on the other hand, the development of additive problem solving abilities can be generalized as follows:

- In early stages of development, students understand the problem in a sequential way and can work with it in a more or less sequential manner, representing numbers using physical objects.

- Those who master additive problem solving can mentally represent known and unknown quantities and their part-whole relations and deal with these relations as a whole system and in a flexible manner (Lesh & Zawojewski, 2007).

The discussed models also have two issues in common:

1. They do not analyze the process of knowledge development[11] in relation to the learning experience.
2. They do not explain why some students go from level 1 (sequential schema) to level 4 (holistic relational schema) very quickly, while others appear to be stuck at the level 1 for a long time.

Almost all the models of knowledge development cited above consider the sequential understanding of addition and subtraction to be a starting point. In Figure 5, I represent this way of knowledge development as a sequence of stages.

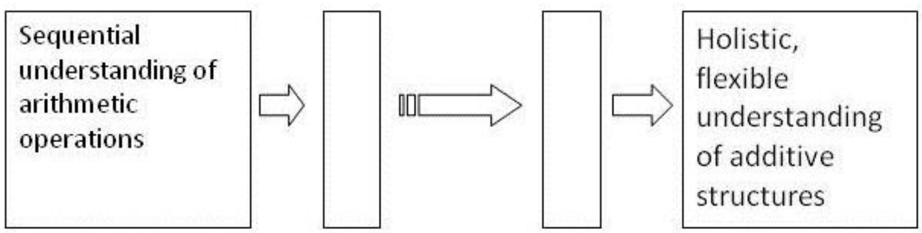


Figure 5. Model of knowledge development within the Operational Paradigm

Sequential understanding is undoubtedly required to construct full and flexible understanding. Different researchers give different numbers of steps of development, but recognize that by the end of the process, holistic and flexible understanding should be developed. The question: Is sequential reasoning the sole and central knowledge the learner can start with in this construction?

2.5.2 Davydov's model

Some researchers (Houdement, 2011; Krutetskii, 1976) have already questioned the fact that some students seem to easily develop the ability to solve word problems,

[11] I understand knowledge development as including, but not equal to cognitive development

while the others can struggle with some types of problems for a long time. Krutetskii (1976) suggests that gifted children seem to see the problem as one whole and do it naturally, without a special prompt. Thus, it is possible that some students have certain relational schemas of reasoning available from the very beginning of their formal education. Otherwise, it is possible that they somehow develop these schemas implicitly, more easily than other students, while being exposed to the problem solving practice.

The Russian school of mathematics education provides us with the powerful idea of additive relationships. Davydov (1982) defines the *additive relationship* as "the law of composition by which the relation between two elements determines a unique third element as a function" (p. 229). This additive relationship can be found in any problem where addition or subtraction is involved. This holistic non-sequential view of the additive relationship is also in line with the ideas of Vergnaud (1982, 2009), who argues that addition is not just the opposite of subtraction, but that there are always three related operations: $a + b = c$; $c - b = a$; $c - a = b$. We can represent the additive relationship mathematically as function $f: N \times N \times N \rightarrow \{true, false\}$

$f(a, b, c) = true$ if $a=c+b$

Davydov (1982) suggests that the *additive relationship* is the basis of knowledge about addition and subtraction. He even insists on teaching this relationship prior to teaching arithmetic operations with numbers. In his experimentations, Davydov (1982) tried to develop in students the relational understanding of situations, where continuous materials were used. For example, students discussed amounts of water in different containers using algebraic language ($A = B + C$ or $A > B$). Although it was not explicitly mentioned in the description of this experimentation (Davydov, 1982), continuous reasoning about quantities and relational reasoning about the situation as a whole appear to be essential aspects of that experimental training. Davydov's (1982) experimentation showed that, if appropriately instructed, very young students are capable of holistic relational and even theoretical reasoning about additive situations.

Currently, a curriculum based on the ideas of Elkonin and Davydov (Davydov et al., 2002) is being implemented as an experimental approach in a number of schools in

Russia as well as other countries. This approach has been recognized as an important advancement in the theory of teaching elementary mathematics (Gusev & Shamsutdinova, 2008; Iannece et al., 2009; Lins & Kaput, 2004; Salmina & Sohina, 1981; Sophian, 2007).

Davydov (1982) pointed out that the relational understanding of additive structures can be grown on another basis: the holistic understanding of physical (not numerical) additive relations between physical objects having length, volume, or area. For example, students can understand that if a liquid in one container is separated into two amounts (two containers), it can be joined together to become the same initial amount. Students can understand that if two ropes are different in length, the difference is a stable length that can be visualized using an appropriate comparison procedure. Thus, Davydov's approach to teaching addition and subtraction can be represented as follows.

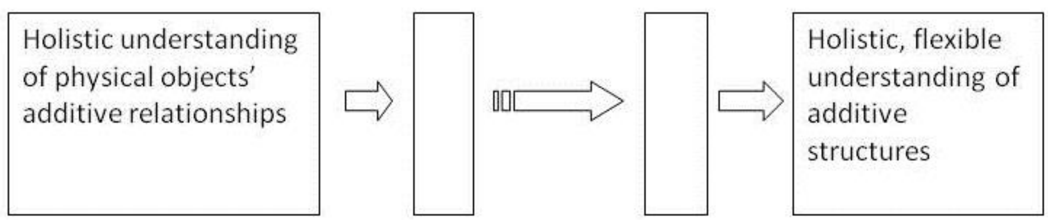


Figure 6. Model of knowledge development within the Relational Paradigm

In Figure 6, rectangles represent steps of development, where the first step is the holistic understanding of the additive relationship in physical objects, and the last step is the holistic and flexible understanding of additive problems.

2.5.3 Research in neuro-education

Contemporary research in neuro-education sheds some light on brain functions in relation to problem solving. Stavy and Babai (2010) used the method of brain imaging to analyze the process of solving certain geometric problems. Their results showed that while the mathematical concepts at play were the same in all problems, there was a tendency for a different part of the brain more activated depending on the geometric condition (Stavy & Babai, 2010). Easy problems were solved primarily via the visual estimation centre. If a more difficult problem was solved correctly, the centres known for their inhibitory control over other centres were mainly involved.

From this perspective, it seems that solving problems of different levels of difficulty can be supported by essentially different brain functions. Sequential reasoning can successfully support the solving of easy (consistent) additive word problems. This reasoning can include thinking about an arithmetic operation as an action or a procedure. The most difficult additive word problems (inconsistent) require an essentially different type of reasoning: holistic perception of the mathematical structure of the problem and thinking about known or unknown numbers as quantities or amounts.

From this point of view, the idea of sequential development of knowledge in additive problem solving—from easy problems (sequential reasoning) to difficult problems (holistic reasoning)—does not seem appropriate. At the same time, the value of the sequential schemas of reasoning should not be underestimated. It is possible that the most attention should be paid to the **equilibrium** between sequential and holistic relational schemas in teaching and learning additive problem solving.

2.5.4 Equilibrated Development model

In the collaborative project (led by Prof. Savard), the third model was considered: the equilibrated development model. Figure 7 presents two starting elements: the sequential understanding of a story and the holistic understanding of the additive relationship in physical objects. These two elements should then be integrated to develop a holistic and flexible understanding of additive problems.

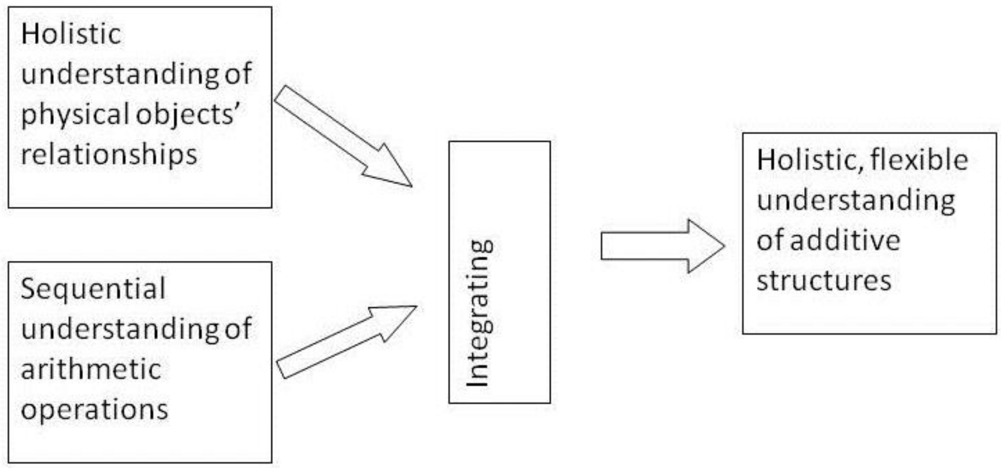


Figure 7. Model of knowledge development, Equilibrated Development Approach

According to this model, the new teaching approach implemented in the collaborative project included first developing in parallel both a basic sense of arithmetic operations as processes of adding or removing and basic holistic reasoning about relationships between physical objects. Second, it included integrating of these two types of reasoning into a process of problem solving to obtain equilibrium.

2.6 Teaching approaches

In this section, I will review different approaches to teaching additive word problem solving in elementary school. This overview will help to better clarify the specific characteristics of teaching approaches that might contribute to the development of sequential and/or holistic relational reasoning related to additive problems.

2.6.1 Existing approaches

Traditional approaches to teaching addition and subtraction fall under the Operational Paradigm. They are developed around the central ideas of the number concept and arithmetic operations with numbers. In North America, the great majority of word problems proposed to students in the beginning (Grades 1 and 2) are consistent problems asking for the final state or the total (Grishchenko, 2009; Xin, 2007). The first experience in solving additive problems involves counting, adding and removing physical objects, mimicking the story using countables or drawing of small circles, etc. (Carpenter et al., 1996).

As I have proposed in the previous sections, consistent problems do not require the implication of the holistic flexible reasoning. The ways students learn to mathematize word problems are mainly based on counting and mimicking. No other ways of mathematizing simple additive word problems are usually proposed to students. Thus, neither the collections of problems nor the available mathematizing method—mimicking-and-counting—really promote the development of holistic flexible reasoning.

In some European and Asian countries (Ng & Lee, 2009), different forms of schematic drawings are used to help students distinguish between problems of different structures and to derive an appropriate mathematical operation. I will discuss this later on. It is important to mention that there is no consensus in research as to whether particular graphic representations or schemas should be taught to students, or whether students should create their own representations (Julo, 2002).

2.6.2 Contemporary approach in Quebec

The math curriculum currently proposed by the Ministry of Education, Leisure, and Sport (MELS, 2004) highlights the importance of additive structures in learning to solve word problems. Teaching approaches currently used in Quebec include solving simple additive word problems to introduce addition and subtraction as mathematical operations and solving complex problem-situations where students should apply and further develop their knowledge of these operations. While solving simple additive problems with students, teachers follow the plan initially proposed by Polya (1973), promoted in the research (Focant, 2003), and adapted by school boards (Gervais, Savard, & Polotskaia, 2013a):

1. Determine what I know from the problem.
2. Determine what I am looking for.
3. Represent the problem by using tokens or drawing circles and count the result (towards the end of the second grade, students represent numbers using ten-blocks or related drawings).

4. Write the answer.

Usually, teachers pay particular attention to the reading of the text. They ask students to highlight important data elements, mimic the story and model the numbers in the problem using tokens or by drawing circles in order to add or subtract.

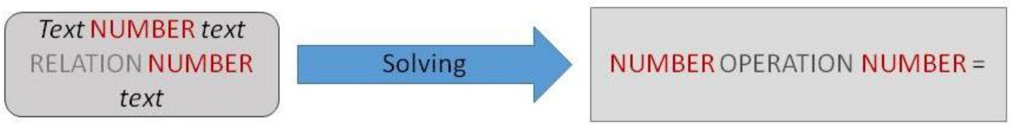


Figure 8. Summary of the traditional approach in simple additive problem solving

Figure 8 presents a summary of the traditional approach to simple additive problem solving. Attention is mainly paid to numbers in the text, and the declared goal is to calculate the answer. The solving process can be seen as the transformation of a text containing numbers into a mathematical operation containing the same numbers.

As the great majority of word problems do not require holistic analysis, the explicit analysis of the mathematical structure of the problem is often omitted. Between the steps two and three of the above-mentioned problem solving plan, there is no explicit modeling of the mathematical structure of the problem. The ways students might mathematize a situation remain the same as in other approaches: mimicking and counting.

2.6.3 Particular teaching strategies

Research proposes different teaching strategies to reinforce the development of problem solving ability (Carpenter & Moser, 1982; Elia et al., 2007; Focant, 2003; Fuson & Willis, 1989; Gamo et al., 2009; Ng & Lee, 2009; Willis & Fuson, 1988) as well as general approaches to teaching mathematics, such as the modeling approach (Lesh et al., 2010) and the realistic approach (De Corte, 2012). I would like to discuss some of these strategies and look at the extent to which these strategies can be related to holistic flexible reasoning development.

Modeling as a teaching approach

In his works, Lesh and colleagues (2003, 2010) propose that modeling should be the central activity in learning mathematics. Particularly with regard to problem solving, "What is central is for the student to develop a model consisting of a conceptual system that is expressed by some representational system and that is useful for some purpose that is understood by the learner" (p. 225). The authors see modeling as the way for students to understand the physical reality of a situation, communicate their understanding to others, grasp the situation and make decisions accordingly. The model can be created through a process of analyzing and mathematizing and explicitly expressed through some form of communication. The authors of the modeling approach are not very specific as to solving simple additive word problems. Yet, it seems that they clearly argue in favour of a holistic view of the problem or, in other words, a mathematical model.

Realistic mathematics

De Corte and colleagues (De Corte, 2012; Verschaffel et al., 1999, 2000) have long studied the contrast between a purely mathematical approach to solving word problems and an approach based on real-life personal experience. According to their research results, the orientation of the problem solving activity towards real life and students' personal experience helps students to better understand the problems and become better problem solvers. This approach is not clear as to what kind of reasoning—sequential or holistic—students should use from their real-life experience.

Manipulatives and mathematical expressions

Many researchers (Carpenter et al., 1999; Gamo et al., 2009; Neef et al., 2003; Ng & Lee, 2009; Xin, 2008) propose modeling problems with students. For example, Carpenter et al. (1999) propose modeling the action and relationship in the problem to use different counting strategies. At the same time, the authors suggest letting students do what comes naturally, that is, not forcing students to use one particular type of modeling. This approach appears to be very close to the one used in Quebec, as students and teachers naturally use tokens or draw circles to represent the numbers and calculate answers. Actually, it is the only way they know how. As the majority of problems discussed in early grades (cycle 1) are consistent, the numerical answer can, in many cases, be obtained by straightforward mimicking of the action described in the problem. In these conditions, it is difficult to say whether students are actually modeling the problem or the calculation.

In this chapter, I discussed how the use of tokens facilitates the solving process while letting students avoid holistic flexible reasoning. Researchers (Carpenter et al., 1988) recognize that “the modeling and counting strategies that children use to solve simple problems with relatively small numbers are too cumbersome to be effective with more complex problems or problems with large numbers” (p. 345). To deal with this obstacle, they propose that students be taught to write open mathematical expressions (e.g., 23 - ? = 17) to mathematize the problem and then transform these expressions into the standard form (23 - 17 = ?). This modeling seems to be close to the algebraic one, where the unknown (usually “x”) is represented by the “?” sign or “□”.

The short form of the model—that is, the mathematical expression—might be instrumental in giving students a holistic view of the situation. Two questions remain unsolved in this approach: a) What should students do to represent a situation if no action is described (e.g., comparison situation)? and b) What should students do to invert an open expression into the standard form? As stated in Chapter 1, the reversibility of arithmetic operations is one of the major learning challenges for elementary students.

Schematic representations

Some studies (Gamo et al., 2009; Neef et al., 2003; Ng & Lee, 2009; Powell, 2011; Xin, 2008) examine the use of different graphic and schematic representations of problems. Gamo et al. (2009) demonstrate that the comparison of problems and use of different graphic representations can help students develop efficient strategies for problem solving. The Singaporean method (Ng & Lee, 2009) uses continuous representation—rectangles of different lengths—to show the quantities involved and their relationships. Powell (2011) argues that teaching students to identify a graphic schema for the problem (from a list of schemas previously taught) helps students, including those at risk of or with learning difficulties, to organize their solution and solve the problem. Any graphic or schematic representation potentially gives the student rapid visual access to the entire system of quantitative relationships described in the problem. Therefore, using diagrams and graphic schemas appears to promote a holistic vision of the problem.

Didactic management and meta-cognitive regulation

There are other studies that propose particular didactic management and organization of classwork. Neef and her colleagues (2003) show that learning about the roles of each data element in a problem greatly improves success in problem solving among students with developmental disabilities. DeBlois (2006) suggests that a request for feedback on the solved problem may provoke coordination between mental representations and procedures and may lead students to reorganize their thoughts. Erdniev (1979) proposed that the direct problem should be solved together with the inverse problem (the same additive situation with the unknown in a different place). Zaitseva and Tselischeva (2010) propose asking students to compose an inverse problem after having solved a direct one. These approaches clearly reflect an effort to reorganize students' reasoning in a holistic and flexible way.

Focant (2003) studied the difficulties students have in organizing their reasoning in problem solving. He argues that the reasoning process should include the following elements, in a cyclic manner:

- Analysis of the problem in its integrity and clarifying the objective
- Calculation planning
- Control over the calculation execution
- Restarting the process in case of inadequate results

Focant proposes that students should be supported in the development of these meta-cognitive skills. It would seem that this approach may contribute to the development of holistic flexible reasoning.

Continuous and discrete reasoning about numbers

Many researchers pay special attention to the child's ability to think of and mentally represent an unknown quantity. Case and Okamoto (1996) hypothesized that this mental representation is similar to a piece of a number line. A similar graphic representation (Arrange All Diagram) was used above in the analysis of problems' structures (please see section 2.2.3). This kind of reasoning can be used for both known and unknown numbers and quantities. It seems that using a continuous mental representation of quantity instead of a one-by-one discrete representation of objects using tokens or dots can help students to liberate the working memory required for the relational holistic schema analysis. This claim is indirectly supported by the empirical results obtained by Gamo et al. (2009) and by the success of Singaporean's method (Beckmann, 2004; Fan & Zhu, 2007; Kaur, 2008). In both cases, the success can be associated with, among others, the continuous graphic representations used to teach problem solving. Davydov's (1982) experiments discussed above also support this idea.

2.6.4 Summary of teaching approaches

I have briefly discussed different approaches to teaching additive word problem solving currently used and proposed in research. Various research groups studying problem solving worked with different groups of students (different grades and levels of cognitive development) and focused their attention on different aspects of teaching and learning (number concept development, modeling, schematic representations, and didactic management). Each of these studies contributes extremely valuable ideas about how the teaching of additive word problems can be organized to promote the development of holistic relational reasoning in learners. However, this specific objective—the development of holistic reasoning for additive structures and additive word problem solving—is not explicitly present in the studies (this refers us to the Operational Paradigm). In the next section, I will briefly describe the equilibrated development approach, proposed and implemented by the team led by Dr. Savard, which put the development of holistic flexible reasoning at the centre of the study (in line with the Relational Paradigm).

2.6.5 Equilibrated Development Approach

In this section, I briefly describe the teaching approach developed within the scope of the collaborative study led by Dr. Savard. (My study of students' reasoning development is a part of this larger project.) To design a teaching approach within the Relational Paradigm, Dr. Savard and her team used ideas from multiple research studies (previously cited in this study). In particular, the team proposed explicitly modeling problems with students, and using a specific graphic representation of the mathematical structure of the problem, called the Arrange All Diagram (AAD) representation. I have used the AAD representation in this chapter to represent and discuss various structures additive problems can have. This representation method is similar to the Singaporean one and to the one used by Davydov. In the Equilibrated Development Approach, the team incorporated the representation method, curriculum content, and elements of didactic management into a whole and continuous process of teaching.

As mentioned earlier, the main idea of equilibrated development is working toward an appropriate balance between two ways in which a word problem can be understood, while paying special attention to the development of holistic relational reasoning. Word problems can be understood as a story with its development in time (supported by sequential reasoning) or as a system of relationships between quantities involved (supported by holistic relational reasoning) (Savard et al., 2013). The main principles:

- Any problem solving task should be an occasion to analyze the additive relationships present in the situation.
- The Arrange All Diagram (AAD) representation method should be used as communication media to express the mathematical structure of the problem, discuss and analyze the structure with students, and figure out the mathematical operation (mathematical expression in standard form) necessary to calculate the answer.
- All categories and classes of additive problems should be present in the curriculum with more attention devoted to the inconsistent problems.
- Activities we proposed were designed to be used as classroom discussions, team work or individual work followed by discussion.

The experimental curriculum comprised three phases: introduction, construction and development. In the introduction phase, students explored the length property of different physical objects (strings, ropes, paper strips, cloths, etc.). They constructed a rigorous procedure to compare the lengths of these objects and visualize the difference, and learned to represent this comparison graphically.

Based on this concrete knowledge of manipulations using lengths, the construction phase proposed that students holistically analyze different situations involving additive comparison. Instead of a word problem with a question, teachers proposed a word description of a mathematically impossible situation; the three quantities involved did not respect the comparison relationship described in the text. Students then explored the situation to find how each value could be changed to satisfy the relationship. To clarify the relationship, students (together with the teacher) constructed a graphic (AAD) representation of the comparison involved. Based on this representation, for each potentially incorrect number, they determined the mathematical operation that could calculate the correct value for this number.

The development phase included solving additive word problems in the following cyclical manner:

1. Read the problem and discuss the context briefly.
2. Construct an AAD representation of the problem.
3. Determine the mathematical expression that can calculate the missing value.
4. Use the available tools to calculate the missing value.
5. Make sense of the calculated value in terms of the initial situation and the AAD representation.
6. Restart from step 1 if needed.

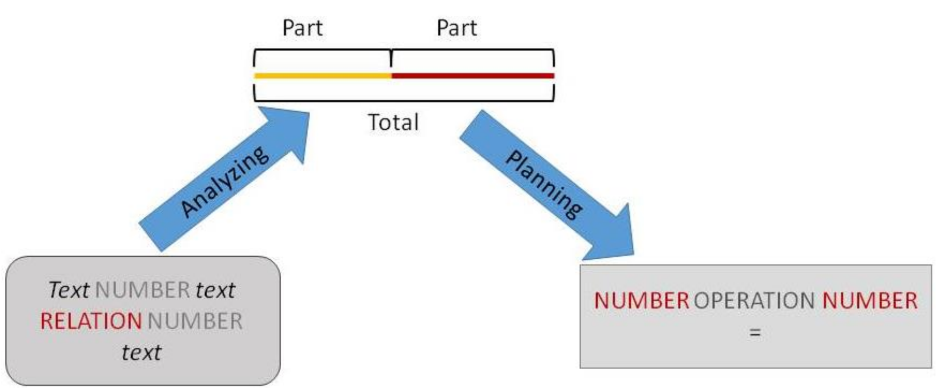


Figure 9. Summary of the Equilibrated Development Approach to solving simple additive word problems

Figure 9 describes how the problem solving is seen in the Equilibrated Development Approach. The text of a problem is analyzed as a description of relationships which should be explicitly represented using the AAD method. This representation then serves to plan the calculation needed.

Other activities were also implemented in the development phase of the curriculum to support students in constructing and making sense of AAD representations.

This is a very brief description of the Equilibrated Development Approach. The objective of this description is to give the necessary information as a background for my own project. The approval or disapproval of the approach is not in the scope of my study. The focus of my study is students mathematizing word problems. In the next sections, I will examine models which can help to interpret students' production in word problem solving.

2.7 Models for interpreting students' production in problem solving

In this section, I will examine several models which can potentially be used to interpret students' production in solving problems and the mathematical modeling process. Being situated in the Relational paradigm, I would like to use models that help interpret students' reasoning about additive relationships. Therefore, I will disregard models of how students think about and represent numbers and operations.

Mukhpadayah and Greer (2001) propose that the mathematical activity (in school settings) be seen in three widening perspectives. At the closest view, the cognitive perspective is the most appropriate. In the case of mathematical problem solving, it would be helpful to see the mathematical cognitive challenges the problem can present. At the second level, the social/cultural perspective becomes important. The situation's social/cultural context can influence students' understanding of the problem and their behaviour in solving it. At the highest level, political perspective comes into play. The content and organization of education are subject to political decisions. I will focus primarily on the social/cultural and cognitive levels in my research as I study problem solving at a very close view to capture more details.

The model proposed by Mukhpadayah and Greer (2001) was further developed by Savard (2008) to look at mathematical activity involving probability and gambling. The model comprises three contexts: mathematical context, socio-cultural context and citizenship context. The model shows that the desirable reasoning moves between different contexts to solve a problem and derive real-life decisions from the solution.

According to this model, in order to solve a problem the student should mathematize the situation by creating a mathematical model. Savard points out that the mathematical modeling process should start in the socio-cultural context as the student uses the known aspects of this context to make sense of the situation. This modeling process should produce a mathematical model of the situation (problem) from which the mathematical results can be derived. The learner should then make sense of these results in terms of the socio-cultural context. This newly obtained sense should be evaluated in relation to the initial understanding of the problem. Thus, the problem solving process is organized in a cycle, potentially supporting the development of critical thinking in learners.

Erik De Corte (2012) proposes a two-level model of the mathematical modeling process. According to him, the process starts with the physical phenomena at hand (described in the problem) and goes through the comprehension stage involving the previous knowledge about the phenomena. This step produces a situation model, which can then be transformed into a mathematical model. The author also points out that the mathematical model is affected by the modeling goals.

All the models mentioned above describe a *desired* way of reasoning for problem-solving and modeling processes. In my research, I am looking for the processes which *actually happen*, including those that are undesired and that fail. Therefore, I can only use these models as general guidance.

In my research, I would like to examine two particular steps in the problem-solving process more closely: the initial understanding of the situation and mathematizing. I would like to see whether the mathematizing that students actually perform can be interpreted as modeling.

I will look at students' productions to understand how students get from the text of the problem to a particular mathematical meaning. DeBlois (2011) proposes a model to interpret students' production in mathematical problem solving and possible learning outcomes. DeBlois explains that students' reasoning in problem solving can be affected by their mental representation of the situation and perception of the teacher's request. Together, these two factors can result in the final production and can thus be reflected and analyzed through this production.

While observing a student solving a problem, this model can help to interpret the student's production as an interplay between their mental representation of the problem and their perception of the teacher's request. However, this model does not explain how students form their mental representation.

From the Construction/Integration model of text comprehension (Kintsch, 2005), it follows that the mental representation of the written problem that the student develops is affected by their previous knowledge and particularities of the text of the problem (see section 2.3.3). These elements are absent from all other models discussed above.

2.7.1 Summary of interpretative models

I have discussed some existing models to interpret students' productions in word problem solving and discover students' ways of mathematizing. Each of these models points out important elements in the problem solving process. None of these models fully describes the possible ways students can mathematize and/or model a problem. The creation of such a model can constitute a valuable contribution to teaching and learning theories. In Chapter 5, I will use the results of my study to construct a more complete model of the mathematizing process elementary students might have while solving word problems.

2.8 Restatement of the research questions

The main purpose of this study is to better understand the development of mathematizing processes in young students when solving simple additive word problems. Although similar questions were raised and studied by many, two aspects make my study unique:

- The Relational Paradigm, in which this study is situated, helps to see an old subject in a new way.
- The experimental teaching approach (Savard et al., 2013) provides a unique opportunity to observe and analyze this knowledge development in fundamentally different conditions.

An examination of students' progress in relation to the main characteristics of the teaching approach can provide researchers and practitioners with valuable information about learners' potential in relational thinking development. The knowledge of how a particular teaching approach can reinforce or inhibit students reasoning potential is critical in providing students with better access to the power of mathematical reasoning.

In Chapter 1, I formulated my research objective as follows:

- To study how elementary students develop their mathematical reasoning in the context of additive word problem solving, while being exposed to the Equilibrated Development Approach.

Based on the developed theoretical framework, this goal can be achieved in two steps. First, I will analyze the learner–mathematics relationship to answer Question 1 within three sub-questions:

1. ***What ways of mathematizing do students use while solving additive word problems?***
    a. ***What are students' mental representations of the problem?***
    b. ***What are students' presumptions about the task, if any?***
    c. ***What mathematizing processes do students use?***

Second, I will analyze the learner-teaching relationship to answer Question 2:

2. ***What are the relationships between the instruction implemented and the students' development of mathematizing processes?***
    a. ***How do students' ways of mathematizing evolve over a particular period of time in the context of a given teaching approach?***

b. ***What elements can be identified in the given curriculum and its implementation in class as potentially affecting the development of students' ways of mathematizing of additive word problems?***

The answer to the first question will provide empirical evidence of how early grade learners mathematize and model simple additive word problems and how this process can be interpreted by the teacher and researcher. The exploration of students' mathematizing processes will shed light on what reasoning elements can facilitate or obscure these processes for them.

The answer to the second question will highlight the relationships between the teaching implemented and the learning outcomes. The latter will help to formulate suggestions about new learning opportunities and teaching principles.

Chapter 3 Research methods

3.1 Research methodology

Researchers agree that the teaching/learning process represents an extreme complexity. Cobb and colleagues (2003) see this complexity as “a hallmark of educational settings” (p. 9). The main goal of this study is delving deeper into the complexity of the mathematical reasoning development. The qualitative methods, such as individual interviews with students, allowed researchers (Krutetskii, 1976; Piaget, 1974; Vygotsky, Luria, & Knox, 1993; Vygotsky, 1997) to create a new vision of their research subjects and contribute to theory development. Thus, the understanding of students’ reasoning processes within the Relational Paradigm requires the use of qualitative methodology.

In the previous chapters, I discussed the need to understand how learners develop mathematical reasoning with regard to additive structures. I did not formulate a hypothesis about this development. Instead, I proposed approaching the subject from the perspective of the Relational Paradigm to shed light on the learners understanding of the additive relationship inherent to a problem. I proposed that the subject be studied at two levels. First, it is important to understand and describe students’ reasoning in problem solving. Second, it is necessary to describe the dynamics of the reasoning development over a period of time and link it to the implemented teaching.

In the first part of my study, I need to observe and understand students’ problem-solving processes from a very close angle. From my theoretical exploration, it follows that there are several models which can help to interpret students’ reasoning in a problem-solving context (De Corte, 2012; DeBlois, 2011; Kintsch, 2005; Savard, 2008). Yet, none of these models describe the phenomena that can potentially be observed in their complexity. According to Clement (2000), insightful explanatory models are urgently needed to better understand the complexity of learners’ reasoning in problem solving.

Researchers (Bruce, 2007; Gerald A. Goldin, 2000; Zazkis, 1999) suggest the grounded theory methodology as an approach to developing new models and theoretical insights based on observations. Bruce (2007) highlights that the grounded theory research is not fully an inductive process. The existing theoretical knowledge can be used to initiate the data collection, which then becomes driven by emergent data.

My main theoretical objectives consist in clarifying the interplay of different elements of mathematical reasoning described in the existing models and discovering other emergent elements, if any. The grounded theory methodology allows me to derive theoretical ideas from observations, while using existing models as general guidance to initiate data collection and inspire data analysis.

