FRACTIONAL TAP-SPACING EQUALIZERS FOR DATA TRANSMISSION

by

©

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ABSTRACT

This thesis presents a study of the theory of conventional and Fractional Tap Spacing Equalizers and outlines their relative benefits and drawbacks. Two special cases of Fractional Tap Spacing Equalizers are emphasized in this work: the T/2-Tap Spacing Equalizer and a new type of equalizer, called a Hybrid Transversal Equalizer, in which the tap spacing is either T or T/2 (where 1/T is the data source symbols rate). A mathematical analysis of these equalizers is carried out and some new results are derived. To support the mathematical analysis, a computer program was used to compare the performance of these models of equalizers and the results obtained are analysed.

SOMMAIRE

Cette thèse présente une étude de la théorie des égaliseurs conventionels et ceux de perforations à espace fractionnel et aussi donne un aperçu de leurs bénifices et inconvénients relatifs. Deux cas spéciaux des égaliseurs de perforations à espace fractional sont mis en relief dans ce travail: l'égaliseur T/2 - de perforation à espace fractionnel et un nouveau type d'égaliseur, appelé l'égaliseur, à hybride transversal, dans lequel l'èspace de la perforation est soit T où T/2 (où 1/T est la vitesse des symboles de la source de données). Une analyse mathématique de ces égaliseurs est exécutée et de nouveaux résultats sont dérivés. Pour supporter l'analyse mathématique, un programme d'ordinateur est employé pour comparer l'accomplissement de ces modèles d'égaliseurs et les résultats obtenus sont analysés.

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1. INTRODUCTION

1.1 The Background and Goal of This Thesis

In Bandlimited data transmissions systems the maximum useful signalling rate is equal to the system bandwidth. At this rate, degradation in system performance is caused by Intersymbol Interference, (ISI), as "tails" of the channel impulse response are superimposed in the receiver, due to previously sent symbols. The ISI makes it more difficult for the detection section to decide which symbol was transmitted at each interval.

The technique used to reduce the degrading influence of Intersymbol Interference is called Equalization. This name originates from a discovery made by Nyquist. Usually the signal is sampled in the receiver. Nyquist showed that if the Fourier Transform of the sampled system impulse response is a constant, ISI is eliminated. Since the Fourier Transform of the sampled system impulse response is seldom constant, some sort of equalization of this function should be performed.

Equalization is achieved by a device usually a part of the receiver, implemented as a Transversal Filter (TF). The TF is built of a tapped delay line and a summer. With each tap there is associated a gain. The outputs of the taps are fed to a summer. The output signal from the TF is the signal at the output of the summer. The only parameters of the TF that can be optimized are the tap gains. Since the sampling process taking place in the receiver is at the symbol rate, every T seconds, this was the tap time spacing in early implementations of equalizers. In recent years it was found out that further improvement of performance can be obtained by increasing the system complexity and making the time spacing between taps smaller than T. Such equalizers are referred to as Fractional-Tap-Spacing-Equalizers.

In this thesis a generalized equalizer model in which tap spacings are arbitrary, is represented. Then, three special cases are examined in detail, namely the conventional T-Spaced, the T/2-Spaced and a Hybrid Transversal Equalizer (HTE). The HTE is a new type of equalizer that is being proposed here. The HTE combines features of the T-Spaced and the T/2-Spaced equalizer.

A study of these three important configurations is carried out here as follows. In Chapter 2 a baseband data transmission system is described. The problem of ISI is discussed, and it is shown how equalization can mitigate its effect. Chapter 3 deals with the topic of optimal (minimum mean square error) equalization. Chapter 4 discusses the important features of the conventional T-Spaced Equalizer.

Chapter 5 deals with the properties of a T/2-Spaced Equalizer and compares them to those of the T-Spaced Equalizer. Next, in Chapter 6 the model of a Hybrid Transversal Equalizer is presented and analysed. In Chapter 7, a computer program is used to compare the three types of equalizers, and the results obtained are analysed. It turns out that the T/2-Spaced Equalizer is better than a T-Spaced Equalizer which spans the same time interval. However, the HTE which spans this time interval but with fewer taps may have satisfactory performance between that of a T/2-Spaced Equalizer and that of a T-Spaced Equalizer. Moreover, in cases where a longer time span is desired a Hybrid Type Equalizer is superior to a pure T/2-Spaced Equalizer with the same number of taps which spans a shorter time interval. The figure of merit for all comparisons is the minimum mean square error. Chapter 8 is a brief study of the subject of Partial Response Signalling (PRS) and Fractional Tap Spacing Equalization. The question posed is whether PRS or correlated levels signalling improves the performance of systems which employ fractional tap spacing equalizers. The conclusion is that PRS or correlated levels signalling do not have such a desired property.

1.2 Previous Work

Extensive material about T-Space Equalization (theory and implementation) is found in references [1] through [7] and in [11], [14], [15]. Selected material about T-Spaced Equalizers which is relevant to the thesis is included in Chapter 2.

The first paper published about Fractional Tap Spacing Equalizers is [8]. The analysis carried out in [8] and in this thesis do not follow the same mathematical lines. A paper which "inspired" this work is [9]. Although written in a very concise manner, it is rich in substance. In this work, among other things we bring the mathematical background and derivations omitted from [9].

2. BASEBAND DATA TRANSMISSION SYSTEM

2.1 The Structure of a Data Transmission System

A baseband data transmission system is shown in Figure 2-1. It consists of three basic subsystems: the transmitter, the channel and the receiver. The transmitter itself has two parts: the data source that emits a symbol every T seconds into a bandlimited filter whose impulse response is $h_T(t)$. The signal at the output of the transmitter, given by:

$$s_T(t) = \sum_i a_i h_T(t-iT)$$

is fed into the channel. The channel is modelled here by a filter with impulse response $h_C(t)$. At the output of $h_C(t)$, random noise $n_R(t)$ is added to the signal. The signal at the output of the channel is:

$$r_R(t) = s_T(t) * h_C(t) + n_R(t)$$

The third part of the system is the receiver. It has three basic components: an input filter $h_R(t)$, a sampler, and a decision unit.

The signal at the output of $h_R(t)$ is given by

$$x(t) = s(t) + n(t) \qquad (2\text{-}1)$$

where: $s(t) = \sum_i a_i h_T(t-iT) * h_C(t) * h_R(t)$

and: $n(t) = n_R(t) * h_R(t)$

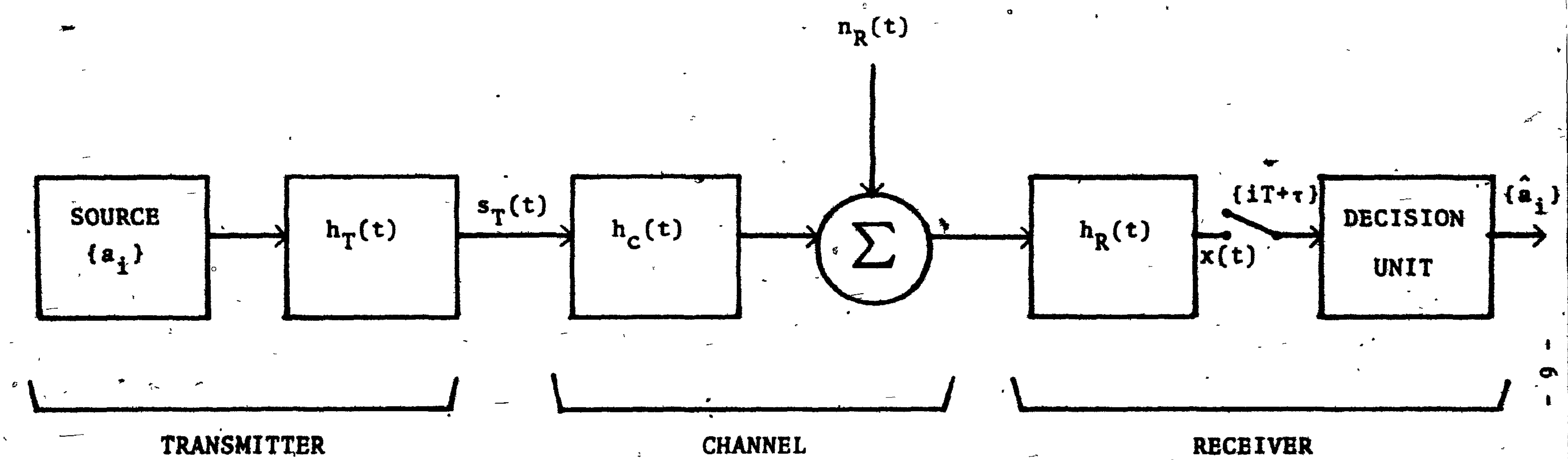

Fig. 2-1: Generalized Baseband Data Transmission System

By defining h(t) as the overall impulse response of the system one can write the signal before the sampler as:

$$x(t) = \sum_i a_i h(t-iT) + n(t) \qquad (2\text{-}2)$$

where:

$$h(t) = h_T(t)*h_C(t)*h_R(t) \qquad (2\text{-}3)$$

The samples at the input to the decision unit are:

$$x(kT+\tau) = \sum_i a_i h(kT-iT+\tau) + n(kT+\tau) \qquad (2\text{-}4)$$

where τ is the sampler time offset with respect to the data source. The decision unit accepts the samples given by Eq. (2-4), and every T seconds emits a symbol $\hat{a}_i$ which is an estimate of a_i, where both a_i and $\hat{a}_i$ usually belong to the same alphabet.

For given transmitter and channel one may seek to optimize the receiver operation (which is estimating a_i). Usually the receiver is optimized so as to improve a system performance index (such as probability of error, output signal-to-noise ratio or mean square error). The optimization itself involves the design of $h_R(t)$ and the decision unit in the receiver.

The additive noise that corrupts the signal in the channel can cause errors in the detection. Another source of degradation is the intersymbol interference (ISI), the nature of which is explained in the next section.

2.2 Intersymbol Interference

Eq. (2-4) can be written as:

$$x_k = \sum_i a_i h_{k-i} + n_k$$

where:

$$x_k \triangleq x(kT+\tau)$$

$$h_{k-i} \triangleq h(kT-iT+\tau)$$

$$n_k \triangleq n(kT+\tau)$$

If we define the present input symbol to have the subscript k we can write:

$$x_k = a_k h_0 + \sum_{i \neq K} a_i h_{k-i} + n_k \qquad (2\text{-}5)$$

One notes that in each sample x_k there are three components. The only desired one is $a_k h_0$; n_k is a noise sample and the sum $\sum_{i \neq k} a_i h_{k-i}$ is a disturbance originating from past and future samples of h(t). This disturbance is referred to as intersymbol interference (ISI).

It is quite easy to derive the Nyquist criterion for the elimination of ISI. Basically, an overall impulse response h(t) is desired, such that:

$$h_i = \begin{cases} h_0 & i=0 \\ 0 & i \neq 0 \end{cases}$$

If this is true for some h(t), then:

$$h(t) \cdot \sum_i \delta(t-iT) = h_0\delta(t)$$

where $\delta(\cdot)$ is a delta function. But $\sum_i \delta(t-iT)$ is a periodic function, thus it has a Fourier series representation, namely:

$$\sum_i \delta(t-iT) = \frac{1}{T}\sum_i e^{j2\pi ti/T}$$

Using this fact, we can write:

$$h(t)\sum_i e^{j2\pi ti/T} = Th_0\delta(t)$$

If we take the Fourier transform of both sides we arrive at:

$$\sum_i H(f-\frac{i}{T}) = T\cdot h_0 \qquad (2\text{-}6)$$

The sum $\sum_i H(f-\frac{i}{T})$ is a periodic function of f and its period is $\frac{1}{T}$. The first period is called the Nyquist equivalent of H(f) and is designated as:

$$Heq(t) \triangleq \sum_i H(f-\frac{i}{T}) \ , \ |f| < \frac{1}{2T}$$

The conclusion drawn from Eq. (2-6) is that for elimination of ISI, Heq(f) should be flat.

This is the Nyquist criterion for ISI cancellation. If h(t) satisfies Eq. (2-6) then at each sampling instant all h_i's are zero except h_0 and there is no ISI.

For a given transmitter and channel there is a result due to Ericson [18] which specifies $H_R(f)$ in terms of the system parameters. This $H_R(t)$ performs at least as well as any other filter.

2.3 Ericson's Result

Given $h_T(t)$, $h_c(t)$ and the noise $n_R(t)$ power spectrum, $Sn_R(f)$.

If: $$\sum_i \frac{G_c(f-{}^i/T)}{Sn_R(f-{}^i/T)} \neq 0 \quad \text{for } |f| < \frac{1}{2T} \tag{2-7}$$

then: $$H_R(f) = \frac{G_c^*(f)}{Sn_R(f)} \cdot \tilde{G}(f) \tag{2-8}$$

where: $$G_c(f) = H_T(f) \cdot H_c(f)$$

and $\tilde{G}(t)$ is periodic with period 1/T. $H_R(f)$ is the receiver input filter. This filter performs at least as well as any other linear filter with respect to any reasonable criterion. A reasonable criterion is a criterion according to which the performance index does not improve when signal to noise ratio is decreased.

$\tilde{G}(f)$ is a periodic frequency response, thus, in the time domain it can be represented by an infinite analog transversal filter. (See Figure 2-2).

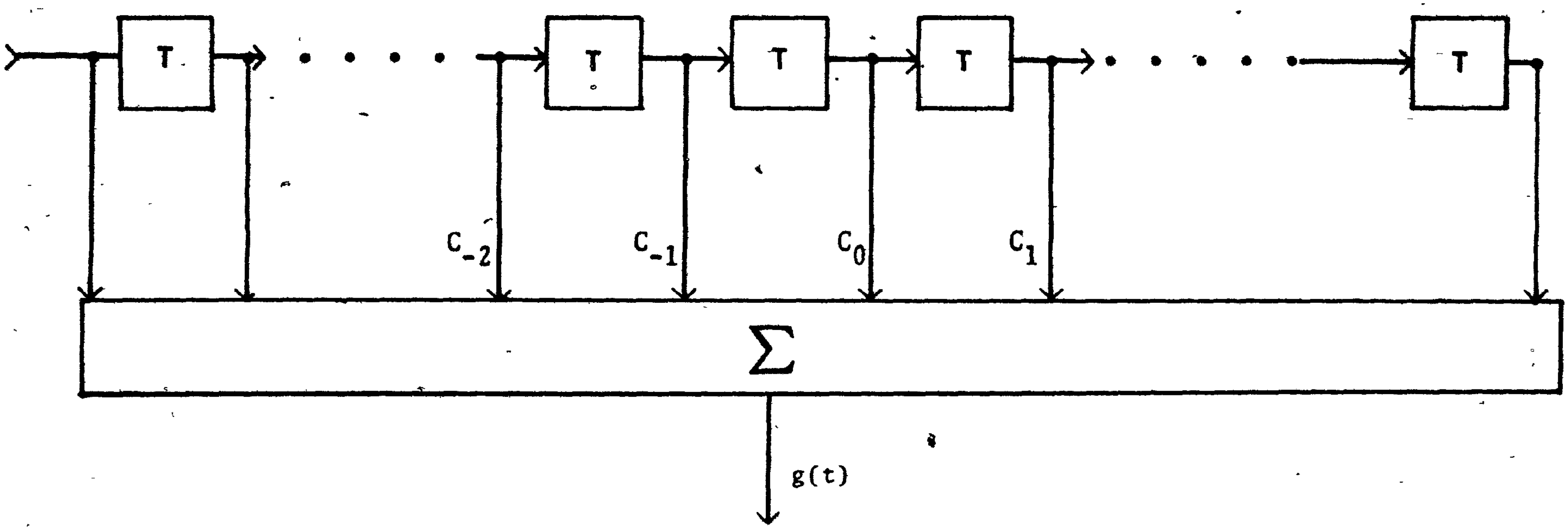

Fig. 2-2: Transversal Filter

$G_c^*(f)/Sn_R(f)$ is the frequency response of a filter matched to the signal in its input. Figure (2-3) depicts the receiver based on Ericson's result.

The following, is an interpretation of Ericson's result; the matched filter maximizes the signal to noise ratio in the decision instants while $\tilde{G}(f)$, the transversal filter (TF), minimizes the ISI that still corrupts the signal in its input.

The above scheme for a receiver is impractical for two reasons:

1. The realization generally calls for an infinite TF which implies an infinite memory.
2. The realization of a matched filter is impractical because the channel is usually unknown or it slowly changes with time.

The compromise is to realize a simple low-pass filter followed by a finite TF. A proper design of the gains of the taps of the TF will result in a suboptimal realizable receiver. Before we discuss the problem of choosing a criterion for optimality we note two points: (1) Instead of using an analog TF we can put the sampler in Figure (2-3) after the matched filter and use a digital transversal filter which can be implemented more easily. (2) The TF

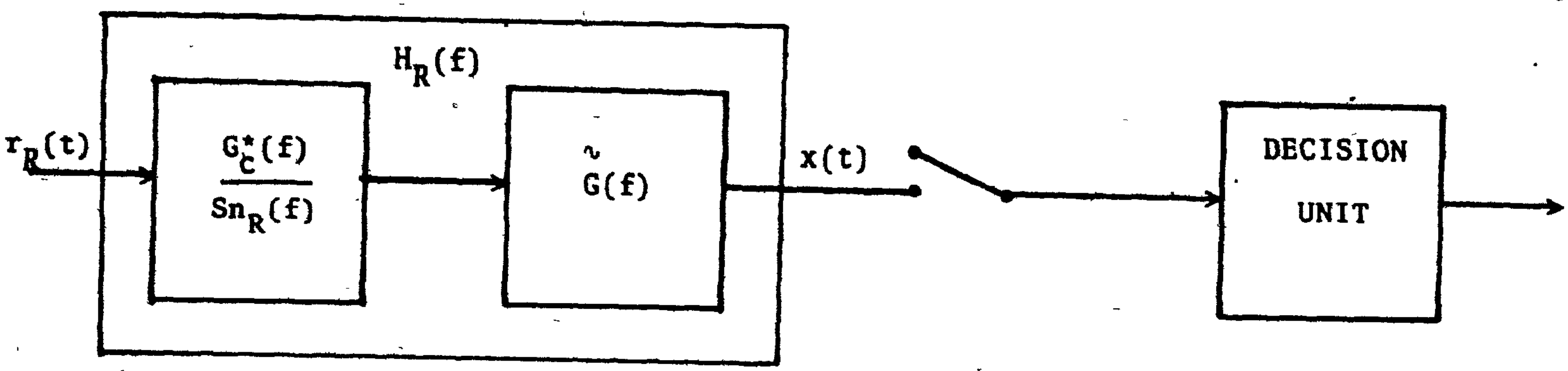

Fig. 2-3: Optimal Receiver

can be used to minimize the ISI by forcing the overall response H(f) to obey Eq. (2-6), namely, it causes the Nyquist equivalent channel Heq(f) to be flat. For this reason the TF is called an equalizer. Fig. (2-4) shows the modified suboptimal receiver, realized with a digital equalizer.

