THE PRIME SPECTRUM OF A RING

Harry Greenspan

Submitted to the faculty of Graduate Studies and Research in partial fulfilment of the requirements for the degree of Master of Science.

McGILL UNIVERSITY

1966

CONTENTS

PREFACE

In the following thesis the prime ideals of a ring are considered as points of a topological space. The topology on this space is called the Zariski topology or the spectral topology.

Many results in this paper are topological ones, but algebraic methods are usually employed in acquiring these results.

Complete proofs are given for all propositions with the exception of those in Chapter I.

The paper presupposes a knowledge of elementary Topology and Modern Algebra as can be found, for example, in [5].

I would like to thank Dr. I. Connell for the great deal of time and help he has given me.

I. Review of basic facts.

In the following paper all rings will be commutative, with an identity.

1. Prime and primary ideals.

Definition 1: Let P be an ideal in a ring R. Then P is said to be prime if whenever xy is in P, then either x is in P or y is in P.

Definition 2: Let R be a ring. Then the prime radical of R, or simply the radical of R, is the intersection of all prime ideals of R. We denote the radical of R by $\rho(R)$.

Definition 3: Let R be a ring and I an ideal of R. Then the prime radical of I, or simply the radical of I, is the intersection of all prime ideals of R which contain I. We denote the radical of I by $\rho(I)$.

Note that $\rho(R) = \rho(0)$.

We will recall and prove the following proposition which will be used often in this paper.

Proposition 1: $\rho(J) = \{x \in R : x^n \in J$ for some positive integer $n\}$.

Proof: Let $x \in \rho(J)$. Suppose $x^n \notin J$ for every positive integer n. Let $\mathcal{L}$ be the set of ideals I such that $J \subset I$ and x^n is not in I for every positive integer n. Since J is in $\mathcal{L}$, $\mathcal{L}$ is not empty. Furthermore, $\mathcal{L}$ is partially ordered by inclusion. Finally, suppose $\mathcal{C}$ is a chain in $\mathcal{L}$. Put $L = \bigcup L_i$, where the L_i are in $\mathcal{C}$. Then L is an ideal which contains J and x^n is not in L for every positive integer n. Also L is an upper bound for $\mathcal{C}$. Hence by

Zorn's Lemma, $\mathcal{L}$ contains a maximal element M.

We will show that M is prime. For suppose that $ab \in M$, but $a \notin M$ and $b \notin M$. Then $M + aR \supsetneqq M$ and $M + bR \supsetneqq M$, so that $x^m \in M + aR$ and $x^n \in M + bR$ for some positive integers m and n. It follows that $x^{m+n} \in (M + aR)(M + bR) \subset M + abR \subset M$. This is a contradiction. Hence M is prime. But $J \subset M$, so $\mathcal{P}(J) \subset \mathcal{P}(M) = M$, and therefore $x \in M$. This is a contradiction.

Conversely, suppose $x^n \in J$. Then $x^n \in P$ for every prime ideal P such that $J \subset P$. It follows that $x \in P$ for all such P, so that $x \in \mathcal{P}(J)$.

The remaining propositions in this chapter will be stated without proofs. (The proofs can be found in (5).)

<u>Proposition 2</u>: If I and J are ideals in a ring R, then the following properties hold:

(1) If $I^k \subset J$ for some positive integer k, then $\mathcal{P}(I) \subset \mathcal{P}(J)$.

(2) $\mathcal{P}(IJ) = \mathcal{P}(I \cap J) = \mathcal{P}(I) \cap \mathcal{P}(J)$.

(3) $\mathcal{P}(\mathcal{P}(I)) = \mathcal{P}(I)$.

<u>Definition 4</u>: Let R be a ring and let Q be an ideal in R. Then Q is said to be <u>primary</u> if for elements $a, b \in R$, whenever $ab \in Q$ and $a \notin Q$, then there exists an integer m such that $b^m \in Q$.

<u>Proposition 3</u>: Let Q be a primary ideal in a ring R. Then $P = \mathcal{P}(Q)$ is a prime ideal.

P is called the associated prime of Q.

2. Primary decomposition theorem.

Definition 5: Let R be a ring. R is said to be a noetherian ring if it satisfies the following three equivalent conditions:

(1) (Ascending chain condition) Every strictly ascending chain $I_1 \subsetneq I_2 \subsetneq \dots$ of ideals of R is finite.

(2) (Maximum condition) In every non-empty family of ideals of R there exists a maximal element. (It is not necessarily a maximal ideal of R.)

(3) (Finite basis condition) Every ideal of R is finitely generated.

Definition 6: An ideal I in a ring R is said to be irreducible if it is not a finite intersection of ideals strictly containing it.

Proposition 4: In a noetherian ring every ideal is a finite intersection of irreducible ideals.

Proposition 5: In a noetherian ring every irreducible ideal is primary.

Hence every ideal in a noetherian ring is a finite intersection of primary ideals.

Definition 7: A representation $I = \bigcap_{i=1}^{n} Q_i$ of an ideal I as a finite intersection of primary ideals Q_i is said to be irredundant (or reduced) if it satisfies the following conditions:

(1) No Q_i contains the intersection of the other ones.

(2) The Q_i's have distinct associated prime ideals.

<u>Proposition 6</u>: In a noetherian ring every ideal admits an irredundant representation as a finite intersection of primary ideals.

<u>Definition 8</u>: The associated prime ideals of the primary ideals occuring in an irredundant primary representation of an ideal I are called the <u>associated prime ideals of I</u>, or simply the prime ideals of I.

<u>Definition 9</u>: A minimal element in the family of associated prime ideals of I is called an <u>isolated</u> prime ideal of I.

<u>Definition 10</u>: If $I = \bigcap_{i=1}^{n} Q_i$ is an irredundant primary representation of I, the ideals Q_i are said to be the <u>primary components</u> of I, and Q_i is called <u>isolated</u> if its associated prime ideal is isolated.

<u>Proposition 7</u>: Let R be an arbitrary ring and I an ideal of R admitting an irredundant primary representation $I = \bigcap_{i=1}^{n} Q_i$, and let $P_i = \mathcal{P}(Q_i)$. Then the P_i are uniquely determimed by I. Hence the isolated primary components of I are uniquely determined by I.

II. Quotient rings and quotient modules.

1. Quotient rings.

Definition 1: Let R be a ring and let S be a subset of R which is closed under multiplication, such that $1 \in S$ and $0 \notin S$. (Such a set is often called a multiplicative system.) Put $D = \left\{\frac{r}{s} : s \in S \text{ and } r \in R\right\}$. Then we define the quotient ring of R, $S^{-1}R$, to be the set of equivalence classes in D of the form $\left[\frac{r}{s}\right]$ with $r \in R$ and $s \in S$, where $\left[\frac{r_1}{s_1}\right] = \left[\frac{r_2}{s_2}\right]$ if and only if there exists an element $s' \in S$ such that $s'(r_1s_2 - r_2s_1) = 0$.

We make $S^{-1}R$ into a ring by defining addition and multiplication as follows:

(1) $\left[\frac{r_1}{s_1}\right] + \left[\frac{r_2}{s_2}\right] = \left[\frac{r_1s_2 + r_2s_1}{s_1s_2}\right]$

(2) $\left[\frac{r_1}{s_1}\right]\left[\frac{r_2}{s_2}\right] = \left[\frac{r_1r_2}{s_1s_2}\right]$

We must show that these operations are well defined. Suppose that $\left[\frac{r_1}{s_1}\right] = \left[\frac{a_1}{b_1}\right]$ and $\left[\frac{r_2}{s_2}\right] = \left[\frac{a_2}{b_2}\right]$, that is,

$s'(r_1b_1 - s_1a_1) = 0$ and $s''(r_2b_2 - s_2a_2) = 0$ for some s' and s'' in S.

We will show that (1) $\left[\frac{a_1}{b_1}\right] + \left[\frac{a_2}{b_2}\right] = \left[\frac{r_1}{s_1}\right] + \left[\frac{r_2}{s_2}\right]$ and that (2) $\left[\frac{a_1}{b_1}\right]\left[\frac{a_2}{b_2}\right] = \left[\frac{r_1}{s_1}\right]\left[\frac{r_2}{s_2}\right]$.

(1) $\left[\frac{a_1}{b_1}\right] + \left[\frac{a_2}{b_2}\right] = \left[\frac{a_1b_2 + a_2b_1}{b_1b_2}\right]$ and $\left[\frac{r_1}{s_1}\right] + \left[\frac{r_2}{s_2}\right] = \left[\frac{r_1s_2 + r_2s_1}{s_1s_2}\right]$.

Now we must find an element $s \in S$ such that

$s(s_1s_2(a_1b_2 + a_2b_1) - b_1b_2(r_1s_2 + r_2s_1)) = 0$, that is, such that $s(s_1s_2a_1b_2 + s_1s_2a_2b_1 - b_1b_2r_1s_2 - b_1b_2r_2s_1) = 0$. Take $s = s's''$. Then, since $s'r_1b_1 = s's_1a_1$ and $s''r_2b_2 = s''s_2a_2$, we have, in fact that the above expression is equal to zero.

In a manner very similar to the above it can be proved that multiplication is also well defined.

Henceforth, instead of working with a class $\left[\frac{r}{s}\right]$ we we will work with one of its representatives, $\frac{r}{s}$.

Note that $\frac{r}{s} = 0$ if and only if there exists an element $s' \in S$ such that $s'r = 0$.

Proposition 1: There exists a ring homomorphism $h : R \to S^{-1}R$ such that

(1) $N = \text{kernel } h = \{x \in R : sx = 0 \text{ for some } s \in S\}$.

(2) the elements in $h(s)$ are units in $S^{-1}R$.

Proof: Define $h : R \to S^{-1}R$ by $h(r) = \frac{r}{1}$. Then h is clearly a ring homomorphism.

(1) $h(r) = 0$ if and only if $\frac{r}{1} = 0$, and this is so if and only if $sr = 0$ for some $s \in S$.

(2) If $h(s)$ is in $h(S)$ then $h(s) = \frac{s}{1}$ is a unit in $S^{-1}R$. ($\frac{1}{s}$ is in $S^{-1}R$.)

h is called the canonical mapping from R into $S^{-1}R$.

We denote by $Sh(J)$ the ideal generated by $h(J)$ in $S^{-1}R$, where J is an ideal in R.

Definition 2: An ideal I of R is said to be a contracted ideal if and only if $h^{-1}(Sh(I)) = I$.

Definition 3: An ideal L of $S^{-1}R$ is said to be an extended ideal if and only if it is of the form Sh(J) for some ideal J in R.

Proposition 2: Let S be a multiplicative system in a ring R, and let $S^{-1}R$ be the quotient ring of R with respect to S. Let h be the canonical mapping from R into $S^{-1}R$.

(1) If I is an ideal in R, then $h^{-1}(Sh(I)) = \{r \in R : sr$ is in I for some $s \in S\}$.

(2) Every ideal L of $S^{-1}R$ is an extended ideal.

Proof: (1) Suppose $x \in h^{-1}(Sh(I))$. Then $h(x) \in Sh(I)$. Hence $\frac{x}{1} = \sum_i \frac{x_i}{s_i}\,\frac{y_i}{1}$, where $x_i \in R$, $s_i \in S$, and $y_i \in I$. Writing the sum over a common denominator, $\prod_i s_i = s \in S$, we see that the numerator is in I, so that $\frac{x}{1} = \frac{y}{s}$, where $y \in I$. It follows that $\frac{xs - y}{s} = 0$, so that there exists an element $s' \in S$ such that $s'(xs - y) = 0$. Therefore $xss' = ys' \in I$, or $xs'' \in I$, where $s'' = ss'$.

Conversely, suppose $xs \in I$. Then $h(xs) \in h(I)$, that is, $\frac{x}{1} \cdot \frac{s}{1} \in h(I) \subset Sh(I)$. Then $\frac{x}{1} \cdot \frac{s}{1} \cdot \frac{1}{s} = \frac{x}{1} \in Sh(I)$. Therefore $h^{-1}(\frac{x}{1}) \in h^{-1}(Sh(I))$ and so $x \in h^{-1}(Sh(I))$.

(2) We will show that $Sh(h^{-1}(L)) = L$ for every ideal L of $S^{-1}R$. Clearly $h(h^{-1}(L)) \subset L$, so $Sh(h^{-1}(L)) \subset L$. (L is an ideal of $S^{-1}R$.)

Conversely, if $a \in L$, then $a = \frac{x}{s}$, where $x \in R$ and $s \in S$. Therefore $\frac{x}{s} \cdot \frac{s}{1} = \frac{x}{1} \in L$, so that $x \in h^{-1}(L)$ and

$a = \frac{x}{s} \in Sh(h^{-1}(L))$.

Remark 1: If $S^{-1}I = \left\{ \frac{a}{s} : a \in I \text{ and } s \in S \right\}$, then we saw in (1) that $Sh(I) = S^{-1}I$ for every ideal I of R.

Corollary: If I is an ideal of R, then $Sh(I) \neq S^{-1}R$ if and only if $I \cap S = \emptyset$.

Proof: $Sh(I) = S^{-1}R \Leftrightarrow h^{-1}(Sh(I)) = h^{-1}(S^{-1}R) = R$ $\Leftrightarrow 1 \in h^{-1}(Sh(I)) \Leftrightarrow$ there exists an element $s \in S$ such that $1.s \in I \Leftrightarrow S \cap I \neq \emptyset$.

