Three topics in gauge/gravity duality

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Abstract

The gauge/gravity duality is extensively studied in recent years, as well as its applications to large N QCD and condensed matter physics. In this thesis, we study the gravity duals of three different gauge theories. The first gauge theory has $\mathcal{N} = 1$ supersymmetry and is conformal, we study the spectrum on its gravity dual solution, i.e., the supergravity modes in anti de Sitter (AdS) space on a 5D Sasaki-Einstein manifold $Y^{p,q}$. The second gauge theory has running couplings and is not conformal. We study the geometric transition as it happens at the far infrared region, which is at the bottom of the cascading RG flow, between resolved conifold with wrapped D5 branes and deformed conifold with only fluxes but no branes. Lastly, we try to find the gravity dual of a certain finite temperature gauge theory which is neither conformal nor supersymmetric.

Résumé

Depuis ces dernières années, la dualité entre les théories de jauge et les théories gravitationnelles ainsi que ses applications dans la mécanique quantique chromodynamique et le domaine de la matière condensée ont été explorées en profondeur. Dans ce mémoire, nous étudions l'homologue gravitationnel de trois différentes théories de jauge. La première théorie possède une supersymétrie de type $\mathcal{N} = 1$ ainsi qu'une symétrie conforme. Nous étudions son spectre à l'aide de son homologue gravitationnelle, i.e., en analysant les modes supersymétriques d'un espace anti de Sitter (AdS) sur une 5-variété de Sasaki-Einstein $Y^{p,q}$. La seconde théorie n'est pas conforme à cause de la variation des constantes de couplage. Nous étudions alors la transition géométrique qui parvient dans la région infrarouge qui se situe au bas du flot de renormalisation cascadant. La géométrie évolue d'une variété conique résolue enveloppée de D5-branes à une variété conique déformée en présence de flux, mais sans aucune membrane enveloppée. Finalement, nous tentons de trouver l'homologue gravitationnel d'une certaine théorie de jauge à température finie qui n'est pas conforme ni supersymétrique.

DEDICATION

To my beloved parents.

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Statement of Originality

This thesis is based on my research done in collaboration with Keshav Dasgupta, Sheldon Katz, Niky Kamran, Alberto Enciso, Radu Tatar, Sachindeo Vaidya, Mohammed Mia, Jihye Seo, Paul Franche and the results presented here constitute work that appeared in the following articles:

• **Fang Chen**, Keshav Dasgupta, Alberto Enciso, Niky Kamran, Jihye Seo, *"On the Scalar Spectrum of the $Y^{p,q}$ Manifolds "*, JHEP 1205 (2012) 009, 41pp, arXiv:1201.5394 [hep-th].

• **Fang Chen**, Keshav Dasgupta, Paul Franche, Sheldon Katz, Radu Tatar, *"Supersymmetric Configurations, Geometric Transitions and New Non-Kahler Manifolds"*, Nucl.Phys. B852 (2011) 553-591, 90pp, arXiv:1007.5316 [hep-th].

• Mohammed Mia, **Fang Chen**, Keshav Dasgupta, Paul Franche, Sachindeo Vaidya, *""Non-Extremality, Chemical Potential and the Infrared limit of Large N Thermal QCD"*, to be appear in PRD, 56pp, arXiv:1202.5321 [hep-th].

Except the scalar spectrum of $Y^{p,q}$ in sections 2.3.3 and 2.3.4 of chapter 2 and the numerical results in section 4.2.1 of chapter 4, I have worked out all the details of the calculations and wrote substantial part of the text for each of the above papers.

Other papers such as

• **Fang Chen**, Keshav Dasgupta, Paul Franche (McGill U.), Radu Tatar, *"Toward the Gravity Dual of Heterotic Small Instantons"*, Phys.Rev. D83 (2011) 046006, 45pp, arXiv:1010.5509 [hep-th].

• James M. Cline, Andrew R. Frey, **Fang Chen**, *"Metastable dark matter mechanisms for INTEGRAL 511 keV γ rays and DAMA/CoGeNT events"*, Phys.Rev. D83 (2011) 083511, 27pp, arXiv:1008.1784 [hep-ph].

• **Fang Chen**, James M. Cline, Anthony Fradette, Andrew R. Frey, Charles Rabideau, *"Exciting dark matter in the galactic center"*, Phys.Rev. D81 (2010) 043523, 16pp, arXiv:0911.2222 [hep-ph].

• **Fang Chen**, James M. Cline, Andrew R. Frey, *"Nonabelian dark matter: Models and constraints"*, Phys.Rev. D80 (2009) 083516, 28pp, arXiv:0907.4746 [hep-ph].

• **Fang Chen**, James M. Cline, Andrew R. Frey, *"A New twist on excited dark matter: Implications for INTEGRAL, PAMELA/ATIC/PPB-BETS, DAMA"*, Phys.Rev. D79 (2009) 063530, 9pp, Phys.Rev. D79 (2009) 063530.

completed during my PhD years will not be used in this thesis.

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Chapter 1

Introduction

Much of what is known about the phases of Quantum chromodynamics (QCD) comes from a variety of techniques, each of which has its own limitations. Perturbative computations can probe a large part of the parameter space of the theory, allowing one to deal with varying numbers of colors N, and flavors N_f. However, these results are valid only at temperatures well above the deconfinement temperature T_c, and at large values of the baryon number chemical potential μ, when the QCD coupling is small. Lattice gauge theory, which provides a rigorous non-perturbative starting point for QCD, has its limitations too. It is difficult to incorporate realistic quark masses, and results are limited to certain regions of T, and μ very small.

In recent years, there has been a considerable studies of using gauge/gravity duality to understand the behavior of $U(N)$ gauge theories at finite temperature. In this chapter we will briefly introduce this duality for three different gauge theories.

1.1 $\mathcal{N} = 4$ conformal field theory and AdS correspondence

In 1997, Juan Maldacena first brought up the idea of large N duality between $\mathcal{N} = 4$ super conformal field theory and supergravity on $AdS \times S^5$ [1]. It was then developed further in [2] and [3] where the states mapping was done. The mass spectrum of type

IIB supergravity on $AdS_5 \times S^5$ was found in [4]. In the following we will briefly describe this duality.

Let us consider a stack of N coincident D-branes. D-branes are sub-spaces where open strings can end. When there are N D-branes, the two end points of each open string can lie on different branes or the same brane, which means there are N^2 extra degrees of freedom. This information is encoded in the Chan-Paton factors. In an oriented string theory, such as type IIA and IIB, the excitations of the open strings on the branes form an $SU(N)$ gauge group, while in an unoriented string theory, such as type I, they form an $SO(N)$ gauge group. D-branes are also BPS states in string theory, meaning they break half of the original supersymmetry.

Now let us look at type IIB theory on 10D flat space-time which has 32 super charges. A stack of N coincident D3 branes will break half of the supersymmetry and leave $\mathcal{N} = 4$ SUSY on the brane world volume. It also turns out that the $SU(N)$ gauge group living on these branes is conformal. When $g_s N \ll 1$, these D3 branes can be considered as perturbations on the flat background. From the world sheet point of view, this is because the effective loop expansion parameter for the open strings is $g_s N$ due to the Chan-Paton factors, not g_s. Thus, under the assumption that $g_s N \ll 1$, the system can be described by the following action,

$$S = S_{brane} + S_{bulk} + S_{interaction}. \tag{1.1}$$

Where S_{brane} is the action of the $SU(N)$ gauge theory living on the brane plus some higher order corrections, S_{bulk} is the 10D supergravity action plus some higher order corrections and $S_{interaction}$ is the action of the interaction between the bulk modes and brane modes. Now if we take low energy limit, i.e. below energy $1/l_s$, where l_s is the string length, only massless modes are generated and the interactions between these two modes can be neglected. Thus there are only two simplified terms in the effective action: $S_{supergravity}$ describes the closed string modes forming a gravity multiplet living in the flat 10D space-time and S_{gauge} describes the open string modes forming an $\mathcal{N} = 4$ $SU(N)$ vector multiplet living on the $3+1$-dimension brane volume.

Now we consider the back reaction of the D3 branes on the 10D space-time. In the supergravity limit it is described by the well known black brane solution

$$ds^2 = H^{-1/2}(r)dx^\mu dx_\mu + H^{1/2}(r)dx^m dx_m \tag{1.2}$$

with the self-dual five form flux $F_5 = (1 + *_{10})dH(r)^{-1}dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$ and other fields vanishing. H is called the warp factor and $H = 1 + \frac{L^4}{r^4}$, where $L^4 \sim g_s N$. The curvature singularity and horizon of the above solution turn out to be at the same place where $r = 0$. These solutions are called extremal solutions while others having different locations of singularity and horizons as non-extremal solutions. When $L \gg 1$, i.e. $g_s N \gg 1$, perturbative classical gravity solution is a good approximation, but it breaks down when $L \ll 1$, or $g_s N \ll 1$. When we take the low energy limit, the closed string excitations decouple from the near horizon geometry, leaving supergravity in the large r flat region and the near horizon geometry in the small r region. The near horizon metric can be obtained by taking the $r \mapsto 0$ limit, which is

$$ds^2 = \frac{r^2}{L^2}dx^4 + \frac{L^2}{r^2}dr^2 + L^2 d^2 S_5. \tag{1.3}$$

The first two terms are the AdS_5 metric, and the last term is the S^5 metric. The curvature radius is L for both of them. In fact $AdS_5 \times S^5$ is also a solution of the type IIB supergravity theory with $F_5 = -4(1 + *_{10})\frac{L^4}{r^5}dr \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$ and other fields zero. When we integrate this F_5 over S^5 we get

$$\int_{S^5} F_5 = N \tag{1.4}$$

which is exactly the same as the number of D3 branes. However, this does not mean there is delta-function singularity in F_5, in other words there is no branes in this solution, since this S^5 has constant radius and does not vanish anywhere. So we again get two non-interacting parts from this point of view: the supergravity in 10D flat space-time and the type IIB $AdS \times S^5$ solution.

We see that the same system has two different descriptions at different limits of the coupling strength $g_s N$. Varying g_s switches from one region to the other. We can further argue that N must be much greater than 1 to get both limits. This is because to keep $g_s N \gg 1$ either g_s or N must be greater than 1, if it is the former, we can do an S-duality to make $g_s < 1$, thus it must be that $N \gg 1$. Now, we can make an assumption that the two descriptions are dual to each other in the limit $N \gg 1$. Neglecting the 10D supergravity which appears in both descriptions, we have a weakly/strongly coupled gauge theory on the branes, and a strongly/weakly coupled type IIB supergravity on $AdS_5 \times S_5$ geometry with fluxes but no branes. The duality means the excitations of the strings on the branes can be mapped to the excitations of the metric and flux.

This is the simplest case of gauge/gravity duality since the gauge theory here is conformal and has $\mathcal{N} = 4$ SUSY. In the real world we have not observed SUSY in any experiment, so we would prefer to study a gauge theory with less or no SUSY.

1.2 $\mathcal{N} = 1$ conformal gauge theories and their gravity dual

Instead of flat space-time, we can put D3 branes in a $M_4 \times K_6$ manifold, where M_4 is the 4D Minkovski space and K_6 is a 6D Ricci flat internal manifold. By the same argument as in the previous section we can get dualities between type IIB string theory on different geometries $AdS \times k_5$, where k_5 is the base of K_6, and different conformal gauge theories. For example, k_5 can be a Sasaki-Einstein manifold of which S^5, S^5/Z_2 and $T^{1,1}$ are the simplest and most widely studied cases. For S^5/Z_2 the gauge group is $SU(N)$ and has $\mathcal{N} = 2$ SUSY, for $T^{1,1}$ the gauge group is $SU(N) \times SU(N)$ and has $\mathcal{N} = 1$ SUSY.

Now let us look at the $T^{1,1}$ case more closely. The cone over $T^{1,1}$ is called conifold, it is a complex manifold. The conifold can be embedded in $\mathbb{C}^4$, via

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0, \tag{1.5}$$

where z_1, z_2, z_3 and z_4 are the complex coordinates of $\mathbb{C}^4$. It is easy to see that there is a singularity at $z_i = 0$. The embedding equation (1.5) can also be written as

$$\det Z_{ij} \equiv \det\begin{pmatrix} z_1 + iz_2 & -z_3 + iz_4 \\ z_3 + iz_4 & z_1 - iz_2 \end{pmatrix} = 0 \tag{1.6}$$

such that $Z_{ij} = \frac{1}{2}\sum_n \sigma^n_{ij} z_n$, and σ^n are Pauli matrices for $n = 1, .., 3$ and $\sigma^4 = i\mathbb{1}$. With a different parametrization we see that the determinant equation is automatically satisfied, i.e.:

$$\det Z_{ij} = \det\begin{pmatrix} z_1 + iz_2 & -z_3 + iz_4 \\ z_3 + iz_4 & z_1 - iz_2 \end{pmatrix} = \det\begin{pmatrix} A_1 B_1 & A_1 B_2 \\ A_2 B_1 & A_2 B_2 \end{pmatrix} = 0 \tag{1.7}$$

where A_i and B_j, $i, j = 1, 2$ are complex numbers. We only need six real degrees of freedom to describe the conifold, but there are eight degrees of freedom in A_i and B_j. To eliminate the extra two degrees of freedom we notice that the combination $A_i B_j$ is invariant under the following transformation:

$$A_i \mapsto \lambda A_i, \;\; B_j \mapsto \frac{1}{\lambda} B_j \tag{1.8}$$

where λ is a complex number. To fix this gauge we first fix the magnitude by demanding

$$|A_1|^2 + |A_2|^2 - |B_1|^2 - |B_2|^2 = 0 \tag{1.9}$$

and then fix the phase by the identification:

$$A_i \;\sim\; e^{\alpha} A_i, \;\; B_j \;\sim\; e^{-\alpha} B_j \tag{1.10}$$

With these two conditions A_i and B_j parametrize the conifold correctly. On the other hand, (1.5) can be rewritten in terms of real coordinates $z_i = x_i + iy_i$, $i = 1, .., 4$ as,

$$\sum_i x_i y_i = 0, \;\; \sum_i (x_i^2 - y_i^2) = 0. \tag{1.11}$$

The coordinates x_i describe an S^3 for any value of y_i, and y_i are orthogonally fibered to them.

The base of the conifold $T^{1,1}$ can be found by intersecting the conifold with an S^7 of radius r,

$$\sum_i^4 |z_i|^2 = r^2 \tag{1.12}$$

From (1.11) and (1.12) we find that the base has an S^3 parametrized by x_i and an S^2 fibered over it parametrized by y_i. Since the fibration is trivial, the base $T^{1,1}$ has a topology of $S^3 \times S^2$.

Now we put N coincident D3 branes at the tip of the conifold. The gauge theory on the branes is a $\mathcal{N} = 1$ supersymmetric gauge theory with a gauge group $SU(N) \times SU(N)$. An easy way to see this is to do a T-duality to type IIA theory. With a properly chosen T-dual direction, the conifold becomes two orthogonal NS5 branes, and the D3 branes becomes two stacks of D4 branes suspended between them on each sides. On each stack of D4 branes these is an $SU(N)$ gauge group, so in total we get $SU(N) \times SU(N)$ gauge group. The field contents are A_i and B_j which are promoted from coordinates to chiral super fields. The superpotential for this theory is

$$W = \lambda \mathrm{tr}(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1) \tag{1.13}$$

and the D-term condition for supersymmetry is

$$D = -\frac{1}{2} \sum [q_{A_i} A_i^* A_i + q_{B_j} B_j^* B_j] + \xi \tag{1.14}$$

where $q_{A_i} = 1$ and $q_{B_j} = -1$ are the charges of A_i and B_j under the $U(1)$ symmetry. When $\xi = 0$, the D-term condition is exactly the same as the defining equation of the conifold.

The back reaction of the branes on the geometry is encoded in the supergravity solution:

$$\begin{aligned} ds^2 &= H^{-1/2}(r)dx^\mu dx_\mu + H^{1/2}(r)\Big[dr^2 + r^2\Big(\frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 \\ &+\frac{1}{6}(d\theta_1^2 + \sin\theta_1 d\phi_1^2) + \frac{1}{6}(d\theta_2^2 + \sin\theta_2 d\phi_2^2)\Big)\Big] \end{aligned} \tag{1.15}$$

with

$$F_5 = (1 + *_{10})dH(r)^{-1}dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \tag{1.16}$$

and other fields vanishing. $H(r)$ is the same as in (1.2). The metric in the square brackets is that of a conifold and can be written as

$$ds_6^2 = dr^2 + r^2\left(\frac{1}{9}(g^5)^2 + \frac{1}{6}\sum_{i=1}^{4}(g^i)^2\right) \tag{1.17}$$

where $g^i, i = 1, .., 5$ are the one forms given by

$$\begin{aligned} g^1 &= \frac{e^1 - e^3}{\sqrt{2}}, \quad g^2 = \frac{e^2 - e^4}{\sqrt{2}} \\ g^3 &= \frac{e^1 + e^3}{\sqrt{2}}, \quad g^4 = \frac{e^2 + e^4}{\sqrt{2}}, \quad g^5 = e^5 \\ e^1 &\equiv -\sin\theta_1\ d\phi_1, \quad e^2 \equiv d\theta_1 \\ e^3 &\equiv \cos\psi\ \sin\theta_2\ d\phi_2 - \sin\psi\ d\theta_2, \\ e^4 &\equiv \sin\psi\ \sin\theta_2\ d\phi_2 + \cos\psi\ d\theta_2, \\ e^5 &\equiv d\psi + \cos\theta_1\ d\phi_1 + \cos\theta_2\ d\phi_2. \end{aligned} \tag{1.18}$$

Taking the 'near horizon' limit the metric becomes $AdS_5 \times T^{1,1}$, which the same as $AdS_5 \times S^5$ is also a supergravity solution with only F_5 fluxes but no branes. The integration of F_5 over $T^{1,1}$ again matches the number of D3 branes N,

$$\int_{T^{1,1}} F_5 = N \tag{1.19}$$

Thus we find that the $\mathcal{N} = 1$ conformal $SU(N) \times SU(N)$ gauge theory is dual to the string theory on $AdS_5 \times T^{1,1}$ in the large N limit.

1.3 $\mathcal{N}=1$ non-conformal gauge theories and their gravity dual

Although till now we only considered D3 branes in the type IIB theory, we can also put other kinds of branes. In fact if we put additional M coincident D5 branes in the conifold case, making them wrap the vanishing S^2 cycle of the conifold, the gauge theory becomes $SU(N+M)\times SU(N)$ and it is no longer conformal. In this section we will discuss this case in detail because it is most closely related to large N QCD.

$D5$ branes wrapping the vanishing two cycle contribute to the D3 charges so they are also called fractional D3 branes. The gauge group $SU(N+M)\times SU(N)$ can be seen by the same T-duality in the previous section. After T-dualizing to type IIA theory, the D5 branes become only one segment of D4 branes suspending between the two NS5 branes, thus only one of the gauge group is enhanced. The $SU(M+N)$ sector has $2N$ effective flavor while the $SU(N)$ sector has $2(M+N)$ effective flavors. Thus it is dual to the $SU(N)\times SU(N-M)$ gauge theory under Seiberg duality. Under a series of such dualities which is called cascading, at the far IR region the gauge theory can be described by $SU(M)\times SU(K)$ group, where $N=lM+K$, and l, k $(0\leqslant K<M)$ are positive integers. Now the number of 'actual' D3 branes N does not make much sense any more, it just describes the effective number of D3 branes at some energy scale. So when one says there are N D3 branes and M D5 branes, one really refers to a certain energy scale where the effective D3 branes is N. If we take $K=0$, at the bottom of the cascade, we are left with $\mathcal{N}=1$ $SU(M)$ strongly coupled gauge theory which looks very much like strongly coupled supersymmetric QCD.

Due to the strong coupling at the IR, the superpotential of the gauge theory receives non-perturbative corrections [5] and becomes

$$W=\lambda N_{ij}N_{kl}\epsilon^{ik}\epsilon^{jl}+(M-1)\Big[\frac{2\Lambda^{3M+1}}{N_{ij}N_{kl}\epsilon^{ik}\epsilon^{jl}}\Big]^{\frac{1}{M}} \tag{1.20}$$

where $N_{ij}=A_iB_j$. The solution for a supersymmetric vacuum is

$$\frac{1}{2}N_{ij}N_{kl}\epsilon^{ik}\epsilon^{jl}=\det N_{ij}=\Big[\frac{\Lambda^{3M+1}}{(2\lambda)^{M-1}}\Big]^{\frac{1}{M}}. \tag{1.21}$$

Comparing to the defining equation of the conifold (1.7) we find that it is no longer a regular conifold, but it is deformed in such a way that its S^3 is finite at $r = 0$. This is called a deformed conifold.

One of the supergravity solution with D3 branes and D5 branes is the following

$$\begin{aligned} ds^2 &= H^{-1/2}(r)dx^\mu dx_\mu + H^{1/2}\Big[dr^2 + r^2\Big(\frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 \\ &\quad + \frac{1}{6}(d\theta_1^2 + \sin\theta_1 d\phi_1^2) + \frac{1}{6}(d\theta_2^2 + \sin\theta_2 d\phi_2^2)\Big)\Big] \end{aligned} \tag{1.22}$$

with the self-dual five form flux in the same form as before and

$$B_2 = \frac{3M\alpha'}{2}\log(r)\,\omega_2, \quad F_3 = \frac{M\alpha'}{2}\omega_3 \tag{1.23}$$

where

$$H(r) = 1 + \frac{1}{r^4}(g_s N + \frac{3}{2\pi}(g_s M)^2 \log\frac{r}{r_0} + \frac{3}{8\pi}(g_s M)^2) \tag{1.24}$$

with r_0 a constant of integration and

$$\omega_2 = \frac{1}{2}\Big(\sin\theta_1 d\theta_1 d\phi_1 - \sin\theta_2 d\theta_2 d\phi_2\Big), \quad \omega_3 = (d\psi - \cos\theta_1 d\phi_1 - \cos\theta_2 d\phi_2)\wedge\omega_2 \tag{1.25}$$

As indicated in the gauge theory side, the internal manifold of the dual gravity solution must not be the regular conifold. So we can not simply take the 'near horizon' limit to get the gravity dual. It was proposed in [6] that the gravity dual of this brane configuration is given by the deformed conifold.

The base of the deformed conifold has the same topology $S^2 \times S^3$ as that of the regular conifold. The unique Kähler metric on the deformed conifold that has vanishing Ricci curvature is

$$\begin{aligned} ds_6^2 &= \frac{1}{2}a^{4/3}K(\rho)\Big[\frac{1}{3K^3(\rho)}\left(d\rho^2 + (g^5)^2\right) + \cosh^2\left(\frac{\rho}{2}\right)\left[(g^3)^2 + (g^4)^2\right] \\ &\quad + \sinh^2\left(\frac{\rho}{2}\right)\left[(g^1)^2 + (g^2)^2\right]\Big] \end{aligned} \tag{1.26}$$

where a is a constant, $g^i, i = 1, .., 5$ are given in (1.18) and

$$K(\rho) = \frac{(\sinh(2\rho) - 2\rho)^{1/3}}{2^{1/3}\sinh\rho}. \tag{1.27}$$

When $a = 0$, (1.26) recovers the metric of the regular conifold (1.17), so a characterizes the deformation of the cone. In fact in terms of complex coordinates, the deformed conifold equation can be written in a way similar to (1.5) as

$$z_1^2 + z_2^2 + z_1^3 + z_2^3 = a \tag{1.28}$$

The difference between the deformed cone and the regular cone is that the regular cone has a $U(1)$ symmetry under which all z_1 in (1.5) are rotated by the same phase, while the deformed cone only has a Z_2 symmetry under which z_i in (1.28) flip signs. This Z_2 symmetry corresponds exactly to the Z_2 symmetry on the gauge theory side, thus it strongly indicates that the correct gravity dual should be the deformed conifold.

Furthermore, observe that with a change of coordinates

$$r^3 = a^2 e^{\rho}, \tag{1.29}$$

for large ρ, the deformed cone metric becomes that of a regular cone (1.17). This implies that at large ρ, the gravity dual is approximately the near horizon geometry of (1.22). This approximation is very useful when we come to the finite temperature fields theory calculations.

Aside from Kähler metrics one can put lots of other metrics on the deformed conifold, in the following chapters we will mainly deal with non-Kähler metric on it.

1.4 Organization of the thesis

In chapter 2, we study the spectra of supergravity modes in AdS_5 on a 5D space with Sasaki-Einstein metrics on $S^2 \times S^3$, given by the $Y^{p,q}$ class. We analyze the full scalar spectrum on these spaces and get both lower and upper bounds on the eigenvalues. We also briefly discuss various other new avenues such as non-commutative and dipole deformations as well as possible non-conformal extensions of these models.

In chapter 3, we first obtain a globally defined supergravity solution of the wrapped D5-branes on the two-cycle of the resolved conifold. And then we use it as a starting point for the geometric transition cycle. We show that the geometric transition is effectively a simple series of mirror transformations followed in between by a flop transition between two intermediate M-theory configurations with different G_2-structures.

In chapter 4, we try to find a non-extremal solution with warped resolved-deformed conifold background which is important to study the infrared limit of large N thermal QCD. We explicitly solve the supergravity equations of motion in the presence of the backreaction from the black-hole, branes and fluxes. The backreactions from the branes and the fluxes are comparatively suppressed to the order that we study. We also study the effect of switching on a chemical potential in the background and, in a particularly simplified scenario, compute the actual value of the chemical potential for our case.

Chapter 5 is summary and discussions.

Chapter 2

Scalar spectrum of the $Y^{p,q}$ manifolds

2.1 Introduction

The gravity dual of $\mathcal{N} = 1$ CFT has been studied earlier from many different perspectives starting with [7] where the associated CFT, endowed with a simple product gauge group and a simple quartic superpotential, appeared from N D3-branes placed at the tip of a conifold geometry. One way to change the gauge group and the superpotential structure is to change the underlying conifold geometry itself by either an orbifolding or an orientifolding action. A subsequent T-duality, mapping these actions to either the *interval* [8, 9] or the *brane-box* models [10, 11], then gives us simple ways to analyse the underlying $\mathcal{N} = 1$ CFTs.

An alternative way to change the gauge group and the superpotential structure is to change the Calabi-Yau condition of the conifold itself, namely, change the Kähler class and the complex structures so as to put different Ricci flat metrics on the conifold. Since there are infinite ways of doing it, there would exist infinite variations of the conifold that are all Calabi-Yau manifolds. All of these would lead to gravity duals of the form $AdS_5 \times Y^{p,q}$ where $Y^{p,q}$ are the so-called Sasaki-Einstein manifolds. These ideas, including the underlying gauge/gravity duality, were developed few years ago in [12, 13, 14, 15].

In this chapter we study spectrum of Sasaki-Einstein manifold $Y^{p,q}$, using spectral-theoretic methods, continuing the work of [16]. More precisely, we study the Laplacian operator of a $Y^{p,q}$ manifold, associated to its scalar spectrum, using the framework laid out in [16]. The authors of [16] analyzed the Cauchy problem, and presented a Fourier-type decomposition for the eigenfunction. In order to use spectral-theoretic methods, they used the Friedrichs extension of the Laplacian operator to rule out logarithmic singularities. This way a self-adjoint extension of unbounded symmetric operator could be determined. Our starting point, in this chapter, is to use this operator to study its eigenmodes.

The *lowest* eigenmodes of the Laplacian were first studied in [17] for $Y^{p,q}$, wherein they also tried to construct an AdS/CFT dictionary. This work was followed by [18] where they studied the lowest eigenmodes for more generic manifolds like the $L^{a,b,c}$ examples. An important progress in [17] was the realization that the Laplacian operator could be expressed in terms of a Heun type operator, whose lowest modes are easily computable. However, for higher modes not much progress has been made in the literature. Even numerical studies do not look simple. In [19], the spectrum is studied numerically for S^5 case, which is the simplest Sasaki-Einstein manifold in 5d, but an equivalent work for the $Y^{p,q}$ case is still lacking.

2.2 $Y^{p,q}$ geometry

The $Y^{p,q}$ metrics are Sasaki-Einstein and therefore a cone over them is Calabi-Yau. We start with the local metric

$$\begin{aligned} ds^2 &= \frac{1-cy}{6}(d\theta^2+\sin^2\theta d\phi^2)+\frac{1-cy}{2f(y)}dy^2+\frac{f(y)}{9(a-y^2)}(d\psi-\cos\theta d\phi)^2 \\ &\quad +\frac{2(a-y^2)}{1-cy}\Big[d\alpha+\frac{ac-2y+y^2c}{6(a-y^2)}(d\psi-\cos\theta d\phi)\Big]^2 \end{aligned} \tag{2.1}$$

where $f(y)=2cy^3-3y^2+a$. As in [13] one can show that $Ric=4g$ for all values of a and c therefore satisfying Einstein condition. For $c=0$ and $a=3$ the metric is exactly the local form of the standard metric on $T^{1,1}$. For $c\neq 0$ one can always rescale y ($y \to y/c$, and also $a \to a/c^2$, $f \to f/c^2$, etc) to set $c = 1$ which we will take in the following.

It is obvious that the first two terms give the metric of an S^2 for a fixed y, if the periodicity of θ and ϕ are π and 2π respectively. To study the (y, ψ) space one first requires

$$\begin{aligned} &1 - y > 0, \quad a - y^2 > 0 \\ &f = a - 3y^2 + 2y^3 \geqslant 0. \end{aligned} \tag{2.2}$$

In order for y to have solutions a must satisfy $0 < a < 1$. The negative solution of $f = 0$ and the smallest positive solution are denoted by y_- and y_+ respectively. Then y needs to take values between $y_- < y < y_+$, (so that all the terms in the metric come with positive sign). When $a = 1$ the metric (2.1) is the local round metric of S^5. If ψ has the period of 2π then (y, ψ) is topologically a 2-sphere[1] .

In order to have a compact manifold one takes the period of α to be $2\pi l$. Then $l^{-1}A$, where A is the last term in the second line of (2.1), becomes a connection on a $U(1)$ bundle over $S^2 \times S^2$ which puts constraints on A. In general such $U(1)$ bundles are completely specified topologically by the gluing on the equator of the two S^2 cycles, C_1 and C_2. These are measured by the corresponding Chern numbers in $H^2(S^2, \mathcal{Z}) = \mathcal{Z}$ which will be labeled as p and q. The Chern numbers are given by

[1] The range of y is taken to be $[y_-, y_+]$. This ensures that w (defined in (2.18)) is strictly positive in this interval and $r \geqslant 0$, vanishing only at the endpoints $y_\pm$. If we identify ψ periodically, the part of g_B (g_B is only defined in [16] but not in this chapter) given by

$$\frac{1 - cy}{2f(y)}\, dy^2 + \frac{f(y)}{9(a - y^2)}\, d\psi^2$$

describes a circle fibered over the interval (y_-, y_+), the size of the circle shrinking to zero at the endpoints. Remarkably, the (y, ψ) fibers are free of conical singularities if the period of ψ is 2π, in which case the circles collapse smoothly and the (y, ψ) fibers are diffeomorphic to a 2-sphere.

the integrals of $l^{-1}A/2\pi$ over C_1 and C_2, namely:

$$p \;=\; \frac{1}{2\pi l}\int_{C_1} A = \frac{y_- - y_+}{6y_-y_+}, \qquad q \;=\; \frac{1}{2\pi l}\int_{C_2} A = \frac{(y_- - y_+)^2}{9y_-y_+} \tag{2.3}$$

From their ratio $\frac{p}{q} = \frac{3}{2(y_+ - y_-)}$, it follows

$$a \;=\; \frac{1}{2} - \frac{p^2 - 3q^2}{4p^3}\sqrt{4p^2 - 3q^2}, \qquad l \;=\; \frac{q}{3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2}} \tag{2.4}$$

Metric (2.1) can be written in a canonical way if one makes the coordinate change

$$\alpha = -\beta/6 - c\psi'/6, \qquad \psi = \psi' \tag{2.5}$$

to (2.1). This converts (2.1) to the following metric:

$$\begin{aligned} ds^2 \;=\;& \frac{1-cy}{6}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{1-cy}{2f(y)}dy^2 + \frac{f(y)}{18(1-cy)}(d\beta + c\cos\theta d\phi)^2 \\ &+\frac{1}{9}(d\psi' - \cos\theta d\phi + y(d\beta + c\cos\theta d\phi))^2. \end{aligned} \tag{2.6}$$

The Killing vector

$$\frac{\partial}{\partial\psi'} = \frac{\partial}{\partial\psi} - \frac{1}{6}\frac{\partial}{\partial\alpha} \tag{2.7}$$

is globally well defined. For a generic value of a its orbit is not closed, in which case the Sasaki-Einstein metric is irregular. It is quasi-regular, if and only if $4p^2 - 3q^2 = m^2, m \in \mathbb{Z}$.

2.3 The spectrum of the $Y^{p,q}$ manifolds

After the brief discussion of the geometry of the $Y^{p,q}$ manifolds, let us come to the main analysis of this chapter: the study of scalar spectrum of these manifolds. We will start by analysing the solution of the Laplacian operator arising from the Fourier decomposition of functions as discussed earlier in [16]. However, as it will turn out, finding the exact solution of the eigenvalue problem in closed form does not seem feasible since the computation of the eigenvalues of the Laplacian boils down to the analysis of a one-dimensional differential operator (which we call S) of Heun type, which has *four* regular singular points. What we will do, therefore, is to find bounds for the eigenvalues of S, which will allow us to approximate the conformal dimensions of the theory. In subsection (2.4), we will study some examples of these modes and discuss cases that may take us beyond the scalar spectra.

2.3.1 Harmonic expansion on $Y^{p,q}$

We will follow the argument in [20] which gives the spectrum of Type IIB on $AdS_5 \times T^{1,1}$. The background solution in Type IIB is

$$ds^2 = \frac{r^2}{R^2}(-dx_0^2 + dx_i^2) + \frac{R^2}{r^2}dr^2 + R^2 ds^2_{Y^{p,q}} \tag{2.8}$$

with the self-dual 5-form flux $F_5 = (1+*)dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge d\left(\frac{r^4}{R^4}\right)$.

When Kaluza−Klein reducing this solution to AdS_5, we first have to compute the fluctuations of the 10-dimensional fields. The fluctuation of the gravitational fields are parametrized as

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} - \frac{1}{3}g_{\mu\nu}h^a_a, \quad \tilde{g}_{\mu a} = h_{\mu a}, \quad \tilde{g}_{ab} = g_{ab} + h_{ab} \tag{2.9}$$

where μ, ν denote the AdS_5 space time while a, b denote the internal space, and g denotes the background metric while h is the fluctuation.

