A linear isoperimetric function for genus 0 diagrams

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Dedication

In memory of Miguel Alcubierre Méndez

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Contribution of Authors

Unless otherwise stated, the results presented in this thesis constitute original joint work with my supervisor, Professor Daniel T Wise.

Abrégé

Nous démontrons l'inégalité isopérimétrique linéaire pour les diagrammes combinatoriques de genus 0. Ainsi, nous généralisons l'inégalité isopérimétrique pour les diagrammes de disques et les diagrammes annulaires.

Abstract

We prove a linear isoperimetric inequality for combinatorial genus 0 diagrams, thus generalising the classical isoperimetric inequalities for disc diagrams and annular diagrams.

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1. Introduction

One of the main characterizations of word-hyperbolic groups is that they are the groups with a linear isoperimetric function. That is, for a compact 2-complex X, the hyperbolicity of $\pi_1 X$ is equivalent to the existence of a linear isoperimetric function for disc diagrams $D \to X$. This means that there is a constant K such that if there exists a disc diagram $D \to X$, then there exists a disc diagram $D' \to X$ with $\partial_{\mathsf{p}} D' = \partial_{\mathsf{p}} D$, and with $\mathsf{Area}(D') \leq K|\partial_{\mathsf{p}} D'|$. It is likewise known that hyperbolic groups have a linear annular isoperimetric function. The goal of this paper is to generalize the linear isoperimetric function to arbitrary genus 0 diagrams. The reader should have in mind "surface" instead of "diagram" below. We clarify the terminology in Definition 2.2.

Theorem 1.1. *Let X be a compact 2-complex with $\pi_1 X$ hyperbolic. For each n, there exists k_n such that the following holds:*

Let $f : S \to X$ be a combinatorial map from a genus 0 diagram to X with $\partial S = \sqcup_{i=1}^{n} C_i$ and where each $C_i \to X$ maps to an infinite order element of $\pi_1 X$.

Then there exists a genus 0 diagram S' and a combinatorial map $f' : S' \to X$ such that:

(1) $\partial S' = \partial S$,
(2) *the restriction of f' to $\partial S'$ equals the restriction of f to ∂S,*
(3) *the number of 2-cells in S' is bounded by $k_n \sum_{i=1}^{n} |C_i|$.*

Gromov explained in [**Gro87**] that if n closed local geodesics $C_i \to M$ in a compact negatively curved manifold M form the boundary circles of a surface S mapping into M, then one can rechoose S so that $\mathsf{Area}(S) = 2(n-2)\pi$. Gromov's argument generalizes immediately to a compact negatively curved space X with π replaced by an upperbound on the area of an ideal triangle in $\widetilde{X}$. We revisit his argument, which builds a surface from ideal triangles in Proposition 1.2. A technical barrier that prevents his argument from applying to general hyperbolic groups, is that the sides of ideal triangles in a δ-hyperbolic complex do not necessarily asymptotically converge - and from a combinatorial viewpoint, ideal triangles do not bound diagrams with finite area.

We nevertheless mimic Gromov's ideal triangle decomposition construction but are then diverted into some interesting technicalities, as the new surface formed from ideal triangles is not a compact surface with boundary. We ultimately explain how to use the decomposition to estimate the combinatorial area and arrive at our goal of a linear isoperimetric function for genus 0 surfaces. At this point we assume, as in Gromov's original version, that the boundary circles of S are essential in X, and moreover, represent conjugacy classes of elements having infinite order in $\pi_1 X$. We add the hypothesis that the surface is planar but expect that this hypothesis can be dropped with a bit of extra care in the argument.

1.a. Gromov's trick. In this section we recall Gromov's use of an ideal triangle decomposition to bound the area of a surface [**Gro87**, p.235]. Thurston originally used such ideal triangle decompositions of hyperbolic surfaces with boundary to build examples, and also glued ideal polyhedra together to form hyperbolic 3-manifolds [**Thu82**].

PROPOSITION 1.2. *Let M be a compact negatively curved manifold with boundary. Let S be a compact planar surface with $\partial S = \sqcup_{i=1}^n C_i$ and $n \geq 3$. There exists a constant K with the following property:*

Let $S \to M$ be a map such that each $C_i \to M$ is essential. Then $S \to M$ can be homotoped to another mapped surface $S' \to M$ such that:

(1) $\mathrm{Int}(S')$ *is built from the union of* $2(n-2)$ *ideal triangles in* $\widetilde{M}$*,*
(2) $\mathsf{Area}(S) \leq K(|\partial S| - \chi(S))$*,*
(3) $\partial S' = \sqcup_{i=1}^n C_i'$ *and each* $C_i' \to M$ *is a local geodesic.*

SKETCH. Build S' in two parts: first homotope each C_i in ∂S to the unique closed geodesic C_i' in its homotopy class to obtain a "collar" composed of cylinders which can be chosen so that the area of the collar is bounded above by a linear function of $|\partial S|$ using the isoperimetric inequality for annuli.