2.4 A Criterion for Optimal Receiver Design

Let P_e be the probability of error at the decision unit output. One would wish to design the receiver so as to minimize the probability of error, P_e. If P_e is chosen as the design optimality criterion the probability density function of the ISI which depends on the specific source and channel must be known. Usually this function is unknown in the receiver, thus, the use of this criterion is very often impractical. A criterion which does not depend on a prior knowledge of the statistical nature of the ISI, but relates easily to input signal-to-noise ratio, and takes into consideration both additive noise and ISI is the mean square error. Under this criterion the receiver design is carried out so as to minimize the mean square error between the receiver and source outputs.

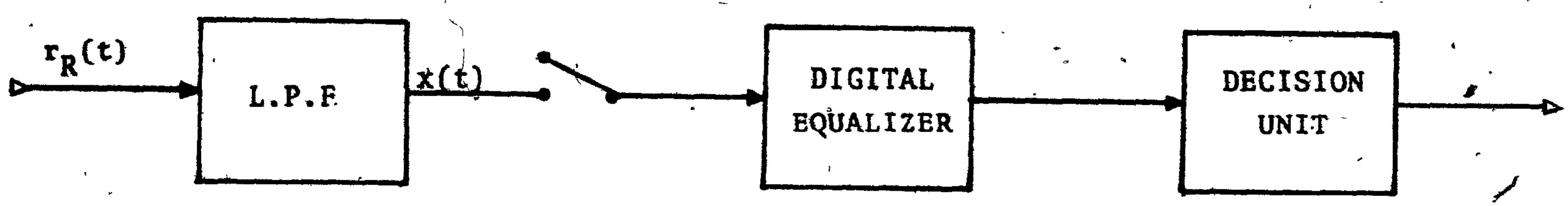

Fig. 2-4: Modified Suboptimal Realizable Receiver

3. OPTIMAL MINIMUM MEAN SQUARE ERROR EQUALIZATION

3.1 The Optimization Problem

As mentioned in Section 2.4 the equalization is achieved by finding a set of gains for the taps of the equalizer. These gain variables can be put in a vector

$$\underline{C} = [C_{-N_1}, \ldots, C_o, \ldots, C_{N_2}]$$

where C_{-N_1} is the gain of the leftmost (see Figure 3-1) tap C_o is the gain of the reference tap and C_{N_2} is the gain of the rightmost tap. The total number of taps is $N=N_1+N_2+1$(†). These gains are chosen so as to minimize the mean square error between the output of the data source and output of the decision unit in the receiver. In the next section this optimization problem is solved for a generalized type of equalizer in which the spacing between the taps is arbitrary, so that the T-spaced, $\frac{T}{2}$-spaced, and Hybrid Transversal Equalizers mentioned in the introduction, are just special cases of this generalized model.

(†) N_1 and N_2 may either be finite or infinite.

3.2 The Optimal Generalized Equalizer

In Figure 3-1 a generalized equalizer is shown, in which the spacing between the taps is arbitrary. Assume, for the sake of mathematical ease, that the equalizer is an analog device (a tapped delay line) and the signal at its input is a continuous one given by Eq. (2-2):

$$x(t) = \sum_i a_i h(t-iT) + n(t) \qquad (3-1)$$

If we assume that the spacing between the taps on the delay line is arbitrary, then, the output of the equalizer is given by:

$$y(t) = \sum_j C_j \, x(t-D_jT)$$

where the D_j is the normalized delay associated with the jth tap on the equalizer's delay line. The kth sample of y(t) as received in the output of the sampler that follows the equalizer (samples at rate of 1/T) is given by:

$$y(kT+\tau) = \sum_j C_j \cdot x(kT-D_jT + \tau)$$

where τ is the constant time offset of the sampler with respect to the data source clock.

In vector notation:

$$y_k = \underline{C}^T \cdot \underline{x}_k \qquad (3-2)$$

where: $\underline{C}$ is the vector of the taps' gains;

$$\underline{x}_k^T \triangleq [\ldots x(kT-D_{-1}T),\ x(kT-D_0T),\ x(kT-D_1T)\ldots]$$

$$y_k \triangleq y(kT+\tau)$$

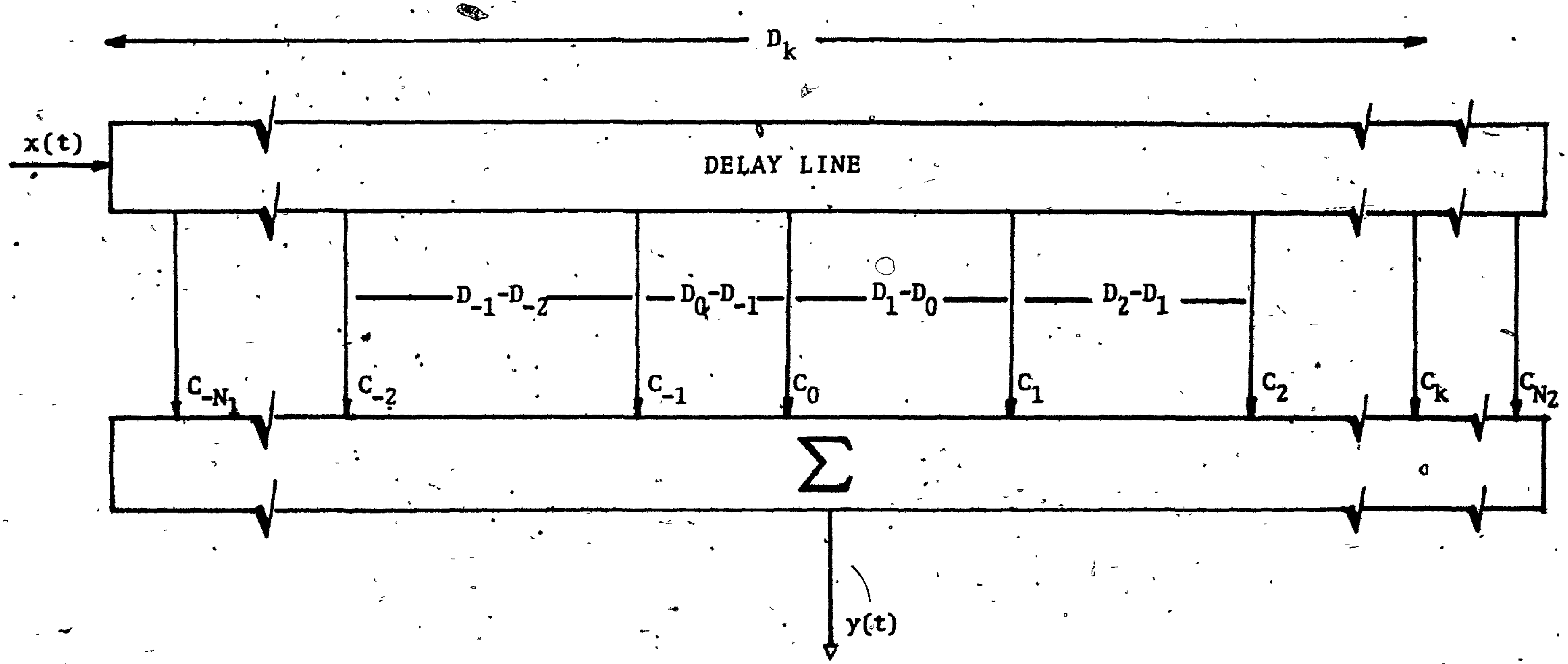

Fig. 3-1: Generalized Equalizer

Let the desired overall response of the system be f(t). If d(t) is the desired output of the equalizer and f(t) is the desired response of the system, then:

$$d(t) = f(t) * \sum_i a_i \delta(t-iT) = \sum_i a_i f(t-iT)$$

The desired output samples are given by:

$$d_k \triangleq d(kT) = \sum_i a_i f(kT-iT) \triangleq \underline{a}^T \cdot \underline{f}_k \qquad (3-3)$$

and $\underline{f}_k \triangleq [\ldots.f[(k-1)T],\ f(kT),\ f[(k+1)T]\ldots]^T$

The error is defined as:

$$e_k \triangleq y_k - d_k$$

The mean square error is:

$$\overline{|e_k|^2} = \overline{(y_k-d_k)(y_k^*-d_k^*)}^{(\dagger)} \qquad (3-4)$$

where the expectation is over the sample space of $\underline{x}_k$. Figure 3-2 shows a block diagram for the generation of e_k. It is shown in Appendix A.I that the vector $\underline{C}$ which minimizes $\overline{|e_k|^2}$ is given by:

$$\underline{C}_{opt} = A^{-1} \cdot \underline{\alpha} \qquad (3-5)$$

where: A is an N x N (positive definite) channel autocovariance matrix whose elements are given by:

$$A_{i,j} = \overline{x^*(kT-D_iT+\tau)\ x\ (kT-D_jT+\tau)} \qquad (3-6)$$

(†) * Superscript means complex conjugate.
All signals and parameters of the equalizer are complex quantities as QAM modulation technique, often used for transmission calls for this convention. (See [Lyon, 15])

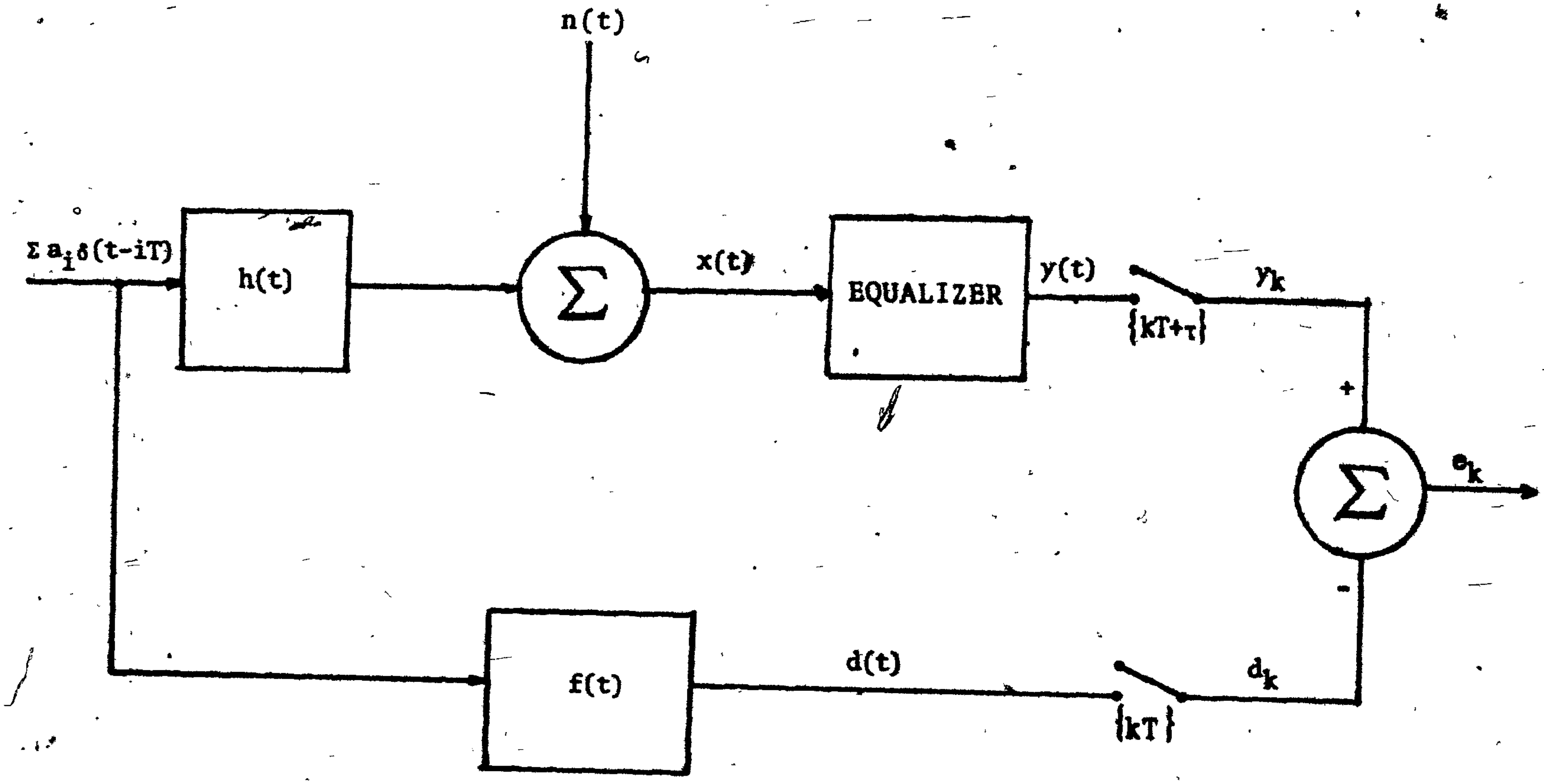

Fig. 3-2: Generating e_k

and $\underline{\alpha}$ is a vector whose elements are given by:

$$\alpha_i = \overline{d_k^* \, x(kT-D_j T+\tau)} \qquad (3\text{-}7)$$

By substituting the expression for $x\,(kT-D_i T+\tau)$, namely:

$$x(kT-D_i T+\tau) = \sum_j a_j h(kT-D_i T+\tau-jT) + n(kT-D_i T+\tau)$$

into Eq. (3-6) and Eq. (3-7) we get (see App. A.II)

$$A_{i,j} = \sum_m \phi_{aa}(m) \sum_n h^*[(n-m-D_i+\tfrac{\tau}{T})T]\, h\,[(n-D_j+\tfrac{\tau}{T})T]$$

$$+ \phi_{nn}[(D_i-D_j)T] \qquad (3\text{-}8)$$

$$\alpha_i = \sum_m \phi_{aa}(m) \sum_n f^*(nT)\cdot h[(n-m-D_i+\tfrac{\tau}{T})T] \qquad (3\text{-}9)$$

where:

$\phi_{aa}(\cdot)$ is the data source autocovariance function

$\phi_{nn}(\cdot)$ is the noise autocovariance function

$f(\cdot)$ is the desired overall impulse response.

For the conventional case, where $D_i = i$, a white data source, white noise with powers σ_a^2 and σ_n^2 respectively, we get:

$$A_{ij} = \sigma_a^2 \sum_n h^*[(n-i+\tfrac{\tau}{T})T]h[(n-j+\tfrac{\tau}{T})T]+ \sigma_n^2\cdot\delta_{ij} \qquad (3\text{-}10)$$

$$\alpha_i = \sigma_a^2 \sum_n f^*(nT)h[(n-i+\tfrac{\tau}{T})T] \qquad (3\text{-}11)$$

Eq. (3-10) can be rewritten as:

$$A_{i,j} = \sigma_a^2 \sum_n h^*[(n+\tfrac{\tau}{T})T]\cdot h[(n+\tfrac{\tau}{T})T+(i-j)T] + \sigma_n^2\delta_{ij} \quad (3\text{-}12)$$

This form emphasizes the fact that, in this case, the A matrix is a Toeplitz matrix (†) (see [Gray, 10] and [Gantmacher, 13]). In general when $D_i \neq i$ A is not Toeplitz.

A more general case is the one in which $D_i = \frac{i}{n}$, namely, the taps are uniformly spaced; n taps on each interval of T sec. Such a case of importance to us is the one in which n = 2. If we use the transform relation

$$h(t) = \int_{-\infty}^{\infty} H(f)e^{j2\pi ft}df$$

to express the samples of h(t) in Eq. (3-8) and Eq. (3-9) it can be shown that Eq. (3-8) can be rewritten as (see App. A.III);

$$A_{k,l} = \frac{1}{T}\int_{-1/2T}^{1/2T} [H^k_{eq}(f)]^* H^l_{eq}(f)\cdot\Phi_{aa}(f)df + \phi_{nn}[(D_k-D_l)T]$$

$$\text{where: } \Phi_{aa}(f) \triangleq \sum_m \phi(m)e^{-j2\pi fmT} \quad (3\text{-}13)$$

(†) A Toeplitz matrix is a matrix in which the $a_{i,j}$ element depends on (i-j) only.

and:

$$H^{k}_{eq}(f) \triangleq \sum_{i} H(f+\tfrac{i}{T})e^{-j2\pi(f+\frac{i}{T})(D_k-\frac{\tau}{T})T} \qquad (3\text{-}14)$$

For, the conventional case discussed earlier:

$$A_{k,1} = \frac{1}{T}\int_{-1/2T}^{1/2T} \phi_{aa}(f)\,|Heq(f)|^2 e^{-j2\pi f(1-k)T}df+\sigma_n^2\delta_{k,1} \quad (\dagger) \qquad (3\text{-}15)$$

where Heq(f) is the Nyquist equivalent channel defined earlier (for $\tau=0$) as:

$$Heq(f) = \sum_{i} H(f+\tfrac{i}{T})e^{j\frac{2\pi\tau}{T}i}$$

By using Eq. (3-9) and the Fourier transform relation of h(t) and d(t) one can show that for the conventional case:

$$\alpha_k = \frac{1}{T}\int_{-1/2T}^{1/2T} Heq^{*}(f)Feq(f)\phi_{aa}(f)e^{-j2\pi f\tau}\cdot e^{j2\pi fkT}df \qquad (3\text{-}16)$$

where Feq(f) is the Nyquist equivalent of the desired overall response.

(†) $\delta_{k,1}$ is the Kroneker delta function.

For the uniform case in which $D_i = \frac{i}{2}$ the elements of the autocovariance matrix can be written as:

$$A_{k,l} = \frac{1}{T} \int_{-1/2T}^{1/2T} \phi_{aa}(f) \cdot |Heq(f)|^2 e^{-j2\pi f(l-k)T/2} df + \sigma_n^2 \cdot \delta_{k,l}$$

where k and l are even. (3-17a)

$$A_{k,l} = \frac{1}{T} \int_{-1/2T}^{1/2T} \phi_{aa}(f) \cdot |\hat{Heq}(f)|^2 e^{-j2\pi f(l-k)T/2} df + \sigma_n^2 \cdot \delta_{k,l}$$

where k and l are odd. (3-17b)

$$A_{k,l} = \frac{1}{T} \int_{-1/2T}^{1/2T} \phi_{aa}(f) \cdot Heq(f)\hat{Heq}^*(f) e^{-j2\pi f(l-k)T/2} df$$

where k is odd and l is even. (3-17c)

$$A_{k,l} = \frac{1}{T} \int_{-1/2T}^{1/2T} \phi_{aa}(f) \cdot Heq^*(f)\hat{Heq}(f) e^{-j2\pi f(l-k)T/2} df$$

where k is even and l is odd, (3-17d)

and:

$$\hat{Heq}(f) \triangleq \sum_i (-1)^i H(f+\frac{i}{T}) e^{j2\pi i\tau/T}$$

One notices that for this case A is not a Toeplitz matrix.