Researchers (Clement, 2000; Gerald A. Goldin, 2000; Zazkis, 1999) recommend the individual task-based interview method of data collection, which has been used by many famous scholars (Carpenter et al., 1993; Nesher et al., 1982; Piaget, 1974; Vygotsky et al., 1993), to answer the types of research questions similar to those of my study:

> What powerful problem solving processes are students learning that go beyond mathematical facts and algorithmic procedures? What kinds of cognitive representations are they developing? What beliefs about mathematics, or affective pathways in relation to mathematics, are children acquiring? What consequences are innovative teaching methods having for their mathematical development? (Goldin, 2000, p. 524)

My research goal is not to prove or disapprove of a research hypothesis. I would like to better describe the development of complex reasoning processes in students. Individual task-based interviews allow me to observe the complexity of students' individual behaviour in the mathematical problem solving context. These detailed observations can potentially lead to scientifically valuable inferences about students' thinking and knowledge development (Gerald A. Goldin, 1997).

The second part of my study involved understanding the relationships between students' reasoning elements, what would emerge from the first part of the study, and the elements of teaching implemented in classrooms. To answer the second question and study the possible links between students' reasoning and classroom instruction, I used the retrospective analysis method (Cobb et al., 2003). This method consists in collecting records of classroom activities through the studied learning period, and analyzing this data as a history of classroom events. This analysis would not be the analysis of teaching per se, but the search for teaching events which could possibly provoke the specific behaviour observed in students. In my case, the main purpose of this retrospective analysis was to find and describe events which could potentially provoke students' ways of problem solving that emerged in the first part of the study.

3.2 Context

The team headed by Dr. Savard is conducting a three-year collaborative research project, funded by the Quebec Ministry of Education, Leisure, and Sport, and involving a school board with rural and urban schools in the French-speaking community. The collaborative project involves additive problem solving in early grades of elementary school. The goals of the collaborative project: 1) develop a pedagogical approach that would promote holistic flexible reasoning in learners with regard to simple additive structures; 2) design and test a set of tasks and didactical scenarios that implement the new approach; 3) propose an appropriate professional development program for teachers.

During the first year of this collaborative project, the team developed a series of didactic scenarios to support learners in the development of relational reasoning in additive problem solving. Six Grade 2 teachers from four different schools participated in the professional development program, where these scenarios were proposed and discussed. Following these discussions, the teachers implemented the didactic scenarios in their classes through the school year.

My study is based on the first year of the collaborative project. Among other things, the observations and analysis of learners' knowledge development at this particular period would help to improve the teaching approach being developed in the collaborative project. All of my study participants also participated in the collaborative study. The results of my study served to inform the further development of didactic scenarios and would help teachers to better implement them in class.

3.3 Data collection methods

According to the chosen methodology, data collection was organized and conducted in the following steps:

- I selected participants from students participating in the collaborative project.
- I created a set of problem-solving tasks.
- I conducted four sessions of semi-structured interviews with my participants, where I asked them to solve experimental tasks.
- I collected instructional materials developed in the collaborative project and used by teachers in the classrooms.
- I regularly observed and video-recorded select classroom sessions to get more data about the implementation of the experimental teaching approach.

Below, I provide more detailed description of each step of data collection.

3.3.1 Participant selection

The selection of my participants was determined by the willingness of teachers to participate in the research project. Thus, two elementary school teachers from the same school accepted to participate in my research. They participated in the training sessions and implemented the new teaching approach in their classes.

The school was a part of a French-speaking community and therefore the entire study was conducted in French.

To ground the theoretical investigation of emergent phenomena, a large number and **variety** of observations is needed. Bruce (2007) explains: “The objective, however, is to gather rich data that do not deplete or over consume valuable research resources” (p.5). To ensure the number and variety of observations, I chose to observe a limited number of students while exposing them to a wide variety of problems. It was also essential to ensure variety in students’ mathematical performance. Thus, I decided to select six students from each class: two for each level of relative achievement in mathematics: high, middle and low.

At the beginning of the school year, I administered a written problem-solving test to all students in both classes. The problem texts used for the test are presented in Appendix 2. I evaluated students' mathematical results in this test. I asked the teachers to evaluate their students' relative level of mathematical achievement: 1 - relatively high, 2 - middle, 3 - relatively low. I compared this evaluation to the results collected through the problem-solving test. From each of two classes, I chose six students to participate in the study as their two evaluations were consistent with one another. This selection method was not very accurate in terms of the actual mathematical strength of participants. However, it helped to establish the diversity of such strengths in the group, which was required by the grounded theory methodology. In the end, five boys and seven girls participated in the study.

McGill's Research Ethics Board approved the study.[12] Parental permission was obtained for all experimental activities and all participants. Consent was also obtained from the two teachers. To protect students' identity, I use pseudonyms in this study and therefore use gender attributes according to the pseudonyms. To protect the teachers' identities, I do not use their names, but the word "teacher."

3.3.2 Instrument: problem-solving tasks

The nature of the grounded theory methodology suggests that the designing and conducting of interviews should be based on existing theories. In the Relational Paradigm, students' reasoning in solving additive word problems should be interpreted in terms of their ability to see and use the additive relationship present in the task (Davydov, 1982). I used my theoretical exploration of the epistemology of additive word problems (described in section 2.2) to design tasks that could potentially provoke students' reasoning about the additive relationship. The following aspects of the problems' text were considered: context, semantic structure, consistency in the language, presence of additional data, and presence of elaborated language. To obtain tasks that really required

[12] The Research Ethics Board certificate are available upon request

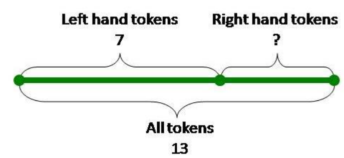
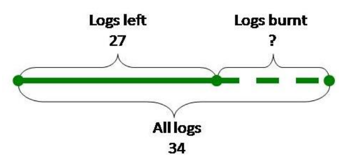
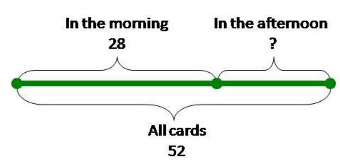
reasoning I chose to create problems with inconsistent language and/or other elements raising the level of difficulty for the student and potentially requiring the analysis of relationships involved. Taking into consideration the need to obtain a wide variety of observations, I designed problems involving different structures and contexts.

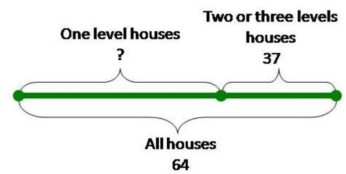
The difficulty of each problem remains, however, relative to the student's knowledge and ability. Thus, in addition to eight problems of medium difficulty, I created five other problems to meet the students' need to work at their appropriate level of difficulty. A total of 13 tasks were used: 8 relatively difficult, 1 easy and 4 very difficult. According to my theoretical exploration of the epistemology of word problems (see section 2.2), the one easy problem (Tokens) does not really require relational reasoning. That is why I only used one problem of this type. This problem gave students the chance to at least start a mathematical discussion in cases where other problems may have been too difficult for them. The four very difficult problems (Pencils, Ski, Butcher, and Snowballs 3) were used to probe students' reasoning in cases where other problems seemed to be too easy for them. These problems comprised two compound additive structures and were not *simple*, strictly speaking.

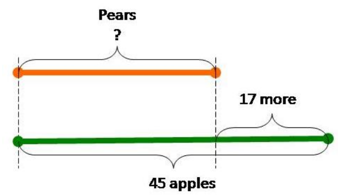
I planned to conduct four interview sessions. It appeared that my participants were able to solve and discuss no more than three problems per interview. In the great majority of cases, we were able to discuss one or two problems. Some variations of the designed problems were proposed to students to meet emergent needs. In the end, I obtained 96 observations, representing the essential amount and variety of data required by the grounded theory methodology.

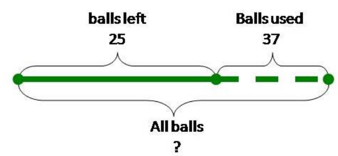
Goldin (2000) suggests providing a detailed description of interview tasks to ensure the reliability and generalizability of the research results. Table **2** presents the wording of the tasks, category and class of the semantic structure(s) involved, additional characteristics affecting relative difficulty of the problem and estimated level of relative difficulty. To clarify the inconsistent language, I highlighted the words in the problem texts that that could potentially lead to an incorrect mathematical operation. These words were not highlighted in the tasks given to students.

Table 2 *Problem Tasks Proposed to Participants*

Name of the problem	Wording	Class and category	Equation	Mathematical expression in standard form	Arrange All Diagram	Additional characteristics	Relative difficulty level
Tokens	I took 13 tokens. I hid 7 of the tokens in my left hand and the rest in my right hand. How many tokens are in my right hand?	Simple negative change, result unknown	13 - 7 = X	13 - 7 = ?			Easy
Logs	There were 34 logs in the pack that dad bought to make a camp fire. The fire burned for 48 minutes. Some logs were already burned. There were 27 logs left in the pack. How many logs were burned?	Simple negative change, change unknown	34 - X = 27	34 - 27 = ?		Elaborate wording One extra piece of data	Relatively difficult
Cards	To organize a school party, Léa must send invitations to 63 people. Before lunch, she prepared 28 invitations. After lunch, she prepared **more** invitations. She now has 52 invitations to send.	Simple change (positive), change unknown	28 + X = 52	52 - 28 = ?		One extra piece of data Elaborate wording Inconsistent language	Relatively difficult

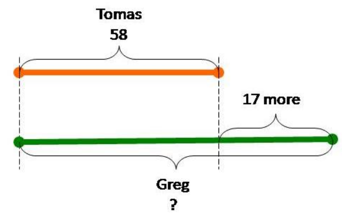
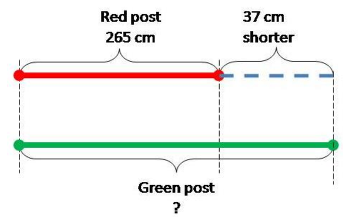
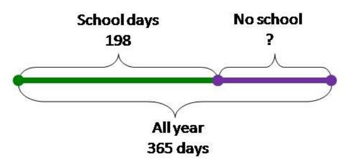
<table>
<tr><td></td><td>How many invitations did Léa prepare after lunch?</td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Houses</td><td>There are 64 houses on our street. Some houses have 1 storey. The other 37 houses are 2- and 3-storey houses. How many 1-storey houses are there on our street?</td><td>Simple combine, part unknown</td><td>64 = X + 37</td><td>64 - 37 = ?</td><td>
</td><td>Two extra numbers

Elaborate wording</td><td>Relatively difficult</td></tr>
<tr><td>Fruits</td><td>There are 45 apples, 34 oranges and some pears in a basket. There are 17 more apples than pears. How many pears are there?</td><td>Simple comparison, referent unknown</td><td>45 = X + 17</td><td>45 - 17 = ?</td><td>
</td><td>One extra number

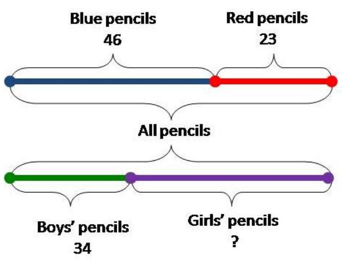
Inconsistent language</td><td>Relatively difficult</td></tr>
<tr><td>Snow balls 1</td><td>Tomas made a bunch of snowballs for his snowball fight with his friend, Greg. After 15 minutes of fighting, Tomas had thrown 37 snowballs. He now has 25 snowballs left. How many snowballs did Tomas make?</td><td>Simple change (negative), initial state unknown</td><td>X - 37 = 25</td><td>37 + 25 = ?</td><td>
</td><td>Elaborate language

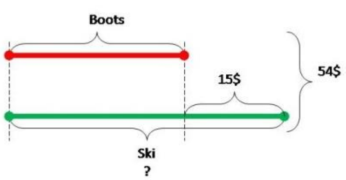
One extra number

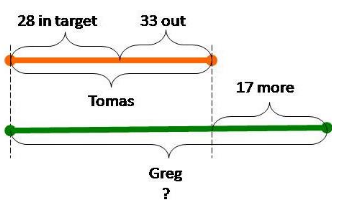
Inconsistent language</td><td>Relatively difficult</td></tr>
</table>

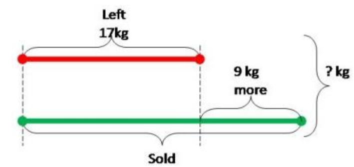
Snow balls 2	Tomas made a bunch of snowballs for his snowball fight. His friend, Greg, said that he made 17 more snowballs than Tomas. Tomas counted his snowballs: there were 58. How many snowballs did Greg make?	Simple comparison, compared unknown	X + 17 = Y X = 58	58 + 17 = ?		Elaborate language	Relatively difficult
Posts	The red post is 37 cm **shorter** than the green post. How long is the green post if the red post is 265 cm?	Simple comparison, referent unknown	X = Y - 37 X = 265	265 + 37		Inconsistent language Continuous context	Relatively difficult
Calendar	In 2011, there were 365 days. In her calendar, Julie marked 198 school days and 10 holidays. How many days were without school in 2011?	Simple combine, part unknown	365 = 198 + X	365 - 198 = ?		Two extra numbers Elaborate language Continuous context	Relatively difficult

Pencils	In a Secondary 5 class, Jeremy counted 46 blue pens and 23 red pens on 21 desks. He knows that 34 of the pens belong to the boys in the class and the others to the girls. How many pens belong to the girls?	Two superposed simple Combine structures, part unknown	46 + 23 = 34 + X	46 + 23 - 34 = ?		One extra number	Very difficult
Ski	A pair of skis and boots cost $54 altogether. The skis cost $15 more than the boots. How much do the skis cost?	Two superposed structures: Simple comparison and simple combine, two quantities unknown	X + Y = 54 X = Y + 15	(54 - 1) / 2 + 15 = ?			Very difficult
Snow balls 3	Tomas made a bunch of snowballs for his snowball fight. His friend Greg said that he made 17 more snowballs than Tomas. Tomas threw all his	Two superposed structures: simple comparis	X + 17 = Y 28 + 33 = X	28 + 33 + 17 = ?		Elaborate language	Very difficult







	snowballs at Greg's fort. 28 snowballs hit the target. The other 33 missed. How many snowballs did Greg prepare?	on and simple combine; compared unknown					
Butcher	The butcher received 3 packs of fresh meat this morning. At noon, he saw that he had 17 kg of meat left. "I sold 9 more kilograms of meat than I have left to sell," said the butcher. How much meat did the butcher receive this morning?	Two superposed structures: Simple change (negative) and simple comparison, total unknown	X - Y = 17 Y = 17 + 9	17 + 17 + 9 = ?		One extra number Elaborate language	Very difficult

3.3.3 Individual interviews with students

I conducted four sessions of individual clinical semi-structured interviews with all participants in November, January, March and May. The time period between each session was about two months.

Table 6 presents the use of problems with participants. For each interview, I proposed at least one relatively difficult problem to each participant. This means that at least one of the following conditions was present in the text of the task:

- additional non-relevant number(s)
- inconsistent language
- elaborate wording

For the first interview session, I wanted to avoid students automatically using the knowledge recently discussed in class (which, at this point, was Comparison Situations). Problems with structures other than that of comparison would better provoke students' reasoning instead of them automatically applying classroom procedures. Thus, for the first session, I used Change and Combine problems. For the other three sessions, I used problems of a variety of classes.

The interviews were conducted outside the classroom, in a small office or empty classroom. In each interview, I explained to the student that it was neither an examination nor an evaluation and that they could return to the classroom at any moment. In each interview, I proposed one to three problems to the student depending on the time available and their work rhythm. Interviews usually lasted 10 to 20 minutes because, at this age (7 to 8), students usually get tired after 20 minutes of intensive mental work.

At the beginning of the year, my participants had no previous experience explaining how they understand the mathematical structure of a problem or what they think about the problem from a mathematical point of view. These types of questions could provoke confusion and obscure students' natural way of reasoning about the problem. That is why I chose to ask a more traditional question: What can we do to solve this problem?

When the third session was conducted in March, the students had been working with the AAD representation method and discussing mathematical expressions in standard form for four months. Starting from the third session, I added two new questions to the interview:

- Can you propose a mathematical expression to solve this problem?
- Can you represent the problem the way you usually do in class (AAD method)?

These questions were asked in cases where the student did not use any diagrams in their solution and/or did not provide a mathematical expression in standard form.

I was looking to see if the student's reasoning was altered by the accomplished calculation process (Vygotsky, 1984). I therefore asked students to describe their strategy orally prior to the execution. However, in cases where students were unclear about their solutions, I advised them to proceed on paper or with tokens.

The work on each task was organized the following way. First, I asked student to read the text of the problem and ensure that they understood it well. I also read the problem for the student myself if necessary. Second, I invited the student to ask questions in case they did not understand certain words or expressions. A careful reading of the text was needed to eliminate the possible influence of reading difficulties (if any) on the participant's mathematical analysis process. Third, I asked the student to think about their solving plan and explain it to me orally prior to any manipulations, drawings or calculations. In cases where the student could not clearly explain their strategy, I advised them to proceed with the solution. I observed the student's strategy and asked questions to clarify the meaning they attributed to various parts of the solution.

My conversations with the students were video-recorded and their drawings and calculations (if any) were collected.

3.3.4 Collecting data about teaching

In order to nourish the retrospective analysis and answer the second question in my study—regarding possible links between teaching and learning—I collected data about the teaching planned and implemented in the classrooms. I collected all instructional materials (paper documents) related to the teaching implemented in the classrooms: the description of the main goals and principles of the teaching approach, descriptions of educational activities, sets of word problems designed by teachers.

To complete the data on the teaching implemented, I observed and video-recorded 10 classroom sessions through all three phases of the curriculum implemented in classrooms: introduction, construction and development.

1. Introduction part 1, October 11
2. Introduction part 2, October 17
3. Introduction part 3, October 24
4. Construction, 360° part 1, November 16
5. Construction, 360° part 2, November 23
6. Construction, 360° part 3, December 15
7. Development, problem solving, January 30
8. Development, Communication game, March 19
9. Development, Communication game, April 4
10. Development, Schema comparison, April 25

3.3.5 Summary of the data collection

The data collected in the study consists of two sets. The first set of data consists of 98 video-recorded observations of students' problem solving. The problems of high levels of difficulty (for the student) with different semantic structures were used to interview the diverse group of students at four different points throughout the school year.

The second set of data includes the instructional materials and 10 video-recorded classroom sessions related to the experimental curriculum.

All the events studied took place in a certain order throughout the school year. Figure 12 presents the approximate calendar of events. The green circles show the video-recordings in class.

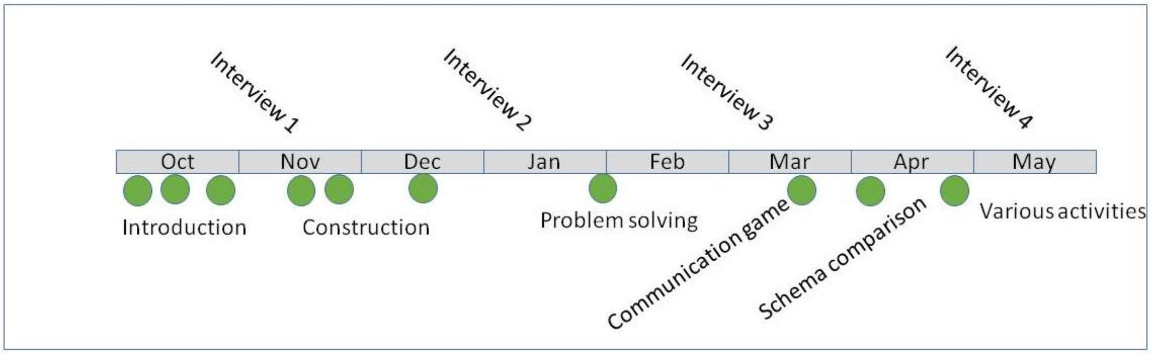


Figure 10. Approximate calendar of events. The green circles show the video-recordings in class.

3.4 Data analysis methods

First research question:

- ***What ways of mathematizing do students use while solving additive word problems?***
    - ***a. What are students' mental representations of the problem?***
    - ***b. What are students' presumptions about the task, if any?***
    - ***c. What is mathematizing processes do students use?***

According to Goldin (1997, p.40), task-based individual interviews

> [...] are used in research for the twin purposes of (a) observing the mathematical behavior of children or adults, usually in an exploratory problem-solving context, and (b) drawing inferences from the observations to allow something to be said about the problem solver's possible meanings, knowledge structures, cognitive processes, affect, or changes in these in the course of the interview.

To answer this question, I analysed the video-recorded interviews in the following manner. I considered students' spoken and written explanations and answers to my questions to be their production in solving a problem. I used the interpretative model developed by DeBlois (2011) to guide my interpretation of students' production in terms of possible mental representations (sub-question a) and their possible perception of the request (sub-question b). Figure 11 presents a simplified version of the DeBlois model.

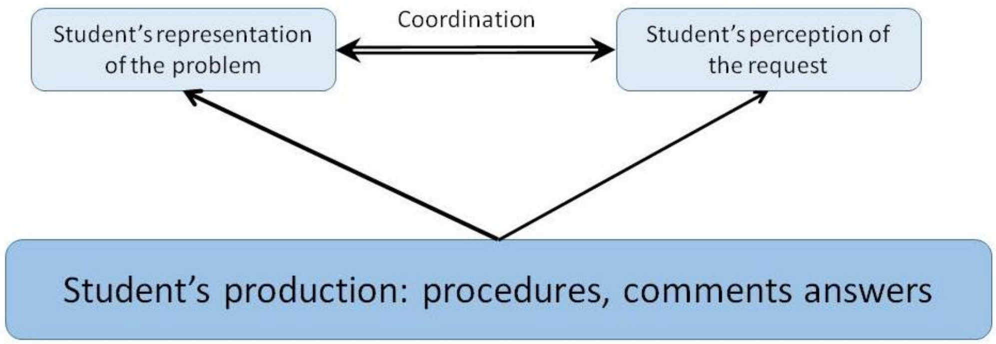


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Figure 11. Simplified model of students' production interpretation

To construct inferences about the students' mental representations, I looked at their production from a Relational Paradigm point of view. I examined whether their production was sequential or holistic in character, whether the order in which data elements were mentioned in their explanations and solution was the same as that of the problem's text and whether they paid attention to the quantitative relationships present in the problem. For example, consider the problem:

> *There were 34 logs in the pack dad bought to make a camp fire. The fire burned for 48 minutes. Some logs burned. There were 27 logs left in the pack. How many logs were burned?*

If the student takes (or draws) 34 objects (logs, tokens, circles) and then removes them one by one until 27 remain, I would suggest that their mental representation corresponds to their solving process and the order of events described in the problem. We can say that the student likely has a sequential mental representation. Conversely, if the order of the student's calculation plan differs from the chronology of the text (34 - 27 =), we can say that there is a possibility that the student has a holistic relational mental representation of the problem. This holistic representation helps them to reorganize the problem's structure to construct the solution (Nesher et al., 1982; Riley et al., 1984; Vergnaud, 1982b).

In relation to the mental representation, I also examined whether the student's production was influenced by their presumption of teacher's or researcher's request (Brousseau, 1988; DeBlois, 2011; Jackson & Cobb, 2010). For example, the sequential solution can be induced if the student feels that the teacher wants them to always proceed in this particular manner.

In my interpretation of students' production, I was also looking for any other elements of reasoning which they might use to mathematize the problem. This analysis was guided by the Construction/Integration model developed by Kintsch (2005) (discussed in section 2.3.1 of Chapter 2) and the model developed by Savard (2008) (discussed in section 2.7 of Chapter 2). I used a simplified version of the Savard's model (Figure 12) with regard to the relationship between social context and mathematical context.

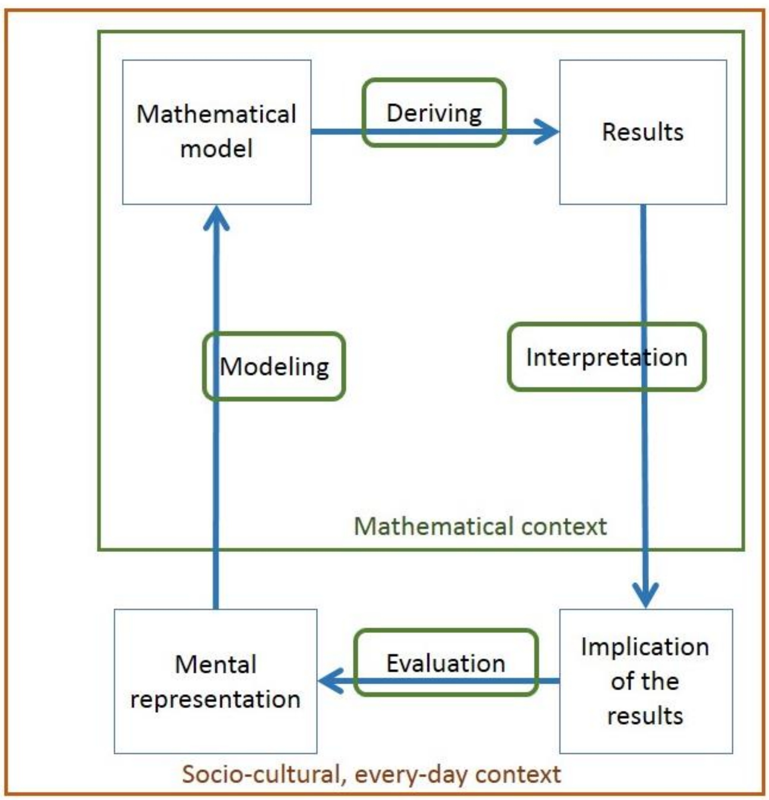

Figure 12. Simplified model of problem solving process according to Savard (2008)

To answer sub-question (c) about students' mathematizing of the problem, I interpreted the way they moved from the text of the problem towards their calculation plan. I was looking at whether their production included explicit mathematical modeling of the quantitative relationships involved, and whether they used this model to derive their calculation plan. For example, the sequential solution for the Logs problem does not include any intermediate steps between the problem and the calculation plan, nor does it present any evidence of the model from which the calculation plan was derived. For the same problem, the equation 34 - ? = 27 can be seen as a mathematical model.

In line with the grounded theory research method, my interpretations were *guided* and not *framed* by the models mentioned above. All reported inferences were made on the basis of multiple viewings of the video-recorded materials. Some parts of these materials were transcribed to give more detailed examples of students' production.

The results of these analyses were categorized and organized in four groups to represent qualitative descriptions of students' ways of mathematizing present at each of the four moments throughout the year (four interview sessions).

Second research question:

- ***What is the relationship between the teaching implemented and students' development of mathematizing processes?***
    a. ***How do students' ways of mathematizing evolve over a particular period of time in the context of the given teaching approach?***
    b. ***What elements can be identified in the experimental curriculum and its implementation in class as potentially affecting the development of students' ways of mathematizing additive word problems?***

To answer this research question, I proceeded in two steps.

First, I colour coded and represented the students' ways of mathematizing, inferred from the four sessions of individual interviews, in a chronological order of the four moments throughout the school year. I was looking for the similarities in how various students changed their ways of mathematizing through the year (change in colour from the first session to the last).

Second, I superposed the images of the learning sequence obtained during the first step and the instruction sequence retrieved from the calendar of learning events and curriculum description, meaning particular activities implemented at certain points of the year.

Cobb et al. (2003) propose a retrospective analysis method to find possible relationships between the students' learning and the teaching implemented. This method consists in creating a history of teaching events through video-recording classroom sessions and looking through this teaching history (video-records) for events possibly related to the mathematical behaviour observed in students. I apply this method of analysis to the curriculum material and activities video-recorded in classes. I analyzed the main didactic characteristics of the experimental curriculum to describe and explain the main learning outcomes in relation to these characteristics. Second, looking closer at how students' ways of mathematizing changed over the year, I analyzed the video-recorded classroom activities. I was looking for particular elements in the curriculum that could explain particular ways in which students approached the problems. I focused considerably on specific teaching elements that may have provoked difficulty for the students.

Figure 13 presents the general design of the study.

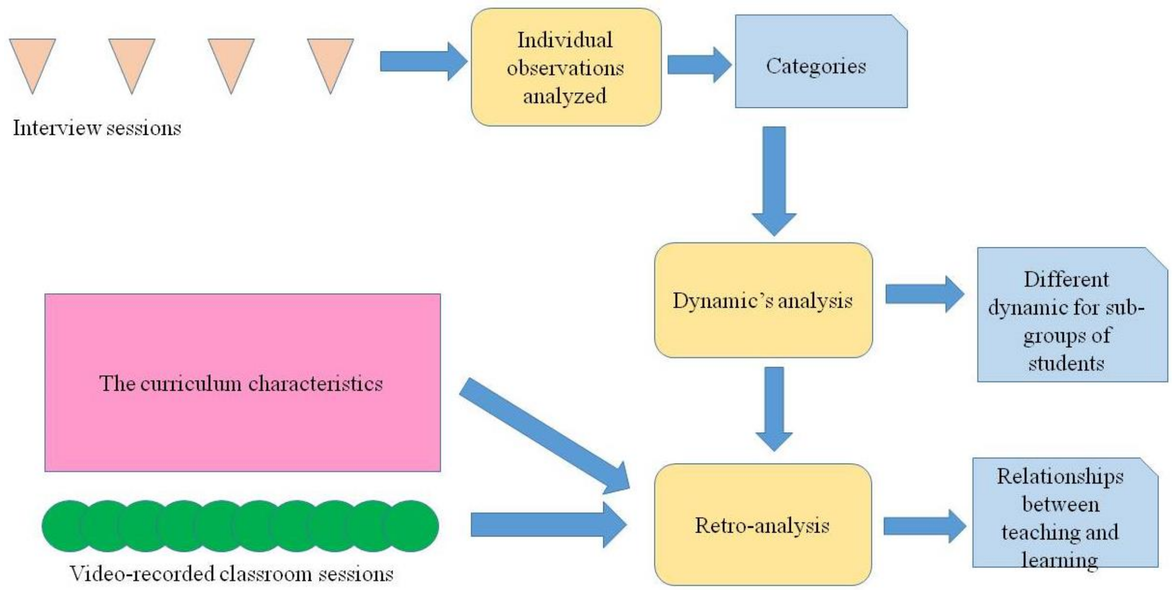

Figure 13. Schema of data collection and analysis

3.5 Limitations of applied methods

The data collection and analysis methods sometimes had limitations.

The participants came from the same school because the study was being conducted in the context of a larger experiment with a limited number of schools, and only teachers from one of those schools agreed to participate in my study.

The participants experienced two different approaches in teaching problem solving: one is currently used in the school board (before the experiment) and the other is experimental (during the studied period). It is possible that other approaches to teaching additive word problem solving could bring out other ways mathematizing in students that were not described in my study.

The period studied was the first year of the larger project, in which the new teaching approach has been developed. At this moment, the first version of the curriculum has been implemented by the teachers for the first time. It is evident that this implementation was a work in progress and needed to be refined based on the results of the study. Therefore, some results of my study (concerning the dynamics of students reasoning development) should be seen in relation to these particular conditions, and not as universal.

Participants were not systematically exposed to all types of semantic structures during each interview session. This was caused by the natural attention span limitations young participants have when doing the intensive mental work that problem solving requires. Yet, the wide variety of the problem-solving observations can compensate for this limitation.

Chapter 4: Results

My study was grounded in the Relational Paradigm, within the context of simple additive word problem solving. According to the chosen grounded theory methodology discussed in Chapter 3, I used important core research in this area and existing theoretical models discussed in Chapter 2 as a source of inspiration and not as a fixed framework. I used the empirical data from my observations to infer new ideas and connect them to existing theories.

