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Classification of hybrid modes in cylindrical dielectric optical waveguides

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The classification of hybrid modes in cylindrical dielectric waveguides consisting of two or three layers is studied. A new mode designation based upon the separation of the characteristic equation is introduced. That is, separate characteristic equations for HE_{nm} and EH_{nm} modes are derived. This new scheme covers dielectric rods, dielectric tubes, cladded optical fibers and any three-layer structure in general. It is analytically shown that no crossover exists between *HE* and *EH* modes with the same order of azimuthal variation (n).

1. INTRODUCTION

Cylindrical dielectric waveguides have, in the past decade, attracted considerable attention because of their important applications in millimeter and optical communications. Their propagation properties have been subjects of extensive investigations in recent years. Cladded optical fibers, dielectric rod and dielectric tube waveguides are the most widely known structures of such kind which have been analyzed by various researchers.

In cylindrical dielectric waveguides, all modes except TE and TM are hybrid, i.e., they have axial components of both electric and magnetic fields. Unlike circularly symmetric modes, the classification of hybrid modes is somewhat complicated. One of the earliest schemes for designating hybrid modes in a dielectric rod was proposed by *Beam et al.* [1949]. It is based upon the relative contributions of E_z and H_z to a transverse component at some reference point. For example, if E_z makes the larger contribution, the corresponding mode is designated *EH*, and so forth. Finding this method of designation arbitrary, *Snitzer* [1961] suggested a scheme based on the value of some amplitude coefficient ratio at frequencies far from cutoff. The modes for which this ratio is $+1$ are designated *EH*, and those for which it is -1 as *HE*.

Snitzer's criterion, although strictly valid far from cutoff, practically settles the question of mode designation for a dielectric rod. It can be verified numerically that the amplitude coefficient ratio has values different from ± 1 at frequencies not far from cutoff, but remains positive for *EH* and negative for *HE* modes. *Kuhn* [1974] has attempted to use the sign of this ratio to classify hybrid modes in a cladded optical fiber. Our investigation, however, reveals that the sign of the amplitude coefficient ratio in a cladded fiber changes along a dispersion curve especially in the cladding mode region and, hence, cannot be utilized to properly classify the hybrid modes.

Clarricoats [1961] also proposed a mode designation, based on the sequence of solutions of the characteristic equation, which works well for a dielectric rod, but has limited success in the case of a cladded fiber. *Kharadly and Lewis* [1969] proposed another method and used it to classify the hybrid modes in a dielectric tube. It is based upon the radial variation of the field components as well as the sequence of solutions. But generally speaking, field configurations do not play a decisive role in the classification of hybrid modes.

From this review, we find that there has not yet been a precise, well defined, and global scheme for the classification of hybrid modes in cylindrical dielectric waveguides. In this paper, we present a new scheme based on the separation of the characteristic equation. In other words, two separate equations are derived, one representing *EH* and the other *HE* modes. This new scheme covers a wide range of structures and, in particular, cladded fibers, dielectric tubes and dielectric rods.

After obtaining separate characteristic equations for *EH* and *HE* modes, it will be analytically proved that no "crossover," as has been observed by *Clarricoats and Chan* [1973], *Kuhn* [1974] and *Yip and Huang* [1975], exists between *EH* and

HE modes, with the same order of azimuthal variation, of a cladded fiber.

2. FORMULATION OF THE PROBLEM

Consider a cylindrical dielectric waveguide consisting of three regions, namely, core, cladding, and the outer region. The three media are assumed to be lossless and homogeneous with the outer region extending to infinity. The ith medium is characterized by a permittivity $\epsilon_i = \epsilon_0 \epsilon_{ri}$ and a permeability $\mu_i = \mu_0 \mu_{ri}$. Figure 1 illustrates the geometry of the problem. Choosing a cylindrical coordinate system r, θ, z and assuming that z and t dependences are given by $\exp[j(\omega t - \beta z)]$, the axial components of the fields can be written as

$$\left.\begin{aligned} E_{z1} &= a_{n1} Z_{n1}(k_1 r) \cdot P_n \\ H_{z1} &= b_{n1} Z_{n1}(k_1 r) \cdot Q_n \end{aligned}\right\} \quad 0 \le r \le r_1 \tag{1}$$

$$\left.\begin{aligned} E_{z2} &= [a_{n2} Z_{n2}(k_2 r) + a_{n3} Z_{n3}(k_2 r)] \cdot P_n \\ H_{z2} &= [b_{n2} Z_{n2}(k_2 r) + b_{n3} Z_{n3}(k_2 r)] \cdot Q_n \end{aligned}\right\} \quad r_1 \le r \le r_2 \tag{2}$$