We thus obtain a surface with geodesic boundary $\partial S'$, and this surface can be homotoped, keeping the boundary fixed, to a surface whose area is $\leq K|\chi(S')|$ by decomposing it into ideal triangles whose sides, in the universal cover, lift to lines that are asymptotic to geodesics covering boundary components. These geodesics project to geodesics in M, so $\partial S'$ can be filled in with m ideal triangles, where the number m depends on $\chi(S') = \chi(S)$. □

REMARK 1.3. The explanation in Proposition 1.2 functions perfectly well for a space that is negatively curved in the sense that it is locally $\mathrm{CAT}(\kappa)$ for some $\kappa < 0$. However, there are two important points to consider. Firstly, there are foundational issues relating the area of the constructed surface to the combinatorial area of a diagram, which is what we shall pursue. (See [**Bri02, BT02**] for work relating classical isoperimetric area to combinatorial area.) Secondly, we are interested in providing a general linear isoperimetric function in the case where M is nonpositively curved but $\pi_1 M$ is word-hyperbolic, and more generally, where M does not even have a locally CAT(0) metric. A substantial technical obstacle is that outside the negatively curved case, there could be flat strips, and hence ideal triangles might not behave in a fashion allowing us to produce a compact surface.

1.b. Related results. There is a stream of research pursuing a homological alternative to Theorem 1.1.

Hyperbolic groups satisfy a *weak* (sometimes called *homological*) linear isoperimetric inequality, in the sense that a k-cycle that bounds a $(k+1)$-chain bounds such a chain whose "area" is linear on the "length" of the k-cycle. This notion was first discussed by Gersten in [**Ger96**]. The existence of a weak linear isoperimetric inequality for hyperbolic groups was proven for $k = 1$ in [**Ger98**] and extended to all $k \geq 1$ by Mineyev [**Min00**] (for homology with $\mathbb{Q}$ and $\mathbb{R}$ coefficients) and by Lang [**Lan00**] (for $\mathbb{Z}$ coefficients). In the $k = 1$ case, which is the case most closely related to this paper, the above result says that a collection of cellular loops that bound a surface, bounds such a surface (possibly with very large genus) whose area is linear on the length of the loops.

Our result is in a sense stronger, since we show that given a genus 0 diagram with prescribed boundary circles, there exists a genus 0 diagram having exactly those circles as boundary components –so, in particular, of the same homotopy type as the original diagram – and whose area is linear on the length of the circles.

Although the homological and ordinary isoperimetric functions are both linear for hyperbolic groups, they can be inequivalent for arbitrary finitely presented groups [**ABDY13**]. A characterization of relative hyperbolicity using a weak isoperimetric function was given in [**MP16**].

In a very different direction, there are results controlling the areas of cylinders associated to simultaneous conjugations in hyperbolic groups [**BH05**].

2. Classical statements

We recall some definitions and classical results that will be needed throughout the paper:

2.a. Slim triangles.

Definition 2.1. Let X be a geodesic metric space and let p_1, p_2, p_3 be any three points in X. A *geodesic triangle* is the union of three geodesic segments $[p_1, p_2], [p_2, p_3], [p_3, p_1]$ with endpoints p_1, p_2, p_3. A geodesic triangle is δ*-slim* if for each $i \in \{1, 2, 3\}$ the segment $[p_i, p_{i+1}]$ lies in the union of the δ-neighbourhoods of $[p_{i+1}, p_{i+2}]$ and $[p_{i+2}, p_i]$ (mod 3). The space X is δ*-hyperbolic* or *Gromov-hyperbolic* if there exists a $\delta \geq 0$ such that every geodesic triangle in X is δ-slim. A group G is *word-hyperbolic* if for some finite generating set S, the Cayley graph $\Upsilon(G, S)$ is δ-hyperbolic and some δ.

2.b. Diagrams.

DEFINITION 2.2. A *genus 0 diagram* S is a compact combinatorial 2-complex with the homotopy type of a genus 0 surface with boundary and a chosen embedding $S \subset \mathbb{S}^2$. The genus 0 diagram has n *boundary paths* $P_1, \ldots, P_n$ which correspond to the attaching maps of the n 2-cells that can be added to S to form the sphere. The *boundary* of S is the disjoint union $\partial S = \sqcup_{i=1}^n C_i$ where each C_i is homeomorphic to a circle and each P_i maps to C_i. A genus 0 diagram is *singular* if it is not homeomorphic to a surface (e.g. S might have cut-vertices). We emphasize that a genus 0 diagram is not necessarily connected. A *disc diagram* is the $n = 1$ case, and an *annular diagram* is the $n = 2$ case when S is connected.

A *genus 0 diagram in a complex* X is a combinatorial map $S \to X$ where S is a genus 0 diagram.