Now we expand the fields $h_{\mu\nu}$, $h_{\mu a}$, h_{ab} and h^a_a into a complete set of harmonic functions on $Y^{p,q}$. With the de Donder and Lorentz-type gauge conditions $D^a h_{(ab)} = 0$

and $D^a h_{a\mu} = 0$ we have the following expansions[2] :

$$\begin{aligned} h_{\mu\nu}(x,y) &= \sum_{\{\lambda\}} H_{\mu\nu}^{\{\lambda\}}(x) Y^{\{\lambda\}}(y), \quad h_a^a(x,y) = \sum_{\{\lambda\}} \pi^{\{\lambda\}}(x) Y^{\{\lambda\}}(y) \\ h_{a\mu}(x,y) &= \sum_{\{\lambda\}} B_\mu^{\{\lambda\}}(x) Y_a^{\{\lambda\}}(y), \quad h_{(ab)}(x,y) = \sum_{\{\lambda\}} \phi^{\{\lambda\}}(x) Y_{(ab)}^{\{\lambda\}}(y) \end{aligned} \tag{2.10}$$

where $[\lambda] \equiv [\lambda_1, \cdots, \lambda_{[5/2]}]$ denotes the SO(5) representation. Similarly with the gauge condition $D^a A_{a\mu} = 0$ and $D^a A_{ab} = 0$ we can expand the type IIB complex zero and the two-forms, B and A_{mn} respectively, as

$$\begin{aligned} A_{\mu\nu}(x,y) &= \sum_{\{\lambda\}} a_{\mu\nu}^{\{\lambda\}}(x) Y^{\{\lambda\}}(y), \quad A_{a\mu}(x,y) = \sum_{\{\lambda\}} a_\mu^{\{\lambda\}}(x) Y_a^{\{\lambda\}}(y) \\ A_{ab}(x,y) &= \sum_{\{\lambda\}} a^{\{\lambda\}}(x) Y_{[ab]}^{\{\lambda\}}(y), \quad B(x,y) = \sum_{\{\lambda\}} B^{\{\lambda\}}(x) Y^{\{\lambda\}}(y) \end{aligned} \tag{2.11}$$

For the four-form flux we can do the same thing by imposing the conditions $D^a a_{abcd} = 0$, $D^a a_{abc\mu} = 0$, $D^a a_{ab\mu\nu} = 0$ and $D^a a_{a\mu\nu\gamma} = 0$,

$$\begin{aligned} a_{abcd} &= \sum_{\{\lambda\}} b^{\{\lambda\}}(x) Y_{abcd}^{\{\lambda\}}(y), \quad a_{abc\mu} = \sum_{\{\lambda\}} b_\mu^{\{\lambda\}}(x) Y_{abc}^{\{\lambda\}}(y) \\ a_{ab\mu\nu} &= \sum_{\{\lambda\}} b_{\mu\nu}^{\{\lambda\}}(x) Y_{ab}^{\{\lambda\}}(y), \quad a_{a\mu\nu\gamma} = \sum_{\{\lambda\}} b_{\mu\nu\gamma}^{\{\lambda\}}(x) Y_a^{\{\lambda\}}(y) \\ a_{\mu\nu\gamma\rho} &= \sum_{\{\lambda\}} b_{\mu\nu\gamma\rho}^{\{\lambda\}}(x) Y^{\{\lambda\}}(y) \end{aligned} \tag{2.12}$$

[2] (x, y) denote coordinates of the AdS_5 and $Y^{p,q}$ spaces respectively and therefore should not be confused with the y coordinates that we will be using to write the metric etc of the $Y^{p,q}$ spaces .

Notice that $Y^{p,q}$ is topologically $S^2 \times S^3$, the same as $T^{1,1}$, so we can argue similarly as in [20] to simplify the expansion,

$$a_{abcd} = \sum_{\{\lambda\}} b^{\{\lambda\}}(x)\epsilon^{e}_{abcd} D_e Y^{\{\lambda\}}(y) \tag{2.13}$$

The full linearlized equation of motion can be found in [4]. In this chapter we are only interested in scalar harmonics which means that we are only looking at the following modes in AdS_5, coming from first line of (2.10), (2.11), and the last line of (2.12):

$$\begin{aligned}
h_{\mu\nu}(x,y) &= \sum_{\{\lambda\}} H^{\{\lambda\}}_{\mu\nu}(x)Y^{\{\lambda\}}(y), \qquad h^a_a(x,y) = \sum_{\{\lambda\}} \pi^{\{\lambda\}}(x)Y^{\{\lambda\}}(y) \\
A^{(i)}_{\mu\nu}(x,y) &= \sum_{\{\lambda,i\}} a^{\{\lambda\}}_{\mu\nu}(x)Y^{\{\lambda\}}(y), \qquad B^{(j)}(x,y) = \sum_{\{\lambda\}} B^{\{\lambda,j\}}(x)Y^{\{\lambda\}}(y) \\
a_{\mu\nu\gamma\rho} &= \sum_{\{\lambda\}} b^{\{\lambda\}}_{\mu\nu\gamma\rho}(x)Y^{\{\lambda\}}(y)
\end{aligned} \tag{2.14}$$

where $A^{(i)}_{\mu\nu}(x,y)$ would be the NS and RR two-forms respectively and $B^{(j)}(x,y)$, where $i,j = 1,2$, would be the axion and the dilaton respectively. The other two quantities π and b that appear respectively from the expansion of h^a_a in (2.10) and from the expansion of a_{abcd} in (2.13), are related to the metric and the four-form respectively. Therefore taking all these into account, we are left with the following equations:

$$\begin{aligned}
&(\Box_x + \boxtimes_y)H^{\{\lambda\}}_{\mu\nu} = 0 \\
&(\Box_x + \boxtimes_y)B^{\{\lambda,j\}} = 0 \\
&(\mathrm{Max} + \boxtimes_y)a^{\{\lambda,i\}}_{\mu\nu} + \frac{2i}{R}\epsilon_{\mu\nu}{}^{\sigma\tau\gamma}\partial_\sigma a^{\{\lambda,i\}}_{\tau\gamma} = 0 \\
&\Box_x \begin{pmatrix} \pi^{\{\lambda\}} \\ b^{\{\lambda\}} \end{pmatrix} + \begin{pmatrix} \boxtimes_y - 32R^{-2} & 80R^{-1}\boxtimes_y \\ -\frac{4}{5}R^{-1} & \boxtimes_y \end{pmatrix} \begin{pmatrix} \pi^{\{\lambda\}} \\ b^{\{\lambda\}} \end{pmatrix} = 0
\end{aligned} \tag{2.15}$$

where Max denotes the Maxwell operator and $\Box_x$, $\boxtimes_y$ are the kinetic operators in the AdS_5 space time and $Y^{p,q}$ spaces respectively. In our case the latter is exactly given by the action of the covariant Laplacian operator on the corresponding $SO(5)$

representation[3] , which can be formally written as

$$\boxtimes_y \equiv \frac{\Box_y Y^{\{\lambda\}}}{Y^{\{\lambda\}}} \tag{2.16}$$

Our next step then is to analyze the eigenvalues of the Laplacian operator in order to find the mass spectrum for these fields.

2.3.2 Scalar modes in $Y^{p,q}$

As we discussed in detail in the above subsection, our goal is to compute the eigenvalues λ_n of the Laplacian in the manifold $Y^{p,q}$. These eigenvalues enter the scalar wave equation on $AdS_5 \times Y^{p,q}$ as masses, so that the conformal dimensions of the associated fields at infinity (i.e for the CFT dual) are given by Witten's formula [3]:

$$\Delta_k = 2 + \sqrt{4 + \lambda_k}\,.$$

It is well known that the Laplacian on $Y^{p,q}$, which we denote by $\Box_y$, defines a non-negative, self-adjoint operator whose domain is the Sobolev space $H^2(Y^{p,q})$ of square-integrable functions with square-integrable second derivatives. The Laplacian is given in local coordinates as [16][4] :

$$\begin{aligned}
\Box_y \equiv \quad & g^{ij}\nabla_i\nabla_j = \frac{1}{\rho(y)}\frac{\partial}{\partial y}\rho(y)\, w(y)\, r(y)\,\frac{\partial}{\partial y} + \frac{1}{w(y)}\frac{\partial^2}{\partial\alpha^2} + \frac{9}{r(y)}\left(\frac{\partial}{\partial\psi} - h(y)\,\frac{\partial}{\partial\alpha}\right)^2 \\
& + \frac{6}{1-y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\left(\frac{\partial}{\partial\phi} + \cos\theta\frac{\partial}{\partial\psi}\right)^2\right]
\end{aligned} \tag{2.17}$$

[3] For more details on the Maxwell and the Laplacian operator see [4, 20].

[4] Note that y denotes different things on LHS and RHS of (2.17). See footnote 2.

where the various coefficients appearing above can be identified from (2.1) after rescaling to set $c = 1$, i.e.,

$$w(y) \equiv \frac{2(a-y^2)}{1-y}, \quad r(y) \equiv \frac{2y^3-3y^2+a}{a-y^2}, \quad h(y) \equiv \frac{y^2-2y+a}{6(a-y^2)}, \quad \rho(y) \equiv \frac{1-y}{18} \tag{2.18}$$

The scalar mode $\Phi(y,\theta,\phi,\psi,\alpha)$ in the internal space now takes the following wave-functional form that was derived in [16]:

$$\Phi = u(y)v(\theta)e^{i(n\phi+2m\psi+l\sigma\alpha/\tau)} \tag{2.19}$$

which means that the Laplacian satisfies:

$$\Box_y\Phi \;=\; \left[S_{nmlj}u(y)\right] v(\theta)e^{i(n\phi+2m\psi+l\sigma\alpha/\tau)} \tag{2.20}$$

where we saw in [16] that the analysis of the eigenvalues of the Laplacian on $Y^{p,q}$ is reduced to that of (the Friedrichs extension of) the one-dimensional operators

$$\begin{aligned} S_{nmlj} \;&\equiv\; -\frac{1}{\rho(y)}\frac{\partial}{\partial y}\rho(y)\,w(y)\,r(y)\,\frac{\partial}{\partial y} + \frac{1}{w(y)}\left(\frac{\sigma l}{\tau}\right)^2 \\ &\quad + \frac{9}{r(y)}\left(2m - h(y)\,\frac{\sigma l}{\tau}\right)^2 + \frac{6\Lambda_{nmj}}{1-y}\,, \\ &= -\frac{2}{1-y}\frac{\partial}{\partial y}(a-3y^2+2y^3)\frac{\partial}{\partial y} + \frac{\gamma^2(1-y)}{4(a-y^2)} + \frac{6\Lambda_{nmj}}{1-y} \\ &\quad + \frac{9(a-y^2)}{a-3y^2+2y^3}\left(2m - \frac{\gamma(a-2y+y^2)}{6(a-y^2)}\right)^2, \end{aligned} \tag{2.21}$$

densely defined on $L^2((y_-,y_+),\rho\, dy)$. We refer to the aforementioned paper for more detailed discussions on the derivation of the above formula[5] . We have set $\gamma \equiv \sigma l/\tau$,

[5] The approach taken in [16] exploits the separability of the $\mathrm{AdS}^5 \times Y^{p,q}$ metrics to compute the eigenfunctions of the Laplace operator in $Y^{p,q}$ in quasi closed form, by expressing them in terms of the eigenfunctions of the Friedrichs extension of a single second-order ordinary differential operator with four regular singular points. The subtle geometry of the spaces $Y^{p,q}$ introduces additional complications in the analysis, since the 'angular' variables in which the metric of $Y^{p,q}$ separates are not

and the function $v(\theta)$ defined in (2.19) satisfies the eigenvalue equation that comes from the angular direction θ as:

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}-\left(\frac{n+2m\cos\theta}{\sin\theta}\right)^2\right]v_{nmj}=-\Lambda_{nmj}v_{nmj}\,. \tag{2.22}$$

The eigenvalues Λ_{nmj} are given by the explicit formula:

$$\Lambda_{nmj}\ \equiv\ 2\Big[2j(j+1)+\big(|n+2m|+|n-2m|\big)(2j+1)+|n+2m||n-2m|+2m^2+n^2\Big]\,. \tag{2.23}$$

In what follows we will drop the indices when there is no risk of confusion.

Before going on, it is worth recalling that the integers n, m, l that label the operators S arise from the (quite subtle) Fourier decomposition of functions that were discussed in [16] and given above in (2.19), while the label j (also an integer) was obtained by explicitly solving an auxiliary eigenvalue problem associated with the geometry of the sphere bundles (which had three regular singular points). However, as we have already mentioned, there is little hope of solving the eigenvalue problem for S in closed form, since the spectral problem for the operator S is governed by a Heun differential equation. What we will do, therefore, is to obtain some estimates for the eigenvalues of S that will allow us to approximate the conformal dimensions of the corresponding CFT.

defined globally. In order to circumvent this problem the steps taken in [16] is to start by constructing a Fourier-type decomposition of the space of square-integrable functions on $Y^{p,q}$ adapted to the global structure of the manifold and to the action of the Laplacian. Once the eigenfunctions of the Laplacian in $Y^{p,q}$ have been computed, the analysis of the Klein–Gordon equation in $\mathrm{AdS}^5 \times Y^{p,q}$ can be reduced to that of a family of linear hyperbolic equations in anti-de Sitter space. In [16] a detailed discussion of the existence and uniqueness of causal propagators for these equations using Ishibashi and Wald's spectral-theoretic approach to wave equations on static space-times based on [21, 22, 23] were presented. Note that for our purpose, this presents several advantages over the classical method of Riesz transforms, since the latter method only yields local solutions to the Cauchy problem in the case in which the underlying space-time is not globally hyperbolic [24].

2.3.3 Behaviour of large eigenvalues (highly excited modes)

In this subsection, we give an asymptotically exact result for large energies (highly excited modes) of the operator $S \equiv S_{nmlj}$. The basic idea is that, if we label the eigenfunctions of this operator by an integer $k = 1, 2, \ldots$, the kth eigenvalue is very close to a constant multiple of k^2 for large k. To put it in a different way, the eigenvalues tend to those of an infinite well, the width of the well determined by the functions P, Q, W that define the Sturm–Liouville operator S. Very informally, the justification would be that at high energies the leading terms are the derivatives; this kind of asymptotic results are usually proved using pseudo-differential operators.

The first observation is that, without any further assumptions, we have an asymptotic formula (for large k, for highly excited modes) for the eigenvalues of S namely: the eigenvalues $\lambda_k \equiv \lambda_k(n, m, l, j)$ of S are asymptotically given by following expansion[6]

$$\lambda_k = C_0 k^2 + o(k^2)\,, \tag{2.24}$$

where the constant

$$C_0 \;\equiv\; 2\pi^2 \left[\int_{y_-}^{y_+} \left(\frac{1-y}{a - 3y^2 + 2y^3} \right)^{1/2} dy \right]^{-2} \tag{2.25}$$

[6] A word of caution about the notation: The error term is $o(k^2)$, and not $\mathcal{O}(k^2)$. The respective notations mean different things, and $o(k^2) \ll \mathcal{O}(k^2)$ for large k. The notation $f(k) = o(g(k))$ means that $\lim_{k\to\infty} \frac{f(k)}{g(k)} = 0$. On the other hand the notation $f(k) = \mathcal{O}(g(k))$ means that that there exists a positive constant C such that for k sufficiently large $|f(k)| \leqslant C|g(k)|$. Simply, $\mathcal{O}(k^n)$ means that a term scaling like k^n in the proper limit of k, as familiar to physicists. Here $k \to \infty$ is appropriate, and later in (2.48) $a \to 0$ is so. The two notions are different and in particular the $o(k^2)$ notation above indicates that the error term grows slower than quadratically in k. If it had a power-law behaviour, it would be $o(k^2) = \mathcal{O}(k^{2-\epsilon})$ with $\epsilon > 0$. In some sense $\mathcal{O}$ is used when we know the power scaling of a term, and o is when we know only the upper bound of the scaling.

depends on the geometry of the manifold (that is, on p and q) through $y_\pm$ but not on the Fourier modes n, m, l, j. (So that only the error term knows about these indices.)

The above statement is a consequence of general results in the theory of singular Sturm–Liouville operators. Indeed, it suffices to note that S is a lower-bounded one-dimensional self-adjoint operator, so it follows from [25, Sec. 10.8] that (2.24) holds true with

$$C_0 \;\equiv\; \left(\frac{1}{\pi}\int_{y_-}^{y_+} \big(w(y)\, r(y)\big)^{-1/2}\, dy\right). \tag{2.26}$$

An easy computation shows that this integral takes the above form, which can in turn be expressed in terms of elliptic functions.

Before ending this subsection, let us make the following remark. Weyl's law[7] ensures that, when the eigenvalues of all the one-dimensional operators corresponding to the various Fourier modes are taken into account, the eigenvalues of the Laplace operator on $Y^{p,q}$ (let's call them $\widetilde{\lambda}_k$) obey the asymptotic law

$$\widetilde{\lambda}_k = (2\pi)^2 \left(\frac{5k}{|\mathbb{S}^4|\mathrm{Vol}(Y^{p,q})}\right)^{2/5} + o(k^{2/5}) \tag{2.27}$$

where $|\mathbb{S}^4|$ denotes the volume of the unit 4-sphere and the volume of the manifold being given by [13]

$$\mathrm{Vol}(Y^{p,q}) = \frac{\pi^2 q^2 [2p + (4p^2 - 3q^2)^{1/2}]}{3p^2[3q^2 - 2p^2 + p(4p^2 - 3q^2)^{1/2}]}\,. \tag{2.28}$$

[7] The Weyl law states that the first term in the asymptotic expansion for the k-th eigenvalue λ_k of the Laplacian on an n-dimensional compact Riemannian manifold is:

$$\lambda_k \;=\; C_n k^{2/n}/(\mathrm{Vol}\ M)^{2/n} + o(k^{2/n})$$

as $k \to \infty$. This was proved by Weyl in [26]. The second term was conjectured by Weyl in 1913 [27] and proved only in 1980 by Ivrii [28].

Eq. (2.24) provides a somewhat more tangible way of presenting this asymptotic result in the sense that the asymptotics is separated into families labeled by additional "quantum numbers". A straightforward but tedious computation shows that, of course, when degeneracies are taken into account, the asymptotics (2.24) can be summed with respect to the additional "quantum numbers" to obtain (2.27).

Let us elaborate this a little bit more. We have seen that the analysis of the eigenvalues of the Laplacian in $Y^{p,q}$ can be reduced to that of the eigenvalues of a family of one-dimensional operators $S = S_{nmlj}$. These operators are labeled by three integers n, m, l and a nonnegative integer j. Notice that if any of the quantum numbers n, m or l is nonzero ("higher Fourier modes"), all the eigenvalues of the Laplacian corresponding to these quantum numbers are necessarily degenerate, as mapping (n, m, l) to $(-n, -m, -l)$ leaves the eigenvalue equation invariant. A convenient way of understanding the behavior of the eigenvalues if the Laplacian in geometric terms is the Weyl's law. For this, let's denote by $\widetilde{\lambda}_k$ the k-th lowest eigenvalue of the Laplacian in $Y^{p,q}$, where each eigenvalue is repeated according to its multiplicity. Obviously, for each k there are "quantum numbers" (n, m, l, j) such that $\widetilde{\lambda}_k = \lambda_{k'}(n, m, l, j)$ for some k'.[8] Weyl's law then ensures that the asymptotic distribution of the eigenvalues $\widetilde{\lambda}_k$ of the Laplacian is related to the volume of the manifold through the relation (2.27).

2.3.4 Bounds for the eigenvalues for small a

In the previous subsection we obtained an asymptotic formula for the eigenvalues, which is asymptotically exact for large energies. It does not provide any information on low-lying eigenvalues, however. So our goal in this subsection is to provide some estimates for the whole spectrum in an appropriate regime. This regime will be the

[8] It is worth emphasizing that one cannot explicitly compute the degeneracy of the eigenvalues, as there could be *non-geometric degeneracies* in the sense that $\lambda_{k_0(n_0,m_0,l_0,j_0)} = \lambda_{k_1(n_1,m_1,l_1,j_1)}$ for some pair of indices *not* related by a symmetry of the equation.

case when the parameter a is small; as we will see, then we can obtain two-sided bounds for the eigenvalues that provide an adequate control of the energies.

The technique we apply here is that, using the fact that a is small $0 < a < 1$, we can Taylor expand the Laplacian operator in terms of small a and drop higher orders of a (as in (2.32)). Obviously this works the best if a is very small, or equivalently when $q \ll p$, but even moderately small a, it is a valid Taylor expansion. Instead of trying to obtain the spectrum of the original Laplacian operator, we use another operator (2.37) whose spectrum is exactly known as in (2.38). With an appropriate constant C which does not depend on the parameters of the equation, we can compute the upper and lower bounds of the eigenvalues of Laplacian. (But we need to know how small C can be, if C has to be large, the bound is very loose. By comparing with the known low-lying scalar spectrum, can we learn something useful about C?)

Before passing to the actual derivation of the bounds, let us discuss the meaning of the smallness of a. It should be noticed that this is in fact a geometric hypothesis on the manifold. In order to see this, let us recall the connection between the parameter a and the integers p, q that controlled the geometry of the bundle. In [13, Sec. 3] it is explained that the relationship between p, q and the endpoints $y_\pm$ is that

$$y_+ - y_- = \frac{3q}{2p} \tag{2.29}$$

The idea now is that it can be easily seen that for any value of the latter quotient we can find an a for which (2.29) is satisfied; indeed, a can be chosen as

$$a = \frac{3}{4}\left[1 - \frac{3q}{2p} - \left(1 - \frac{1}{3}\left(\frac{3q}{2p}\right)^2\right)^{1/2}\right]^2 - \frac{1}{4}\left[1 - \frac{3q}{2p} - \left(1 - \frac{1}{3}\left(\frac{3q}{2p}\right)^2\right)^{1/2}\right]^3 \tag{2.30}$$

Hence it is not hard to see that $a \ll 1$ is equivalent to $q \ll p$, so this condition translates immediately as a condition on the geometry of the bundles. It this case,

$$a = \frac{27q^2}{16p^2} + \mathcal{O}(q^3/p^3) \tag{2.31}$$

A closer look at the subsection on rational roots in [13] reveals that there is also an infinite number of solutions with rational roots and arbitrarily small values of a (recall that in this case the Sasaki–Einstein structure is quasi-regular.)

The idea now is that, for very small a, the operator $-S$ should be very similar to the one we obtain by dropping higher powers of a (e.g. in the Taylor expansion of the coefficients), namely

$$-2\frac{\partial}{\partial y}(a-3y^2)\frac{\partial}{\partial y}+\frac{\gamma^2}{2(a-y^2)}+6\Lambda+\frac{18(a-y^2)}{a-3y^2}\left(m+\frac{\gamma y}{6(a-y^2)}\right)^2 \tag{2.32}$$

This expression defines a self-adjoint operator on $L^2(-(a/3)^{1/2},(a/3)^{1/2})$ via its Friedrichs extension (notice we still have too many singular points to solve the eigenvalue equation for S). It is convenient to make things independent of a by rescaling. For future convenience, we introduce the variable $t\equiv a^{-1/2}y$ and, noticing that

$$\gamma=\sigma lq(3a)^{1/2}\,(1+\mathcal{O}(a)) \tag{2.33}$$

we set $\bar\gamma\equiv a^{-1/2}\gamma$ (observe that $\bar\gamma$ still depends on a, although it tends to a well-defined nonzero limit as $a\to 0$). Here and in what follows, by $\mathcal{O}(a)$ we will denote quantities bounded by a constant (independent of any labels and of the geometry) times a, and whose derivatives satisfy analogous bounds (i.e., behave like symbols with respect to these bounds). We are thus led to consider (the Friedrichs extension of) the operator

$$T\equiv\ -\frac{\partial}{\partial t}P(t)\frac{\partial}{\partial t}+Q(t) \tag{2.34}$$

in $L^2(I)$, with $I\equiv(-3^{-1/2},3^{-1/2})$ and

$$\begin{aligned}P(t)&\equiv 2(1-3t^2)\,,\\ Q(t)&\equiv \frac{\bar\gamma^2}{2(1-t^2)}+6\Lambda+\frac{18(1-t^2)}{1-3t^2}\left(m+\frac{\bar\gamma t}{6(1-t^2)}\right)^2.\end{aligned}$$

In order to relate the spectral properties of S (as an unbounded self-adjoint operator on $L^2((y_-,y_+),\rho\,dy)$ to those of T (on the space $L^2(I)$ with the standard Lebesgue measure dt), it is convenient to start by relating these two L^2 spaces. An obvious way to do so is through the following a-dependent change of variables:

$$t \;\equiv\; -\frac{1}{\sqrt{3}}+\frac{2}{\sqrt{3}}\frac{\int_{y_-}^{y}\rho(y')\,dy'}{\int_{y_-}^{y_+}\rho(y')\,dy'} \;\equiv\; \mathcal{T}_a(y)\,. \tag{2.35}$$

This induces a unitary transformation $L^2((y_-,y_+),\rho\,dy) \to L^2(I,dt)$, which transforms S into the Sturm–Liouville operator of the form:

$$\tilde{S} \equiv \; -\frac{\partial}{\partial t}\tilde{P}(t)\frac{\partial}{\partial t}+\tilde{Q}(t)\,. \tag{2.36}$$

To derive the bounds, we start with the following observation: the spectrum of the auxiliary operator

$$T_\mu \;\equiv\; -\frac{\partial}{\partial t}P(t)\frac{\partial}{\partial t}+\frac{\mu}{1-3t^2} \tag{2.37}$$

on $L^2(I)$, as a function of the parameter μ, is given by

$$\ell_k(\mu) \;\equiv\; \frac{3}{2}\left(1+\sqrt{\frac{8\mu}{3}}+2k\right)^2-\frac{3}{2} \tag{2.38}$$

The proof of the above statement can be argued using a straightforward computation. To start, observe that it suffices to see that the exponents of the equation $T_\mu f = -\ell f$ are

$$\pm\sqrt{\frac{\mu}{24}}, \qquad\qquad \frac{1}{2}\left(1\pm\sqrt{1+\frac{2\lambda}{3}}\right) \tag{2.39}$$

at (0 and at 1) and at ∞ respectively. The eigenvalues then arise as the necessary condition for

$$(1-3t^2)^{-(\mu/24)^{1/2}}f(t) \tag{2.40}$$

to be a polynomial in t, thus proving the required statement.

After developing the necessary mathematical preliminaries, we are now ready to compute bounds for the eigenvalues of S (which coincide with those of $\tilde{S}$, by definition). Notice that we cannot obtain bounds using a relative compactness argument, as any perturbation of the function P will lead to corrections that are not relatively compact with respect to the original operator (because they have the same number of derivatives as the initial operator). What we can do is to exploit monotonicity using the following two observations. The first observation is that there is a constant C, which does not depend on the parameters of the equation, such that the following bounds for $\tilde{P}(t)$ hold for all $t \in I$:

$$(1 - Ca)P(t) \leqslant \tilde{P}(t) \leqslant (1 + Ca)P(t)\,. \tag{2.41}$$

This inequality is obvious in view of the formula (2.35) for the map $y \mapsto t$, and simply asserts (roughly speaking) that the map does not alter the singularities too much.

Our second observation is somewhat similar to the first one in the sense that we again claim that there is a constant C, which does not depend on the parameters of the equation, such that the following bounds for $\tilde{Q}(t)$ hold for all $t \in I$:

$$\begin{aligned} \tilde{Q}(t) &\geqslant (1 - Ca)\left(\frac{\mu_-}{1 - 3t^2} + \frac{1 + \bar{\gamma}^2}{2} + 6\Lambda - Ca\right), \\ \tilde{Q}(t) &\leqslant (1 + Ca)\left(\frac{\mu_+}{1 - 3t^2} + \frac{3(1 + \bar{\gamma}^2)}{4} + 6\Lambda + Ca\right), \end{aligned} \tag{2.42}$$

where μ_+ and μ_- are defined in the following way:

$$\mu_- \;\equiv\; 12 \max\left\{0, m - \frac{\bar{\gamma}}{4\sqrt{3}}\right\}^2, \qquad \mu_+ : \;\equiv\; 18\left(m + \frac{\bar{\gamma}}{4\sqrt{3}}\right)^2. \tag{2.43}$$

The proof of the above two inequalities are a straightforward consequence of the fact that

$$(1 - Ca)(Q(t) - Ca) \leqslant \tilde{Q}(t) \leqslant (1 + Ca)(Q(t) + Ca)\,. \tag{2.44}$$

(One might wonder why we included an additive error Ca here and not in the estimate for $\tilde{P}$. The reason is that $\tilde{P}$ does not vanish in the interval I, and this is enough for us to control the error via a multiplicative constant.)

It is standard that if we take nicely behaved functions $P_j(t)$ and $Q_j(t)$ on I, with $j = 1, 2$ and $P_j(t) > 0$, and suppose that $P_1(t) \geqslant P_2(t)$ and $Q_1(t) \geqslant Q_2(t)$ (resp. $P_1(t) \leqslant P_2(t)$ and $Q_1(t) \leqslant Q_2(t)$), then the k-th eigenvalue of (the Friedrichs extension of) the operator $-\frac{d}{dt}P_1(t)\frac{d}{dt}+Q_1(t)$ is larger or equal (resp. smaller or equal) than those of $-\frac{d}{dt}P_2(t)\frac{d}{dt}+Q_2(t)$. Hence it is elementary to derive the bounds

$$\Lambda_k^{[-]} \leqslant \lambda_k \leqslant \Lambda_k^{[+]} \tag{2.45}$$

where

$$\begin{aligned} \Lambda_k^{[-]} &= (1-Ca)\left(\ell_k(\mu_-) + \frac{1+\bar{\gamma}^2}{2} + 6\Lambda - Ca\right), \\ \Lambda_k^{[+]} &= (1+Ca)\left(\ell_k(\mu_+) + \frac{3(1+\bar{\gamma}^2)}{4} + 6\Lambda + Ca\right), \end{aligned} \tag{2.46}$$

from the inequalities (2.41) and (2.42), the formula for the eigenvalues $\ell_k(\mu)$ of the auxiliary operator T_μ derived in (2.38) and elementary inequalities in I such as

$$1 \leqslant (1-t^2)^{-1} \leqslant 3/2\,. \tag{2.47}$$

The bounds (2.45), in which C stands for an a-independent constant and $\ell_k(\mu)$ is given by (2.38), constitute the main result of this subsection[9] .

[9] One might worry about the strength of our bound. For example a question would be whether the bound could be loose if constant such as C is large. To answer this we first note that the constants do not arise exactly from a power series expansion, but rather as the Taylor formula with estimates for the remainder (which is essentially the mean value theorem). Therefore, the constant C can be explicitly computed as the (sum of the) supremum (for t and a between certain values) of the derivative of some functions appearing in P or Q with respect to the parameter a. For this reason, the behavior of this constant is controlled, and can be computed explicitly. For example, a preliminary computation reveals that the constant C can be chosen to be of order

As a remark, notice that the above bounds also ensure that the eigenvalues have the asymptotic behavior

$$\lambda_k = 6(1 + \mathcal{O}(a))k^2 + \mathcal{O}(k)\,. \tag{2.48}$$

This is precisely the growth rate computed in (2.24), since it is easy to see that the constant

$$C_0 \;\equiv 2\pi^2 \left[\int_{y_-}^{y_+} \left(\frac{1-y}{a - 3y^2 + 2y^3} \right)^{1/2} dy \right]^{-2} \tag{2.49}$$

entering Weyl's law (2.25) tends to 6 as the constant a tends to 0.

2.4 Examples of scalar and other modes

Now that we have discussed the spectrum of scalar modes in the internal $Y^{p,q}$ space, it is time to study some examples of these modes. However before moving ahead we should point out that in this section (and also the next) we will *not* address the spectra of theory. To analyse the spectra (for example along the lines of [29, 30, 20]) we not only need to go beyond the scalar fields, but would also require exact eigenvalues of the KK modes for all spin-states of the theory. The advances that we made in the previous section do not help us in getting these details, and therefore we will suffice ourselves by studying some basics aspects of scalar and other modes from supergravity perspective in this section. In the next section we will discuss possible non-conformal extensions of our model. Again the emphasis therein would be to study the supergravity background and not the matching of spectra.

The simplest examples of scalar and other modes that appear for our case are from the decomposition of the 2-forms in (2.11). These decompositions lead to two possible theories on the boundary where we define the CFTs.

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10 when a is smaller than 0.1, so the relative error is at most of order 10^{-n} when $p/q < 10^{-n-1}$. (These estimates can be refined easily.)

• Non-commutative geometry: Let us consider the NS B field with both components along the boundary, i.e we can switch on $B_{ij}(x)$ where $i, j = 1, 2, 3$ and x_μ specify coordinates in AdS_5 space, leading to non-commutative geometry in the dual gauge theory. For example, a B-field component of the form $B_{ij}(r)$, with r being the radial direction in the AdS_5 space, would be able to generate non-commutative theory on the boundary. Clearly this mode is a scalar mode in the internal $Y^{p,q}$ space.

• Dipole theory: This time we consider the NS B-field which has one component along the boundary and the other component either along the radial r direction or along the internal $Y^{p,q}$ directions. Consider first a component of the NS B field of the form B_{ir}. However if this component is only a function of x^μ, then we can make a gauge transformation to rotate the NS B field components along the boundary which in turn will convert the boundary theory to a non-commutative theory. The other alternative is to make it gauge equivalent to zero for the B field component of the form $B_{ir}(r)$. Thus the only non-trivial cases appear to be of the form $B_{ir}(y), B_{ia}(x, y)$ and they both lead to the dipole theories. However none of these are scalar modes in the internal $Y^{p,q}$. The special case where the NS B field is of the form $B_{ia}(x, y)$ fits in with our decomposition (2.11), and leads to a simple vector decomposition of the boundary theory.

Thus the simplest scalar mode leading to noncommutativity can be specified by a 2-form θ^{ij} such that the commutator of the coordinates on the boundary theory is $[x^i, x^j] = i\theta^{ij}$. The parameter θ^{ij} has dimensions -2. At low-energies, noncommutative super Yang-Mills theory (NCSYM) can be described by augmenting the action with:

$$\int \theta^{ij} \mathcal{O}_{ij}(x) d^4x, \tag{2.50}$$

where $\mathcal{O}_{ij}$ is an operator of dimension 6 in the superconformal SYM on a commutative space. In the conventions such that the SYM Lagrangian is:

$$\mathcal{L}_{\text{SYM}} = \text{tr}\left[\frac{1}{2g^2}\sum_{I=1}^{6}\partial_i\phi^I\partial^i\phi^I + \frac{1}{4g^2}F_{ij}F^{ij} + \frac{1}{2g^2}\sum_{I<J}\left[\phi^I, \phi^J\right]^2\right] + \text{fermions}, \tag{2.51}$$

the bosonic part of the operator $\mathcal{O}_{ij}$ can be written as:

$$\mathrm{tr}\left[\frac{1}{2g^2}F_{jk}F^{kl}F_{li}-\frac{1}{2g^2}F_{ij}F^{kl}F_{kl}+\frac{1}{g^2}F_{ik}\sum_{I=1}^{6}\partial_j\phi^I\partial^k\phi^I-\frac{1}{4g^2}F_{ij}\sum_{I=1}^{6}\partial_k\phi^I\partial_k\phi^I\right]. \tag{2.52}$$

Here, g is the SYM coupling constant, F_{ij} is the $U(N)$ field-strength, and ϕ^I ($I = 1\ldots 6$) are the scalars.