$$\left.\begin{aligned} E_{z3} &= a_{n4} Z_{n4}(k_3 r) \cdot P_n \\ H_{z3} &= b_{n4} Z_{n4}(k_3 r) \cdot Q_n \end{aligned}\right\} \quad r \ge r_2 \tag{3}$$

where $P_n = \cos(n\theta + \psi_n) \exp[j(\omega t - \beta z)]$, $Q_n = \sin(n\theta + \psi_n) \exp[j(\omega t - \beta z)]$ and

$$(k_i)^2 = \begin{cases} k_0^2(\mu_{ri}\epsilon_{ri} - \bar{\beta}^2), & \mu_{ri}\epsilon_{ri} \ge \bar{\beta}^2 \\ k_0^2(\bar{\beta}^2 - \mu_{ri}\epsilon_{ri}), & \mu_{ri}\epsilon_{ri} \le \bar{\beta}^2 \end{cases}$$

with $\bar{\beta} = \beta/k_0$ as the normalized propagation constant. ψ_n is a phase constant and Z_{ni} ($i = 1, \ldots, 4$) are the Bessel and the modified Hankel functions given in Table 1 in appendix A.

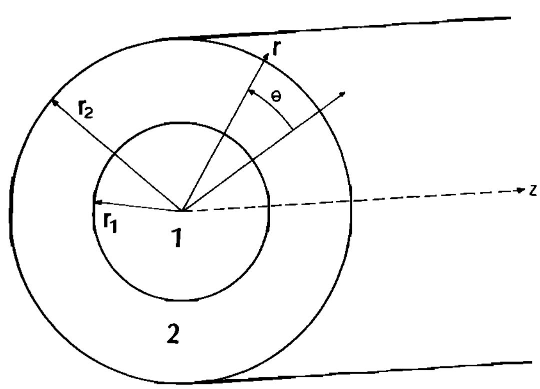


Fig. 1. Geometry of a three-layer cylindrical dielectric waveguide.

The transverse components of the fields in each region are readily obtained from their respective axial components. At the boundaries $r = r_1$ and $r = r_2$, the tangential components of the fields must be continuous, resulting in a set of eight equations with eight unknowns. The characteristic equation is obtained by setting the determinant of the 8×8 coefficient matrix to zero. The result is

$$G_1 \eta_1^2 + G_2 \eta_1 + G_3 = 0 \tag{4}$$

where

$$G_1 = ad - bc,$$
$$G_2 = (ad' - cb') + (da' - bc'),$$
$$G_3 = a'd' - b'c'$$

with

$$a = \mu_{r2}\epsilon_{r1}(\epsilon_{r2}\Delta_2 - \epsilon_{r3}\Delta_5)$$
$$a' = \epsilon_{r2} AB(\xi - 1) - \mu_{r2}\epsilon_{r2}(\epsilon_{r2}\Delta_3 + \epsilon_{r3}\Delta_1\eta_6)$$
$$b = \mu_{r1}\epsilon_{r2}(\xi - 1)B$$
$$b' = \mu_{r2} A(\epsilon_{r2}\Delta_2 - \epsilon_{r3}\Delta_5) + \mu_{r2}\epsilon_{r2}\Delta_1 B$$
$$c = \mu_{r2}\epsilon_{r1}(\xi - 1)B$$
$$c' = \epsilon_{r2} A(\mu_{r2}\Delta_2 - \mu_{r3}\Delta_5) + \mu_{r2}\epsilon_{r2}\Delta_1 B$$
$$d = \mu_{r1}\epsilon_{r2}(\mu_{r2}\Delta_2 - \mu_{r3}\Delta_5)$$
$$d' = \mu_{r2} AB(\xi - 1) - \mu_{r2}\epsilon_{r2}(\mu_{r2}\Delta_3 + \mu_{r3}\Delta_1\eta_6)$$

and

$$\Delta_1 = \eta_2 - \xi\eta_3,$$
$$\Delta_2 = \xi\eta_4 - \eta_5$$
$$\Delta_3 = \xi\eta_3\eta_4 - \eta_2\eta_5$$
$$\Delta_4 = \xi(\eta_2 - \eta_3)(\eta_4 - \eta_5)$$
$$\Delta_5 = (\xi - 1)\eta_6$$

where η_i ($i = 1, 2, \ldots, 6$), ξ, A and B are defined in appendix A. Moreover, the notations $x = k_1 r_1$, $u_1 = k_2 r_1$, $u_2 = k_2 r_2$ and $w = k_3 r_2$ have been used in the derivation of (4).