When D is a disc diagram, we use the notation $\partial_{\mathsf{p}} D$ for the *boundary path* of D, which is the path travelling around D that corresponds to the attaching map of the 2-cell $\mathbb{S}^2 - D$.

LEMMA 2.3 (Van Kampen). *Let X be a combinatorial 2-complex. Let $P \to X^1$ be a closed combinatorial path. Then P is nullhomotopic if and only if there exists a disc diagram D in X with $\partial_{\mathsf{p}} D \cong P$ so that there is a commutative diagram:*

$$\begin{array}{ccc} \partial_{\mathsf{p}} D & \longrightarrow & D \\ \downarrow & & \downarrow \\ P & \longrightarrow & X \end{array}$$

Let X be a compact 2-complex whose universal cover $\widetilde{X}$ has 1-skeleton that is δ-hyperbolic for some $\delta \geq 0$. Proofs of the following results can be found in [**BH99**, p.417 and p.454]:

THEOREM 2.4 (Disc isoperimetry). *There is a constant $N = N(\delta)$ such that for every null-homotopic closed combinatorial path $\sigma \to X$, there exists a disc diagram $D \to X$ with $\partial D = \sigma$ and $\mathsf{Area}(D) \leq N|\sigma|$.*

PROPOSITION 2.5 (Annular isoperimetry). *There is a constant $M = M(\delta)$ such that if two essential closed combinatorial paths σ and σ' are homotopic in X, there exists an annular diagram $A \to X$ with $\partial A = \sigma \cup \sigma'$ and $\mathsf{Area}(A) \leq M \cdot max\{|\sigma|, |\sigma'|\}$.*

2.c. Exponential divergence. Another property that characterises Gromov-hyperbolicity is exponential divergence.

THEOREM 2.6. *Let X be a δ-hyperbolic geodesic metric space, then there exists an exponential function $e : \mathbb{N} \to \mathbb{R}$ with the following property.*

For all $R, r \in \mathbb{N}$, all $x \in X$, and all geodesics $c_1 : [0, a_1] \to X, c_2 : [0, a_2] \to X$ with $c_1(0) = c_2(0) = x$, if $R + r \leq min\{a_1, a_2\}$ and $\mathsf{d}(c_1(R), c_2(R)) > e(0)$, then any path connecting $c_1(R + r)$ to $c_2(R + r)$ outside the ball $B(x, R + r)$ must have length at least $e(r)$.

DEFINITION 2.7. An (a, b)-*quasigeodesic* (where $a > 0$ and $b \geq 0$) is a function $\varphi : \mathbb{R} \to X$ satisfying the following for all $s, t \in \mathbb{R}$:

$$\frac{1}{a}|s - t| - b \ \leq \ d(\varphi(t), \varphi(s)) \ \leq \ a|s - t| + b.$$

THEOREM 2.8. *Let p and q be points of a δ-hyperbolic geodesic metric space X. For each $a, b > 0$, there exists a constant $L = L(a, b)$ such that the following holds: If σ and σ' are (a, b)-quasi geodesics with the same endpoints, then $\sigma \in N_L(\sigma')$.*

A path $\eta : [0, \infty) \to X$ is a *ray*. It is *geodesic* if for each $t \geq 0$, the subpath $\eta|_{[0,t]}$ is a geodesic. Two geodesic rays η, η' are equivalent if there is a constant $\mathcal{L}$ such that $\mathsf{d}(\eta(t), \eta'(t)) \leq \mathcal{L}$ for all t. The *Gromov boundary* of X is the set $\partial X = \{[\eta] | \eta$ is a geodesic ray in $X\}$.

We employ the following consequence of Theorem 2.6:

COROLLARY 2.9. *Let X be δ-hyperbolic, and let γ, γ' be geodesic rays representing different points on ∂X, and for which there exists $\mathcal{O}$ with $\mathsf{d}(\gamma(0), \gamma'(0)) \leq \mathcal{O}$. Then for each $T \geq 0$ there is an $R = R(T) \geq 0$ with $\mathsf{d}(\gamma(r), \gamma'(r)) \geq T$ for all $r \geq R$.*

PROOF. Let γ, γ' be geodesic rays with $d(\gamma(0), \gamma'(0)) \leq \mathcal{O}$ for some $\mathcal{O} > 0$ and representing different points on ∂X, and let s be a geodesic path of length $\leq \mathcal{O}$ with endpoints $\gamma(0), \gamma'(0)$. Then the path $c = s\gamma'$ is a $(1, \mathcal{O})$-quasigeodesic ray with $c(0) = \gamma(0)$, by Theorem 2.8 there exists a constant L and a geodesic ray γ'' with $c \subset N_L(\gamma'')$ and $\gamma''(0) = \gamma(0)$, and by Theorem 2.6 there exists an exponential function e with $\mathsf{d}(\gamma(t), \gamma''(t)) \geq e(t)$, but $c \subset N_L(\gamma'')$, so c – and hence γ'– is also at distance $e(t)$ from $\gamma(t)$. ☐

Later we will also make use of the following local-to-global criterion for quasigeodesics. A proof can be found in [**HW15**] in a slightly different setting.