For the second case we expect the boundary theory to be deformed by an operator of the form $\mathcal{O}_i$. The deformation by $L^i\mathcal{O}_i$ (where L^i is a constant vector) is the low-energy expansion of a nonlocal field-theory, the so-called dipole-theory, described in [31, 32, 33].

Furthermore, as discussed in [31] (see also [32, 33, 34]), the bosonic part of the SYM operator $\mathcal{O}_i$ can be calculated by changing to local variables (see [31] for more details). We can write it in $\mathcal{N}=1$ superfield notation as [31]:

$$\mathcal{O}_i=\frac{i}{g^2_{YM}}\int d^2\theta\epsilon^{ab}\mathrm{tr}\left[\sigma_i^{\alpha\dot{\alpha}}W_\alpha\Phi_a D_{\dot{\alpha}}\Phi_b+\Phi\Phi_a D_i\Phi_b\right]+\mathrm{c.c.} \tag{2.53}$$

Here, we denote the $\mathcal{N}=1$ chiral field as Φ and the $\mathcal{N}=1$ vector-multiplet with the field-strength W_α. The original $\mathcal{N}=2$ hypermultiplet is now written in terms of the two $\mathcal{N}=1$ chiral multiplets Φ_a ($a=1,2$). Finally, $\sigma_i^{\alpha\dot{\alpha}}$ are Pauli matrices. As expected, the operator $\mathcal{O}_i$ has conformal dimension 5.

2.4.1 Possible type IIA brane realisation

In the following we will discuss these backgrounds in somewhat more details by switching on appropriate B fields. This is slightly different from allowing the B field as a *fluctuation*. A non-trivial background B field will change the geometry in some particular way which would reflect the corresponding backreactions. To analyse the corresponding backreactions we have to study the scenario directly from N D3-branes probing the geometry given by a cone over the $Y^{p,q}$ spaces. This starting point in fact has many intriguing possibilities in addition to the ones related to generating non-local field theories. One of the possibilities is to see whether a brane realisation of the form [9] in type IIA can also be made for our case. We will therefore start by analysing this interesting possibility first and then go for the non-local theories.

To study D3-branes at the tip of a cone over the $Y^{p,q}$ manifolds, we will assume the usual ansatz for the D3-brane metric given in terms of a harmonic function H which is typically a function of r and the $Y^{p,q}$ coordinates. Let us therefore take the following metric ansatz:

$$ds^2_{\rm IIB} = H^{-1/2} ds^2_{0123} + H^{1/2}(dr^2 + r^2 dM_5^2), \tag{2.54}$$

where dM_5^2 is the same in eq. (2.1) and $F_5 = (1+*)d\beta_0 \wedge dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3$ with $H = 1 + \frac{r_0^4}{r^4} \equiv \beta_0$. We also assume the dilaton is zero. As in [9], the internal metric has three isometries along the α, ψ and ϕ directions. We first do a T-duality along α direction. The metric becomes

$$\begin{aligned} ds^2_{\rm IIA} &= H^{-\frac{1}{2}} ds^2_{0123} + H^{\frac{1}{2}} \Bigg\{ dr^2 + r^2 \Big[\frac{1-cy}{6}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{1-cy}{2f(y)} dy^2 \\ &\quad + \frac{f(y)}{9(a-y^2)}(d\psi^2 - \cos\theta d\phi)^2 + \frac{(1-cy)}{2Hr^4(a-y^2)} d\alpha^2 \Big] \Bigg\} \\ &= H^{-\frac{1}{2}} \left[dx^2_{0123} + \frac{1-cy}{2r^2(a-y^2)} d\alpha^2 \right] + H^{\frac{1}{2}} \Bigg\{ dr^2 + r^2 \Big[\frac{1-cy}{6}(d\theta^2 + \sin^2\theta d\phi)^2 \\ &\quad + \frac{1-cy}{2f(y)} dy^2 + \frac{f(y)}{9(a-y^2)}(d\psi - \cos\theta d\phi)^2 \Big] \Bigg\}, \end{aligned} \tag{2.55}$$

with the following two components of the B-fields:

$$B_{\alpha\psi} = \frac{ac - 2y + y^2 c}{6(a-y^2)}, \qquad B_{\alpha\phi} = -\frac{ac - 2y + y^2 c}{6(a-y^2)} \cos\theta, \tag{2.56}$$

and the original D3 branes become D4 branes. The existence of the two B-fields might indicate the possibility of two NS5 branes, provided $H_{\rm NS} = dB$ is a source term and the integral of $H_{\rm NS}$ over a three-cycle is an integer. The first one is harder to determine because the knowledge of the global behavior of the two B-field components is lacking, although the metric that we are dealing with is global. This is because we *delocalized* along the α direction to make the harmonic function H independent of that direction so that T-duality rules of [35] could be implemented. This is of course a slight oversimplification as this works well for some purposes, but not others. The harmonic function should be taken to be a function of α as well, and then one may T-dualise the background using the technique illustrated in [36]. Under such a T-duality both the B-field components will pick up dependences on H as well. We will discuss more on this a little later.

For the second case, one may do better by converting the three-forms to two-forms and integrating over two-cycles. This can be easily achieved by making a U-duality transformation of the form $T_\alpha S T_3$ where S denotes a S-duality transformation and T_m denotes a T-duality along x^m direction. Thus making a T-duality along x_3 direction we get the following metric in type IIB theory:

$$\begin{aligned} ds^2 &= H^{-\frac{1}{2}} dx_{012}^2 + H^{\frac{1}{2}} \Bigg\{ dx_3^2 + dr^2 + r^2 \Big[\frac{1-cy}{6}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{1-cy}{2f(y)} dy^2 \\ &\quad + \frac{f(y)}{9(a-y^2)}(d\psi^2 - \cos\theta d\phi)^2 + \frac{(1-cy)}{2Hr^4(a-y^2)} d\alpha^2 \Big] \Bigg\}. \end{aligned} \tag{2.57}$$

Under this T-duality the D4 branes become D3 branes but extending along x^0, x^1, x^2 and α directions. However the B-fields remain unchanged. If these B-fields are coming from some source NS5-branes, then the NS5-branes would not change under the T-duality.

Let us now do the S-duality under which the NS B-fields become RR B-fields and the metric gets an overall factor from the dilaton field $\sqrt{\frac{2r^2(a-y^2)}{1-cy}}$ while the D3 branes remain the same. When we T-dualize this background along α direction, the metric becomes

$$\begin{aligned} ds^2 &= \mathcal{H}^{-\frac{1}{2}} dx_{012}^2 + H\mathcal{H}^{-\frac{1}{2}} \Bigg\{ dx_3^2 + dr^2 + r^2 \Big[\frac{1-cy}{6}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{1-cy}{2f(y)} dy^2 \\ &\quad + \frac{f(y)}{9(a-y^2)}(d\psi^2 - \cos\theta d\phi)^2 + d\alpha^2 \Big] \Bigg\}, \end{aligned} \tag{2.58}$$

and the RR three-form fields become the type IIA gauge fields. We have also defined $\mathcal{H} = \frac{H}{2r^2}\frac{1-cy}{a-y^2}$ as our modified harmonic function. If these gauge fields are sourced by D6 branes then they are the ones that come from the type IIB D5 branes. The D3 branes on the other hand become D2 branes. Lifting this configuration to M-theory the eleventh direction has the required local ALE fibration with M2 branes at a point on the four-fold.

The above set of manipulation is suggestive of NS5 branes in the original type IIA configuration provided the gauge field EOM has a source term. Thus if we write the local type IIA gauge field over a patch as:

$$A = A_\psi d\psi + A_\phi d\phi \equiv \frac{ac - 2y + y^2 c}{6a - 6y^2}\Big[\mathcal{F}_1(H)d\psi - \mathcal{F}_2(H)\cos\theta\ d\phi\Big], \tag{2.59}$$

where we have inserted the correction from the harmonic function as $\mathcal{F}_{1,2}(H)$, then there exists a global field strength $F = dA$. Now if it satisfies the two conditions mentioned earlier, namely

$$d * F = \text{sources}, \qquad \int_{S^2} F = \text{integer}, \tag{2.60}$$

then this would not only help us to identify the NS5 branes in the original type IIA set-up, but also help us to *count* the number of the NS5 branes. Such a source term in (2.60) may not be too difficult to see from our analysis if we take (2.59) seriously. The LHS of (2.60) will involve terms like $d * d\mathcal{F}_1(H)$ and $d * d\mathcal{F}_2(H)$. Since $\Box H = *d * dH$ lead to source terms in the supergravity solution, it should be no surprise if the above two terms in (2.60) coming from $\mathcal{F}_{1,2}(H)$ lead to D6 brane source terms in our model.

The above analysis is definitely suggestive of this scenario, although the precise orientations of the NS5 branes are not clear to us at this stage. Furthermore there is the subtlety pointed out in [37] which we might have to consider too. Note also that from (2.55) the D4 branes are wrapped along a non-trivial S^1_α cycle. More details on this will be relegated to future works.

Before we end this subsection, we would like to point out another scenario related to the type IIB metric (2.6). As has been described earlier, (2.6) is related to (2.1) by a series of coordinate transformations. Interestingly the metric (2.6) is closely related to the conifold metric if one makes the following substitutions in (2.6):

$$c = 0, \qquad a = 3, \qquad y = -\cos\,\theta_2, \qquad \beta = \phi_2, \qquad \theta = \theta_1, \qquad \phi = \phi_1 \tag{2.61}$$

where β was defined in (2.5). So a natural question to ask would be what happens if one makes a T-duality along the ψ direction. It is of course well known that, in the limit (2.61), a T-duality along ψ direction leads to an orthogonal (not necessarily intersecting) NS5 branes configuration [9]. If we now make a T-duality along ψ direction, the metric that we get in type IIA side is the following:

$$\begin{aligned} ds^2 \;&=\; H^{-1/2}\left[dx_{0123}^2 + \frac{18(1-cy)}{r^2 W}d\psi^2\right] + H^{1/2}\left[dr^2 + r^2\Big(\frac{1-cy}{6}(d\theta^2 + \sin^2\theta d\phi^2)\right. \\ &\qquad\qquad \left. + \frac{1-cy}{2f}dy^2 + \frac{4f}{W}d\alpha^2\Big)\right], \end{aligned} \tag{2.62}$$

where $W = 3c^2y^2 - 6cy + 2 + ac^2$. Interestingly, we find the metric has the simpler form without cross-terms at all. This is again reminiscent of [9]. We also find two NS B fields whose components are given as:

$$B_{\psi\alpha} = \frac{6(ac - 2y + cy^2)}{W}, \qquad B_{\psi\phi} = -\cos\theta. \tag{2.63}$$

The absence of a cross-term is not a big surprise because we can rewrite (2.6) in a suggestive way using the coordinates (2.61) and taking (c, a) away from the conifold value $(0, 3)$. The metric (2.6) becomes:

$$\begin{aligned} ds^2 \;&=\; a_1(d\theta_1^2 + \sin^2\,\theta_1 d\phi_1^2) + \left[a_2\,\sin^2\,\theta_2 d\theta_2^2 + a_3\,(d\phi_2 + c\,\cos\,\theta_1 d\phi_1)^2\right] \\ &\qquad + \frac{1}{9}\Big[d\psi + (1 + c\,\cos\,\theta_2)\cos\,\theta_1 d\phi_1 - \cos\,\theta_2 d\phi_2\Big]^2, \end{aligned} \tag{2.64}$$

where a_1, a_2 and a_3 are given by the following expressions:

$$a_1 = \frac{1 + c\,\cos\,\theta_2}{6}, \quad a_2 = \frac{1}{2}\cdot\frac{1 + c\,\cos\,\theta_2}{a - 3\cos^2\theta_2 - 2c\,\cos^3\theta_2}, \quad a_3 = \frac{1}{18}\cdot\frac{a - 3\cos^2\theta_2 - 2c\cos^3\theta_2}{1 + c\,\cos\,\theta_2}. \tag{2.65}$$

A T-duality along ψ direction will give us the configuration that we discussed above (using non-canonical coordinates)[10] . To see what (2.62) and (2.63) imply, let us again go to the limit where $c = 0$ and $a = 3$. In this limit[11] we recover the exact brane picture of type IIA discussed in [9]. This may mean that we have some NS5 branes along the (θ, ϕ) directions and some NS5 branes along (α, y) directions (or in a more canonical language, we have a set of NS5 branes along (θ_1, ϕ_1) directions and another set of NS5 branes along (θ_2, ϕ_2) directions). These two set of NS5 branes are locally orthogonal to each other, so as to preserve $\mathcal{N} = 1$ supersymmetry. The $d\psi$ fibration structure in (2.64) also tells us that there are two local B-fields in type IIA side that would T-dualise to give us the required background (2.64). The N type IIB D3-branes become N of D4 branes along ψ direction suspended between these NS5 branes.

Unfortunately the $c \neq 0$ scenario is not quite the same as the simpler $(c, a) = (0, 3)$ scenario. In particular[12] at $y = y_1$ and $y = y_2$ the metric (2.62) develops conical singularities, in other words now y and α no longer form a sphere. This can be easily seen by taking the limit $y \to y_i$ where $i = 1, 2$. In this limit we can write the metric along the y and α directions as:

$$\frac{1 - y_i}{f_i'(y - y_i)} dy^2 \; + \; \frac{4 f_i'(y - y_i)}{W_i} d\alpha^2. \tag{2.66}$$

[10] Note however that (2.62) and the T-dual of (2.64) may look different because in (2.62) one cannot substitute the coordinate transformation directly as the coordinates of (2.62) are the T-dual coordinates of (2.1). Thus a simple substitution of $\alpha = -\frac{1}{6}(\phi_2 + c\psi)$ in (2.62) cannot be done.

[11] For all other purposes we set $c = 1$.

[12] We will henceforth use only the non-canonical coordinates by choice. An equivalent construction could be easily done with the canonical coordinates (2.61).

This is not quite the metric of a 2-sphere. To see this more clearly, let us define a quantity R in the following way:

$$R \equiv 2\sqrt{\frac{(1-y_i)(x-y_i)}{f_i'}}. \tag{2.67}$$

Using this defination we can rewrite the metric (2.66) in a bit more suggestive way:

$$dR^2 + \frac{f_i'^2 R^2}{(1-y_i)W_i} d\alpha^2. \tag{2.68}$$

Clearly the above metric becomes the metric of a 2-sphere only when α is periodic with a period of $L \equiv 2\pi\sqrt{(1-x_i)W_i/f_i'}$. However recall that instead α has a period of $l \neq L$. This means we will always have two conical singularities at $y = y_i$.

Let us now prove that there are no other singularities in this metric. Notice that other singularities can happen only at $W = 0$, which has two roots:

$$y_\pm = 1 \pm \sqrt{\frac{1-a}{3}}. \tag{2.69}$$

Since $y_+ > 1$, it is clear that y_+ is already out of the range of y, while it is not so obvious for $0 < y_- < 1$. To see the range of y_-, we substitute y_- into f to get:

$$f(y_-) = -\frac{2}{3\sqrt{3}}(1-a)\sqrt{1-a} < 0, \tag{2.70}$$

which means $y_- > y_2$ and therefore it is also out of the range. Therefore there are no other singularities in this metric.

The above picture gives us an indication how the brane dual could be constructed although the actual details are much harder to present than our previous construction. It is also true that the delocalization effects are again present in the harmonic function but this time, thanks to the canonical representation of the metric (2.64), a direct mapping to the intersecting brane configuration for $c = 0, a = 3$ gives us a hope that similar brane dual description does exist for generic cases (although at this stage one may need to consider the subtleties pointed out in [37]). The interesting thing however is that a T-duality along α also seems to lead to a similar configuration provided of course (2.60) holds. This shouldn't be a surprise because α and ψ are related by a linear coordinate transformation for $c \neq 0$.

2.4.2 Non-commutative and dipole deformations

The above T-duality arguments give us a way to study the underlying $\mathcal{N} = 1$ gauge theory from two different point of views: one directly from N D3 branes at the tip of the cone in type IIB theory, and other from N D4 branes in a configuration of two orthogonal set of NS5 branes in type IIA theory; although for the latter case the precise orientations of the two NS5 branes still need to be determined.

The non-commutative and the dipole deformations could also be studied from these two viewpoints. However in this chapter we will not consider the type IIA brane interpretations of these deformations. Here we will suffice with only the type IIB description and a fuller picture will be elaborated in a forthcoming work.

Our starting point is the well known observation that once we have a solution we can use TsT to deform it into various different solutions, where T is a T-duality transformation and s is a shift.

Given the background metric (2.54) with D3 branes we have three kinds of deformations which is studied in Chapter 3.

In this chapter we are only interested in the first kind of deformation which is shown in detail in Chapter 3, whose advantage is that the internal metric remains unchanged so our scalar modes analysis in $Y^{p,q}$ is still valid. Of course this still doesn't help us to get the exact matching of spectra as we pointed out earlier. For the rest two kinds of deformations our analysis generally cannot be applied as the internal metric will change quite a bit.

In the following section we will discuss the non-conformal extensions of the above models. We will specifically concentrate on the possibility of geometric transitions in these models.

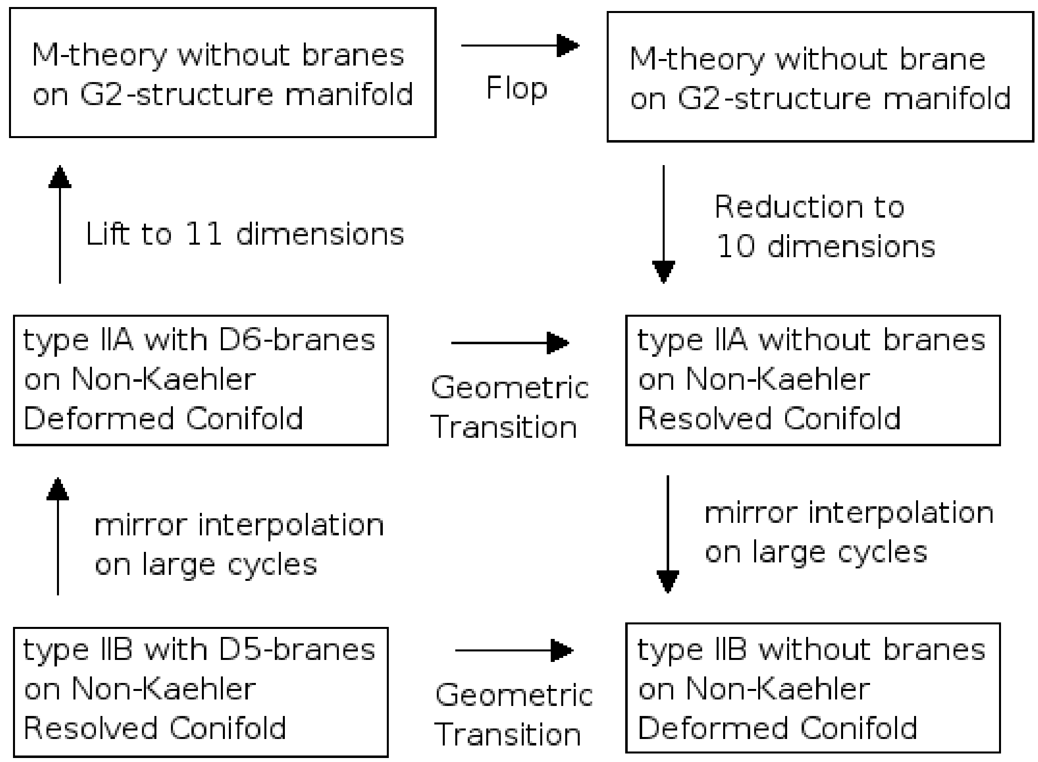

Figure 2.1: The duality map to generate the full geometric transitions in the supersymmetric global set-up of type IIA and type IIB theories.

2.5 Non-conformal duals and geometric transitions

The non-conformal duals to the $Y^{p,q}$ spaces, along the lines of the cascading model of [6], have already been addressed in the literature (see for example [38, 39] etc). The UV gauge groups for $Y^{p,1}$ and $Y^{p,p-1}$ are respectively given in equations (75) and (87) of [38]. For both the cases the IR gauge group is:

$$SU(M) \times SU(2M) \times ... \times SU(2pM) \tag{2.71}$$

where M denotes the number of D5 branes wrapping the two-cycles of $Y^{p,1}$ and $Y^{p,p-1}$ spaces. Such a gauge group is more complicated than the simple picture that we had for [6] and therefore the *far* IR picture could be more involved: there could be non-trivial baryonic branches. This story has not yet been fully clarified, and therefore it gives hope that the brane picture that we developed here may help us to study the far IR picture in more details[13] . We will however not pursue the cascading story anymore here. Instead we go to a slightly different direction that may provide us with an alternative way to study the far IR physics of these models [41, 42].

Our starting point would be to ask whether the far IR physics of the non-conformal set-up could be likened to the geometric transtion story [43] that we developed in the series of papers starting with [44] and culminating with [42] (Chapter 2). For the geometric transition picture to hold, we need few essential ingredients:

• Resolution and deformation for the cone over $Y^{p,q}$. These resolved and deformed spaces are not required to be Calabi-Yau spaces, but they should have at least $SU(3)$ structures (in the presence of branes and fluxes) so that supersymmetric models could be constructed.

• Supersymmetric configurations with D5 branes wrapped on two-cyles of the resolved $Y^{p,q}$ and D6 branes wrapped on three-cycles of the deformed $Y^{p,q}$ including supersymmetric configurations *without* branes but with fluxes. Again the *overall* pictures for both cases should preserve $SU(3)$ structures.

• Two kinds of G_2 structure manifolds should exist in M-theory. One, the lift of the deformed $Y^{p,q}$ space with wrapped D6 branes in type IIA, and two, the lift of the resolved $Y^{p,q}$ space with fluxes but no branes also in type IIA. Additionally these two G_2 structure manifolds should be related by a flop transition, similar to the one constructed for the $T^{1,1}$ case in [45].

If all the three ingredients discussed above are present then one would be able to describe geometric transition using the resolved and the deformed $Y^{p,q}$ manifolds

[13] For example the brane picture developed for the $T^{1,1}$ case in [40] clearly showed how the far IR physics for cascading theory could be understood. We expect similar story to unfold here too.

via the duality map given in figure 2.5. In the following we will describe possible realisation of these scenarios. Our starting point would be the resolution and the deformation of the cones over $Y^{p,q}$ manifolds, that lie in the heart of these scenarios.

2.5.1 Resolution and deformation of the cones over $Y^{p,q}$

A natural question is whether there can be resolutions for the cone over $Y^{p,q}$ as the resolved conifold. The answer is in the affirmative and the metric on the resolved cone over $Y^{p,q}$ was obtained explicitly in [46], [47] and [48]. The metric is,

$$\begin{aligned} ds^2_{RS} &= \frac{(1-y)(1-x)}{3}(d\theta^2+\sin^2\theta d\phi^2)+\frac{(y-x)(1-y)}{h(y)}dy^2+\frac{(x-y)(1-x)}{f(x)}dx^2 \\ &+ \frac{f(x)}{9(1-x)(x-y)}\Big[d\psi-\cos\theta d\phi+y(d\beta+\cos\theta d\phi)\Big]^2 \\ &+ \frac{h(y)}{9(1-x)(y-x)}\Big[d\psi-\cos\theta d\phi+x(d\beta+\cos\theta d\phi)\Big]^2 \end{aligned} \quad (2.72)$$

where $f(x) = 2x^3-3x^2+a$ and $h(y) = 2y^3-3y^2+b$. We will also take the sechsbein e^a to be the ones given in eq (2.8) of [41] with appropriate redefinations of the variables therein.

As explained in the earlier subsection, $x_1 < x < x_2$. One can take y to be non-compact and denote two consecutive roots of $h(y)$ by y_1 and y_2. We focus on the case where the resolution is obtained by blowing up a CP^1, referred to as small partial resolutions in [48]. For this type of resolution we have $x_1 = y_1$ which requires $a = b$. Thus,

$$-\infty < y < x_1, \quad x_1 < x < x_2, \quad a = b \quad (2.73)$$

If one take $y = -r^2/2$ and expand the metric (2.72) in the large r it becomes $ds^2_{RS} \to dr^2 + r^2 ds^2$ where ds^2 is exactly (2.6), so it is a cone over $Y^{p,q}$.

Having got the resolution of the cone over $Y^{p,q}$, we now want to study the deformation of the cone over $Y^{p,q}$, which should be a mirror of the resolved cone over $Y^{p,q}$. Strominger, Yau and Zaslow conjectured in [80] that the mirror manifold can be obtained by three T-dualities. There are three isometric directions ψ, β and ϕ, so we will first do T-dualities along these directions. The metric we get after three T-dualities is:

$$
\begin{aligned}
ds^2_{\rm SYZ} &= \frac{(1-y)(1-x)}{3}(d\theta^2+\sin^2\theta d\phi^2) + \frac{(y-x)(1-y)}{h(y)}dy^2 + \frac{(x-y)(1-x)}{f(x)}dx^2\\
&+ \frac{f(x)h(y)(x-y)\cos\theta}{9(f(x)y(1-y)^2)-h(y)x(1-x)^2}\Big(d\psi + \frac{f(x)y^2(1-y)-h(y)x^2(1-x)}{f(x)y(1-y)-h(y)x(1-x)d\beta}\\
&+ \frac{1}{\cos\theta}d\phi\Big) + \frac{9h(y)(1-x)}{f(x)h(y)(x-y)\cos^2\theta}\Big[(1-x)\cos\theta d\beta + xd\phi\Big]^2\\
&+ \frac{9f(x)(1-y)}{f(x)h(y)(y-x)\cos^2\theta}\Big[(1-y)\cos\theta d\beta + yd\phi\Big]^2 \qquad (2.74)
\end{aligned}
$$

The above metric however cannot be the full answer as T-dualities *a la* [80] require us to take the base to be very large. In [42] (or Chapter 2) (see also [44]) we saw that making the base large actually *mixes* the isometry directions, leading eventually to the generation of additional cross-terms missing from the metric obtained by making naive T-dualities. Thus the actual mirror metric will have cross-terms in addition to what we already have in (2.74).

The complete picture is rather involved as the recipe for making the base bigger using coordinate transformations *a la* [42] is not readily available now. However despite this obstacle, one thing is clear from the analysis of [42]: the resultant metric will not be a Kähler manifold, in fact, it may not even be a complex manifold. This is consistent with the result of [50, 51] (see also [52] where certain obstructions to the existence of Sasaki-Einstein metrics on this manifold is shown). It will also be interesting to compare our result with the one got in [53].

2.5.2 D5 branes on the resolved $Y^{p,q}$ manifold

The technical obstacle that we encountered in the previous subsection doesn't prohibit us to write the metric of N D5 branes wrapped on the two-cycle of the resolved cone over $Y^{p,q}$ manifold. Recently the NS5 brane picture has been studied in [41]. The analysis of [41] is similar in spirit to the one discussed in [42], both the analyses being motivated by the work of [54]. The complete background for N D5 branes wrapped on the resolution two-cycle is given by:

$$
\begin{aligned}
&F_3 = h\,\cosh\,\beta\;e^{-2\phi} * d\left(e^{2\phi}J\right), \qquad H_3 = -hF_0^2\sinh\,\beta\;e^{-2\phi}d\left(e^{2\phi}J\right)\\
&F_5 = -\frac{1}{4}(1+*)dA_0\wedge dx^0\wedge dx^1\wedge dx^2\wedge dx^3\\
&ds^2 = F_0 ds^2_{0123} + \sum_{a=1}^{6} f_a e^{2a}, \qquad \phi = \log\,F_0 + \frac{1}{2}\log\,h,
\end{aligned}
\tag{2.75}
$$

where e^a are the sechsbein defined in [41] and J is the fundamental form associated with the internal metric. The above background is supersymmetric by construction and since the RR three-form F_3 is not closed, it represents precisely the IR configuration of wrapped D5-branes on warped non-Kähler resolved $Y^{p,q}$ manifold. The two warp factors (h, F_0) as well as the coefficients f_a in the internal metric are all functions of (r, y, x) which, in turn, preserve the three isometries of the internal space. Notice also that the background has a non-trivial dilaton, with the internal space being a non-Kähler resolved cone over $Y^{p,q}$. The form of the background (3.15) is similar to the one that we had in [42] except now the internal space is different. This is of course expected if one had to preserve $\mathcal{N} = 1$ supersymmetry. The five-form, which is switched on to preserve the susy, has the form F_5 in (3.15) with:

$$
\begin{aligned}
A_0 \quad &= \quad \frac{\cosh\,\beta\,\sinh\,\beta(1-e^{-2\phi}h^{-2}F_0^{-4})}{e^{2\phi}h^{-2}F_0^{-4}\cosh^2\beta - \sinh^2\beta}\\
&= \quad (F_0^2-1)\tanh\,\beta\left[1+\left(\frac{1-F_0^2}{F_0^2}\right)\mathrm{sech}^2\beta + \left(\frac{1-F_0^2}{F_0^2}\right)^2\mathrm{sech}^4\beta\right].
\end{aligned}
\tag{2.76}
$$

Let us now make a few observations. The parameter β that we have in the background is in general constant and could take any value. This means that there is a class of allowed backgrounds satisfying the supersymmetry condition. Imagine also that we define a six-dimensional internal space in the following way:

$$
ds_6^2 \;=\; \left(\frac{NF_0\cosh^2\beta}{1+F_0^2\sinh^2\beta}\right)\sum_{a=1}^{6} f_a e^{2a}, \tag{2.77}
$$

then one could easily argue that there are a series of dualities[14] that would convert the following background

$$ds^2 = ds^2_{0123} + N ds^2_6, \qquad H_{\rm NS} = e^{-2\Phi} * d\left(e^{2\Phi} J\right), \qquad \Phi = -\phi \tag{2.78}$$

to the one given earlier in (3.15). The above background (2.78) is of course the one studied in [41]. Although this is no big surprise, but it is satisfying to see that our picture can be made consistent with both [41] as well as [42].

2.5.3 Toward geometric transitions for $Y^{p,q}$ manifolds

Once we have the background (3.15) and (3.16) we should be able to use directly the duality cycle shown in figure 2.5. This however will turn out to be more subtle than the story that we developed in [42]. But before we go about elucidating the

[14] Starting with the background (2.78), we perform a S-duality that transforms the NS three-form to RR three-form F_3 and converts the dilaton Φ to ϕ without changing the metric in the Einstein frame. We now make three T-dualities along the spacetime directions $x^{1,2,3}$ that takes us to type IIA theory. Observe that this is *not* the mirror construction. We then lift the type IIA configuration to M-theory and perform a boost (with a parameter β) along the eleventh direction. This boost is crucial in generating D0-brane *gauge* charges in M-theory. A dimensional reduction back to IIA theory does exactly what we wanted: it generates the necessary number of D0-brane charges from the boost, without breaking the underlying supersymmetry of the system. Finally, once we have the IIA configuration, we go back to type IIB by performing the three T-dualities along $x^{1,2,3}$ directions. From the D0-brane charges, we get back our three-brane charges namely the five-form. The duality cycle also gives us NS three-form H_3 as well as the expected RR three-form F_3. Therefore the final configuration is exactly what we required for IR geometric transition: wrapped D5s with necessary sources on a non-Kähler globally defined resolved $Y^{p,q}$ background (3.15). Also as expected, the background preserves supersymmetry and therefore should be our starting point. One may also note that the thee-forms that we get in (3.15) satisfy

$$\cosh\beta \; H_3 \; + \; F_0^2 \; \sinh\beta * F_3 \; = \; 0$$

which is the modified ISD (imaginary self-duality) condition. For more details, see [54, 42].

issues, let us clarify certain things about *generalized* SYZ. The original work of SYZ [80] is based on two facts: (a) all Calabi-Yau manifolds can be written in terms of a T^3 fibration over a base $\mathcal{B}$, and (b) in the limit where $\mathcal{B}$ is much larger than the T^3 fiber, mirror of the given CY manifold is given by three simultaneous T-dualities along the T^3 fiber directions.

For our case, the starting manifold (3.15) is not a CY manifold but instead is a six-dimensional manifold with an $SU(3)$ structure and torsion H_3. For this case there does exist a generalisation of the SYZ technique: it is again given by three T-dualities along the T^3 fiber [55, 56, 57]. The difference now is that we cannot claim that *all* $SU(3)$ structure manifolds can be expressed in terms of T^3 fibrations over some base manifolds (although [58, 56, 57] has discussed more generic cases by applying *local* T-dualities). This generalization of the SYZ technique is called the generalized mirror rule[15] .

Our method now would be to use the generalized SYZ technique to go to the type IIA mirror manifold with wrapped D6-branes. Unfortunately now there are two subtleties that make the analysis much more non-trivial than the one that we had in [42]. The first one is already been discussed earlier: we don't know exactly what kind of coordinate transformations we should do to make the base bigger than the T^3 fiber. Recall that in [42], out of infinite possible coordinate transformations available, we could find a class of transformations that can not only make the base bigger but also lead us to the right mirror manifold. The main reason why we could find that particular class of transformations earlier was solely based on the fact that we *knew* the existence of a deformed conifold solution. This privileged information, unfortunately, is not available to us now.

[15] For more details as to why the generalized mirror rule would lead to another $SU(3)$ structure manifold that is the *mirror* of the original manifold is discussed in [56, 57].

The second issue is even more non-trivial. Looking at the background (3.15) and from $H_3 = dB_{\rm NS}$, we see that the $B_{\rm NS}$ fields will have components that are *parallel* to the directions of the T^3 fiber. T-dualities with B_{NS} fields along the directions of duality lead to non-geometric manifolds! Therefore the type IIA dual manifold will most likely be a non-geometric space which in turn means that the duality cycle depicted in figure 2.5 cannot be very straightforward[16] .

Existence of non-geometric space, however, does not mean that there is no underlying gauge/gravity duality. In fact in the geometric transition set-up there were already indications, even for the simplest resolved conifold case, that the full gauge/gravity duality will involve non-geometric manifolds [60], although we argued in [42] that there is small configuration space of fluxes where we expect the duality to be captured by purely geometric manifolds. The question now is whether such a scenario, with only geometric spaces, could be realised for the present case also. We will leave this for future work.