Proposition 3: Let R be a ring and let S be a multiplicative system in R. Let h be the canonical homomorphism from R into $S^{-1}R$. If $P \cap S = \emptyset$, where P is a prime ideal of R, then P is a contracted ideal and $Sh(P)$ is a prime ideal of $S^{-1}R$.

Proof: Clearly $P \subset h^{-1}(Sh(P))$.

Conversely, if $a \in h^{-1}(Sh(P))$, then $as \in P$ for some $s \in S$. Therefore $a \in P$ (since $S \cap P = \emptyset$).

Let $\frac{a}{s} \cdot \frac{b}{s} \in Sh(P)$, so that $ab \in P$. (See Remark 1.)

Then $a \in P$ or $b \in P$, say $a \in P$. Therefore $\frac{a}{1} \in Sh(P)$, so $\frac{a}{s} \in Sh(P)$. Hence $Sh(P)$ is a prime ideal of $S^{-1}R$.

Corollary: The mapping $P \longrightarrow Sh(P)$ is a one to one mapping of the set of all contracted prime ideals of R (or equivalently: the set of all prime ideals of R which are disjoint from S) onto the set of all prime ideals of $S^{-1}R$.

Proof: If $Sh(P_1) = Sh(P_2)$, then $h^{-1}(Sh(P_1)) = h^{-1}(Sh(P_2))$.

Since P_1 and P_2 are contracted ideals, it follows that $P_1 = P_2$. Therefore the mapping is one to one.

Let L be a prime ideal of $S^{-1}R$. Then $P = h^{-1}(L)$ is a contracted ideal in R. So we have $Sh(P) = L$ and the mapping is onto.

Note: $P \cap S = \emptyset$. For if $P \cap S \neq \emptyset$, then $Sh(P) = S^{-1}R$. (See Corollary of Proposition 2.) But $Sh(P) = L$ so that $L = S^{-1}R$, which is a contradiction.

Remark 2: If P is a prime ideal in R then $S = R - P$ is a multiplicative system. We denote $S^{-1}R$ by R_P.

2. Quotient modules.

Definition 4: Let R be a ring and let S be a multiplicative system in R. Let A be an R-module. Put $F = \left\{ \frac{a}{s} : a \in A \text{ and } s \in S \right\}$. Then we define the quotient module of A, $S^{-1}A$, to be the set of equivalence classes in F of the form $\left[\frac{a}{s}\right]$ with $a \in A$ and $s \in S$, where $\left[\frac{a_1}{s_1}\right] = \left[\frac{a_2}{s_2}\right]$ if and only if there exists an element $s' \in S$ such that $s'(a_1 s_2 - a_2 s_1) = 0$.

We make $S^{-1}A$ into an $S^{-1}R$-module by defining addition and multiplication by an element of $S^{-1}R$ as follows:

(1) $\left[\frac{a_1}{s_1}\right] + \left[\frac{a_2}{s_2}\right] = \left[\frac{a_1 s_2 + a_2 s_1}{s_1 s_2}\right]$

(2) $\left[\frac{a_1}{s_1}\right]\left[\frac{r_1}{s_2}\right] = \left[\frac{a_1 r_1}{s_1 s_2}\right]$

As in section 1, it can easily be shown that these operations are well defined. Again we work with a representative $\frac{a}{s}$ of the class $\left[\frac{a}{s}\right]$ instead of with the whole class.

Note that $\frac{a}{s} = 0$ if and only if there exists an element $s' \in S$ such that $s'a = 0$.

We have the canonical group homomorphism $h : A \rightarrow S^{-1}A$ defined by $h(a) = \frac{a}{1}$, which satisfies the two conditions of Proposition 1 in section 1, with R replaced by A and $S^{-1}R$ replaced by $S^{-1}A$. The image of an R-submodule of A is made into an R-submodule of $S^{-1}A$ by defining $rh(a)$ to be equal to $\frac{r}{1}h(a)$.

Remark 3: If $S = R - P$, where P is a prime ideal of R, we denote $S^{-1}A$ by A_P.

Proposition 4: If A is a finitely generated R-module, then $S^{-1}A = 0$ if and only if there exists an element $s \in S$ such that $sA = 0$.

Proof: If $sA = 0$ then clearly $S^{-1}A = 0$.

Conversely, since A is finitely generated, there exist elements $a_1, \ldots, a_n$ in A such that $A = a_1R + \ldots + a_nR$. Since $S^{-1}A = 0$, for all a_i in A, $i = 1, \ldots, n$ there exists an element s_i in S such that $s_i a_i = 0$. Put $s = s_1 \ldots s_n$. Then $sa = 0$ for all $a \in A$, and so $sA = 0$.

Lemma 1: Let R be a ring. If I is an ideal in R, then the set $S = \{1 + x : x \in I\}$ is a multiplicative system of R. The ideal $S^{-1}I$ of $S^{-1}R$ is contained in the Jacobson radical of $S^{-1}R$; $\mathcal{R}(S^{-1}R)$.

Proof: The first assertion is clear.

To show that $S^{-1}I \subset \mathcal{R}(S^{-1}R)$ it is sufficient to show that for all $\frac{x}{s} \in S^{-1}I$, $\frac{1}{1} - \frac{x}{s}$ is a unit in $S^{-1}R$. Now $\frac{1}{1} - \frac{x}{s} = \frac{s + x}{s} = \frac{1 + x \;\; + x}{s} = \frac{1 + x''}{s}$ and since $1 + x''$ is in S, by the definition of S, therefore $\frac{s}{1 + x''}$ is in $S^{-1}R$. Thus $\frac{1}{1} - \frac{x}{s}$ is a unit in $S^{-1}R$.

Proposition 5: If A is a finitely generated R-module and I is an ideal in R, then $IA = A$ if and only if there exists an element $x \in I$ such that $(1 + x)A = 0$.

Proof: If there exists an element $x \in I$ such that $(1 + x)A = 0$ then clearly $IA = A$.

Conversely, let $S = \{1 + x : x \in I\}$. Since A is a finitely generated R-module, therefore $S^{-1}A$ is a finitely generated $S^{-1}R$-module. For if $a_1, \ldots, a_n$ is a system of generators for A then $\frac{a_1}{1}, \ldots, \frac{a_n}{1}$ is a system of generators for $S^{-1}A$. Since $IA = A$, therefore $S^{-1}I.S^{-1}A = S^{-1}A$. For if $\frac{a}{s} \in S^{-1}A$, where $a \in A$ and $s \in S$, then $a = y_1a_1 + \ldots + y_na_n$ where y_i is in A, $i = 1, \ldots, n$. Hence $\frac{a}{s} = \frac{y_1a_1}{s} + \ldots + \frac{y_na_n}{s}$ $= \frac{y_1}{s}\frac{a_1}{1} + \ldots + \frac{y_n}{s}\frac{a_n}{1} \in S^{-1}I.S^{-1}A$. But by Lemma 1, $S^{-1}I \subset \mathcal{R}(S^{-1}R)$. Therefore, by Nakayama's Lemma, $S^{-1}A = 0$. ($\mathcal{R}(S^{-1}R).S^{-1}A$ is contained in $\mathcal{R}(S^{-1}A)$ and $\mathcal{R}(S^{-1}A) = S^{-1}A$ imply that $S^{-1}A = 0$.) Hence, by Proposition 4, there exists an element $1 + x \in S$ such that $(1 + x)A = 0$.

II. Topology

1. Irreducible topological spaces.

Definition 1: A topological space X is said to be irreducible if every finite intersection of non-empty open sets is non-empty.

For a topological space X to be irreducible it is necessary and sufficient that it be non-empty and that the intersection of two non-empty, open sets in X be non-empty (or what is the same, that the union of two closed sets different from X be different from X).

Proposition 1: Let X be a non-empty topological space. The following conditions are equivalent:

(1) X is irreducible.

(2) Every non-empty, open set in X is dense in X.

(3) Every open set in X is connected.

Proof: (1)$\Longleftrightarrow$(2). By definition, A is dense in X if and only if $A \cap G \neq \emptyset$ for every non-empty, open set G in X.

(3)$\Longrightarrow$(1). Suppose X is not irreducible. Then there exist non-empty, open sets U_1 and U_2 in X such that $U_1 \cap U_2 = \emptyset$. Then $U_1 \cup U_2$ is an open set in X which is not connected.

(1)$\Longrightarrow$(3). Suppose U is an open set in X which is not connected. Then there exists a non-empty subset of U, not equal to U, say A, which is both open and closed in U. $\mathcal{C}(A)$ in U is also both open and closed in U (hence in X) and $A \cap \mathcal{C}(A) = \emptyset$. Hence X is not irreducible.

Remark 1: A Hausdorff space is irreducible only if it consists of a single point.

Definition 2: In an irreducible space X, a point x is said to be a generator if $\overline{\{x\}} = X$.

Remark 2: If X is a T_0-space (that is, for every two distinct points of X there exists a neighborhood of at least one which does not contain the other) then X has at most one generator. For if x and y are two distinct generators of X, that is, $\overline{\{x\}} = \overline{\{y\}} = X$, then clearly every neighborhood of x meets $\{y\}$ and conversely.

Remark 3: If X is a T_1-space (that is, for every two distinct points of X there is a neighborhood of each which does not contain the other) then X has no generators, unless it consists of only one point. For if x and y are two distinct points of X, then $\overline{\{x\}} = X$ implies that every neighborhood of y meets $\{x\}$. This is a contradiction.

Proposition 2: Let X and Y be two irreducible spaces, each with at least one generator. Let $f : X \to Y$ be a continuous function. Then $\overline{f(X)} = Y$ if and only if for every generator x in X, $f(x) = y$ is a generator in Y.

Proof: Let $\overline{f(X)} = Y$. Suppose $f(x) = y$, where $\overline{\{x\}} = X$. Then $f(X) = f(\overline{\{x\}}) \subset \overline{\{f(x)\}}$. (See (3), page 86.) Hence $\overline{f(X)} \subset \overline{\{f(x)\}}$, that is, $\overline{\{f(x)\}} = Y$.

Conversely, there exists a point x in X such that $\overline{\{x\}} = X$ and $\overline{\{f(x)\}} = \overline{\{y\}} = Y$. Now f(x) is in f(X), so $\overline{\{f(x)\}} \subset \overline{f(X)}$ and $\overline{f(X)} = Y$.

A subset E of a topological space X is an irreducible set if the subspace E is irreducible.

Let E be a subset of X. Then E is irreducible if and only if for every two open sets U and V in X, such that $U \cap E \neq \emptyset$ and $V \cap E \neq \emptyset$, we have that $(U \cap V) \cap E \neq \emptyset$, or (what is the same) for every two closed sets F and G in X such that $E \subset F \cup G$, we have either $E \subset F$ or $E \subset G$.

The proof is as follows. If U and V are open in X and $U' = U \cap E \neq \emptyset$, $V' = V \cap E \neq \emptyset$, then U' and V' are both open in E. Hence $U' \cap V' \neq \emptyset$. Therefore $(U \cap V) \cap E \neq \emptyset$.

Conversely, let U and V be two non-empty open sets in E. We must show that $U \cap V \neq \emptyset$. Now $U \cap E \neq \emptyset$ and $V \cap E \neq \emptyset$, so $(U \cap V) \cap E \neq \emptyset$ and clearly $U \cap V \neq \emptyset$.

By induction on n we deduce that if $\{F_i\}_{1 \leq i \leq n}$ is a family of closed sets in X such that $E \subset \bigcup_{i=1}^{n} F_i$ then $E \subset F_i$ for some i, $1 \leq i \leq n$.

<u>Proposition 3</u>: In a topological space X, a subset E is irreducible if and only if $\bar{E}$ is irreducible.

<u>Proof</u>: If G is open in X then $G \cap E \neq \emptyset$ if and only if $G \cap \bar{E} \neq \emptyset$. For if $x \in \bar{E} \cap G$ then every neighborhood of x meets E. But there exists a neighborhood of x, say N_x, such that $N_x \subset G$, so G meets E, that is, $G \cap E \neq \emptyset$. The proposition follows immediately.

<u>Proposition 4</u>: (1) If X is an irreducible space, every non-empty, open set in X is irreducible.

(2) Let $\{U_a\}_{a \in A}$ be a non-empty, open covering of a topological space X such that $U_a \cap U_b \neq \emptyset$ for all a, $b \in A$.

If the sets U_a are irreducible then X is irreducible.

Proof: (1) If X is irreducible, U is a non-empty, open subset of X, and V is a non-empty, open subset of U, then V is also open in X and hence dense in X. Therefore V is dense in U, so U is irreducible by Proposition 1.

(2) We will show that for every non-empty, open set V in X, $V \cap U_a \neq \emptyset$ for all $a \in A$. Now since $X \subset \bigcup_{a \in A} U_a$, there exists at least one $c \in A$ such that $V \cap U_c \neq \emptyset$. Since $U_a \cap U_c \neq \emptyset$ for all $a \in A$ and $V \cap U_c$ is dense in U_c, (It is open in U_c and U_c is irreducible.) therefore $V \cap U_c \cap U_a \neq \emptyset$ for all $a \in A$. Hence $U_a \cap V \neq \emptyset$ for all $a \in A$.

