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Intersymbol Interference in digital systems

PERFORMANCE OF DIGITAL COMMUNICATION SYSTEMS

IN

NOISE AND INTERSYMBOL INTERFERENCE

by

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ABSTRACT

The very interesting problem of the evaluation of the probability of error of digital communication systems in the presence of noise and intersymbol interference is considered here. Known methods of solution (the exhaustive method, the power series method, the Gaussian Quadrature rule method...) are presented and their properties compared and discussed.

The Chebyshev expansion method (CEM) is next introduced. We think that the CEM is the one which should logically follow the power series expansion (PSE) method. Thus it will be seen that in the CEM the same information, i.e. the moments of the random variable which represents the intersymbol interference process, are used to compute the probability of error, as in the PSE method. Furthermore, the weak point of the PSE method i.e., its slow convergence is corrected in the CEM.

Specific results are finally shown and discussed.

RESUME

Le très intéressant problème de l'évaluation de la probabilité d'erreur des systèmes de communication digitale en présence du bruit et de l'interférence entre les symboles est considéré ici. Les méthodes de solution connues (La méthode exhaustive, la méthode d'expansion en Séries de Taylor, la méthode des règles de quadrature Gaussienne...) sont présentées et leurs particularités discutées.

La méthode de l'expansion en polynômes de Chebyshev (EPC) qui est ensuite introduite, nous paraît être la suite logique à celle de l'expansion en Séries de Taylor (EST). Ainsi on va voir que dans la méthode EPC, les même données que dans la méthode EST, sont utilisées pour produire la probabilité d'erreur. De plus, la principale faiblesse de la méthode EST, càd la lenteur de la convergence, se trouve corrigée dans la méthode EPC.

Des résultats spécifiques sont finalement présentés.

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TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Introduction to the Problem

The most meaningful way to measure the performance of many digital communication systems is to evaluate their probabilito of digit (or symbol) error, P (e). Since the efficiency as well as the accuracy required from different communication systems may vary considerably depending on the desired application, one may point out the importance of such an appropriate measure of efficiency. Furthermore, different systems can be compared by this crite - rion of symbol error.

The probability of symbol error may be defined as the probability of an erroneous decision about the nature of the symbol which has been sent. Thus, one out of many possible symbols is transmitted at a time by means of waveforms which should be somehow distinct from one another. At the receiving end, from the received waveform, the memoryless decision device should decide which symbol has been sent. There is an average probability of error connected with this decision process.

The probability of symbol error P (e) can be easily evaluated if one assumes that additive noise represents the only cause of defect. In fact, many communication systems have been designed with the above assumption, which is not true at all in digital systems.

Although noise is an important perturbator, usually it is not the only one. Another cause of disturbance which should be carefully considered is the intersymbol interference.

We know that in digital systems a transmitted narrow-pulse waveform (which represents a certain symbol) is actually received as a smeared waveform of longer duration. Output waveforms are commonly detected by periodic sampling after filtering. Intersymbol interference (ISI), as its name shows, results from the overlapping of the received waveform of a digit into the time-slots reserved for other digits. This mix-up may prevent the decision device from correctly deciding which digit has been sent.

From the designer viewpoint, intersymbol interference results from many different causes: distortion due to a dispersive transmission medium, nonoptimal choice of sampling instants, imperfect design of transmitted signals and / or receiving filters.

ISI can be reduced by carefully waveshaping the transmitted waveforms and/ or designing eleborate filters, or by equalization of the system.

Equalization can be considered from two points of view: minimization of peak distortion, and mean square distortion minimization. The main goal of equalization consists in achieving minimum error in the transmission scheme; but until recently, no convenient way of computing the probability of error due to intersymbol interference (even in the absence of noise) was provided, so that one had to be content with the above criterions for equalization. Hence, an easy scheme for evaluating the probability of error in the presence of noise and intersymbol interference is most desirable.

We note here that the probability of error need not be known exactly. An accurate estimate would be sufficient, provided that the estimate does not differ from the exact result by more than, say, 1%.

In the following, we shall first state the problem in its context, i.e., as it occurs in a digital system and then we shall summarize known methods to compute P (e).

1.2 Statement of the Problem

We shall consider only a 2L-level digital pulse amplitude modulation (PAM) system in this thesis. Extension to other systems such as partial response coded (PRC) or phase shift keyed (PSK) systems is possible, but requires some more work.

A 2L-level PAM system is shown in Fig. 1.1, where the $\{a_h\}_{h=-\infty}^{\infty}$ form the sequence of input symbols, and each of the a_h takes on values from the set $\left\{ (-2L+1)d, (-2L+3)d, \ldots, -d, d, \ldots, (2L-1)d \right\}$, with probability $p_k = Pr\left\{ a_h = kd \right\}$ for any h. We note that the distance between two adjacent levels is 2d. T (ω), C (ω) represent, respectively, the transfer functions of the transmitting filter, the channel and the receiving filter. We have:

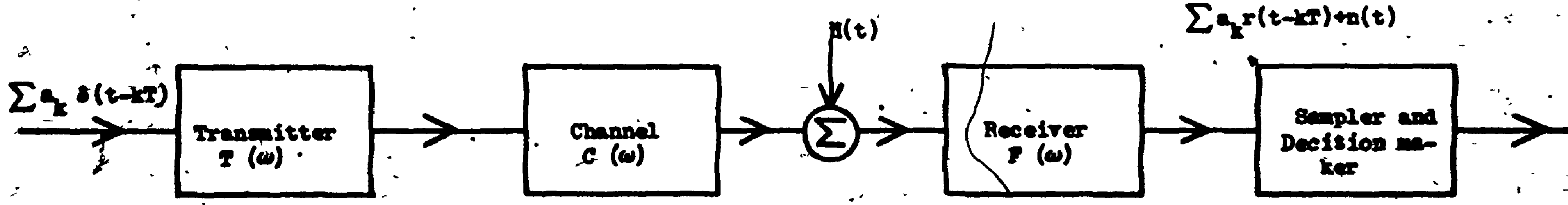

Fig. 1.1 Block diagram of a 2L-level PAM System

$$R(\omega) = T(\omega)\, C(\omega)\, F(\omega) \qquad (1.1.1)$$

$$r(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} R(\omega)\, d\omega \qquad (1.1.2)$$

where r (t) may be called the noiseless overall impulse response of the time-invariant channel. Furthermore, n(t) is the additive noise, T is the signaling interval, and the decision thresholds are set at levels [2]

$$0, \pm 2\, d_r(t_o), \pm 4\, dr(t_o), \ldots, \pm(2L - 2)\, dr\,(t_o)$$

The received signal being sampled every T seconds, i.e., at times $t_o + kT$, the sampled signal at time t_o can then be represented as:

$$y(t_o) = a_o r(t_o) + \sum_{n=-\infty}^{\infty}{}' a_n r(t_o - nT) + n(t_o) \qquad (1.2.1)$$

The prime in the summation denotes the exclusion of the term with $n = 0$.

One may denote $y(t_o) = y_o$, $r(t_o) = r_o$, $r(t_o - nT) = r_n$, $n(t_o) = n_o$, for notational convenience. In (1.2.1), the desired signal is $a_o r_o$, n_o is the noise and the summation represents the intersymbol interference term. Indeed, $a_{-n} r_{-n}$ in (1.2.1) results from the symbol a_{-n} which has been sent nT units of times prior to the transmitting of symbol a_o. It may be seen then that, as previously said, ISI results from the spreading of r (t) over more than one signaling interval. Note also that ISI are caused by the past transmitted symbols, as well as the "future" transmitted symbols, due to the time delay in the system.

If we now consider $\sum_{n=-\infty}^{\infty}{}' a_n r_n$ as a random variable (r.v.) X, the conditional probability of error given that the kth level is transmitted is:

$$P\left\{ e|a_o = kd \right\} = P\left\{ |X + n_o| > dr_o \right\} \text{ for } k \neq \pm (2L - 1) \quad (1.2.2)$$

$$\begin{aligned} P\left\{ e|a_o = (-2L + 1)d \right\} &= P\left\{ (X + n_o) > dr_o \right\} \\ P\left\{ e|a_o = (+2L - 1)d \right\} &= P\left\{ (X + n_o) < -dr_o \right\} \end{aligned} \quad (1.2.3)$$

In (1.2.2), the condition $|X + n_o| > dr_o$ means that the sampled received signal falls in a different slice than in the one it is supposed to be. This causes the decision device to draw an erroneous conclusion. For the outermost levels, as in (1.2.3), error occurs only if the received signal falls in an inner slice.

We have then:

$$P(e) = \sum_{k=-2L+1,2}^{2L-1} P\left\{ a_o = kd \right\} P\left\{ e \mid a_o = kd \right\}$$

$$\begin{aligned} &= \sum_{k=-2L+3,2}^{2L-3} p_k P\left\{ |X + n_o| > dr_o \right\} + p_{2L-1} P\left\{ X + n_o < -dr_o \right\} \\ &\quad + p_{-2L+1} P\left\{ X + n_o > dr_o \right\} \end{aligned} \quad (1.2.4)$$

We can easily see then that X is an even random variable if we assume equiprobability of all the levels. This is a usual assumption. Then, $P\left\{X + n_o > dr_o\right\} = P\left\{X + n_o < - dr_o\right\}$, and (1.2.4) can be written as:

$$P(e) = (1 - p_{2L-1}) P\left\{ |X + n_o| > dr_o \right\}$$
$$= 2(1 - \frac{1}{2L}) P\left\{ (X + n_o) > dr_o \right\} \quad (1.2.5)$$
$$= \frac{2L-1}{L} P\left\{ X + n_o > dr_o \right\}$$

The above equation is true for binary as well as for 2L-ary systems.

The problem of evaluating P (e) in the presence of both noise and intersymbol interference is intimately linked with the RHS of (1.2.5). In this RHS, dr_o being a constant, we have thus to consider the distribution of a sum of two random variables, the noise r.v. and the ISI r.v.; the probability distribution function (pdf) of the noise r.v. is assumed known and even, but the pdf of the ISI r.v. cannot be described explicitly.

If we denote by $F(\cdot)$, the distribution function of the noise r.v., (1.2.5) becomes then:

$$P(e) = \frac{2L-1}{L} E\left\{ P_r\left\{ X + n_o > dr_o \right\} \mid X = x \right\}$$
$$= \frac{2L-1}{L} E\left\{ 1 - F(dr_o - x) \right\} \quad (1.2.6)$$

The expectation $E\{\cdot\}$ should be taken over all values of X.

From (1.2.6) one may note that theoretically the ISI may "help" the decision-device. This can be seen most easily in a binary system, when the ISI is added to the signal, resulting in a signal with more power. But, practically ISI almost always has an undesirable effect.

Without intersymbol interference, the system is optimaly-designed to cope with noise; the presence of ISI destroys this optimality.

Equation (1.2.6) would be very easy to compute if the pdf of X, the ISI r.v. was known. Unfortunately, the problem of evaluating the pdf of X is a problem of long standing, and a very difficult one [10]. In fact, in most practical cases, the limiting probability density, obtained when one considers an arbitrarily large number of interfering samples, is singular. This is due to the fact that the asymptotic decay of impulse responses in physical channels is exponential [2].

In order to be able to compute (1.2.6) numerically, one then must suppose that the impulse response r (t) has only a finite number of interfering samples say M. That is, we shall suppose:

$$r_h = 0, \; h < -p \; , \; h < q, \; p + q = M$$

This truncation is common to all numerical methods even though it introduces a certain error. This is a very reasonable approximation for any real channel.

Moreover, we can vary the number M with the channel, e.g., in some cases a large value of M is used, in order to accomodate all the interfering samples of respectable size. Actually the value of M depends on the accuracy required for the computation of P(e).

Another way to offset the error introduced by the above truncation consists in gradually augmenting the value of M (this corresponds to taking into account smaller and smaller samples of r (t), until the computed P (e) stops changing.

Under the above hypothesis, the pdf i(x) of X always exists; as X takes on a finite number of values which are discrete, i (x) can be represented by a weighted sum of Dirac delta functions:

$$i\ (x) = \sum_{i=1}^{(2L)^M} q_i \ \delta\,(x - xi) \qquad (1.2.7)$$

where q_i is the probability that X equals xi,.

We can then write (1.2.5) as:

$$P(e) = \frac{2L - 1}{L} \int_D \left\{ 1 - F(dr_o - x) \right\} i(x)\ dx \qquad (1.2.8)$$

$$= \frac{2L - 1}{L} \sum_{i=1}^{(2L)^M} q_i \left\{ 1 - F(dr_o - xi) \right\} \qquad (1.2.9)$$

Here, although we consider only a finite number of interfering samples it remains very difficult to compute i(x), the pdf of X. Therefore, being unable to express i (x) directly we should replace the information that i (x) conveys by some alternate information that is related to i (x). Furthermore this alternate information should have a form suitable for use with the integral in (1.2.8). We shall see that the required information is provided by the moments of X. The use of the moments is appropriate here, due to the following reasons.

i) From the moments of X, one can form an idea of the shape of its distribution.

ii) In this case the moments are available as they can be computed iteratively.

That is why all the recent methods of computing P (e) use the moments of i (x). In the following when one refers to the moments, these are the moments of the ISI r.v.

1.3 Review of known methods

The first attempt to solve the problem was to derive bounds on the probability of error. More elaborate computational methods followed, when it was observed that these bounds were not sufficiently accurate for practical purposes.

i) Exhaustive method:

By truncating r (t) to M samples, we may be able to compute the probability of error by the direct enumeration of all possible cases. But, as the number of cases grows exponentially with M, this method is too time consuming and can be applied only to a few simple systems. For a 4-level, 11- pulse truncation approximation, for example, we would need more than an hour of computer time to derive the probability of error.

The only utility of such a method is that it provides an accurate result for some binary systems that can be served to check results achieved by other methods.

ii) Worst case bounds

As its name implies, this bound is computed by giving the ISI its worst value, i.e. the largest one. Thus in (1.2.5), instead of $\Pr\{ X + n_o > dr_o \}$, we would have $\Pr\{(\max_{x \in D} X + n_o) > dr_o\}$.

In binary systems, the probability that this worst-case occurs is very small, and it is negligible in M-ary ($M > 2$) systems. Thus, this method leads to very pessimistic results.

This bound is related to the so-called "eye-pattern" which is the superposition of all possible signals presented to the sampler. The eye-pattern can provide some interesting information about the performance characteristics of data transmission systems, but its use is not adequate to estimate P (e).

An open-eye pattern means that ISI and noise are small. A system with closed eye pattern may suffer bad performance resulting from large ISI.

But, the eye pattern method is too intuitive to be accurate and designers cannot afford to base their designs on such results.

iii) Chernoff bounds:

Upon noticing that our problem may be reduced to the estimation of the probability distribution function (pdf) of two independent random variables, a generalized Chebyshev inequality has been initially used by Saltzberg [3] to obtain an upper bound on P (e). Here, the noise is assumed to be white Gaussian. Derivation of Saltzberg bound involves a problem of optimization, and does not require too much computational work.

Although this bound is better than the worst case bound, it is still too loose to be useful in practical applications. But, one of the results achieved by Saltzberg has been to open the way and give a direction to more research in the domain of deriving tighter bounds on P (e).

An upper bound, which also involves an optimization of a Chernoff inequality and thus is related to Saltzberg bound, was derived by Lugannani [4] who claimed that his bound is simpler to use. Recently, Falconer and Gitlin [19] and McLane [20] have also generalized the Saltzberg bound for single error to given patterns of double or more errors.

iv) Power-Series expansion methods: [5-7 , 11]

. In order to compute P (e), we have to deal with the sum of two random variables. This method expands into power series the noise process (in the probability domain) or the characteristic function of the ISI r.v. (in the transformed domain), and leads to successive differentiation of the distribution function of the noise random process. In the case of Gaussian noise, we then obtain a series of Hermite polynomials. The absolute convergence of this series has been theorically proven.

This method presents thus a great improvement over preceding methods. In the case where the channel distortion is not severe, it gives satisfactory results. But, whenever the channel suffers severe distortions, i.e., when the intersymbol interference (or co-channel interference) is important this method gives oscillating results. The main drawback associated with this method is that the power series does not converge fast enough to the result.

v) GQR method [8; 18]

This method is based upon nonclassical Gaussian quadrature rule (GQR). GQR gives an approximation to definite integrals of the following form:

$$I = \int_a^b f(x)\,\omega(x)\,dx \tag{1.3.1}$$

The approximation consists of a linear combination of values of the function f (x).

$$I \simeq \sum_{i=1}^{m} \omega_i \; f(x_i) \tag{1.3.2}$$

The set $\left\{\omega_i, x_i\right\}_{i=1}^{m}$ is a quadrature rule corresponding to the weight function $\omega(x)$. Recalling that in our problem we have an integral of the same form as the one in (1.3.1), and upon identifying $1 - F(dr_o - x)$ and $i(x)$ in (1.2.8), respectively, with $f(x)$ and $\omega(x)$ in (1.3.1), we can produce a set

$\{i_i, x_i\}_{i=1}^{M}$ which is used in conjunction with formula (1.3.2) to give an approximation to P(e):

$$P(e) \simeq \frac{2L-1}{L} \sum_{i=1}^{m} i_i \left(1-F(dr_o-x_i)\right) \tag{1.3.3}$$

The x_i's, called the nodes or abscissas of the QR are generally computed from the knowledge of i(x), the ISI r.v.

Here, i(x) being unknown, Benedetto et al. [8] take opportunity of a mathematical scheme to derive the x_i's, by using simply the moments of the ISI. This scheme does not always succeed to produce the abscissas and weights, due to round-off errors.

vi) <u>Other methods:</u>

One may mention here the method suggested by Hill [17] who his applied the idea underlying the concept of fast-Fourier transform, to the problem of computing P(e). His method involves the decomposition of interfering samples into blocks of fewer samples; the pdf of each block is then found (this can be done easily) and they are convolved together to give the pdf of the ISI r.v. Thus this method constructs an approximation to the pdf of the ISI r.v. One may also prove that GQR method also constructs such an approximation.

In all the above cases, independence of successive symbols has been implicitly assumed. Glave [16] has developped an upper bound for correlated signals, called the variance constrained bound. Surprisingly enough, the variance constrained bound appears to give tight results, although it has been conceived in a "worst - case" idea. The peak and RMS values of the ISI are the only two elements required for this bound.

In the following, in Chapters 11 and 111, we shall first present and discuss in detail the power series expansion and the GQR methods, from an explanatory point of view, with merits and drawbacks being mentionned. The differences between Ho and Yeh method, Shimbo and Celebiler method, and Prabhu method are stressed. A physical interpretation of the GQR method is given.

In Chapter 1V, the new method introduced in this thesis called the Chebyshev series expansion method,an improved version of the power series expansion method, is derived. Finally results achieved by power series expansion, GQR and Chebyshev series expansion methods are presented and compared.

CHAPTER II

THE POWER SERIES EXPANSION METHOD

As shown in (1.2.8) one is primarily concerned with an integral of the form

$$\int_D c(x)\ i(x)\ dx$$

where i(x) is an unknown pdf and the domain D is bounded.

The difficulty here lies in the fact that i(x) cannot be classified in any category of functions. The distribution of X the ISI r.v., may theorically be uniform or it may decrease exponentially. In fact, up to the present time no method has been derived to compute i(x), except in special cases. Thus all the methods to compute the above integral use in a way or another, an approximation for i(x).

2.1. Shimbo and Celebiler Method [7]

Shimbo and Celebiler probably started from the fact that the ISI results from the combination of a large number of independent sources, thus it should present a normal-like distribution. But, they have also shown that the ISI r.v. is definitely not normal. Therefore they have then applied the Gram Charlier series expansion [15] to the characteristic function of the ISI r.v.