If $\mu_{r1}\epsilon_{r1} > \mu_{r2}\epsilon_{r2} > \mu_{r3}\epsilon_{r3}$, (4) represents the characteristic equation of a cladded optical fiber. The case $\mu_{r1}\epsilon_{r1} \ge \bar{\beta}^2 \ge \mu_{r2}\epsilon_{r2} > \mu_{r3}\epsilon_{r3}$ corresponds to core modes, while for $\mu_{r1}\epsilon_{r1} > \mu_{r2}\epsilon_{r2} \ge \bar{\beta}^2 \ge \mu_{r3}\epsilon_{r3}$, (4) represents the cladding modes. The condition $\mu_{r2}\epsilon_{r2} > \mu_{r1}\epsilon_{r1} \geq \mu_{r3}\epsilon_{r3}$ applies to dielectric tubes. The characteristic equation of a dielectric rod is contained in (4), for if $r_2 \to r_1$ and $\mu_{r1}\epsilon_{r1} \geq \bar{\beta}^2$, the three-layer waveguide shrinks to a rod. Letting $r_2 \to r_1$, then $u_2 \to u_1$, $\eta_4 \to \eta_2$, $\eta_5 \to \eta_3$, $\xi \to 1$, and as a result (4) reduces to

$$(\epsilon_{r1}\eta_1 - \epsilon_{r3}\eta_6)(\mu_{r1}\eta_1 - \mu_{r3}\eta_6) = n^2\bar{\beta}^2(1/x^2 + 1/w^2)^2 \quad (5)$$

which is the well known characteristic equation for a rod, and

$$\eta_1 = J_n'(x)/xJ_n(x), \quad \eta_6 = -K_n'(w)/wK_n(w)$$

3. CLASSIFICATION OF MODES

When $n = 0$, the characteristic equation (4) splits into two equations corresponding to the circularly symmetric TE and TM modes. The results are

$$\epsilon_{r1}\eta_1(\epsilon_{r2}\Delta_2 - \epsilon_{r3}\Delta_5) - \epsilon_{r2}^2\Delta_3 - \epsilon_{r2}\epsilon_{r3}\Delta_1\eta_6 = 0, \quad \text{for TM modes} \quad (6)$$

$$\mu_{r1}\eta_1(\mu_{r2}\Delta_2 - \mu_{r3}\Delta_5) - \mu_{r2}^2\Delta_3 - \mu_{r2}\mu_{r3}\Delta_1\eta_6 = 0, \quad \text{for TE modes} \quad (7)$$

In order to get some insight into the classification of hybrid modes, we first look at the dielectric rod situation. Equation (5) can be regarded as quadratic in η_1; thus, solving it for η_1 yields

$$\eta_1 = (1/2\mu_{r1}\epsilon_{r1})\{(\mu_{r1}\epsilon_{r3} + \mu_{r3}\epsilon_{r1})\eta_6 \pm [(\mu_{r3}\epsilon_{r1} - \mu_{r1}\epsilon_{r3})^2\eta_6^2 + 4\mu_{r1}\epsilon_{r1}\bar{\beta}^2 n^2(1/x^2 + 1/w^2)^2]^{1/2}\} \quad (8)$$

On the other hand, the amplitude coefficient ratio defined by *Snitzer* [1961] as $P = -(\mu_0\omega/\beta)\cdot(b_{n1}/a_{n1})$ is rewritten

$$P = \frac{\epsilon_{r1}\eta_1 - \epsilon_{r3}\eta_6}{n\bar{\beta}^2(1/x^2 + 1/w^2)} = \frac{n(1/x^2 + 1/w^2)}{\mu_{r1}\eta_1 - \mu_{r3}\eta_6} \quad (9)$$

Substituting for η_1 from (8) results in

$$P = [1/2n\mu_{r1}\bar{\beta}^2(1/x^2 + 1/w^2)]\{(\mu_{r3}\epsilon_{r1} - \mu_{r1}\epsilon_{r3})\eta_6 \pm [(\mu_{r3}\epsilon_{r1} - \mu_{r1}\epsilon_{r3})^2\eta_6^2 + 4\mu_{r1}\epsilon_{r1}n^2\bar{\beta}^2(1/x^2 + 1/w^2)^2]^{1/2}\} \quad (10)$$

It is evident that in (10), P is always positive for a + sign and negative for a − sign. Besides, at frequencies far from cutoff, i.e., when $w \to \infty$, $\bar{\beta}^2 \to \mu_{r1}\epsilon_{r1}$ with x remaining finite, $P \to \pm 1/\mu_{r1}$. Assuming that the relative permeabilities are all unit, P is then equal to ± 1, a result which was first derived by *Snitzer* [1961]. It is now quite clear that (8) with a + sign represents *EH* and with a − sign *HE* modes.