In this chapter, I present the analysis results obtained in two phases. In phase 1, I analyzed observations of students solving problems during four interview sessions to understand what they do to mathematize problems. For each of the four interview sessions, I discuss possible inferences about the students' reasoning and present categories of mathematizing methods. At the end of each session's description, I give an approximate image of the session as a whole (all observations made during the session). In phase 2, I analyzed how students' strategies changed over time and the teaching implemented, in order to understand the process of learning and knowledge development. I present the dynamics of the group over time and discuss the changes from session to session. I present links between the reasoning developed in students and the teaching implemented.

4.1 Phase 1: Ways of mathematizing

This phase of analysis provides the information to answer the first question of my study: ***What ways of mathematizing do students use to solve additive word problems?***

For each of the four interview sessions, I present examples of observations, my inferences related to mental representations and presumptions students might have about the tasks. I describe the categories of mathematizing that emerged in each interview session. To summarize each session, I give a graphic representation all the ways of mathematizing that emerged. All reported inferences are made on the basis of multiple viewings of the video-recorded materials. Some parts of these materials were transcribed to provide more detailed examples of students' production.

4.1.1 First session, November

For the first interview session, three problems were used: Logs (Change, medium difficulty), Tokens (Change, easy) and Pencils (Combine, difficult).[13]

> **Logs**: There were 34 logs in the pack that dad bought to make a camp fire. The fire burned for 48 minutes. Some logs were already burned. There were 27 logs left in the pack. How many logs were burned?
>
> **Tokens**: I took 13 tokens. I hid 7 of the tokens in my left hand and the rest in my right hand. How many tokens are in my right hand?
>
> **Pencils**: In a Secondary 5 class, Jeremy counted 46 blue pens and 23 red pens on 21 desks. He knows that 34 of the pens belong to the boys in the class and the others to the girls. How many pens belong to the girls?

I analyzed a total of 27 observations for this session. All students were asked to present their solution plan orally and then execute their plan in the event that they were unable to explain.

4.1.1.1 Mental representation

In the first interview session, it was possible to infer three specific ways students might represent the problem mentally: sequential mental representation, structure substitution and tacit holistic representation. In many cases, it seemed that their reasoning was affected by their presumption about the task. Thus, their production could not be attributed exclusively to their mental representations. Below, I describe my findings in more detail.

Sequential mental representation

The most popular way of solving (solution plan explained by student and/or student's actions) is following the story and representing the objects involved by small circles or tokens that can be added or removed (crossed out). This strategy (way of solving) was observed in 18 cases out of 27 (Logs - 11/12, Tokens - 5/6, Pencils - 2/9). The Pencils problem is about static relationships, not change. By "following the story" in this case, I understand that students were trying to follow the order of information in the text. This way of problem solving was observed in each of the 12 students at least once.

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[13] For more details about problems, see Chapter 3.

In each case, the students really followed the story the way it unfolded in the text. A typical solution plan looked like this:

> Researcher: What do we need to do?
>
> Elodie (Logs): I will draw 34 logs and then remove till there are 27. So, I will see how many were burned.

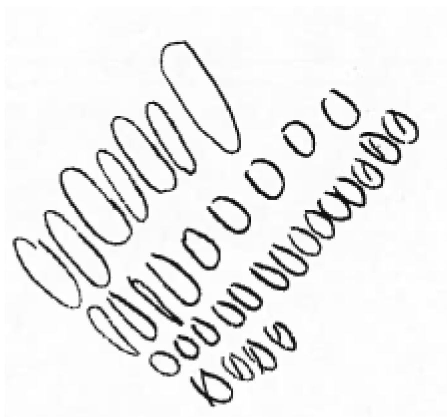

Figure 14. Elodie's solution to the Logs problem

> Researcher: What do we need to do?
>
> Philippe (Tokens): I take 13 tokens. I put 7 in my right hand and the rest into my left hand.

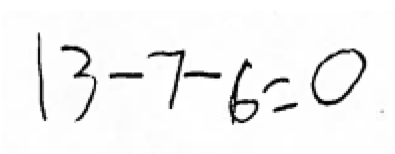

Figure 15. Philippe's solution to the Tokens problem

> Researcher: What do we need to do?
>
> Cathy (Pencils): You do 46 pencils, and then 23 others, then you give 21 to boys, and you give the remaining pencils to girls. … I can't find the mathematical expression.

Based on these and other observations, it seems that the mental representation students have of a problem is the story itself with a sequence of events or portions of information in the order that they are presented in the text. That is why I call this representation the *sequential mental representation*.

In the cases of Cathy and Nicolas (Pencils), they were able to transform the complex combine structure of the problem into a sequence of drawing actions. Thus, it is possible that they saw the problem as a whole. However, their mental representation of the problem was not holistic and flexible enough to let them formulate a mathematical expression or an equation. Cathy said: "I cannot find the mathematical expression."

Structure substitution or shift of meaning

From some of the students' explanations, it appeared as though they were referring to a different problem, not the one given in the task. Similar productions were observed in six cases.

A typical example looks like this (Logs):

Researcher: First, try to explain. What do we need to do?
Nicolas: Well, at the beginning, there were 34 logs. I will draw 34 circles for the logs. Then, I will make a line. Then here, there were 27 logs that were burned. So here, I will cross out 27. Then after, I will draw an equal sign and I will write the answer.
Researcher: Why do you think you should draw 37 um … draw 34 and cross out 27? Why do you think so?
Nicolas: Well. Because they say that, well, 27 in the pack were burned. That means … need to remove 27, there were 34.

My theoretical explorations in section 2.3.3 of Chapter 2 allow me to interpret the students' performance in these cases as follows. Their strong knowledge of the *known-remove-known* mathematical structure affected their interpretation of the text before they actually started to solve the task. As a result, they replaced the Change structure with an unknown change with a Change structure with an unknown result. The intended situation was thus modified to correspond to the well-known, however inappropriate, mathematical structure. The students did not critically review their reasoning with regard to the text of the given task. In the case of the Logs problem, these students actually solved another task without noticing it.

Substituted task: *There were 34 logs in the pack that dad bought for the camp fire. The fire burned for 48 minutes. **27 logs burned**. How many logs are left in the pack?*

In cases of structure substitution, it seems that the mental representation students had of a problem was a story with the same context as the original, but a different mathematical structure. The students **believed** that the task was about this (modified) situation. The quality of this mental representation can be sequential or holistic.

Holistic representation

Only one student proposed a mathematical expression as a solution plan. Viktor's solution to the Tokens problem looked like this: 13 - 7 = 7.

> Researcher: What do we need to do?
>
> Viktor: It's like you have your fingers. Then, if you see, you remove 7, on the other side. Then you see on your right if they are equal. We can do a mathematical expression.
>
> Researcher: Which one?
>
> Viktor: 13 - 7 = 7

When prompted, Viktor drew 13 circles and crossed out 7 of them. He explained that when the answer was found, the expression could be changed, so he did: 13 - 7 = 6.

Viktor created an **equation** using an approximate number instead of a variable (= 7), thus expressing his reasoning in terms of a system. Viktor mentioned his fingers while explaining his solution. It is possible that he was referring to using fingers as a method of calculation. It is also possible that, in his head, he "saw" the tokens in his hands and coordinated this vision with the story described in the problem. This likely helped him to formulate his equation. We can say that Viktor transformed the sequence of the story into a systemic representation.

Another student (Philippe) created a somewhat holistic representation of the Logs problem using tokens. He set out 34 tokens and separated 27 of them from the rest. His explanations were not coherent with his actions or his final answer to the problem. He counted 27 tokens out of 34, explained that he removed 7, and finally declared that the answer was 48. However, it is quite probable that Philippe used a holistic mental representation to deal with the problem because he did not follow the story when constructing his solution using tokens.

Complex structure and its mental representation

Three students, Cathy, Rosa and Nicolas, proposed an adequate solution plan for the Pencils problem, in which two superposed structures were present. This potentially means that the students had a full and flexible mental representation of the problem's complex mathematical structure. However, only one of these students (Rosa) gave an explanation which differs in order from the sequence of original story.

> Rosa (Pencils): They say the 34 pencils are for boys. The number of remaining pencils is for girls. I will draw the blue pencils and the red pencils and then cross out 34 because they say it's for boys.

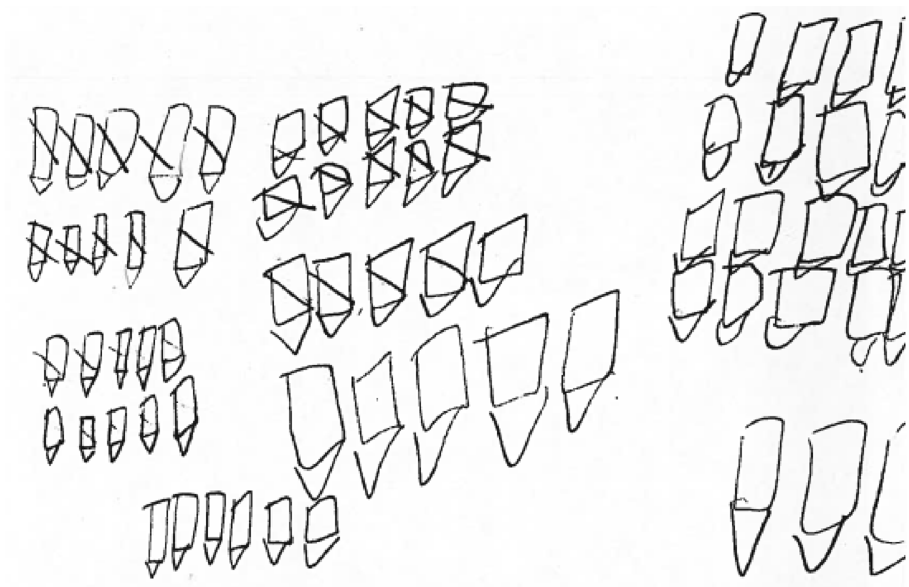

Figure 16. Rosa's solution to the Pencils problem

Rosa likely summarized the problem and created a holistic mental representation before providing a solution plan. Another interesting detail about Rosa's production is that she did not use numbers to refer to the number of blue and red pencils, but explained in terms of the relationship instead of just citing words from the task.

The other two students who solved the Pencils problem described how they would manipulate objects or circles step by step to get to the solution, respecting the order of the information in the task. For the following observation, it is not possible to say whether Nicolas had any mental representations other than a sequential one in his head.

> Nicolas (Pencils): You make a [vertical] line. You do 46 circles on one side and 23 on the other. We can writ R for red and B for bleu. Then you cross our 34. 34 are for boys. You

can encircle them and write ‘boys’. Then you count the rest. You can encircle them and write ‘girls’.

4.1.1.2 Students’ presumptions about the task

Draw-and-count presumption

Additional information from the students’ explanations can be interpreted based on how they perceived the teacher’s request or their presumptions about the task. Almost all students explained that the drawing (circles) **should** be used to solve the problem and the result **should** be counted.

> Elodie (Logs): One **should draw** … the 34 logs … 27 logs still in the package … After we will know how many were burned.
>
> Nicolas (Pencils): We need to cross out 34, and then you will circle them and write “boys” on it. Then you will **count** how many for girls.

The way the students presume they should solve the task could look like this: draw objects, count the answer, write the formula and give the answer. During the first interview session, even students who were able to calculate the answer in other ways still suggested drawing circles.

> Michael (Logs): There were 34 logs in the package, and then 27 were burned.
> [Michael tries to calculate something using his fingers.]
> Researcher: Please, explain what you are doing.
> Michael: I am doing with my fingers. When number [inaudible] do the expression. Then I will draw it.
>
> Nicolas (Tokens): It will be more or less the same thing as for the Logs except that the numbers are smaller. They say we had 13 tokens, then we hid 7 in the left hand. So we need to take away 7. We will do the same kind of drawing as for the logs but with a smaller number of tokens … of things. So, that’s it. Then, we will take away 7. We will draw 13 circles, and then cross out 7. Then we will know how many it equals to.

Michael explained that he would calculate using his fingers and then, nevertheless, drew circles. Nicolas also suggested drawing circles, despite being able to mentally calculate 13 - 7. Evidently, drawing and counting is the method most of the students thought **they should use**, and actually did so successfully in many cases.

The draw-and-count presumption did not affect every student. For example, when Viktor **explained** the Tokens problem, he proposed an equation. He then used the circles drawing to **calculate** the unknown.

Numbers-are-important presumption

The method of drawing-and-counting so highly respected by many students does not guarantee a success. Example:

> Nadia (Tokens): Here they say 13 tokens, so I will draw 13 tokens. [Draws 13 circles] Then, I will hide 7 in my left hand. [Draws another 7 circles] So, in my right hand there are … this many [points to the first 13 circles] and in my left hand this many [points to the 7 circles]. So 13 are left in my right hand.

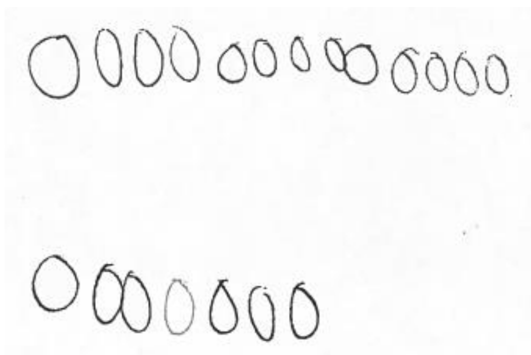

Figure 17. Nadia's solution to the Tokens problem

Nadia does not revise her own production critically. After having drawn 13 and 7 circles, she continues solving now based on this drawing and not on the initial task. She counts the first 13 circles still visually present on her picture as tokens in the right hand.

Another interpretation of this case can be the following. Nadia believes that the most important part of the problem is numerical data. While solving the Logs problem she says: "First of all I look at the numbers." So, to solve the Tokens problem, she looks for the numbers in the text and represents each number as a set of circles. The particular presumption that Nadia likely has helps her focus her attention on numbers as important elements of the task. At the same time, the semantic meaning relating these numbers to the problem got lost in her reasoning. This will be discussed more in Chapter 5.

Grouping presumption

Two students mentioned that they will use "packages" or "groups of ten" (Cathy and Philippe).

> Cathy (Logs problem): I will start to make groups, groups of 34 and then I will remove until 27 remain.
>
> Philippe (Logs problem): [puts tokens on the table] I will make logs, I need 34, I did one ten, I am doing the other ten … and 4 ones. I did it in groups of ten.

The influence of recent class work is likely the cause. The students presume that they need to use what they recently learned in class about grouping. They are not too critical of this knowledge in relation to the actual task.

4.1.1.3 Mathematizing

In the first interview session, I identified four types of mathematizing that students possibly used: mimicking, equation, tacit mental modeling and object modeling.

Mimicking

The main strategy that students used to solve problems, which we can qualify as modeling, is that of mimicking the story: following the flow of the story or the order in which the information is presented in the task and adding or removing objects (tokens, circles). When analyzing this type of production, we cannot say whether students model the problem or the calculation (numbers and operations), as it is the same process for them. Therefore, this modeling process cannot be distinguished from the calculation process.

This type of mathematizing is not modeling per se, as there is no clear distinction between the model and the calculation plan. This method can help students formulate the appropriate mathematical expression as a calculation plan, but only for problems where the unknown is the final state (such as in the Tokens problem). In the case of the Logs problem, most students described their solution as "removing logs until 27 are left." No one actually transformed the structure “34 - ? = 27” into the expression “34 – 27 = ?”. We can say that mimicking as a way of mathematizing does not allow this transformation.

Tacit mental model

It is very difficult to know whether students modeled the problems mentally prior to proposing solutions. In most cases, they explained their solutions using mimicking strategy, and thus the structure of the solution did not differ from the structure of the problem. One student (Rosa) explicitly summarized a problem (Pencils) before explaining her solution. She explained it using data elements in an order different from the one given in the task. It seems that she constructed a tacit mental model that might have been holistic and different from a Sequential mental representation. Unfortunately, we cannot describe this model in more detail because the only way for Rosa to communicate her reasoning was to explain her calculation process: the drawing and counting process. It appears that Rosa used her holistic mental representation as a model to construct her solution.

It is possible that other students created tacit mental models as well and then derived their solutions from these models. However, this hypothesis cannot be confirmed based on my observations.

Mathematical equation

One student (Viktor) constructed an equation (13 - 7 = 7) to express his understanding of the Tokens problem. This can be interpreted as a mathematical model because it was an explicit step that was independent from the calculation process and occurred before the calculation. At the same time, this model (equation) was not transformed to derive the calculation plan, because the transformation was not really needed. The tokens problem is a Change problem where the result is unknown. This mathematizing method was only observed in one case in the whole study.

Objects model

One student (Philippe) used tokens to represent and solve the Logs problem. He set out 34 tokens and divided them into two groups: 27 and "others." This display of tokens can be interpreted as an explicit model of the problem from which the student derived the result. He counted the "other" tokens to determine the unknown number. Philippe did not explicitly describe a calculation plan. Thus, he used his model to derive the numerical result and not the calculation plan (no arithmetic operations or equivalent actions).

Another interpretation could be that he created a tacit holistic mental model of the situation, which allowed him to transform the story's sequence of events (34 - ? = 27) into a different calculation sequence (34 - 27 = ?) that he implemented using tokens. In this interpretation, what Philippe did with tokens should be seen as a calculation plan derived from a mental model. However, he failed to explain his calculation plan. Instead, he said that he removed 7 tokens. This leads me to suggest that the first interpretation is more plausible.

4.1.1.4 Summary of the first session

Table **8** of Appendix 1 contains a summary of the observations and inferences made during the first interview session (number of observations/cases = 27).

From the first interview session's observations, we can see that the majority of students view tasks sequentially and use the mimicking process to go directly from the text to the numerical answer. This way of mathematizing might be provoked by their presumption that they **need** to draw and count to solve a problem. There were three cases in which this presumption was not critically balanced with the mathematical realities of the task and the students' abilities. In all cases of mimicking, we cannot identify the mathematical model potentially present in the problem-solving process. In a same student's production, drawing and manipulating represent two things at the same time: their vision of the problem and their calculation process.

Figure 18 presents possible ways students mathematized problems during the first interview session. This picture is an approximation, because in some cases, different interpretations were possible.

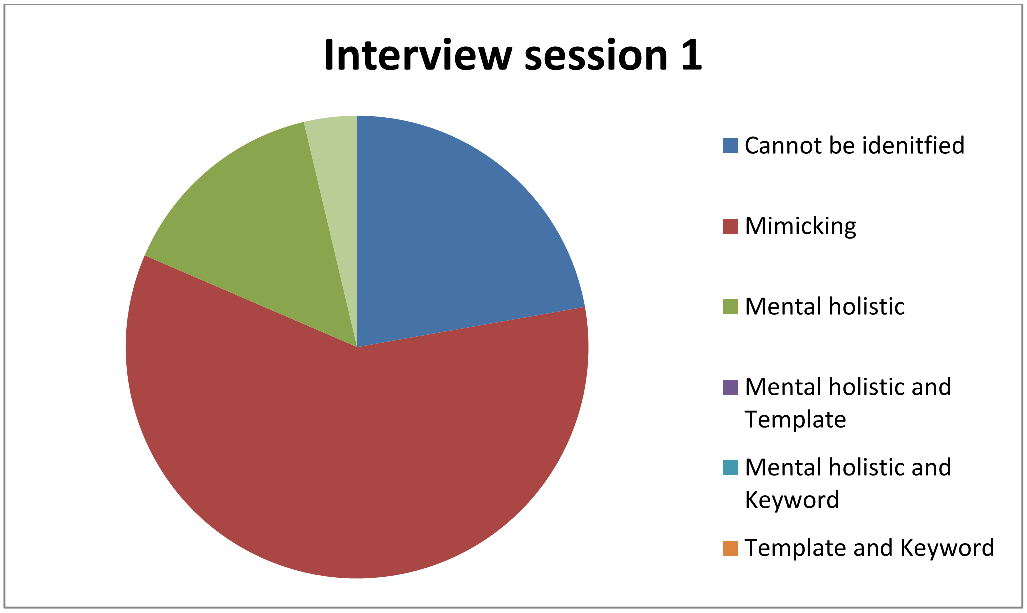

Figure 18. Ways of mathematizing in the first session

Students' difficulties

Several students had difficulty analyzing the Pencils problem and proposing a solution plan (n = 5). This problem is based on two superposed Combine structures and is generally more difficult for students. It is difficult to describe the mental representation in these cases because students did not express coherent opinions in this regard. However, there are several observations from these cases that can be valuable.

Borrowing from larger social context

I observed students having great difficulty with the Pencils problem and trying to add extra information to the situation. The information they used was taken from their classroom context.

> Maria (Pencils): It does not work ... Because there are 21 desks and 34 boys. ... You need to have 2 per desk.
>
> Emma (Pencils): [We need to] divide the blue pencils equally into two parts, for boys and girls, then red ones. If there is anything left, give it to Jeremy.

We can interpret these statements in two ways. It is likely that the social aspect of the situation dominated the students' reasoning (affective plan), and that "2 per desk" and "equal share" therefore became more important than the information in the task. It is also possible that the students failed to mathematize the problem in any form (cognitive plan) and returned to the larger (social) context in an effort to get more information. In my study, this behaviour was only observed in cases where students had difficulty mathematizing the problem. Thus, the second interpretation appears more viable. Both students were not very critical of the information they took from the social context, so their misuse of the social context became visible.

Quantity of information

In some cases (n = 3) students (Nicolas, Rosa, Cathy) were able to keep multiple data in mind and correctly use portions of it one by one (sequential or tacit holistic mental representations). In other cases (n = 6), the quantity of information may have affected the students' ability to recall all the information needed for the solution.

> Maria (Pencils): I don't understand anything: How many boys were there? … I will take 48 ... oops, 46, they are blues. But I don't know how many students. So I take them and give to each student. It does not work? … I take this and I give to boys, so I give 34 to boys. ... It doesn't work."

Maria tried to apply the mimicking (drawing and counting) strategy, but it did not help her to make sense of the situation as a whole. She may have failed to retain and mentally manipulate the relatively large quantity of data.

4.1.2 Second session, January

The second interview session was conducted in January, a week after the winter break. To probe students' reasoning in problem-solving situations I used the Houses problem (Combine, part unknown), Cards problem (Positive Change, change unknown), and Fruits problem (Comparison, inconsistent language).

> **Houses**: There are 64 houses on our street. Some houses have 1 storey. The other 37 houses are 2- and 3-storey houses. How many 1-storey houses are there on our street?
>
> **Cards**: To organize a school party, Léa must send invitations to 63 people. Before lunch, she prepared 28 invitations. After lunch, she prepared more invitations. She now has 52 invitations to send. How many invitations did Léa prepare after lunch?
>
> **Fruits**: There are 45 apples, 34 oranges and some pears in a basket. There are 17 more apples than pears. How many pears are there?

Thus, all types of structures were used. The Houses problem in its original version appeared to be very difficult for students. To deal with this situation, I additionally proposed a simplified version of the problem for some students. Example:

There are 8 houses on my street. Some of them are small. The other 3 houses are big. How many small houses are there?

In some cases, I used the Ski problem (Combine and Compare, both quantities unknown) to challenge stronger students with something more difficult. All students were asked to present their solution plan orally and execute their plan in the event that they were unable to explain.

A total of 22 observations were analyzed.

4.1.2.1 Mental representation

Sequential representation

As in the first interview session, many students used or proposed drawing and counting circles or objects to solve problems. However, in only two cases (Michael and Cathy) can we interpret the solution plan as being based on a Sequential mental representation of the problem (Cards). Example:

> Cathy (Cards): I think we need to do 28 cards and then count up to 52. This way we will be able to see how many she did. … I will draw circles until I get to 52.

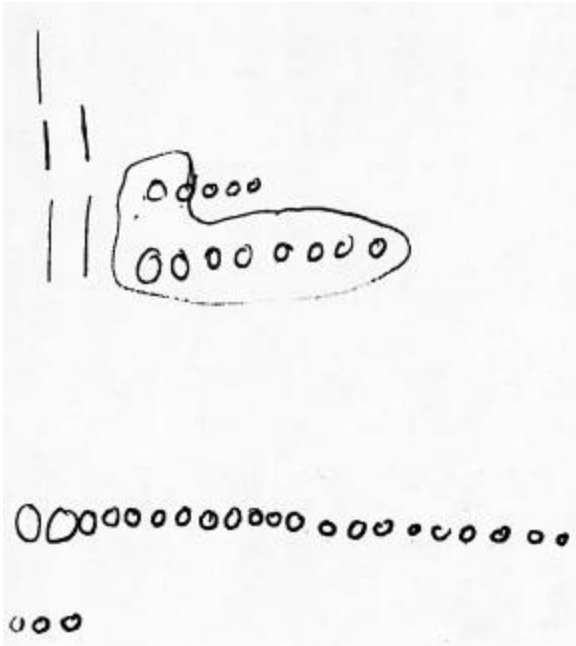

Figure 19. Cathy’s solution to the Cards problem

Just like in the first interview session, we can say that these two students used the story itself as a mental representation. It is possible that the draw-and-count presumption still affected students’ reasoning and solving strategy.

In many other cases where drawing circles was observed, other mental representations were inferred.

Volatile mental representation

A volatile mental representation was identified in two students’ productions for the Fruits problem. Below, I propose an interpretation of one of these observations: the case of Nadia.

At first, it seems that Nadia grasps the qualitative relationship described in the problem well. She repeats many times that there are 17 more apples than pears and explains that it is impossible to have more pears than apples. This can be interpreted as a holistic relational mental representation of the problem. However, the final solution plan she proposes does not correspond to this mental representation.

> Nadia (Fruits): I will draw 45 apples, and then they say there are 17 apples more, so I will draw 17 more apples.

This solution plan can be interpreted as having been derived from a Sequential representation. We can also say that Nadia did not use her mental representation of the problem when proposing a solution plan. Instead, she followed her presumption about drawing and counting and used the keyword "more." Any of these two interpretations suggests her reasoning about the problem is fragmented, disrupted and volatile—it changes easily based on the circumstances.

Holistic representation

In seven cases, we can identify students' solutions as descriptions of a calculation process and not as a chronology of the situation: two out of three cases for the Ski problem, three out of seven cases for the Fruits problem and two out of seven cases for the Houses problem. In all seven cases, it is possible that the students summarized the situation for themselves prior to proposing a calculation plan.

> Michael (Fruits): I will draw 45 apples and then I will make 17 lines [moves his finger as if to cross out] and I will see how many pears.

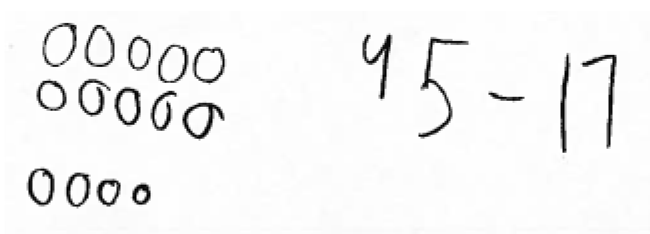

Figure 20. Michael's solution to the Fruits problem

Michael describes the calculation process by saying "I will draw 17 lines," which means crossing out or taking away 17. The problem states: "17 more apples than pears." According to Nesher et al., students who can read "more" and perform subtraction are certainly able to see the structure of the problem as a whole and in a flexible way.

> Nicolas (Ski, after some discussion about the "more" expression): Half will be 27 ... then we will take 15 from the boots, 27 - 15 = will be the number for the boots. So the boots cost 12. For the ski, we will take 27 and add 15. This will give me the answer.

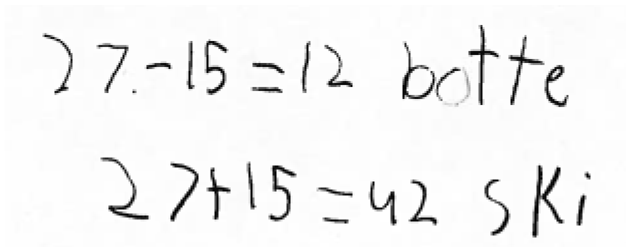

Figure 21. Nicolas's solution to the Ski problem

Nicolas tried to share the total price between ski and boots at the same time and obtain the required difference between the two prices. We can therefore conclude that, prior to proposing a solution strategy, he considered the complex structure presented in the Ski problem as a whole.

> Emma (simplified version of the Houses problem): We can draw eight houses. Then we need to circle three houses and count the remaining small houses.

Emma transformed the Combine problem, which is a **state** problem (no action mentioned) into a calculation **process**. No student did this transformation for the full version of the Houses problem. Two students successfully solved the simpler version of the problem. It is evident that the small numbers and simplified wording of the problem allowed the students to see the problem in a holistic way and transform this vision into a calculation process.

In none of the seven cases did students explicitly express their understanding of the quantitative relationships they discovered in the problem. We can only judge their understanding based on their solutions. Once again, we should point out the lack of adequate communication tools used to express the vision of the quantitative relationships they used to find a solution strategy.

Emergent mental representation

A new way to solve problems—using an AA diagram—was observed during the second interview. Only one student used this approach to solve the Fruits problem.

> Maria (Fruits): The oranges, we do not need them in the problem. There are apples and pears that are good in the problem. … You can draw 35 apples … oops, 45 apples, and

after I will count how many, ummm, 17 apples more, so I will count how many, ummm, how many, ummm, how many, ummm, pears. If I use strings it will be easier.

Maria: [Draws the first line] This will be the apples, because I know that there are more apples. [Draws the second line smaller] This is the line for pears. They say there are 45 apples and ummm 17 apples more in the basket. ... I don't know really... I know that this [shows the difference part on the diagram] is the difference.

Researcher: Do you know the difference?

Maria: No.

Researcher rereads the text to clarify the “17 apples more” expression.

Maria shows 45 as a part of the apples’ line. Answering the researcher’s question, she says that 45 is the whole line. Answering the researcher’s question about pears

Maria says: We know that this is equal to this [shows the shorter segment (pears) and the part of the longer segment (apples)] so it is equal to 45.

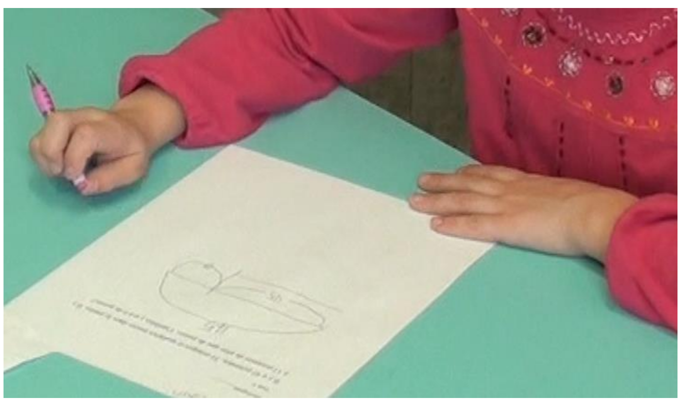

Figure 22. Maria solves the Fruits problem. In her drawing, the larger segment is marked “45,” and part of this segment is also marked “45.”

Maria first proposed drawing and counting the apples, but changed her strategy saying that it would be easier with strings. She tried to represent apples and pears as two segments of different lengths. Her diagram corresponded well to the Comparison structure of the problem. Maria had difficulty connecting her qualitatively correct representation with numerical data. For example, she correctly pointed out the difference between the two segments (apples and pears) in her diagram, but did not associate it with the number 17.

