One dimensional discrete Schrödinger operators

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Abstract

We consider the one dimensional discrete Schrödinger operator $h = h_0 + V$ on the full line and half line, where h_0 is the discrete Laplacian and V is a real-valued potential. We explain the Spectral theorem for the operator and give explicit formulas of the Green's function and spectral measures in case of the Laplacian. We explore the rank one potentials and compute their scattering operator. We also explore periodic potentials on the full line. We introduce random Schrödinger operators, and reproduce the proof of the celebrated theorem of Pastur that the spectrum is almost surely the same set. To illustrate ergodic families of random operators, we study the Anderson model in one dimension.

L'objet de la thèse est l'opérateur de Schrödinger discret $h = h_0 + V$ en une dimension, sur la ligne et la demi-ligne, où h_0 est le Laplacien discret et V est un potential à valeurs réelles. Nous expliquons le théorème spectral pour cet opérateur et donnons des formules explicites dans le cas du Laplacien. Nous explorons les perturbations du premier ordre. Nous explorons aussi les potentiels périodiques sur la ligne. Après avoir introduit les opérateurs de Schrödinger aléatoirs, nous reproduisons le célèbre théorème de Pastur établissant l'éxistence d'un spectre identique presque partout. Afin d'illustrer les familles d'opérateurs de Schrödinger aléatoirs ergodiques, nous étudions le modèle d'Anderson.

Contents

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Preface

This document is a thesis submitted in partial requirements of the degree of M. Sc. in Mathematics, sought by the author, Marc-Adrien Mandich. The author was supervised by Dr. Vojkan Jakšić. The thesis has been written by the author, and many parts of the theorems were explained by the supervisor.

1 Introduction

1.1 Outline of paper

This thesis exposes many tools that are used in mathematical physics research and is written at a level that other graduate students can understand. It is almost self-contained, and most of the missing elements can be found in [**Ja**]. Because this thesis contains very little physics, there is not much interpretation to be discussed. We will therefore proceed with basics that will come in handy in all sections of the thesis.

In the second part of the introduction, we define the Schrödinger equation, and introduce all the essential tools of the thesis : the transfer matrices, the Wronskian, the Green's Function and Weyl-m functions, the Borel and Poisson transforms, the boundary values of the resolvent, and the Helffer-Sjöstrand formula for the Spectral theorem.

The second section analyzes the Laplacian on the the full line, not only as an operator from $\ell^2(\mathbb{Z})$ to itself but also from $\ell^p(\mathbb{Z})$ to itself. We state the Spectral theorem for the operator and show that its spectrum is $[-2, 2]$. We calculate the Green's function and boundary values of the resolvent, as well as the spectral measures μ_{δ_n} for h and δ_n. We then move on to the Schrödinger operator and show that the Green's function decays exponentially as we move away from the diagonal. This result allows us in particular to show that the spectrum of the Schrödinger operator as an operator from $\ell^p(\mathbb{Z})$ to itself is the same. We then describe the direct integral decomposition of operator and discuss the generalized eigenfunction expansion.

In the third section we repeat much of the same analysis for the half line case.

In the fourth section, we return to the full line operator and show that rank one perturbation has the same spectrum as the Laplacian plus an eigenvalue outside $[-2, 2]$. After introducing the basics of Scattering Theory and proving Pearson's theorem, we directly show that the wave operators exist and are complete for rank one perturbations, and compute the scattering matrix.

The fifth section deals with various basic potentials that have mainly absolutely continuous spectra. We show that periodic operators have absolutely continuous spectra composed of finitely many bands. Finally we reproduce Simon's proof ([**Si1**]) that having all eigenfunctions bounded implies purely absolutely continuous spectrum.

In the sixth section, we discuss random Schrödinger operators and prove Pastur's theorem. We discuss issues of measurability that become necessary, and then further discuss minimally and uniquely ergodic operators. We end by showing that the Anderson model is ergodic and show that its spectrum is almost surely $[-2, 2] + \text{supp}\ \nu$, where ν is the probability distribution of the model.

Lastly, the appendix contains many solutions to exercises in [**Ja**] that a reader unfamiliar with Spectral Theory might want to read.

1.2 Transfer Matrices, Wronskian, Green's Function, Borel Tranform and some Spectral Theory

We begin by establishing notation that will be recurrent throughout the thesis.

Notation 1.1.

(i) *Let* $\ell(\mathbb{Z})$ *denote the vector space of all sequences* $u = \{u(n)\}_{n\in\mathbb{Z}}$ *with coefficients in* $\mathbb{C}$.

(ii) *For* $1 \le p < \infty$, *denote the Banach space* $\ell^p(\mathbb{Z}) = \{u \in \ell(\mathbb{Z}) : \|u\|_p = (\sum_{n\in\mathbb{Z}} |u(n)|^p)^{1/p} < \infty\}$.

(iii) *For* $p = \infty$, *denote the Banach space* $\ell^\infty(\mathbb{Z}) = \{u \in \ell(\mathbb{Z}) : \|u\|_\infty = \sup_{n\in\mathbb{Z}} |u(n)| < \infty\}$.

(iv) *Let* $\ell_0(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z})$ *denote the vector space of all sequences with finitely many non-zero terms.*

(v) $\ell(\mathbb{Z}_+), \ell^p(\mathbb{Z}_+), \ell^\infty(\mathbb{Z}_+)$ *and* $\ell_0(\mathbb{Z}_+)$ *are defined similarly for* $\mathbb{Z}_+ := \{1, 2, ...\}$.

(vi) $C_0(\mathbb{R})$ *are the continuous functions on* $\mathbb{R}$ *that vanish at infinity.*

(vii) $C_c(\mathbb{R})$ *are the continuous functions on* $\mathbb{R}$ *with compact support.*

(viii) $B_b(\mathbb{R})$ *are the bounded Borel functions.*

(ix) *$\rho(A)$ and $\sigma(A) = \mathbb{C}\backslash\rho(A)$ denote resolvent set and spectrum of a bounded operator A. The resolvent at z is denoted $R(A, z)$ or $(A - z)^{-1}$.*

Note that $\ell_0(\mathbb{Z})$ (resp. $\ell_0(\mathbb{Z}_+)$) is dense in $\ell^p(\mathbb{Z})$ (resp. $\ell^p(\mathbb{Z}_+)$) for all $1 \leq p < \infty$. We will be working chiefly on $\ell^2(\mathbb{Z})$ (resp. $\ell^2(\mathbb{Z}_+)$) , and we will denote by $\{\delta_n\}_{n\in\mathbb{Z}}$ (resp. $\{\delta_n\}_{n\in\mathbb{Z}_+}$) the canonical orthonormal basis, in the sense of a Hilbert spaces.

Depending on the section, $\mathcal{H}$ will typically stand for $\ell^2(\mathbb{Z})$ or $\ell^2(\mathbb{Z}_+)$, but will also be used sometimes to refer to any Hilbert space. $\mathcal{B}(\mathcal{H})$ will denote the bounded operators on $\mathcal{H}$.

Definition 1.2. *The discrete Schrödinger operator on the full line ($\mathbb{Z}$) is defined as follows:*

(i) *Let $V : \mathbb{Z} \to \mathbb{R}$, which we will refer to as the potential. To such a function we associate the linear map, again denoted by V, $V : \ell(\mathbb{Z}) \to \ell(\mathbb{Z})$, $(Vu)(n) = V(n)u(n)$.*

(ii) *The discrete Schrödinger operator on $\ell(\mathbb{Z})$ is the map $h = h_0 + V : \ell(\mathbb{Z}) \to \ell(\mathbb{Z})$,*

$$(hu)(n) = u(n-1) + u(n+1) + V(n)u(n). \tag{1.1}$$

By the discrete Schrödinger equation on $\mathbb{Z}$ we mean the difference equation:

$$hu = zu, \quad u \in \ell(\mathbb{Z}), z \in \mathbb{C} \tag{1.2}$$

Definition 1.3. *The discrete Schrödinger operator on the half line $\mathbb{Z}_+$ is defined analogously, with a Dirichlet boundary condition: $h = h_0 + V : \ell(\mathbb{Z}_+) \to \ell(\mathbb{Z}_+)$,*

$$(hu)(n) = u(n-1) + u(n+1) + V(n)u(n) \tag{1.3}$$

$$u(0) = 0 \tag{1.4}$$

Equivalently, it can be written as:

$$(hu)(n) = \begin{cases} u(n-1) + u(n+1) + V(n)u(n) & n \geq 2 \\ u(n+1) + V(n)u(n) & n = 1 \end{cases} \tag{1.5}$$

By the discrete Schrödinger equation on $\mathbb{Z}_+$ we mean the difference equation:

$$hu = zu, \quad u \in \ell(\mathbb{Z}_+), z \in \mathbb{C} \tag{1.6}$$

When the potential V is identically zero, h is called the Laplacian and is denoted h_0. It is also denoted $-\Delta$ in the literature. We also introduce the shift operators on $\ell(\mathbb{Z})$ (resp. $\ell(\mathbb{Z}_+)$):

(i) to the right $(Ru)(n) = u(n-1)$.

(ii) to the left $L : (Lu)(n) = u(n+1)$.

Consequently $h = L+R+V$. L, R and V are linear operators so that the full line and half line Schrödinger operators are linear operators.

We define the eigenspaces associated to $z \in \mathbb{C}$ as:

$$E(z) := \{u \in \ell(\mathbb{Z}) : hu = zu\} \text{ for the full line Schrödinger operator} \tag{1.7}$$
$$E(z) := \{u \in \ell(\mathbb{Z}_+) : hu = zu\} \text{ for the half line Schrödinger operator} \tag{1.8}$$

Note that if $u \in E(z)$ and $z \in \mathbb{R}$, then $\overline{u} \in E(z)$, where $\overline{u} = \{\overline{u(n)}\}_{n\in\mathbb{Z}}$.

It is also useful to look at the Schrödinger equation from the point of view of dynamical systems:

Definition 1.4. *Define for $z \in \mathbb{C}$ and $u \in \ell(\mathbb{Z})$ (resp. $u \in \ell(\mathbb{Z}_+)$):*

$$A(z,n) := \begin{pmatrix} z - V(n) & -1 \\ 1 & 0 \end{pmatrix} \quad \textit{and} \quad \Phi(n) := \begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix}$$

Then $\det A(z,n) = 1$ and $A(z,n)^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & z - V(n) \end{pmatrix}$.

A key observation is that u satisfies the Scrödinger equation if and only if $\forall n \in \mathbb{Z}$ (resp. $\forall n \in \mathbb{Z}_+$):

$$\Phi(n) = A(z,n)\Phi(n-1). \tag{1.9}$$

For this reason the matrix $A(z,n)$ is called the transfer matrix. The corresponding flow on $\ell(\mathbb{Z};\mathbb{C}^2)$ is given by the fundamental matrix:

$$T(z,n,m) := \begin{cases} A(z,n)\cdots A(z,m+1) & n > m \\ \mathbb{1} & n = m \\ A(z,n+1)^{-1}\cdots A(z,m)^{-1} & n < m \end{cases}$$

Then T satisfies the following identities: $\forall n,m,i \in \mathbb{Z}$ (resp. $\forall n,m,i \in \mathbb{Z}_+$),

(i) $T(z,n,n-1) = A(z,n)$.

(ii) $\Phi(n) = T(z,n,m)\Phi(m)$.

(iii) $T(z,n,m) = T(z,n,i)T(z,i,m)$.

(iv) $T(z,n,m)^{-1} = T(z,m,n)$.

We later use, we will be using the operator norm of T, i.e.

$$\|T(z,n,m)\| := \sup_{\phi\in\mathbb{C}^2\setminus\{0\}} \frac{\|T(z,n,m)\phi\|_{\mathbb{C}^2}}{\|\phi\|_{\mathbb{C}^2}}. \tag{1.10}$$

The Wronskian comes in handy sometimes:

Definition 1.5. *The Wronskian of two sequences $u, v \in \ell(\mathbb{Z})$ (resp. $\ell(\mathbb{Z}_+)$) is the sequence*

$$W_n(u,v) := u(n)v(n+1) - u(n+1)v(n). \tag{1.11}$$

It satisfies $W_n(u,v) = -W_n(v,u)$.

Lemma 1.6. *Let $u, v \in E(z)$. Then:*

(i) *$W(u,v) := W_n(u,v)$ is a constant sequence.*

(ii) *$W(u,v)$ is zero if and only if u and v are linearly dependent.*

Proof.

(i) The Wronskian is constant since for all n:

$$\begin{aligned} W_n(u,v) &= u(n)v(n+1) - u(n+1)v(n) \\ &= u(n)(zv(n) - V(n)v(n) - v(n+1)) - (zu(n) - V(n)u(n) - u(n+1))v(n) \\ &= W_{n-1}(u,v). \end{aligned}$$

(ii) The Wronskian is obviously zero if u and v are linearly dependent. Conversely, if $W(u,v) = 0$, in particular $W_1(u,v) = 0$ and so the matrix $\begin{pmatrix} v(2) & u(2) \\ v(1) & u(1) \end{pmatrix}$ has determinant equal to zero. Thus it has a nonzero kernel, i.e. there are α, β, not both zero, such that $\begin{pmatrix} v(2) & u(2) \\ v(1) & u(1) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Applying $T(z, 1 \pm n, 1)$ for $n = 1, 2, ...$ shows that $\alpha v(n) + \beta u(n) = 0$ for all n.

□

Proposition 1.7. *On the full line* $\dim E(z) = 2$ $\forall z \in \mathbb{C}$, *whereas on the half line* $\dim E(z) = 1$ $\forall z \in \mathbb{C}$.

Proof. We first consider the full line case. If $u \in E(z)$, it is enough to know any two consecutive terms of u to completely determine the other terms. For instance if $\Phi(0)$ is known, then applying $T(z, n, 0)$ determines all the other terms uniquely. Hence there exist at least two linearly independent solutions $c(z), s(z) \in \ell(\mathbb{Z})$ to the equation $hu = zu$ satisfying:

$$\begin{aligned} s(0,z) &= 0 \qquad & s(1,z) &= 1 \\ c(0,z) &= 1 \qquad & c(1,z) &= 0 \end{aligned}$$

$W(c(z), s(z)) = 1 \neq 0$ shows that $c(z)$ and $s(z)$ are indeed linearly independent.
Now suppose $u \in E(z)$ is nonzero. In particular, $u(0), u(1)$ cannot both be zero.
The determinant of $\begin{pmatrix} s(1,z) & c(1,z) \\ s(0,z) & c(0,z) \end{pmatrix}$ being nonzero, there exist α, β, not both zero, such that

$$\begin{pmatrix} s(1,z) & c(1,z) \\ s(0,z) & c(0,z) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} u(1) \\ u(0) \end{pmatrix}.$$

Applying $T(z, \pm n, 0)$ for $n = 1, 2, ...$ shows that $u(n) = \alpha s(n,z) + \beta c(n,z)$ for all n.

For the half line case, note that for $u \in E(z)$, it is enough to know $u(1)$ to completely determine the other terms. Hence there exists at least a solution $s(z) \in \ell(\mathbb{Z}_+)$ to the equation $hu = zu$ satisfying:

$$s(0,z) = 0 \qquad s(1,z) = 1$$

Now if $u \in E(z)$ is nonzero, then in particular, $u(1) \neq 0$ and so $\exists \alpha$ such that $s(1,z) = \alpha u(1)$. An easy induction shows that in fact $s(n,z) = \alpha u(n)$ for all $n \in \mathbb{Z}_+$. □

Definition 1.8. *In both the full line and half line cases, we call the solutions $c(z)$ and $s(z)$ to the formal difference equation $hu = zu$ with boundary conditions:*

$$\begin{pmatrix} s(0,z) & s(1,z) \\ c(0,z) & c(1,z) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.12}$$

the fundamental solutions.

The following lemma for the full line case trivially holds in the half line case since $u(0) = 0$.

Lemma 1.9. *For $u \in E(z)$: $u(n) = u(0)c(n,z) + u(1)s(n,z)$ for all $n \in \mathbb{Z}$. In particular,*

$$T(z,n,0) = \begin{pmatrix} s(n+1,z) & c(n+1,z) \\ s(n,z) & c(n,z) \end{pmatrix} \tag{1.13}$$

Proof. Let $u(n) = \alpha c(n) + \beta s(n)$ for every n. Then:

$$\begin{aligned} \begin{pmatrix} W(c,u) \\ W(s,u) \end{pmatrix} &= \begin{pmatrix} c(n) & c(n+1) \\ s(n) & s(n+1) \end{pmatrix} \begin{pmatrix} u(n+1) \\ -u(n) \end{pmatrix} \\ &= \alpha \begin{pmatrix} c(n) & c(n+1) \\ s(n) & s(n+1) \end{pmatrix} \begin{pmatrix} c(n+1) \\ -c(n) \end{pmatrix} + \beta \begin{pmatrix} c(n) & c(n+1) \\ s(n) & s(n+1) \end{pmatrix} \begin{pmatrix} s(n+1) \\ -s(n) \end{pmatrix} \\ &= \alpha \begin{pmatrix} 0 \\ W(s,c) \end{pmatrix} + \beta \begin{pmatrix} W(c,s) \\ 0 \end{pmatrix} \end{aligned}$$

So that $\alpha = \frac{W(s,u)}{W(s,c)} = u(0)$ and $\beta = \frac{W(c,u)}{W(c,s)} = u(1)$. □

Lemma 1.10. $\|T(z,n,0)^{-1}\| = \|T(z,n,0)\|$.

Proof. Using lemma 1.9, observe that

$$T(z,n,0)^{-1} = J\, T(z,n,0)^{\mathrm{T}} J^{-1}, \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

where T^{T} denotes the transpose of T. Since J is unitary,

$$\|T(z,n,0)^{-1}\| = \|T(z,n,0)^{\mathrm{T}}\| = \|T(z,n,0)^*\| = \|T(z,n,0)\|.$$

Note that we have used the fact that for bounded operators A on a Hilbert space, $\|A\| = \|A^*\|$ and that for square matrices $\|A\| = \|\overline{A}\|$. □

The Green's function will play a key role:

Definition 1.11. *For $z \in \rho(h)$, let $G(z,n,m) := \langle \delta_n, (h-z)^{-1}\delta_m \rangle$, $n,m \in \mathbb{Z}$ for the full line case and $n,m \in \mathbb{Z}_+$ for the half line case. They are the matrix elements of $(h-z)^{-1}$.*

Because the resolvent map $\rho(h) \ni z \to R(h,z) \in \mathcal{B}(\mathcal{H})$ is analytic, $\rho(h) \ni z \to \langle \delta_n, (h-z)^{-1}\delta_m \rangle \in \mathbb{C}$ is analytic for every n,m. The Green's function also satisfies the following key relation: if $G_m := \{G_m(n) = G(z,n,m)\}_{n \in \mathbb{Z}/\mathbb{Z}_+}$ and $G_n := \{G_n(m) = G(z,n,m)\}_{m \in \mathbb{Z}/\mathbb{Z}_+}$, then

$$((h-z)G_m)(n) = \delta_{n,m} \quad \text{and} \quad ((h-z)G_n)(m) = \delta_{n,m}. \tag{1.14}$$

A simple application of the first resolvent identity gives the following useful formula in both the full line and half line cases:

Lemma 1.12. *For all $n \in \mathbb{Z}$:* $\operatorname{Im}\langle\delta_n,(h-z)^{-1}\delta_n\rangle = (\operatorname{Im} z)\,\|(h-z)^{-1}\delta_n\|^2$.

Proof. $\operatorname{Im}\langle\delta_n,(h-z)^{-1}\delta_n\rangle = \frac{1}{2i}\left(\langle\delta_n,(h-z)^{-1}\delta_n\rangle - \overline{\langle\delta_n,(h-z)^{-1}\delta_n\rangle}\right)$

$$= \frac{1}{2i}\langle\delta_n,\left[(h-z)^{-1}-(h-\overline{z})^{-1}\right]\delta_n\rangle = \frac{1}{2i}\langle\delta_n,\left[(h-\overline{z})^{-1}(z-\overline{z})(h-z)^{-1}\right]\delta_n\rangle = (\operatorname{Im} z)\,\|(h-z)^{-1}\delta_n\|^2.$$

□

Lemma 1.13. *Let h be the full line Schrödinger operator and $z \in \rho(h)$. There are linearly independent sequences $u_+(z)$ and $u_-(z) \in E(z)$ that are square summable near $+\infty$ and $-\infty$ respectively. Moreover, the subspaces $E(z,\pm\infty)$ of $E(z)$ consisting of square summable sequences at $\pm\infty$ are one-dimensional.*

Proof. Let $u_+(n,z) := \langle\delta_n,(h-z)^{-1}\delta_1\rangle$ for $n \geq 1$ and $u_-(n,z) := \langle\delta_n,(h-z)^{-1}\delta_{-1}\rangle$ for $n \leq -1$. The remaining terms are obtained by applying the transfer matrix.

By construction $(hu_+)(n) = zu_+(n)$ for $n \leq 1$. For $n > 1$ we have

$$\begin{aligned}(hu_+)(n) &= G(z,n+1,1) + G(z,n-1,1) + V(n)G(z,n,1)\\ &= \langle h\delta_n,(h-z)^{-1}\delta_1\rangle\\ &= \langle(h-\overline{z})\delta_n,(h-z)^{-1}\delta_1\rangle + z\langle\delta_n,(h-z)^{-1}\delta_1\rangle\\ &= zu_+(n).\end{aligned}$$

u_+ is square summable at $+\infty$ since $\sum_{n\geq1}|\langle\delta_n,(h-z)^{-1}\delta_1\rangle|^2 \leq \|(h-z)^{-1}\delta_1\|^2 < \infty$. If $u_+(z)$ and $u_-(z)$ were linearly dependent, then they would be in $E(z)\cap\ell^2(\mathbb{Z})$, i.e. eigenvectors of h, contradicting $z \in \rho(h)$. If $u_{\pm,1}$ and $u_{\pm,2} \in E(z,\pm\infty)$, then $W(u_{\pm,1},u_{\pm,2}) = \lim_{n\to\pm\infty} W_n(u_{\pm,1},u_{\pm,2}) = 0$. □

Lemma 1.14. *Let u_+ and u_- be as in lemma 1.13.*

(i) *For the full line Schrödinger operator, $\forall n,m \in \mathbb{Z}$:*

$$G(z,n,m) = \begin{cases}\frac{u_+(n,z)u_-(m,z)}{W(u_-(z),u_+(z))} & m \leq n\\[2mm] \frac{u_+(m,z)u_-(n,z)}{W(u_-(z),u_+(z))} & n \leq m\end{cases} \tag{1.15}$$

(ii) *For the half line Schrödinger operator, $\forall n,m \in \mathbb{Z}_+$:*

$$G(z,n,m) = \begin{cases}\frac{u_+(n,z)s(m,z)}{W(s(z),u_+(z))} & m \leq n\\[2mm] \frac{u_+(m,z)s(n,z)}{W(s(z),u_+(z))} & n \leq m\end{cases} \tag{1.16}$$

In particular the Green's function is symmetric.

Proof.

1. Let

$$H(z,n,m) := \begin{cases}\frac{u_+(n,z)u_-(m,z)}{W(u_-(z),u_+(z))} & m \leq n\\[2mm] \frac{u_+(m,z)u_-(n,z)}{W(u_-(z),u_+(z))} & n \leq m\end{cases} \tag{1.17}$$

Fix m. If $H_1 := \{H_1(n) = H(z,n,m)\}_{n\in\mathbb{Z}}$, then $((h-z)H_1)(n) = \delta_{n,m}$. Moreover H_1 is square summable. Therefore $((h-z)(H_1-G_1))(n) = 0$ and $H_1 - G_1$ is square summable. Hence $H_1 = G_1$. Since this is true for all m, $G(z,n,m) = H(z,n,m)$.

2. This is shown using the same argument, and uses the fact that $s(z)$ satisfies the half line Schrödinger equation at $n = 1$.

□

We will now introduce the Weyl m-functions. First we need to do an observation. Let $z \in \mathbb{C}\backslash\mathbb{R}$. Then $u_+(1,z) := \langle \delta_1, (h-z)^{-1}\delta_1\rangle$ is not zero by lemma 1.12. Also, $u_+(0,z)$ is not zero. To see this, consider the sequence $\tilde{u}_+(z)$ defined by $\tilde{u}_+(n,z) := \langle \delta_n, (h-z)^{-1}\delta_1\rangle$ for $n \geq 1$ and where h is the half line Schrödinger operator, and the other terms are obtained by applying the transfer matrix of the full line Schrödinger operator. Then $\tilde{u}_+(z) \in E(z,+\infty)$ and so $u_+(z)$ and $\tilde{u}_+(z)$ are linearly dependent. Now $\tilde{u}_+(0,z) = 0$ would imply that $\tilde{u}_+(z)$ is an eigenvector of the half line Schrödinger operator with eigenvalue z, which contradicts $z \in \mathbb{C}\backslash\mathbb{R}$. Hence $\tilde{u}_+(0,z) \neq 0$, and $u_+(0,z) \neq 0$.

Definition 1.15. *The Weyl-m function is defined on $\mathbb{C}\backslash\mathbb{R}$ by $m_+(z) := \langle \delta_1, (h-z)^{-1}\delta_1\rangle$, where h is the half line Schrödinger operator.*

By lemma 1.14, $m_+(z) = -\frac{u_+(1)}{u_+(0)}$. It is a holomorphic function.

Many of the tools that we will use are originally proved using techniques of harmonic analysis. The reader is encouraged to consult the notes on Topics in Spectral Theory, [**Ja**], for more details.

The Borel transform of a complex or positive measure μ satisfying $\int_{\mathbb{R}} \frac{d\mu(t)}{1+|t|} < \infty$ is defined by

$$F_\mu(z) := \int_{\mathbb{R}} \frac{d\mu(t)}{t-z}, \qquad z \in \mathbb{C}_+. \tag{1.18}$$

The Poisson transform of μ satisfying $\int_{\mathbb{R}} \frac{d\mu(t)}{1+t^2} < \infty$ is defined by

$$P_\mu(E+i\varepsilon) := \int_{\mathbb{R}} \frac{\varepsilon d\mu(t)}{(E-t)^2+\varepsilon^2}, \qquad \varepsilon > 0. \tag{1.19}$$

The functions $F_\mu(z)$ and $P_\mu(z)$ are analytic in $\mathbb{C}_+$. If μ is the Lebesgue measure, then $P_\mu(z) = \pi$ for all $z \in \mathbb{C}_+$. If μ is a positive or signed measure, then $\operatorname{Im} F_\mu = P_\mu$. We will need the following result on the differentiation of measures:

Theorem 1.16. *Let ν be a complex measure and μ a positive measure. Let $\nu = f\mu + \nu_{\mathrm{s}}$ be the Radon-Nikodym decomposition. Let μ_{sing} be the part of μ singular with respect to the Lebesgue measure. Then:*

(i)
$$\lim_{\varepsilon\downarrow 0} \frac{P_\nu(E+i\varepsilon)}{P_\mu(E+i\varepsilon)} = f(E), \qquad \text{for } \mu-a.e.\ E. \tag{1.20}$$

(ii)
$$\lim_{\varepsilon\downarrow 0} \frac{P_\nu(E+i\varepsilon)}{P_\mu(E+i\varepsilon)} = \infty, \qquad \text{for } \nu_{\mathrm{s}}-a.e.\ E. \tag{1.21}$$

(iii)
$$\lim_{\varepsilon\downarrow 0} \frac{F_\nu(E+i\varepsilon)}{F_\mu(E+i\varepsilon)} = f(E), \qquad \text{for } \mu_{\mathrm{sing}}-a.e.\ E. \tag{1.22}$$

Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$, $\psi \in \mathcal{H}$, and μ_ψ the spectral measure for A and ψ. Let F_{μ_ψ} and P_{μ_ψ} be the Borel and Poisson transform of μ_ψ. The important fact is that for all $z \in \mathbb{C}\backslash\mathbb{R}$:

$$\langle \psi, (A-z)^{-1}\psi\rangle = \int_{\mathbb{R}} \frac{d\mu_\psi(x)}{x-z} = F_{\mu_\psi}. \tag{1.23}$$

We will need the following theorem, which may be referred to in the literature as the theorem of De la Vallée Poussin:

Theorem 1.17.

(i) *For Lebesgue a.e. $E \in \mathbb{R}$ the following limit exists and is finite and non-zero:*

$$\lim_{\varepsilon\downarrow 0} F_{\mu_\psi}(E+i\varepsilon) = \lim_{\varepsilon\downarrow 0}\langle \psi, (A-E-i\varepsilon)^{-1}\psi\rangle := \langle \psi, (A-E-i0)^{-1}\psi\rangle.$$

(ii) $d\mu_{\psi,\mathrm{ac}}(E) = \frac{1}{\pi}\mathrm{Im}\,\langle \psi, (A-E-i0)^{-1}\psi\rangle dE.$

(iii) $\mu_{\psi,\mathrm{sing}}$ *is concentrated on the set* $\{E \in \mathbb{R} : \lim_{\varepsilon\downarrow 0}\mathrm{Im}\,\langle \psi, (A-E-i\varepsilon)^{-1}\psi\rangle = \infty\}$.

Finally to end this section, we briefly describe a more sophisticated Functional Calculus based on the Helffer and Sjöstrand formula. The reader is referred to Chapter 2 of [**D**] for the complete exposition. The reader unfamiliar with Spectral Theory might want to look first at Chapters VII and VIII of [**RS1**] for the standard approach to the Functional Calculus. We let $\langle z\rangle := \sqrt{1+|z|^2}$. For $\beta \in \mathbb{R}$, let S^β be the set of complex-valued $C^\infty(\mathbb{R})$ functions such that there exists a c_n so that:

$$|f^{(n)}(x)| \le c_n \langle x\rangle^{\beta-n}, \quad \forall x \in \mathbb{R}, \forall n \in \mathbb{N}. \tag{1.24}$$

We set $\mathcal{A} := \bigcup_{\beta<0} S^\beta$ and equip $\mathcal{A}$ with a family of norms: for $n \ge 1$:

$$\|f\|_n = \sum_{k=0}^{n} \int_{-\infty}^{\infty} |f^{(k)}(x)|\langle x\rangle^{k-1} dx. \tag{1.25}$$

$\mathcal{A}$ is an algebra for the multiplication of functions. It contains rational functions that vanish at $\pm\infty$ and have non-vanishing denominator on the real axis, in particular functions of the form $f_z(x) := 1/(x-z)$. In fact $C_c^\infty(\mathbb{R})$ is dense in $\mathcal{A}$ for the norms $\|\cdot\|_n$.

For $f \in C^\infty(\mathbb{R})$, we define its quasi-analytic extension $\tilde{f} : \mathbb{C} \to \mathbb{C}$ by

$$\tilde{f}(z) = \left(\sum_{k=0}^{n} f^{(k)}(x)\frac{(iy)^k}{k!}\right)\sigma(x,y) \tag{1.26}$$

with $z = x+iy$, $n \ge 1$, $\sigma(x,y) = \tau(y/\langle x\rangle)$, where $\tau \in C_c^\infty(\mathbb{R})$ is equal to one on $[-1,1]$ and has support on $[-2,2]$. Note that $\tilde{f} \in C^\infty(\mathbb{C})$. Its support is on the set $|y| \le 2\langle x\rangle$. The choice of τ and n turn out to have no importance. An explicit computation gives:

$$\frac{\partial}{\partial \overline{z}}\tilde{f}(z) := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\tilde{f}(z) \tag{1.27}$$

$$= \left(\sum_{k=0}^{n} f^{(k)}(x)\frac{(iy)^k}{k!}\right)\frac{\sigma_x(x,y)+i\sigma_y(x,y)}{2} + f^{(n+1)}(x)\frac{(iy)^n}{n!}\frac{\sigma(x,y)}{2}.$$

Since the support of $\sigma_x(x,y)$ and $\sigma_y(x,y)$ are included in the set $\langle x\rangle \le |y| \le 2\langle x\rangle$, if x is fixed and $y \to 0$ we see that

$$\left|\frac{\partial}{\partial \bar{z}}\tilde{f}(z)\right| = \mathcal{O}(|y|^n). \tag{1.28}$$

It is in that sense that we mean that $\tilde{f}$ is quasi-analytic.

Definition 1.18. *For any $f \in \mathcal{A}$ and any self-adjoint operator A on a Hilbert space $\mathcal{H}$, the Hilffer-Sjöstrand formula for $f(A)$ reads:*

$$f(A) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial}{\partial \bar{z}}\tilde{f}(z)(A-z)^{-1}dxdy \in \mathcal{B}(\mathcal{H}). \tag{1.29}$$

This usefulness of this formula is that it gives an explicit formula for $f(A)$ and allows to compute functions of operators by means of their resolvent. It can be shown that the expression (1.29) converges in norm for all $f \in \mathcal{A}$ and that it satisfies the bound $\|f(A)\| \le c_n\|f\|_{n+1}$ for all $f \in \mathcal{A}$ and $n \ge 1$. It can be shown that $f(A)$ is independent of the choice of σ and n. The important consequence of formula (1.29) is that it possesses the properties of a Functional Calculus.

Theorem 1.19.

(i) *If $f \in C_c^\infty(\mathbb{R})$ and* supp $f \cap \sigma(A) = \emptyset$*, then $f(A) = 0$.*

(ii) *$(fg)(A) = f(A)g(A)$ for all $f, g \in \mathcal{A}$.*

(iii) *$\overline{f}(A) = f(A)^*$ and $\|f(A)\| \le \|f\|_\infty$.*

(iv) *$f_z(A) = (A-z)^{-1}$ for all $z \in \mathbb{C}\backslash\mathbb{R}$.*

In particular, there exists a unique linear map $C_0(\mathbb{R}) \ni f \to f(A) \in \mathcal{B}(\mathcal{H})$ which is a Functional Calculus, i.e. it coincides with the standard Functional Calculus (the uniqueness being a consequence of the Stone-Weierstrass theorem).

2 The Discrete Schrödinger Operator on the Full Line

2.1 The Laplacian, its Spectral theorem and Spectrum

As we saw in proposition 1.7, $E(z)$ is a two dimensional subspace. For the Laplacian, we can in fact give an explicit basis:

Proposition 2.1. *The following are a basis for $E(z)$:*

(i) *For $z \in \mathbb{C}\backslash\{-2, 2\}$: $u = \{u(n) = (\frac{z+\sqrt{z^2-4}}{2})^n\}_{n\in\mathbb{Z}}$ and $v = \{v(n) = (\frac{z+\sqrt{z^2-4}}{2})^{-n}\}_{n\in\mathbb{Z}}$.*

(ii) *For $z = -2$: $u = \{u(n) = (-1)^n\}_{n\in\mathbb{Z}}$ and $v = \{v(n) = -(-1)^n n\}_{n\in\mathbb{Z}}$.*

(iii) *For $z = 2$: $u = \{u(n) = 1\}_{n\in\mathbb{Z}}$ and $v = \{v(n) = n\}_{n\in\mathbb{Z}}$.*

Proof. Fix $\lambda \in \mathbb{C}$, $\lambda \ne 0$, and consider the sequences $\{u(n) = \lambda^n\}_{n\in\mathbb{Z}}$ and $\{v(n) = \lambda^{-n}\}_{n\in\mathbb{Z}}$. Then $(h_0u)(n) = \lambda^{n+1} + \lambda^{n-1} = (\lambda + \frac{1}{\lambda})u(n) = zu(n)$ and similarly $(h_0v)(n) = (\lambda + \frac{1}{\lambda})v(n)$ and so we see that $u, v \in E(\lambda + \frac{1}{\lambda})$. u and v defined as such will be linearly dependent if and only if $\exists\alpha$ such that

$\alpha\lambda^n = \lambda^{-n}\ \forall n \in \mathbb{Z}$. That is, if and only if $\lambda \in \{-1, 1\}$, or equivalently if and only if $z \in \{-2, 2\}$. Moreover,

$$z = \lambda + \frac{1}{\lambda} \Longleftrightarrow \lambda^2 - z\lambda + 1 = 0 \Longleftrightarrow \lambda = \frac{z \pm \sqrt{z^2-4}}{2}.$$

Therefore for $z \in \mathbb{C}\backslash\{-2, 2\}$, $\{u(n) = \lambda^n = (\frac{z+\sqrt{z^2-4}}{2})^n\}$ and $\{v(n) = \lambda^{-n} = (\frac{z+\sqrt{z^2-4}}{2})^{-n}\}$ (we could also have taken the other root) are linearly independent in $E(z)$. For $z = -2$ and $z = 2$, the above gives one eigenvector, namely $\{u(n) = (-1)^n\}$ and $\{u(n) = 1\}$ respectively. To find another eigenvector, we propagate $(s(0), s(1)) = (0, 1)$. ☐

Corollary 2.2.

(i) *For* $z \in \mathbb{C}\backslash[-2, 2]$, *every sequence in* $E(z)$ *is unbounded, that is,* $\dim(E(z) \cap \ell^\infty(\mathbb{Z})) = 0$.

(ii) *For* $z \in (-2, 2)$, *every sequence in* $E(z)$ *is bounded, that is,* $\dim(E(z) \cap \ell^\infty(\mathbb{Z})) = 2$.

(iii) *For* $z \in \{-2, 2\}$, $\dim(E(z) \cap \ell^\infty(\mathbb{Z})) = 1$.

Proof. Let λ and z be related as in proposition 2.1, that is $z = \lambda + \frac{1}{\lambda}$, $u(n) = \lambda^n$ and $v(n) = \lambda^{-n}$.

(i) If $|\lambda| > 1$ then $\lim_{n\to\infty} |u(n)| = \infty$; $\lim_{n\to-\infty} |u(n)| = 0$; $\lim_{n\to-\infty} |v(n)| = \infty$; $\lim_{n\to\infty} |v(n)| = 0$. If w is a non trivial linear combination of u and v, say $w = \alpha u + \beta v$, then $\Big||\alpha u(n)| - |\beta v(n)|\Big| \le |w(n)|$ shows that w is also unbounded. The same argument works for the case $|\lambda| < 1$.

(ii) If $|\lambda| = 1$, $\lambda \notin \{-1, 1\}$, say $\lambda = e^{i\theta}$ for some $\theta \in (0, \pi) \cup (\pi, 2\pi)$, then $z = \lambda + \frac{1}{\lambda} = 2\cos\theta$. $|u(n)| = |e^{i\theta n}| \le 1$ and $|v(n)| = |e^{-i\theta n}| \le 1$. If $w = \alpha u + \beta v$, then $|w(n)| \le |\alpha| + |\beta|$.

(iii) Finally for $\lambda \in \{-1, 1\}$ we exhibited in proposition 2.1 bases containing a bounded sequence and an unbounded sequence, and so it must be that every basis for $E(-2)$ and $E(2)$ contains exactly one unbounded sequence.

☐

From the definitions, it is obvious that for $1 \le p \le \infty$, the shift operators $R, L : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ are isometries. Consequently $h_0\ell^p(\mathbb{Z}) \subset \ell^p(\mathbb{Z})$ and $\|h_0\|_{\ell^p} \le \|R\|_{\ell^p} + \|L\|_{\ell^p} = 2$. In fact:

Proposition 2.3. *For* $1 \le p \le \infty$, $h_0 : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ *is a bounded linear operator and* $\|h_0\|_{\ell^p} = 2$.

Proof. On $\ell^\infty(\mathbb{Z})$, take $u(n) = 1, \forall n$. Then $\|u\|_\infty = 1$, $(h_0u)(n) = 2, \forall n$, and $\|h_0u\|_\infty = 2$, so $\|h_0\|_{\ell^\infty} = 2$. On $\ell^p(\mathbb{Z})$, $1 \le p < \infty$, consider the sequence of sequences for $N \in \mathbb{N}$:

$$u^{(N)}(n) = \begin{cases} 1 & 0 \le n \le N \\ 0 & \text{otherwise} \end{cases} \qquad \text{Then} \qquad (h_0u^{(N)})(n) = \begin{cases} 1 & n = -1, 0, N, N+1 \\ 2 & 1 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Then} \quad \|u^{(N)}\|_p = (N+1)^{1/p} \quad \text{and} \quad \|h_0u^{(N)}\|_p = (4 + (N-1)2^p)^{1/p}$$

so that

$$\lim_{N\to\infty} \frac{\|h_0u^{(N)}\|_p}{\|u^{(N)}\|_p} = \lim_{N\to\infty} \frac{(4 + (N-1)2^p)^{1/p}}{(N+1)^{1/p}} = 2 \quad \text{and} \quad \|h_0\|_{\ell^p} := \sup_{\substack{u\in\ell^p(\mathbb{Z})\\ u\neq 0}} \frac{\|h_0u\|_p}{\|u\|_p} = 2.$$

□

Among the $\ell^p(\mathbb{Z})$ spaces only $\ell^2(\mathbb{Z})$ is a Hilbert space. So the following characterization of the discrete Laplacian is applicable only to $\ell^2(\mathbb{Z})$:

Proposition 2.4. *$h_0 : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is self-adjoint.*

Proof. Since R, L are unitary operators, $L^* = L^{-1} = R$ and $R^* = R^{-1} = L$. □

We now investigate the spectrum of h_0. Without any Spectral Theory, we can already find that

Proposition 2.5. *For all $1 \leq p \leq \infty$, $[-2, 2] \subset \sigma(h_0)$ as an operator from $\ell^p(\mathbb{Z})$ to $\ell^p(\mathbb{Z})$.*

Proof. We essentially use the Weyl criterion (theorem 8.8). Let $E \in [-2, 2]$ and let $\theta \in [0, 2\pi)$ be such that $2\cos\theta = E$. Consider the sequence in $N \in \mathbb{N}$ of truncated plane waves travelling to the right:

$$u^{(N)}(n) = \begin{cases} e^{in\theta} & 0 \leq n \leq N \\ 0 & \text{otherwise} \end{cases}$$

$$(h_0 u^{(N)})(n) = \begin{cases} 1 & n = -1 \\ e^{i\theta} & n = 0 \\ 2\cos\theta e^{in\theta} & 1 \leq n \leq N-1 \\ e^{i(N-1)\theta} & n = N \\ e^{iN\theta} & n = N+1 \\ 0 & \text{otherwise} \end{cases} \qquad ((h_0 - E)u^{(N)})(n) = \begin{cases} 1 & n = -1 \\ e^{i\theta} - E & n = 0 \\ 0 & 1 \leq n \leq N-1 \\ e^{i(N-1)\theta} - Ee^{iN\theta} & n = N \\ e^{iN\theta} & n = N+1 \\ 0 & \text{otherwise} \end{cases}$$

Then for $1 \leq p < \infty$:

$$\lim_{N\to\infty} \frac{\|h_0 u^{(N)} - Eu^{(N)}\|_p}{\|u^{(N)}\|_p} = \lim_{N\to\infty} \frac{(2 + |e^{i\theta} - E|^p + |e^{i(N-1)\theta} - Ee^{iN\theta}|^p)^{1/p}}{N^{1/p}} = 0.$$

Hence $\lim_{N\to\infty} (h_0 - E)\frac{u^{(N)}}{\|u^{(N)}\|_p} = 0$.

If E were not in the spectrum of h_0, then by continuity of $(h_0 - E)^{-1}$ we would have:

$$\lim_{N\to\infty} (h_0 - E)^{-1}(h_0 - E)\frac{u^{(N)}}{\|u^{(N)}\|_p} = \lim_{N\to\infty} \frac{u^{(N)}}{\|u^{(N)}\|_p} = 0$$

in contradiction to the fact that $\frac{u^{(N)}}{\|u^{(N)}\|_p}$ are unit vectors.

For the case $p = \infty$, Corollary 2.2 gives $[-2, 2] = \sigma_{\mathrm{p}}(h_0)$, the point spectrum of h_0. □

To fully determine the spectrum of h_0 as an operator from $\ell^2(\mathbb{Z})$ to $\ell^2(\mathbb{Z})$, it is convenient to invoke the Spectral theorem for (bounded) self-adjoint operators. We know of the existence of a unitary $U : \ell^2(\mathbb{Z}) \to L^2(M, d\mu)$ and a bounded real-valued function f such that h_0 is unitarily equivalent to the operator of multiplication by f on $L^2(M, d\mu)$. For the Laplacian, the unitary map is precisely the Fourier transform:

$$\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2\left([-\pi, \pi], \frac{d\theta}{2\pi}\right) \qquad (\mathcal{F}u)(\theta) = \sum_{n\in\mathbb{Z}} u(n)e^{in\theta}$$

with inverse given by

$$\mathcal{F}^{-1} : L^2\left([-\pi, \pi], \frac{d\theta}{2\pi}\right) \to \ell^2(\mathbb{Z}) \qquad (\mathcal{F}^{-1}f)(n) = \left\langle e^{in\theta}, f(\theta)\right\rangle.$$

Proposition 2.6. *$h_0 : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is unitarily equivalent via the Fourier transform $\mathcal{F}$ to the multiplication operator $M_g : L^2\left([-\pi,\pi], \frac{d\theta}{2\pi}\right) \to L^2\left([-\pi,\pi], \frac{d\theta}{2\pi}\right)$ by the function $g(\theta) = 2\cos(\theta)$, that is:*

$$\Big([\mathcal{F}h_0\mathcal{F}^{-1}]f\Big)(\theta) = 2\cos(\theta)f(\theta).$$

In particular, $\sigma(h_0) = \text{ess ran}\,(2\cos(\theta)) = [-2,2]$.

Proof. For $f \in L^2\left([-\pi,\pi], \frac{d\theta}{2\pi}\right)$, we have :

$$\Big([h_0\mathcal{F}^{-1}]f\Big)(n) = \Big\langle e^{i(n-1)\theta}, f(\theta)\Big\rangle + \Big\langle e^{i(n+1)\theta}, f(\theta)\Big\rangle = \Big\langle e^{in\theta}, 2\cos(\theta)f(\theta)\Big\rangle$$

and so

$$\Big([\mathcal{F}h_0\mathcal{F}^{-1}]f\Big)(\theta) = \sum_{n\in\mathbb{Z}} \Big\langle e^{in\theta}, 2\cos(\theta)f(\theta)\Big\rangle e^{in\theta} = 2\cos(\theta)f(\theta).$$

By proposition 8.22, $\sigma(h_0) = \sigma(M_g) = \text{ess ran}\,(2\cos(\theta)) = [-2,2]$. □

We now look to extend the result about the spectrum to all $\ell^p(\mathbb{Z})$ spaces.

Given an infinite array $(a_{nm})_{i,j=-\infty}^{\infty}$, we can form a formal operator A from $\ell(\mathbb{Z})$ to itself acting as $(Au)(n) := \sum_{m\in\mathbb{Z}} a_{nm}u(m)$. Recall that we have the following standard results:

$$\|A\|_1 = \sup_m \Big(\sum_n |a_{nm}|\Big); \quad \|A\|_\infty = \sup_n \Big(\sum_m |a_{nm}|\Big); \quad \|A\|_p \le \|A\|_1^{1/p}\|A\|_\infty^{1-1/p}.$$

$\|A\|_1$ and $\|A\|_\infty$ deal with sums along columns and rows of A. We add another similar result which deals with sums along diagonals of A:

Lemma 2.7. *Let $A = (a_{ij})_{i,j=-\infty}^{\infty}$ be an infinite matrix satisfying $M := \sum_{k\in\mathbb{Z}} \sup_{i-j=k} |a_{ij}| < \infty$. Then $A : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ is a bounded operator for $1 \le p \le \infty$.*

Proof. Let x be a unit vector. For $1 \le p < \infty$:

$$\|Ax\|_p = \Big(\sum_i \Big|\sum_j a_{ij}x_j\Big|^p\Big)^{1/p} \le \Big(\sum_i \Big(\sum_j |a_{ij}x_j|\Big)^p\Big)^{1/p} = \Big(\sum_i \Big(\sum_j |a_{ij}|^{1-1/p}|a_{ij}|^{1/p}|x_j|\Big)^p\Big)^{1/p}$$

$$\le \Big(\sum_i \Big(\sum_j |a_{ij}|^{(1-1/p)q}\Big)^{p/q}\Big(\sum_j |a_{ij}||x_j|^p\Big)\Big)^{1/p} \le \Big(\sum_i \Big(\sum_j \sup_{k-l=i-j} |a_{kl}|\Big)^{p/q}\Big(\sum_j |a_{ij}||x_j|^p\Big)\Big)^{1/p}$$

$$= \Big(\sum_i \Big(\sum_{j'} \sup_{k-l=j'} |a_{kl}|\Big)^{p/q}\Big(\sum_j |a_{ij}||x_j|^p\Big)\Big)^{1/p} = M^{1/q}\Big(\sum_i\sum_j |a_{ij}||x_j|^p\Big)^{1/p}$$

$$= M^{1/q}\Big(\sum_j |x_j|^p \sum_i |a_{ij}|\Big)^{1/p} \le M^{1/q}\Big(\sum_j |x_j|^p \sum_i \sup_{k-l=i-j} |a_{kl}|\Big)^{1/p} = M^{1/q+1/p}\Big(\sum_j |x_j|^p\Big)^{1/p} = M.$$

We have used Hölder's inequality and the fact that one can interchange the order of summation for positive doubly indexed sequences. For $p = \infty$:

$$\|Ax\|_\infty = \sup_{i\in\mathbb{Z}} \Big|\sum_{j\in\mathbb{Z}} a_{ij}x_j\Big| \le \sup_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} |a_{ij}x_j| \le \sup_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} |a_{ij}| \le M.$$

□

Notation 2.8. *By definition, for $z \in \rho(h_0)$, $(h_0 - z) : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is invertible with bounded inverse. Let $A(z)$ be the matrix operator with elements given by $a_{nm}(z) = \langle \delta_n, (h_0 - z)^{-1}\delta_m\rangle$.*

Theorem 2.9. *Let $z \in \rho(h_0)$ as an operator from $\ell^2(\mathbb{Z})$ to itself. Then the matrix elements of $A(z)$ satisfy:*

$$|a_{nm}(z)| := |\langle \delta_n, (h_0 - z)^{-1}\delta_m\rangle| \le Ce^{-\gamma|m-n|} \tag{2.1}$$

$\forall m, n \in \mathbb{Z}$ and for some constants $\gamma, C > 0$ which depend on z. In particular $A(z) : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ is a bounded linear operator for $1 \le p \le \infty$.