In the probability domain, i(x) is thus expanded into the following:

$$i(x) = N_s(x) + \sum_{k=1}^{\infty} \frac{C_k}{k!} N_s^{(k)}(x) \qquad (2.1.1)$$

where $N_s(x)$ is the density function of a zero mean normal r.v. having variance s^2, which is to be determined, and where $N_s^{(k)}$ represents the k^{th} derivative of N_s. In fact, in (2.1.1) s^2 represents the "normal" or "Gaussian" part of the intersymbol interference. Thus in one of their first papers on the subject, Shimbo and Celebiler gave s^2 a value related to the "power" of the ISI.

The derivation of the C_k's in (2.1.1) follows. It is known that:

$$N_s^{(k)}(x) = \frac{1}{s\sqrt{2}} N^{(k)}\left(\frac{x}{s\sqrt{2}}\right) \qquad (2.1.2)$$

with

$$N(x) = \exp(-x^2) \qquad (2.1.3)$$

One then has

$$N_s^{(k)}(x) = \frac{(-1)^k}{s\sqrt{2\pi}} H_k\left(\frac{x}{s\sqrt{2}}\right)\left(\frac{1}{s\sqrt{2}}\right)^k \exp\left(-\frac{x^2}{2s^2}\right) \qquad (2.1.4)$$

where $H_k(.)$ is the Hermite polynomial of degree k.

It is known that the Hermite polynomials are orthogonal with respect to the Gaussian function, i.e.

$$\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2s^2}\right) H_m\left(\frac{x}{s\sqrt{2}}\right) H_n\left(\frac{x}{s\sqrt{2}}\right) dx = s\sqrt{2\pi}\ 2^n\ n!\ \delta_{mn} \qquad (2.1.5)$$

where δ_{mn} is The Kronecker delta.

In the following it is assumed that $i(x)$ is even, which is usually the case. Then upon multiplying both sides of (2.1.1), with $N_s^{(k)}(x)$ replaced by the RHS of (2.1.4), by $H_{2k}\left(\frac{x}{s\sqrt{2}}\right)$ and making the integration, we have

$$\int_{-\infty}^{\infty} i(x) H_{2k}\left(\frac{x}{s\sqrt{2}}\right) dx = \int_{-\infty}^{\infty} H_{2k}\left(\frac{x}{s\sqrt{2}}\right)\left(N_s(x) + \sum_{k=1}^{\infty} \frac{C_{2k}}{(2k)!} N_s^{(2k)}(x)\right) dx$$

$$= \int_{-\infty}^{\infty} H_{2k}\left(\frac{x}{s\sqrt{2}}\right)\left(N_s(x) + \sum_{k=1}^{\infty} \frac{C_{2k}}{(2k)!} \frac{(-1)^{2k}}{s\sqrt{2\pi}} H_{2k}\left(\frac{x}{s\sqrt{2}}\right)\left(\frac{1}{s\sqrt{2}}\right)^{2k} \exp\left(\frac{-x^2}{2s^2}\right)\right) dx$$

$$= \int_{-\infty}^{\infty} H_{2k}\left(\frac{x}{s\sqrt{2}}\right) \frac{C_{2k}}{(2k)!} \frac{1}{\sqrt{\pi}} \left(\frac{1}{s\sqrt{2}}\right)^{2k+1} H_{2k}\left(\frac{x}{s\sqrt{2}}\right) \exp\left(\frac{-x^2}{2s^2}\right) dx$$

$$= \frac{C_{2k}}{(2k)!} s\sqrt{2\pi}\, 2^{2k} (2k)! \frac{1}{\sqrt{\pi}} \left(\frac{1}{s\sqrt{2}}\right)^{2k+1}$$

$$= \frac{2^k}{s^{2k}} C_{2k} \tag{2.1.6}$$

Hence,

$$C_{2k} = \frac{s^{2k}}{2^k} \int_{-\infty}^{\infty} i(x)\, H_{2k}\left(\frac{x}{s\sqrt{2}}\right) dx \tag{2.1.7}$$

The C_{2k+1}'s are null due to the assumption of eveness for $i(x)$.

In (2.1.7) $H_{2k}\left(\frac{x}{s\sqrt{2}}\right)$ being a polynomial of degree $2k$, the C_{2k}'s can be evaluated from the above if the moments of $i(x)$ can be computed. As expected, the C_{2k}'s depend on the chosen value for s.

In fact, Shimbo and Celebiler use the following Gram Charlier expansion for I(u), the characteristic function corresponding to i(x).

$$I(u) = \exp(-\frac{1}{2} s^2 u^2)(1 + \sum_{k=1}^{\infty} b_{2k} u^{2k}) \qquad (2.1.8)$$

Equation (2.1.8) is in fact the direct dual of (2.1.1), the transform of which is

$$I(u) = \exp(-\frac{1}{2} s^2 u^2)(1 + \sum_{k=1}^{\infty} \frac{C_{2k}}{(2k)!} (-1)^k u^{2k}) \qquad (2.1.9)$$

which means that upon identification:

$$b_{2k} = (-1)^k \frac{C_{2k}}{(2k)!}$$

$$= \frac{(-1)^k}{(2k)!} \frac{s^{2k}}{2^k} \int_{-\infty}^{\infty} i(x)\, H_{2k}(\frac{x}{s\sqrt{2}})\, dx \qquad (2.1.10)$$

One knows that the Hermite polynomials are related via the following recurrence relation

$$H_{n+1}(x) = 2xH_n(x) - 2_n H_{n-1}(x).$$

The above thus provides a way for computing recursively the b_{2k} coefficients. The derivation will not be given here; Shimbo and Celebiler also gave a scheme to compute recursively the b_{2k} coefficients.

2.2. Ho and Yeh method:

It will be seen here how Ho and Yeh's method differs from Shimbo and Celebiler's method, and how they resemble each other in numerical computation.

Shimbo and Celebiler use a Gram-Charlier expansion to approximate the ISI r.v. by a normal distribution function followed by an infinite series of weighted derivatives. We note that this infinite series has destroyed the Gaussian-like property of the distribution which cannot be easily identified.

We also note that the value of s^2 has been determined somewhat arbitrarily. Then, what should the optimum value for s^2 be? By optimum one means the value of s^2 which allows one to truncate the infinite series (for one is not going to use the whole series but just the first few terms) after the minimum number of terms.

It can be seen then that one is going away from the physical aspect of the problem and is considering the computational aspect.

It is easy to see why one is led to give more importance to the computational aspect. First of all, it may be noted that what one ultimately wants to know is the probability of error, which is a number. The knowledge of the distribution of the ISI is not of primary importance, although it can give much more insight into the problem.

Also at present time, it seems that no convenient way exists which can be used to determine the shape of the distribution function of the ISI r.v.

Therefore it is necessary to abandon the idea of computing i(x) and to find a way of computing the integral without full knowledge of i(x). This is a numerical process.

On the other hand, if it is noticed that the determination of i(x) is desirable but not indispensable for the problem, because a computed P(e) that differs by no more than 1% from the exact value is very satisfactory, one may now try to approximate the ISI r.v. in such a way as to be able to compute our integral adequately.

Thus one of the main merits of the power series expansion method has been to introduce the use of the moments of the ISI r.v. It is in fact well known that under certain conditions, the moments of a random variable completely describe its distribution function.

More precisely, let it be supposed that the Taylor series expansion of the characteristic function of a random variable exists throughout some interval in u which contains the origin and that one may write

$$\Phi_x(u) = \sum_{k=0}^{\infty} \Phi_x^{(k)}(0) \frac{u^k}{k!} \quad , \qquad (2.2.1)$$

Φ_x denoting the characteristic function. Then by the moment-generating property of the characteristic function, one has:

$$E\{X^k\} = (-i)^k \Phi_x^{(k)}(o)$$

so that (2.2.1) becomes:

$$\Phi_x(u) = \sum_{k=o}^{\infty} E\{X^k\} \frac{(iu)^k}{k!}$$

or in the case of an even random variable

$$\Phi_x(u) = \sum_{k=o}^{\infty} E\{X^{2k}\} \frac{(-1)^k}{(2k)!} u^{2k} \qquad (2.2.2)$$

Under the above conditions, the characteristic function (and hence the corresponding pdf) is uniquely determined by the moments of the given random variable.

Necessarily, as in all other expansions, it is hoped that the expansion in (2.2.2) gives a valid approximation after the first few terms.

In their derivation, Ho and Yeh have used (at least im plicitly) (2.2.2). Thus the characteristic function I(u) of the ISI r.v. is expanded into:

$$I_x(u) = \sum_{k=o}^{\infty} \frac{\mu_{2k}}{(2k)!} (-1)^k u^{2k}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{\mu_{2k}}{(2k)!} (-1)^k u^{2k} \qquad (2.2.3)$$

where μ_{2k} is the 2kth moment.

If one compares (2.1.8) used by Shimbo and Celebiler with the above equation, one may see the "minor" difference between the two methods.

Shimbo and Celebiler use a Charlier expansion for the characteristic function of the ISI r.v., which leads to the product of an exponential Gaussian term (with variance s^2, to be determined) and a power series expansion. This gives the ISI a Gaussian-like term, corrected by successive weighted derivatives.

As to Ho and Yeh, as shown by relations (2.2.2) and (2.2.3), they restrict themselves to a straight Taylor series expansion around the origin, so that no Gaussian like term appears here ($s = o$).

The difference comes about probably from the fact that Shimbo and Celebiler were initially trying to attach a physical interpretation to their derivation, which was not Ho and Yeh's concern.

Numerically speaking, the two methods are similar, and this can be seen as in the following.

Consider the c. f. of the combined random variable (gaussian noise + ISI) as used respectively by Shimbo and Celebiler and Ho and Yeh. From (2.1.9) and (2.2.3) one may write:

$$\Phi_1(u) = \exp(-\frac{1}{2} s^2 u^2) \left(1 + \sum_{k=1}^{\infty} a_{2k} u^{2k}\right) \exp(-\frac{1}{2} \sigma_n^2 u^2)$$

$$\Phi_2(u) = \left(1 + \sum_{k=1}^{\infty} d_{2k} u^{2k}\right) \exp(-\frac{1}{2} \sigma_n^2 u^2)$$

The a_{2k} and d_{2k} are both computed from the moments. Furthermore by rewriting $\Phi_1(u)$ as

$$\Phi_1(u) = \exp(-\frac{1}{2} \sigma^2 u^2) \left(1 + \sum_{k=1}^{\infty} a_{2k} u^{2k}\right)$$

with $\sigma^2 = \sigma_n^2 + s^2$, it may be seen that $\Phi_1(u)$ and $\Phi_2(u)$ have the same form. Although $a_{2k} \neq d_{2k}$ and $\sigma^2 = \sigma_n^2 + s^2$, this does not affect the numerical process, as the independent variable here is u. The rates of convergence of the two methods are, however, different.

In fact, in the Shimbo and Celebiler derivation, if the value given to s is zero, one derives the Ho and Yeh result.

Thus rewriting (2.1.10)

$$b_{2k} = \frac{(-1)^k}{(2k)!} \frac{s^{2k}}{2^k} \int_{-\infty}^{\infty} i(x)\, H_{2k}\left(\frac{x}{s\sqrt{2}}\right) dx$$

and upon nullifying the value of s, which means that $s^{2k} H_{2k}\left(\frac{x}{s\sqrt{2}}\right)$ is reduced to its first term,

$$b_{2k} = \frac{(-1)^k}{(2k)!} \int_{-\infty}^{\infty} \left(\frac{s^{2k}}{2^k}\right) i(x) \frac{x^{2k}}{s^{2k}} \frac{2^{2k}}{2^k} dx \qquad (2.2.4)$$

$$= \frac{(-1)^k}{(2k)!} \int_{-\infty}^{\infty} i(x) x^{2k} dx$$

$$= \frac{(-1)^k}{(2k)!} \mu_{2k}$$

This is exactly the coefficient of u^{2k} in (2.2.3), the equation used by Ho and Yeh.

Thus theorically one may say that the Shimbo and Celebiler method is superior in the case of the white Gaussian noise, because of the existence of one degree of freedom (the value of s). But in the case of nonwhite Gaussian noise, the Ho and Yeh method is simpler, unless we use the value s=o for the Shimbo and Celebiler method.

2.3. Derivation of the probability of error

The characteristic function (c.f.) of the combined random variable (additive Gaussian noise + ISI) can be expressed as the product of its components

$$\Phi_c(u) = \exp\left(-\frac{1}{2} \sigma_n^2 u^2\right) \exp\left(-\frac{1}{2} s^2 u^2\right)\left(1 + \sum_{k=1}^{\infty} b_{2k} u^{2k}\right) \qquad (2.3.1)$$

As was mentioned earlier, the b_{2k}'s can be computed iteratively and are weighted moments of the ISI r.v. By letting $\sigma^2 = \sigma_n^2 + s^2$, one obtains

$$\Phi_c(u)=\exp(-\frac{\sigma^2 u^2}{2})\ (1+\sum_{k=1}^{\infty} b_{2k}u^{2k}) \qquad (2.3.2)$$

Taking the inverse of the above,

$$\Omega_c(x)=(1+\sum_{k=1}^{\infty} b_{2k}(-1)^k \frac{d^{2k}}{dx^{2k}}\left[\frac{1}{\sqrt{2\pi}\sigma}\exp(-\frac{x^2}{2\sigma^2})\right]$$

Therefore

$$\Omega_c(x)=\frac{1}{\sigma\sqrt{2\pi}}\exp(-\frac{x^2}{2\sigma^2})+\sum_{k=1}^{\infty} b_{2k}(-1)^k(\frac{1}{\sigma\sqrt{2}})^{2k}H_{2k}(\frac{x}{\sigma\sqrt{2}})\exp(-\frac{x^2}{2\sigma^2})$$

The above represents the combined random variable. Applying it to (1.2.5) one obtains

$$P(e) = \frac{2L-1}{L}\int_{dro}^{\infty}\Omega_c(x)\,dx$$

$$= \frac{2L-1}{L}\left(\frac{1}{2}\operatorname{erfc}(\frac{dro}{\sigma\sqrt{2}})+\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{\pi}}b_{2k}H_{2k-1}(\frac{dro}{\sigma\sqrt{2}})(\frac{1}{\sigma\sqrt{2}})^{2k}\exp(-\frac{d^2r_o^2}{2\sigma^2})\right) \qquad (2.3.3)$$

Equations (12) of reference [7] and (8) of reference [6] have exactly the same form. In the former 2L=2 (binary), while in the latter $\sigma^2=\sigma_n^2$ (i.e. with s^2=o).

With s^2 = o one may derive from (2.2.3) the value of b_{2k} which is put into (2.3.3) to give:

$$P_e = \frac{2L-1}{L}\left(\frac{1}{2}\,\mathrm{erfc}\left(\frac{dr_o}{\sigma\sqrt{2}}\right) + \sum_{k=1}^{\infty}\left(\frac{1}{2\sigma^2}\right)^k \exp\left(-\frac{(dr_o)^2}{2\sigma^2}\right) H_{2k-1}\left(\frac{dr_o}{\sigma\sqrt{2}}\right)\right)$$

where

$$\frac{1}{2}\,\mathrm{erfc}(a) = \frac{1}{\sqrt{\pi}}\int_a^{\infty}\exp(-t^2)\,dt$$

The above derivation seems somewhat complex, because it was first necessary to transform the random variables, multiply them and then return to the probability domain.

A more straight forward method which yields the same result consists of expanding into Taylor series (in the probability domain) the function $1-F(dr_o - x)$ around dr_o.

In the case of the white Gaussian noise,

$$1-F(dr_o - X) = \int_{dr_o - x}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2\sigma^2}\right) dx$$

$$= \frac{1}{2}\,\mathrm{erfc}\left(\frac{dr_o - x}{\sigma\sqrt{2}}\right)$$

$$= \frac{1}{2}\left[\mathrm{erfc}\left(\frac{dr_o}{\sqrt{2}}\right) + \sum_{k=1}^{\infty}\frac{(-x)^k}{(\sigma\sqrt{2})^k}\,\frac{1}{k!}\,\frac{d^k}{du^k}\,\mathrm{erfc}(u)\Big|_{u=\frac{dr_o}{\sqrt{2}}}\right]$$

Differentiation of erfc (u) gives rise to Hermite polynomials:

$$\frac{d^k}{du^k}\,\mathrm{erfc}(u)\Big|_{u=\frac{dr_o}{\sigma\sqrt{2}}} = (-1)^k H_{k-1}\left(\frac{dr_o}{\sigma\sqrt{2}}\right) \exp\left(\frac{-d^2 r_o^2}{2\sigma^2}\right)\frac{2}{\sqrt{\pi}}$$

Assuming the eveness of the noise and of the ISI r.v., it follows that

$$\int_D \frac{1}{2}\,\mathrm{erfc}\left(\frac{dr_o - x}{\sigma\sqrt{2}}\right) i(x)\,dx$$

$$= \frac{1}{2}\,\mathrm{erfc}\left(\frac{dr_o}{\sigma\sqrt{2}}\right) + \sum_{k=1}^{\infty} \frac{1}{\sqrt{\pi}} \frac{2k}{(\sigma\sqrt{2})^{2k}} \frac{1}{(2k)!} H_{2k-1}\left(\frac{dr_o}{\sigma\sqrt{2}}\right) \exp\left(-\frac{d^2 r_o^2}{2\sigma^2}\right) \quad (2.3.4)$$

In the above formulations, the first term represents the P(e) due to white noise alone. The following terms describe the effect of the ISI in an additive Gaussian noise environment. In another environment the effect of the "same" ISI will be different. It can also be seen from (2.3.3) that the ISI cannot be described merely as white Gaussian noise.

In case of nonwhite Gaussian noise, $\Phi_c(u)$, the c.f. given by (2.3.1) contains a part which is formed by the convolution of the nonGaussian noise and the "equivalent" Gaussian part of the ISI. We note that in the case where s is different from zero, the result of this convolution cannot always be expressed in a closed form and this will add a new dimension of difficulty to the problem. Thus to avoid this difficulty the use of s=o, which corresponds merely to a Taylor series expansion of the noise process in the probability domain, is advised, if we want to use the power series expansion method, unless the convolution is simple to perform.

One of the disadvantages of the above formulation is that it precludes any physical interpretation of the intersymbol interference.

Equations (2.3.3) and (2.3.4) can be shown to be absolutely convergent. But, this convergence is usually slow, so that quite a few terms have to be used in order to get a satisfactory result. The convergence of the power series is directly linked to the relative magnitude of the maximum deviation caused by the ISI, with respect to the magnitude of the signal. For small ISI, the ratio S/I_m, where I_m is the max deviation caused by the ISI, is larger and the convergence is fast. For small values of S/I_m the series converge very slowly.

It is probable that for values of S/I_m larger than 3 or 4, the power series expansion method would diverge due to the round-off errors in the computer. Thus, it can hardly be applied to M-ary PAM systems, with M larger than 8 or 10.

Shimbo and Celebiler have suggested a method to improve the convergence of the series. They emmerate directly all possible interference that may be caused by say N terms (the N largest terms) and use their formula for each case separately. This has a limited applicability due to the amount of computer time involved. In fact, this scheme combines the power series expansion method with the emmerative method.

2.4. Convergence property

As was mentioned earlier, there are two kinds of errors associated with the series expansion scheme:

i) The truncation of the overall impulse response.

This is common to all numerical methods. This kind of error is not considered here for it is theorical rather than practical. That is, the number of interfering samples can almost always be increased until the computed P(e) stops changing. Furthermore this problem is very difficult and can itself be a topic of theorical research.

The interested reader may look at Ref. 7 . A partial solution is also given in 2.6. There an upper bound on P(e), computed with all of the interfering samples taken into account, is derived. The difference between this upper bound and the P(e) computed with a certain number, say M, of interfering samples, may give an idea of the magnitude of the error induced by the truncation of the overall impulse response.

ii) The truncation of the series expansion.