By using Eq. (3-9) with $D_i = \frac{i}{2}$ and the transform relations for f(t) and h(t) it can be shown that the elements of the $\underline{\alpha}$ vector are given by:

$$\alpha_k = \frac{1}{T}\int_{-1/2T}^{1/2T} H^*eq(f)Feq(f)\phi_{aa}(f)e^{j2\pi f\tau}\cdot e^{j2\pi kfT/2}df$$

for k even. (3-18)

$$\alpha_k = \frac{1}{T}\int_{-1/2T}^{1/2T} \hat{H}^*eq(f)Feq(f)\phi_{aa}(f)e^{j2\pi f\tau}\cdot e^{j2\pi kfT/2}df$$

for k odd. (3-19)

In the next two chapters the properties of T-spaced and T/2-spaced equalizers are discussed in detail.

4. IMPLEMENTATION AND PROPERTIES OF A T-SPACED EQUALIZER

4.1 An Iterative Method for Equalization

In Section 2.4 it was mentioned that equalizers are implemented at the receiver end as decision directed adaptive devices. In this section we discuss briefly the theory of Iterative-Adaptive-Equalization and show how such an equalizer is implemented.

In order to equalize a given channel, Eq. (3-5) must be solved for $\underline{C}$opt. The solution of Eq. (3-5) involves the inversion of the NxN A matrix, where N may be quite large (a typical number may range between 32 to 64). Fortunately, there is an iterative method to solve Eq. (3-5) (see [Proakis, 1], [Ungerboeck, 6]).

We look for a vector $\underline{C}$opt that minimizes $\overline{|e|^2}$. This vector can be found iteratively by:

$$\underline{C}^{i+1} = \underline{C}^{i} - D^{-1}\nabla_{\underline{C}^i} \overline{|e|^2}\Big|_{\underline{C}=\underline{C}^i} \tag{4-1}$$

D is a matrix whose elements are given by:

$$D_{i,j} = \frac{\delta^2 \overline{|e|^2}}{\delta C_i \, \delta C_j} \tag{4-2}$$

It can be easily verified that (see App. A.I).

$$\nabla_{\underline{C}} \overline{|e|^2} = 2A\underline{C} - 2\underline{\alpha} \tag{4-3}$$

If instead of computing D we take a constant a/2, which is called the iteration step, we get a simplified iterative formula:

$$\underline{C}^{i+1} = \underline{C}^{i} - a(A\underline{C}^{i} - \alpha) \qquad (4\text{-}4)$$

We shall prove the following theorem:

Theorem: Let A be a positive definite matrix, then it is possible to choose a so that

$$\lim_{i \to \infty} \underline{C}^{i} = \underline{C}\mathrm{opt}$$

Proof:

Recall that for A positive definite, we have $\underline{u}^T A \underline{u} > 0$ for any vector $\underline{u}$, and the eigenvalues of A are all positive. If we subtract $\underline{C}$opt from both sides of Eq. (4-4) we get:

$$\Delta\underline{C}^{i+1} = \Delta\underline{C}^{i} - aA\Delta\underline{C}^{i} = (I - aA)\Delta\underline{C}^{i} \qquad (4\text{-}5)$$

Define: $B \triangleq I - aA$

Note: If $a_i = \sum_j A_{i,j} b_j$ then by Schwartz's inequality we get:

$$\sum_i a_i^2 \le \sum_{i,j} A_{i,j}^2 \cdot \sum_j b_j^2 \qquad (4\text{-}6)$$

On both sides of Eq. (4-6) we identify the following norms:

$$||\underline{a}|| \triangleq {}_{i}\; a_i^2$$

$$||A|| \triangleq \sum_{i,j} A_{i,j}^2$$

$$||\underline{b}|| \triangleq \sum_j b_j^2$$

With these notations at hand we conclude from Eq. (4-5) that:

$$||\underline{\Delta C}^{i+1}|| < ||B|| \quad ||\underline{\Delta C}^{i}|| \tag{4-7}$$

This means that in each iteration the error vector gets smaller. Now we make use of another norm definition for B, which is:

$$||\hat{B}|| = \max_{\{\lambda_B\}} |\lambda_B| \triangleq \lambda$$

where: $\{\lambda_B\}$ is the set of eigenvalues of B.

By using the last definition in Eq. (4-7) one gets:

$$||\underline{\Delta C}^{i+1}|| \leq \lambda \; ||\underline{\Delta C}^{i}||$$

If $\lambda < 1$ the solution of this inequality is

$$||\underline{\Delta C}^{i+1}|| \leq \lambda^{i} ||\underline{\Delta C}^{o}||$$

λ can be made smaller than 1 by properly choosing the parameter a.

It is quite obvious that $\lambda_B = 1 - a\lambda_A$, thus

$$\lambda = \max_{\{\lambda_A\}} \{|1-a\lambda_{A_{min}}|, |1-a\lambda_{A_{max}}|\}$$

By choosing: $a = \dfrac{2}{\lambda_{A_{min}} + \lambda_{A_{max}}} > 0$

we get:

$$\lambda = |1-a\lambda_{A_{min}}| = \frac{\lambda_{A_{max}} - \lambda_{A_{min}}}{\lambda_{A_{max}} + \lambda_{A_{min}}} < 1$$

Conclusions:

1. a can be chosen so as to ensure that

$$\lim_{i \to \infty} ||\underline{\Delta C}^i|| = 0$$

2. It can be shown that this choice of a brings about the tightest bound on convergence of $\underline{C}^i$ to $\underline{C}$opt (see [Gersho, 14]) and that fastest convergence takes place.

3. A smaller spread of the eigenvalues results in faster convergence.

The following is a brief description of an equalizer model in which the iterative solution of Eq. (2-5) is practically implemented. In Sec. (4-1) we saw that $\underline{C}^{i+1}=\underline{C}^i-a\nabla_{\underline{C}^i}\overline{|e|^2}$

where: $\nabla_{\underline{C}^i}\overline{|e|^2} = a(A\underline{c}^i - \underline{\alpha})$

$$\alpha = \overline{d_k^* \cdot x_k}$$

$$A = \overline{\underline{x}_k^* \cdot \underline{x}_k^T}$$

Thus, by Eq. (4-3), and by assuming x(t) is real we get:

$$\nabla_{\underline{C}^i}\overline{|e|^2} = 2 \cdot \overline{[x_k \cdot (\underline{x}_k^T \cdot \underline{C}^i - d_k)]}$$

We note that $\underline{x}_k^T \cdot \underline{C}^i$ is the k'th output of the system during the i'th updating cycle of the taps, and that d_k is the desired output, thus, $\underline{x}_k^T\,\underline{C}^i - d_k$ is the error, and we can write:

$$\nabla_{\underline{C}^i} \overline{|e|^2} = 2 \cdot \overline{[\underline{x}_k \cdot e_k^i]}$$

where each component of $\nabla_{\underline{C}^i} \overline{|e|^2}$ can be written as:

$$\frac{\partial \overline{|e|^2}}{\partial C_j} = \overline{2\{x[(k-D_j)T] \cdot e_k^i\}}$$

If we could calculate $\overline{[e_k^i \cdot \underline{X}_k]}$ in the receiver it would yield an optimal value for $\frac{\partial \overline{|e|^2}}{\partial C_j}$. Unfortunately the receiver does not have the knowledge about the statistics of $e_k^i \cdot \underline{x}_k$ and it uses an unbiased estimate of this mean namely: $e_k^i \cdot \underline{x}_k$ thus, in practice the updating procedure is carried out according to:

$$\underline{C}^{i+1} = \underline{C}^i - a \cdot e_k^i \cdot \underline{x}_k \qquad (4\text{-}8)$$

Figure 4-1 shows an automatic adaptive equalizer. Extensive material about the implementation problems is found in the references.

In the light of Eq. (4-8) Figure 4-1 is quite clear. The only part that deserves a few words of explanation is the switch. At the beginning of a transmission, the probability of error in the receiver is assumed to be high, thus a fixed sequence of symbols, known to the receiver is used to sound the system after carrier synchronization has been established. This symbol sequence is locally generated in the receiver and used to generate e_i. During this period the switch is on position "a". After a while, probability of error reduces drastically and a decision directed mode is established by changing the position to "b" automatically.

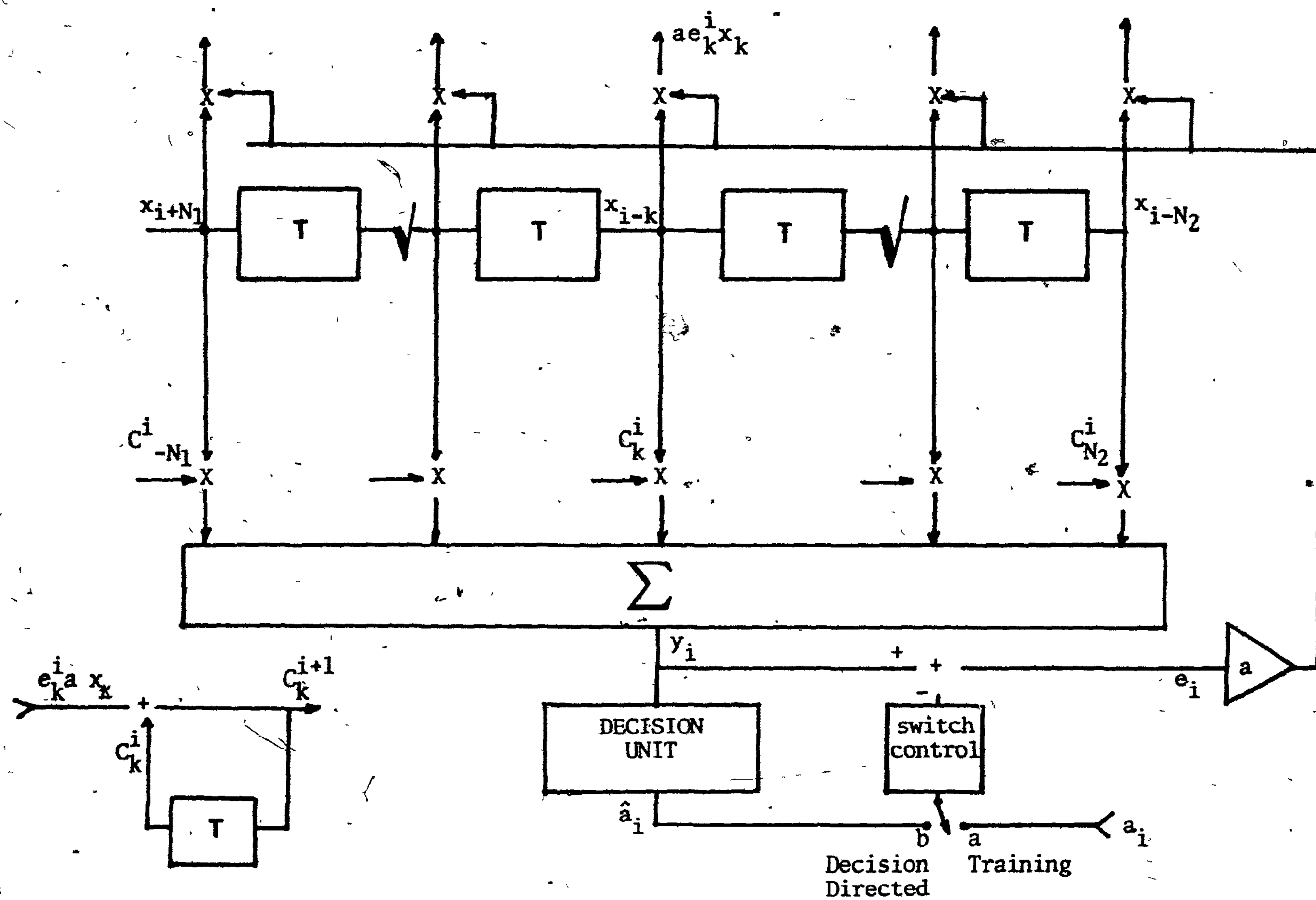

Fig. 4-1: Automatic Adaptive Equalizer

4.2 On the Eigenvalues of the Autocovariance Matrix

We begin this section by stating and proving the following theorem:

Theorem: The eigenvalues of the system autocovariance matrix are bounded by the maximum value (M) and the minimum value (m) of $|Heq(f)|^2$; $(\sigma_a^2 = 1)$.

Proof: Assume that λ_A is an eigenvalue of A, and that $\underline{u}$ is its corresponding eigenvector.

By definition: $$A = \overline{\underline{x}_k^* \, \underline{x}_k^T}$$

Note that: $$\underline{u}^H A \underline{u} = \lambda_A \underline{u}^H \underline{u} \quad (\dagger)$$

Using the definition of A we get: $$\underline{u}^H \overline{\underline{x}_k^* \underline{x}_k^T} \underline{u} = \lambda_A \underline{u}^H \underline{u}.$$

Define: $$q_k = \underline{x}_k^T \underline{u}$$

Thus: $$\overline{|q_k|^2} = \lambda_A \underline{u}^H \underline{u} \qquad (4\text{-}9)$$

If Q(f) is the Z-transform of $\{q_k\}$ computed around the unit circle in the Z-plane then:

$$Q(f) = U(f)Xeq(f)$$

where as before: $$Xeq(f) \triangleq \sum_i X(f+\frac{i}{T})$$

($\dagger$) H superscript means conjugate - transpose operator.

By using Eq. (2-2) and Parseval's theorem in Eq. (4-9), we get:

$$\overline{|q_k|^2} = \sigma_a^2 \cdot \int_{-\frac{1}{2T}}^{\frac{1}{2T}} |U(f)Heq(f)|^2 df - \lambda_A \int_{-\frac{1}{2T}}^{\frac{1}{2T}} |U(f)|^2 df$$

But it was given that: $m \le |Heq(f)|^2 \le M$,

thus, we arrive at the following result:

$$m \le \lambda_A \le M$$

We may conclude that the larger the spread of the eigenvalues, the farther the channel's Nyquist equivalent response is from being flat. As was mentioned in Sec. 4.1 this fact implies longer convergence time of the taps in the iterative model previously discussed.

Next, we find expressions for the eigenvalues and eigenvectors of the autocovariance matrix of a model employing an infinite T-spaced equalizer.

We previously got that [Eq.(3-15)]

$$A_{k,1} = \frac{1}{T} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} |Heq(f)|^2 \Phi_{aa}(f) e^{-j2\pi f(1-k)T} df + \sigma_n^2 \delta_{k,1}$$

We note again that A is Toeplitz. For a general row, s, of A, we write: (for $\sigma_n^2=0$)

$$\sum_1 A_{s,1} e^{j2\pi\lambda T1} = \sum_1 \frac{1}{T} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} |Heq(f)|^2 \Phi_{aa}(f) e^{j2\pi fsT} e^{-j2\pi(f-\lambda)1T} df$$

$$= \frac{1}{T}\int_{-\frac{1}{2T}}^{\frac{1}{2T}} |Heq(f)|^2 \Phi_{aa}(f) e^{j2\pi fsT} \sum_{l} e^{-j2\pi(f-\lambda)lT} df$$

$$= |Heq(\lambda)|^2 \Phi_{aa}(\lambda) e^{j2\pi\lambda sT}$$

Thus, the vector whose components are $\{e^{j2\pi fsT}\}$ is an eigenvector of A and

$$|Heq(f)|^2 \Phi_{aa}(f) \tag{4-10}$$

is its corresponding eigenvalue.

The above result is somewhat obvious once one regards an infinite Toeplitz matrix as a circulant matrix in the limiting case, and uses the fact that the eigenvalues of a circulant matrix are given by the Discrete Fourier Transform (D.F.T.) of its rows, [Gray, 10], [Noble, 16].

4.3 The Frequency Response of a T-Spaced Equalizer

In Sec. 3.2 it was shown that the optimal taps' gains $\{C_i\}_{i=-N_1}^{i=N_2}$ are given by

$$A\underline{c} = \underline{\alpha}$$

Starting from this equation we can write another equation.

$$\sum_k \sum_l A_{k,l} C_l e^{-j2\pi\lambda kT} = \sum_k \alpha_k e^{-j2\pi\lambda kT} \tag{4-11}$$

By substituting Eq. (3-15) for $A_{k,l}$ and Eq. (3-16) for α_k, into Eq. (4-11), one can show that the first period of the periodic frequency response, of an infinite T-Spaced equalizer is given by

$$C(f) = \frac{\Phi_{aa}(f)\cdot Feq(f)\cdot Heq^*(f)\cdot e^{j2\pi f\tau}}{\Phi_{aa}(f)\cdot|Heq(f)|^2 \quad + \quad \sigma_n^{\ 2}} \quad , \quad |f| \leq \frac{1}{2T} \qquad (4\text{-}12)$$

In the noiseless case Eq. (4-12) simplifies to

$$C(f) = \frac{Feq(f)}{Heq(f)} \cdot e^{j2\pi f\tau} \quad , \quad |f| \leq \frac{1}{2T} \qquad (4\text{-}13)$$

We see that any zero of $H_{eq}(f)$ within the Nyquist range is a pole of C(f).

Note that although H(f) may have no zeroes (or-near-zeroes) in $|f| \leq \frac{1}{2T}$, Heq(f) may have zeroes because of the superposition of terms such as $H(f+\frac{i}{T})e^{j2\pi i\tau/T}$ in $H_{eq}(f)$.

In case dips are introduced into Heq(f) by a certain choice of τ, C(f) tends to be very large and huge values for C_i's may be required, which are difficult to implement. Large values for taps' gains may also cause severe noise enhancement in certain frequencies, increasing probability of error in the system.

In order to overcome the problem of sampling phase dependence of the system's performance there should be some form of sampling phase control which chooses a good sampling phase in the receiver and only heuristic methods are available in practice to do it [Qureshi, 11].

4.4 The Minimum Mean Square Error of an Infinite T-Spaced Equalizer

The minimum mean square error of an equalizer is defined by Eq.(3-4) and is given in App. A.I. as:

$$\overline{|e|^2}_{min} = \overline{\underline{a}^H G\cdot\underline{a}} - \underline{\alpha}^H\cdot\underline{C}_{opt} = \sum_j\sum_i \overline{a_i^* a_j} f_i^* f_j - \sum_i \alpha_i C_{i\ opt} \quad (4\text{-}14)$$

where: $G = \{G_{i,j}\}$ and $G_{i,j} = f_i^* f_j$

The first term can be expressed as

$$\underline{a}^H\cdot G\cdot\underline{a} = \frac{1}{T}\int_{-1/2T}^{1/2T} |Feq(f)|^2 \Phi_{aa}(f)df \quad (4\text{-}15)$$

The second term can be expressed as

$$\underline{\alpha}\cdot\underline{C}opt = \frac{1}{T}\int_{-1/2T}^{1/2T} Heq(f)\cdot F^*eq(f)\cdot e^{-j2\pi f\tau}\Phi_{aa}(f)C(f)df \quad (4\text{-}16)$$

By subtracting Eq. (4-16) from Eq. (4-15) we arrive at:

$$\overline{|e|^2}_{min} = \frac{\sigma_n^2}{T}\int_{-1/2T}^{1/2T} \frac{|Feq(f)|^2\Phi_{aa}(f)}{|Heq(f)|^2\Phi_{aa}(f) + \sigma_n^2} df \quad (4\text{-}17)$$

Eq. (4-17) shows that for a noiseless case an infinite optimum equalizer gives zero mean square error. One can also see that once there is noise in the channel, its significance is highly dependent on τ - the sampling phase which is hidden in $|H_{eq}(f)|^2$. For some values of τ a null or near-null may be introduced in Heq(f) within the Nyquist range at some frequencies and by Eq. (4-17) this may cause a larger value for the integrand and thus a larger minimum mean square error.