Proof. Assume $m \ge n$:

$$\langle \delta_n, (h_0 - z)^{-1}\delta_m\rangle = \Big\langle \delta_n, \Big[\mathcal{F}^{-1}M_{(2\cos(\theta)-z)^{-1}}\mathcal{F}\Big]\delta_m\Big\rangle = \Big\langle \delta_n, \mathcal{F}^{-1}\Big(\frac{1}{2\cos\theta - z}\frac{1}{\sqrt{2\pi}}e^{im\theta}\Big)\Big\rangle$$

$$= \Big\langle \delta_n, \sum_{k\in\mathbb{Z}} \frac{\delta_k}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \frac{e^{-ik\theta}}{2\cos\theta - z}\frac{e^{im\theta}}{\sqrt{2\pi}}\Big\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i(m-n)\theta}}{2\cos\theta - z}d\theta = \frac{1}{2\pi i}\oint_{|\lambda|=1} \frac{\lambda^{m-n}}{\lambda^2 - \lambda z + 1}d\lambda.$$

Now $\lambda^2 - \lambda z + 1 = 0 \implies \lambda_{1,2} = \frac{z\pm\sqrt{z^2-4}}{2}$. Notice that $\lambda_1\lambda_2 = 1$, so that either both roots are on the unit circle, or one is inside and the other outside. In the first case, suppose that $\lambda_1 = e^{i\phi}$. Then $2\cos\phi = e^{i\phi} + e^{-i\phi} = z \Rightarrow z \in [-2, 2]$. However $[-2, 2] = \sigma(h_0)$, so it never happens that both roots are on the unit circle. In the other case, suppose for definiteness that $|\lambda_1| < 1$ and $|\lambda_2| > 1$. Then an application of Cauchy's integral formula gives:

$$\langle \delta_n, (h_0 - z)^{-1}\delta_m\rangle = \frac{1}{2\pi i}\oint_{|\lambda|=1} \frac{\lambda^{m-n}}{(\lambda - \lambda_1)(\lambda - \lambda_2)}d\lambda = \frac{\lambda_1^{m-n}}{\lambda_1 - \lambda_2} = \frac{e^{(m-n)(\ln|\lambda_1|+i\arg\lambda_1)}}{\lambda_1 - \lambda_2}.$$

Define $\gamma(z) := -\ln|\lambda_1|$ and $C(z) := |\frac{1}{\lambda_1-\lambda_2}|$. Then $\gamma(z), C(z) > 0$ and $|\langle \delta_n, (h_0-z)^{-1}\delta_m\rangle| = C(z)e^{-\gamma(z)|m-n|}$. If $n > m$, then $|\langle \delta_n, (h_0 - z)^{-1}\delta_m\rangle| = |\langle \delta_m, (h_0 - \overline{z})^{-1}\delta_n\rangle| = C(\overline{z})e^{-\gamma(\overline{z})|m-n|}$. Equation (2.1) follows by taking $C = \max\{C(z), C(\overline{z})\}$ and $\gamma = \min\{\gamma(z), \gamma(\overline{z})\}$.

That $A(z) : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ is a bounded linear operator for $1 \le p \le \infty$ now follows from lemma 2.7 and the fact that $\sum_{k\in\mathbb{Z}} e^{-\gamma|k|} < \infty$. □

The proof of the preceding theorem contains a lot of useful information that we will get back to after; but to finish what we had started, we have:

Theorem 2.10. *For all $1 \le p \le \infty$, $\sigma(h_0) = [-2, 2]$ as an operator from $\ell^p(\mathbb{Z})$ to $\ell^p(\mathbb{Z})$. In all cases $R(h_0, z) = A(z)$.*

Proof. By proposition 2.5, it remains to prove that $[-2, 2]^c \subset \rho(h_0)$. We know that the result holds for $h_0 : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$. Fix $z \in [-2, 2]^c$ and let $u \in \ell^p(\mathbb{Z})$ $(1 \le p \le \infty)$. Then $\Big([(h_0 - z)A]u\Big)(n) =$

$$= \sum_{m\in\mathbb{Z}} \langle \delta_{n-1}, (h_0 - z)^{-1}\delta_m\rangle u(m) + \sum_{m\in\mathbb{Z}} \langle \delta_{n+1}, (h_0 - z)^{-1}\delta_m\rangle u(m) - z\sum_{m\in\mathbb{Z}} \langle \delta_n, (h_0 - z)^{-1}\delta_m\rangle u(m)$$

$$= \sum_{m\in\mathbb{Z}} \Big(\langle \delta_{n-1}, (h_0 - z)^{-1}\delta_m\rangle + \langle \delta_{n+1}, (h_0 - z)^{-1}\delta_m\rangle - z\langle \delta_n, (h_0 - z)^{-1}\delta_m\rangle\Big)u(m)$$

$$= \sum_{m\in\mathbb{Z}} \langle (h_0 - \overline{z})\delta_n, (h_0 - z)^{-1}\delta_m\rangle u(m) = u(n).$$

Furthermore:

$$\Big([A(h_0 - z)]u\Big)(n) = \sum_{m\in\mathbb{Z}} \langle \delta_n, (h_0 - z)^{-1}\delta_m\rangle \big(u(m-1) + u(m+1) - zu(m)\big)$$

$$= \sum_{m\in\mathbb{Z}} \langle \delta_n, (h_0 - z)^{-1}\delta_m\rangle u(m-1) + \sum_{m\in\mathbb{Z}} \langle \delta_n, (h_0 - z)^{-1}\delta_m\rangle u(m+1) - z\sum_{m\in\mathbb{Z}} \langle \delta_n, (h_0 - z)^{-1}\delta_m\rangle u(m)$$

$$= \sum_{m\in\mathbb{Z}} \langle \delta_n, (h_0 - z)^{-1}\delta_{m+1}\rangle u(m) + \sum_{m\in\mathbb{Z}} \langle \delta_n, (h_0 - z)^{-1}\delta_{m-1}\rangle u(m) - z\sum_{m\in\mathbb{Z}} \langle \delta_n, (h_0 - z)^{-1}\delta_m\rangle u(m)$$

$$= \sum_{m\in\mathbb{Z}} \Big(\langle \delta_n, (h_0 - z)^{-1}\delta_{m+1}\rangle + \langle \delta_n, (h_0 - z)^{-1}\delta_{m-1}\rangle u(m) - z\langle \delta_{n+1}, (h_0 - z)^{-1}\delta_m\rangle\Big) u(m)$$

$$= \sum_{m\in\mathbb{Z}} \langle \delta_n, (h_0 - z)^{-1}(h_0 - z)\delta_m\rangle u(m) = u(n).$$

The above calculations are justified since all sums converge (absolutely). They show that $(h_0 - z) : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ is a bijection with inverse A, which we know to be bounded. Hence $z \in \rho(h_0)$ as an operator from $\ell^p(\mathbb{Z})$ to $\ell^p(\mathbb{Z})$. □

2.2 Boundary Values of the Resolvent and Green's Function for the Laplacian

We now turn our attention to those values z in the resolvent of h_0 that have Im $z \neq 0$.

For later use, we advise the reader that we will be using the convention that

$$\sqrt{z} := +\frac{1}{\sqrt{2}}\left(\text{sign(y)}\sqrt{|z| + x} + i\sqrt{|z| - x}\right). \tag{2.2}$$

Proposition 2.11. *For $E \in \mathbb{R}$ and for all $n \in \mathbb{Z}$:*

$$\lim_{\varepsilon\to\pm 0} \langle \delta_n, (h_0 - E - i\varepsilon)^{-1}\delta_n\rangle = \pm\frac{i}{\sqrt{4 - E^2}} \tag{2.3}$$

Proof. From the calculations of theorem 2.9, we know that $\langle \delta_n, (h_0 - z)^{-1}\delta_n\rangle = \frac{1}{\lambda_1 - \lambda_2} = \pm\frac{1}{\sqrt{z^2-4}}$ depending on whether $\lambda_1 = \frac{z+\sqrt{z^2-4}}{2}$ or $\lambda_1 = \frac{z-\sqrt{z^2-4}}{2}$. Let $z = E + i\varepsilon$, $\varepsilon \neq 0$. Applying formula (2.2) gives:

$$\frac{1}{\sqrt{z^2-4}} = \begin{cases} \frac{-i}{\sqrt{\varepsilon^2+4}} & \text{when } E = 0 \\ \frac{1}{\sqrt{2}} \frac{\text{sign}(E\varepsilon)\sqrt{\sqrt{(E^2-\varepsilon^2-4)^2+(2E\varepsilon)^2}+(E^2-\varepsilon^2-4)} \;-\; i\sqrt{\sqrt{(E^2-\varepsilon^2-4)^2+(2E\varepsilon)^2}-(E^2-\varepsilon^2-4)}}{\sqrt{(E^2-\varepsilon^2-4)^2+(2E\varepsilon)^2}} & \text{when } E \neq 0 \end{cases}$$

From lemma 1.12, $\text{sign}\big(\text{Im}\,\langle \delta_n, (h_0 - z)^{-1}\delta_n\rangle\big) = \text{sign}(\text{Im}\, z)$ which allows us to adjust the signs:

$$\langle \delta_n, (h_0 - z)^{-1}\delta_n\rangle = -\frac{\text{sign}(\varepsilon)}{\sqrt{z^2-4}}$$

Finally the result follows by taking the limit. □

Proposition 2.12. *For all $n, m \in \mathbb{Z}$, the Green's function is given by:*

$$G(z,n,m) = \langle \delta_n, (h_0 - z)^{-1}\delta_m \rangle = -\frac{\text{sign}(\text{Im } z)}{\sqrt{z^2-4}} \frac{(z - \text{sign}(\text{Im } z)\sqrt{z^2-4})^{|m-n|}}{2^{|m-n|}} \tag{2.4}$$

and

$$\lim_{\varepsilon \to \pm 0} G(E + i\varepsilon, n, m) := \langle \delta_n, (h_0 - E \mp i0)^{-1}\delta_m \rangle = \frac{\pm i}{\sqrt{4-E^2}} \frac{(E \mp i\sqrt{4-E^2})^{|m-n|}}{2^{|m-n|}}. \tag{2.5}$$

Moreover,

$$G(\overline{z}, n, m) = \overline{G(z,n,m)}. \tag{2.6}$$

Proof. From proposition 2.11 we see that $\lambda_1 = \frac{z+\sqrt{z^2-4}}{2} \Leftrightarrow \text{sign}(\varepsilon) < 0$.

The result again follows from the calculations of proposition 2.9, namely $\langle \delta_n, (h_0 - z)^{-1}\delta_m \rangle = \frac{\lambda_1^{m-n}}{\lambda_1 - \lambda_2}$ for $m > n$. Note that $-\overline{\sqrt{\overline{z}^2 - 4}} = \sqrt{z^2-4}$ and $-\overline{\frac{1}{\sqrt{\overline{z}^2-4}}} = \frac{1}{\sqrt{z^2-4}}$. That shows the identity (2.6). □

The Green's functions tend to different limits as $\varepsilon \to 0$, which reflects the jump discontinuity of the resolvent through the spectrum of h_0. For further use, we'll need in particular the formula:

$$\lim_{\varepsilon \downarrow 0} \langle \delta_0, (h_0 - z)^{-1}\delta_1 \rangle = \begin{cases} \frac{iE}{2\sqrt{4-E^2}} + \frac{1}{2} & E \in [-2,2] \\ \frac{-E\text{sign}(E)}{2\sqrt{E^2-4}} + \frac{1}{2} & E \in [-2,2]^c \end{cases} \tag{2.7}$$

We can find another interesting way to express the Green's function for $|z| > 2$:

Proposition 2.13. *For $|z| > 2$, $\langle \delta_0 | (h_0 - z)^{-1}\delta_n \rangle = -\sum_{\kappa=0}^{\infty} \frac{1}{z^{\kappa+1}} \langle \delta_0 | h_0^\kappa \delta_n \rangle$, where*

$$\langle \delta_m, h_0^\kappa \delta_{n+m} \rangle = \langle \delta_0, h_0^\kappa \delta_n \rangle = \begin{cases} \binom{\kappa}{\frac{\kappa-n}{2}} & \kappa \geq |n| \text{ and } \kappa - n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

Proof.

For the first part, note that for $|z| > 2$, $(h-z)^{-1} = -\frac{1}{z}(1 - \frac{h}{z})^{-1} = -\frac{1}{z}\sum_{\kappa=0}^{\infty} \frac{h^\kappa}{z^\kappa}$. For the second part we give two different proofs. The first goes as follows:

$\langle \delta_0, h^\kappa \delta_n \rangle$ is equal to the number of paths $P = (P(1), P(2), ..., P(\kappa))$ on the lattice of length κ starting at $P(1) = 0$ and ending at $P(\kappa) = n$ and satisfying $|P(i) - P(i+1)| = 1$ for $1 \leq i \leq \kappa - 1$. For a given path P, let κ_L and κ_R be the number of displacements to the left and right respectively. Then $\kappa_L + \kappa_R = \kappa$ and $\kappa_L(-1) + \kappa_R(1) = n$. So the number of such paths is 0 if $\kappa < |n|$ or if κ and n don't have the same parity, and is equal to $\binom{\kappa}{\frac{\kappa+n}{2}} = \binom{\kappa}{\frac{\kappa-n}{2}}$ otherwise. Note that reasoning in terms of paths shows that we have translation invariance, i.e. $\langle \delta_0 | h^\kappa \delta_n \rangle = \langle \delta_m | h^\kappa \delta_{n+m} \rangle \ \forall n, m \in \mathbb{Z}$, so we have in fact calculated all matrix elements of h^κ.

The second proof uses the fact that the Fourier transform is inner product preserving, namely:

$$\langle \delta_0 | h^\kappa \delta_n \rangle = \langle \frac{1}{\sqrt{2\pi}} | (2\cos\theta)^\kappa \frac{e^{in\theta}}{\sqrt{2\pi}} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} (2\cos\theta)^\kappa e^{in\theta} d\theta$$

$$= \frac{1}{2\pi i} \oint_{|z|=1} (z + z^{-1})^\kappa \frac{z^n}{z} dz = \frac{1}{2\pi i} \sum_{j=0}^{\kappa} \binom{\kappa}{j} \oint_{|z|=1} z^{2j-\kappa+n-1} dz.$$

By the fundamental result $\frac{1}{2\pi i}\oint_{|z|=1} z^m dz = 1$ if $m = -1$ and 0 if $m \in \mathbb{Z}\backslash\{-1\}$, we get that $\langle \delta_0 | h^\kappa \delta_n \rangle = \binom{\kappa}{j_0}$ if there exists j_0, $0 \leq j_0 \leq \kappa$ satisfying $2j_0 = \kappa - n$, and equals 0 otherwise. This is equivalent to the formulation given before. □

We end the section by computing the spectral measures μ_{δ_n} for the canonical basis elements $\{\delta_n\}$. Note that by the formula of the Green's function we see that $\mu_{\delta_n} = \mu_{\delta_m}$ for all n, m.

Proposition 2.14. *Let μ_0 be the spectral measure for δ_0 and h_0. Then μ_0 is purely absolutely continuous and*

$$\frac{d\mu_0}{dx}(x) = \frac{1}{\pi}\frac{1}{\sqrt{4-x^2}}\mathbb{1}_{[-2,2]}(x). \tag{2.8}$$

Proof. We know that h_0 is unitarily equivalent to the operator of multiplication by $2\cos\theta$ (proposition 3.5). Hence $\langle \delta_0, f(h_0)\delta_0\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(2\cos\theta)d\theta$. Letting $2\cos\theta = x$, $d\theta = \frac{dx}{\sqrt{4-x^2}}$ for $\theta \in [-\pi, 0]$ and $d\theta = -\frac{dx}{\sqrt{4-x^2}}$ for $\theta \in [0,\pi]$. Hence $\int_{\mathbb{R}} f(x)d\mu_0(x) = \langle \delta_0, f(h_0)\delta_0\rangle = \frac{1}{2\pi}\int_{-2}^{2} f(x)\frac{dx}{\sqrt{4-x^2}} - \frac{1}{2\pi}\int_{2}^{-2} f(x)\frac{dx}{\sqrt{4-x^2}} = \int_{\mathbb{R}} f(x)\frac{1}{\pi}\frac{1}{\sqrt{4-x^2}}\mathbb{1}_{[-2,2]}(x)dx$. Since this holds for all bounded Borel functions f, the result follows. □

2.3 The Green's Function for the Schrödinger Operator

The goal of this section is to show that regardless the potential, the Green's function $G(z, n, m)$ for the Schrödinger operator decay exponentially.

Proposition 2.15. *For $1 \leq p \leq \infty$, $h : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ is a bounded operator if and only if $\|V\|_\infty := \sup_{n\in\mathbb{Z}} |V(n)| < \infty$. Furthermore: $\|h\|_{\ell^p} \leq 2 + \|V\|_\infty$.*

Proof. If there is a sequence $V(n_k)$ such that $\lim_{k\to\infty} |V(n_k)| \to \infty$, then $\|h\delta_{n_k}\|_p \to \infty$ and so h is unbounded. The rest follows from $\|h\|_{\ell^p} \leq \|h_0\|_{\ell^p} + \|V\|_{\ell^p} \leq 2 + \|V\|_\infty$. □

In the most part of this thesis, we will assume tht $\|V\|_\infty < \infty$. Consequently V is a bounded operator and $\langle u, Vv\rangle = \langle Vu, v\rangle$ for all $u, v \in \ell^2(\mathbb{Z})$ shows that $V : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is a self-adjoint operator. If V is an unbounded operator, self-adjointness is a consequence of proposition 8.21. In any case Dom (h) = Dom $(V) = \ell^2(\mathbb{Z})$ if and only if $\|V\|_\infty < \infty$ and $h :$ Dom $(V) \to \ell^2(\mathbb{Z})$ is self-adjoint.

Later on we will be interested in characterizing the spectrum of $h : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ for various potentials V. We can already mention that $\sigma(h) \subset [-2 - \|V\|_\infty, 2 + \|V\|_\infty]$, the formula being obviously true if V is unbounded and follows from the fact that for bounded self-adjoint operators, the spectral radius is equal to the operator's norm.

We will prove a fundamental estimate on $\langle \delta_n, (h-z)^{-1}\delta_m\rangle$ for $z \in \rho(h)$. This will allow us to show, among other things, that for $1 \leq p < \infty$, the spectrum of h as an operator from $\ell^p(\mathbb{Z})$ to itself is the same as its spectrum as an operator from $\ell^2(\mathbb{Z})$ to itself. The argument we employ is a Combes-Thomas type argument. Of course we have an a priori estimate

$$|\langle \delta_n, (h-z)^{-1}\delta_m\rangle| \leq \frac{1}{\text{dist}(z, \sigma(h))} := \frac{1}{d}. \tag{2.9}$$

Definition 2.16. *For $a \in \mathbb{R}$, define the maps:*

(i) $T_a : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, $(T_a u)(n) = e^{ian}u(n)$.

(ii) $h_a : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, $h_a = T_a \circ h \circ T_a^*$.

(iii) $R_a : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, $(R_a u)(n) = (e^{-ia} - 1)u(n+1) + (e^{ia} - 1)u(n-1)$.

Equivalently, we can write $R_a = (e^{-ia} - 1)R + (e^{ia} - 1)L$. It is obvious that T_a and R_a are bounded operators and R_a is self-adjoint. We really need a to be real in the definition of h_a otherwise T_a is not unitary. However we can allow a to be complex in the definition of R_a. The proof of the following lemma is staightforward.

Lemma 2.17. *The map T_a satisfies:*

(i) *T_a is unitary.*

(ii) *$T_a^* = T_{-a}$.*

(iii) *$T_{a_1+a_2} = T_{a_1} \circ T_{a_2}$.*

Proposition 2.18. *Let $z \in \rho(h)$ as an operator from $\ell^2(\mathbb{Z})$ to itself. Then $\forall a \in \mathbb{R}$:*

(i) *h_a is unitarily equivalent to h.*

(ii) *$h_a = h + R_a$.*

(iii) *$(h_a - z) = (h - z)[1 + (h - z)^{-1}R_a]$ and $(h_a - z)^{-1} = [1 + (h - z)^{-1}R_a]^{-1}(h - z)^{-1}$.*

(iv) *$\langle \delta_n, (h - z)^{-1}\delta_m\rangle = e^{-ia(n-m)}\langle \delta_n, [1 + (h - z)^{-1}R_a]^{-1}(h - z)^{-1}\delta_m\rangle$.*

Proof.

(i) $h_a = T_a h T_a^*$ and T_a is unitary by lemma 2.17.

(ii) First, $h_a = T_a h T_a^* = T_a h_0 T_a^* + T_a V T_a^* = T_a h_0 T_a^* + V$. Then:

$$\begin{aligned}
([T_a^*]u)(n) &= e^{-ian}u(n)\\
([h_0 T_a^*]u)(n) &= e^{-ia(n+1)}u(n+1) + e^{-ia(n-1)}u(n-1)\\
([T_a h_0 T_a^*]u)(n) &= e^{-ia}u(n+1) + e^{ia}u(n-1).
\end{aligned}$$

Finally:

$$\begin{aligned}
(h_a u)(n) &= e^{-ia}u(n+1) + e^{ia}u(n-1) + V(n)u(n)\\
&= (u(n+1) + u(n-1) + V(n)u(n)) + (e^{-ia} - 1)u(n+1) + (e^{ia} - 1)u(n-1)\\
&= (hu)(n) + (R_a u)(n).
\end{aligned}$$

(iii) $(h_a - z) = (h - z) + R_a = (h - z)[1 + (h - z)^{-1}R_a]$. Since h_a and h are unitarily equivalent, $z \in \rho(h) \iff z \in \rho(h_a)$ and it follows that $[1 + (h - z)^{-1}R_a]$ is a bijection.

(iv)
$$\begin{aligned}
&\langle \delta_n, (h - z)^{-1}\delta_m\rangle = \langle T_a\delta_n, T_a(h - z)^{-1}T_a^* T_a\delta_m\rangle = e^{-ia(n-m)}\langle \delta_n, T_a(h - z)^{-1}T_a^*\delta_m\rangle\\
&= e^{-ia(n-m)}\langle \delta_n, (h_a - z)^{-1}\delta_m\rangle = e^{-ia(n-m)}\langle \delta_n, [1 + (h - z)^{-1}R_a]^{-1}(h - z)^{-1}\delta_m\rangle.
\end{aligned}$$

□

We can reformulate the last item of the previous theorem as follows: the map

$$\mathbb{R} \ni a \to e^{-ia(n-m)}\langle \delta_n, [1 + (h - z)^{-1} R_a]^{-1}(h - z)^{-1}\delta_m\rangle \in \mathbb{C} \tag{2.10}$$

is constant and equals $\langle \delta_n, (h - z)^{-1}\delta_m\rangle$. We proceed to extend this map analytically. We remind the reader of two useful facts from complex analysis.

Theorem 2.19.

(i) (Identity theorem) *Given functions f and g holomorphic on a connected open set $D \subset \mathbb{C}$, and a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in D converging to $x \in D$, if $f(x_n) = g(x_n)$ $\forall n \in \mathbb{N}$, then $f = g$ on D.*

(ii) *Let $\{f_n(z)\}_{n=1}^{\infty}$ be a sequence of holomorphic functions on an open connected set $D \subset \mathbb{C}$ which converge uniformly to $f(z)$. Then $f(z)$ is holomorphic on D.*

Notation 2.20. *Denote $D_r(\mathbb{C}) = \{a \in \mathbb{C} : |a| < r\}$ and $D_r(\mathbb{R}) = \{a \in \mathbb{R} : |a| < r\}$.*

Theorem 2.21. *Let $z \in \rho(h)$ as an operator from $\ell^2(\mathbb{Z})$ to itself. Then for $a \in \mathbb{C}$:*

(i) $\|R_a\|_{\ell^2} \leq 2|a|e^{|a|}$.

(ii) *The map $\mathbb{C} \ni a \to R_a \in \mathcal{B}(\ell^2(\mathbb{Z}))$ is entire analytic.*

(iii) *Let $r_0 > 0$ be the unique number such that $2r_0e^{r_0} = d := \mathrm{dist}(z, \sigma(h))$. Then for $|a| < r_0$, we have:*

$$[1 + (h - z)^{-1}R_a]^{-1} = \sum_{j=0}^{\infty}(-1)^j[(h - z)^{-1}R_a]^j, \quad \|[1 + (h - z)^{-1}R_a]^{-1}\| \leq \frac{d}{d - 2|a|e^{|a|}}.$$

(iv) *The map $D_{r_0}(\mathbb{C}) \ni a \to [1 + (h - z)^{-1}R_a]^{-1} \in \mathcal{B}(\ell^2(\mathbb{Z}))$ is analytic.*

(v) *The map*

$$D_{r_0}(\mathbb{C}) \ni a \to e^{-ia(n-m)}\langle \delta_n, [1 + (h - z)^{-1}R_a]^{-1}(h - z)^{-1}\delta_m\rangle$$

is the analytic continuation of the map

$$D_{r_0}(\mathbb{R}) \ni a \to e^{-ia(n-m)}\langle \delta_n, [1 + (h - z)^{-1}R_a]^{-1}(h - z)^{-1}\delta_m\rangle.$$

In particular, for all $a \in D_{r_0}(\mathbb{C})$: $\langle \delta_n, (h-z)^{-1}\delta_m\rangle = e^{-ia(n-m)}\langle \delta_n, [1+(h-z)^{-1}R_a]^{-1}(h-z)^{-1}\delta_m\rangle$.

Proof.

(i) If $a \in \mathbb{R}$, the following argument works:

$$\|R_a\| \leq \|(e^{-ia} - 1)R\| + \|(e^{ia} - 1)L\| \leq |e^{-ia} - 1| + |e^{ia} - 1| \leq 4|\sin(a/2)| \leq 2|a| \leq 2|a|e^{|a|}.$$

However in general we need the following inequality: for $t \in \mathbb{C}$, $|e^t - 1| = \left|\sum_{n\geq 0} \frac{t^n}{n!} - 1\right| \leq \sum_{n\geq 1} \frac{|t|^n}{n!} = e^{|t|} - 1$. By the Mean Value theorem, $\frac{e^{|t|}-e^0}{|t|-0} = e^c$ for some $c \in [0, |t|]$, so $|e^t - 1| \leq e^{|t|} - 1 = |t|e^c \leq |t|e^{|t|}$.

(ii) This is equivalent to verifying that the map $\mathbb{C} \ni a \to \lambda(R_a) \in \mathbb{C}$ is entire analytic for every linear functional $\lambda \in (\mathcal{B}(\ell^2(\mathbb{Z})))^*$. But $\lambda(R_a) = (e^{-ia} - 1)\lambda(R) + (e^{ia} - 1)\lambda(L)$ which is just the sum of two entire analytic functions ($\lambda(R)$ and $\lambda(L)$ are just constants for a given λ).

(iii) If $|a| < r_0$, then $\|(h-z)^{-1}R_a\| \leq \frac{1}{d}2|a|e^{|a|} < \frac{1}{d}2r_0e^{r_0} = 1$. So $\sum_{j=0}^{\infty}(-1)^j[(h-z)^{-1}R_a]^j$ is well defined and equals $[1 + (h - z)^{-1}R_a]^{-1}$. For the second part, $\|[1 + (h - z)^{-1}R_a]^{-1}\| =$

$$\left\|\sum_{j=0}^{\infty}(-1)^j[(h-z)^{-1}R_a]^j\right\| \leq \sum_{j=0}^{\infty}\|(h-z)^{-1}R_a\|^j \leq \sum_{j=0}^{\infty}\left(\frac{1}{d}2|a|e^{|a|}\right)^j = \frac{d}{d - 2|a|e^{|a|}}.$$

(iv) We show that the map $D_{r_0-\varepsilon} \ni a \to [1 + (h - z)^{-1}R_a]^{-1} \in \mathcal{B}(\ell^2(\mathbb{Z}))$ is weakly analytic for every $\varepsilon > 0$. Let λ be any linear functional on $\mathcal{B}(\ell^2(\mathbb{Z}))$. By continuity of λ, for every $a \in D_{r_0-\varepsilon}$:

$$\lambda\left([1 + (h-z)^{-1}R_a]^{-1}\right) = \lambda\left(\sum_{j=0}^{\infty}(-1)^j[(h-z)^{-1}R_a]^j\right) = \sum_{j=0}^{\infty}\lambda\Big((-1)^j[(h-z)^{-1}R_a]^j\Big).$$

Since $a \to R_a$ analytic and the product of operator-valued analytic functions are analytic, we obtain a sequence

$$\left\{\lambda\Big((-1)^j[(h-z)^{-1}R_a]^j\Big)\right\}_{j=0}^{\infty}$$

of complex-valued analytic functions. For $a \in D_{r_0-\varepsilon}$, we have:

$$\left|\lambda\Big((-1)^j[(h-z)^{-1}R_a]^j\Big)\right| \leq \|\lambda\|\ \|[(h-z)^{-1}R_a]\|^j \leq \|\lambda\|\left(\frac{2(r_0-\varepsilon)e^{(r_0-\varepsilon)}}{d}\right)^j := M_j.$$

Since $\frac{2(r_0-\varepsilon)e^{(r_0-\varepsilon)}}{d} < 1$, the series $\sum_{j=0}^{\infty} M_j$ converges and so by the Weierstrass M-test, the sequence $S_N(a) = \sum_{j=0}^{N}\lambda\Big((-1)^j[(h-z)^{-1}R_a]^j\Big)$ converges uniformly to $\sum_{j=0}^{\infty}\lambda\Big((-1)^j[(h-z)^{-1}R_a]^j\Big)$. By theorem 2.19 we conclude that $\sum_{j=0}^{\infty}\lambda\Big((-1)^j[(h-z)^{-1}R_a]^j\Big)$ is analytic on $D_{r_0-\varepsilon}$. The result follows by taking $\varepsilon \to 0$.

(v) From (iv), we get that the map $D_{r_0}(\mathbb{C}) \ni a \to [1 + (h-z)^{-1}R_a]^{-1}(h-z)^{-1} \in \mathcal{B}(\ell^2(\mathbb{Z}))$ is weakly analytic. In particular the map $D_{r_0}(\mathbb{C}) \ni a \to \langle\delta_n, [1 + (h-z)^{-1}R_a]^{-1}(h-z)^{-1}\delta_m\rangle \in \mathbb{C}$ is analytic. Therefore, the maps $D_{r_0}(\mathbb{C}) \ni a \to e^{-ia(n-m)}\langle\delta_n, [1 + (h-z)^{-1}R_a]^{-1}(h-z)^{-1}\delta_m\rangle$ and $D_{r_0}(\mathbb{C}) \ni a \to \langle\delta_n, (h-z)^{-1}\delta_m\rangle$ are two analytic maps that agree on $D_{r_0}(\mathbb{R})$ by (2.10). It follows by the Identity theorem that these two maps agree on $D_{r_0}(\mathbb{C})$.

□

We are now ready to prove our fundamental estimate, the analogous result to theorem 2.9. For $z \in \rho(h)$ as an operator from $\ell^2(\mathbb{Z})$ to itself, let $A(z)$ be the infinite matrix operator whose entries are $a_{nm} := \langle\delta_n, (h-z)^{-1}\delta_m\rangle$.

Theorem 2.22. *The matrix elements of $A(z)$ satisfy*

$$|a_{nm}| := |\langle\delta_n, (h-z)^{-1}\delta_m\rangle| \leq Ce^{-\gamma|n-m|}$$

where $C, \gamma > 0$ depend only on z. In particular $A(z) : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ is a bounded linear operator for $1 \le p \le \infty$.

Proof.

By part (v) of theorem 2.21, we have that for all $a \in D_{r_0}(\mathbb{C})$:

$$|\langle \delta_n, (h-z)^{-1}\delta_m \rangle| = |e^{-ia(n-m)}||\langle \delta_n, [1+(h-z)^{-1}R_a]^{-1}(h-z)^{-1}\delta_m \rangle| \le |e^{-ia(n-m)}|\frac{1}{d-2|a|e^{|a|}}.$$

The result follows by choosing $a = -i\gamma \text{sign}(n-m)$ where $0 < \gamma < r_0$. □

An identical computation as in proposition 2.10 shows that if $z \in \rho(h)$ as an operator from $\ell^2(\mathbb{Z})$ to itself, then $z \in \rho(h)$ as an operator from $\ell^p(\mathbb{Z})$ to itself for all $1 \le p \le \infty$. To show that the resolvent sets are in fact the same, we will use the following fact concerning Banach space adjoints:

Definition 2.23. *Given Banach spaces X, Y and a bounded linear operator $A : X \to Y$, the Banach space adjoint of A, denoted A', is defined to be the bounded linear operator from Y^* to X^* such that for all $\lambda \in Y^*, x \in X$:*

$$(A'\lambda)(x) = \lambda(Ax).$$

We explicitly compute the Banach space adjoint of $h - z : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ for $1 \le p < \infty$:

Proposition 2.24. *Let $1 \le p < \infty$, q its conjugate exponent and denote $U : (\ell^p(\mathbb{Z}))^* \to \ell^q(\mathbb{Z})$ the isometric isomorphism. Then $(h-z)' : (\ell^p(\mathbb{Z}))^* \to (\ell^p(\mathbb{Z}))^*$ satisfies*

$$(h-z)' = U^{-1}(h-z)U. \tag{2.11}$$

Proof. Let $u \in \ell^p(\mathbb{Z})$, $\lambda \in (\ell^p(\mathbb{Z}))^*$, $U\lambda = \{\lambda(n)\}_{n\in\mathbb{Z}}$. Then:

$$[(h-z)'\lambda]u = \lambda((h-z)u) = \sum_{n\in\mathbb{Z}} \lambda(n)\left(u(n+1) + u(n-1) + V(n)u(n) - zu(n)\right)$$

$$= \sum_{n\in\mathbb{Z}} \left(\lambda(n+1) + \lambda(n-1) + V(n)\lambda(n) - z\lambda(n)\right)u(n).$$

The above calculation is justified since the sums converge (absolutely) by Hölder's inequality. For the sake of clarity, define $\tilde{\lambda} := (h-z)'\lambda \in (\ell^p(\mathbb{Z}))^*$ and $U\tilde{\lambda} = \{\tilde{\lambda}(n)\}_{n\in\mathbb{Z}} \in \ell^q(\mathbb{Z})$. Then we have

$$[(h-z)'\lambda]u = \sum_{n\in\mathbb{Z}} \tilde{\lambda}(n)u(n) = \sum_{n\in\mathbb{Z}} \left(\lambda(n+1) + \lambda(n-1) + V(n)\lambda(n) - z\lambda(n)\right)u(n).$$

Since this is true for all $u \in \ell^p(\mathbb{Z})$, we must have

$$\left(\lambda(n+1) + \lambda(n+1) + V(n)\lambda(n) - z\lambda(n)\right) = \tilde{\lambda}(n)$$

for all n. In other words, $(h-z)U\lambda = U\tilde{\lambda} = U(h-z)'\lambda$. Finally, since this is true for all $\lambda \in (\ell^p(\mathbb{Z}))^*$, we get $(h-z)' = U^{-1}(h-z)U$. □

In our situation the Banach space adjoint A' is of interest because it enjoys the following relationship with A:

Theorem 2.25. *Let X be a Banach space, $A \in \mathcal{B}(X)$. Then $\sigma(A) = \sigma(A')$ and $R(A', z) = R(A, z)'$.*

Reed-Simon attribute the theorem to Phillips.

We will also need a Riesz-Thorin interpolation type result:

Theorem 2.26. *Let $1 \le p_0 < p_1 \le \infty$. For $t \in [0,1]$, let $p_t^{-1} := (1-t)p_0^{-1} + tp_1^{-1}$. If $A : \ell^{p_0}(\mathbb{Z}) \to \ell^{p_0}(\mathbb{Z})$ and $A : \ell^{p_1}(\mathbb{Z}) \to \ell^{p_1}(\mathbb{Z})$ are bounded linear maps with respective norms M_0 and M_1, then $A : \ell^{p_t}(\mathbb{Z}) \to \ell^{p_t}(\mathbb{Z})$ is a bounded linear operator with norm less than $M_0^{1-t}M_1^t$.*

Note that p_t takes all values between p_0 and p_1 as t varies from 0 to 1.

Notation 2.27. *For $1 \le p \le \infty$, denote $\sigma(h,p)$ and $\rho(h,p)$ the spectrum and resolvent set of h as an operator from $\ell^p(\mathbb{Z})$ to itself.*

Theorem 2.28. *For all $1 \le p < \infty$, $\sigma(h,p) = \sigma(h,2)$. For $p = \infty$, $\sigma(h,\infty) \subset \sigma(h,2)$.*

Proof. Remains to show that $\rho(h,p) \subset \rho(h,2)$ for $1 \le p < \infty$. Denote q the conjugate exponent of p. We have:

$$\begin{aligned} z \in \rho(h,p) &\Leftrightarrow (h-z) : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z}) \text{ is a bijection and } (h-z)^{-1} \in \mathcal{B}(\ell^p(\mathbb{Z})) \\ &\Leftrightarrow (h-z)' : (\ell^p(\mathbb{Z}))^* \to (\ell^p(\mathbb{Z}))^* \text{ is a bijection and } [(h-z)']^{-1} \in \mathcal{B}((\ell^p(\mathbb{Z}))^*) \\ &\Leftrightarrow (h-z) : \ell^q(\mathbb{Z}) \to \ell^q(\mathbb{Z}) \text{ is a bijection and } (h-z)^{-1} \in \mathcal{B}(\ell^q(\mathbb{Z})) \\ &\Leftrightarrow z \in \rho(h,q) \end{aligned}$$

Here we have used theorem 2.25 and proposition 2.24. We apply theorem 2.26 to get that $(h-z)^{-1} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is a bounded operator. Hence $z \in \rho(h,2)$. □

We end this section by giving another consequence of the exponential decay of the Green's function.

Proposition 2.29. *Let $z \in \rho(h)$. Then the sequences $u_+(z)$ and $u_-(z) \in E(z)$ introduced in lemma 1.13 are in fact exponentially decaying at $+\infty$ and $-\infty$ respectively. Moreover they are exponentially growing at $-\infty$ and $+\infty$ respectively.*

Proof. Recall that $u_+(n,z) = \langle \delta_n, (h-z)^{-1}\delta_1 \rangle$ for $n \ge 1$ and $u_-(n,z) = \langle \delta_n, (h-z)^{-1}\delta_{-1} \rangle$ for $n \le 1$. Their Wronskian $W(u_+, u_-)$ is constant. By theorem 2.22 we know that u_+ and u_- in fact decay exponentially at $+\infty$ and $-\infty$ respectively. The Wronskian cannot be zero, since if they were linearly dependent, i.e. $u_+ = \alpha u_-$ for some $\alpha \neq 0$, then they would be eigenvectors in $\ell^2(\mathbb{Z})$, contradicting $z \in \rho(h)$. For $n \ge 1$, we get

$$\begin{aligned} |W(u_+,u_-)| &\le |u_+(n)u_-(n+1)| + |u_-(n)u_+(n+1)| \\ &\le C\left(e^{-\gamma(n-1)}|u_-(n+1)| + e^{-\gamma n}|u_-(n)|\right) = C\left(e^{2\gamma}\frac{|u_-(n+1)|}{e^{\gamma(n+1)}} + \frac{|u_-(n)|}{e^{\gamma n}}\right). \end{aligned}$$

So $\frac{|W(u_+,u_-)|}{C(e^{2\gamma}+1)} \le \liminf_{n\to+\infty} \frac{|u_-(n)|}{e^{\gamma n}}$. This shows that u_- is eventually exponentially growing at $+\infty$. A similar calculation shows the result for u_+ at $-\infty$. □

2.4 The Spectral theorem for the Schrödinger Operator

In this section we will denote $\mathcal{H} := \ell^2(\mathbb{Z})$ and we will assume that V is a bounded potential so that h is a bounded operator.

Lemma 2.30. *For all $k \in \mathbb{Z}$ the pair of vectors $\{\delta_k, \delta_{k+1}\}$ is a cyclic set for h and $\mathcal{H}$.*

Proof. Let $\mathcal{L}$ be the linear span of $\{h^n\delta_k, h^n\delta_{k+1} : n \geq 0\}$. We need to show that $\mathcal{L}$ is dense in $\ell^2(\mathbb{Z})$. $\delta_{k+2} = h\delta_{k+1} - \delta_k - V_{k+1}\delta_{k+1}$ so $\delta_{k+2} \in \mathcal{L}$. Inductively we get $\delta_m \in \mathcal{L}$ for all $m \geq k+2$. A similar induction shows that $\delta_m \in \mathcal{L}$ for all $m \leq k-1$. $\square$

In what follows we will be working with the cyclic set $\{\delta_0, \delta_1\}$. It follows from the general theory that h has spectral multiplicity two.

For $i = 0, 1$, let $\mathcal{H}_i$ be the cyclic subspace generated by h and δ_i. Let $U_i : \mathcal{H}_i \to L^2(\mathbb{R}, d\mu_{\delta_i})$ be the unique spectral unitary operators and μ_{δ_i} the spectral measures for $\mathcal{H}_i$ and δ_i. We have $(U_i h)f = E(U_i f)$ for all $f \in \mathcal{H}_i$ and $(U_i\delta_i)(E) = \mathbb{1}_{\text{supp } \mu_{\delta_i}}(E)$. A similar analysis done in the half line case (theorem 3.11) shows that the following limits

$$\lim_{\varepsilon\downarrow 0} \frac{\operatorname{Im}\langle \delta_i, (h - E - i\varepsilon)^{-1}\delta_n\rangle}{\operatorname{Im}\langle \delta_i, (h - E - i\varepsilon)^{-1}\delta_i\rangle}, \qquad \lim_{\varepsilon\downarrow 0} \frac{\langle \delta_i, (h - E - i\varepsilon)^{-1}\delta_n\rangle}{\langle \delta_i, (h - E - i\varepsilon)^{-1}\delta_i\rangle}$$

exist and are finite for $\mu_{\delta_i,\text{ac}} - a.e.\ E$ and $\mu_{\delta_i,\text{sing}} - a.e.\ E$ respectively. They define functions $\delta_n^{(i)}(E)$ for every $n \in \mathbb{Z}$. If $\mathbb{1}_i$ denotes the orthogonal projection on $\mathcal{H}_i$, then we have $(U_i\mathbb{1}_i\delta_n)(E) = \delta_n^{(i)}(E)$. For $\mu_{\delta_i} - a.e.\ E$ the sequence $\{\delta_n^{(i)}(E)\}_{n\in\mathbb{Z}}$ satisfies the Schrödinger equation:

$$\delta_{n+1}^{(i)}(E) + \delta_{n-1}^{(i)}(E) + V(n)\delta_n^{(i)}(E) = E\delta_n^{(i)}(E).$$

If we follow the general theory, the next step is to glue the subspaces $\mathcal{H}_0$ and $\mathcal{H}_1$ together as follows. Since $\mathcal{H}_0 \subsetneq \mathcal{H}$, we have $\mathbb{1}_0^\perp\delta_1 = \delta_1 - \mathbb{1}_0\delta_1 \neq 0$, where $\mathbb{1}_0^\perp$ denotes the orthogonal projection on $\mathcal{H}_0^\perp$. Let $\overline{\mathcal{H}_0}$ be the cyclic space spanned by h and $\mathbb{1}_0^\perp\delta_1$.

Lemma 2.31.

(i) *$\mathcal{H}_0$ and $\overline{\mathcal{H}_0}$ are h-invariant.*

(ii) *$\mathbb{1}_0 h = h\mathbb{1}_0$ and $\mathbb{1}_0^\perp h = h\mathbb{1}_0^\perp$.*

(iii) *$\overline{\mathcal{H}_0} = \mathcal{H}_0^\perp$ and $\mathcal{H} = \mathcal{H}_0 \oplus \overline{\mathcal{H}_0}$.*

Proof.

(i) This is obvious.

(ii) Let $f \in \mathcal{H}$. $(\mathbb{1}_0 h)f = (\mathbb{1}_0 h)(\mathbb{1}_0 f + \mathbb{1}_0^\perp f) = (h\mathbb{1}_0)f$ and so $\mathbb{1}_0 h = h\mathbb{1}_0$ by invariance.

(iii) We only need to show that $\overline{\mathcal{H}_0} = \mathcal{H}_0^\perp$. $\langle h^n\delta_0, h^m\mathbb{1}_0^\perp\delta_1\rangle = \langle h^{n+m}\delta_0, \mathbb{1}_0^\perp\delta_1\rangle = 0$ for all $n, m \geq 0$. By linearity and continuity of the inner product we get that $\overline{\mathcal{H}_0} \subset \mathcal{H}_0^\perp$. Since $\overline{\mathcal{H}_0}$ is a closed subspace of $\mathcal{H}_0^\perp$, we can decompose $\mathcal{H}_0^\perp$ into $\overline{\mathcal{H}_0} \oplus (\overline{\mathcal{H}_0})^\perp$. Now $w \in (\overline{\mathcal{H}_0})^\perp \Leftrightarrow \langle u + v, w\rangle$ for all $u \in \mathcal{H}_0$ and $v \in \overline{\mathcal{H}_0}$. However $\{\delta_0, \delta_1\}$ are cyclic for h so there exists a sequence $(w_n)_{n=1}^\infty$ converging to w, where each w_n is of the form

$$\sum_{j=0}^{N(n)} a_j(h^j\delta_0) + \sum_{j=0}^{N(n)} b_j(h^j\delta_1) = \sum_{j=0}^{N(n)} a_j(h^j\delta_0) + \sum_{j=0}^{N(n)} b_j(h^j\mathbb{1}_0\delta_1) + \sum_{j=0}^{N(n)} b_j(h^j\mathbb{1}_0^\perp\delta_1)$$

for some $a_j, b_j \in \mathbb{C}$. Using the fact that $\mathcal{H}_0$ and $\overline{\mathcal{H}_0}$ are invariant, we can more simply write $w_n = u_n + v_n$ where $u_n \in \mathcal{H}_0$ and $v_n \in \overline{\mathcal{H}_0}$. Then $\|w\|^2 = \langle \lim_{n\to\infty} w_n, w\rangle = \lim_{n\to\infty}\langle u_n + v_n, w\rangle = 0$ and so $(\overline{\mathcal{H}_0})^\perp = \{0\}$.

□

Let $\overline{\mu_{\delta_0}}$ be the spectral measure for h and $\mathbb{1}_0^\perp \delta_1$ and $\overline{U} : \overline{\mathcal{H}_0} \to L^2(\mathbb{R}, d\overline{\mu_{\delta_0}})$ be the corresponding spectral unitary. The limits

$$\lim_{\varepsilon\downarrow 0} \frac{\mathrm{Im}\langle \mathbb{1}_0^\perp\delta_1, (h-E-i\varepsilon)^{-1}\mathbb{1}_0^\perp\delta_n\rangle}{\mathrm{Im}\langle \mathbb{1}_0^\perp\delta_1, (h-E-i\varepsilon)^{-1}\mathbb{1}_0^\perp\delta_1\rangle}, \qquad \lim_{\varepsilon\downarrow 0} \frac{\langle \mathbb{1}_0^\perp\delta_1, (h-E-i\varepsilon)^{-1}\mathbb{1}_0^\perp\delta_n\rangle}{\langle \mathbb{1}_0^\perp\delta_1, (h-E-i\varepsilon)^{-1}\mathbb{1}_0^\perp\delta_1\rangle}$$

exist and are finite for $\overline{\mu_{\delta_0,\mathrm{ac}}} - a.e.\ E$ and $\overline{\mu_{\delta_0,\mathrm{sing}}} - a.e.\ E$ respectively and allow us to define functions $\overline{\delta_n}(E)$ for $\overline{\mu_{\delta_0}} - a.e.\ E$. Moreover $(\overline{U}\mathbb{1}_0^\perp\delta_n)(E) = \overline{\delta_n}(E)$. However this development fails to be useful because the functions $\overline{\delta_n}(E)$ do not satisfy the Schrödinger equation.

We will now describe the direct integral decomposition of h on $\mathcal{H}$, and describe the eigenfunction expansion.

Consider a family signed measures $\{\mu_{i,j} : i, j = 0, 1\}$ on $\mathbb{R}$ and the corresponding matrix valued measure :

$$\mu := \begin{pmatrix} \mu_{0,0} & \mu_{0,1} \\ \mu_{1,0} & \mu_{1,1} \end{pmatrix}, \qquad d\mu := \begin{pmatrix} d\mu_{0,0} & d\mu_{0,1} \\ d\mu_{1,0} & d\mu_{1,1} \end{pmatrix} \tag{2.12}$$

For the following general procedure we will assume that μ is a positive semi-definite and hermitian matrix, that is, $\mu(B)$ is positive semi-definite and hermitian for all Borel sets B. Let e_1, e_2 be the standard basis for $\mathbb{C}^2$. Then $0 \le e_1^{\mathrm{T}}\mu(B)e_1 = \mu_{0,0}(B)$ and $0 \le e_2^{\mathrm{T}}\mu(B)e_2 = \mu_{1,1}(B)$ shows that $\mu_{0,0}$ and $\mu_{1,1}$ are positive measures on $\mathbb{R}$. Hermicity implies $\mu_{0,1} = \mu_{1,0}$. We define the trace measure $\mu^{tr} := \mu_{0,0} + \mu_{1,1}$, which is positive. Another consequence of positive semi-definiteness is that $\det\mu(B) \ge 0$ so that $|\mu_{0,1}(B)| \le \sqrt{\mu_{0,0}(B)\mu_{1,1}(B)} \le \frac{\mu^{tr}(B)}{2}$ $[(\sqrt{\mu_{0,0}} - \sqrt{\mu_{1,1}})^2 \ge 0$]. Hence $\mu_{i,j} \ll \mu^{tr}$ for $i, j = 0, 1$.

We introduce the corresponding matrix of Radon-Nikodym derivatives:

$$R(E) = (R_{i,j}(E))_{i,j=0,1} := \begin{pmatrix} \frac{d\mu_{0,0}}{d\mu^{tr}}(E) & \frac{d\mu_{0,1}}{d\mu^{tr}}(E) \\ \frac{d\mu_{1,0}}{d\mu^{tr}}(E) & \frac{d\mu_{1,1}}{d\mu^{tr}}(E) \end{pmatrix} \tag{2.13}$$

Lemma 2.32. *$R(E)$ is positive semi-definite for μ^{tr} a.e. E.*

Proof. Let $Q = \{\xi \in \mathbb{C} : \xi$ has rational coordinates$\}$. First note that

$$P := \Big\{E \in \mathbb{R} : R(E) \text{ is positive definite } \Big\} = \bigcap_{\xi_0,\xi_1 \in Q} \Big\{E \in \mathbb{R} : \sum_{i,j=0,1} \overline{\xi_i} R_{i,j}(E)\xi_j \ge 0\Big\}$$

together with the fact that $\sum_{i,j=0,1} \overline{\xi_i} R_{i,j}(E)\xi_j$ is a measurable function shows that P is a measurable set. Now suppose by contradiction that $R(E)$ is not positive definite μ^{tr} a.e. Then

$$\mu^{tr}\left(\bigcup_{\xi_0,\xi_1 \in Q} \Big\{E \in \mathbb{R} : \sum_{i,j=0,1} \overline{\xi_i} R_{i,j}(E)\xi_j < 0\Big\}\right) > 0$$

and so there is $\zeta = \begin{pmatrix} \zeta_0 \\ \zeta_1 \end{pmatrix} \in Q^2$ such that the set $B(\zeta_0, \zeta_1) := \Big\{E \in \mathbb{R} : \sum_{i,j=0,1} \overline{\zeta_i} R_{i,j}(E)\zeta_j < 0\Big\}$ satisfies

$\mu^{tr}(B(\zeta_0,\zeta_1)) > 0$. Then the following yields a contradiction:

$$\zeta^{\mathrm{T}}\mu(B(\zeta_0,\zeta_1))\zeta = \int_{B(\zeta_0,\zeta_1)} \sum_{i,j=0,1} \overline{\zeta_i} R_{i,j}(E)\zeta_j d\mu^{tr}(E) < 0.$$

□

Let $U(E)$ be the unitary matrix which diagonalizes $R(E)$, that is:

$$R(E) = U(E)^* \begin{pmatrix} r_1(E) & 0 \\ 0 & r_2(E) \end{pmatrix} U(E). \tag{2.14}$$

Note that $0 \le r_1(E), r_2(E) \le 1$ and tr $R(E) = r_1(E) + r_2(E) = 1$.