The representation Maria created for the Fruits problem can be seen as a model. After having created this representation, she used it to draw conclusions about the unknown. She said, “We know that this equals to this [shows the Pears segment and the part of the Apples segment with 45 written above], so it is equal to 45." Since 45 is visually present above part of the apples segment, which is equivalent to the pears segment, Maria used this drawing to conclude that there were 45 pears. The answer 45 is actually incorrect. This fact helps us to confirm that the student based her reasoning on her diagram and not on her previous understanding of the problem. Maria did not critically revise the information she derived from the diagram in relation to the initial task.

Maria tried to communicate her vision of the problem using the method of AA diagrams, which was new to her, and failed to make connections between the problem and the diagram. The idea of using diagrams (possibly a presumption that this method **should** be used) affected her reasoning about the problem and made her mental representation unstable and volatile.

Another interpretation could be that Maria tried to make sense of the problem by using the diagram, which she knows better, and which just occasionally corresponds to the problem's structure (Comparison). To verify this last hypothesis, I asked her to solve the Houses problem (Combine structure), also using a diagram. I advised her to think carefully about the type of diagram to use, because the new problem was different from the previous one. For the Houses problem, Maria used the same two-segment representation, which was incoherent with the problem's structure. This confirmed that her choice of diagram was affected by reasons other than the problem's structure.

Both times that Maria tried to use the diagram, she had difficulty coordinating the problem with the representation. This means that the graphic representation (diagram) she created was not just her vision of the problem. The representation method she chose (because of her presumption) affected the way she reasoned about the problem. We can say that what she ended up thinking about the mathematics of the problem emerged from the process of explicit modeling. The same can likely be said about the mimicking method observed in the first session. The students see the problem in a sequential way, partially because they presume they should do so, and the method of representation thus affects their reasoning. This corresponds to the model proposed by DeBlois (2011).

Misinterpretations (structure substitution)

In two cases of the Cards problem, students seem to misinterpret the problem structurally. In one case, it was difficult to interpret the student's explanations to identify her mental representation of the problem. In the other case, the student seemed to be looking for the number of cards to make to complete the job. The solution proposed by the student followed the chronology of the text and suggested removing cards already made from the number of invited persons.

Eva (Cards): We need to take 63 and remove 28 and 52 and this will be the number.

Eva proposed subtracting two known numbers from the big total. It is possible that she misinterpreted the problem as follows: "We need to make 63 cards. We did 28 in the morning and 52 in the afternoon. How many cards do we need to make to complete the job?"

As I explained earlier (section 2.3.3 of Chapter 2), students who misinterpret a problem structurally, possibly see a different problem in their head. They are sure that they understand the problem and propose an adequate solution. If we accept that Eva substituted the problem's structure, we need to interpret her solution as being derived from a sequential mental representation.

No solution

In 11 out of 22 cases, students did not provide coherent explanations, so it was not possible to interpret their mental representation of the problem.

4.1.2.2 Presumptions about the task

In seven cases during the second interview session, students continued to propose drawing and counting or manipulating objects as solution methods. It is possible that students didn't know other methods to obtain the numerical result. However, it is also possible that this behaviour is produced by the old classroom tradition (Jackson, Shahan, Gibbons, & Cobb, 2012) of representing numbers by small circles, institutionalized in Grade 1. This tradition was no longer promoted by teachers in Grade 2 classes.

There were only two cases where the participant behaved according to the new request of the teacher and used line diagrams. It is possible that the other students did not trust this new method.

In several other cases, students tried to directly propose the calculation algorithm expressed in numbers when they were able to do so. There are two possible interpretations of this behaviour. First, the explicit request for considering the whole problem with quantitative relationships, now promoted in class, brought about this new behaviour in learners in the problem-solving situation—that is, thinking out the whole problem prior to proposing the calculation plan. Second, the students always thought about the whole problem in conjunction with the quantitative relationships involved, but previous presumptions about the draw-and-count method inhibited the explicit demonstration of this reasoning by students, imposing the mimicking strategy. Thus, when teachers no longer required drawing and counting, the learners behaved according to their natural way of reasoning.

4.1.2.3 Mathematizing

The mimicking method of mathematizing the task, described in section 4.1.1, continued to be observed (2 cases). At the same time, in 12 out of 22 cases, students failed to propose a clear solution plan. This led us think that at least some of the students who failed possibly tried to mimic the problem, but that this way of mathematizing was inadequate for or difficult to apply to the problems of this term. Other ways of mathematizing inferred were: incorrect use of keywords, using a holistic mental representation and using AA diagrams.

Incorrect use of keywords

In one case (Nadia solving the Fruits problem), the solution the student proposed and her explanations led us to think that she used the keyword “more” to derive her calculation plan and saw it as an operation, not a relationship.

> Nadia (Fruits): I will draw 45 apples, and then they say there are 17 apples more, so I will draw 17 more apples.

As explained above (p. 106), the calculation plan does not correspond to her general understanding of the problem as a comparison situation. I will further discuss the disruption phenomenon later in this chapter.

Tacit holistic model

In seven cases, students' solutions differed in order from the initial text of the problem. As was described in the term 1 analysis, the students mentally represented the problem in a holistic manner and this helped them to manipulate the structure and/or create a calculation plan (Nesher et al., 1982). We can say that this mental model is their way of mathematizing problems. Their mental model cannot be interpreted further because of lack of convenient communication methods. Students did not master the AA diagram method and did not use it to communicate their understanding of the problem.

AA diagrams

During the second interview session, only one student spontaneously used the AA diagram method to represent and analyze problems (Maria, Fruits and Houses). As described above when I discussed emergent mental representations, Maria constructed her mathematical understanding of problems through the process of representation. We can say that the use of AA diagrams is her way of mathematizing problems.

Maria inappropriately used the same type of diagram for two different problems. At a closer glance, the question becomes whether Maria constructed her diagrams based on the information from the problem and just failed to do it properly, or used the same type of diagram for another reason. Both interpretations are possible. In the first interpretation, we could say that the student **constructed** the diagram to mathematize the problem. She likely failed to construct a different diagram for the Houses problem because of her lack of knowledge about this type of problem. The second interpretation is that Maria used the two-segment diagram because she presumed that this was the **only way** to mathematize problems. In this interpretation, we could say that she used the diagram as a **template** and tried to fit the problem into the diagram.

4.1.2.4 Summary of the second session

Table **9** of Appendix 1 contains a summary of the observations made during the second interview session (number of observations/cases = 22). Figure 23 presents an approximate image of the second interview session relative to the type of mathematizing students used.

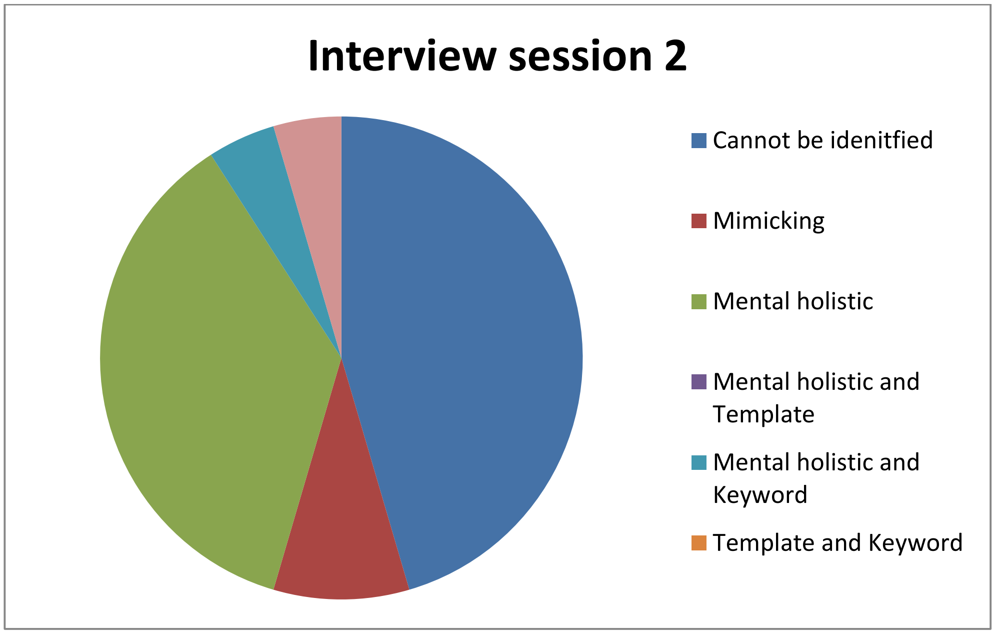

Figure 23. Session 2, summary of ways of mathematizing

During the second interview, a variety of mental representations emerged: two cases of sequential, seven cases of holistic relational and two cases of volatile. The second interview session analysis shows that the students' presumptions about what should be done to solve a problem changed. The majority of them did not use mimicking and draw-and-count strategies anymore. The students who constructed a holistic mental representation of the problem used this representation as a model. The model was hidden from the researcher because the students did not use any communication methods to clearly express their understanding. However, the structure of the solution plans they proposed differed from the chronology of the problem.

Only one student explicitly and spontaneously used AA diagram method and used it to derive a solution plan.

In 10 cases, it was difficult to interpret students' productions. If we take into consideration the amount of mimicking present during the previous (first) term, their failure to construct a sound explanation can be explained by a contradiction between the nature of the problems (the mathematical structures) and the way the students tried to think about them (most likely sequentially).

In one case, various parts of a student's production were not coherent with one other. We can conjecture that the student's mental representation of the problem was volatile, composed of disrupted parts. This means that the student likely tried to use multiple different (partial) ways to mathematize the problem. One of these ways is a mental holistic representation of the comparison relationship. The other way is the use of the draw-and-count method applied in a sequential manner.

Students' difficulties

In many cases, problems seemed to be too difficult for students, and their previously learned strategies of mimicking, drawing, and counting did not help them to understand the problem.

Long and complex determinants

One problem seemed to be particularly difficult for many students: the Houses problem. Students mainly struggled with the long and complex compound adjectives in the data description: "1-storey houses" and "2- or 3-storey houses." For example, some proposed to count storeys instead of houses. Later on in the interviews, the researcher used "big houses" and "small houses." This also created confusion in some students. They thought that the number of small houses was fewer or used a shorter line to represent the quantity of small houses. This shows that the purely linguistic characteristics of a text can have a significant influence on students' mathematical interpretation of a problem. Complex linguistic structures affect the structural perception of the text that they construct.

Hidden structures

Two of the problems used in the second interview session had a hidden additive structure, in addition to the one intended by the task: the Fruits and Cards problems. Each of them has additional data: 34 oranges in the Fruits problem and 63 cards to be made in the Cards problem. Each of these numbers can potentially be related to other data via an additive structure, different from the problem's actual structure. In the Fruits problem, the total number of fruits, which is unknown, is equal to the sum of apples, oranges, and pears. This would be a Combine structure with two unknowns: the total and the pears. The intended structure of this problem is Comparison (apples and pears). In the Cards problem, two different structures are present: 1) all the cards that need to be made (63) are equal to the sum of the cards already prepared (52) and the cards yet to prepare (Negative change, result unknown); 2) the prepared cards (52) are equal to the sum of cards prepared in the morning (28) and the cards prepared in the afternoon (Combine, part unknown).

These two problems were also difficult for some students. Three students solving the Fruits problem and three solving the Cards problem first focused on the hidden structure instead of the intended structure of the problem.

It would appear that the difficulty in the Fruits problem and Cards problem was created by the presence of extra data. However, the case of the Logs problem (first interview session) shows that the element of "48 minutes" did not confuse the majority of students. This means that the difficulty does not come from the fact that extra data is present, but from the fact that this extra data creates additional relationships with other data. Thus, the solver needs to deal with two structures instead of one.

4.1.3 Third session, March

This interview session was conducted in March. For this set of interviews, I used the following problems: Snowballs 1 (Negative change, initial state unknown), Snowballs2 (Comparison, compared unknown, consistent), and Snowballs 3 (Comparison and Combine).

> **Snowballs 1**: Tomas made a bunch of snowballs for his snowball fight with his friend Greg. After 15 minutes of fighting, Tomas had thrown 37 snowballs. He now has 25 snowballs left. How many snowballs did Tomas make?

> **Snowballs 2**: Tomas made a bunch of snowballs for his snowball fight. His friend Greg said that he made 17 more snowballs than Tomas. Tomas counted his snowballs: there were 58. How many snowballs did Greg make?
>
> **Snowballs 3**: Tomas made a bunch of snowballs for his snowball fight. His friend Greg said that he made 17 more snowballs than Tomas. Tomas threw all his snowballs at Greg's fort. 28 snowballs hit the target. The other 33 missed. How many snowballs did Greg prepare?

One student was asked to solve the Houses problem (Combine), which she had not seen in the previous interviews. Another student was asked to solve the Tokens problem (easy) to encourage her participation in the dialogue.

In the previous two interview sessions, I observed and interpreted students' reasoning, which I identified as volatile. These observations led me to propose a hypothesis: Students' production, and even their reasoning about a problem, can be affected by the communication tool they choose or are asked to use. To test this hypothesis, I decided to modify the way I conducted the interviews. I added two questions: Can you provide a mathematical expression? Can you represent the problem the way you usually do in class?

These questions were not appropriate for the previous sessions because the notion of "mathematical expression" as a calculation plan and "representation" as a quantitative relationship had not been worked on in class. At the time of the third interview session, these questions could be used as additional tools to obtain a more precise picture. Like in the first two interviews, students were invited to explain their solution plan as they wished. Only after were the additional questions asked, if needed.

4.1.3.1 Mental representation

Mental holistic representation

During the third interview session, a mathematical expression or verbal explanation of the calculation with numbers was the most popular way of explaining the solution (n = 12). This happened in four cases for the Snowballs 1 problem, five cases for the Snowballs 2 problem, two cases for the Snowballs 3 problem and one case for the Tokens problem. Here is an example:

Philippe (Snowballs 1): 37 …Yes … No, he threw 37… At the beginning, he threw 37 and he still has 25. He threw 37 in 15 minutes, and 25 remain. [Thinks in his head] 69?
Researcher: How do you know?
Philippe: I calculated. Instead of calculating on my fingers, I did three plus two equals five. If I take 59, it will not fit because I have 7 and 5. This will be bigger than 59."

Philippe repeated the elements of the situation aloud and immediately started to calculate in his head. He explained how he added tens and units mentally. The operation he used was 37 + 25, which is correct.

Two more students explained that they would "make 37 balls" and then add 25.This can be interpreted as a reference to the draw-and-count strategy. For example:

Michael (Snowballs 1): We make 37 balls, and then we add 25, and then we count them and we write the answer.
Researcher: Can you write the mathematical expression?
Michael: 37 + 25

In 10 out of 14 cases in which students explained the calculation to use or gave the mathematical expression, it seemed that they had summarized the problem for themselves and/or created a holistic mental representation prior to giving their explanations. In all 10 cases, the order of numbers and/or operation used were not the same as in the text of the problem (Nesher et al., 1982). Without additional communication, it was impossible to further clarify their mental representation of the problem.

To further probe their understanding of the problem, I asked students to represent it using the AA diagram method. In eight cases, students created adequate diagrams (three cases for Snowballs 1, four cases for Snowballs 2 and one case for Snowballs 3). The following example confirms the inference about students' full mental representation of the problem.

Josef (Snowballs 1): We need to draw cords to write the mathematical expression. [Writes: 37 - 25 …] Oh no … 37 + 25 … I am sure because at the beginning he took … Oh no, it should be subtraction … We need to do + because he made more balls …

Researcher: Think carefully, read the problem one more time.

Josef [after reading the problem]: Yes we need to do + because we need to know how many there were at the beginning.

Researcher: Well, can you represent it now?

[Josef draws a segment.]

Researcher: What does that represent?

[Josef continues with the diagram, adds a separation point and draws tree arches. He puts 37 over the left part of the arch and 25 over the right part.]

Researcher: What are you looking for?

Josef: How many there are. How many balls he made at the beginning.

Researcher: Put the question mark please.

[Josef puts the question mark on the arch's total.]

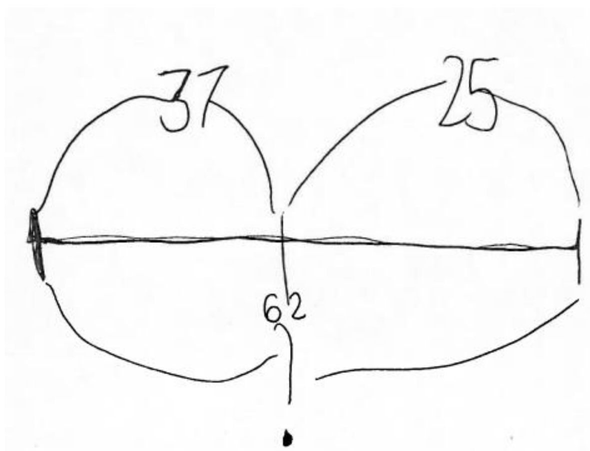

Figure 24. Josef's solution to the Snowballs 1 problem

In four other cases, the diagram students created upon request did not correspond to the problem's requirement and they had difficulty connecting the diagram to the problem. Example:

Cathy (Snowballs 2): You take the 58, you add 17, you see how many. [Makes a tens representation to calculate. Calculates the answer.]

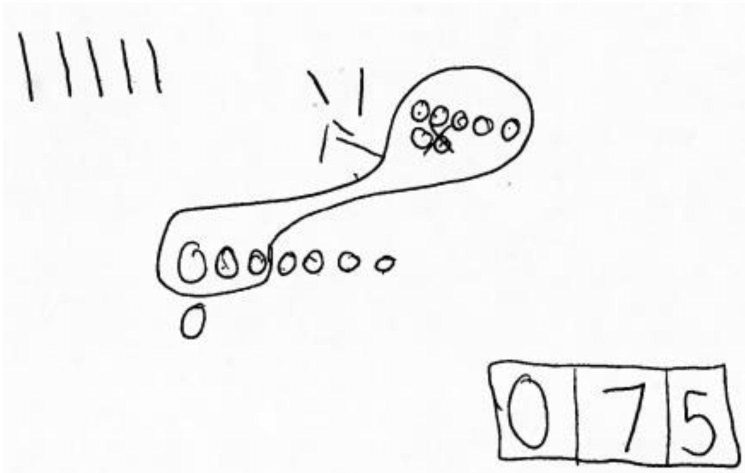

Figure 25. Cathy’s solution to the Snowballs 2 problem

Researcher: Can you represent this as you did in class?

Cathy: Yes. [Draws a segment and makes a separation point, puts question mark on the total, writes 58 above the left part and 17 above the right part]

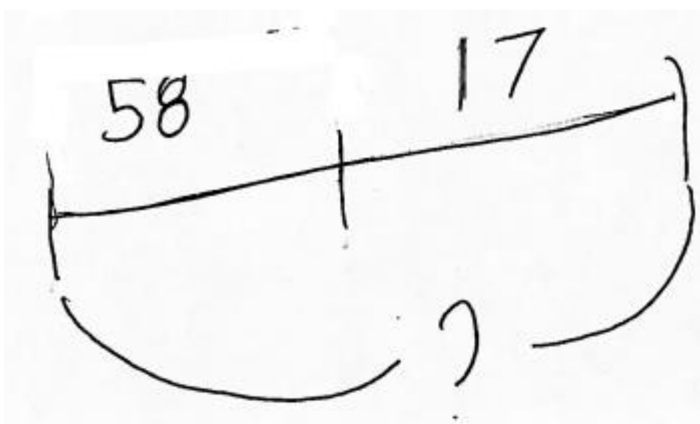

Figure 26. Cathy’s representation using AA diagram

Researcher: Can you explain?

Cathy: This [points to 58] is what Thomas made. This [points to 17] is what Greg ... what Greg made more than him. We are looking for how many in total.

Researcher: [Points to the left part] Are these the made by Thomas or by Greg?

Cathy: Made by Thomas.

Researcher: These? And where are Greg's snowballs?

Cathy: [Thinks] But we don't know how many?

Researcher: I am not asking you how many. I am asking you where Greg's are represented.

Cathy: [Points the total line] Here.

Researcher: [Points the whole line] Is it all of this? But you said that this part [points to the left part] is for Thomas.

Cathy: Yes. And these [points to the right part] are Greg's.

Researcher: Is it just this for Greg?

Cathy: Yes.

Cathy proposed how to calculate the answer to the problem and executed this plan using a tens/units representation. The operation "58 + 17" that she used is correct. Answering my question, Cathy drew a single-segment diagram which did not correspond to the Comparison structure of the problem, but corresponded to the calculation plan well. She then had trouble coordinating the diagram with the problem. One possible interpretation is that Cathy had an incorrect understanding of the problem. For example, she could interpret the problem as a Combine situation, where the total of Thomas and Greg's snowballs is requested. In this case, the diagram and calculation plan would correspond perfectly to this understanding. However, we see that Cathy mentioned "This [pointing to 17] is what Greg ... what Greg made **more** than him [Thomas]." This statement indicates that she was aware of the comparison described in the situation. The second interpretation is that Cathy had an appropriate holistic mental representation of the problem, but when constructing a diagram, had trouble with the diagram method itself as a communication tool. She failed to communicate her understanding of the problem through a diagram.

Volatile mental representation

In order to further analyze students' reasoning, I asked students to write a mathematical expression and represent the problem the way they did in class. In eight cases, students did it spontaneously without being prompted. This helped me to observe multiple cases (n = 12) where different parts of students' production did not correspond to each other. Example:

Maria (Snowballs 1): I know that he used 37 balls and 25 remain, so we need to put the two numbers together to figure out how many there were at the beginning.

Researcher: I would now like you to write the mathematical expression and then draw to represent the problem.

[Maria writes “37 – 25.”]

Researcher: You just said that we need to put two numbers together.

Maria: Yes, that’s true [puts + sign].

Researcher: I will ask you to draw your representation and then we will return to the mathematical expression.

Maria: I think it is minus.

[Maria draws a big segment with an arch.]

Maria: There are a number of balls. [Rereads the problem]

Maria: I do not understand ... There is no total number.

Researcher: Yes. That is what we are looking for.

[Maria put an arc on the first part of the line.]

Researcher: What is it?

Maria: This is at the beginning, what he threw.

Researcher: And how many are there?

Maria: Thirty-seven. And only 25 balls remain. [Puts 25 above the second part]

Researcher: And the question mark?

Maria: Here. [Puts question mark on the total part]

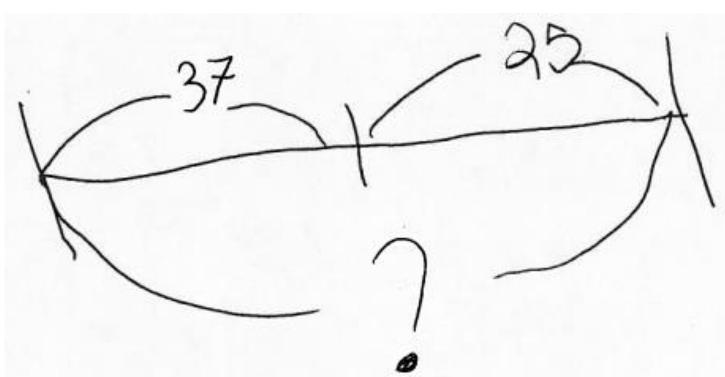

Figure 27. Maria’s solution to the Snowballs 1 problem

Maria: We are looking for how many balls there are in total.

Researcher: And what should we do to find it? What can be the mathematical expression?

Maria [showing the two numbers on the diagram]: 37 minus 25 is equal to the answer?

Maria first summarized the problem and proposed putting two numbers together, which is correct. She then proposed a mathematical expression where one number is subtracted from another. Answering my question, Maria drew an AA diagram in which she clearly, and correctly, identified the balls thrown and balls remaining as two parts of a total. However, she insisted on using subtraction in her mathematical expression. What could be the mental representation Maria had of the problem? Does this mental image correspond to the diagram Maria drew or to the mathematical expression she proposed?

We can assume that Maria had some understanding of the problem, but her productions (oral explanation, diagram and mathematical expression) varied and depended on the form of communication requested from her. For some reason, these productions became disjointed and uncoordinated. It seems that the diagram corresponded well to the initial summary Maria made. The mathematical expression was the product of a different thought process related to the formalization of the expression "threw out," which she transformed directly into a subtraction operation.

Emergent mental representations

In eight cases, students chose to start their explanation by drawing a diagram. These were three cases for Snowballs 1, three for Snowballs 2, one for Snowballs 3, and one for the House problem. However, in only three cases did the constructed diagram, explanations and mathematical expression correspond to each other. Example:

> Eva (Snowballs 1): [Starts to draw a segment diagram. Draws one big segment.] I draw all the balls …
>
> Eva: [Draws an arch over the left part] This is the 15 at the beginning.
>
> Eva: [Draws another arch over the right part of the segment] This is the 25 remaining.
>
> Researcher: Please, one more time. What is 15? [Pointing on the left part]
>
> Eva: I made a mistake, it should be 37. … 37 are the balls remaining … the balls that Thomas threw. Others are what remains. We should find the total. Eva draws an arch for the total

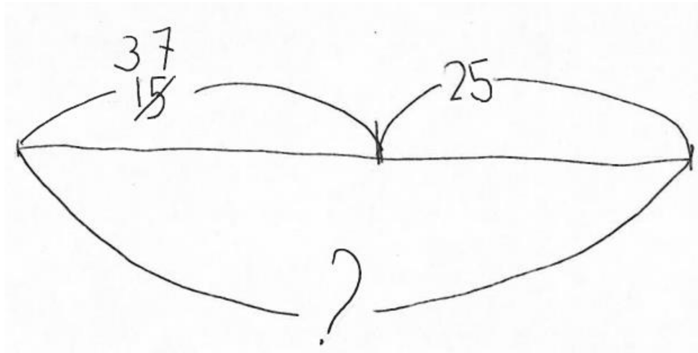

Figure 28. Eva's solution to the Snowballs 1 problem

Researcher: What next?

Eva: The mathematical expression. [37 + 25]

In the example above, it seems as though the student constructs her mental representation of the problem in parallel with the diagram. Eva transformed the Negative-Change structure of the problem into the Combine structure of the diagram. Then, she used the diagram to construct the mathematical expression. Every part of her production was well coordinated because they were the result of the same continuous thinking process. We do not know what mental representation Eva had after reading the text and before starting the diagram. We might conclude that the holistic relational representation she had at the end emerged from the process of explicit modeling. Similar observations were made in two more cases.

In the other five cases, the diagram that students constructed did not correspond to the mathematical structure of the problem, the mathematical expression or the explanations students gave orally. In these cases, we might say that their mental representation was fragmented and volatile (see the previous section).

4.1.3.2 Students' presumptions about the task

The most popular presumption that can be inferred from the students' comments is well expressed by Josef (Snowballs 1): "We need to draw cords to write the mathematical expression." This means that to solve a problem, students need to represent it using the AA diagram method and then deduce mathematical expressions from it as a calculation plan. This corresponds to the teaching approach being implemented in class.

From my observations (n = 5), some students started their solution by drawing a diagram, but did not use the diagram to derive a calculation plan from it. One of the possible interpretations is that the students presumed that the diagram was required, but did not really need it for their reasoning. The other possibility is described above as volatile mental representation.

Not all the learners demonstrated a presumption about diagrams. At least one student (Philippe) immediately tried to calculate mentally or guess the number. We can suggest that Philippe presumed that the numerical answer was most important. Many other students provided their calculation plan first and drew a diagram upon the researcher's request only. Very few students occasionally started to draw circles.

It seemed that learners had started to grasp the new way of problem solving. Taking into consideration the multiple difficulties learners have with diagrams, we can imagine that many of them did not rely on this new tool and preferred to work mentally until the calculation plan was clear for them.

None of the students really used circle drawings to understand the problem. We can say that the draw-and-count presumption was no longer present.

4.1.3.3 Mathematizing

Mimicking

During this session, only one student mimicked the Tokens problem using her hands to understand the situation. It did not help her with the mathematical expression. Mimicking as a way of mathematizing was discussed in section 4.1.1.2 of this chapter.

Using diagrams as templates

In section 4.1.2.3, I proposed that students can use diagrams differently. In eight cases during the third session, students used diagrams in a special way. They first drew segment(s) without any indication of how the segments corresponded to the problem. Sometimes, the diagram corresponded to the need of the problem, sometimes it did not. Sometimes students were able to correctly identify the diagram's parts, sometimes they were not. What was common for almost all the students was the way they drew their diagram. Almost all of them used the one-segment diagram independently from the actual mathematical structure of the problem at hand. They always started with the big segment, then put a separation point and drew three arcs. Only then did they try to put numbers on the diagram. It looked like they were trying to use a template they recently worked on in class. They did not create a representation of the problem, but tried to adjust the template to the problem at hand.

Below, I interpret the example of Emma and the Snowballs 1 problem to better describe this.

> Emma (Snowballs 1): At the beginning we don't know because they say here that he made many, but they don't say how many he made in total. After, he threw 37, there were already 37, then he removed, because he threw them, he removed them. So we don't know how many are there ... in total.
>
> Researcher: What else do we know?
>
> Emma: There are 25 remaining. We don't know how many in total.
>
> Researcher: So, what should we do?
>
> Emma: We will take all ... I will draw. [Draws a big segment and makes a separation point. Makes arcs above right and left parts, then under the whole line. Starts to represent the numbers as tens and units.]
>
> Researcher: Please explain what you drew
>
> Emma: [Touching the segment] In total ... We do not know how many in total.
>
> Researcher: Can you put the question mark on what we do not know?
>
> Emma: [Puts the question mark on total] There are already 37.
>
> Researcher: Where are they?
>
> Emma: [Points to the left part] Here. [Writes 37] Then 25, we removed them, so it goes here. [Writes 25 above the right part]
>
> Researcher: What should we do now?
>
> Emma: Like, I draw 3 tens ... I will do the mathematical expression ... I will write “37 - 25 =” and this will give the answer.
>
> Researcher: Is it minus you will do?
>
> Emma: Plus?
>
> Researcher: What do you think?

Emma: Minus, because he removed. Because when he throws, it makes minus.

First, Emma explains her understanding of the problem and it seems to be quite adequate. Then she tries to draw a diagram. She draws a template of a one-segment diagram without numbers. **Immediately afterwards**, Emma starts her calculation using tens and units representation. It seems that Emma does not use the constructed diagram to continue her reasoning. Upon my request, Emma correctly points out that the total is unknown and identifies two parts with appropriate numbers. However, she names the left part as "**There are already** 37" and not as "balls thrown." She also explains "Then 25, **we removed them**, so **it goes here** [on the diagram]." The diagram constructed seemed to correspond to the need of the problem. Yet, there were some indications in Emma's explanations that she used it as a template and, upon my request only, she fit numbers from the problem into this template. Emma's reasoning about this diagram is dissociated with her oral explanations given before. The same can be said about her mathematical expression because it does not correspond to the diagram or to her explanation of the problem. Later she explains: "Minus, because he **removed**. Because when he throws, it makes minus."

Mental modeling

In 16 cases, students started their explanation by giving their calculation plan or a mathematical expression. The correct mathematical expressions, which differed from the sequential flow of the data in the problem, allowed me to infer that the solvers used a holistic mental model of the problem prior to proposing their solutions. They derived their mathematical expressions and explanations from this model. Without appropriate communication tools, it is impossible to describe these models (see the case of Cathy on the pages 119-120). Responding to my request, students tried to draw an AA diagram to show their hidden understanding. However, the process of diagramming itself affected their reasoning, so the diagrams they created cannot be seen as reproduction of their previous thinking, and certainly not as their mental model.