4.5 The Analysis of a finite T-Spaced Equalizer with Periodic Data Source

The previous sections dealt with the general case of an infinite T-Spaced equalizer. We were unable to get a useful closed form expression for the finite equalizer frequency response. However, it is possible to derive useful results if the data is assumed to be a periodic sequence with autocorrelation function $\phi_{aa}(m)$, given by:

$$\phi_{aa}(m) = \begin{cases} T \text{ for } m = kN & k = 0,\pm1,\pm2,\ldots \\ 0 \text{ otherwise} \end{cases} \tag{4-18}$$

where NT is the time span of the equalizer.

It would be expected that the results to be derived here will coincide with those derived for the infinite equalizer if the period of the data is large. Short periodic sequences are used for pseudo-random channel sounding, i.e. periodic sequences are used to sound the channel frequency response at N dense discrete frequencies since the spectrum of the sequence consists of equally spaced, equal height spectral lines [Muller, 3].

For such periodic input it is possible to show (using Eq. (3-15), Eq. (3-16)) that:

$$A_{k,l} = \frac{1}{NT}\sum_{n=-N_1}^{N_1} \left|Heq\left(\frac{n}{NT}\right)\right|^2 e^{-j2\pi n(l-k)/N} \tag{4-19}$$

$$\alpha_k = \frac{1}{NT}\sum_{n=-N_1}^{N_1} Heq^*\left(\frac{n}{NT}\right) Feq\left(\frac{n}{NT}\right)\, e^{-j2\pi nk/N} \tag{4-20}$$

where the number of taps is $N = 2N_1+1$, $(N_1 = N_2)$.

By constructing the equation

$$\sum_{k=-N_1}^{N_1}\ \sum_{l=-N_1}^{N_1} A_{k,l}\cdot C_l \cdot e^{-j2\pi mk/N} = \sum_{k=-N_1}^{N_1} \alpha_k \cdot e^{-j2\pi mk/N} \tag{4-21}$$

and substituting equations Eq. (4-19) and Eq. (4-20) into Eq. (4-21) one arrives at the following result, giving the taps weights:

$$C\left(\frac{m}{NT}\right) = \frac{Heq^*\left(\frac{m}{NT}\right)\cdot Feq\left(\frac{m}{NT}\right)\cdot e^{j2\pi m\tau/NT}}{\left|Heq\left(\frac{m}{NT}\right)\right|^2 + \sigma_n^2} \quad , \ -N_1 \le m \le N_1 \tag{4-22}$$

This result shows that the frequency response of a finite equalizer with periodic input is completely determined by N equally spaced samples of the response of the infinite equalizer given by Eq. (4-12).

It can be shown, following the same development as in Sec. 4.2 that the N eigenvalues of the system are given by: $(\sigma_n^2=0)$

$$\lambda_n = \left|Heq\left(\frac{n}{NT}\right)\right|^2 \quad , \ -N_1 \le n \le N_1 \tag{4-23}$$

This result shows that the eigenvalues depend on τ

since $H_{eq}(f)$ depends on τ. This τ-dependency may cause a large spread in the eigenvalues and as a result a large convergence time for the adaptive iterative structure discussed in Chapter 2.

5. PROPERTIES OF A T/2-SPACED EQUALIZER

5.1 The Frequency Response of an Infinite T/2-Equalizer

The basic equation that governs the equalizer is $A\underline{c} = \underline{\alpha}$ where the elements of A and $\underline{\alpha}$ are given by Eq. (3-17), Eq. (3-18) and Eq. (3-19).

In order to derive an expression for the frequency response of an infinite T/2 equalizer, we make the following definitions:

Let $\{c_k\}_{k=-\infty}^{\infty}$ represent the gains of an infinite T-Spaced Equalizer, and let $\{d_k\}_{\substack{k=-\infty \\ k\neq o}}^{\infty}$ be the gains of additional taps inserted in between the previous taps as shown in Fig. 5-1. By definition, the frequency response of this equalizer is given by:

$$C(f) \triangleq c(f) + d(f)$$

where: $$c(f) \triangleq \sum_{\substack{k \\ \text{even}}} c_k e^{j2\pi fkT/2}$$

and: $$d(f) \triangleq \sum_{\substack{k \\ \text{odd}}} d_k e^{j2\pi fkT/2}$$

We also write down the following two equations:

$$\sum_{\substack{k \\ \text{even}}} \sum_{\substack{l \\ \text{even}}} A_{k,l} c_l e^{-j2\pi\lambda kT/2} + \sum_{\substack{k \\ \text{even}}} \sum_{\substack{l \\ \text{odd}}} A_{k,l} d_l e^{-j2\pi\lambda kT/2} = \sum_{\substack{k \\ \text{even}}} \alpha_k e^{-j2\pi\lambda kT/2} \quad (5\text{-}1)$$

$$\sum_{\substack{k \\ \text{odd}}} \sum_{\substack{l \\ \text{even}}} A_{k,l} c_l e^{-j2\pi\lambda kT/2} + \sum_{\substack{k \\ \text{odd}}} \sum_{\substack{l \\ \text{odd}}} A_{k,l} d_l e^{-j2\pi\lambda kT/2} = \sum_{\substack{k \\ \text{odd}}} \alpha_k e^{-j2\pi\lambda kT/2} \quad (5\text{-}2)$$

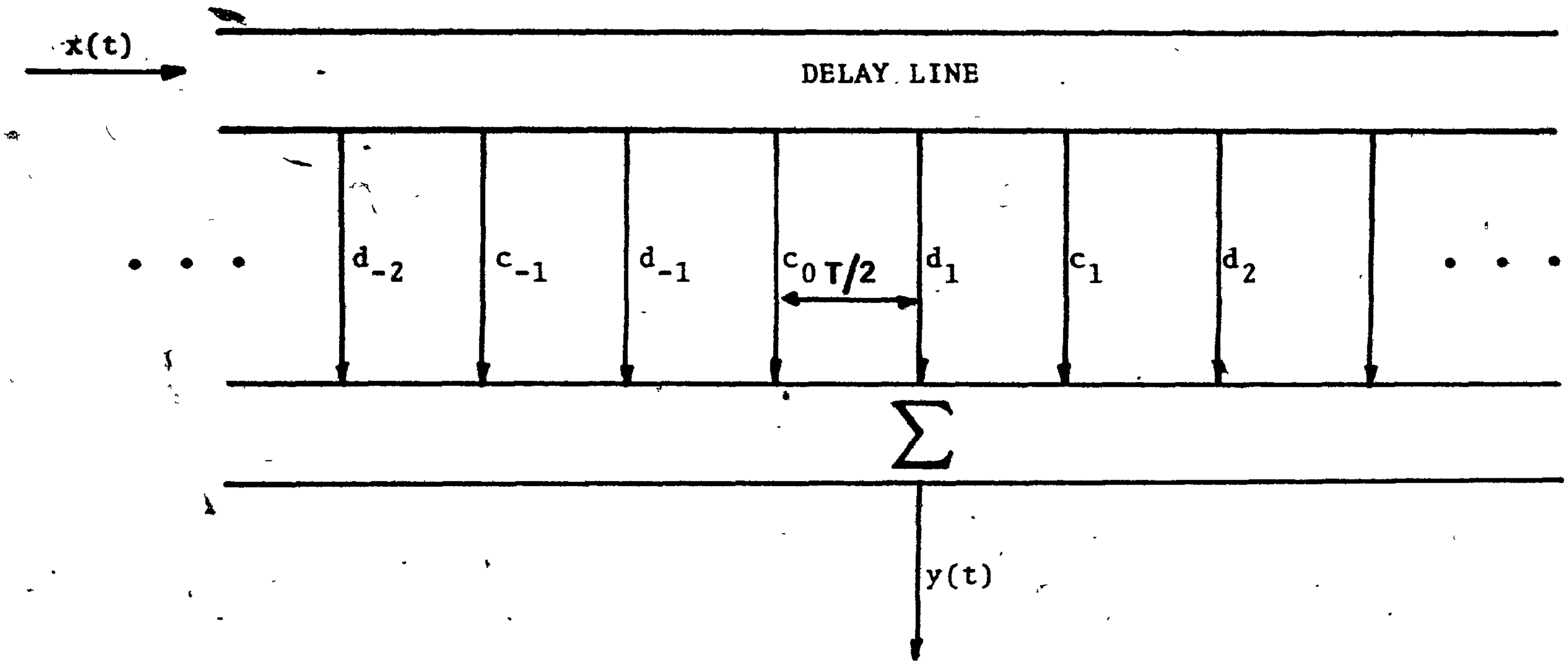

Fig. 5-1: A T/2-Spaced Equalizer

If we insert into these equations the expressions given by Eq. (3-17), Eq. (3-18) and Eq. (3-19) we arrive at two equations for c(f) and d(f).

By solving these equations and forming the sum c(f) + d(f) we get the following expression for the frequency response of a T/2-Equalizer:

$$C(f) = \frac{2 \cdot Feq(f) \cdot \Phi_{aa}(f) \cdot H^{*}(f) \cdot e^{j2\pi f\tau}}{\Phi_{aa}(f)[|\hat{H}eq(f)|^2 + |Heq(f)|^2] + \sigma_n^2} \tag{5-3}$$

The expression $|\hat{H}eq(f)|^2 + |Heq(f)|^2$ is equal to the folded power spectrum of the overall response once the assumption that H(f) is bandlimited to $|f| \leq \frac{1}{T}$ is made, and we may write:

$$C(f) = H^{*}(f) \cdot e^{j2\pi f\tau} \cdot \frac{\Phi_{aa}(f) \cdot Feq(f)}{\Phi_{aa}(f)[|H(f+\frac{1}{T})|^2 + |H(f)|^2 + |H(f-\frac{1}{T})|^2] + \sigma_n^2} \qquad |f| \leq 1/2T \tag{5-4}$$

From Eq. (5-4) it is obvious that the optimal infinite T/2 equalizer may be viewed as having two parts in cascade: the first one is a matched filter, matched to the overall frequency response of the system up to the equalizer. This part as is well known [Schwartz, 17], maximizes the signal-to-noise ratio at the sampling instants in the receiver.

The task of the second part is to minimize the mean square error due to intersymbol interference which still corrupts the output of the matched filter.

We find that in contrast to the situation in the case of a T-Spaced Equalizer no poles (or near-poles) can be caused by the denominator of C(f) within the Nyquist range by the sampler timing τ. In fact, the denominator of C(f) does not depend on τ, and can be expressed in terms of the folded power spectrum of the unequalized channel. Moreover, one may note an interesting result if the data symbols are uncorrelated and the desired response, f(t), is a unit pulse. In this case, once the folded power spectrum is constant, the equalizer turns to be a matched filter which maximizes the signal to noise ratio at sampling instants and minimizes ISI as well.

5.2 The Eigenvalues of a T/2 Equalizer

Using the experience gained in deriving Eq. (4-10) one can verify that the eigenvectors and eigenvalues of an infinite T/2-Equalizer are given by (see: [Qureshi, Forney, 9]) two eigenvectors, expressed as:

$$\underline{V}_{1,2}(f) = [\ldots, \pm\hat{H}eq^*(f)e^{-j2\pi fT/2}, Heq^*(f), \pm\hat{H}eq^*(f)e^{j2\pi fT/2}, \pm Heq^*(f)e^{j2\pi fT}, \ldots]^T \quad (5\text{-}5)$$

with corresponding eigenvalues

$$\lambda_1(f) = |Heq(f)|^2 + |\hat{H}eq(f)|^2 \qquad \text{when } (+) \text{ sign holds} \qquad (5\text{-}6)$$

and

$$\lambda_2(f) = 0 \qquad \text{when } (-) \text{ sign holds.} \qquad (5\text{-}7)$$

As shown before $\lambda_1(f)$ can be expressed as the folded power spectrum when the assumption that H(f) is band limited holds. Thus:

$$\lambda_1(f) = \sum_i |H(f-\frac{i}{T})|^2 \qquad (5\text{-}8)$$

and for $f < 1/T$ we have;

$$\lambda_1(f) = |H(f-\frac{1}{T})|^2 + |H(f)|^2 + |H(f+\frac{1}{T})|^2 \qquad (5\text{-}9)$$

We see that a constant folded power spectrum in the T/2 case has the same effect as constant folded spectrum in the T case: in both cases it is possible, by a judicious choice of the step size to have the taps gains reach their optimal values in one iteration.

One may also note that while in the T-case the eigenvalues spread is subject to changes due to the sampling timing offset, τ, in the T/2 case, where λ_1 (f) does not depend on τ, the convergence process does not depend on the sampler timing.

5.3 A Finite T/2-Equalizer with Periodic Data Source

For the case of a channel equalized by a finite T/2 Equalizer which spans a time interval NT and a periodic data source with period NT, one can show in a way similar to that employed in Sec. 5.1 that:

$$C\left(\frac{n}{NT}\right)=H^*\left(\frac{n}{NT}\right)\cdot e^{j2\pi n\tau/N}\cdot\frac{2Feq(f)}{2[|H(\frac{n}{NT}-\frac{1}{T})|^2+|H(\frac{n}{NT})|^2+|H(\frac{n}{NT}+\frac{1}{T})|^2]+\sigma_n^2} \quad -N/2\leq n\leq N/2 \tag{5-10}$$

The above result shows that the periodic frequency response in this case is completely determined by N samples of the infinite T/2-Equalizer frequency response.

5.4 The Eigenvectors and Eigenvalues of a T/2 Equalizer with Periodic Data Source

For the case of a finite T/2 Equalizer and a periodic data source the N x N autocorrelation matrix has N independent eigenvectors and N different eigenvalues whose form is given by [9]:

$$\lambda_m = \frac{1}{T}\left(|Heq(\frac{m}{NT})|^2+|\hat{H}eq(\frac{m}{NT})|^2\right) \qquad 0\leq m\leq N-1 \tag{5-11}$$

$$\underline{V_m}=\frac{1}{T}\left(Heq^*(\frac{m}{NT}),\hat{H}eq^*(\frac{m}{NT})\cdot e^{j2\pi m/2N},\ldots,\hat{H}^*(f)\cdot e^{j2\pi(2N-1)/2N}\right)^T$$

The other N eigenvalues of A are identically zero. We have already seen that λ_n is a sample of the folded power spectrum when H(f) is bandlimited. One can see from Eq. (5-11) that in this case, once the eigenvalues' spread is small, the folded power spectrum is almost Nyquist and the convergence process described in Sec. 4.1 is fast. Moreover, the optimal equalizer constitutes a matched filter with respect to channel noise.

5.5 The Minimum Mean Square Error of an Infinite T/2 Equalizer

By applying very much the same procedure outlined in

Sec. 3.5 one can show that for a T/2 equalizer, the minimum mean square error given by:

$$\overline{|e|^2 min} = \underline{a}^H \cdot G \cdot \underline{a}^* - \underline{\alpha}^H \cdot \underline{C}opt$$

$$= \sum_i \sum_j a_j^* a_i G_{i,j} - \sum_{k,even} \alpha_k c_k opt - \sum_{k,odd} \alpha_k d_k opt$$

can be expressed as:

$$\overline{|e|^2 min} = \frac{\sigma_n^2}{T} \int_{-1/2T}^{1/2T} \frac{|Feq(f)|^2 \Phi_{aa}(f)}{\Phi_{aa}(f)[|\hat{H}eq(f)|^2 + |Heq(f)|^2] + \sigma_n^2} df \quad (5\text{-}13)$$

One notes that here $\overline{|e|^2 min}$ is not influenced by τ. Moreover, by comparison with the expression derived for the T-case one can see that

$$\overline{|e|^2 min}_{\frac{T}{2}} \leq \overline{|e|^2 min}_T \quad (5\text{-}14)$$

which proves that the T/2 equalizer has better performance which is independent of τ. In [Ungerboeck, 8] Ungerboeck shows by simulation that Eq. (5-14) also holds for a finite 3T/4 equalizer which proves to be free from τ changes influence over a large time interval. In [9] a similar simulation was carried out for a T/2 finite equalizer with similar results.

6. A HYBRID TRANSVERSAL EQUALIZER (HTE)

6.1 A Hybrid Type Equalizer is a T-spaced equalizer with some additional taps inserted around the reference tap in between the T-spaced taps. This type is a special case of the general one presented in Sec. 3.2. Fig. 6-1 shows a finite length Hybrid-Type Equalizer. Such an equalizer is expected to have many of the benefits of a T/2-equalizer, but with the same number of taps can be made to span a larger time interval. This enables the equalizer to take care of impulse responses which have significant energy over the whole time span of the HTE. The more additional taps we insert into a given T-spaced equalizer, the more the HTE behaviour will resemble that of a pure T/2-equalizer.

The hope is that the T/2 section of the HTE can avoid creation of nulls, or near nulls in the Nyquist equivalent spectrum of the system.

It has been shown in literature (see: [6],[9]) that in the iterative adaptive model discussed in Chapter 4, there is an additional noise component, $\overline{e^2_\Delta}$, due to the taps gains fluctuations. This noise power is linearly proportional to N, the number of taps. In order to reduce this excess noise it is desirable to reduce the number of taps in the equalizer. The HTE is expected to suffer less than a pure

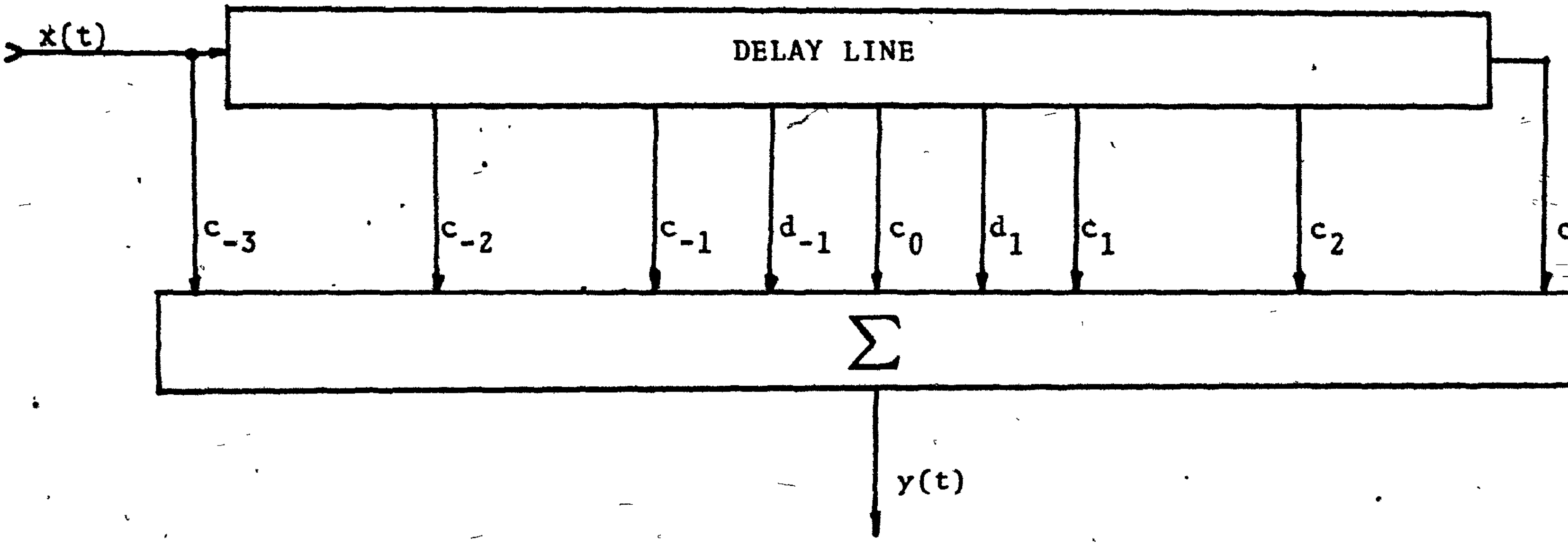

Fig. 6-1: Hybrid Transversal Equalizer
$N_1=N_2=3$

T/2 equalizer from taps fluctuations noise as it has fewer taps.