The following refers then to the truncation error introduced by the truncation of the series expansion, when one considers a N-term truncation of the channel impulse response.

The truncation of the RHS of (2.3.4) after n terms introduces the following truncation error E_n:

$$E_n = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{d^2 r_o^2}{2\sigma^2}\right) \sum_{k=n+1}^{\infty} H_{2k-1}\left(\frac{dr_o}{\sigma\sqrt{2}}\right)\left(\frac{1}{\sigma\sqrt{2}}\right)^{2k} \frac{\mu_{2k}}{(2k)!} \qquad (2.4.1)$$

For notational convenience, let $l = \frac{dr_o}{\sigma\sqrt{2}}$. The range of X, the ISI r.v., being finite, define

$$d_m = \sum_{h=1}^{M} |r_h|$$

Then,

$$\mu_{2k+2s} \leq m_{2k}\, d_m^{2s}, \quad k \text{ and } s \geq 0$$

$$m_{2k} = \mu_{2k}$$

It is known that [13]

$$\left|H_{2k-1}(t)\right| \leq b\, 2^{\frac{2k-1}{2}} \sqrt{(2k-1)!}\ \exp\left(\frac{t^2}{2}\right),\ b = 1.086435 \qquad (2.4.2)$$

Upon substituting (2.4.2) into (2.4.1),

$$E_n \leq \frac{b}{\sqrt{\pi}} \exp(-l^2) \exp\left(+\frac{l^2}{2}\right) \sum_{k=n+1}^{\infty} \frac{(\sqrt{2})^{2k-1}}{(\sqrt{2})^{2k}} \frac{\sqrt{(2k-1)!}}{(2k)!} \frac{\mu_{2k}}{\sigma^{2k}}$$

$$\leq \frac{b}{\sqrt{2\pi}} \exp\left(-\frac{l^2}{2}\right) \frac{\mu_{2n}}{\sigma^{2n}} \sum_{k=n+1}^{\infty} \frac{d_m^{2(k-n)}}{\sigma^{2(k-n)}} \frac{1}{2k} \frac{1}{\sqrt{(2k-1)!}}$$

Upon changing the limits of the summation,

$$E_n \leq \frac{b}{\sqrt{2\pi}} \exp\left(-\frac{l^2}{2}\right) \frac{\mu_{2n}}{\sigma^{2n}} \sum_{k=1}^{\infty} \frac{d_m^{2k}}{\sigma^{2k}} \frac{1}{2(n+k)} \frac{1}{\sqrt{(2n+2k-1)!}}$$

Replacing now the summation by a larger term,

$$En \leq \frac{b}{\sqrt{2\pi}} \exp(-\frac{1^2}{2}) \frac{\mu_{2n}}{\sigma^{2n}} \frac{1}{2n+2} \frac{d_m^2}{\sigma^2} \left[1 + \sum_{k=1}^{\infty} (\frac{d_m^2}{\sigma^2})^k (\frac{1}{\sqrt{(2n+2)(2n+3)}} \right]$$

Assuming now $\frac{d_m^2}{\sigma^2(\sqrt{(2n+2)(2n+3)})} < 1$, the geometric series in the RHS converges to a limit which is substituted into the above to give

$$En \leq \frac{b}{\sqrt{2\pi}} \exp(\frac{d^2 r_o^2}{4}) \frac{1}{2n+2} \frac{\mu_{2n}}{\sigma^{2n}} \frac{d_m^2}{\sigma^2} \frac{1}{1 - \frac{(d_m/\sigma)^2}{\sqrt{(2n+2)(2n+3)}}} \tag{2.4.3}$$

This is exactly (40) of ref. 11 .

From the above derivation it can be seen that (2.3.3) and (2.3.4) theorically can be determined with as great an accuracy as desired if d_m is finite. Thus the series expansion method converges in case of white Gaussian noise.

2.5. Determination of the moments:

Although many schemes [5,8] have been devised to iteratively compute the moments needed for the evaluation of P(e), the simplest scheme is probably, Prabhu's one [11] , which will be given here.

We are interested in the evaluation of the moments of a sum of discrete independent random r.v.'s numbered say from 1 to M.

$$X = \sum_{m=1}^{M} a_m r_n = \sum_{m=1}^{M} X_n$$

For a 2L-level PAM system, a_n takes on values from the set

$\pm d_o, \pm 3 d_o, \ldots, \pm (2L-1)d_o$. If one lets

$$Y_N = \sum_{n=1}^{N} X_n$$

then,

$$Y_M = X$$

Thus it is desired to evaluate $\langle Y_M^{\,n} \rangle$ for different values of n. In fact, as the X_n's are even random variables ($Pr(a_n = i) = Pr(a_n = -i)$), only the even moments of Y_M are nonzero. Necessarily, they are all positive.

Now

$$\langle Y_N^{2m} \rangle = \langle (Y_{N-1} + X_N)^{2m} \rangle$$

which means

$$\langle Y_N^{2m} \rangle = \sum_{i=o}^{m} \binom{2m}{2i} \langle Y_{N-1} \rangle^{2i} \langle X_N \rangle^{2m-2i} \qquad (2.5.1)$$

Y_{N-1} and X_n being independent for all values of N. In formula (2.5.1)

$$\langle X_N^{o} \rangle = 1 \text{ for all values of N}$$

$$\langle Y_1^{2m} \rangle = \langle X_1^{2m} \rangle \text{ for all values of m,}$$

and the $\langle X_N^{2m} \rangle$ can be computed enumeratively for all values of N and m. In fact,

$$\langle X_N^{2m} \rangle = \sum_{i=-(2L-1),2}^{2L-1} \Pr(A_N=i)\ (ir_N)^{2m}$$

$$= \sum_{i=1,2}^{2L-1} 2\Pr(a_N=i)(ir_N)^{2m} \qquad (2.5.2)$$

because we have assumed $\Pr(a_N=i) = \Pr(a_N=-i)$.

If all the levels are equiprobable

$$\langle X_N^{2m} \rangle = \frac{1}{L} \sum_{i=1,2}^{2L-1} (ir_N)^{2m} \qquad (2.5.3)$$

Thus (2.5.1) provides a recurrence formula to compute the moments of X. To do that, we first use (2.5.2) or (2.5.3) to compute the $\langle X_N^{2m} \rangle$ for all values of N from 1 to M; and for m up to the highest order of the desired moments. Then one may repeatedly use (2.5.1) for successively increasing values of N until M' is reached.

2.6. Prabhu's Method:

In the following Prabhu's method will be considered which, in fact, can be described as a more complete (but not necessarily more efficient) power series expansion method, for we shall see that he also uses the Taylor series expansion of the noise process to compute the probability of error.

In the problem at hand most of the difficulty arises from the estimation of the distribution function of the ISI r.v.

which is composed theorically of an infinite series of r.v. Not being able to deal directly with this infinite series (the central limit theorem does not apply here), but knowing that the effect of the ISI is usually, and practically, limited to its few largest terms, one is led to consider following two random variables.

i) The r.v. formed by the sum of the noise r.v. n_o and the r.v. associated with the N largest interfering samples; This is called X_n.

ii) The r.v. associated with the remaining (theorically infinite) interfering samples, X_r.

Thus one wishes to compute:

$$P_x(a) = \Pr\left\{ X \leq a \right\} = \Pr\left\{ X_n + X_r \leq a \right\} \qquad (2.6.1)$$

Since $P_{Xn}(a) = \Pr\left\{ X_n \leq a \right\}$ can theorically be computed via the power series expansion method, it may be possible to get both upper and lower bounds to $P_x(a)$ in terms of $P_{Xn}(a)$ and some known parameters associated with X_r. One has thus avoided dealing with the infinite series directly. Now, if the difference between the two bounds (upper and lower) decreases monotonically with N, $P_X(a)$ can be computed with arbitrarily small errors.

Denoting the pdf by f and the distribution function by F,

$$P_x(a) = \int_{-\infty}^{\infty} \int_{-\infty}^{a-y} f_{X_n, X_r}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} Fx_n(a-y)\, dF_{xr}(y)$$

$$= \langle Fx_n(a-y) \rangle_{Xr} \qquad (2.6.2)$$

As such (2.6.2) cannot take opportunity of the power series expansion method. Therefore Prabhu has used the following derivation to transform (2.6.2) into a more suitable formulation.

By selecting an interval (-lo,up), lo and up being positive, from the range of Xr we can write

$$P_X(a) = \int_{-\infty}^{-lo} F_{xn}(a-y)\, dF_{xr}(y) + \int_{-lo}^{up} Fxn(a-y)\, dF_{xr}(y) + \int_{up}^{\infty} F_{xn}(a-y)\, dF_{xr}(y).$$

$$= I_1 + I_2 + I_3 \qquad (2.6.3)$$

One then has

$$0 \le I_1 \le F_{xr}(-lo)$$

$$0 \le I_3 \le F_{xn}(a-up) \int_{up}^{\infty} dF_{xn}(y) \le F_{xn}(a+lo) \left(1-F_{xr}(up)\right)$$

$$I_2 \le F_{xn}(a-up) \int_{-lo}^{up} dF_{xr}(y) = F_{xn}(a-up)\left(F_{xr}(up) - F_{xr}(-lo)\right)$$

and

$$I_2 \le F_{xn}(a+lo) \int_{-lo}^{up} dF_{xr}(y) = F_{xn}(a+lo)\left(F_{xr}(up) - F_{xr}(-lo)\right)$$

Combining (2.6.3) with the above, one obtains

$$P_x(a) \geq F_{xn}(a-up)\left(F_{xr}(up) - F_{xr}(-lo)\right)$$

$$P_x(a) \leq F_{xr}(-lo) + F_{xn}(a+lo)\left(1-F_{xr}(-lo)\right)$$

$$\leq F_{xr}(-lo) + F_{xn}(a+lo)$$

Upper bounds on $F_{xr}(-lo)$ and $F_{xr}(up)$ should then be derived as they cannot be evaluated directly. These bounds should be tight in order to get an accurate estimate of $P_x(a)$. Therefore

$$0 \leq F_{xr}(-lo) = Pr\left\{Xr \leq -lo\right\} \leq L(-lo) \leq 1$$

$$0 \leq 1 - F_{xr}(up) = Pr\left\{Xr \geq up\right\} \leq U(up) \leq 1$$

One then has:

$$1 \geq F_{xr}(up) - F_{xn}(-lo) = Pr\left\{-lo \leq X_r \leq up\right\}$$

$$\geq 1 - L_{xr}(-lo) - U_{xr}(up) \geq 0$$

If such bounds can be found, it may be seen that

$$F_{xn}(a-up)\left(1-L(-lo) - U(up)\right) \leq P_x(a)$$

$$P_x(a) \leq F_{xn}(a+lo) + L(-lo) \qquad (2.6.4)$$

Derivations of L(-lo) and U(up):

Prabhu suggested the Chernoff bounding technique to compute the above bounds. That is

$$Pr\left\{X_r \leq -lo\right\} \leq \exp(-\lambda lo)\Phi_{xr}(-\lambda),\ \lambda \geq 0,$$

$$\Pr\left\{ X_r \geq up \right\} \leq \exp(-\lambda up)\, \Phi_{xr}(\lambda), \quad \lambda \geq 0,$$

where Φ_{xr} denotes the moment-generating function of Xr.

Define two functions Θ_{xr} and Ψ_{xr} which have the property

$$0 \leq \Phi_{xr}(\lambda) \leq \Theta_{xr}(\lambda), \quad \lambda \geq 0$$

$$0 \leq \Phi_{xr}(-\lambda) \leq \Psi_{xr}(-\lambda), \quad \lambda \geq 0$$

It is then possible to optimize $\exp(-\lambda\, lo)\, \Psi_{xr}(-\lambda)$ and $\exp(-\lambda\, up)\, \Theta_{xr}(\lambda)$. $\Psi_{xr}(-\lambda)$ and $\Theta_{xr}(\lambda)$ are chosen such that the above is easy to optimize.

That is,

$$L(-lo) = \exp(-\lambda_{opt}\; lo)\, \Psi_{xr}(-\lambda_{opt})$$

$$U(up) = \exp(-\lambda_{opt}\; up)\, \Theta_{xr}(+\lambda_{opt})$$

If one now refers to (2.6.4), it can be seen that (2.6.4) together with the above provide the bounds one is looking for.

Practically, in a given numerical situation, Prabhu's method involves the following works in addition to the use of the power series expansion method for computing $F_{xn}(a-up)$ and $F_{xn}(a+lo)$:

i) The optimization of the values for L(-lo) and U(up) by choosing the optimum value for λ. Note that lo and up are also chosen values.

ii) The use of the optimum value for λ to express the difference D_N between the upper and lower bounds as a function of N, -lo, up.

iii) The gradual increase of the value used for N, while choosing for each value of N satisfactory values for -lo and up, until D_N becomes sufficiently small.

This procedure, if it can be implemented adequately, has the merit to give the exact P(e) as accurately as wanted. But it cannot be easily implemented and in general the value of N that should be used is quite large.

Thus Prabhu seems to give more importance to the evaluation of the effect of the residual pulse train, than other authors do. It can be said that he has explored more fully the problem of the residual pulse train, and has shown that it is possible to derive bounds on the effects of the residual pulse train, at least in the white Gaussian noise case, which is not of minor importance.

CHAPTER 111

GAUSSIAN QUADRATURE RULE METHOD

Although originally devised in the North American continent, the power series expansion method has also received attention from many researchers in Europe who are interested in the problem of intersymbol interference. Amongst these researchers are Benedetto et al. [8] who have applied the series expansion method to the ISI problem. But, they found that the series expansion, although it is adequate, at least theorically, presents the disadvantage of slow convergence to the result. To overcome this difficulty, Benedetto et al. have suggested the use of Gaussian quadrature rules (GQR) to the ISI problem.

3.1. GQR:

GQR is the most widely investigated method for approximating definite integrals of the form

$$\int_a^b f(x)\, i(x)\, dx \tag{3.1.1}$$

As was said earlier, the above integral is intimately linked to our problem, where $f(x) = 1-F(dr_o-x)$, and where $F(.)$ is the distribution function of the noise process, $i(x)$ the ISI r.v. pdf, and the interval (a,b) is the domain of definition of X.

With GQR the above definite integral is approximated by a linear combination of values of the function $f(x)$

$$\int_a^b f(x)\ i(x)\ dx \cong \sum_{k=1}^{m} i_k\ f(x_k) \qquad (3.1.2)$$

The set $\left\{i_k, x_k\right\}_{k=1}^{m}$ is called a quadrature rule (QR) of order 2m-1, corresponding to the weight function $i(x)$. The x_k's are called the nodes (or abscissas) and the i_k's the weights (or coefficients).

A QR has degree of precision m if it is exact whenever $f(x)$ is a polynomial of degree less than or equal to m, or in other words if it is exact for $f(x) = 1, x, x^2, \ldots x^m$. In the case of a nonnegative $i(x)$, n nodes and n weights can be found to make the GQR exact for all polynomials of degree $\leq 2n-1$. This is the highest degree of precision that can be achieved by an n-point QR. GQR is closely related to orthogonal polynomials whose properties are summarized in the following.

Orthogonal polynomials

Consider a sequence of monic polynomials $P_n(x)$ for n=0,1,2,... which satisfy the following relation

$$\int_a^b i(x)\ k_n P_n(x)\ k_m P_m(x)\ dx = \delta_{mn},$$

where δ_{mn} is the Kronecker delta, and the k_i's the normalizing factors. Then the sequence is said to be orthogonal with respect to the weight function i(x). If i(x) is nonnegative on (a, b), as in this case, the sequence can be easily shown to be unique, from the following consideration

$$\int_a^b i(x)\, x^n\, P_m(x)\, dx = 0, \text{ for } n < m$$

Obviously, then

$$\int_a^b i(x)\, Pm^2(x) dx \neq 0$$

In fact, the above integral equals $1/km^2$.

Any three consecutive polynomials in the sequence can be related via the following expression

$$x\, P_{n-1}(x) = \beta_{n-1}\, P_{n-2}(x) + \alpha_n P_{n-1}(x) + \beta_n P_n(x) \qquad (3.1.3)$$

with

$$P_{-1}(x) = 0\ , \quad P_o(x) = 1$$

This is an important property of orthogonal polynomials. The coefficients α_n's, β_n's, will play an important role in Benedetto et al.'s derivation.

From (3.1.3) one can derive the Christoffel-Darboux identity, which is useful for deriving the coefficients of the GQR

$$\sum_{l=o}^{n} k_l^2\, P_l(x) P_l(y) = k_n^2\, \frac{P_{n+1}(x)\, P_n(y) - P_n(x)\, P_{n+1}(y)}{x - y} \qquad (3.1.4)$$

One can now state (without proof) the following theorem.

Assume m different values of x in the interval (a,b), $x_1, x_2, \ldots x_m$ and let $n > 1$, then m coefficients can be found such that the GQR gives an exact result for all polynomials of degree $\leq m+n-1$ iff the polynomial $P_m(x) = \prod_{i=1}^{m} (x-x_i)$ is orthogonal to all polynomials of degree $\leq n-1$.

The above theorem relates the orthogonal polynomials with the GQR. It says that the roots of the polynomials, which are orthogonal with respect to i(x), form the nodes of the GQR. The weights can then be deduced to be interpolary and are computed from

$$i_k = \int_a^b \frac{P_m(x)}{(x-x_k)\,P_m'(x_k)}\, i(x)\, dx \tag{3.1.5}$$

$$P_m'(x_k) = \left.\frac{dP_m(x)}{dx}\right|_{x=x_k}$$

The above can be transformed by the Christoffel Darboux identity

$$i_k = -\frac{1}{k_m^2\, P_m'(x_k)\, P_{m+1}(x_k)}$$

or

$$i_k = \left(\sum_{l=0}^{m-1} k_l^2\, P_l^2(x_k)\right)^{-1}$$

The above formulas are derived for an m-point QR.

One very important property of the GQR is its convergence; in fact, for f(x) continuous and i(x) non-negative on (a,b), it can be shown that

$$\lim_{m \to \infty} \sum_{k=1}^{m} i_k f(x_k) = \int_a^b f(x)\, i(x)\, dx$$

Although this property is not equivalent to the absolute convergence property of the power series expansion method, in the case of white Gaussian noise, it confers on the GQR method one strong advantage: it applies to almost any f(x).

The usual procedure for finding the coefficients i_k's consists of generating the set of orthogonal polynomials associated with the weight function i(x) by first calculating the three-term recurrence formula. The nodes of the QR are then determined by computing the roots of these polynomials. As these roots are real and simple, Newton-Raphson iteration is advised.

The m nodes of the QR of precision order 2m-1 are thus the m roots of the orthogonal polynomial of degree m. The weights are then deduced from associated formulas. This scheme cannot be applied directly in our case, for i(x) is unknown.

Fortunately, Golub and Welsch [9] have devised a method to compute the m nodes and m weights, which uses the moments of i(x) alone. This can be done because orthogonality relations involve only linear combinations of weighted moments of i(x) and not i(x) itself.

Their method involves the following:

i) Evaluation of the recurrence terms $\{\alpha_n\}_{n=1}^{m}$ and $\{\beta_n\}_{n=1}^{m}$ of (3.1.3) by means of the first 2m moments of x, the ISI r.v. This involves the Choleski decomposition of the matrix of moments M (which in our case is the Hankel matrix in the moments) into $R^T R$, R being an upper triangular matrix. The α_n's and β_n's are then simply derived from the R matrix.

ii) Generation of a tridiagonal matrix involving the α's and the β's. Eigenvalues and eigenvectors of this matrix are then computed via the Q-R algorithm. While the nodes are given by the above eigenvalues, the first components of the eigenvectors form the weights.