Consider the sesquilinear form on the collection of Borel measurable functions from $\mathbb{R}$ to $\mathbb{C}^2$: let $F_0(E), F_1(E), G_0(E), G_1(E) : \mathbb{R} \to \mathbb{C}$ be measurable functions and let

$$F(E) = \begin{pmatrix} F_0(E) \\ F_1(E) \end{pmatrix} \quad \text{and} \quad G(E) = \begin{pmatrix} G_0(E) \\ G_1(E) \end{pmatrix}.$$

Define:

$$\begin{aligned} \Big\langle F, G \Big\rangle &:= \int_{\mathbb{R}} \sum_{i,j=0,1} \overline{F_i(E)} G_j(E) d\mu_{i,j}(E) \qquad (2.15) \\ &= \int_{\mathbb{R}} \sum_{i,j=0,1} \overline{F_i(E)} G_j(E) \frac{d\mu_{i,j}}{d\mu^{tr}}(E) d\mu^{tr}(E) \\ &= \int_{\mathbb{R}} \langle F(E), R(E)G(E) \rangle_{\mathrm{Std}} \, d\mu^{tr}(E) \end{aligned}$$

Since $R(E)$ is positive semi-definite and Hermitian, the sesquilinear form satisfies:

1. $\Big\langle F, \alpha G + \beta H \Big\rangle = \alpha \Big\langle F, G \Big\rangle + \beta \Big\langle F, H \Big\rangle$ for all $\alpha, \beta \in \mathbb{C}$.

2. $\Big\langle F, G \Big\rangle = \overline{\Big\langle G, F \Big\rangle}$.

3. $\Big\langle F, F \Big\rangle \ge 0$.

Denote by $\mathcal{L}^2(\mathbb{R}, \mathbb{C}^2, d\mu)$ the collection of Borel measurable functions $F(E) = \begin{pmatrix} F_0(E) \\ F_1(E) \end{pmatrix}$ for which

$$\|F\| := \sqrt{\Big\langle F, F \Big\rangle} < \infty$$

and define $L^2(\mathbb{R}, \mathbb{C}^2, d\mu)$ as the set of all equivalence classes in $\mathcal{L}^2(\mathbb{R}, \mathbb{C}^2, d\mu)$ modulo the set of null function, that is, F and G belong to the same equivalence class if and only if $\|F - G\| = 0$. Then $L^2(\mathbb{R}, \mathbb{C}^2, d\mu)$ is a pre-Hilbert space with inner product given by 2.15. To show that $L^2(\mathbb{R}, \mathbb{C}^2, d\mu)$ is in fact complete, note that the map

$$\mathcal{U} : L^2(\mathbb{R}, \mathbb{C}^2, d\mu) \to L^2(\mathbb{R}, \mathbb{C}^2, \left(\begin{smallmatrix} r_1(E) & 0 \\ 0 & r_2(E) \end{smallmatrix}\right) d\mu^{tr}) \tag{2.16}$$

$$\mathcal{U} : F(E) \to U(E)F(E) \tag{2.17}$$

is unitary since

$$\int_{\mathbb{R}} \langle F(E), R(E)G(E)\rangle_{\text{Std}}\ d\mu^{tr}(E) = \int_{\mathbb{R}} \langle U(E)F(E), \begin{pmatrix} r_1(E) & 0 \\ 0 & r_2(E) \end{pmatrix} U(E)G(E)\rangle_{\text{Std}}\ d\mu^{tr}(E).$$

Moreover, it is evident that $L^2(\mathbb{R}, \mathbb{C}^2, \begin{pmatrix} r_1(E) & 0 \\ 0 & r_2(E) \end{pmatrix} d\mu^{tr}) = L^2\Big(\mathbb{R}, r_1(E) d\mu^{tr}(E)\Big) \oplus L^2\Big(\mathbb{R}, r_1(E) d\mu^{tr}(E)\Big)$, and the latter being a Hilbert space, because the direct sum of two Hilbert spaces is again a Hilbert space, it follows that $L^2(\mathbb{R}, \mathbb{C}^2, d\mu)$ is complete.

We now construct a unitary map from $\ell^2(\mathbb{Z})$ to $L^2(\mathbb{R}, \mathbb{C}^2, d\mu)$ such that h acts as multiplication by E in $L^2(\mathbb{R}, \mathbb{C}^2, d\mu)$. A direct proof of completeness can be found in Chapter 1.4 of [**C**].

We now come back to the Schrödinger operator.

Definition 2.33. *For $i = 0, 1$, let $\mu_{i,i}$ denote the spectral measure for h and δ_i, and $\mathcal{H}_i$ the corresponding cyclic subspaces. Let $\rho := \mu_{0,0} + \mu_{1,1}$. These are positive measures and $\mu_{i,i} \ll \rho$.*

As a consequence of proposition 8.14, we have:

Proposition 2.34. $\text{supp}\ \rho = \sigma(h),\ \text{supp}\ \rho_{\text{ac}} = \sigma_{\text{ac}}(h),\ \text{supp}\ \rho_{\text{sing}} = \sigma_{\text{sing}}(h)$.

By the polarization identity there are spectral measures $\mu_{0,1}$ and $\mu_{1,0}$ for h such that $\forall z \in \mathbb{C}\backslash\mathbb{R}$:

$$\langle \delta_0, (h - z)^{-1}\delta_1 \rangle = \int_{\mathbb{R}} \frac{d\mu_{0,1}(E)}{E - z} = F_{\mu_{0,1}}(z)$$
$$\langle \delta_1, (h - z)^{-1}\delta_0 \rangle = \int_{\mathbb{R}} \frac{d\mu_{1,0}(E)}{E - z} = F_{\mu_{1,0}}(z)$$

We show that the matrix measure

$$\mu = \begin{pmatrix} \mu_{0,0} & \mu_{0,1} \\ \mu_{1,0} & \mu_{1,1} \end{pmatrix} \tag{2.18}$$

satisfies the assumptions given in (2.12). But first we need the following lemma:

Lemma 2.35. *$(U_0 \mathbb{1}_0 \delta_1)(E)$ and $(U_1 \mathbb{1}_1 \delta_0)(E)$ are real valued functions.*

Proof. For all $f \in \mathcal{H}_0$, $[(U_0 h)f](E) = E[U_0 f](E)$. In particular, we have $(U_0\delta_0)(E) = \mathbb{1}(E)$ and for $k \geq 1$, $(U_0 h^k \delta_0) = E^k \mathbb{1}(E)$. Hence $\{h^k\delta_0 : k \geq 0\}$ spans $\mathcal{H}_0$ while $\{E^k : k \geq 0\}$ spans $U_0\mathcal{H}_0$. If we applied the Gram-Schmidt orthonormalizaton procedure to both of these collections we would obtain sequences in $\ell^2(\mathbb{Z})$ with real coefficients on the one hand and polynomials in $L^2(\mathbb{R}, d\mu_{0,0})$ with real coefficients. So $\mathbb{1}_0\delta_1$ is a linear combination (or limit) of sequences with real coefficients and $(U_0\mathbb{1}_0\delta_1)(E)$ is a linear combination (or limit) of polynomials with real coefficients. ☐

Proposition 2.36. *$\mu_{0,1} = \mu_{1,0}$ and these are signed measures (not complex). Moreover $\mu_{0,1} \ll \rho$. More specifically,*

$$d\mu_{0,1}(E) = (U_0\mathbb{1}_0\delta_1)(E) d\mu_{0,0}(E) = d\mu_{1,0}(E) = (U_1\mathbb{1}_1\delta_0)(E) d\mu_{1,1}(E).$$

Proof. $\forall z \in \mathbb{C}\backslash\mathbb{R}$:

$$\int_{\mathbb{R}} \frac{d\mu_{0,1}(E)}{E - z} = \langle \delta_0, (h - z)^{-1}\delta_1 \rangle = \langle \mathbb{1}_0\delta_0, (h - z)^{-1}\delta_1 \rangle = \langle \delta_0, (h - z)^{-1}\mathbb{1}_0\delta_1 \rangle = \int_{\mathbb{R}} \frac{(U_0\mathbb{1}_0\delta_1)(E) d\mu_{0,0}(E)}{E - z}.$$

Because $\left\{\frac{1}{E-z} : z \in \mathbb{C}\backslash\mathbb{R}\right\}$ is dense in $C_0(\mathbb{R})$, by approximating characteristic functions and applying the Dominated Convergence theorem allows us to conclude $d\mu_{0,1}(E) = (U_0\mathbb{1}_0\delta_1)(E)d\mu_{0,0}(E)$ and thus $\mu_{0,1} \ll \rho$.

Also, by lemma 2.35, $\mu_{0,1}$ is indeed a signed measure. Next,

$$\langle\delta_1, (h-z)^{-1}\delta_0\rangle = \overline{\langle\delta_0, (h-\overline{z})^{-1}\delta_1\rangle} = \overline{\int_{\mathbb{R}} \frac{U_0(\mathbb{1}_0\delta_1)(E)}{E-\overline{z}} d\mu_{0,0}(E)} = \int_{\mathbb{R}} \frac{U_0(\mathbb{1}_0\delta_1)(E)}{E-z} d\mu_{0,0}(E).$$

So we also have $d\mu_{1,0}(E) = (U_0\mathbb{1}_0\delta_1)(E)d\mu_{0,0}(E)$. □

As a result, the spectral matrix measure μ is Hermitian. Remains to verify that:

Lemma 2.37. *The spectral matrix measure μ associated to h is positive semi-definite.*

Proof. Let B be an arbitrary Borel set and $\xi_0, \xi_1 \in \mathbb{C}$ and consider its corresponding orthogonal projector $\mathbb{1}_B(h)$. Then:

$$\sum_{i,j=0,1} \overline{\xi}_i\xi_j\mu_{i,j}(B) = \sum_{i,j=0,1} \overline{\xi}_i\xi_j\langle\delta_i, \mathbb{1}_B(h)\delta_j\rangle = \left\|\mathbb{1}_B(h)\sum_{i=0,1}\xi_i\delta_i\right\|^2 \geq 0.$$

□

Moving forward the fundamental solutions $c(E)$ and $s(E) \in \ell(\mathbb{Z})$ to the Schrödinger equation satisfying the initial conditions $(c(0,E), c(1,E)) = (1,0)$ and $(s(0,E), s(1,E)) = (0,1)$ will play a key role.

Proposition 2.38.

(i) *For fixed $n \in \mathbb{Z}$, $c(n,E)$ and $s(n,E)$ are polynomials in E with real coefficients and of degree at most $|n|$.*

(ii) *For fixed $n \in \mathbb{Z}$, $\delta_n = c(n,h)\delta_0 + s(n,h)\delta_1$.*

(iii) $\ell_0(\mathbb{Z}) = \{P_0(h)\delta_0 + P_1(h)\delta_1 : P_0, P_1$ are polynomials$\}$.

(iv) *Every $f \in \ell_0(\mathbb{Z})$ decomposes into $P_0(h)\delta_0 + P_1(h)\delta_1$ for some unique polynomials P_0 and P_1.*

Proof.

(i) By definition, $c(0,E)$ is the function identically equal to 1 and $c(1,E)$ is the function identically equal to 0. The functions $c(n,E)$ for $n > 1$ and $n < 0$ are inductively obtained by the Schrödinger equation. So $c(2,E) = -1$, $c(3,E) = -(E - V(2))$, etc.

(ii) The identity is easily checked for $n = 0,1$. We now proceed by induction. Assume that the formula holds for $0, 1, ..., n$. Then:

$$\begin{aligned} c(n+1,h)\delta_0 + s(n+1,h)\delta_1 &= [(h-V(n))c(n,h) - c(n-1,h)]\delta_0 + [(h-V(n))s(n,h) - s(n-1,h)]\delta_1 \\ &= (h-V(n))[c(n,h)\delta_0 + s(n,h)\delta_1] - [c(n-1,h)\delta_0 + s(n-1,h)\delta_1] \\ &= (h-V(n))\delta_n - \delta_{n-1} \\ &= \delta_{n+1}. \end{aligned}$$

The identity is also proved inductively for $n < 0$.

(iii) Any $f \in \ell_0(\mathbb{Z})$ can be written as $\sum_n f(n)\delta_n = \left(\sum_n f(n)c(n,h)\right)\delta_0 + \left(\sum_n f(n)s(n,h)\right)\delta_1$. This shows that $\ell_0(\mathbb{Z}) \subset \{P_0(h)\delta_0 + P_1(h)\delta_1 : P_0, P_1 \text{ are polynomials}\}$ and the reverse inclusion is obvious.

(iv) To show that the decomposition is unique, suppose that $P_0(h)\delta_0 + P_1(h)\delta_1 = Q_0(h)\delta_0 + Q_1(h)\delta_1 \Leftrightarrow [P_0(h) - Q_0(h)]\delta_0 = [P_1(h) - Q_1(h)]\delta_1$. If $P_0(h) - Q_0(h)$ is monic of degree $n > 0$, then $[P_0(h) - Q_0(h)]\delta_0 = \delta_{-n} + \delta_n + \sum_{-n<k<n} \langle \delta_k, [P_0(h) - Q_0(h)]\delta_0 \rangle \delta_k$, which forces the degree of $P_1(h) - Q_1(h)$ to be simultaneously equal to $n-1$ and $n+1$, which is not possible. Hence $P_0(h) - Q_0(h) = P_1(h) - Q_1(h) = 0$.

□

Denote $P^2(\mathbb{R}, \mathbb{C}^2, d\mu)$ the subset of $L^2(\mathbb{R}, \mathbb{C}^2, d\mu)$ consisting of vector valued polynomials. In particular

$$\delta_n(E) := \begin{pmatrix} c(n,E) \\ s(n,E) \end{pmatrix} \in P^2(\mathbb{R}, \mathbb{C}^2, d\mu).$$

Moreover,

$$\begin{aligned} \Big\langle \delta_n(E), \delta_m(E) \Big\rangle &= \int_{\mathbb{R}} c(n,E)c(m,E)d\mu_{0,0}(E) + \int_{\mathbb{R}} c(n,E)s(m,E)d\mu_{0,1}(E) \\ &+ \int_{\mathbb{R}} s(n,E)c(m,E)d\mu_{1,0}(E) + \int_{\mathbb{R}} s(n,E)s(m,E)d\mu_{1,1}(E) \\ &= \langle c(n,h)\delta_0 + s(n,h)\delta_1, c(m,h)\delta_0 + s(m,h)\delta_1 \rangle \\ &= \langle \delta_n, \delta_m \rangle \end{aligned} \tag{2.19}$$

shows that $\{\delta_n(E) : n \in \mathbb{Z}\}$ is an orthonormal family with respect to ρ.

Theorem 2.39.

(i) *The map*

$$\mathcal{U} : \ell_0(\mathbb{Z}) \to P^2(\mathbb{R}, \mathbb{C}^2, d\mu)$$

$$\mathcal{U} : P_0(h)\delta_0 + P_1(h)\delta_1 \to P(E) = \begin{pmatrix} P_0(E) \\ P_1(E) \end{pmatrix}$$

or equivalently,

$$\mathcal{U} : f = \sum_n f(n)\delta_n \to \sum_n f(n)\delta_n(E)$$

is a surjective isometry with inverse given by

$$\mathcal{U}^{-1} : P(E) \to \Big\{ \Big\langle \delta_n(E), P(E) \Big\rangle \Big\}_n.$$

(ii) *The map extends uniquely to a unitary map*

$$\mathcal{U} : \ell^2(\mathbb{Z}) \to L^2(\mathbb{R}, \mathbb{C}^2, d\mu)$$

$$\mathcal{U} : f = \sum_{n \in \mathbb{Z}} f(n)\delta_n \to \sum_{n \in Z} f(n)\delta_n(E)$$

with the property that

$$[(\mathcal{U}h\mathcal{U}^{-1})f](E) = Ef(E). \tag{2.20}$$

Proof.

(i) The isometry comes from the fact that:

$$\langle P_0(h)\delta_0 + P_1(h)\delta_1, P_0(h)\delta_0 + P_1(h)\delta_1\rangle = \sum_{i,j=0,1}\int_{\mathbb{R}} \overline{P_i(E)}P_j(E)d\mu_{i,j}(E)$$

To show that the two different versions agree, note that if $f = P_0(h)\delta_0 + P_1(h)\delta_1 \in \ell_0(\mathbb{Z})$ and $f = \sum_n f(n)\delta_n = \left(\sum_n f(n)c(n,h)\right)\delta_0 + \left(\sum_n f(n)s(n,h)\right)\delta_1$, then $P_0(h) = \sum_n f(n)c(n,h)$ and $P_1(h) = \sum_n f(n)s(n,h)$, so that $\sum_n f(n)\delta_n(E) = P(E)$.

To show surjectivity, note that

$$\Big\langle \delta_n(E), P(E)\Big\rangle = \langle c(n,h)\delta_0 + s(n,h)\delta_1, P_0(h)\delta_0 + P_1(h)\delta_1\rangle = \langle \delta_n, P_0(h)\delta_0 + P_1(h)\delta_1\rangle$$

and $P_0(h)\delta_0 + P_1(h)\delta_1$ obviously maps to $P(E)$.

Finally, let $P(E) \in P^2(\mathbb{R}, \mathbb{C}^2, d\rho)$, $P(E) = \sum_n \Big\langle \delta_n(E'), P(E')\Big\rangle \delta_n(E)$. Then

$$\begin{aligned}
[(\mathcal{U}h\mathcal{U}^{-1})P](E) &= \sum_n \left(\Big\langle \delta_{n-1}(E'), P(E')\Big\rangle + \Big\langle \delta_{n+1}(E'), P(E')\Big\rangle + V(n)\Big\langle \delta_n(E'), P(E')\Big\rangle\right)\delta_n(E) \\
&= \sum_n \left(\Big\langle \delta_{n-1}(E') + \delta_{n+1}(E') + V(n)\delta_n(E'), P(E')\Big\rangle\right)\delta_n(E) \\
&= \sum_n \left(\Big\langle E'\delta_n(E'), P(E')\Big\rangle\right)\delta_n(E) = EP(E).
\end{aligned}$$

(ii) The extension follows by density of $\ell_0(\mathbb{Z})$ in $\ell^2(\mathbb{Z})$ and density of $P^2(\mathbb{R}, \mathbb{C}^2, d\mu)$ in $L^2(\mathbb{R}, \mathbb{C}^2, d\mu)$ (see lemma 8.3).

□

Corollary 2.40. *For all* $\phi \in C(\mathbb{R})$ *or* $B_b(\mathbb{R})$*:*

$$[(\mathcal{U}\phi(h)\mathcal{U}^{-1})f](E) = \phi(E)f(E). \tag{2.21}$$

Proof. Iterating formula (2.20) and using linearity shows that for any polynomial ϕ and $f, g \in \ell^2(\mathbb{Z})$:

$$\langle f, \phi(h)g\rangle = \Big\langle (\mathcal{U}f)(E), (\mathcal{U}\phi(h)\mathcal{U}^{-1})(E)(\mathcal{U}g)(E)\Big\rangle = \Big\langle (\mathcal{U}f)(E), \phi(E)(\mathcal{U}g)(E)\Big\rangle.$$

[Note that if V were an unbounded potential, then instead of considering polynomials, we would be considering functions of the form $f_z(x) = 1/(x-z)$.] We show that this relation holds for all $\phi \in B_b(\mathbb{R})$ or $C(\mathbb{R})$. Expanding the last relation gives:

$$\int_{\mathbb{R}} \phi(E)d\mu_{f,g}(E) = \langle f, \phi(h)g\rangle = \Big\langle (\mathcal{U}f)(E), \phi(E)(\mathcal{U}g)(E)\Big\rangle = \int_{\mathbb{R}} \phi(E)d\mu_{\mathcal{U}f,\mathcal{U}g}(E). \tag{2.22}$$

Although $\mu_{f,g}$ and $\mu_{\mathcal{U}f,\mathcal{U}g}$ are complex measures, we can split them up into a linear combination of four positive measures and apply the Dominated Convergence theorem to each component. Let $\tilde{\phi} \in C(\mathbb{R})$ and choose a sequence of polynomials converging uniformly to $\tilde{\phi}$. [As $\mu_{f,g}$ and $\mu_{\mathcal{U}f,\mathcal{U}g}$ are concentrated on $\sigma(h)$ we may restrict the analysis to a large enough interval]. It is easy to see that $\tilde{\phi} \in L^1(\mathbb{R}, \mu_{f,g}) \cap L^1(\mathbb{R}, \mu_{\mathcal{U}f,\mathcal{U}g})$ and an application of the Dominated Convergence theorem shows that (2.22) holds for $\tilde{\phi}$. Then approximating characteristic functions with continuous functions shows that in fact $\mu_{f,g} = \mu_{\mathcal{U}f,\mathcal{U}g}$. Therefore (2.22) also holds for $\phi \in B_b(\mathbb{R})$. □

Definition 2.41. *For $E \in \operatorname{supp} \rho$, we form the matrix*

$$R(E) = \begin{pmatrix} \frac{d\mu_{0,0}}{d\rho}(E) & \frac{d\mu_{0,1}}{d\rho}(E) \\ \frac{d\mu_{1,0}}{d\rho}(E) & \frac{d\mu_{1,1}}{d\rho}(E) \end{pmatrix} \tag{2.23}$$

The theorem on the differentiation of measures gives for $i, j = 0, 1$:

$$R_{i,j}(E) = \lim_{\varepsilon \downarrow 0} \frac{P_{\mu_{i,j}}(E + i\varepsilon)}{P_\rho(E + i\varepsilon)} = \lim_{\varepsilon \downarrow 0} \frac{\operatorname{Im} \langle \delta_i, (h - E - i\varepsilon)^{-1} \delta_j \rangle}{\operatorname{Im} \left[\langle \delta_0, (h - E - i\varepsilon)^{-1} \delta_0 \rangle + \langle \delta_1, (h - E - i\varepsilon)^{-1} \delta_1 \rangle \right]} \quad \rho - a.e.\ E \tag{2.24}$$

Let $U(E)$ be the unitary matrix such that $R(E) = U(E) \begin{pmatrix} r_1(E) & 0 \\ 0 & r_2(E) \end{pmatrix} U^*(E)$. As discussed at the beginning of this section, we have the following isomorphism:

$$\tilde{\mathcal{U}} : L^2(\mathbb{R}, \mathbb{C}^2, d\mu) \to L^2(\mathbb{R}, \mathbb{C}^2, \begin{pmatrix} r_1(E) & 0 \\ 0 & r_2(E) \end{pmatrix} d\rho) = L^2(\mathbb{R}, r_1(E) d\rho(E)) \oplus L^2(\mathbb{R}, r_2(E) d\rho(E)) \tag{2.25}$$

$$(\tilde{\mathcal{U}} F)(E) = U(E) F(E) \tag{2.26}$$

Definition 2.42. *Let $\mathcal{R}(n, m, E) := \langle \delta_n(E), R(E) \delta_m(E) \rangle_{\mathrm{Std}}$.*

A reformulation of equation (2.19) reads $\int_{\mathbb{R}} \mathcal{R}(n, m, E) d\rho(E) = \langle \delta_n, \delta_m \rangle$. Moreover, by formula (2.21), one easily works out that for all $\phi \in C(\mathbb{R})$ or $B_b(\mathbb{R})$:

$$\langle f, \phi(h) g \rangle = \sum_{n,m \in \mathbb{Z}} \overline{f(n)} g(n) \int_{\mathbb{R}} \phi(E) \mathcal{R}(n, m, E) d\rho(E). \tag{2.27}$$

We can give a precise characterisation of the multiplicity of h as follows:

Definition 2.43. *For $i = 1, 2$, let $\varepsilon_i := \{E \in \operatorname{supp} \rho : R(E)$ has rank $i\}$.*

Note that the ε_i are Borel measurable sets since $\varepsilon_1 = \{\det R(E) = 0\}$ and $\varepsilon_2 = \{\det R(E) > 0\}$ and $\det R(E) = \frac{d\mu_{0,0}}{d\rho}(E) \frac{d\mu_{1,1}}{d\rho}(E) - \frac{d\mu_{0,1}}{d\rho}(E) \frac{d\mu_{1,0}}{d\rho}(E)$ is a measurable function.

$R(E)$ is hermitian so it has a decomposition in terms of its eigenvectors $\{e_i(E)\}_{i=1}^{N(E)}$ of the form

$$R(E) = \sum_{i=1}^{N(E)} r_i(E) e_i(E) \langle e_i(E), \cdot \rangle, \quad \text{where } N(E) = \begin{cases} 1 & \text{on } \varepsilon_1 \\ 2 & \text{on } \varepsilon_2 \end{cases} \tag{2.28}$$

We introduce the generalized eigenfunctions:

Definition 2.44. *Let $f_i(E) \in \ell(\mathbb{Z})$ be defined by $f_i(n, E) := \sqrt{r_i(E)} \langle \delta_n(E), e_i(E) \rangle_{\mathrm{Std}}$.*

Consequently

$$\mathcal{R}(n,m,E) = \sum_{i=1}^{N(E)} r_i(E)\langle \delta_n(E), e_i(E)\rangle_{\text{Std}} \langle e_i(E), \delta_m(E)\rangle_{\text{Std}} = \sum_{i=1}^{N(E)} \overline{f_i(n,E)} f_i(m,E). \tag{2.29}$$

Note that we have used the fact that $\mathcal{R}(n,m,E) = \overline{\mathcal{R}(n,m,E)}$.

Proposition 2.45. *The $f_i(E)$ solve the Schrödinger equation and for $E \in \varepsilon_2$, $f_1(E)$ and $f_2(E)$ are linearly independent.*

Proof.

$$f_i(n-1,E) + f_i(n+1,E) + V(n) f_i(n,E) = \sqrt{r_i(E)}\langle \delta_{n-1}(E) + \delta_{n+1}(E) + V(n)\delta_n(E), e_i(E)\rangle_{\text{Std}}$$

$$= \sqrt{r_i(E)}\langle E\delta_n(E), e_i(E)\rangle_{\text{Std}} = E f_i(n,E).$$

Next we show that the Wronskian is non-zero for $E \in \varepsilon_2$:

$$W(f_1(E), f_2(E)) = f_1(0,E) f_2(1,E) - f_1(1,E) f_2(0,E)$$

$$= \sqrt{r_1(E) r_2(E)}\Big(\langle \delta_0(E), e_1(E)\rangle_{\text{Std}}\langle \delta_1(E), e_2(E)\rangle_{\text{Std}} - \langle \delta_1(E), e_1(E)\rangle_{\text{Std}}\langle \delta_0(E), e_2(E)\rangle_{\text{Std}}\Big)$$

$$= \sqrt{r_1(E) r_2(E)} \det U(E).$$

where we used the fact that the columns of $U(E)$ are $e_1(E)$ and $e_2(E)$. □

Let $\mathbb{1}_{\varepsilon_i}(h)$ denote the orthogonal projections onto the subspaces correpsonding to ε_i. Note that $\varepsilon_1 \cup \varepsilon_2 = \sigma(h)$ and $\varepsilon_1 \cap \varepsilon_2 = \varnothing$ implies that $\mathbb{1}_{\varepsilon_1}(h) \oplus \mathbb{1}_{\varepsilon_2}(h)$ is the identity operator on $\ell^2(\mathbb{Z})$.

Theorem 2.46. *$h = h\mathbb{1}_{\varepsilon_1}(h) \oplus h\mathbb{1}_{\varepsilon_2}(h)$ and the spectral multiplicity of $h\mathbb{1}_{\varepsilon_1}(h)$ is one whereas the spectral multiplicity of $h\mathbb{1}_{\varepsilon_2}(h)$ is two. The unitary map*

$$\ell^2(\mathbb{Z}) \to L^2(\mathbb{R}, \mathbb{1}_{\varepsilon_1} d\rho) \bigoplus L^2(\mathbb{R}, \mathbb{C}^2, \mathbb{1}_{\varepsilon_2} d\rho) \tag{2.30}$$

$$g = \{g(n)\}_{n\in\mathbb{Z}} \to \begin{cases} \sum_n f_1(n,E) g(n) & \text{on } \varepsilon_1 \\ \Big(\sum_n f_1(n,E) g(n), \sum_n f_2(n,E) g(n)\Big) & \text{on } \varepsilon_2 \end{cases} \tag{2.31}$$

is such that $h\mathbb{1}_{\varepsilon_1}(h)$ is unitarily equivalent to multiplication by E on $L^2(\mathbb{R}, \mathbb{1}_{\varepsilon_1} d\rho)$ and $h\mathbb{1}_{\varepsilon_2}(h)$ is unitarily equivalent to multiplication by E on $L^2(\mathbb{R}, \mathbb{C}^2, \mathbb{1}_{\varepsilon_2} d\rho)$.

The proof of this theorem is essentially the result of combining equations (2.27) and (2.29). Namely:

$$\begin{aligned}\langle u, \phi(h)v\rangle &= \sum_{n,m} \overline{u(n)}v(m)\int_{\varepsilon_1} \phi(E)\mathcal{R}(n,m,E)d\rho + \sum_{n,m} \overline{u(n)}v(m)\int_{\varepsilon_2} \phi(E)\mathcal{R}(n,m,E)d\rho \\ &= \sum_{n,m} \overline{u(n)}v(m)\int_{\varepsilon_1} \phi(E)\overline{f_1(n,E)}f_1(m,E)d\rho \\ &+ \sum_{n,m} \overline{u(n)}v(m)\int_{\varepsilon_2} \phi(E)\overline{f_1(n,E)}f_1(m,E)d\rho \\ &+ \sum_{n,m} \overline{u(n)}v(m)\int_{\varepsilon_2} \phi(E)\overline{f_2(n,E)}f_2(m,E)d\rho.\end{aligned}$$

Finally, we have the equivalent of proposition 3.14, proved in the same way:

Proposition 2.47. *For every $\varepsilon > 0$, there exists for $\rho - a.e.$ E a constant $c = c(E) > 0$ such that $|f_i(n,E)| \le c\langle n\rangle^{1/2+\varepsilon}$ for all $n \in \mathbb{Z}$.*

2.5 The Generalized Eigenfunction Expansion for the Laplacian

We apply the machinery developed in the previous section in the case of the Laplacian. For all $n \in \mathbb{Z}$, $E \in [-2, 2]$:

$$\begin{pmatrix} c(n+1,E) \\ c(n,E) \end{pmatrix} = \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} c(1,E) \\ c(0,E) \end{pmatrix} = \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} s(n+1,E) \\ s(n,E) \end{pmatrix} = \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} s(1,E) \\ s(0,E) \end{pmatrix} = \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The eigenvalues of the transfer matrix $A(E,n) = \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix}$ are $\frac{E \pm i\sqrt{4-E^2}}{2}$. Let $\theta(E) = \arctan\left(\frac{\sqrt{4-E^2}}{E}\right)$. Then

$$\frac{E \pm i\sqrt{4-E^2}}{2} = \begin{cases} e^{\pm i\theta(E)} & \text{if } E \in (0,2) \\ e^{\pm i\theta(E)+i\pi} & \text{if } E \in (-2,0) \\ e^{\pm i\pi/2} & \text{if } E = 0 \end{cases}$$

If the decomposition is $A(E,n) = P^{-1}(E)D(E)P(E)$, then for $E \in (0,2)$:

$$P^{-1}(E) = \begin{pmatrix} 1 & 1 \\ e^{i\theta(E)} & e^{-i\theta(E)} \end{pmatrix} \quad D(E) = \begin{pmatrix} e^{-i\theta(E)} & 0 \\ 0 & e^{i\theta(E)} \end{pmatrix} \quad P(E) = \frac{i}{\sqrt{4-E^2}} \begin{pmatrix} e^{-i\theta(E)} & -1 \\ -e^{i\theta(E)} & 1 \end{pmatrix}$$

Therefore the fundamental solutions for $E \in (0,2)$:

$$s(n,E) = \frac{i}{\sqrt{4-E^2}}\left(e^{-i\theta(E)n} - e^{i\theta(E)n}\right) = \frac{2\sin(\theta(E)n)}{\sqrt{4-E^2}} \tag{2.32}$$

$$c(n,E) = -\frac{2\sin(\theta(E)(n-1))}{\sqrt{4-E^2}} \tag{2.33}$$

Similarly for $E \in (-2, 0)$, they are:

$$s(n, E) = (-1)^n \frac{2\sin(\theta(E)n)}{\sqrt{4 - E^2}} \tag{2.34}$$

$$c(n, E) = -(-1)^{n-1} \frac{2\sin(\theta(E)(n-1))}{\sqrt{4 - E^2}} \tag{2.35}$$

and for $E = 0$:

$$s(n, 0) = \sin(\pi n/2) \tag{2.36}$$

$$c(n, 0) = -\sin(\pi(n-1)/2) \tag{2.37}$$

Note that we have the following relationships:

$$s(n, E) = -s(-n, E), \qquad c(n, E) = -s(n-1, E). \tag{2.38}$$

Moreover:

$$\lim_{E \to +2} s(n, E) = \lim_{E \to +2} \frac{2\sin(\theta(E)n)}{\theta(E)n} \frac{\theta(E)n}{\tan\theta(E)} \frac{\frac{\sqrt{4-E^2}}{E}}{\sqrt{4 - E^2}} = n \tag{2.39}$$

shows that $s(n, 2) = n$. Similarly

$$\begin{aligned} c(n, 2) &= -(n-1) \\ s(n, -2) &= -(-1)^n n \\ c(n, -2) &= (-1)^{n-1}(n-1). \end{aligned}$$

For $n > 0$:

$$\begin{aligned} s(n, E) &= \frac{-1}{\sqrt{E^2 - 4}} \left(\left(\frac{E - \sqrt{E^2 - 4}}{2} \right)^n - \left(\frac{E + \sqrt{E^2 - 4}}{2} \right)^n \right) \\ &= \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} E^i \sqrt{E^2 - 4}^{\,n-i-1} \left(1 - (-1)^{n-i}\right). \end{aligned}$$

For n even,

$$\begin{aligned} s(n, E) &= \frac{2}{2^n} \sum_{i=0}^{n/2-1} \binom{n}{2i+1} E^{2i+1} (E^2 - 4)^{n/2-1-i} \\ &= \sum_{i=0}^{n/2-1} \left[\sum_{j=0}^{n/2-1-i} \binom{n}{2i+1} \binom{n/2-1-i}{j} \left(\frac{E}{2}\right)^{n-2j-1} (-1)^j \right] \\ &= \sum_{k=0}^{n/2-1} \left[(-1)^k \sum_{l=0}^{n/2-1-k} \binom{n}{2l+1} \binom{n/2-1-l}{k} \right] \left(\frac{E}{2}\right)^{n-2k-1} \\ &= \sum_{k=0}^{n/2-1} \left[(-1)^k \sum_{l=k}^{n/2-1} \binom{n}{2l+1} \binom{l}{k} \right] \left(\frac{E}{2}\right)^{n-2k-1} \end{aligned}$$

Notice that we have used the fact that:

$$\begin{aligned}
&\sum_{l=0}^{n/2-1-k} \binom{n}{2l+1}\binom{n/2-1-l}{k} \\
&= \binom{n}{1}\binom{n/2-1}{k} + \binom{n}{3}\binom{n/2-2}{k} + \ldots + \binom{n}{n-2k-3}\binom{k+1}{k} + \binom{n}{n-2k-1}\binom{k}{k} \\
&= \binom{n}{2k+1}\binom{k}{k} + \binom{n}{2k+3}\binom{k+1}{k} + \ldots + \binom{n}{n-3}\binom{n/2-2}{k} + \binom{n}{n-1}\binom{n/2-1}{k} \\
&= \sum_{l=k}^{n/2-1} \binom{n}{2l+1}\binom{l}{k}
\end{aligned}$$

For n odd,

$$\begin{aligned}
s(n,E) &= \frac{2}{2^n} \sum_{i=0}^{(n-1)/2} \binom{n}{2i} E^{2i}(E^2-4)^{(n-1)/2-i} \\
&= \sum_{i=0}^{(n-1)/2} \left[\sum_{j=0}^{(n-1)/2-i} \binom{n}{2i}\binom{(n-1)/2-i}{j} \left(\frac{E}{2}\right)^{n-2j-1} (-1)^j \right] \\
&= \sum_{k=0}^{(n-1)/2} \left[(-1)^k \sum_{l=0}^{(n-1)/2-k} \binom{n}{2l}\binom{(n-1)/2-l}{k} \right] \left(\frac{E}{2}\right)^{n-2k-1} \\
&= \sum_{k=0}^{(n-1)/2} \left[(-1)^k \sum_{l=k}^{(n-1)/2} \binom{n}{2l+1}\binom{l}{k} \right] \left(\frac{E}{2}\right)^{n-2k-1}
\end{aligned}$$

Hence for all $n > 0$

$$s(n,E) = \sum_{k=0}^{[\![n/2]\!]} \left[(-1)^k \sum_{l=k}^{[\![n/2]\!]} \binom{n}{2l+1}\binom{l}{k} \right] \left(\frac{E}{2}\right)^{n-2k-1} \tag{2.40}$$

where $[\![x]\!] = \sup\{n \in \mathbb{Z} : n < x\}$.

We use the boundary values to calculate the spectral measures. For $E \in [-2,2]$:

$$\frac{d\mu_{i,j}}{dE} = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \operatorname{Im} \langle \delta_i, (h_0 - E - i\varepsilon)^{-1}\delta_j \rangle = \begin{pmatrix} \frac{1}{\pi}\frac{1}{\sqrt{4-E^2}} & \frac{1}{2\pi}\frac{E}{\sqrt{4-E^2}} \\ \frac{1}{2\pi}\frac{E}{\sqrt{4-E^2}} & \frac{1}{\pi}\frac{1}{\sqrt{4-E^2}} \end{pmatrix} \tag{2.41}$$

with

$$\rho(B) = \mu_{0,0}(B) + \mu_{1,1}(B) = \int_B \mathbb{1}_{[-2,2]}(E) \frac{2dE}{\pi\sqrt{4-E^2}}. \tag{2.42}$$

Using $\frac{d\mu_{i,j}}{d\rho} = \frac{d\mu_{i,j}}{dE}\frac{dE}{d\rho}$ gives:

$$R(E) = \frac{d\mu_{i,j}}{d\rho}(E) = \begin{pmatrix} 1/2 & E/4 \\ E/4 & 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{2} - \frac{E}{4} & 0 \\ 0 & \frac{1}{2} + \frac{E}{4} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \sum_{i=1}^{N(E)} r_i(E) e_i(E) \langle e_i(E), \cdot \rangle \tag{2.43}$$

where

$$e_1(E) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \quad e_2(E) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad r_1(E) = \frac{1}{2} - \frac{E}{4} \quad r_2(E) = \frac{1}{2} + \frac{E}{4} \tag{2.44}$$

$$\varepsilon_1 = \{E \in [-2, 2] : \det R(E) = 0\} = \{\pm 2\} \quad \varepsilon_2 = \{E \in [-2, 2] : \det R(E) > 0\} = (-2, 2) \tag{2.45}$$

And the generalized eigenfunctions are:

$$f_1(n, E) = \sqrt{r_1(E)} \langle \delta_n(E), e_1(E) \rangle = \sqrt{1/2 - E/4} \frac{c_n(E) - s_n(E)}{\sqrt{2}}. \tag{2.46}$$

$$f_2(n, E) = \sqrt{r_2(E)} \langle \delta_n(E), e_2(E) \rangle = \sqrt{1/2 + E/4} \frac{c_n(E) + s_n(E)}{\sqrt{2}}. \tag{2.47}$$

3 The Discrete Schrödinger Operator on the Half Line

3.1 The Laplacian, its Spectrum and Boundary Values of the Resolvent

It is obvious that for $1 \leq p \leq \infty$, the shift operator $R : \ell^p(\mathbb{Z}_+) \to \ell^p(\mathbb{Z}_+)$ is an isometry. $L : \ell^p(\mathbb{Z}_+) \to \ell^p(\mathbb{Z}_+)$ is not an isometry however, but satisfies $\|L\|_{\ell^p} = 1$. Consequently $h_0 \ell^p(\mathbb{Z}_+) \subset \ell^p(\mathbb{Z}_+)$ and $\|h_0\|_{\ell^p} \leq \|R\|_{\ell^p} + \|L\|_{\ell^p} = 2$. In fact:

Proposition 3.1. *For $1 \leq p \leq \infty$, $h_0 : \ell^p(\mathbb{Z}_+) \to \ell^p(\mathbb{Z}_+)$ is a bounded linear operator and $\|h_0\|_{\ell^p} = 2$.*

This proposition is proven exactly in the same as for the full line case (proposition 2.3). We obviously need to check that:

Proposition 3.2. *$h_0 : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ is self-adjoint.*

Proof. To show self-adjointness, it is enough to show that $\langle \delta_n | h \delta_m \rangle = \langle h \delta_n | \delta_m \rangle$ for all $n, m \in \mathbb{Z}_+$ because of linearity and continuity of the inner product and of h_0. For $|n - m| \geq 2$ or $n = m$, clearly $\langle \delta_n | h \delta_m \rangle = \langle h \delta_n | \delta_m \rangle = 0$, while if $|n - m| = 1$, $\langle \delta_n | h \delta_m \rangle = \langle h \delta_n | \delta_m \rangle = 1$. □

We can also give an explicit basis for $E(z)$.

Proposition 3.3. *The following are a basis for $E(z)$:*

(i) *For $z \in \mathbb{C} \backslash [-2, 2]$: $u = \{u(n) = (\frac{z + \sqrt{z^2 - 4}}{2})^n - (\frac{z + \sqrt{z^2 - 4}}{2})^{-n}\}_{n \in \mathbb{Z}_+}$.*

(ii) *For $z \in (-2, 2)$: $u = \{u(n) = \sin(n\theta)\}_{n \in \mathbb{Z}_+}$, where θ is such that $z = 2\cos(\theta)$.*

(iii) *For $z = 2$: $u = \{u(n) = n\}_{n \in \mathbb{Z}_+}$.*

(iv) *For $z = -2$: $u = \{u(n) = (-1)^{n+1} n\}_{n \in \mathbb{Z}_+}$.*

Proof. For $\lambda \in \mathbb{C}$ consider the sequence $\{u(n) = \lambda^n - \lambda^{-n}\}_{n \in \mathbb{Z}_+}$. Then $(h_0 u)(n) = (\lambda + \lambda^{-1}) u(n) = z u(n)$. Solving for λ in terms of z gives $\lambda = \frac{z \pm \sqrt{z^2 - 4}}{2}$. The solution for $z \in [-2, 2]$ can easily be verified directly. □

Remark 3.1. *On the modified half line* $\Gamma := \{N, N+1, N+2, ...\}$, *the basis for* $E(z)$ *would be* $\{u(n) = (\frac{z+\sqrt{z^2-4}}{2})^{n-N+1} - (\frac{z+\sqrt{z^2-4}}{2})^{-n+N-1}\}_{n\in\Gamma}$ *for* $z \in \mathbb{C}\backslash[-2,2]$ *and* $\{u(n) = \sin\Big((n-N+1)\theta\Big)\}_{n\in\Gamma}$ *for* $z \in (-2,2)$.

We investigate the spectrum.

Proposition 3.4. *For* $1 \leq p \leq \infty$, $[-2,2] \subset \sigma(h_0)$ *as an operator from* $\ell^p(\mathbb{Z}_+)$ *to itself.*

Proof. The argument is analogous to proposition 2.5.

Let $E \in (-2,2)$ and let θ be such that $2\cos\theta = E$. Consider the truncated Weyl sequence:

$$\forall M \in \mathbb{Z}_+ : \quad u^{(m)}(n) := \begin{cases} \sin(n\theta) & 1 \leq n \leq m \\ 0 & n > m \end{cases}$$

Then

$$((h_0 - E)u^{(m)})(n) = \begin{cases} 0 & n = 1 \\ 0 & 1 < n < m \\ -\sin\Big((m+1)\theta\Big) & n = m \\ \sin(m\theta) & n = m+1 \\ 0 & n > m+1 \end{cases}$$

As a result of lemma 8.2, $\lim_{m\to\infty} \|u^{(m)}\|_p^p = \lim_{m\to\infty} \sum_{n=1}^{m} |\sin(n\theta)|^p = \infty$ for $1 \leq p < \infty$. Hence $\lim_{m\to\infty} \frac{\|(h-E)u^{(m)}\|_p}{\|u^{(m)}\|_p} = 0$. The case $p = \infty$ follows from proposition 3.3, because $\sigma_{\mathrm{p}}(h_0) = (-2,2)$. □

For the next result, we recall a few basic facts about Fourier series. Every $f \in L^2([-\pi,\pi], \frac{d\theta}{2\pi})$, admits the following decomposition:

$$f(\theta) = \sum_{n=0}^{\infty} \Big(a_n \cos(n\theta) + b_n \sin(n\theta)\Big)$$

for some $a_n, b_n \in \mathbb{C}$. Moreover any function can be decomposed as the sum of an even and odd function : $f(\theta) = [f(\theta) + f(-\theta)]/2 + [f(\theta) - f(-\theta)]/2$. Let $L^2_{\mathrm{e}}([-\pi,\pi], \frac{d\theta}{2\pi})$ and $L^2_{\mathrm{o}}([-\pi,\pi], \frac{d\theta}{2\pi})$ be the subspaces of $L^2([-\pi,\pi], \frac{d\theta}{2\pi})$ of even and odd functions. These subspaces are closed and orthogonal. In fact the map $e^{in\theta} \to (\cos(n\theta), \sin(n\theta))$ shows that $L^2([-\pi,\pi], \frac{d\theta}{2\pi}) = L^2_{\mathrm{e}}([-\pi,\pi], \frac{d\theta}{2\pi}) \oplus L^2_{\mathrm{o}}([-\pi,\pi], \frac{d\theta}{2\pi})$. $\{1, \sqrt{2}\cos(\theta), \sqrt{2}\cos(2\theta), ...\}$ and $\{\sqrt{2}\sin(\theta), \sqrt{2}\sin(2\theta), ...\}$ are complete orthonormal sets for $L^2_{\mathrm{e}}([-\pi,\pi], \frac{d\theta}{2\pi})$ and $L^2_{\mathrm{o}}([-\pi,\pi], \frac{d\theta}{2\pi})$.

Define the map

$$\tilde{\mathcal{F}}_s : \ell_0(Z_+) \to L^2_{\mathrm{o}}\bigg([-\pi,\pi], \frac{d\theta}{2\pi}\bigg)$$
$$\tilde{\mathcal{F}}_s : u \to \hat{u}(\theta) = \sum_{n\geq 1} u(n)\sqrt{2}\sin(n\theta).$$

The map is well defined and inner product preserving:

$$\langle u, v\rangle = \sum_{n\geq 1} \overline{u_n} v_n = \sum_{n,m\geq 1} \overline{u_n} v_m \delta_{nm} = \sum_{n,m\geq 1} \overline{u_n} v_m \langle\sqrt{2}\sin(n\theta), \sqrt{2}\sin(m\theta)\rangle = \langle\hat{u}(\theta), \hat{v}(\theta)\rangle.$$

Therefore $\tilde{\mathcal{F}}_s$ extends uniquely to a map from $\ell^2(\mathbb{Z}_+)$ to $L^2_{\text{o}}([-\pi,\pi],\frac{d\theta}{2\pi})$ and is surjective from the general theory of Fourier series. The inverse is:

$$\tilde{\mathcal{F}}_s^{-1} : L^2_{\text{o}}\Big([-\pi,\pi],\frac{d\theta}{2\pi}\Big) \to \ell^2(Z_+)$$
$$\tilde{\mathcal{F}}_s^{-1} : f \to f(n) = \langle\sqrt{2}\sin(n\theta), f(\theta)\rangle = \int_{-\pi}^{\pi}\sqrt{2}\sin(n\theta)f(\theta)\frac{d\theta}{2\pi}.$$

However since the map $L^2_{\text{o}}\Big([-\pi,\pi],\frac{d\theta}{2\pi}\Big) \to L^2([0,\pi],d\theta)$, $\sqrt{2}\sin(n\theta) \to \sqrt{2/\pi}\sin(n\theta)$ is unitary, it is more customary to write the Fourier sine transform as follows:

$$\mathcal{F}_s : \ell_0(Z_+) \to L^2([0,\pi],d\theta)$$
$$\mathcal{F}_s : u \to \hat{u}(\theta) = \sum_{n\geq 1} u(n)\sqrt{2/\pi}\sin(n\theta).$$

with inverse

$$\mathcal{F}_s^{-1} : L^2([0,\pi],d\theta) \to \ell^2(Z_+)$$
$$\mathcal{F}_s^{-1} : f \to f(n) = \langle\sqrt{2/\pi}\sin(n\theta), f(\theta)\rangle = \int_0^{\pi}\sqrt{2/\pi}\sin(n\theta)f(\theta)d\theta.$$

Proposition 3.5. *$h_0 : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ is unitarily equivalent via the Fourier sine transform $\mathcal{F}_s$ to the multiplication operator $M_g : L^2([0,\pi],d\theta) \to L^2([0,\pi],d\theta)$ by the function $g(\theta) = 2\cos(\theta)$, that is:*

$$\Big([\mathcal{F}_s h_0 \mathcal{F}_s^{-1}]f\Big)(\theta) = 2\cos(\theta)f(\theta).$$

In particular, $\sigma(h_0) = \text{ess ran }(2\cos(\theta)) = [-2,2]$.