Incorrect keyword use

Similarly to the second interview session, in four cases of the third session, the mathematical expressions students proposed did not correspond to their oral explanations, the problem or the diagram. Some students clearly explained that they used “minus” because something was “thrown” in the problem. This means that the mathematical expression student proposed was not logically derived from the mathematical model of the problem (if any), but lexically from the text of the task (see the case of Emma on the pages 125-126).

AA Diagram modeling

In only three cases could the observations be interpreted as explicit modeling using the AA Diagram method. The students gradually constructed and explained their diagram, then derived a mathematical expression or answer from the diagram. This shows: 1) the process of diagramming served students to mathematize the problem; 2) the diagram itself served as a model for drawing conclusions about the calculation needed (see the case of Eva on the page 123). Although diagrams and conclusions were incorrect in some cases, the process of explicit modeling was clearly observable.

Use of multiple ways of mathematizing

In the third session, some students seemed to use multiple ways of reasoning for the same problem, as had already been observed in the second session. In some cases, the whole process of solving seemed to be a dissociated set of productions: oral explanation, adjusting of the diagram template, translating the keyword from the text into the arithmetic operation (see the case of Maria on the page 122). During this session, multiple cases of this type were observed. I will come back to this phenomenon later in the chapter.

4.1.3.4 Summary of the third interview session

Table **10** of Appendix 1 presents observations from the third interview session (Number of observations/cases = 23). The third interview session is characterized by the variety of communication forms/tools students used to show their reasoning about problems.

During the third interview session, in more than half of the cases, students used a mathematical expression to express their solution (n = 12). They tried to understand the problem working in their minds and then directly derived a calculation plan or a mathematical expression. In one case, a student used her hands to make sense of the problem. In the other cases (n = 10), students used diagrams first and only then proposed a mathematical expression. They chose the diagram approach although they had not yet mastered it. It appeared that students no longer consider the draw-and-count strategy to be an option.

From my analysis, it follows that in at least 10 cases, students used a mental holistic representation of the problem as model to derive a solution plan. In three cases, it seemed that the students' mental representation of the problem emerged from the explicit modeling process. In nine cases, parts of the students' production (explanation, diagram, and mathematical expression) were dissociated and uncoordinated; the students' mental representation of the problem was volatile.

This important dissociation of students' production observed in the third interview session can be explained in different ways. First, each part of this production can be associated with a different mental representation that a student has of a problem. In this case, it is difficult to explain the relationship between these different representations and their simultaneous existence for the student. Alternatively, we can propose that they have one mental representation of the problem, but this representation is not very clear for them and not very stable. It can be seen as a network of reasoning elements, not well connected and not well organized. The student's production is not only the product of their mental representation and presumptions about the task, but also by the communication or representation tool (diagram, verbal explanation, mathematical expression). Thus, the request for a particular form of communication and the communication process itself can affect and modify a student's reasoning.

Although AA diagrams were produced in some cases, students did not use them as models. In eight cases, students used a diagram as a template to complete their solution.

In four cases, students relied on keywords from the text to derive (incorrectly) the arithmetic operation to use.

Figure 29 presents the approximate picture of the third session.

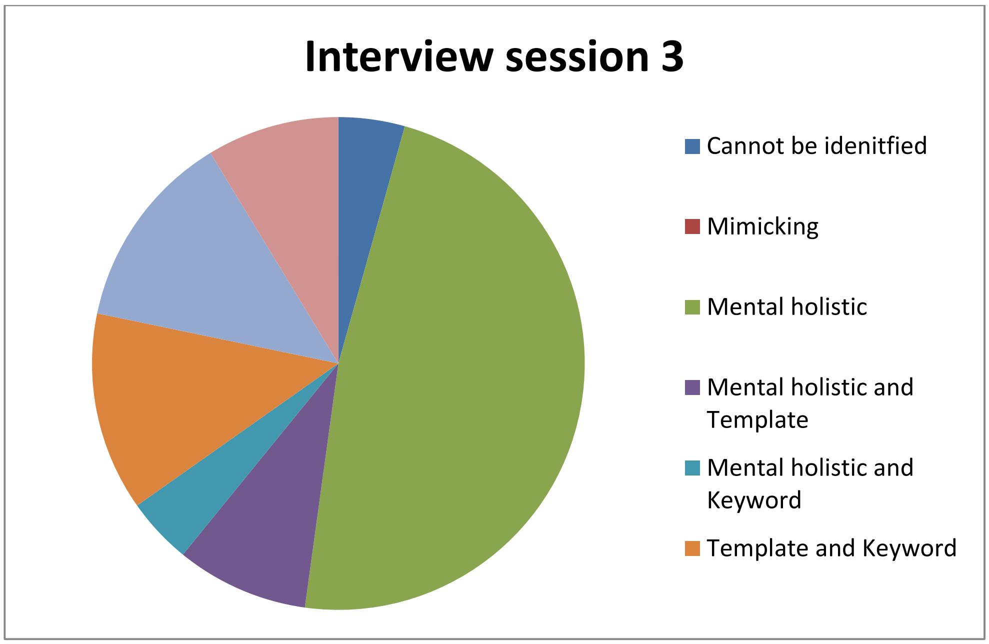

Figure 29. Session 3, ways of mathematizing

Students' difficulties

Summarizing observations made in the third session, I can say that the students had two main difficulties. The first difficulty can be associated with the use of diagram as a template. The diagram that students constructed, **with its particular form**, was not in relation to the problem or students' understanding of the problem. It likely came from their memory as a required element they usually use in class. When constructing this diagram, students had difficulty associating it with the problem at hand. This hypothesis will be verified in the third phase of my analysis.

The second difficulty is the lack of coordination between different means students used to mathematize the same problem. In only three cases was the process of explicit AAD representation coherent and logical. In all other cases of AAD representation, the solving process looked like a set of dissociated thought processes. It seems that the appropriate process of explicit modeling using AA diagrams was not established yet for great majority of students. Thus, I have observed an intermediate state of learning to model.

4.1.4 Fourth session, May

The fourth interview session took place in May. By that time of the year, students had been exposed to problems and situations with all types of simple additive structures. The following problems were used: Posts (Comparison, referred unknown, inconsistent language) and Calendar (Combine, part unknown). These two problems were given to majority of students. The Butcher problem (Negative Change and Comparison, total unknown) was used for two students to probe their behaviour in a more difficult situation.

> **Posts**: The red post is 37 cm shorter than the green post. How long is the green post if the red post is 265 cm?
>
> **Calendar**: In 2011, there were 365 days. In her calendar, Julie marked 198 school days and 10 holidays. How many days were without school in 2011?
>
> **Butcher**: The butcher received 3 packs of fresh meat this morning. At noon, he saw that he had 17 kg of meat left. “I sold 9 more kilograms of meat than I have left to sell,” said the butcher. How much meat did the butcher receive this morning?

The Fruits problem (Comparison, inconsistent language) and Cards problem (Positive Change, change unknown) were given to three students, who had not done them before. In total, I analyzed 19 observations.

4.1.4.1 Mental representation

Mental holistic representation

Only 4 out of 19 observations from the fourth session can be interpreted as students using tacit mental representation to derive a correct calculation plan (1 for the Posts problem, 3 for the Calendar problem). Neither the Calendar problem nor the Posts problem directly hints at the correct choice of arithmetic operation to be used. In all four cases when the mathematical expression was proposed by students, it seemed that the students had developed a holistic flexible mental representation of the problem to be able to solve correctly. For detailed analysis of similar cases please see section 4.1.2.2.

Emergent mental representation

In six cases, the observations can be interpreted as students constructing their understanding of the problem through the process of explicit representation (AAD). One successful process of this type was discussed in 4.1.3.1. The following example shows a not very successful process.

> Viktor (Posts): We need to write the mathematical expression.
>
> Researcher: Can you write it?
>
> [Viktor starts to draw two parallel segments of different length (comparison diagram, correct).]
>
> Researcher: Can you please explain what you're drawing?
>
> Viktor: [I'm drawing a] mathematical expression.
>
> Researcher: Is it a mathematical expression?
>
> Viktor: Yes.
>
> Researcher: I think it is a representation.
>
> Viktor: Yes.
>
> Researcher: So, what did you represent?
>
> Viktor: 37 cm shorter, so 37 goes here [points to the diagram in the first part of the longer segment, which is incorrect, as the 37 is the difference].
>
> Viktor puts 37 for the left part.
>
> Viktor: Here is also 37 as they are the same [points to the shorter segment]. Now the difference, it is more, so it will be 265. [Points to the difference part of the longer segment]
>
> Researcher: Do you think that 265 cm is the difference?
>
> Viktor: Yes.
>
> Researcher re-reads the problem and asks about the length of the red post.
>
> Viktor: So 37 is the difference [writes 37 for the difference part and 265 for the short segment].
>
> Viktor: We know that here is 265 [points to the left part of the long segment, which is equivalent to the short segment and put 265 on it].
>
> Viktor: [We are] looking for the difference.

Researcher: Are we looking for the difference? [Points to the text of the question]

Viktor reads the question.

Researcher: Where is the green post?

[Viktor points to the bigger line (this gesture is unclear).]

Researcher: [Shows the left part] Is it just this? Or all this [shows the whole big segment].

Viktor: It is all this [shows the whole segment].

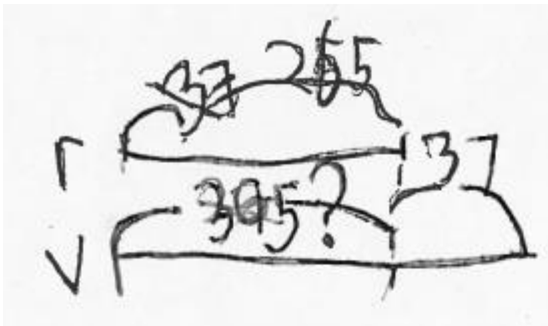

Figure 30. Viktor's solution to the Posts problem

Researcher: What should we do to find this?

Viktor: We do 265 minus 37.

Researcher: 265 - 37. Why?

Viktor: Plus 37.

Researcher: Plus 37? Why?

Viktor: Because we need to know ALL this. [Writes 265 + 37]

Researcher: This way we will find which post?

Viktor: Green.

At the beginning, Viktor correctly represented the comparison nature of the situation. We can say that he had a holistic mental representation of the relationship described in the problem. Viktor incorrectly put the number 37 (which is the difference) on the left part of the greater segment, which is equivalent to the shorter segment. Then he declared that 265 was the difference. This is incorrect. Viktor likely had difficulty remembering the text and navigating through the data and relations at the same time. However, when the diagram was created, Viktor used it to draw conclusions: "Here is also 37, as they are the same [points to the shorter segment]." When the diagram was corrected, to create the mathematical expression, Viktor first used an incorrect operation (265 - 37). This mathematical expression did not correspond to the diagram. It is possible that the operation “subtraction” came from the word “shorter” used in the text of the problem. Viktor did not explain this decision. When challenged by the researcher, he changed his opinion and confirmed his new choice using the diagram: “Because we need to know ALL this” (this expression is related to our discussion about the green post’s segment).

Summarizing this interview, I propose that Viktor did not have a clear understanding of the problem at the beginning; he only grasped the comparison relationship in it. The further use of the constructed AA diagram was not perfect and could easily fail without the researcher’s interventions. However, the presence of an AA diagram made the questioning possible. The researcher’s questions about the text and the diagram directed the student’s attention to different parts of his reasoning. This was enough to help Viktor clarify his mental representation and formulate the mathematical expression.

In the other five cases where students used diagrams, they started by representing the comparison relationship (two-segment diagram) without numbers. When numbers came into play, not all students were able to use them correctly with the diagram. In two cases, few questions were enough to clarify the solution. We can conclude that the mental representation students finally developed (and used to derive the mathematical expression) was the product of the representation process.

Volatile mental representation

Only two cases from the fourth session can be interpreted as students having volatile mental representation (please see the section 4.1.2.1 or 4.1.3.1 for detailed description of this way of reasoning). The different parts of the students’ productions seemed to come from different reasoning processes.

Structure substitution

In at least two cases, students initially interpreted the problem’s structure incorrectly (Posts). The problem states that the red post is shorter than the green. In the first case (Michael), the student immediately proposed the mathematical expression

"265 - 37 =," which is incorrect. It appeared as though the student just directly transformed the word "shorter" into subtraction operation. However, responding to the researcher's request, the student drew two vertical rectangles (posts) and identified the shorter as green and the longer as red. This confirmed that the student has substituted the structure of the problem and not misused the word "shorter."

In the second case (Eva), the student started her explanation by drawing a comparison diagram (two horizontal parallel segments) and explained that the longer one represented the red post and the shorter represented the green post. It was obvious that she had misinterpreted the problem.

Summarizing these two cases, I propose that the use of graphic representation (AAD) helped to identify the structure substitution in the students' reasoning.

4.1.4.2 Students' presumptions about the task

Many of the observations made in the fourth interview (n = 9) can be interpreted as the students having presumed that a diagram was requested. The majority of students started their explanations by drawing a diagram or simply declaring that they need to draw a diagram.

Maria (Posts): You **need** to make two strings.

Different students acted on their presumption in different ways. Some of them used a diagram as a mandatory element that was not really related to their understanding of the problem (template). Other students used the diagram to reason about the problem and derive the mathematical expression. At least one student (Maria) used the diagram when she was comfortable with a problem (Posts) and abandoned the diagram method as soon as a problem (Butcher) appeared to be difficult. For the last problem, Maria drew circles to represent kilograms.

Three students used a mathematical expression while doing all analyses in their head. One of these students mentioned draw-and-count method, but did not really use it or a diagram to solve the problem. We can say that all three students likely were not affected by the “diagram” presumption because they did not feel they needed a diagram to understand the problem.

One student (Nadia) clearly favoured numbers. When asked to represent what was in the problem, Nadia explained that “here are two numbers” and circled 37 and 265 in the text.

4.1.4.3 Mathematizing

AA diagram as template

The majority of students spontaneously used AA diagrams in solving problems (12 observations out of 19). Nonetheless, six of these observations revealed that students used the diagram as a template. Below, I analyze several examples.

> Rosa (Calendar): I need to write a mathematical expression. … The strings? I will do a representation.
> Researcher: As you wish, as long as you explain what you are doing.
> Rosa: [Draws a comparison diagram template without numbers] This is holidays, 10 days [points to the small segment]. This is 198 school days [points to the big segment, makes the two segments equal].
> [Rosa starts to represent numbers by tens and units and then stops.]
> Researcher: Please, explain to me your representation.
> Rosa: This is 10 days for holidays and this is 198 days of school.
> Researcher: Can you show me the school days? [Rosa shows the line] And the holidays? [Rosa shows the other line] Are they the same quantity?
> Rosa: No.
> Researcher: In your representation they seem to be the same.
> Rosa: I did a bad representation [with prompt, draws another representation in which one line is partitioned for 198 school days and 10 holidays].

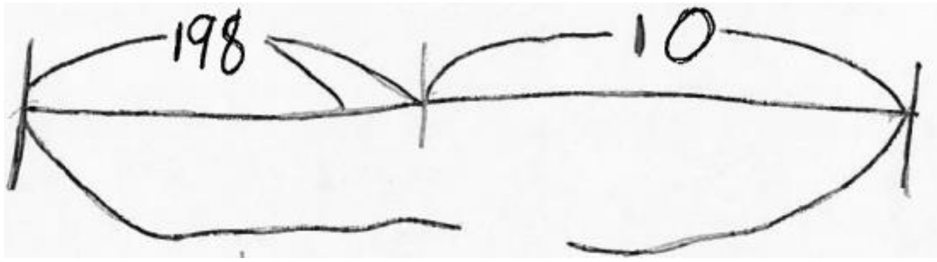

Figure 31. Rosa's second representation of the Calendar problem

Rosa tried to explain and said that neither of the representations helped her to find the answer. The researcher re-read the problem one more time and explained each term. Later in the discussion, Rosa tried to calculate before the representation was ready. Having constructed an appropriate diagram, Rosa hesitated when choosing the operation for the mathematical expression.

In this interview, Rosa tried two different templates (of a diagram) one after the other to represent two numbers she had chosen (incorrectly) from the problem: school days and holidays. For each template, she tried to input numbers arbitrarily. Neither of these representations helped her to better understand the problem. It is clear from this example that Rosa did not construct diagram based on her understanding of the problem. She tried to put numbers into one of the known templates and got blocked. The fact that she started with the comparison template, shows that the choice of template was not critically evaluated before use.

> Nadia (Posts): We will do this. [Draws a one-segment representation template] This is the representation. It represents what it is, what we are looking for.
>
> Researcher: What we are looking for?
>
> Nadia: The green post
>
> Researcher: Show me in your representation how you represented the green post.
>
> Nadia: [Looks at the diagram] We don't know.
>
> Researcher: Yes, we do not know. But is it represented somewhere in your diagram?
>
> Nadia: Usually in class we put a question mark if we are looking for that.
>
> Researcher: We create a representation to represent something about the problem. Can you please explain what your representation represents?
>
> Nadia: I don't know what it represents.

Researcher: Can you make a representation to represent what you have in the problem?

Nadia: There are two numbers here. [Circles the two numbers in the text] Say that red post measures 37 cm.

Researcher: Does it measure 37 cm?

Nadia: It is shorter than the green.

Researcher: Can you represent this?

[Nadia tries to use her first drawing to represent the situation. She puts 37 for the right part and 265 for the total.]

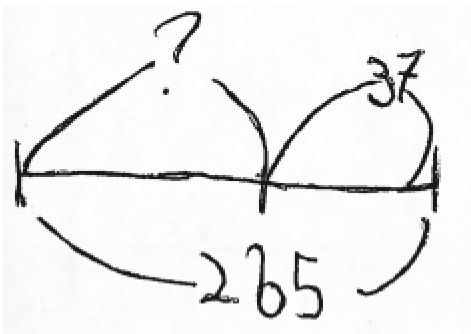

Figure 32. Nadia's solution to the Posts problem

In this example, Nadia proposed a template without any relation to the problem and accepted that she did not know what the drawing represented. Even after some discussion, in which the comparison situation was mentioned, Nadia continued to use the one-segment template and put numbers in it.

AA diagram as model

In many cases (n = 6) I observed the use of the AA diagram as a model. Students started their explanation by constructing a diagram. They consulted the text or gave explanations for each piece of their drawing. They derived the mathematical expression and other mathematical conclusions from the diagram. In these cases, all forms of communication were coherent and adequately related to the text of the problem. Thus, we can conclude that the students really modeled the problem and derived their calculation plan from this model. Below is an example of successful modeling of the Posts problem.

Maria (Posts): You need to do two strings. The difference says that the red post is smaller than the green. It is 37 cm shorter than the green. So the smaller is 37 … And we are looking for how long is the green post.

Researcher: Can you draw it all?

Maria draws two horizontal segments one above the other not well aligned, but the second is longer than the first. Maria put a separation point on the longest segment at the level where the shortest ends.

Maria: Here is the red post [points to the shortest segment] and here is the green post [points to the longer segment]. Here is [points to the "difference"] 30 cm more ... This one [points to the shortest segment] is not 30 cm. This [draws an arch for the left part of the longest segment] ... We know that this ... because these two are equal [points the shortest and to the left part of the longest]. So this is 265 [writes 265 for the left part]. This is also 265 [makes an arch for the shortest and writes 265], because these parts are equal. And here is a small difference [points to the difference part, makes an arch and put 37].

Researcher: So what should we do?

Maria: Mathematical expression "265 + 37 =".

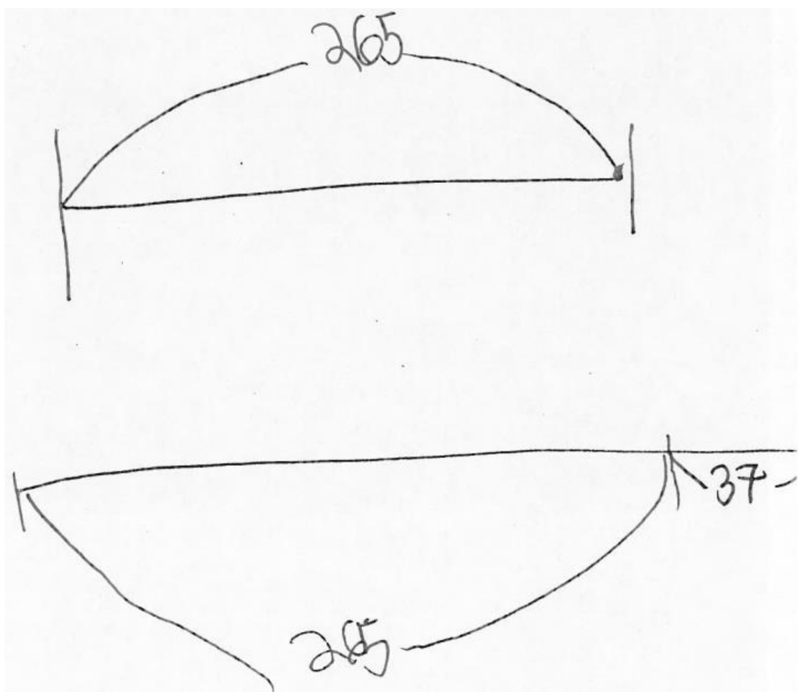

Figure 33. Maria's solution to the Posts problem

It seems that the idea of "strings" helped Maria to grasp the mathematical structure of the problem. In this dialogue, Maria carefully explains her representation (which is correct) and draws conclusions from her drawing. She says, "So this is 265 [writes 265 for the left part]. This is also 265 [makes an arc for the shortest and writes 265], because these parts are equal." Her mathematical expression is fully coherent with her representation and with her explanations.

Mental holistic model

In the fourth session, as in previous sessions, some students (n = 4) derived the calculation plan directly from their mental model (please see section 4.1.2.2 for examples). What is more interesting is that in one case where a student used multiple ways to mathematize the problem, she favoured her mental model, and not other pieces of her knowledge, to derive the mathematical expression from it. Below, I analyze this case.

> Emma (Posts): [Immediately draws a one-segment diagram] I'm drawing … like the total.
> Researcher: What is the total?
> Emma: It is 265.
> Researcher: But what is 265?
> Emma: It is the red post measures 265 cm. [Puts a separation point and makes an arc for the total with 265 on it, then makes an arc for the left part]
> Researcher: What is it?
> Emma: This is red post that measures 37 cm.
> Researcher: Where is the red post?
> Emma shows in the text on the 37.
> Researcher: Does it measure 37cm?
> Emma: No, because they say shorter than the green post.
> Researcher: Please explain your drawing.
> Emma: Here, first of all, I will put "don't know" for the green post and "already there" will be 37. [Draws an arc over the right part and puts 37, then puts question mark over the left part]
> Emma: Now, the mathematical expression [writes 265 - 37].
> [Emma thinks a while and corrects to "+". Emma starts to calculate using hundreds, tens and units representation. She incorrectly interprets the final number in her calculation (it should be 302) and put 32 for the left part of the line.]

Emma used the one-segment template, even though she clearly mentioned the comparison expression "because they say shorter than the green post." She then put the known numbers on the diagram and named them "the total," "don't know," and "already there." These names do not correspond to the context of the problem. It is quite possible that they arrived in her reasoning together with the template, which was part of her previous in-class experience (I will verify this hypothesis in the third phase of my analysis). When Emma created the mathematical expression, the first version corresponded to the diagram (learned prototype). Nonetheless, Emma changed it immediately to one corresponding to the problem and, likely, her understanding of the situation of comparison, where the bigger quantity was unknown.

The diagram Emma used did not reflect her mental representation of the problem or only represented a part of her reasoning about the problem. She did not choose a template according to her understanding of the problem. Instead, by using a diagram she tried to adapt her mental representation of the problem to the template.

In Emma's case, her initial understanding of the problem appeared to be more important to her than the diagram. She therefore used her understanding, not the diagram, to create mathematical expression. We can conclude that she had created a holistic understanding of the problem, which was flexible enough to let her transform the comparison situation into an appropriate arithmetic operation. However, this mental representation was not coordinated with the other part of her reasoning.

4.1.4.4 Summary of the fourth interview session

Table **11** in Appendix 1 presents a summary of the 19 observations made during the fourth interview session. The fourth interview session revealed that in half of the cases (n = 10 out of 19), students developed a holistic relational mental representation of the problem, and in the four of these cases they used their mental representation as a tacit model.

Many students (n = 12) recognized the diagram as a required tool for solving problems. However, only six of them really relied on this tool to analyze and solve the problem. Three students used multiple tools to mathematize the same problem and were not able to coordinate these tools altogether. Three other students used the diagram as template (only), without success. Two students tried to represent the problem (Butcher) using circle drawings. It seemed that in these cases, the students returned to their old knowledge because the problem was too difficult for them and they did not trust the new AAD tool.

In 2 cases, it was difficult to interpret students' reasoning.

Figure 34 presents an approximate picture of the session in relation to the ways students used to mathematize problems.

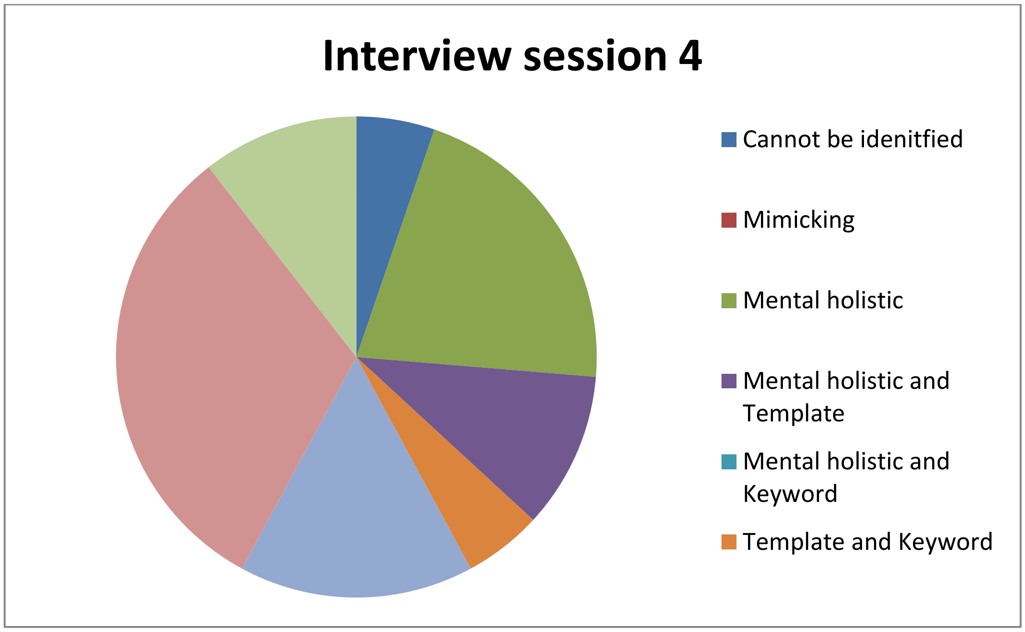

Figure 34. Session 4 overview

Students' difficulties

The relatively unknown context of the Calendar problem created difficulty for many students. However, the difficulty was not related to the mathematical structure of this problem, which was well understood when the context was clarified.

As in the third session, many students encountered difficulty while trying to adapt a template to the data from the problem. In my analysis of this difficulty in section 4.1.3.4, I formulated the hypothesis that the students generalized the diagram method together with a particular type and disposition of the segments. The case of Rosa described above shows that another interpretation can also be possible. To solve the Calendar problem, Rosa tried two different templates: with one segment and with two parallel segments. Neither of these tries was successful. It seems that it was not a particular template that caused difficulty for the student. It was how it was used: *diagram-as-template.* The idea of starting with a template and then putting numbers on it, could damage the desired reasoning process, which is supposed to be grounded in the problem's text (not in the template). I will verify these two interpretations in the second phase of my analysis.

4.1.5 Summary of the interviews

Summarizing my analysis of the four sessions of individual interviews with students I will first present an overview of the inferred types of mental representations and related presumptions about the problem-solving tasks. Second, I will present a description of all ways of mathematizing inferred from my observations and partially based on my theoretical explorations. Finally, I will conclude with an overall view of the reasoning process in problem solving.

4.1.5.1 Mental representations and presumptions

According to researchers (Brousseau, 1988; DeBlois, 2011), the mental representations learners construct of a problem are related to and/or induced by the presumptions they have about problem solving. From my observations it was possible to infer the following mental representations and possibly related presumptions.

Table 3 *Mental Representations and Presumptions*

Mental representations	Possibly related presumptions
Sequential: The solver follows the order of events or data described in the text.	Need to draw and count objects, small circles as the situation (text) unfolds Need to consider numbers
Tacit holistic: The solver thinks about the problem as a whole in their head and mentally transforms the problems structure to calculate the answer or to formulate the calculation plan.	Need to find the numerical answer Need to consider the problem as a whole

Structure substitution: The solver understands the text as another problem not intended by the teacher/researcher.	
Volatile: The solver's understanding of the problem seems to depend on the required communication tool.	Need to draw an AA diagram The arithmetic operation to use in the mathematical expression is described by the keyword
Emergent: The solver constructs their understanding of the problem through the process of explicit modeling (using AAD method).	Need to draw an AA diagram

There is no way of determining whether the presumptions directly induce specific mental representations or some mental representations are results of students' natural way of reasoning. Structure substitution, which happens at the implicit level of the first interpretation of the text of the problem, can create important deviations in the process of solving, making the interpretation of students' production much more difficult. In multiple cases, students' productions depended on the communication tool used. Different parts of these productions could be interpreted as based on different mental representations and different presumptions.

My analysis shows that describing students' reasoning in terms of "mental representation" being influenced by their "presumptions" is useful but not detailed enough. This terminology can only apply to cases where a solver's production is coherent in all its parts and through all communications.

4.1.5.2 Mathematizing

From my observations, I inferred the following categories of mathematizing.

Mimicking: The solver uses the flow of the text of the problem to sequentially represent numbers and events (keywords) as they would happen in the real situation. At the end of this process, the result can be counted directly from the representation. This way of mathematizing is described by Carpenter et al. (1999). However, these researchers pay more attention to how students calculate the numbers (Operational Paradigm) than how they deal with the relationships (Relational Paradigm).

Use of tacit mental model: The solver analyses the problem mentally and mentally transforms the flow of the text, or the mathematical structure of the problem, into an essentially different structure: a mathematical expression in a standard form.

Use of AA diagram as template: The solver uses an AA diagram template without taking into account the actual structure of the problem. The solver tries to adapt the data from the problem to the arbitrarily chosen template.

Use of keywords (inappropriate): The solver transforms the keyword(s) from the problem into an arithmetic operation in a straightforward way and uses numbers from the problem to formulate the mathematical expression. This way of mathematizing is described by Hegarty and colleagues (Hegarty, Mayer, & Monk, 1995). These researchers pay attention to how students transform the text into arithmetic operation (Operational Paradigm).

Constructing AA diagram as model: The solver constructs AA diagram based on their understanding of the relationships described in the problem. The solver derives the mathematical expression and other mathematical conclusions from the AA diagram.

Use of mathematical equation: The solver translates the text of the problem into an equation identifying the element of the equation which is unknown. The solver uses the equation to derive the calculation plan. This way of mathematizing is described by Carpenter et al. (1988).