THEOREM 2.10. *Let X be δ-hyperbolic. Consider a piecewise geodesic path $\sigma_1\lambda_1\sigma_2\lambda_2 \cdots \lambda_k\sigma_{k+1}$. For each $L > 0$ there exists $\alpha, \beta > 0$ such that $\sigma_1\lambda_1 \cdots \lambda_k\sigma_{k+1}$ is a (α, β)-quasigeodesic provided that:*

(1) $\frac{1}{2}|\lambda_i| \geq 6(L + \delta)$ *for each* i.
(2) $diam\big(\lambda_i \cap \mathcal{N}_{3\delta}(\lambda_{i+1})\big) \leq L$ *for each* i.
(3) $diam\big(\lambda_i \cap \mathcal{N}_{3\delta}(\sigma_{i+1})\big) \leq L$ *for each* i.
(4) $diam\big(\sigma_i \cap \mathcal{N}_{3\delta}(\lambda_i)\big) \leq L$ *for each* i.

3. Linear isoperimetric function

3.a. Genus 0 generalization. The goal of this subsection is to prove the following statement:

THEOREM 3.1. *Let X be a compact 2-complex such that the 1-skeleton of $\widetilde{X}$ is δ-hyperbolic. For each $n \geq 1$ there is a constant k_n such that the following holds: Let $S \to X$ be a genus 0 diagram in X with boundary circles $C_1, \ldots, C_n$ and suppose each C_i is either null-homotopic or represents an infinite-order element of $\pi_1 X$. There exists a genus 0 diagram $S' \to X$ with $\partial S = \partial S'$ and $\mathsf{Area}(S') \leq k_n|\partial S'|$.*

PROOF. **Organization of the proof:** We are most interested in the case where each C_i is essential, in which case we obtain a connected S' if we start with a connected S. In the interest of better organising the proof for this case, we first handle the following situations:

(1) The case $n = 1$ is Theorem 2.4.
(2) If S has several connected components and every boundary circle in each connected component is essential, then we can prove the Theorem separately for each of the components, and if some component has two boundary circles then Proposition 2.5 applies.
(3) If S has a component S_o with a null-homotopic boundary circle C_n, then by Theorem 2.4 there is a disc diagram $D \to X$ with $\partial D = C_n$ and $\mathsf{Area}(D) \leq k_1|C_n|$. Letting $T = D \cup_{C_n} S$, by induction there exists T' with $\partial T' = \partial T$ and $\mathsf{Area}(T') \leq k_{n-1}|\partial T'|$. Hence letting $S' = D \sqcup T'$ the result holds provided $k_n \geq \max\{k_1, k_{n-1}\}$.

It now suffices to proceed with the proof assuming: S is connected, $n \geq 3$, and S has no null-homotopic boundary circles. This case is handled in Corollary 4.22 under the assumption that each boundary circle maps to an infinite order element in X. □

REMARK 3.2. In some scenarios it is possible to inductively obtain a low area diagram for S by first producing a closely related diagram with fewer boundary circles and then gluing to obtain a low area diagram S'' with $\partial S = \partial S''$.

Suppose S has a path $\sigma \to S$ whose projection $\sigma \to X$ is path-homotopic to a path $\sigma' \to X$ with $|\sigma'| < 2\delta$ that joins distinct boundary circles C_{n-1}, C_n (after reindexing).

Homotope S to a diagram S' containing σ' and cut S' along σ' to obtain a genus 0 diagram R with $\partial R = \{C_1 \sqcup \cdots \sqcup C_{n-2} \sqcup C''_{n-1}\}$ such that the cycle around C''_{n-1} is of the form $c_{n-1}\sigma' c_n {\sigma'}^{-1}$ (where c_i represents a closed path around C_i at an endpoint of σ') as in Figure 1. By induction there is a genus 0 diagram

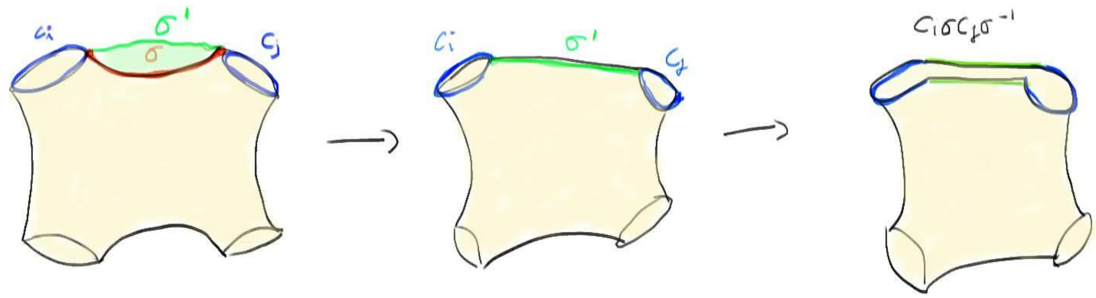

FIGURE 1. Simplification of S described in Remark 3.2.