Now $V \cap U_a$ is open in U_a and so is dense in U_a for all $a \in A$. We will show that $\overline{V} = X$. Let $x \in X$. Then $x \in U_a$ for some $a \in A$. But $\overline{V \cap U_a} = U_a$. Therefore for every neighborhood N_x of x, $N_x \cap (V \cap U_a) \neq \emptyset$. In particular $N_x \cap V \neq \emptyset$. Hence $x \in \overline{V}$ and $\overline{V} = X$, that is, V is dense in X, so X is irreducible.

Proposition 5: Let X and Y be two topological spaces and f a continuous function from X into Y. Then for every irreducible set E in X, f(E) is irreducible in Y.

Proof: Suppose U and V are open sets in Y such that $U \cap f(E) \neq \emptyset$ and $V \cap f(E) \neq \emptyset$. Then $f^{-1}(U) \cap E \neq \emptyset$; for if $x \in U \cap f(E)$, then $x \in f(E)$, so $f^{-1}(x) \cap E \neq \emptyset$. But $f^{-1}(x) \subset f^{-1}(U)$, so $f^{-1}(U) \cap E \neq \emptyset$.

Similarly, $f^{-1}(V) \cap E \neq \emptyset$. Therefore $(f^{-1}(U) \cap f^{-1}(V)) \cap E \neq \emptyset$, that is, $f^{-1}(U \cap V) \cap E \neq \emptyset$.

Hence $(U \cap V) \cap f(E) \neq \emptyset$.

Definition 3: A maximal irreducible set in a topological space is called an irreducible component.

By Proposition 3 every irreducible component of X is closed in X.

Proposition 6: Let X be a topological space. Every irreducible set in X is contained in an irreducible component of X, and X is the union of its irreducible components.

Proof: Let G be an irreducible set in X. Let $\mathfrak{J}$ be the family of irreducible sets which contain G. Since G is in $\mathfrak{J}$, $\mathfrak{J}$ is not empty. Furthermore, $\mathfrak{J}$ is partially ordered by inclusion. Finally, suppose $\mathfrak{C}$ is a chain in $\mathfrak{J}$. Put $E = \bigcup F_i$, where the F_i are in $\mathfrak{C}$.

We will show that E is irreducible. Let U and V be two open sets in X such that $U \cap E \neq \emptyset$ and $V \cap E \neq \emptyset$. Since $\mathfrak{C}$ is totally ordered, there is a set F_i in $\mathfrak{C}$ such that $F_i \cap U \neq \emptyset$ and $F_i \cap V \neq \emptyset$. Since F_i is irreducible, $F_i \cap (U \cap V) \neq \emptyset$, so $E \cap (U \cap V) \neq \emptyset$. Hence E is irreducible. Clearly E is an upper bound for $\mathfrak{C}$. It follows by Zorn's Lemma that $\mathfrak{J}$ contains a maximal element T, which is clearly an irreducible component, and $G \subset T$.

The second assertion follows from the first and the fact that every set consisting of a single point is irreducible.

Corollary: Every connected component of a topological space X is the union of irreducible components of X.

Proof: Let D be a connected component of X. Let $\{F_j\}_{j \in J}$

be the family of irreducible components of X such that $D \subset \bigcup_{j \in J} F_j$ and such that for each $j \in J$ some x in F_j is also in D. (If $F_i \cap D = \emptyset$, $D \subset \bigcup_{j \neq i} F_j$.)

We will show that $D = \bigcup_{j \in J} F_j$. For each j, F_j is irreducible and hence connected by Proposition 1. It is therefore contained in a connected component which must be D (since some x in F_j is also in D). Hence $F_j \subset D$ for all $j \in J$ and $\bigcup_{j \in J} F_j \subset D$.

Remark 4: Two distinct irreducible components of X may have points in common. In fact, as we have seen above, they may both be contained in the same connected component. An example of such a case will be given later.

Proposition 7: Let X be a topological space and $\{P_i\}_{1 \le i \le n}$ a finite covering of X formed with closed irreducible sets. Then the irreducible components of X are the maximal elements (by inclusion) of the set of P_i's.

Proof: We may assume that the P_i's are pairwise incomparable. Let E be an irreducible set in X, then $E \subset \bigcup_{i=1}^{n} P_i$. Since the P_i are closed, therefore $E \subset P_i$ for some i, $1 \le i \le n$, so the P_i are the only maximal irreducible sets in X and hence are the only possible irreducible components. Clearly the maximal sets of $\{P_i\}_{1 \le i \le n}$ are irreducible components.

Corollary: Let X be a topological space and let E be a subspace of X having only a finite number of distinct irreducible components, $\{Q_i\}_{1\le i\le n}$. Then the irreducible components of the closure $\overline{E}$ in X are the closures $\overline{Q_i}$ $(1\le i\le n)$ of Q_i $(1\le i\le n)$ and $\overline{Q_i} \ne \overline{Q_j}$ if $i \ne j$.

Proof: Since $E = \bigcup_{i=1}^{n} Q_i$, therefore $\overline{E} = \bigcup_{i=1}^{n} \overline{Q_i}$ and $\overline{Q_i}$ is irreducible $(1\le i\le n)$. It remains to show that each $\overline{Q_i}$ is an irreducible component in $\overline{E}$. It suffices to show that $\overline{Q_i} \not\subset \overline{Q_j}$ for $i \ne j$. Now Q_i is closed in E, so $\overline{Q_i} \cap E = Q_i$. If $\overline{Q_i} \subset \overline{Q_j}$, then $\overline{Q_i} \cap E \subset \overline{Q_j} \cap E$ and $Q_i \subset Q_j$. This is a contradiction.

Proposition 8: Let U be an open set in a topological space X. The mapping $V \to \overline{V}$ (closure in X) is a bijection from the family of closed irreducible subsets of U onto the family of closed irreducible sets in X which meet U. The inverse mapping is $Z \to Z \cap U$. In particular, this bijection maps the set of irreducible components of U onto the set of irreducible components of X which meet U.

Proof: If V is a closed, irreducible subset of U, then $\overline{V}$ is irreducible, (See Proposition 3.) and $\overline{V}$ is closed in X. Also $\overline{V} \cap U = V \ne \emptyset$ (since V is closed in U).

Suppose $\overline{V_1} = \overline{V_2}$. Then $\overline{V_1} \cap U = \overline{V_2} \cap U$ and $V_1 = V_2$.

Therefore the mapping is one to one. If Z is a closed, irreducible subset of X and $Z \cap U \ne \emptyset$, then $Z \cap U$ is a non-empty, open subset of Z and so is irreducible. (See Proposition 4.) Also $Z \cap U$ is dense in Z, by Proposition 1. Furthermore, since Z is closed, $\overline{Z \cap U} = Z$. Finally,

$Z \cap U$ is closed in U. Hence the mapping is onto.

2. Noetherian spaces.

Definition 4: A topological space X is said to be noetherian if every non-empty family of closed sets in X, ordered by inclusion, has a minimal element.

Equivalently, every non-empty family of open sets in X, ordered by inclusion, has a maximal element; or every decreasing (respectively increasing) sequence of closed (respectively open) sets in X is stationary.

Proposition 9: (1) Every subspace of a noetherian space is noetherian.

(2) Let $\{A_i\}_{i \in I}$ be a finite covering of a topological space X. If the subspaces A_i of X are noetherian for all $i \in I$, then X is noetherian.

Proof: (1) Let X be a noetherian space, let A be a subspace of X, and let $\{F_n\}_{n \geq 0}$ be a decreasing sequence of subsets of A, closed in A. Then $F_n = \overline{F}_n \cap A$ for all n, and the closures $\overline{F}_n$ of F_n form a decreasing sequence of closed sets in X. This sequence is stationary since X is noetherian. Hence the sequence $\{F_n\}_{n \geq 0}$ is stationary.

(2) Let $\{G_n\}_{n \geq 0}$ be a decreasing sequence of closed sets in X. For each n, $G_n \cap A_i$ is closed in A_i for all $i \in I$ and hence $\{G_n \cap A_i\}_{n \geq 0}$ is stationary for all $i \in I$. Since I is finite, there exists an integer n_0 such that for $n \geq n_0$ $G_n \cap A_i = G_{n_0} \cap A_i$ for all $i \in I$. But for each n,

$G_n = \bigcup_{i \in I} (G_n \cap A_i)$, therefore for $n \geq n_0$, $G_n = G_{n_0}$ and $\{G_n\}_{n \geq 0}$ is stationary, so X is noetherian.

<u>Proposition 10</u>: A topological space X is noetherian if and only if every open set in X is compact.

<u>Proof</u>: Suppose X is noetherian. By Proposition 9 it is sufficient to show that every noetherian subspace of X is compact. Suppose Y is a noetherian subspace of X. Let $\{U_i\}_{i \in I}$ be an open covering of Y. Let $\mathfrak{J}$ be the family of all finite unions of the U_i. $\mathfrak{J}$ is not empty and $\mathfrak{J}$ is ordered by inclusion, so $\mathfrak{J}$ has a maximal element, say $V = \bigcup_{i \in H} U_i$, where H is a finite subset of I. Now $V \cup U_i$ is in $\mathfrak{J}$ and $V \subset V \cup U_i$ for all $i \in I$. Hence $V = V \cup U_i$ for all $i \in I$. If $x \in Y$, $x \in U_i$ for some $i \in I$, so $x \in V \cup U_i$. Therefore $x \in V$ and $V = Y$.

Conversely, suppose that every open set in X is compact, and let $\{U_n\}_{n \geq 0}$ be an increasing sequence of open sets in X. $V = \bigcup_{n=0}^{\infty} U_n$ is open and hence compact. Since $\{U_n\}_{n \geq 0}$ is an open covering of V, there exists a finite sub-family of $\{U_n\}_{n \geq 0}$ which covers V, say $U_1, \ldots, U_m$. Therefore $V = U_r$ for some index r ($U_1 \subset U_2 \subset \ldots \subset U_m$) and $U_r = U_{r+1} = \ldots$, so $\{U_n\}_{n \geq 0}$ is stationary.

<u>Lemma 1</u>: (Principle of noetherian induction). Let E be an ordered set such that every subset of E has a minimal element. Let $F \subset E$ with the following property: If a in E is such that the relation $x < a$ implies that $x \in F$, then a is in F. We have then that $F = E$.

Proof: Suppose $F \neq E$. Then $\mathcal{C}(F) \neq \emptyset$, so it has a minimal element b. Now $b \in E$ and $x < b$, so $x \in F$. Hence $b \in F$, which is a contradiction.

Proposition 11: If X is a noetherian space, the set of irreducible components of X (and a fortiori, the set of connected components of X) is finite.

Proof: We will show that X is a finite union of closed irreducible sets. (The proposition will follow from Proposition 8.) Let E be the family of closed sets in X (ordered by inclusion) and let F be the family of finite unions of closed irreducible sets. ($F \subset E$.) Let Y be a closed set in X such that every closed subset of Y (not equal to Y) belongs to F. We will show that $Y \in F$.

If Y is irreducible, then Y is in F by the definition of F. If Y is not irreducible, there exist closed sets Y_1' and Y_2' in X such that $Y \subset Y_1' \cup Y_2'$ but $Y \not\subset Y_1'$ and $Y \not\subset Y_2'$. Let $Y_1 = Y_1' \cap Y$ and $Y_2 = Y_2' \cap Y$. Both Y_1 and Y_2 are closed in X (and Y). Then $Y = Y_1 \cup Y_2$, but $Y \neq Y_1$ and $Y \neq Y_2$. Now $Y_1 \in F$ and $Y_2 \in F$, so $Y = Y_1 \cup Y_2 \in F$ and $F = E$ by Lemma 1. Hence X is a finite union of closed irreducible sets.

Remark 5: Suppose X is a noetherian Hausdorff space. Then X is finite. This will follow if we can show that every point in X is an irreducible component. But if $\{x\} \subsetneq F$, there exists an element $y \neq x$, in F and the subspace F is Hausdorff, hence not irreducible. (See Remark 1.)

IV. The prime spectrum and support of a module.

1. The prime spectrum of a ring.

Let R be a ring and let X be the set of prime ideals of R. For every subset M of R we write $V(M) = \{P \in X : M \subset P\}$. It is clear that if I is the ideal generated by M, then $V(M) = V(I)$. If M consists of one point a, we write $V(a) = V(\{a\})$, and we have $V(a) = V(Ra)$.

The mapping $M \to V(M)$ is monotone decreasing for the relation of inclusion in R and X. Moreover, we have the following formulas:

(1) $V(0) = X$ and $V(1) = \emptyset$.

(2) $V(\bigcup_{i \in I} M_i) = \bigcap_{i \in I} V(M_i)$, M_i being subsets of R.

(3) $V(I \cap J) = V(IJ) = V(I) \cup V(J)$, where I and J are ideals in R.

(4) $V(\sum_{a \in A} I_a) = \bigcap_{a \in A} V(I_a)$, where the I_a are ideals in R.

Remark 1: If I is an ideal in R such that $V(I) = \emptyset$, then $I = R$. For if $I \neq R$ then $I \subset M$, a maximal ideal which is also prime.

Remark 2: If I is an ideal in R, then $V(I) = V(\mathcal{P}(I))$. For if $I \subset P$ then $\mathcal{P}(I) \subset \mathcal{P}(P) = P$.

Formulas (1) to (3) show that the family of sets V(M) in X satisfy the axioms of closed sets for a topology.

Definition 1: Let R be a ring. Let X be the set of prime ideals of R with the topology whose closed sets are precisely the sets V(M), where M runs through the set of subsets of R. We call X the prime spectrum of R and we denote it by Spec(R). The topology so defined is called the spectral topology or the Zariski topology on X.