In a similar manner, (4), the characteristic equation of a three-layer structure, may be split into two equations, each representing one class of modes. Solving (4) for η_1 we get

$$\eta_1 = (1/2G_1)[-G_2 \pm (\delta)^{1/2}] \quad (11)$$

where $\delta = G_2^2 - 4G_1G_3$. It is proved in appendix A that

$$\delta = [(ad' - cb') - (da' - bc')]^2 + (4\epsilon_{r1}/\mu_{r1})(db' - bd')^2 \quad (12)$$

Since δ is always greater than zero, (11) with either sign represents a set of curves in the $(\omega - \beta)$ plane which are continuous and have continuous slopes too. Moreover, the fact that δ does not become zero implies that there are no crossovers between *EH* and *HE* modes.

It is now evident that (11) with a + or a − sign inevitably represents one class of modes. The mode with zero cutoff frequency has been traditionally referred to as the HE_{11} mode. For a particular structure, this mode is always contained in one of the equations in (11). Thus, the following scheme for the classification of hybrid modes is proposed.

Split the dispersion equation as in (11). Let the equation which includes the dominant HE_{11} mode represent HE_{nm} and the other equation EH_{nm} modes. Accordingly, it can be verified that for a cladded optical fiber (11) with a + sign represents *EH* and with a − sign *HE* modes. In the case of a dielectric tube, whenever $\bar{\beta}^2 \leq \mu_{r1}\epsilon_{r1}$, + should be used for *EH* and − for *HE* modes, while if $\bar{\beta}^2 \geq \mu_{r1}\epsilon_{r1}$, + will be for *HE* and − for *EH* modes.

The amplitude coefficient ratio is given by

$$P = -\frac{\mu_0\omega}{\beta}\frac{b_{n1}}{a_{n1}} = \frac{1}{2\bar{\beta}(db' - bd')} \cdot [(da' - bc') - (ad' - cb') \pm (\delta)^{1/2}] \quad (13)$$

We observe that the numerator of the right-hand side expression in (13) is always positive for a + sign and negative for a − sign. The sign of its denominator, however, depends on the sign of $(db' - bd')$. For the rod problem, $(db' - bd')$ reduces to $n\bar{\beta}(1/x^2 + 1/w^2) > 0$, and hence the sign of P can be used for mode designations. For a cladded fiber and tube, $(db' - bd')$ takes both positive and negative values. Consequently, the sign of P cannot be utilized for the classification of hybrid modes in three-layer structures. This point will be discussed further in the following section.

4. NUMERICAL RESULTS

To obtain the normalized propagation coefficient $\bar{\beta}$ as a function of frequency, the dispersion equations derived in section 3 were solved numerically using a root search technique. Several typical values for permittivities were chosen and all permeabilities were assumed to be unity. Dispersion characteristics for several lower-order hybrid modes have been plotted. These plots illustrate $\bar{\beta}$ versus v, the normalized frequency defined as

$$v = \begin{cases} (2\pi r_1/\lambda)(\epsilon_{r1} - 1)^{1/2}, & \text{for a dielectric rod} \\ (2\pi r_1/\lambda)(\epsilon_{r1} - \epsilon_{r2})^{1/2}, & \text{for a cladded fiber} \\ (2\pi r_1/\lambda)(\epsilon_{r2} - \epsilon_{r1})^{1/2}, & \text{for a dielectric tube} \end{cases}$$

Figure 2 shows the characteristics of a dielectric rod with $\epsilon_{r1} = 2.25$. The amplitude coefficient ratios for the HE_{11} and EH_{11} modes are plotted versus $\bar{\beta}$ in Figure 3. It is observed that P is positive for the EH_{11}, while it is negative for the HE_{11} mode. This property, that P is positive for EH_{nm} and negative for HE_{nm} modes of a dielectric rod, has been analytically verified in section 3.

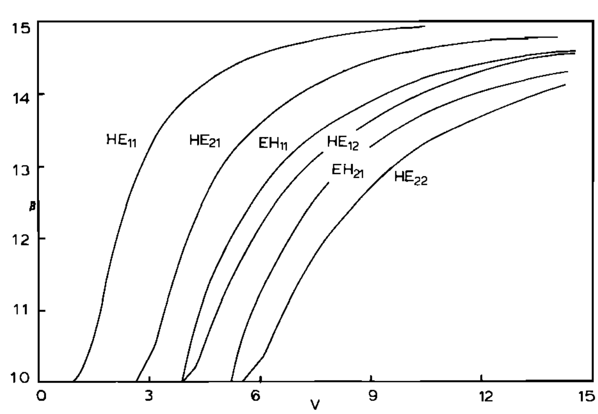


Fig. 2. Characteristics of a dielectric rod with $\epsilon_r = 2.25$.