Use of object model: The solver represents the whole situation described in the problem as a system while using objects (or circle drawings). The solver derives the calculation plan or other mathematical conclusions from this representation. This way of mathematizing is very similar to the mimicking. The difference is in the order of use of the data from the text.

4.1.5.3 Overview

The process of reasoning in word problem solving depends on and can be influenced by:

1. specific linguistic characteristics of the text (for example, long or short determinants)
2. the problem's context (known or unknown for the solver)
3. knowledge about specific expressions of quantitative relationships ("has 35 more than")
4. knowledge of ways and tools of mathematizing
5. presumptions about what is requested and what **should** be used
6. knowledge of a meta-cognitive process to be applied
7. the ability to coordinate multiple elements of the information and ways of reasoning

From my observations, it follows that in the third and the fourth sessions, the process of creating a calculation plan can be fully independent of the knowledge of how to carry out concrete operations with concrete numbers. This is coherent with the algebraic way of reasoning described in research (Gerhard, 2009; Schmittau, 2005; Xin et al., 2011).

The AAD method of representation can potentially be a great support in the organization of the problem solving process. This method allows the mathematical structure of the problem to be represented and discussed without representing the numbers per se. As a researcher, this method (when known by students) opened new opportunities to dig deeper into the students' mathematical reasoning and reveal the complexity of the mathematical reasoning in additive word problem solving.

4.2 Phase 2: Learning

The first phase of my study allowed me to make inferences about various means students used to mathematize additive word problems. In the second phase, I used this data to answer the second question of my study: ***What are the relationships between instruction implemented and students' development of mathematizing processes?***

First, I analyzed how students' problem solving changed with time. Second, I used a retrospective analysis method to discover links between the observed change in learners' behaviour and the curriculum implemented in the classrooms.

4.2.1 Dynamics of knowledge development

In this section, I use the previously inferred categories of students' ways of mathematizing problems to describe the dynamics of learning for each student and for the group from session to session.

Figure 35 represents the types of mathematizing that the students used in all four interview sessions. I represent observations by sessions from left to right. I have grouped students according to how their ways of mathematizing changed throughout the observation period. With regard to this change, it was possible to distinguish three groups of students: slow, medium, and indifferent. As I will explain later, these names of the groups approximately reflect the students' relative achievements in learning new way of mathematizing (AAD modeling).

The students in the first group (slower group in Figure 35) mainly used mimicking at the beginning of the year. By the next stage, they tried to rely on mental work, but most of the time unsuccessfully (no mathematizing could be observed). By the following stage, they started to use various ways to mathematize problems including AAD, but failed to coordinate all elements of their reasoning.

The students in second group (middle group in Figure 35) mainly used mimicking at the beginning of the year. By the next stage, they tried to rely on mental work, however unsuccessfully (only one student successfully used AAD). By the following stage, they tried to integrate AAD in their strategy, but often failed to coordinate all elements of reasoning. Finally, they succeeded in modeling problems by using AAD.

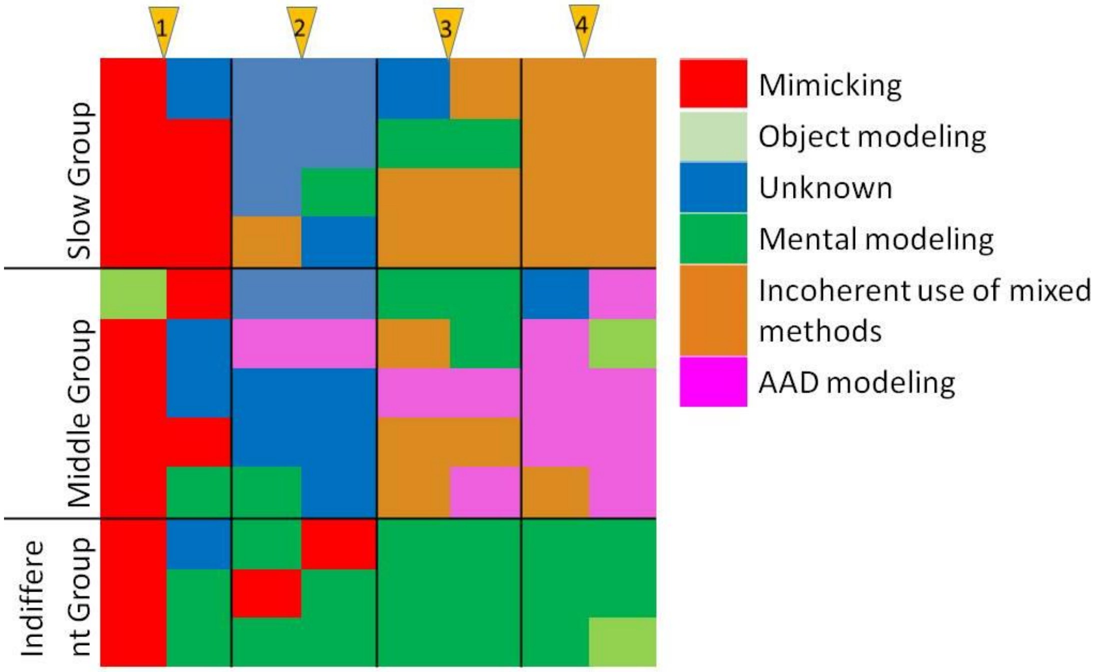

Figure 35. Dynamics of knowledge development

The students in the third group (indifferent group in Figure 35) also used mimicking at the beginning of the year. By the next stage, they mainly relied on their mental work to solve all problems. They continued with this strategy until the end. They did not spontaneously use AAD, even when problems were very difficult for them.

The results show constant change in students' reasoning development. All students went from mimicking towards more advanced methods of mathematizing. However, they did it differently and with different success. Gaining a better understanding this difference requires going back to the approximate evaluation of students' relative achievement in mathematics that I used to select my participants. The three groups I have identified based on my observations correspond more or less to the students' relative math achievement at the beginning of the year. There are only three exceptions: one student from the "lower achievers" group appears in the medium group; one from the "average achievers" group appears in the slow group; and one from "higher achievers" group appears in the medium group. The grouping I identified based on my observations more represents how knowledge was developed, rather than if there was actual achievement in problem solving (or in mathematics in general). However, there is some correlation between the approximate evaluation made at the beginning of the year and the final groups based on knowledge development. I named groups “slow”, “medium” and “indifferent.” These terms describe the students in each group in terms of how their reasoning developed and not in terms of success in finding the answer to a problem.

From my analysis, it follows that students in the middle group (n = 5) benefitted the best from the experimental teaching approach. At the end of the observation period, they demonstrated adequate use of the AAD method to model problems and derive solutions. The students in the slow group (n = 4) also benefitted from the experimental teaching and tried to integrate AAD into their reasoning. However, they seemed to have more difficulty (compared to the middle group) doing this. By the last interview session, none of these students demonstrated appropriate use of AAD. However, three of them demonstrated some strategies that allowed for a correct solution (correct calculation plan). The students in the indifferent group (n = 3) seemed not to have benefitted from the experimental teaching. They were always very successful in solving simple additive word problems. However, they did not learn a lot of new things in relation to problem modeling. When a more difficult problem was given to them, they analyzed it mentally or they tried to draw circles to represent and count objects (kilograms), which was unsuccessful.

The analysis of the dynamics of knowledge development reveals that the learning process was not easy or straightforward for students. At the beginning, the group demonstrated success in using mimicking method. Then, there was a period of failure where students mostly relied on their mental work and not mimicking or modeling. The next period can be described as trying out modeling, where many students struggled to coordinate different ways of reasoning. The last period is that of appropriate use of the AAD method (not for all students). To better understand the mechanisms and causes of this change, and especially figure out the main causes of students’ difficulties, I need to analyze the experimental curriculum and the instructions implemented in the classes.

4.2.2 Teaching implemented

In this section, I give a short overview of the curriculum planned and implemented during the period studied. The main idea of the teaching approach was that any problem solving task should be an opportunity to analyze the additive relationships present in the situation. The AAD representation method was used as a communication media to express the structure of the problem and organize the discussion and analysis of this structure. The goal of the approach was to support students in the development of holistic relational reasoning in the context of additive word problem solving.

The implemented curriculum comprised three phases. In the *introduction* phase, students explored the length property of different physical objects (strings, ropes, paper strips, cloths, etc.). Students constructed a rigorous procedure to compare the lengths of these objects, visualized the difference and learned to represent this comparison graphically.

Based on this concrete knowledge of manipulations with lengths, the *construction* phase required students to holistically analyze various situations involving additive comparison. Instead of a word problem with a question, teachers proposed a word description of a mathematically impossible situation: the three quantities involved did not respect the comparison relationship described in the text. Students then explored the situation to find how each value could be changed to satisfy the relationship. To clarify the relationship, students (together with the teacher) constructed a graphic representation of the comparison involved (AAD). Based on this representation, for each potentially incorrect number, they determined the mathematical operation to calculate the correct value for this number.

The *development* phase included solving additive word problems in the following manner:

1. Read the problem and discuss the context briefly.
2. Construct an AAD representation of the problem.
3. Find the mathematical expression that can calculate the missing value.
4. Use available tools to calculate the missing value.

5. Make sense of the calculated value in terms of the initial situation.

Some other activities were implemented in the development phase to support students in constructing and making sense of the AAD representations.

Table **4** presents the description of the sequence of teaching/learning activities developed and implemented in classrooms.

Table 4 *Experimental Curriculum*

Phase	Activity type	Instructional goals	Time period
Introduction	Manipulations, analysis of lengths	Discuss and analyze with students different situations which involve the **comparison** of physical objects by their length. Develop and use a rigorous procedure of lengths comparison through the juxtaposition of objects (ropes, belts, cheese sticks …) to demonstrate the difference. Communicate the procedure orally and by graphic representation. Correctly use symbols >, <, = together with appropriate language.	October
Construction	360° analysis	Develop the understanding that the three quantities are related. Link the **comparison of numbers to the comparison of lengths**. Construct graphic representation of the situation as a whole using Arrange All Diagram. Connect to the comparison of lengths. Analyze the situation using diagram, figure out appropriate mathematical operations to find the bigger quantity, the smaller quantity and the difference.	November, December
Development	Simple **Combine** and Simple **Change** problems, solving with 360° analysis at the end	Construct graphic representation of the situation as a whole using AAD method. Analyze the situation using diagram, figure out appropriate mathematical operation to find the unknown quantity (part or total). Develop the understanding that the three quantities are related.	January, February
	Problem Communication game	Develop the ability to represent a problem using the AAD method.	March
	Diagram comparison activity	Develop the ability to analyze the graphic representation (AAD) of a problem.	April

4.2.3 Links between learning and teaching

In this section, I analyze relationship between the curriculum and teaching implemented during the observed period and the dynamics of the knowledge development in students. This will help to answer the second question of the study about the possible relationships between teaching and learning.

In order to find the relationship between students' performance through the interviews and the teaching approach to which they were exposed, I use the calendar of activities implemented in class, as well as the retrospective analysis of classroom activities videotaped during the experiment. I tried to find events, sequences of events and characteristics of the events which could provoke (or can be related to) the behaviour observed in students.

The following image describes the development of the teaching events and the students' performance over time, throughout the interview sessions. We can see that the period when students mainly rely on mental work corresponds to the end of the Construction phase and the beginning of the Development phase of the curriculum. During the Development phase, the students start to use elements of the AAD method. To explain this observation, I need to look deeper into the teaching implemented.

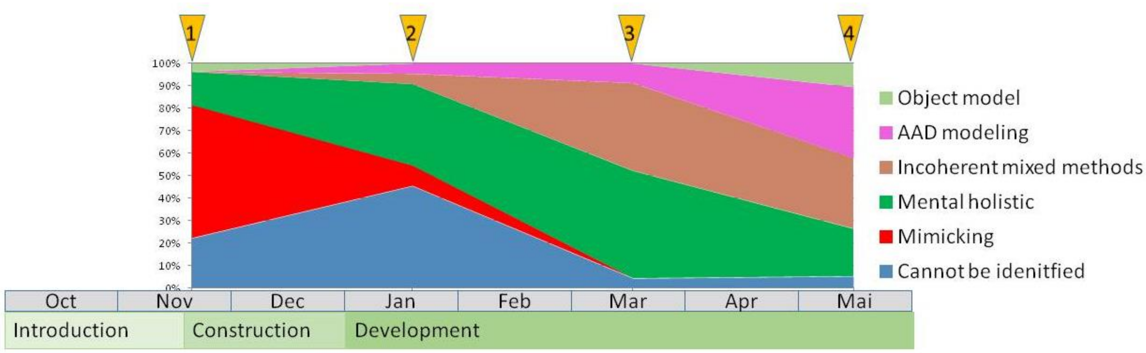

Figure 36. Calendar of teaching and learning events (yellow triangles represent the interview sessions)

Table **5** below describes the four teaching/learning periods observed in the study. I present the following aspects of each period: types of learning activities implemented, types of students involvement, categories of additive structures involved, and students' performance during the interview session at the end of each period: successful and unsuccessful.

Table 5 *Students' Performance in Relation to Teaching Events*

Period	Before the first interview (November 7)[14]	Before the second interview (January 13)	Before the third interview (March 26)	Before the last interview (May 9)
Activity type	**Introduction** Manipulations with objects having length	**Construction** 360° activities	**Development** Discrete to continuous Problem solving	**Development** Problem solving Captain's game Schema comparison
Student involvement	Whole-class discussion Individual work with objects	Whole class discussion Students do not construct the diagram themselves	Class discussion Students construct diagram of the same structure as previously discussed with the whole class Individual work followed by a whole-class discussion	Individual work followed by a whole-class discussion Students construct diagrams for a new problems themselves
Mathematical structures involved	Comparison	Comparison	Combine Change	Combine Change

[14] Four students were absent on this date. They were interviewed on December 5.

				Comparison
Video-recorded lessons	October 11, 17, and 24	November 16 and 23, December 15	January 30, March 19	April 4 and 25
Student success during the interview	Coherent use of the mimicking strategy for change problems	More students try to summarize the problem and propose adequate mathematical expression	Students do very well at the comparison problem with referent unknown (inconsistent)	More adequate use of AAD of different structures
Student difficulties during the interview	When mimicking cannot be applied, students get blocked or are unable to construct an adequate mathematical expression	Fail to apply mimicking to comparison situation Fail to analyze problem mentally	Use the recent diagram type as a template without critical analysis	Non-critical use of template

In the first interview session, I probed students in solving two Change problems and one problem with a complex Combine structure. In Figure 36, the main colour present at the time is red, for mimicking. Simple Change structures had not yet been discussed in class using the AAD representation method. These problems are semantically close to the mimicking representation (red in the illustration), because an action is present and well indicated in the text. We should also take into consideration the students' previous year experience with mimicking for adding and subtracting. I conjecture that in these conditions, students did not even try to recall and apply their new knowledge about diagrams (under construction), a method they only experienced in connection with the Comparison structure. The transfer of knowledge seems to be impossible at least for two reasons: previous knowledge (in Grade 1, students were taught according to the traditional approach) is strong and strongly associated with Change situations; new knowledge is weak and only associated with Comparison situations. An important factor can also be the students' presumption that they "need to draw and count." We can see that majority of students use this strategy in the first session (red color in Figure 36).

In the second interview session, I used Compare and Combine problems. In Figure 36, the main colors present at this time are green and blue, for mental modeling and unknown. For a long time before the second session, students had not been required to draw and count to solve a problem. Instead, the teachers always constructed an AAD representation for each problem they discussed with students. First, they thoroughly discussed the problem's structure using the AAD representation, and then they proposed doing calculations if needed (lessons video recorded on November 16 and 23 as well as December 15). According to the Relational Paradigm, where the teaching approach belongs, the main importance was given to relationships.

It is likely for these reasons that many students did not use the mimicking method any more (red on the graphic). We can propose that they did not really feel that the draw-and-count strategy was necessary for their reasoning. They likely used it before because it was required by the teacher. Students did not use diagrams either (brown or pink). Prior to that moment, the students had only seen Comparison structures and had not been constructing diagrams by themselves (please see Table **5**). Thus, the transfer of knowledge about AAD was difficult for them. The video recordings (November16 and 23 as well as December 15) show that teachers did not ask students to construct an AAD representation by themselves prior to the class discussion. Instead, teachers constructed AAD representations on the blackboard, sometimes asking one or two students to do it for the class. The absence of individual practice of constructing AAD representations might also explain why students did not try to use the diagrams in the second session.

Some students continued to use mimicking during the second session (red color in Figure 36). Reasonable explanations are that the old presumption still dominated the students' reasoning or they preferred to rely on old well-known methods than to experiment with new, weak knowledge.

In the third interview session, I used Comparison and Change problems. In Figure 34, the main colours present are green and brown, for mental modeling and incoherent use of mixed methods. Previously in class, students had been using the AAD method to represent various structures. However in the last months, they had not been working with the Comparison problems (see Table **5**). That means that in the previous month, the students mainly used one-segment diagrams. Students had practised constructing the diagrams individually to solve a problem and play the Captain's game of representation. Observations from the third interview session show that half of the students try to use diagrams (brown and pink). This can be explained by the new presumption of the "need to use diagram," which started to take effect. At the same time, many students failed to coordinate their understanding of the problem's text, the diagram and the mathematical expression (brown in the illustration). I can propose three possible explanations for this failure.

First, the knowledge of the diagram method is still weak in students, as this method is more difficult to learn than mimicking.

Second, through their learning experience in Grade 1 and 2, the students accumulated multiple pieces of knowledge about what could be done to mathematize the problem.

- The numbers, which are important (one of the students' presumption), can be represented by small circles.
- The numbers might be counted or calculated to find *the answer*. Some words in the text of the problem are important and might signify adding or taking away.
- The diagrams, or particular template, should be used to represent the problem.

The multiplicity of pieces of knowledge that do not connect into a reasoning network might create additional cognitive demand for students, significantly complicating their thinking.

The third explanation can be found if we look closer at how teachers work with diagrams in class. It includes two characteristics of implemented teaching.

First characteristic: The same type of diagram (one-segment) has been presented to students for a long time (see Table **5**). This could provoke over-generalization of the diagram as a form instead of diagram as a method. Thus in some students, this learning experience provoked a specific presumption: "need to use one-segment diagram, and put numbers on it." In the interview, we see students drawing a segment separated into two parts without any reference to the problem (see sections 4.1.3.3 and 4.1.4.3 for examples).

Second characteristic: Teachers used designated words to name parts of the diagram: "in total," "already there," "added," etc. The following example is taken from the lesson video-recorded on January 30:

> Teacher proposes a problem to students. They read the problem and discuss the context. The teacher put an image of a diagram on the blackboard:

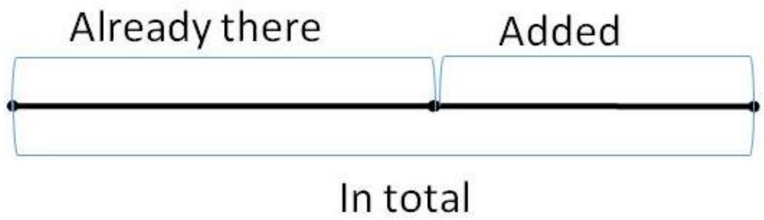

Figure 37. The diagram template used by the teachers

> Teacher: I will draw a larger diagram here, so you will be able to see the words we put on the small diagram [points to the diagram prototype and draws a big line below on the blackboard].
> Teacher: We worked with another diagram before [points to the comparison prototype (two-segments) in the corner of the class], and now we will work more with this one [points to the one-segment prototype]. So, if I put the same words here [writes the words “Already there,” “Added,” and “In total” on her large diagram].
> Teacher: Can anybody explain to me what it means? When you see this, what do you understand?

In the videos recorded on January 30 in two classrooms, the teachers proposed two problems of the same structure (Simple Positive Change, change unknown) while using the same diagram prototype. We can conjecture that some students have generalized the diagram as a template together with the designated words. We can hear these designated words in students’ explanations (see example of Emma on page 140, and example of Nadia on page 136). The formality of the diagram takes the place of reasoning and the meaning of the elements of the problem.

In the last interview session, I used the problems with continuous context and Comparison structures. In Figure 36, the main colors are green, brown and pink, for mental modeling, incoherent use of mixed methods and modeling with AAD. At that time, students had experienced various structures and practised drawing diagrams by themselves. The majority of students used diagrams, and five students used diagrams adequately. Some disconnections between text, diagram and math expression are still present for some students in some problems. I observed students from middle group using diagrams and successfully solving Simple Compare problems with extra data and inconsistent language (type of problems recognized as the most difficult). This means that in general, the AAD method was accessible for students of this age and could potentially be mastered by majority of them. Still, some of the students continued to have difficulty or just did not use the AAD method.

Only three students (from the indifferent group) did not use diagrams, even when probed with difficult problems. They relied solely on mental analysis to derive a solution plan (green colour in Figure 36). They could construct a correct diagram, when asked, for problems easier for them, but did not do it spontaneously. When challenged with problems that were too difficult, they reverted back to the circle drawing, which was not successful. These problems could potentially be solved using AA diagrams. However, the students did not try to use the AAD method as a means to solve difficult problems. The analysis of the instructional materials implemented throughout the year revealed that all problems and situations proposed in class were easy for these students to solve mentally. Thus, the students from the indifferent group were never challenged with problems that were difficult for them. They never experienced the need for any tools other than mental work.

4.2.4 Summary

Summarizing this analysis, I can say that the implemented teaching approach had a strong positive impact on the students' learning to explicitly mathematize problems and model using AAD representation. After five months of experimenting, almost all students tried to summarize the problem in a holistic and flexible way to create a mathematical expression. Half of them spontaneously used AAD representations. Two months later, the great majority of students used AAD representations and half of them did it adequately for at least one problem.

A more detailed analysis shows that many students experienced difficulties in learning to use this method. From my analysis, it follows that there are three main sources of difficulty. First, the AAD method of modeling represents, by itself, a challenge for students and requires important instructional effort.

Second, some characteristics of the curriculum and the way it was implemented possibly provoked additional challenges for some students. Students working with the same type of representation for a long period of time could provoke over-generalization of a particular layout. The way the teachers used diagrams in class and their use of designated words to name parts of the diagram could cause students to view the diagram as a template, where the numbers from the text should be plugged in. This method reversed students' reasoning from "situation-grounded" to "template-grounded." The absence of very challenging problems in the teaching implemented prevented students from the indifferent group from appreciating and developing new knowledge.

Third, some parts of previous knowledge and presumptions students brought to class (keyword use, representation of numbers, looking for the numerical answer) obscured and complicated their reasoning.

For the students in the slow group, the combination of all three challenges could be critical, seriously slowing down the development of new knowledge.

4.3 Summary of Chapter 4

I presented the analysis results obtained in two phases. The results in phase 1, contribute to understanding of students' ways to mathematize problems. The analysis presented in phase 2, explores students' strategies changed over time and under the influence of the teaching implemented. The detailed analysis provided in this chapter jets light on the complex process of learning and knowledge development in the context of additive problems. The next chapter presents a discussion of obtained results in the effort to propose new theoretical vision of the phenomena explored.

Chapter 5 Discussion

In this chapter, I will review the research questions formulated for this study using the theoretical framework developed in Chapter 2 and the obtained results described in Chapter 4. The main objective of the project was to study students' learning process in relation to the teaching implemented throughout the year. More precisely, I studied students' ways of mathematizing additive word problems, and how these ways evolved over a given period and why.

5.1 Answering Question 1

> ***What ways of mathematizing do students use to solve additive word problems?***

The analysis of my observations helped me to identify the following ways students mathematize a problem.

5.1.1 Mimicking

Mimicking was the most popular way of doing problems observed at the beginning of the studied learning period. It consists in:

- treating the problem sequentially as the text is read
- drawing several small circles to represent a number as soon as it appears in the problem's text
- directly implementing the semantic meaning of the verb or the relational expression by adding more circles or crossing out circles
- counting the result

Mimicking was a problem solving method that was encouraged in the classroom before the experiment. There are at least two explanations as to why students used mimicking at the beginning of the experimental period. According to DeBlois (2011), a student's way of problem solving is affected by the mental representation they have of the situation and their interpretation of the teacher's (or researcher's) request. First, students could believe that mimicking is what is requested from them. Second, students could mentally represent the problem as a sequence of events, and mimicking could thus be their way of expressing this mental representation. Two questions arise: Do sequential reasoning and mimicking come naturally to students or were these strategies generated by previous learning experiences and, more specifically, by their presumptions about this method? As can be seen from the summary of the second session (see section 4.1.2.4), half of the students abandoned mimicking, even though they did not use the diagram method that had recently been promoted in the classroom. It is quite possible that strictly sequential reasoning is not the only way students can think about a problem. Thus, for many students, the teacher's request, implicit classroom norms, or didactic contract (Brousseau, 1988; DeBlois, 2011; Jackson & Cobb, 2010) might be the only reason to use mimicking and counting as solving strategy.

According to researchers (Lesh & Zawojewski, 2007; Savard, 2001; Verschaffel et al., 2000; Verschaffel, Greer, Van Dooren, & Mukhopadhyay, 2009), in order to be considered mathematical modeling, the reasoning process should include two distinct steps: creating a model and using the model to draw a solution plan or calculate. It is very difficult to interpret the mimicking observed in the experiment as a two-step process. Rather, it should be interpreted as the **simultaneous superposition and full integration of mathematizing and calculation**. Once a student finishes mimicking, they can observe and count the result.

The mimicking process is difficult to generalize as a method of solving for all types of additive problems. For example, it is almost impossible to mimic inconsistent comparison problems (see Table **7** in Appendix 1 for examples of such problems). At the same time, mimicking can be seen as an appropriate calculation tool on the condition that what is mimicked is the arithmetic operation and not the events of the word problem.

Within the Operational Paradigm, the distinction between analysis of the word problem and the execution of the chosen operation is often missing or not clear. Researchers usually observe young children solving additive word problems using manipulatives or circle drawing (Carpenter et al., 1999; Giroux & Lemoyne, 1998). Within the Relational Paradigm, this distinction is the key element. In this paradigm, the mimicking cannot be recognized as an appropriate method of solving word problems.

5.1.2 Use of tacit mental model

In different interview sessions, I have observed students analyzing the problem mentally and immediately proposing solution plans with a structure that was different from the initial mathematical structure of the problem. According to researchers (Nesher et al., 1982; Riley et al., 1984; Vergnaud, 1982a), to create a correct mathematical expression or another type of calculation plan, in all cases, students have to see the problem in a holistic way and transform the initial structure of the problem. This leads us to think that the modeling step in these cases was implicitly present, but hidden from the direct observation. Students likely derived the calculation plan from their tacit mental mathematical model of the problem. Nonetheless, the presence of such model can be confirmed by the fact that the students transformed the initial mathematical structure.

The fact that this method is apparent without the teacher or researcher having requested it shows that, for some students, holistic reasoning about the problem comes naturally. It could also mean that the new teaching approach generated this way of reasoning in some students before they mastered the use of AA diagrams. Any of these inferences confirms that the **holistic analysis of a situation is accessible for many young students** (Davydov, 2008; Salmina & Sohina, 1981; Sophian, 2007)**.**

5.1.3 Using diagram as template

Many students tried to use diagrams as templates when representing and analyzing a problem. They chose a one- or two-segment diagram, immediately drew the template and tried to fit numbers in it. In these cases, the type of the diagram was usually inappropriate and/or the numbers were placed incorrectly. Thus, there was no coherence between the mathematical structure of the problem and the diagram. This process shows that the students' reasoning (mathematizing) was **inversed**. Instead of basing their reasoning on the situation described in the task and looking for convenient elements within the mathematical context, they chose a mathematical element, which they presumed was required, and then looked for data within the task description to satisfy the chosen element.

The phenomenon of disruption between the real-world meaning of the problem and it's mathematical solution have been studied by researchers (Mellone, Verschaffel, & Van Dooren, 2014). The researchers concentrated their attention on how to remediate the problem without questioning the sources of this trouble. To my knowledge, the way of using diagrams as a template as a manifestation of the disruption phenomenon has not yet been described in literature.

Many authors (Jitendra & Hoff, 1996; Levain, Le Borgne, & Simard, 2006; Willis & Fuson, 1988; Xin, 2008) propose using representational templates and graphic schemas to support students' mathematizing of problems. My analysis let me conjecture that the positive effects of using graphic schemas (diagrams) depend on the very careful teaching of how and why these graphic schemas should be constructed and used in different problems. Otherwise, the use of graphic schemas can provoke **the inversion of the mathematizing process in some students**. The use of diagrams as templates can be induced by the over-generalization of the practice of using the same type of diagram over a long period of time. It can also be reinforced by the use of keywords to name parts of the template. These two conditions appear to facilitate the disruption between the problem's socio-cultural context and the mathematics for some students.

5.1.4 Multiple uncoordinated means

When students used AA diagrams, even if not successfully, the distinct and explicit step of modeling was visible. The diagram drawn on paper can potentially become subject to revision, questioning, and further refining. In multiple cases in my study, students never refined their diagrams, neither did they use them to derive a calculation plan. The diagram was incoherent with the text and even with the student's explanation of the text. The mathematical expression created afterward was incoherent with the diagram and coherent with the task or vice versa. This particular phenomenon of dissociation of multiple mathematizing processes cannot be interpreted by any model discussed in Chapter 2.

Looking closer at the data analysis, I can propose that students' productions be seen as a result of the communication process. Students try to communicate their understanding of the problem, along with their understanding of the teacher's request by using of specific media, which in my case are: natural language (with some mathematical elements incorporated), mathematical expression, drawing circles and mimicking as well as AA diagrams. If we adopt this perspective, we can see that in cases of incoherent use of diagrams, the three distinct acts of communication seem to be independent from each other. For example, a student used the recently studied diagram as a template (which is incoherent with the problem) because she only felt comfortable with this type of diagram. At the same time, she created a mathematical expression based on the keyword she recognized in the text (which is inappropriate). Nevertheless, the student was able to reformulate the problem in her own words (natural language) in a way which was coherent with the text. This leads us to the idea that **the student's production can be strongly affected by the communication media requested and/or used by the student.** Some students generalize each media with particular—however not always correct—rules of production. For example, "the operation required in the mathematical expression should correspond to the keyword in the text", "the diagram should be of a particular form", etc. In cases of young children or when the media has not yet been mastered by the student, the communication process alone can be a challenge.

The mismatches in communication via different media that I observed in the study occurred in different combinations, individually or together. For example, the students constructed a coherent diagram and then explained that the operation to use should be subtraction (which is incorrect), "because when he throws, it's minus." We can see that communication via diagram and communication via mathematical expression are guided by completely different rules. These rules, usually implicit, govern the process of transformation of students' reasoning and mental representation into a concrete form of communication. The fact that the same student could produce essentially different communications via different media caused us to question the existence of a stable mental representation of the problem for the student. **Can this representation be understood as a kind of static picture, or should it be seen as a dynamic and constantly fluctuating process**? When only one way of communication is requested from a student, can their production be adequately interpreted? If we see mental representation as a process, what should be done, from a didactic point of view, to minimize the fluctuations and obtain better stability so that the mental representation can be transformed into an appropriate, concrete form in a coherent way via any requested media?