Clearly Spec(R) = $\emptyset$ if and only if R = $\{0\}$.

Let X be the prime spectrum of a ring R. For all $r \in R$, let $X_r = \{P \in X : r \notin P\}$. Then $X_r = X - V(r)$, so that X_r is open in X. By (2) above every closed set in X is the intersection of closed sets of the form V(r). Hence the X_r form a base for the spectral topology on X. Moreover it follows immediately from the definitions that $X_0 = \emptyset$, $X_1 = X$ and more generally $X_r = X$ for every unit $r \in R$. ($X_r = X - V(r)$, but $r \in P$, so $rr^{-1} \in P$ and $1 \in P$, which is a contradiction. Hence $V(r) = \emptyset$.)

Remark 3: $X_{rs} = X_r \cap X_s$ for r and s in R. For $X_{rs} = \complement(V(rs))$ and $X_r \cap X_s = \complement(V(r) \cup V(s))$, and $rs \in P$ if and only if $r \in P$ or $s \in P$.

Proposition 1: Let R be a ring and let I be a finitely generated ideal in R. Then the following are equivalent:

(1) $I^2 = I$.

(2) I = eR where $e^2 = e \in I$.

(3) V(I) is open and the two conditions $\rho(J) = \rho(I)$ and $J \subset I$ imply that J = I.

Proof: (1) $\Rightarrow$ (2). By Proposition 5, Chapter II, Section 2 $I^2 = I$ implies that there exists an element $f \in I$ such that $(1 + f)I = 0$, that is, for all $a \in I$, $(1 + f)a = 0$. Hence $a = -fa$ for all $a \in I$. Take $e = -f$. Then $a = ea$ for all $a \in I$. In particular $e^2 = e$ and I = eR.

(2) $\Rightarrow$ (3). Let $P \in V(I)$. We will show that $P \in X_{1-e} \subset V(I)$. Now $1 - e \notin P$. For $e \in I \subset P$; so $P \in X_{1-e}$. Furthemore, $X_{1-e} \subset V(I)$. For let $Q \in X_{1-e}$, that is, $1 - e \notin Q$. We

will show that $I \subset Q$. It suffices to show that $e \in Q$. Now if $e \notin Q$ then $e(1 - e) = 0 \notin Q$. This is a contradiction. Let $\rho(J) = \rho(eR)$ and suppose that $J \subset eR$. We will show that $eR \subset J$ by showing that $e \in J$. Now $e \in \rho(eR) = \rho(J)$, so $e^n \in J$ for some positive integer n, that is, $e \in J$.

(3) $\Longrightarrow$(1). $\rho(I^2) = \rho(I) \cap \rho(I) = \rho(I)$ by Proposition 2, Chapter I, Section 1. Also $I^2 \subset I$, so $I^2 = I$.

For every subset Y of X, let $\mathfrak{J}(Y) = \cap \{P : P \in Y\}$. Clearly $\mathfrak{J}(Y)$ is an ideal in R. The mapping $Y \longrightarrow \mathfrak{J}(Y)$ is monotone decreasing for the relation of inclusion in X and in R. Moreover, we have $\mathfrak{J}(\emptyset) = R$ and $\mathfrak{J}(\bigcup_{a \in A} Y_a) = \bigcap_{a \in A} \mathfrak{J}(Y_a)$ for every family $\{Y_a\}_{a \in A}$ of subsets of X.

<u>Proposition 2</u>: Let R be a ring, let I be an ideal in R, and let $Y \subset X = \mathrm{Spec}(R)$.

(1) V(I) is closed in X and $\mathfrak{J}(Y)$ is an ideal in R equal to its radical.

(2) $\mathfrak{J}(V(I)) = \rho(I)$ and $V(\mathfrak{J}(Y)) = \overline{Y}$.

(3) The mappings $\mathfrak{J}$ and V define inverse monotone decreasing bijections (that is, $V^{-1} = \mathfrak{J}$) between the set of closed subsets of X and the set of ideals in R equal to their radicals.

<u>Proof</u>: (1) V(I) is closed by definition.

$\mathfrak{J}(Y) = \cap \{P : P \in Y\}$ is an ideal and $\rho(\mathfrak{J}(Y))$ $= \rho(\cap \{P : P \in Y\}) = \{x : x^n \in P \text{ for all } P \in Y\} = \bigcap_{P \in Y} \rho(P)$ $= \bigcap_{P \in Y} P = \mathfrak{J}(Y)$.

(2) $\mathfrak{J}(V(I)) = \cap \{P : P \in V(I)\} = \cap \{P : I \subset P\} = \mathcal{P}(I)$. To show that $V(\mathfrak{J}(Y)) = \overline{Y}$ we will show that $V(\mathfrak{J}(Y))$ is the smallest closed set containing Y. Let $V(M) \supset Y$. If $P \in Y$ then $P \in V(M)$ and $M \subset P$, that is, $M \subset P$ for all $P \in Y$. Hence $M \subset \mathfrak{J}(Y)$, so $V(\mathfrak{J}(Y)) \subset V(M)$. But $Y \subset V(\mathfrak{J}(Y))$ (since $V(\mathfrak{J}(Y)) = \{P : \cap \{P : P \in Y\} \subset P\} \supset Y$), therefore $V(\mathfrak{J}(Y))$ is the smallest closed set containing Y.

(3) By (1) $\mathfrak{J}$ is a mapping from the set of closed subsets of X to the set of ideals in R equal to their radicals; V is a mapping from the set of ideals in R equal to their radicals to the set of closed subsets of X.

It remains to show that $V\mathfrak{J} = \mathfrak{J}V = 1$. By (2), if $I = \mathcal{P}(I)$ then $\mathfrak{J}(V(I)) = \mathcal{P}(I) = I$. If Y is closed in X then $V(\mathfrak{J}(Y)) = \overline{Y} = Y$.

<u>Remark 4</u>: If $M \subset R$, then $V(M) = V(I)$ where I is the ideal generated by M. Now V(I) is closed so $V(\mathfrak{J}(V(I))) = V(I)$ by Proposition 2. Hence $V(\mathfrak{J}(V(M))) = V(M)$.

Similarly, $\mathfrak{J}(V(\mathfrak{J}(Y))) = \mathfrak{J}(Y)$ for any $Y \subset X$.

<u>Corollary 1</u>: For every family $\{Y_a\}_{a \in A}$ of closed subsets of X, $\mathfrak{J}(\bigcap_{a \in A} Y_a) = \mathcal{P}(\sum_{a \in A} \mathfrak{J}(Y_a))$.

<u>Proof</u>: Since Y_a is closed for all $a \in A$, therefore $\bigcap_{a \in A} Y_a$ is closed. We will show that $\mathfrak{J}(\bigcap_{a \in A} Y_a)$ is the smallest ideal equal to its radical and containing all the $\mathfrak{J}(Y_a)$. Suppose that $I = \mathcal{P}(I)$ and $\mathfrak{J}(Y_a) \subset I$ for all $a \in A$. Then

$V(I) \subset V(\mathcal{I}(Y_a)) = Y_a$ for all $a \in A$, that is, $V(I) \subset \bigcap_{a \in A} Y_a$.

Hence $\mathcal{I}(V(I)) \supset \mathcal{I}(\bigcap_{a \in A} Y_a)$ and $I \supset \mathcal{I}(\bigcap_{a \in A} Y_a)$. Thus

$\sum_{a \in A} \mathcal{I}(Y_a) \subset \mathcal{I}(\bigcap_{a \in A} Y_a)$ and $\mathcal{P}(\sum_{a \in A} \mathcal{I}(Y_a)) = \mathcal{P}(\mathcal{I}(\bigcap_{a \in A} Y_a))$

$= \mathcal{I}(\bigcap_{a \in A} Y_a)$. But $\mathcal{P}(\sum_{a \in A} \mathcal{I}(Y_a)) = \mathcal{P}(\mathcal{P}(\sum_{a \in A} \mathcal{I}(Y_a)))$ and $\mathcal{I}(Y_a) \subset \mathcal{P}(\sum_{a \in A} \mathcal{I}(Y_a))$ for all $a \in A$, therefore

$\mathcal{I}(\bigcap_{a \in A} Y_a) \subset \mathcal{P}(\sum_{a \in A} \mathcal{I}(Y_a))$, so $\mathcal{P}(\sum_{a \in A} \mathcal{I}(Y_a)) = \mathcal{I}(\bigcap_{a \in A} Y_a)$.

Corollary 2: If I and J are two ideals in R, then the following are equivalent:

(1) $V(I) \subset V(J)$.

(2) $J \subset \mathcal{P}(I)$.

(3) $\mathcal{P}(J) \subset \mathcal{P}(I)$.

Proof: (2)$\Leftrightarrow$(3). This is clear.

(1)$\Leftrightarrow$(3). $V(I) = V(\mathcal{P}(I))$ and $V(J) = V(\mathcal{P}(J))$ by Remark 2, so $V(I) \subset V(J) \Leftrightarrow \mathcal{I}(V(\mathcal{P}(J))) \subset \mathcal{I}(V(\mathcal{P}(I)))$

$\Leftrightarrow \mathcal{P}(J) \subset \mathcal{P}(I)$.

Corollary 3: Let $\{f_a\}_{a \in A}$ be a family of elements of R.

If $g \in R$, then a necessary and sufficient condition for $X_g \subset \bigcup_{a \in A} X_{f_a}$ is that there exists an integer n such that g^n belongs to an ideal generated by the f_a.

Proof: $X_g \subset \bigcup_{a \in A} X_{f_a} \Leftrightarrow V(g) \supset \bigcap_{a \in A} V(f_a) \Leftrightarrow V(\bigcup_{a \in A} f_a) \subset V(g)$

$\Leftrightarrow V(I) \subset V(Rg)$ where I is the ideal generated by the f_a

$\Leftrightarrow Rg \subset \mathcal{P}(I)$ (See Corollary 2.) $\Leftrightarrow g \in \mathcal{P}(I) \Leftrightarrow g^n \in I$ for some integer n.

<u>Corollary 4</u>: $X_f = X_g$ if and only if there exist integers m and $n > 0$ such that $f^m \in Rg$ and $g^n \in Rf$.

<u>Proof</u>: $X_f \subset X_g$ if and only if $f^m \in Rg$ for some integer m and $X_g \subset X_f$ if and only if $g^n \in Rf$ for some integer n.

<u>Corollary 5</u>: $X_f = \emptyset$ if and only if f is nilpotent.

<u>Proof</u>: $X_f \subset X_0 = \emptyset$ if and only if $f^n = \emptyset$ for some integer n.

<u>Corollary 6</u>: (1) $\overline{\{P\}} = V(P)$, where $P \in X = \text{Spec}(R)$.

(2) $\{P\}$ is closed in X if and only if P is maximal.

<u>Proof</u>: (1) $\mathfrak{I}(\{P\}) = \cap \{P : P \in \{P\}\} = P$, so $V(\mathfrak{I}(\{P\})) = \overline{\{P\}} = V(P)$ by Proposition 2.

(2) $\overline{\{P\}} = \{P\} \Leftrightarrow V(P) = \{P\} \Leftrightarrow \{Q : P \subset Q\} = \{P\} \Leftrightarrow$ P is maximal.

<u>Corollary 7</u>: If R is a noetherian ring, X = Spec(R) is a noetherian space.

<u>Proof</u>: Let $\{Y_n\}_{n>0}$ be a decreasing sequence of closed sets. Then $\{\mathfrak{I}(Y_n)\}_{n>0}$ is an increasing sequence of ideals in R. Hence there exists an integer n_0 such that $\mathfrak{I}(Y_n) = \mathfrak{I}(Y_{n_0})$ for $n > n_0$, so $V(\mathfrak{I}(Y_n)) = V(\mathfrak{I}(Y_{n_0}))$ for $n > n_0$ and $Y_n = Y_{n_0}$ for $n > n_0$. Therefore X is a noetherian space.

<u>Proposition 3</u>: Let R be a ring. Then for every $r \in R$ the open set X_r in X = Spec(R) is compact. In particular the space X is compact.

<u>Proof</u>: Since the X_r form a base for the topology, it is sufficient to show that if $\{r_a\}_{a \in A}$ is a set of elements in R such that $X_r \subset \bigcup_{a \in A} X_{r_a}$ then there exists a finite subset

$\{r_a\}_{a \in H}$ such that $X_r \subset \bigcup_{a \in H} X_{r_a}$. Since $X_r \subset \bigcup_{a \in A} X_{r_a}$, therefore there exists an integer $n > 0$ such that r^n is in the ideal generated by the r_a, by Corollary 3, Proposition 2. Hence r^n is in the ideal generated by a finite number of the r_a, say $\{r_a\}_{a \in H}$. Therefore $X_r \subset \bigcup_{a \in H} X_{r_a}$ (again by Corollary 3, Proposition 2).

In particular, $X = X_1$ is compact.

Proposition 4: Let R be a ring and let $\mathcal{P}$ be its radical. Then X = Spec(R) is discrete if and only if $R/\mathcal{P}$ is a direct sum of a finite number of fields.

Proof: Suppose X = Spec(R) is discrete. Then $\{P\}$ is open for every prime ideal P in R, and $\{P\} = X_r$ for some $r \in R$, that is, $r \notin P$, and if $r \notin Q$ then $Q = P$. Now $\bigcup_{P \subset R} \{P\}$ is an open cover of X and since X is compact, by Proposition 3, there exists a finite subcover, that is, there exist only finitely many prime ideals in R. Also since X is discrete each prime ideal is maximal, by Corollary 6, Proposition 2. ($\{P\}$ is closed for every prime ideal P.)