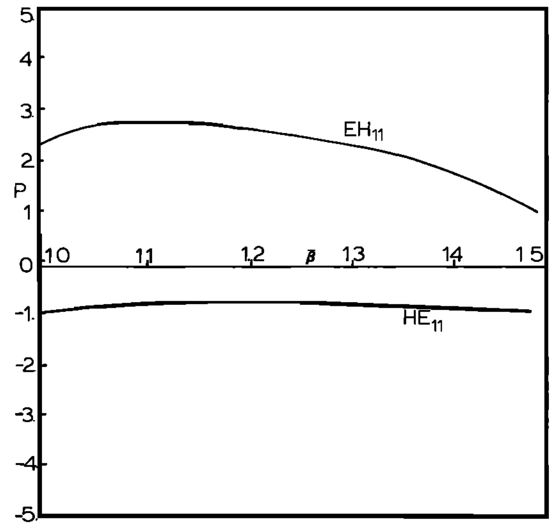
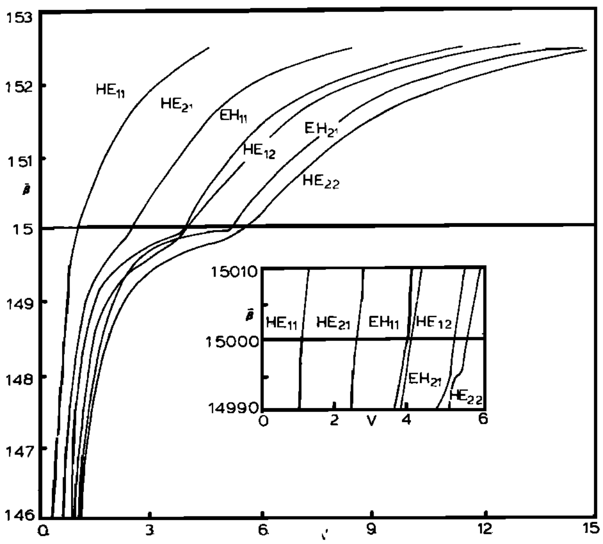


Fig. 3. Amplitude coefficient ratios for the HE_{11} and EH_{11} modes of a rod with $\epsilon_r = 2.25$.

The characteristics of a cladded fiber with $\epsilon_{r1} = 2.341$, $\epsilon_{r2} = 2.25$, $\epsilon_{r3} = 1$ and $r_2/r_1 = 5$ are shown in Figure 4. Figure 4 (bottom) shows the characteristics in the cladding mode region only. In the inset to Figure 4 (top) the behavior of $\bar{\beta}$ in the neighborhood of $1.499 < \bar{\beta} < 1.501$ is examined. It is observed that the characteristics of the HE_{12} and EH_{11} modes get very close to each other [see also *Yip and Huang*, 1975] near $V = 3.82$, but do not cross each other. A similar phenomenon takes place between the HE_{22} and EH_{21} modes near $V = 5.1$. *Clarricoats and Chan* [1973] and *Kuhn* [1974] have considered such points as being crossovers, but our analytical investigation showed that no crossover can exist between HE_{nm} and EH_{nm} modes. What then is the reason for such strange behaviour? A quick examination of (4) reveals that in the neighborhood of $V = 3.82$ (or 5.1), $J_n(x) \simeq 0$; $n = 1$ (or 2), or equivalently η_1 becomes very large. Therefore, G_1 should be very small and at the same time δ happens to be small too (G_2 small). Consequently the two roots of (4) become approximately equal, both satisfying $J_n(x) \simeq 0$.

The amplitude coefficient ratios for the HE_{11} and EH_{11} modes are illustrated in Figure 5. We observe that P is always negative for the HE_{11} mode, while for the EH_{11} mode it varies from $-\infty$ to $+\infty$ in the cladding mode region. In the core mode region $(1.5 < \bar{\beta} < 1.53)$, however, P is positive



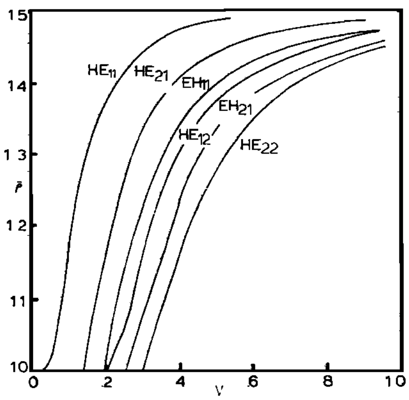


Fig. 4. Characteristics of a cladded fiber with $\epsilon_{r1} = 2.341$, $\epsilon_{r2} = 2.25$, $\epsilon_{r3} = 1$, and $r_2/r_1 = 5$.

for the EH_{11} mode approaching +1 at high frequencies and it is negative for the HE_{11} mode approaching −1 as frequency increases. It can be analytically verified that these properties hold for all modes in the core mode region.