[16] There is a third subtlety that has to do with the size of the T^3 fiber in the mirror manifold. If the size of the fiber is small i.e of $\mathcal{O}(\alpha')$, then supergravity description may not be possible, and one might have to go to a Gepner type sigma model description. For the model studied in [44, 42] this was not an issue because we could study a class of manifolds parametrised by choice of warp factors that not only satisfy EOMs but also lie in subspaces, where sugra descriptions are valid, on both sides of figure 3 in [59]. These subspaces are related by geometric transitions. For generic choices of the warp factors in [42, 59], it would be interesting to see if the subspaces could incorporate the $Y^{p,q}$ manifolds.

Chapter 3

Geometric transitions at the bottom of cascading RG flow

3.1 Introduction

The original gauge/gravity duality [1], [3] deals exclusively with theories that have no running of the coupling constants, or with theories that have some running of the coupling constants but eventually fall into fixed point surfaces, for example [7]. The first kind of dualities that consider the actual running of the coupling constants leading to, say, confining theories were discussed some time back in [6], [43], [61] and its extension to include fundamental flavors in [62]. The type IIA brane constructions for theories like [7] were first discussed in [8], and for theories with running couplings were discussed in [69]. In fact in the fourth reference of [69] the precise distinctions between [6] and [43] were pointed out in details.

In recent times we have seen many new advantages of studying theories like [6] and [43] that deal with running couplings. The confining behavior of these theories in the far IR is of course very powerful in extending them to more realistic scenarios like high temperature QCD [70, 71]. The cascading nature of these theories allow them to remain strongly coupled throughout the RG flow from UV to IR, and therefore *supergravity* duals can describe the full dynamics of the corresponding gauge theories. For the Klebanov-Strassler (KS) theory [6] even the full UV completion, that allow no Landau poles or UV divergences of the Wilson loops, *can* be achieved by attaching an UV cap to the KS geometry[70]. An example of the full UV completion of the KS geometry both at zero and non-zero temperatures has been recently accomplished in [71]. The UV cap therein is given by an asympototic AdS space that, in the dual gauge theory, will allow for an asymptotic conformal behavior in the UV and linear confinement in the far IR.

For the model studied by Vafa [43] the full UV completion would be more non-trivial. We expect the UV to be a six-dimensional theory instead of a four-dimensional one. A six-dimensional UV completion that allows no Landau poles in the presence of fundamental flavors has not been constructed so far. In fact a proper study of fundamental flavors *a-la* [62] is yet to be done for this case. The F-theory [106, 74, 75] embedding of this model would be crucial in analysing the full UV completion. However some aspects of an intemediate UV behavior, for example cascading dynamics, have been discussed in the past [72] where the cascade is likened to an infinite sequence of flop transitions. The IR dynamics of the theory where we expect geometric transition to happen is actually the last stage in this sequence of transformations where the flop is immediately followed by a conifold transition. At this point we should expect the wrapped D5-branes to be completely replaced by fluxes (at least in the absence of fundamental flavors) [72]. What happens in the presence of fundamental flavors is rather subtle, and we will not discuss this here anymore. In fact we will only concentrate on the last stage of the transition, namely, the geometric transition in this chapter. The intermediate cascading dynamics or the UV completion will be discussed elsewhere [76].

Since the geometric transition leads to a confining theory, the corresponding gauge dynamics is strongly coupled. Therefore the physics of this transition can be captured exclusively by supergravity backgrounds. In some of our earlier works [44, 112] we managed to study this purely using the supergravity backgrounds in the *local* limit, meaning that the sugra background was studied around a specific chosen point in the internal six-dimensional space. The reason for this was the absence of a known globally defined supergravity solution of the wrapped D5-branes on the two-cycle of the resolved conifold. The only known global solution i.e [77] was unfortunately not supersymmetric (see [78, 79] for details) although it satisfied the type IIB EOMs. In this chapter, among other things, we will be able to solve this problem and provide a fully supersymmetric globally defined solution for the wrapped D5-branes on a certain resolved conifold. What we will argue soon is that the resolved conifold should have a non-Kähler metric to allow for supersymmetric solutions. This non-Kählerity appears exactly from the back-reactions of the wrapped D5-branes.

Despite the absence of supersymetric solutions, in [44, 112] we managed to show, at least locally, the full geometric transitons in type II theories. The gravity duals for the IR confining gauge theories on the wrapped D6-branes in type IIA and wrapped D5-branes in type IIB were completely captured by non-Kähler deformations of the resolved and the deformed conifolds respectively. In this chapter we will show that globally under some simplifying assumptions this conclusion remains unchanged, but generically these manifolds would become non-geometric (see [60] for a recent discussion on this). In the following sub-section we will briefly review the state of geometric transition using local supergravity analysis before we proceed to compute the full global picture.

3.1.1 Geometric transition and supersymmetric solution

Let us begin with a bit of historical notes. The original study of open-closed string duality in type II theory starts with D6 branes wrapping a three cycle of a non-compact deformed conifold. Naively one might expect the deformed conifold to be a complex Kähler manifold with a non-zero three cycle. However as discussed earlier in [44, 112] this is not quite correct, and the manifold that actually would solve the string equations of motion is a non-Kähler deformation of the deformed conifold. It also turns out that the manifold has no integrable complex structure, but only has an almost complex structure. This is consistent with the prediction of [106].

However, as one may recall, in all our earlier papers we managed to study only the *local* behavior of the manifolds. This is because the full global picture was hard to construct, and any naive procedure always tend to lead to non-supersymmetric solutions. In deriving the local metric, we took a simpler model where all the spheres were replaced by tori with periodic coordinates (x, θ_1) and (y, θ_2). The coordinate z formed a non-trivial $U(1)$ fibration over the T^2 base. The replacement of spheres by two tori was directly motivated from the corresponding brane constructions of [8], where non-compact NS5 branes required the existence of tori instead of spheres in the T-dual picture.

Locally the non-Kählerity of the underlying metric can be easily seen from its explicit form:

$$\begin{aligned} ds^2_{IIA} \; = \; & g_1 \Big[(dz - b_{z\mu}\; dx^\mu) + \Delta_1 \cot\hat{\theta}_1 \;(dx - b_{x\theta_i}\; d\theta_i) \\ & + \Delta_2 \cot\hat{\theta}_2 \;(dy - b_{y\theta_j}\; d\theta_j) + ..\Big]^2 \\ & + g_2\; [d\theta_1^2 + (dx - b_{x\theta_i}\; d\theta_i)^2] + g_3\; [d\theta_2^2 + (dy - b_{y\theta_j}\; d\theta_j)^2] \\ & + g_4 \sin\psi\; [(dx - b_{x\theta_i}\; d\theta_i)\; d\theta_2 + (dy - b_{y\theta_j}\; d\theta_j)\; d\theta_1] \\ & + g_4 \cos\psi\; [d\theta_1\; d\theta_2 - (dx - b_{x\theta_i}\; d\theta_i)(dy - b_{y\theta_j}\; d\theta_j)] \end{aligned} \tag{3.1}$$

where the coefficients g_i and the coordinates $\theta_i, \hat{\theta}_i$ etc. are defined in [44, 112]. The background has non-trivial gauge fields (that form the sources of the wrapped D6 branes) and a non-zero string coupling (which could in principle be small).

Existence of such an exact supergravity background helps us to obtain the corresponding mirror type IIB background. One would expect that this can be easily achieved using the mirror rules of [80]. It turns out however that the mirror rules of [80], as discussed in [44, 112], do not quite suffice. A detailed analysis of this is presented in [44, 112]. As discussed therein, we have to be careful about various subtle issues while doing the mirror transformations:

(a) The mirror rules of [80] tells us that *any* Calabi-Yau manifold with a mirror admits, at least *locally*, a T^3 fibration over a three dimensional base. This seems to fail for the deformed conifold as it does not possess enough isometries to represent it as a T^3 fibration. On the other hand, a resolved conifold does have a well defined

T^3 torus over a three-dimensional base, which can be exploited to get the mirror (see also [81]). It also turns out that the T^3 torus is a lagrangian submanifold, so a mirror transformations will not break any supersymmetry.

(b) Viewing the mirror transformation naively as three T-dualities along the T^3 torus *does not* give the right mirror metric. There are various issues here. The rules of [80] tell us that the mirror transformation would only work when the three dimensional base is very large. The configuration that we have is exactly opposite of the case [80]. Recall that our configuration lies at the end of a much larger cascading theory. By UV/IR correspondences, this means that the base manifold is very small. Furthermore we are at the *tip* of the geometric transition and therefore we have to be in a situation with very small base (in fact very small fiber too). In [44, 112] we showed that we could still apply the rules of [80] if we impose a non-trivial large complex structure on the underlying T^3 torus. The complex structure can be integrable or non-integrable. Using an integrable complex structure, we showed in [44, 112] that we can come remarkably close to getting the right mirror metric. Our conjecture there was that if we use a non-integrable complex structure we can get the right mirror manifold.

It seems therefore natural to start with the manifold that exhibits three isometry directions — the resolved conifold. We can, however, not use the metric for D5 branes wrapping the S^2 of a resolved conifold as derived in [77], because it breaks all supersymmetry [78]. The metric that we proposed in [44, 112] (where we kept the harmonic functions undetermined) is very close to the metric of [77] but differs in some subtle way:

(a) The type IIB resolved conifold metric that we proposed in [44, 112] is a D5 wrapping a two cycle that *preserves* supersymmetry. We will discuss this issue in more detail below.

(b) As explained in [44, 112], our IIB manifold also has seven branes (and possibly orientifold planes) along with the type IIB three-form fluxes. The metric constructed in [77] doesn't have seven branes but allows three-form fluxes.

The *local* behavior of the type IIB metric is expressed in terms of non-trivial complex structures τ_1 and τ_2 as $dz_1 = dx - \tau_1 d\theta_1$ and $dz_2 = dy - \tau_2 d\theta_2$. The local metric then reads

$$ds^2 = (dz + \Delta_1 \cot\theta_1 \, dx + \Delta_2 \cot\theta_2 \, dy)^2 + |dz_1|^2 + |dz_2|^2 \tag{3.2}$$

where all the warp factors can locally be absorbed in to the coordinate differentials. In this formalism the metric may naively look similar to the one studied in [77] but the global picture is completely different from the one proposed by [77]. Our aim in this chapter is therefore two-fold: to determine the full global picture (at least without the inclusion of UV caps), and to follow the duality cycle that will lead us to analyse geometric transitions in type II theories.

3.2 Analysis of the global picture and the cycle of geometric transitions

With all the mathematical construction at hand, it is time now to discuss the geometrical aspect of the problem i.e the supergravity metric and the fluxes in type II theories. Our starting point would be the issue of supersymmetry in the usual resolved conifold background with fluxes and branes in type IIB background. Once we obtain this, it will prepare us for all the subsequent stages of the duality cycle for the geometric transition [44, 112].

3.2.1 Analysis of the global picture in type IIB

From our earlier works we know that there are two ways of extending our local configuration of [44, 112] to study supersymmetric cases in the full global picture:

(a) The full global geometry is a six-dimensional Kähler manifold with F-theory seven-branes distributed in some particular way. These seven-branes contribute to massive fundamental flavors in the gauge theory. Orientation of these seven-branes are the generalised version of the Ouyang [62] (or the Kuperstein [84]) embeddings.

(b) The full global geometry is non-Kähler with or without F-theory seven-branes. The seven-branes could be embedded in this picture via Ouyang or the Kuperstein embedding, which in turn would provide fundamental matters in the gauge theory. In fact the possibility of such a global completion was already hinted in the second paper of [44, 112].

Let us see how from our local picture studied in [44, 112] these two possibilities can be realised. In the first paper of [44, 112], the local metric was argued to be of the following form:

$$\begin{aligned} ds^2 \;=\; & dr^2 + \left(dz + \sqrt{\frac{\gamma'}{\gamma}}\, r_0 \cot\langle\theta_1\rangle\, dx + \sqrt{\frac{\gamma'}{(\gamma+4a^2)}}\, r_0 \cot\langle\theta_2\rangle\, dy\right)^2 + \\ & + \left[\frac{\gamma\sqrt{h}}{4}\, d\theta_1^2 + dx^2\right] + \left[\frac{(\gamma+a^2)\sqrt{h}}{4}\, d\theta_2^2 + dy^2\right] + ... \end{aligned} \tag{3.3}$$

where all the coefficients are measured at a fixed chosen point $(r_0, \langle\psi\rangle, \langle\phi_i\rangle, \langle\theta_i\rangle)$. For more details see [44, 112]. The local $B_{\rm NS}$ field was taken to be:

$$B_{\rm NS} = b_{x\theta_i} dx \wedge d\theta_i + b_{y\theta_i} dy \wedge d\theta_i \tag{3.4}$$

where $i = 1, 2$. The above background is invariant under the orbifold operation:

$$\mathcal{I}_{xy}: \quad x \;\to\; -x, \qquad y \;\to\; -y \tag{3.5}$$

and therefore can support D7/O7 states at the following orientifold points:

$$\frac{\mathbf{T}^2}{\mathcal{I}_{xy}\ \Omega\ (-1)^{F_L}} \tag{3.6}$$

It is interesting to note that, at the orientifold point, a component like b_{xy} is projected out. However the orientifold projection may allow components like b_{xz}, b_{yz} which could in principle make our mirror manifold non-geometric. In the local picture advocated in [44, 112] we only see components like (3.4) so the local mirror is non-Kähler and geometric.

More interestingly, the orientifolding operation (3.6) allows, along with the wrapped D5-branes, the D7-branes and O7-planes along the internal directions $(r, z, \theta_1, \theta_2)$ located at the four fixed points of the torus $\mathbf{T^2}$ along (x, y) directions. Therefore, in the local picture, a possible susy preserving Ouyang-type configuration would be D5-branes wrapped on the two-torus (θ_2, ϕ_2) and the seven branes wrapping $(\theta_1, \theta_2, \psi)$ and stretched along the radial direction r. On the other hand, globally in a resolved conifold background the seven-branes are in a configuration that is the *union* of branch 1 and branch 2 (see [62, 89, 70, 71] for details). Recall that in branch 1 the seven-branes wrap the $\mathbf{P^1}$ parametrised by (θ_2, ϕ_2) and are embedded along (r, ψ) directions at a point on the other $\mathbf{P^1}$ parametrised by (θ_1, ϕ_1); whereas in branch 2 the seven-branes are at a point on the $\mathbf{P^1}$ parametrised by (θ_2, ϕ_2). Thus globally a susy configuration of seven-branes is a *two*-dimensional surface in $\mathbf{P^1} \times \mathbf{P^1}$ and stretched along (r, ψ) directions determined by the appropriate embedding equation. Therefore in the local limit the two-dimensional susy preserving surface in $\mathbf{T^2} \times \mathbf{T^2}$ should be the two-cycle parametrised by (θ_1, θ_2) as prescribed in [44, 112].

Away from the orientifold point, the local metric takes the following fibration form:

$$\begin{aligned}
ds^2 = \quad & h^{-1/2} ds^2_{0123} + \gamma' \sqrt{h}\ dr^2 + (dz + \Delta_1 \cot\theta_1\ dx + \Delta_2 \cot\theta_2\ dy)^2 + \\
& + \left(\frac{\gamma\sqrt{h}}{4} d\theta_1^2 + dx^2\right) + \left(\frac{(\gamma + 4a^2)\sqrt{h}}{4} d\theta_2^2 + dy^2\right) \\
H_3 = \quad & d\mathcal{J}_1 \wedge d\theta_1 \wedge dx + d\mathcal{J}_2 \wedge d\theta_2 \wedge dy \\
F_5 = \quad & K(r)\ (1 + *)\ dx \wedge dy \wedge dz \wedge d\theta_1 \wedge d\theta_2 \\
F_3 = \quad & c_1\ (dz \wedge d\theta_2 \wedge dy - dz \wedge d\theta_1 \wedge dx)
\end{aligned} \tag{3.7}$$

with additional axio-dilaton that appear from the seven-brane sources. The above form of orientifold projection only allows a non-trivial fibration structure *away* from the orientifold point. However there exist another orientifold operation that may be more well suited at the orientifold point. This can be applied locally via:

$$\mathcal{I}_{x\theta_1} : \quad x \ \to \ -x, \qquad \theta_1 \ \to \ \pi - \theta_1 \tag{3.8}$$

The above action gives rise to the following orientifold action:

$$\frac{\mathbf{T}^2}{\mathcal{I}_{x\theta_1}\Omega(-1)^{F_L}} \tag{3.9}$$

that will keep the wrapped D5 branes and the $B_{\rm NS}$ field with the following components at the orientifold point:

$$\begin{aligned} B_{\rm NS} \;\; = \;\; & b_{x\theta_2}\; dx \wedge d\theta_2 + b_{y\theta_1}\; dy \wedge d\theta_1 + b_{xy}\; dx \wedge dy + b_{xz}\; dx \wedge dz \; + \\ & b_{rx}\; dr \wedge dx + b_{r\theta_1}\; dr \wedge d\theta_1 + b_{\theta_1\theta_2}\; d\theta_1 \wedge d\theta_2 + b_{z\theta_1}\; dz \wedge d\theta_1 \end{aligned} \tag{3.10}$$

which means that at the orientifold point not only is the IIB metric non-trivial, the mirror can also be non-Kähler and non-geometric. The seven-branes and the orientifold-planes are parallel to the wrapped D5 branes. In the following we will argue how susy is preserved in the global set-up when the seven-branes are moved away from the wrapped D5 branes. This is the case where the fundamental hypermultiplets are infinitely massive.

The naive global extension of the above configuration along the lines of [77] will lead to a non-susy configuration. This is because we have assumed that the global extension of a configuration like (3.7) is Kähler in the presence of a $B_{\rm NS}$ field like (3.4) *away* from the orientifold point. The simplest global extension that we will study here as the starting point for the IIB geometric transition is a non-Kähler manifold with D5-branes wrapping two cycles of this manifold. Of course it may be possible to add other branes and fluxes to make the ambient space conformally Kähler, but we will not do so here. We will use the following set of duality transformations, recently proposed by [54], to get our type IIB intial configuration.

• Our starting point would be a non-Kähler type IIB metric with a background dilaton ϕ and NS three-form H_3 that satisfies the standard relation $H_3 = e^{2\phi} * d(e^{-2\phi}J)$ with J being the fundamental (1,1) form.

• On this background we perform a S-duality that transforms the NS three-form to RR three-form F_3, and in the process converts the dilaton to $-\phi$ without changing the metric in the Einstein frame.

• We now make three T-dualities along the spacetime directions $x^{1,2,3}$ that takes us to type IIA theory. Observe that this is *not* the mirror construction.

• We lift the type IIA configuration to M-theory and perform a boost along the eleventh direction. This boost is crucial in generating D0-brane *gauge* charges in M-theory.

• A dimensional reduction back to IIA theory does exactly what we wanted: it generates the necessary number of D0-brane charges from the boost, without breaking the underlying supersymmetry of the system.

• Once we have the IIA configuration, we go back to type IIB by performing the three T-dualities along $x^{1,2,3}$ directions. From the D0-branes, we get back our three-brane charges namely the five-form. The duality cycle also gives us NS three-form H_3 as well as the expected RR three-form F_3. Therefore the final configuration is exactly what we required for IR geometric transition: wrapped D5s with necessary sources on a non-Kähler globally defined "resolved" conifold background. Also as expected, the background preserves supersymmetry and therefore should be our starting point. This background should also be compared with the one given in [77] that solves EOM but do not preserve supersymmetry. To start off, we switch on a non-trivial background dilaton $\phi(r)$ and a NS three-form $H_{\rm NS}$ on a background outlined by the following metric:

$$ds^2 = h^{1/2}e^{\phi}d\widetilde{s}^2_{0123} + h^{-1/2}e^{\phi}ds^2_6 \tag{3.11}$$

where we have defined the variables in the following way:

$$h = \frac{e^{-2\phi}F_0^{-4}}{e^{-2\phi}h^{-2}F_0^{-4}{\rm cosh}^2\beta - {\rm sinh}^2\beta}, \qquad d\widetilde{s}^2_{0123} = F_0 ds^2_{0123} \tag{3.12}$$

$$ds^2_6 = F_1\, dr^2 + F_2(d\psi + \cos\,\theta_1 d\phi_1 + \cos\,\theta_2 d\phi_2)^2 + \sum_{i=1}^{2} F_{2+i}(d\theta_i^2 + \sin^2\theta_i d\phi_i^2)$$

with β being an arbitrary constant and $F_i \equiv F_i(r)$ are functions of the radial coordinate for simplicity. Observe that in the first equation of (3.12), h appears on both sides, and so we need to solve a cubic equation whose real root will give us the actual warp factor h. In general we expect F_i to be functions of all the internal coordinates. We will give an example of this soon when we derive a more precise initial metric. For the time being we will consider (3.11) to be our starting point. Also to preserve supersymmetry[1] , we expect:

$$H_{\rm NS} = e^{2\phi} * d\left(e^{-2\phi}J\right) \tag{3.13}$$

where J is the fundamental form associated with the metric, and we can choose to impose one of the following two conditions on the NS three-form:

$$\begin{aligned} dH_{\rm NS} &\equiv d * dJ - d * (d\phi \wedge J) = \text{sources} \\ dH_{\rm NS} &\equiv d * dJ - d * (d\phi \wedge J) = \alpha'(\text{tr } R \wedge R - \text{Tr } F \wedge F) \end{aligned} \tag{3.14}$$

The first condition is what we require here. This will give rise to the IR wrapped D5 branes theory on non-Kähler resolved conifold set-up (after the chain of dualities mentioned above). The latter case will be for the heterotic theory. We can use the non-closure of $H_{\rm NS}$ to study not only the vector bundles F on the heterotic side, but also the possibility of geometric transition in the heterotic theory! We have alluded to this possibility in our earlier papers [44, 112]. We will try to complete that side of the story in our follow-up paper [76].

Now following the chain of dualities mentioned above, we can get the following type IIB background:

$$\begin{aligned} & F_3 = h \cosh \beta \; e^{2\phi} * d\left(e^{-2\phi}J\right), \qquad H_3 = -hF_0^2 \sinh \beta \; e^{2\phi} d\left(e^{-2\phi}J\right) \\ & F_5 = -\frac{1}{4}(1+*)dA_0 \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \qquad \phi_{\rm now} = -\phi \\ & ds^2 = F_0 ds^2_{0123} + F_1 \; dr^2 + F_2(d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2 \\ & \qquad + \sum_{i=1}^{2} F_{2+i}(d\theta_i^2 + \sin^2\theta_i d\phi_i^2) \end{aligned} \tag{3.15}$$

[1] Or, equivalently, preserving $SU(3)$ structure.

which is, by construction, supersymmetric and since F_3 is not closed it represents precisely the IR configuration of wrapped D5-branes on warped non-Kähler resolved conifold. The five-form is switched on to satify the equation of motion with

$$A_0 \;=\; \frac{\cosh\,\beta\,\sinh\,\beta(1 - e^{2\phi}h^{-2}F_0^{-4})}{e^{-2\phi}h^{-2}F_0^{-4}\cosh^2\beta - \sinh^2\beta} \tag{3.16}$$

Therefore this is starting metric, whose local forms we studied in details in [44, 112], should be taken instead of the metric derived in [77]. The ISD condition for our case gets modified to the following condition on the fluxes:

$$\cosh\beta\;H_3 \;+\; F_0^2\;\sinh\beta * F_3 \;=\; 0 \tag{3.17}$$

which may be compared to [54]. Our derivation could also solve the long standing problem of finding the supersymmetric configuration of wrapped D5-branes on a resolved conifold set-up.

3.2.2 More explicit type IIB background before geometric transition

In the above section we saw how one could derive the precise intial metric that not only serves as starting point for geometric transition, but is also supersymmetric. One may make this more specific by solving the $SU(3)$ structure condition specified in section 2.4. This is worked out in Appendix A of [42]. The metric derived this way has many non-trivial components compared to our initial ansatze (3.15). This is not a problem in itself, because we can always do some coordinate transformations to bring the metric that doesn't have components like $g_{r\mu}$ where $\mu = \theta_i, \phi_i, \psi$. But the metric will have other non-trivial cross-terms. It may be possible to make further coordinate transformations to bring the above metric in a form that closely resembles (3.15), but we will not pursue this here as this doesn't change the underlying physics. Instead we will continue using the background (3.15) and assume that the values of the coefficients F_i are to be fixed using our above metric configuration. Other possible cross-terms, not considered in (3.15), will only make the IIA background more non-trivial without revealing new physics. Henceforth our starting point would be (3.15) with the assumption that the coefficients are to be derived from the metric discussed in the above subsection. Once we know the metric, we can follow up the steps described earlier to compute the three-forms. The NS three-form H_3 has the form:

$$\begin{aligned}
\frac{H_3}{hF_0^2\sinh\beta} = & + (2\phi_{\theta_1}\sqrt{F_1F_2}\cos\theta_1 + \sqrt{F_1F_2}\sin\theta_1 + 2\phi_r F_3\sin\theta_1 \\
& - F_{3r}\sin\theta_1)dr\wedge d\theta_1\wedge d\phi_1 \\
& + (2\phi_{\theta_2}\sqrt{F_1F_2}\cos\theta_2 + \sqrt{F_1F_2}\sin\theta_2 + 2\phi_r F_4\sin\theta_2 \\
& - F_{4r}\sin\theta_2)dr\wedge d\theta_2\wedge d\phi_2 \\
& - 2\phi_{\theta_1}\sqrt{F_1F_2}dr\wedge d\psi\wedge d\theta_1 - 2\phi_{\theta_2}\sqrt{F_1F_2}dr\wedge d\psi\wedge d\theta_2 \\
& + 2\phi_{\theta_1}\sqrt{F_1F_2}\cos\theta_2 dr\wedge d\theta_1\wedge d\phi_2 + 2\phi_{\theta_2}\sqrt{F_1F_2}\cos\theta_1 dr\wedge d\theta_2\wedge d\phi_1 \\
& - 2\phi_{\theta_2}F_3\sin\theta_1 d\theta_1\wedge d\theta_2\wedge d\phi_1 + 2\phi_{\theta_1}F_4\sin\theta_2 d\theta_1\wedge d\theta_2\wedge d\phi_2 \qquad (3.18)
\end{aligned}$$

where we have defined $\phi_\alpha \equiv \partial_\alpha\phi$ with $\alpha = \theta_i, r$ as $\phi \equiv \phi(r,\theta_1,\theta_2)$ for simplicity. A constant ϕ is not good for us, and also leads to certain issues detailed in [90]. Once we have H_3, we can get dH_3 as:

$$\begin{aligned}
\frac{dH_3}{\sinh\beta} = & \Big[(hF_0^2)_{\theta_2}(2\phi_{\theta_1}\sqrt{F_1F_2}\cos\theta_1 + \sqrt{F_1F_2}\sin\theta_1 + 2\phi_r F_3\sin\theta_1 - F_{3r}\sin\theta_1) \\
& -2(hF_0^2)_{\theta_1}\phi_{\theta_2}\sqrt{F_1F_2}\cos\theta_1 - 2(hF_0^2)_r\phi_{\theta_2}F_3\sin\theta_1\Big]dr\wedge d\theta_1\wedge d\theta_2\wedge d\phi_1 \\
& +\Big[-(hF_0^2)_{\theta_1}(2\phi_{\theta_2}\sqrt{F_1F_2}\cos\theta_2 + \sqrt{F_1F_2}\sin\theta_2 + 2\phi_r F_4\sin\theta_2 - F_{4r}\sin\theta_2) \\
& +2(hF_0^2)_{\theta_2}\phi_{\theta_1}\sqrt{F_1F_2}\cos\theta_2 + 2(hF_0^2)_r\phi_{\theta_1}F_4\sin\theta_2\Big]dr\wedge d\theta_1\wedge d\theta_2\wedge d\phi_2 \\
& +2\Big[(hF_0^2)_{\theta_2}\phi_{\theta_1}\sqrt{F_1F_2} - (hF_0^2)_{\theta_1}\phi_{\theta_2}\sqrt{F_1F_2}\Big]dr\wedge d\theta_1\wedge d\theta_2\wedge d\psi \\
& +2hF_0^2\phi_{\theta_2}\sin\theta_1(\sqrt{F_1F_2} - F_{3r})dr\wedge d\theta_1\wedge d\theta_2\wedge d\phi_1 \\
& +2hF_0^2\phi_{\theta_1}\sin\theta_2(F_{4r} - \sqrt{F_1F_2})dr\wedge d\theta_1\wedge d\theta_2\wedge d\phi_2 \qquad (3.19)
\end{aligned}$$

with $F_{ir} \equiv \partial_r F_i$ and $(hF_0^2)_i = \partial_i(hF_0^2)$. From (3.19) it means that if we make $(\sqrt{F_1F_2} - F_{3r})$, $(F_{4r} - \sqrt{F_1F_2})$, $\Big((hF_0^2)_{\theta_1}\phi_{\theta_2} - (hF_0^2)_{\theta_2}\phi_{\theta_1}\Big)$ and $\Big((hF_0^2)_{\theta_1}\phi_r - (hF_0^2)_r\phi_{\theta_1}\Big)$ identically zero then H_3 will be closed. One however may worry that making H_3 closed implies too much constraints on the F_i's. For the present case this may still be okay because the initial choice of the background (3.12) forms a class of solutions parametrised by our choice of F_i and the dilaton ϕ. A specific choice of the background with a specified complex structure and Kähler class is exemplified in Appendix A of [42]. For this case we can define a closed three-form with appropriate choice of the dilaton, so that our choice remains generic enough. Thus the B_{NS} field can be gauge transformed to have only the following components:

$$\begin{aligned} b_{r\psi} &= \int -2hF_0^2 \sinh\beta\ \phi_{\theta_1}\sqrt{F_1F_2}\ d\theta_1, & b_{\theta_1\phi_1} &= \int 2hF_0^2\sinh\beta\ \phi_{\theta_2}F_3\sin\theta_1\ d\theta_2, \\ b_{r\phi_1} &= \int -2hF_0^2 \sinh\beta\ \phi_{\theta_2}\sqrt{F_1F_2}\cos\theta_1\ d\theta_2, & b_{\theta_2\phi_2} &= \int 2hF_0^2\sinh\beta\ \phi_{\theta_1}F_4\sin\theta_2\ d\theta_1, \\ b_{r\phi_2} &= \int -2hF_0^2 \sinh\beta\ \phi_{\theta_1}\sqrt{F_1F_2}\cos\theta_2\ d\theta_2 \end{aligned} \tag{3.20}$$

where we see that there are three new components of the form $b_{r\alpha}$ compared to the local case [44, 112]. This is expected because we are no longer fixed to $r = r_0$, but have global access. However before moving ahead we will pause to comment on switching on other possible components of the B_{NS} field of the form:

$$b_{\phi_1\phi_2}\ d\phi_1 \wedge d\phi_2\ +\ \sum_{i=1}^{2} b_{\phi_i\psi}\ d\phi_i \wedge d\psi \tag{3.21}$$

Such choices of B_{NS} fields would make the type IIA background *non-geometric*. So far locally we saw that the type IIA backgrounds remains geometric but does become non-Kähler. Is there a possibility that the IIA background globally is non-geometric also? We will reflect on this point later, but for the time being we will assume that the B_{NS} field is only of the form (3.20) and doesn't have additional components like (3.21).

Next comes the RR three-form F_3. From our previous set of duality arguments, this is given by:

$$\begin{aligned}
\frac{F_3}{hF_0^2\cosh\beta} = \quad & 2KF_1F_2F_3F_4\sin\theta_2\sin\theta_1(\phi_{\theta_1}\sin\theta_1\cos\theta_2 - \phi_{\theta_2}\sin\theta_2\cos\theta_1)dr\wedge d\phi_1\wedge d\phi_2 \\
& + KF_3^2\sin^2\theta_1\sin\theta_2(2\phi_{\theta_2}\sqrt{F_1F_2}F_4\sin\theta_2 - F_2\sqrt{F_1F_2}\cos\theta_2 \\
& - 2\phi_rF_2F_4\cos\theta_2 + F_2F_{4r}\cos\theta_2)d\theta_1\wedge d\phi_1\wedge d\phi_2 \\
& + KF_4^2\sin^2\theta_2\sin\theta_1(-2\phi_{\theta_1}\sqrt{F_1F_2}\sin\theta_1 + F_2\sqrt{F_1F_2}\cos\theta_1 \\
& + 2\phi_rF_2F_3\cos\theta_1 - F_2F_{3r}\cos\theta_1)d\theta_2\wedge d\phi_1\wedge d\phi_2 \\
& - KF_2F_3^2\sin^2\theta_1(2\phi_rF_4\sin\theta_2 + \sqrt{F_1F_2}\sin\theta_2 - F_{4r}\sin\theta_2)d\psi\wedge d\theta_1\wedge d\phi_1 \\
& - KF_2F_4^2\sin^2\theta_2(2\phi_rF_3\sin\theta_1 + \sqrt{F_1F_2}\sin\theta_1 - F_{3r}\sin\theta_1)d\psi\wedge d\theta_2\wedge d\phi_2 \\
& - 2\phi_{\theta_2}KF_1F_2F_3F_4\sin\theta_1\sin^2\theta_2 dr\wedge d\psi\wedge d\phi_2 \\
& - 2\phi_{\theta_1}KF_1F_2F_3F_4\sin\theta_2\sin^2\theta_1 dr\wedge d\psi\wedge d\phi_1 \qquad (3.22)
\end{aligned}$$

where as before ϕ_α should be understood as derivatives on ϕ i.e $\partial_\alpha\phi$, and we have defined K as:

$$K = \frac{\text{cosec }\theta_1\text{cosec }\theta_2}{\sqrt{F_1F_2}F_3F_4} \qquad (3.23)$$

Note that dF_3 is no longer closed, and will be related to delta function sources coming from the wrapped D5-branes.

Once we have the explicit forms for the three-forms, to satisy the type IIB EOMs we will now require RR five-form. This is easy to work out, and is given by:

$$\begin{aligned}
F_5 \quad = \quad & \frac{1}{4}\Big[- A_{0r}dr\wedge dt\wedge dx\wedge dy\wedge dz - A_{0\theta_1}d\theta_1\wedge dt\wedge dx\wedge dy\wedge dz \\
& - A_{0\theta_2}d\theta_2\wedge dt\wedge dx\wedge dy\wedge dz - PF_2F_3F_4\sin^2\theta_1\sin^2\theta_2 \\
& \times(A_{0r}F_3F_4 d\psi\wedge d\theta_1\wedge d\theta_2\wedge d\phi_1\wedge d\phi_2 + A_{0\theta_1}F_1F_4 dr\wedge d\psi\wedge d\theta_2\wedge d\phi_1\wedge d\phi_2 \\
& + A_{0\theta_2}F_1F_3 dr\wedge d\psi\wedge d\theta_1\wedge d\phi_1\wedge d\phi_2)\Big] \qquad (3.24)
\end{aligned}$$

where $A_{0\alpha} \equiv \partial_\alpha A_0$ and A_0 is given in (3.16). We have also defined P as:

$$P = \frac{\text{cosec } \theta_1 \text{cosec } \theta_2}{\sqrt{F_1 F_2} F_0^2 F_3 F_4} \tag{3.25}$$

Thus with (3.18), (3.22), (4.44) and (3.15) we have the complete susy background in type IIB before geometric transition. In the following subsection, we will use this to compute the type IIA mirror configuration.