We will show that $R/\mathcal{P} = R/\bigcap_{i=1}^{n} P_i \cong R/P_1 \oplus \ldots \oplus R/P_n$ by induction on n. Define f from $R/P_1 \cap P_2$ to $R/P_1 \oplus R/P_2$ by $f(r + P_1 \cap P_2) = r + P_1 + r + P_2$.

Clearly f is a homomorphism.

If $f(r + P_1 \cap P_2) = 0$ then $r + P_1 + r + P_2 = 0$ so $r + P_1 = 0$ and $r + P_2 = 0$. Hence $r \in P_1$ and $r \in P_2$ and

$r \in P_1 \cap P_2$, so that $r + P_1 \cap P_2 = 0$. Therefore f is one to one.

Now $P_1 + P_2 = R$. Let $r = p_1 + p_2$. Then

$f(r + P_1 \cap P_2) = p_2 + P_1 + p_1 + P_2$. If $r_1 + P_1 + r_2 + P_2$

is in $R/P_1 \oplus R/P_2$, say $r_1 = p_1 + p_2$ and $r_2 = q_1 + q_2$, where

p_1 and q_1 are in P_1 and p_2 and q_2 are in P_2, then

$r_1 + P_1 + r_2 + P_2 = p_2 + P_1 + q_1 + P_2$ and

$f(q_1 + p_2 + P_1 \cap P_2) = p_2 + P_1 + q_1 + P_2$. Hence f is onto.

Suppose $R/P_1 \cap \ldots \cap P_{n-1} \cong R/P_1 \oplus \ldots \oplus R/P_{n-1}$.

Now $P_n + \bigcap_{i=1}^{n-1} P_i = R$. For there exists an element $p_i \notin P_n$

such that $p_i \in P_i$, $i = 1,\ldots,n-1$, so $p = p_1 \ldots p_{n-1} \notin P_n$

but $p \in \bigcap_{i=1}^{n-1} P_i$. Hence $R/\bigcap_{i=1}^{n-1} P_i \cap P_n \cong R/\bigcap_{i=1}^{n-1} P_i + R/P_n$ by the

case $n = 2$. Therefore $R/\bigcap_{i=1}^{n} P_n \cong R/P_1 \oplus \ldots \oplus R/P_n$ by induction.

Conversely, suppose $R/\mathcal{P} \cong F_1 \oplus \ldots \oplus F_n$, where

the F_i are fields. Then $R/\mathcal{P}$ has only finitely many prime

ideals (all of the form $F_1 \oplus \ldots \oplus F_{i-1} \oplus 0 \oplus F_{i+1} \oplus \ldots$

$\oplus F_n$) and hence so does R. These are clearly all maximal.

Suppose the prime ideals are $P_1,\ldots,P_n$. We will show that

these are all open. We will show, for example, that P_1

is open. Since P_1 is maximal, therefore $P_1 \not\subset P_j$ for

$j = 2,\ldots,n$ so there exists an element $r_j \notin P_1$ such that $r_j \in P_j$ for $j = 2,\ldots,n$. Then $r = r_2 \ldots r_n \notin P_1$ and $r \in P_j$ for $j = 2,\ldots,n$. Hence $\{P_1\} = X_r$ is open.

Proposition 5: Let R and S be two rings and let $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$. Suppose h is a homomorphism from R into S. Then the mapping $\mathrm{Spec}\ h : Y \to X$ defined by $\mathrm{Spec}\ h(Q) = h^{-1}(Q)$ is continuous.

Proof: Let $V(M)$ be closed in X, where M is a subset of R. We will show that $\mathrm{Spec}\ h^{-1}(V(M))$ is closed in Y. Now

$$\begin{aligned}\mathrm{Spec}\ h^{-1}(V(M)) &= \{Q \in Y : \mathrm{Spec}\ h(Q) \in V(M)\}\\ &= \{Q \in Y : h^{-1}(Q) \in V(M)\}\\ &= \{Q \in Y : M \subset h^{-1}(Q)\}\\ &= \{Q \in Y : h(M) \subset Q\}\\ &= V(h(M)), \text{ which is closed in } Y.\end{aligned}$$

The function Spec h is called the function associated with the homomorphism h.

Remark 5: Spec is a contravariant functor from the category of commutative rings to the category of topological spaces.

Proposition 6: Let $h : R \to S$ be a homomorphism such that for all $s \in S$, $s = uh(r)$, where u is a unit in S and $r \in R$. Then there exists a subspace V of $X = \mathrm{Spec}(R)$ such that $\mathrm{Spec}\ h : \mathrm{Spec}(S) \to V$ is a homeomorphism.

Proof: (1) Spec h is continuous by Proposition 5.

(2) Spec h is clearly onto.

(3) Let $\mathrm{Spec}\ h(Q_1) = \mathrm{Spec}\ h(Q_2)$, where Q_1 and Q_2 are prime ideals in S, that is, $h^{-1}(Q_1) = h^{-1}(Q_2)$. We will show

that $Q_1 = Q_2$. Let $q \in Q_1$, then $q = uh(r)$, where $r \in R$ and u is a unit in S. Hence $uh(r) \in Q_1$ so $h(r) \in Q_1$, since u is a unit in S. It follows that $r \in h^{-1}(Q_1) = h^{-1}(Q_2)$ and so $h(r) \in Q_2$. Therefore $uh(r) = q \in Q_2$, so $Q_1 \subset Q_2$. Similarly, $Q_2 \subset Q_1$. Hence $Q_1 = Q_2$ and Spec h is one to one.

(4) It remains to show that Spec h^{-1} is continuous. It suffices to prove that Spec h is an open mapping. Let X_s be a base member in Spec(S). We will show that Spec $h(X_s)$ is open in V by showing that Spec $h(X_s) = V \cap U$, where U is open in Spec(R). Now $s = uh(r)$, where u is a unit in S and $r \in R$. We claim that Spec $h(X_s) = V \cap X_r$.

Let $P \in$ Spec $h(X_s)$, that is, $P \in$ Spec $h(\{Q : s \notin Q\})$ $= \{h^{-1}(Q) : s \notin Q\}$, that is, $P = h^{-1}(Q)$ where $s \notin Q$, so clearly $P \in V$. We will show that $r \notin P$. Since $s \notin Q$, $uh(r) \notin Q$ so that $h(r) \notin Q$ and $r \notin h^{-1}(Q) = P$.

Conversely, if $P \in V \cap X_r$ then $P = h^{-1}(Q)$ and $r \notin P$. Then $s \notin Q$. For if $s \in Q$ then $uh(r) \in Q$ so that $h(r) \in Q$ and $r \in h^{-1}(Q) = P$, which is a contradiction.

<u>Corollary</u>: Suppose h is an epimorphism from R onto S and suppose K = kernel h. Then Spec h is a homeomorphism from Y = Spec(S) onto the closed subspace V(K) of X = Spec(R).

<u>Proof</u>: For all $s \in S$, $s = 1h(r)$, where $1 \in S$ and $r \in R$, so by Proposition 6, there exists a subspace V of X such that Spec h : Spec(S) $\longrightarrow$ V is a homeomorphism. We will show that V = V(K). Let $P \in V$. Then $P = h^{-1}(Q)$ so $K \subset P$. Hence

$V(P) \subset V(K)$ and $P \in V(K)$.

Conversely, if $P \in V(K)$ then $K \subset P$ so that $h(P) = Q$, a prime ideal of S. Therefore $h^{-1}(Q) = h^{-1}(h(P)) = P$ so that $P \in V$.

Proposition 7: Let $h : R \to S$ be a ring homomorphism. Then for every ideal J of S, $\overline{\text{Spec } h(V(J))} = V(h^{-1}(J))$.

Proof: If $P \in \text{Spec } h(V(J))$ then $P = h^{-1}(Q)$, where $J \subset Q$, Q being a prime ideal in S, that is, $h^{-1}(J) \subset h^{-1}(Q) = P$. Hence $P \in V(h^{-1}(J))$. Therefore $\text{Spec } h(V(J)) \subset V(h^{-1}(J))$ and since $V(h^{-1}(J))$ is closed, it follows that $\overline{\text{Spec } h(V(J))} \subset V(h^{-1}(J))$.

Conversely, let $P \in V(h^{-1}(J))$, that is, $h^{-1}(J) \subset P$. We must show that if P is not in Spec $h(V(J))$ then P is a limit point of Spec $h(V(J))$, that is, for all r such that $P \in X_r$, $X_r \cap \text{Spec } h(V(J)) \neq \emptyset$, or, for all r such that $r \notin P$, there exists a prime ideal Q of R such that $r \notin Q$ and $Q = h^{-1}(P')$, where $J \subset P'$, P' being a prime ideal in S.

Now $J \subset M$, where M is a maximal ideal and hence a prime ideal in S. Suppose that $r \notin P$. We claim that there exists a prime ideal P' of S such that $J \subset P'$ and such that $h(r) \notin P'$. For if $h(r) \in P$ for all P such that $J \subset P$, then $h(r) \in \mathcal{P}(J) = \{x \in R : x^n \in J\}$, that is, $h(r^n) = h(r)^n \in J$ so $h^{-1}(h(r^n)) \subset h^{-1}(J)$ and $r^n \in h^{-1}(J) \subset P$. Therefore $r \in P$, which is a contradiction.

Let $Q = h^{-1}(P')$. Since $h(r) \notin P'$, therefore $r \notin Q = h^{-1}(P')$. (If $r \in Q$ then $h(r) \in h(Q) = h(h^{-1}(P')) \subset P'$.) Hence P is a limit point of Spec $h(V(J))$. Therefore

$P \in \overline{\text{Spec } h(V(J))}$ and $\overline{\text{Spec } h(V(J))} = V(h^{-1}(J))$.

Corollary: Let $h : R \rightarrow S$ be a ring homomorphism. Then $\overline{\text{Spec } h(\text{Spec}(S))} = \text{Spec}(R)$ if and only if kernel h is a nil ideal.

Proof: Suppose kernel h is a nil ideal, so that kernel $h \subset \mathcal{P}(0_R)$, where 0_R is the zero of R. Then $\overline{\text{Spec } h(\text{Spec}(S))} = \overline{\text{Spec } h(V(0_S))} = V(h^{-1}(0_S))$ by Proposition 7. Now $V(h^{-1}(0_S)) = V(\text{kernel } h) \supset V(\mathcal{P}(0_R)) = V(0_R)$ (See Remark 2.) = Spec(R). Therefore $\overline{\text{Spec } h(\text{Spec}(S))} = \text{Spec}(R)$.

Conversely, suppose $\overline{\text{Spec } h(\text{Spec}(S))} = \text{Spec}(R)$, that is, $\overline{\text{Spec } h(V(0_S))} = V(0_R)$. Then $V(h^{-1}(0_S)) = V(0_R)$ by Proposition 7, so that $\mathcal{J}(V(h^{-1}(0_S))) = \mathcal{J}(V(0_R))$. It follows that $\mathcal{P}(h^{-1}(0_S)) = \mathcal{P}(0_R)$ by Proposition 2, so that $\mathcal{P}(\text{kernel } h) = \mathcal{P}(0_R)$. Hence kernel h is a nil ideal.

Proposition 8: Let R be a ring, let S be a multiplicative system in R, and let h be the canonical homomorphism from R into $S^{-1}R$. Then Spec h is a homeomorphism from Y = $\text{Spec}(S^{-1}R)$ onto the subspace of X = Spec(R) consisting of those prime ideals in R which do not intersect S.

Proof: (1) Spec h is continuous by Proposition 5.

(2) Suppose Spec $h(Q_1)$ = Spec $h(Q_2)$ or $h^{-1}(Q_1) = h^{-1}(Q_2)$. Then $Sh(h^{-1}(Q_1)) = Sh(h^{-1}(Q_2))$ so that $Q_1 = Q_2$ by Proposition 2, Chapter II, Section 1. Hence Spec h is one to one.

(3) Suppose P is an element of X such that $P \cap S = \emptyset$. Then Sh(P) is a prime ideal of $S^{-1}R$ and $h^{-1}(Sh(P) = P$ by

Proposition 3, Chapter II, Section 1. Put $Q = Sh(P)$. Then $\text{Spec } h(Q) = P$ so that Spec h is onto.

(4) It remains to prove that $\text{Spec } h^{-1}$ is continuous. Let $r' = \frac{r}{s} \in S^{-1}R$, where $r \in R$ and $s \in S$. Then $Y_{r'} = Y_{\frac{r}{1}}$. For $Y_{r'} = \{Q \in Y : r' \notin Q\} = \{Q \in Y : \frac{r}{s} \notin Q\} = \{Q \in Y : \frac{r}{s} \cdot \frac{s}{1} \notin Q\}$ $= \{Q \in Y : \frac{r}{1} \notin Q\} = Y_{\frac{r}{1}}$. ($\frac{r}{s} \cdot \frac{s}{1} \in Q$ if and only if $\frac{r}{s} \in Q$, since $\frac{s}{1}$ is a unit in $S^{-1}R$ and hence is not in Q.)