5.1.5 AA diagram as model

During the third and fourth sessions, many students demonstrated the coherent use of the diagram as a mathematical model of the problem. They explained that they needed to draw a diagram, choose an appropriate diagram type, put values and determinants on the diagram according to the mathematical structure of the problem, draw conclusions about the calculation to perform from the diagram and create a correct mathematical expression. They also correctly explained their reasoning when asked about the mathematical expression and their interpretation of the diagram. We can suggest that in these cases, students truly had a stable understanding of the problem and that this understanding helped them to communicate their reasoning coherently via different media. We can also suggest that the process of constructing and refining the diagram, a process students had been practising in class, helped them to construct and stabilize a coherent mental representation of the problem. Both interpretations suggest that the use of AA diagrams in these cases affected their reasoning in a strongly positive way. This view on the use of graphic representations and diagrams is in line with the works of many researchers studied similar teaching techniques (Ferrucci, Yeap, & Carter, 2003; Fuson & Willis, 1989; Levain et al., 2006; Powell, 2011; Willis & Fuson, 1988; Wolters, 1983; Xin et al., 2008; Xin, 2012).

5.1.6 Theoretical outcomes

5.1.6.1 Factors affecting students' production in word problem solving

The data analysis provided in Chapter 4 concerning Question 1 was inspired by the DeBlois model (2011) of students' production interpretation. DeBlois proposed that student's production in problem solving should be interpreted as an interplay between students' representation of the problem and their perception of the teacher's request. The results obtained through this study suggest the following precisions to the DeBlois model.

First, the interplay between the particularities of the text, reading ability, previous knowledge and presumption about the task can affect the reading/interpretation process

(Kintsch, 2005) and thus provoke the creation of an inappropriate mental representation (structure substitution).

Second, we should take into consideration communication media as an important factor directly affecting students' communication/production. In cases where students master the mathematical knowledge and media or where only one type of media is used, such effects cannot be observed. In other cases, the incoherence of productions via different media can unveil the instability of the mental representation, lack of mathematical knowledge or difficulty using one of the media.

I propose the following model of how different factors affect students' production in word problem solving. Figure 38 shows that multiple factors, such as particularities in the text, reading ability, previous mathematical knowledge and presumptions about the task directly and simultaneously affect the mental representation of the problem formed by the solver. This mental representation can be volatile and fluctuate in time and depending on the requested communication form.

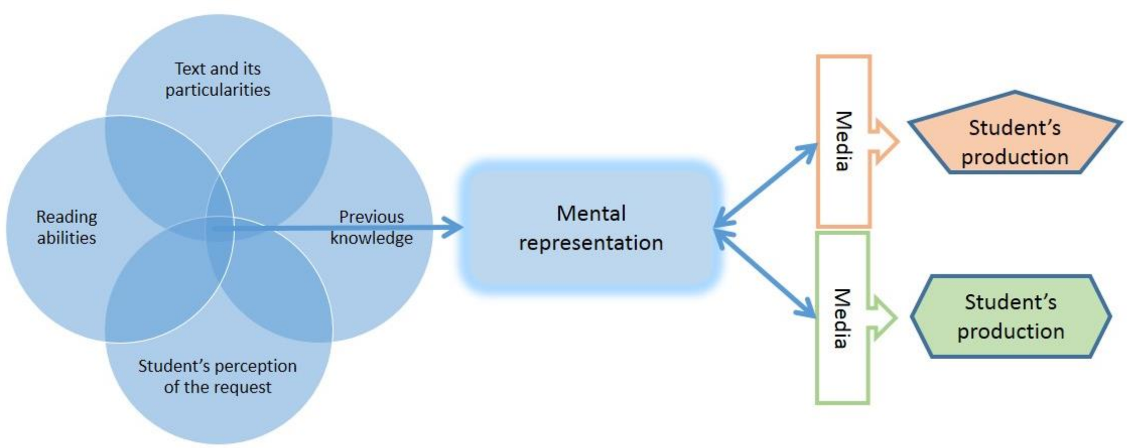

Figure 38. Factors affecting students' production in word problem solving

5.1.6.2 Students' mathematizing processes

I would also like to review models of mathematizing and modeling discussed in Chapter 2: the model developed by De Corte (2012) and the model developed by Savard

(2008). The both of these models propose that students should move from a real-life situation or socio-cultural context towards mathematical context to mathematize problems.

De Corte (2012) and Savard (2008) propose that students' mathematizing of a problem should be based on an understanding of the real-life phenomena described in the task (see the example of Savard's model on Figure 39).

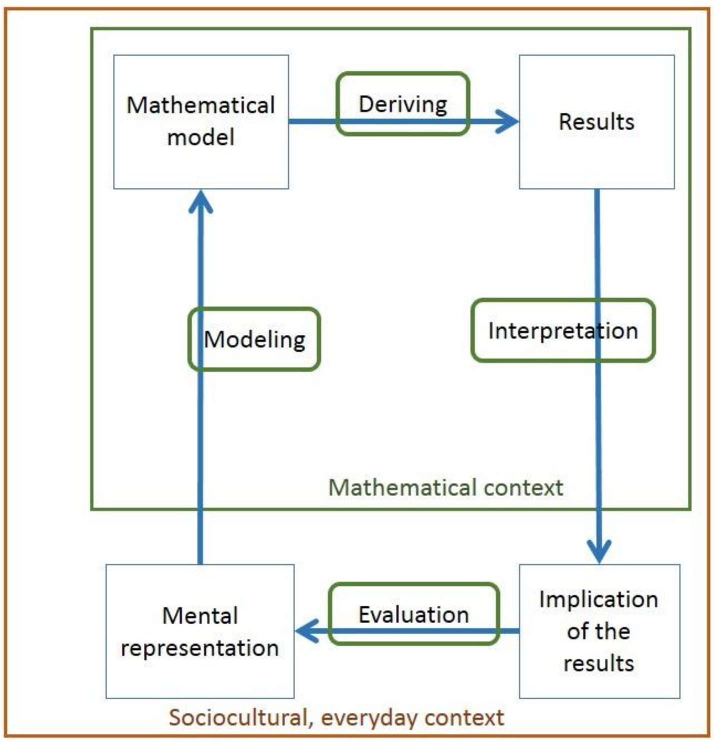

Figure 39. Savard's model simplified (2008)

In my experiment, students first solved problems presented as a text, in which I used very simple contexts and the phenomena of everyday life for students. The students did not demonstrate any difficulty understanding the phenomena per se. Second, as explained above, the mental representation of the problem (situation) students created was mostly affected by the reading and interpretation process, and not by the knowledge about related real-life things, such as fruits or logs burning. Finally, in the rare occasions where students used their previous real-life experience (equal share for all students in the class, two pencils per desk, see section 4.1.1.3), students turned to this knowledge because they failed to understand the problem's structure, and this extra knowledge did not help them to do so. These three reasons lead me to think that, even though both De Corte and Savard models effectively describe the modeling process in general, none of them includes the reading/interpretation part, which in my case is very important. The model developed by Savard seems to be simpler to work with in cases of simple additive word problems. Another aspect in favour of this model is that it represents the mathematical context enclosed inside the sociocultural context. As I explained above, one of the difficulties students experience is the rupture between the situation and the mathematics. Savard's model better address this issue.

I use the central part of the Savard model and adapt it to the case of mathematizing simple additive word problems.

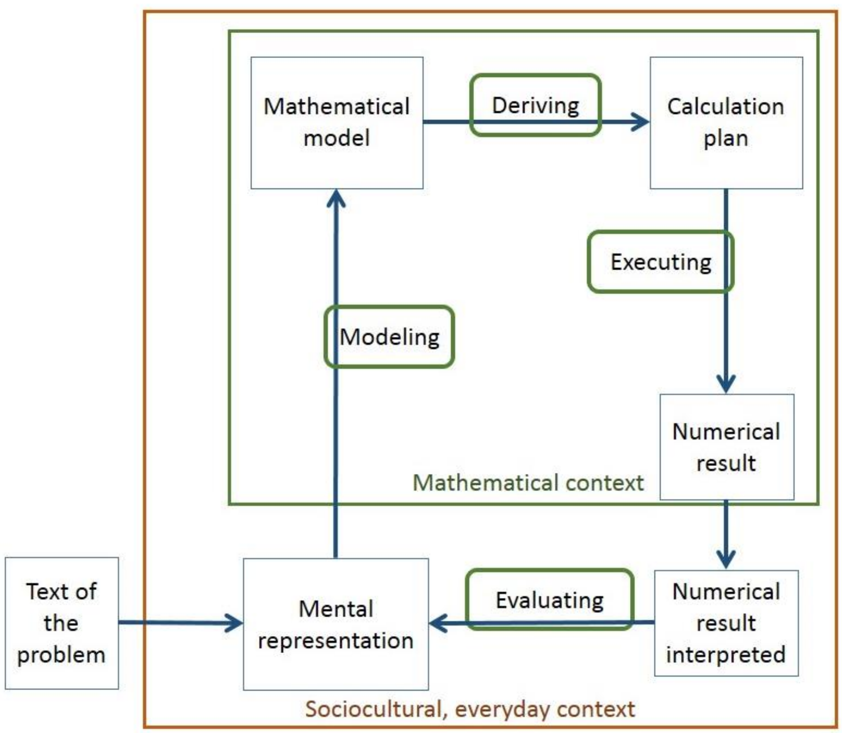

Figure 40. Model of solving simple word problems via mathematical modeling

This adapted model (see Figure 40) includes the text of the problem, which does not belong to any context and should be read and interpreted to form a mental representation related to the sociocultural and everyday context. The result derived from the mathematical model is represented as calculation plan and numerical result. These two new elements better describe what can happen inside the mathematical context.

Ideally, the mental representation should be transformed into the mathematical model through the modeling process. The calculation plan should be derived from this mathematical model and then executed. The numerical answer should be interpreted in the everyday context and evaluated in relation to the mental representation. The entire process can be restarted in case of incoherence. Revising and restarting should include reading over the text again. This is a desired problem solving process.

The mathematical model can be implicit (mental) or explicit (AA diagram, equation, or objects organization). In the cases of adequate use of AA diagrams, we can clearly identify the two steps: mathematical modeling and calculation planning.

In the case of mimicking, the mathematizing process cannot really be seen as modeling. This is a shortcut from the situation to the calculation process or even directly to the numerical result (see Figure 41).

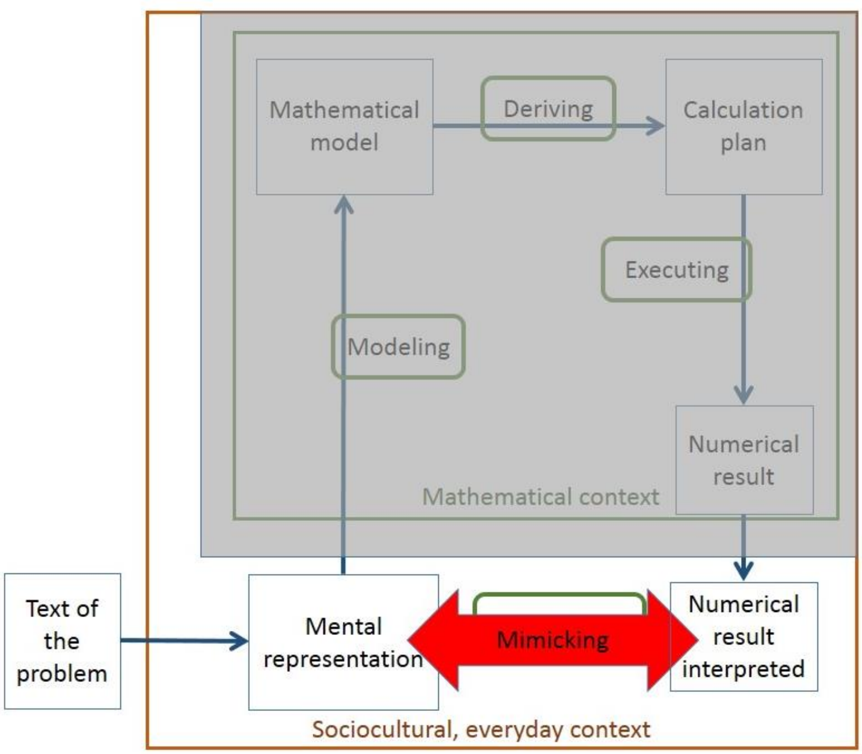

Figure 41. Model of solving simple word problems via mimicking

In cases where multiple means were used to mathematize problems, we cannot say that students really modeled the problem. Rather, they tried to connect their understanding of the situation to different pieces of mathematical context by using different uncoordinated means. Figure 42 presents this process.

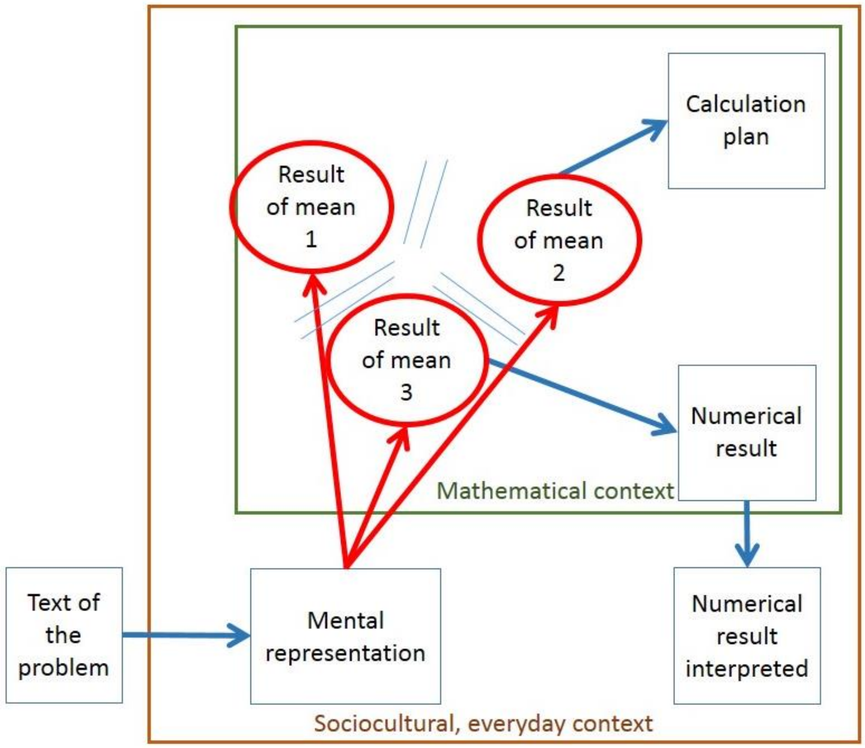

Figure 42. Model of disrupted processes of mathematizing

In some specific cases, such as the direct use of a keyword or a diagram as a template, the mathematizing process can be inverted (please see Figure 43).

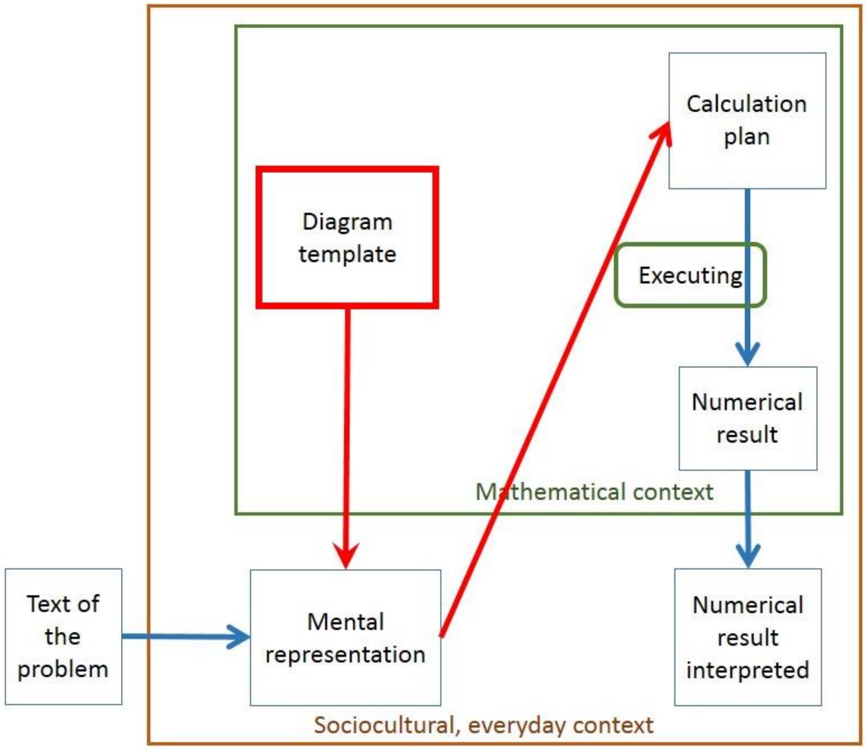

Figure 43. Model of inverted mathematizing process

5.2 Answering Question 2

What are the relationships between the teaching implemented and students' development of mathematizing processes?

5.2.1 Dynamics of students' reasoning development

The students' ways of solving word problems observed in the study gradually changed from one session to another. We can therefore suppose that ways of reasoning also changing accordingly. The number of students who used mimicking was highest in the first session (n = 11). It decreased in the second session (n = 7), and disappeared in the third session. This can be explained by the gradual disappearance of the students' presumptions generate by the previous teaching approach. According to the new teaching approach in the classroom, teachers asked students to construct a diagram to solve the problem. However, during the second session of interviews, no students tried to use diagrams; instead, many of them relied on their mental work only, and many failed to propose a solution. The old way of reasoning was in conflict with the reality of the tasks. The new way of reasoning had not yet been established.

Further along in the study, students increasingly tried to use diagrams to explicitly represent the mathematical structure of the problem. This new way of mathematizing and communicating their thoughts appeared to be a challenge for many. They failed to coordinate their multiple fragments of knowledge of ways of mathematizing accumulated until that moment. This failure to coordination strongly suggests that, at this stage, the mental representation of the problem students had was not a stable construct, but a volatile or fluctuating process.

As learning progresses, more and more students are able to produce coherent diagrams and mathematical expressions and explain their reasoning clearly and coherently. Using diagrams helps students to focus their attention on the mathematics of the problem and explicitly coordinate their analysis of the problem's structure. Thus, they learn to arrange their reasoning to obtain a stable and coherent mental representation of the problem.

Summarizing the knowledge development dynamic, I suggest distinguishing four stages:

1. Old way of mathematizing: mimicking
2. The loss of rules: relying on mental work
3. Trying new rules: using various uncoordinated ways of mathematizing
4. Appropriation of a new way of mathematizing: modeling with AA diagram

I suggest that this dynamic is not universal in learning to problem solve, but depends on the teaching implemented by teachers and experienced by learners.

This dynamic is interesting from a cognitive psychological point of view, because it describes a "revolutionary" (non-evolutionary) type of knowledge development. The old way of reasoning is in conflict with the Relational Paradigm and thus, with the new way of teaching. It is possible that the same type of cognitive conflict happens when students, who have been learning arithmetic for many years, start to learn algebra. Arithmetic is essentially about numbers and calculations, and algebra is about relationships and structures. To avoid this conflict, researchers (Cai & Knuth, 2011; Carraher & Martinez, 2008; Kieran, 1989; Lins & Kaput, 2004; Radford, 2011) propose starting to learn some elements of algebra in early grades of elementary school. Other research (Davydov & Kudriavtsev, 1998; Schmittau & Morris, 2004) confirms that relational reasoning is accessible for very young students. Thus, to avoid the conflict observed in my study, it could be appropriate to start the new way of teaching at grade one.

5.2.2 The dynamic of the "indifferent" group

The development dynamic observed in students from the indifferent group is not the same as for other students. They successfully went from the mimicking stage to mental work stage, but then remained at this stage until the end of the observation period. The absence of challenge in class-work inhibited students from appropriating the new method as their own tool for reasoning and solving. They used it as a way to communicate their understanding, if required by teacher.

5.2.3 Causes of the particular dynamics observed in the study

From my analysis, it follows that the particular knowledge development dynamic can be partially explained by the teaching implemented in classes and experienced by students.

First, the overall progress from mimicking to explicit modeling via AA diagrams and progress in the coordination of reasoning about different parts of the students' production should be explained by the implementation of the Equilibrated Development Approach. This approach aims to organize mathematical reasoning in problem solving according to the cycle of word problem solving described above (see section 2.6.5, p. 51). Continuous practice of reasoning in this way helped many students to acquire the desired knowledge and skills.

Second, in my study, it was impossible to look at each element of the experimental approach separately. All of the approach's characteristics contributed to the students' progress. However, one important didactic element became more visible throughout the study because it was absent at the beginning and only incorporated into the classwork later on. At the beginning, teachers did not ask students to construct diagrams by themselves. The teachers constructed diagrams on the blackboard and discussed them with the whole class. From my analysis, it follows that the students started to spontaneously use diagrams to solve problems only after they were asked to construct diagrams by themselves in class. The request for more active use of diagrams (construction) certainly became the pivot point in the process of learning, turning it from implicit to explicit modeling.

Third, the stages of mimicking and the loss of rules can be caused by students' previous experience in the first grade and their practice of adding and subtracting concrete numbers, which is usually done through mimicking. The time for these two steps can be significantly reduced by adapting and reorganizing the whole curriculum of additive problem solving for the first and second grade.

Fourth, the stage of trying new ways of mathematizing seems inevitable as the students need time to develop the ability to coordinate their reasoning in specific situations of explicit modeling. However, the time required for this development also depends on the teaching implemented in class. From my analysis, it follows that the use of the same diagram template over a long period of time and keywords associated with the parts of such templates prolong the period of inappropriate and inefficient coordination of reasoning. This explanation is in line with the explanations proposed by Wolters (1983). In her experiment, Wolters proposed several activities for students involving part-whole relationships. She explains that this teaching program had positive effects on solving part-whole problems and negative effects on solving join/separate and comparison problems.

Fifth, the lack of challenging problems in the curriculum can partially explain the absence of progress in some students, who appear to be good problem solvers. Researchers points out the necessity of challenging problems to foster students mathematical development (Freiman, 2006; Jackson & Cobb, 2010; Krutetskii, 1976).

My study shows how, for some students, the absence of a real challenge has a negative effect on the appropriation of a new reasoning tool.

5.2.4 Theoretical outcome

My observations and analysis of students solving problems and developing their reasoning are in line with previous research in this area concerning their difficulties (Giroux & Ste-Marie, 2001; Kintsch & Greeno, 1985; Nesher et al., 1982). The majority of researchers associated these difficulties with a lack of cognitive resources available to the learner, including limited working memory and inefficient coordination of data elements. Thus, the development of the problem solving knowledge and ability was conceptualized as a gradual development of these resources through the practice of problem solving. My study raises two issues related to this conceptualization.

First, my results indirectly support the hypothesis of independence and even some competition between sequential and holistic flexible reasoning. I have clearly observed that a strongly developed mimicking strategy prevented students from developing more sophisticated ways of reasoning: relational, structural, holistic. This fact puts the developmental order described by researchers into question. It is not that this order is not correct, but that it is not the only one desired and possible. In the Relational Paradigm, knowledge about relations described in the situation can be studied and learned independently from calculation methods (Davydov, 2008; Sophian, 2007).

Second, from my analysis, it follows that one of the challenges students encountered in the study was the coordination of their reasoning and how to communicate it. This inference is also in line with the previous research as it can be seen as general lack of coordination. However, the practice of such coordination in an explicit way, supported by the use of AA diagrams, helped many students to master (to some extent) the process of solving problems through a specific meta-cognitive process. I have observed students successfully solving difficult problems (according to their level) thanks to this process and likely in spite of limited cognitive resources. This is in line with the conclusions of Focant (2003), who associates students' difficulties in solving word problems with the lack of meta-cognitive skills. In my study, an actual measurement of students' cognitive resources was not performed. However, my data may suggest that the lack of cognitive resources can potentially be compensated for by the development of appropriate meta-cognitive skills. The appropriate meta-cognitive process might then serve as a basis for further development or better use of cognitive resources. Thus, the knowledge (to be developed) associated with additive problem solving should not be limited to mathematical concepts and procedures, but should include particular, well-defined meta-cognitive skills and strategies. The modeling and using AA diagrams appears to be a meta-cognitive tool to be taught and learned.

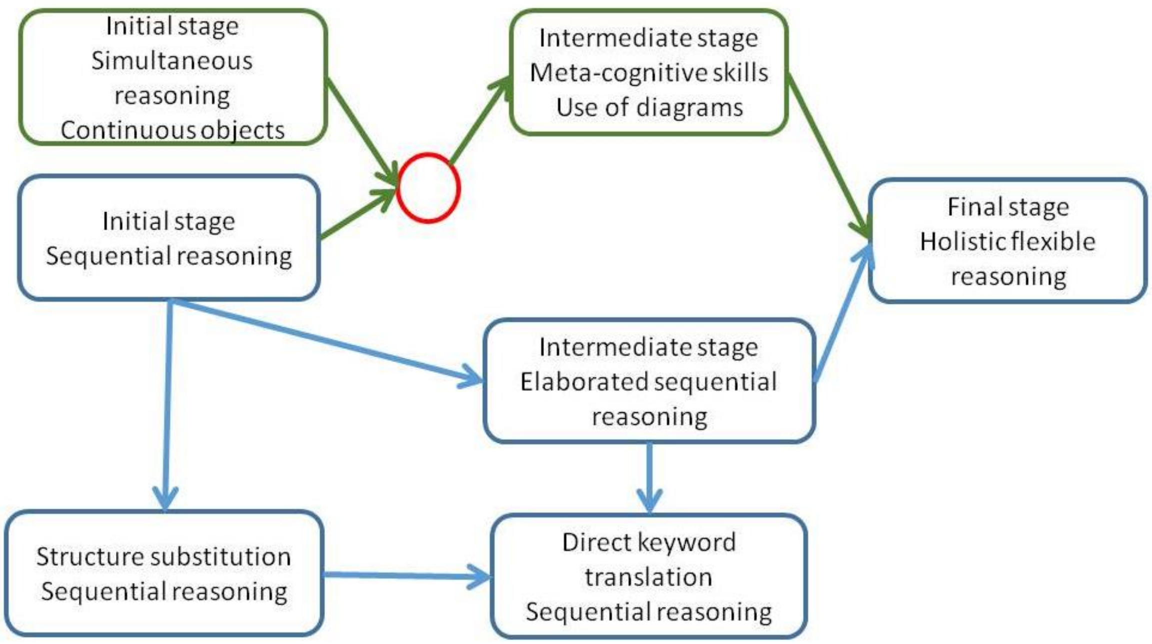

Figure 44. Two ways of developing additive problem solving knowledge

Figure 44 represents two possible ways of developing additive problem solving knowledge. The traditional approach[15] (in blue) relies on sequential reasoning about addition and subtraction. Researchers consider this way of sequential reasoning to be natural and intuitive for students. This approach strongly associates the reasoning about the problem with numbers and calculation (Operational Paradigm). The difficulties

[15] Traditionally used by many, including schools where the study was organized.

associated with this approach, such as the structure substitution phenomenon and direct keyword translation, were discussed in Chapter 2.

The Equilibrated Development Approach (in green) relies on the dissociation of relational reasoning from calculation of numbers (Relational Paradigm). My study shows that holistic relational reasoning can be also natural for students and thus, the teaching approach based on this reasoning can be quite successful.

Chapter 6 Conclusions

Mathematical problem solving in early grades, the knowledge related to solving word problems involving addition and subtraction, as well as the ways students develop this knowledge has held the attention of researchers in mathematics education for a long time. According to research (Lesh et al., 2003), the ability to mathematically analyze and model a situation directly contributes to success in problem solving and the development of critical thinking (Mukhopadhyay & Greer, 2001). Yet, it is widely accepted by researchers that this ability is one of the most difficult to develop. More research is needed to better understand the cognitive processes involved in mathematizing and modeling at a phenomenological level (Hestenes, 2010).

My study aimed to provide a more in-depth understanding of how students mathematize simple additive word problems, describe the dynamics of related knowledge development throughout one school year, as well as find and understand certain relationships between the implemented teaching and the students' success and challenges on this journey. This study was part of a larger project that aims to develop and implement a new approach to teaching additive word problem solving. This approach, the Equilibrated Development Approach, is described in further detail in section 2.6.5.

Grounded in the Relational Paradigm (section 2.1), my study questioned whether students understand and how they come to understand, communicate and use the quantitative relationships described in a written problem. I used the grounded theory research methodology. While drawing inspiration from important core of research in this area (see Chapter 2 for the theoretical exploration), I did not want to limit myself to any particular theory or model. To construct my theoretical inferences, I used 96 observations of students solving and discussing various additive word problems in clinical interviews. I then conducted a retrospective analysis of video-recorded classroom activities to better understand the logic of the students' knowledge development progress.

6.1 How students mathematize problems

My analysis of students' productions during clinical interviews let me distinguish seven strategies students used to mathematize problems, five of which I described in detail. Four strategies—mimicking, using object models, using equations and using keywords—have been previously discussed in the literature (Carpenter et al., 1999; Hegarty et al., 1995). In my study, these ways of solving are described more precisely from the Relational Paradigm point of view. For example, I propose that a distinction be made between mimicking, which is a sequential way of representing a problem, and object modeling, which is based on a holistic vision of the system of quantitative relationships.

The two other mathematizing strategies I described (using an AA diagram as a template and using an AA diagram as an explicit model) are related to the new teaching approach and, to my knowledge, have yet to be described in literature.

Two more phenomena described in my study deserve special attention from researchers and practitioners. *Structure substitution* is the complete and more or less stable structural misinterpretation of a problem that students can construct through the reading of the task. This phenomenon was partially described by Giroux and Ste-Marie (2001) as shift of meaning. Structure substitution redirects students' attention towards a completely different mathematical structure and thus prevents them from learning from the initial task.

The *use of multiple uncoordinated strategies* to mathematize the same word problem is another interesting phenomenon I observed. I explained it by the instability of the mental representation in students affected by the need to use different communication media. The request for various forms of communication (oral explanation, AA diagram, mathematical expression) makes it more difficult for students to reason mathematically, but helps to reveal the lack of stability and vulnerability of this reasoning. More research is required to understand how to help students to learn to coordinate different means of mathematizing and how to avoid the conflicts between these means.

I have also described four stages in students' knowledge development in relation to the implementation of the new teaching approach. These stages are quite specific to the implementation and conditions of the study. However, one generalization can be made: the stage of the loss of rules (described in section 5.2.1) was provoked by a drastic shift in the teaching paradigm from Operational to Relational. More research is needed to understand how the change in teaching approaches and philosophies, for example from arithmetic to algebra or from discrete objects to continuous, can affect students' learning.

6.2 The Equilibrated Development Approach as the positive cause of knowledge development in students

The approval or disapproval of the experimental teaching approach is not within the scope of my study. However, I analyzed some casual relations between the implementation of this approach and students' learning. My data analysis shows that the Equilibrated Development Approach generally had a strong positive effect on students' development of ways to mathematize and model additive problems. The following didactic elements distinguish this approach from all other approaches discussed in Chapter 2:

- Separating the analysis of the problem from calculation
- Explicit request for the representation of the mathematical structure of the problem and not numbers or operations
- Using AA diagrams and the explicit request for the constructing of such diagrams
- Explicit request for the coordination between text, diagram and mathematical expression

My analysis shows that the implementation of these elements in teaching took students from mimicking, past implicit modeling, towards explicit mathematical modeling in problem solving. In this study, it was impossible to evaluate the positive contribution of each didactic element separately, nor was it the objective. Learning is a very complex phenomenon and only an appropriate combination and interplay of multiple didactic elements can explain a positive learning outcome (Cobb et al., 2003).