In the following sections the HTE is mathematically analysed, and some interesting results are presented relating an HTE to the pure T/2-spaced equalizer, both spanning the same time interval.

6.2 The Optimal HTE

In order to analyze the HTE model we refer to Fig. 6-2. It is obvious that every HTE can be decomposed into sections as shown in the figure.

From Eq. (2-2) we know that:

$$x(t) = \sum_i a_i h(t-iT) + n(t)$$

and from the figure

$$y_k = y_{k_1} + y_{k_2} + y_{k_3}$$

where:

$$y_{k_1} = \sum_{i=-N_0}^{-N_1} c_i \cdot x_{k-i} \triangleq \underline{c}^T \cdot \underline{x}_k \qquad (6\text{-}1)$$

$$y_{k_2} = \sum_{i=-2N_1+1}^{2N_2} d_i \cdot x_{k-d_1-i/2} = \underline{d}^T \cdot \underline{x}_{k-d_1} \qquad (6\text{-}2)$$

$$y_{k_3} = \sum_{i=N_2+1}^{N_3} e_i \cdot x_{k-d_1-d_2-i} = \underline{e}^T \cdot \underline{x}_{k-d_1-d_2} \qquad (6\text{-}3)$$

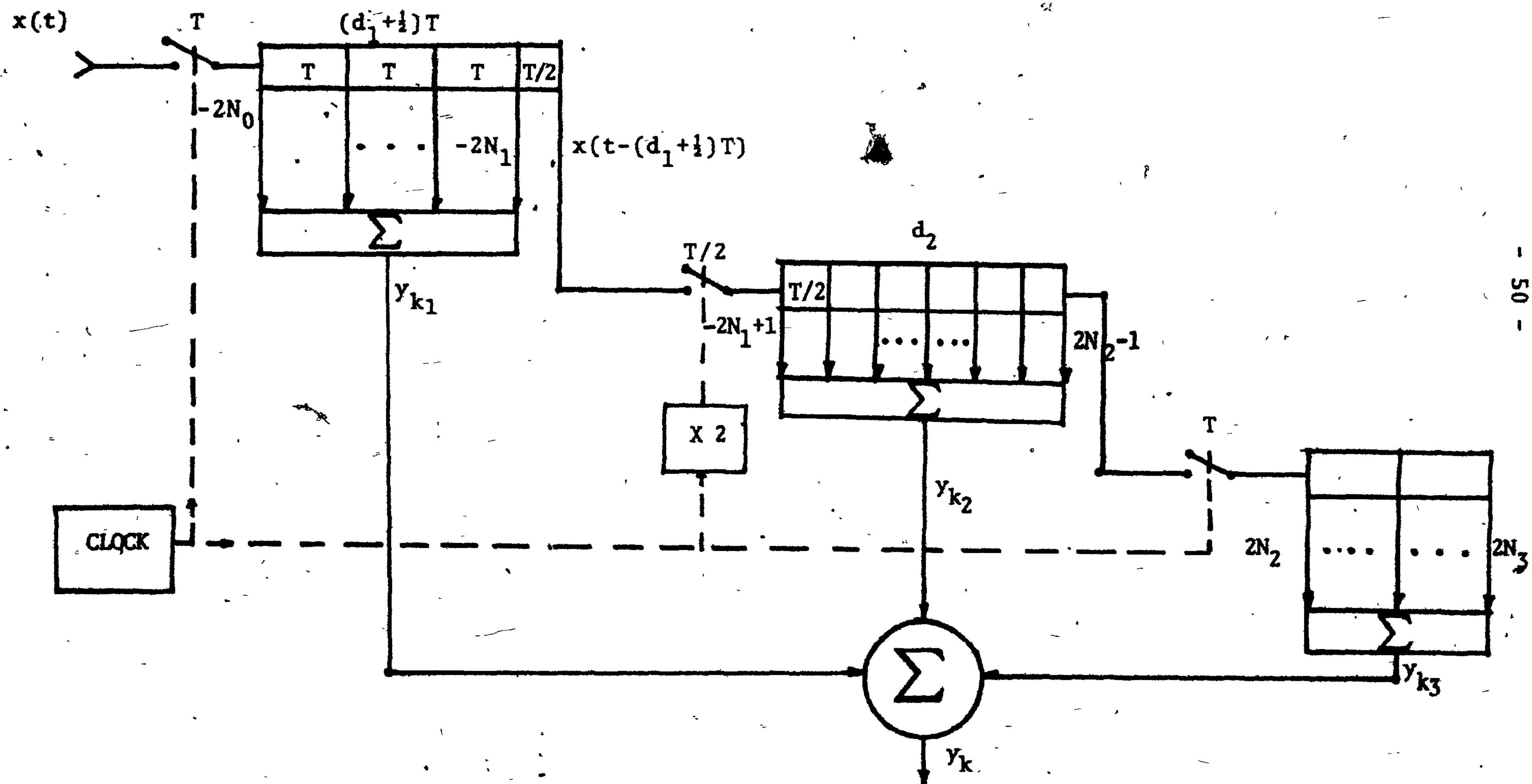

Fig. 6-2: HTE Model

We have also defined the desired output as $d_k = \underline{a}^T \cdot \underline{f}_k$ where $\{f_k\}$ are samples of a desired overall impulse response.

The mean square error is:

$$\overline{|e_k|^2} = (y_k^* - \underline{a}^H \cdot \underline{f}_k^*) \cdot (y_k - \underline{a}^T \cdot \underline{f}_k) \tag{6-4}$$

By substituting Eq. (6-1), Eq. (6-2), Eq. (6-3) into Eq. (6-4) and making the following assertions:

$$A_1 \triangleq \overline{\underline{x}_k^* \, \underline{x}_k^T} \tag{6-5}$$

$$A_2 \triangleq \overline{\underline{x}_{k-d_1}^* \cdot \underline{x}_{k-d_1}^T}$$

$$A_3 \triangleq \overline{\underline{x}_{k-d_1-d_2}^* \cdot \underline{x}_{k-d_1-d_2}^T}$$

$$B \triangleq \overline{\underline{x}_k^* \cdot \underline{x}_{k-d_1}^T}$$

$$V \triangleq \overline{\underline{x}_{k-d_1}^* \cdot \underline{x}_{k-d_1-d_2}^T}$$

$$W \triangleq \overline{\underline{x}_k^* \cdot \underline{x}_{k-d_1-d_2}^T}$$

$$\underline{\alpha}_1 \triangleq \overline{\underline{x}_k^* \cdot d_k} \tag{6-6}$$

$$\underline{\alpha}_2 \triangleq \overline{\underline{x}_{k-d_1}^* \cdot d_k}$$

$$\underline{\alpha}_3 \triangleq \overline{\underline{x}_{k-d_1-d_2}^* \cdot d_k}$$

$$G \triangleq \underline{f}_k^* \cdot \underline{f}_k^T$$

one gets:

$$\overline{|e|^2} = \underline{c}^H \cdot A_1 \cdot \underline{c} + \underline{c}^H \cdot B \cdot \underline{d} + \underline{c}^H \cdot W \cdot \underline{e} - \underline{c}^H \cdot \underline{\alpha}_1 \tag{6-7}$$

$$+ \underline{d}^H \cdot A_2 \cdot \underline{d} + \underline{d}^H \cdot B^H \cdot \underline{c} + \underline{d}^H \cdot V \cdot \underline{e} - \underline{d}^H \cdot \underline{\alpha}_2$$

$$+ \underline{e}^H \cdot A_3 \cdot \underline{e} + \underline{e}^H \cdot W^H \cdot \underline{c} + \underline{e}^H \cdot V^H \cdot \underline{d} - \underline{e}^H \cdot \underline{\alpha}_3$$

$$- \underline{\alpha}_1^H \cdot \underline{c} - \underline{\alpha}_2^H \cdot \underline{d} - \underline{\alpha}_3^H \cdot \underline{e} + \underline{a}^H \cdot G \cdot \underline{a}$$

By differentiating Eq. (6-7) with respect to $\underline{c}$, $\underline{e}$ and $\underline{d}$ we arrive at the following set of linear equations for $\underline{c}_{opt}$, $\underline{e}_{opt}$ and $\underline{d}_{opt}$:

$$\begin{bmatrix} A_1 & B & W \\ B^H & A_2 & V \\ W^H & V^H & A_3 \end{bmatrix} \begin{bmatrix} \underline{c} \\ \underline{d} \\ \underline{e} \end{bmatrix}_{opt} = \begin{bmatrix} \underline{\alpha}_1 \\ \underline{\alpha}_2 \\ \underline{\alpha}_3 \end{bmatrix} \tag{6-8}$$

Our task now is to identify the elements of the matrices A_1, A_2, A_3, B, V, W, and the vectors $\underline{\alpha}_1$, $\underline{\alpha}_2$ and $\underline{\alpha}_3$.

One can quite easily verify that the elements of these matrices are related to the elements of the T/2 Equalizer autocovariance matrix as follows:

$A_1 = \{A_{i,j}\}$ for: $-2N_0 \le i,j \le -2N_1$, i,j even

$A_2 = \{A_{i,j}\}$ for: $-2N_1+1 \le i,j \le 2N_2-1$

$A_3 = \{A_{i,j}\}$ for: $2N_2 \le i,j \le 2N_3$, i,j even

B = $\{A_{i,j}\}$ for: $-2N_0 \le i,j \le -2N_1$, i even; $-2N_1+1 \le j \le 2N_2-1$, j odd.

W = $\{A_{i,j}\}$ for: $2N_2+1 \le i,j \le 2N_3 \le i,j$ even

V = $\{A_{i,j}\}$ for: $-2N_1+1 \le i \le 2N_2-1, 2N_2 \le j \le 2N_3$, j even

Also:

$$\alpha_{1i} = \{\alpha_i\} \quad -N_0 \le i \le -N_1$$

$$\alpha_{2i} = \{\alpha_i\} \quad -2N_1+1 \le i \le 2N_2-1$$

$$\alpha_{3i} = \{\sigma_i\} \quad 2N_2 \le i \le 2N_3$$

The conclusion from the above is that the autocovariance matrix for the HTE can be derived from the matrix of the T/2 case by deleting those rows and columns which correspond to in between taps which are not used in the Hybrid version. A similar result holds for the $\underline{\alpha}$-vector of the HTE.

6.3 The Frequency Response of an HTE

Assume that the T-spaced sections of the HTE shown in Fig. 6-2 are infinite. If one denotes the T-spaced taps by $\{v_i\}$ and the in between taps by $\{w_i\}$, then the frequency response of the HTE is given by:

$$C(f) = \sum_i v_i e^{j2\pi fTi} + \sum_{i=-N_1}^{N_1} w_i \, e^{j2\pi fT(i+\frac{1}{2})} \tag{6-9}$$

In Section 6.2 we described the structure of the autocovariance matrix for a system equalized by an HTE. Having at hand this knowledge, we can follow the procedures described in Sec. 4.3 and in Sec. 5.1 (for the derivations of the frequency-response of T and T/2 equalizers resp.) and arrive at the following two equations for W(f) and V(f):

$$\sum_{\substack{k\\ \text{even}}} \sum_{\substack{l\\ \text{even}}} A_{k,l} v_l e^{-j2\pi k\lambda T/2} + \sum_{\substack{k\\ \text{even}}} \sum_{\substack{l=-2N_1+1\\ \text{odd}}}^{2N_1-1} A_{k,l} w_l e^{-j2\pi k\lambda T/2} = \sum_{\substack{k\\ \text{even}}} \alpha_k e^{-j2\pi\lambda kT/2} \qquad (6\text{-}10a)$$

$$\sum_{\substack{k=-2N_1+1\\ \text{odd}}}^{2N_1-1} \sum_{\substack{l\\ \text{even}}} A_{k,l} v_l e^{-j2\pi\lambda kT/2} + \sum_{\substack{k=-2N_1+1\\ \text{odd}}}^{2N_1-1} \sum_{\substack{l=-2N_1+1\\ \text{odd}}}^{2N_1-1} A_{k,l} w_l e^{-j2\pi\lambda kT/2} = \sum_{\substack{k=-2N_1+1\\ \text{odd}}}^{2N_1-1} \alpha_k e^{j2\pi\lambda kT/2} \qquad (6\text{-}10b)$$

By substituting Eq. (3-17) into Eq. (6-10) and by making the following definitions

$$W(f) \triangleq \sum_{\substack{i=-N_1\\ \text{odd}}}^{N_1} w_i e^{j2\pi fiT/2}$$

$$V(f) \triangleq \sum_{\substack{i\\ \text{even}}} v_i e^{j2\pi fiT/2}$$

$$Y(f) \triangleq \sum_{k=-N_1}^{N_1} e^{j2\pi f(k+\frac{1}{2})T}$$

one arrives at the following two equations for W(f) and V(f):

$$[|Heq(f)|^2 \Phi_{aa}(f)+\sigma_n^2]\cdot V(f)+\overset{*}{Heq}(f)\widehat{Heq}(f)\Phi_{aa}(f)W(f) = \overset{*}{Heq}(f)Feq(f)\Phi_{aa}(f)e^{j2\pi f\tau} \qquad (6\text{-}11a)$$

$$\int_{-1/2T}^{1/2T} \hat{H}eq^{*}(\lambda)Heq(\lambda)\Phi_{aa}(\lambda)\; V(\lambda)\cdot Y(f-\lambda)d\lambda \; + \tag{6-11b}$$

$$\int_{-1/2T}^{1/2T} |\hat{H}eq(\lambda)|^2\Phi_{aa}(\lambda)W(\lambda)\; Y(f-\lambda)d\lambda+\sigma_n^2W(f) =$$

$$\int_{-1/2T}^{1/2T} \hat{H}eq^{*}(\lambda)Feq(\lambda)\Phi_{aa}(\lambda)e^{j2\pi\lambda\tau}\cdot Y(f-\lambda)d\lambda$$

Unfortunately, it is impossible to continue from this point towards solving (6-11) for V(f) and W(f) without making additional assumptions. First, we note that each of the integrals in (6-11b) is a convolution in the frequency domain. Then one can see that when $N_1 \to \infty$, Y(f) approaches an impulse $\delta(f)$ reducing our HTE case to the infinite T/2-equalization case, which was treated in Sec. 5.1.

When N_1 is finite the function of f generated by each of the integrals in (6-11b) is a smeared version of the part of the integrand convolved with Y(f), (see Fig. 6-3) and the degree of smearing, depends on N_1.

Assuming that N_1 is not too small we get that Eq. (5-4) is still a good approximation for C(f) in this case.

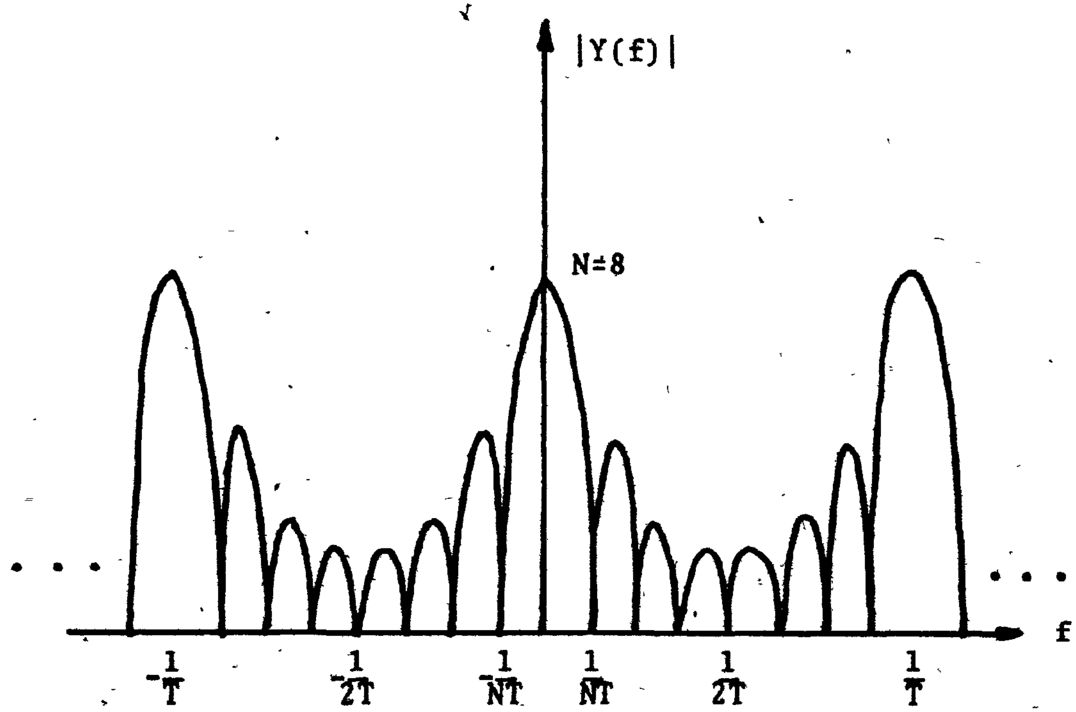

Fig. 6-3: The Magnitude of Y(f) for N=8

7. COMPARISON BETWEEN FINITE LENGTH T, T/2 AND HYBRID TYPE EQUALIZERS

7.1 Computer Program for Comparison

A Fortran IV program was used to compare these three cases. The structure of the program is as follows:

The program reads in the channel samples, the index of reference sample, along with an indication whether the samples are T or T/2-spaced. Then, the program reads in the parameters of the equalizer; i.e., the number of taps, the location of the reference tap and the input signal to noise ratio. The program computes and prints the channel autocovariance matrix, the eigenvalues, the resulting equalizer optimal taps gains, and the minimum mean square error.