3.2.3 The type IIA mirror configuration

As it stands, the metric in (3.15) has three obvious isometries associated with translation along the three angular directions ϕ_1, ϕ_2 and ψ. So there is a natural $\mathbf{T}^3$ embedded in our configuration, and one might naively think that the mirror would be three T-dualities along $\mathbf{T}^3$. Such a simple transformation doesn't work for our case because our configuration represents the IR limit of a cascading gauge theory where the base of the three torus is *small*. Mirror transformation *a la* SYZ [80] works exactly in the opposite limit! So naive T-dualities will not give us the mirror metric, and we need to first make the base, paramerised by θ_1, θ_2 and r, very large[2] . The simplest way to do this would be to make the following transformation on the background (3.15):

$$\begin{aligned}
&d\psi \ \rightarrow \ d\psi \ + \ f_1 \cos\theta_1 \ d\theta_1 \ + \ f_2 \cos\theta_2 \ d\theta_2 \\
&d\phi_1 \ \rightarrow \ d\phi_1 \ - \ f_1 \ d\theta_1, \qquad d\phi_2 \ \rightarrow \ d\phi_2 \ - \ f_2 \ d\theta_2
\end{aligned} \tag{3.26}$$

with the assumption that $f_i = f_i(\theta_i)$ so that the transformations (3.26) would be integrable. Note that these transformations are similar in form as in the first reference of [44, 112] and would change the complex structure of the base accordingly.

[2] This effectively means that the distances along the θ_i directions have to be made very large, as r is non-compact. See also our earlier works [44, 112] where this has been explained in more details.

Under these transformations the $B_{\rm NS}$ field generates extra components $b_{r\theta_1}$, $b_{r\theta_2}$. It is however interesting to note that they vanish as follows:

$$b_{r\theta_1} = f_1(b_{r\psi}\cos\theta_1 - b_{r\phi_1}) = 0, \;\; b_{r\theta_2} = f_2(b_{r\psi}\cos\theta_2 - b_{r\phi_2}) = 0 \tag{3.27}$$

implying that the $B_{\rm NS}$ field do not change under the transformation (3.26). This is similar to the local case also [44, 112].

On the other hand the RR three-form *does* change under the coordinate transformation (3.26). The extra components of the three-form are the following:

$$\begin{aligned}
&F_{r\theta_1\theta_2} = f_1 f_2(F_{r\phi_1\phi_2} - \cos\theta_1 F_{r\psi\phi_2} + \cos\theta_2 F_{r\psi\phi_1}), \quad F_{r\psi\theta_1} = -f_1 F_{r\psi\phi_1}\\
&F_{r\theta_1\phi_2} = -f_1(F_{r\phi_1\phi_2} - \cos\theta_1 F_{r\psi\phi_2}), \quad F_{\theta_1\theta_2\phi_1} = f_2(F_{\theta_1\phi_1\phi_2} - \cos\theta_2 F_{\psi\theta_1\phi_1}),\\
&F_{r\theta_2\phi_1} = f_2(F_{r\phi_1\phi_2} + \cos\theta_2 F_{r\psi\phi_1}), \quad F_{\theta_1\theta_2\phi_2} = f_1(F_{\theta_2\phi_1\phi_2} + \cos\theta_1 F_{\psi\theta_2\phi_2}),\\
&F_{r\theta_2\phi_2} = f_2\cos\theta_2 F_{r\psi\phi_2}, \quad F_{r\psi\theta_2} = -f_2 F_{r\psi\phi_2}, \quad F_{r\theta_1\phi_1} = f_1\cos\theta_1 F_{r\psi\phi_1}
\end{aligned} \tag{3.28}$$

A physical reason for this change can be easily understood: under the coordinate transformation (3.26) the base parametrised by (θ_1, θ_2) become large. This means that the associated RR three-form field strengths increase simultaneously, which is of course what we see in (3.28). Note that the component $F_{r\theta_1\theta_2}$ dominates over all other extra components in (3.28) because this lies exclusively on the base parametrised by the coordinates (r, θ_1, θ_2) which is made much bigger than the $\mathbf{T}^3$ fibre parametrised by the coordinates (ψ, ϕ_1, ϕ_2).

Once the three-form F_3 changes, the RR five-form also has to change. Its is easy to show that the extra components of the five-form are:

$$\begin{aligned}
&F_{r\theta_1\theta_2\phi_1\phi_2} = f_1\cos\theta_1 F_{r\psi\theta_2\phi_1\phi_2} - f_2\cos\theta_2 F_{r\psi\theta_1\phi_1\phi_2},\\
&F_{r\psi\theta_1\theta_2\phi_2} = f_1 F_{r\psi\theta_2\phi_1\phi_2}, \quad F_{r\psi\theta_1\theta_2\phi_1} = f_2 F_{r\psi\theta_1\phi_1\phi_2}
\end{aligned} \tag{3.29}$$

satisfying the background EOMs. All these extra components will give rise to RR four-form in Type IIA after mirror transformation, as we will show soon. But before that let us infer how the metric changes. Under the transformation (3.26) the metric

(3.15) takes the following form:

$$
\begin{aligned}
ds^2 = \quad & F_0 ds^2_{0123} + F_1\ dr^2 + F_2(d\psi + \cos\ \theta_1 d\phi_1 + \cos\ \theta_2 d\phi_2)^2 + \sum_{i=1}^{2} F_{2+i}\ \sin^2\theta_i d\phi_i^2 \\
& + \sum_{i=1}^{2}\left[F_{2+i}\left(1 + f_i^2\ \sin^2\theta_i\right) d\theta_i^2 - 2 f_i F_{2+i}\ \sin^2\theta_i\ d\phi_i d\theta_i\right]
\end{aligned}
\tag{3.30}
$$

which in fact does exactly what we wanted: it enlarges the θ_i-cycles, but doesn't change the $B_{\rm NS}$ field. For SYZ to work properly, we require the base size to be very large, and therefore we will require f_i also to be large. This conclusion fits well with the local picture that we had in [44, 112]. Note that we have also generated cross terms. These cross terms will be useful soon. The eleven metric components are:

$$
\begin{aligned}
& j_{rr} = F_1, \quad j_{\phi_1\theta_1} = -f_1 F_3 \sin^2\theta_1, \quad j_{\phi_2\theta_2} = -f_2 F_4 \sin^2\theta_2 \\
& j_{\psi\psi} = F_2(1-\epsilon), \quad j_{\phi_1\psi} = F_2 \cos\ \theta_1, \quad j_{\phi_2\psi} = F_2 \cos\ \theta_2 \\
& j_{\phi_1\phi_1} = F_2\cos^2\theta_1 + F_3\sin^2\theta_1, \quad j_{\phi_2\phi_2} = F_2\cos^2\theta_2 + F_4\sin^2\theta_2 \\
& j_{\phi_1\phi_2} = F_2 \cos\ \theta_1\theta_2, \quad j_{\theta_1\theta_1} = F_3(1 + f_1^2\sin^2\theta_1), \quad j_{\theta_2\theta_2} = F_4(1 + f_2^2\sin^2\theta_2)
\end{aligned}
\tag{3.31}
$$

where ϵ is a very small number. Let us also define another quantity α in the following way:

$$
\alpha^{-1}\ \equiv\ F_3F_4\sin^2\theta_1\sin^2\theta_2 + F_2F_4\cos^2\theta_1\sin^2\theta_2 + F_2F_3\sin^2\theta_1\cos^2\theta_2 \tag{3.32}
$$

away from the point $(\theta_1, \theta_2) = 0$. Now assuming that f_1, f_2 are very large, we can perform the mirror transformation along (ψ, ϕ_1, ϕ_2) directions. The mirror metric in type IIA takes the following form:

$$
ds^2_{\rm mirror} = F_0 ds^2_{0123} + ds^2_6 \tag{3.33}
$$

where the six-dimensional internal space is a non-Kähler deformation of the deformed conifold in the following way:

$$\begin{aligned}
ds_6^2 = F_1 dr^2 + \frac{\alpha F_2}{\Delta_1\Delta_2}\Big[d\psi - b_{\psi r}dr + \Delta_1 \cos\,\theta_1\Big(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr\Big) \\
+\Delta_2 \cos\,\theta_2\Big(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr\Big)\Big]^2 \\
+\,\alpha j_{\phi_2\phi_2}\Big(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr\Big)^2 + \alpha j_{\phi_1\phi_1}\Big(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr\Big)^2 \\
-\,2\alpha j_{\phi_1\phi_2}\Big(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr\Big)\Big(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr\Big) \\
-\,2\alpha j_{\phi_1\phi_2}\left(\frac{f_1 f_2\epsilon}{\alpha}\right)d\theta_1 d\theta_2 + \Big(F_3 - \epsilon\ F_2 f_1^2\cos^2\theta_1\Big)d\theta_1^2 + \Big(F_4 - \epsilon\ F_2 f_2^2\cos^2\theta_2\Big)d\theta_2^2
\end{aligned} \tag{3.34}$$

and we have defined Δ_i in the following way:

$$\Delta_1 \ = \ \alpha F_2 F_4 \sin^2\theta_2, \quad \Delta_2 \ = \ \alpha F_2 F_3 \sin^2\theta_1 \tag{3.35}$$

At this stage we can extract the consequence of the fact that both f_1 and f_2 are very large. This fits perfectly well with the mirror metric because f_i^2 as well as $f_1 f_2$ come with the coefficient ϵ. This means that if we impose the following constraint:

$$f_1 f_2 \epsilon \ \equiv \ -\alpha \tag{3.36}$$

i.e both f_i proportional to $\epsilon^{-1/2}$, it will bring the cross-terms in the metric to the following suggestive form:

$$2\alpha j_{\phi_1\phi_2}\Big[d\theta_1 d\theta_2 - \Big(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr\Big)\Big(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr\Big)\Big] \tag{3.37}$$

In the limit $b_{\phi_1\alpha} = b_{\phi_2\alpha} = 0$ with $\alpha = r, \theta_i$, (3.37) is in fact a term of the deformed conifold! The above conclusion seems rather encouraging, provided of course (3.36) is satisfied. In the local limit, similar condition also arose (see the first reference of [44, 112]) and we argued therein that as long as we can define

$$f_i \ \propto \ \frac{(-1)^i\langle\alpha\rangle_i}{\sqrt{\epsilon}} \tag{3.38}$$

where $\langle\alpha\rangle_i$ depend only on θ_i the constraint (3.36) is satisfied. Therefore a condition like (3.36) works perfectly well in the local case. Question is, can we satisfy (3.36) also for the global case?

The answer is now tricky. We demanded that $f_i = f_i(\theta_i)$, otherwise global coordinate transformation like (3.26) *cannot* be defined. This means that F_i appearing in the definition of α in (3.36) will have to be highly constrained. Generically this is not possible[3] , but in special case this may happen.

The special case arises if we allow F_2 to depend on the angular coordiate θ_i also in such a way that

$$F_2(r,\theta_1,\theta_2) \;=\; -\,\frac{(\beta_1\beta_2)^{-1} + F_3F_4\sin^2\theta_1\sin^2\theta_2}{F_4\cos^2\theta_1\sin^2\theta_2 + F_3\sin^2\theta_1\cos^2\theta_2} \tag{3.39}$$

where $f_i \equiv \frac{\beta_i}{\sqrt{\epsilon}}$. This tells us that the radial dependence of F_2 is fixed by $F_3(r)$ and $F_4(r)$, but the angular dependences are pretty much unfixed because $\beta_i(\theta_i)$ are arbitrary functions of θ_i respectively. However the above relation (3.39) already looks tight, but let us move on and see how far we can go with these kind of arguments. Our next question would therefore be: is there a way to fix the angular dependences also?

To see how to fix the angular dependences, we can go back to the equivalent local limit of (3.37) where the particular way of writing the metric allows us to make a coordinate rotation to bring the term (3.37) into the more familar deformed conifold form [91]. This, as we know from [91, 44, 112], is only possible iff *other* terms in the metric remain invariant under the coordinate transformation. If this condition is

[3] For the local case α was defined at $r = r_0$ so this subtlety did not arise and, as we discussed above, we used $\langle\alpha\rangle_i$ to define f_i so things were perfectly consistent there.

imposed globally, then it would imply the following two relations:

$$\begin{aligned}\beta_1 &= \pm\sqrt{\frac{F_3-\alpha j_{\phi_2\phi_2}}{F_2\cos^2\theta_1}}\\ \beta_2 &= \mp\sqrt{\frac{F_4-\alpha j_{\phi_1\phi_1}}{F_2\cos^2\theta_1}}\end{aligned}\tag{3.40}$$

In the local case, studied in the first reference of [44, 112], relations like (3.40) are consistent in the sense that (3.36) is satisfied. Unfortunately, this is no longer true for the global case generically because the above relation along with (3.36) would lead to inconsistent set of equations. Therefore in general the mirror metric will take the following form:

$$\begin{aligned}ds_6^2 = F_1 dr^2 + \frac{\alpha F_2}{\Delta_1\Delta_2}\Big[d\psi - b_{\psi r}dr + \Delta_1\cos\,\theta_1\Big(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr\Big)\\ +\Delta_2\cos\,\theta_2\Big(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr\Big)\Big]^2\\ + \alpha j_{\phi_2\phi_2}\Big(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr\Big)^2 + \alpha j_{\phi_1\phi_1}\Big(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr\Big)^2\\ - 2\alpha j_{\phi_1\phi_2}\Big(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr\Big)\Big(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr\Big)\\ - 2j_{\phi_1\phi_2}\beta_1\beta_2\; d\theta_1 d\theta_2 + \Big(F_3 - F_2\beta_1^2\cos^2\theta_1\Big)d\theta_1^2 + \Big(F_4 - F_2\beta_2^2\cos^2\theta_2\Big)d\theta_2^2\end{aligned}\tag{3.41}$$

Only in very special cases, where (3.40) and (3.36) are both simultaneously satisfied, we expect the mirror to take the following symmetric form:

$$\begin{aligned}ds_6^2 = \quad & F_1 dr^2 + \frac{\alpha F_2}{\Delta_1\Delta_2}\Big[d\psi - b_{\psi r}dr + \Delta_1\cos\,\theta_1\Big(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr\Big)\\ & +\Delta_2\cos\,\theta_2\Big(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr\Big)\Big]^2\\ & +\alpha j_{\phi_2\phi_2}\Big[d\theta_1^2 + \Big(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr\Big)^2\Big]\\ & +\alpha j_{\phi_1\phi_1}\Big[d\theta_2^2 + \Big(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr\Big)^2\Big]\\ & +2\alpha j_{\phi_1\phi_2}\Big[d\theta_1 d\theta_2 - \Big(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr\Big)\Big(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr\Big)\Big]\end{aligned}\tag{3.42}$$

which is strongly reminiscent of the deformed conifold! Observe that both the forms of the metrics are finite and well defined. This tells us that our procedure of making the base large before performing SYZ [80] is logical and correct.

On the other hand, the cross term that we developed in the metric appears as the $B_{\rm NS}$ field in type IIA theory. Expectedly, this B-field is large and is given by the following form:

$$
\begin{aligned}
\widetilde{B} \quad &= \alpha f_1 F_3 \sin^2\theta_1 \left(F_2 \cos^2\theta_2 + F_4 \sin^2\theta_2\right) d\theta_1 \wedge d\phi_1 \\
&+ \alpha f_2 F_4 \sin^2\theta_2 \left(F_2 \cos^2\theta_1 + F_3 \sin^2\theta_1\right) d\theta_2 \wedge d\phi_2 \\
&+ \left(1 - \frac{\epsilon}{\alpha F_2 F_4 \sin^2\theta_1 \sin^2\theta_2}\right)(f_1 \cos\theta_1 d\theta_1 + f_2 \cos\theta_2 d\theta_2) \wedge d\psi
\end{aligned} \tag{3.43}
$$

In the limit $\epsilon \to 0$, the last two terms are pure gauge. In the local limit (see the first paper of [112]) all the F_i were constants, and so $\widetilde{B}$ became a pure gauge when written in terms of $\langle\alpha\rangle_i$. This doesn't seem to be the case globally, unless of course F_i's are of some specific forms.

The wrapped D6 brane two-form charges now come partly from the type IIB three-forms and partly from the five-forms. The three-forms contributions to the IIA two-forms are given by the following components:

$$
\begin{aligned}
&\widetilde{F}_{\psi\theta_1} = F_{\phi_1\phi_2\theta_1}, \quad \widetilde{F}_{\psi\theta_2} = F_{\phi_1\phi_2\theta_2}, \quad \widetilde{F}_{\psi r} = F_{\phi_1\phi_2 r}, \\
&\widetilde{F}_{\phi_1 r} = F_{r\phi_2\psi} + \frac{2j_{\phi_1\phi_2}}{j_{\phi_1\phi_1}} F_{\phi_1 r\psi} + \frac{2j_{\psi\phi_1}}{j_{\phi_1\phi_1}} F_{r\phi_1\phi_2}, \\
&\widetilde{F}_{\phi_2 r} = F_{\phi_1 r\psi} + 2\alpha(j_{\phi_2\psi} j_{\phi_1\phi_1} - j_{\phi_1\phi_2} j_{\phi_1\psi}) F_{r\phi_1\phi_2}, \\
&\widetilde{F}_{\phi_1\theta_2} = F_{\psi\theta_2\phi_2} + 2\frac{j_{\phi_1\psi}}{j_{\phi_1\phi_1}} F_{\phi_1\phi_2\theta_2}, \\
&\widetilde{F}_{\phi_1\theta_1} = 2\frac{j_{\phi_1\phi_2}}{j_{\phi_1\phi_1}} F_{\phi_1\theta_1\psi} + 2\frac{j_{\psi\phi_1}}{j_{\phi_1\phi_1}} F_{\phi_1\phi_2\theta_1}, \\
&\widetilde{F}_{\phi_2\theta_1} = F_{\psi\phi_1\theta_1} + 2\alpha(j_{\phi_2\psi} j_{\phi_1\phi_1} - j_{\phi_1\phi_2} j_{\phi_1\psi}) F_{\phi_1\phi_2\theta_1}, \\
&\widetilde{F}_{\phi_2\theta_2} = 2\alpha(j_{\phi_2\psi} j_{\phi_1\phi_1} - j_{\phi_1\phi_2} j_{\phi_1\psi}) F_{\phi_1\phi_2\theta_2}
\end{aligned} \tag{3.44}
$$

Similarly, the five-forms contributions to the type IIA two-forms are given in terms of the following components:

$$\begin{aligned}
\widetilde{F}_{\theta_1\theta_2} &= F_{\psi\theta_1\theta_2\phi_1\phi_2} + b_{\theta_2\phi_2}F_{\psi\theta_1\phi_1} + b_{\theta_1\phi_1}F_{\psi\theta_2\phi_2} \\
\widetilde{F}_{r\theta_1} &= F_{r\theta_1\phi_1\phi_2\psi} + b_{theta_1\phi_1}F_{r\psi\phi_2} + \left(b_{\phi_2 r} + \frac{j_{\phi_1\phi_2}}{j_{\phi_1\phi_1}}b_{r\phi_1}\right)F_{\psi\theta_1\phi_1} + \frac{j_{\psi\phi_1}}{j_{\phi_1\phi_1}}b_{\theta_1\phi_1}F_{r\phi_1\phi_2} \\
&\quad + \left(b_{r\psi} - \frac{j_{\psi\phi_1}}{j_{\phi_1\phi_1}}b_{r\phi_1}\right)F_{\theta_1\phi_1\phi_2} + \frac{j_{\phi_1\phi_2}}{j_{\phi_1\phi_1}}b_{\theta_1\phi_1}F_{r\psi\phi_1} \\
\widetilde{F}_{r\theta_2} &= F_{r\theta_2\phi_1\phi_2\psi} + b_{r\phi_1}F_{\psi\theta_2\phi_2} + b_{r\psi}F_{\theta_2\phi_1\phi_2} - b_{\theta_2\phi_2}F_{r\psi\phi_1} \qquad (3.45)
\end{aligned}$$

All the above components are finite and give rise to the required D6-branes charges. However since the B-field is large, to compensate this in the EOMs we need large G-fluxes in type IIA. These fluxes come exactly from the extra three- and five-form components (3.28) and (3.29) respectively. These three- and five-form components give rise to twelve components of the four-form fluxes in IIA namely:

$$\begin{aligned}
&\widetilde{F}_{r\psi\theta_1\theta_2}, \quad \widetilde{F}_{r\psi\theta_1\phi_1}, \quad \widetilde{F}_{r\psi\theta_1\phi_2}, \quad \widetilde{F}_{r\psi\theta_2\phi_1} \\
&\widetilde{F}_{r\psi\theta_2\phi_2}, \quad \widetilde{F}_{r\theta_1\theta_2\phi_1}, \quad \widetilde{F}_{r\theta_1\theta_2\phi_2}, \quad \widetilde{F}_{r\theta_1\phi_1\phi_2} \\
&\widetilde{F}_{r\theta_2\phi_1\phi_2}, \quad \widetilde{F}_{\psi\theta_1\theta_2\phi_1}, \quad \widetilde{F}_{\psi\theta_1\theta_2\phi_2}, \quad \widetilde{F}_{\theta_1\theta_2\phi_1\phi_2} \qquad (3.46)
\end{aligned}$$

These components are listed in Appendix B of [42] which the readers may refer to for details. Combined with (3.43), these fluxes lift to M-theory as G-fluxes with components along the spacetime and the eleventh directions respectively. Interestingly, both the metric (3.41) or (3.42) along with the two-form flux components (3.44) and (3.45) lift to a geometrical configuration in M-theory, which we expect to have a G_2 structure. This is of course expected because both the non-Kähler deformed conifold as well as the wrapped D6-branes tend to become geometrical configurations when the type IIA coupling is made very large. In the following sub-section we will dwell on this in more details.

3.2.4 M theory lift, flop transition and type IIA reduction

The lift of our type IIA mirror configuration to M-theory is rather straighforward. The eleven-directional fibration is given by gauge fluxes derived from the two-form components (3.44) and (3.45). It is easy to show that we need only A_{ϕ_i}, A_{θ_i} and A_r components. Using these, the M-theory lift of our IIA symmetric mirror metric (3.42) is:

$$
\begin{aligned}
ds^2_{11} = \quad & e^{-\frac{2\phi}{3}}\Bigg\{F_0 ds^2_{0123} + F_1 dr^2 + \frac{\alpha F_2}{\Delta_1\Delta_2}\Big[d\psi - b_{\psi r}dr \\
& +\Delta_1 \cos\,\theta_1\Big(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr\Big) + \Delta_2 \cos\,\theta_2\Big(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr\Big)\Big]^2 \\
& +\alpha j_{\phi_2\phi_2}\Big[d\theta_1^2 + \Big(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr\Big)^2\Big] \\
& +\alpha j_{\phi_1\phi_1}\Big[d\theta_2^2 + \Big(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr\Big)^2\Big] \\
& +2\alpha j_{\phi_1\phi_2}\Big[d\theta_1 d\theta_2 - \Big(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr\Big)\Big(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr\Big)\Big]\Bigg\} \\
& +e^{\frac{4\phi}{3}}\Big[dx_{11} + A_{\phi_1}d\phi_1 + A_{\phi_2}d\phi_2 + A_{\theta_1} + A_{\theta_2}d\theta_2 + A_r dr\Big]^2 \qquad (3.47)
\end{aligned}
$$

It is easy to see that the non-symmetric mirror metric (3.41) will also lift to M-theory in an identical way. The local limit of the (3.47) is precisely the one discussed in the first paper of [44, 112] and, as discussed therein, we expect the manifold (3.47) to have a G_2 structure to preserve supersymmetry. To see this for our case, we have to express the metric (3.47) in terms of certain one-forms similar to the ones given in [112] (see also [92]). Following the first paper of [112] we first express the B-fields appearing in the fibration of (3.47) in terms of periodic angular coordinates λ_i in the following way:

$$
\begin{aligned}
&\tan\lambda_1 \;\equiv\; b_{\phi_1\theta_1}, \quad \tan\lambda_2 \;\equiv\; b_{\phi_2\theta_2}, \quad \tan\lambda_3 \;\equiv\; b_{\psi r} \\
&\tan\lambda_4 \;\equiv\; b_{\phi_1 r}, \quad\; \tan\lambda_5 \;\equiv\; b_{\phi_2 r} \qquad (3.48)
\end{aligned}
$$

Using these we can define two set of one-forms. The first set, called σ_i with $i = 1, .., 3$, can be expressed in terms of λ_i as:

$$\begin{aligned}
\sigma_1 &= \sin\psi_1(d\phi_1 - \tan\lambda_4 dr) + \sec\lambda_1\cos(\psi_1+\lambda_1)d\theta_1, \\
\sigma_2 &= \cos\psi_1(d\phi_1 - \tan\lambda_4 dr) - \sec\lambda_1\sin(\psi_1+\lambda_1)d\theta_1, \\
\sigma_3 &= d\psi_1 - \frac{1}{2}\tan\lambda_3 dr + \Delta_1\cos\theta_1(d\phi_1 - \tan\lambda_1 d\theta_1 - \tan\lambda_4 dr)
\end{aligned} \tag{3.49}$$

and the second set can be expressed in terms of λ_i as:

$$\begin{aligned}
\Sigma_1 &= -\sin\psi_2(d\phi_2 - \tan\lambda_5 dr) + \sec\lambda_2\cos(\psi_2+\lambda_2)d\theta_2, \\
\Sigma_2 &= -\cos\psi_2(d\phi_2 - \tan\lambda_5 dr) - \sec\lambda_2\sin(\psi_2+\lambda_2)d\theta_2, \\
\Sigma_3 &= d\psi_2 + \frac{1}{2}\tan\lambda_3 dr - \Delta_2\cos\theta_2(d\phi_2 - \tan\lambda_2 d\theta_2 - \tan\lambda_5 dr)
\end{aligned} \tag{3.50}$$

At this stage one may compare these two set of one-forms to the ones given by eq. (6.2) and eq. (6.3) in the first paper of [112]. The definition of ψ_1 and ψ_2 follow exactly as in [112], i.e

$$d\psi = d\psi_1 - d\psi_2, \qquad dx_{11} = d\psi_1 + d\psi_2 \tag{3.51}$$

Furthermore we can perform the following rotation of the coordinates:

$$\begin{pmatrix} \mathcal{D}\phi_2 \\ d\theta_2 \end{pmatrix} \to \begin{pmatrix} \cos\ \psi_0 & -\sin\ \psi_0 \\ \sin\ \psi_0 & \cos\ \psi_0 \end{pmatrix} \begin{pmatrix} \mathcal{D}\phi_2 \\ d\theta_2 \end{pmatrix} \tag{3.52}$$

with $\mathcal{D}\phi_2 \equiv d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr$ and ψ_0 a constant. If we make this transformation to the symmetric mirror metric of type IIA (3.42), this will lift to M-theory not as

(3.47), but to a more *suggestive* configuration:

$$
\begin{aligned}
ds^2_{11} = \quad & e^{-\frac{2\phi}{3}}\Bigg\{F_0 ds^2_{0123} + F_1 dr^2 + \frac{\alpha F_2}{\Delta_1\Delta_2}\Big[d\psi - b_{\psi r}dr - b_{\psi\theta_2}d\theta_2 \\
& +\Delta_1\cos\,\theta_1\Big(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr\Big) + \Delta_2\cos\,\theta_2\cos\,\psi_0\Big(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr\Big)\Big]^2 \\
& +\alpha j_{\phi_2\phi_2}\Big[d\theta_1^2 + \Big(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr\Big)^2\Big] \\
& +\alpha j_{\phi_1\phi_1}\Big[d\theta_2^2 + \Big(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr\Big)^2\Big] \\
& +2\alpha j_{\phi_1\phi_2}\cos\,\psi_0\Big[d\theta_1 d\theta_2 - \Big(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr\Big)\Big(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr\Big)\Big] \\
& +2\alpha j_{\phi_1\phi_2}\sin\,\psi_0\Big[\Big(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr\Big)d\theta_2 + \Big(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr\Big)d\theta_1\Big]\Bigg\} \\
& +e^{\frac{4\phi}{3}}\Big[dx_{11} + \widetilde{A}_{\phi_1}d\phi_1 + \widetilde{A}_{\phi_2}d\phi_2 + \widetilde{A}_{\theta_1} + \widetilde{A}_{\theta_2}d\theta_2 + \widetilde{A}_r dr\Big]^2
\end{aligned}
\tag{3.53}
$$

where we have introduced a B-field fibration using $b_{\psi\theta_2} \equiv \Delta_2\,\sin\,\psi_0\cos\,\theta_2$ to modify the $d\psi$ fibration structure. The eleven-dimensional fibration structure will also change accordingly because we can always express the $A_{\phi_2}d\phi_2$ term in dx_{11} of (3.47) using $\mathcal{D}\phi_2$. Thus the overall eleven-dimensional fibration will retain its form but with shifted A_μ fields denoted above by the $\widetilde{A}_\mu$ fields. In terms of the fibration components of (3.47) one can show that $\widetilde{A}_{\phi_1} = A_{\phi_1}, \widetilde{A}_{\theta_1} = A_{\theta_1}$ and the rest of the components can be presented in the following matrix form:

$$
\begin{pmatrix} \widetilde{A}_{\phi_2} \\ \\ \widetilde{A}_{\theta_2} \\ \\ \widetilde{A}_r \end{pmatrix} = \begin{pmatrix} \cos\psi_0 + b_{\phi_2\theta_2}\sin\psi_0 & \sin\psi_0 & 0 \\ \\ -(1 + b^2_{\phi_2\theta_2})\sin\psi_0 & \cos\psi_0 - b_{\phi_2\theta_2}\sin\psi_0 & 0 \\ \\ b_{\phi_2 r}(1 - \cos\psi_0 - b_{\phi_2\theta_2}\sin\psi_0) & -b_{\phi_2 r}\sin\psi_0 & 1 \end{pmatrix} \begin{pmatrix} A_{\phi_2} \\ \\ A_{\theta_2} \\ \\ A_r \end{pmatrix}
\tag{3.54}
$$

Additionally with the above modification, the above metric is surprisingly close to the uplift of a non-Kähler deformed conifold metric with wrapped D6-branes to M-theory provided we can make an additional substitution in (3.53):

$$\psi_0 \quad \to \quad \psi \tag{3.55}$$

Making such a substitution may lead one to think that the ψ isometry that we have in (3.47) is removed. This is *not* the case with the non-Kähler deformed conifold because the extra B-field component $b_{\psi\theta_2}$ in the $d\psi$ fibration structure of (3.53) as well as the vector fields $\widetilde{A}_\mu$ in the dx_{11} fibration structure transform non-trivially under shift in ψ to restore the isometry! One may also do a somewhat similar rotation like (3.52) to the non-symmetric type IIA metric (3.41) and bring it in a more suggestive format.