R' with $\mathsf{Area}(R') \leq k_{n-1}|\partial R'|$, and S'' is obtained from R' by identifying the two copies of σ' in $C''_{n-1} \subset \partial R'$.

Such a situation arises if some $\gamma \in \cup\Delta_i$ is periodic (see Construction 3.10). Recall that a bi-infinite embedded path $\gamma \subset \widetilde{X}$ is *periodic* if Stabiliser$(\gamma) \neq 1$.

3.b. Ideal triangles. The aim of this section is to prove Lemma 3.7, which is a straightforward generalisation to ideal geodesic triangles of the slim triangle property. To this end, we prove first a few technical Lemmas.

For a geodesic or geodesic ray η we will frequently use the notation $\eta_t = \eta(t)$.

LEMMA 3.3. *Let η, η' be geodesic rays such that each one lies in a finite neighbourhood of the other. Then there exist $q, r \geq 0$ such that $\mathsf{d}(\eta_q, \eta'_r) \leq 2\delta$. Moreover, q, r can be chosen arbitrarily large.*

PROOF. Let $F > 0$ with $\eta \subset N_F(\eta'), \eta' \subset N_F(\eta)$, and let η'_r be such that $\mathsf{d}(\eta_0, \eta'_r) \leq F$. Choose η_p at distance $> F + 2\delta$ from η_0 and η'_r. The rectangle in Figure 3 shows that a point on $\overline{\eta_0\eta_p}$ at distance more than $F + 2\delta$ from both $\overline{\eta_0\eta'_r}$ and $\overline{\eta_p\eta'_q}$ must be within 2δ of a point on $\overline{\eta'_r\eta'_q}$. ☐

LEMMA 3.4. *Let η, η', η'' be as in the statement of Lemma 3.7. Then there exist points $\eta_0, \eta'_0, \eta''_0$ at distance $\leq 7\delta$.*

PROOF. By Lemma 3.3, there exist q, q', r, r', s, s' such that

$$\begin{aligned} \mathsf{d}\big(\eta_r, \eta'_{r'}\big) &\leq 2\delta \\ \mathsf{d}\big(\eta_q, \eta''_{q'}\big) &\leq 2\delta \\ \mathsf{d}\big(\eta'_s, \eta''_{s'}\big) &\leq 2\delta \end{aligned}$$

Consider the geodesic hexagon with sides $\overline{\eta_q\eta''_{q'}}, \overline{\eta''_{q'}\eta''_s}, \overline{\eta''_s\eta'_{s'}}, \overline{\eta'_{s'}\eta'_{r'}}, \overline{\eta'_{r'}\eta_r}, \overline{\eta_r\eta_q}$ and subdivide it by taking geodesics β, β', β'' as illustrated in Figure 2. Reparametrising if necessary, let $\eta_0 \in \overline{\eta_q\eta_r}$ be the vertex of an intriangle corresponding to the triangle with sides $\beta\overline{\eta_q\eta_r}\beta'$. The point η_0 is at distance $\leq \delta$ from a point $x \in \beta$, and x is at most 3δ away from a point $\eta''_0 \in \eta''$. Similarly, η_0 is at distance $\leq \delta$ from a point $y \in \beta'$, and y is at most 4δ away from a point $\eta'_0 \in \eta'$ (the various possibilities are sketched in Figure 2). □

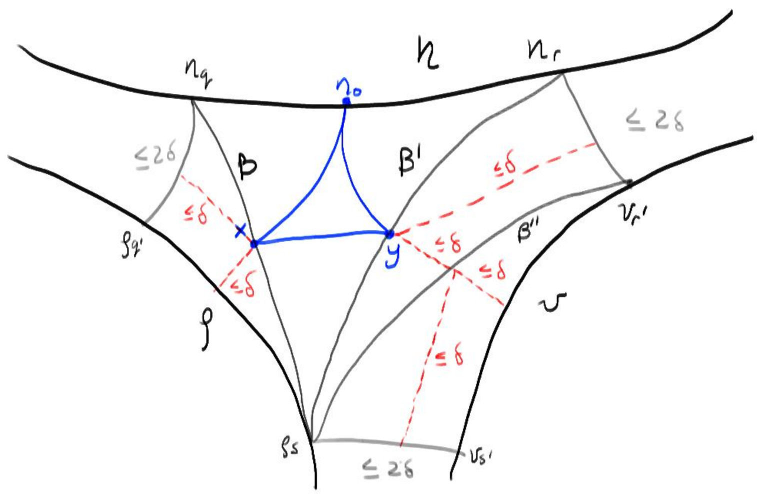

FIGURE 2. Configuration involved in the proof of Lemma 3.4. As the dotted lines indicate, either y is 2δ away from η', or y is 2δ away from $\overline{\eta''_s\eta'_{s'}}$, in which case y is 4δ away from η', or y is 2δ away from $\overline{\eta'_{r'}\eta_r}$, in which case y is 4δ away from η'.