When a T/2 equalizer is run, any HTE's performance can be computed. Moreover, the program is used to find the optimal location of the in between additional taps for an HTE and a given fixed time span equalizer. Also, for a fixed number of taps, the program finds the optimal time span, and thus the number of in between taps. The program is listed in Appendix B.

In the next sections, the results for two typical channels are presented.

7.2 Optimizing a Fixed-Time-Span Equalizer (†)

The channel chosen for optimization is the channel used in [Ungerboeck, 8]. The channel impulse response is shown in Fig. 7-1..

For this channel the program computed the minimum mean square error of a 7T-time span equalizer, starting with a pure T-equalizer. Then, one T/2-tap at a time was inserted among the T-taps and all possible T/2-taps positions were tried. This was done for a high signal to noise ratio in order to bring out the differences between the possible hybrid configurations.

In Fig. 7.2 one can see the minimum mean square error vs the number of additional taps. For each additional tap, the best and worst HTE configurations are shown. This yields a "contour" within the limits of which, all possible configurations lie. The arrays of ones and zeroes on the graph represent the related configurations; a "1" stands for a tap which is used and "0" stands for a tap which is not used in the HTE.

In Table 7-1 we give the improvement in minimum mean square error, achieved by adding taps, with respect to the pure T-spaced equalizer performance.

The improvement achieved by optimally inserting only one additional T/2 tap is remarkable.

(†) In each of section 7.2 and section 7.3, results obtained for one typical channel response are represented. Similar results were obtained for other practical channel responses.

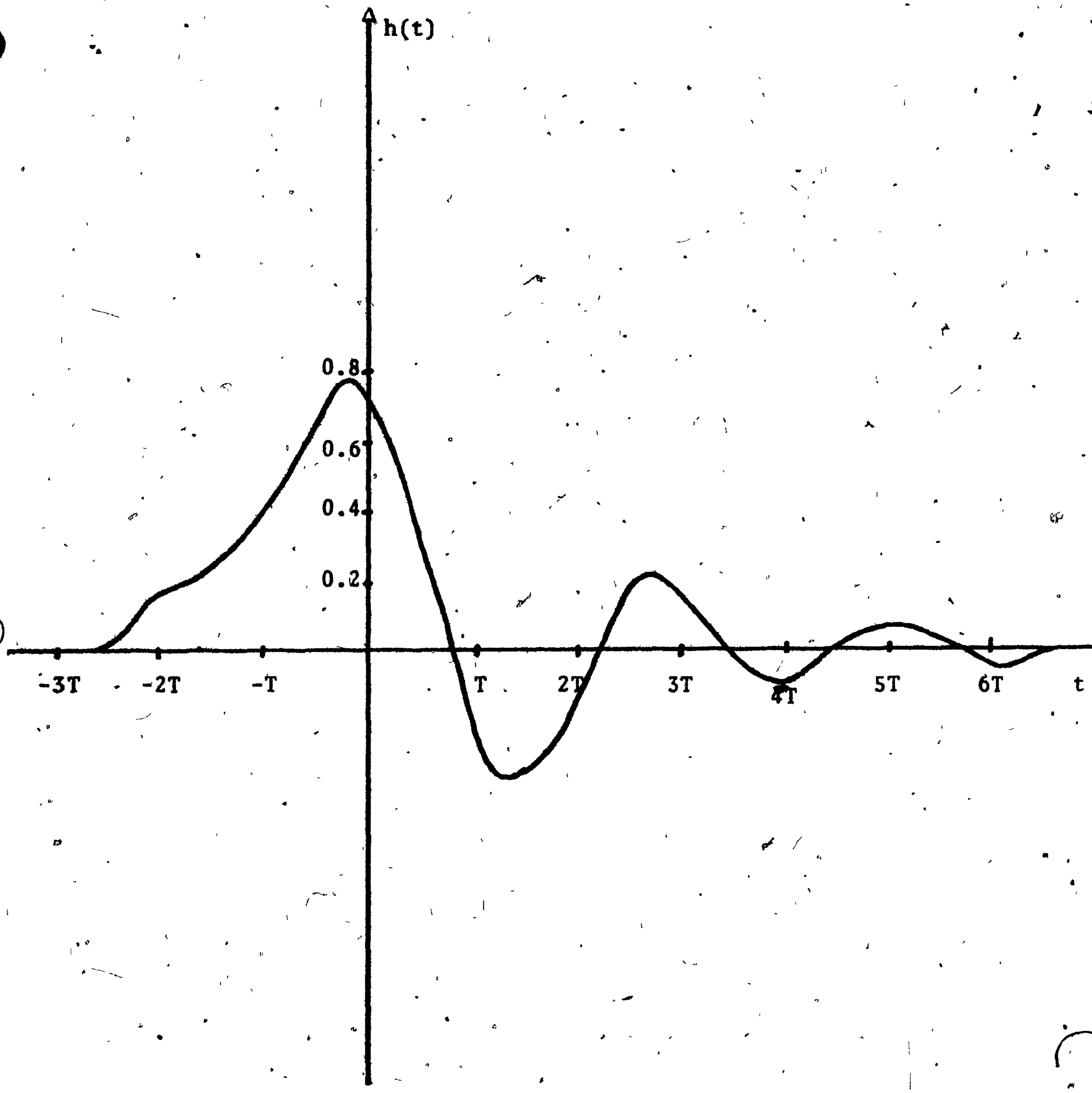

Fig. 7-1: Channel Impulse Response [8]

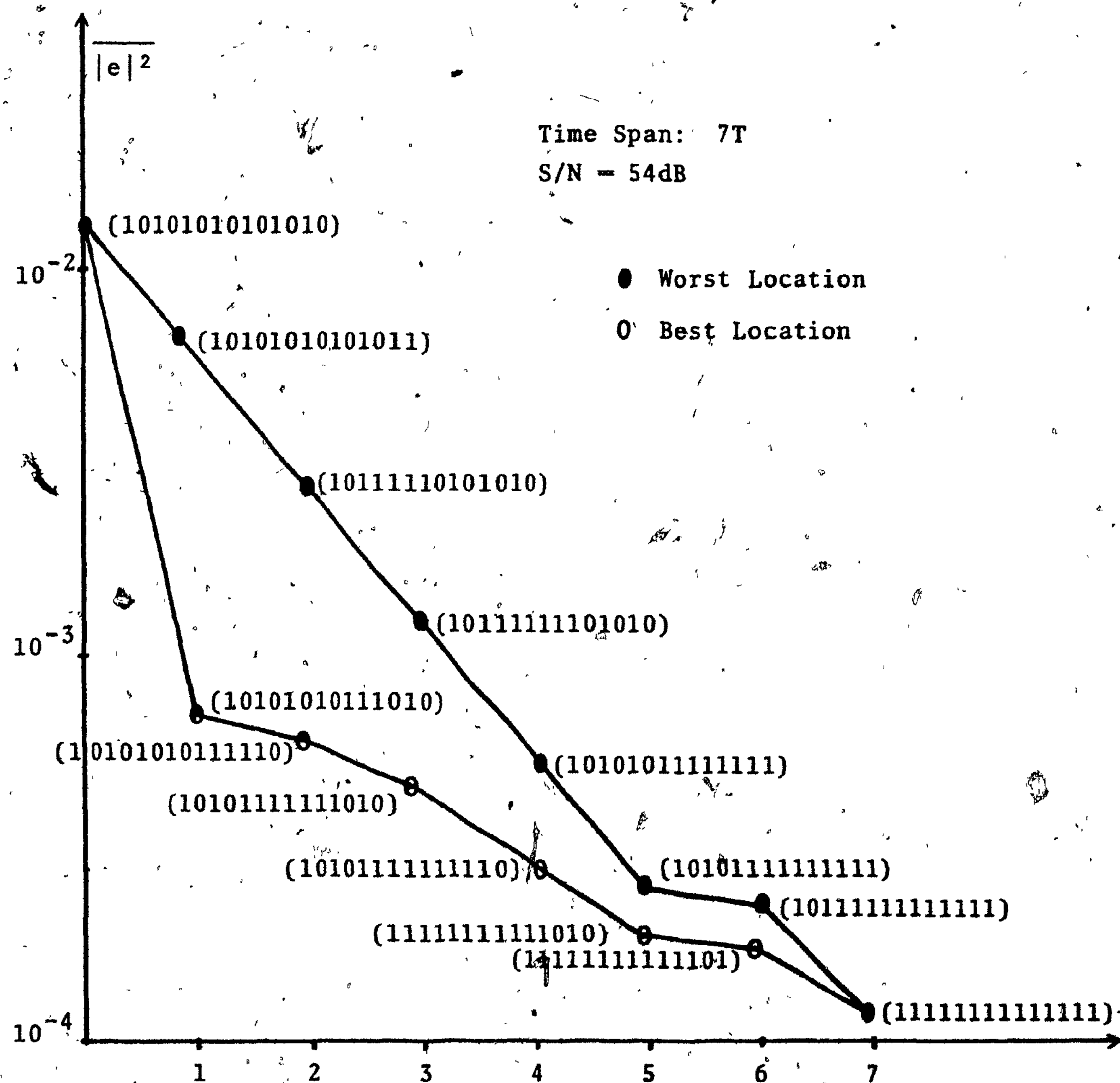

Fig. 7-2: Minimum Mean Square Error vs Number of Additional Taps

No. of additional taps	Improvement min	max
1	3.7 dB	12.9 dB
2	7.6 dB	13.6 dB
3	11.0 dB	15.0 dB
4	14.3 dB	17.0 dB
5	17.5 dB	18.8 dB
6	17.7 dB	19.1 dB
7	-	20.8 dB

Table 7-1

HTE Performance Improvement
vs Number of Additional Taps

The difference in improvement between the best location of the additional tap and the worst location is significant.

7.3 Optimization of a Fixed Number of Taps HTE

The program was used to find the time span of an Hybrid Type Equalizer having 10 taps, for which the least minimum mean square error is obtained. The channel used in this section is shown in Fig. 7-3. This is an interpolated version of the sampled impulse response used in [7] and in [9].

In Fig. 7-4 one can see that for a 10-taps equalizer the optimal time span is 7T. The additional T/2 taps were inserted in a symmetrical manner around the reference tap which is located in the middle of the equalizer's delay line. The ratio between the minimum mean square error of a pure T/2 equalizer with 10 taps and an HTE which spans 7T is about 19.3 in this case. We may conclude that in cases where the channel impulse response is long, and has significant energy over most of its duration. A longer HTE is to be preferred over a pure T/2 equalizer with the same number of taps.

7.4 Sampling Timing Sensitivity

In this section we compare the sampling time offset sensitivity of a T-spaced, T/2-spaced and a Hybrid Transversal Equalizer, all having the same time span but the complexity is increasing: the T-spaced equalizer have 7 taps, the hybrid equalizer has 10 taps, and the T/2-spaced equalizer has 14 taps.

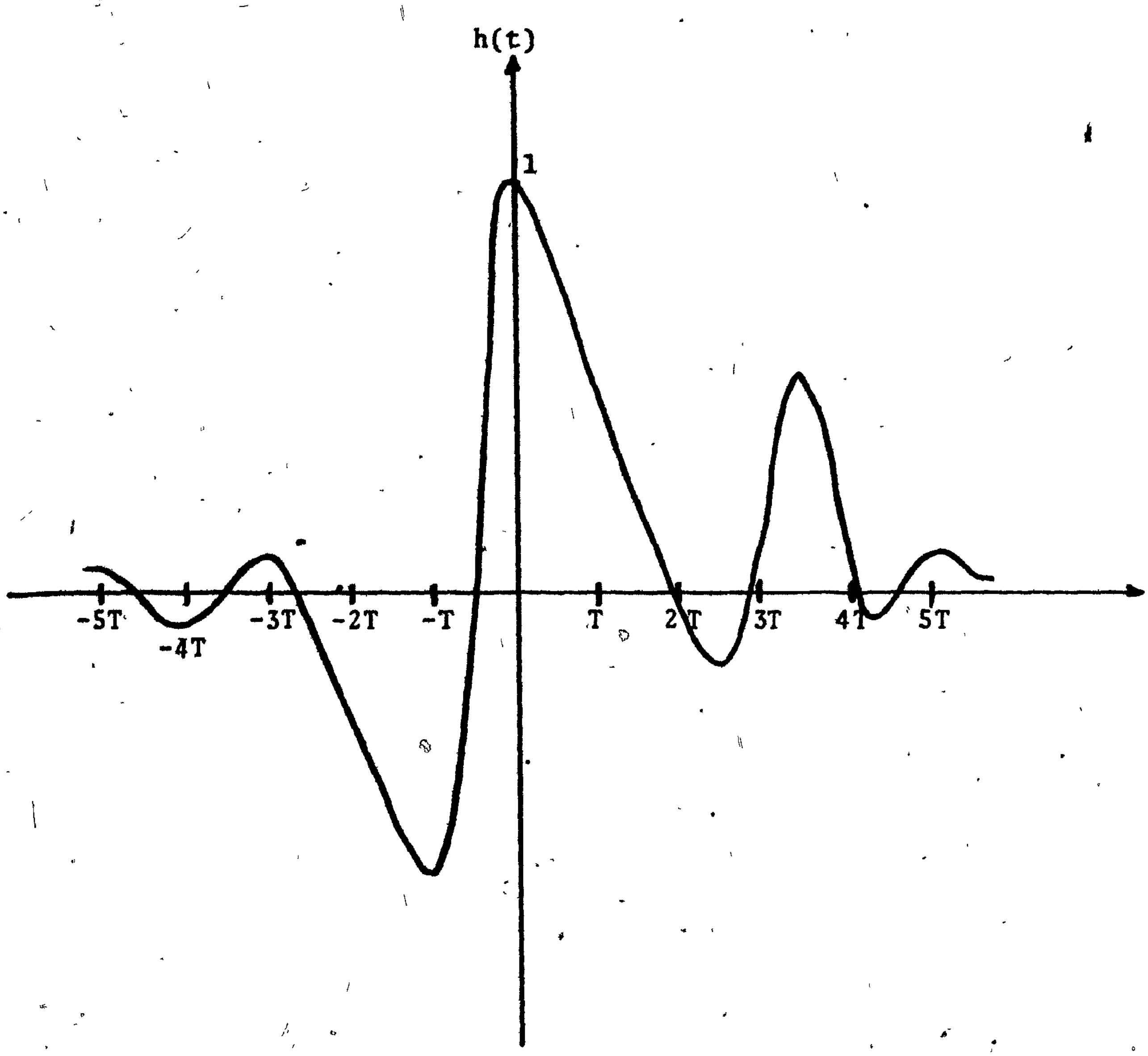

Fig. 7-3: Channel Impulse Response

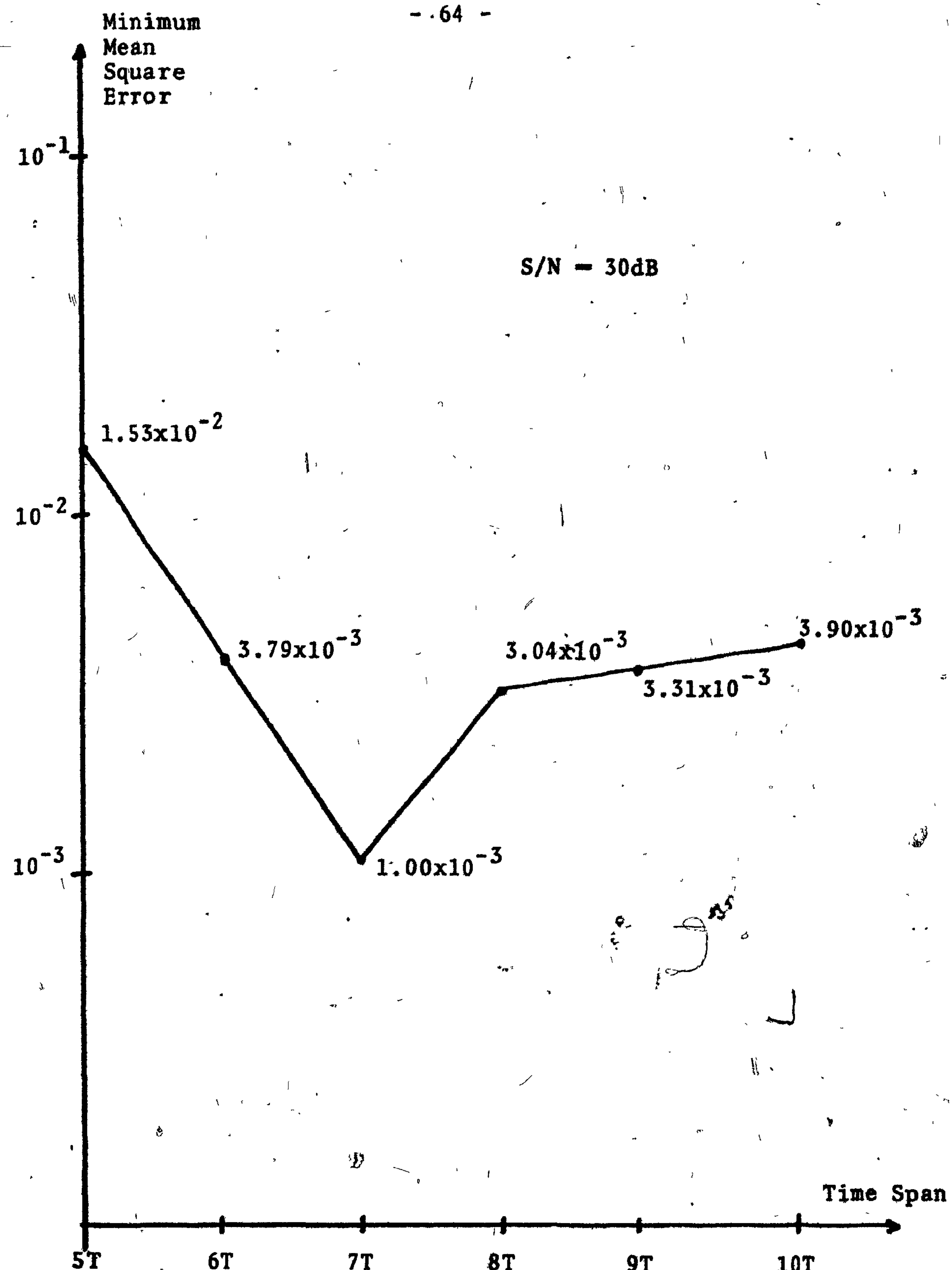

Fig. 7-4: Time Span vs Minimum Mean Square Error (10-Taps)

In order to check the sampling time offset sensitivity, the channel in Sec. 7.2 was sampled in various phases with T spaces and with T/2 spaces. For each phase the minimum mean square error was computed. The results are shown in Fig. 7-5. The T/2-spaced equalizer proves to be superior to T-spaced equalizer; one notes the big changes in performance in the T-case, and the modest changes in the T/2-case with sampling timing changes over an interval of [-T, +T]. The ratio between maximum and minimum values of mean square error in the T-spaced equalizer is 18 while the same ratio for a T/2-spaced equalizer that spans the same time interval is about 2. For a hybrid configuration represented by (10101111111010), (three additional taps. The reference tap is in the middle of the equalizer) the sensitivity is smaller than that of a T-spaced equalizer but worse than that of the T/2-equalizer as expected.