Under the rotation (3.52), the one-forms (3.49) and (3.50) should also change. In fact only the second set of one-forms (3.50) changes under (3.52). These changes can be easily worked out and, to avoid cluttering of formulae, we will rename the changed one-forms (3.50) as Σ_i also. Thus the one-forms for our purposes will be (σ_i, Σ_i) with Σ_i to be viewed as the rotated one-forms. Using these one-forms we can rewrite the M-theory metric in two possible ways. The first one is the lift of the non-symmetric type IIA metric (3.41):

$$\begin{aligned}
ds_7^2 \quad &= g_r dr^2 + g_1(\sigma_3 + \Sigma_3)^2 + g_2(\sigma_3 - \Sigma_3)^2 \\
&+ g_3(\sin\psi_1\sigma_1 + \cos\psi_1\sigma_2)^2 + \widetilde{g}_3(\cos\psi_1\sigma_1 - \sin\psi_1\sigma_2)^2 \\
&+ g_4(\sin\psi_2\Sigma_1 + \cos\psi_2\Sigma_2)^2 + \widetilde{g}_4(\cos\psi_2\Sigma_1 - \sin\psi_2\Sigma_2)^2 \\
&+ g_5(\sin\psi_1\sigma_1 + \cos\psi_1\sigma_2)(\sin\psi_2\Sigma_1 + \cos\psi_2\Sigma_2) \\
&- \widetilde{g}_5(\cos\psi\sigma_1 - \sin\psi_1\sigma_2)(\cos\psi_2\Sigma_1 - \sin\psi_2\Sigma_2)
\end{aligned} \tag{3.56}$$

where we have defined the coefficients $g_i, \widetilde{g}_i$ as:

$$\begin{aligned}
&g_r = e^{-2\phi/3}F_1, \quad g_1 = e^{4\phi/3}, \quad g_2 = e^{-2\phi/3}\frac{\alpha F_2}{\Delta_1\Delta_2}, \quad g_3 = \alpha j_{\phi_2\phi_2}, \\
&\widetilde{g}_3 = F_3 - F_2\beta_1^2\cos^2\theta_1, \quad g_4 = \alpha j_{\phi_1\phi_1}, \quad \widetilde{g}_4 = F_4 - F_2\beta_2^2\cos^2\theta_2 \\
&g_5 = 2\alpha j_{\phi_1\phi_2}, \quad \widetilde{g}_5 = 2\beta_1\beta_2 j_{\phi_1\phi_2}
\end{aligned} \tag{3.57}$$

The second way to rewrite the metric is a little more suggestive of the way to perform the flop operation on the M-theory manifold and has a nice form for the symmetric case (3.42). The local form of this has already appeared in the first reference of [112], and the readers may want to look at that for more details. Here we will simply quote the result:

$$ds_7^2 = \alpha_1^2 \sum_{a=1}^{2} (\sigma_a + \zeta \Sigma_a)^2 + \alpha_2^2 \sum_{a=1}^{2} (\sigma_a - \zeta \Sigma_a)^2 + \alpha_3^2 (\sigma_3 + \Sigma_3)^2 + \alpha_4^2 (\sigma_3 - \Sigma_3)^2 + \alpha_5^2 dr^2 \tag{3.58}$$

The above is a familiar form by which any G_2 structure metric could be expressed. Once we switch off λ_i the manifolds has a G_2 holonomy. The coefficients α_i and ζ are not arbitrary. They are fixed by the EOM and, for our case, they take the following values:

$$\begin{gathered}\alpha_1 = \frac{1}{2} e^{-\frac{\phi}{3}} \sqrt{2\alpha \left(j_{\phi_2\phi_2} + \frac{j_{\phi_1\phi_2}}{\zeta} \right)}, \qquad \alpha_2 = \frac{1}{2} e^{-\frac{\phi}{3}} \sqrt{2\alpha \left(j_{\phi_2\phi_2} - \frac{j_{\phi_1\phi_2}}{\zeta} \right)} \\ \alpha_3 = e^{\frac{2\phi}{3}}, \qquad \alpha_4 = e^{-\frac{\phi}{3}} \sqrt{\frac{\alpha F_2}{\Delta_1 \Delta_2}}, \qquad \alpha_5 = e^{-\frac{\phi}{3}} \sqrt{F_1}, \qquad \zeta = \sqrt{\frac{j_{\phi_1\phi_1}}{j_{\phi_2\phi_2}}}\end{gathered} \tag{3.59}$$

The operation of flop on the above metric (3.58) has already been discussed in details in sec. 7 of the first reference of [112]. Using similar techniques, after the flop we expect the metric to look like:

$$ds_7^2 = a_1(\sigma_1^2 + \sigma_2^2) + a_2(\Sigma_1^2 + \Sigma_2^2) + a_3(\sigma_3 + \Sigma_3)^2 + a_4(\sigma_3 - \Sigma_3)^2 + a_5 dr^2 \tag{3.60}$$

with a_i, $i = 1, ..., 5$ are some coefficients to be determined. Due to the global nature of our metric, the operation of flop can be performed by a class of transformations parametrized by the values of a, b etc. in the following way:

$$\begin{aligned}
&\sigma_1 \mapsto a\sigma_1 + b\Sigma_1, \quad \Sigma_1 \mapsto e\sigma_1 + f\Sigma_1,\\
&\sigma_2 \mapsto c\sigma_2 + d\Sigma_2, \quad \Sigma_2 \mapsto g\sigma_2 + h\Sigma_2,\\
&\sigma_3 + \Sigma_3 \mapsto \sigma_3 - \Sigma_3, \quad \sigma_3 - \Sigma_3 \mapsto \sigma_3 + \Sigma_3
\end{aligned} \tag{3.61}$$

Now comparing (3.58) and (3.56) one can pretty much fix the coefficients c, d etc. in terms of a, b in the following way:

$$\begin{aligned}
&c = a\sqrt{\frac{k^2G_2 + kG_3 + G_1}{\omega^2G_5 + \omega G_6 + G_4}}, \qquad d = b\sqrt{\frac{\mu^2G_2 + \mu G_3 + G_1}{\tau^2G_5 + \tau G_6 + G_4}}\\
&e = ak, \quad g = c\omega, \quad f = b\mu, \quad h = d\tau
\end{aligned} \tag{3.62}$$

The other coefficients appearing above, namely, k, ω, μ, τ satisfy the following equations:

$$\begin{aligned}
&2G_1 + 2k\mu G_2 + (k+\mu)G_3 = 0, \quad G_7 + k\tau G_8 + \tau G_9 + kG_{10} = 0,\\
&2G_4 + 2\tau\omega G_5 + (\tau+\omega)G_6 = 0, \quad G_7 + \omega\mu G_8 + \omega G_9 + \mu G_{10} = 0.
\end{aligned} \tag{3.63}$$

whose solutions are fixed by the following values of G_i determined from the G_2 structure metric (3.56) or (3.58) using $(g_i, \widetilde{g}_i)$ defined earlier in (3.57):

$$\begin{aligned}
&G_1 = g_3\sin^2\psi_1 + \widetilde{g}_3\cos^2\psi_1, \qquad G_2 = g_4\sin^2\psi_2 + \widetilde{g}_4\cos^2\psi_2,\\
&G_3 = g_5\sin\psi_1\sin\psi_2 - \widetilde{g}_5\cos\psi_1\cos\psi_2,\\
&G_4 = g_3\cos^2\psi_1 + \widetilde{g}_3\sin^2\psi_1, \qquad G_5 = g_4\cos^2\psi_2 + \widetilde{g}_4\sin^2\psi_2,\\
&G_6 = g_5\cos\psi_1\cos\psi_2 - \widetilde{g}_5\sin\psi_1\sin\psi_2,\\
&G_7 = (g_3 - \widetilde{g}_3)\sin\psi_1\cos\psi_1, \quad G_9 = g_5\sin\psi_1\cos\psi_2 + \widetilde{g}_5\cos\psi_1\sin\psi_2,\\
&G_8 = (g_4 - \widetilde{g}_4)\sin\psi_2\cos\psi_2, \quad G_{10} = g_5\cos\psi_1\sin\psi_2 + \widetilde{g}_5\sin\psi_1\cos\psi_2.
\end{aligned} \tag{3.64}$$

Using all the above relations, the a_i coefficients in the M-theory metric after flop transition (3.61) can be determined in terms of (a, b). The final form of the metric therefore is given by:

$$\begin{aligned} ds^2_{11} &= e^{-\frac{2\phi}{3}} F_0 ds^2_{0,1,2,3} + g_r dr^2 + g_1(\sigma_3 - \Sigma_3)^2 + g_2(\sigma_3 + \Sigma_3)^2 \\ &+ a^2(k^2 G_2 + kG_3 + G_1)(\sigma_1^2 + \sigma_2^2) + b^2(\mu^2 G_2 + \mu G_3 + G_1)(\Sigma_1^2 + \Sigma_2^2) \end{aligned} \tag{3.65}$$

We are now one step away from getting the type IIA metric from the above metric. Reducing along x_{11} the metric takes the following form in type IIA theory:

$$\begin{aligned} ds^2_{10} &= F_0 ds^2_{0,1,2,3} + F_1 dr^2 + e^{2\phi}\Big[d\psi - b_{\psi\mu}dx^\mu + \Delta_1 \cos\,\theta_1\Big(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr\Big) \\ &\qquad + \widetilde{\Delta}_2 \cos\,\theta_2\Big(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr\Big)\Big]^2 \\ &+ e^{\frac{2\phi}{3}} a^2(k^2 G_2 + kG_3 + G_1)\Big[d\theta_1^2 + (d\phi_1^2 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr)^2\Big] \\ &+ e^{\frac{2\phi}{3}} b^2(\mu^2 G_2 + \mu G_3 + G_1)\Big[d\theta_2^2 + (d\phi_2^2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr)^2\Big] \end{aligned} \tag{3.66}$$

which has an amazing similarity with warped resolved conifold! The above metric is completely global and supersymmetric, and should be viewed as the gravity dual in the IR for the gauge theory on wrapped D6-branes before geometric transition. In this background there are no six-branes. The wrapped D6-branes have *dissolved* in the geometry, and is replaced by the following one-form flux components:

$$A = \Delta_1 \cos\,\theta_1\Big(d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr\Big) - \widetilde{\Delta}_2 \cos\,\theta_2\Big(d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr\Big) \tag{3.67}$$

with $\widetilde{\Delta}_2$ is a slight deformation of Δ_2 appearing from the rotation (3.52) before the flop operation. The type IIA background also supports an effective dilaton, that measures the IIA coupling, and is given by:

$$\phi_{\rm eff} \;=\; \frac{3}{4}\ln(g_2) \tag{3.68}$$

Before we end this section, there are a few loose ends that need to be tied up. The first one is related to the M-theory G-fluxes. These G-fluxes stem from (3.46) and

(3.43) in type IIA, and they are in general large[4] . In the local picture both (3.43) as well as (3.46) components were all pure gauges, and therefore they did not contribute to the background. Here we expect they would, and therefore we need to see how these fluxes behave under:

- The rotation of coordinates (3.52), and
- The flop transformation (3.61).

Both these effects can be easily worked out if we can express our fluxes (3.46) and (3.43) completely in terms of the one-forms (3.49) and (3.50). As we noted before, under the rotation (3.52), the one-forms (3.50) changes accordingly. Therefore to compensate both the changes, namely rotation (3.52) and the flop (3.61), all we need is to express the M-theory lift of the fluxes in terms of (3.49) and the transformed (3.50). The latter can be easily performed by first expressing the G-fluxes in terms of (3.50) and then change Σ_i to the transformed Σ_i (recall that we are using the same notation for Σ_i and its transformed version).

To achieve all this, we can express the differential coordinates completely in terms of σ_i and Σ_i. Since there are seven differential coordinates $(dr, d\theta_1, d\phi_1, d\theta_2, d\phi_2, d\psi_1, d\psi_2)$ but six one-forms (σ_i, Σ_i), we can assume dr goes to itself, and then the rest of the

[4] In the limit where ϵ in (3.31) or (3.36) is a small but *finite* number, the type IIA flux components (3.43) and (3.46) will be large but finite. In the following analysis we will allude to this case only.

differential forms map to the (σ_i, Σ_i) in the following way:

$$\begin{aligned}
& d\theta_1 = \cos\psi_1\sigma_1 - \sin\psi_1\sigma_2, \quad d\theta_2 = \cos\psi_2\Sigma_1 - \sin\psi_2\Sigma_2, \\
& d\phi_1 - \tan\lambda_4 dr = \sec\lambda_1\Big[\sin(\psi_1+\lambda_1)\sigma_1 + \cos(\psi_1+\lambda_1)\sigma_2\Big], \\
& d\phi_2 - \tan\lambda_5 dr = -\sec\lambda_2\Big[\sin(\psi_2-\lambda_2)\Sigma_1 + \cos(\psi_2-\lambda_2)\Sigma_2\Big], \\
& d\psi - \tan\lambda_3 dr = \sigma_3 - \Sigma_3 - \Delta_1\cos\,\theta_1\Big(\sin\psi_1\sigma_1 + \cos\psi_1\sigma_2\Big) \\
& \qquad\qquad\qquad\qquad + \Delta_2\cos\,\theta_2\Big(\sin\psi_2\Sigma_1 + \cos\psi_2\Sigma_2\Big) \\
& dx_{11} = \sigma_3 + \Sigma_3 + \Delta_1\cos\,\theta_1\Big(\sin\psi_1\sigma_1 + \cos\psi_1\sigma_2\Big) \\
& \qquad\qquad - \Delta_2\cos\,\theta_2\Big(\sin\psi_2\Sigma_1 + \cos\psi_2\Sigma_2\Big)
\end{aligned} \tag{3.69}$$

Under the rotation and flop the flux components mix in rather non-trivial way. We therefore expect, after the IIA reduction from M-theory, the three-form and four-form flux components of type IIA (3.43) and (3.46) respectively before geometric transition go to *new* three- and four-form flux components. They can be expressed as:

$$B_{\text{now}} = \bar{b}_{ij} dx^i \wedge dx^j, \qquad F_{\text{now}} = \bar{f}_{ijkl} dx^i \wedge dx^j \wedge dx^k \wedge dx^l \tag{3.70}$$

where $x^{i,j,k,l} = r, \theta_i, \phi_i, \psi$ and $\bar{b}_{ij}$ and $\bar{f}_{ijkl}$ are functions of (r, θ_i) and not of (ϕ_i, ψ) because of the underlying T-duality symmetry.

3.2.5 Type IIB after mirror transition

With the type IIA picture at hand, we are now at the last chain of the duality transformation that will give us the supergravity dual of the confining gauge theory on the wrapped D5-branes. Our starting points are now:

- The type IIA metric (3.66).
- The remnant of the D6-brane charges, i.e the one-form fluxes (3.67).
- The type IIA string coupling, or the dilaton (3.68), and
- The B_{NS} and the F_4 fluxes (3.70) from the remnant of the IIB shift transformations.

Before moving ahead, let us make two observations. The first is that now we *do* have components like $\bar{b}_{\phi_1\phi_2}, \bar{b}_{\psi\phi_i}$ in (3.70). This would mean that the mirror type IIB should become non-geometric! This is what one would have expected generically, and our analysis does confirm this. Observe that locally, as in [112], this aspect of non-geometricity was not visible because most of the extra fluxes were pure gauges. In the global case, the system is rather non-trivial and the dual gravitational description may become non-geometric. Question now is whether we can look for a special case where we can study the system as a *geometric* manifold. It turns out at the orientifold point there might be a situation where we can switch off the extra flux components and consider only the standard B-field components. Recall that due to various rotations (3.52) and shifting (3.26) the orientifolding is more involved, as all the internal coordinates are mixed up in these transformations. However this may not generically remove all the necessary components. Therefore to simplify the situation, in the following, we will study the type IIB mirror by first keeping:

$$\bar{b}_{\phi_1\phi_2} ~=~ \bar{b}_{\psi\phi_i} ~=~ 0 \tag{3.71}$$

so that the mirror could be geometric. Switching on (3.71) in the IIA scenario will then make the system non-geometric.

Secondly, note that except for the $B_{\rm NS}$ and the F_4 fluxes, rest of the components for the metric or the one-form fluxes, or even the dilaton are all finite. The $B_{\rm NS}$ and the F_4 fluxes are large, and in the limit ϵ in (3.31) is a small but finite integer, these would also be finite (but large). To proceed further let us define:

$$\mathcal{D}\phi_1 \equiv d\phi_1 - b_{\phi_1\theta_1}d\theta_1 - b_{\phi_1 r}dr, \qquad \mathcal{D}\phi_2 \equiv d\phi_2 - b_{\phi_2\theta_2}d\theta_2 - b_{\phi_2 r}dr \tag{3.72}$$

Using this, the type IIA metric can be rewritten as:

$$\begin{aligned} ds^2_{10} \quad &= F_0 ds^2_{0,1,2,3} + F_1 dr^2 + e^{2\phi}\Big(d\psi - b_{\psi\mu}dx^\mu + \Delta_1 \cos\,\theta_1 ~\mathcal{D}\phi_1 + \widetilde{\Delta}_2 \cos\,\theta_2 ~\mathcal{D}\phi_2\Big)^2 \\ &+ \mathcal{F}_1\Big(d\theta_1^2 + \mathcal{D}\phi_1^2\Big) + \mathcal{F}_2\Big(d\theta_2^2 + \mathcal{D}\phi_2^2\Big) \end{aligned} \tag{3.73}$$

In this form the non-Kählerity is obvious in terms of the fibrations $\mathcal{D}\phi_i$ and the resolution parameters of the two two-cycles are determined completely in terms of $\mathcal{F}_i$ as:

$$\mathcal{F}_1 = e^{\frac{2\phi}{3}} a^2 (k^2 G_2 + kG_3 + G_1), \qquad \mathcal{F}_2 = e^{\frac{2\phi}{3}} b^2 (\mu^2 G_2 + \mu G_3 + G_1) \tag{3.74}$$

It is now clear that to determine the type IIB mirror using SYZ [80] we have to make the base bigger as before. The manifold (3.73) still retains isometries along (ϕ_i, ψ), so after we enlarge the base we can perform SYZ in the usual way. Since these details are rather straightforward to work out, we will not redo them again now. To put the type IIB metric in some suggestive format, let us define the following quantities:

$$\begin{aligned}
&\bar{\alpha} = \left(\bar{j}_{\phi_1\phi_1}\bar{j}_{\phi_2\phi_2} - \bar{j}_{\phi_1\phi_2} + \bar{b}^2_{\phi_1\phi_2}\right)^{-1}, \qquad \widetilde{\mathcal{D}}\psi \equiv d\psi + g_{r\psi} dr \\
&\widetilde{\mathcal{D}}\phi_1 \equiv \sqrt{\frac{g_{\phi_1\phi_1}}{g_{\theta_1\theta_1}}}\Big(d\phi_1 + g_{\phi_1\theta_1} d\theta_1 + g_{r\phi_1} dr\Big), \quad \widetilde{\mathcal{D}}\phi_2 \equiv \sqrt{\frac{g_{\phi_2\phi_2}}{g_{\theta_2\theta_2}}}\Big(d\phi_2 + g_{\phi_2\theta_2} d\theta_2 + g_{r\phi_2} dr\Big)
\end{aligned} \tag{3.75}$$

where $\bar{j}_{\mu\nu}$ denote the components of the IIA metric (3.73), and $g_{\mu\nu}$ are defined in terms of $\bar{j}_{\mu\nu}$ in Appendix C of [42]. Using these definitions, and taking $b_{r\phi_i} = b_{r\psi} = 0$, the mirror manifold in type IIB theory takes the following form:

$$\begin{aligned}
ds^2 \;=\; & F_0^2 ds^2_{0,1,2,3} + g_{rr} dr^2 + g_{\psi\psi}\Big(\widetilde{\mathcal{D}}\psi + \widehat{\Delta}_1\ \widetilde{\mathcal{D}}\phi_1 + \widehat{\Delta}_2\ \widetilde{\mathcal{D}}\phi_2\Big)^2 \\
& + g_{\theta_1\theta_1}\Big(d\theta_1^2 + \widetilde{\mathcal{D}}\phi_1^2\Big) + g_{\theta_2\theta_2}\Big(d\theta_2^2 + \widetilde{\mathcal{D}}\phi_2^2\Big) + g_{\theta_1\theta_2}\Big(d\theta_1 d\theta_2 + \widehat{\Delta}_3\ \widetilde{\mathcal{D}}\phi_1\widetilde{\mathcal{D}}\phi_2\Big)
\end{aligned} \tag{3.76}$$

which looks surprisingly close to the warped resolved-deformed conifold! Clearly the manifold is non-Kähler and $\hat{\Delta}_i$ are defined as:

$$\widehat{\Delta}_1 \equiv \sqrt{\frac{g_{\theta_1\theta_1} g^2_{\psi\phi_1}}{g_{\phi_1\phi_1}}}, \qquad \widehat{\Delta}_2 \equiv \sqrt{\frac{g_{\theta_2\theta_2} g^2_{\psi\phi_2}}{g_{\phi_2\phi_2}}}, \qquad \widehat{\Delta}_3 \equiv \sqrt{\frac{g_{\theta_1\theta_1} g_{\theta_2\theta_2} g^2_{\phi_1\phi_2}}{g_{\phi_1\phi_1} g_{\phi_2\phi_2} g^2_{\theta_1\theta_2}}} \tag{3.77}$$

The type IIB fluxes are rather involved, but they could be worked out exactly as in Appendix B of [42]. We will not do so here, but discuss their implications in our follow-up paper [76]. It is interesting that the solutions that we get in type IIA as well as type IIB for the gravity duals look very close to what have been advocated in the literature so far in the limit where we switch off certain components of the $\bar{b}$-fields as well as $b_{r\phi_i}, b_{r\psi}$. Once we keep these components then the metric (3.76) *cannot* be the global description. The global description will have to be a non-geometric manifold. In the present chapter we will not discuss the non-geometric aspect anymore, and details on this will be presented in our upcoming paper [76]. We end this section by noting that the duality cycle that we advocated here (and also in [44, 112] earlier) does lead to the correct gauge/gravity dualities for the confining theories.

Chapter 4

Gravity dual of finite temperature field theory

4.1 Introduction

Lots of analytic results coming from gauge/gravity duality are derived for conformal gauge theories with $\mathcal{N} = 4$ supersymmetry in the large N limit. One may thus genuinely be concerned about their applicability to QCD for which all these are not true. Recent progress in this area, however, has provided us with strong hints to overcome these limitations, and move towards models of gauge-gravity duality that are not supersymmetric, and are non-conformal (in a sense that will be made precise later).

The first set of models that managed to expand the original AdS/CFT construction to incorporate renormalization group runnings are [66] and [67] that connected conformal fixed points at IR and UV, and [68] that connected the UV $\mathcal{N} = 4$ conformal fixed point to a $\mathcal{N} = 1$ confining theory. The next set of models, that we would be mostly interested in, do not have any fixed points (or fixed surfaces) in the paths of the RG flows. The key example in this set is the Klebanov-Strassler (KS) model [6] (with an extension by Ouyang [115] to incorporate fundamental matters) that provided an IR dual of, although not exactly QCD, but at least its closest cousin: large N supersymmetric QCD. The UV of the original Klebanov-Strassler model is now known to have some issues, like the divergences of the Wilson loops at high energies, and additional Landau poles once fundamental matters have been introduced. This means that UV completion is necessary, and to have the full gravity dual of the corresponding gauge theory that behaves well at high energies, the KS geometry has to be augmented by a proper asymptotic manifold.

Other extensions to the original KS model quickly followed. For example in [104, 97] the cascading picture of the original KS model was extended to incorporate black-hole without any fundamental matter, which was then further extended to incorporate matter in [98]. However none of the above models actually considered the full UV completion as most of the analysis of these works were directed towards unravelling the IR physics. Therefore issues like UV divergences of Wilson loops and Landau poles were not investigated.

In a series of works [70, 71, 107] done over the last couple of years, the authors tried to address these concerns. The aim therein was to incorporate the backreactions from the black-hole, fluxes, and branes consistently so as to have a well defined UV completion that not only allow one to get rid of all the poles etc., but also give one a model that could come *closest* to what one might have expected from a large N thermal QCD. The authors did manage to at least successfully generate such a UV completed dual picture, but many of the backreactions turned out to be too difficult to incorporate fully. The main aim of this chapter is to make progress in this direction.

4.2 Analysis of the background

Let us start with the model that was studied in [70]. The IR physics is captured by the Ouyang-Klebanov-Strassler-black-hole (OKS-BH) geometry, namely, the small r physics is determined by a warped resolved-deformed conifold with fluxes, seven-branes and a black hole in the ten-dimensional spacetime. On the other hand the UV physics is conformal, and is captured by an asymptotically AdS geometry with fluxes and seven-branes.

As discussed in [71], these two geometries, namely the asymptotic AdS and OKS, can be connected by an intermediate configuration with brane sources and fluxes. These branes sources were elaborated in details in [71], although many coefficients in the background geometry were left undetermined therein. In the following we will fill up some of these missing steps.

Let us begin with the basic ansatze for the metric in the three regions. For all the three regions we assume that the radial coordinate r spans $b < r < r_{\rm min}$ for Region 1 where we expect all the confining dynamics to take place; $r_{\rm min} < r < r_o$ for the intermediate region called Region 2; and $r_o < r < \infty$ for Region 3 which captures the asymptotically conformal region. The minimum radius $r = b$, which signifies the cut-off coming from the blown-up S^3 (as well as S^2, although for most of the calculations in this chapter we will only consider a warped resolved conifold instead of a warped resolved-deformed conifold), maps to the expectation of the gluino condensates of the dual gauge theory at zero temperature. Considering all these regions, the non-extremal metric takes the following form:

$$
\begin{aligned}
ds^2 &= \frac{1}{\sqrt{h}}\Big[-g_1(r)dt^2 + dx^2 + dy^2 + dz^2\Big] + \sqrt{h}\Big[g_2(r)^{-1}dr^2 + d\mathcal{M}_5^2\Big] \\
&\equiv -e^{2A+2B}dt^2 + e^{2A}\delta_{ij}dx^i dx^j + e^{-2A-2B}\widetilde{g}_{mn}dx^m dx^n \qquad (4.1)
\end{aligned}
$$

where $g_i(r)$ are the black-hole factors and we have taken $g_1 = g_2$, the components go as $i, j = 1, 2, 3$ and $m, n = 4, ..., 9$, the warp factors A, B are defined as:

$$
A = -\frac{1}{4}\log h, \qquad B = \frac{1}{2}\log g_1 \qquad (4.2)
$$

$d\mathcal{M}_5^2$ is typically the metric of warped resolved-deformed conifold and h is the warp factor that behaves differently in the three regions as shown in [71].

Observe that in the extremal limit, $g_1 = g_2 \approx 1$ and the extremal metric is dual to the low temperature confining phase of the gauge theory. To see this, note that in the absence of any seven branes, Region 1 of the geometry of [71] in the extremal limit is identical to the IR geometry of Klebanov-Strassler (KS) model [6]. If seven branes are placed far away from Region 1, that is $r_{\rm min} \gg b$, we can neglect their back-reactions and consider the axion-dilaton field to be effectively constant as in [6]. Hence in the extremal limit, Region 1 of [71] is identical to the IR region of KS which, in turn, is dual to the low temperature confining phase of the $SU(M)$ gauge theory wherein chiral symmetry is broken. The extremal geometry can incorporate temperature of the field theory once we analytically continue to Euclidean signature with $it \to \tau$ and impose periodic and anti-periodic boundary conditions for the bosons and fermions on the closed time circle. Furthermore, in extremal case the entropy will vanish. This is expected as the entropy from the dual geometry arises from the fluxes which are at least $\mathcal{O}(N_{\text{eff}})$, where N_{eff} is effective brane charge. As the deformed cone represents confinement of charge, we expect to get $N_{\text{eff}} = 0$ from the dual geometry. This is indeed what happens as energy scale for a thermal field theory is set by the temperature and at low temperature, only the IR degrees of freedom are excited. This means in the dual geometry, all we need is the region near $r \sim b$ of the deformed cone $-$ but in this region the five-form flux vanishes [6] and we get $N_{\text{eff}} = 0$.

As the temperature is increased, we expect that the non-extremal solution will have less free energy than the extremal solution, just as in the case for the AdS-black holes [96], and Hawking-Page phase transition will take place [101]. The focus of this work will be to analyze the non-extremal solution which is dual to the deconfined phase of large N thermal QCD, while a detailed analysis of phase transitions will be presented in a follow up paper[102].

The non-extremal solutions we present in this chapter are precisely dual to the high temperature regime of the gauge theory $-$ where chiral symmetry is restored and the light degrees of freedom are deconfined. However, heavy quarkonium states arising from the seven branes placed in the UV region can coexist with the chirally symmetric phase above the deconfinement temperature. But as temperature is raised even further, the heavy quarkonium states will eventually melt [108, 71].

For both extremal and non-extremal cases, typically h would have logarithmic factors in Region 1 whereas it would have inverse r behavior in Region 3. In the intermediate region, the warp factor will typically have both the logarithmic and the inverse r behavior. Therefore to summarise, the background should satisfy the following properties:

• Fluxes are non imaginary self-dual i.e non-ISD, and become ISD once the black-hole factors g_i in the metric are removed. Therefore the deviation for ISD property is proportional to the horizon radius r_h.

• The gravity dual of the deconfined phase is given by resolved warped-deformed conifold with a black-hole. In the limit where the deformation parameter is small, the background can be succinctly captured by a resolved conifold with fluxes and black hole.

• The resolution parameter is no longer constant because of the various back-reactions. In fact the resolution parameter becomes function of r_h/r as well as g_sN_f, and g_sM^2/N where g_s is the string coupling, M is the number of bi-fundamental matter, N is the number of colors, and N_f is the number of fundamental flavors.

From the above set of arguments, we can use the following ansatze for the internal metric:

$$\begin{aligned}\widetilde{g}_{mn}dx^m dx^n &= dr^2 + r^2 e^{2B}\Big[A(d\psi + \cos\,\theta_1 d\phi_1 + \cos\,\theta_2 d\phi_2)^2 + \mathcal{O}(g_sM^2/N, r_h^4/r^4)\\ &+ B(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \frac{1}{6}(1+F)(1+\mathcal{G})\left(\frac{d\theta_2^2}{1+\mathcal{G}} + \sin^2\theta_2 d\phi_2^2\right)\Big]\\ &+ 2f_b\Big[\cos\,\psi(d\theta_1 d\theta_2 + \sin\,\theta_1\,\sin\,\theta_2 d\phi_1 d\phi_2) - \sin\,\psi(\sin\,\theta_2 d\phi_2 d\theta_1 - \sin\,\theta_1 d\phi_1 d\theta_2)\Big]\end{aligned} \tag{4.3}$$

where we will only consider the resolved conifold limit, with F being related to the resolution parameter (whose value will be determined later). In other words, we take:

$$\begin{aligned}f_b &\to 0, \qquad F \equiv \frac{6a^2}{r^2}, \qquad \mathcal{G} \to 0\\ A &= \frac{1}{9} + \mathcal{O}(g_sM^2/N, r_h^4/r^4), \qquad B = \frac{1}{6} + \mathcal{O}(g_sM^2/N, r_h^4/r^4)\end{aligned} \tag{4.4}$$

where the numerical factor of 6 is inserted to bring certain expressions in a better format. As we will see, this F (or equivalently a) determines the squashing factor *between* the two spheres, and we can consistently keep the second squashing factor, $\mathcal{G}$, to be zero.

The resolution parameter discussed above needs a bit more elaboration. First of all, as we mentioned earlier, a^2 is not a constant in our model. As we will show in (4.54), the resolution parameter takes the following form:

$$a^2 \; = \; a_0^2 + r_h^2 \mathcal{O}(g_s M^2/N) + r_h^4 \mathcal{O}(g_s^2 M^2 N_f/N) \tag{4.5}$$

where we have switched on a *bare* resolution parameter a_0^2 to allow for the theory to have a baryonic branch [99]. However even if we switch off the bare resolution parameter, the background EOMs will still generate a resolution parameter proportional to the horizon radius r_h. This not a contradiction with the result of [85] wherein it was argued that one may not be able to simultaneously resolve and deform a Calabi-Yau cone. The fact that our metric is non-Kähler takes us away from the constraints imposed in [85].

In the following section we will argue for these parameters and their dependences on the horizon radius by analysing the non-extremal limit of the warped resolved-deformed conifold background[1] .

---

[1] We will continue calling this background as the Klebanov-Strassler background as they all fall in the same class of supergravity solution.

4.2.1 Derivation of the non-extremal BH solution for the Klebanov-Strassler model

We first compute the non-extremal metric arising from Type IIB supergravity action given, in the notations of [103], in the following way[2] :

$$\begin{aligned} S_{\rm IIB} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\,\sqrt{-g}\left[R - \frac{\partial_a\tau\partial^a\bar\tau}{2|{\rm Im}\tau|^2} - \frac{G_3\cdot\bar G_3}{12{\rm Im}\tau} - \frac{\widetilde F_5^2}{4\cdot 5!}\right] \\ &+ \frac{1}{8i\kappa_{10}^2}\int\frac{C_4\wedge G_3\wedge\bar G_3}{{\rm Im}\tau} + S_{\rm loc} \end{aligned} \tag{4.6}$$

where $S_{\rm loc}$ is the action for all the localized sources in ten dimensional geometry i.e five-branes and seven-branes mostly from Region 2 onwards. Our aim is to re-analyse the non-extremal Klebanov-Strassler solution. Recall that for Klebanov-Tseytlin model the non-extremal solutions were analyzed in [104], while in [70] there have been studies of gravity duals of finite temperature cascading gauge theory with fundamental matters[3] . However in [70] precise background fluxes and the warp factors taking into the backreactions of the BH geometry were only conjectured. Here we will derive the non-extremal metric dual to a UV complete gauge theory that mimics features of large N QCD at the lowest energies, justifying the proposals made in [70, 71]. One immediate outcome of this would be the verification of the conjectured dependence of the resolution parameter a^2 on the horizon radius r_h.

Our ansatz for the metric is (4.1). We look for solutions with regular Schwarzschild horizon at $r = r_h$. This is achieved by imposing $e^{B(r_h)} = 0$ and considering solutions to A such that $e^{A(r_h)} \neq 0$, which guarantees a non-singular horizon [104]. By solving the Einstein equations along with the flux equations with these boundary conditions, we will find the non-extremal solutions with regular horizons.

[2] Although in this section we will use the Einstein frame to express the metric, we will however not distinguish between the two frames in later sections because the dilaton will be considered constant, unless mentioned otherwise.

[3] See also [105] where somewhat similar analyses were also done.

Observe that we have warped Minkowski four directions, a non-compact radial direction r and a compact five manifold $\mathcal{M}_5$. The back reactions of the fluxes G_3, $\widetilde{F}_5$ and axion-dilaton field τ will modify the warp factor e^{A+B} while $\widetilde{g}_{mn}$ will be altered due to the presence of a black hole and the various sources. In particular $\widetilde{g}_{mn}$ will be a warped resolved-deformed conifold with a bare resolution parameter a_0. Note however that only the warp factor e^{A+B} will be essential to analyze the confinement/deconfinement mechanism for the boundary field theory [71]. The linear confinement of quarks and the string breaking mechanism which eventually describes the deconfinement of $Q\overline{Q}$ pair, is only sensitive to the warp factor. The exact solutions for the internal metric in the non-extremal limit taking into account the back reaction of the various fluxes is not essential to study free energy of the $Q\overline{Q}$ pair. Nevertheless we will find the exact form of the internal metric up to linear order in resolution function F.