LEMMA 3.5. *Let η, η' be geodesic rays lying in finite neighbourhoods of each other in a δ-hyperbolic geodesic metric space. If $\mathsf{d}(\eta_0, \eta'_0) \leq K$ then $\mathsf{d}(\eta_t, \eta'_t) \leq K'(\delta) = K'$ for all $t \geq 0$.*

PROOF. By Lemma 3.3, there are $q, r > m$ with $\mathsf{d}(\eta_q, \eta'_r) \leq 2\delta$. Let Q be the quadrilateral with sides $\overline{\eta_0\eta'_0}$, $\overline{\eta_q\eta'_r}$, $\overline{\eta_0\eta_q}$ and $\overline{\eta'_0\eta'_r}$. We claim that for any n, m with $\eta_n, \eta_m \in \overline{\eta_0\eta_q}$, $d(\eta_n, \eta_m) \leq K + 2\delta$, similarly, $d(\eta'_n, \eta'_m) \leq K + 2\delta$ for $\eta'_n, \eta'_m \in \overline{\eta'_0\eta'_r}$. Indeed: $m \leq K + |\eta_n| + 2\delta = K + n + 2\delta$, so $m - n \leq K + 2\delta$ and $n \leq K + |\eta_m| + 2\delta = K + m + 2\delta$, so $|n - m| \leq K + 2\delta$. Hence $|m - n| = d(\eta_n, \eta_m) \leq K + 2\delta$ and similarly for $d(\eta'_n, \eta'_m)$. Now we will bound $\mathsf{d}(\eta_m, \eta'_m)$. There are 2 cases:

(1) If there exists n with $d(\eta_m, \eta'_n) \leq 2\delta$, then $\mathsf{d}(\eta_m, \eta'_m) \leq \mathsf{d}(\eta_m, \eta'_n) + \mathsf{d}(\eta'_n, \eta'_m) \leq 2\delta + K + 2\delta$. The same holds by a symmetric argument if there exists n with $d(\eta'_m, \eta_n) \leq 2\delta$.
(2) Otherwise, both η_m and η'_m are within distance 2δ of $\overline{\eta_0\eta'_0}$. Since $|\overline{\eta_0\eta'_0}| = K$, it follows that $\mathsf{d}(\eta_m, \eta'_m) \leq 2\delta + K + 2\delta$.

Either way, $d(\eta, \eta'_m) \leq K + 4\delta := K'$. See Figure 3 for a sketch of the various cases. □

REMARK 3.6. Better constants can be obtained for Lemmas 3.3, 3.4 and 3.5 by a more thorough, albeit not much harder, case-by-case analysis. We opted to skip such an analysis, since it constitutes an unnecessary distraction from the more substantial arguments in the rest of the paper.

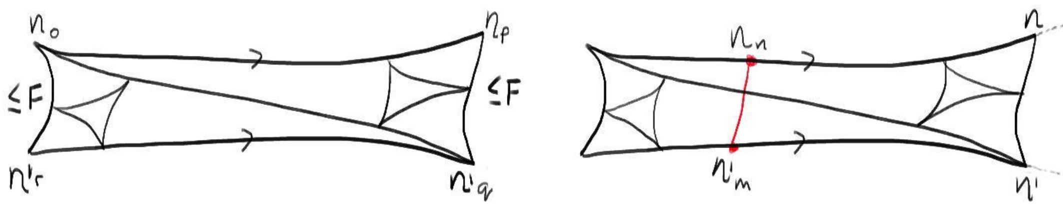

FIGURE 3. Rectangles used in the proofs of Lemma 3.5 and Lemma 3.3.

LEMMA 3.7 (δ-ideal triangles). *Let $\widetilde{X}$ be a δ-hyperbolic metric space. Let η, η', η'' be bi-infinite geodesics in $\widetilde{X}$, that form an ideal triangle in the sense that there are three points $\{a, b, c\} \subset \partial X$ with $\partial\eta = \{b, c\}$ and $\partial\eta' = \{a, c\}$ and $\partial\eta'' = \{a, b\}$.*

There exist isometric reparametrizations of η, η', and η'' with:

$$\begin{aligned} \mathsf{d}\big(\eta_t, \eta'_t\big) &\leq K' \quad \textit{for} \quad t \geq 0 \\ \mathsf{d}\big(\eta_{-t}, \eta''_{-t}\big) &\leq K' \quad \textit{for} \quad t \geq 0 \\ \mathsf{d}\big(\eta'_{-t}, \eta''_t\big) &\leq K' \quad \textit{for} \quad t \geq 0 \end{aligned}$$

DEFINITION 3.8. Let $\{\eta_0, \eta'_0, \eta''_0\}$ be as in Lemma 3.7, an *intriangle* is the set Λ determined by three geodesics arcs s, s', s'' –called *sides*– with endpoints $\{\eta_0, \eta'_0, \eta''_0\}$. See Figure 4.