7.5 <u>Calculation of the Autocovariance Matrix Eigenvalues</u>

In this section the eigenvalues of the autocovariance matrix for the channel used in Sec. 7.1 (Fig. 7-1), are computed. The eigenvalues were calculated for both the periodic and the white data source cases, for a T-spaced equalizer, T/2-spaced equalizer and the hybrid configuration used in Sec. 7.4. By examining the results (summerized in Table 7-2) the following observations are made:

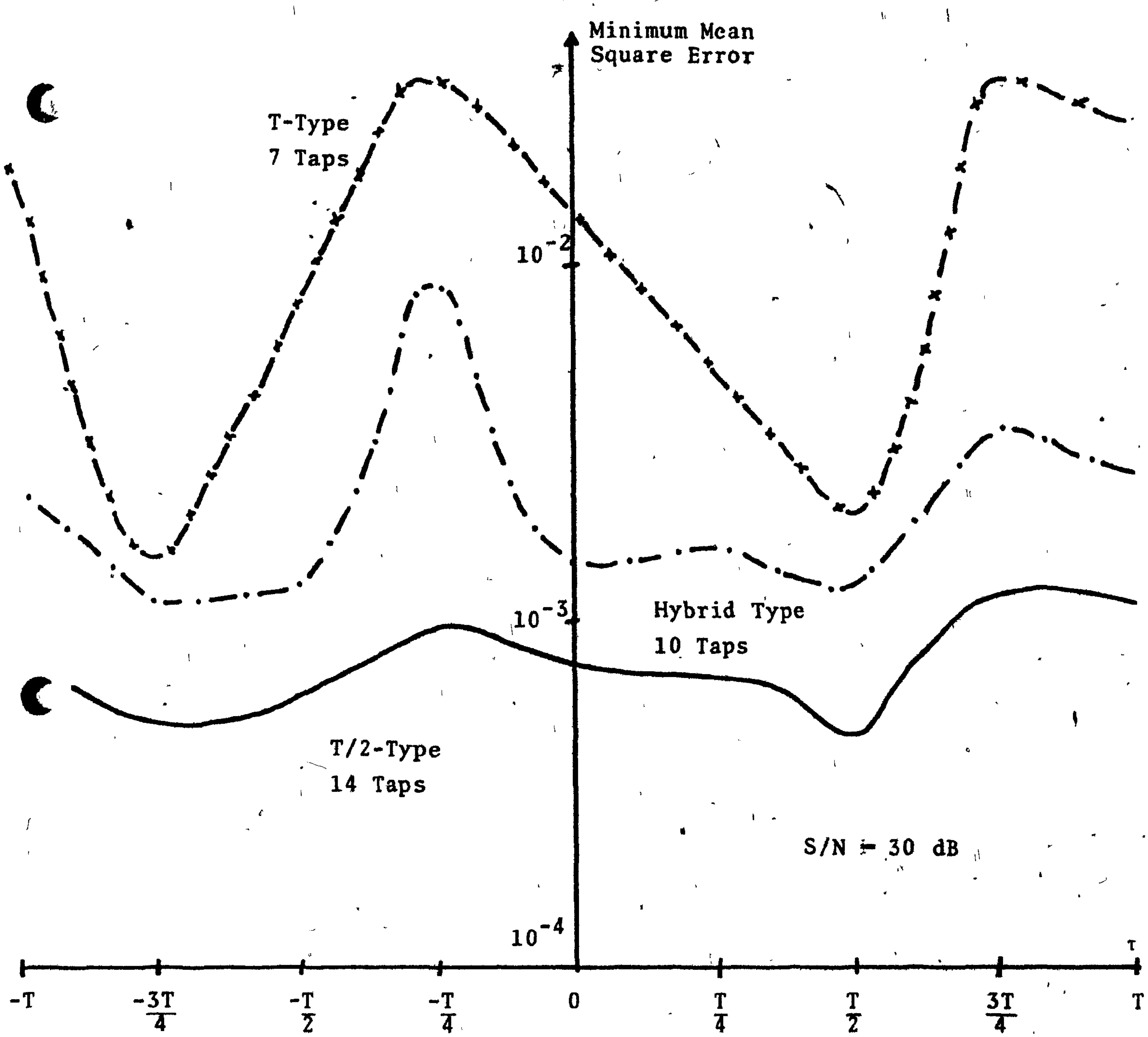

Fig. 7-5: Sampling Timing Offset Sensitivity

T-Equalizer	
Periodic s/n = 54dB	White s/n = 30dB
0.4789	0.2861
0.4789	0.6155
0.7333	0.7297
0.7333	0.838
0.9881	0.9608
0.9881	1.050
1.228	1.175

Ref. tap: 4

T/2-Equalizer	
Periodic s/n = 54dB	White s/n = 30dB
0.4×10^{-5}	0.1066×10^{-2}
- " -	0.1245×10^{-2}
- " -	0.1536×10^{-1}
- " -	0.2146×10^{-2}
- " -	0.6979×10^{-2}
- " -	0.2197×10^{-1}
- " -	0.2962×10^{-1}
0.8275	0.5382
0.8275	0.9921
1.502	1.502
1.502	1.502
1.973	1.973
1.973	1.973
2.900	2.093

HTE (0101111111010)	
Periodic s/n = 54dB	White s/n = 30dB
0.4×10^{-5}	0.1356×10^{-2}
0.4×10^{-5}	0.1598×10^{-2}
0.4×10^{-5}	0.5207×10^{-1}
0.3630	0.2931
0.4678	0.7176
0.5879	0.8727
0.6994	1.008
1.191	1.185
1.592	1.701
1.759	2.023

Ref. tap: 7

Table 7-2: Eigenvalues Results

1. For the T-spaced samples of a real input response we get a circulant autocovariance matrix. Its eigenvalues are real and come in equal pairs (except for the largest one, when the matrix dimension is odd). This originates from the fact that the eigenvalues of a circulant matrix are given by the D.F.T. of its rows [Noble, 16].
2. For the T/2-spaced equalizer with periodic source, half the eigenvalues are equal to the noise to signal ratio in the channel. The values of these eigenvalues is zero once there is no noise in the system. This implies that in this case the system given by $A\underline{c}=\underline{\alpha}$ is overdetermined and it may have many different solutions for $\underline{C}$opt.

   The remaining half are in equal pairs. The reason is that they are equally spaced samples of the channel folded power spectrum (as proved in Sec. 5.3) which is an even function. One may notice that the eigenvalues of the T/2-case with white data source split into two groups. The seven small ones may be interpreted as smeared values corresponding to the seven small ones computed for the T/2-case with periodic input. A similar observation can be made for the HTE case.
3. One can see that the eigenvalues spread for all equalizers is about the same. This implies about equal tap gains convergence time in the iterative model discussed in

Chapter 4. This idea is supported by simulations results in [Ungerboeck, 8] carried out for a non-periodic case.

8. CORRELATED LEVEL SIGNALLING AND FRACTIONAL TAP SPACING EQUALIZATION

In previous chapters the data source was assumed to be either white or periodic. It is interesting to verify how correlated level signalling performs with Fractional Tap-Spacing-Equalizers.

8.1 Correlated Level or Partial Response Signalling

The usual constraint on signals chosen for signalling over a channel is that they do not give rise to intersymbol interference. Sometimes, signal design based on this criterion is very difficult, if not impossible and may turn the system to be very sensitive to sampling timing.

A design which allows for a certain amount of controlled intersymbol interference while the transmission bandwidth is confined to the Nyquist bandwidth is referred to as Partial Response Signalling (PRS) or, Correlated Level Signalling (CLS). The controlled intersymbol interference can be removed from the incoming signal in the receiver. On the other hand, because the number of received levels is larger for PRS it has a narrower noise margin for a constant signal power.

The first PRS that was employed is called duobinary and will be discussed below. An extensive study of PRS is in [Kabal, Pasupathy, 7].

It is interesting to verify how PRS influences the performance of a channel equalized with a T/2 equalizer.

8.2 The Duobinary PRS and T/2-Equalization

In Fig. 8-1 we show the impulse response and frequency response of a channel that allows duobinary PRS.

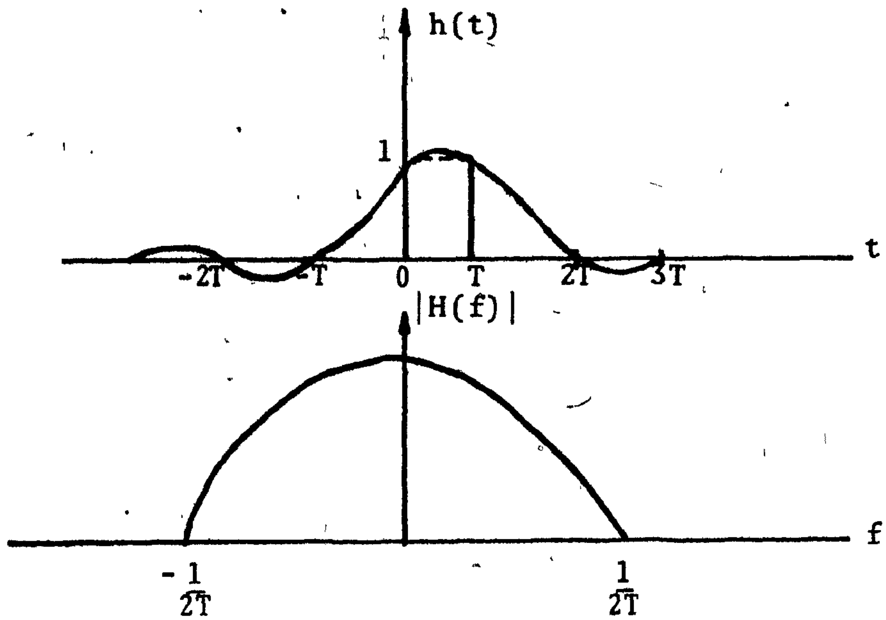

Fig. 8-1: Duobinary Impulse and Frequency Response

In [7] it is shown that any PRS system has frequency response which can be expressed as: $H(f) = F(f) \cdot G(f)$

where $G(f)$ obeys Nyquist's criterion, and $F(f) = \sum_{n=o}^{N-1} f_n e^{-j2\pi fTn}$

where $\{f_n\}$ are the desired samples of the channel's impulse response. For duobinary: $f_o = f_1 = 1$

thus: $F(f)=1+e^{-j2\pi fT}$ (8-1)

In order to have a channel with duobinary response the binary data stream is precoded by the filter given by Eq. (8-1). Moreover the rest of the channel's response should satisfy

Nyquist's criterion. For the binary input with levels -1 and 1 we may get at F(f) output the levels: -2, 0, 2; three levels instead of two. This fact increases the probability of error in the detection [7]. This is the trade-off between the narrow transmission bandwidth and performance quality.

Assuming that the original data source has power spectrum $\Phi_{aa}(f)$, after precoding it changes to $\Phi_{bb}(f)$, where $\Phi_{bb}(f) = \Phi_{aa}(f)\cdot|F(f)|^2$

By Eq. (8-1), we get

$$\Phi_{bb}(f) = \Phi_{aa}(f)\cdot 4\cdot\cos^2\pi fT \qquad (8\text{-}2)$$

If we substitute $\Phi_{bb}(f)$ for $\Phi_{aa}(f)$ in Eq. (3-17) and define:

$$Heqc(f) \triangleq 2\cdot Heq(f)\cdot\cos\pi fT$$

$$\widehat{Heqc}(f) \triangleq 2\cdot\widehat{Heq}(f)\cdot\cos\pi fT$$

we get for the eigenvalues of an infinite T/2 equalizer the following expression:

$$\lambda(f) = \{|Heq(f)|^2 + |\widehat{Heq}(f)|^2\}\cdot\cos^2\pi fT \qquad (8\text{-}3)$$

We recall that the expression in brackets is the folded power spectrum of the channel (under the assumption that H(f) is bandlimited). From Eq. (8-3) one may conclude that duo-binary precoding tends to increase the spread of the eigenvalues of the system.

Larger spread of the eigenvalues results in longer convergence time in the iterative model discussed in Chapter 4.

9. SUMMARY

We started by presenting a generalized data transmission system model and showed how an optimally designed generalized equalizer can minimize the mean square error in such a system. Through Chapters 3 to 6 we dealt with three special cases of equalizers: the T-Spaced Equalizer, the T/2-Spaced Equalizer and a Hybrid Type Equalizer. We discussed and compared the properties of these three models. The T-spaced equalizer's properties are extensively discussed in literature and its review, brought here, prepares the ground for the discussion of the T/2-spaced equalizer. The T/2-spaced equalizer is not that extensively discussed in literature although it is known to be superior to T-spaced equalizer in certain features. Here we derived closed form expressions characterizing the T/2-spaced equalizers. By these expressions we could show <u>why</u> the T/2 equalizer is superior to a T-spaced equalizer in some respects.

Next we suggested a new model, namely, the HTE, that possesses some of the benefits of both the T-spaced and the T/2-spaced equalizers. The three models were compared by a computer program. The results obtained confirmed previous derivations and assumptions. The discussion through Chapters 2 to 7 show that a T/2-spaced equalizer gives a much smaller minimum mean square error than that given by a T-spaced equalizer that spans the same time interval. The improvement can easily reach 10dB. Moreover, the sensitivity to sampling timing in the receiver is much smaller in the T/2-spaced equalizer. Convergence time of taps gains in the iterative model is about the same, as shown by simulation results contained in other papers and by a similar eigenvalues spread obtained here, for these two cases. The performances of the HTE lie between those of the previous two equalizers. Its use can be important when a compromise has to be done between performances and time span, given a constraint on the number of taps. Larger time span can be vital for cases in which the channel impulse response is long. In such cases the longer HTE can be superior to a shorter pure T/2-Equalizer with the same number of taps. The HTE's sensitivity to sampling timing is less than that of a pure T-spaced equalizer that spans the same time interval. Noise enhancement due to channel noise is the smaller in a T/2-Equalizer while the HTE is again in between them. The HTE has the benefit of a lower complexity relative to a pure T/2-Equalizer that spans the same time interval, as complexity is proportional to N, the total number of taps.

In Chapter 8 a brief discussion reveals that PRS has no inherent benefits for fractional tap spacing equalization.

§

10. LITERATURE

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APPENDIX A

A.I The Derivation of Eq. (3-5)

We start from Eq. (3-4) (which is the definition of the mean square error):

$$\overline{|e_k|^2} = \overline{(y_k-d_k)(y_k^*-d_k^*)}$$

Using the vector notations defined in Sec. 2.2 and in Sec. 3.2 we get:

$$\overline{|e_k|^2} = (\underline{c}^H \underline{x}_k^* - d_k^*)(\underline{x}_k^T \underline{c} - d_k)$$

By defining the following matrix and vector:

$$A \triangleq \overline{\underline{x}_k^* \cdot \underline{x}_k^T}$$

$$\underline{\alpha} \triangleq \overline{\underline{x}_k \cdot d_k^*}$$

we can write

$$\overline{|e_k|^2} = \underline{c}^H \cdot A \cdot \underline{c} - \underline{\alpha}^T \cdot \underline{c} - \underline{c}^H \cdot \underline{\alpha}^* + \overline{|d_k|^2}$$

$\underline{c}$ is a complex vector; $\underline{c} = Re[\underline{c}]+Im[\underline{c}]$. To minimize $\overline{|e_k|^2}$ with respect to $\underline{c}$ we have to differentiate it with respect to $Re[\underline{c}]$ and $Im[\underline{c}]$. However it can be shown that

$$\frac{\partial}{\partial Re(\underline{c})}\overline{|e_k|^2} + j \cdot \frac{\partial}{\partial Im(\underline{c})}\overline{|e_k|^2} = \frac{\partial}{\partial \underline{c}}\overline{|e_k|^2}$$

With this result at hand, we get:

$$\frac{\partial}{\partial \underline{c}}\overline{|e_k|^2} = 2 \cdot A \cdot \underline{c} - 2 \cdot \underline{\alpha} = 0$$

or: $A \cdot \underline{c} = \underline{\alpha}$ and the minimum mean square error can be written

as: $\overline{|e_k|^2}_{min} = \overline{|d_k|^2} - \underline{\alpha}^H \cdot \underline{c}opt$.

A.II The Derivation of Eq. (3-8)

For convenience we start from Eq. (3-1), which is repeated here:

$$x(t) = \sum_i a_i h(t-iT) + n(t) \qquad \text{(A.II-1)}$$

By substituting Eq. (A.II-1) in Eq. (3-6) we get:

$$A_{k,l} = \sum_{ij} \overline{a_i^* a_j} \cdot h^*[(k-D_k-i+\tau/T)T] \cdot h[(k-D_l-j+\tau/T)T] + n^*[(k-D_k+\tau/T)T] \cdot n[(k-D_l+\tau/T)T] \qquad \text{(A.II-2)}$$

By defining: $\Phi_{aa}(i-j) \triangleq \overline{a_i^* a_j}$, the last term in Eq. (A.II-2) as $\Phi_{nn}[(D_k-D_l)T]$, $m \triangleq i-j$, and at last, $n \triangleq k-j$, we arrive at:

$$A_{k,l} = \sum_m \phi_{aa}(m) \sum_n h^*[(n-m-D_k+\tau/T)T] \cdot h[(n-D_l+\tau/T)T] + \phi_{nn}[(D_k-D_l)T] \qquad \text{(A.II-3)}$$

which is Eq. (3-8).

Eq. (3-9) is derived in a similar manner starting from Eq. (3-7).