Now $\frac{r}{1} \in Q$ if and only if $r \in h^{-1}(Q) = \text{Spec } h(Q)$. For if $r \in h^{-1}(Q)$ then $h(r) \in Q$ so $\frac{r}{1} \in Q$. If $h(r) = \frac{r}{1} \in Q$ then $h^{-1}(h(r)) \subset h^{-1}(Q)$ and $r \in h^{-1}(Q)$. Hence $\frac{r}{1} \notin Q$ if and only if $r \notin h^{-1}(Q) = \text{Spec } h(Q)$, that is, $Q \in Y_{\frac{r}{1}}$ if and only if $\text{Spec } h(Q) \in X_r$. Therefore $\text{Spec } h(Y_{r'}) = X_r \cap \text{Spec } h(Y)$. For if $P \in X_r \cap \text{Spec } h(Y)$ then $P = \text{Spec } h(Q)$, where $Q \in Y$, and $Q \in Y_{\frac{r}{1}} = Y_{r'}$ so that $P = \text{Spec } h(Q) \in \text{Spec } h(Y_{r'})$.

Conversely, if $P \in \text{Spec } h(Y_{r'})$ then $P = \text{Spec } h(Q)$, where $Q \in Y_{r'} = Y_{\frac{r}{1}}$ and so $P \in X_r$.

Therefore the image of a member of the base in Y is the intersection of Spec h(Y) and a member of the base in X. Hence $\text{Spec } h^{-1}$ is continuous.

<u>Proposition 9</u>: Let R be a ring. Then $Y \subset X = \text{Spec}(R)$ is irreducible if and only if $\mathfrak{J}(Y)$ is prime.

<u>Proof</u>: Let $P = \mathfrak{J}(Y)$. We claim that if $r \in R$, then $r \in P$

if and only if $Y \subset V(r)$, that is, $r \in \bigcap \{P' : P' \in Y\}$ if and only if $Y \subset \{Q : r \in Q\}$. For suppose that $r \in \mathfrak{J}(Y)$. Let $P' \in Y$. Then $r \in P'$ so $P' \in \{Q : r \in Q\}$.

Conversely, suppose that $Y \subset \{Q : r \in Q\}$, that is, for all $Q \in Y$, $r \in Q$. Then $r \in \bigcap \{Q : Q \in Y\}$.

Now suppose Y is irreducible and suppose that $rs \in P$, where r and s are elements of R. Then $Y \subset V(rs) = V(r) \cup V(s)$, and since Y is irreducible, and V(r) and V(s) are closed, therefore $Y \subset V(r)$ or $Y \subset V(s)$, that is, $r \in P$ or $s \in P$. Hence P is prime.

Conversely, suppose P is prime. Now $\overline{Y} = V(\mathfrak{J}(Y)) = V(P)$ by Proposition 2, and since P is prime, $P = \mathfrak{J}(P) = \bigcap \{P : P \in \{P\}\}$. Therefore $\overline{Y} = V(\mathfrak{J}(P)) = \overline{\{P\}}$. Now $\{P\}$ is irreducible (since every set consisting of a single point is irreducible), therefore so is $\overline{\{P\}} = \overline{Y}$ and hence so is Y. (See Proposition 3, Chapter III, Section 1.)

<u>Corollary 1</u>: Let R be a ring. Then X = Spec(R) is irreducible if and only if $R/\mathfrak{P}(0)$ is an integral domain.

<u>Proof</u>: $\mathfrak{J}(X) = \mathfrak{P}(0)$, so X is irreducible if and only if $\mathfrak{P}(0)$ is prime by Proposition 9, that is, if and only if $R/\mathfrak{P}(0)$ is an integral domain.

<u>Corollary 2</u>: The mapping $P \to V(P)$ is a bijection from X = Spec(R) onto the set of closed, irreducible subsets of X. In particular, the irreducible components of a closed subset Y of X are the sets V(P), where P runs through the set of minimal elements in the set of prime ideals of R which contain $\mathfrak{J}(Y)$.

Proof: If $P \in X$ then $V(P)$ is irreducible, since $\mathfrak{I}(V(P)) = \wp(P) = P$ is prime by Proposition 9. Clearly $V(P)$ is closed.

If $V(P_1) = V(P_2)$ then $\mathfrak{I}(V(P_1)) = \mathfrak{I}(V(P_2))$ and $P_1 = P_2$, so the mapping is one to one.

If Y is a closed, irreducible subset of X then $\mathfrak{I}(Y)$ is prime by Proposition 9, and $V(\mathfrak{I}(Y)) = \overline{Y} = Y$ by Proposition 2. Hence the mapping is onto.

Let Y be closed in X. Its irreducible components are among the $V(P)$, where P is prime. Now $V(P) \subset Y$ if and only if $\mathfrak{I}(V(P)) \supset \mathfrak{I}(Y)$, that is, if and only if $P \supset \mathfrak{I}(Y)$ by Proposition 2. Also $V(P)$ is a maximal irreducible set if and only if P is a minimal prime ideal. For suppose $V(P)$ is maximal. Then if $Q \subset P$, $V(P) \subset V(Q)$, so that $V(P) = V(Q)$. Therefore $\mathfrak{I}(V(P)) = \mathfrak{I}(V(Q))$ and $Q = P$. Hence P is a minimal prime ideal.

Conversely, if P is a minimal prime ideal, suppose $V(P) \subset V(Q)$. Then $\mathfrak{I}(V(P)) \supset \mathfrak{I}(V(Q))$ so $Q \subset P$ and $Q = P$. Hence $V(Q) = V(P)$. Therefore $V(P)$ is a maximal irreducible set.

Corollary 3: The set of minimal prime ideals of a noetherian ring R is finite.

Proof: Since R is a noetherian ring, therefore $X = \mathrm{Spec}(R)$ is a noetherian space by Proposition 2, Corollary 7. Hence X has only a finite number of irreducible components. But the irreducible components of X are the sets $V(P)$, where P runs through the set of minimal prime ideals of R which contain $\mathcal{P}(0)$, that is, all minimal prime ideals of R. (See Proposition 9.) Hence R has only a finite number of minimal prime ideals.

Proposition 10: Let R be a ring. Then

(1) X = Spec(R) is a T_0 space.

(2) every irreducible component of X has a unique generator.

Proof: (1) Suppose P_1 and P_2 are two points in X such that $P_1 \neq P_2$. Then either $P_1 \not\subset P_2$ or $P_2 \not\subset P_1$, say $P_1 \not\subset P_2$. We then have that there exists an element $r \in P_1$ such that $r \notin P_2$. It follows that X_r is a neighborhood of P_2 which does not contain P_1.

(2) Let Y be an irreducible component of X. Then Y = V(P) for some $P \in Y$ by Proposition 9, Corollary 2, and $\overline{\{P\}} = V(\mathcal{J}(P)) = V(P) = Y$. Hence Y has at least one generator. That it is unique follows from the fact that X is a T_0 space and Remark 2, Chapter III, Section 1.

Corollary: If R is an integral domain and X = Spec(R), then

(1) X is irreducible and its generator is $\{(0)\}$.

(2) $\{(0)\}$ is an isolated point of X if and only if the intersection of all non-zero prime ideals of R is not equal to zero.

Proof: (1) Since R is an integral domain, therefore (0) is a prime ideal, so $(0) = \mathcal{P}(0)$ and X is irreducible by Proposition 9, Corollary 1. Also $\overline{\{(0)\}} = V(\mathcal{J}(\{(0)\}))$ $= V(0) = X$. (See Proposition 2.)

(2) Let $M = \bigcap \{P \in X : P \neq (0)\}$. Suppose $\{(0)\}$ is an

isolated point of X, that is, $\{(0)\}$ is open in X. Then $\{(0)\} = X_r$ for some $r \in R$ and $r \in M$. For if there exists some $P \neq (0)$ such that $r \notin P$ then $P \in X_r$, which is a contradiction.

Conversely, if $M \neq (0)$ then there exists an element $r \in M$ such that $r \neq 0$ and $X_r = \{(0)\}$. For if $P \in X_r$ then $r \notin P$ and $P = (0)$. Hence $\{(0)\}$ is an isolated point of X.

<u>Proposition 11</u>: Let R be a noetherian ring and let X = Spec(R). Then a subset F of X is closed if and only if it satisfies the following two properties:

(1) For all $P \in F$, $V(P) \subset F$.

(2) For all $P \notin F$, there exists a closed set V(N), where N is a subset of R, such that $F \cap V(P) \subset V(N) \subset V(P)$ and such that $P \notin V(N)$.

<u>Proof</u>: Suppose F is closed in X. Then F = V(M), where M is a subset of R.

(1) If $P \in V(M)$ then $M \subset P$, so if $Q \in V(P)$, that is, $P \subset Q$, then $M \subset Q$ and $Q \in V(M)$. Hence $V(P) \subset V(M) = F$.

(2) Suppose $P \notin V(M)$. Take $N = M \cup P$. $M \not\subset P$ so $N \neq P$. Hence $N \not\subset P$ and $P \notin V(N)$. Furthermore, if $Q \in V(N)$ then $N \subset Q$ so that $P \subset N \subset Q$ and $Q \in V(P)$. Therefore $V(N) \subset V(P)$. Finally, if $Q \in V(M) \cap V(P)$ then $Q \in V(M \cup P) = V(N)$. Hence $F \cap V(P) \subset V(N)$.

Conversely, suppose F satisfies conditions (1) and (2). Since $\overline{F}$ is a closed subset of X, its irreducible components are of the form V(P), where P is a minimal prime ideal in $\overline{F}$. (See Proposition 9, Corollary 2.)

Also since X is a noetherian space, the subspace $\overline{F}$ is also noetherian by Proposition 9, Chapter III, Section 2, and hence $\overline{F}$ has only finitely many irreducible components. (See Proposition 11, Chapter III, Section 2.) Suppose $\overline{F} = \bigcup_{i=1}^{n} V(P_i)$, where the $V(P_i)$ are the irreducible components of $\overline{F}$. Now for each i there exists a closed set $V(N_i)$, N_i being a subset of R, such that $F \cap V(P_i) \subset V(N_i) \subset V(P_i) \subset \overline{F}$. For if $P_i \in F$, we may take $N_i = P_i$ and if $P_i \notin F$, then by (2), there exists $N_i \subset R$ such that $F \cap V(P_i) \subset V(N_i) \subset V(P_i) \subset \overline{F}$. Hence $\bigcup_{i=1}^{n}(F \cap V(P_i)) \subset \bigcup_{i=1}^{n} V(N_i) \subset \overline{F}$. It follows that $F \cap (\bigcup_{i=1}^{n} V(P_i)) \subset \bigcup_{i=1}^{n} V(N_i) \subset \overline{F}$. Therefore $F \cap \overline{F} = F \subset \bigcup_{i=1}^{n} V(N_i) \subset \overline{F}$, so that $\bigcup_{i=1}^{n} V(N_i) = \overline{F} = \bigcup_{i=1}^{n} V(P_i)$. Now for each i, $V(P_i)$ is irreducible and $V(P_i) \subset V(N_1) \cup \ldots \cup V(N_n)$. Therefore $V(P_i) \subset V(N_j)$ for some j, $j = 1,\ldots,n$. If $j \neq i$ it follows that $V(P_i) \subset V(N_j) \subset V(P_j)$, which is a contradiction. Hence $j = i$ and $V(P_i) \subset V(N_i)$ so that $V(N_i) = V(P_i)$. Therefore $P_i \in V(N_i)$. Now if $P_i \notin F$ then by (2) $P_i \notin V(N_i)$, which is a contradiction. Hence $P_i \in F$, $i = 1,\ldots,n$. Therefore $V(P_i) \subset F$, $i = 1,\ldots,n$ by (1), so that $\bigcup_{i=1}^{n} V(P_i) \subset F$ or $\overline{F} \subset F$. It follows that $\overline{F} = F$ and F is closed.

2. Support of a module.

Definition 2: Let R be a ring and let A be an R-module. Then the set of prime ideals P in R such that $A_P \neq 0$ (See Chapter II, Section 2, Remark 3.) is called the support of A and is denoted by Supp(A).

Proposition 12: If I is an ideal in R, then V(I) = Supp(R/I).

Proof: We first show that if S is a multiplicative system in R, then $S^{-1}(R/I) \cong S^{-1}R/S^{-1}I$.

Define $f : S^{-1}(R/I) \longrightarrow S^{-1}R/S^{-1}I$ as follows:

$$f\left(\frac{r + I}{s}\right) = \frac{r}{s} + S^{-1}I.$$

(1) f is well defined:

Suppose $\frac{r_1 + I}{s_1} = \frac{r_2 + I}{s_2}$. Then there exists an element s' in S such that $s'(s_2(r_1 + I) - s_1(r_2 + I)) = 0$ in R/I, that is, $s's_2r_1 - s's_1r_2 \in I$. Hence $\frac{s's_2r_1 - s's_1r_2}{s's_1s_2} \in S^{-1}I$, so that $\frac{r_1}{s_1} - \frac{r_2}{s_2} \in S^{-1}I$ and $\frac{r_1}{s_1} + S^{-1}I = \frac{r_2}{s_2} + S^{-1}I$

(2) f is clearly an $S^{-1}R$-homomorphism.

(3) f is one to one:

If $f\left(\frac{r + I}{s}\right) = 0$ then $\frac{r}{s} + S^{-1}I = 0$, so that $\frac{r}{s} \in S^{-1}I$ and $r \in I$. Hence $\frac{r + I}{s} = 0$.

(4) f is clearly onto.

Therefore $S^{-1}(R/I) \cong S^{-1}R/S^{-1}I$.