3.2. Benedetto et al.'s derivation

First note that, as Golub and Welsch's scheme involves the knowledge of moments of i(x) which can be computed iteratively only if the number of interfering samples taken into account is finite, the GQR method automatically applies only to the case of a truncated overall channel impulse response.

Through the use of Golub and Welsch's scheme to derive the QR set $\{i_k, x_k\}_{k=1}^{m}$, Benedetto et al. apply the GQR to (1.2.8) to get

$$P(e) \cong \frac{2L-1}{L} \sum_{k=1}^{m} i_k \left(1-F(dr_o - x_k)\right) \tag{3.2.1}$$

Thus, in the case of Gaussian white noise with variance σ^2, we have $1-F(dr_o-x_k) = \frac{1}{2}\,\mathrm{erfc}\,(dr_o-x_k)$ and (3.2.1) becomes then

$$P(e) \cong \frac{2L-1}{2L} \sum_{k=1}^{m} i_k \,\mathrm{erfc}\left(\frac{dr_o - x_k}{\sigma\sqrt{2}}\right) \tag{3.2.2}$$

The error committed in the above formula is of the form $\frac{f^{2m}(\xi)}{k_m^2\,(2m)!}$

i.e., $\frac{1}{2}\;\mathrm{erfc}^{2m}\left(\frac{dr_o-\xi}{\sigma\sqrt{2}}\right)\frac{1}{k_m^2\,(2m)!}$, where $\xi \in (a,b)$ and k_m is the leading coefficient in $\hat{H}_m(x) \triangleq k_m P_m(x)$.

Thus, numerically speaking, in the GQR method the number of interfering samples is first determined. Depending then on the order of accuracy wanted for the GQR, the number of moments are computed; following that, Golub and Welsch's scheme is applied to produce the nodes and weights required.

Formulas (3.2.1) or (3.2.2) then give an approximation to P(e).

Note that with this method the whole procedure in the above has to be repeated if one wishes to change the number of interfering samples taken into account, or if it is desired to increase the order of precision of the QR.

Computational errors

In this part the error introduced by the truncation of the channel impulse response, will be ignored, as it is exactly the same in other methods.

i) Error committed by the QR

In the case of white Gaussian noise, Benedetto et al. [8] have derived an upper bound on this error.

$$E_n < A \exp -\frac{(d\ r_o - I_m)}{4\ \sigma_n^2}\ , \quad Im < d\ r_o$$

$$E_n < A \quad , \quad Im \geq d\ r_o.$$

where Im is the largest deviation caused by ISI,

$$A = \frac{B(2L-1)\ 2^n\ \left[(2n-1)!\right]^{\frac{1}{2}}}{(2n)!\ (2\ \sigma_n^2\ )^n\ k_n^2 \sqrt{\pi}}\ , \quad B = 1.086435,$$

and En corresponds to an n-point GQR. This bound is not always tight.

ii) Round-off errors

Round-off errors, which occur due to the limited accuracy of the digital computer, grow with the increase of the order of precision of the quadrature rule. Thus when high order precision is needed (this happens when the ISI is large), we should give more attention to round-off errors. Double precision computation is advised.

The accuracy of the values computed for the set

$\{i_k, x_k\}_{k=1}^{n}$ is of primary importance in GQR. In the GQR method, this set is derived from the knowledge of the moments, which thus, should be computed with the greatest accuracy possible.

As these moments are computed iteratively, errors are unfortunately cumulative. Due to this fact, the scheme devised to compute the recurrence coefficients α_n and β_n may fail. Benedetto et al. [8] have proposed the following methods to reduce the effect of round off errors.

1. The use of numerical differentiation of analytic functions, using the fact that the k^{th} moment of the r.v. can be derived directly from the corresponding derivation of the characteristic function of the random variable X.

2. The use of a recently devised scheme to compute the GQR starting from "modified moments" of the form

$$m_k = \int_D P_k(x)\ i(x)\ dx$$

where $P_k(x)$ is a member of a set of orthogonal polynomials suitably chosen.

Although the GQR method appears to be suitable in many cases, the implementation of the above suggestions may prove to be necessary when the ISI is large.

A physical interpretation

In the following, it will be seen that GQR method as applied to this problem consists of giving to the r.v. X an approximation, say $V_1(x)$, of its distribution function denoted here by $V(x)$.

It is known from Chebyshev's work in probability that there exists a unique m-step distribution $V_1(x)$ which has the same first 2m moments as $V(x)$, (which in this case is the distribution function of the ISI r.v.)

Referring now to Fig. 3.1, we can see that m values of the abscissae, a_i, and m values of the magnitudes, A_i, are needed to determine the step-wise function $V_1(x)$.

Thus

$$\frac{dV_1(x)}{dx} = \sum_{i=1}^{m} A_i \ \delta(x-a_i) \qquad (3.2.4)$$

One can proceed to determine the a_i's and A_i's.

Given $m_0, m_1, \ldots m_{2m-1}$, the moments, we know that $V_1(x)$ and $V(x)$ are related via

$$m_k = \int_D x^k \, dV_1(x) = \int_D x^k \sum_{i=1}^{m} A_i \, \delta(x-a_i) \, dx$$

$$= \sum_{n=1}^{m} A_n a_n^k$$

for $k=0,1,2,\ldots 2m-1$.

That is,

$$\int_D x^k dV(x) = \sum_{n=1}^{m} A_n a_n^k \qquad (3.2.5)$$

Equations (3.2.5) for different values of k can be multiplied by arbitrary constants and upon the summation we get

$$\int Q_{2m-1}(x) dV_1(x) = \sum_{n=1}^{m} Q_{2m-1}(a_n) A_n = \int_D Q_{2m-1}(x) dV(x) \qquad (3.2.6)$$

where $Q_{2m-1}(.)$ denotes any polynomial of order not higher than $2m-1$.

Consider now the polynomial whose roots are the abscissae we are looking for, i.e.,

$$D_m(x) = \prod_{i=1}^{m} (x-a_i) = x^m + \sum_{i=0}^{m-1} C_i \, x^i$$

Consider then the set : $\left\{x^i D_m(x)\right\}_{i=0}^{m-1}$. It is clear that $x^i D_m(x) \Big|_{x=a_j} = 0$ for any i,j, $i \leq m-1$, $j \leq m$.

Thus by using (3.2.6)

$$\int x^{\lambda} D_m(x) dV_1(x) = \sum_{n=1}^{m} A_n(x^{\lambda} D_m(x))\Big|_{x=a_n} = 0$$

$$= \int_D x^{\lambda} Dm(x)\ dV(x) \quad , \text{ for } \lambda \leq m-1$$

So that

$$0 = \int_D x^{\lambda} D_m(x)\ dV(x) = \int x^{m+\lambda} + \sum_{i=o}^{m-1} C_i\ x^{i+\lambda} \quad dV(x)$$

$$= M_{m+\lambda} + \sum_{i=o}^{m-1} C_i\ M_{i+\lambda}$$

Upon writing the above explicitly for $\lambda = 0 \longrightarrow m-1$, one has

$$\text{(A)} \quad \begin{aligned} &C_o M_o + C_1 M_1 + \ldots\ldots\ldots\ldots + C_{m-1} M_{m-1} = -M_m \\ &C_o M_1 + C_1 M_2 + \ldots\ldots\ldots\ldots + C_{m-1} M_m = -M_{m+1} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &C_o M_{m-1} + C_1 M_m + \ldots\ldots\ldots\ldots + C_{m-1} M_{2m-2} = M_{2m-1} \end{aligned}$$

The determinant of system (A) can be shown to be positive. One can then solve this linear system for the coefficients C_i's and thus $D_m(x) = x^m + \sum_{i=o}^{m-1} C_i x^i$ is known. The roots of $D_m(x)$ give the a_i's (absissae for $V_1(x)$).

As for the A_i's, once the a_i's are known they can

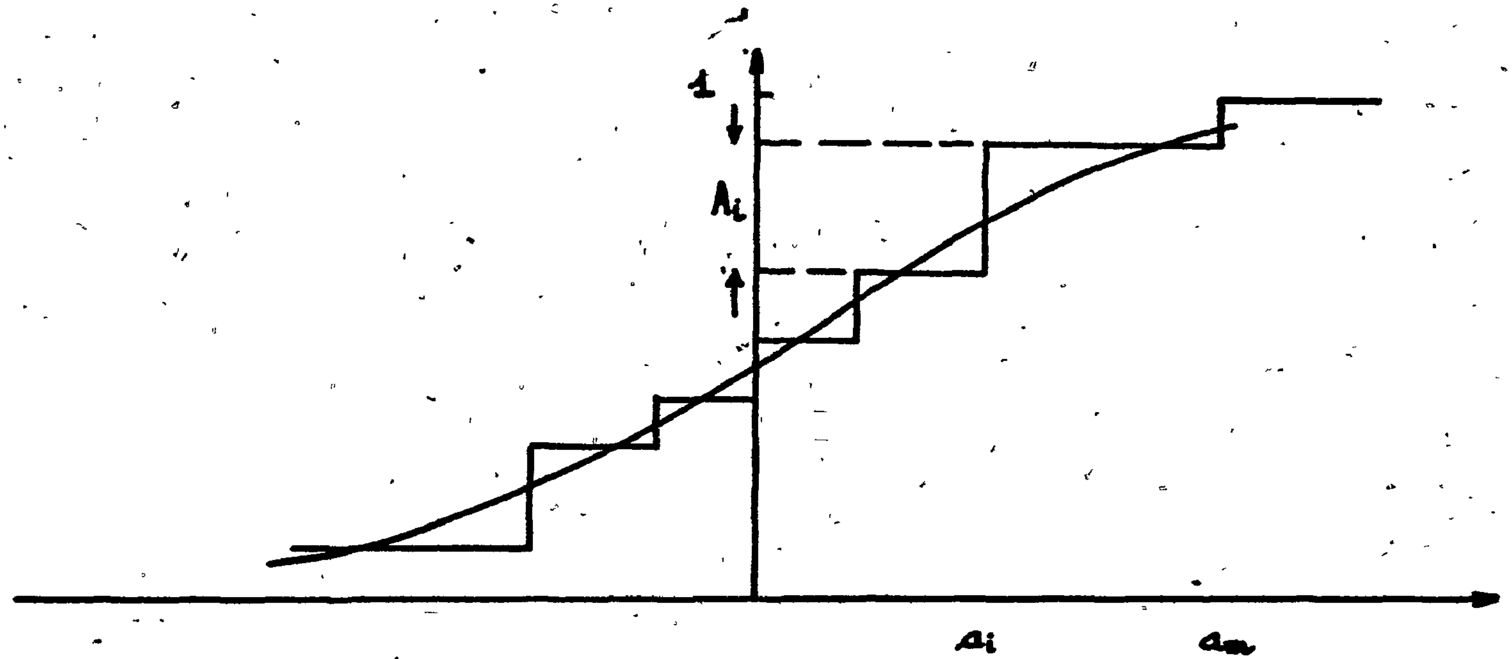

Fig. 3.1. A physical interpretation of GQR

be computed from the m equations:

$$\text{(B)}\qquad \begin{aligned} &\sum_{n=1}^{m} A_n = 1 \\ &\sum_{n=1}^{m} a_n^i A_n = m_i \quad \text{for } i = 1,2,\ldots,m-1 \end{aligned}$$

The determinant of the above equations is different from zero if all the a_i's are unique, which is the case here. Consequently, the above system can be solved for $A_1, A_2, \ldots A_m$, which are all positive.

One can now proceed to show that the A_i's and a_i's just found are the same as the i_i's and x_i's used in the GQR.

The monic polynomial $D_m(x) \triangleq x^m + \sum_{i=o}^{m-1} C_i x_i$ is such that the c_i's are uniquely determined by the system of linear equations (A), which results from the relation $\int_D x^i D_m(x) dV(x) = 0$ for $i=0,1,2. \ldots, m-1$.

On the other hand, $H_m(x) = k_m \prod_{i=1}^{m} (x-x_i)$. the x_i's here being the nodes of the QR used by Benedetto et al., also satisfies the above relation as can be seen from

$$\int_D x^i H_m(x) dV(x) = \sum_{k=1}^{m} i_k x_k^i H_m(x_k) = 0$$

for i such that $i+m \leq 2m-1$ (the GQR being exact for all polynomials of degree $\leq 2m-1$), i.e., for $i=0,1,2,\ldots,m-1$.

As the system of linear equations (A) uniquely determine $D_m(x)$, one may conclude that $D_m(x) = \frac{1}{\lambda_m} H_m(x)$.

The same reasoning can be then applied to system (B) to show that the i_k's and the A_k's are exactly the same.

Thus we have shown that the GQR method, as applied to the ISI problem, consists of giving to $i(x)$, the pdf of the ISI r.v. X, an approximation $i_1(x)$ which is of the form

$$i_1(x) = \sum_{k=1}^{m} i_k \ \delta(x-x_k).$$

Indeed, upon replacing $i(x)$ in (1.2.8) by $i_1(x)$, we get

$$P(e) \simeq \frac{2L-1}{L} \int_D 1-F(dr_o-x) \ i_1(x)\,dx$$

$$= \frac{2L-1}{L} \int_D 1-F(dr_o-x) \ \sum_{k=1}^{m} i_k \,\delta(x-x_k) \ dx$$

$$= \frac{2L-1}{L} \sum_{k=1}^{m} i_k \left(1-F(dr_o-x_k)\right)$$

which is exactly (3.2.1).

The m-step distribution, which is used to approximate the distribution of the ISI, possess the same first 2m moments as the distribution of the ISI. Higher-order moments differ, but are not null in both cases.

3.3 Comparisons of the two methods

As Benedetto et al. have already used the power series expansion method themselves, it is natural to expect that the GQR method offers improvement over the power series expansion (P.S.E) method. This is the case in the examples given by Benedetto et al. in [8].

The main improvement consists of increasing the rate of convergence of the GQR; that is, the convergence of the successive values computed by the GQR (or by the P.S. E.) to the exact result as the order of precision (or the number of terms in the expansion) is increased.

From the examples given by Benedetto et al. in [8], it appears that the GQR converges fast enough to the result to be of practical use in situations where the effect of ISI is limited. We cannot infer from these examples that the GQR method will be adequate in case of large ISI. We are assured that by increasing the order of precision we will get the exact result (assuming that round-off errors are not important), but nothing can be said of how fast this convergence is. In practice, the GQR is faster than the P.S.E., but no criterion has been given to justify this statement.

The criterion in favour of the GQR method is that it offers the highest degree of precision in the area of definite integration using a prefixed number of points. But, this is not a criterion of convergence. It is then interesting to apply the GQR method in systems with large ISI, for example, in an 8-ary or

16-ary PAM systems to see the effect of round-off errors on the convergence of the method. One cannot be certain that in those cases the GQR method converges rapidly to the result.

Alternatively, we know that the P.S.E. method oscillates more when the amplitude of the signal is large than when it is small. This is due to the fact that an increase in the signal amplitude is followed by a corresponding increase in the range of the ISI; more terms are then needed for an adequate approximation. This characteristic is common to all series expansion.

One may now say that the computational difficulty associated with the GQR method grows more than linearly with the increase of the order of precision, due to all the complex mechanisms needed to compute the nodes and weights, whereas numerical implementation of the P.S.E. method is much more straightforward.

Finally, we mention that the GQR method is directly applicable to other types of channels as is the P.S.E. method. This property has not been accredited to the P.S.E. method by Benedetto et al.[8]. But, it has been seen how to apply the P.S.E. method to other types of channels and noise statistics.

The GQR applies equally well to multiple integrals; this property justifies its use for other types of disturbances (e.g., phase jitter) in addition to noise and I S I

In conclusion, one may note a subtle but fundamental difference between the power series expansion and the GQR methods, as applied to the ISI problem. The P.S.E. method uses a Taylor series expansion of the noise process. Thus in computing the value given by an expansion truncated after 2n terms, we only use the first 2n moments and implicitly assume the nullity of all the moments of order higher than 2n, although we know that this is not true. This incorrect implicit assumption is inherent in the method. On the other hand, the GQR method does not use the moments "crudely". It combines them before using them. One important result which follows is that the moments of order higher than 2n(in case of an n point GQR) are no longer implicitly assumed to be null. This can be seen from the fact that the even moments of the approximating pdf [refer to $V_1(x)$ in Sec. 3.2] are positive. Thus GQR method "interpolates" the higher order moments from the low order moments and uses the entire expression to compute P(e). This may be why it appears to converge faster than the P.S.E. method. But, this does not mean that the "interpolation" thus introduced always gives an improvement over a series expansion method. It will be seen that the Chebyshev series appears to converge as fast as, if not faster than, the GQR method.

CHAPTER IV

THE CHEBYSHEV EXPANSION METHOD

In Chapter II, it has been seen that the power series expansion method consists mainly in a Taylor series expansion of the distribution function of the noise process, which is thus approximated by a polynomial. This substitution allows moments of the intersymbol interference (ISI) to be used to compute P(e). Therefore if one wishes to use a series expansion method, it should be able to give an accurate approximating polynomial after a small number of terms. This is required to ensure that the minimum number of moments is needed, which in turn, minimizes truncation errors. The Chebyshev series expansion possess the quality just mentioned.

In the following, the rate of convergence of series expansion into Chebyshev polynomials will also be studied before they are applied to the problem. Results achieved by this method are then presented and compared with other methods. Some comments follow.

4.1 The Chebyshev polynomials

Since the problem can now be reduced to the approximation of a certain probability distribution function, the following criterion may be applied to test the accuracy of the approximating process: The maximum deviation provided by the approximation should be minimum and not exceed a certain limit. This criterion forms the foundation of the minimax theory.

Consider the Lagrangian interpolation formula for functions $f(x)$ with bounded derivatives of order $n+1$ in $-1 \leq x \leq 1$; $f(x)$ is then expressed as a polynomial of degree n plus a remainder term, the polynomial having the same values as $f(x)$ at $n+1$ points, $x_0, x_1, \ldots, x_n$.

$$f(x) = p_n(x) + \frac{\Pi(x)\, f^{(n+1)}(\xi)}{(n+1)!}, \quad -1 \leq \xi \leq 1 \qquad (4.1.1)$$

$$\Pi(x) = (x-x_0)(x-x_1)\ldots(x-x_n)$$

The polynomial $p_n(x)$ is then given by

$$p_n(x) = \sum_{k=0}^{n} l_k(x) f(x_k), \quad l_k(x) = \frac{\Pi'(x)}{\Pi'(x_k)} \qquad (4.1.2)$$

where in the second of the above, the primes denote that the terms $(x-x_k)$ and (x_k-x_k) are omitted, respectively, from the products $\Pi(x)$ and $\Pi(x_k)$. It is easily seen that $l_k(x_i) = \delta_{ki}$, so that effectively $p_n(x_k) = f(x_k)$ for all k. Moreover, $\Pi(x)$, a polynomial with unity leading coefficient vanishes at the above points x_k.

The question now is how to choose the points of agreement, x_k's, so that the maximum modulus of the error $e_n(x) =$.

$f(x) - p_n(x)$ is as small as possible in the standardized range $-1 \leq x \leq 1$. This is the condition imposed by the minimax theory.

The solution of the above problem clearly depends on $f(x)$ and no single $p_n(x)$ will suffice for the set of all functions with $(n + 1)$ continuous derivatives in that range. One may note that there exists many methods to find the minimax polynomial in the literature, so these techniques will not be presented here.