The dispersion characteristics for a tube with $\epsilon_{r1} = \epsilon_{r3} = 1$, $\epsilon_{r2} = 2.25$ and $r_2/r_1 = 2$ are shown in Figure 6. The amplitude coefficient ratios for the EH_{11} and HE_{11} modes are plotted in Figure 7. The ratio P remains positive for the EH_{11} mode, but it varies from $-\infty$ to $+\infty$ for the HE_{11} mode. The analysis of the coefficient P reaffirms the previously stated point that the sign of P cannot be utilized for the designation of hybrid modes.

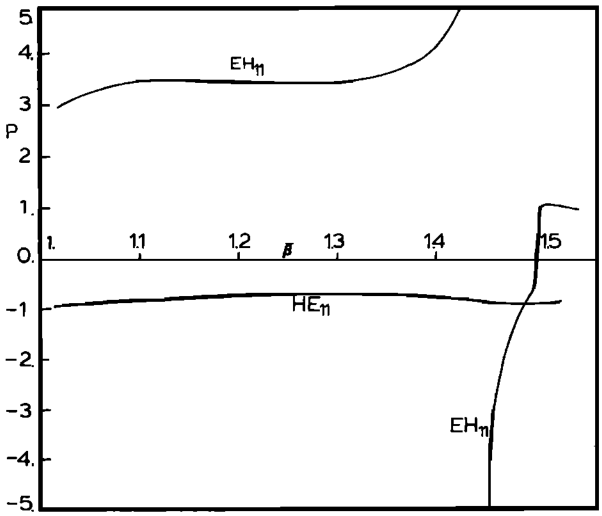


Fig. 5. Amplitude coefficient ratios for the HE_{11} and EH_{11} modes of a cladded fiber with $\epsilon_{r1} = 2.341$, $\epsilon_{r2} = 2.25$, $\epsilon_{r3} = 1$, and $r_2/r_1 = 5$.

5. DISCUSSION

The manner in which the characteristic equation can be split is not unique. For example, (5) which was considered to be quadratic in η_1 may likewise be regarded to be quadratic in η_6. Solving (5) for η_6 yields

$$\eta_6 = (1/2\mu_{r3}\epsilon_{r3})\{(\mu_{r1}\epsilon_{r3} + \mu_{r3}\epsilon_{r1})\eta_1 \pm [(\mu_{r1}\epsilon_{r3} - \mu_{r3}\epsilon_{r1})^2\eta_1^2 + 4\mu_{r3}\epsilon_{r3}n^2\bar{\beta}^2(1/x^2 + 1/w^2)^2]^{1/2}\} \tag{14}$$

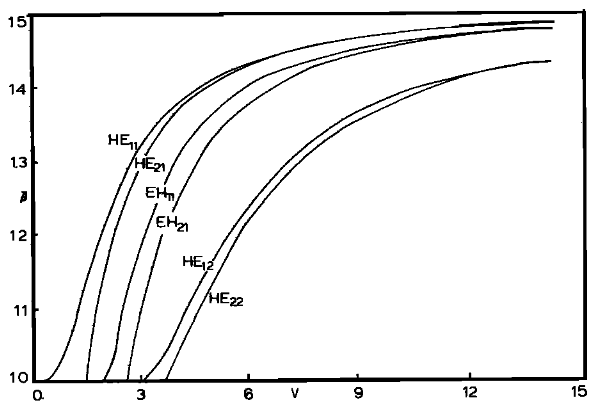


Fig. 6. Characteristics of a dielectric tube with $\epsilon_{r1} = \epsilon_{r3} = 1$, $\epsilon_{r2} = 2.25$, and $r_2/r_1 = 2$.