Proof. For $f \in L^2([0,\pi],d\theta)$, we have :

$$\Big([h_0\mathcal{F}_s^{-1}]f\Big)(n) = \Big\langle\sqrt{2/\pi}\sin\Big((n-1)\theta\Big), f(\theta)\Big\rangle + \Big\langle\sqrt{2/\pi}\sin\Big((n+1)\theta\Big), f(\theta)\Big\rangle$$
$$= \Big\langle\sqrt{2/\pi}\sin(n\theta), 2\cos(\theta)f(\theta)\Big\rangle$$

and so

$$\Big([\mathcal{F}_s h_0\mathcal{F}_s^{-1}]f\Big)(\theta) = \sum_{n\in\mathbb{Z}}\Big\langle\sqrt{2/\pi}\sin(n\theta), 2\cos(\theta)f(\theta)\Big\rangle\sqrt{2/\pi}\sin(n\theta) = 2\cos(\theta)f(\theta).$$

□

Remark 3.2. *On the modified half line $\Gamma = \{N, N+1, N+2, ...\}$, the Fourier sine transform is $\hat{u}(\theta) = \sqrt{2/\pi}\sin\Big((n-N+1)\theta\Big)$.*

There is another way of writing out the Spectral theorem for h_0. The change of variable $\theta = \arccos(x)$ shows that there is a unitary map

$$\tilde{U} : \ell^2(\mathbb{Z}_+) \to L^2([-1,1], dx) \tag{3.1}$$

$$u = \{u(n)\}_{n\in\mathbb{Z}} \to (\tilde{U}u)(x) = \sqrt{\frac{2}{\pi\sqrt{1-x^2}}} \sum_{n\geq 1} u(n)\sin(n\arccos(x)). \tag{3.2}$$

and h_0 is unitarily equivalent to multiplication by x.

Proposition 3.6. *Let μ_{δ_1} be the spectral measure for δ_1 and h_0. Then μ_{δ_1} is purely absolutely continuous and*

$$\frac{d\mu_{\delta_1}}{dx}(x) = \frac{1}{2\pi}\sqrt{4-x^2}\mathbb{1}_{[-2,2]}(x). \tag{3.3}$$

Proof. $\langle \delta_1, f(h_0)\delta_1\rangle = \int_0^\pi \frac{2}{\pi}\sin^2(\theta) f(2\cos\theta)d\theta$. Letting $2\cos\theta = x$, $d\theta = -\frac{dx}{\sqrt{4-x^2}}$ for $\theta \in [0,\pi]$. Hence $\int_{\mathbb{R}} f(x)d\mu_{\delta_1}(x) = \langle \delta_1, f(h_0)\delta_1\rangle = \int_{-2}^2 f(x)\sqrt{4-x^2}dx/(2\pi) = \int_{\mathbb{R}} f(x)\sqrt{4-x^2}\mathbb{1}_{[-2,2]}(x)dx/(2\pi)$. □

Note that μ_{δ_1} is a probability measure and the Lebesgue measure is absolutely continuous w.r.t. μ_{δ_1}. In particular $dy = \frac{dy}{d\mu_{\delta_1}}d\mu_{\delta_1}(y) = \frac{2\pi}{\sqrt{4-y^2}}d\mu_{\delta_1}(y)$. Going back to 3.2, if we do the change of variable $x = y/2$, then

$$\begin{aligned}\int_{-1}^1 (\tilde{U}u)(x)(\tilde{U}v)(x)dx &= \sum_{n,m\geq 1} u(n)v(m)\int_{-1}^1 \frac{2}{\pi}\frac{\sin(n\arccos(x))\sin(m\arccos(x))}{\sqrt{1-x^2}}dx \\ &= \sum_{n,m\geq 1} u(n)v(m)\int_{-2}^2 \frac{2}{\pi}\frac{\sin(n\arccos(y/2))\sin(m\arccos(y/2))}{\sqrt{4-y^2}}dy \\ &= \sum_{n,m\geq 1} u(n)v(m)\int_{-2}^2 \frac{\sin(n\arccos(y/2))\sin(m\arccos(y/2)}{1-(y/2)^2}d\mu_{\delta_1}(y)\end{aligned}$$

Also note that $\frac{\sin(\arccos(y/2))}{\sqrt{1-(y/2)^2}} = 1$. Therefore we have established the unique unitary satisfying $[(Uh_0U^{-1})f](y) = yf(y)$ for $f \in L^2(\mathbb{R}, d\mu_{\delta_1})$ and $(U\delta_1)(y) = \mathbb{1}(y)$:

Theorem 3.7. *The unitary map $U : \ell^2(\mathbb{Z}_+) \to L^2([-2,2], \mu_{\delta_1})$*

$$u = \{u(n)\} \to (Uu)(y) = \sum_{n\geq 1} \frac{\sin(n\arccos(y/2))}{\sqrt{1-(y/2)^2}}. \tag{3.4}$$

is the spectral theorem for the cyclic vector δ_1 of h_0 on $\ell^2(\mathbb{Z}_+)$.

We now investigate the boundary values of the resolvent.

Proposition 3.8. *For $E \in \mathbb{R}$ and for all $n \in \mathbb{Z}$:*

$$\lim_{\varepsilon\to\pm 0}\langle \delta_1, (h_0 - E - i\varepsilon)^{-1}\delta_1\rangle = \pm\frac{i}{2}\sqrt{4-E^2}. \tag{3.5}$$

Proof.

Let $z = E + i\varepsilon$, $\varepsilon \neq 0$. We have:

$$\langle \delta_1, (h_0-z)^{-1}\delta_1\rangle = \langle \sqrt{2}\sin(\theta), \frac{1}{2\cos\theta - z}\sqrt{2}\sin(\theta)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin^2(\theta)}{2\cos\theta - z}d\theta = \frac{-1}{2}\frac{1}{2\pi i}\oint_{|\omega|=1}\frac{(\omega-\omega^{-1})^2}{\omega^2-\omega z+1}d\omega.$$

Let ω_1 and ω_2 be the two roots of $\omega^2 - \omega z + 1 = 0$. We showed in proposition 2.11 that $\omega_1\omega_2 = 1$, together with the fact that $\sigma(h_0) = [-2, 2]$ implies that $|\omega_1| < 1$ and $|\omega_2| > 1$. Applying Cauchy's integral formula gives:

$$\langle \delta_1, (h_0 - z)^{-1}\delta_1 \rangle = \frac{-1}{2}\frac{(\omega_1 - \omega_1^{-1})^2}{\omega_1 - \omega_2} = \frac{-1}{2}\frac{(\omega_1 + \omega_1^{-1})^2 - 4}{\omega_1 - \omega_2} = \frac{-1}{2}\frac{z^2 - 4}{\omega_1 - \omega_2} = \pm\frac{1}{2}\sqrt{z^2 - 4}$$

depending on whether $\omega_1 = \frac{z+\sqrt{z^2-4}}{2}$ or $\omega_1 = \frac{z-\sqrt{z^2-4}}{2}$.

$$\sqrt{z^2 - 4} = \begin{cases} i\sqrt{\varepsilon^2 + 4} & \text{when } E = 0 \\ \frac{1}{\sqrt{2}}\left(\operatorname{sign}(E\varepsilon)\sqrt{\sqrt{(E^2 - \varepsilon^2 - 4)^2 + (2E\varepsilon)^2} + (E^2 - \varepsilon^2 - 4)} + i\sqrt{\sqrt{(E^2 - \varepsilon^2 - 4)^2 + (2E\varepsilon)^2} - (E^2 - \varepsilon^2 - 4)}\right) & \text{when } E \neq 0 \end{cases}$$

Using lemma 1.12 to adjust the signs gives:

$$\langle \delta_1, (h_0 - z)^{-1}\delta_1 \rangle = \frac{\operatorname{sign}(\varepsilon)\sqrt{z^2 - 4}}{2}.$$

The result follows by taking the limit. □

Proposition 3.9. *Let $z = E + i\varepsilon$, $E \in [-2, 2], \varepsilon \neq 0$. Then $\forall n, m \geq 1$:*

$$\lim_{\varepsilon \downarrow 0}\langle \delta_n, (h_0 - E - i\varepsilon)^{-1}\delta_m \rangle = 2i\frac{\sin(n\theta(E))\sin(m\theta(E))}{\sqrt{4 - E^2}} \tag{3.6}$$

where $\theta(E)$ is the angle such that $e^{i\theta(E)} = \frac{E - i\sqrt{4-E^2}}{2}$.

Proof. According to the proof proposition 3.8:

$$\begin{aligned} \langle \delta_n, (h_0 - E - i\varepsilon)^{-1}\delta_m \rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin(n\theta)\sin(m\theta)}{2\cos\theta - z}d\theta \\ &= \frac{-1}{2}\frac{1}{2\pi i}\oint_{|\omega|=1}\frac{(\omega^n - \omega^{-n})(\omega^m - \omega^{-m})}{\omega^2 - \omega z + 1}d\omega \\ &= \frac{-1}{2}\frac{(\omega_1^n - \omega_1^{-n})(\omega_1^m - \omega_1^{-m})}{\omega_1 - \omega_2}. \end{aligned}$$

Now $\omega_1 = \frac{z-\sqrt{z^2-4}}{2} \Leftrightarrow \varepsilon > 0$. Let $\lambda := \frac{z-\sqrt{z^2-4}}{2}$. Then for $\varepsilon > 0$:

$$\langle \delta_n, (h_0 - E - i\varepsilon)^{-1}\delta_m \rangle = \frac{1}{2}\frac{(\lambda^n - \lambda^{-n})(\lambda^m - \lambda^{-m})}{\sqrt{z^2 - 4}}.$$

Note that

$$\lim_{\varepsilon \downarrow 0}\frac{z - \sqrt{z^2 - 4}}{2} = \frac{E - i\sqrt{4 - E^2}}{2} = e^{i\theta(E)}.$$

Therefore

$$\lim_{\varepsilon \downarrow 0}\langle \delta_n, (h_0 - E - i\varepsilon)^{-1}\delta_m \rangle = \frac{1}{2}\frac{(e^{in\theta(E)} - e^{-in\theta(E)})(e^{im\theta(E)} - e^{-im\theta(E)})}{i\sqrt{4 - E^2}} = 2i\frac{\sin(n\theta(E))\sin(m\theta(E))}{\sqrt{4 - E^2}}.$$

□

We have to mention that we have correctly verified that

$$d\mu_{\delta_1,\mathrm{ac}} = \frac{1}{2\pi}\sqrt{4-E^2}\mathbb{1}_{[-2,2]}(E)dE = \frac{1}{\pi}\mathrm{Im}\,\langle\delta_1,(h_0-E-i0)^{-1}\delta_1\rangle dE$$

and it is a probability measure.

3.2 Spectral Theory of the Schrödinger Operator

In this section, we consider the Schrödinger operator h on $\mathcal{H} := \ell^2(\mathbb{Z}_+)$ given by:

$$(hu)(n) = u(n+1) + u(n-1) + V(n)u(n) \tag{3.7}$$

$$u(0) = 0 \tag{3.8}$$

Here $V : \mathbb{Z}_+ \to \mathbb{R}$ is a priori any function. It follows that h is a linear self-adjoint operator and we will assume that V is a bounded potential so that h is a bounded operator.

Lemma 3.10. *δ_1 is a cyclic vector for $h : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$.*

Proof. We need to show that the linear span of $\{h^k\delta_1 : k \geq 0\}$ is dense in $\ell^2(\mathbb{Z}_+)$. An easy induction shows that for any $k \geq 0$, $h^k\delta_1 = \delta_{k+1} + \sum_{j=1}^{k} c_j\delta_j$ for some constants c_j. As a result the linear span of $\{h^k\delta_1 : k \geq 0\}$ is the same as the linear span of $\{\delta_k : k \geq 1\}$, which is obviously dense in $\ell^2(\mathbb{Z}_+)$. □

We apply the machinery of the spectral theorem for self-adjoint operators: let μ_{δ_1} be the spectral measure for h and δ_1. Since δ_1 is a cyclic vector, h is of multiplicity one, we trivially have a direct integral decomposition and $\sigma(h) = \mathrm{supp}\,\mu_{\delta_1}$. Let $U : \ell^2(\mathbb{Z}_+) \to L^2(\mathbb{R}, d\mu_{\delta_1})$ be the unique unitary map satisfying $[(UhU^{-1})f](E) = Ef(E)$ for $f \in L^2(\mathbb{R}, d\mu_{\delta_1})$ and $U\delta_1 = \mathbb{1}(E)$. The Borel transform $F_{\mu_{\delta_1}}(z)$ satisfies the important relation $\langle\delta_1,(h-z)^{-1}\delta_1\rangle = \int_{\mathbb{R}} \frac{d\mu_{\delta_1}(x)}{x-z}$ for all $z \in \mathbb{C}\backslash\mathbb{R}$.

We introduce functions $\delta_n(E)$ that are defined with the boundary values of the resolvent and that turn out to be equal to $(U\delta_n)(E)$ almost everywhere. As a result, the collection of functions $\{\delta_n(E)\}_{n\in\mathbb{Z}_+}$ form an orthonormal basis for $L^2(\mathbb{R}, d\mu_{\delta_1})$ and for any $f = \{f(n)\}_{n\in\mathbb{Z}_+} \in \ell^2(\mathbb{Z}_+)$, $(Uf)(E) = \sum_n f(n)\delta_n(E)$. The formula in terms of boundary values of the resolvent will give us a better insight on the functions $(U\delta_n)(E)$.

Theorem 3.11. *For all $n \geq 1$:*

(i) *$(U\delta_n)(E)$ is a polynomial of degree $n-1$ in E with real coefficients.*

(ii) $(U\mathbb{1}_{ac}\delta_n)(E) = \lim_{\varepsilon\downarrow 0} \frac{\mathrm{Im}\,\langle\delta_1,(h-E-i\varepsilon)^{-1}\delta_n\rangle}{\mathrm{Im}\,\langle\delta_1,(h-E-i\varepsilon)^{-1}\delta_1\rangle}$ *for $\mu_{\delta_1,\mathrm{ac}}$ - a.e. E.*

(iii) $(U\mathbb{1}_{\mathrm{sing}}\delta_n)(E) = \lim_{\varepsilon\downarrow 0} \frac{\langle\delta_1,(h-E-i\varepsilon)^{-1}\delta_n\rangle}{\langle\delta_1,(h-E-i\varepsilon)^{-1}\delta_1\rangle}$ *for $\mu_{\delta_1,\mathrm{sing}}$ - a.e. E.*

(iv) *In* (ii) *and* (iii) *the limits exist and are finite for $\mu_{\delta_1,\mathrm{ac}}$ - a.e. E and $\mu_{\delta_1,\mathrm{sing}}$ - a.e. E respectively.*

(v) *Let I_n be the set of E for which the limits in* (ii) *and* (iii) *exist and are finite. Define the following function on I_n:*

$$\delta_n(E) := \begin{cases} \lim_{\varepsilon\downarrow 0} \frac{\mathrm{Im}\,\langle \delta_1,(h-E-i\varepsilon)^{-1}\delta_n\rangle}{\mathrm{Im}\,\langle \delta_1,(h-E-i\varepsilon)^{-1}\delta_1\rangle} & \text{if } E \in \text{supp } \mu_{\delta_1,\text{ac}} \\ \lim_{\varepsilon\downarrow 0} \frac{\langle \delta_1,(h-E-i\varepsilon)^{-1}\delta_n\rangle}{\langle \delta_1,(h-E-i\varepsilon)^{-1}\delta_1\rangle} & \text{if } E \in \text{supp } \mu_{\delta_1,\text{sing}} \end{cases}$$

We obtain a function which is defined for μ_{δ_1} - a.e. E and satisfies

$$(U\delta_n)(E) = \delta_n(E) \;\; \textit{for } \mu_{\delta_1} - a.e. E.$$

(vi) *The collection of functions $\{\delta_n(E)\}_{n\in\mathbb{Z}_+}$ can be defined on a common subset $I = \cap_n I_n$ of $\mathbb{R}$ so that $\mu_{\delta_1}(I) = 1$ and for all $n \in \mathbb{Z}_+$, $(U\delta_n)(E) = \delta_n(E)$ for $\mu_{\delta_1} - a.e.E$.*

Proof.

(i) $(U\delta_0)(E)$ is the zero function and $(U\delta_1)(E) = \mathbb{1}(E)$. $\delta_{n+1} = h\delta_n - V(n)\delta_n - \delta_{n-1}$ gives

$$(U\delta_{n+1})(E) = (U(h\delta_n))(E) - V(n)(U\delta_n)(E) - (U\delta_{n-1})(E) = E(U\delta_n)(E) - V(n)(U\delta_n)(E) - (U\delta_{n-1})(E).$$

So $(U\delta_2)(E) = E - V(1)$, a polynomial of degree 1. An easy induction establishes the result.

(ii-iii) This is an application of theorem 1.16. Let $z = E + i\varepsilon$,

$$\begin{aligned}\mathrm{Im}\,\langle \delta_1, (h-z)^{-1}\delta_n\rangle &= \frac{1}{2i}\left(\int_{\mathbb{R}} \frac{(U\delta_n)(t)}{t-z} d\mu_{\delta_1}(t) - \int_{\mathbb{R}} \frac{(U\delta_n)(t)}{t-\overline{z}} d\mu_{\delta_1}(t)\right) \\ &= \int_{\mathbb{R}} \frac{\varepsilon}{(t-E)^2+\varepsilon^2} (U\delta_n)(t)\; d\mu_{\delta_1}(t)\end{aligned}$$

So

$$\begin{aligned}\lim_{\varepsilon\downarrow 0} \frac{\mathrm{Im}\,\langle \delta_1, (h-E-i\varepsilon)^{-1}\delta_n\rangle}{\mathrm{Im}\,\langle \delta_1, (h-E-i\varepsilon)^{-1}\delta_1\rangle} &= \lim_{\varepsilon\downarrow 0} \frac{P_{(U\delta_n)\mu_{\delta_1}}(E+i\varepsilon)}{P_{\mu_{\delta_1}}(E+i\varepsilon)} \\ &= (U\delta_n)(E) \text{ for } \mu_{\delta_1,\text{ac}} - a.e.\ E \\ &= (U(\mathbb{1}_{\text{ac}} + \mathbb{1}_{\text{sing}})\delta_n)(E) \text{ for } \mu_{\delta_1,\text{ac}} - a.e.\ E \\ &= (U\mathbb{1}_{\text{ac}}\delta_n)(E) \text{ for } \mu_{\delta_1,\text{ac}} - a.e.\ E.\end{aligned}$$

Similarly

$$\begin{aligned}\lim_{\varepsilon\downarrow 0} \frac{\langle \delta_1, (h-E-i\varepsilon)^{-1}\delta_n\rangle}{\langle \delta_1, (h-E-i\varepsilon)^{-1}\delta_1\rangle} &= \lim_{\varepsilon\downarrow 0} \frac{F_{(U\delta_n)\mu_{\delta_1}}(E+i\varepsilon)}{F_{\mu_{\delta_1}}(E+i\varepsilon)} \\ &= (U\delta_n)(E) \text{ for } \mu_{\delta_1,\text{sing}} - a.e.\ E \\ &= (U\mathbb{1}_{\text{sing}}\delta_n)(E) \text{ for } \mu_{\delta_1,\text{sing}} - a.e.\ E.\end{aligned}$$

(iv) This is due to the fact that $(U\mathbb{1}_{\text{ac}}\delta_n)(E) \in L^2(\mathbb{R}, d\mu_{\delta_0,\text{ac}})$ and $(U\mathbb{1}_{\text{sing}}\delta_n)(E) \in L^2(\mathbb{R}, d\mu_{\delta_0,\text{sing}})$.

(v-vi) Are easily verified.

□

Of course, the spectral theorem provides us with the following formula:

$$(U\delta_{n+1})(E) + (U\delta_{n-1})(E) + V(n)(U\delta_n)(E) = E(U\delta_n)(E) \text{ for } \mu_{\delta_1} - a.e.\ E. \tag{3.9}$$

Here we explicitly carry out the calculation for $\delta_n(E)$. The advantage of defining all the $\delta_n(E)$ on a common set I is that for every fixed $E \in I$ we can make sense of the following sequence: $\delta(E) := \{\delta(n,E) = \delta_n(E)\}_{n\in\mathbb{Z}_+} \in \ell(\mathbb{Z}_+)$. By convention we also define for $E \in I$, $\delta(0,E) := 0$. Then $(\delta(0,E), \delta(1,E)) = (0,1)$.

Proposition 3.12. *There is a set $I' \subset I$, $\mu_{\delta_1}(I') = 1$, such that for all $n \geq 1$ and $E \in I'$:*

$$\delta_{n+1}(E) + \delta_{n-1}(E) + V(n)\delta_n(E) = E\delta_n(E). \tag{3.10}$$

In particular, $\delta(E)$ is the fundamental solution to the Schrödinger equation, namely $\delta(E) = s(E)$.

Proof. We first show the result for $E \in I \cap \operatorname{supp} \mu_{\delta_1,\mathrm{ac}}$:

$$\begin{aligned}
\delta_{n+1}(E) + \delta_{n-1}(E) + V(n)\delta_n(E) &= \lim_{\varepsilon\downarrow 0} \frac{\operatorname{Im}\langle \delta_1, (h-E-i\varepsilon)^{-1}(\delta_{n+1}+\delta_{n-1}+V\delta_n)\rangle}{\operatorname{Im}\langle \delta_1, (h-E-i\varepsilon)^{-1}\delta_1\rangle} \\
&= \lim_{\varepsilon\downarrow 0} \frac{\operatorname{Im}\langle \delta_1, (h-E-i\varepsilon)^{-1}h\delta_n\rangle}{\operatorname{Im}\langle \delta_1, (h-E-i\varepsilon)^{-1}\delta_1\rangle} \\
&= \lim_{\varepsilon\downarrow 0} \frac{\operatorname{Im}\Big(\langle \delta_1, \delta_n\rangle + (E+i\varepsilon)\langle \delta_1, (h-E-i\varepsilon)^{-1}\delta_n\rangle\Big)}{\operatorname{Im}\langle \delta_1, (h-E-i\varepsilon)^{-1}\delta_1\rangle} \\
&= \lim_{\varepsilon\downarrow 0} E\frac{\operatorname{Im}\langle \delta_1, (h-E-i\varepsilon)^{-1}\delta_n\rangle}{\operatorname{Im}\langle \delta_1, (h-E-i\varepsilon)^{-1}\delta_1\rangle} + \lim_{\varepsilon\downarrow 0} \varepsilon\frac{\operatorname{Re}\langle \delta_1, (h-E-i\varepsilon)^{-1}\delta_n\rangle}{\operatorname{Im}\langle \delta_1, (h-E-i\varepsilon)^{-1}\delta_1\rangle}.
\end{aligned}$$

By theorem 1.16, $\lim_{\varepsilon\downarrow 0} \operatorname{Re}\langle \delta_1, (h-E-i\varepsilon)^{-1}\delta_n\rangle$ exists and is finite for $\mu_{\delta_1,\mathrm{ac}} - a.e.E$, and $\lim_{\varepsilon\downarrow 0} \operatorname{Im}\langle \delta_1, (h-E-i\varepsilon)^{-1}\delta_1\rangle$ exists and is finite and non-zero for $\mu_{\delta_1,\mathrm{ac}} - a.e.E$, so that

$$\lim_{\varepsilon\downarrow 0} \varepsilon\frac{\operatorname{Re}\langle \delta_1, (h-E-i\varepsilon)^{-1}\delta_n\rangle}{\operatorname{Im}\langle \delta_1, (h-E-i\varepsilon)^{-1}\delta_1\rangle} = 0.$$

A similar calculation gives that for $E \in I \cap \operatorname{supp} \mu_{\delta_1,\mathrm{sing}}$:

$$\begin{aligned}
&\delta_{n+1}(E) + \delta_{n-1}(E) + V(n)\delta_n(E) = \\
&= \lim_{\varepsilon\downarrow 0} \frac{\langle \delta_1, \delta_n\rangle}{\langle \delta_1, (h-E-i\varepsilon)^{-1}\delta_1\rangle} + E\lim_{\varepsilon\downarrow 0} \frac{\langle \delta_1, (h-E-i\varepsilon)^{-1}\delta_n\rangle}{\langle \delta_1, (h-E-i\varepsilon)^{-1}\delta_1\rangle} + \lim_{\varepsilon\downarrow 0} i\varepsilon\frac{\langle \delta_1, (h-E-i\varepsilon)^{-1}\delta_n\rangle}{\langle \delta_1, (h-E-i\varepsilon)^{-1}\delta_1\rangle}.
\end{aligned}$$

Since $\lim_{\varepsilon\downarrow 0} |\langle \delta_1, (h-E-i\varepsilon)^{-1}\delta_1\rangle| = \infty$ for $\mu_{\delta_1,\mathrm{sing}}$-a.e. E, the first limit goes to 0. As for third limit, it also goes to 0 since by theorem 3.11, $\lim_{\varepsilon\downarrow 0} \frac{\langle \delta_1, (h-E-i\varepsilon)^{-1}\delta_n\rangle}{\langle \delta_1, (h-E-i\varepsilon)^{-1}\delta_1\rangle}$ exists and is finite for $\mu_{\delta_1,\mathrm{sing}}$-a.e. E. ☐

Notation 3.13. *We use the Japanese bracket convention $\langle n\rangle = \sqrt{1+n^2}$.*

Proposition 3.14. *Let $\varepsilon > 0$. Then there is a set $I''(\varepsilon) \subset I$ such that $\mu_{\delta_1}(I''(\varepsilon)) = 1$ with the following property: for every $E \in I''(\varepsilon)$, there exists a constant $c = c(E) > 0$ such that $|\delta_n(E)| \leq c\langle n\rangle^{1/2+\varepsilon}$ for all $n \in \mathbb{Z}_+$.*

Proof. Let $f = \{f(n)\}_{n\in\mathbb{Z}_+} \in \ell^2(\mathbb{Z}_+)$ be arbitrary. The Monotone Convergence theorem gives:

$$\infty > \langle f, f\rangle = \langle Uf, Uf\rangle = \sum_{n\in\mathbb{Z}_+} \int_{\mathbb{R}} |f(n)|^2 |\delta_n(E)|^2 d\mu_{\delta_1}(E) = \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}_+} |f(n)|^2 |\delta_n(E)|^2 d\mu_{\delta_1}(E).$$

From this we conclude that for μ_{δ_1} - a.e. E there exists a constant $c(E) < \infty$ such that $\sum_{n\in\mathbb{Z}_+} |f(n)|^2|\delta_n(E)|^2 < c(E)^2$. In particular, $|f(n)||\delta_n(E)| < c(E)$ for μ_{δ_1} - a.e. E. Considering the sequence

$$f = \left\{ f(n) = \frac{1}{\langle n\rangle^{1/2+\varepsilon}} \right\}_{n\in\mathbb{Z}_+} \in \ell^2(\mathbb{Z}_+)$$

yields the result. □

Proposition 3.15. *Let $E \in \mathbb{R}$ and suppose that the fundamental solution $s(E)$ satisfies the following: there exists $\varepsilon > 0$ and $c > 0$ such that $|s(n, E)| < c\langle n\rangle^{1-\varepsilon}$ for all $n \in \mathbb{Z}_+$. Then $E \in \sigma(h)$.*

Proof. We construct a Weyl sequence:

$$s^{(m)}(n) = \begin{cases} \frac{s(n)}{\left(\sum_{j=1}^m |s(j)|^2\right)^{1/2}} & 0 \le n \le m \\ 0 & n > m \end{cases}$$

Then $\|s^{(m)}\| = 1$ and using the fact that $s(E)$ solves the Schrödinger equation we have:

$$((h-E)s^{(m)})(n) = \begin{cases} 0 & 0 \le n < m \\ \frac{-s(m+1)}{\left(\sum_{j=1}^m |s(j)|^2\right)^{1/2}} & n = m \\ \frac{s(m)}{\left(\sum_{j=1}^m |u_j|^2\right)^{1/2}} & n = m+1 \\ 0 & n > m+1 \end{cases}$$

Then $\|(h-E)s^{(m)}\|_2^2 = \frac{|s(m)|^2+|s(m+1)|^2}{\sum_{j=1}^m |s(j)|^2}$. We claim that this sequence has a subsequence converging to 0. If this is not the case then there is exists $m_0 > 0$ and $\alpha > 0$ such that $m \ge m_0$ implies $|s(m)|^2 + |s(m+1)|^2 \ge \alpha \sum_{j=1}^m |s(j)|^2 \ge \alpha$. For $m \ge m_0$, we have :

$$|s(m)|^2 + |s(m+1)|^2 \ge \alpha \sum_{j=1}^m |s(j)|^2 \ge \frac{\alpha}{2} \sum_{j=1}^{m-1} \left(|s(j)|^2 + |s(j+1)|^2\right) \ge \frac{\alpha^2}{2}(m - m_0)$$

and so

$$\sum_{j=1}^m |s(j)|^2 \ge \frac{1}{2} \sum_{j=1}^{m-1} \left(|s(j)|^2 + |s(j+1)|^2\right) \ge \frac{\alpha^2}{4} \sum_{j=m_0}^{m-1} (j - m_0) = \frac{\alpha^2}{4} \frac{(m-m_0-1)(m-m_0)}{2}$$

Hence for $m \geq m_0$ we have :

$$\liminf_{m\to\infty} \frac{|s(m)|^2 + |s(m+1)|^2}{\sum_{j=1}^{m} |s(j)|^2} \leq \liminf_{m\to\infty} \frac{|s(m)|^2 + |s(m+1)|^2}{\frac{\alpha^2}{4}\frac{(m-m_0-1)(m-m_0)}{2}} = 0.$$

This contradiction establishes the existence of a subsequence $\{m_k\}_{k=1}^{\infty}$ with $\lim_{m\to\infty} \|(h-E)s^{(m_k)}\|_2^2 = 0$. ☐

Definition 3.16. *Let $S = \{E \in \mathbb{R} :$ there exists $\varepsilon > 0$ such that $|s(n,E)| < c\langle n\rangle^{1-\varepsilon}$ for all $n \in \mathbb{Z}_+\}$.*

Theorem 3.17. *$\sigma(h) = \overline{S}$.*

Proof. proposition 3.15 gives $\overline{S} \subset \sigma(h)$. For the reverse inclusion, fix some $0 < \varepsilon < 1/2$ and let $\mathcal{I} = I' \cap I''(\varepsilon)$. Then by propositions 3.14 and 3.12, $\mathcal{I} \subset S$. Since $\sigma(h) = \operatorname{supp} \mu_{\delta_1}$, it is enough to check that $\operatorname{supp} \mu_{\delta_1} \subset \overline{\mathcal{I}}$. Let $E \in \operatorname{supp} \mu_{\delta_1}$ and suppose that $E \notin \overline{\mathcal{I}}$. Then there exists $B_\varepsilon(E)$ such that $B_\varepsilon(E) \cap \mathcal{I} = \varnothing$. So $0 < \mu_{\delta_1}(B_\varepsilon(E)) \leq \mu_{\delta_1}(\mathbb{R}\backslash\mathcal{I}) = 0$, where the strict inequality is by definition of the support and the equality is due to the fact that $\mathcal{I}$ is the intersection of two sets of full measure. That's a contradiction. ☐

4 Rank one Perturbations of the Laplacian on the Full Line

4.1 Overview of Rank one Perturbations

In this section we focus on potentials which are nonzero at exactly one point on the line. We will assume $V(0) \neq 0$ and $V(n) = 0$ for $n \neq 0$. If $\lambda := V(0)$, $\lambda \in \mathbb{R}$, then the corresponding Schrödinger operator is $h_\lambda = h + \lambda\langle\delta_0, \cdot\rangle\delta_0$. The goal of the spectral theory is to describe h_λ in terms of λ. This section is based on Chapter 5 of [**Ja**].

Lemma 4.1. *The cyclic space spanned by h_λ and δ_0 is the same as the cyclic space spanned by h_0 and δ_0. In particular it doesn't depend on λ.*

Proof. Let $\mathcal{L}_\lambda$ denote the linear span of $\{(h_\lambda - z)^{-1}\delta_0 : z \in \mathbb{C}\backslash\mathbb{R}\}$ and $\mathcal{L}_0$ the linear span of $\{(h_0 - z)^{-1}\delta_0 : z \in \mathbb{C}\backslash\mathbb{R}\}$. Formula (ii) of lemma 4.2 gives $\mathcal{L}_0 \subset \mathcal{L}_\lambda$ and formula (iii) gives the reverse inclusion. ☐

If $\mathcal{H}_0$ denotes this cyclic subspace, then $h_\lambda|_{\mathcal{H}_0^\perp} = h_0|_{\mathcal{H}_0^\perp}$. Therefore we may focus on $h_\lambda|_{\mathcal{H}_0}$. Let μ_λ be the spectral measure for h_λ and δ_0. Recall the Borel transform $F_\lambda(z) = \int_{\mathbb{R}} \frac{d\mu_\lambda(E)}{E-z} = \langle\delta_0, (h_\lambda - z)^{-1}\delta_0\rangle$. The following lemma relates the rank one perturbation to the Laplacian and is easily proved.

Lemma 4.2. *For any $z \in \mathbb{C}\backslash\mathbb{R}$:*

(i) $(h_\lambda - z)^{-1} - (h_0 - z)^{-1} = (h_\lambda - z)^{-1}(h_0 - h_\lambda)(h_0 - z)^{-1} = (h_0 - z)^{-1}(h_0 - h_\lambda)(h_\lambda - z)^{-1}$.

(ii) $(h_\lambda - z)^{-1}\delta_0 = (h_0 - z)^{-1}\delta_0 - \lambda\langle\delta_0, (h_\lambda - z)^{-1}\delta_0\rangle(h_0 - z)^{-1}\delta_0$.

(iii) $(h_0 - z)^{-1}\delta_0 = (h_\lambda - z)^{-1}\delta_0 - \lambda\langle\delta_0, (h_0 - z)^{-1}\delta_0\rangle(h_\lambda - z)^{-1}\delta_0$.

(iv) $F_\lambda(z) = \frac{F_0(z)}{1+\lambda F_0(z)}$.

(v) $\operatorname{Im} F_\lambda(z) = \frac{\operatorname{Im} F_0(z)}{|1+\lambda F_0(z)|^2}$.

Recall that we had already calculated the Borel transform $F_0(z)$ in proposition 2.11. We had found that for $z = x + iy$, $x \in \mathbb{R}, y > 0$:

$$F_0(x+iy) = \langle \delta_0, (h_0 - z)^{-1}\delta_0 \rangle = -\frac{1}{\sqrt{z^2-4}} \tag{4.1}$$

$$F_0(x) = \lim_{y \downarrow 0} F_0(x+iy) = \begin{cases} \frac{i}{\sqrt{4-x^2}} & x \in [-2,2] \\ \frac{-\text{sign}(x)}{\sqrt{x^2-4}} & x \in \mathbb{R} \backslash [-2,2] \end{cases} \tag{4.2}$$

from which we conclude that

$$F_\lambda(x+iy) = \langle \delta_0, (h_\lambda - z)^{-1}\delta_0 \rangle = \frac{1}{\lambda - \sqrt{z^2-4}} \tag{4.3}$$

$$F_\lambda(x) = \lim_{y \downarrow 0} F_\lambda(x+iy) = \begin{cases} \frac{\lambda + i\sqrt{4-x^2}}{\lambda^2+(4-x^2)} & x \in [-2,2] \\ \frac{1}{\lambda - \text{sign}(x)\sqrt{x^2-4}} & x \in \mathbb{R} \backslash [-2,2] \end{cases} \tag{4.4}$$

Definition 4.3. *For $\lambda \neq 0$, let*

(i) $G(x) := \int_{\mathbb{R}} \frac{d\mu_0(t)}{(x-t)^2}$.

(ii) $S_\lambda := \{x \in \mathbb{R} : F_0(x) = -1/\lambda \ , \ G(x) = \infty\}$.

(iii) $T_\lambda := \{x \in \mathbb{R} : F_0(x) = -1/\lambda \ , \ G(x) < \infty\}$.

(iv) $L := \{x \in \mathbb{R} : \text{Im } F_0(x) > 0\}$.

Note that S_λ, T_λ and L are mutually disjoint. It can be shown (see theorem 21 of Chapter 3 of [**Ja**]) that S_λ and T_λ have zero Lebesgue measure. The following lemma gives more information on $G(x)$:

Lemma 4.4. *Let μ be a positive σ-finite measure. Then*

$$G(x) := \int_{\mathbb{R}} \frac{d\mu(t)}{(s-t)^2} = \infty \ \mu - a.e. \ s \in \mathbb{R}. \tag{4.5}$$

Proof. First suppose that $d\mu(t) = f(t)dt + d\mu_s(t)$ is a finite measure. We have:

$$\int_{\mathbb{R}} \frac{d\mu(t)}{(s-t)^2} = 2\int_0^\infty t\ \mu\left(\left\{x \in \mathbb{R} : \frac{1}{|s-x|} > t\right\}\right) dt = 4\int_0^\infty \frac{\mu(s-1/t, s+1/t)}{\text{Leb }(s-1/t, s+1/t)} dt.$$

The first equality is proved by interchaning the order of integration and invoking Fubini's theorem (see theorem 8.16 in [**Ru**]). Let $D(s,t) := \frac{\mu(s-1/t,s+1/t)}{\text{Leb }(s-1/t,s+1/t)}$. By the theorem on the differentiation of measures, $\lim_{t\to\infty} D(s,t) = \infty$ for $\mu_s - a.e.\ s \in \mathbb{R}$. Hence $\int_{\mathbb{R}} \frac{d\mu(t)}{(s-t)^2} = \infty$ for $\mu_s - a.e.\ s \in \mathbb{R}$. Also, $\lim_{t\to\infty} D(s,t) = f(s)$ for Leb $- a.e.\ s \in \mathbb{R}$, hence for $\mu_{\text{ac}} - a.e.\ s \in \mathbb{R}$, and combined with the fact that $f(s) > 0$ for $\mu_{\text{ac}} - a.e.\ s \in \mathbb{R}$, gives $\lim_{t\to\infty} D(s,t) > 0$ for $\mu_{\text{ac}} - a.e.\ s \in \mathbb{R}$. Hence $\int_{\mathbb{R}} \frac{d\mu(t)}{(s-t)^2} = \infty$ for $\mu_{\text{ac}} - a.e.\ s \in \mathbb{R}$.

If μ is σ-finite, we partition $\mathbb{R}$ into a countable collection of disjoint sets $\{X_i\}_{i=1}^\infty$ with $\mu(X_i) < \infty$ for all i. Considering the collection $\{\mu_{X_i}\}$ of finite measures on $\mathbb{R}$ defined by $\mu_{X_i}(E) = \mu(E \cap X_i)$ puts us back into the previous case. □

Theorem 4.5.

(i) T_λ *is the set of eigenvalues of* h_λ *and* $\mu_{\lambda,\mathrm{pp}} = \sum_{x\in T_\lambda} \frac{\delta_{\{x\}}}{\lambda^2 G(x)}$.

(ii) $\mu_{\lambda,\mathrm{sc}}$ *is concentrated on* S_λ.

(iii) *For all* λ, $L = \Sigma_{ac}^{ess}(h_\lambda)$ *and* $\sigma_{\mathrm{ac}}(h_\lambda) = \sigma_{\mathrm{ac}}(h_0)$.

(iv) *The measures* $\{\mu_{\lambda,\mathrm{sing}}\}_{\lambda\in\mathbb{R}}$ *are mutually singular.*

Applying this theorem to the full line Schrödinger operator gives:

Theorem 4.6. *For* $h_\lambda = h_0 + \lambda\langle\delta_0,\cdot\rangle\delta_0$ *:*

(i) $\Sigma_{\mathrm{ac}}^{\mathrm{ess}}(h_\lambda) = \sigma_{\mathrm{ac}}(h_\lambda) = [-2,2]$.

(ii) *For every* $\lambda \neq 0$, h_λ *has a unique eigenvalue at* $E(\lambda) = \mathrm{sign}(\lambda)\sqrt{\lambda^2+4}$.

(iii) $d\mu_\lambda(E) = d\mu_{\lambda,\mathrm{pp}}(E) + h_\lambda(E)dE$, *where*

$$\mu_{\lambda,\mathrm{pp}} = \frac{1}{\lambda^2 G(E(\lambda))}\delta_{\{E(\lambda)\}}, \quad h_\lambda(E) = \frac{1}{\pi}\frac{\sqrt{4-E^2}}{\lambda^2+(4-E^2)}\mathbb{1}_{[-2,2]}(E)$$

and

$$G(E(\lambda)) = \int_{-2}^{2} \frac{1}{(E(\lambda)-t)^2}\frac{dt}{\pi\sqrt{4-t^2}}.$$

(iv) $\sigma(h_\lambda) = [-2,2] \cup E(\lambda)$.

Proof.

(i) Referring to formula (4.2), we have that Im $F_0(x) > 0$ if and only if $x \in [-2,2]$. Therefore $L = \Sigma_{\mathrm{ac}}^{\mathrm{ess}}(h_\lambda) = [-2,2] = \sigma_{\mathrm{ac}}(h_0) = \sigma_{\mathrm{ac}}(h_\lambda)$.

(ii) Referring to formula (4.2), $F_0(x) = -1/\lambda$ has no solution in $[-2,2]$. $F_0(x) = -1/\lambda$ if and only if $\frac{-\mathrm{sign}(x)}{\sqrt{x^2-4}} = -1/\lambda$. Solving for $x > 0$, we have a solution only if $\lambda > 0$, in which case $x = \sqrt{\lambda^2+4}$. For $x < 0$ a solution exists only if $\lambda < 0$ and then $x = -\sqrt{\lambda^2+4}$. Hence for a given $\lambda \neq 0$, there is a unique x solving $F_0(x) = -1/\lambda$ and $x = \mathrm{sign}(\lambda)\sqrt{\lambda^2+4}$. Since $|x| > 2$, $G(x) = \int_{[-2,2]} \frac{1}{(x-t)^2}\frac{dt}{\pi\sqrt{4-t^2}} < \infty$. Therefore the set of eigenvalues $T_\lambda = \{\mathrm{sign}(\lambda)\sqrt{\lambda^2+4}\}$ consists of one point and $S_\lambda = \varnothing$.

(iii) We have that $\mu_{\lambda,\mathrm{sc}} = 0$. By the de la Vallée Poussin theorem, $h_\lambda(E) = \frac{d\mu_{\lambda,\mathrm{ac}}}{dE} = \frac{1}{\pi}\mathrm{Im}\, F_\lambda(E) = \frac{1}{\pi}\frac{\sqrt{4-E^2}}{\lambda^2+(4-E^2)}\mathbb{1}_{[-2,2]}(E)$.

(iv) Follows from the previous items.

□

4.2 Elements of Scattering Theory

In this section we will introduce the basic concepts of Scattering Theory and prove Pearson's theorem (the proof we present was given by Claude-Alain Pillet). The reader is referred to Section XI.3 of [**RS3**] for a more complete exposition on the subject.

Definition 4.7. *Let A, B be self-adjoint operators on a Hilbert space $\mathcal{H}$. The wave operators $\Omega_\pm(A,B)$ exist if the limits*

$$\Omega^\pm(A,B) := \underset{t\to\pm\infty}{\text{s-lim}}\, e^{itA}e^{-itB}\mathbb{1}_{\text{ac}}(B)$$

exist for all vectors in $\psi \in \mathcal{H}$.

When there is no confusion we shall write $\Omega_\pm$ instead of $\Omega_\pm(A,B)$. We denote $\mathcal{H}_{\text{ac}}(A/B) := \mathbb{1}_{\text{ac}}(A/B)\mathcal{H}$.

Theorem 4.8. (Stone's theorem) *For any self-adjoint operator A, the group $\{e^{itA}\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter unitary group. Moreover, for all $\psi \in \text{Dom}(A)$, the map $t \to e^{itA}\psi$ is strongly differentiable and*

$$\lim_{t\to 0} \frac{e^{itA}\psi - \psi}{t} = iA\psi. \tag{4.6}$$

Proposition 4.9. *Suppose that the wave operators exist. Then:*

(i) $\Omega_\pm$ *are partial isometries with initial subspace $\mathcal{H}_{\text{ac}}(B)$ and final subspace contained in $\mathcal{H}_{\text{ac}}(A)$.*

(ii) Ran $\Omega_\pm$ *are $A-$invariant subspaces; $\Omega_\pm[\text{Dom}(B)] \subset \text{Dom}(A)$; and*

$$A\Omega_\pm(A,B) = \Omega_\pm(A,B)B. \tag{4.7}$$

(iii) *For any $\varphi \in B_b(\mathbb{R})$, $\varphi(A)\Omega_\pm(A,B) = \Omega_\pm(A,B)\varphi(B)$.*

Proof. Obviously $\mathcal{H}_{\text{ac}}(B)^\perp \subset \text{Ker}\ \Omega_\pm$. If $u \in \mathcal{H}_{\text{ac}}(B) = (\mathcal{H}_{\text{ac}}(B)^\perp)^\perp$, then $\|e^{itA}e^{-itB}\mathbb{1}_{\text{ac}}(B)u\| = \|u\|$ for every t and so $\|\Omega_\pm u\| = \|u\|$. Note that for every $s \in \mathbb{R}$, $\Omega_\pm = e^{isA}\Omega_\pm e^{-isB}$, or equivalently,

$$s^{-1}(e^{-isA} - \mathbb{1})\Omega_\pm = \Omega_\pm(e^{-isB} - \mathbb{1})s^{-1}.$$

Taking the limit $s \to 0$ and applying Stone's theorem proves equation (4.7). Equation (4.7) also shows that Ran $\Omega_\pm$ are $A-$invariant.

Also note that $A\restriction_{\text{Ran}\ \Omega_\pm}$ is unitarily equivalent to $B\restriction_{\mathcal{H}_{\text{ac}}(B)}$. Since the ac/sc/pp components of the Hilbert space are preserved under unitaries, it follows that Ran $\Omega_\pm \subset \mathcal{H}_{\text{ac}}(A)$.

By virtue of $e^{-isA}\Omega_\pm = \Omega_\pm e^{-isB}$, we have for $f \in \mathcal{H}$, $\langle f, e^{-isA}\Omega_\pm f\rangle = \langle \Omega_\pm^* f, e^{-isB} f\rangle$ or equivalently, $\int_{\mathbb{R}} e^{-is\lambda} d\mu_A(\lambda) = \int_{\mathbb{R}} e^{-is\lambda} d\mu_B(\lambda)$ for some spectral measures μ_A and μ_B. By the uniqueness theorem for the Fourier-Stieltjes transform of finite measures, it follows that $\mu_A = \mu_B$. Hence $\langle f, \varphi(A)\Omega_\pm f\rangle = \langle \Omega_\pm^* f, \varphi(B) f\rangle$ for all $\varphi \in B_b(\mathbb{R})$ and the result follows. □

Remark 4.1. *Since $A\restriction_{\text{Ran}\ \Omega^\pm}$ is unitarily equivalent to $B\restriction_{\mathcal{H}_{\text{ac}}(B)}$, we have that $\sigma_{\text{ac}}(B) \subset \sigma_{\text{ac}}(A)$ when the wave operators exist.*

Wave operators also satisfy a chain rule, namely that if $\Omega_\pm(A,B)$ and $\Omega_\pm(B,C)$ exist then $\Omega_\pm(A,C)$ exist and $\Omega_\pm(A,C) = \Omega_\pm(A,B)\Omega_\pm(B,C)$.

Definition 4.10. *If the wave operators exist then they are said to be complete if Ran $\Omega_\pm(A,B) = \mathcal{H}_{\text{ac}}(A)$.*

Proposition 4.11. *The wave operators $\Omega_\pm(A,B)$ are complete if and only if $\Omega_\pm(B,A)$ exist.*

Proof. If both $\Omega_\pm(A,B)$ and $\Omega_\pm(B,A)$ exist, then by the chain rule

$$\mathbb{1}_{\rm ac}(A) = \Omega_\pm(A,A) = \Omega_\pm(A,B)\Omega_\pm(B,A)$$

and so $\mathcal{H}_{\rm ac}(A) \subset \text{Ran } \Omega_\pm(A,B)$; the reverse inclusion was shown in proposition 4.9. Conversely, suppose that $\Omega_\pm(A,B)$ exist and $\text{Ran } \Omega_\pm(A,B) = \mathcal{H}_{\rm ac}(A)$. Let $\psi \in \mathcal{H}_{\rm ac}(A)$. We want to prove the existence of the limit

$$\lim_{t\to\pm\infty} e^{itB}e^{-itA}\psi.$$

By assumption there is a ϕ such that $\psi = \Omega_\pm(A,B)\phi$. Then

$$\lim_{t\to\pm\infty} \|\mathbb{1}_{\rm ac}(B)\phi - e^{itB}e^{-itA}\psi\| = \lim_{t\to\pm\infty} \|e^{itA}e^{-itB}\mathbb{1}_{\rm ac}(B)\phi - \Omega_\pm(A,B)\phi\| = 0.$$

□

Adjoints of partial isometries are known to act like inverses, namely:

$$\mathbb{1}_{\rm ac}(B) = \Omega_\pm^*(A,B)\Omega_\pm(A,B), \quad \mathbb{1}_{\rm ac}(A) = \Omega_\pm(A,B)\Omega_\pm^*(A,B), \tag{4.8}$$

If the wave operators are complete, then the chain rule shows that the adjoints are given by

$$\Omega_\pm^*(A,B) = \Omega_\pm(B,A) = \underset{t\to\pm\infty}{\text{s-lim}}\, e^{itB}e^{-itA}\mathbb{1}_{ac}(A). \tag{4.9}$$

We now introduce the scattering operator - the fundamental object of Scattering Theory:

Definition 4.12. *The scattering operator is defined by*

$$S(A,B) := \Omega_+^*(A,B)\Omega_-(A,B) \tag{4.10}$$

It is a unitary operator on $\mathcal{H}_{\rm ac}(B)$.

Our next goal is to prove the basic existence and completeness theorem of the wave operators, namely Pearson's theorem. First we introduce a few technical tools. Recall the classical Riemann-Lebesgue lemma, namely that if μ is a complex measure absolutely continuous w.r.t. the Lebesgue measure, then $\lim_{t\to\pm\infty} \int_{\mathbb{R}} e^{itx}d\mu(x) = 0$. We have a Riemann-Lebesgue type lemma for operators:

Lemma 4.13. *Let A be a self-adjoint operator, then* $\underset{t\to\pm\infty}{\text{w-lim}}\, e^{itA}\mathbb{1}_{\rm ac}(A) = 0$.

Proof. Let $\psi \in \mathcal{H}_{\rm ac}(A)$, and let $\mathcal{H}_\psi$ be the cyclic subspace generated by A and ψ, $\mathbb{1}_\psi$ the orthogonal projection onto $\mathcal{H}_\psi$. Note that $\mathcal{H}_\psi \subset \mathcal{H}_{\rm ac}(A)$. Also $\psi \in \mathcal{H}_{\rm ac}(A)$ implies that μ_ψ is purely absolutely continuous. Then:

$$\langle \phi, e^{itA}\psi\rangle = \langle \mathbb{1}_\psi\phi, e^{itA}\psi\rangle = \int_{\mathbb{R}} \overline{(U_\psi \mathbb{1}_\psi\phi)(x)}e^{itx}d\mu_\psi(x).$$

Note that $\overline{(U_\psi\mathbb{1}_\psi\phi)} \in L^2(\mathbb{R}, d\mu_\psi) \subset L^1(\mathbb{R}, d\mu_\psi)$. Hence $\overline{(U_\psi\mathbb{1}_\psi\phi)(x)}d\mu_\psi(x)$ defines a complex measure absolutely continuous w.r.t. the Lebesgue measure, and so the result follows from the classical Riemann-Lebesgue lemma. □

Corollary 4.14. *If C is a compact operator on $\mathcal{H}$ and A self-adjoint, then* $\underset{t\to\pm\infty}{\text{s-lim}}\, Ce^{itA}\mathbb{1}_{\rm ac}(A) = 0$.