Now, if a maximum absolute value of $f^{(n+1)}(\xi)$, say M, is assumed, one can then minimize the quantity.

$$\max\left|\frac{e_n(x)}{M}\right| = \max\left|\frac{\Pi(x)}{(n+1)!}\right| \tag{4.1.3}$$

One then must find that polynomial $\Pi(x)$ of degree n+1, with unity leading coefficient, for which $\max|\Pi(x)|$ is a minimum in $-1 \leq x \leq 1$. It can be seen then that $\Pi(x)$ must alternate maxima and minima, +N and -N, at n+2 successive points $-1 < y_1, < y_2 \ldots \ldots y_n < 1$, including the end points. If $q(x)$ is another polynomial of degree n+1 with unity leading coefficient which has smaller extreme values in $-1 \leq x \leq 1$, the difference $\Pi(x) - q(x)$ will alternate positive and negative values at n+2 points and, therefore, have n+1 zeros. But, $\Pi(x) - q(x)$ is a polynomial of degree n or less, hence this is not true and $\Pi(x) = q(x)$.

The most obvious functions with successive, equal and opposite extremum values are the trigonometric functions sin and cos. Since $\cos(n+1)\theta$ is a polynomial of degree n+1 in $\cos\theta$ with equal and opposite values of ± 1 at n+2 points in $0 \le \theta \le \pi$, including the end points, we deduce that the required unique polynomial $\Pi(x)$ is some multiple of the Chebyshev polynomial defined by

$$T_n(x) = \cos n\theta, \quad \cos\theta = x \quad , -1 \le x \le 1, \qquad (4.1.4)$$

with $T_0(x) = 1$, $T_1(x) = x$.

From the trigonometric identity

$$\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos n\theta,$$

we find the important relation

$$T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x) \qquad (4.1.5)$$

with $T_0(x) = 1$, $T_1(x) = x$.

Now from the relation

$$\int_0^\pi \cos n\theta \cos m\theta \, d\theta = \frac{\pi}{2}\delta_{mn}, \quad m \text{ and } n \ne 0,$$

we get

$$\int_{-1}^{1} T_n(x)\, T_m(x)(1-x^2)^{-\frac{1}{2}} dx = \frac{\pi}{2}\delta_{mn}, \quad m \text{ and } n \ne 0 \qquad (4.1.6)$$

i.e. the polynomials $T_n(x)$ and $T_m(x)$ are orthogonal with respect to $(1-x^2)^{-\frac{1}{2}}$ in the range $-1 \le x \le 1$.

Thus, according to the above, $\Pi(x)$ is chosen to be

$$\Pi(x) = 2^{-n}T_{n+1}(x),$$

the coefficient 2^{-n} being necessary to get an unity leading coefficient. As $T_{n+1} \leq 1$, $\max |\Pi(x)| = 2^{-n}$.

The points $x_0, x_1, \ldots, x_n$ at which $p_n(x)$ is equal to $f(x)$ are the zeros of $T_{n+1}(x)$, namely,

$$x_i = \cos\left[\frac{2i+1}{n+1}\,\frac{\pi}{2}\right], \quad i=0,1,\ldots,n \tag{4.1.7}$$

It is then ensured that

$$e_n(x) \leq \frac{M}{2^n (n+1)!}$$

which means that if M is finite for all n, increasing n allows one to achieve an arbitrarily small error.

One may conclude, therefore, that analytic functions, such as erfc(x), can be approximated as closely as desired at all points in a certain interval by choosing polynomials of successively higher degrees which are equal to f(x) at the zeros of the relevant Chebyshev polynomials.

What has been obtained from the above consideration of the minimax theory is very important. In a sense, one may state that the use of Chebyshev polynomials allows one to approximate functions in a certain interval with an error which never exceeds a certain ϵ in all the interval. In the problem in this thesis, one can replace the minimax criterion by some less stringent condition. For that purpose, consider the series expansion of f(x) into Chebyshev polynomials,

$$f(x) = \frac{1}{2} a_0 + \sum_{k=1}^{\infty} a_k T_k(x) \qquad (4.1.8)$$

$$a_k = \frac{2}{\pi} \int_{-1}^{1} T_k(x)\ f(x)\ (1-x^2)^{-\frac{1}{2}}\ dx \qquad (4.1.9)$$

(4.1.9) is obtained from the orthogonality of Chebyshev polynomials with respect to $(1-x^2)^{-\frac{1}{2}}$ in $-1 \leq x \leq 1$.

If the above series is truncated after the term $a_n T_n(x)$, the remainder will often be dominated by its first term; thus we expect the truncated series to be close to the minimax polynomial, though not identical to it. It has been shown [14] that in all practical cases the truncated Chebyshev series is almost as good as the minimax polynomial.

One more very important property of the Chebyshev expansion is its very fast rate of convergence.

Taylor series expansion as used by the power series expansion method has been seen to give oscillating results due to the rather slow rate of convergence. In the case where quite a few terms are needed in the expansion - for example when the ISI is large- truncation errors, which occur in the digital computer, may prevent us from obtaining the exact P(e).

Chebyshev series expansion, as it is related to the Fourier theory via the cosine function, eliminates that slow convergence. Fewer terms will be needed for an accurate approximation, and effects of truncation errors diminish accordingly.

Thus, before applying the Chebyshev expansion to the problem, its convergence will be studied.

4.2 Convergence of the Chebyshev expansion:

In the following, the convergence of the expansion in terms of Chebyshev polynomials, will be studied, first by relating the Chebyshev expansion to the Fourier theory, then by showing that expansion in terms of Chebyshev polynomials has the fastest rate of convergence in a certain class of orthogonal polynomials which include the Legendre and Taylor "polynomials".

It is well known that the expansion into Fourier series is a very valuable tool for the approximation of functions of period 2π. Indeed, for certain classes of such functions $f(x)$, the Fourier series will converge for most values of x in the complete range $-\infty < x < \infty$. Such classes include functions with the following properties:

i) $f(x)$ is defined arbitrarily in $-\pi < x < \pi$ and for all other values of x, by the periodicity condition $f(x+2\pi)=f(x)$.

ii) $f(x)$ is absolutely integrable, that is, $\int_{-\pi}^{\pi} |f(x)|\, dx$ exists.

Then at any interior point of any interval, in which $f(x)$ is bounded and has a finite number of maxima and minima, and a finite number of non-coincident discontinuities, the Fourier series converges to $\frac{1}{2}\left\{f(x^+) + f(x^-)\right\}$ which reduces to $f(x)$ at a point of continuity.

This class includes most of the functions whose expansions are needed in practical problems.

But, our main concern for the problem in hand is expansions valid over a finite interval, which can be taken, without loss of generality, to be $-\pi \leq x \leq \pi$. The Fourier series are computed from values of $f(x)$ in this interval only and to this extent periodicity gives no particular advantages.

On the other hand, in this case, unless $f(x)$ and all its derivatives have the same values at $-\pi$ and π, there is a terminal discontinuity of some order at these points. The rate of convergence of the Fourier series, that is the rate of decrease of its coefficients, depends on the degree of smoothness of the function, measured by the order of the derivative which first becomes discontinuous at any point in the closed interval $-\pi \leq x \leq \pi$. Even if $f(x)$ is perfectly smooth, it may have terminal discontinuity in relation to the Fourier theory and this will affect the rate of convergence of its Fourier series.

Consider a function $f(x)$ defined in $-\pi \leq x \leq \pi$, one can represent $f(x)$ as the sum of the respective even and odd functions $f_1(x) = \frac{1}{2}\left\{f(x) + f(-x)\right\}$, $f_2(x) = \frac{1}{2}\left\{f(x) - f(-x)\right\}$. Then the cosine series for $f_1(x)$ is

$$f_1(x) = \frac{1}{2} a_0 + \sum_{k=1}^{\infty} a_k \cos kx, \quad a_k = \frac{2}{\pi}\int_0^{\pi} f_1(x) \cos kx\,dx$$

and the sine series for $f_2(x)$

$$f_2(x) = \sum_{k=1}^{\infty} b_k \sin kx, \quad b_k = \frac{2}{\pi}\int_0^{\pi} f_2(x) \sin kx dx$$

At $x=0$ and $x=\pi$, all the terms in the sine series vanish, while $f_2(x)$ at $x=\pi$ equals $\frac{1}{2}\left\{ f(\pi) - f(-\pi) \right\}$. Thus, unless $f(\pi) = f(-\pi)$ the sine series can never give the correct answer at π and $-\pi$ and may converge only slowly at intermediate points. The cosine series can converge correctly to any $f(\pi)$ and $f(-\pi)$ but its first derivative vanishes at the terminal points and unless $f_1(x)$ also has this property, a 'discontinuity' in the first derivative at the two endpoints is encountered. This is less serious than a discontinuity in the function value and one would expect the cosine series to converge faster than the sine series for smooth functions.

If one is now interested in a function $f(x)$ defined only in $0 \leq x \leq \pi$, then its definition in $-\pi \leq x \leq 0$, can be chosen arbitrarily.

Thus $f(x)$ can be considered as an odd or even function and the sine or cosine series can be found. Now unless $f(o)=f(\pi)=o$, there is a discontimuity at o and π in the sine series. In the other case, a discontinuity is probably encountered in the first derivative.

Now consider

$$a_k = \int_0^{\pi} f(x)\cos kx dx = \frac{1}{k^2}\Big[f'(x)\cos kx\Big]_0^{\pi} - \frac{1}{k^2}\int_0^{\pi} f''(x)\cos kx dx$$

$$b_k = \int_0^{\pi} f(x)\sin kx dx = -\frac{1}{k}\Big[f(x)\cos kx\Big]_0^{\pi} + \frac{1}{k}\int_0^{\pi} f'(x)\cos kx dx$$

For large values of k, the integrals are likely to be small, and are dominated by the oscillation in coskx. One may deduce that the cosine series converge ultimately as k^{-2} and the sine series as k^{-1} unless f(x) has special properties.

It has been seen that with Fourier expansion of a non periodic f(x), problems arise from terminal discontinuities. These can be avoided with the Chebyshev form of the Fourier series. Consider the range $-1 \leq x \leq 1$ and make the change of variables $x = \cos\theta$ so that

$$f(x) = f(\cos\theta) = g(\theta), \quad 0 \leq \theta \leq \pi$$

the function $g(\theta)$ is genuinely periodic since $g(\theta + 2\pi) = g(\theta)$. We should then expect the cosine Fourier series

$$g(\theta) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos k\theta, \quad a_k = \frac{2}{\pi}\int_0^{\pi} g(\theta)\cos k\theta \, d\theta$$

to converge quite rapidly.

Letting $x = \cos\theta$, one gets the Chebyshev expansion, as shown in (4.1.8) and (4.1.9), which thus has the same convergence property of the Fourier series for f(x) with the advantage that the terminal discontinuities are eliminated, because $g(\theta)$ is even and genuinely periodic.

Now recall that the coefficient of x^{n+1} in $T_{n+1}(x)$ is 2^n. The maximum absolute value of $T_{n+1}(x)$ being unity, the monic polynomial $2^{-n}\ T_{n+1}(x)$ has maximum absolute value 2^{-n}. One very important property of this monic polynomial is that of all monic polynomials of degree n+1, it has the smallest deviation from zero. This has been proved already.

It can then be shown from all the above considerations that the coefficient a_k in (4.1.9) has order of magnitude $\frac{1}{2^{k-1}\ k!}$, which is considerably smaller for large values of k, than the k^{-1} or k^{-3} of the best Fourier series. This should establish the Chebyshev series expansion as one of the best methods.

Consider now the class of ultra-spherical polynomials which are orthogonal with respect to $(1-x^2)^\alpha$ for various values of α. More specifically, compare the rates of convergence of series expansions into these polynomials.

Let $\left\{\Lambda_r\right\}_{r=1}^{\infty}$ be a set of ultra spherical polynomials orthogonal with respect to some function v(x) having the above form.

One has, then

$$\int_{-1}^{1} v(x)\ \Lambda_r(x)\ q_{r-1}(x)dx = 0$$

where $q_{r-1}(x)$ is any polynomial of degree r-1 or less. By integrating this expression by parts, using the fact that the r^{th} derivative of $q_{r-1}(x)$ is identically zero, and letting

$$v(x)\,\Lambda_r(x) = \frac{d^r Vr(x)}{dx^r} = Vr^{(r)}\,(x),$$

it can be shown that one gets the following differential system,

$$\frac{d^{r+1}}{dx^{r+1}}\left\{ \frac{1}{w(x)} \quad \frac{d^r Vr(x)}{dx^r} \right\} = 0 \qquad (4.2.1)$$

$$V_r(\pm 1) = V_r'(\pm 1) = \ldots\ldots = V_r^{(r-1)}\,(\pm 1) = 0$$

When one solves the above system for particular values of α , one gets many cases of interest. For example the Legendre polynomials are associated with $v(x) = 1$, i.e. for $\alpha = 0$. In this case we find

$$Vr(x) = Cr(x^2-1)^r \quad , \Lambda_r(x) = Cr\,\frac{d^r}{dx^r}\,(x^2-1)^r \qquad (4.2.2)$$

Cr is chosen to be equal to $1/2^r(r)!$ to give the standard form of the Legendre polynomial with maximum absolute value of unity in (-1,1).

The coefficient A_r of x^r of $\Lambda_r(x)$ can be easily deduced from (4.2.2) to be equal to $\dfrac{(2r)!}{2^r\,(r!)^2}$. This coefficient will play an important role in the following.

Now, the Chebyshev polynomials can be produced with $w(x) = (1-x^2)^{-\frac{1}{2}}$. Solving (4.2.1) for $\alpha = -\frac{1}{2}$ gives then

$$\Lambda_r(x) = T_r(x),$$

$T_r(x)$ being the r^{th} order Chebyshev polynomial. It has been seen that the leading coefficient, Ar, in this case is 2^{r-1}, which is always larger than the corresponding coefficient in the Legendre polynomial for $r \geq 2$.

Finally, it can be shown that the Taylor series with orthogonal polynomials $\Lambda_r(x) = x^r$ correspond to the limiting ultra spherical orthogonal polynomials given by $\alpha \longrightarrow \infty$. Thus A_r in this case is always unity, which is always less than the corresponding coefficients in the Legendre and Chebyshev expansions.

One can show that the coefficients in the different series expansions are inversely proportional to A_r, and that, therefore, a faster rate of convergence corresponds to a larger A_r.

From (4.2.1) one has

$$V_r(x)=C_r(1-x^2)^{r+\alpha}, \Lambda_r(x)=C_r(1-x^2)^{-\alpha}\frac{d^r}{dx^r}\ (1-x^2)^{r+\alpha}$$

Now the coefficient a_r in the orthogonal expansion of $f(x)$ is given by

$$a_r=\int_{-1}^{1} v(x)\Lambda_r(x)f(x)dx \Big/ \int_{-1}^{1} v(x)\Lambda_r^2\ (x)dx$$

which can be transformed into

$$a_r=(-1)^r \frac{1}{A_r r!}\ \frac{\int_{-1}^{1} f(x)\ V_r^{(r)}(x)dx.}{\int_{-1}^{1} V_r(x)\ dx}$$

which can be simplified into,

$$r!\,A_r a_r = \frac{\int_{-1}^{1} f^{(r)}(x)\; V_r(x)dx}{\int_{-1}^{1} V_r(x)\,dx} \tag{4.2.3}$$

Since $V_r(x)$ does not change sign in $-1 \leq x \leq 1$, one can invoke the second law of the mean and write

$$r!\,A_r a_r = f^{(r)}(\xi) \quad , \quad -1 \leq \xi \leq 1 \tag{4.2.4}$$

Now consider the nature of the function $V_r(x)$. For large values of $r+\alpha$, one can conclude that the major contribution to the integrals in (4.2.3) comes from the region near $x=0$ so that to a good approximation one can write finally

$$r!\,A_r a_r \simeq f^{(r)}(o)$$

$$a_r \simeq \frac{f^{(r)}(o)}{r!\,A_r} \tag{4.2.5}$$

Before drawing conclusions about the rate of convergence of the Chebyshev series, recall that of all the polynomials with unity leading coefficient, $2^{1-r}\,T_r(x)$ has the smallest deviation from zero in $-1 \leq x \leq 1$. Correspondingly the Chebyshev polynomial $T_r(x)$ has the largest possible leading coefficient of all the polynomials with unity maximum absolute value in $-1 \leq x \leq 1$, Also for the Taylor series, the orthogonal polynomial Λ_r with unity maximum absolute value being x^r, the leading coefficient is always one; and this is the smallest possible value for a leading coefficient.

From all the above considerations, one may conclude that of all expansions in terms of ultra spherical polynomials the Chebyshev series will generally have the fastest rate of convergence and the Taylor series the slowest. The Legendre series falls in between those two.

4.3. Application to our problem

Rewriting (1.2.8) for the case of non-equiprobability of the levels, one has

$$P(e) = 2(1-p_{2L-1}) \int_D F\,(dr_o-x)\; i(x)\; dx \qquad (4.3.1)$$

$F(dr_o-x)$ is then expanded into a truncated, say after n terms, series of Chebyshev polynomials. This series can then be transformed into an ordered polynomial. Note that this transformation is not an optimum process as truncation errors may augment when numerical computation is done with this ordered polynomial form.

The function $F(dr_o-x)$ is thus replaced by the ordered polynomial which once substituted into (4.3.1) produces a weighted sum of the moments for an approximation of the integral.

The following shows the different steps which occur in the numerical process in case of the white Gaussian noise with variance σ^2.

The function $\frac{1}{2}$ erfc $(\frac{dr_o-x}{\sigma\sqrt{2}})$ should be expanded into Chebyshev series. Let the domain D of X, the ISI random variable be $(a,-a)$ and let $x=ay$ where $y \in (-1,1)$. Then

$$\frac{1}{2}\,\mathrm{erfc}\,\left(\frac{dr_o - ay}{\sigma\sqrt{2}}\right) \simeq \sum_{r=0}^{n} b_r T_r(y) \qquad (4.3.2)$$

where $T_r(y)$ is the Chebyshev polynomial of order r. and the coefficient b_r is given by

$$b_r = \frac{2}{\pi}\int_{-1}^{1} (1-y^2)^{-\frac{1}{2}}\ \frac{1}{2}\,\mathrm{erfc}\,\left(\frac{dr_o - ay}{\sigma\sqrt{2}}\right) T_r(y)\,dy \qquad (4.3.3)$$

The evaluation of the function erfc is provided by a standard routine in the digital computer. Note that the value of a varies proportionally with the maximum deviation caused by the ISI. The larger a is, the more terms that will be needed for the truncated series to approximate erfc suitably in the interval of interest. With the above one gets

$$P(e) = 2(1-p_{2L-1})\int_{-1}^{1} \frac{a}{2}\,\mathrm{erfc}\,\left(\frac{dr_o - ay}{\sigma\sqrt{2}}\right) i(ay)dy$$

then replacing the noise process by its approximation,

$$P(e) \cong 2(1-p_{2L-1})\int_{-1}^{1} a\left(\frac{b_o}{2} + \sum_{r=1}^{n} b_r T_r(y)\right) i(ay)dy$$

With

$$\frac{b_o}{2} + \sum_{r=1}^{n} b_r T_r(y) = \sum_{r=0}^{n} b_r T_r(y) = \sum_{r=0}^{n} c_r \left(\frac{x}{a}\right)^r ,$$

the above becomes

$$P(e) \simeq 2(1-p_{2L-1})\int_{-a}^{a} \sum_{r=0}^{n} c_r \left(\frac{x}{a}\right)^r i(x)dx$$

$$=2(1-p_{2L-1})\left(c_0+c_1\frac{\mu_1}{a}+c_2\frac{\mu_2}{a^2}+\ldots+c_n\frac{\mu_n}{a^n}\right) \qquad (4.3.4)$$

With the above formula one is ready to evaluate an approximation for P(e). In order to apply (4.3.4) one needs to know the coefficients c's, as well as the moments μ's.

For the computation to the moments, refer to Chapter 2.