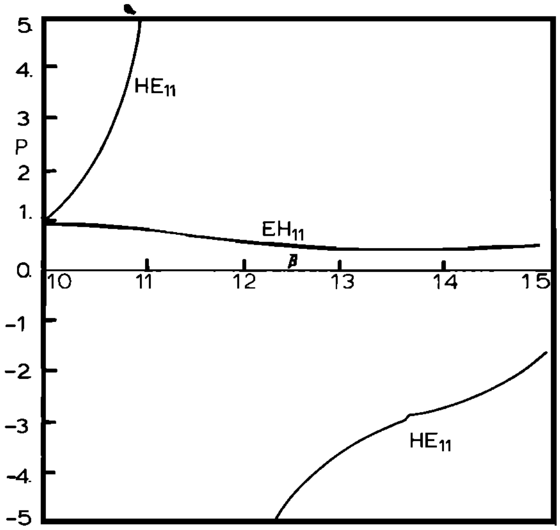


Fig. 7. Amplitude coefficient ratios for the HE_{11} and EH_{11} modes of a tube with $\epsilon_{r1} = \epsilon_{r3} = 1$, $\epsilon_{r2} = 2.25$, and $r_2/r_1 = 2$.

Does (14) give a mode classification different from that of (8)? To answer this question we derive the corresponding expression for the amplitude coefficient ratio. From (9) and (14),

$$P = [-1/2n\bar{\beta}^2 \mu_{r3}(1/x^2 + 1/w^2)]\{(\mu_{r1}\epsilon_{r3} - \mu_{r3}\epsilon_{r1})\eta_1 \pm [(\mu_{r1}\epsilon_{r3} - \mu_{r3}\epsilon_{r1})^2 \eta_1^2 + 4\mu_{r3}\epsilon_{r3} n^2 \bar{\beta}^2 (1/x^2 + 1/w^2)^2]^{1/2}\} \quad (15)$$

It is evident that in (15), $P > 0$ for a $-$ sign and $P < 0$ for a $+$ sign. On the other hand, $P < 0$ corresponds to HE and $P > 0$ corresponds to EH modes. Thus, (14) with a $+$ sign represents HE and with a $-$ sign EH modes. In other words, (14) with a $+$ sign is equivalent to (8) with a $-$ sign and vice versa.

In a three-layer problem one faces a more complicated situation. Here we confine ourselves to cladded fibers which are of particular interest to us. Using large argument approximations, given by (A8)-(A9), in (13), it is readily found that the coefficient P approaches $+1$ for EH and -1 for HE modes at frequencies far from cutoff. Thus, if the dispersion equation of a cladded fiber is separated in a manner different from (11) and the two new equations are such that each represents a set of continuous curves in the $(\omega - \beta)$ plane, and moreover, if the corresponding expression for P approaches $+1$ for EH and -1 for HE modes at frequencies far from cutoff, these new equations will necessarily yield the same mode classification as (11). It is thus concluded that, although the equation representing one class of modes may not have a unique form, the classification of modes is unique. A similar analysis can also be carried out for a dielectric tube.

6. CONCLUSIONS

A new scheme for the classification of hybrid modes in cylindrical dielectric waveguides has been proposed. Two separate equations, one representing EH and the other HE modes, have been derived. This new scheme is precise, well defined, universal, and yields a unique classification. It has been shown that no crossovers exist between HE and EH modes of the same order of azimuthal variation. It has also been shown that the amplitude coefficient ratio cannot be used for hybrid mode designations in three-layer structures. Dispersion curves of several lower-order modes for a rod, a cladded fiber and a tube with some specified parameters have been presented.

APPENDIX A

Introducing

$$\nu_i = \begin{cases} 1, & \mu_{ri}\epsilon_{ri} > \bar{\beta}^2 \\ -1, & \mu_{ri}\epsilon_{ri} < \bar{\beta}^2 \end{cases}, \quad i = 1,2,3$$

the Z_n functions are summarized in Table 1. In Table 1, J_n and Y_n are the Bessel and I_n and K_n are the modified Hankel functions. The quantities η_i $(i = 1,2,\ldots,6)$, ξ, A, and B are defined as

$$\eta_1 = \nu_1 Z'_{n1}(x)/xZ_{n1}(x)$$

$$\eta_2 = \nu_2 Z'_{n2}(u_1)/u_1 Z_{n2}(u_1)$$

$$\eta_3 = \nu_2 Z'_{n3}(u_1)/u_1 Z_{n3}(u_1)$$

$$\eta_4 = \nu_2 Z'_{n2}(u_2)/u_2 Z_{n2}(u_2)$$

$$\eta_5 = \nu_2 Z'_{n3}(u_2)/u_2 Z_{n3}(u_2)$$

TABLE 1. Definitions of functions Z_n.