PROOF OF LEMMA 3.7. By Lemma 3.4, there are points $\eta_0 \in \eta, \eta'_0 \in \eta', \eta''_0 \in \eta''$ at distance $\leq 7\delta$. Hence by Lemma 3.5 with $K = 7\delta$ we have:

$$\begin{aligned} \mathsf{d}\big(\eta_t, \eta'_t\big) &\leq K' \\ \mathsf{d}\big(\eta_{-t}, \eta''_{-t}\big) &\leq K' \\ \mathsf{d}\big(\eta'_{-t}, \eta''_t\big) &\leq K' \end{aligned}$$

□

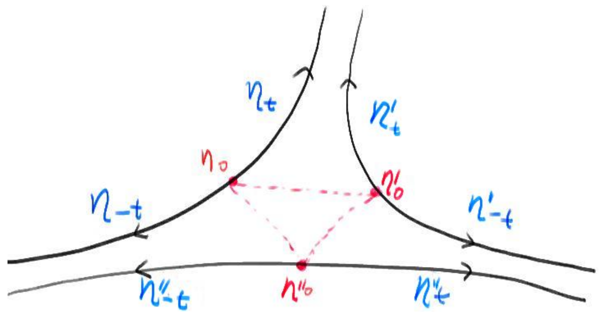

FIGURE 4. The configuration described in Lemma 3.7.

3.c. The ideal retriangulation.

REMARK 3.9. For what follows, it will be necessary to assume that genus 0 diagrams are homeomorphic to genus 0 surfaces. This implies no loss of generality, since every genus 0 diagram deformation retracts to a genus 0 surface which can also be assumed to carry a cell structure.

CONSTRUCTION 3.10 (δ-ideal triangulation). Let $S \to X$ be a genus 0 diagram with n boundary components $C_1, \ldots, C_n$ whose universal covers embed as uniform quasigeodesics $\widetilde{C}_i \subset \widetilde{X}$. We construct from S an "infinite genus 0 diagram" $S^\delta \to X$ such that its universal cover is built from geodesic δ-ideal triangles $\{\widetilde{\Delta}_j\}$.

Recall that, since S is homeomorphic to a genus 0 surface, the interior of S can be decomposed into topological ideal triangles $\{\triangle_j\}$ in such a way that pairs of sides converge to the boundary circles of S, moreover, these ideal triangles can be taken to be geodesic ideal triangles by endowing S with a complete hyperbolic metric. It follows readily from an Euler characteristic calculation (viewing the boundary circles as vertices of a triangulation of S^2) that $\#\{\triangle_j\} = 2(n-2)$ and that there are $3(n-2)$ sides in the triangulation.

Each side g maps to a quasigeodesic $\tilde{g}$ in $\widetilde{X}$ under the induced map $\widetilde{S} \to \widetilde{X}$. Choose a geodesic $\tilde{\gamma}$ with the same endpoints as $\tilde{g}$ in $\partial\widetilde{X}$. For each lift $\widetilde{\triangle}_j$ of an ideal triangle in $\{\triangle_j\}$, a choice of a triple of geodesics $(\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3)$ defines a *δ-ideal triangle* $\widetilde{\Delta}_j$ in $\widetilde{X}$ as in Lemma 3.7. Repeating this procedure for each lift of each $\triangle_j$ yields a set of δ-ideal triangles $\{\widetilde{\Delta}_j\}$ associated to $\{\widetilde{\triangle}_j\}$.

REMARK 3.11 (Choices). The construction described above is not canonical: the topological ideal triangulation of S is not unique, and therefore neither is the induced δ-ideal triangulation, nor the intriangles. For the remainder of the paper, we assume that one such choice has been made and is kept throughout.

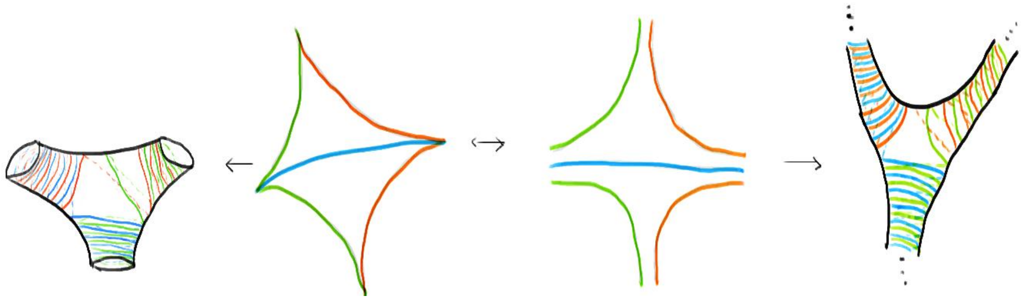

FIGURE 5. Heuristics of Construction 3.10 for a "pair of pants" genus 0 diagram.