Two methods of computing the coefficients of the Chebyshev expansion will be presented here, as formula (4.3.3) is not easy to compute. The first method, called the method of exact coefficients, is very general and can be applied whenever the definition of the noise process allows the value of its distribution function to be computed point by point, in the interval of interest. This is an interesting feature of the method as distribution functions of the noise process cannot often be defined analytically. This method depends on the orthogonality properties of the Chebyshev polynomials, which are expressed here in the familiar cosine form:

$$\int_0^{\pi}\cos r\theta\,\cos s\theta\,d\theta = \pi/2\;\delta_{rs},\; r \text{ and } s \neq 0$$

$$\sum_{k=0}^{N}{}'' \cos\frac{\pi rk}{N}\,\cos\frac{\pi sk}{N} = \frac{N}{2}\,\delta_{sr},\; r \text{ and } s \neq 0 \qquad (4.3.5)$$

where the double prime denotes that the first and last terms are halved.

$$\sum_{k=0}^{N-1} \cos \frac{\pi r(k+\frac{1}{2})}{N} \cos \frac{\pi s(k+\frac{1}{2})}{N} = \frac{N}{2} \delta_{rs}, \text{ r and } s \neq 0 \quad (4.3.6)$$

The cosine form of the coefficient a_r of the Chebyshev expansion is

$$a_r = \frac{2}{\pi}\int_0^{\pi} f(\cos\theta) \cos r\theta \, d\theta$$

Recall that

$$f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty} a_r T_r(x) = \sum_{r=0}^{\infty}{}' a_r T_r(x) \quad (4.3.7)$$

Let us define coefficients a_{rN} and u_{rN} by

$$a_{rN} = \frac{2}{N}\sum_{k=0}^{N}{}'' f(\cos \frac{\pi k}{N}) \cos \frac{\pi rk}{N} \quad (4.3.8)$$

$$u_{rN} = \frac{2}{N}\sum_{k=0}^{N-1} f(\cos \frac{\pi (k+\frac{1}{2})}{N}) \cos \frac{\pi r(k+\frac{1}{2})}{N} \quad (4.3.9)$$

In (4.3.8) and (4.3.9), replace, respectively, $f(\cos\frac{\pi k}{N})$ and $f(\cos\frac{\pi(k+\frac{1}{2})}{N})$ by their expansion and get

$$a_{rN} = \frac{2}{N} \sum_{k=0}^{N}{}'' \left[\sum_{s=0}^{\infty}{}' a_s \cos \frac{s\pi k}{N} \right] \cos \frac{\pi rk}{N}$$

$$u_{rN} = \frac{2}{N} \sum_{k=0}^{N-1} \left[\sum_{s=0}^{\infty}{}' a_s \cos \frac{s\,\pi(k+\frac{1}{2})}{N} \right] \cos \frac{r\,\pi(k+\frac{1}{2})}{N}$$

Making use now of the orthogonality relationships (4.3.5) and (4.3.6),

$$a_{rN} = a_r + a_{2N-r} + a_{2N+r} + a_{4N-r} + a_{4N+r} + \ldots\ldots \quad (4.3.10)$$

$$u_{rN} = a_r - a_{2N-r} - a_{2N+r} + a_{4N-r} + a_{4N+r} - \ldots\ldots \quad (4.3.11)$$

In particular,

$$a_{rN} + u_{rN} = 2a_{r2N} \tag{4.3.12}$$

$$a_{rN} - u_{rN} = 2a_{2N-r,2N} \tag{4.3.13}$$

An algarithm for finding the coefficients a_r to any required degree of accuracy then has the following steps: a_{rN} is first computed enumeratively for some small value of N, say 2, through formula (4.3.8).

i) Use formula(4.3.9) to compute

u_{rN} for $r=0,1,\ldots,N-1$,

noting that $u_{NN}=0$

ii) Use then (4.3.12) in the form

$$a_{r2N} = \frac{1}{2}(a_{rN} + u_{rN})$$

to find a_{r2N} for $r=0,1,\ldots.N$.

iii) Use then (4.3.13) in the form

$$a_{r2N} = \frac{1}{2}(a_{2N-r,N} - u_{2N-r,N})$$

to find a_{r2N} for $r= N+1, N+2,\ldots,2N$. One is then ready to perform another cycle with N replaced by 2N.

The above procedure might be stopped as soon as a value of N is reached at which the last few coefficients a_{NN}, a_{NN-1}, a_{NN-2} are less than the maximum permissible error, say ϵ, in the a_r. By this time we expect each u_{rN} to differ from the corresponding a_r by an amount less than ϵ.

The second method of computing the coefficients of the

Chebyshev expansion does not actually evaluate these coefficients but it approximates them, and therefore is called the method of approximate coefficients. This method is, then, less efficient than the last one, but the additional approximation thus introduced is almost always valid, in the sense that whenever the initial truncated series (the one with the exact coefficients) is a satisfactory approximation for the noise function, this second approximation does not introduce further noticeable error.

With this method one uses the interpolation formula

$$p_N(x) = \sum_{r=o}^{N}{}' d_r T_r(x)$$

$$d_r = \frac{2}{N+1} \sum_{k=o}^{n} f(x_k) T_r(x_k), \quad x_k = \cos\left(\frac{2k+1}{n+1}\frac{\pi}{2}\right) \quad , N=n$$

Note that with this formula, the error $e_n(x) = f(x) - p_N(x)$ satisfies for sufficiently smooth functions.

$$\max \frac{e_n(x)}{f^{N+1}(\xi)} = \min, \quad |\xi| < 1$$

The d_r's can be computed very easily and are related to the true a_r's coefficients via

$$d_r = \frac{2}{n+1} \sum_{k=o}^{n} f(x_k)\, T_r(x_k)$$

Replacing $f(x_k)$ by its infinite series

$$d_r = \frac{2}{n+1} \sum_{k=o}^{n} T_r(x_k) \left\{ \sum_{s=o}^{\infty} a_s T_s(x_k) \right\}$$

Noting that

$$T_r(x_k) = (-1)^p \, T_{2p(n+1)\pm r}(x_k)$$

and upon using the orthogonality property of the Chebyshev polynomials

$$\sum_{k=0}^{n} T_s(x_k)\, T_r(x_k) = \frac{n+1}{2}\,\delta_{sr},$$

one derives

$$d_r = a_r - (a_{2n+2-r} + a_{2n+2+r}) + (a_{4n+4-r} + a_{4n+4+r}) - \ldots$$

One can conclude from the above that whenever the initial truncation is satisfactory, i.e., a_i for $i > n$ is negligible, d_r differs then from c_r by negligible amounts.

In fact, it can be shown, by writing explicitly the difference $f(x) - \sum_{r=0}^{n}{}' \, d_r T_r(x)$, that if the second method is used, the error never exceeds twice the error of the initial truncated series.

The two methods are not entirely different and are related one another. We can consider the first method as a refined version of the second one.

Thus an evaluation of the P(e) can now be performed via (4.3.4) as the moments and the coefficients of the Chebyshev expansion are computed. A summary of the computational procedure follows.

After all of the interfering samples one wishes to take into account are determined, the S/N ratio is chosen and the maximum deviation caused by the ISI is obtained as the 'product' of the sum of all the maximum interfering samples by the S/N ratio. An initial program is then run to produce the moments, according to the iterative relation (2.5.1). Note that the maximum deviation grows proportionally with the number of levels.

A second program is then run to produce the coefficients of the expansion. Note the following points:

i) The Chebyshev polynomials are generated via their iterative relationship.

ii) The S/N ratio as well as the maximum deviation are combined to give the "central" value as well as the interval in which the noise process is to be approximated.

iii) A routine should be provided or written to compute the values of the noise function at desired points.

iv) The coefficients are more easily obtained by the method of approximate coefficients, the other method gives better results, but can be avoided if approximate results are adequate. However, its use is suggested whenever the range to be considered is large. Note that the values of the coefficients as provided by the exact method do not change appreciably when the number of terms in the series expansion is increased, whereas with the second method, computed values for the coefficients will change slightly when the number of terms in the series is augmented.

v) The coefficients and the moments are finally combined to produce an estimate of P(e).

vi) Double precision arithmetic is required throughout.

Before presenting the numerical results, it now remains to estimate truncation errors, as was done for other methods.

As in any other scheme, two kinds of truncation errors are to be considered here.

i) The error due to the truncation of the overall channel impulse response. This analysis has been treated in the preceding chapters in connection with other methods. One can always augment the number M of interfering samples until the computed probability of error stops changing.

ii) The error introduced by the Chebyshev expansion. An analysis of this second kind of error appears in Appendix for the white Gaussian noise. There an upper bound on the deviation between the noise function and its approximation as given by the Chebyshev series is derived. One can then take opportunity of the fact that $\int_D i(x)dx=1$ to conclude that the maximum possible error is certainly less than $2(1-p_{2L-1})$ times the upper bound just derived. This can not be considered as a tight bound and is actually many times larger than the actual error. It is shown that

$$|\delta_n| < A\exp\left[-\frac{(d\ |r_0| - a)^2}{4\sigma^2}\right],\ \text{if } a < d\ |r_0|$$

$$\delta_n < A \qquad , \text{ if } a \geq d\,|r_o| \qquad (4.3.14)$$

where

$$A = \frac{Ba^{n+1}\,(1-p_{2L-1})}{2^{n-1/2}(n+1)\,\sqrt{n!}\sqrt{\pi}\,\sigma^{n+1}} \;,\; B = 1.086435$$

It can be verified that for sufficiently large n, the righthand sides of (4.3.14) go to zero, and this is always the case for all values of a/σ.

In fact, it is merely repetitious to state that the Chebyshev series is absolutely convergent in the case of white Gaussian noise, for it has been seen that it converges much faster than the Taylor series which already possess this property.

CHAPTER V

NUMERICAL RESULTS

Some of the results obtained by the Chebyshev expansion method are presented here. The purpose is to show that the method is a valid one which can be used confidently to evaluate the effect of ISI. To support this statement, in the first examples, comparisons are made with other methods: the power series expansion method, the GQR method, our's and also the very recent Murphy's method [18] of Legendre expansion. We have thus chosen "popular" cases i.e., the bandlimited signal and the fourth-order Chebyshev pulse, both in white Gaussian noise.

To further verify the adequacy of this method, it has been applied in cases where other methods may not suceed very well, i.e., the importance of ISI has been substantially increased by augmenting the number of levels. In all the following examples, the second method of computing coefficients in the Chebyshev expansion has been used, which has already been proven adequate. Note that the method of exact coefficients has also been used and has given better results, which are not presented here. Gaussian noise and equiprobability of all the levels are assumed for convenience.

5.1. Binary cases:

A) The bandlimited signal:

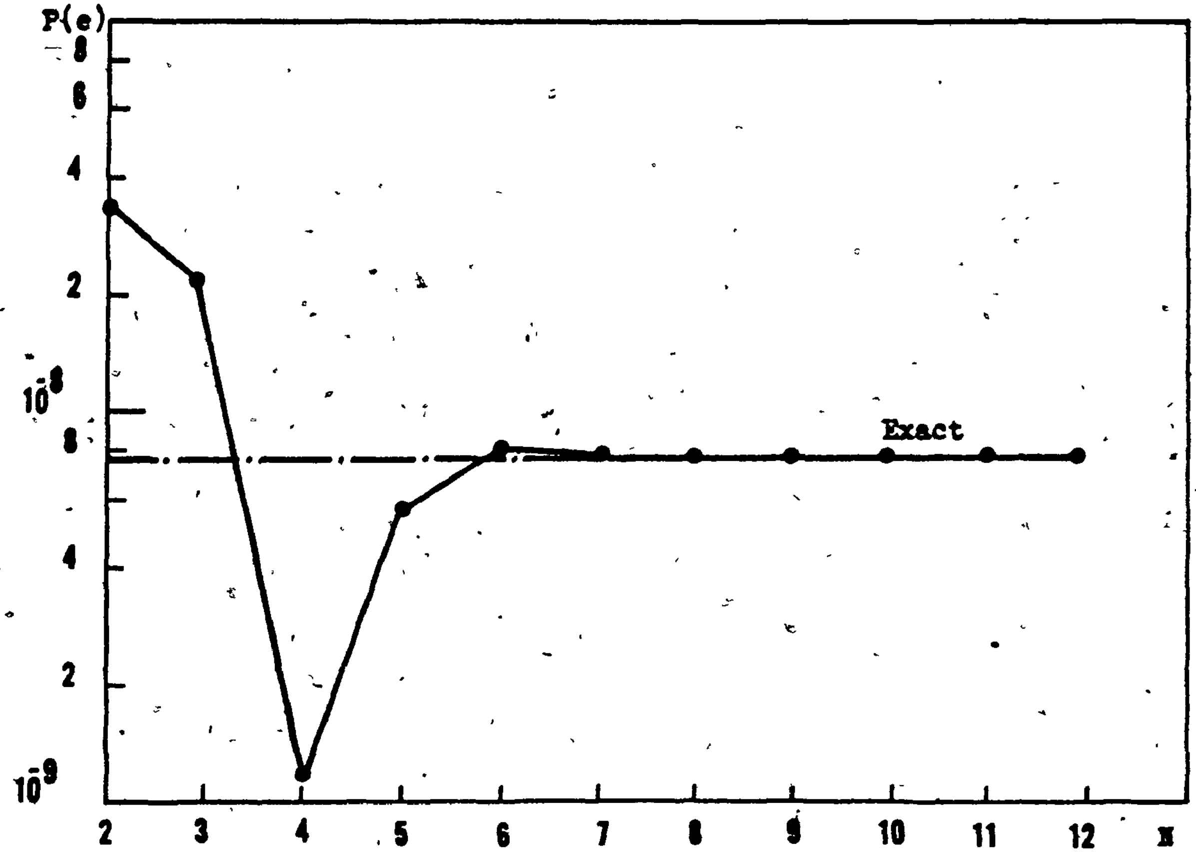

Fig. 5.1 - Chebyshev Expansion for the ideal bandlimited signal; Δ = 0.05T, S/N = 16 db; N represents the number of moments used.

Initially consider a binary PAM system when the received pulse is ideally bandlimited and truncated after the 11-most important samples.

Fig 5.1 shows the error probability computed by the Chebyshev polynomial expansion method as a function of the number N of terms used in the expansion as well as the known exact result based on the exhaustive method which is not time-consuming here. The sampling time deviation from the nominal value is 0.05T and the signal to noise ratio is 16 db.

In Fig. 5.2 the same result as obtained by Murphy's method (Legendre expansion) is shown. In his expansion, 11 terms are needed to obtain the result, while the Chebyshev method required only 7.

In Fig. 5.3, comparison is made between the GQR method and the Chebyshev method. Here, the two methods give similar results, but the GQR method needs a few more moments (9) than our's to derive the result.

We mention here that the power series expansion method uses many more terms than any of the above methods.

The reason why either the number of moments or the number of terms in the expansion is used as a basis for comparison can be explained by the following consideration.

i) The convergence of a series expansion is measured by the number of terms that should be used to get an accurate

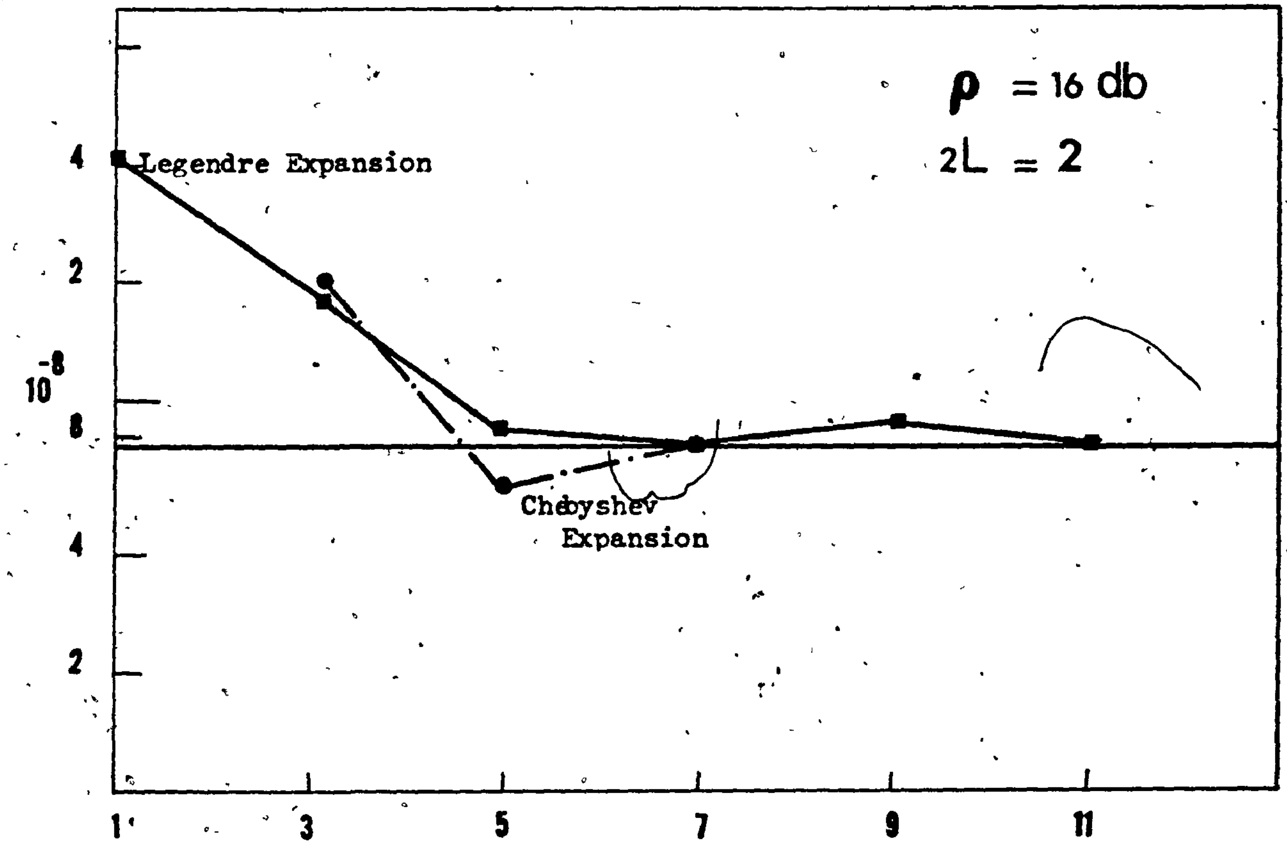

Fig. 5.2 - Legendre expansion for the ideal bandlimited case; $\Delta = 0.05T$

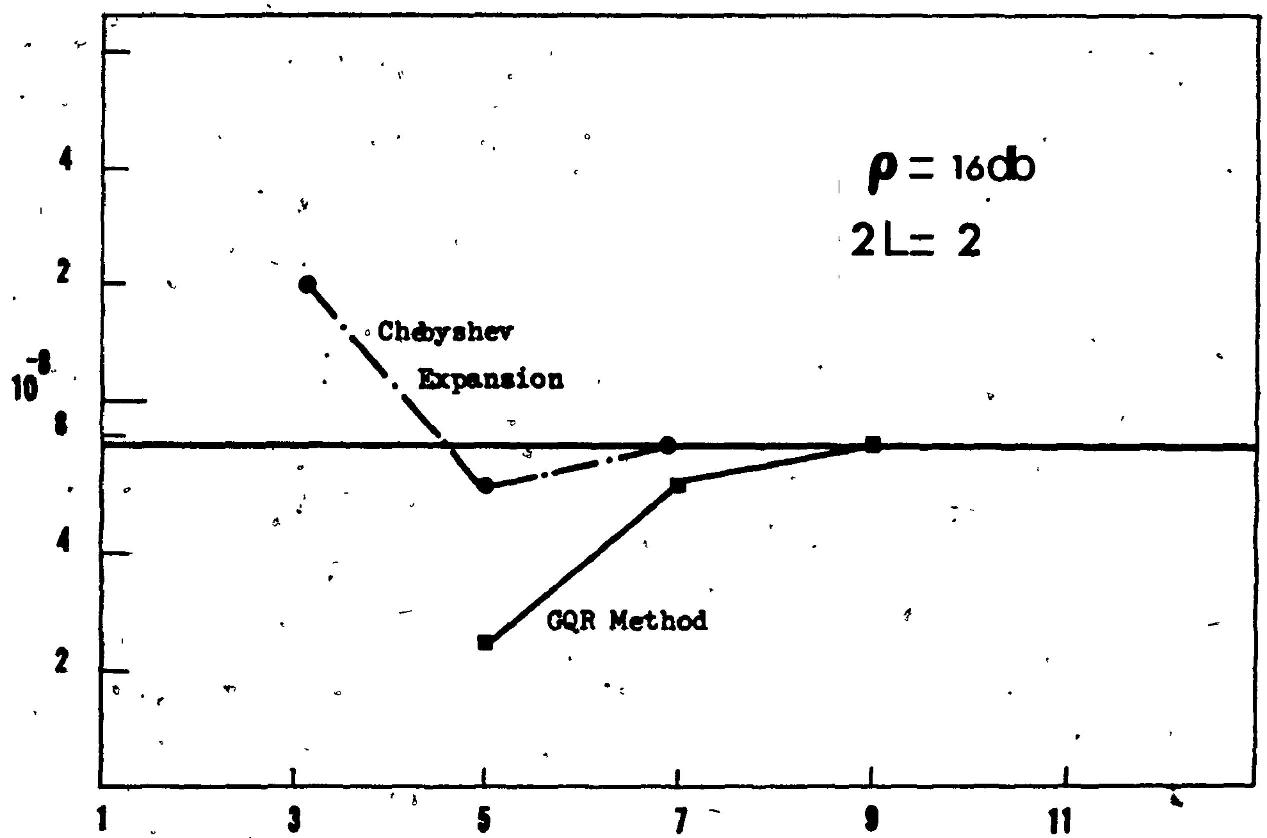

Fig. 5.3 - GQR method for the ideal bandlimited case with $\Delta = 0.05T$

approximate. Therefore one may compare Legendre, Taylor and Chebyshev series methods on basis of this quantity. Note that the number of terms in the series expansions equals the order of the approximating polynomial; it also equals the number of moments of the ISI r.v. needed to get the result.

ii) Since GQR is not a series expansion method, the number of moments used was chosen as basis for comparison with the Chebyshev method. One may object to such a basis of comparison and ask why the two methods were not compared on the number of "points" used? In fact, there is no absolute basis of comparison between the GQR and the series expansion methods; but it may be mentioned that the time used to set up the programs as well as the amount of computation associated with the derivation of results via the GQR method, with a certain number of "points", exceeds that of all the series expansion methods using the same number of moments.