We restrict to fluxes and axion-dilaton field τ which only depend on x^m and not on the Minkowski coordinates x^μ. Then the Einstein equations can be written as

$$\begin{aligned}
R_{\mu\nu} &= -g_{\mu\nu}\left[\frac{G_3\cdot\bar{G}_3}{48\ \mathrm{Im}\tau}+\frac{\widetilde{F}_5^2}{8\cdot 5!}\right]+\frac{\widetilde{F}_{\mu abcd}\widetilde{F}_\nu^{\ abcd}}{4\cdot 4!}+\kappa_{10}^2\left(T_{\mu\nu}^{\mathrm{loc}}-\frac{1}{8}g_{\mu\nu}T^{\mathrm{loc}}\right)\\
R_{mn} &= -g_{mn}\left[\frac{G_3\cdot\bar{G}_3}{48\ \mathrm{Im}\tau}+\frac{\widetilde{F}_5^2}{8\cdot 5!}\right]+\frac{\widetilde{F}_{mabcd}\widetilde{F}_n^{\ abcd}}{4\cdot 4!}+\frac{G_m^{\ bc}\bar{G}_{nbc}}{4\ \mathrm{Im}\tau}+\frac{\partial_m\tau\partial_n\bar{\tau}}{2\ |\mathrm{Im}\tau|^2}\\
&+ \kappa_{10}^2\left(T_{mn}^{\mathrm{loc}}-\frac{1}{8}g_{mn}T^{\mathrm{loc}}\right)
\end{aligned}\tag{4.7}$$

where $\widetilde{F}_5$ is given by the following self dual form

$$\widetilde{F}_5=(1+*_{10})d\alpha\wedge dx^0\wedge dx^1\wedge dx^2\wedge dx^3\tag{4.8}$$

with $\alpha=e^{4A}$ and T^{loc} being the trace of

$$T_{ab}^{\mathrm{loc}}=-\frac{2}{\sqrt{-g}}\frac{\delta S_{\mathrm{loc}}}{\delta g^{ab}}\tag{4.9}$$

Using the form of the five-form flux (4.8), the first equation in (4.7) becomes

$$R_{\mu\nu} = -g_{\mu\nu}\left[\frac{G_3\cdot\bar{G}_3}{48\ \mathrm{Im}\tau}+\frac{e^{-8A-2B}\partial_m\alpha\partial^m\alpha}{4}\right]+\kappa_{10}^2\left(T_{\mu\nu}^{\mathrm{loc}}-\frac{1}{8}g_{\mu\nu}T^{\mathrm{loc}}\right)\tag{4.10}$$

On the other hand, the Ricci tensor in the Minkowski direction takes the following simple form

$$R_{\mu\nu} \;=\; -\frac{1}{2}\left[\partial_m(g^{mn}\partial_n g_{\mu\nu}) + g^{mn}\Gamma^M_{nM}\partial_m g_{\mu\nu} - g^{mn}g^{\nu'\mu'}\partial_m g_{\mu'\mu}\partial_n g_{\nu'\nu}\right] \quad (4.11)$$

where $\nu', \mu' = 0,..,3$ and Γ^M_{nM} is the Christoffel symbol. Now using the ansatz (4.1) for the metric, (4.11) can be written as

$$\begin{aligned} R_{tt} &= e^{4(A+B)}\left[\widetilde{\nabla}^2(A+B) - 3\widetilde{g}^{mn}\partial_n B\partial_m(A+B)\right] \\ R_{ij} &= -\eta_{ij}e^{2(2A+B)}\left[\widetilde{\nabla}^2 A - 3\widetilde{g}^{mn}\partial_n B\partial_m A\right] \end{aligned} \quad (4.12)$$

where we have defined the Laplacian as:

$$\widetilde{\nabla}^2 = \widetilde{g}^{mn}\partial_m\partial_n + \partial_m\widetilde{g}^{mn}\partial_n + \frac{1}{2}\widetilde{g}^{mn}\widetilde{g}^{pq}\partial_n\widetilde{g}_{pq}\partial_m \quad (4.13)$$

The set of equations can be simplified by taking the trace of the first equation in (4.7) and using (4.12). Doing this we get

$$\begin{aligned} \widetilde{\nabla}^2(4A+B) - 3\widetilde{g}^{mn}\partial_n B\partial_m(4A+B) \;&=\; e^{-2A-2B}\frac{G_{mnp}\bar{G}^{mnp}}{12\mathrm{Im}\tau} + e^{-10A-4B}\partial_m\alpha\partial^m\alpha \\ &+\frac{k_{10}^2}{2}e^{-2A-2B}(T^m_m - T^\mu_\mu)^{loc} \end{aligned} \quad (4.14)$$

On the other hand using (4.10) in (4.12), one gets

$$R^t_t - R^x_x = 0 \quad (4.15)$$

which in turn would immediately imply

$$\widetilde{\nabla}^2 B - 3\widetilde{g}^{mn}\partial_m B\partial_n B = 0 \quad (4.16)$$

Minimizing the action (4.6) also gives the Bianchi identity for the five-form flux, namely

$$d\widetilde{F}_5 = H_3 \wedge F_3 + 2\kappa_{10}^2 T_3\rho_3 \quad (4.17)$$

where ρ_3 is the D3 charge density from the localized sources [103]. Using (4.8) in (4.17) and subtracting it from (4.14) one gets the following

$$\begin{aligned}\widetilde{\nabla}^2(e^{4A+B}-\alpha) &= \frac{e^{2A-B}}{6\mathrm{Im}\tau}|iG_3 - *_6G_3|^2 + e^{-6A-3B}|\partial(e^{4A+B}-\alpha)|^2 \\ &+3e^{-2A-2B}\partial_m B\partial^m(e^{4A+B}-\alpha) + \text{local source}\end{aligned} \tag{4.18}$$

The Ricci tensor, on the other hand, for the $x^m, m = 4,..,9$ directions takes the following form

$$\begin{aligned}R_{mn} &= \widetilde{R}_{mn} + \widetilde{g}_{mn}\widetilde{\nabla}^2(A+B) - 3\widetilde{g}_{mn}\widetilde{g}^{\lambda k}\partial_\lambda B\partial_k(A+B) \\ &+ 3\widetilde{\nabla}_m\partial_n B + \partial_m B\partial_n B - 8\partial_m A\partial_n A - 2\partial_{(m}A\partial_{n)}B\end{aligned} \tag{4.19}$$

where $\widetilde{\nabla}_m$ is the covariant derivative given by

$$\widetilde{\nabla}_m V_c = \partial_m V_c - \widetilde{\Gamma}^b_{mc} V_b \tag{4.20}$$

for any vector V_b. Here $\widetilde{R}_{mn}$ is the Ricci tensor and $\widetilde{\Gamma}^b_{mc}$ is the Christoffel symbol for the metric $\widetilde{g}_{mn}$. The equation for $\widetilde{R}_{mn}$ is given by:

$$\begin{aligned}\widetilde{R}_{mn} &= -g_{mn}\frac{G_3\cdot\bar{G}_3}{24\mathrm{Im}\tau} + \frac{G_{mab}\cdot\bar{G}^{ab}_n}{4\mathrm{Im}\tau} + \frac{\partial_m\tau\partial_n\bar{\tau}}{2\mid\mathrm{Im}\tau\mid^2} \\ &+\frac{F_{mabcd}F^{abcd}_n}{4\cdot 4!} + g_{mn}\frac{F_{\mu abcd}F^{\mu abcd}}{16\cdot 4!} + 8\partial_m A\partial_n A \\ &-3\widetilde{\nabla}_m\partial_n B - \partial_m B\partial_n B + 2\partial_{(m}A\partial_{n)}B\end{aligned} \tag{4.21}$$

which means, in general, this could lead to twenty different equations in six-dimensions (including another one for the trace). On the other hand the equation of motion for G_3 can be expressed in terms of a seven-form $\Lambda_7 \equiv *_{10}G_3 - iC_4\wedge G_3$ in the following way:

$$d\Lambda_7 + \frac{i}{\mathrm{Im}\tau}d\tau\wedge\mathrm{Re}\Lambda_7 = 0 \tag{4.22}$$

where typically Λ_7 would study the deviations from the ISD behavior. For example, using our metric ansatz we can express Λ_7 as

$$\Lambda_7 = \left[e^{4A+B} *_6 G_3 - i\alpha G_3\right]\wedge dt\wedge dx\wedge dy\wedge dz \tag{4.23}$$

The above choice of Λ_7 leads us to three different classes of solutions from the G_3 EOM (4.22). These three classes can be tabulated in the following way:

• If $\alpha = e^{4A+B}$ in (4.23) and $\Lambda_7 = d\Lambda_7 = 0$ then G_3 must be ISD. When $B = 0$ then this is the same as GKP solution [103], and in this case τ is not restricted[4] .

• If $\alpha \neq e^{4A+B}$ then we can take $\Lambda_7 \neq 0$ but keep $d\Lambda_7 = 0$ and $d\tau = 0$. This means Λ_7 is closed but not necessarily exact, and τ is a constant[5] .

• If $\alpha \neq e^{4A+B}$ then we can again take $\Lambda_7 \neq 0$ but now $d\Lambda_7 \neq 0$ and $d\tau \neq 0$ such that (4.22) is satisfied. This means both axion and the dilaton could run in this scenario.

In this chapter we are taking $\alpha = e^{4A}$, so we have to consider the last two cases. Expressing Λ_7 as $\Lambda_7 = T_3 \wedge dt \wedge dx \wedge dy \wedge dz$ we have $e^B *_6 G_3 - iG_3 = T_3$ where T_3 is non-zero as long as B is non-zero. A simple solution then would be to restrict oneself to the second case, i.e

$$dT_3 \; = \; 0, \qquad\qquad \tau \; = \; \text{constant} \tag{4.24}$$

Notice also that at far infinity, i.e $r \to \infty$, $B \to 0$, therefore $T_3 \to 0$ as well[6] . Using the above argument, G_3 can then be expressed in terms of T_3 as

$$G_3 = \frac{e^B *_6 T_3 + iT_3}{1 - e^{2B}} \tag{4.25}$$

[4] One can find solutions for $\alpha = e^{4A+B}$ case when $B \neq 0$, but this solution doesn't have correct conformal limit, i.e. when we switch off G_3, it doesn't reduce to the KW solution. In the dual gauge theory the charge obviously varies with the temperature which is not the case in the ordinary gauge theory.

[5] Or $\tau = d\lambda_{-1}$ i.e d of a (-1)-form. The functional form for the (-1)-form is non-trivial, so this option is more cumbersome to use.

[6] This is of course *without* considering the UV completion. With UV completion the large r behavior is non-trivial as discussed in [70, 71].

Since $\tau =$ constant, this means the closure of G_3 will involve a non-trivial constraint connecting the internal metric components with B and T_3. However in this chapter we will not be solving these equations explicitly but approximating G_3 by Ouyang-Klebanov-Strassler flux $G_3^{(0)}$ which is ISD in their metric. This approximation suffices for our case, as we show below.

Let us substitute $G_3 = G_3^{(0)}$ and $\alpha = e^{4A}$ into (4.23). This will convert Λ_7 to a simpler seven-form in the following way:

$$
\begin{aligned}
\Lambda_7 &= e^{4A}(e^B *_6 G_3^0 - iG_3^0) \wedge dt \wedge dx \wedge dy \wedge dz \\
&\approx 3e^{4A}(e^{2B}-1) g^{rr} \epsilon_{rabc}{}^{de} G^{(0)}_{rde}\, dx^a \wedge dx^b \wedge dx^c \wedge dt \wedge dx \wedge dy \wedge dz \\
&\approx \frac{3r_h^4}{N} g^{rr} \epsilon_{rabc}{}^{de} G^{(0)}_{rde}\, dx^a \wedge dx^b \wedge dx^c \wedge dt \wedge dx \wedge dy \wedge dz
\end{aligned} \tag{4.26}
$$

At large N the right hand side is small and therefore deviation from OKS flux is of $\mathcal{O}(r_h, 1/N)$ so one may consider $\Lambda \approx 0$. This means $G_3 = G_3^{(0)}$ is a good approximation. Additionally, since F_5 is self-dual, $\widetilde{R}_{mn}$ can be simplified as

$$
\begin{aligned}
\widetilde{R}_{mn} &= -g_{mn}\frac{G_3 \cdot \bar{G}_3}{24\mathrm{Im}\tau} + \frac{G_{mab} \cdot \bar{G}_n^{ab}}{4\mathrm{Im}\tau} + \frac{\partial_m \tau \partial_n \bar{\tau}}{2 \mid \mathrm{Im}\tau \mid^2} + 8(1-e^{-2B})\partial_m A \partial_n A \\
&\quad -3\widetilde{\nabla}_m \partial_n B - \partial_m B \partial_n B + 2\partial_{(m} A \partial_{n)} B
\end{aligned} \tag{4.27}
$$

We see the first two terms are suppressed by $g_s M^2/N$ and the third term is removed because τ is a constant. So we can ignore these contributions for the time being. Then, assuming A and B only depends on r, (4.27) will lead to

$$
\begin{aligned}
\widetilde{R}_{rr} &= 8(1-e^{-2B})\partial_r A \partial_r A - 3\widetilde{\nabla}_r \partial_r B - \partial_r B \partial_r B + 2\partial_{(r} A \partial_{r)} B \\
\widetilde{R}_{ab} &= -\frac{3}{2}\partial_r \widetilde{g}_{ab} \partial_r B
\end{aligned} \tag{4.28}
$$

where (a, b) denote the angular directions. We now see that for $r > r_h$, the $\widetilde{R}_{ab}$ contribution is suppressed equivalently as the $\widetilde{R}_{rr}$ contribution, therefore we need to keep both the parts. This conclusion can also be extended to R_{mn} in (4.19), which implies that the R_{rr} and R_{ab} contributions are equally suppressed. All this then further implies that we need to solve the twenty-one metric equations. This is a formidable exercise. Is there a way by which we can avoid doing this?

A possible way out would be to study the relative suppressions of various terms in the system of equations. This criteria was already anticipated in [70]. For example, as we discussed in [70], we can equivalently take:

$$(g_s, N, M, N_f) \;\to\; (\epsilon^c, \epsilon^{-a}, \epsilon^{-b}, \epsilon^{-d}) \tag{4.29}$$

This would clearly show that $(g_s N, g_s M)$ are very large but $(g_s N_f, g_s M^2/N, g_s^2 M N_f)$ as well as M/N are suppressed in the following way:

$$\begin{aligned}
&(g_s N, g_s M) \;\to\; (\epsilon^{c-a}, \epsilon^{c-b})\\
&(g_s N_f, g_s M^2/N, g_s^2 M N_f, M/N) \;\to\; (\epsilon^{c-d}, \epsilon^{c-2b+a}, \epsilon^{2c-b-d}, \epsilon^{a-b})
\end{aligned} \tag{4.30}$$

provided (a, b, c, d) satisfy the following inequalities[7] :

$$a \;>\; b \;>\; c \;>\; d, \qquad a + c \;>\; 2b, \qquad 2c \;>\; b + d \tag{4.31}$$

Let the smallest scale in our problem be the ratio M/N. Then if the argument of the relative suppressions of various terms in R_{mn} has to make sense, one would require the precise range of r where our approximations hold water. This gives us:

$$r \;\geq\; r_h \left(\frac{N}{M}\right)^{1/4} \tag{4.32}$$

Thus if we are in this range, we can see that the curvature terms simplify drastically. This would give us a hint that if we solve the simplest trace equation along with the flux equations (4.16), (4.17), and (4.18) we would be reasonably close to the correct

[7] A solution to the inequalities is $a = 8, b = 3, c = 5/2, d = 1$, as given in [70]. One can of course allow other values of (a, b, c, d) that satisfy the inequalities.

answer because the other twenty component equations would only change the results[8] to $\mathcal{O}(r_h^4/r^4)$. So once we are in the range (4.32) the only corrections to our simplified trace equation will be to $\mathcal{O}(g_s M^2/N)$ and $\mathcal{O}(r_h^4/r^4)$. This is not so bad because if we choose ϵ in (4.29) to be $\epsilon = 0.1$, then

$$N = 10^8, \quad M = 10^3, \quad N_f = 10, \quad g_s = 0.0032, \quad r > 17.78 r_h \tag{4.33}$$

which means for r beyond $17.78 r_h$ the contributions coming from the individual component equations to the solution generated using only the trace equation will not be too drastic.

Therefore, once the dust settles, tracing the second equation in (4.7), using (4.14), (4.16) and (4.19), we get

$$\begin{aligned} &\frac{\widetilde{R}}{6} + \frac{4}{3}\widetilde{g}^{mn}\partial_m A \partial_n A \left(e^{-2B} - 1\right) + \frac{\widetilde{g}^{mn}}{6}\left(3\widetilde{\nabla}_m \partial_n B + \partial_m B \partial_n B\right) \\ &\qquad - \frac{\widetilde{g}^{mn}}{3}\partial_{(m}A\partial_{n)}B = \frac{\widetilde{g}^{mn}\partial_m \tau \partial_n \bar{\tau}}{12|\mathrm{Im}\tau|^2} \end{aligned} \tag{4.34}$$

where $\widetilde{R} = \widetilde{g}^{mn}\widetilde{R}_{mn}$ and we have ignored all local sources.

Our goal now is to solve the system of four equations (4.16), (4.17), (4.18) and (4.34) and find solutions for the warp factors A, B, the internal metric $\widetilde{g}_{mn}$ and the fluxes. In obtaining the solutions, we will be working in the limit where there is no local sources, G_3 is closed while the explicit form of the fluxes that solve the flux equations are described in the following subsection[9] . As we mentioned earlier, if we

[8] This in particular means that not only the coefficients of all the terms of the internal metric will change to $\mathcal{O}(r_h^4/r^4)$ but also any *new* component will appear to $\mathcal{O}(r_h^4/r^4)$. This is exactly how we choose our initial metric ansatze (4.3) and therefore the system is self-consistent.

[9] It is of course possible to consider additional sources to obtain a UV complete solution as done in [71]. But for the purpose of the current section, which is to analyze the non-extremal limit for the IR geometry, we will ignore local sources and discuss their effects briefly towards the end.

choose $\alpha = e^{4A}$, (4.18) will imply that G_3 is ISD, in the extremal limit i.e $e^B = 1$. On the other hand, G_3 is not ISD on a deformed cone in the presence of a black hole, and the terms in G_3 which make it non-ISD are precisely proportional to the blackhole horizon and the deformation function F that appears in $\widetilde{g}_{mn}$. With these considerations and our choice of internal metric $\widetilde{g}_{mn}$ we get

$$|iG_3 - *_6 G_3|^2 \;=\; \left|\frac{i *_6 T_3 + T_3}{1+e^B}\right|^2 \;\sim\; \mathcal{O}(F^2, r_h^8/r^8) \tag{4.35}$$

Thus with a choice of $\alpha = e^{4A} + \mathcal{O}(F^2)$, (4.18) can be solved exactly. But if $F \ll 1$, we can ignore $\mathcal{O}(F^2)$ terms which means up to linear order in F, (4.18) becomes

$$\widetilde{\nabla}^2(e^{4A+B} - e^{4A}) \;=\; e^{-6A-3B}|\partial(e^{4A+B} - e^{4A})|^2 + 3e^{-2A-2B}\partial_m B\partial^m(e^{4A+B} - e^{4A}) \tag{4.36}$$

Thus ignoring $\mathcal{O}(F^2)$ in (4.18)[10] , we are essentialy solving (4.16), (4.17), (4.34) and (4.36). In fact we will show that (4.17) dictates $F \ll 1$ and our explicit numerical solutions will also be consistent with this assumption, justifying our perturbative analysis.

Now only considering up to linear order terms in F, we get $\alpha = e^{4A}$ which relates the warp factor to the five-form field strength which in turn depends on G_3 by the Bianchi identity (4.17). Thus e^A depends on the non-ISD G_3 as G_3 is modified in the presence of a black hole. But the choice of $\alpha = e^{4A}$ also means that the dependence of G_3 on blackhole horizon r_h appears in the form of a resolution parameter $a = a(r_h)$, a crucial fact that was first conjectured in [70] and will be further illustrated in the next subsection.

[10] The term in (4.35) appearing in (4.18) contributes as $\sim \mathcal{O}(F^2(g_sM^2/N)^l), l \geq 1$ which can be easily obtained by using $e^{-4A} \sim \mathcal{O}(g_sN)\,[1 + \mathcal{O}(g_sM^2/N)]$. As $g_sM^2/N \ll 1$, we can ignore $\mathcal{O}(F^2 g_sM^2/N)$ terms. See also (4.29), (4.30) and (4.31) for more details on the various scaling limits.

As already mentioned, equation (4.17) determining e^A also depends on the internal metric $\widetilde{g}_{mn}$. In the absence of any flux and axion-dilaton field, $\widetilde{g}_{mn}$ is the metric of base of the deformed conifold $T^{1,1}$ which has the topology of $S^2 \times S^3$. In the presence of a black hole horizon and various sources, the internal metric will be modified in the following way:

$$\widetilde{g}_{mn} = \widetilde{g}^{[0]}_{mn} + \widetilde{g}^{[1]}_{mn} \tag{4.37}$$

where $\widetilde{g}^{[0]}_{mn}$ is the metric of a resolved deformed cone (or more appropriately, here, the resolved cone) with base $T^{1,1}$ and therefore $\widetilde{g}^{[1]}_{mn}$ denotes all the corrections due the black hole and all other sources. This means that $\widetilde{g}^{[1]}_{mn}$ contains all the informations of the resolution factor and its subsequent dependence on the horizon radius etc. Note also that, as we have a horizon at $r = r_h$ with M units of fluxes[11] and N_f number of seven branes, $\widetilde{g}^{[1]}_{mn}$ must at least be of $\mathcal{O}(M, N_f, r_h^4/r^4)$. We will evaluate e^A and $\widetilde{g}^{[1]}_{mn}$ to lowest order in $\frac{g_s M^2}{N}$ and $g_s N_f$ which in turn will drastically simplify our analysis. Our choice of $\widetilde{g}^{[0]}_{mn}$ and $\widetilde{g}^{[1]}_{mn}$ will be such that we have (4.3) for the internal metric.

The Bianchi identity for the five-form flux, in the absence of any three-brane sources, reads

$$d\widetilde{F}_5 = H_3 \wedge F_3 \tag{4.38}$$

where F_3 and H_3 are the RR and the NS three-form fluxes. They are given as

$$F_3 \; = \; F_3^{(0)} + \mathcal{O}(F), \qquad H_3 \; = \; H_3^{(0)} + \mathcal{O}(F) \tag{4.39}$$

where $F_3^{(0)}, H_3^{(0)}$ are the fluxes in the absence of any squashing, that is for $F = 0$ (we expressed this earlier as $G_3^{(0)} \equiv F_3^{(0)} - \tau H_3^{(0)}$). For the regular cone, taking into account the running of the τ field, $F_3^{(0)}$ and $G_3^{(0)}$ are exactly the Ouyang fluxes [115],

[11] In the intermediate region, i.e Region 2 of the geometry, we will also have (p, q) five-brane sources.

while the exact form of the fluxes in a deformed conifold were discussed in [89] [6]. Now from the form of the fluxes on deformed cone one gets that

$$F_3 \ \sim \ M[1 + \mathcal{O}(F)], \qquad H_3 \ \sim \ g_s M[1 + \mathcal{O}(F)] \tag{4.40}$$

Using this and (4.39) one readily gets that

$$H_3 \wedge F_3 = F_3^{(0)} \wedge H_3^{(0)} + \mathcal{O}(M^2 F) \tag{4.41}$$

An immediate question is: what can be said about the squashing function F? In the absence of the three-form fluxes, i.e $M = 0$, there is no squashing as the Klebanov-Witten solution [7] with running dilaton [115] needs no squashing. This remains true even when we introduce temperature. To see this, observe that the non-extremal limit of Klebanov-Witten(KW) model does not require any modification of the internal space: which means $F = 0$ with $e^{2B} = 1 - \bar{r}_h^4/r^4$ and the internal space is exactly $T^{1,1}$. There could be squashing due to the running of τ field in the KW blackground, but squashing would be at $\mathcal{O}(g_s^2 N_f^2)$, so we can ignore it as we will only consider up to linear order in $g_s N_f$. These behaviors indicate that F must be at least proportional to M. In the following subsection, we will justify this claim.

4.2.2 Behavior of F and various scaling limits

Let us go to the case when there is no blackhole but we have non-zero three-form flux i.e $M \neq 0$. For this case we are back to Klebanov-Strassler-Ouyang background with no squahing and $F = 0$. This means, F must also be proportional to the blackhole horizon r_h. Combining this with the form of the Ouyang fluxes, taking into account of the back reactions of the seven branes, we expect

$$F \sim \mathcal{O}(a_0^2, r_h g_s M^\alpha / N^\beta, r_h g_s^2 N_f^2) \tag{4.42}$$

with a_0 being the bare resolution parameter discussed earlier and (α, β) are some integers. Notice that we have inserted a suppression factor of $N^{-\beta}$ assuming $\beta > 0$ in anticipation of a possible perturbative expansion. Therefore using our ansatz (4.42) in (4.41) gives us

$$F_3 \wedge H_3 = F_3^{(0)} \wedge H_3^{(0)} + \mathcal{O}(a_0^4, r_h^2 g_s^2 M^{\alpha+2}/N^{\beta+1}, r_h^2 g_s^2 N_f^2) \tag{4.43}$$

implying that up to quadratic order in M, we only need Ouyang fluxes to solve (4.38). But to guarantee that we only need to consider up to quadratic order in M, we must show that higher order i.e $\mathcal{O}(g_s^2 M^{\alpha+2}/N^{\beta+1})$ terms are small compared to the $g_s M^2/N$ terms coming from $F_3^{(0)} \wedge H_3^{(0)}$. This will indeed be the case once we solve (4.38) up to $\mathcal{O}(M^2)$ [12] . We will see $F \sim M/N$ where $N \gg M$ and this justifies ignoring the second term in (4.43). In fact solving (4.38) with our ansatz for the warp factor shows that $\frac{1}{g_s N}(F_3 \wedge H_3)$ is the relevant term that enters into the equaton of motion (see Appendix A). Hence in solving (4.38) with our choice of warp factor, we are really ignoring $\mathcal{O}(g_s M^3/N^2)$ and keeping terms only up to $\mathcal{O}(g_s M^2/N)$. This truncation is consistent for $N \gg M$ which is achievable as we showed in (4.29) and (4.30). However one might question the suppression terms in (4.42) and in (4.43) if (α, β) exponents are arbitrary compared to the range that (4.29) would impose. That this will not be the case will become apparent from the following discussions.

To start then we shall continue using the following five-form flux:

$$\widetilde{F}_5 = (1 + *_{10}) d\alpha \wedge d^4x \tag{4.44}$$

With this form of $\widetilde{F}_5$ and $\alpha = e^{4A} = 1/h$, (4.38) becomes an equation involving h, e^{2B} and F. We already know that in the AdS limit $e^{2B} = 1 - \bar{r}_h^4/r^4$. In our non-AdS geometry we expect:

$$e^{2B} = 1 - \frac{\bar{r}_h^4}{r^4} + G \tag{4.45}$$

[12] If the solution to (4.38) up to $\mathcal{O}(M^2)$ tells us that $F > 1$, then we cannot ignore the second term in (4.43) and therefore have to include $\mathcal{O}(M^3)$ and higher in solving (4.38). But, as we will argue soon, our solutions show that $F \ll 1$, which justifies our truncation.

where G is at least $\mathcal{O}(M, N_f)$. Using this expansion for e^{2B}, along with the precise form of the Ouyang three-form fluxes $F_3^{(0)}, H_3^{(0)}$ and only considering up to $\mathcal{O}(M^2)$ terms [13] , (4.38) reads

$$\left[\partial_r\partial_r h^1 + \frac{1}{g}\partial_{\theta_i}\left(\bar{g}_0^{\theta_i\theta_i}\partial_{\theta_i}h^1\right) + \frac{r_h^4/r^4}{g}\partial_{\theta_i}\left(\bar{g}_0^{\theta_i\theta_i}\partial_{\theta_i}h^0\right)\right]r^5 + 5r^4\partial_r h^1 = 4L^4\partial_r F \quad (4.46)$$

where $\bar{g}_0^{mn}$ is proportional to the deformed conifold metric (see Appendix A), $h = h^0 + h^1$ with h^0 being the Ouyang warp factor

$$\begin{aligned} h^0 &= \frac{L^4}{r^4}\left\{1 + \frac{3g_sM^2}{2\pi N}\,\log r\left[1 + \frac{3g_sN_f}{2\pi}\left(\log r + \frac{1}{2}\right)\right]\right. \\ &+ \left.\frac{3g_s^2M^2N_f}{8\pi^2 N}\,\log r\,\log\left(\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right)\right\} \end{aligned} \quad (4.47)$$

and h^1 is the contribution due to the presence of the black hole.

We can readily see from (4.46) why $F \sim M/N$. First note that the non-extremal limit of Klebanov-Witten model has an exact solution, $h = L^4/r^4$ with $h^1 = 0$. h^1 is only non-trivial due to the presence of three form fluxes, the black hole and other sources. Thus, $h^1 \sim \mathcal{O}(M, g_sN_f, r_h^4/r^4)$. On the other hand $L^4 = g_sN\alpha'^2$ and thus one gets from (4.46) that $F \sim \mathcal{O}(M/N, g_sM^2/N)$. But $L^4/\alpha'^2 \gg 1$ and we can choose it large enough such that $N \gg M$ which guarantees that $F \ll 1$. This is of course consistent with (4.29)[14] .

[13] Again in ignoring higher order terms in M, we are assuming that $F, G \sim \mathcal{O}(M/N) < 1$, which will be consistent with our solution. On the other hand, the $\mathcal{O}(M^2)$ term that enters into (4.46) from the Ouyang warp factor should be understood to be of $\mathcal{O}(g_sM^2/N)$. Terms of $\mathcal{O}(M^3)$ in (4.46) come from products of g_sM^2/N with F and since $F \sim \mathcal{O}(M/N) < 1$, the $\mathcal{O}(M^3) \ll \mathcal{O}(M^2)$ can be ignored. Thus we have sometimes ignored the $1/N$ factor or g_s/N factor, but they can always be inserted back in appropriate context.

[14] Note that the third term in (4.46), because of the θ_i derivative, is suppressed as $g_s^3M^2N_f$. Using (4.29) and footnote 7 this would go to zero as $\epsilon^{1/2}$. Also comparing

The key point in the above argument came from $L^4/\alpha'^2 \gg 1$ appearing in the Ouyang solution, which is on a regular cone while we have a deformed cone. How can we use the form of h^0 as given by (4.47) for the case of a deformed cone? The answer lies in the fact that for large radial distances, the deformed cone coincides with the regular cone. The Klebanov-Strassler solution in the large r regime behaves as the Klebanov-Tseytlin solution, i.e the warp factor for KS model becomes

$$\begin{aligned} h_{KS} &\sim \frac{\alpha'^2}{r^4}\left[g_s^2 M^2 \log\left(\frac{r}{r_*}\right)\right] \\ &= \frac{\alpha'^2}{r^4}\left[g_s^2 M^2 \log\, b + g_s^2 M^2 \log\left(\frac{r}{br_*}\right)\right] \\ &= \frac{L^4}{r^4}\left[1 + \frac{g_s M^2}{N}\log\left(\frac{r}{r_0}\right)\right] \end{aligned} \tag{4.48}$$

where b is some scale and $L^4 = g_s N \alpha'^2$ with $N = g_s M^2 \log\, b, r_0 = br_*$. The above expansion shows that the KS warp factor in the deformed cone can really coincide with the Klebanov-Tseytlin solution. Once back-reactions of the flavor D7 branes are taken into account, KS solution in the deformed cone background will take the form of the Ouyang solution. We can of course choose $\log\, b \gg 1$ such that $L^4/\alpha'^2 \gg 1$, so our argument that $M/N \ll 1$ holds even if we started with KS solution and not the Ouyang solution[15] . Hence it is justified to use the Ouyang solution even for the deformed cone.

Also note that, although there were no D3 branes in the KS solution, an effective $N = g_s M^2 \log\, b$ reappears in the warp factor of KS model in the large r region. This N can be identified with the N appearing in the Ouyang solution which also justifies using the Ouyang solution on the deformed cone background for large r region. For

this term with $g_s M$, the fall-of is $g_s^2 M N_f$ which from (4.30) goes to zero as ϵ. Therefore from all criteria in (4.46), $h^1 \sim \mathcal{O}(M)$ seems consistent.

[15] Incidentally, using (4.29), we would require b to go to infinity as $\exp\left(\epsilon^{-9/2}\right)$.

small radial distances, we cannot use the h^0 as given in (4.47) $-$ hence the non-extremal solutions we consider are only valid for large radial distances. This also means, we are considering large horizon r_h and the geometry is dual to the high temperature regime of the gauge theory. A conclusion that is consistent with our earlier works.

4.2.3 Analysis of the full background with backreactions

Once the behavior of F and the suppression orders for various terms are laid out, we are ready to tackle the backreactions to order g_sN_f and g_sM^2/N. We start from the equation of motion for τ given in the following way:

$$\widetilde{\nabla}^2\tau \sim \widetilde{g}^{mn}\partial_m\tau\partial_n\bar{\tau} \tag{4.49}$$

However, the underlying F-theory picture [106] on which we based our solution [71], dictates that $\partial\tau \sim \mathcal{O}(g_sN_f)$ and therefore we will ignore terms of $\mathcal{O}(g_s^2N_f^2)$. So the precise form of τ will not appear in any of the equations (4.46), (4.16), (4.34) and (4.36).

Thus with our ansatz for the metric (4.1), (4.3) and choice of fluxes, we have *four* equations (4.46), (4.16), (4.34) and (4.36) that we need to solve and *three* unknown functions h^1, G and F. However, it is more convenient to write $h^1 \sim A^1L^4/r^4$ and then from (4.46) one readily sees that

$$A^1 \;\sim\; \mathcal{O}(M/N) \ll 1, \quad \text{with} \quad F \sim \mathcal{O}(M/N) + \mathcal{O}(g_sM^2/N) + \mathcal{O}(g_s^2M^2N_f/N) \ll 1 \tag{4.50}$$

and so the third term in F is even more suppressed. Now what can we say about G? As already pointed out, $G \sim \mathcal{O}(M, g_sN_f)$. But using the form of F as given above in (4.50), one readily gets from expanding (4.16), that

$$G = \mathcal{O}(F) \sim \mathcal{O}(M/N) + \mathcal{O}(g_sM^2/N) + \mathcal{O}(g_s^2M^2N_f/N) \ll 1 \tag{4.51}$$

Thus it is reasonable to consider only up to linear order terms in A^1, G and F. But (4.36) is a trivial equation up to linear order (see Appendix A) and hence the only non-trivial equations we are solving are (4.46), (4.16) and (4.34). Thus we have a system of three equations and three functions A^1, G and F $-$ which can be easily solved.

Note that once the above three equations are solved, the corrections from *all* the other Einstein equations are automatically suppressed, as long as we are in the range (4.32), and the precise functional form of the axion-dilaton field τ and the non-ISD three-form flux G_3 do not influence the four equations up to linear order in A^1, G and F. This is because (4.46) is obtained from (4.17) which is identical to (4.14) (up to linear order in A^1, G and F) which in turn is obtained by tracing Einstein equations in the Minkowski directions. On the other hand, (4.34) is obtained from tracing the Einstein equations in the internal directions. Hence a solution to (4.46) and (4.34) along with the background Ouyang warp factor h^0 and three form fluxes G_3 minimizes the action (4.6) where only Ricci scalar and the flux strength appear for the radial range (4.32). Thus solving (4.46) and (4.34) really means putting the action on shell which guarantees that individual Einstein equations change the metric only to order r_h^4/r^4 as depicted in (4.3).

The form of the solutions to the three equations along with the boundary conditions that dictate the behavior of the warp factor A, B near the horizon is discussed in Appendix A. Here we only quote the functional form of the solutions

$$\begin{aligned} h^1 &= \frac{L^4}{r^4}\left(A_0 + A_1 \log r + A_2 \log^2 r\right) \\ e^{2B} &\equiv g = 1 - \frac{\bar{r}_h^4}{r^4} + G \equiv 1 - \frac{\bar{r}_h^4}{r^4} + g_0 + g_1 \log r + g_2 \log^2 r \\ F &= F_0 + F_1 \log r + F_2 \log^2 r \end{aligned} \tag{4.52}$$

where A_i, g_i, F_i for $i = 0, 1, 2$ are in general functions of r and the internal coordinates θ_j, ϕ_j, ψ, with $j = 1, 2$. In Appendix A we have worked out the simplest case where

A_i, g_i, F_i are assumed to be functions of r only by neglecting $\mathcal{O}(g_s N_f)$ terms[16] . This is a reasonable assumption for small number of flavors. Furthermore, the thermodynamics of the field theory is dictated by the behaviour of the dual geometry near the black hole horizon (4.32). If we keep all the seven branes away from the black hole, we can ignore running of τ near the black hole. On the other hand, for constant τ we expect a Klebanov-Strassler type solution which essentially means the warp factors A, B and squashing factor F are only functions of r. Hence, as long as we are dealing with the light degrees of freedom that arise from the deformed cone ignoring the back reaction of seven branes, we can neglect the contributions from the seven branes far away from the black hole and consider the solution in (4.52) to be functions of r only.