Proof. It is enough to show for finite rank operators since they are dense in the compact operators with respect to the operator norm. Let $C_N = \sum_{n=1}^{N} \lambda_n \langle \psi_n, \cdot \rangle \phi_n$ be finite rank ($N < \infty$). Then:

$$\lim_{t\to\pm\infty} C_N e^{itA} \mathbb{1}_{\mathrm{ac}}(A) f = \lim_{t\to\pm\infty} \sum_{n=1}^{N} \lambda_n \langle \psi_n, e^{itA} \mathbb{1}_{\mathrm{ac}}(A) f \rangle \phi_n = 0.$$

□

We now introduce a dense subset $L^\infty(A)$ of $\mathcal{H}_{\mathrm{ac}}(A)$ that will be useful for the following proofs.

Definition 4.15. *Let A be a self-adjoint operator and denote by $L^\infty(A)$ the set of all $\psi \in \mathcal{H}$ such that $d\mu_\psi(x) = |f(x)|^2 dx$ with $f \in L^\infty(\mathbb{R})$. Let $|||f|||$ be the L^∞-norm of f.*

It can be verified that $|||\cdot|||$ is a norm on $L^\infty(A)$. If $\psi \in \mathcal{H}_{\mathrm{ac}}(A)$, then $d\mu_\psi(x) = |f(x)|^2 dx$ for some $f \in L^2(\mathbb{R})$. If $f_n(x) := f(x)\mathbb{1}_{\{|f|\le n\}}(x)$, then by the Dominated Convergence theorem $f_n \to f$ in $L^2(\mathbb{R})$. Since supp $f_n \subset$ supp $f \subset \sigma_{\mathrm{ac}}(A)$, there are $\psi_n \in \mathcal{H}_{\mathrm{ac}}(A)$, namely $\psi_n = \mathbb{1}_{\{|f|\le n\}}(A)\psi$, such that $d\mu_{\psi_n}(x) = |f_n(x)|^2 dx$. Furthermore ψ_n converges to ψ in the usual norm on $\mathcal{H}$ and so $L^\infty(A)$ is dense in $\mathcal{H}_{\mathrm{ac}}(A)$.

Lemma 4.16. *Let A be self-adjoint, $f \in L^\infty(A)$, and let T be a Hilbert-Schmidt operator on $\mathcal{H}$. Then*

$$\int_{\mathbb{R}} \|Te^{itA} f\|^2 dt \le 2\pi |||f|||^2 \|T\|_2^2, \tag{4.11}$$

where $\|T\|_2^2 = \mathrm{tr}(T^*T)$ denotes the Hilbert-Schmidt norm.

Proof. Since T is Hilbert-Schmidt, in particular it is compact and so it has a representation $T = \sum_{n=1}^{\infty} \lambda_n \langle \varphi_n, \cdot \rangle \psi_n$ with $\sum_n \lambda_n^2 = \|T\|_2^2$ and orthonormal families $\{\varphi_n\}$ and $\{\psi_n\}$. We have

$$\|Te^{itA} f\|^2 = \sum_n \lambda_n^2 |\langle \varphi_n, e^{itA} f \rangle|^2$$

and $\langle \varphi_n, e^{itA} f \rangle = \int_{\mathbb{R}} \overline{(U_f\varphi_n)(x)} |f(x)|^2 e^{itx} dx$ is the Fourier transform of $(2\pi)^{1/2}\overline{(U_f\varphi_n)}|f|^2 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. An application of the Monotone Convergence and Plancherel theorems gives:

$$\int_{\mathbb{R}} \|Te^{itA} f\|^2 dt = \sum_n \lambda_n^2 \int_{\mathbb{R}} |\langle \varphi_n, e^{itA} f \rangle|^2 dt = 2\pi \sum_n \lambda_n^2 \int_{\mathbb{R}} |\overline{(U_f\varphi_n)(x)}|^2 |f(x)|^4 dx$$

$$2\pi \le \sum_n \lambda_n^2 |||f|||^2 \int_{\mathbb{R}} |\overline{(U_f\varphi_n)(x)}|^2 |f(x)|^2 dx = 2\pi |||f|||^2 \|T\|_2^2 \|U_f\varphi_n\|^2 \le 2\pi |||f|||^2 \|T\|_2^2.$$

□

We now prove Pearson's theorem. The proof we provide was given by Claude-Alain Pillet and is substantially more transparent than that given in [**RS3**].

Theorem 4.17. (Pearson's theorem) *Let A and B be self-adjoint and J be a bounded operator such that $J(\mathrm{Dom}(B)) \subset \mathrm{Dom}(A)$. Suppose that there is a trace-class operator C so that $C = AJ - JB$ in form, that is, for $\psi \in \mathrm{Dom}(A)$ and $\phi \in \mathrm{Dom}(B)$ we have:*

$$\langle \psi, C\phi \rangle = \langle A\psi, J\phi \rangle - \langle \psi, JB\phi \rangle. \tag{4.12}$$

In particular we also have:

$$\langle \phi, C^* \psi \rangle = \langle \phi, J^* A \psi \rangle - \langle B\phi, J^* \psi \rangle. \tag{4.13}$$

Then the following strong limits exist:

$$\Omega_\pm(A, B; J) := \operatorname*{s-lim}_{t \to \pm\infty} e^{itA} J e^{-itB} \mathbb{1}_{\rm ac}(B). \tag{4.14}$$

Proof. Let $W(t) := e^{itA} J e^{-itB}$. By the Cauchy criterion of completeness, we need to show that

$$\lim_{t,s \to \pm\infty} \left\| \Big[W(t) - W(s) \Big] f \right\| = 0 \tag{4.15}$$

for all $f \in \mathcal{H}_{\rm ac}(B)$, and in fact we only need to show (4.15) for a dense subset, namely $L^\infty(B)$.

Since

$$\left\| \Big[W(t) - W(s) \Big] f \right\|^2 = \langle f, W(t)^* \Big[W(t) - W(s) \Big] f \rangle - \langle f, W(s)^* \Big[W(t) - W(s) \Big] f \rangle$$

it is enough (by symmetry) to show that

$$\lim_{t,s \to \pm\infty} \langle f, W(t)^* \Big[W(t) - W(s) \Big] f \rangle = 0. \tag{4.16}$$

Consider

$$\langle f, e^{iuB} W(t)^* \Big[W(t) - W(s) \Big] e^{-iuB} f \rangle. \tag{4.17}$$

Since $\Big[W(t) - W(s) \Big] = \int_s^t \frac{dW(r)}{dr} dr = i \int_s^t e^{irA} C e^{-irB} dr$ and C is trace-class, it follows that $e^{irA} C e^{-irB}$ is compact and so $\Big[W(t) - W(s) \Big]$ is compact as a norm limit of compact operators. Thus $\lim\limits_{u \to \pm\infty} \Big[W(t) - W(s) \Big] e^{-iuB} f = 0$ by Corollary 4.14. In particular (4.17) goes to zero as $u \to \pm\infty$.

Hence if we can show that

$$\lim_{t,s \to \pm\infty} \langle f, \Big[e^{iuB} W(t)^* [W(t) - W(s)] e^{-iuB} - W(t)^* [W(t) - W(s)] \Big] f \rangle = 0 \tag{4.18}$$

uniformly in u, (4.16) will follow immediately.

$$\begin{aligned} &\langle f, \Big[e^{iuB} W(t)^* [W(t) - W(s)] e^{-iuB} - W(t)^* [W(t) - W(s)] \Big] f \rangle \\ &= \langle f, \Big[e^{iuB} W(t)^* W(t) e^{-iuB} - W(t)^* W(t) \Big] f \rangle - \langle f, \Big[e^{iuB} W(t)^* W(s) e^{-iuB} - W(t)^* W(s) \Big] f \rangle \\ &:= X_1(t, s, u) - X_2(t, s, u). \end{aligned}$$

We will show that $\lim\limits_{t,s \to \pm\infty} X_2(t, s, u) = 0$ uniformly in u, and the same method will hold for $X_1(t, s, u)$.

Note that $\frac{d}{dr} \left(e^{irB} W(t)^* W(s) e^{-irB} \right) = i e^{irB} \Big[B, W(t)^* W(s) \Big] e^{-irB}$ and so

$$X_2(t, s, u) = i \int_0^u \langle f, e^{irB} \Big[B, W(t)^* W(s) \Big] e^{-irB} f \rangle dr.$$

We show that $\left[B, W(t)^*W(s)\right] = e^{itB}\left(J^*e^{-i(t-s)A}C - C^*e^{-i(t-s)A}J\right)e^{-isB}$ by computing the matrix elements. Let $\psi, \phi \in \mathrm{Dom}(B)$. Then:

$$\begin{aligned}
&\left\langle \psi, \left[B, W(t)^*W(s)\right]\phi \right\rangle \\
&= \langle \psi, Be^{itB}J^*e^{-itA}e^{isA}Je^{-isB}\phi\rangle - \langle \psi, e^{itB}J^*e^{-itA}e^{isA}Je^{-isB}B\phi\rangle \\
&= \langle Be^{-itB}\psi, J^*e^{-i(t-s)A}Je^{-isB}\phi\rangle - \langle e^{i(t-s)A}Je^{-itB}\psi, JBe^{-isB}\phi\rangle \\
&= \left[\langle e^{-itB}\psi, J^*Ae^{-i(t-s)A}Je^{-isB}\phi\rangle - \langle e^{-itB}\psi, C^*e^{-i(t-s)A}Je^{-isB}\phi\rangle\right] \\
&\quad - \left[\langle Ae^{i(t-s)A}Je^{-itB}\psi, Je^{-isB}\phi\rangle - \langle e^{i(t-s)A}Je^{-itB}\psi, Ce^{-isB}\phi\rangle\right] \\
&= \langle \psi, e^{itB}[J^*e^{-i(t-s)A}C - C^*e^{-i(t-s)A}J]e^{-isB}\phi\rangle
\end{aligned}$$

Moreover since C is trace-class, $C = F^*G$ for some F, G Hilbert-Schmidt. Therefore:

$$\begin{aligned}
X_2(t,s,u) &= i\int_0^u \langle Fe^{i(t-s)A}Je^{-i(r+t)B}f, Ge^{-i(r+s)B}f\rangle dr - i\int_0^u \langle Ge^{-i(r+t)B}f, Fe^{-i(t-s)A}Je^{-i(r+s)B}f\rangle dr \\
&:= iX_{2,1}(t,s,u) - iX_{2,2}(t,s,u).
\end{aligned}$$

We will show that $\lim_{t,s\to\pm\infty} X_{2,1}(t,s,u) = 0$ uniformly in u, and the same method will hold for $X_{2,2}(t,s,u)$.

$$\begin{aligned}
X_{2,1}(t,s,u) &\leq \int_0^\infty \|Fe^{i(t-s)A}Je^{-i(r+t)B}f\|\|Ge^{-i(r+s)B}f\|dr \\
&\leq \left(\int_0^\infty \|Fe^{i(t-s)A}Je^{-i(r+t)B}f\|^2 dr\right)^{1/2}\left(\int_0^\infty \|Ge^{-i(r+s)B}f\|^2 dr\right)^{1/2} \\
&= \left(\int_t^\infty \|Fe^{i(t-s)A}Je^{-irB}f\|^2 dr\right)^{1/2}\left(\int_s^\infty \|Ge^{-irB}f\|^2 dr\right)^{1/2} \\
&\leq \sqrt{2\pi}\|Fe^{i(t-s)A}J\|_2 |||f||| \left(\int_s^\infty \|Ge^{-irB}f\|^2 dr\right)^{1/2} \\
&\leq \sqrt{2\pi}\|F\|_2\|J\| |||f||| \left(\int_s^\infty \|Ge^{-irB}f\|^2 dr\right)^{1/2}
\end{aligned}$$

Note that we could apply lemma 4.16 because $Fe^{i(t-s)A}J$ is Hilbert-Schmidt. Again by the same lemma we know that the integral on the LHS goes to zero as $s \to \infty$ and the rate of convergence is independent of u. This completes the proof. □

If $\mathrm{Dom}(A) = \mathrm{Dom}(B)$, and $A - B$ is trace-class, this also proves the

Theorem 4.18. (Kato-Rosenblum theorem) *If A and B are self-adjoint operators and $A - B$ are trace-class, then the wave operators $\Omega_\pm(A, B)$ exist and are complete.*

It is worth mentioning that if $h = h_0 + V$ where V is in $\ell^1(\mathbb{Z})$, then $h - h_0$ is trace-class, i.e. $\mathrm{tr}(|V|) < \infty$, so that $\Omega_\pm(h, h_0)$ exist and are complete.

4.3 Scattering Operator for the Rank One Perturbation

We will deal with Rank one perturbations of the Laplacian operator ($h = h_0 + \lambda\langle\delta_n, \cdot\rangle\delta_n$), show directly the existence and completeness of the wave operators (instead of just quoting the results of the previous section), compute the wave and scattering operators.

For computational purposes, it is useful to rewrite the wave operators in the following way:

$$\Omega_\pm(A, B) = \mathbb{1}_{\mathrm{ac}}(B) + i \operatorname*{s-lim}_{t\to\pm\infty} \int_0^t e^{isA}(A - B)e^{-isB}\mathbb{1}_{\mathrm{ac}}(B)ds \tag{4.19}$$

which is a consequence of

$$\int_0^t \frac{d}{ds}\left(e^{isA}e^{-isB}\mathbb{1}_{\mathrm{ac}}(B)\right) ds = e^{itA}e^{-itB}\mathbb{1}_{\mathrm{ac}}(B) - \mathbb{1}_{\mathrm{ac}}(B) = i\int_0^t e^{isA}(A - B)e^{-isB}\mathbb{1}_{\mathrm{ac}}(B)ds.$$

Let $C_c^\infty(-2, 2)$ denote the smooth functions supported on $(-2, 2)$. Recall that the absolutely continuous part of the spectral measure μ_λ for δ_n and h_λ is $\frac{d\mu_\lambda}{dx} = \frac{\sqrt{4-x^2}}{\pi(\lambda^2+(4-x^2))}$.

Lemma 4.19. *$C_c^\infty(-2, 2)$ is dense in $L^2\left([-2, 2], \frac{\sqrt{4-x^2}}{\lambda^2+(4-x^2)}dx\right)$ for all $\lambda \in \mathbb{R}$.*

Proof. We take for granted that $C_c^\infty(-2, 2)$ is dense in $L^2([-2, 2], dx)$ and for simplicity of notation, we consider the case $\lambda = 0$.

$$\begin{aligned}
\psi(x) \in L^2\left([-2, 2], \frac{1}{\sqrt{4 - x^2}}dx\right) &\Leftrightarrow \frac{\psi(x)}{\sqrt[4]{4 - x^2}} \in L^2([-2, 2], dx) \\
&\Rightarrow \exists\ \phi(x) \in C_c^\infty(-2, 2) \text{ such that } \int_{-2}^2 \left|\frac{\psi(x)}{\sqrt[4]{4 - x^2}} - \phi(x)\right|^2 dx < \varepsilon \\
&\Rightarrow \int_{-2}^2 \left|\psi(x) - \sqrt[4]{4 - x^2}\phi(x)\right|^2 \frac{1}{\sqrt{4 - x^2}}dx < \varepsilon \\
&\Rightarrow C_c^\infty(-2, 2) \text{ is dense in } L^2\left([-2, 2], \frac{1}{\sqrt{4 - x^2}}dx\right)
\end{aligned}$$

since $\sqrt[4]{4 - x^2}\phi(x) \in C_c^\infty(-2, 2)$. □

Proposition 4.20. *$\Omega^\pm(h_\lambda, h_0)$ exist and are complete.*

Proof. We have to show that the following limit exists for every $\psi \in \mathcal{H}_{ac}(h_0)$:

$$\lim_{t\to\pm\infty} e^{ith_\lambda}e^{-ith_0}\psi = \psi + i\lambda \lim_{t\to\pm\infty}\int_0^t \langle\delta_n, e^{-ish_0}\psi\rangle e^{ish_\lambda}\delta_n ds. \tag{4.20}$$

It is enough to show that

$$\left\|\int_0^\infty \langle\delta_n, e^{-ish_0}\psi\rangle e^{ish_\lambda}\delta_n ds\right\| \le \int_0^\infty |\langle\delta_n, e^{-ish_0}\psi\rangle| ds < \infty \tag{4.21}$$

since it implies that $\int_0^t\langle\delta_n, e^{-ish_0}\psi\rangle e^{ish_\lambda}\delta_n ds$ is a Cauchy sequence.

Consider the vector $\mathbb{1}_{\mathrm{ac}}(h_0)\delta_n$. Its spectral measure is purely absolutely continuous and is given by

$$d\mu_{\delta_n,\mathrm{ac}}(E) = \frac{1}{\pi}\mathrm{Im}\,\langle\delta_n, (h_0 - E - i0)^{-1}\delta_n\rangle dE = \frac{1}{\pi}\frac{1}{\sqrt{4 - E^2}}\mathbb{1}_{[-2,2]}(E)dE.$$

Denote by $\mathcal{H}_{\delta_n}$ the cyclic space generated by δ_n and $\mathbb{1}_{\delta_n}$ the projection onto it. Then the cyclic space generated by $\mathbb{1}_{\rm ac}(h_0)\delta_n$ is equal to $\mathbb{1}_{\rm ac}(h_0)\mathcal{H}_{\delta_n}$. If $\psi \in \mathcal{H}_{\rm ac}(h_0)$, then $\mathbb{1}_{\delta_n}\psi \in \mathbb{1}_{\rm ac}(h_0)\mathcal{H}_{\delta_n}$. By lemma 4.19, $\mathcal{D} := \{\phi \in \mathcal{H}_{\rm ac}(h_0) : (U_{\delta_n}\mathbb{1}_{\delta_n}\phi)(E) \in C_c^\infty(-2,2)\}$ is dense in $\mathcal{H}_{\rm ac}(h_0)$.

Assume for now that $\psi \in \mathcal{D}$. In particular we have

$$\lim_{E\to\pm 2} \frac{(U_{\delta_n}\mathbb{1}_{\delta_n}\psi)(E)}{\pi\sqrt{4-E^2}} = 0, \qquad \lim_{E\to\pm 2} \frac{d}{dE}\frac{(U_{\delta_n}\mathbb{1}_{\delta_n}\psi)(E)}{\pi\sqrt{4-E^2}} = 0.$$

Integrating twice by parts shows that

$$\begin{aligned}
\langle \delta_n, e^{-ish_0}\psi\rangle &= \langle \mathbb{1}_{\rm ac}(h_0)\delta_n, e^{-ish_0}\psi\rangle = \langle \delta_n, e^{-ish_0}\mathbb{1}_{\delta_n}\psi\rangle \\
&= \int_{\mathbb{R}} e^{-isE}(U_{\delta_n}\mathbb{1}_{\delta_n}\psi)(E)d\mu_{\delta_n,\rm ac} \\
&= \int_{-2}^{2} e^{-isE}\frac{(U_{\delta_n}\mathbb{1}_{\delta_n}\psi)(E)}{\pi\sqrt{4-E^2}}dE \\
&= \int_{-2}^{2} \frac{e^{-isE}}{(-is)^2}\frac{d^2}{dE^2}\left(\frac{(U_{\delta_n}\mathbb{1}_{\delta_n}\psi)(E)}{\pi\sqrt{4-E^2}}\right)dE.
\end{aligned}$$

Hence $|\langle \delta_n, e^{-ish_0}\psi\rangle| \lesssim \{\frac{1}{s^2}, \|\psi\|\}$ and it follows by (4.21) that $\int_0^\infty |\langle \delta_n, e^{-ish_0}\psi\rangle| ds \le \|\psi\| + \int_1^\infty \frac{1}{s^2}ds < \infty$.

Now if $\psi \in \mathcal{H}_{\rm ac}(h_0)$ is arbitrary, then choose a sequence $\psi_n \in \mathcal{D}$ converging to ψ, and use an $\varepsilon/3$ argument to show that the sequence $e^{ith_\lambda}e^{-ith_0}\phi$ is Cauchy, namely, if $W(t) := e^{ith_\lambda}e^{-ith_0}$, then:

$$\|W(t)\psi - W(t')\psi\| \le \|W(t)\psi - W(t)\psi_n\| + \|W(t)\psi_n - W(t')\psi_n\| + \|W(t')\psi_n - W(t')\psi\|$$

which goes to zero as $n, t, t' \to \infty$.

To show that $\Omega_\pm(h_0, h_\lambda)$ exist, the same proof works by considering the cyclic space generated by $\mathbb{1}_{\rm ac}(h_\lambda)\delta_n$ and its spectral measure $d\mu_{\delta_n,\rm ac}(E) = \frac{\sqrt{4-E^2}}{\pi(\lambda^2+(4-E^2))}\mathbb{1}_{[-2,2]}dE$. □

Moving forward the following useful will be result.

Lemma 4.21. *Let $f(x)$ be a bounded measurable function and suppose that $\lim_{t\to\infty}\int_0^t f(x)dx < \infty$. Then*

$$\lim_{t\to\infty}\int_0^t f(x)dx = \lim_{\varepsilon\downarrow 0}\int_0^\infty e^{-\varepsilon s}f(s)ds. \tag{4.22}$$

In particular, $\lim_{t\to\infty}\int_{-t}^t f(x)dx = \lim_{\varepsilon\downarrow 0}\int_{-\infty}^\infty e^{-\varepsilon|s|}f(s)ds$.

Proof. Let $\lim_{t\to\infty}\int_0^t f(x)dx := a$, $g(t) := \int_0^t f(s)ds$, and $q(\varepsilon) := \int_0^\infty e^{-\varepsilon s}f(s)ds$. Then $g'(t) = f(t)$ a.e., so integration by parts shows that $q(\varepsilon) = \int_0^\infty \varepsilon e^{-\varepsilon s}g(s)ds$. Since $\int_0^\infty \varepsilon e^{-\varepsilon s}ds = 1$ for all $\varepsilon > 0$, we have

$$|q(\varepsilon) - a| \le \int_0^T \varepsilon e^{-\varepsilon s}|g(s) - a|ds + \int_T^\infty \varepsilon e^{-\varepsilon s}|g(s) - a|ds.$$

Let $\delta > 0$ be given and choose $T > 0$ such that $|g(t) - a| < \delta/2$ for $t > T$, and then choose $e > 0$ so that $\left(\sup_{t\in\mathbb{R}}|g(t)| + |a|\right)Te < \delta/2$ (note that g is bounded). Then $|q(\varepsilon) - a| < \delta$ for all $\varepsilon < e$.

A similar identity on the negative half line holds by symmetry and the identity on the full line is a combination of the two identities. □

Let $\mathcal{H}_{\delta_n}$ denote the cyclic space generated by h_0 and δ_n. Recall that the cyclic space generated by h_λ and δ_n is equal to $\mathcal{H}_{\delta_n}$, and moreover, $h_\lambda\restriction_{\mathcal{H}_{\delta_n}^\perp} = h_0\restriction_{\mathcal{H}_{\delta_n}^\perp}$. Since for a bounded operator A $e^{itA} = \sum_n (itA)^n/n!$, it follows that $\mathcal{H}_{\delta_n}$ and $\mathcal{H}_{\delta_n}^\perp$ are invariant subspaces for $e^{\pm ith_0}$ and $e^{\pm ith_\lambda}$. Also $\mathcal{H}_{\delta_n}$ and $\mathcal{H}_{\delta_n}^\perp$ are invariant subspaces for $\mathbb{1}_{\rm ac}(h_0)$ and $\mathbb{1}_{\rm ac}(h_\lambda)$. Therefore when we write $\langle f, \Omega_\pm(h_\lambda, h_0)g\rangle$, or $\langle g, \Omega_\pm(h_\lambda, h_0)f\rangle$ we may assume that $f, g \in \mathcal{H}_{\delta_n}$. Let us denote $\mu := \mu_{\delta_n, h_0, ac}$ the absolutely continuous part of the spectral measure for δ_n and h_0; $d\mu(E) = \frac{1}{\pi\sqrt{4-E^2}}dE$. For simplicity of notation, we will denote the element of $L^2(\mathbb{R}, d\mu)$ corresponding to $\chi \in \mathcal{H}_{\delta_n}$ simply by $\chi(E)$.

Proposition 4.22. *Let $g \in \mathcal{H}$ be given and let $[\Omega_\pm^*(h_\lambda, h_0)g](E)$ be the element of $L^2(\mathbb{R}, d\mu)$ corresponding to $\Omega_\pm^*(h_\lambda, h_0)g$. Then*

$$[\Omega_\pm^*(h_\lambda, h_0)g](E) = g(E) - \lambda\langle \delta_n, (h_\lambda - E \mp i0)^{-1}g\rangle \tag{4.23}$$

Proof. Let $f \in \mathcal{H}_{\rm ac}(h_0) = \mathcal{H}$. We will compute $\langle f, \Omega_\pm^*(h_\lambda, h_0)g\rangle$, the computation for $\Omega_-^*(h_\lambda, h_0)$ is identical.

$$\begin{aligned}\langle f, \Omega_+^*(h_\lambda, h_0)g\rangle =& \langle f, \Omega_+(h_0, h_\lambda)g\rangle\\ =& \langle f, g\rangle - i\lambda \lim_{t\to\infty}\int_0^t \langle f, e^{ish_0}\delta_n\rangle\langle\delta_n, e^{-ish_\lambda}g\rangle ds\\ =& \langle f, g\rangle - i\lambda \lim_{\varepsilon\downarrow 0}\int_0^\infty e^{-\varepsilon s}\langle f, e^{ish_0}\delta_n\rangle\langle\delta_n, e^{-ish_\lambda}g\rangle ds\\ =& \langle f, g\rangle - i\lambda \lim_{\varepsilon\downarrow 0}\int_0^\infty e^{-\varepsilon s}\left[\int_{\mathbb{R}} \overline{f(E)}e^{isE}d\mu(E)\right]\langle\delta_n, e^{-ish_\lambda}g\rangle ds\\ =& \langle f, g\rangle - i\lambda \lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}} \overline{f(E)}\left[\int_0^\infty \langle\delta_n, e^{-is(h_\lambda - E - i\varepsilon)}g\rangle ds\right]d\mu(E).\end{aligned}$$

The interchange of order of intergration is justified by Fubini's theorem. Also for fixed $\varepsilon > 0$,

$$\int_0^\infty \langle\delta_n, e^{-is(h_\lambda - E - i\varepsilon)}g\rangle ds = \langle\delta_n, \frac{e^{-is(h_\lambda - E - i\varepsilon)}}{-i(h_\lambda - E - i\varepsilon)}g\rangle\Big|_0^\infty$$

and $e^{-is(h_\lambda - E - i\varepsilon)} = e^{-is(h_\lambda - E)}e^{-s\varepsilon}$ converges strongly to the zero operator, so:

$$\langle f, \Omega_+^*(h_\lambda, h_0)g\rangle = \langle f, g\rangle - \lambda\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}} \overline{f(E)}\langle\delta_n, (h_\lambda - E - i\varepsilon)^{-1}g\rangle d\mu(E).$$

At this point we have to justify that we can take the limit inside the integral. Since $\langle\delta_n, (h_\lambda - E - i0)^{-1}g\rangle$ exists and is finite for Lebesgue a.e. E, we have by Egoroff's theorem that for any n there is a measurable set M_n with $|\mathbb{R}\backslash M_n| < 1/n$ and $\langle\delta_n, (h_\lambda - E - i\varepsilon)^{-1}g\rangle \to \langle\delta_n, (h_\lambda - E - i0)^{-1}g\rangle$ uniformly on M_n. It is not hard to see that the set

$$\bigcup_{n>0}\{f \in \mathcal{H}_{\rm ac}(h_0) : \operatorname{supp} f \subset M_n\} \tag{4.24}$$

is dense in $\mathcal{H}_{\rm ac}(h_0)$. Suppose that f belongs to this set. Then $\langle\delta_n, (h_\lambda - E - i\varepsilon)^{-1}g\rangle \to \langle\delta_n, (h_\lambda -$

$E-i0)^{-1}g\rangle$ on the support of f. Therefore since $f\in L^1(\mathbb{R},d\mu)$, by the Dominated Convergence theorem,

$$\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}}|\overline{f(E)}||\langle\delta_n,(h_\lambda-E-i\varepsilon)^{-1}g\rangle-\langle\delta_n,(h_\lambda-E-i0)^{-1}g\rangle|d\mu(E)=0. \tag{4.25}$$

As a uniform limit of analytic functions, $\langle\delta_n,(h_\lambda-E-i0)^{-1}g\rangle$ is bounded on the support of f so $f\langle\delta_n,(h_\lambda-E-i0)^{-1}g\rangle\in L^1(\mathbb{R},d\mu)$. Therefore

$$\langle f,\Omega_+^*(h_\lambda,h_0)g\rangle=\langle f,g\rangle-\lambda\int_{\mathbb{R}}\overline{f(E)}\langle\delta_n,(h_\lambda-E-i0)^{-1}g\rangle d\mu(E) \tag{4.26}$$

holds for a dense set of f in $\mathcal{H}_{\mathrm{ac}}(h_0)$. □

Proposition 4.23. *Let $g\in\mathcal{H}$ be given and let $[S(h_\lambda,h_0)g](E)$ be the element of $L^2(\mathbb{R},d\mu)$ corresponding to $S(h_\lambda,h_0)g$ mapping $\mathcal{H}_{\mathrm{ac}}(h_0)=\mathcal{H}$ to itself. Then*

$$[S(h_\lambda,h_0)g](E)=g(E)-\lambda\left(1-\lambda\langle\delta_n,(h_\lambda-E-i0)^{-1}\delta_n\rangle\right)\langle\delta_n,M(E)g\rangle. \tag{4.27}$$

where $M(E)=(h_0-E-i0)^{-1}-(h_0-E+i0)^{-1}$.

Proof. Recall that $\Omega_+^*(h_\lambda,h_0)\Omega_+(h_\lambda,h_0)=\mathbb{1}_{\mathrm{ac}}(h_0)$. Let $f\in\mathcal{H}_{\mathrm{ac}}(h_0)$. Then

$$\begin{aligned}\langle f,(S-\mathbb{1})g\rangle=&\langle f,\Omega_+(h_\lambda,h_0)^*(\Omega_-(h_\lambda,h_0)-\Omega_+(h_\lambda,h_0))g\rangle\\=&\langle\Omega_+(h_\lambda,h_0)f,(\Omega_-(h_\lambda,h_0)-\Omega_+(h_\lambda,h_0)g\rangle\end{aligned}$$

In virtue of relation (4.19), $\Omega_--\Omega_+=-i\operatorname*{s-lim}_{t\to\infty}\int_{-t}^t e^{ish_\lambda}(h_\lambda-h_0)e^{-ish_0}ds$. Therefore

$$\begin{aligned}\langle f,(S-\mathbb{1})g\rangle=&-i\lim_{t\to\infty}\langle\Omega_+(h_\lambda,h_0)f,\int_{-t}^t e^{ish_\lambda}(h_\lambda-h_0)e^{-ish_0}g\,ds\rangle\\=&-i\lambda\lim_{t\to\infty}\int_{-t}^t\langle\Omega_+(h_\lambda,h_0)f,e^{ish_\lambda}\delta_n\rangle\langle\delta_n,e^{-ish_0}g\rangle ds\\=&-i\lambda\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}}e^{-\varepsilon|s|}\langle\Omega_+(h_\lambda,h_0)f,e^{ish_\lambda}\delta_n\rangle\langle\delta_n,e^{-ish_0}g\rangle ds\end{aligned}$$

where we have applied lemma 4.21. We use the intertwining property $e^{-ish_\lambda}\Omega_+=\Omega_+e^{-ish_0}$, and apply the result of proposition 4.22:

$$\begin{aligned}\langle f,(S-\mathbb{1})g\rangle=&-i\lambda\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}}e^{-\varepsilon|s|}\langle e^{-ish_0}f,\Omega_+^*(h_\lambda,h_0)\delta_n\rangle\langle\delta_n,e^{-ish_0}g\rangle ds\\=&-i\lambda\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}}e^{-\varepsilon|s|}\left[\int_{\mathbb{R}}\overline{f(E)}e^{isE}\left(\mathbb{1}(E)-\lambda\langle\delta_n,(h_\lambda-E-i0)^{-1}\delta_n\rangle\right)d\mu(E)\right]\langle\delta_n,e^{-ish_0}g\rangle ds\\=&-i\lambda\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}}\overline{f(E)}\left(\mathbb{1}(E)-\lambda\langle\delta_n,(h_\lambda-E-i0)^{-1}\delta_n\rangle\right)\left[\int_{\mathbb{R}}\langle\delta_n,e^{-is(h_0-E-i\varepsilon\mathrm{sign}(s)}g\rangle ds\right]d\mu(E)\end{aligned}$$

Here $d\mu(E)=\frac{\mathbb{1}_{[-2,2]}(E)}{\pi\sqrt{4-E^2}}dE$.

$$\left[\int_{\mathbb{R}}\langle\delta_n,e^{-is(h_0-E-i\varepsilon\mathrm{sign}(s)}g\rangle ds\right]=\frac{1}{i}\left(\langle\delta_n,(h_0-E-i\varepsilon)^{-1}g\rangle-\langle\delta_n,(h_0-E+i\varepsilon)^{-1}g\rangle\right).$$

A similar density argument as in proposition 4.22 shows that we may take the limit inside the integral.

It follows that

$$\langle f, (S - \mathbb{1})g\rangle = -\lambda \int_{\mathbb{R}} \overline{f(E)} \left(\mathbb{1}(E) - \lambda\langle\delta_n, (h_\lambda - E - i0)^{-1}\delta_n\rangle\right) \langle\delta_n, M(E)g\rangle d\mu(E)$$

where $M(E) = (h_0 - E - i0)^{-1} - (h_0 - E + i0)^{-1}$. □

5 Analysis of Absolutely Continuous Spectra

5.1 Periodic Schrödinger Operators on the Full Line

5.1.1 The Case of a Finite Interval

In this section we investigate the Laplacian on a finite interval with Dirichlet, periodic and antiperiodic boundary conditions. The results will be partly useful for the next section.

Notation 5.1.

(i) *Fix $N, M \in \mathbb{Z}$, $N < M$, let $\Gamma = [N, M] \subset \mathbb{Z}$.*

(ii) *We denote $p := (M - N + 1)$ the length of the interval $[N, M]$.*

(iii) *Let $\ell(\Gamma)$ denote the Hilbert space of all sequences $u = \{u(n)\}_{N \leq n \leq M}$ with coefficients in $\mathbb{C}$.*

Note that $\mathcal{H} := \mathbb{C}^p = \ell(\Gamma)$ is a Hilbert space of dimension p.

Definition 5.2. *The Laplacian operator h_0 on $\mathcal{H}$ with Dirichlet boundary conditions is given by:*

$$(h_0 u)(n) = u(n+1) + u(n-1) \tag{5.1}$$

$$u(N-1) = 0 \tag{5.2}$$

$$u(M+1) = 0 \tag{5.3}$$

h_0 is easily verified to be a linear operator and therefore it may be represented by a matrix. The matrix representing h_0 in the canonical basis for $\ell(\Gamma)$ is self-adjoint $p \times p$ matrix of the form:

$$[h_0] = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & 1 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \ddots & & 0 \\ & \ddots & \ddots & \ddots & \ddots & \\ 0 & & \ddots & \ddots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 1 & 0 \end{pmatrix} \tag{5.4}$$

The spectrum $\sigma(h_0)$ is pure point. In fact:

Proposition 5.3. *The eigenvalues and corresponding eigenvectors of h_0 are:*

$$\lambda_k = 2\cos\left(\frac{\pi k}{p+1}\right), \quad k = 1, 2, ..., p$$

$$u_k = \left\{ (u_k)(n) = \sin\left(\frac{(n-N+1)\pi k}{p+1} \right) \right\}.$$

Proof. It is a simple matter to verify that $h_0 u_k = \lambda_k u_k$ as given above. We have found $p = \dim \mathcal{H}$ distinct eigenvalues, so we have found all of them. □

Definition 5.4. *The Laplacian operator h_0 on $\mathcal{H}$ with periodic boundary conditions is given by:*

$$(h_0 u)(n) = u(n+1) + u(n-1) \tag{5.5}$$

$$u(N-1) = u(M) \tag{5.6}$$

$$u(M+1) = u(N) \tag{5.7}$$

The matrix representing h_0 in the canonical basis for $\ell(\Gamma)$ is self-adjoint $p \times p$ of the form:

$$[h_0] = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 1 \\ 1 & 0 & 1 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \ddots & & 0 \\ & \ddots & \ddots & \ddots & \ddots & \\ 0 & & \ddots & \ddots & 0 & 1 \\ 1 & \cdots & \cdots & \cdots & 1 & 0 \end{pmatrix} \tag{5.8}$$

Proposition 5.5. *The eigenvalues and corresponding eigenvectors of h_0 are (including repitition):*

$$\lambda_k = 2\cos\left(\frac{2k\pi}{p}\right), \quad k = 0, 1, ..., p-1$$

$$u_k = \left\{ (u_k)(n) = e^{i2k\pi n/p} \right\}.$$

More precisely:

(i) *If p is even: the $\frac{p}{2}+1$ distinct eigenvalues are:*

$$2 = 2\cos\left(2\pi\frac{0}{p}\right) > 2\cos\left(\pm 2\pi\frac{1}{p}\right) > 2\cos\left(\pm 2\pi\frac{2}{p}\right) > ... > 2\cos\left(\pm 2\pi\frac{(p/2-1)}{p}\right) > 2\cos\left(2\pi\frac{p/2}{p}\right) = -2.$$

(ii) *If p is odd: the $\frac{p+1}{2}$ distinct eigenvalues are:*

$$2 = 2\cos\left(2\pi\frac{0}{p}\right) > 2\cos\left(\pm 2\pi\frac{1}{p}\right) > 2\cos\left(\pm 2\pi\frac{2}{p}\right) > ... > 2\cos\left(\pm 2\pi\frac{(p-3)/2}{p}\right) > 2\cos\left(\pm 2\pi\frac{(p-1)/2}{p}\right).$$

Definition 5.6. *The Laplacian operator h_0 on $\mathcal{H}$ with antiperiodic boundary conditions is given by:*

$$(h_0 u)(n) = u(n+1) + u(n-1) \tag{5.9}$$

$$u(N-1) = -u(M) \tag{5.10}$$

$$u(M+1) = -u(N) \tag{5.11}$$

The matrix representing h_0 in the canonical basis for $\ell(\Gamma)$ is self-adjoint $p \times p$ of the form:

$$[h_0] = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & -1 \\ 1 & 0 & 1 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \ddots & & 0 \\ & \ddots & \ddots & \ddots & \ddots & \\ 0 & & \ddots & \ddots & 0 & 1 \\ -1 & \cdots & \cdots & \cdots & 1 & 0 \end{pmatrix} \tag{5.12}$$

Proposition 5.7. *The eigenvalues and corresponding eigenvectors of h_0 are (including repitition):*

$$\lambda_k = 2\cos\left(\frac{(2k+1)\pi}{p}\right), \quad k = 0, 1, ..., p-1$$

$$u_k = \left\{(u_k)(n) = e^{i(2k+1)\pi n/p}\right\}.$$

More precisely:

(i) *If p is even: the $\frac{p}{2}$ distinct eigenvalues are:*

$$2\cos\left(\pm\pi\frac{1}{p}\right) > 2\cos\left(\pm\pi\frac{3}{p}\right) > 2\cos\left(\pm\pi\frac{5}{p}\right) > ... > 2\cos\left(\pm\pi\frac{p-3}{p}\right) > 2\cos\left(\pm\pi\frac{p-1}{p}\right).$$

(ii) *If p is odd: the $\frac{p+1}{2}$ distinct eigenvalues are:*

$$2\cos\left(\pm\pi\frac{1}{p}\right) > 2\cos\left(\pm\pi\frac{3}{p}\right) > 2\cos\left(\pm\pi\frac{5}{p}\right) > ... > 2\cos\left(\pm\pi\frac{p-2}{p}\right) > 2\cos\left(\pi\frac{p}{p}\right) = -2.$$

5.1.2 The Full Line Periodic Operator

We explore period p periodic Schrödinger operators on $\mathcal{H} := \ell^2(\mathbb{Z})$, namely:

$$(hu)(n) = u(n-1) + u(n+1) + V(n)u(n) \tag{5.13}$$

$$V(n+p) = V(n) \ \forall n \in \mathbb{Z} \tag{5.14}$$

We reproduce the proofs given in Chapter 5 of [**Si**]. The main goal of this section is to show that the spectrum of periodic operators is absolutely continuous and composed of at most p disjoint bands, or intervals. The analysis of such operators is part of Floquet theory. Again we remind the reader that the potential V is real-valued, so that h is a self-adjoint operator. Periodicity of V also implies that h is a bounded operator not only from $\ell^2(\mathbb{Z})$ to $\ell^2(\mathbb{Z})$ but also from $\ell^\infty(\mathbb{Z})$ to $\ell^\infty(\mathbb{Z})$. If L denotes the shift to the left, periodicity implies

$$hL^p = L^p h. \tag{5.15}$$

Definition 5.8. *For $\theta \in [0, 2\pi)$, we set:*

$$\ell^\infty_\theta(\mathbb{Z}) := \{u \in \ell^\infty(\mathbb{Z}) : u(n+p) = e^{-i\theta}u(n)\ \forall n\} = \{u \in \ell^\infty(\mathbb{Z}) : L^p u = e^{-i\theta}u\}. \tag{5.16}$$

Solutions to (5.13) in $\ell^\infty_\theta(\mathbb{Z})$ are called Floquet solutions. They belong to $\ell^\infty(\mathbb{Z})\backslash\ell^2(\mathbb{Z})$. Recall that any $m \in \mathbb{Z}$ can be decomposed into $m = n + kp$ for some $n \in \{1, ..., p\}$ and $k \in \mathbb{Z}$, that is, $m = n\ (p)$.

Lemma 5.9. *$\ell^\infty_\theta(\mathbb{Z})$ is a vector subspace of $\ell^\infty(\mathbb{Z})$ of dimension p.*

Proof. The map $\ell^\infty_\theta(\mathbb{Z}) \ni u \to \{u(n)\}_{n=1}^p \in \mathbb{C}^p$ is easily seen to be a linear bijection. In particular it maps the basis $\{\delta_\theta^{(j)} : j = 1, ..., p\}$ to the basis $\{\delta_j : j = 1, ...p\}$, where $\delta_\theta^{(j)}$ is the sequence defined as $\delta_\theta^{(j)}(n + kp) = e^{-ik\theta}\delta_{jn}$, $n = 1, ..., p$; $k \in \mathbb{Z}$. □

We fix the basis of $\ell^\infty_\theta(\mathbb{Z})$ to be

$$\{\delta_\theta^{(j)} : j = 1, ..., p\} \text{ where } \delta_\theta^{(j)}(n + kp) = e^{-ik\theta}\delta_{jn}, n = 1, ..., p; k \in \mathbb{Z}.$$

Proposition 5.10. *$\ell^\infty_\theta(\mathbb{Z})$ is h-invariant and its matrix representation in the basis $\{\delta_\theta^{(j)}\}_{j=1}^p$ is:*

$$[h(\theta)] = \begin{pmatrix} V(1) & 1 & 0 & \cdots & \cdots & e^{i\theta} \\ 1 & V(2) & 1 & \cdots & \cdots & 0 \\ 0 & 1 & V(3) & \ddots & & 0 \\ & \ddots & \ddots & \ddots & \ddots & \\ 0 & & \ddots & \ddots & V(p-1) & 1 \\ e^{-i\theta} & \cdots & \cdots & \cdots & 1 & V(p) \end{pmatrix}$$

Proof. Combining (5.15) and (5.16) gives $L^p(hu) = h(L^p u) = h(e^{-i\theta}u) = e^{-i\theta}(hu)$, showing that h takes $\ell^\infty_\theta(\mathbb{Z})$ to itself. An easy calculation of $h\delta_\theta^{(1)}$, $h\delta_\theta^{(p)}$ and $h\delta_\theta^{(j)}$, $1 < j < p$ yields the above matrix. □

To simplify the notation, we will write $h(\theta)$ instead of $[h(\theta)]$.

Lemma 5.11. *If $u^{(j)} \in \ell^\infty_{\theta_j}(\mathbb{Z})$ for $j = 1, ..., q$ are nonzero with the θ_j distinct, then $\{u^{(j)}\}_{j=1}^q$ are linearly independent.*

Proof. Notice that for all $k \in \mathbb{Z}$, $e^{ik\theta_j}u^{(i)}(n + kp) = e^{ik(\theta_j-\theta_i)}u^{(i)}(n)$, and so

$$\sum_{k=-L}^{L} e^{ik\theta_j}u^{(i)}(n+kp) = \begin{cases} u^{(i)}(n)\left(\frac{e^{i(\theta_j-\theta_i)(L+1)}-1}{e^{i(\theta_j-\theta_i)}-1} + \frac{e^{i(\theta_i-\theta_j)(L+1)}-1}{e^{i(\theta_i-\theta_j)}-1} - 1\right) & j \neq i \\ (2L+1)u^{(i)}(n) & j = i \end{cases}$$

Consequently:

$$\lim_{L\to\infty} \frac{1}{2L+1} \sum_{k=-L}^{L} e^{ik\theta_j}u^{(i)}(n+kp) = \delta_{ij}u^{(i)}(n).$$

Now if $\sum_{i=1}^{q} \alpha_i u^{(i)} = 0$, then $\forall n \in \mathbb{Z}$:

$$\alpha_j u^{(j)}(n) = \sum_{i=1}^{q} \alpha_i \left(\delta_{ij}u^{(i)}(n)\right) = \lim_{L\to\infty} \frac{1}{2L+1} \sum_{k=-L}^{L} e^{ik\theta_j} \left(\sum_{i=1}^{q} \alpha_i u^{(i)}(n+kp)\right) = 0$$

and so $\alpha_j = 0$ for $j = 1, ..., q$. □

Notation 5.12. *We denote the p eigenvalues (counting multiplicity) of $h(\theta)$ by:*

$$e_1(\theta) \le e_2(\theta) \le \ldots \le e_p(\theta). \tag{5.17}$$

The following theorem is of fundamental importance in the analysis of the periodic Schrödinger operator.

Theorem 5.13. *Let h be a periodic Schrödinger operator. Then with the notation above, we have:*

(i) *$e_j(2\pi - \theta) = e_j(\theta)$ for $\theta \in [0, \pi]$.*

(ii) *For $\theta \in (0, \pi)$, the $e_j(\theta)$ are simple eigenvalues.*

(iii) *Each $e_j(\theta)$ is real analytic on $(0, \pi)$.*

(iv) *For $\theta, \theta' \in [0, 2\pi]$, $h(\theta)$ and $h(\theta')$ have disjoint eigenvalues unless $\theta = \theta'$ or $\theta = 2\pi - \theta'$.*

(v) *The eigenvalues of $h(0)$ and $h(\pi)$ satisfy the following hierarchy:*

$$e_p(0) > e_p(\pi) \ge e_{p-1}(\pi) > e_{p-1}(0) \ge \ldots \tag{5.18}$$

(vi) *On $(0, \pi)$, $(-1)^{p-j} e_j(\theta)$ is strictly monotone decreasing.*

Proof.

(i) Suppose that the diagonalisation of $h(\theta)$ is $h(\theta) = P(\theta)D(\theta)P^{-1}(\theta)$. Then:

$$h(2\pi - \theta) = \overline{h(\theta)} = \overline{P(\theta)}\ \overline{D(\theta)}\ \overline{P(\theta)}^{-1} = \overline{P(\theta)} D(\theta)\ \overline{P(\theta)}^{-1}$$

since the eigenvalues of $h(\theta)$ are real. We have used the fact that for matrices A, B: $\overline{AB} = \overline{A}\ \overline{B}$ and $(\overline{A})^{-1} = \overline{A^{-1}}$.

(ii) Suppose by contradiction that λ is a degenerate eigenvalue of $h(\theta)$, that is, there are $u^{(1)}, u^{(2)} \in \ell^\infty_\theta(\mathbb{Z})$ and linearly independent such that $hu^{(1)} = \lambda u^{(1)}$ and $hu^{(2)} = \lambda u^{(2)}$. By (i), λ is also an eigenvalue of $[h(2\pi-\theta)]$ and so there is $u^{(3)} \in \ell^\infty_{2\pi-\theta}(\mathbb{Z})$ such that $hu^{(3)} = \lambda u^{(3)}$. By lemma 5.11, $u^{(3)}$ is linearly independent with any linear combination of $u^{(1)}$ and $u^{(2)}$, hence we have $\dim E(\lambda) \ge 3$, which violates proposition 1.7.

(iii) Let $\chi(\theta, \lambda)$ be the characteristic polynomial of $h(\theta)$. It is a polynomial of degree p in λ with coefficients being analytic functions of θ. The functions $e_j(\theta)$ are defined implicitely through the relation $\chi(\theta, e_j(\theta)) = 0$. For $\theta \in (0, \pi)$, the $e_j(\theta)$ are simple roots, and therefore $\frac{\partial \chi}{\partial \lambda}(\theta, e_j(\theta)) \neq 0$. Hence by the analytic implicit function theorem, the $e_j(\theta)$ are analytic as simple roots of a polynomial with analytic coefficients.

(iv) All cases of $\theta, \theta' \in [0, 2\pi)$ can be handled in the same way as in (ii) by finding three linearly independent eigenvectors, except for the case $\{\theta, \theta'\} = \{0, \pi\}$, which will be shown in (v).

(v-vi) We let $h^{(0)}$ denote the free Laplacian, that is $(h^{(0)}u)(n) = u(n+1) + u(n-1)$, and $h^{(0)}(\theta)$ its restriction to $\ell^\infty_\theta(\mathbb{Z})$. $h^{(0)}$ is trivially a periodic Schrödinger operator and so all of the previous analysis applies. Referring to propositions 5.5 and 5.7, we see that equation (5.18) is valid for $h^{(0)}$.

In particular, $h^{(0)}(0)$ and $h^{(0)}(\pi)$ have different eigenvalues, so that (iv) is fully verified. (v) is also true for $h^{(0)}$.