Furthermore, one can look at all the different methods as different processes of estimating or replacing missing information: the probability density function of the ISI r.v. Thus, in all the methods one is substituting for the missing information some less descriptive and less concise information, i.e., the moments of the distribution. This is why it sounds reasonable to measure the efficiency of [illegible] methods by the amount of information they need. The less information they need, the more powerful they are.

Now, in Fig. 5.4, we change the sampling time deviation to 0.2T. The probability of error in the otherwise similar conditions jumps from 10^{-9} to 10^{-3}. Our methods then produces the result (as computed once more by the emmerative method) with an eight-term series expansion. The same scenario occurs here, as in Fig.5.1, i.e., the oscillation occurs at the first few terms (as in any other series expansion), but the convergence is very fast; differences between the computed value and the exact one decrease sensibly with more terms, and once the result is reached, the values computed do not change.

The same result is obtained with the GQR method (Fig. 5.5). Again more terms are needed here.

Expansion in terms of Legendre polynomials proves to be adequate after 5 terms. In the same time oscillating results are obtained with the power series expansion method, probably due to round off errors.

B) The 4th order Chebyshev pulse

These are results achieved by the Chebyshev method with another overall impulse response: a 4th-order Chebyshev pulse truncated after 11 terms with a sampling time deviation of 0.2T.

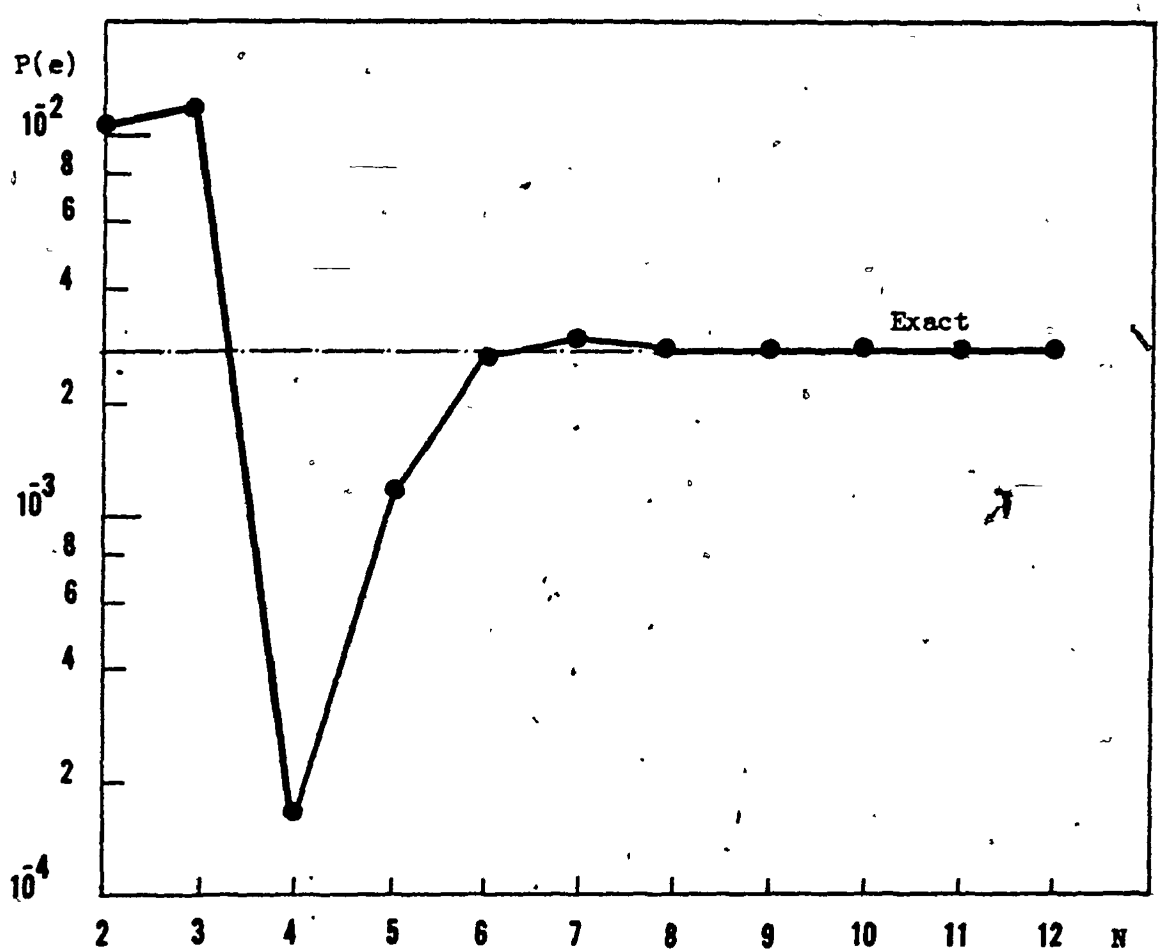

Fig. 5.4 - Chebyshev expansion for the ideal bandlimited pulse with $\Delta = 0.2T$; S/N = 16 db; N represents the number of moments used.

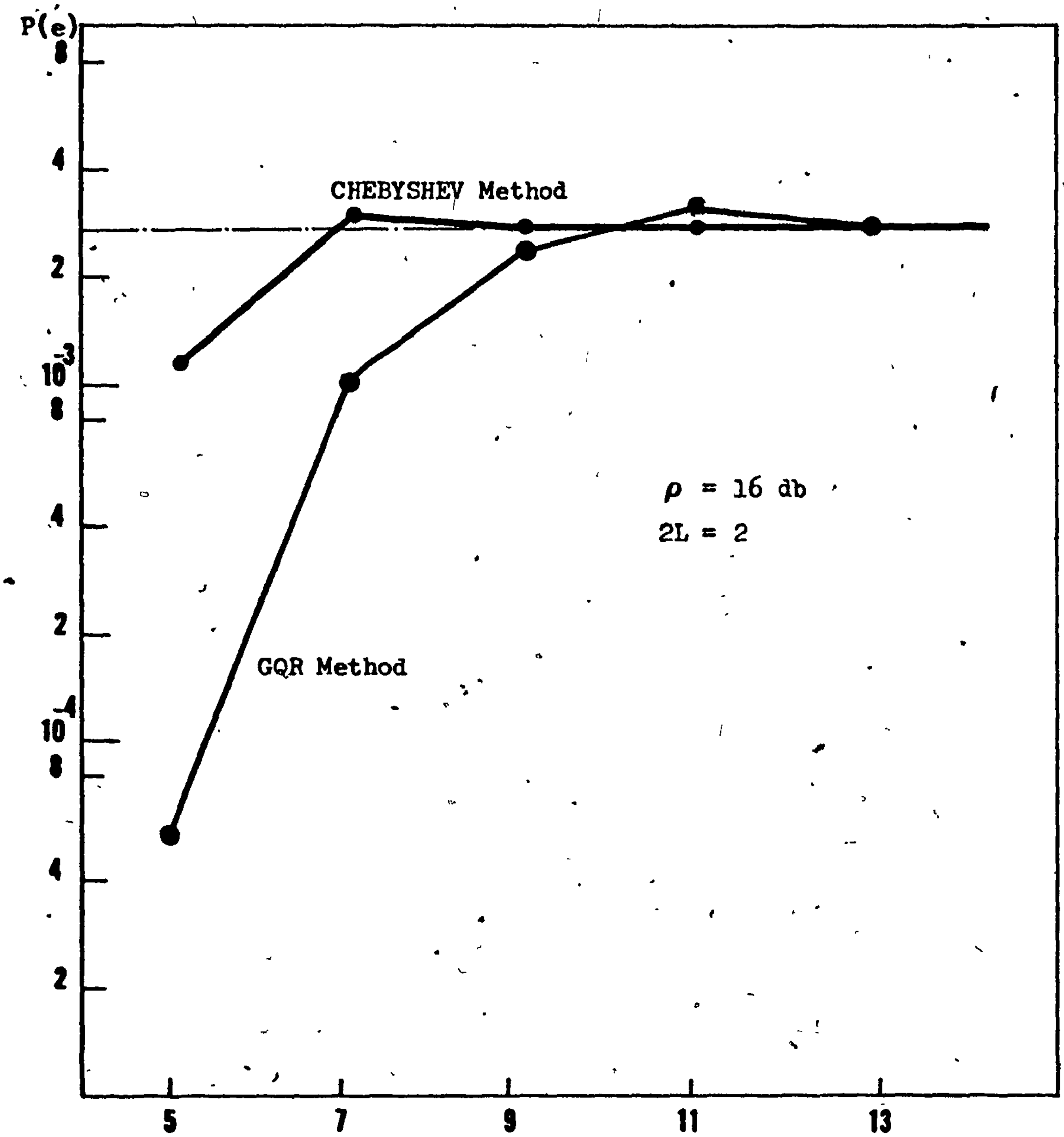

Fig. 5.5 - Comparison between GQR and Chebyshev expansion methods for the ideal bandlimited case, $\Delta = 0.2T$.

Once more, the convergence is very rapid. In Fig. 5.6, values computed with twice the number of terms by the power series expansion method oscillate considerably.

In Fig.5.7, a comparison is made with the Legendre expansion. The two methods once more appear equivalent.

If we now take into account the fact that in all these results, only approximate values are used for the coefficients in the Chebyshev series expansion (i.e., a more accurate and faster result can be reached if the exact coefficients are computed and used instead of the approximate ones) one may deduce that the Chebyshev expansion method is slightly superior than all the others, whenever one goes through the process of computing the exact values of the coefficients in the expansion.

5.2 Multilevel cases:

The effect of the intersymbol interference increases as the number of levels increases. In fact, the maximum deviation caused by the ISI becomes, respectively, 3 times and 7 times as large in 4-level and 8-level PAM systems as in the corresponding binary system. An ideal bandlimited pulse with the sampling time deviation of 0.05T is used in Figs. 5.8 , 5.9 and 5.10.

Fig.5.8, shows the error probability computed by the Chebyshev expansion method for different values of the S/N ratio for the 4-level PAM system; notice the convexity of this curve

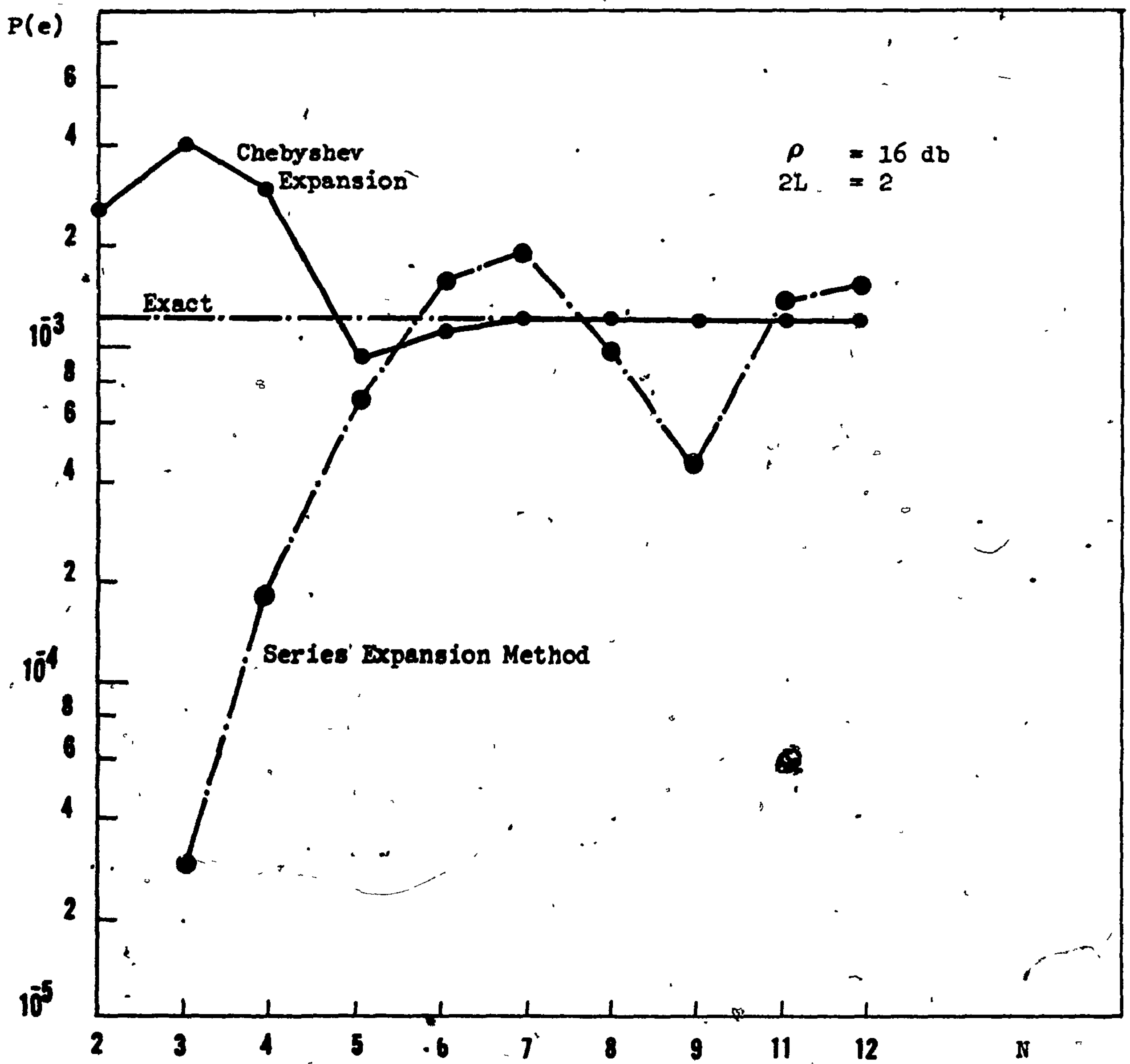

Fig. 5.6 - Probability of error for fourth order Chebyshev pulse with $\Delta = 0.2T$.

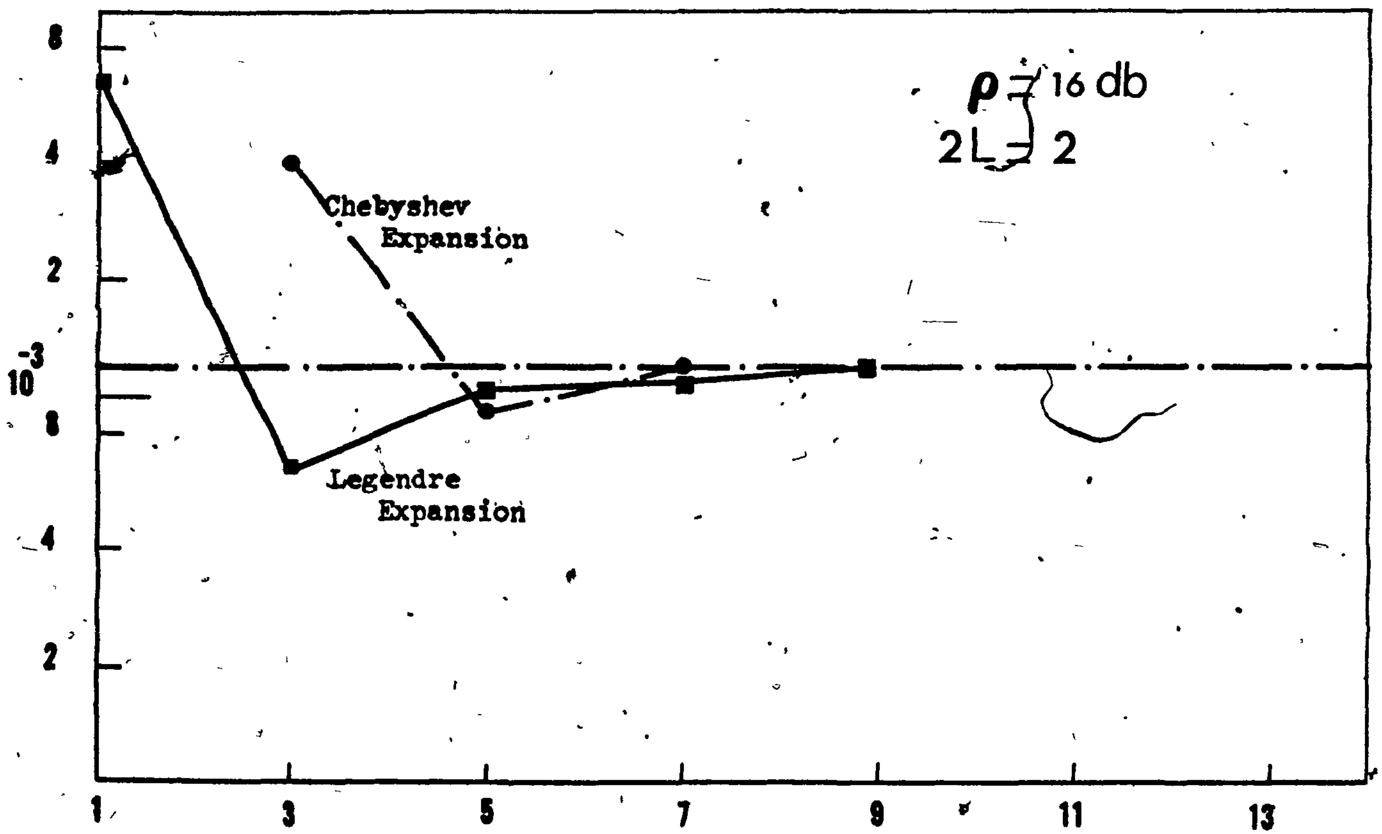

Fig. 5.7 - Comparison between Legendre and Chebyshev expansion methods for fourth-order Chebyshev pulse with $\Delta = 0.2T$.