A.III The Derivation of Eq. (3-13)

Start with the transform definition

$$h(t) = \int_{-\infty}^{\infty} H(f)e^{j2\pi ft}df$$

to substitute in Eq. (A.II-3) . By this substitution and by carrying out the integrations first and then the summation over m and n, we can write:

$$A_{k,1} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H^*(f)\cdot H(\lambda)\cdot \sum_m \phi_{aa}(m)e^{j2\pi fmT}\cdot \sum_n e^{j2\pi n(\lambda-f)T}\cdot e^{-j2\pi(f-\lambda)\tau}\cdot$$

$$\cdot\, e^{j2\pi fD_kT}\cdot e^{-j2\pi\lambda D_1T}d\lambda df+\sigma_n^2\delta_{k,1} \qquad \text{(A.III-1)}$$

Define the data source power spectrum as:

$$\Phi_{aa}(f) \triangleq \sum_m \phi_{aa}(m)e^{j2\pi fmT}$$

and note that: $\sum_n e^{-j2\pi(f-\lambda)Tn} = \frac{1}{T}\sum_i \delta(\lambda-f-\frac{i}{T})$

In light of the above, if the integration in (A.III-1) is carried out on successive intervals of length $\frac{1}{T}$ and if some careful manipulations are made we arrive at:

$$A_{k,l} = \frac{1}{T}\int_{-1/2T}^{1/2T} [\overset{k}{Heq}(f)]^{*}\cdot\overset{l}{Heq}(f)\cdot\Phi_{aa}(f)\,df$$

which is Eq. (3-13), where:

$$\overset{l}{Heq}(f) \triangleq \sum_{n} H(f+\tfrac{n}{T})\cdot e^{-j2\pi(f+\frac{n}{T})D_l T}\cdot e^{j2\pi n\tau/T} .$$

APPENDIX B

PROGRAM LIST

```
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION X( 0),G(50),B( 0,50),C(50),NO(50),BETA(50),MD( 0)
      DIMENSION D(50),Z(50,50),WK(50)  ,H(1500)  ,ALFA(50),P(50,50)
      DIMENSION Q(50,50),BB(50,50),F(50)
      DATA IIN,IOUT/5,6/
      CALL NOUFL
      READ(IIN,2400) VARN,VARS
 2400 FORMAT(2F10.0)
      READ(IIN,2300) LG,(G(I),I=1,LG)
 2300 FORMAT(I10/(8F10.0))
      DO 6 KI=1,10
      KT=1
      READ(IIN,2100)LX,(X(I),I=1,LX)
 2100 FORMAT(I10/(8E10.0))
      READ(IIN,2200)ISAMPL,N,NTAP,IREF
 2200 FORMAT( I10)
      IF( N.EQ.1) GO TO 8
   13 READ(IIN,2110)ND,(MD(I),I=1,NTAP)
 2110 FORMAT(I10/20I1)
      ND1=NTAP-ND
    8 CONTINUE
      CALL GTAP(X,LX,ISAMPL,N,C,NTAP,IREF,G,LG,VARN,VARS,OMSE,B,KI,D,Z,K
     *T,ALFA,BB,MD,ND,ND1,P,Q,BETA)
      IF( N.GE.2.AND.ND.GT.0) GO TO 12
      GO TO 6
   12 KT=KT+1
      GO TO 13
    6 CONTINUE
      STOP
      END
```

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```
      SUBROUTINE GTAP(X,LX,ISAMPL,N,C,NTAP,IREF,G,LG,VARN,VARS,OMSE,B,KI
     *,D,Z,KT,ALFA,BB,MD,ND,ND1,P,Q,BETA)
C  THIS SUBROUTINE CALCULATES AND PRINTS OUT THE TAP COEFFICIENTS OF
C  THE TRANSVERSAL FILTER WHICH MINIMIZES THE MEAN SQUARE ERROR GIVEN
C  THE CHANNEL PULSE RESPONSE AND THE DESIRED CHANNEL-EQUALIZER PULSE
C  RESPONSE. THE TRANSVERSAL FILTER HAS A TAP SPACING WHICH MAY BE A
C  SUBMULTIPLE OF THE SYMBOL SPACING. THIS SUBROUTINE ALSO CALCULATES
C  AND PRINTS THE RESULTING MEAN SQUARE ERROR. IT ASSUMES THAT THE
C  INPUT SYMBOLS ARE UNCORRELATED WITH VARIANCE VARS AND THAT THE NOISE
C  IS WHITE WITH VARIANCE VARN. SUBROUTINE GELG IS USED TO SOLVE
C  SIMULTANEOUS EQUATIONS.
C
C  X        - INPUT ARRAY CONTAINING THE CHANNEL PUSE RESPONSE SAMPLES
C             (WITH THE SAME SAMPLE SPACING AS THE EQUALIZER TAP SPACING)
C  LX       - NUMBER OF SAMPLES IN X
C  ISAMPL   - SUBSCRIPT OF THE REFERENCE SAMPLE (1 TO LX)
C  N        - RATIO OF THE SYMBOL SPACING TO THE TAP SPACING (AN INTEGER)
C  C        - OUTPUT ARRAY OF TAP COEFFICIENTS
C  NTAP       NUMBER OF TAP COEFFICIENTS (MAXIMUM 50)
C  IREF     - DESIRED POSITION OF REFERENCE TAP
C  G        - ARRAY CONTAINING THE DESIRED PULSE RESPONSE SAMPLES (WITH
C             SPACING EQUAL TO THE SYMBOL INTERVAL)
C  LG       - NUMBER OF SAMPLES IN G
C  VARN     - VARIANCE OF THE NOISE SAMPLES
C  VARS     - VARIANCE OF THE INPUT SYMBOLS
C  OMSE     - MEAN SQUARE ERROR AFTER EQUALIZATION
C  B        - WORK ARRAY WITH NTAP*NTAP ELEMENTS
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION Z(NTAP,NTAP),WK(50),H(1500),F(50)
      DIMENSION X(LX),C(NTAP),G(50),B(NTAP,NTAP),NO(50),D(NTAP),ALFA(NTA
     *P),BB(NTAP,NTAP),BETA(ND1),P(ND1,ND1),Q(ND1,NTAP)
      DIMENSION MD(NTAP)
      DATA LETRC/2HC(/,IOUT/6/
C
      IF(KT.GT.1) GO TO 21
      IS=ISAMPL+IREF
      ISNT=IS+N*NTAP
C
C  FILL IN THE SQUARE ARRAY B STARTING WITH TERMS ABOVE THE DIAGONAL
      RVNVS=VARN/VARS
      DO 60 I=1,NTAP
      C(I)=0.0D0
      ALFA(I)=C(I)
       SUM=RVNVS
       DO 30 J=I,NTAP
         IBEG=ISNT-I+((J+1-ISNT)/N)*N
         IF(IBEG.GT.LX) GO TO 20
         IMJ=I-J
         DO 10 K=IBEG,LX,N
           KK=K+IMJ
           SUM=SUM+X(K)*X(KK)
   10    CONTINUE
   20    B(I,J)=SUM
```

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      B(I,J)=
      SUM=
 70     CONTINUE
 40   CONTINUE
C  FILL IN THE REST OF THE ARRAY B
      IF(NTAP.LE.1) GO TO 70
      DO 60 I=2,NTAP
        JEND=I-1
        DO 50 J=1,JEND
          B(I,J)=B(J,I)
      BB(I,J)=B(J,I)
 50     CONTINUE
 60   CONTINUE
C
   70 IJOB=1
      IB=NTAP
      IZ=NTAP
      IF(KT.GT.1) GO TO 71
      IF( N.EQ.1) GO TO 75
      IF( N.EQ.2) GO TO 85
   75 WRITE(IOUT,3100) ((B(I,J),I=1,NTAP),J=1,NTAP)
 3100 FORMAT(1H1,10X,30HCHANNEL AUTOCORRELATION MATRIX/
     *10X,15(2H--)/(1H0, 7(1X,F10.3)))
      GO TO 101
   85 WRITE(IOUT,3110) ((B(I,J),I=1,NTAP),J=1,NTAP)
 3110 FORMAT(1H1,10X,30HCHANNEL AUTOCORRELATION MATRIX/
     *10X,15(2H--)/(1H0,14(1X,F7.3)))
  101 WRITE(IOUT,3200) N
 3200 FORMAT(1H0,10X,31HTHE SYMBOL-TAP SPACING RATIO IS,2X,I10/
     *10X,22(2H--))
   71 CALL VCVTFS(B,NTAP,IB,H)
      CALL EIGRS(H,NTAP,IJOB,D,Z,IZ,WK,IER)
      WRITE(IOUT,3300) (D(I),I=1,NTAP)
 3300 FORMAT(1H0,3X,16HTHE EIGENVALUES:/
     *5(E12.4))
C  FILL IN THE RIGHT HAND SIDE VECTOR C (X CROSS-CORRELATED WITH G)
      DO 90 J=1,LG
        ISJ=IS+(J-1)*N
        IBEG=MAX0(1,ISJ-LX)
        IF(IBEG.GT.NTAP) GO TO 100
        IEND=MIN0(NTAP,ISJ-1)
        DO 80 I=IBEG,IEND
          KK=ISJ-I
          C(I)=C(I)+G(J)*X(KK)
      ALFA(I)=C(I)
 80     CONTINUE
 90   CONTINUE
 100  CALL GELG(C,B,NTAP,1,1.0D-7,IER)
      IF(IER.NE.0) WRITE(IOUT,4000) IER
      DO 110 I=1,NTAP
 110  ND(I)=I-IREF
      WRITE(IOUT,2000) (LETRC,ND(I),C(I),I=1,NTAP)
      MCTR=(IS-2)/N+1
      MEND=(LX+NTAP-IS)/N+MCTR
      SUMSQ=0.0
      DO 130 M=1, MEND
      AUX=0.0
      K=M-MCTR
      ISK=IS+K*N
      JBEG=MAX0(1,ISK-LX)
      JEND=MIN0( NTAP,ISK-1)
      DO 120 J=JBEG,JEND
      KK=ISK-J
      AUX=AUX+C(J)*X(KK)
 120  CONTINUE
      KG=K+1
      IF(KG.GE.1.AND.KG.LE.LG) AUX=AUX-G(KG)
      SUMSQ=SUMSQ+AUX*AUX
 130  CONTINUE
      IF(KG.GE.LG) GO TO 150
      IBEG=KG+1
      DO 140 KG=IBEG,LG
      SUMSQ=SUMSQ+G(KG)**2
 140  CONTINUE
 150  SUMC2=0.0
      DO 160    I=1,NTAP
 160  SUMC2=SUMC2+C(I)**2
      OMSE=VARS*SUMSQ+VARN*SUMC2
      WRITE(IOUT,3000) OMSE,NTAP,IREF
      IF(N.NE.2) GO TO 2001
   21 I1=0
      DO711 I=1,NTAP
      IF(MD(I).EQ.0) GO TO711
      I1=I1+1
      BETA(I1)=ALFA(I)
```

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```
      DO 710 J=1,NTAP
      O(I1,J)=BB(I,J)
  710 CONTINUE
  711 CONTINUE
      J1=0
      DO713 J=1,NTAP
      IF(MD(J).EQ.0) GO TO713
      J1=J1+1
      ND1=NTAP-ND
      DO712 I=1,ND1
      P(I,J1)=O(I,J)
  712 CONTINUE
  713 CONTINUE
      ID=ND1
      IZ=ND1
      CALL VCVTFS(P,ND1,ID,H)
      CALL EIGRS(H,ND1,1JOB,D,Z,IZ,WK,IER)
      WRITE(IOUT,3000) (D(I),I=1,ND1)
      CALL GELG(BETA,P,ND1,1,1.0D-7,IER)
      DO 209 I=1,NTAP
  209 C(I)=0.0D0
      J=0
      DO 210 I=1,NTAP
        IF(MD(I).EQ.1) GO TO 211
        GO TO 210
  211   J=J+1
        C(I)=BETA(J)
  210 CONTINUE
      WRITE(IOUT,5001) (MD(I),I=1,NTAP)
 5001 FORMAT(1H1,3X,21HHYBRID TYPE EQUALIZER/3X,17(1H-)/
     *1H0,3X,14HT/2 TAPS USED:,20I2)
      WRITE(IOUT,1900) (LETRC,NO(I),    C(I),I=1,NTAP)
 1900 FORMAT(1H0,42X,33HTAP GAINS OF THE HYBRID EQUALIZER/
     *(1H0,4X,5(5X,A2,I3,3H) =,1PE11.4)))
      MCTR=(IS-2)/N+1
      MEND=(LX+NTAP-IS)/N+MCTR
      SUMSQ=0.0
      DO 1301M=1,MEND
        AUX=0.0
        K=M-MCTR
        ISK=IS+K*N
        JBEG=MAX0(1,ISK-LX)
        JEND=MIN0(NTAP,ISK-1)
        DO 1201J=JBEG,JEND
          KK=ISK-J
          AUX=AUX+C(J)*X(KK)
 1201   CONTINUE
        KG=K+1
        IF(KG.GE.1 .AND. KG.LE.LG) AUX=AUX-G(KG)
        SUMSQ=SUMSQ+AUX*AUX
 1301 CONTINUE
C
      IF(KG.GE.LG) GO TO 1501
      IBEG=KG+1
      DO 1401KG=IBEG,LG
        SUMSQ=SUMSQ+G(KG)**2
 1401 CONTINUE
C
C  FIND THE SQUARED DISTORTION DUE TO NOISE
 1501 SUMC2=0.0
      DO 1601I=1,NTAP
 1601 SUMC2=SUMC2+C(I)**2
C
      DMSE=VARS*SUMSQ+VARN*SUMC2
      WRITE(IOUT,3000) DMSE,ND1,IREF
 2001 CONTINUE
      RETURN
C
 2000 FORMAT(1H0,45X,33HLEAST MEAN SQUARE ERROR EQUALIZER/
     *         1H0,42X,38HTAP GAINS OF THE TRANSVERSAL EQUALIZER/
     *       (1H ,4X,5(5X,A2,I3,3H) =,1PE11.4)))
 3000 FORMAT(1H0,40X,19HMEAN SQUARE ERROR =,1PE11.4/
     *         1H ,40X,16HNUMBER OF TAPS =,I3,10X,15HREFERENCE TAP =,I3)
 4000 FORMAT(1H0,25X,35H***ERROR IN GTAP-GELG, ERROR CODE =,I3,3H***)
      END
```

```
      SUBROUTINE GELG(R,A,M,N,EPS,IER)
C                                                      18/11/75 P. KABAL
C
C     THIS SUBROUTINE SOLVES A SYSTEM OF SIMULTANEOUS EQUATIONS USING
C     GAUSSIAN ELIMINATION WITH COMPLETE PIVOTING. THE INPUT MATRICES ARE
C     STORED COLUMNWISE IN SUCCESSIVE LOCATIONS. ON RETURN THE SOLUTION IS
C     STORED COLUMNWISE ALSO. THE PROCEDURE GIVES RESULTS IF THE NUMBER
C     OF EQUATIONS M IS GREATER THAN ZERO AND THE PIVOT ELEMENTS AT ALL
C     ELIMINATION STEPS ARE DIFFERENT FROM ZERO. A WARNING (IER=K), IF
C     GIVEN, INDICATES A POSSIBLE LOSS OF SIGNIFICANCE. IN THE CASE OF A
C     WELL SCALED MATRIX A AND AN APPROPRIATE TOLERANCE EPS, IER=K MAY BE
C     INTERPRETED TO MEAN THAT MATRIX A HAS THE RANK K.
C
C     R       - THE M BY N MATRIX OF RIGHT HAND SIDE VECTORS. EACH VECTOR
C               IS A COLUMN OF R. ON RETURN R CONTAINS THE SOLUTION OF THE
C               EQUATIONS.
C     A       - THE M BY M COEFFICIENT MATRIX (DESTROYED).
C     M       - THE NUMBER OF EQUATIONS IN THE SYSTEM.
C     N       - THE NUMBER OF VECTORS IN R.
C     EPS     - AN INPUT PARAMETER WHICH IS USED AS A RELATIVE TOLERANCE IN
C               TESTING FOR LOSS OF SIGNIFICANCE.
C     IER     - RESULTING ERROR CODE.
C               IER= 0 - NO ERROR.
C               IER=-1 - NO RESULT BECAUSE M IS LESS THAN 1 OR A PIVOT
C                        ELEMENT AT ANY ELIMINATION STEP IS EQUAL TO 0.
C               IER= K - WARNING OF A POSSIBLE LOSS OF SIGNIFICANCE AT
C                        ELIMINATION STEP K+1 (THE PIVOT ELEMENT WAS LESS
C                        THAN OR EQUAL TO THE RELATIVE TOLERANCE EPS TIMES
C                        THE GREATEST ELEMENT (ABSOLUTE VALUE) OF MATRIX A).
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(M,M),R(M,1)
C
      IF(M.LE.0) GO TO 260
      IER=0
C
C     FIND THE LARGEST ELEMENT IN MATRIX A
      APIV=0.0
      DO 110 I=1,M
        DO 100 J=1,M
      TEMP=DABS(A(I,J))
          IF(APIV.GE.TEMP) GO TO 100
          APIV=TEMP
          IROW=I
          JCOL=J
100     CONTINUE
110   CONTINUE
C
C     A(IROW,JCOL) IS THE PIVOT ELEMENT
C     APIV CONTAINS THE ABSOLUTE VALUE OF A(IROW,JCOL)
      TOL=EPS*APIV
C
C
C     ELIMINATION LOOP
```

```
      DO 200 K=1,M
C
C   TEST ON SINGULARITY
      IF(APIV.LE.0.0) GO TO 260
      IF(IER.EQ.0 .AND. APIV.LE.TOL) IER=K-1
C
      PIVI=1.0/A(IROW,JCOL)
C
C   ROW REDUCTION AND ROW INTERCHANGE IN THE RIGHT HAND SIDE MATRIX R
      DO 120 J=1,N
        TEMP=PIVI*R(IROW,J)
        R(IROW,J)=R(K,J)
        R(K,J)=TEMP
120   CONTINUE
C
      IF(K.GE.M) GO TO 210
      IF(JCOL.LE.K) GO TO 140
C
C   COLUMN INTERCHANGE IN MATRIX A
      DO 130 I=K,M
        TEMP=A(I,K)
        A(I,K)=A(I,JCOL)
        A(I,JCOL)=TEMP
130   CONTINUE
C
C   ROW INTERCHANGE AND PIVOT ROW REDUCTION IN MATRIX A
140   DO 150 J=K,M
        TEMP=PIVI*A(IROW,J)
        A(IROW,J)=A(K,J)
        A(K,J)=TEMP
150   CONTINUE
C
C   SAVE COLUMN INTERCHANGE INFORMATION ON THE DIAGONAL OF MATRIX A
      A(K,K)=JCOL
C
C   ELEMENT REDUCTION IN MATRICES A AND R AND NEXT PIVOT SEARCH
      APIV=0.0
      KP1=K+1
      DO 190 I=KP1,M
        PIVN=-A(I,K)
C
        DO 170 J=KP1,M
          A(I,J)=A(I,J)+PIVN*A(K,J)
      TEMP=DABS(A(I,J))
          IF(APIV.GE.TEMP) GO TO 170
          APIV=TEMP
          IROW=I
          JCOL=J
170     CONTINUE
C
        DO 180 J=1,N
180     R(I,J)=R(I,J)+PIVN*R(K,J)
C
190   CONTINUE
```

```
200     CONTINUE
C
C
210     IF(M.EQ.1) GO TO 250
C
C  BACK SUBSTITUTION AND BACK EXCHANGE
        II=M
        DO 240 I=2,M
          III=II
          II=II-1
          JCOL=A(MI,II)+0.5
          DO 230 J=1,N
            TEMP=R(II,J)
C
            DO 220 L=III,M
220         TEMP=TEMP-A(II,L)*R(L,J)
C
            R(II,J)=R(JCOL,J)
            R(JCOL,J)=TEMP
230       CONTINUE
240     CONTINUE
C
250     RETURN
C
C  ERROR RETURN
260     IER=-1
        RETURN
        END
```