In particular, when S = R - P, where P is a prime ideal in R, then $(R/I)_P \cong R_P/I_P$. Now by the corollary of Proposition 2, Chapter II, Section 1, $I_P = R_P$ if and only if $S \cap I \neq \emptyset$, that is, if and only if $I \not\subset P$. Hence $(R/I)_P = 0$ if and only if $I \not\subset P$. Therefore $P \in V(I)$ if and only if $P \in \mathrm{Supp}(R/I)$.

In particular, $\mathrm{Supp}(R) = V(0) = \mathrm{Spec}(R)$.

Proposition 13: Let R be a ring and let A be an R-module.

(1) If B is a submodule of A, then $\mathrm{Supp}(A) = \mathrm{Supp}(B) \cup \mathrm{Supp}(A/B)$.

(2) If A is the sum of a family $\{B_i\}_{i \in I}$ of submodules, then $\mathrm{Supp}(A) = \bigcup_{i \in I} \mathrm{Supp}(B_i)$.

(3) If $\{B_i\}_{i=1}^{n}$ is a finite family of submodules of A, then

$$\mathrm{Supp}(A/\bigcap_{i=1}^{n} B_i) = \bigcup_{i=1}^{n} \mathrm{Supp}(A/B_i).$$

Proof: (1) Suppose $P \in \mathrm{Supp}(A)$, that is, $A_P \neq 0$. Then there exists an element $a \in A$ such that for all $s \in S$, $as \neq 0$. If $P \notin \mathrm{Supp}(B)$ then $B_P = 0$. We will show that $(a + B)s \neq 0$ for all $s \in S$. If $(a + B)s = 0$ for some $s \in S$, then $as \in B$ and $(as)s' = as'' \neq 0$ for all $s' \in S$, that is $B_P \neq 0$. This is a contradiction. Hence $(A/B)_P \neq 0$.

Conversely, if $P \in \mathrm{Supp}(B)$ then $B_P \neq 0$ and therefore $A_P \neq 0$ (since $B_P \subset A_P$). If $P \in \mathrm{Supp}(A/B)$ then $(A/B)_P \neq 0$ so $A_P/B_P \neq 0$ and therefore $A_P \neq 0$. Hence in both cases $P \in \mathrm{Supp}(A)$.

(2) If $A_P \neq 0$ and $A = \sum_{i \in I} B_i$, then there exists $i \in I$ such that $(B_i)_P \neq 0$. For suppose $(B_i)_P = 0$ for all $i \in I$. Let

$a \in A$, then $a = b_1 + \dots + b_n$. Now there exists $s_i \in S = R - P$ such that $b_i s_i = 0$, $i = 1,\dots,n$. Take $s = s_1 \dots s_n$. Then $as = 0$ and $A_P = 0$. This is a contradiction.

Conversely, if $(B_i)_P \neq 0$ for some $i \in I$, then there exists an element $b_i \in B_i$ such that $b_i s \neq 0$ for all $s \in S$. Consider $a = 0 + \dots + 0 + b_i + 0 + \dots$. Now $as \neq 0$ for all $s \in S$ so that $A_P \neq 0$.

(3) Let $P \in \operatorname{Supp}(A/\bigcap_{i=1}^{n} B_i)$, that is, $(A/\bigcap_{i=1}^{n} B_i)_P \neq 0$. Then there exists an element $a + \bigcap_{i=1}^{n} B_i$ such that $(a + \bigcap_{i=1}^{n} B_i)s \neq 0$ for all $s \in S$, that is, $as \notin \bigcap_{i=1}^{n} B_i$ for all $s \in S$. We will show that there exists B_j such that $as \notin B_j$ for all $s \in S$. If not, for all B_i there exists $s_i \in S$ such that $as_i \in B_i$, $i = 1,\dots,n$. Let $s = s_1 \dots s_n$. Then $as \in B_i$, $i = 1,\dots,n$ so that $as \in \bigcap_{i=1}^{n} B_i$. This is a contradiction. Hence for some j, $as \notin B_j$ for all $s \in S$ and therefore $as + B_j \neq 0$ for all $s \in S$. Therefore $(A/B_j)_P \neq 0$ and $P \in \operatorname{Supp}(A/B_j)$.

Conversely, let $P \in \bigcup_{i=1}^{n} \operatorname{Supp}(A/B_i)$, say $P \in \operatorname{Supp}(A/B_j)$. Then $(A/B_j)_P \neq 0$ so there exists an element $a + B_j$ such that $(a + B_j)s \neq 0$ for all $s \in S$. Hence $as \notin B_j$ for all $s \in S$, so that $as \notin \bigcap_{i=1}^{n} B_i$ for all $s \in S$ and $as + \bigcap_{i=1}^{n} B_i \neq 0$ for all $s \in S$, that is, $(A/\bigcap_{i=1}^{n} B_i)_P \neq 0$. Therefore $P \in \operatorname{Supp}(A/\bigcap_{i=1}^{n} B_i)$.

Notice that in case (3) we required a <u>finite</u> family of submodules of A. We will show that this is in fact necessary. Before we can do this, however, we shall require a few more results.

<u>Corollary</u>: Let R be a ring and let A be an R-module. Let $\{m_i\}_{i\in I}$ be a system of generators for A, and let $J_i = \text{Ann } m_i = \{r \in R : rm_i = 0\}$. Then $\text{Supp}(A) = \bigcup_{i\in I} V(J_i)$.

<u>Proof</u>: $A = \sum_{i\in I} Rm_i$ so $\text{Supp}(A) = \bigcup_{i\in I} \text{Supp}(Rm_i)$ by Proposition 13. Now $Rm_i \cong R/J_i$, where $r + J_i \to rm_i$. Hence

$\text{Supp}(A) = \bigcup_{i\in I} \text{Supp}(R/J_i) = \bigcup_{i\in I} V(J_i)$ by Proposition 12.

<u>Proposition 14</u>: Let R be a ring, let A be an R-module, and let J = Ann A. If A is finitely generated then Supp(A) = V(J).

<u>Proof</u>: Let $\{a_i\}_{i=1}^{n}$ be a system of generators for A and let $J_i = \text{Ann } a_i$, $i = 1,\ldots,n$. Then $J = \bigcap_{i=1}^{n} J_i$. (For $j \in J \Leftrightarrow jA = 0 \Leftrightarrow j(\sum_{i=1}^{n} Ra_i) = 0 \Leftrightarrow Rja_i = 0,\ i = 1,\ldots,n$ $\Leftrightarrow ja_i = 0,\ i = 1,\ldots,n \Leftrightarrow j \in \text{Ann } a_i,\ i = 1,\ldots,n.$) Hence $V(J) = V(\bigcap_{i=1}^{n} J_i) = \bigcup_{i=1}^{n} V(J_i) = \text{Supp}(A)$ by the corollary of Proposition 13.

Supp(A) is thus a closed set in Spec(R) = Supp(R).

<u>Corollary 1</u>: Let R be a ring, let A be a finitely generated R-module, and let r be an element of R. Then $r \in P$ for

all $P \in \mathrm{Supp}(A)$ if and only if $r^n A = 0$ for some integer n.

<u>Proof</u>: $\bigcap \{P : P \in \mathrm{Supp}(A)\} = \bigcap \{P : P \in V(J)\}$ where $J = \mathrm{Ann}\ A$. (See Proposition 14.) Now $\bigcap \{P : P \in V(J)\} = \mathcal{J}(V(J))$ $= \mathcal{P}(J)$ by Proposition 2, Section 1. But $r \in \mathcal{P}(J)$ if and only if $r^n \in J$ for some integer n, that is, if and only if $r^n A = 0$. Hence the proposition follows.

<u>Lemma 1</u>: Let R be a ring, let J be an ideal in R, and let I be a finitely generated ideal in R such that $I \subset \mathcal{P}(J)$. Then there exists an integer $k > 0$ such that $I^k \subset J$.

<u>Proof</u>: Let I be generated by $\{x_j\}_{j=1}^{n}$. Now there exists an integer h such that $x_j^h \in J$, $1 \leq j \leq n$. Take $k = nh$. Then if $x \in I$, $x = Rx_1 + \ldots + Rx_n$ and $x^k = (Rx_1 + \ldots + Rx_n)^k$ is in J.

<u>Corollary 2</u>: Let R be a noetherian ring, let A be a finitely generated R-module, and let I be an ideal in R. Then $\mathrm{Supp}(A) \subset V(I)$ if and only if there exists an integer $k > 0$ such that $I^k A = 0$.

<u>Proof</u>: Let $J = \mathrm{Ann}\ A$. Then by Proposition 14, $\mathrm{Supp}(A)$ $= V(J)$. Hence $\mathrm{Supp}(A) \subset V(I)$ if and only if $V(J) \subset V(I)$ and this is true if and only if $I \subset \mathcal{P}(J)$ by Proposition 2, Corollary 2, Section 1. Now since R is noetherian, I is finitely generated and so $I \subset \mathcal{P}(J)$ if and only if there exists an integer $k > 0$ such that $I^k \subset J$ by Lemma 1, that is, if and only if $I^k A = 0$.

We can now show that case (3) of Proposition 13 holds only for a finite number of submodules of A. Consider the case where $R = A = Z$, the set of integers. Let p be a prime number. We will show that $\mathrm{Supp}(Z/\bigcap_{k=1}^{\infty} p^k Z) \neq \bigcup_{k=1}^{\infty} \mathrm{Supp}(Z/p^k Z)$. Now $\bigcap_{k=1}^{\infty} p^k Z = (0)$ so that $\mathrm{Supp}(Z/\bigcap_{k=1}^{\infty} p^k Z) = \mathrm{Supp}(Z) = \mathrm{Spec}(Z)$ and therefore contains qZ for every prime number q. On the other hand $Z/p^k Z$ is finitely generated. (It is generated by $1 + p^k Z$.) Hence by Proposition 14, $\mathrm{Supp}(Z/p^k Z) = V(p^k Z)$ (since $p^k Z = \mathrm{Ann}\,(Z/p^k Z)$). Therefore $\bigcup_{k=1}^{\infty} \mathrm{Supp}(Z/p^k Z) = \bigcup_{k=1}^{\infty} V(p^k Z)$. Now suppose $qZ \in \bigcup_{k=1}^{\infty} \mathrm{Supp}(Z/p^k Z)$, that is, $qZ \in V(p^k Z)$ for some integer k. Then $p^k Z \subset qZ$ and $q \mid p^k$. Hence $q = p$. The only prime ideal in $\bigcup_{k=1}^{\infty} \mathrm{Supp}(Z/p^k Z)$ is therefore pZ.

We recall that in Proposition 14 we proved that if $J = \mathrm{Ann}\, A$, where A is a <u>finitely generated</u> R-module, then $\mathrm{Supp}(A) = V(J)$. We will now show that the condition that A be finitely generated is actually necessary.

Consider again the case where $R = A = Z$ and let p be a prime number. Put $M = Z/pZ \oplus Z/p^2 Z \oplus \ldots$. M is clearly not finitely generated. Now $\mathrm{Supp}(M) = \bigcup_{k=1}^{\infty} \mathrm{Supp}(Z/p^k Z)$ by Proposition 13. If $J = \mathrm{Ann}\, M$, then

$$\begin{aligned} J &= \{r \in Z : r(Z/pZ \oplus \ldots) = 0\} \\ &= \{r \in Z : rZ \subset p^k Z,\ k = 1, 2, \ldots\} \\ &= \{r \in Z : p^k \mid r,\ k = 1, 2, \ldots\} \\ &= (0) \end{aligned}$$

So $V(J) = \mathrm{Supp}(Z)$. But in the example above we saw that

$\mathrm{Supp}(Z) \neq \bigcup_{k=1}^{\infty} \mathrm{Supp}(Z/p^kZ)$. Therefore $V(J) \neq \mathrm{Supp}(M)$.

We will show, however, that in this case Supp(M) is closed. Let $K = \mathrm{Supp}(M)$ and suppose qZ is a limit point of K, q being a prime number, that is, for all $r \in Z$ such that $qZ \in X_r$, $X_r \cap K \neq \emptyset$. This means that if $r \notin qZ$ then there exists q_1Z, where q_1 is a prime number, such that $r \notin q_1Z$ and $p^kZ \subset q_1Z$ for some integer k (See above example.), that is, $q_1 | p^k$ so that $q_1 = p$. Hence we have if $r \notin qZ$ then $r \notin pZ$. Now if $q \neq p$ then $p \notin qZ$ and it would follow that $p \notin pZ$ which is clearly impossible. Therefore $q = p$ and $qZ = pZ \in K = \mathrm{Supp}(M)$. Hence Supp(M) is closed.

It may seem that for any R-module A, Supp(A) is closed in Spec(R). (For example, this is always true when A is finitely generated.) However we will now give an example where this is not so.

Let $N = Z/Z \oplus Z/2Z \oplus Z/3Z \oplus \ldots$. Then

$\mathrm{Supp}(N) = \bigcup_{n=1}^{\infty} \mathrm{Supp}(Z/nZ)$ (SeeProposition 13.) $= \bigcup_{n=1}^{\infty} V(nZ)$

by Proposition 14. Now (0) is a prime ideal in Z and

$(0) \notin \bigcup_{n=1}^{\infty} V(nZ)$. We will show that (0) is a limit point

of Supp(N).