Z_{n1}		Z_{n2}		Z_{n3}		Z_{n4}
$\nu_1 = 1$	$\nu_1 = -1$	$\nu_2 = 1$	$\nu_2 = -1$	$\nu_2 = 1$	$\nu_2 = -1$	$\nu_3 = -1$
J_n	I_n	J_n	I_n	Y_n	K_n	K_n

$$\eta_6 = \nu_3 Z'_{n4}(w)/wZ_{n4}(w)$$

$$\xi = \nu_2 \frac{Z_{n2}(u_2)}{Z_{n2}(u_1)} \nu_2 \frac{Z_{n3}(u_1)}{Z_{n3}(u_2)},$$

$$A = n\bar{\beta}(\nu_1/x^2 - \nu_2/u_1^2)$$

$$B = n\bar{\beta}(\nu_2/u_2^2 - \nu_3/w^2)$$

To prove that $\delta > 0$, we may write

$$G_1 G_3 = (ad - bc)(a'd' - b'c') = (ad' - cb') \cdot (da' - bc') - (db' - bd')(ac' - ca') \tag{A1}$$

Furthermore,

$$db' - bd' = \mu_{r1}\mu_{r2}\epsilon_{r2} T$$

$$ac' - ca' = \epsilon_{r1}\epsilon_{r2}\mu_{r2} T \tag{A2}$$

with

$$T = A[(\epsilon_{r2}\Delta_2 - \epsilon_{r3}\Delta_5)(\mu_{r2}\Delta_2 - \mu_{r3}\Delta_5) - B^2(\xi - 1)^2] + \mu_{r2}\epsilon_{r2}B\Delta_4$$

Using (A1) and (A2) in $\delta = G_2^2 - 4G_1G_3$ we obtain

$$\delta = [(ad' - cb') - (da' - bc')]^2 + 4\mu_{r1}\epsilon_{r1}(\mu_{r2}\epsilon_{r2}T)^2 \tag{A3}$$

Although the square root of δ exists, this is not sufficient for (11) with either sign to have continuous slope in the $(\omega - \beta)$ plane, unless δ does not become zero. If δ is to be zero at some point (ω_0, β_0), then from (11) and (A3),

$$\eta_1 + G_2/2G_1 = 0 \tag{A4}$$

$$db' - bd' = 0 \tag{A5}$$

$$(ad' - cb') - (da' - bc') = 0 \tag{A6}$$

Using (A5) and (A6) in (A4) yields

$$x^2\eta_1 + w^2\eta_6 = 0 \tag{A7}$$

Here, there are three independent equations, (A5) to (A7), with two unknowns ω_0, β_0. Further, (A7) is independent of μ_{r2} and ϵ_{r2}, while (A5) and (A6) depend on them both explicitly and implicitly in the arguments of η_2 to η_5. Hence, any solution of (A5) and (A6) which depends on μ_{r2} and ϵ_{r2} does not necessarily satisfy (A7). It is then concluded that δ does not generally become zero. In some special cases $\delta = 0$ gives rise to obvious contradictions. For example, in the case of a dielectric tube with $\mu_{r1}\epsilon_{r1} = \mu_{r3}\epsilon_{r3}$, (A7) becomes

$$xI_{n-1}(x)/I_n(x) + wK_{n-1}(w)/K_n(w) = 0$$

which is not true.

The large argument approximations are:

$$I_n(t) \rightarrow e^t/(2\pi t)^{1/2}, \quad t \rightarrow \infty \tag{A8}$$

$$K_n(t) \rightarrow e^{-t}/(2\pi t)^{1/2}, \quad t \rightarrow \infty \tag{A9}$$

REFERENCES

Beam, R. E., M. M. Astrahan, W. C. Jakes, H. M. Wachowski, and W. L. Firestone (1949), Dielectric tube waveguides, *Rep. ATI* 94929, chap. 5, Northwestern University, Evanston, Illinois.

Clarricoats, P. J. B. (1961), Propagation along unbounded and bounded dielectric rods, *Proc. Inst. Elec. Eng.*, *108C*, 170-176.

Clarricoats, P. J. B., and K. B. Chan (1973), Propagation behaviour of cylindrical dielectric-rod waveguides, *Proc. Inst. Elec. Eng.*, *120*(11), 1371-1378.

Kharadly, M. M. Z., and J. E. Lewis (1969), Properties of dielectric tube waveguides, *Proc. Inst. Elec. Eng.*, *116*(2), 214-224.

Kuhn, M. H. (1974), The influence of the refractive index step due to the finite cladding of homogeneous fibers on the hybrid properties of modes, *Arch. Elek. Übertrag.*, *10*, 393-401.

Snitzer, E. (1961), Cylindrical dielectric waveguide modes, *J. Opt. Soc. Amer.*, *51*, 491-498.

Yip, G. L., and J. C. Y. Huang (1975), Mode conversion due to a localized inhomogeneity in a cladded fiber, paper presented at Colloquium on Optics of Guided Waves, Paris, France.