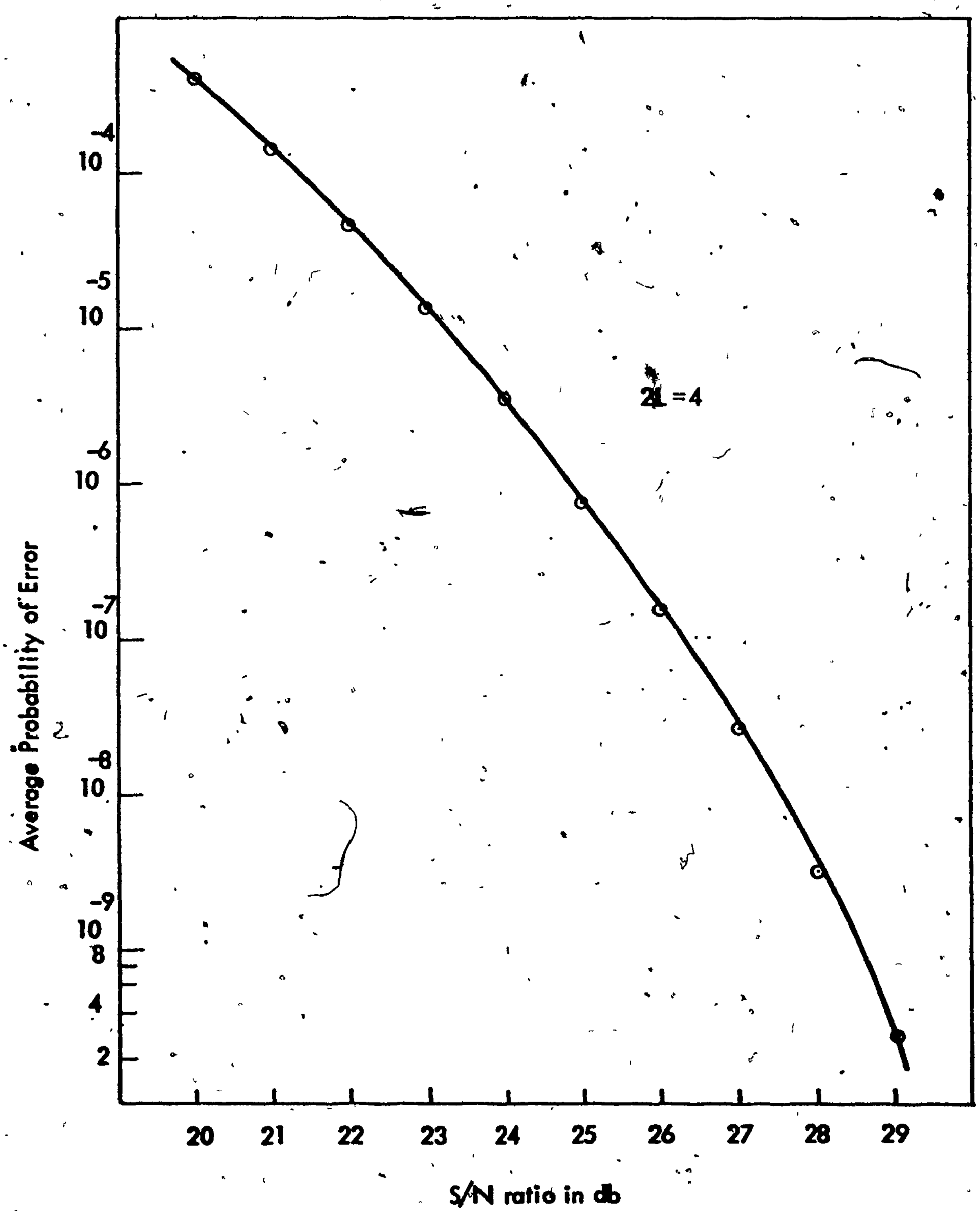

Fig. 5.8 - Probabilities of error versus S/N ratio for a 4-level PAM System, when the received pulse is ideally band-limited, with $\Delta = 0.05T$

which should result from the fact that the error function also drops steadily as the S/N ratio is increased. Thus ISI in this case can theorically be overcome, i.e., any desired value of P(e), however small, can be obtained by increasing the signal power. This result is particular to the Gaussian channel and of less importance to the ideal bandlimited pulse. We can expect existence of channels in which the combined noise ISI limits the P(e) to be larger than a certain value, independently of the S/N ratio. But this is not the case here.

The error probability for a typical value of the S/N ratio in the same example is plotted in Fig. 5.9 as a function of the number of terms in the Chebyshev expansion, to illustrate the efficiency of the method. Notice there is no substantial increase in the number of terms of the Chebyshev expansion to be used to reach the result. Also, as expected, a fast convergence to the result is observed. On the other hand, one should not worry about the negative value obtained with 8 terms: this simply means that the expansion is not yet sufficient.

By increasing the number of levels to 8, in Fig. 5.10, twice as many terms are required before converging to the result. The signal power here is taken such that the square of the distance between two adjacent signal levels divided by the noise power equals 20 db. Observe the following points:

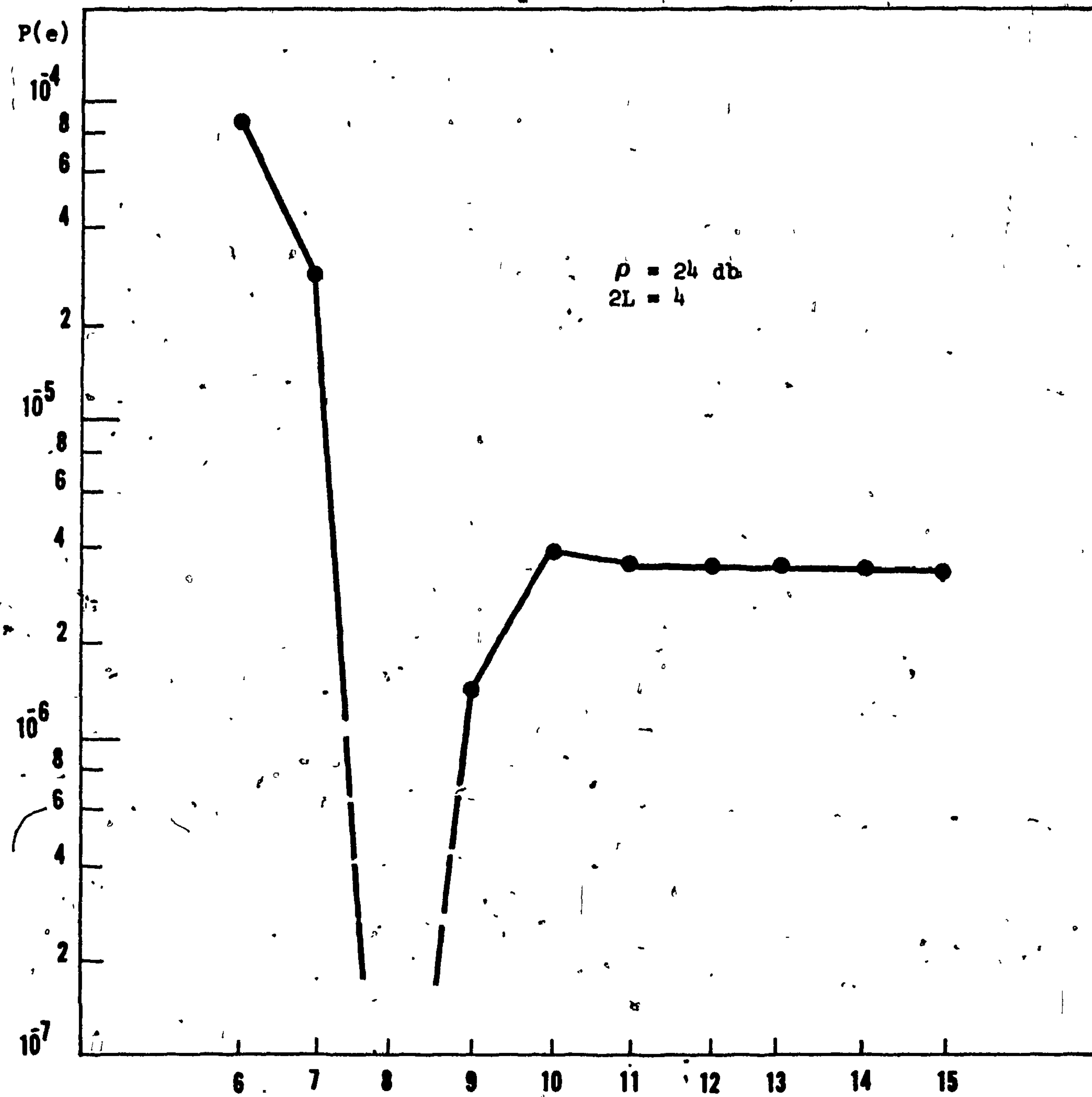

Fig. 5.9 - Chebyshev expansion for a 4-level PAM System.

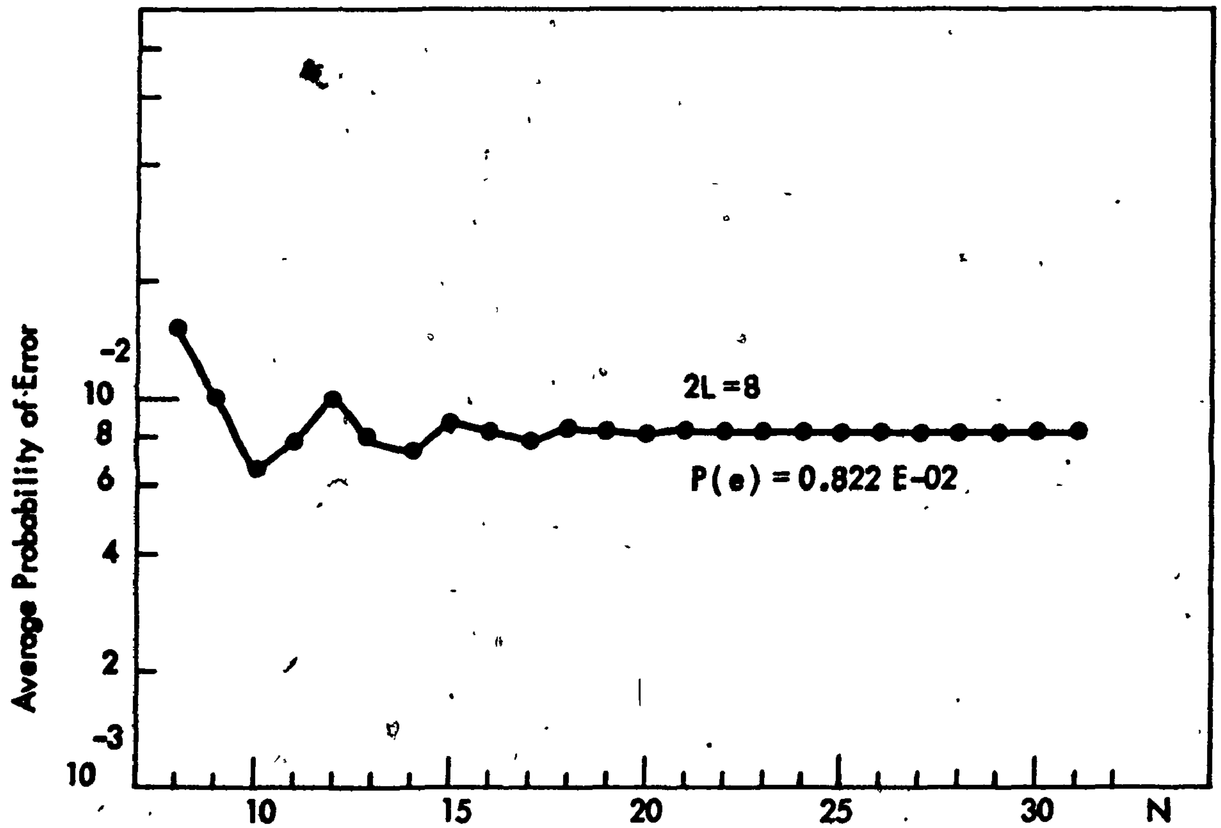

Fig. 5.10 - Convergence of the Chebyshev expansion for an 8-level PAM system. N is the # of moments used.

i) The curve has the form of a damped sinusoidal oscillation with a high damping coefficient. The result is reached after about 3 periods, each period consisting of about 6 or 7 terms in the expansion. No doubt far fewer terms are needed here if exact coefficients of the expansion are used.

ii) Once the value of n(which corresponds to the number of terms used in the Chebyshev expansion), used in (4.3.4) is sufficiently large to give an adequate result, increasing n thereafter does not produce any appreciable change in the computed value. This thus provides a valid stopping criterion in the number of terms to be used in the Chebyshev expansion.

5.3. Comments

Due to the surprising complexity associated with the estimation of the probability distribution of the ISI random variable, the problem of the ISI has been transformed into an approximation problem.

All the methods, therefore, either approximate the distribution of the ISI itself, or transform the noise function in a form more suitable to be used in conjunction with the information computable from the ISI process. Numerical computation thus becomes indispensable. That is why the efficiency of different methods is measured in terms of standards associated with numerical process, such as the ease of numerical implementation, the computer time, as well as the diversity of applications...

More specifically, of the three series expansion methods the power series expansion is the most direct, i.e., its numerical implementation is straight forward. However, this "quality" prevents it from being the most efficient, because the rate of convergence of such a straight forward method cannot be fast enough here. It is then natural that the Legendre and Chebyshev series which are based on results derived from the power series method, prove to be more efficient.

The Legendre and Chebyshev polynomials, as well as the Taylor polynomials all belong to the class of ultra-spherical polynomials. Of this class the Chebyshev polynomials possess the most rapid rate of convergence. Implementation of the Chebyshev expansion is somewhat more difficult than the other two, which are either more direct or more 'natural' (The Legendre polynomials are orthogonal with respect to the unity function in (-1,1)). Exact coefficients of the series should be found indirectly through a computational procedure, which, has been verified to be easily implemented numerically [21] .

An absolute comparison between series expansions and GQR methods cannot be made. However, it is possible to observe that the GQR method in this context is more difficult to implement, since the determination of weights and nodes is highly indirect and rather complex. Furthermore, its results are condi -

tionned by the accuracy of the computed values of the moments, while series expansion methods depend much less on that accuracy.

The difficulty in comparing the series expansion methods and the GQR method can be seen to arise from the fact that they approximate two different 'physical' entities ! From this point of view, the GQR method has an advantage over the series expansion as it reconstructs an elementary distribution for the intersymbol interference. By doing so, it gives more insight into the physical aspect of the ISI phenomena.

Although they seem to be two parallel ways of solving the ISI problem, it is possible to bind the series expansion and the GQR methods via the mathematical consideration that they all minimize some weighted error of the form

$$\int_D \left(f(x) - p_n(x) \right)^2 w(x)\, dx \qquad (5.3.1)$$

where $f(x)$ is the noise function, $p_n(x)$ the approximating polynomial, and $w(x)$ is: the unity function, $(1-x^2)^{-\frac{1}{2}}$, and $i(x)$ (the probability density function of the ISI) for the Legendre, Chebyshev expansion methods and the GQR method respectively. Thus the error is uniformly minimized with Legendre expansion, Chebyshev expansion is designed to minimize errors near the ends of the range, while the GQR method minimizes the error in proportional relation with the density of the ISI. The superiority of one or other of the above minimization criteria depends strongly on the function f(x). Therefore the consideration of the above criteria alone cannot tell us which one is the most suitable for a problem.

In fact, the ideal approximation should minimize the quantity

$$\int_D \left(f(x) - p_n(x)\right) i(x)\, dx$$

In summary, one may conclude that accurate enough methods have been devised by which one may confidently compute the probability of error in the presence of ISI. Thus, the problem of ISI can now be explored more fully. That is one may now study the degradation induced by the ISI in the system: how sampling time deviations affect the system, whether a barrier exists in the system performance when signal power is increased, whether there exists an "optimum" signal power, etc. We can also explore the conditions in which the degradation is the worst, in order to avoid them. Efficiency of equalizers can now be rated in terms of the probability of error itself.

CHAPTER VI

CONCLUSION

A new method of evaluating digital communication systems performance in the presence of noise and intersymbol interference has been presented. The method is based on the expansion of the noise process into Chebyshev polynomials and permits to use the moments of the intersymbol interference random variables. As predicted by the theory, the numerical results have confirmed that the Chebyshev expansion appears to be one of the most rapidly convergent expansion methods, so that its range of applicability extends to cases where the intersymbol interference is important. The proposed method can equally be applied to the cases of non Gaussian noise and its extension to partial response coded data transmission systems is straight forward [8]. The method can also be extended to phase shift keyed systems.

APPENDIX

In this Appendix, we derive an upper bound on the error introduced by truncating the Chebyshev series. Let

$$\delta_n = f(y) - p_n(y),\ y \in [-1,1]$$

By the Lagrangian interpolation formula,

$$\delta_n = \frac{\pi(y)\, f^{(n+1)}(N)}{(n+1)!} \quad \text{for some } N \in [-1,1]$$

where

$$\pi(y) = \prod_{i=0}^{n} (y - y_i)$$

Choosing the points $y_0, \ldots, y_n$ as

$$y_i = \cos\left(\frac{2i+1}{n+1} \cdot \frac{\pi}{2}\right),\ i=0,1,\ldots,n,$$

we have [12,p.6]

$$\pi(y) = 2^{-n}\, T_{n+1}(y)$$

where $T_{n+1}(y)$ is the Chebyshev polynomial of order n+1. Thus,

$$|\pi(y)| \leq 2^{-n} \text{ for } y \in [-1,1]$$

Hence, δ_n is upper bounded by

$$|\delta_n| \leq \frac{1}{2^n (n+1)!} \max_{y\in(-1,1)} f^{(n+1)}(y) \tag{1}$$

where

$$f(y) = (1-p_{2L-1}) \operatorname{erfc}\left(\frac{dr_0-ay}{\sigma\sqrt{2}}\right) \tag{2}$$

But [13,p.298]

$$\operatorname{erfc}^{(n+1)}(z) = (-1)^n 2(\pi)^{-1/2} H_n(z) \exp(-z^2). \tag{3}$$

where $H_n(\cdot)$ is the Hermite polynomial of order n, and [13,p.787]

$$|H_n(z)| < B \exp(z^2/2) 2^{n/2}(n!)^{1/2}, \quad B=1.086435. \tag{4}$$

Substituting (2) -(4) in (1), we have

$$|\delta_n| < \frac{B\, a^{n-1}(1-p_{2L-1})}{2^{n-1/2}(n+1)\sqrt{n!}\sqrt{\pi}\,\sigma^{n+1}} \max_y \exp\left(-\frac{(dr_0-ay)^2}{4\sigma^2}\right)$$

Therefore,

$$|\delta_n| < A \exp\left(-\frac{(d\, r_0 - a)^2}{4\sigma^2}\right) \text{ if } a < d\,|r_0| \tag{4.3.14}$$

$$|\delta_n| < A \qquad \text{if } a > d\,|r_0|$$

where

$$A = \frac{B\, a^{(n+1)}(1-p_{2L-1})}{2^{n-1/2}(n+1)\sqrt{n!}\sqrt{\pi}\,\sigma^{n+1}}, \quad B= 1.086435$$

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