Suppose $(0) \in X_r$, that is, $r \neq 0$. We must show that $X_r \cap \mathrm{Supp}(N) \neq \emptyset$, that is, there exists qZ, q being a prime number, such that $r \notin qZ$ and $nZ \subset qZ$ for some integer n. Take q to be any prime number greater than r. Then $r \notin qZ$ (since $q \nmid r$) and $qZ \in V(qZ) \subset \bigcup_{n=1}^{\infty} V(nZ)$. Therefore Supp(N) is not closed in Spec(Z).

Finally, we proved in Proposition 13 that if $A = \sum_{i \in I} B_i$, where A is an R-module and the B_i are submodules, that $\text{Supp}(A) = \bigcup_{i \in I} \text{Supp}(B_i)$. We will show that this proposition does not necessarily hold if $A = \prod_{i \in I} B_i$.

Consider the case where $A = \prod_{k=1}^{\infty} Z/p^k Z$, where Z is the set of integers and p is a prime number. We will show that Supp(A) = Supp(Z), that is, $qZ \in \text{Supp}(A)$ or $(A)_{qZ} \neq 0$ for every prime number q. Now $(1 + pZ, 1 + p^2Z, \ldots)$ is in $A = \prod_{k=1}^{\infty} Z/p^k Z$ and $(1 + pZ, 1 + p^2Z, \ldots)s \neq 0$ for all $s \in S = Z - qZ$. For if $(1 + pZ, 1 + p^2Z, \ldots)s = 0$ for some $s \in S$, then $s \in p^n Z$, $n = 1, 2, \ldots$ so that $p^n | s$, $n = 1, 2, \ldots$. This is of course impossible. Therefore Supp(A) = Supp(Z). However, as we have seen in an earlier example, $\bigcup_{k=1}^{\infty} \text{Supp}(Z/p^k Z) \neq \text{Supp}(Z)$. Hence the counter-example is established.

<u>Proposition 15</u>: Let R be a ring and let A and B be two R-modules such that A is finitely generated. Then $\text{Supp}(\text{Hom}_R(A,B)) \subset \text{Supp}(A) \cap \text{Supp}(B)$.

<u>Proof</u>: Let $P \in \text{Supp}(H)$, where $H = \text{Hom}_R(A,B)$, that is, $H_P \neq 0$. Then there exists $f \in H$ such that $sf \neq 0$ for all $s \in S = R - P$. Hence $sf(A) \neq 0$ for all $s \in S$. If $A_P = 0$ then there exists an element $s \in S$ such that $sA = 0$ by Proposition 4, Chapter

II, Section 2. Therefore $f(sA) = 0$ for all $f \in H$, or $sf(A) = 0$ for all $f \in H$. This is a contradiction. Hence $A_P \neq 0$ so $P \in \mathrm{Supp}(A)$.

Now for any $f \in H$, $f(A)$ is finitely generated. Therefore $\mathrm{Supp}(H) \subset \mathrm{Supp}(f(A)) \subset \mathrm{Supp}(B)$ (since $f(A) \subset B$). Hence $\mathrm{Supp}(\mathrm{Hom}_R(A,B)) \subset \mathrm{Supp}(A) \cap \mathrm{Supp}(B)$.

Proposition 15 does not hold if A is not finitely generated. For let $A = Z/pZ \oplus Z/p^2Z \oplus \dots$, where Z is the set of integers and p is a prime number. We will show that $\mathrm{Supp}(\mathrm{Hom}_Z(A,A)) \not\subset \mathrm{Supp}(A)$. Recall that the only element in Supp(A) is pZ. Suppose $q \neq p$. Then $qZ \in \mathrm{Supp}(\mathrm{Hom}_Z(A,A))$, that is, $(\mathrm{Hom}_Z(A,A))_{qZ} \neq 0$. For $1 \in \mathrm{Hom}_Z(A,A)$ and $1s \neq 0$ for all $s \in S = Z - qZ$. Therefore $qZ \in \mathrm{Supp}(\mathrm{Hom}_Z(A,A))$ but $qZ \notin \mathrm{Supp}(A)$.

<u>Definition 3</u>: Two ideals I and J of a ring R are said to be <u>co-maximal</u> if $I + J = R$ or if there exist elements $a \in I$ and $b \in J$ such that $a + b = 1$.

<u>Proposition 16</u>: Let R be a ring and let $J_1,\dots,J_n$ be ideals in R.

(1) If I is an ideal in R such that I and J_k are co-maximal, $k = 1,\dots,n$, then I and $J_1 \cap \dots \cap J_n$ are co-maximal. Also I and $J_1 \dots J_n$ are co-maximal.

(2) If $J_1,\dots,J_n$ are <u>pairwise co-maximal</u> (that is, $J_i + J_k = R$ for $i \neq k$), then $J_1 \cap \dots \cap J_n = J_1 \dots J_n$.

<u>Proof</u>: (1) $R = R^n = \prod_{k=1}^{n} (I + J_k) = I + \prod_{k=1}^{n} J_k \subset R$. Hence $R = I + \prod_{k=1}^{n} J_k$. Now $\prod_{k=1}^{n} J_k \subset \bigcap_{k=1}^{n} J_k$, so $I + \bigcap_{k=1}^{n} J_k = R$.

(2) We use induction on n. Suppose J_1 and J_2 are co-maximal. Then $J_1 \cap J_2 = (J_1 \cap J_2)(J_1 + J_2) = J_1(J_1 \cap J_2) + J_2(J_1 \cap J_2)$

$\subset J_1J_2 + J_2J_1 = J_1J_2$.

Clearly $J_1J_2 \subset J_1 \cap J_2$.

Assume that the result holds for n - 1 J_i's. By (1) J_n is co-maximal with $J_1 \cap \ldots \cap J_{n-1}$. Therefore

$(J_1 \cap \ldots \cap J_{n-1}) \cap J_n = (J_1 \cap \ldots \cap J_{n-1})J_n$

$= (J_1 \ldots J_{n-1})J_n$.

<u>Remark 6</u>: I and J are co-maximal ideals of R if and only if $V(I) \cap V(J) = \emptyset$. For if I + J = R then $V(I) \cap V(J) = V(I + J) = V(R) = \emptyset$, and if $V(I) \cap V(J) = \emptyset$ then $V(I + J) = \emptyset$ so I + J = R. (See Remark 1, Section 1.)

We conclude this paper with the following rather lengthy but quite important proposition.

<u>Proposition 17</u>: Let R be a noetherian ring and let A be a finitely generated R-module. Then A admits a decomposition as a direct sum of modules $A_1, \ldots, A_s$ ($A = A_1 \oplus \ldots \oplus A_s$), where Ann $A_i = J_i$, i = 1,...,s, and the J_i are pairwise co-maximal (i = 1,...,s). Each A_i can be decomposed no further in the above manner. If I = Ann A, then $I = J_1 \cap \ldots \cap J_s = J_1 \ldots J_s$ and we thus obtain a representation of I as an intersection of pairwise co-maximal ideals. Each J_i can no longer be represented as such an intersection.

Proof: Since R is a noetherian ring, X = Spec(R) is a noetherian space, by Corollary 7, Proposition 2, Section 1, and hence so is Supp(A). (See Proposition 10, Chapter III, Section 2.) Therefore Supp(A) has only a finite number of connected components, say Supp(A) = $V_1 \cup \dots \cup V_s$, where the V_i are the connected components, $i = 1,\dots,s$. (See Proposition 12, Chapter III, Section 2.)

Let $I = J_{1,1} \cap \dots \cap J_{1,t_1} \cap \dots \cap J_{s,1} \cap \dots \cap J_{s,t_s}$

be an irredundant primary decomposition of I (See Chapter I, Section 2.) with $Q_{i,j} = \mathcal{P}(J_{i,j})$ and such that $Q_{i,j} \in V_i$

($j = 1,\dots,t_i$ and $i = 1,\dots,s$). Put $J_i = J_{i,1} \cap \dots \cap J_{i,t_i}$.

Then $I = J_1 \cap \dots \cap J_s$ and $\text{Supp}(A) = V(I) = V(J_1 \cap \dots \cap J_s)$

$= V(J_1) \cup \dots \cup V(J_s)$. (A is finitely generated: See Proposition 14.) Now $V(J_i) = V(J_{i,1} \cap \dots \cap J_{i,t_i})$

$= V(J_{i,1}) \cup \dots \cup V(J_{i,t_i}) = V(Q_{i,1}) \cup \dots \cup V(Q_{i,t_i})$

(See Remark 2, Section 1.) $\subset V_i$. ($Q_{i,j} \in V_i$, $j = 1,\dots,t_i$ and V_i is closed; see Proposition 11, Section 1.) Furthermore $V(J_i) \cap V(J_k) \subset V_i \cap V_k = \emptyset$ for $i \neq k$ and since $V(J_i)$ and $V(J_k)$ are both closed they are separated. We will show that $V(J_i)$ is a connected component of Supp(A), $i = 1,\dots,s$. Suppose $V(J_i) \subset Y$, where Y is connected and $Y \subset \text{Supp}(A)$. Then $Y \subset V(J_1) \cup \dots \cup V(J_s)$ so that $Y \subset V(J_k)$

for some k. Hence $V(J_i) \subset V(J_k)$. It follows that $V(J_i) = Y$. Therefore $V(J_i)$ is a connected component in Supp(A). By the uniqueness of connected components $V(J_i) = V_i$, $i = 1,\ldots,s$. Also since $V(J_i) \cap V(J_k) = \emptyset$ for $i \neq k$, J_i and J_k are co-maximal by Remark 6.

Let $L_i = \bigcap_{k \neq i} J_k$, let $A_i = L_i A$, and let $B_i = \sum_{k \neq i} A_k$, $i = 1,\ldots,s$. We will show that $A = A_1 \oplus \cdots \oplus A_s$ and $J_i = \text{Ann } A_i$.

(1) $J_i = \text{Ann } A_i$:

$J_i A_i = J_i L_i A = (J_i \cap L_i)A$ (since J_i and L_i are co-maximal; see Proposition 16) $= IA = 0$. Therefore $\text{Ann } A_i \supset J_i$.

Conversely, suppose $xA_i = 0$. We will show that $x \in J_i$. Now $xL_i A = 0$ so that $xL_i \subset I \subset J_i$. Since J_i and L_i are co-maximal, therefore there exist elements $a_i \in J_i$ and $b_i \in L_i$ such that $1 = a_i + b_i$. Hence $x = xa_i + xb_i \in J_i$.

(2) The A_i generate A:

Since the J_k are pairwise co-maximal, $k = 1,\ldots,s$, therefore J_i and $\prod_{k \neq i} J_k = \bigcap_{k \neq i} J_k = L_i$ are co-maximal, by Proposition 16. Hence for every i there exist elements $c_i \in J_i$ and $d_i \in L_i$ such that $c_i + d_i = 1$. It follows that

$$1 = d_1 + c_1(d_2 + c_2(d_3 + \cdots + c_{s-1}(d_s + c_s)))\ldots))),$$

that is, $1 = x_1 + \cdots + x_s + y$, where $x_i \in L_i$, $i = 1,\ldots,s$

and $y \in I$. Hence $A = x_1A + \dots + x_sA + yA \subset L_1A + \dots + L_sA$ (since $yA = 0$) $= A_1 + \dots + A_s$ so $A = A_1 + \dots + A_s$.

(3) The sum is direct:

If $i \neq j$, $L_iA_j = (\bigcap_{k \neq i} J_k)A_j \quad J_jA_j = 0$ by (1). Therefore $L_iB_i = L_i(\sum'_{j \neq i} A_j) = 0$. Hence if $x \in A_i \cap B_i$, then $(J_i + L_i)x = J_ix + L_ix = Rx = 0$. It follows that $x = 0$.

We will now show that (i) for each i, $A_i \neq A_i' \oplus A_i''$ with $J_i' + J_i'' = R$, where $J_i' = \text{Ann } A_i'$ and $J_i'' = \text{Ann } A_i''$ and (ii) for each k, $J_k \neq J_k' \cap J_k''$ such that $J_k' + J_k'' = R$.

(i) $\text{Supp}(A_i' \oplus A_i'') = V(\text{Ann}(A_i' \oplus A_i'')) = V(\text{Ann } A_i' \cap \text{Ann } A_i'') = V(J_i' \cap J_i'') = V(J_i') \cup V(J_i'')$. But $V(J_i') \cap V(J_i'') = \emptyset$, since J_i' and J_i'' are co-maximal by Remark 6. Therefore $\text{Supp}(A_i) = V(J_i) = V_i$ is not connected. This is a contradiction.

(ii) If $L_k' = J_k''$, $L_k'' = J_k'$, $A_i' = L_i'A$, and $A_i'' = L_i''A$, we obtain $A_i = A_i' \oplus A_i''$, $J_i' = \text{Ann } A_i'$, and $J_i'' = \text{Ann } A_i''$ by an argument similar to that in (1), (2) and (3) above. However this is impossible as we have just seen.

BIBLIOGRAPHY

1. Bourbaki, N., Algèbre Commutative, Chapitre 2, Hermann, Paris, 1961.
2. Bourbaki, N., Topologie Générale, Chapitre 1, Hermann, Paris, 1961.
3. Kelley, J. L., General Topology, D. Van Nostrand Co., Princeton, 1955.
4. Serre, J-P., Algèbre Locale - Multiplicités, Lecture Notes in Mathematics, 11 (1965).
5. Zariski, O. and Samuel, P., Commutative Algebra, Volume I, D. Van Nostrand Company, Princeton, 1958.