We continuously propagate all the properties that $h^{(0)}$ has for h as follows: For $y \in [0,1]$, we let $h^{(y)} := (1-y)h^{(0)} + yh$. Notice that $h^{(y)}$ is again a periodic Schrödinger operator with potential yV and so all of the previous analysis may be applied to $h^{(y)}$. If $h^{(y)}(\theta)$ denotes the matrix restricting $h^{(y)}$ to $\ell^\infty_\theta(\mathbb{Z})$ and $\chi(y,\theta,\lambda)$ is the characteristic polynomial of $h^{(y)}(\theta)$, which is a polynomial of degree p in λ with coefficients being continuous functions of y and θ, it follows that the eigenvalues $e_j(y,\theta)$ are continous functions of θ and y on the square $[0,\pi]\times[0,1]$. They determine p continuous surfaces on the square $(\theta,y) \in [0,\pi]\times[0,1]$. These surfaces cannot intersect over $(0,\pi)\times[0,1]$ since for each $\theta \in (0,\pi)$, the eigenvalues $e_j(y,\theta)$ of $h^{(y)}_\theta$ are simple by (ii). Also, there is no way for the eigenvalue $e_j(y,0)$ of $h^{(y)}(0)$ to cross the eigenvalue $e_j(y,\pi)$ of $h^{(y)}(\pi)$ as y varies without crossing $e_j(y,\theta)$ for some $\theta \in (0,\pi)$, which cannot happen by (iv). In particular, (v) and (vi) must hold for all $y \in [0,1)$. By continuity, (iv), (v) and (vi) must also hold at $y = 1$.

□

We can now define the important notions of bands and gaps:

Definition 5.14. *For $j = 1, 2, ..., p$:*

$$b_j := \{e_j(\theta) : \theta \in [0,2\pi]\} = \{e_j(\theta) : \theta \in [0,\pi]\}$$

and

$$b_j^{int} := \{e_j(\theta) : \theta \in (0,\pi)\}$$

The b_j are called bands. For convenience, we will furthermore use the following notation: b_j stands for $[e_j(0), e_j(\pi)]$ if $e_j(0) < e_j(\pi)$ and stands for $[e_j(\pi), e_j(0)]$ if $e_j(0) > e_j(\pi)$.

By theorem 5.13, $b_j^{\text{int}} \cap b_{j'}^{\text{int}} = \varnothing$ for $j \neq j'$ and two different bands may intersect only at their common endpoint. The gaps are the open intervals between the bands. Thus there will be $p-1$ gaps if and only if the bands are disjoint. We have the mod p Fourier transform:

$$\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2\left([0,2\pi], \frac{d\theta}{2\pi}; \mathbb{C}^p\right) \tag{5.19}$$

$$(\mathcal{F}u)_n(\theta) = \sum_{k=-\infty}^{\infty} u(n+kp)e^{ik\theta} \tag{5.20}$$

The map is unitary since:

$$\|\mathcal{F}u\|^2_{L^2} = \sum_{n=1}^{p} \int_{[0,2\pi]} |(\mathcal{F}u)_n(\theta)|^2 \frac{d\theta}{2\pi} = \sum_{n=1}^{p} \sum_{k=-\infty}^{\infty} |u(n+kp)|^2 = \sum_{n\in\mathbb{Z}} |u(n)|^2. \tag{5.21}$$

It is a simple matter to check that the inverse is given by:

$$\mathcal{F}^{-1} : L^2\left([0,2\pi], \frac{d\theta}{2\pi}; \mathbb{C}^p\right) \to \ell^2(\mathbb{Z}) \tag{5.22}$$

$$(\mathcal{F}^{-1}f)(n+kp) = \langle e^{ik\theta}, f_n(\theta)\rangle = \int_{[0,2\pi]} e^{-ik\theta} f_n(\theta) \frac{d\theta}{2\pi}. \tag{5.23}$$

By the spectral theorem for finite matrices, there are unitaries $U(\theta) : \mathbb{C}^p \to \mathbb{C}^p$ such that:

$$U^{-1}(\theta)h(\theta)U(\theta) = \begin{pmatrix} e_1(\theta) & & & \\ & \ddots & & \\ & & \ddots & \\ & & & e_p(\theta) \end{pmatrix} \tag{5.24}$$

Lemma 5.15. *The unitary matrices $U(\theta)$ induce a unitary map $\mathcal{U} : L^2\left([0,2\pi], \frac{d\theta}{2\pi}; \mathbb{C}^p\right) \to L^2\left([0,2\pi], \frac{d\theta}{2\pi}; \mathbb{C}^p\right)$*

$$(\mathcal{U}f)(\theta) = U(\theta)f(\theta) \tag{5.25}$$

Proof. $\mathcal{U}$ is unitary since:

$$\|f\|_{L^2}^2 = \sum_{n=1}^{p} \int_{[0,2\pi]} |f_n(\theta)|^2 \frac{d\theta}{2\pi} = \int_{[0,2\pi]} \|f.(\theta)\|_{\mathbb{C}^p}^2 \frac{d\theta}{2\pi}$$
$$= \int_{[0,2\pi]} \|(Uf).(\theta)\|_{\mathbb{C}^p}^2 \frac{d\theta}{2\pi} = \sum_{n=1}^{p} \int_{[0,2\pi]} |(Uf)_n(\theta)|^2 \frac{d\theta}{2\pi} = \|\mathcal{U}f\|_{L^2}^2$$

Its inverse $\mathcal{U}^{-1} : L^2\left([0,2\pi], \frac{d\theta}{2\pi}; \mathbb{C}^p\right) \to L^2\left([0,2\pi], \frac{d\theta}{2\pi}; \mathbb{C}^p\right)$ is obviously given by:

$$(\mathcal{U}f)(\theta) = U^{-1}(\theta)f(\theta) \tag{5.26}$$

□

It is important to remind the reader that in order for $U(\theta)f(\theta) \in L^2\left([0,2\pi], \frac{d\theta}{2\pi}; \mathbb{C}^p\right)$ to make sense, we need $U(\theta)$ to be a measurable function of θ. By that we mean that the entries of $U(\theta)$ are measurable functions of θ. In fact, it is not hard to see that the entries of $U(\theta)$ are in fact continuous functions of θ.

Theorem 5.16. *Let h be a periodic Schrödinger operator. Then with the notation above, we have:*

(i)
$$[(\mathcal{F}h\mathcal{F}^{-1})f]_n(\theta) = [h(\theta)f]_n(\theta) \tag{5.27}$$

(ii)
$$[(\mathcal{U}\mathcal{F})h(\mathcal{U}\mathcal{F})^{-1})f]_n(\theta) = e_n(\theta)f_n(\theta) \tag{5.28}$$

Proof.

(i) For $1 < n < p$:

$$[(h\mathcal{F}^{-1})f](n+kp) = \langle e^{ik\theta}, f_{n-1}(\theta)\rangle + \langle e^{ik\theta}, f_{n+1}(\theta)\rangle + V(n)\langle e^{ik\theta}, f_n(\theta)\rangle$$

$$\begin{aligned}[(\mathcal{F}h\mathcal{F}^{-1})f]_n(\theta) &= \sum_{k=-\infty}^{\infty} \left(\langle e^{ik\theta'}, f_{n-1}(\theta')\rangle + \langle e^{ik\theta'}, f_{n+1}(\theta')\rangle + V(n)\langle e^{ik\theta'}, f_n(\theta')\rangle\right) e^{ik\theta} \\ &= f_{n-1}(\theta) + f_{n+1}(\theta) + V(n)f_n(\theta) = [h(\theta)f]_n(\theta).\end{aligned}$$

For $n = 1$ (the case $n = p$ is a similar calculation):

$$[(h\mathcal{F}^{-1})f](1+kp) = \langle e^{i(k-1)\theta}, f_p(\theta)\rangle + \langle e^{ik\theta}, f_2(\theta)\rangle + V(1)\langle e^{ik\theta}, f_1(\theta)\rangle$$

$$[(\mathcal{F}h\mathcal{F}^{-1})f]_1(\theta) = \sum_{k=-\infty}^{\infty} \left(\langle e^{i(k-1)\theta'}, f_p(\theta')\rangle + \langle e^{ik\theta'}, f_2(\theta')\rangle + V(1)\langle e^{ik\theta'}, f_1(\theta')\rangle\right) e^{ik\theta}$$

$$= e^{ik\theta} f_p(\theta) + f_2(\theta) + V(1)f_1(\theta) = [h(\theta)f]_1(\theta).$$

(ii) We apply equation (5.27) to $\mathcal{U}^{-1}f$ and get that for $n = 1, ..., p$:

$$[(\mathcal{F})h(\mathcal{U}\mathcal{F})^{-1})f]_n(\theta) = [h(\theta)U^{-1}(\theta)f]_n(\theta)$$

Thus equation (5.24) gives:

$$[(\mathcal{U}\mathcal{F})h(\mathcal{U}\mathcal{F})^{-1})f]_n(\theta) = [U(\theta)h(\theta)U^{-1}(\theta)f]_n(\theta) = e_n(\theta)f_n(\theta).$$

□

Lemma 5.17. *Let $a : [\alpha, \beta] \to \mathbb{R}$ be strictly monotone and continuous and $A : L^2([\alpha, \beta], dx) \to L^2([\alpha, \beta], dx)$ the self-adjoint operator of multiplication by a:*

$$(Af)(x) = a(x)f(x).$$

Then A is unitarily equivalent to the operator of multiplication

$$B : L^2([a(\alpha), a(\beta)], da^{-1}) \to L^2([a(\alpha), a(\beta)], da^{-1}), \quad (Bg)(y) = yg(y).$$

The unitary map is $\mathcal{V} : L^2([a(\alpha), a(\beta)], da^{-1}) \to L^2([\alpha, \beta], dx)$, $\mathcal{V}g = g \circ a$.

Remark 5.1. *a is a bijection and a^{-1} denotes its inverse function, not $(1/a)$. da^{-1} is the Stieltjes measure associated with a^{-1}.*

Proof. The change of variables formula for Stieltjes integrals gives

$$\int_{a(\alpha)}^{a(\beta)} |g(y)|^2 da^{-1}(y) = \int_{\alpha}^{\beta} |g(a(x))|^2 da^{-1}(a(x)) = \int_{\alpha}^{\beta} |(\mathcal{V}g)(x)|^2 dx$$

which shows that $\mathcal{V}$ is an isometry. It is also surjective since $\mathcal{V}(f \circ a^{-1}) = f$. Finally one easily verifies $\mathcal{V}B\mathcal{V}^{-1} = A$. □

Theorem 5.18. *Let $h : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be a period p Schrödinger operator with bands $\{b_j\}_{j=1}^p$. Then $\sigma(h) = b := \cup_j b_j$ and the spectrum is purely absolutely continuous with multiplicity two.*

Proof. Note that

$$L^2\left([0, 2\pi], \frac{d\theta}{2\pi}\right) = L^2\left([0, \pi], \frac{d\theta}{2\pi}\right) \oplus L^2\left([\pi, 2\pi], \frac{d\theta}{2\pi}\right)$$

and

$$L^2\left([0, 2\pi], \frac{d\theta}{2\pi}; \mathbb{C}^p\right) = L^2\left([0, 2\pi], \frac{d\theta}{2\pi}\right) \bigoplus_1 ... \bigoplus_{p-1} L^2\left([0, 2\pi], \frac{d\theta}{2\pi}\right).$$

Hence by lemma 5.17:

$$L^2\left([0,2\pi],\frac{d\theta}{2\pi};\mathbb{C}^p\right)=\left(\bigoplus_{j=1}^{p}L^2\left([e_j(0),e_j(\pi)],\frac{de_j^{-1}(\lambda)}{2\pi}\right)\right)\bigoplus\left(\bigoplus_{j=1}^{p}L^2\left([e_j(\pi),e_j(2\pi)],\frac{de_j^{-1}(\lambda)}{2\pi}\right)\right)$$

Let $\lambda\in b_j^{\mathrm{int}}$. There is a unique $\theta\in(0,\pi)$ such that $\lambda=e_j(\theta)=e_j(2\pi-\theta)$. We write $\theta(\lambda)$. In particular $\theta'(\lambda)=\frac{1}{e_j'(\theta(\lambda))}$. Since $e_j(\theta)$ is real analytic on $(0,\pi)$, its inverse $e_j^{-1}(\lambda)$ is real analytic on b_j^{int} and so in particular

$$de_j^{-1}(\lambda)=(e_j^{-1})'(\lambda)d\lambda=\frac{1}{e_j'(\theta(\lambda))}d\lambda=\theta'(\lambda)d\lambda. \tag{5.29}$$

There is also a sign issue depending on whether $e_j'(\theta)\gtrless 0$:

$$e_j(0)<e_j(\pi)\iff e_j'(\theta)>0,\ \theta\in(0,\pi)\iff(e_j^{-1})'(\lambda)>0,\ \lambda\in b_j^{\mathrm{int}}.$$

From this we conclude that:

$$L^2\left([e_j(0),e_j(\pi)],\frac{de_j^{-1}(\lambda)}{2\pi}\right)=\begin{cases}L^2\left(b_j,\frac{1}{2\pi}\theta'(\lambda)d\lambda\right) & \text{if } e_j(0)<e_j(\pi)\\ L^2\left(b_j,\frac{1}{2\pi}|\theta'(\lambda)|d\lambda\right) & \text{if } e_j(0)>e_j(\pi)\end{cases}$$

The same analysis can be carried out for $\theta\in(\pi,2\pi)$. We define

$$d\nu(\lambda):=\frac{1}{2\pi}|\theta'(\lambda)|d\lambda. \tag{5.30}$$

Then

$$\begin{aligned}L^2\left([0,2\pi],\frac{d\theta}{2\pi};\mathbb{C}^p\right)&=\left(\bigoplus_{j=1}^{p}L^2\left(b_j,d\nu(\lambda)\right)\right)\bigoplus\left(\bigoplus_{j=1}^{p}L^2\left(b_j,d\nu(\lambda)\right)\right)\\&=L^2\left(b,d\nu(\lambda)\right)\oplus L^2\left(b,d\nu(\lambda)\right)\\&=L^2\left(b,d\nu(\lambda);\mathbb{C}^2\right)\end{aligned}$$

By composing the various unitary maps of theorem 5.16 and lemma 5.17, we have a unitary map

$$\hat{\mathcal{U}}:\ell^2(\mathbb{Z})\to L^2(b,d\nu(\lambda);\mathbb{C}^2)$$

satisfying for $f(\lambda)=(f^+(\lambda),f^-(\lambda))$:

$$[\hat{\mathcal{U}}h\hat{\mathcal{U}}^{-1}f]^{\pm}(\lambda)=\lambda f^{\pm}(\lambda).$$

The spectrum of $M_\theta:L^2\left(b,d\nu(\lambda)\right)\to L^2\left(b,d\nu(\lambda)\right)$, the operator of mulitplication by the independent variable θ, is the support of the measure ν, namely b. Thus $\sigma(h)=b$. Formula (5.29) show that the spectrum is purely absolutely continuous. □

In the proof of the previous theorem, we were pretty vague about the unitary map $\hat{\mathcal{U}}:\ell^2(\mathbb{Z})\to L^2\left(\sigma(h),d\nu(\lambda);\mathbb{C}^2\right)$. We make this map more explicit.

Let $\lambda\in b_j^{\mathrm{int}}$ and $\theta=\theta(\lambda)$ be such that $\lambda=e_j(\theta)=e_j(2\pi-\theta)$. Let φ_λ^+ denote an eigenvector for $h(\theta)$

and λ and φ_λ^- denote an eigenvector for $h(2\pi - \theta)$ and λ. We have:

$$(h-\lambda)\varphi_\lambda^\pm = 0 \tag{5.31}$$

and

$$\varphi_\lambda^\pm(n+kp) = e^{\mp ik\theta(\lambda)}\varphi_\lambda^\pm(n). \tag{5.32}$$

Lemma 5.19. *$\varphi_\lambda^\pm(n) \neq 0$ for all $n \in \mathbb{Z}$.*

Proof. $\varphi_\lambda^+ \in E(\lambda)$ and $\lambda \in \mathbb{R}$ implies that $\overline{\varphi_\lambda^+} \in E(\lambda)$. Moreover

$$\overline{\varphi_\lambda^+(n+kp)} = e^{+ik\theta(\lambda)}\overline{\varphi_\lambda^+(n)} = e^{-ik(2\pi-\theta(\lambda))}\overline{\varphi_\lambda^+(n)}.$$

So $\overline{\varphi_\lambda^+} \in \ell^\infty_{2\pi-\theta(\lambda)}$. If $\varphi_\lambda^+(n) = 0$, then $\overline{\varphi_\lambda^+(n)} = 0$ and so $W(\varphi_\lambda^+, \overline{\varphi_\lambda^+}) = 0$, implying that φ_λ^+ and $\overline{\varphi_\lambda^+}$ are linearly dependent. However, they are also linearly independent by lemma 5.11, so we get a contradiction. □

As a consequence of lemma 5.19, we may normalize $\varphi_\lambda^\pm$ by requiring:

$$\varphi_\lambda^\pm(0) > 0 \tag{5.33}$$

and

$$\sum_{j=1}^{p} |\varphi_\lambda^\pm(j)|^2 = 1. \tag{5.34}$$

Lemma 5.20. *With this normalization: $\varphi_\lambda^- = \overline{\varphi_\lambda^+}$.*

Proof. Since $\varphi_\lambda^-, \overline{\varphi_\lambda^+} \in E(\lambda) \cap \ell^\infty_{2\pi-\theta(\lambda)}$ and λ is a simple eigenvalue of $h(2\pi - \theta(\lambda))$, it follows that φ_λ^- and $\overline{\varphi_\lambda^+}$ are linearly dependent. Normalization (5.34) shows that in fact they must be equal. □

Recall the measure $d\nu(\lambda) = \frac{1}{2\pi}|\theta'(\lambda)|d\lambda$ defined on $\cup_j b_j^{\text{int}}$ from theorem 5.18. We finally have a complete picture of the direct integral decomposition:

Theorem 5.21. *The unitary map $\hat{\mathcal{U}} : \ell^2(\mathbb{Z}) \to L^2\left(\sigma(h), d\nu(\lambda); \mathbb{C}^2\right)$*

$$\hat{\mathcal{U}} : u \to \hat{u}^\pm(\lambda) = \sum_{n\in\mathbb{Z}} \overline{\varphi_\lambda^\pm(n)}u(n) \tag{5.35}$$

satisfies

$$[\hat{\mathcal{U}}(hu)]^\pm(\lambda) = \lambda[\hat{\mathcal{U}}u]^\pm(\lambda). \tag{5.36}$$

Its inverse is given by

$$\hat{\mathcal{U}}^{-1} : (f^+(\lambda), f^-(\lambda)) \to f(n) = \int_{\sigma(h)} \left(\varphi_\lambda^+(n)f^+(\lambda) + \varphi_\lambda^-(n)f^-(\lambda)\right)d\nu(\lambda) \tag{5.37}$$

Proof. Note that $\tilde{\varphi}_\lambda^+ := \{\varphi_\lambda^+(n)\}_{n=1}^p$ is an eigenvector of $h(\theta(\lambda))$ normalized by (5.34), so if $\lambda_1, ..., \lambda_p$ are the λ's with a given θ, $\{\tilde{\varphi}^+_{\lambda_j}\}_{j=1}^p$ is an orthonormal basis for $\mathbb{C}^p$. Thus the following matrix is unitary:

$$\tilde{U}^\pm = \begin{pmatrix} \varphi^\pm_{\lambda_1}(1) & \varphi^\pm_{\lambda_2}(1) & \cdots & \cdots & \varphi^\pm_{\lambda_p}(1) \\ \varphi^\pm_{\lambda_1}(2) & \varphi^\pm_{\lambda_2}(2) & \cdots & \cdots & \varphi^\pm_{\lambda_p}(2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \varphi^\pm_{\lambda_1}(p) & \varphi^\pm_{\lambda_2}(p) & \cdots & \cdots & \varphi^\pm_{\lambda_p}(p) \end{pmatrix} \tag{5.38}$$

First we have:

$$\sum_{n\in\mathbb{Z}} \overline{\varphi^\pm_\lambda(n)} u(n) = \sum_{n=1}^p \overline{\varphi^\pm_\lambda(n)} \sum_{k\in\mathbb{Z}} e^{ik\theta(\lambda)} u(n+kp) = \sum_{n=1}^p \overline{\varphi^\pm_\lambda(n)} (\mathcal{F}u)_n(\theta(\lambda)).$$

$$\begin{aligned}
\|\hat{u}\|^2_{L^2} &= \int_{\sigma(h)} |\hat{u}^+(\lambda)|^2 d\nu(\lambda) + \int_{\sigma(h)} |\hat{u}^-(\lambda)|^2 d\nu(\lambda) \\
&= \sum_{j=1}^p \int_{b_j} \left| \sum_{n=1}^p \overline{\varphi^+_\lambda(n)} (\mathcal{F}u)_n(\theta(\lambda)) \right|^2 d\nu(\lambda) + \sum_{j=1}^p \int_{b_j} \left| \sum_{n=1}^p \overline{\varphi^-_\lambda(n)} (\mathcal{F}u)_n(\theta(\lambda)) \right|^2 d\nu(\lambda) \\
&= \int_0^\pi \sum_{j=1}^p \left| \sum_{n=1}^p \overline{\varphi^+_{\lambda_j(\theta)}(n)} (\mathcal{F}u)_n(\theta) \right|^2 \frac{d\theta}{2\pi} + \int_\pi^{2\pi} \sum_{j=1}^p \left| \sum_{n=1}^p \overline{\varphi^-_{\lambda_j(\theta)}(n)} (\mathcal{F}u)_n(\theta) \right|^2 \frac{d\theta}{2\pi} \\
&= \int_0^\pi \|(\tilde{U}^+)^*(\mathcal{F}u).(\theta)\|^2 \frac{d\theta}{2\pi} + \int_\pi^{2\pi} \sum_{j=1}^p \|(\tilde{U}^-)^*(\mathcal{F}u).(\theta)\|^2 \frac{d\theta}{2\pi} \\
&= \int_0^\pi \|(\mathcal{F}u).(\theta)\|^2 \frac{d\theta}{2\pi} + \int_\pi^{2\pi} \|(\mathcal{F}u).(\theta)\|^2 \frac{d\theta}{2\pi} = \sum_{n\in\mathbb{Z}} |u(n)|^2
\end{aligned}$$

by (5.21).

(5.36) is a consequence of (5.31) because:

$$\begin{aligned}
[\hat{\mathcal{U}}(hu)]^\pm(\lambda) &= \sum_{n\in\mathbb{Z}} \overline{\varphi^\pm_\lambda(n)} (u(n+1) + u(n-1) + V(n)u(n)) \\
&= \sum_{n\in\mathbb{Z}} \left(\overline{\varphi^\pm_\lambda(n-1)} + \overline{\varphi^\pm_\lambda(n+1)} + V(n)\overline{\varphi^\pm_\lambda(n)} \right) u(n) \\
&= \sum_{n\in\mathbb{Z}} \overline{(h\varphi^\pm_\lambda)(n)} u(n) = \lambda [\hat{\mathcal{U}}u]^\pm(\lambda).
\end{aligned}$$

□

5.2 Bounded Eigenfunctions and Absolutely Continuous Spectrum

In this section, we will consider the Schrödinger operator h on $\ell^2(\mathbb{Z}_+)$ given by

$$(hu)(n) = u(n+1) + u(n-1) + V(n)u(n) \tag{5.39}$$

$$u(0) = 0 \tag{5.40}$$

We reproduce the proof in Simon's paper [**Si2**] to show that if all the eigenfunctions of $h = h_0 + V$ are bounded, then h has purely absolutely continuous spectrum. We show that if V is of bounded variation, then the spectrum of h is absolutely continuous on $(-2, 2)$. For any $z \in \mathbb{C}$, recall the two fundamental solutions $c(z), s(z)$ with boundary conditions:

$$\begin{pmatrix} s(0,z) & s(1,z) \\ c(0,z) & c(1,z) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5.41}$$

Recall from definition 1.15 that the Weyl m-function is defined for $z \in \mathbb{C} \backslash \mathbb{R}$ as $m_+(z) := -u(1, z)$, where $u(z) = \frac{u_+(z)}{u_+(0,z)}$ is the unique sequence satisfying the full line Schrödinger equation, being square summable at $+\infty$ and normalized by $u(0, z) = 1$. lemma 1.12 is so useful we give another proof of it:

Lemma 5.22. *Let $z \in \mathbb{C} \backslash \mathbb{R}$. Then* $\text{Im } m_+(z) = (\text{Im } z) \sum_{n=1}^{\infty} |u(n, z)|^2$.

Proof. We look at the Wronskian of u and $\overline{u}$:

$$W_0(u(z), \overline{u(z)}) = u(0)\overline{u(1)} - u(1)\overline{u(0)} = -\overline{m_+(z)} + m_+(z) = 2i \text{ Im } m_+(z).$$

$$\begin{aligned} W_{n+1}(u(z), \overline{u(z)}) &= u(n+1)\overline{(zu(n+1) - V(n+1)u(n+1) - u(n))} \\ &\quad - (zu(n+1) - V(n+1)u(n+1) - u(n))\,\overline{u(n+1)} \\ &= W_n(u(z), \overline{u(z)}) - 2i\ (\text{Im } z)|u(n+1)|^2. \end{aligned}$$

So $W_{n+1}(u(z), \overline{u(z)}) = W_0(u(z), \overline{u(z)}) - 2i\ (\text{Im } z) \sum_{i=1}^{n+1} |u(i)|^2$. The result follows by taking $n \to \infty$. □

Recall the transfer matrix

$$A(E, n) := T(E, n, n-1) = \begin{pmatrix} E - V(n) & -1 \\ 1 & 0 \end{pmatrix} \tag{5.42}$$

where T is the matrix satisfying

$$T(E, n, 0) = \begin{pmatrix} s(n+1, E) & c(n+1, E) \\ s(n, E) & c(n, E) \end{pmatrix} \quad \text{and} \quad T(E, n, m) = T(E, n, 0)T(E, m, 0)^{-1}. \tag{5.43}$$

Note that $\|T(E, n, n-1)\| = \|A(E, n)\| \geq 1$ so that

$$1 \leq C(E) := \sup_{n,m \in \mathbb{Z}_+} \|T(E, n, m)\| \leq \sup_{n \in \mathbb{Z}_+} \|T(E, n, 0)\|^2 \tag{5.44}$$

is finite if and only if both $s(E)$ and $c(E)$ are bounded.

Definition 5.23. *Let $S := \{E \in \mathbb{R} : c(E), s(E)$ are bounded on $\mathbb{Z}_+\}$.*

Theorem 5.24. *On S, the spectral measure μ for h is purely absolutely continuous in the sense that*

(i) $\mu_{\text{ac}}(T) > 0$ *for any $T \subset S$ with $|T| > 0$.*

(ii) $\mu_{\text{sing}}(S) = 0$.

In fact, if $E \in S$, then:

(iii) $\liminf_{\varepsilon \downarrow 0} \operatorname{Im} m_+(E + i\varepsilon) \geq \frac{1}{4} C(E)^{-3} > 0$.

(iv) $\limsup_{\varepsilon \downarrow 0} |m_+(E + i\varepsilon)| \leq 4C(E)^3 < \infty$.

Proof. It is known that $d\mu(E) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \operatorname{Im} m_+(E + i\varepsilon) dE$, see for example [**Si3**]. By the theorem of de la Vallée Poussin,

$$d\mu_{\text{ac}}(E) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \operatorname{Im} m_+(E + i\varepsilon) dE \tag{5.45}$$

and

$$\mu_{\text{sing}} \text{ is supported on } \{E \in \mathbb{R} : \lim_{\varepsilon \downarrow 0} \operatorname{Im} m_+(E + i\varepsilon) = \infty\}. \tag{5.46}$$

Therefore (iii),(iv) imply (i), (ii) respectively.

$$\begin{aligned} T(E + i\varepsilon, n, 0) &= \left[\begin{pmatrix} i\varepsilon & 0 \\ 0 & 0 \end{pmatrix} + A(E, n) \right] \left[\begin{pmatrix} i\varepsilon & 0 \\ 0 & 0 \end{pmatrix} + A(E, n-1) \right] ... \left[\begin{pmatrix} i\varepsilon & 0 \\ 0 & 0 \end{pmatrix} + A(E, 1) \right] \\ &= T(E, n, 0) + \sum_{j=0}^{n-1} T(E, n, j+1) \begin{pmatrix} i\varepsilon & 0 \\ 0 & 0 \end{pmatrix} \left[\begin{pmatrix} i\varepsilon & 0 \\ 0 & 0 \end{pmatrix} + A(E, j) \right] ... \left[\begin{pmatrix} i\varepsilon & 0 \\ 0 & 0 \end{pmatrix} + A(E, 1) \right] \\ &= T(E, n, 0) + \sum_{j=0}^{n-1} (i\varepsilon) T(E, n, j+1) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T(E + i\varepsilon, j, 0) \end{aligned} \tag{5.47}$$

By induction on n we show using (5.47) that

$$\|T(E + i\varepsilon, n, 0)\| \leq C(1 + C\varepsilon)^n \leq Ce^{\varepsilon Cn} \tag{5.48}$$

For $n = 1$, we have

$$\|T(E + i\varepsilon, 1, 0)\| = \left\| \begin{pmatrix} i\varepsilon & 0 \\ 0 & 0 \end{pmatrix} + A(E, 1) \right\| \leq \varepsilon + C \leq C(1 + C\varepsilon).$$

Suppose that (5.48) holds for $n - 1$, then

$$\|T(E + i\varepsilon, n, 0)\| \leq \left\| \left[\begin{pmatrix} i\varepsilon & 0 \\ 0 & 0 \end{pmatrix} + A(E, n) \right] \right\| \|T(E + i\varepsilon, n-1, 0)\| \leq (\varepsilon + C) C(1 + C\varepsilon)^{n-1} = C(1 + C\varepsilon)^n.$$

The identity

$$T(E + i\varepsilon, n, 0)^{-1} \begin{pmatrix} u(E + i\varepsilon, n+1) \\ u(E + i\varepsilon, n) \end{pmatrix} = \begin{pmatrix} u(E + i\varepsilon, 1) \\ u(E + i\varepsilon, 0) \end{pmatrix}$$

together with $\|T^{-1}\| = \|T\|$ and (5.48) leads to

$$|u(E + i\varepsilon, n+1)|^2 + |u(E + i\varepsilon, n)|^2 \geq C^{-2} e^{-2\varepsilon Cn} \left(1 + |m_+(E + i\varepsilon)|^2\right)$$

Summing over $n = 1, 3, 5, ...$ yields:

$$\sum_{n=1}^{\infty} |u(E+i\varepsilon,n)|^2 \geq \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} C^{-2}e^{-2\varepsilon Cn}\left(1+|m_+(E+i\varepsilon)|^2\right) = C^{-2}e^{-2\varepsilon C}\left(1-e^{-4\varepsilon C}\right)^{-1}\left(1+|m_+(E+i\varepsilon)|^2\right)$$

The identity $\text{Im } m_+(E+i\varepsilon) = \varepsilon \sum_{n=1}^{\infty} |u(E+i\varepsilon,n)|^2$ (lemma 5.22) implies:

$$\text{Im } m_+(E+i\varepsilon) \geq \frac{1}{4}C^{-3}e^{-2\varepsilon C}(4\varepsilon C)\left(1-e^{-4\varepsilon C}\right)^{-1}\left(1+|m_+(E+i\varepsilon)|^2\right)$$

and a simple application of l'Hôpital's rule gives

$$\liminf_{\varepsilon\downarrow 0} \frac{\text{Im } m_+(E+i\varepsilon)}{1+|m_+(E+i\varepsilon)|^2} \geq \frac{1}{4}C^{-3} \tag{5.49}$$

Since $1 \geq \frac{1}{(1+|m_+(E+i\varepsilon)|^2)}$, (5.49) implies $\liminf_{\varepsilon\downarrow 0} \text{Im } m_+(E+i\varepsilon) \geq \frac{1}{4}C^{-3}$.
(5.49) also implies $\limsup_{\varepsilon\downarrow 0} |m_+(E+i\varepsilon)| \leq 4C^3$ using the fact that

$$\frac{1}{|m_+(E+i\varepsilon)|} \geq \frac{\text{Im } m_+(E+i\varepsilon)}{1+|m_+(E+i\varepsilon)|^2} \quad \text{and} \quad \liminf_{\varepsilon\downarrow 0} \frac{1}{|m_+(E+i\varepsilon)|} = \frac{1}{\limsup_{\varepsilon\downarrow 0} |m_+(E+i\varepsilon)|}.$$

□

We now apply this result to potentials with bounded total variation.

Theorem 5.25. *Suppose that the potential of the Schrödinger operator $h = h_0 + V$ vanishes at infinity and is of bounded variation. That is:*

1. $V(n) \to 0$ *as* $n \to \infty$.
2. $\sum_{n=1}^{\infty} |V(n+1) - V(n)| < \infty$.

Then $\sigma(h)$ is purely absolutely continuous on $(-2, 2)$. In particular, the result holds for $V \in \ell^1(\mathbb{Z}_+)$.

Remark 5.2. *Slightly more generally, if V is of bounded variation, then $V_\infty := \lim_{n\to\infty} V(n)$ exists since*

$$|V(n) - V(m)| \leq \sum_{j=n}^{m-1} |V(j+1) - V(j)| \to 0 \text{ as } n, m \to \infty.$$

So by writing $h = (h_0 + V_\infty) + (V - V_\infty)$ we see that it is no loss of generality to assume $V(n) \to 0$, i.e. $\sigma(h)$ will be purely absolutely continuous in $(-2 + V_\infty, 2 + V_\infty)$.

Proof. Suppose that x satisfies $hx = Ex$ with $E \in \mathbb{R}$. Moreover we may assume that $x(n) \in \mathbb{R}, \forall n$, since that is the case for $c(E)$ and $s(E)$. Let

$$K(n) := x^2(n+1) + x^2(n) + [V(n) - E]x(n)x(n+1).$$

We show that K is a bounded sequence whenever $E \in (-2, 2)$. (5.51) will then imply that x is bounded and the result will follow by theorem 5.24.

$$K(n+1)-K(n) = [x(n+2)-x(n)][x(n+2)+x(n)+(V(n+1)-E)x(n+1)]+[V(n+1)-V(n)]x(n)x(n+1).$$

Thus

$$|K(n+1)-K(n)| \leq |V(n+1)-V(n)||x(n)x(n+1)| \leq |V(n+1)-V(n)|(x^2(n+1)+x^2(n)). \quad (5.50)$$

Suppose that $E \in (-2,2)$. Then since $V(n) \to 0$, $\exists N_0$ such that for all $n \geq N_0$, $2-|V(n)-E| \geq \delta > 0$. For such n:

$$\begin{aligned} K(n) &\geq x^2(n+1) + x^2(n) - |V(n)-E||x(n)x(n+1)| \\ &\geq x^2(n+1) + x^2(n) + (\delta-2)|x(n)x(n+1)| \\ &= \frac{\delta}{2}(x^2(n+1)+x^2(n)) + \left(1-\frac{\delta}{2}\right)(|x(n+1)|-|x(n)|)^2 \\ &\geq \frac{\delta}{2}(x^2(n+1)+x^2(n)) \end{aligned} \quad (5.51)$$

so (5.50) becomes

$$|K(n+1)-K(n)| \leq \frac{2}{\delta}|V(n+1)-V(n)|K(n)$$

so that

$$K(n+1) \leq \left(1+\frac{2}{\delta}|V(n+1)-V(n)|\right)K(n).$$

Inductively, we get that for all $n \geq N_0$:

$$K(n) \leq K(N_0)\prod_{j=N_0}^{n}\left(1+\frac{2}{\delta}|V(n+1)-V(n)|\right) \leq K(N_0)\prod_{j=N_0}^{\infty}\left(1+\frac{2}{\delta}|V(n+1)-V(n)|\right).$$

This product is convergent is convergent since:

$$\ln\prod_{j=N_0}^{\infty}\left(1+\frac{2}{\delta}|V(n+1)-V(n)|\right) = \sum_{j=N_0}^{\infty}\ln\left(1+\frac{2}{\delta}|V(n+1)-V(n)|\right) \quad (5.52)$$

$$\leq \sum_{j=N_0}^{\infty}\frac{2}{\delta}|V(n+1)-V(n)| < \infty. \quad (5.53)$$

□

6 One Dimensional Random Schrödinger Operators

6.1 Pastur's theorem, Minimally and Uniquely Ergodic Operators

We now deal with families of random Hamiltonians. The traditional setup is as follows: let $\Omega = \mathbb{R}^{\mathbb{Z}}$ and for each $\omega \in \Omega$, let $V_\omega := \sum_{n\in\mathbb{Z}} \omega(n)\langle\delta_n, \cdot\rangle\delta_n$. We write $h_\omega = h_0 + V_\omega$, and h_ω is an operator on $\ell(\mathbb{Z})$. Since V_ω is real-valued, h_ω is self-adjoint on its natural domain $\mathrm{Dom}(h_\omega) = \{f \in \ell^2(\mathbb{Z}) : h_\omega f \in \ell^2(\mathbb{Z})\} = \mathrm{Dom}(V_\omega)$. Of course if instead Ω were equal to $S^{\mathbb{Z}}$ for some bounded Borel subset S of $\mathbb{R}$, then $\mathrm{Dom}(h_\omega) = \ell^2(\mathbb{Z})$. For many applications Ω is taken to be $S^{\mathbb{Z}}$ with S bounded, and we will stick with this convention.

The structure on Ω is that of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ is the $\sigma-$algebra generated by the cylinder sets, i.e. sets of the form $\{\omega : \omega_{i_1} \in A_1, ..., \omega_{i_n} \in A_n\}$ for $i_1, ..., i_n \in \mathbb{Z}$ and $A_1, ..., A_n$ Borel sets in S.

We define the shift operators T_i on Ω by $(T_i\omega)(n) = \omega(n-i)$ (i.e. shifts to the right by i). Note that $T_i : \Omega \to \Omega$ is a bijection and $(T_i)^{-1} = T^{-i}$. It is also a measurable transformation since the inverse image of cylinder set is again a cylinder set, namely $T_{-1}\{\omega : \omega(n_1) \in A_1, \omega(n_2) \in A_2, ..., \omega(n_k) \in A_k\} = \{\omega : \omega(n_1 - 1) \in A_1, \omega(n_2 - 1) \in A_2, ..., \omega(n_k - 1) \in A_k\}$.

Definition 6.1. *A probability measure $\mathbb{P}$ on Ω is called stationary if $\mathbb{P}(T_{-1}A) = \mathbb{P}(A)$ for any $A \in \mathcal{F}$. Accordingly the $\{T_i\}$ are called measure preserving transformations for $\mathbb{P}$. A stationary probability measure is called ergodic w.r.t to T if $T_{-1}(A) = A$ implies $\mathbb{P}(A) \in \{0, 1\}$.*

An extended random variable $X : \Omega \to \mathbb{R} \cup \{+\infty\} := \overline{R}$ is called invariant under T_i if $X \circ T_i = X$ $\mathbb{P}$-a.s. for all $i \in \mathbb{Z}$.

Proposition 6.2. *Suppose that $\mathbb{P}$ is ergodic and that the extended random variable X is invariant under T. Then X is constant $\mathbb{P}$-a.s.*

Proof. Let $\Omega_M := \{\omega : X(\omega) \le M\}$. Since X is invariant, that is, $X(T_i\omega) = X(\omega)$, $T_{-1}\Omega_M = \Omega_M$, and ergodicity implies that $\mathbb{P}(\Omega_M) \in \{0, 1\}$. Notice that $M \le M'$ implies $\Omega_M \subset \Omega_{M'}$. Denote $Z \cup \{+\infty\} := \overline{Z}$. We have

$$\bigcup_{M\in\overline{R}} \Omega_M = \bigcup_{M\in\overline{Z}} \Omega_M = \Omega \quad \text{and} \quad \bigcap_{M\in\overline{R}} \Omega_M = \bigcap_{M\in\overline{Z}} \Omega_M = \emptyset.$$

Now $\mathbb{P}(\bigcap_{M\in\overline{Z}} \Omega_M) = 0$ forces $M_0 := \inf_{\mathbb{P}(\Omega_M)=1} M$ to be finite.

We have $\Omega_{M_0} = \bigcap_{n\in\mathbb{N}} \Omega_{M_0+(1/n)}$ and $\tilde{\Omega}_{M_0} := \{X < M_0\} = \bigcup_{n\in\mathbb{N}} \Omega_{M_0-(1/n)}$

and so $\mathbb{P}(\Omega_{M_0}) = 1$, $\mathbb{P}(\tilde{\Omega}_{M_0}) = 0$. It follows that $\mathbb{P}(X = M_0) = \mathbb{P}(\Omega_{M_0} \backslash \tilde{\Omega}_{M_0}) = 1$. □

Similarly to the T_i, define unitary shift operators U_i on $\ell^2(\mathbb{Z})$ by $(U_i u)(n) = u(n-i)$. The following lemma provides a key relation for families of random operators.

Lemma 6.3. *If $\{h_\omega\}_{\omega\in\Omega}$ is a family of self-adjoint operators that is ergodic with respect to the family $\{T_i\}_{i\in\mathbb{Z}}$ of measure preserving transformations, then $f(h_{T_i\omega}) = U_i f(h_\omega) U_i^*$ for all $f \in B_b(\mathbb{R})$, the bounded Borel functions. Moreover, if h_ω is bounded, then the relation holds for all polynomials and continuous functions as well.*

Proof. We will assume that the h_ω are bounded operators. We show explicitly that $h_{T_i\omega} = U_i h_\omega U_i^*$.

$$(U_i^* u)(n) = u(n+i)$$

$$(h_\omega U_i^* u)(n) = (h_0 u)(n+i) + \omega(n) u(n+i)$$

$$(U_i h_\omega U_i^* u)(n) = (h_0 u)(n) + \omega(n-i) u(n)$$

Therefore, $U_i h_\omega U_i^* u = h_0 + \sum_{n \in \mathbb{Z}} \omega_{n-i} \langle \delta_n, \cdot \rangle \delta_n = h_{T_i \omega}$. The relation easily extends to polynomials.

If h_ω were unbounded, the relation can be verified directly for resolvents, i.e. for functions $f_z(x) = 1/(x-z)$, $z \in \mathbb{C} \backslash \mathbb{R}$. Linear combinations of the f_z are dense in $C_0(\mathbb{R})$, the continuous functions vanishing at infinity, and so by the Stone-Weierstrass theorem, the relation holds for $f \in C_0(\mathbb{R})$.

Finally since h_ω and $U h_\omega U^*$ are self-adjoint, we have for $f \in C_0(\mathbb{R})$:

$$\int_{\mathbb{R}} f(E) d\mu_{\psi,\phi}(E) = \langle \psi, f(U h_\omega U^*) \phi \rangle = \langle \psi, U f(h_\omega) U^* \phi \rangle$$

$$= \langle U^* \psi, f(h_\omega) U^* \phi \rangle = \int_{\mathbb{R}} f(E) d\nu_{U^*\psi, U^*\phi}(E).$$

This shows that $\mu_{\psi,\phi}$ and $\nu_{U^*\psi,U^*\phi}$ agree so that the relation holds in fact for all $f \in B_b(\mathbb{R})$. □

We now come to the celebrated theorem of Pastur, namely that the spectrum of random Schrödinger operators is almost surely the same set.

Theorem 6.4. *There exist deterministic sets Σ_{ac}, Σ_{sc}, Σ_{pp} such that for $\mathbb{P}$-a.e. ω,*

$$\sigma_{\text{ac}}(h_\omega) = \Sigma_{\text{ac}} \qquad \sigma_{\text{sc}}(h_\omega) = \Sigma_{\text{sc}} \qquad \sigma_{\text{pp}}(h_\omega) = \Sigma_{\text{pp}}.$$

Proof. By lemma 6.3, we have

$$\int_{\mathbb{R}} \frac{d\mu_n^{T_i\omega}(t)}{t-z} = \langle \delta_n, (U_i h_\omega U_i^* - z)^{-1} \delta_n \rangle$$

$$= \langle U_i^* \delta_n, (h_\omega - z)^{-1} U_i^* \delta_n \rangle = \langle \delta_{n+i}, (h_\omega - z)^{-1} \delta_{n+i} \rangle = \int_{\mathbb{R}} \frac{d\mu_{n+i}^\omega(t)}{t-z},$$

hence $\mu_{n+i}^\omega = \mu_n^{T_i\omega}$.

Let $\mu^{T_i\omega} = \sum_{n \in \mathbb{Z}} a_n \mu_n^{T_i\omega}$ be a spectral measure for $h_{T_i\omega}$. Then $\mu^{T_i\omega} = \sum_{n \in \mathbb{Z}} a_n \mu_{n+i}^\omega = \sum_{n \in \mathbb{Z}} a_{n-i} \mu_n^\omega$. So $\mu^{T_i\omega}$ is also a spectral measure for h_ω. By proposition 8.14, $\sigma_{\text{ac}}(h_{T_i\omega}) = \text{supp } \mu_{\text{ac}}^{T_i\omega} = \sigma_{\text{ac}}(h_\omega) = \text{supp } \mu_{\text{ac}}^\omega$. The same goes for the sc/pp components.

In what follows # stands for ac/sc/pp. For $r_1, r_2 \in \mathbb{Q}$, $r_1 < r_2$, consider the set

$$E_{r_1,r_2}^{\#} = \{\omega : (r_1, r_2) \cap \text{supp } \mu_{\#}^\omega = \varnothing\}.$$

supp $\mu_{\#}^{T_i\omega} =$ supp $\mu_{\#}^\omega$ means that $T_i(E_{r_1,r_2}^{\#}) = E_{r_1,r_2}^{\#}$ for all i and since $\mathbb{P}$ is ergodic, we have $\mathbb{P}(E_{r_1,r_2}^{\#}) \in \{0, 1\}$.

Let

$$\Omega_{r_1,r_2}^{\#} = \begin{cases} E_{r_1,r_2}^{\#} & \text{if } \mathbb{P}(E_{r_1,r_2}^{\#}) = 1 \\ \Omega \backslash E_{r_1,r_2}^{\#} & \text{if } \mathbb{P}(E_{r_1,r_2}^{\#}) = 0 \end{cases} \quad \text{and} \quad \tilde{\Omega}^{\#} = \bigcap_{r_1, r_2 \in \mathbb{Q}} \Omega_{r_1,r_2}^{\#}.$$

Note that $\mathbb{P}(\tilde{\Omega}^{\#}) = 1$ since $\tilde{\Omega}^{\#}$ is the countable intersection of sets of full measure. We claim that for $\omega_1, \omega_2 \in \tilde{\Omega}^{\#}$, $\sigma_{\#}(h_{\omega_1}) = \sigma_{\#}(h_{\omega_2})$. Indeed, if $e \notin \sigma_{\#}(h_{\omega_1})$, then there are rationals r_1, r_2 such that $r_1 < e < r_2$ and $\omega_1 \in E^{\#}_{r_1,r_2}$. Then $\omega_2 \in E^{\#}_{r_1,r_2}$ and so $e \notin \sigma_{\#}(h_{\omega_2})$. Reversing the roles of ω_1 and ω_2 yields the statement. □

In the last theorem, we overlooked an important step, namely verifying that the sets $E^{\#}_{r_1,r_2}$ are measurable. We will provide the necessary details to convince ourselves that the sets are measurable.

Definition 6.5. *We say that the family of operators $\{A_\omega\}$ is weakly measurable if the mapping $\omega \to \langle\psi, A_\omega\phi\rangle$ is measurable for all $\psi, \phi \in \mathcal{H}$ (of course it is assumed here that the A_ω are bounded operators). For a family of self-adjoint operators, we say that the family $\{A_\omega\}$ is self-adjoint measurable if the mapping $\omega \to \langle\psi, (A_\omega - z)^{-1}\phi\rangle$ is measurable for all $\psi, \phi \in \mathcal{H}$ and $z \in \mathbb{C}\backslash\mathbb{R}$.*

Note that $\langle\psi, A_\omega B_\omega\phi\rangle = \sum_n \langle\psi, A_\omega\delta_n\rangle\langle\delta_n, B_\omega\phi\rangle$ shows that the product of bounded weakly measurable operators is again weakly measurable.

We recall some facts about the measurability of projections and spectral measures. The reader is referred to [Ja1] for more details. If the map $\{h_\omega\}$ is self-adjoint measurable, then $f(h_\omega)$ is weakly measurable for all $f \in B_b(\mathbb{R})$, in particular for $f(t) = e^{it}$ and $f(t) = \mathbb{1}_B(t)$, the characteristic function of a Borel set. By the Trotter product formula, the family $\{h_\omega\}$ is self-adjoint measurable.

Proposition 6.6. *The projections $\mathbb{1}_{\text{cont}}(h_\omega)$ and $\mathbb{1}_{\text{ac}}(h_\omega)$ are weakly measurable.*

Proof. Let $\chi_N(n) = 1$ if $|n| \le N$ and $\chi_N(n) = 0$ if $|n| > N$. It follows by the RAGE theorem (theorem 46, [Ja]) and the polarization identity that for every N:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \langle e^{-ith_\omega}\mathbb{1}_{\text{cont}}(h_\omega)f, \chi_N \mathbb{1}_{\text{cont}}(h_\omega)e^{-ith_\omega}g\rangle dt = 0,$$

Therefore (assuming wlog that $f \in \mathbb{1}_{\text{cont}}(h_\omega)\mathcal{H}$)

$$\begin{aligned}
\langle f, \mathbb{1}_{\text{cont}}(h_\omega)g\rangle &= \frac{1}{T}\int_0^T \langle f, e^{ith_\omega}(\mathbb{1} - \chi_N + \chi_N)\mathbb{1}_{\text{cont}}(h_\omega)e^{-ith_\omega}g\rangle dt \\
&= \lim_{T\to\infty}\frac{1}{T}\int_0^T \langle f, e^{ith_\omega}(\mathbb{1} - \chi_N)\mathbb{1}_{\text{cont}}(h_\omega)e^{-ith_\omega}g\rangle dt \\
&= \lim_{T\to\infty}\frac{1}{T}\int_0^T \langle f, e^{ith_\omega}(\mathbb{1} - \chi_N)e^{-ith_\omega}g\rangle dt \\
&- \lim_{T\to\infty}\frac{1}{T}\int_0^T \langle f, e^{ith_\omega}(\mathbb{1} - \chi_N)\mathbb{1}_{\text{pp}}(h_\omega)e^{-ith_\omega}g\rangle dt.
\end{aligned}$$

The last term is estimated using $\mathbb{1}_{\text{pp}}(h_\omega)g = \sum_j g_j$, where the g_j are eigenfunctions of h_ω, i.e. $h_\omega g_j = \lambda_j g_j$.

$$\begin{aligned}
\langle f, e^{ith_\omega}(\mathbb{1} - \chi_N)\mathbb{1}_{\text{pp}}(h_\omega)e^{-ith_\omega}g\rangle &\le \|e^{-ith_\omega}f\|\|(\mathbb{1} - \chi_N)\mathbb{1}_{\text{pp}}(h_\omega)e^{-ith_\omega}g\| \\
&\le \|f\|\Big\|\sum_j(\mathbb{1} - \chi_N)e^{-it\lambda_j}g_j\Big\| \le \|f\|\sum_j\Big\|(\mathbb{1} - \chi_N)g_j\Big\|.
\end{aligned}$$

Therefore we have

$$\langle f, \mathbb{1}_{\text{cont}}(h_\omega)g\rangle = \lim_{N\to\infty}\lim_{T\to\infty}\frac{1}{T}\int_0^T \langle f, e^{ith_\omega}(\mathbb{1} - \chi_N)e^{-ith_\omega}g\rangle dt$$

which is measurable.