To account for the heavy quarks, we have to include $\mathcal{O}(g_s N_f)$ terms but our ansatz (4.52) remains the same with the understanding that now A_i, g_i, F_i are additionally funtions of the internal coordinates. Interestingly, however, to analyze the melting of the heavy quarkonium states, we can consider string world sheets that are *fixed* in the internal directions which results in evaluating the warp factors A, B only for fixed values of the angles θ_j, ϕ_j, ψ. This means our above analysis would suffice. Hence, even for the study of linear confinement and melting of heavy $Q\overline{Q}$ pairs, it is sufficient enough to treat the solutions in (4.52) as being functions of the radial coordinate only (see [118, 119] for related works in this direction).

In Figures 4.1, 4.2 and 4.3 we have plotted $g(u), A_0(u)$ and $F_0(u)$ where $u \equiv r/\bar{r}_h$ using the numerical solutions to equations (4.46), (4.16) and (4.34). As discussed in Appendix A, at the lowest order of perturbation, only keeping up to linear order terms in $g_s M^2/N$, equations (4.46), (4.16) and (4.34) drastically simplify. We obtain a solution with only A_0, g_0 and F_0 non trivial while $A_1 = A_2 = g_1 = g_2 = F_1 = F_2 = 0$. For the plots, we have chosen $3g_s M^2/2\pi N = 1/2$ and the following boundary

[16] It should also be clear that $A_i \sim \mathcal{O}(M/N)$ from (4.50).

conditions[17]

$$
\begin{aligned}
&A_0(\infty) = 0, \quad A_0'(\infty) = 0 \\
&g_0(\infty) = 0, \quad g_0'(\infty) = 0 \\
&F_0(\infty) = 0, \quad F_0'(\infty) = 0
\end{aligned} \tag{4.53}
$$

Note that $g(1.04) \sim 0$, indicating that the horizon has shifted from the AdS black hole value of $\bar{r}_h$ and we have obtained a larger black hole with horizon $r_h \sim 1.04\,\bar{r}_h$. The fact that the black hole is of larger size than the AdS limit is consistent with the underlying gauge theory structure[18] . The presence of the fractional branes has increased the effective mass of the black hole. In fact, the black hole entropy is larger than the corresponding AdS limit since $A_0(r_h) > 0$ and using Walds formula, one readily gets that $s/T^3 \sim N_{\rm eff}^2 > N^2$ where we have defined $g_s N_{\rm eff} = r_h^4 h(r_h)$. However, we should be careful about these numerical results. Notice that there is an additional condition eq.(4.32) to satisfy. Suppose we take the same value of N, g_s as in eq.(4.33), then $M \simeq 1.8 \times 10^5$ to get $3g_s M^2/2\pi N = 1/2$. According to eq.(4.32) we get $r \gtrsim 5r_h$, which means we can only trust our numerical results in the regime $r > 5r_h$. With different values of N, M and g_s, we will find different minimum ratios of r and r_h according to eq.(4.32), nevertheless the ratio must always be greater than 1. To find geometry near the black hole horizon we need to consider higher order corrections which is not studied here.

[17] Let us assume, for simplicity and for performing the numerical analysis, $3g_s M^2/2\pi N = 1/2$ to be the smallest scale in the theory (instead of M/N that we took earlier). Then the argument used earlier in (4.32) will imply that we should trust our result for $r > 1.19 r_h$.

[18] Also note that the result is consistent with the first law of black hole thermodynamics which states that the increase in horizon radius is related to the increase in the mass of the black hole. The addition of five-branes have increased the effective mass of the black hole compared to the AdS limit.

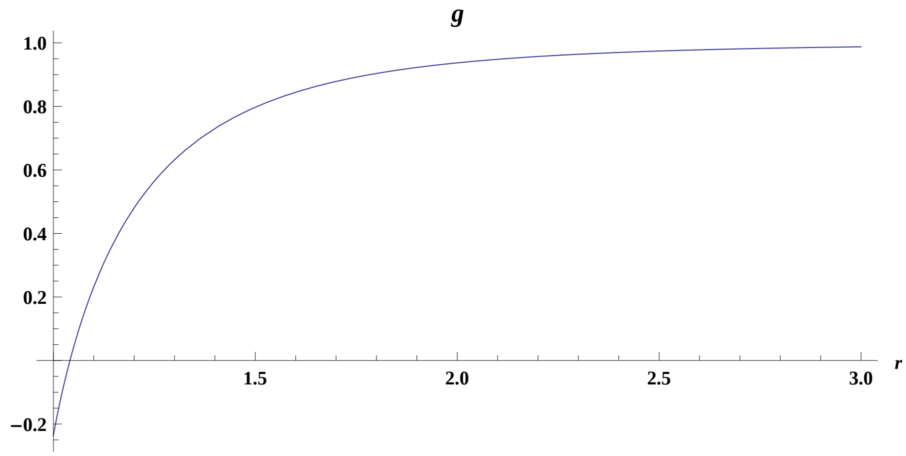

Figure 4.1: The blackhole factor g as a function of $u \equiv r/\bar{r}_h$. We have plotted g along the y-axis and u along the x-axis.

Finally, note that the identification of F with a^2 in (4.4) implies that the resolution parameter is given by

$$\begin{aligned} a^2 = \quad & a_0^2 + \frac{5g_sM^2p_{11}r_h^2}{32\pi N} + \frac{g_sM^2}{N}\frac{r_h^2}{4\pi}\left[p_{12}\log\, r + p_{13}\log^2 r\right] \\ & + \frac{1}{4\pi}\left(\frac{g_sM^2}{N}\right)(g_sN_f)\, r_h^4\left(p_{14}\frac{\log\, r}{r^2} + \frac{p_{15}}{r^2}\right)\log\left(\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right) \end{aligned} \tag{4.54}$$

where we show the bare resolution parameter[19] in F and a^2. The coefficients p_{ij} are constant numbers that could be determined from (4.52) and Appendix A. The above representation of the resolution parameter is perfectly consistent with our conjecture in [70]: the resolution parameter will pick up dependence on the horizon radius r_h. Interestingly we now have managed to get the leading order $\frac{g_sM^2}{N}\log\, r$ corrections to the result.

[19] In the limit where the bare resolution parameter vanishes, which is the Klebanov-Tseytlin solution, we see that the g_sM^2/N corrections actually make the small r regions non-singular creating an apparent resolution parameter proportional to the horizon radius.

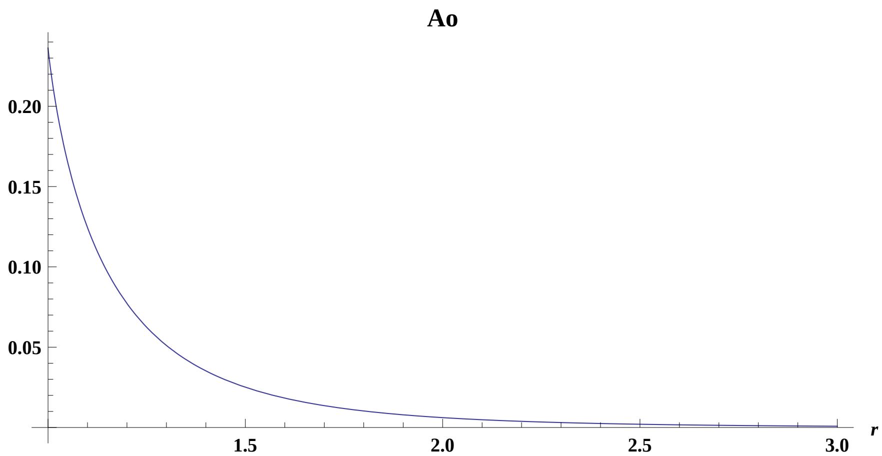

Figure 4.2: Non-extremal contribution to the warp factor given by A_0 plotted as a function of $u \equiv r/\bar{r}_h$. A_0 is plotted along y-axis, and u is still along the x-axis.

However there is one issue that might be confusing the reader. From Figure 4.3 we see that F_0 is always negative for all values of r in the range $r_h \leq r < \infty$. Our identification of F with a^2 would then imply a to be a purely imaginary number. However surprisingly this *does not* create a problem. As we will show in the next subsection, all the fluxes etc. are completely expressed in terms of a^2, so that a does not appear anywhere. Even terms with logarithms appear as $\log |a^2|$, so that $a^2 < 0$ do not create any inconsistencies. This is of course shouldn't come as a surprise because the resolution parameter appear in the metric (4.3) as $1 + F_0$ and since $|F_0| < 1$ it shouldn't lead to any inconsistencies no matter how we relate F_0 to a.

In our opinion the result that we presented above is probably the first time where the backreaction effects from black hole, including the resolution factor, are taken into account in a self-consistent way to lowest orders in $g_s N_f$ and $g_s M^2/N$. To this order, as we showed above, the backreactions from fluxes and branes could be consistently ignored in the near horizon limit (4.32). One may now take this background and compute the IR effects for large N thermal QCD. However before we go about studying these effects we would like to dwell, just for the sake of completeness, on the corrections to the Klebanov-Strassler three-form fluxes that arise from the backreactions of the black-hole, local brane sources, and the resolution parameter. Readers wishing to

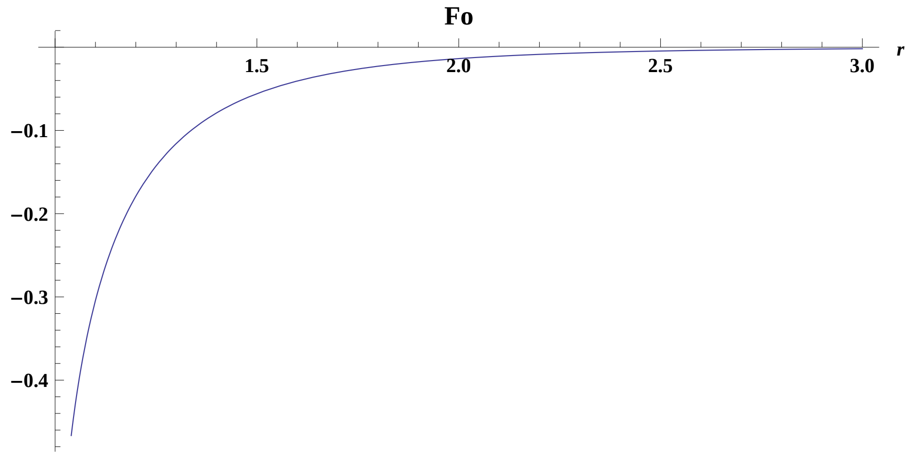

Figure 4.3: The squashing factor given by the resolution function F_0 as a function of $u \equiv r/\bar{r}_h$. Note that F_0 is always negative for all distances outside the black hole horizon. We plot F_0 along the y-axis, and u is along x-axis.

know our results may however skip the next sub-section altogether and proceed on with the calculations of the RG flows and the effects of the chemical potential.

4.2.4 Short detour on dualities and dipole deformations

Our final aim of this section would be to take a short detour and study the effect of the dipole deformations on the flavor seven-branes in the gravity picture. This dipole deformation, since it affects the seven-branes, should also have some effect on the fundamental quarks in the gauge theory. We will make some speculations how the dipole deformations effect the far IR picture.

Our starting assumption would be that the solutions presented in the earlier subsections have isometries along ϕ_1, ϕ_2 and ψ directions. This in particular means that the coefficients appearing in (4.52) i.e (A_i, F_i, G_i, g_i) are all functions of (r, θ_i) only and not of (ϕ_i, ψ). This is not a strong assumption as we saw earlier that even to $\mathcal{O}(g_s^2 M^2 N_f/N)$ the (ϕ_i, ψ) dependences do not show up. It could be that the background retains its isometry along (ϕ_i, ψ) directions to all orders in $g_s N_f$ and $g_s M^2/N$, but we haven't shown this here.

Before moving ahead let us clarify a point here. Dipole (or non-commutative) deformations can be studied in two possible ways. In the conformal case, one takes the D3-brane metric written in terms of its harmonic functions, and then use TsT (T-duality, followed by a shift s, and then another T-duality) to generate new solution. The new solution is still given in terms of D3-branes and harmonic functions, but now there is a background $B_{\rm NS}$ field. One then takes the near horizon limit to determine the gravity dual of this scenario. The gravity dual has no D3-branes, but both F_5 as well as $H_3 = dB_{\rm NS}$ fluxes are still present. The near horizon geometry do not change the internal metric too much, and therefore analysis on both sides of the story is somewhat similar.

The above criteria changes quite a bit once we go to the non-conformal case. The gravity dual is not simply given by taking the near-horizon limits of the D3 and the wrapped D5-branes. To avoid naked singularities of the Klebanov-Tseytlin form, one now has to deform the internal space also. This means making a TsT transformation on the brane side, one may not necessarily get the full gravity dual picture easily. This is also clear in the geometric transition set-up, whose supergravity solution is developed in [112, 42]. So we could do TsT transformations on two sides of the picture, leading to two possible different interpretations.

Thus, once we have solutions for both sides, namely the gauge-theory and the gravity sides, we can use TsT transformations to deform them into various different solutions. In this chapter we will not consider the dipole (or non-commutative) deformations on the gauge-theory side of the story[20] , but concentrate only on the gravity side. This means, given the background metric (4.1) with fluxes, five-branes

[20] The dipole deformations on the gauge theory side, at least in the far IR and in the *local* case, has been discussed earlier in [113]. The readers may refer to those papers for more details on the multiply allowed dipole deformations.

and seven-branes, the TsT transformed backgrounds will be related to some interesting deformations of the four-dimensional thermal gauge theories. These deformations can be classified to fall into four categories. They are listed as follows[21] :

• T-dualize along one space direction say x_3 then shift along another space direction say x_2 mixing (x_2, x_3) and then T-dualize back along x_3 direction.

• T-dualize along x_3 and then shift[22] along one of the internal directions that are isometries of the background, namely along ϕ_1, ϕ_2 or ψ directions[23] , and then T-dualize back along x_3 direction.

• T-dualize, shift and then T-dualize along internal directions. The shift will mix two of the internal directions in some appropriate way.

The first operation will lead to a non-commutative gauge theory on the D7-branes with $[x_2, x_3] = iB_{23}$ as our algebra. The second one is more interesting. T-dualizing along x_3 but making a shift on the directions along which the D7-branes are oriented i.e along ϕ_1, θ_1 and ψ (recall that the D7-branes wrap the two-sphere parametrised by (θ_1, ϕ_1) and are spread along (r, ψ) directions) will lead again to a non-commutative gauge theory on the D7-branes. On the other hand, if we make a shift along the orthogonal direction parametrised by ϕ_2, then the theory on the D7-branes will be a dipole gauge theory. For the last case, one is T-dualizing and shifting along the directions of the D7-branes. This will again lead to non-commutative theory on the D7-branes. On the other hand if we shift along ϕ_2 but T-dualise along the D7-brane directions, we will get dipole theory on the D7-branes.

---

[21] We will use (x_0, x_1, x_2, x_3) as a convenient reparametrization of (t, x, y, z) used earlier. The former will be more convenient for the next couple of sections.

[22] Again mixing x_3 with one of the internal directions.

[23] For simplicity we will only consider the isometry directions.

To analyse these in case-by-case basis, let us study the first kind of deformation first. We choose the shift to be

$$x_2 \mapsto \frac{x_2}{\cos\theta} + \sin\theta x_3, \quad x_3 \mapsto \cos\theta x_3 \tag{4.55}$$

After the series of transformations discussed above, i.e TsT, the metric (4.1), becomes[24] :

$$ds^2 = \frac{1}{\sqrt{h}}\left[-g_1 dx_0^2 + dx_1^2 + J(dx_2^2 + dx_3^2)\right] + \sqrt{h}(g_2^{-1}dr^2 + d\mathcal{M}_5^2) \tag{4.56}$$

with the Lorentz breaking deformations along (x^2, x^3) directions specified by J. There is also a background $B_{\rm NS}$ field that accounts for the non-commutativity. Both J and the $B_{\rm NS}$ field are defined as:

$$J^{-1} \;=\; \sin^2\theta h^{-1} + \cos^2\theta, \qquad B_{23} \;=\; \tan\theta h^{-1} J \tag{4.57}$$

The metric has the same form as in [64] and the gauge theory on the D7-branes become non-commutative in the x^2 and x^3 directions.

For the second kind of deformation we follow similar procedure as above except that now we shift along ψ direction and T-dualise along x^3 direction. The resulting metric is

$$\begin{aligned} ds^2 \;=\; & \frac{1}{\sqrt{h}}\left(-g_1 dx_0^2 + dx_1^2 + dx_2^2 + \frac{9}{9\cos^2\theta + r^2\sin^2\theta}dx_3^2\right) \\ & + \sqrt{h}\left[\frac{r^2(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2}{9\cos^2\theta + r^2\sin^2\theta} + ...\right] \end{aligned} \tag{4.58}$$

where note that the x^3 and the ψ circle is non-trivially warped. The dotted terms are unchanged from the original metric (4.1). However the $B_{\rm NS}$ field now is non-trivial

[24] For this section we will ignore the $\mathcal{O}(g_s M^2/N, r_h^4/r^4)$ corrections to the internal metric (4.3). A more precise result will not change the physics to the order that we are studying here.

because of the ψ fibration structure:

$$B = \frac{r^2 \tan\theta}{9\cos^2\theta \ + \ r^2\sin^2\theta} \ dx_3 \wedge (d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2) \tag{4.59}$$

The scenario now is interesting because we have three components of the $B_{\rm NS}$ field, with two of the components $B_{3\psi}$ and $B_{3\phi_1}$ parallel to the D7-branes and one component $B_{3\phi_2}$ having one leg orthogonal to the D7-branes. Existence of these three components would lead to a complicated theory on the D7-branes that in some limit may be considered as a combination of both dipole and non-commutative deformations of the world-volume theory on the D7-branes.

As an example for the third kind of deformation[25] we first T-dualize along ψ direction and then shift along ϕ_2 direction. The resulting metric, keeping both the squashing factors $(F, \mathcal{G})$ in (4.1), will look like

$$\begin{aligned} ds^2 \ &= \ \frac{1}{\sqrt{h}}(-g_1 dt^2 + dx^2 + dy^2 + dz^2) + \sqrt{h}\Big[r^2 J(d\psi + \cos\theta_1\cos\theta d\phi_1 + \cos\theta_2 d\phi_2)^2 \\ &+ \frac{3}{2} r^2 J(1+F)(1+\mathcal{G})\sin\theta_2^2 d\phi_2^2 + ...\Big] \end{aligned} \tag{4.60}$$

with the dotted terms being the terms unchanged from the original metric (4.1). As expected, note that both the ψ fibration structure as well as the ϕ_2 directions get warped by J. As before there is also a $B_{\rm NS}$ field. Both J and $B_{\rm NS}$ are given by:

$$\begin{aligned} B_{\psi\phi_2} \ &= \ \frac{1}{6}\tan\theta \ \sin^2\theta_2 \ (1+F)(1+\mathcal{G})J \\ J^{-1} \ &= \ 9\cos^2\theta \ + \ \frac{1}{6} h r^2 (1+F)(1+\mathcal{G})\sin^2\theta_2 \sin^2\theta \end{aligned} \tag{4.61}$$

[25] Note that we cannot construct another theory by shifting along θ_2 direction and then T-dualing along ϕ_2 direction because θ_2 is not an isometric direction. A shift along the non-isometry directions, like the θ_2 direction, will destroy the existing isometry directions making the T-duality operations highly non-trivial.

This solution shows that the gauge theory on the D7-branes has become a non-local dipole theory[26] .

All these solutions generated from our TsT duality operations lead to new gauge theories on the D7-branes. As mentioned earlier, it is not clear to us whether these deformations are the corresponding gravity duals of the respective deformations on the gauge theory side of the picture. One thing however is clear: due to the dipole deformations on the D7-branes, the KK masses of the fluctuations are different from the original theory. In fact the dipole deformations (along appropriate directions) tend to make the KK states *heavier* [113]. Therefore we would expect operator dimensions on the field theory side to also change accordingly.

The TsT duality operation doesn't change the warp factor nor the BH factor. Naively applying the criteria from [71] one would think this doesn't change the thermal behavior of the theory. However, notice that now there is an extra non-constant $B_{\rm NS}$ field that cannot be gauged away. This means when we write down the Nambo-Goto action for the strings, the effect of the $B_{\rm NS}$ field can no longer be ignored and it'll definitely change the criteria of the confinement/deconfinement transition studied in [71]. This shouldn't be surprising because one would expect the thermal behavior of the dipole deformed quarks to be different from the un-deformed ones.

---

[26] On the other hand, the shifts and the duality directions that we choose are not the most generic ones. We can make numerous other shifts. One simple example could be as follows: we T-dualize along space direction x_3, then shift as $z \mapsto z + \lambda\theta_2^2/2$ and finally T-dualize back to generate a non-trivial background with the metric

$$ds^2 \;=\; \frac{1}{\sqrt{h}}(-g_1 dt^2 + dx^2 + dy^2 + dz^2) + \sqrt{h}(g_2^{-1} dr^2 + d\mathcal{M}_5^2 + \lambda^2\ \theta_2^2\ d\theta_2^2)$$

and a $B_{\rm NS}$ field, $B_{3\theta_2} \;=\; \lambda\theta_2$. This would generate dipole deformation on the D7-branes.

Chapter 5

Summary and discussion

In this thesis, we studied the gravity duals of three gauge theories with different properties that we list below.

In Chapter 2 we studied the scalar spectrum of the $Y^{p,q}$ manifold and obtained both upper and lower bounds for all the eigenmodes λ_k of the scalar Laplacian. We also tried to extend this gravity dual to that of the non-conformal gauge theories by resolving and deforming the $Y^{p,q}$ manifold. It is clear that there is much that can be done for this case. First, we would like to find the exact eigenvalues of the scalar Laplacian, this might be achieved by following numerical methods, and then matching them with the operators at the gauge theory side; secondly, we would like to find the deformed $Y^{p,q}$ manifold and study the geometric transition between the resolved and deformed $Y^{p,q}$ as in Chapter 3; thirdly, we would like to find out the brane constructions on the gauge theory side.

In Chapter 3, we studied the duality between a non-conformal gauge theory and type IIB supergravity on a deformed conifold at the bottom of the cascade. This is described by a geometric transition between resolved and deformed conifold in type IIB theory or a flop transition between two G_2 structure manifolds when lifted to M theory. We find that generally the final gravity duals have non-geometric configurations. One of the issue that we left unstudied is to understand the non-geometric aspects of the mirror configurations. Understanding these far IR regions of the gravity solutions can help us understand certain properties of the dual gauge theories at strong couplings, such as confinement.

In chapter 4, we studied the backreactions from black hole, branes and fluxes on the background geometry and on the various UV completions. Two challenges still remain: one, to study the equations at $r = r_h$ in Region 1 and two, to study them in the intermediate buffering region i.e Region 2. In both cases the analysis may get very involved because for the first case one would now have to solve all the twenty internal Einstein equations; and for the second case the (p, q) five-brane sources and fluxes will further complicate the scenario. These details are left for future works.

In summary, our aim is to understand the gauge theories in strongly coupled regimes. Gauge/gravity duality provides a good tool. However, the gravity duals can easily get complicated when gauge theories lose some nice properties such as supersymmetry and conformality, or when the structure of the gauge group becomes complicated. Nevertheless, as depicted in this thesis, many interesting properties of the corresponding gauge theories may in fact be extracted in these solutions. The story, however, is far from being complete and there are many more interesting avenues to investigate now.

Appendices

Appendix A

Details on the squashing and the warp factor computations

In the absence of blackhole that is $e^{2B} = 1$, the warp factor $\alpha = e^{4A} = 1/h$ only depends on r, θ_1 and θ_2 even when D7 back reaction is taken into account by considering the running axion-dilaton field [115]. In this extremal limit $r_h = 0$, we have ISD three-form fluxes G_3 and the internal metric $\widetilde{g}_{mn}$ describes a Ricci flat deformed cone. For the non-extremal case, we will demand similar behavior for the warp factor h and will find that such solutions do exist. Using $h \equiv h(r, \theta_1, \theta_2)$ only, in the non-extremal case we get,

$$\begin{aligned} d\widetilde{F}_5 &= d\left(\left[\partial_r h \; \zeta + e^{-2B} \left(\bar{g}^{\theta_1\theta_1} \partial_{\theta_1} h \; \eta_1 + \bar{g}^{\theta_2\theta_2} \partial_{\theta_2} h \; \eta_2 \right) \right] \frac{r^5(1 + F + \mathcal{G}/2)}{108} \sin\theta_1 \sin\theta_2 \right) \\ &\equiv d\mathcal{D} \end{aligned} \tag{A.1}$$

where we have used our metric ansatz (4.1, 4.3) and definition of the five-form flux (4.44). We have only kept linear terms in $F, \mathcal{G}$ and this is justified as we look for solutions $F, \mathcal{G} \ll 1$ and ignore terms higher order term. In the above we have also defined

$$\bar{g}_{pq} = e^{-2B} \widetilde{g}_{pq} \tag{A.2}$$

where p, q run over the compact directions and thus (A.2) implies $\bar{g}_{pq}$ is indepndent of B. Here ζ, η_i are five-forms given by

$$\begin{aligned}
\zeta &= d\psi \wedge d\phi_1 \wedge d\phi_2 \wedge d\theta_1 \wedge d\theta_2 \\
\eta_1 &= d\psi \wedge dr \wedge d\phi_1 \wedge d\phi_2 \wedge d\theta_2 \\
\eta_2 &= d\psi \wedge dr \wedge d\phi_1 \wedge d\phi_2 \wedge d\theta_1
\end{aligned} \tag{A.3}$$

Just like the extremal case, we will assume that $\partial_{\theta_i} h \sim \mathcal{O}(g_s^2 N_f M^2/N)$ and we will find that this choice is consistent with all the Einstein equations and equations for the fluxes . With this assumption, we readily get up to $\mathcal{O}(g_s M^2/N)$, and ignoring $\mathcal{O}(F, G, \mathcal{G})\mathcal{O}(g_s M^2/N)$ (since $F, G, \mathcal{G} \ll 1$)

$$\bar{g}^{\theta_i\theta_i}\partial_{\theta_i} h = \bar{g}_0^{\theta_i\theta_i}\partial_{\theta_i} h \tag{A.4}$$

where $i = 1, 2$ and $\bar{g}_0^{pq}$ is zeroth order in M, N_f. But at zeroth order in M, N_f, the compact five dimensional internal space $\mathcal{M}_5$ is exactly the deformed cone and thus $\bar{g}_0^{pq}$ is precisely the metric of deformed $T^{1,1}$. Our ansatz for the black hole factor e^{2B} is given in (4.45) where G is at least $\mathcal{O}(M/N, g_s M^2/N, g_s N_f)$. This is a sufficient condition as in the absence of five-branes and seven branes, we have $AdS \times T^{1,1}$ with black hole where $e^{2B} = 1 - \frac{\bar{r}_h^4}{r^4}$ and $\bar{r}_h \gg b$. This is because for large r, the deformed cone becomes the regular cone and considering $\bar{r}_h \gg a$, we are effectively putting a black hole in a regular cone. In other words, the non-extremal limit of the geometry only 'sees' the regular cone and deformation of the cone is hidden behind the black hole horizon. This also means our non-extremal solution is valid only for large horizon, that is the non-extremal solution only captures the large temperature deconfined chirally symmetric phase of the gauge theory. The extremal solution without any black hole is dual to the confined phase.

Now using (4.45) in $\mathcal{D}$ reads

$$\begin{aligned}
\mathcal{D} &= \left[\partial_r h\,\zeta + \frac{1}{1-\frac{r_h^4}{r^4}}\left(\bar{g}_0^{\theta_i\theta_i}\partial_{\theta_i} h\,\eta_i\right)\right] r^5\frac{\sin\theta_1\sin\theta_2}{108}\left(1+F+G/2\right)\\
&= \left[\partial_r h^0\,\zeta + \bar{g}_0^{\theta_i\theta_i}\partial_{\theta_i} h^0\,\eta_i\right] r^5\frac{\sin\theta_1\sin\theta_2}{108} + \left[\partial_r h^1\,\zeta + \frac{1}{1-\frac{r_h^4}{r^4}}\bar{g}_0^{\theta_i\theta_i}\partial_{\theta_i} h^1\,\eta_i\right.\\
&+ \left.\frac{r_h^4/r^4}{1-\frac{r_h^4}{r^4}}\bar{g}_0^{\theta_i\theta_i}\partial_{\theta_i} h^0\,\eta_i\right] r^5\frac{\sin\theta_1\sin\theta_2}{108} + r^5\frac{(F+G/2)\sin\theta_1\sin\theta_2}{108}\partial_r h^0\,\zeta \qquad \text{(A.5)}
\end{aligned}$$

where we have only considered up to $\mathcal{O}(g_s M^2/N)$ terms. Here h^0 is the Ouyang solution and h^1 is the correction due to the black hole which alters the internal compact space $\mathcal{M}_5$. But the Ouyang solution satisfies Bianchi identity exactly as:

$$d\left[\left(\partial_r h^0\zeta + \bar{g}_0^{\theta_i\theta_i}\partial_{\theta_i} h^0\eta_i\right) r^5\frac{\sin\theta_1\sin\theta_2}{108}\right] = H_3^{(0)}\wedge F_3^{(0)} \qquad \text{(A.6)}$$

Using this in (4.38) we get

$$d\left[\left(\partial_r h^1\zeta + \frac{1}{g}\bar{g}_0^{\theta_i\theta_i}\partial_{\theta_i} h^1\eta_i + \frac{r_h^4/r^4}{g}\bar{g}_0^{\theta_i\theta_i}\partial_{\theta_i} h^0\eta_i\right) r^5\frac{\sin\theta_1\sin\theta_2}{108} + r^5\frac{(F+G/2)\sin\theta_1\sin\theta_2}{108}\partial_r h^0\zeta\right] = 0$$

which gives us (4.46). The derivations of (4.16), (4.34) and (4.36) have already been discussed in section 2.2.

We will now solve the four equations (4.46), (4.16), (4.34) and (4.36) by ignoring all terms of $\mathcal{O}(g_s N_f)$. In this limit, all angular dependences vanish and all the functions A, F and G are only functions of the radial coordinate r. This also means we are ignoring the back reaction of the seven branes and our solution should be considered as the non-extremal generalization of Klebanov-Strassler theory with modified UV behavior. For $N_f = 0$, with $e^{-4A} = h = h^0 + h^1$, we take the following ansatz

$$\begin{aligned} h^1 &= \frac{L^4}{r^4}\left(A_0(r) + A_1(r)\log\, r + A_2(r)\log^2 r\right) \\ e^{2B} &\equiv g = 1 - \frac{r_h^4}{r^4} + g_0(r) + g_1(r)\log\, r + g_2(r)\log^2 r \\ F &= F_0(r) + F_1(r)\log\, r + F_2(r)\log^2 r \end{aligned} \tag{A.7}$$

With our ansatz, only taking up to linear order terms in A_i, F_i and g_i one obtains that the equation derived from (4.36) is trivial. Also up to linear order, $A_1 = A_2 = F_1 = F_2 = g_1 = g_2 = 0$ is a solution with A_0, F_0, g_0 being the only non-trivial functions. The equations resulting from (4.46),(4.16) and (4.34) are as follows

$$\begin{aligned} &(i)\ rA_0'' - 3A_0' - 4F_0' = 0 \\ &(ii)\ 5r^4 g_0' + 4\bar{r}_h^4 F_0' + r^5 g_0'' = 0 \\ &(iii)\ \frac{6g_sM^2}{N\pi}\bar{r}_h^4 + 56r^4 g_0 + 16r^4 F_0 + 4r\bar{r}_h^4 A_0' + 49r^5 g_0' + 24r^5 F_0' + 12rr_h^4 F_0' \\ &\quad + 7r^6 g_0'' + 4r^6 F_0'' - 4r^2 r_h^4 F_0'' = 0 \end{aligned} \tag{A.8}$$

To solve these second order differential equations, all we need to do now is specify the boundary conditions. As we have second order differential equations, we can choose two boundary conditions. A priori we do not know where the horizon is, that is we do not r_h such that $e^{2B(r_h)=0}$, so we cannot specify the boundary condition at the horizon. Additionally we cannot take r to be smaller than the range (4.32). However, since we are looking for solution such that asymptotically we recover the extremal geometry, we can impose the following boundary conditions:

$$\begin{aligned} &\lim_{r\to\infty} A_0(r) = 0 \\ &\lim_{r\to\infty} g_0(r) = 0 \\ &\lim_{r\to\infty} F_0(r) = 0 \end{aligned} \tag{A.9}$$

From the form of the equations in (A.8), we see that inverse power series in r is a possible candidate for the solutions that obey the boundary conditions (A.9). On the other hand, as already discussed in section 2.1, we expect A_i, g_i and F_i to be proportional to the horizon r_h. Thus our anstaz is

$$A_0(r) = \bar{a}_k^0 \left(\frac{r_h}{r}\right)^k, \qquad F_0(r) = \bar{f}_k^0 \left(\frac{r_h}{r}\right)^k$$
$$g_0(r) = \bar{\zeta}_k^0 \left(\frac{r_h}{r}\right)^k, \tag{A.10}$$

where $\bar{a}_k^0, \bar{f}_k^0$ and $\bar{\zeta}_k^0$ are atleast $\mathcal{O}(M/N, g_s M^2/N)$, and the radial coordinate r is assumed in the range (4.32). The boundary condition (A.9) implies

$$\bar{a}_0^0 = \bar{f}_0^0 = \bar{\zeta}_0^0 = 0 \tag{A.11}$$

We can further choose three other boundary conditions. Again since (a) we do not know where the horizon is and (b) the radial coordinate is constrained by (4.32), we will choose the following boundary conditions: at $r = \infty$ and choose

$$\begin{aligned} &\lim_{r\to\infty} A_0'(r) = 0 \\ &\lim_{r\to\infty} g_0'(r) = 0 \\ &\lim_{r\to\infty} F_0'(r) = 0 \end{aligned} \tag{A.12}$$

which is automatically solved by our ansatz (A.10). With the set of boundary conditions (A.9) and (A.12), we solve (A.8) numerically. The exact solution (whose validity should be considered for $r > (N/M)^{1/4} r_h$) is plotted in Figures 4.2.3, 4.2.3, 4.2.3. Observe that the numerical solutions are consistent with the analytic behavior in (A.10). When $M = 0$, equations (A.8) imply that we have the trivial solution, i.e. $A_0 = g_0 = F_0 = 0$. But since $M \neq 0$, we must have non-trivial solutions to satisfy (A.8).

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