6.3 Various learning challenges and their causes

My observations helped me to identify various challenges students can face while solving problems and learning to use AA diagrams. First of all, the reading and interpreting of the text of a problem deserves special attention. Even though I tried to avoid unknown words and contexts in the interview tasks, I observed other challenges that reading can bring about for the learner:

- The use of long and complex determinants when referring to quantities in the text of a problem can obscure the general semantic structure of the situation described (case of the Houses problem).
- Some verbs (for example "burned" in the Logs problem) can be strongly associated with a particular well-known semantic meaning (for example remove) and thus induce the structure substitution phenomenon, preventing students from appropriate interpreting the intended task.
- Additional irrelevant data, which becomes a part of the additional semantic structure, can prevent learner from recognizing the semantic structure relevant to the question of the problem (case of the Fruits problem).

These challenges come from students' lack of related knowledge and/or ability. According to the CI model (Kintsch, 2005) and in line with the conclusions made by Voyer (2009), the presence or absence of these challenges in a problem can directly affect the reading/interpretation process and thus, the accessibility of the problem for the learner. At the same time, practising facing such challenges can help learners to develop appropriate knowledge/ability. Thus, teachers should carefully manage the use of these challenges in the teaching/learning process based on the various needs of their students.

When students try to mathematize a problem, other challenges can arise:

- A previously mastered way of representation—mimicking the story—as well as a student's associated presumption can induce a sequentialization of the reasoning process and prevent students from performing a holistic flexible analysis of the mathematical structure of the problem. Thus, students are unable to produce the mathematical expression in a standard form for inconsistent problems.

- The diagram representation method itself as well as the multiple forms of communication requested from the learners proposes an additional challenge in coordinating their reasoning and communication.

The roles of these two challenges are different. The first one, related to mimicking, is mainly caused by the teaching approach and can be significantly reduced, if not avoided, with the appropriate teaching.

The second one, related to the coordination of different media, seems to be caused by the absence of appropriate knowledge and ability and might thus be considered a learning target (Vygotsky, 1984). From my observations, it follows that the practice of this specific coordination of reasoning is the main engine of the development of the explicit holistic analysis and modeling in students. The meta-cognitive process of solving a word problem organized as a problem-solving cycle can provide the effective and efficient support for such development.

Some other learning challenges worth mentioning:

- Too much time spent studying the same type of problems (for example, comparison problems) and the related type of diagram can provoke the idea that the particular diagram can be used as a universal template. This can prevent students from more profound reasoning and knowledge transfer.
- The use of designated words associated with the diagram representations can obscure for learners the meaningful links between the story (situation) and its representation.
- The absence of an adequate challenge in the proposed tasks can prevent learners from appreciating the newly developed knowledge as a useful tool and more profound knowledge development.

All these challenges can be reduced if not avoided via reorganization of the curriculum content and adaptations of the classroom work.

6.4 Theoretical value of the study

Based on the theoretical inferences produced through the study, I proposed two theoretical models to understand students' ways of mathematizing word problems. First, I proposed a model (see section 5.1.6.1) to explain factors affecting how students construct mental representations and their production in the process of problem solving.

Second, I adapted the model of mathematical modeling proposed by Savard (2008) to explain various means students use to mathematize problems (see section 5.1.6.2).

The problem solving cycle proposed in the second model can be seen as a meta-cognitive organization of the problem-solving process. The use of the AA diagrams through this cycle makes this meta-cognitive organization a powerful didactic tool, providing students and teachers with the effective communication media to discuss the mathematical relationships and mathematical structure of a problem. This way of communication helps students to liberate their reasoning from concerns about calculation and concentrate their mental effort on the structure and relationships.

The phenomenon of using AA diagram as template described in this study helped to understand inverse mathematizing in students' reasoning (see section 5.1.6.2). This phenomenon should be considered when any type of graphic schema or diagram is used to help young students solve problems. Educators should pay special attention to students' reasoning to keep it grounded in the situation described in the task and not in graphic schemas or templates.

6.5 Value of the study to the larger experiment

My study was organized within the larger scope of a collaborative project led by Professor Savard. Thus, my study met multiple needs and requests of the larger experiment. The analysis and conclusions of my study helped to rearrange the experimental curriculum to avoid overgeneralization of particular types of diagrams and other learning challenges. Following my recommendations issued from this study, the implementation of experimental didactic scenarios was elaborated and adjusted. More educational activities were designed. The teacher education program related to experimental teaching was also revamped.

6.6 Limitations of the study

This study was limited in several terms. First, the study was based on observations made in only one school located in a rural area. Second, all observed students experienced the same teaching approach in word problem solving. Observations of students learning in any other teaching conditions were not analyzed. Third, the number of participants (n = 12) is small and the participants were not exposed to all types of additive problems in a systematic way. Thus, an adequate statistical analysis cannot be made using this data. For the same reason, a comparative gender analysis was not performed. All these limitations should be considered while generalizing and using the results of this study.

6.7 Perspectives on teaching and future research

In spite of some limitations, the study opens new perspectives on teaching additive problem solving and future research in this area. The analysis provided in the study highlights the value of the Relational Paradigm as the imperative tool for the research and practice.

In the area of mathematical reasoning development in early grades, the Relational Paradigm helped to see the gap between sequential and relational reasoning learners might have in relation to word problems with additive structures. Approaching some other areas of mathematics education, such as multiplicative relationships and fractions, within the Relational Paradigm might also produce new valuable scientific results.

Viewing classroom practices through the lens of the Relational Paradigm helped to identify pedagogical elements that support and hinder the development of relational mathematical reasoning in students. The study suggests that the development of relational reasoning should be initiated from the very beginning of the mathematical instruction in school, in parallel with other types of reasoning to prevent later cognitive conflicts and obstacles. Particular attention should be paid to providing learners of all levels in mathematics with adequate challenges to support their reasoning development.

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Appendix 1 Tables

Table 6 *Using Tasks for Participants*

Participant name	**Participant relative force in mathematics**	**First interview session**			**Second interview session**		**Third interview session**		**Fourth interview session**	
		Problem 1	Problem 2	Problem 3	Problem 1	Problem 2	Problem 1	Problem 2	Problem 1	Problem 2
Elodie	Weak	Logs	Pencils *		Houses*		Tokens	Snow balls 1	Cards	
Josef	Weak	Tokens	Logs		Fruits	Houses	Snow balls 1		Posts	
Michael	Strong	Logs	Pencils		Fruits	Cards	Snow balls 1	Snow balls 2	Year	Posts
Maria	Average	Logs	Pencils		Fruits	Houses	Snow balls 1	Snow balls 2	Posts	Butcher
Rosa	Strong	Logs	Pencils		Fruits	Ski	Snow balls 2	Snow balls 3	Year	Posts
Eva	Average	Logs	Pencils		Cards		Snow balls 1	Houses	Fruits	Posts
Cathy	Strong	Logs	Pencils		Cards	Ski	Snow balls 2	Snow balls 3	Year	
Emma	Weak	Logs	Tokens	Pencils	Houses		Snow balls 1	Snow balls 2	Posts	

Philippe	Weak	Logs	Tokens		Houses		Snow balls 1	Snow balls 2	Cards	
Nicolas	Strong	Logs	Tokens	Pencils	Fruits	Ski	Snow balls 2	Snow balls 3	Year	Boucher
Nadia	Average	Logs	Tokens		Fruits	Houses	Snow balls 1	Snow balls 2	Year	
Viktor	Average	Logs	Tokens	Pencils	Fruits	Cards	Snow balls 1	Snow balls 2	Posts	

Table 7 *Comparative Table of Problems' Classifications*

Categories	Place of unknown	Classification (1)	Classification (2)	Classification (3)	Classification (4)	Example
Simple positive change	Result unknown	Transformation links two measures	Change	Change	Join	*Peter had 17 marbles before playing. Tom gave him 4 marbles. How many marbles does he have now?*
	Start unknown	Transformation links two measures	Change	Change	Join	*Peter had some marbles before playing. He won 4 marbles. He now has 13 marbles. How many marbles did he have before playing?*
	Change unknown	Transformation links two measures	Change	Change	Join	*Peter had 7 marbles before playing. He won some marbles. He now has 13 marbles. How many marbles did he win?*
Simple negative change	Result unknown	Transformation links two measures	Change	Change	Separate	*Peter had 17 marbles before playing. He lost 4 marbles. How many marbles does he have now?*
	Start unknown	Transformation links two measures	Change	Change	Separate	*Peter had some marbles before playing. He lost 4 marbles. He now has 13 marbles. How many marbles did he have before playing?*
	Change unknown	Transformation links two measures	Change	Change	Separate	*Peter had 17 marbles before playing. He lost some marbles. He now has 13 marbles. How many marbles did he lose?*
Simple compare	Positive action (difference) unknown	Static relationship links two measures	Compare	Equalizing	Compare	*Joe has 3 marbles, Tom has 8 marbles. How many marbles does Joe need to have as many marbles as Tom?*
	Negative action (difference) unknown	Static relationship links two measures	Compare	Equalizing	Compare	*Joe has 8 marbles, Tom has 3 marbles. How many marbles does Joe need to have as many marbles as Tom?*

Simple combine	Total unknown	Composition of two measures	Combine	Combine	Part-part-whole	*Peter has 6 marbles in his right pocket and 8 marbles in his left pocket. How many marbles does Peter have altogether?*
	Part unknown	Composition of two measures	Combine	Combine	Part-part-whole	*Peter has 6 marbles in his right pocket and some marbles in his left pocket. He has 14 marbles altogether. How many marbles does Peter have in his left pocket?*
Simple compare	Difference unknown	A static relationship links two measures	Compare	Compare	Compare	*Tom has 6 marbles. Joe has 4 marbles. How many more marbles does Tom have than Joe?*
	Compared unknown	A static relationship links two measures	Compare	Compare	Compare	*Tom has 6 marbles. Joe has 4 marbles more than Tom. How many marbles does Joe have?*
	Referent unknown	A static relationship links two measures	Compare	Compare	Compare	*Tom has 6 marbles. He has 4 marbles more than Joe. How many marbles does Joe have?*

Table 8 *Summary of the First Interview Session*

Student	Problem	Observations	Mathematizing
Elodie	Logs 34 - ? = 27	Orally proposes drawing and removing circles until 27 are left	Mimicking
	Pencils 46 + 23 = 34 + ?	No solution	Cannot be identified
Josef	Logs 34 - ? = 27	Orally proposes drawing 34 circles and then crossing out some, may be 20	Mimicking
	Tokens 13 - 7 = ?	Orally proposes comparing left and right hands. Then draws, removes, and displaces circles according to the original story.	Structure substitution + Mimicking
Emma	Logs 34 - ? = 27	Orally explains that 27 logs were burned. She draws 34 circles and crosses out 27.	Structure substitution + Mimicking
	Tokens 13 - 7 = ?	Draws 13 circles then divides them into two big circles 7 (on her left) and 6 (on her right)	Mimicking
	Pencils 46 + 23 = 34 + ?	Re-reads the data from the text many times. Starts to draw the first essential number (46) using circles. Proposes distributing blue and red pencils to boys and girls equally and giving the remainder to Jeremy.	Cannot be identified
Philippe	Logs 34 - ? = 27	Orally explains that he does not know how to solve. Using tokens, constructs 34, then counts 27 many times. Finally, arranges tokens to obtain 27 and "others," then counts others. Explains that he removed 7. For the question "how many logs were burned?" answers "48".	Object model
	Tokens 13 - 7 = ?	Manipulates tokens according to the story. Puts them in his left and right hands.	Mimicking
Maria	Logs 34-?=27	Orally proposes drawing 34 logs and removing until 24 are left	Mimicking

	Pencils 46 + 23 = 34 + ?	Tries to make sense of the problem by applying a process of giving pencils to students. No solution.	Cannot be identified
Eva	Logs 34 - ? = 27	Proposes drawing 34 and removing 27. Explains that 27 were burned	Structure substitution + Mimicking
	Pencils 46 + 23 = 34 + ?	Tries to make sense of the problem by applying a process (drawing desks and giving pencils to students). No solution.	Cannot be identified
Nadia	Logs 34 - ? = 27	Explains that she looks at numbers first. Draws 34 circles, then crosses out by counting backward (34 - 33 - 32 -...)	Mimicking
	Tokens 13 - 7 = ?	Orally follows the story. Draws circles to represent numbers (13 and 7). Counts 13. Declares that there are 13 in the right hand.	Mimicking
Viktor	Logs 34 - ? = 27	Orally proposes drawing 34 and removing until 27 are left	Mimicking
	Tokens 13 - 7 = ?	Proposes seeing on fingers on the right and on the left, then proposes an equation: 13 — 7 = 7. Explains that one should calculate the correct value to replace "= 7".	Mimicking
	Pencils 46 + 23 = 34 + ?	Proposes drawing circles for "whatever in the problem" and counting the answer. No solution.	Cannot be identified
Michael	Logs 34 - ? = 27	Explains that 27 logs were burned. Tries to calculate on his fingers. Explains that he needs to draw. Draws 34 circles and crosses out 27.	Structure substitution + Mimicking
	Pencils 46 + 23 = 34 + ?	Explains that blue pencils are for boys because 34 are for boys. No solution	Cannot be identified
Rosa	Logs 34 - ? = 27	Orally proposes drawing 34 logs and crossing out until 27 are left	Mimicking

	Pencils 46 + 23 = 34 + ?	Orally explains that 34 pencils are for boys and the rest is for girls. Proposes a solution plan: draw blue pencils and red pencils then remove 34 for boys.	Mental holistic
Cathy	Logs 34 - ? = 27	Orally proposes starting with packages, a package of 34, then removing until 27 are left	Mimicking
	Pencils 46 + 23 = 34 + ?	Orally proposes doing 46 pencils and 23 other pencils then to give 21 to boys	Possibility of mental holistic model
Nicolas	Logs 34 - ? = 27	Orally proposes that 27 logs were burned, thus needing to draw 34 and cross out 27	Structure substitution + Mimicking
	Tokens 13 - 7 = ?	Orally proposes drawing 13 and removing 7	Mimicking
	Pencils 46 + 23 = 34 + ?	Orally proposes drawing 46 circles and 23 circles then crossing out 34 circles for boys	Possibility of mental holistic model

Note: I briefly describe my observations and the mathematizing students possibly used. This data about students' reasoning should be seen as the most probable interpretation and not as only one possible.

Table 9 *Summary of the Second Interview Session*

Student	Problem	Observations	Mathematizing
Emma	Houses 64 = X + 37	Proposes to separate 37 houses into two stories and three stories. “There are not many houses on my street.” No solution for the initial problem.	Cannot identify
	Houses 8 = X + 3	Proposes counting houses, and then proposes drawing 8 houses, circling 3 and counting the others	Mental Holistic
Elodie	Cards 28 + X = 52	No solution	Cannot be identified
Josef	Fruits 45 = X + 17	Proposes putting pears into the basket. No solution	Difficult to interpret
	Houses 5 = X + 2	Proposes the answer (=3) first. Explains that he counted back	Mental
	Houses 64 = X + 37	No solution	Cannot be identified
Philippe	Houses 64 = X + 37	No solution	Cannot be identified
	Houses 8 = X + 3	Proposes the answer 5 first. Then explains his calculation as 8 - 3 = 5.	Mental holistic for small numbers

Maria	Fruits 45 = X + 17	Misinterprets the expression “17 apples more.” Uses the line diagram to represent the problem. Explains that the bigger line is for apples, because there are more apples than pears. Has trouble recalling data from the problem. Has trouble attributing data to the diagram. Concludes from the diagram that there are 45 pears.	Diagram
Nadia	Fruits 45 = X + 17	Repeats “17 more apples than pears” many times. Explains that it is impossible to have 20. It is impossible to have more pears than apples. She draws 17 circles (apples). Finally, she proposes to draw 45 apples and then 17 apples more. Says she does not remember anything they did in autumn.	Mental Holistic + Keyword
	Houses 64 = X + 37	Has difficulty understanding the problem. Finally, proposes to draw 37 big houses, explains that we cannot count small houses, but “then we should put them.” Gets blocked.	Cannot be identified
Eva	Cards 28 + X = 52	Possibly misinterprets the problem, proposes to calculate: (63 - 28 - 52).	Difficult to interpret
Viktor	Fruits 45 = X + 17	Proposes calculating on fingers or mentally. No solution.	Difficult to interpret
	Cards 28 + X = 52	Proposes considering 52 cards and the answer, and says that there are 63 invitation cards. No solution.	Cannot be identified
Michael	Fruits 45 = X + 17	First, sees the problem as Combine (45 + 17). After the clarification, proposes drawing 45 circles (apples) and then crossing out 17.	Mental holistic
	Cards 28 + X = 52	First, tries to count on his fingers. Second, proposes drawing 28 cards and continuing to obtain 52.	Mimicking

Rosa	Fruits 45 = X + 17	Proposes drawing apples and then circling 17 of them. Then, counts the rest (for pears).	Mental holistic model
	Ski X + Y = 54 X = Y + 15	No solution. Has difficulty representing unknown quantity.	Cannot be identified
Cathy	Cards 28 + X = 52	Proposes drawing 28 cards and continuing to obtain 52. She draws 2 tens and 8 units. Then draws circles to obtain 52 in total.	Mimicking
	Ski X + Y = 54 X = Y + 15	Proposes taking 54 and distributing between ski and boots to obtain the price of ski 15 more than boots. Obtains (20 + 34).	Balance
Nicolas	Fruits 45 = X + 17	Proposes doing 45 - 17	Mental holistic model
	Ski X + Y = 54 X = Y + 15	First, proposes taking 15 and counting up to obtain 54. Then proposes sharing equally and then taking some from boots and giving to ski to obtain the difference of 15. Calculates the half of 54 (27) then removes 15 from 27 and adds it to the other 27.	Balance

Table 10 *Summary of the third interview session*

Student	Problem	Observations	Mathematizing
Elodie	Tokens 13 - 7 = ?	Explains the problem using her hands. Explains that we need a mathematical expression to solve the problem. Writes 13 + 7 = (incorrect, does not correspond to her oral explanation of the problem). For the representation, draws a template and put the two numbers for two parts and the "?" for the total. Representation corresponds to the math expression.	Cannot be identified
	Snowballs 1 ? - 37 = 25	Proposes the Math expression 37 + 25 = (correct). For the representation, draws a segment and put 37. Put 25 for the part of the segment. Representation does not correspond to the math expression.	Mental holistic + Template
Josef	Snowballs 1 ? - 37 = 25	Proposes mathematical expression 37 + 25. Explains that we need to know what was at the beginning. Representation corresponds to the math expression and the problem.	Mental holistic
Emma	Snowballs 1 ? - 37 = 25	Proposes drawing diagram. Draws a template, on my request put numbers correctly. Explains that 37 are "already there" and 25 "remain." Proposes the math expression 37 - 25 = (incorrect). Explains that when one throws out, it makes minus.	Template+ Keyword
Emma	Snowballs 2 ? = 17 + 58	Proposes to draw diagram. Draws a big segment, writes 58 on it and explains this is for Thomas. Put 17 on a part of this segment (also for Thomas). Proposes to find the other part of the segment (incorrect). Has difficulty to explain the representation in terms of the problem. Gets blocked.	Template
Philippe	Snowballs 1 ? - 37 = 25	Explains that he mentally put 3 +2 together and 7+5. Explains that he did minus, because "he threw 37." Then explains that we need to do + to find what was at the beginning. Does not remember how to draw diagrams.	Mental holistic

	Snowballs 2 ?=17+58	Explains that Greg has more than Thomas. Proposes the math expression 58 + 17 =. For the representation, draws a big segment and divides it in two parts: left for Thomas and right for Greg (does not correspond to the need of the problem—comparison).	Mental holistic
Maria	Snowballs 1 ? - 37 = 25	Explains that we need to put two numbers together to find out how many there were at the beginning. For the math expression: 37 - 25 (incorrect). For the representation: corresponds to the problem, well explained. After drawing the diagram, confirms her math expression, which is incorrect and does not correspond to the diagram.	Mental holistic + Keyword
	Snowballs 2 ? = 17 + 58	Explains that we need to count "this" (58) and "this" (17) because Greg has 17 more than Thomas. Mathematical expression. For the math expression: 58 + 17 (correct). For the representation: draws a one-segment template divided into two parts (does not correspond to the need of the problem—comparison). Has difficulty to associate the representation with the problem.	Mental holistic
Eva	Snowballs 1 ? - 37 = 25	Immediately starts to draw a diagram. Re-consult the text many times. Diagram corresponds to the problem, well explained. Math expression 37 + 25 (correct).	AAD as model
	Houses 64 = ? + 37	Diagram corresponds to the problem, well explained. Math expression: 64 - 37 (correct).	AAD as model
Nadia	Snowballs 1 ? - 37 = 25	Immediately starts to draw prototype without numbers. Finally, diagram corresponds to the problem. Math expression: 37 - 25= (incorrect).	Template + Keyword
	Snowballs 2 ? = 17 + 58	Immediately starts to draw template without numbers (one segment). The diagram does not end up corresponding to the problem's need. Math expression: 58 + 17 = (correct and corresponds to the diagram). Has difficulty explaining her diagram in terms of the problem. Explains that Greg made 17 more Thomas, but on the diagram points to the 17 segment and says that this is Greg's part and the other part is for Thomas.	Template + Keyword

Viktor	Snowballs 1	Immediately starts to draw template without numbers (one segment). Uses the keywords “already there” and “added.” Has difficulty associating the template with the problem. Gets blocked.	Template
Viktor	Snowballs 2 ? = 17 + 58	Immediately starts to draw a segment, says that it is 58 for Thomas. Then continues the line, says it is 17 for Greg (does not correspond to the need of the problem—comparison). Math expression: 58 + 17 = (correct)	Template
Michael	Snowballs 1 ? - 37 = 25	Proposes to do 37 balls and add 25. Math expression: 37 + 25 = (correct). For the diagram: draws a template, then puts numbers correctly and explains well.	Mental holistic
	Snowballs 2 ? = 17 + 58	Immediately starts to write a mathematical expression. Thinks a bit about the operation, first puts “-”,then corrects to “+”: 58 + 17. Explains the expression correctly. For the diagram: starts with the template, has difficulty inserting numbers. Finally proposes to put 58 for the left part of the segment (Thomas) and 72 for the right part of the segment (Greg).	Mental holistic
Rosa	Snowballs 2 ? = 17 + 58	Explains that she cannot explain without diagram. Starts with a one-segment template. Puts 17 for the left part and 58 for the right part (does not correspond to the need of the problem—comparison). Math expression: 17 + 58 = (correct and corresponds to the diagram). Explains that Greg has 17 more than Thomas. Explains that the diagram represents the balls for both boys. Finally affirms that the boys have 75 altogether.	Template + Mental holistic
	Snowballs 3 Y = x + 17 X = 28 + 33	Starts with two-segment diagram without numbers. Explains that the bigger segment is for Greg because he has more than Thomas. Explains that she will add 28 and 33 to know how many Thomas has, then will calculate for Greg. Has difficulty associating the data with the diagram: Put “?” on the left part of the Greg’s segment (should be the whole segment).	AA diagram
Cathy	Snowballs 2 ? = 17 + 58	Proposes to take 58 and add 17. For the representation, starts with a one-segment template. Explains that the left part is 58 for Thomas and the right part represents how many more Greg has than Thomas.	Mental holistic

	Snowballs 3 Y = x + 17 X = 28 + 33	Refers to the previous problem to find information. After clarification explains that she first needs to find for Thomas 28 + 33, then she will find for Greg, as he has 17 more than Thomas. For the representation: first starts to calculate using tens and units. Finally draws a two segment diagram (correct) with the shorter segment divided into two parts. Explains her diagram correctly.	Mental holistic
Nicolas	Snowballs 2 ? = 17 + 58	Explains that we need to add 58 and 17 to find for Greg. For the representation: starts with tens and units. Then explains that he cannot do for Greg immediately because he does not know how many Greg has. Starts to draw circles. Finally draws a correct diagram (two segments for comparison), which he explains well.	Mental holistic
	Snowballs 3	Explains that to find for Greg we need to add 28, 33 and 17, because 23 + 33 equals Thomas's snowballs and Greg has 17 more than Thomas. For the representation: says that he does not know. Finally draws a correct diagram which he explains well.	Mental holistic

Note: I briefly describe my observations and mathematizing students possibly used. This data about students' reasoning should be seen as the most probable interpretation, and not as only one possible.

Table 11 *Summary of the fourth interview session*

Student	Problem	Observations	Modeling
Elodie	Cards	Says that she needs to draw a line (diagram). Draws a diagram which corresponds to the problem. Identifies the elements well. Math expression: 28 + 52. Proposes to use tens blocks. Calculates the sum = 80. Proposes that it should be subtraction. Writes 28 - 52 =, thinks a few seconds, erases and writes 52 - 28 =. Starts to calculate using tens blocks. Adds instead of subtracts. Recalculates correctly after a prompt. Correctly identifies the role of the found number in the problem and on the diagram.	Template + Keyword
Emma	Posts	Starts to draw one segment diagram template. Explains that she represented “Total” which is 265 for the red post. Shows the left part and explains that it is for red post, which is 37 cm. Explains that for green posts “we don’t know,” and “already there” will be 37 (diagram incorrect). Mathematical expression: 265 + 37 = (corresponds to the problem, does not correspond to the diagram).	Template +Mental holistic
Philippe	Cards	Says that one needs to make more than 10 cards to complete the work (structure substitution). Has difficulty understanding the problem. Proposes to add 28 + 52, then to subtract, then to add. Gets blocked. For the representation, asks whether he should draw cards as squares. Has difficulty constructing a diagram.	Cannot be identified
	Posts	Immediately proposes to draw posts. Draws two vertical lines and explains that the smaller one is red and the bigger one is green. When prompted, put the numbers on the diagram correctly. For the math expression: 265 + 37 =	AAD as model
Josef	Posts	Explains that he will use ten-blocks and “strings”, first “strings”, then blocks. Starts to draw one-segment template and proposes orally the math expression 265 - 37. Explains that 37 goes here (left part of the segment), because 37 posts were “already there,” put 37 on the right part. Explains that “at the beginning” there were 265 posts and put 265 on the left part (diagram does not correspond to the need of the problem. Writes the math expression 265 – 37 = (does not correspond to the problem or to the diagram).	Template
Maria	Posts	Explains that we need to draw two segments, that red post is smaller than the green one. For the representation: draws two horizontal segments, short for the red posts and long for the green. Shows left part of the bigger segment and explains that this is 265 which is equal to the short segment (red). Shows	AAD as model

		right part of the bigger segment and explains that this is the difference = 37cm. For the math expression: 265 + 37 =.	
	Butcher	First, draws three big circles to represent three packs of meat. Then draws 9 small circles to represent 9 kg. Gets blocked.	Object model
Nadia	Posts	Immediately starts to draw one-segment template. Has difficulty showing where the green post is represented on her drawing. When asked to represent what is in the problem, explains that here are two numbers and circles 37 and 265. Explains that red post is smaller than green one (correct). Put 37 on the left part of the segment and 265 for the total (incorrect). When asked to show on her representation that red post is smaller than green one, says that she does not know.	Template
	Calendar	Has difficulty understanding the problem. After clarification, draws the one segment template. Says that the segment represents “the total.” Circles 198 in the text and put it on the right part of the segment. Explains that it is school days. For the mathematical expression: 365 - 198 = (this conclusion comes from the text, not from diagram).	Template
Eva	Fruits	Proposes to draw two segments diagram. Identifies elements in coherence with the problem. Writes the math expression: 45 - 17 =.	AAD as model
	Posts	Draws two horizontal segments and shows the difference. Explains that the bigger one is for the red post and the smaller one is for the green. Explains that she does not know the difference. I advise her to re-read the problem. After re-reading the problem, she explains that on the contrary, the big segment represents the green post and the small one the red post. Puts the “?” on the big segment. Puts 37 on the small segment and 265 for the difference part (incorrect). I advise her to re-read the problem one more time. When asked whether the red posts measures 37 cm, recognizes the error. For the mathematical expression: 265 + 37.	AAD as model
Viktor	Posts	Proposes to do math expression and starts to draw two segments diagram. Explains that “here is 37 shorter, so 37 goes here” (left part of the bigger segment (incorrect)). Explains that the smaller segment is equal to the left part of the bigger one (correct), put 37 on the small segment. Explains that the	AAD as model

		difference is more, so "265 goes here" (right part of the bigger segment, which represents the difference on the diagram).	
Michael	Calendar	Proposes the math expression 365 – 198 =. Explains that it will be the days without school.	Mental holistic model
	Posts	Proposes the math expression: 265 - 37 = (incorrect, structure substitution). For the representation, draws two posts (vertical rectangles). Explains that the bigger one is red and the shorter one is green (incorrect). I advise him to re-read the problem. After reading the first sentence, recognizes the error and corrects the math expression: 265 + 37. When prompted, correctly identifies elements of the diagram.	Mental holistic model
Rosa	Calendar	Draws two-segment diagram. Puts 10 for the small segment and 198 for the big segment, then makes small segment longer to obtain the same length as for big one. When asked whether the two segments are equal, says that her representation is not good. Draws one segment template, puts 198 on the left part and 10 on the right part. Explains that the total segment represents all days of the year. Says that representation is not good. Explains that from both representations she cannot find what is requested. I re-read the problem and ensure that she understands each expression.	Template
	Posts	Draws two-segment diagram template, put 37 on the difference part. Put "?" on the small segment, then on the left part of the big segment (which is equal to the small segment). Explains that small segment is the red post, the smaller one. Explains that the big segment is the green post. Explains that 37 means "bigger than the red post." When asked about "?", recognises that this is known and put 265. For the math expression: 265 + 37 = (correct).	AAD as model
Cathy	Calendar	Proposes to do 365 and then "cross out" 198 to obtain the number of days without school. Calculates using ten-blocks drawing.	Mental holistic model
Nicolas	Calendar	Proposes to do 365 - 198, because 198 are days without school.	Mental holistic model
	Butcher	Has difficulty understanding the problem. Proposes to separate in two parts. Proposes to put one on the left and one on the right and continue this way to separate the 17. Gets blocked. For the representation:	Object model

	draws one segment template, put "?" for the total. Explains that the total is what we are looking for. Explains that there is no difference where to put 17, on the left part or on the right part. Puts 17 on the left. Gets blocked.

Note: I briefly describe my observations and mathematizing students possibly used. This data about students' reasoning should be seen as the most probable interpretation, and not as only one possible.

Appendix 2 Written problem-solving test

1. There were 13 ants on a maple leaf. Then 28 ants crawled onto the same leaf. How many ants are there on the leaf now?
2. A 3-year-old lady bug has 17 little spots on its back. Matthew counted 8 spots on the right side of the lady bug's back. How many spots does the lady bug have on the left side of its back?
3. There were 48 children on the school bus. Some children got off at the first 3 stops. There are now 28 children on the bus. How many children got off the bus?
4. Mom is making cinnamon buns. She has to make 36 buns. After 25 minutes of working, she made 28 buns. How many cinnamon buns does she still have to make?
5. Danielle has some math problems to solve for homework this week. After 3 days of working, she solved 14 problems. She still has 7 problems left to solve. How many problems did Danielle have for homework?
6. There are some marbles and 36 small cubes in a box. There are 14 red marbles and 27 green marbles. How many marbles are there in the box?
7. Mélanie bought 17 apples and 7 yellow pears. Some of the apples are yellow and the other 8 are green. How many yellow fruits did Mélanie buy?

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