As for $\mathbb{1}_{\mathrm{ac}}(h_\omega)$, we have the following formula:

$$\langle \psi, \mathbb{1}_{\mathrm{ac}}(h_\omega)\psi\rangle = \lim_{N\to\infty}\lim_{p\uparrow 1}\lim_{\varepsilon\downarrow 0}\frac{1}{\pi^p}\int_{-N}^{N}[\mathrm{Im}\,\langle\psi,(h_\omega - E - i\varepsilon)^{-1}\psi\rangle]^p dE. \tag{6.1}$$

which shows that $\omega \to \langle\psi, \mathbb{1}_{\mathrm{ac}}(h_\omega)\psi\rangle$ is weakly measurable. The result follows by the polarization identity. □

As a consequence of this proposition, it follows that

$$\omega \to \mathbb{1}_{\mathrm{sc}}(h_\omega) = \mathbb{1}_{\mathrm{cont}}(h_\omega) - \mathbb{1}_{\mathrm{ac}}(h_\omega) \quad \text{and} \quad \omega \to \mathbb{1}_{\mathrm{pp}}(h_\omega) = \mathbb{1} - \mathbb{1}_{\mathrm{cont}}(h_\omega) \tag{6.2}$$

are weakly measurable.

Finally to show that the sets $E^{\#}_{r_1,r_2}$ of theorem 6.4 are measurable, we note that for any Borel set B,

$$\omega \to \langle \mathbb{1}_B(h_\omega)\psi, \mathbb{1}_{\#}(h_\omega)\psi\rangle = \mu^{\omega}_{\psi,\#}(B) \tag{6.3}$$

are weakly measurable functions, and therefore the maps

$$\omega \to \mu^{\omega}_{\#}(B) = \sum_n a_n \mu^{\omega}_{\psi,\#}(B) \tag{6.4}$$

are weakly measurable. Consequently the identity

$$E^{\#}_{r_1,r_2} = \{\omega : (r_1,r_2)\cap \mathrm{supp}\ \mu^{\omega}_{\#} = \varnothing\} = \cap_n\{\omega : \mu^{\omega}_{\#}(r_1 - 1/n, r_2 - 1/n) = 0\} \tag{6.5}$$

shows that $E^{\#}_{r_1,r_2}$ are measurable sets.

Let us make the setup of the random Schrödinger operators even more general. Let $f \in \Omega = S^{\mathbb{Z}} \to \mathbb{R}$ be a continuous function and define the potential $V_\omega(n) = f(T_n\omega)$. In many applications, Ω is compact, so that $f(T_n\omega)$ has bounded range, and the h_ω are uniformly bounded operators on $\ell^2(\mathbb{Z})$. The standard metric on Ω is then $d(\omega,\omega') = \sum_n 2^{-n}\frac{|\omega(n)-\omega'(n)|}{1+|\omega(n)-\omega'(n)|}$. Note that if f is taken to be the projection onto the zeroth coorinate, we retrieve the potential described earlier.

Definition 6.7. *If the orbit $\{T_n\omega : n \in \mathbb{Z}\}$ is dense in Ω for every ω, we say that T is minimal for Ω.*

If further we assume that T is minimal, we can give a worthy improvement on Pastur's theorem. If Σ denotes the deterministic set such that $\sigma(h_\omega) = \Sigma$ for almost every ω, then we have in fact (see [**Da**]):

Theorem 6.8. *Suppose that T is minimal for Ω. Then for every $\omega \in \Omega$, $\sigma(h_\omega) = \Sigma$.*

Proof. Let $\omega_1, \omega_2 \in \Omega$ be given. By minimality, there is a sequence n_j such that $T_{n_j}\omega_2 \to \omega_1$ as $j \to \infty$. Note that $T_m T_{n_j}\omega_2 \to T_m\omega_1$ uniformly in $m \in \mathbb{Z}$, and so:

$$\|h_{T_{n_j}\omega_2}\psi - h_{\omega_1}\psi\| = \sum_m |f(T_m T_{n_j}\omega_2) - f(T_m\omega_1)|^2|\psi(m)|^2\delta_m\Big\| \le \varepsilon\|\psi\|$$

where by continuity of f, we have chosen j sufficiently large so that $|f(T_mT_{n_j}\omega_2) - f(T_m\omega_1)|^2 \le \varepsilon$. Therefore $h_{T_{n_j}\omega_2}$ converges strongly to h_{ω_1}. Denote by $\mu_\psi^{T_{n_j}\omega_2}$ and $\mu_\psi^{\omega_1}$ the spectral measures for ψ and $h_{T_{n_j}\omega_2}$ and h_{ω_1} respectively. Then using the Borel transform representation shows that $\mu_\psi^{T_{n_j}\omega_2}$ converges weakly to $\mu_\psi^{\omega_1}$, and so by the Portmanteau theorem, $\liminf \mu_\psi^{T_{n_j}\omega_2}(I) \geq \mu_\psi^{\omega_1}(I)$ for every open interval I. Taking a point $x \in \sigma(h_{\omega_1})$ and picking intervals I_n centered at x and converging to x shows that $\liminf \mu_\psi^{T_{n_j}\omega_2}(I_n) \geq \mu_\psi^{\omega_1}(I_n) > 0$ and we can find a sequence $(x_n) \in \cup_{j\geq 1}\sigma(h_{T_{n_j}\omega_2})$ converging to x. Therefore

$$\sigma(h_{\omega_1}) \subset \overline{\bigcup_{j\geq 1} \sigma(h_{T_{n_j}\omega_2})}.$$

Finally each of the operators $h_{T_{n_j}\omega_2}$ is unitarily equivalent to h_{ω_2}, therefore we have

$$\sigma(h_{\omega_1}) \subset \overline{\bigcup_{j\geq 1} \sigma(h_{T_{n_j}\omega_2})} = \sigma(h_{\omega_2}).$$

Reversing the roles of ω_1 and ω_2 completes the proof. □

Definition 6.9. *A measurable transformation $T : \Omega \to \Omega$ is said to be uniquely ergodic if it admits only one invariant probability measure.*

Recall the Ergodic theorem, the strong form of the Strong Law of Large Numbers: if $f \in L^1(\Omega, d\mathbb{P})$, then $\mathbb{P}$ - almost surely:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T_k\omega) = \int_\Omega f(\omega) d\mathbb{P}(\omega) \quad \text{(constant)}. \tag{6.6}$$

Theorem 6.10. *If T is uniquely ergodic, then for all $f \in C(\Omega)$, the collection of continuous functions, $\frac{1}{n}\sum_{k=0}^{n-1} f(T_k\omega)$ converges uniformly in $\omega \in \Omega$ to $c_f := \int_\Omega f d\mathbb{P}$.*

Proof. Suppose not. Then there exists $f_0 \in C(\Omega)$ for which the limit does not exist or for which the convergence is not uniform. In either case, there exists $\varepsilon > 0$ and a sequence (ω_j) such that for all $j \geq 1$,

$$\left| \frac{1}{n_j} \sum_{k=0}^{n_j-1} f_0(T_k\omega_j) - c_{f_0} \right| \geq \varepsilon.$$

Let $\Phi := \{\varphi_m\}$ be a countable dense subset of $C(\Omega)$. Consider $\left\{\frac{1}{n_j}\sum_{k=0}^{n_j-1} \varphi_1(T_k\omega_j)\right\}_{j=1}^{\infty}$. This is a bounded sequence of real numbers (bounded above by sup $|\varphi_1(\omega)|$) and so it has a convergent subsequence. Using a diagonal argument, there is a common subsequence (for simplicity also denoted by (ω_j)) such that for all $\varphi_m \in \Phi$:

$$\frac{1}{n_j} \sum_{k=0}^{n_j-1} \varphi_m(T_k\omega_j) \quad \text{converges}. \tag{6.7}$$

We now show that by density of $\Phi \in C(\Omega)$ we can extend (6.7) to all $g \in C(\Omega)$. Let $\varepsilon > 0$ and choose m such that $\sup_{\omega\in\Omega} |g(\omega) - \varphi_m(\omega)| < \varepsilon$. Then

$$\left| \frac{1}{n_j} \sum_{k=0}^{n_j-1} g(T_k\omega_j) - \frac{1}{n_j} \sum_{k=0}^{n_j-1} \varphi_m(T_k\omega_j) \right| < \varepsilon. \tag{6.8}$$

An $\varepsilon/3$ argument shows that $\left\{\frac{1}{n_j}\sum_{k=0}^{n_j-1} g(T_k\omega_j)\right\}_{j=1}^{\infty}$ is a Cauchy sequence, namely:

$$\left|\frac{1}{n_j}\sum_{k=0}^{n_j-1} g(T_k\omega_j) - \frac{1}{n_{j'}}\sum_{k=0}^{n_{j'}-1} g(T_k\omega_{j'})\right| \le \left|\frac{1}{n_j}\sum_{k=0}^{n_j-1} g(T_k\omega_j) - \frac{1}{n_j}\sum_{k=0}^{n_j-1} \varphi_m(T_k\omega_j)\right|$$

$$+\left|\frac{1}{n_j}\sum_{k=0}^{n_j-1} \varphi_m(T_k\omega_j) - \frac{1}{n_{j'}}\sum_{k=0}^{n_{j'}-1} \varphi_m(T_k\omega_{j'})\right| + \left|\frac{1}{n_{j'}}\sum_{k=0}^{n_{j'}-1} g(T_k\omega_{j'}) - \frac{1}{n_{j'}}\sum_{k=0}^{n_{j'}-1} \varphi_m(T_k\omega_{j'})\right|.$$

The first and third terms on the RHS are controlled by (6.8) while the middle term is controlled because of (6.7). Therefore for all $g \in C(\Omega)$,

$$\frac{1}{n_j}\sum_{k=0}^{n_j-1} g(T_k\omega_j) \text{ converges.} \tag{6.9}$$

On the other hand, we know by the Ergodic theorem that for all $g \in C(\Omega)$, $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} g(T_k\omega) = c_g$ almost surely. Therefore there exists $\tilde{\omega}$ such that $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f_0(T_k\tilde{\omega}) = c_{f_0}$.

Considering the dense set Φ, the sequence $\left\{\frac{1}{n}\sum_{k=0}^{n-1} \varphi_1(T_k\tilde{\omega})\right\}_{n=1}^{\infty}$ and repeating the diagonal argument yields a common subsequence (n_i) such that for all $g \in C(\Omega)$,

$$\frac{1}{n_i}\sum_{k=0}^{n_i-1} g(T_k\tilde{\omega}) \text{ converges.} \tag{6.10}$$

Now (6.9) and (6.10) define respectively two bounded linear functionals $J_1, J_2 : C(\Omega) \to \mathbb{R}$ satisfying $J_1(\mathbb{1}) = J_2(\mathbb{1}) = 1$ and $\|J_1\|, \|J_2\| \le 1$. By the Riesz Representation theorem, there exist Borel measures μ_1, μ_2 on Ω such that

$$J_1(g) = \int_\Omega g d\mu_1, \qquad J_2(g) = \int_\Omega g d\mu_2.$$

The identity

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} g(T_k\omega) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n-1} g(T_k\omega)$$

shows that μ_1, μ_2 are T-invariant. However as $|J_1(f_0) - c_{f_0}| \ge \varepsilon$ and $J_2(f_0) = c_{f_0}$, $\mu_1 \ne \mu_2$ and so T is not uniquely ergodic, a contradiction. □

Definition 6.11. *We say that $\mathbb{P}$ has full measure if $\mathbb{P}(O) > 0$ for every open set in Ω.*

Proposition 6.12. *If T is uniquely ergodic, and its invariant measure has full measure, then T is minimal.*

Proof. Let f be supported on an arbitrary open set $O \neq \varnothing$, and we may suppose wlog that $\int_\Omega f d\mathbb{P} = 1$. Then by theorem 6.10, $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(T_k\omega) = \int_\Omega f d\mathbb{P} = 1$ for every ω. Therefore the orbit of ω must enter O. □

Again we are in the setting where Ω is compact and $V_\omega(n) = f(T_n(\omega))$, f continuous.

Proposition 6.13. *The map $\Omega \ni \omega \to (h_\omega - z)^{-1}\phi$ is continuous for every $\phi \in \ell^2(\mathbb{Z})$ and $z \in \mathbb{C}\backslash\mathbb{R}$.*

Proof. By the first resolvent identity, $(h_\omega - z)^{-1}\phi - (h_{\omega'} - z)^{-1}\phi = (h_\omega - z)^{-1}(V_\omega - V_{\omega'})(h_{\omega'} - z)^{-1}\phi$. Therefore

$$\|(h_\omega - z)^{-1}\phi - (h_{\omega'} - z)^{-1}\phi\| \leq \frac{1}{|\text{Im } z|}\|(V_\omega - V_{\omega'})(h_{\omega'} - z)^{-1}\phi\|.$$

Now

$$\begin{aligned}(V_\omega - V_{\omega'})(h_{\omega'} - z)^{-1}\phi\|^2 &= \sum_{k\in\mathbb{Z}} |\langle (V_\omega - V_{\omega'})(h_{\omega'} - z)^{-1}\phi, \delta_k\rangle|^2 \\ &= \sum_{k\in\mathbb{Z}} |V_\omega(k) - V_{\omega'}(k)|^2 |\langle (h_{\omega'} - z)^{-1}\phi, \delta_k\rangle|^2\end{aligned}$$

Fix $\varepsilon > 0$, there exists M such that $|V_\omega(k) - V_{\omega'}(k)| \leq M$ for all $k \in \mathbb{Z}, \omega, \omega'$, and there exists K_0 such that $\sum_{|k|\geq K_0} M^2 |\langle (h_{\omega'} - z)^{-1}\phi, \delta_k\rangle|^2 \leq \varepsilon/2$. By continuity of f, we can choose $\delta > 0$, such that $d(\omega, \omega') < \delta$ implies $\sum_{k<K_0} |V_\omega(k) - V_{\omega'}(k)|^2 |\langle (h_{\omega'} - z)^{-1}\phi, \delta_k\rangle|^2 \leq \varepsilon/2$. The result follows. □

Let Σ be the set such that $\sigma(h_\omega) = \Sigma$, $\mathbb{P}$-a.s.

Proposition 6.14. *If $\mathbb{P}$ is of full measure, then $\sigma(h_\omega) \subset \Sigma$ for all $\omega \in \Omega$.*

Proof. Let $x \notin \Sigma$. Consider a smooth bump function $f \in C^\infty(\mathbb{R})$ that is supported around x, but $\text{supp } f \cap \Sigma = \emptyset$. Then by the Helffer-Sjöstrand formula, $f(h_\omega) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial f}{\partial \bar{z}} \tilde{f}(z)(h_\omega - z)^{-1} dxdy$ for all ω. Therefore $\langle \phi, f(h_\omega)\phi\rangle = \frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial f}{\partial \bar{z}} \tilde{f}(z)\langle \phi, (h_\omega - z)^{-1}\phi\rangle dxdy$ for all ω. Applying proposition 6.13 and the fact that $\mathbb{P}$ has full measure shows that the spectral measure for h_ω is not supported near x. □

6.2 Example of Ergodic Operators: The Anderson Model

The setup for the Anderson model is as follows: We have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The real-valued potentials $\{V_\omega(n)\}_{n\in\mathbb{Z}}$ are independent and identically distributed random variables with common probability distribution ν, that is, for any Borel set $A \subset \mathbb{R}$ and any n,

$$\mathbb{P}(\{\omega : V_\omega(n) \in A\}) = \nu(A)$$

and for any finite collection of Borel sets $A_1, ..., A_k$ of $\mathbb{R}$

$$\mathbb{P}\Big(\{\omega : V_\omega(n_1) \in A_1, ..., V_\omega(n_k) \in A_k\}\Big) = \prod_{i=1}^{k} \mathbb{P}\left(\{\omega : V_\omega(n_i) \in A_i\}\right) = \prod_{i=1}^{k} \nu(A_i)$$

For every $\omega \in \Omega$ we define a multiplication operator

$$V_\omega := \sum_{n\in\mathbb{Z}} V_\omega(n)\langle \delta_n, \cdot\rangle \delta_n$$

with domain $\text{Dom}(V_\omega) := \{\psi \in \ell^2(\mathbb{Z}) : V_\omega\psi \in \ell^2(\mathbb{Z})\}$. By the Anderson model we mean the hamiltonian

$$h_\omega = h_0 + V_\omega$$

where $h_0 = -\Delta$ is the discrete Laplacian on the lattice. We obviously have $\text{Dom}(h_\omega) = \text{Dom}(V_\omega)$.

Proposition 6.15. *For every $\omega \in \Omega$, V_ω is self-adjoint on $\text{Dom}(V_\omega)$ and essentially self-adjoint on $\ell_0(\mathbb{Z})$. h_ω is self-adjoint and essentially self-adjoint on $\ell_0(\mathbb{Z})$.*

Proof. The statement for V_ω is proposition 8.21. Since h_0 is a bounded self-adjoint operator, the statement for h_ω follows by the Kato-Rellich theorem (theorem 8.24). □

If supp ν is compact, say supp $\nu \subset [-M, M]$, then almost surely we have $|V_\omega(n)| \leq M$ for all n. In that case $\mathrm{Dom}(h_\omega)$ is all of $\ell^2(Z)$ and the family of random operators $\{h_\omega = h_0 + V_\omega\}$ is almost surely uniformly bounded by $2 + M$.

Example 6.16. *In the literature it is common to take the following probability space: We start with a probability space $(S, \mathcal{B}_S, \nu)$, where $S \subset \mathbb{R}$ and $\mathcal{B}_S$ denotes the Borel sigma-algebra on S. Then the standard construction is the infinite product space*

$$(\Omega, \mathcal{F}, \mathbb{P}) = \bigotimes_{n\in\mathbb{Z}} (S, \mathcal{B}_S, \nu)$$

$\bigotimes_{n\in\mathbb{Z}} \mathcal{B}_S$ is the sigma-algebra generated by the cylinder sets $\{\omega : \omega(n_1) \in A_1, ..., \omega(n_k) \in A_k\}$ for $n_1, ..., n_k \in \mathbb{Z}$ and $A_1, ..., A_k$ Borel sets in $\mathbb{R}$. Recall $\bigotimes_{n\in\mathbb{Z}} \mathcal{B}_S = \mathcal{B}_{S^\mathbb{Z}}$. $\mathbb{P}$ satisfies $\mathbb{P}\Big(\{\omega : \omega(n_1) \in A_1, ..., \omega(n_k) \in A_k\}\Big) = \prod_{i=1}^{k} \nu(A_i)$. Consequently the random variables $V_\omega(n) := \omega(n)$ are independent and identically distributed by construction. The Anderson model takes the form :

$$h_\omega = h_0 + \sum_{n\in\mathbb{Z}} \omega(n) \langle \delta_n, \cdot \rangle \delta_n$$

Theorem 6.17. *The product measure $\mathbb{P} = \bigotimes_{n\in\mathbb{Z}} \nu$ is ergodic w.r.t. the shifts $\{T_j\}$, $(T_j\omega)(n) = \omega(n-j)$.*

Notation 6.18. *We denote $\mathcal{F}_0$ the collection of all cylinder sets in $S^\mathbb{Z}$.*

Proof. First we show that $\{T_j\}$ is a measure-preserving family. For $A_1, ..., A_k \subset S$, let $A = \{\omega : \omega(n_1) \in A_1, ..., \omega(n_k) \in A_k\} \in \mathcal{F}_0$. Then:

$$\mathbb{P}\left(T_{-j}A\right) = \mathbb{P}\big(\{\omega : \omega(n_1 - j) \in A_1, ..., \omega(n_k - j) \in A_k\}\big) = \prod_{i=1}^{k} \nu(A_i) = \mathbb{P}(A)$$

Moreover one easily verifies using the relations $T_{-j}(A^c) = (T_{-j}A)^c$ and $T_{-j}(\cup_n A_n) = \cup_n (T_{-j}A_n)$ that the collection $\mathcal{F}_1$ of all sets $A \in \mathcal{F}$ satisfying $\mathbb{P}\left(T_{-j}A\right) = \mathbb{P}(A)$ is a sigma-algebra. Thus $\mathcal{F}_1 = \mathcal{F}$.

Now we show that for all $A, B \in \mathcal{F}$:

$$\mathbb{P}\left((T_{-j}A) \cap B\right) \to \mathbb{P}(A)\mathbb{P}(B) \quad \text{as } |j| \to \infty \tag{6.11}$$

If $A, B \in \mathcal{F}_0$, then $\mathbb{P}((T_{-j}A) \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ for all $|j|$ sufficiently large so that (6.11) holds. Moreover

$$\mathbb{P}((T_{-j}S^\mathbb{Z}) \cap A) = \mathbb{P}(S^\mathbb{Z})\mathbb{P}(A)$$

and

$$\mathbb{P}((T_{-j}A) \cap S^\mathbb{Z}) = \mathbb{P}(A)\mathbb{P}(S^\mathbb{Z})$$

for all $A \in \mathcal{F}_0$.

Consider the collection $\mathcal{M}$ of all sets $\mathcal{F}_2 \subset \mathcal{F}$ such that :

(i) $\mathcal{F}_2 \supset \mathcal{F}_0 \cup \{S^\mathbb{Z}\}$.

(ii) (6.11) hold for all $A, B \in \mathcal{F}_2$.

Then $(\mathcal{M}, \subset)$ is a non-empty partially ordered set. If $\mathcal{C} = \{C_\alpha\}$ is a chain in $\mathcal{M}$, then $\cup_\alpha C_\alpha$ is easily seen to an upper bound for the chain $\mathcal{C}$ belonging to $\mathcal{M}$. Hence by Zorn's lemma, $\mathcal{M}$ has a maximal element, say $\overline{\mathcal{F}}$.

Remains to show that $\overline{\mathcal{F}}$ is a sigma-algebra. It will then follow that (6.11) holds for all $A, B \in \mathcal{F}$.

If $A, B \in \overline{\mathcal{F}}$, then

$$\mathbb{P}((T_{-j}A^c) \cap B) = \mathbb{P}((T_{-j}A)^c \cap B) = \mathbb{P}(B) - \mathbb{P}((T_{-j}A) \cap B) \to \mathbb{P}(A^c)\mathbb{P}(B)$$

and

$$\mathbb{P}((T_{-j}B) \cap A^c) = \mathbb{P}(T_{-j}B) - \mathbb{P}((T_{-j}B) \cap A) \to \mathbb{P}(B) - \mathbb{P}(B)\mathbb{P}(A) = \mathbb{P}(B)\mathbb{P}(A^c)$$

Since $\overline{\mathcal{F}}$ is maximal, then $A^c \in \overline{\mathcal{F}}$. If $\{A_n\}$ are disjoint in $\overline{\mathcal{F}}$, $B \in \overline{\mathcal{F}}$, then

$$\mathbb{P}((T_{-j}(\cup_n A_n)) \cap B) = \sum_n \mathbb{P}((T_{-j}A_n) \cap B) = \int_{\mathbb{N}} \mathbb{P}((T_{-j}A_n) \cap B) d\mu_c(n)$$

where μ_c is the counting measure on $\mathbb{N}$. Now $\mathbb{P}(T_{-j}A_n \cap B) \le \mathbb{P}(T_{-j}A_n) = \mathbb{P}(A_n)$, and $\int_{\mathbb{N}} \mathbb{P}(A_n) d\mu_c(n) = \sum_n \mathbb{P}(A_n) = \mathbb{P}(\cup_n A_n) < \infty$. So the Dominated Convergence theorem gives $\lim_{|j|\to\infty} \mathbb{P}((T_{-j}(\cup_n A_n)) \cap B) = \mathbb{P}(\cup_n A_n)\mathbb{P}(B)$. Similarly $\lim_{|j|\to\infty} \mathbb{P}((T_{-j}B) \cap (\cup_n A_n)) = \mathbb{P}(B)\mathbb{P}(\cup_n A_n)$. So $\cup_n A_n \in \overline{\mathcal{F}}$.

Finally, if $M \in \mathcal{F}$ is an invariant set, then $\mathbb{P}(M) = \lim_{|j|\to\infty} \mathbb{P}((T_{-j}M) \cap M) = \mathbb{P}(M)^2$, so that $\mathbb{P}(M) \in \{0, 1\}$. $\square$

Proposition 6.19. *$\sigma(V_\omega) = \operatorname{supp} \nu \quad \mathbb{P}$-a.s.*

Proof. Since $\ell^2(\mathbb{Z}) = L^2(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), \mu_c)$ where μ_c is the counting measure on $\mathbb{Z}$, it follows by proposition 8.22 that for all ω:

$$\sigma(V_\omega) = \text{ess ran } (V_\omega) = \{x \in \mathbb{R} : \mu_c(\{n \in \mathbb{Z} : |V_\omega(n) - x| < \varepsilon\}) > 0, \forall \varepsilon > 0\} = \overline{\{V_\omega(n) : n \in \mathbb{Z}\}}.$$

Remains to show that $\operatorname{supp} \nu = \overline{\{V_\omega(n) : n \in \mathbb{Z}\}}$ almost surely.

$$\begin{aligned}
\lambda \notin \operatorname{supp} \nu &\Rightarrow \exists \varepsilon > 0 : \nu(\lambda - \varepsilon, \lambda + \varepsilon) = 0 \\
&\Rightarrow \exists \varepsilon > 0 : E_n := \{\omega : V_\omega(n) \in (\lambda - \varepsilon, \lambda + \varepsilon)\} \text{ satisfies } P(E_n) = 0 \ \ \forall n \\
&\Rightarrow \exists \varepsilon > 0 : E := \cup_n E_n = \{\omega : V_\omega(n) \in (\lambda - \varepsilon, \lambda + \varepsilon) \text{ for some } n\} \text{ satisfies } P(E) = 0 \\
&\Rightarrow \exists \varepsilon > 0 : E^c = \{\omega : V_\omega(n) \notin (\lambda - \varepsilon, \lambda + \varepsilon) \text{ for all } n\} \text{ satisfies } P(E^c) = 1 \\
&\Rightarrow \lambda \notin \overline{\{V_\omega(n) : n \in \mathbb{Z}\}} \text{ for all } \omega \in E^c
\end{aligned}$$

Consequently $\overline{\{V_\omega(n) : n \in \mathbb{Z}\}} \subset \operatorname{supp} \nu$ for all $\omega \in E^c$. For the reverse inclusion we use the Borel-

Cantelli lemma. Let D be a countable dense subset of supp ν:

$$\begin{aligned}\lambda \in D &\Rightarrow \forall m > 0 : \nu(\lambda - 1/m, \lambda + 1/m) > 0 \\ &\Rightarrow \forall m > 0, \forall n : E_{n,m,\lambda} := \{\omega : V_\omega(n) \in (\lambda - 1/m, \lambda + 1/m)\} \text{ satisfies } \mathbb{P}(E_{n,m,\lambda}) > 0 \\ &\Rightarrow \forall m > 0 : E_{m,\lambda} := \limsup_n E_{n,m,\lambda} \text{ satisfies } \mathbb{P}(E_{m,\lambda}) = 1 \\ &\Rightarrow E_\lambda := \cap_m E_{m,\lambda} \text{ satisfies } \mathbb{P}(E_\lambda) = 1\end{aligned}$$

At this point, $\lambda \in \overline{\{V_\omega(n) : n \in \mathbb{Z}\}}$ for all $\omega \in E_\lambda$. Finally, let $E := \cap_\lambda E_\lambda$. Then $\mathbb{P}(E) = 1$ and $D \subset \overline{\{V_\omega(n) : n \in \mathbb{Z}\}}$ for all $\omega \in E$. Hence

$$\overline{D} = \text{supp } \nu \subset \overline{\{V_\omega(n) : n \in \mathbb{Z}\}} \text{ for all } \omega \in E.$$

□

In example (6.16), since the measure $\mathbb{P} = \otimes_{n\in\mathbb{Z}}\nu$ is ergodic, there exists a deterministic set Σ such that $\sigma(h_\omega) = \Sigma$ $\mathbb{P}$-a.s. In fact, we have more generally:

Theorem 6.20. $\sigma(h_\omega) = [-2, 2] + \text{supp } \nu \quad \mathbb{P}$*-a.s.*

The proof of the theorem requires some preliminary results.

Proposition 6.21. *Let A and B be self-adjoint operators on a Hilbert space $\mathcal{H}$. Assume that A is bounded and that $\sigma(A)$ is a connected set. Then $\sigma(A+B) \subset \sigma(A) + \sigma(B) := \{a + b : a \in \sigma(A), b \in \sigma(B)\}$.*

Remark 6.1. *By lemma 8.15, we may assume that the* $\text{Dom}(A) = \mathcal{H}$. *$B$ however may be unbounded.*

Proof. We start by showing that $\sigma(A) + \sigma(B)$ is a closed set. Indeed, since A is bounded and self-adjoint, its spectrum is a compact subset of the real line, whereas since B is self-adjoint and possibly unbounded, its spectrum is a closed subset of the real line. Now suppose $a_n + b_n \to x$ for some $a_n, b_n \in \sigma(A), \sigma(B)$. By compactness a_n has a subsequence a_{n_k} converging to $a \in \sigma(A)$. Then b_{n_k} converges to $x - a$, and so $x - a \in \sigma(B)$. Thus $x \in \sigma(A) + \sigma(B)$.

Since $\sigma(A)$ is a connected set, proposition 8.4 says that $\sigma(A) = [m, M]$ for some $m \leq M$ and $\|A\| = \max(M, -m)$.

By shifting the operator and the spectrum, and by the proof of proposition 8.4, we may assume wlog that $\sigma(A) = [-\|A\|, \|A\|]$. We show the contrapositive. Suppose that $z \notin \sigma(A) + \sigma(B) = \Big\{\Big[b - \|A\|, b + \|A\|\Big] : b \in \sigma(B)\Big\}$. This with the fact that $\sigma(A) + \sigma(B)$ is closed gives $\text{dist}(z, \sigma(B)) > \|A\|$. Also notice that $z \notin \sigma(B)$. Then:

$$\|(z - B)^{-1}A\| \leq \|(z - B)^{-1}\|\|A\| \leq \frac{\|A\|}{\text{dist}(z, \sigma(B))} < 1.$$

Thus $1 \in \rho((z - B)^{-1}A)$. Finally the identity

$$z - (A + B) = (z - B)(1 - (z - B)^{-1}A)$$

shows that $z \notin \sigma(A+B)$.

□

Lemma 6.22. *There is a measurable set Ω_0 with $\mathbb{P}(\Omega_0) = 1$ such that the following is true: For any $\omega \in \Omega_0$, any finite set $\Lambda \subset \mathbb{Z}$, any collection $\{a_i\}_{i\in\Lambda} \subset \operatorname{supp} \nu$ and any $\varepsilon > 0$, there is a sequence $\{j_n\} \subset \mathbb{Z}$ with $|j_n| \to \infty$ such that for all n*

$$\sup_{i\in\Lambda} |a_i - V_\omega(i + j_n)| < \varepsilon$$

Proof. Fix a finite set $\Lambda \subset \mathbb{Z}$, a collection $\{a_i\}_{i\in\Lambda} \subset \operatorname{supp} \nu$ and an $\varepsilon > 0$.

Let $A = \{\omega : \sup_{i\in\Lambda} |a_i - V_\omega(i)| < \varepsilon\} = \bigcap_{i\in\Lambda} \{\omega : |a_i - V_\omega(i)| < \varepsilon\}$. Notice that A is measurable since $\{V_\omega(n)\}$ are assumed to be random variables. Then $\mathbb{P}(A) = \prod_{i\in\Lambda} \mathbb{P}\{\omega : |a_i - V_\omega(i)| < \varepsilon\} = \prod_{i\in\Lambda} \nu(a_i - \varepsilon, a_i + \varepsilon) > 0$.

Now choose a sequence $j_n \in \mathbb{Z}$ such that the distance between any j_n, j_m $(n \neq m)$ is greater than twice the diameter of Λ. Then the events

$$A_n = A_n(\Lambda, \{a_i\}, \varepsilon) = \{\omega : \sup_{i\in\Lambda} |a_i - V_\omega(i + j_n)| < \varepsilon\}$$

are independent and $\mathbb{P}(A_n) = \mathbb{P}(A) > 0$. So by the Borel-Cantelli lemma we have that the set

$$\Omega_{\Lambda,\{a_i\},\varepsilon} = \{\omega : \omega \in A_n \text{ for infinitely many } n\}$$

has probability one.

Now since subsets of separable metric spaces are separable, $\operatorname{supp} \nu$ contains a countable dense set D. Moreover, the collection S of all finite subsets of $\mathbb{Z}$ is countable. Thus the measurable set

$$\Omega_0 := \bigcap_{\substack{\Lambda\in S\\ \{a_i\}\subset D, n\in\mathbb{N}}} \Omega_{\Lambda,\{a_i\},\frac{1}{n}}$$

has probability one. By construction and density of D in $\operatorname{supp} \nu$, one easily verifies that Ω_0 satisfies the requirements of the assertion. □

Proposition 6.23. *Let $a \in \operatorname{supp} \nu$ and denote $h_a = h_0 + a$. Then for all $y \neq 0$, $e \in \mathbb{R}$, and $\omega \in \Omega_0$:*

$$\|(h_\omega - e - iy)^{-1}\| \geq \|(h_a - e - iy)^{-1}\|$$

Proof. Fix $\omega \in \Omega_0$. By lemma, 6.22, for any arbitrary finite set $\Lambda \subset \mathbb{Z}$, we can find a sequence $\{j_n^{(m)}\} \subset \mathbb{Z}$ such that $|j_n^{(m)}| \to \infty$ and $\sup_{i\in\Lambda} |a - V_\omega(i + j_n^{(m)})| < \frac{1}{m}$ for all n. Using a diagonal argument we can extract a sequence $\{j_n\} \subset \mathbb{Z}$ such that $|j_n| \to \infty$ and $\lim_{n\to\infty} \sup_{i\in\Lambda} |a - V_\omega(i + j_n)| = 0$.

Now let ψ be a unit vector in $\ell^2(\mathbb{Z})$.

$$\psi_n := U_{j_n}\psi, \qquad V_\omega^{(n)} := U_{j_n}^* V_\omega U_{j_n} \qquad h_\omega^{(n)} := h_0 + V_\omega^{(n)}$$

For $\phi \in \ell_0(\mathbb{Z})$, let $\Lambda = \{k \in \mathbb{Z} : \langle \delta_k, \phi \rangle \neq 0\}$. Then $|\Lambda| < \infty$ and choose $\{j_n\}$ such that $\lim_{n\to\infty} \sup_{k\in\Lambda} |a - V_\omega(k + j_n)| = 0$. Then

$$\lim_{n\to\infty} V_\omega^{(n)}\phi = \sum_{k\in\Lambda} \langle \delta_k, \phi \rangle \lim_{n\to\infty} V_\omega(k + j_n)(\delta_k) = a\phi$$

The key relation to prove is that for $\varphi \in \ell^2(\mathbb{Z})$, $z \in \mathbb{C} \subset \mathbb{R}$:

$$\lim_{n\to\infty} (h_\omega^{(n)} - z)^{-1}\varphi = (h_a - z)^{-1}\varphi. \tag{6.12}$$

This is known as strong resolvent convergence. It is well known (see e.g. Chapter 1.1 of [**C**]) that if we have self-adjoint operators $(A, \mathrm{Dom}A), (A_k, \mathrm{Dom}A_k)_{k=1}^\infty$ and $\mathrm{Dom}A$ is a core for $\mathcal{H}$ which is contained in all of the $\mathrm{Dom}A_k$, and for which we have:

$$\lim_{k\to\infty} A_k f = Af \tag{6.13}$$

for all $f \in \mathrm{Dom}A$, then A_k converges to A in the sense of strong resolvent convergence. We will prove this fact here directly. First we show that this relation holds for a dense set. Let $\phi \in D := \{\phi \in \ell^2(\mathbb{Z}) : (h_a - z)^{-1}\phi \in \ell_0(\mathbb{Z})\}$. It is not hard to see that D is dense in $\ell^2(\mathbb{Z})$.

$$\begin{aligned}\lim_{n\to\infty} \|(h_\omega^{(n)} - z)^{-1}\phi - (h_a - z)^{-1}\phi\| &= \lim_{n\to\infty} \|(h_\omega^{(n)} - z)^{-1}(a - V_\omega^{(n)})(h_a - z)^{-1}\phi\| \\ &\le \lim_{n\to\infty} \frac{1}{|\mathrm{Im}\ z|}\|(a - V_\omega^{(n)})(h_a - z)^{-1}\phi\| = 0.\end{aligned}$$

Hence $(h_\omega^{(n)} - z)^{-1}$ converges strongly to $(h_a - z)^{-1}$ on D. As the family $\{(h_\omega^{(n)} - z)^{-1}\}$ is uniformly bounded, a standard $\frac{\varepsilon}{3}$ argument shows that the strong convergence can be extended to the closure.

The identity

$$U_{j_n}(h_\omega^{(n)} - e - iy)^{-1}\psi = (h_\omega - e - iy)^{-1}\psi_n$$

combined with (6.12) yields

$$\lim_{n\to\infty} \|(h_\omega - e - iy)^{-1}\psi_n\| = \|(h_a - e - iy)^{-1}\psi\|$$

Hence

$$\|(h_\omega - e - iy)^{-1}\| \ge \|(h_a - e - iy)^{-1}\|$$

□

Lemma 6.24. *For any self-adjoint operator A, $e \in \sigma(A) \Leftrightarrow \|(A - e - iy)^{-1}\| = |y|^{-1}$ for every $y > 0$.*

Proof. If $e \in \sigma(A)$, we have a Weyl sequence $\psi_n \in \mathrm{Dom}(A)$ of unit vectors, that is, $\lim_{n\to\infty} \|(A - e)\psi_n\| = 0$. Now $1 = \|\psi_n\|^2 \le \|(A - e - iy)^{-1}\|^2\|(A - e - iy)\psi_n\|^2 = \|(A - e - iy)^{-1}\|^2 \left(\|(A - e)\psi_n\|^2 + y^2\|\psi_n\|^2\right) \le y^{-2}\left(\|(A - e)\psi_n\|^2 + y^2\right)$. Thus $\lim_{n\to\infty} \|(A - e - iy)^{-1}\|^2\|(A - e - iy)\psi_n\|^2 = \|(A - e - iy)^{-1}\|^2 y^2 = 1$, i.e. $\|(A - e - iy)^{-1}\| = |y|^{-1}$.
Conversely, $|y|^{-1} = \|(A - e - iy)^{-1}\| \le \mathrm{dist}(e + iy, \sigma(A))^{-1} \le |y|^{-1}$, hence $\mathrm{dist}(e + iy, \sigma(A))^{-1} = |y|^{-1}$, and so $\mathrm{dist}(e, \sigma(A)) = 0$. □

We can now complete the proof of theorem 6.20.

Proof. proposition 6.19 and proposition 6.21 together imply $\sigma(h_\omega) \subset [-2, 2] + \mathrm{supp}\ \nu$ $\mathbb{P}$-a.s.

For the reverse inclusion, let $\lambda \in [-2, 2]$, $a \in \mathrm{supp}\ \nu$. Then by lemma 6.24, $\|(h_0 - \lambda - iy)^{-1}\| = |y|^{-1}$. By proposition 6.23,

$$|y|^{-1} = \|(h_0 - \lambda - iy)^{-1}\| = \|(h_a - (\lambda + a) - iy)^{-1}\| \le \|(h_\omega - (\lambda + a) - iy)^{-1}\| \le |y|^{-1} \qquad \mathbb{P} - a.s.$$

Hence $|y|^{-1} = \|(h_\omega - (\lambda + a) - iy)^{-1}\|$ and $\lambda + a \in \sigma(h_\omega)$ follows from lemma 6.24. $\square$

7 Conclusion

This overview of one dimensional discrete Schrödinger operators was an excellent way leads to many interesting questions, such as the analysis of absolutely, singular, and essential spectra for various potentials. To progress in the direction of random and ergodic potentials, one might be interested in the recent work of Artur Avila on quasi-periodic Schrödinger operators.

8 Appendix

The appendix is a collection of useful facts for the thesis, together with solutions to exercises in [**Ja**].

Definition 8.1. (Rotations on the unit circle) *Let* $M := \{z \in \mathbb{C} : |z| = 1\}$. *Let* $\alpha \in [0, 1)$ *be given and set*

$$T_\alpha(z) = e^{2\pi i\alpha} z. \tag{8.1}$$

The orbit of $z \in M$ *is the set* $\mathcal{O}(z) := \{T_\alpha^n(z) : n \in \mathbb{N}\}$.

Lemma 8.2. *If* α *is rational,* $\mathcal{O}(z)$ *is a finite set for all* $z \in M$. *If* α *is irrational,* $\mathcal{O}(z)$ *is dense in* M *for all* $z \in M$.

Proof. We use the following metric on M: for $\omega, \omega' \in \mathbb{R}$,

$$d(e^{2\pi i\omega}, e^{2\pi i\omega'}) = |2\pi\omega - 2\pi\omega'| \pmod{2\pi}.$$

If α is rational, say $\alpha = p/q$, with p, q coprime, then

$$\mathcal{O}(e^{2\pi i\omega}) = \{e^{2\pi i\omega}, e^{2\pi i(\omega+p/q)}, e^{2\pi i(\omega+2p/q)}, ..., e^{2\pi i(\omega+(q-1)p/q)}\}.$$

Let α be irrational. Then $T_\alpha^n(e^{2\pi i\omega}) = T_\alpha^m(e^{2\pi i\omega}) \Leftrightarrow |2\pi(\omega + n\alpha) - 2\pi(\omega + m\alpha)| = 0 \pmod{2\pi} \Leftrightarrow m = n$. Hence the trajectory never repeats itself. Also $\{T_\alpha^n(e^{2\pi i\omega}) : n \in \mathbb{N}\} = e^{2\pi i\omega}\{T_\alpha^n(1) : n \in \mathbb{N}\}$ shows that it is enough to show that the orbit of $z = 1$ is dense in M. Suppose by contradiction that $\mathcal{O}(1)$ is not dense, that is, there is $\varepsilon > 0$ and $\phi \in [0, 1)$ such that $d(e^{2\pi i\phi}, T_\alpha^n(1)) > \varepsilon$ for all n.

If $m > n$ are such that $d(e^{2\pi in\alpha}, e^{2\pi im\alpha}) < \varepsilon$, then $|2\pi(m-n)\alpha| \pmod{2\pi} < \varepsilon$ and so $d(1, e^{2\pi i(m-n)\alpha}) < \varepsilon$. The angle $2\pi(m - n)\alpha$ is small enough that there exists $k \in N$ such that $d(e^{2\pi i\phi}, e^{2\pi ik(n-m)\alpha}) < \varepsilon$. However,since $e^{2\pi ik(n-m)\alpha} \in \mathcal{O}(1)$, it must be that $d(e^{2\pi in\alpha}, e^{2\pi im\alpha}) \geq \varepsilon$ for all m, n. Hence there must be at most $2\pi/\varepsilon < \infty$ elements of the form $e^{2\pi in\alpha}$, contradicting the fact that $\mathcal{O}(1)$ is an infinite set. $\square$

Lemma 8.3. *Suppose that* ρ *is a positive finite measure on* $\mathbb{R}$. *Then the set of all continuous functions and the set of all bounded Borel functions are dense subsets of* $L^2(\mathbb{R}, d\rho)$. *If in addition* ρ *is compactly supported, then the set of polynomials is also dense in* $L^2(\mathbb{R}, d\rho)$.

Proof. Let $f \in L^2(\mathbb{R}, d\rho)$ and consider $f_n(E) = \begin{cases} f(E) & |f(E)| \leq n \\ 0 & |f(E)| > n \end{cases}$

$|f(E) - f_n(E)| \to 0$ a.e. since $|f(E)| < \infty$ a.e. Moreover $|f(E) - f_n(E)| \leq 2|f(E)| \in L^2(\mathbb{R}, d\rho)$ so $\|f - f_n\|_2^2 = \int_{\mathbb{R}} |f(E) - f_n(E)|^2 d\rho \to 0$ by DCT. Next we show that bounded Borel functions can be approximated by continuous functions. Let $\varepsilon > 0$ be given. First choose N such that $\rho([-N,N]^c) < \varepsilon$. By Lusin's theorem, there exists a compact $K \subset [-N,N]$ such that $f|_K$ is continuous and $\rho([-N,N]\backslash K) < \varepsilon$. Note that K is a disjoint union of closed intervals together with countably many singletons. We define a continuous function g on $\mathbb{R}$ as follows : g is equal to f on K, piecewise linear on $[-N,N]\backslash K$ in such a way that g is continuous on $[-N,N]$ and $\|g\|_{\infty,[-N,N]} \leq \|f\|_{\infty,[-N,N]}$, g is identically 0 on $[-N-1,N+1]$ and linear on $[-N-1,-N]$ and $[N,N+1]$. Then $\|f-g\|_2^2 = \int_{K^c} |f-g|^2 d\rho \leq 4\|f\|_\infty^2 \varepsilon$.

Finally, if ρ is compactly supported on $[a,b]$ and f is continuous on $\mathbb{R}$, then there is a polynomial P such that $\|f-P\|_{\infty,[a,b]} \leq \varepsilon$. Then $\|f-P\|_2^2 \leq \rho([a,b])\|f-P\|_{\infty,[a,b]}^2 \leq \rho([a,b])\varepsilon^2$. □

Proposition 8.4. *If A is a bounded self-adjoint linear operator on a Hilbert space $\mathcal{H}$, then $\|A\| = \sup_{\|\psi\|=1} |\langle \psi, A\psi\rangle|$. Moreover, if $M := \sup_{\|\psi\|=1} \langle \psi, A\psi\rangle$ and $m := \inf_{\|\psi\|=1} \langle \psi, A\psi\rangle$, then*

$$\{m\} \cup \{M\} \subset \sigma(A) \subset [m, M].$$

Proof. Let $\alpha := \sup_{\|\psi\|=1} |\langle \psi, A\psi\rangle|$. Cauchy-Schwarz gives $\alpha \leq \|A\|$. For the reverse inequality, note that

$$\begin{aligned}
\|A\| &= \sup_{\|\psi\|=1} \|A\psi\| \\
&= \sup_{\|\psi\|=1} \sup_{\|\phi\|=1} |\langle \phi, A\psi\rangle| \\
&= \sup \{|\langle \phi, A\psi\rangle| : \|\phi\| = 1, \|\psi\| = 1\} \\
&= \sup \{|\langle \phi, A\psi\rangle| : \|\phi\| = 1, \|\psi\| = 1, \langle \phi, A\psi\rangle \text{ is real } \}
\end{aligned}$$

By the polarization identity,

$$4\langle \phi, A\psi\rangle = \langle \phi+\psi, A(\phi+\psi)\rangle - \langle \phi-\psi, A(\phi-\psi)\rangle - i\langle \phi+i\psi, A(\phi+i\psi)\rangle + i\langle \phi-i\psi, A(\phi-i\psi)\rangle$$

Since A is self-adjoint, the first two terms on the RHS are real while the two last terms are purely imaginary. In particular, considering ϕ and ψ so that $\langle \phi, A\psi\rangle$ is real, we have,

$$\begin{aligned}
|\langle \phi, A\psi\rangle|^2 &= \frac{1}{16}\Big|\langle \phi+\psi, A(\phi+\psi)\rangle - \langle \phi-\psi, A(\phi-\psi)\rangle\Big|^2 \\
&\leq \frac{1}{16}\left(\|\phi+\psi\|^2\Big|\Big\langle \frac{\phi+\psi}{\|\phi+\psi\|}, A(\frac{\phi+\psi}{\|\phi+\psi\|})\Big\rangle\Big| + \|\phi-\psi\|^2\Big|\Big\langle \frac{\phi-\psi}{\|\phi-\psi\|}, A(\frac{\phi-\psi}{\|\phi-\psi\|})\Big\rangle\Big|\right)^2 \\
&\leq \frac{1}{16}\left(\|\phi+\psi\|^2\alpha + \|\phi-\psi\|^2\alpha\right)^2 \\
&= \frac{1}{4}\alpha^2\left(\|\phi\|^2 + \|\psi\|^2\right)^2
\end{aligned}$$

where we have used the parallelogram identity in the last step. Finally taking the supremum over all ϕ and ψ satisfying $\|\phi\| = \|\psi\| = 1$ and $\langle \phi, A\psi\rangle$ is real yields $\|A\| \leq \alpha$.

We define $M := \sup_{\|\phi\|=1} \langle \phi, A\phi\rangle$, $m := \inf_{\|\phi\|=1} \langle \phi, A\phi\rangle$. Let ϕ_n be a sequence of unit vectors such that $|\langle \phi_n, A\phi_n\rangle| \to \|A\|$ and $|\langle \phi_n, A\phi_n\rangle|$ is an increasing sequence. Let $\{\phi_n^+\} = \{\phi_n : \langle \phi_n, A\phi_n\rangle \geq 0\}$ and

$\{\phi_n^-\} = \{\phi_n : \langle \phi_n, A\phi_n \rangle < 0\}$. If $\{\phi_n^-\}$ is a finite collection, it follows that $\|A\| = \sup_{\|\phi\|=1} \langle \phi, A\phi \rangle$, or if $\{\phi_n^+\}$ is a finite collection, it follows that $\|A\| = \sup_{\|\phi\|=1} -\langle \phi, A\phi \rangle = -\inf_{\|\phi\|=1} \langle \phi, A\phi \rangle$. Hence $\|A\| = \max(M, -m)$.

For any $t \in \mathbb{R}$, let $A_t = A - t$. Then $\|A_t\| = \max(M - t, \ -m + t)$.

Choose t so that $M - t = -m + t$, namely, $t^* = \frac{m+M}{2}$. We have $\|A_{t^*}\| = \sup_{\|\phi\|=1} \langle \phi, A_{t^*}\phi \rangle = -\inf_{\|\phi\|=1} \langle \phi, A_{t^*}\phi \rangle$.

Since A_{t^*} is self-ajoint, its spectrum is real and its spectral radius is equal to $\|A_{t^*}\| = M - t = -m + t$. Therefore $\sigma(A_{t^*}) \subset [m - t, M - t]$. Therefore $\sigma(A) \subset [m, M]$.

We claim that $\pm\|A_{t^*}\| \in \sigma(A_{t^*})$ and prove it using the criterion of Weyl (theorem 8.8). Since $\|A_{t^*}\| = \sup_{\|\phi\|=1} \langle \phi, A'\phi \rangle$, there is a sequence of unit vectors $\{\phi_n\}$ such that $\langle \phi_n, (A_{t^*} - \|A_{t^*}\|)\phi_n \rangle \to 0$. Then $\|(A_{t^*} - \|A_{t^*}\|)\phi_n\|^2 = -2\|A_{t^*}\|\langle \phi_n, A_{t^*}\phi_n \rangle + \|A_{t^*}\phi_n\|^2 + \|A_{t^*}\|^2 \leq -2\|A_{t^*}\|\langle \phi_n, A_{t^*}\phi_n \rangle + 2\|A_{t^*}\|^2 \to 0$. Hence $\|A_{t^*}\| \in \sigma(A_{t^*})$ and a similar argument using $\|A_{t^*}\| = -\inf_{\|\phi\|=1} \langle \phi, A_{t^*}\phi \rangle$ shows that $-\|A_{t^*}\| \in \sigma(A_{t^*})$.

Finally, we have $A_{t^*} - \|A_{t^*}\| = (A - t) - (M - t) = A - M$ and $A_{t^*} + \|A_{t^*}\| = (A - t) + (-m + t) = A - m$. We conclude that $m, M \in \sigma(A)$. □

Proposition 8.5. *Let $C_0(\mathbb{R})$ be the Banach space of continous functions from $\mathbb{R}$ to $\mathbb{C}$ that vanish at infinity with norm $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$. For $f \in C_0(\mathbb{R})$, let $f_y(x) = \frac{y}{\pi}\int_{\mathbb{R}} \frac{f(t)}{(x-t)^2+y^2}dt = f * g_y$ where $g_y(x) = \frac{y}{\pi}\frac{1}{x^2+y^2}$. Then:*

(i) $\lim_{y\downarrow 0} \|f - f_y\| = 0$

(ii) *the linear span of* $\{\frac{1}{(x-a)^2+b^2} : a \in \mathbb{R}, b > 0\}$ *is dense in* $C_0(\mathbb{R})$.

(iii) *the linear span of* $\{\frac{1}{x-z} : z \in \mathbb{C}\backslash\mathbb{R}\}$ *is dense in* $C_0(\mathbb{R})$.

Proof.

(i) One can evaluate the integral to find that $\int_{\mathbb{R}} \frac{1}{(x-t)^2+y^2}dt = \frac{\pi}{y}$ for all x. Let $\varepsilon > 0$ be given and choose $\delta > 0$ such that $|f(s) - f(t)| < \varepsilon$ whenever $|s - t| < \delta$. Then for fixed x and arbitrary y:

$$\begin{aligned} |f(x) - f_y(x)| &= \Big|\frac{y}{\pi}\int_{\mathbb{R}} \frac{f(x) - f(t)}{(x-t)^2 + y^2}dt\Big| \\ &\leq \frac{y}{\pi}\int_{\mathbb{R}} \frac{|f(x) - f(t)|}{(x-t)^2 + y^2}dt \\ &\leq \varepsilon + 2\|f\|\frac{y}{\pi}\int_{|t-x|>\delta} \frac{1}{(x-t)^2 + y^2}dt \\ &\leq \varepsilon + 2\|f\|\frac{y}{\pi}\int_{|t-x|>\delta} \frac{1}{(x-t)^2}dt \\ &= \varepsilon + 4\|f\|\frac{y}{\pi\delta} \end{aligned}$$

Hence $\limsup_{y\downarrow 0} \|f - f_y\| \leq \varepsilon$, and so $\limsup_{y\downarrow 0} \|f - f_y\| = 0$.

(ii) The idea is to approximate $f_y(x)$ by Riemann sums of the form $\sum_{k=0}^{n} \frac{\lambda_k}{(x-t_k)^2+y^2}$ where λ_k and t_k are independent of x. Let $\varepsilon > 0$ be given and fix $y > 0$ so that $\|f - f_y\| \le \varepsilon/3$. Then fix $A > 0$ so that $|f(t)| < \varepsilon/3$ for all $t \in (-\infty, -A) \cup (A, \infty)$. Partition $[-A, A]$ into n segments of equal length and choose the tags $t_k = -A + \frac{2Ak}{n}$, $k = 0, 1, ..., n-1$. Define the functions

$$g_y(x) = \frac{y}{\pi}\int_{-A}^{A} \frac{f(t)}{(x-t)^2+y^2}dt = \frac{y}{\pi}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \frac{f(t)}{(x-t)^2+y^2}dt$$

and

$$h_y^n(x) = \frac{y}{\pi}\sum_{k=0}^{n-1} \frac{f(t_k)}{(x-t_k)^2+y^2}\frac{2A}{n} = \frac{y}{\pi}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \frac{f(t_k)}{(x-t_k)^2+y^2}dt$$

It follows that $\|f_y - g_y\| \le \varepsilon/3$ and it remains to show that n can be chosen large enough so that $\|g_y - h_y^n\| \le \varepsilon/3$. [Then let $\lambda_k = \frac{y}{\pi}\frac{2A}{n}f(t_k)$ to finish the proof of 2.]

$$\|g_y - h_y^n\| \le \frac{y}{\pi}\sum_{k=0}^{n-1}\left\|\int_{t_k}^{t_{k+1}} \frac{f(t)-f(t_k)}{(x-t)^2+y^2}dt\right\| + \frac{y}{\pi}\sum_{k=0}^{n-1}\left\|\int_{t_k}^{t_{k+1}} f(t_k)\left(\frac{1}{(x-t)^2+y^2} - \frac{1}{(x-t_k)^2+y^2}\right)dt\right\| \tag{8.2}$$

By uniform continuity of f we may choose n large enough so that $|f(t) - f(t_k)| \le \frac{\varepsilon}{6}\frac{\pi y}{2A}$ whenever $t \in [t_k, t_{k+1}]$.

Then

$$\left\|\int_{t_k}^{t_{k+1}} \frac{f(t)-f(t_k)}{(x-t)^2+y^2}dt\right\| \le \frac{\varepsilon}{6}\frac{\pi y}{2A}\sup_{x\in\mathbb{R}}\int_{t_k}^{t_{k+1}} \frac{1}{(x-t)^2+y^2}dt = \frac{\varepsilon}{6}\frac{\pi y}{2A}\int_{-A/n}^{A/n}\frac{1}{t^2+y^2}dt \le \frac{\varepsilon}{6}\frac{\pi}{yn}$$

and so the first term on the RHS of (8.2) satisfies

$$\frac{y}{\pi}\sum_{k=0}^{n-1}\left\|\int_{t_k}^{t_{k+1}} \frac{f(t)-f(t_k)}{(x-t)^2+y^2}dt\right\| \le \varepsilon/6$$

For the second term on the RHS of (8.2), we use the MVT : $\left|\frac{1}{(x-t)^2+y^2} - \frac{1}{(x-t_k)^2+y^2}\right| \le \|w\|(t-t_k)$ where $w(x,t) = \frac{d}{dt}\frac{1}{(x-t)^2+y^2} = \frac{x-t}{((x-t)^2+y^2)^2}$. So

$$\begin{aligned}\frac{y}{\pi}\sum_{k=0}^{n-1}\left\|\int_{t_k}^{t_{k+1}} f(t_k)\left(\frac{1}{(x-t)^2+y^2} - \frac{1}{(x-t_k)^2+y^2}\right)dt\right\| &\le \frac{y}{\pi}\|f\|\|w\|\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}(t-t_k)dt \\ &= \frac{y}{\pi}\|f\|\|w\|\sum_{k=0}^{n-1}\frac{(t_{k+1}-t_k)^2}{2} \\ &= O(\frac{1}{n})\end{aligned}$$

Hence we may choose n even larger as previously chosen to make

$$\frac{y}{\pi}\sum_{k=0}^{n-1}\left\|\int_{t_k}^{t_{k+1}} f(t_k)\left(\frac{1}{(x-t)^2+y^2}-\frac{1}{(x-t_k)^2+y^2}\right)dt\right\| \leq \varepsilon/6$$

(iii) $\frac{1}{(x-a)^2+b^2} = \frac{-i/(2b)}{x-(a+ib)} + \frac{i/(2b)}{x-(a-ib)}$.

□

Lemma 8.6. *Let $(A, D(A))$ be a self-adjoint operator on $\mathcal{H}$. Let F be a Borel set in $\mathbb{R}$, $\mathbb{1}_F$ its characteristic function and $\mathbb{1}_F(A)$ the corresponding orthogonal projection induced by the Functional Calculus. Then*

$$\overline{\text{int}(F)\cap\sigma(A)} \;\subset\; \sigma(A|_{\text{Ran }\mathbb{1}_F(A)}) \;\subset\; \sigma(A)\cap\overline{F}$$

Remark 8.1. *Since $\mathbb{1}_F(A)$ is an orthogonal projection, $\mathcal{H} = Ran\ \mathbb{1}_F(A) \oplus Ker\ \mathbb{1}_F(A)$ is a direct sum of two closed and A-invariant subspaces of $\mathcal{H}$. If we let*

$$D(A|_{\text{Ran }\mathbb{1}_F}) := \mathbb{1}_F(A)D(A)$$

$$D(A|_{\text{Ker }\mathbb{1}_F}) := (\mathbb{1}-\mathbb{1}_F(A))D(A)$$

then by $A|_{\text{Ran }\mathbb{1}_F(A)}$ we mean the operator A with domain $D(A|_{\text{Ran }\mathbb{1}_F})$ and $A|_{\text{Ker }\mathbb{1}_F(A)}$ the operator A with domain $D(A|_{\text{Ker }\mathbb{1}_F})$. Moreover,

$$D(A|_{\text{Ran }\mathbb{1}_F}) \oplus D(A|_{\text{Ker }\mathbb{1}_F}) = D(A)$$

$$A|_{\text{Ran }\mathbb{1}_F(A)} \oplus A|_{\text{Ker }\mathbb{1}_F(A)} = A$$

is a direct sum of two self-adjoint operators.

Proof. Note that by $\sigma(A) = \sigma(A|_{\text{Ran }\mathbb{1}_F(A)}) \cup \sigma(A|_{\text{Ker }\mathbb{1}_F(A)})$.

For the first inclusion, note that $\text{int}(F)\cap\sigma(A) = \Big[\text{int}(F)\cap\sigma(A|_{\text{Ran }\mathbb{1}_F(A)})\Big]\cup\Big[\text{int}(F)\cap\sigma(A|_{\text{Ker }\mathbb{1}_F(A)})\Big]$, so it is enough to show that $\text{int}(F)\cap\sigma(A|_{\text{Ker }\mathbb{1}_F(A)}) = \varnothing$, or equivalently, $\sigma(A|_{\text{Ker }\mathbb{1}_F(A)}) \subset \overline{F^c}$. Choosing an orthonormal basis $\{\phi_n\}$ for Ran $\mathbb{1}_F(A)$ so that $\mathbb{1}_F(A) = \sum_n \langle\phi_n,\cdot\rangle\phi_n$ and a cyclic set of vectors $\{\varphi_k\}$ such that the direct sum of the cyclic subspaces generated by A and $\{\varphi_k\}$ equals Ker $\mathbb{1}_F(A)$ (as in the statement of the spectral theorem), we have that $\sigma(A|_{\text{Ker }\mathbb{1}_F(A)}) = \overline{\bigcup_k \text{supp }\mu_{\varphi_k}}$. But $\mu_{\varphi_k}(F) = \langle\varphi_k, \mathbb{1}_F(A)\varphi_k\rangle = 0$ for every k, hence supp $\mu_{\varphi_k} \subset \overline{F^c}$. We conclude $\overline{\bigcup_k \text{supp }\mu_{\varphi_k}} \subset \overline{F^c}$.

For the second inclusion, we show that $\sigma(A|_{\text{Ran }\mathbb{1}_F(A)}) \subset \overline{F}$. As before, choosing an orthonormal basis $\{\varphi'_k\}$ for Ker $\mathbb{1}_F(A)$ and a cyclic set of vectors $\{\phi'_n\}$ such that the direct sum of the cyclic subspaces generated by A and $\{\phi'_n\}$ equals Ran $\mathbb{1}_F(A)$, we have that $\sigma(A|_{\text{Ran }\mathbb{1}_F(A)}) = \overline{\bigcup_n \text{supp }\mu_{\phi'_n}}$. But $\mu_{\phi'_n}(F^c) = \langle\phi'_n, \mathbb{1}_{F^c}(A)\phi'_n\rangle = 0$, hence supp $\mu_{\phi'_n} \subset \overline{F}$ and $\overline{\bigcup_n \text{supp }\mu_{\phi'_n}} \subset \overline{F}$. □

The following lemma gives a characterisation of the spectrum and the point spectrum in terms of orthogonal projections.

Lemma 8.7. *Let A be self-adjoint on $\mathcal{H}$. Then:*

(i) $e \in \sigma(A)$ *if and only if for all* $\varepsilon > 0$, $\mathbb{1}_{(e-\varepsilon,e+\varepsilon)}(A) \neq \{0\}$.

(ii) $e \in \sigma_{\mathrm{p}}(A)$ *if and only if* $\mathbb{1}_{\{e\}}(A) \neq \{0\}$.

Proof.

(i) If $e \in \sigma(A)$, then $\forall \varepsilon > 0$, $e \in \overline{(e-\varepsilon, e+\varepsilon) \cap \sigma(A)} \subset \sigma(A|_{\mathrm{Ran}\ \mathbb{1}_{(e-\varepsilon,e+\varepsilon)}(A)})$ by lemma 8.6, which forces $\mathrm{Ran}\ \mathbb{1}_{(e-\varepsilon,e+\varepsilon)}(A) \neq \{0\}$. Indeed, the spectrum of $A : \{0\} \to \{0\}$ is empty.

Conversely, if $e \notin \sigma(A)$, then since $\sigma(A)$ is closed, there exists $\varepsilon > 0$ such that $\sigma(A) \cap \overline{(e-\varepsilon, e+\varepsilon)} = \varnothing$. But then lemma 8.6 implies that $\sigma(A|_{\mathrm{Ran}\ \mathbb{1}_F(A)}) = \varnothing$, which is not possible (see theorem 8.20).

(ii) By the Functional Calculus, if $A\psi = e\psi$ for some non zero ψ, then $\mathbb{1}_{\{e\}}(A)\psi = 1 \cdot \psi$, so that $\mathbb{1}_{\{e\}}(A) \neq 0$. Conversely, for all $x \in \mathbb{R}$, $x \cdot \mathbb{1}_{\{e\}}(x) = e \cdot \mathbb{1}_{\{e\}}(x)$. Hence $\hat{\phi}(x \cdot \mathbb{1}_{\{e\}}(x)) = A \cdot \mathbb{1}_{\{e\}}(A) = e \cdot \mathbb{1}_{\{e\}}(A)$ where $\hat{\phi} : \mathcal{B}_{\mathrm{b}}(\mathbb{R}) \to \mathcal{B}(\mathcal{H})$ is the $*$-homomorphism of the Functional Calculus, so that if $\mathbb{1}_{\{e\}}(A) \neq 0$, e is an eigenvalue of A.

□

A clear proof of the following useful theorem is in chapter 4 of [**Ja**].

Theorem 8.8. *(Weyl's Criterion) Let $(A, D(A))$ be a self-adjoint operator on $\mathcal{H}$. Then $\lambda \in \sigma(A)$ if and only if there exists a sequence of unit vectors $\{\psi_n\}$ in $D(A)$ such that* $\lim_{n\to\infty} \|(A-\lambda)\psi_n\| = 0$.

Theorem 8.9. *Let A be self-adjoint. Then:*

(i) *For $z = x + iy$ and $\psi \in \mathrm{Dom}(A)$:* $\|(A-z)\psi\|^2 = \|(A-x)\psi\|^2 + y^2\|\psi\|^2$.

(ii) *For any $x \in \mathbb{R}$ and $\psi \in \mathcal{H}$:* $\lim_{y\to\infty} iy(A-x-iy)^{-1}\psi = -\psi$.

(iii) *If $\lambda_1, \lambda_2 \in \sigma_{\mathrm{p}}(A)$, $\lambda_1 \neq \lambda_2$, and ψ_1, ψ_2 are corresponding eigenvectors, then $\psi_1 \perp \psi_2$.*

Definition 8.10. *The cyclic space generated by A and $\psi \in \mathcal{H}$ is the closure of the linear span of the set*

$$\{(A-z)^{-1}\psi_1 : z \in \mathbb{C}\backslash\mathbb{R}\}.$$

It is denoted $\mathcal{H}_\psi$.

The identity $\lim_{y\to\infty} iy(A-x-iy)^{-1}\psi = -\psi$ of theorem 8.9 shows that $\psi \in \mathcal{H}_\psi$.

Lemma 8.11. *Let μ_n be positive finite Borel measures, $a_n > 0$, such that*

$$\mu(\mathbb{R}) = \sum_{n=1}^{\infty} a_n \mu_n(\mathbb{R}) < \infty.$$

For $B \in \mathbb{R}$, let

$$\mu(B) := \sum_{n=1}^{\infty} a_n \mu_n(B).$$

Then μ is a positive finite Borel measure and $\mathrm{supp}\ \mu = \overline{\bigcup_{n=1}^{\infty} \mathrm{supp}\ \mu_{\psi_n}}$.

Proof. Clearly, supp $\mu \supset \overline{\bigcup_{n=1}^{\infty} \text{supp}\, \mu_n}$. To show the reverse inclusion, if $x \in \text{supp}\, \mu$, then for every $\varepsilon > 0$, $\mu(x-\varepsilon, x+\varepsilon) > 0$, hence there exists $n = n(\varepsilon) \in \mathbb{N}$ such that $\mu_n(x-\varepsilon, x+\varepsilon) > 0$. For such n, there is $x_n \in (x-\varepsilon, x+\varepsilon)$ with $x_n \in \text{supp}\, \mu_n$. Then $x_n \to x$ as $\varepsilon \to 0$, so $x \in \overline{\bigcup_{n=1}^{\infty} \text{supp}\, \mu_n}$. □

Lemma 8.12. *Let A be a self-adjoint operator and $\psi \in \mathcal{H}$. Then the cyclic subspace $\mathcal{H}_\psi$ generated by A and ψ is A-invariant, that is, $\forall z \in \mathbb{C}\backslash\mathbb{R}$, $(A-z)^{-1}\mathcal{H}_\psi \subset \mathcal{H}_\psi$.*

Proof. Since $(A-z)^{-1}$ is a continuous and linear operator, it is enough to show that $(A-z)^{-1}(A-z')^{-1}\psi \in \mathcal{H}_\psi$ for all $z, z' \in \mathbb{C}\backslash\mathbb{R}$. If $z \neq z'$, then this is equal to $\frac{1}{z-z'}\left((A-z)^{-1} - (A-z')^{-1}\right)\psi \in \mathcal{H}_\psi$. If $z = z'$, then choose a sequence $z_i \in \mathbb{C}\backslash\mathbb{R}$ such that $z_i \neq z$ and $z_i \to z$, so that $\frac{1}{x-z_i} \to \frac{1}{x-z}$ pointwise. Then by the Functional Calculus $(A-z)^{-1} = \underset{i\to\infty}{\text{s-lim}}(A-z_i)^{-1}$, so $(A-z)^{-2}\psi = \lim_{i\to\infty}(A-z_i)^{-1}(A-z)^{-1}\psi \in \mathcal{H}_\psi$ since the subspace is closed. □

Let A be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$ and let $\{\phi_n\}_{n=1}^{\infty}$ be a cyclic set for A. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence such that $a_n > 0$ and

$$\sum_{n=1}^{\infty} a_n \|\phi_n\|^2 < \infty.$$

If $\mathcal{H}_n$ denotes the cyclic subspace generated by $\sqrt{a_n}\phi_n$ and A, and if $\mathcal{H}_i \perp \mathcal{H}_j$ for all i, j then

$$\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n = \bigoplus_{n=1}^{\infty} L^2(\mathbb{R}, d\mu_{\sqrt{a_n}\phi_n}) = L^2(M, d\mu)$$

where

$$M = \bigcup_{n=1}^{\infty} \mathbb{R}_n$$

and

$$\mu(F) = \sum_{n=1}^{\infty} \mu_{\sqrt{a_n}\phi_n}(F \cap \mathbb{R}_n).$$

Note that for any Borel set E, $\langle \sqrt{a_n}\phi_n, \mathbb{1}_E \sqrt{a_n}\phi_n \rangle = \mu_{\sqrt{a_n}\phi_n}(E) = a_n \langle \phi_n, \mathbb{1}_E \phi_n \rangle = a_n \mu_{\phi_n}(E)$. Hence $\mu_{\sqrt{a_n}\phi_n} = a_n \mu_{\phi_n}$. μ is a finite measure on M, since $\mu(M) = \sum_{n=1}^{\infty} \|\sqrt{a_n}\phi_n\|^2 < \infty$. However μ can also naturally be interpreted as a finite measure on $\mathbb{R}$. This gives rise to the following definition:

Definition 8.13. *The spectral measure for A, denoted μ_A, is a Borel measure on $\mathbb{R}$ defined by*

$$\mu_A := \sum_{n=1}^{\infty} a_n \mu_{\phi_n}.$$

Obviously, μ_A depends on the choice of $\{\phi_n\}$ and $\{a_n\}$. The following theorem explains in what sense we mean that μ_A is *the* spectral measure for A.

Recall that two positive Borel measures μ_1 and μ_2 are called equivalent (we write $\mu_1 \sim \mu_2$) if μ_1 and μ_2 have the same sets of measure zero, or equivalently, if $\mu_1 \ll \mu_2 \ll \mu_1$.

Proposition 8.14. *Let A be self-adjoint on $\mathcal{H}$ and $\{\phi_n\}_{n=1}^{\infty}$, $\{\varphi_n\}_{n=1}^{\infty}$ be cyclic sets for $\mathcal{H}$. Let μ_A and*

ν_A be the spectral measures corresponding to $\{\phi_n\}_{n=1}^{\infty}$ and $\{\varphi_n\}_{n=1}^{\infty}$ respectively, that is,

$$\mu_A = \sum_{n=1}^{\infty} a_n \mu_{\phi_n} \quad \textit{and} \quad \nu_A = \sum_{n=1}^{\infty} b_n \nu_{\varphi_n}.$$

Then $\mu_A \sim \nu_A,\ \mu_{A,\mathrm{ac}} \sim \nu_{A,\mathrm{ac}},\ \mu_{A,\mathrm{sc}} \sim \nu_{A,\mathrm{sc}},\ \mu_{A,\mathrm{pp}} \sim \nu_{A,\mathrm{pp}}$.

Moreover, $\operatorname{supp} \mu_{A,\mathrm{ac}} = \sigma_{\mathrm{ac}}(A),\ \operatorname{supp} \mu_{A,\mathrm{sc}} = \sigma_{\mathrm{sc}}(A),\ \operatorname{supp} \mu_{A,\mathrm{pp}} = \sigma_{\mathrm{pp}}(A)$.

Proof. Set $\psi_1 = \phi_1$ and let $\mathcal{H}_1$ be the cyclic space generated by A and ψ_1.

We set $A_1 = A\restriction_{\mathcal{H}_1}$. □

Define $\psi_n, \mathcal{H}_n$ and A_n inductively as follows. For simplicity let us assume that $\{\phi_n\}_{n=1}^{\infty}$ is a minimal set of generators. Decompose $\phi_2 = \phi_2^{(1)} + \phi_2^{(2)}$, where $\phi_2^{(1)} \in \mathcal{H}_1$ and $\phi_2^{(2)} \in \mathcal{H}_1^{\perp}$. Set $\psi_2 = \phi_2^{(2)}$ and let $\mathcal{H}_2$ be the cyclic space generated by A and ψ_2. It is easy to check that $\mathcal{H}_1 \perp \mathcal{H}_2$. Set $A_2 = A\restriction_{\mathcal{H}_2}$. In this way define $\psi_n, \mathcal{H}_n$ and A_n inductively to obtain a sequence of mutually orthogonal spaces $\{\mathcal{H}_n\}_{n=1}^{\infty}$ and a seqence of operators $\{A_n\}_{n=1}^{\infty}$, such that each $\mathcal{H}_n$ is generated by A and ψ_n.

Now form the structure $\bigoplus_{n=1}^{\infty} \mathcal{H}_n = \{(x_1, x_2, ...) \in \prod_{n=1}^{\infty} \mathcal{H}_n : \sum_{n=1}^{\infty} \|x_n\|^2 < \infty\}$. One checks that the map $U : \bigoplus_{n=1}^{\infty} \mathcal{H}_n \to \mathcal{H}$, $U(x_1, x_2, ...) \to \sum_{n=1}^{\infty} x_n$ is unitary, so that we have in fact $\bigoplus_{n=1}^{\infty} \mathcal{H}_n = \mathcal{H}$ and that $U \bigoplus_{n=1}^{\infty} A_n U^{-1} = A$. We also see that $A - z$ is invertible iff each $A_n - z$ is invertible, and $(A-z)^{-1}$ is bounded iff $(A_n - z)^{-1}$ are uniformly bounded. Hence we have $\sigma(A) \supset \bigcup_{n=1}^{\infty} \mathrm{sp}(A_n)$ and taking closures gives $\sigma(A) \supset \overline{\bigcup_{n=1}^{\infty} \sigma(A_n)}$. To show the reverse inclusion it remains to show that if $(A-z)^{-1}$ exists but is not bounded then $z \in \overline{\bigcup_{n=1}^{\infty} \sigma(A_n)}$. Now $\|(A-z)^{-1}\| = \infty$ iff $\sup_{1 \le n \le \infty} \|(A_n - z)^{-1}\| = \infty$. If there is n such that $(A_n - z)^{-1}$ is unbounded then we are done. Otherwise each $(A_n - z)^{-1}$ are bounded self-adjoint operators and so as a result of the functional calculus we have $\|(A_n - z)^{-1}\| = \frac{1}{\mathrm{dist}(z, \sigma(A_n))}$, which shows the reverse inclusion. Hence $\sigma(A) = \overline{\bigcup_{n=1}^{\infty} \sigma(A_n)}$.

For each n let $\mu_{\phi_n}, \mu_{\psi_n}$ and ν_{φ_n} be the spectral measures for A and $\phi_n, \psi_n, \varphi_n$ respectively. We want to show that $\nu_A \ll \mu_A$, equivalently that $\nu_{\varphi_n} \ll \mu_A$ for all n. First notice that $\mu_{\psi_n} \ll \mu_{\phi_n}$ for all n. Indeed, for any Borel measurable set E, let $\mathbb{1}_E = \mathbb{1}_E(A)$ be the orthogonal projection corresponding to E; we have

$$\begin{aligned}
\mu_{\phi_n}(E) &= \langle \phi_n, \mathbb{1}_E \phi_n \rangle \\
&= \langle \phi_n^{(1)} + ... + \phi_n^{(n-1)} + \psi_n, \mathbb{1}_E \left(\phi_n^{(1)} + ... + \phi_n^{(n-1)} + \psi_n \right) \rangle \\
&= \sum_{i=1}^{(n-1)} \mu_{\phi_n^{(i)}}(E) + \mu_{\psi_n}(E)
\end{aligned}$$

It follows that $\nu_{\varphi_n} \ll \mu_A$, since

$$\begin{aligned}\nu_{\varphi_n}(E) &= \langle \varphi_n, \mathbb{1}_E \varphi_n \rangle \\ &= \langle \sum_{i=1}^{\infty} c_i \psi_i, \mathbb{1}_E \sum_{j=1}^{\infty} c_j \psi_j \rangle \\ &= \sum_{i=1}^{\infty} |c_i|^2 \langle \psi_i, \mathbb{1}_E \psi_i \rangle \\ &= \sum_{i=1}^{\infty} |c_i|^2 \mu_{\psi_n}(E)\end{aligned}$$

The rest follows by measure theory. Since $\mu_A \ll \nu_A \ll \mu_A$, there exists $f \in L^1(\mathbb{R}, \mu_A)$ such that $\nu_A = \int f d\mu_A$. Then

$$\nu_{A,ac} - \int f d\mu_{A,ac} = \int f d\mu_{A,s} - \nu_{A,s}$$

The signed measure $\int f d\mu_{A,s} - \nu_{A,s}$ is simultaneously absolutely continuous and singular with respect to the Lebesgue measure, hence $\nu_{A,s} = \int f d\mu_{A,s}$ and $\nu_{A,ac} = \int f d\mu_{A,ac}$. Reversing the argument gives $\mu_{A,ac} \sim \nu_{A,ac}$ and $\mu_{A,s} \sim \nu_{A,s}$. Now $\nu_A(\{a\}) = f(a)\mu_A(\{a\})$, for every $a \in \mathbb{R}$. Reversing the role of μ_A and ν_A shows that they have the same set of atoms. Hence $\mu_{A,pp} \sim \nu_{A,pp}$ and $\mu_{A,sc} \sim \nu_{A,sc}$.

Now $\{\sqrt{a_n}\psi_n\}_{n=1}^{\infty}$ are mutually orthogonal, generate the subspaces $\mathcal{H}_n$ and $\sum_{n=1}^{\infty} \|\sqrt{a_n}\psi_n\|^2 \leq \sum_{n=1}^{\infty} a_n \|\phi_n\|^2 < \infty$. So by definition

$$\sigma_{ac/sc/pp}(A) = \overline{\bigcup_{n=1}^{\infty} \text{supp}\mu_{\sqrt{a_n}\psi_n}, ac/sc/pp}.$$

The measure $\mu'_A = \sum_{n=1}^{\infty} \mu_{\sqrt{a_n}\psi_n} = \sum_{n=1}^{\infty} a_n \mu_{\psi_n}$ is therefore a spectral measure for A.

Hence $\text{supp}\mu_{A,ac/sc/pp} = \text{supp}\mu'_{A,ac/sc/pp}$.

Finally, lemma 8.11 proves that

$$\overline{\bigcup_{n=1}^{\infty} \text{supp}\mu_{\sqrt{a_n}\psi_n}, ac/sc/pp} = \text{supp}_{\mu'_{A,ac/sc/pp}}$$

. □

A linear operator on a separable Hilbert space $\mathcal{H}$ is a pair (A, D) where $D \subset \mathcal{H}$ is a linear subspace and $A : D \to \mathcal{H}$ is linear. D is called the domain of A and will be denoted $\text{Dom}(A)$. When $\text{Dom}(A)$ is dense in $\mathcal{H}$ we say that A is densely defined. Densely defined operators are convenient because they admit well defined adjoint operators.

An operator $(A, \text{Dom}(A))$ is bounded on its domain if

$$\sup_{\psi \in D, \psi \neq 0} \frac{\|A\psi\|}{\|\psi\|} < \infty$$

An extension $(\tilde{A}, \text{Dom}(\tilde{A}))$ of $(A, \text{Dom}(A))$ is another linear operator such that $\text{Dom}(\tilde{A}) \supset \text{Dom}(A)$ and $\tilde{A}\Big|_{\text{Dom}(A)} = A$. In this case it is customary to simply write $\tilde{A} \supset A$.

The linear operator $(A, \text{Dom}(A))$ is said to be closed if the graph $\Gamma(A) := \{(\varphi, A\varphi) \in \mathcal{H} \oplus \mathcal{H} : \varphi \in$

Dom$(A)\}$ is closed with respect to the norm $\|(\varphi,\psi)\|^2 = \|\varphi\|^2 + \|\psi\|^2$ on $\mathcal{H}\bigoplus\mathcal{H}$. $\Gamma(A)$ is a linear subset of $\mathcal{H}\oplus\mathcal{H}$. A linear operator is said to be closable if it admits a closed extension. In that case it admits a unique "smallest" closed extension $(\overline{A}, \mathrm{Dom}(\overline{A}))$, called the closure of $(A, \mathrm{Dom}(A))$, and characterized by the fact that $\overline{A} \subset \tilde{A}$ for any other closed extension. Moreover, the graph of the closure satisfies $\Gamma(\overline{A}) = \overline{\Gamma(A)}$. Let A be closed. Then a set $D \subset \mathrm{Dom}(A)$ is called a core for A if $\overline{\tilde{A}\big|_D} = A$. The importance of closed operators is twofold: first, one can define the spectrum of such operators in a way that is consistent with the definition for bounded operators and second, they generalize bounded operators. Indeed, by the Closed Graph theorem, linear operators defined on a Banach space are closed if and only if they are bounded.

Lemma 8.15. *Suppose that $(A, \mathrm{Dom}(A))$ is densely defined and bounded on its domain. Then A extends to a unique bounded operator defined on all of $\mathcal{H}$.*

Proof. Let $\psi_n \in \mathrm{Dom}(A)$, $\psi \in \mathcal{H}\backslash\mathrm{Dom}(A)$ be such that $\psi_n \to \psi$. $A\psi_n$ is a Cauchy sequence and hence converges to $\xi \in \mathcal{H}$. Let $A\psi := \xi$. The norm of the extended operator is the same as original, since $\frac{\|A\psi\|}{\|\psi\|} \le \|A\|\frac{\|\psi_n\|}{\|\psi\|} + \frac{\|A\|}{\|\psi\|}\|\psi - \psi_n\| \to \|A\|$ as $n \to \infty$. □

Lemma 8.16. *If $A : D \to D' \subset \mathcal{H}$ is a bijection, then (A, D) is closed if and only if (A^{-1}, D') is closed.*

Proof. Suppose that (A, D) is closed and let $\psi_n \in D'$, $\varphi_n = A^{-1}\psi_n \in D$ be such that $\psi_n \to \psi$ and $\varphi_n \to \varphi$. Then $\|(\psi_n, A^{-1}\psi_n) - (\psi, \varphi)\| \to 0$ if and only if $\|(\varphi_n, A\varphi_n) - (\varphi, \psi)\| \to 0$ so that $\psi \in D'$ and $\varphi \in D$. □

The spectrum $\sigma(A)$ of a closed linear operator (A, D) is the complement of the resolvent set

$$\rho(A) := \{z \in \mathbb{C} : (A - z) : D \to \mathcal{H} \text{ is a bijection, and } (A - z)^{-1} : \mathcal{H} \to D \text{ is a bounded operator}\}$$

Note that the condition that $(A - z)^{-1} : \mathcal{H} \to D$ be a bounded operator together with the Closed Graph theorem and lemma 8.16 requires that (A, D) be a closed operator, and conversely, the assumption that (A, D) be closed implies that $(A - z)^{-1} : \mathcal{H} \to D$ is bounded.

The spectrum of a linear operator is always a closed set, and if A is a bounded operator on its domain, then $\sigma(A)$ is compact. Furthermore,

Lemma 8.17. *If $\sigma(A) \neq \mathbb{C}$, then A is closed.*

Proof. If $z \in \rho(A)$, then $(A - z)^{-1} : \mathcal{H} \to D$ is a bounded operator and so by the Closed Graph theorem $\big((A - z)^{-1}, \mathcal{H}\big)$ is closed. It follows by lemma 8.16 that $((A - z), D)$ and hence (A, D) is closed. □

The adjoint (A^*, D^*) of (A, D) is the linear operator determined as follows: D^* is set of all $\psi \in \mathcal{H}$ such that there exists a $\xi \in \mathcal{H}$ so that

$$\langle \psi, A\varphi \rangle = \langle \xi, \varphi \rangle, \qquad \text{for all } \varphi \in D \tag{8.3}$$

If D is dense, ξ is unique so that one sets $A^*\psi := \xi$. An operator and its adjoint are thus related by

$$\langle \psi, A\varphi \rangle = \langle A^*\psi, \varphi \rangle, \qquad \text{for all } \varphi \in D, \psi \in D^*$$

If D^* is also dense, we may further define $A^{**} := (A^*)^*$, and so on.

Theorem 8.18. *Let (A, D) be densely defined. Then:*

1. *(A^*, D^*) is a closed operator.*
2. *If A is closable, then D^* is dense and $\overline{A} = A^{**}$.*
3. *If A is closable, $\overline{A}^* = A^*$.*

Proof.

1. Introduce the unitary operator $e^{-i\frac{\pi}{2}} : \mathcal{H} \bigoplus \mathcal{H} \ni \langle \psi, \varphi \rangle \to \langle \varphi, -\psi \rangle \in \mathcal{H} \bigoplus \mathcal{H}$ and the set $e^{-i\frac{\pi}{2}}\Gamma(A) := \{(A\varphi, -\varphi) : \varphi \in D\}$. Then $(\psi, \xi) \in \Gamma(A^*)$ if and only if $\langle \psi, A\varphi \rangle = \langle \xi, \varphi \rangle$ for all $\varphi \in D$ if and only if $\Big\langle (\psi, \xi), (A\varphi, -\varphi) \Big\rangle = 0$ for all $\varphi \in D$ if and only if $(\psi, \xi) \in [e^{-i\frac{\pi}{2}}\Gamma(A)]^\perp$, thus $\Gamma(A^*) = [e^{-i\frac{\pi}{2}}\Gamma(A)]^\perp$.

2. Suppose that $\eta \in \mathcal{H}$ is such that $\langle \psi, \eta \rangle = 0$ for all $\psi \in D^*$. Then $(\eta, 0) \in [\Gamma(A^*)]^\perp = [e^{-i\frac{\pi}{2}}\Gamma(A)]^{\perp\perp} = \overline{[e^{-i\frac{\pi}{2}}\Gamma(A)]}$. Thus there exists $\varphi_n \in D$ such that $\varphi_n \to 0$ and $A\varphi_n \to \eta$. Since A is closable, $\eta = \overline{A}0 = 0$ and so $(D^*)^\perp = 0$, i.e. $\overline{D^*} = (D^*)^{\perp\perp} = \mathcal{H}$.

   To show that $\overline{A} = A^{**}$, we note that $e^{-i\frac{\pi}{2}}[E^\perp] = [e^{-i\frac{\pi}{2}}E]^\perp$ and $e^{-i\pi}E = (e^{-i\frac{\pi}{2}})^2 E = -E = E$ for any linear subspace E of $\mathcal{H} \bigoplus \mathcal{H}$. Then:

   $$\Gamma(\overline{A}) = \overline{\Gamma(A)} = [\Gamma(A)^\perp]^\perp = [[e^{-i\pi}\Gamma(A)]^\perp]^\perp = [e^{-i\frac{\pi}{2}}[e^{-i\frac{\pi}{2}}\Gamma(A)]^\perp]^\perp = [e^{-i\frac{\pi}{2}}\Gamma(A^*)]^\perp = \Gamma(A^{**})$$

   Hence $\overline{A} = A^{**}$.

3. $A^* = \overline{A^*} = (A^*)^{**} = (A^{**})^* = \overline{A}^*$

□

An operator (A, D) is symmetric if for all $\varphi, \psi \in D$, $\langle \varphi, A\psi \rangle = \langle A\varphi, \psi \rangle$. In addition, we assume that symmetric operators are densely defined. If (A, D) is symmetric and $\psi \in D$, then taking $\xi = A\psi$ in 8.3 shows that $D^* \supset D$. Moreover

$$\langle A\psi, \varphi \rangle = \langle \psi, A\varphi \rangle = \langle A^*\psi, \varphi \rangle, \qquad \text{for all } \psi, \varphi \in D$$

shows that (A^*, D^*) is a (densely defined) closed extension of (A, D).

A symmetric operator is self-adjoint if it is equal to its adjoint. A symmetric operator is essentially self-adjoint if its closure is self-adjoint. In that case $A^{**} = \overline{A} = \overline{A}^* = A^*$, where the first and last equalities follow from theorem 8.18. We therefore have the following scenarios:

For symmetric operators

$$A \subset A^{**} \subset A^*$$

For closed symmetric operators

$$A = A^{**} \subset A^*$$

For essentially self-adjoint operators

$$A \subset A^{**} = A^*$$

For self-adjoint operators

$$A = A^{**} = A^*$$

For (A, D) symmetric we define the defiency subspaces $L^\pm$ by

$$L^\pm := \{\varphi \in D^* : A^*\varphi = \pm i\varphi\} = \text{Ker}(A^* \mp i) = \text{Ran}(H \pm i)^\perp = \{\varphi \in \mathcal{H} : \langle A\psi, \varphi\rangle = \pm i\psi, \varphi\rangle \;\; \forall \psi \in D\}.$$

Note that the definition of $L^\pm$ is invariant if one replaces A by its closure $\overline{A}$. The deficiency indices are the dimensions of $L^\pm$. The following theorem characterizes symmetric operators with self-adjoint extensions:

Theorem 8.19. *Let (A, D) be a symmetric operator. Then it has a self-adjoint extension if and only if the deficiency indicies are equal. Moreover, the following are equivalent:*

1. *A is essentially self-adjoint.*
2. *Both deficiency indices are zero.*
3. *$\overline{A}$ is the only self-adjoint extension of A.*

Theorem 8.20. *Let A be self-adjoint. Then*

1. *$\sigma(A) \subset \mathbb{R}$ and $\sigma(A) \neq \varnothing$.*
2. *$\|(A - z)^{-1}\| \leq \frac{1}{|\text{Im } z|}$ for $z \in \mathbb{C}\backslash\mathbb{R}$.*
3. *$[(A - z)^{-1}]^* = (A - \overline{z})^{-1}$for $z \in \rho(A)$.*

Proof. We only prove that the spectrum is non empty. If $\sigma(A)$ were empty, then $A^{-1} : \mathcal{H} \to D$ would be a bounded operator. Moreover it is self-adjoint by the third part of this theorem. Let us determine its resolvent. If $z \neq 0$, then the operator $A^{-1} - z : \mathcal{H} \to \mathcal{H}$ is bounded. It is injective since $(A^{-1} - z)\varphi = (A^{-1} - z)\psi \Leftrightarrow A^{-1}(\varphi - \psi) = z(\varphi - \psi) \Rightarrow (\varphi - \psi) = 0$. It is surjective since $[A^{-1} - z](-z^{-1}A(A - z^{-1})^{-1}\psi) = \psi$ for all $\psi \in \mathcal{H}$. Therefore by the Inverse Mapping theorem $(A^{-1} - z)^{-1}$ is bounded. Hence $z \in \rho(A^{-1})$. However since the spectrum of a bounded operator is non-empty we must have $\sigma(A^{-1}) = \{0\}$. However, the spectral radius of a bounded self-adjoint operator is equal to its norm, thus $\|A^{-1}\| = 0$, i.e. $A^{-1} = 0$ which is a contradiction. □

Let $\Omega \subset \mathbb{R}^n$ and $\mathcal{B}(\Omega)$ be the Borel sigma-algebra on Ω. For any positive measure μ we have the Hilbert space $\mathcal{H} = L^2(\Omega, \mathcal{B}(\Omega), \mu)$. Consider a measurable function $a : \Omega \to \mathbb{R}$ that is bounded on bounded subsets of Ω. We set

$$D := \{f \in \mathcal{H} : \int_\Omega (1 + a^2(x))|f(x)|^2 d\mu(x) < \infty\}$$

D is dense in $\mathcal{H}$ since it contains all characteristic functions 1_E supported on bounded sets $E \subset \Omega$ and all their finite superpositions. Thus the multiplication operator (A, D) defined by

$$(Af)(x) = a(x)f(x) \quad \text{for all } f \in D$$

is densely defined. We also introduce

$$L^2_c := \{f \in \mathcal{H} : \text{supp}(f) \text{ is compact}\}$$

Then L_c^2 is also dense in $\mathcal{H}$ and $L_c^2 \subset D$.

Proposition 8.21. *(A, D) is self-adjoint and (A, L_c^2) is essentially self-adjoint. (A, D) is therefore the closure of (A, L_c^2).*

Proof. Since a is real-valued, A is easily shown to be a symmetric operator. Let $z \in \mathbb{C}\backslash\mathbb{R}$. Then the bounded operator $R(z) : \mathcal{H} \to \mathcal{H}$ given by

$$(R(z)f)(x) = (a(x) - z)^{-1} f(x)$$

is obviously the inverse of $(A - z) : D \to \mathcal{H}$. Hence $\sigma(A) \neq \mathbb{C}$, and so A is a closed symmetric operator by lemma 8.17. Moreover the deficiency indices of A are both zero: For $f \in D^*$,

$$\begin{aligned}
A^* f = \pm i f &\Leftrightarrow \langle g, A^* f\rangle = \pm i\langle g, f\rangle \ , \forall g \in D \\
&\Leftrightarrow \langle Ag, f\rangle = \pm i\langle g, f\rangle \ , \forall g \in D \\
&\Leftrightarrow \int_\Omega \overline{Ag} f d\mu = \pm i \int_\Omega \overline{g} f d\mu \ , \forall g \in D \\
&\Leftrightarrow \int_\Omega (a(x) \mp i)\overline{g(x)} f(x) d\mu(x) \ , \forall g \in D \\
&\Leftrightarrow f = 0 \quad \mu - a.e.
\end{aligned}$$

Hence (A, D) is essentially self-adjoint by theorem 8.19, hence self-adjoint. To show that (A, L_c^2) is essentially self-adjoint, we need to show that $\overline{\Gamma(A, L_c^2)} = \Gamma(A, D)$. For $f \in D$ and $j \in \mathbb{N}$, let $f_j(x) = \mathbb{1}_{[-j,j]^n}(x) f(x)$. Then $f_j \in L_c^2$ and the Lebesgue dominated convergence theorem implies that $(f_j, Af_j) \to (f, Af)$ as $j \to \infty$. ☐

Proposition 8.22. *The spectrum and resolvent of the multiplication operator (A, D) satisfy:*

1. $\sigma(A) = \text{ess ran } (a) := \{z \in \mathbb{C} : \mu(\{|a(x) - z| < \varepsilon\}) > 0, \forall \varepsilon > 0\}$.
2. $\|(A - z)^{-1}\| = \|(a(x) - z)^{-1}\|_\infty = \frac{1}{\text{dist}(z, \sigma(A))}$ *for* $z \in \rho(A)$.

Proof.

1. If λ is not in the essential range of a, then the operator of multiplication by $(a(x) - z)^{-1}$ from $\mathcal{H} \to \mathcal{H}$ is bounded. This operator is also the inverse of the operator of multiplication by $(a(x) - z)$ from $D \to \mathcal{H}$, and so $z \in \rho(A)$. Conversely, if z is in the essential range of a, let

   $$F_j = F_j(z) = \{x \in \Omega : |a(x) - z| < 2^{-j}\}$$

   Then
   $$\|(A - z)\mathbb{1}_{F_j}\|^2 = \int_\Omega |a(x) - z|^2 \mathbb{1}_{F_j}(x) d\mu(x) \leq 2^{-2j} \int_\Omega \mathbb{1}_{F_j} d\mu = 2^{-2j} \|\mathbb{1}_{F_j}\|^2$$

   Thus $\|(A-z)\mathbb{1}_{F_j}\| \leq 2^{-j}\|\mathbb{1}_{F_j}\|$ which shows that $(A-z)^{-1}$, if it exists, cannot be a bounded operator. Indeed, $(A - z)^{-1}$ would map the unit element $\frac{(A-z)\mathbb{1}_{F_j}}{\|(A-z)\mathbb{1}_{F_j}\|}$ to an element with norm greater than 2^j. Thus $z \in \sigma(A)$.

2. If $z \in \rho(A)$, then $z \notin \text{ess ran}\,(a)$ and so $\exists \varepsilon > 0$ such that $\mu(\{|a(x)-\lambda| < \varepsilon\}) = 0 \Leftrightarrow \mu(\{|(a(x)-\lambda)^{-1}| > \varepsilon^{-1}\}) = 0$ and so $\|(a(x)-z)^{-1}\|_\infty < \infty$. For $f \in L^2(\Omega, \mu)$,

$$\|(A-z)^{-1}f\|^2 = \int_\Omega |a(x)-z|^{-2}|f(x)|^2 d\mu(x) \le \|(a(x)-z)^{-1}\|_\infty^2 \|f\|^2$$

and so $\|(A-z)^{-1}\| \le \|(a(x)-z)^{-1}\|_\infty$.

Conversely, let $\varepsilon > 0$ and $F := \left\{|a(x)-z|^{-1} \ge \|(a(x)-z)^{-1}\|_\infty - \varepsilon\right\}$. Then

$$\|(A-z)^{-1}\mathbb{1}_F\|^2 = \int_\Omega |a(x)-z|^{-2}\mathbb{1}_F(x) d\mu(x) \ge \left(\|(a(x)-z)^{-1}\|_\infty - \varepsilon\right)^2 \|\mathbb{1}_F\|^2$$

which shows that $\|(A-z)^{-1}\| \ge \|(a(x)-z)^{-1}\|_\infty$.

If $z \in \rho(A)$, then $d := \inf\limits_{y \in \text{ess ran }(a)} |z-y| = \inf\limits_{y \in \sigma(A)} |z-y| = \text{dist}(z, \sigma(A)) > 0$. If $U_n = \{|a(x)-z| < d - \frac{1}{n}\}$, we have $\mu(U_n) = 0$ for all $n \in \mathbb{N}$ sufficiently large. Hence $\mu(\cup_n U_n) = \mu(\{|a(x)-z| < d\}) = 0$. This shows that $\|(a(x)-z)^{-1}\|_\infty \le \frac{1}{d}$.

To show the reverse inequality, for all $\frac{1}{d} > \varepsilon > 0$ we have $\mu(\{|a(x)-z| < \frac{d}{1-d\varepsilon}\}) > 0 \Leftrightarrow \mu(\{|a(x)-z|^{-1} > \frac{1}{d} - \varepsilon\}) > 0$ which implies that $\|(a(x)-z)^{-1}\|_\infty \ge \frac{1}{d}$.

□

In general, if $(A, \text{Dom}(A))$ and $(B, \text{Dom}(B)$ self-adjoint operators, then $(A+B, \text{Dom}(A) \cap \text{Dom}(B))$ need not be self-adjoint. The following result on the perturbation of self-adjoint explains when $A+B$ may be self-adjoint. The following is based on Chapter X of [RS2].

Let A and B be densely defined operators on a Hilbert space $\mathcal{H}$. B is said to be A-bounded if:

1. $\text{Dom}(B) \supset \text{Dom}(A)$.

2. There exist $a \ge 0$ and $b < \infty$ such that for all $\psi \in \text{Dom}(A)$

$$\|B\psi\| \le a\|A\psi\| + b\|\psi\|$$

In this case we say that B has relative bound a with respect to A. The infimum over such relative bounds is the relative bound of B with respect to A. Typically b is taken larger as a is chosen smaller.

Example 8.23. *If B is a bounded operator, then B is A-bounded and its relative bound is zero.*

Theorem 8.24. *(Kato-Rellich) Let A and B be linear operators on $\mathcal{H}$. Suppose that A is self-adjoint, B is symmetric, and B is A-bounded with relative bound $a < 1$. Then $A+B$ is self-adjoint on* Dom(A) *and essentially self-adjoint on any core of A. In particular, if*

Corollary 8.25. *If A and B are self-adjoint operators and B is bounded, then $A+B$ is self-adjoint on* Dom(A) *and essentially self-adjoint on any core of A.*

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