DEFINITION 3.12. An *infinite disc diagram* is a contractible 2-complex that embeds in a punctured sphere, and whose boundary paths are bi-infinite. An *infinite genus* 0 *diagram* is a 2-complex that deformation retracts to a genus 0 surface, and that is homeomorphic to a genus 0 surface except perhaps at cut-vertices.

LEMMA 3.13. *Each $\widetilde{\Delta}_j$ bounds an infinite disc diagram in $\widetilde{X}$.*

PROOF. Let τ_1, τ_2, τ_3 be arcs joining the three sides of $\widetilde{\Delta}_j$, so $H_0 := \widetilde{\Delta}_j \cup (\cup_{i=1}^{3} \tau_i)$ forms an hexagon (possibly degenerate). By Van Kampen's Lemma, H_0 bounds a disc diagram D_0. Define H_n inductively by taking arcs $\tau_1^n, \tau_2^n, \tau_3^n$ joining the three sides of $\widetilde{\Delta}_j$ and such that $H_n := \widetilde{\Delta}_j \cup (\cup_{i=1}^{3} \tau_i^n)$ bounds a disc diagram D_n that properly contains D_{n-1}. Then $\cup_1^\infty H_i = \widetilde{\Delta}_j$, and by Kőnig's Infinity Lemma, $\widetilde{\Delta}_j$ bounds an infinite disc diagram $D_\infty = \cup_1^\infty D_i$. □

DEFINITION 3.14. Choose basepoints $p_{ij} \in g_{ij} \subset \triangle_j$ for each $i \in \{1, 2, 3\}$ and $j \in \{1, \ldots, 2(n-2)\}$. The map $\cup_j \widetilde{\triangle}_j \to \cup_j \widetilde{\Delta}_j$ that assigns a geodesic $\tilde{\gamma}$ to each g, sends the p_{ij} to basepoints $\hat{p}_{ij}$. Therefore the triangles of $\{\widetilde{\Delta}_j\}$ can be glued together unambiguously to form $S^\delta = (\sqcup \widetilde{\Delta}_j)/\sim$.

Explicitly, if $\{\gamma_k\}_{k \in K}$ are the various geodesics and $\phi_{k0} : \gamma_k \to \Delta_{i(k0)}$ and $\phi_{k1} : \gamma_k \to \Delta_{i(k1)}$ are the two identifications between γ_k and the sides of δ-ideal triangles, then we have:

$$S^\delta = (\sqcup \widetilde{\Delta}_j) \;/\; \{\phi_{k0}(x) \sim \phi_{k1}(x) \;:\; k \in K \text{ and } x \in \gamma_k\} \tag{1}$$

REMARK 3.15. There is a vexing difference between a hyperbolic ideal triangle and its corresponding δ-ideal triangle. While the former is homeomorphic to a disc minus 3 points, the later could be singular and even contain infinitely many cut points. In the extreme case where $\widetilde{X}$ is a tree, every δ-ideal triangle is an infinite tripod.

Because of these cut points, it is a priori possible that the quotient space $S^\delta = (\sqcup \widetilde{\Delta}_j)/\sim$ is not planar, as in Figure 6.

This possibility is not directly covered by the method in Section 4, and it will be convenient to adopt a workaround. For this purpose we "buffer" S^δ in a way that avoids singularities and actually maintains the homeomorphism type of the interior of S.

Construction 3.16 (Buffer). The product of a geodesic γ_k with the interval $[0,1]$ is called a *buffer*. We create a new diagram by taking a buffer for each $k \in K$, and identifying its "left" and "right" sides with the sides of δ-ideal triangles that γ_k maps to in S^δ.

Explicitly, we replace Equation (1) with the following:

$$S^\times = (\bigsqcup_j \widetilde{\Delta}_j \sqcup \bigsqcup_k (\gamma_k \times [0,1])) \;/\; \{\phi_{k0}(x) \sim (x,0),\ \phi_{k1}(x) \sim (x,1) \ :\ k \in K \text{ and } x \in \gamma_k\}$$

By relaxing the condition of diagrams in X to allow the map $S^\times \to X$ to be cellular instead of combinatorial, we obtain a genus 0 infinite diagram in X via the composition $S^\times \to S^\delta \to X$, where the map $S^\times \to S^\delta$ is the cellular map that collapses each buffer via projection to the first factor to a bi-infinite line in S^δ.

While it might seem counterproductive to replace a diagram with one that has larger area, the reader should keep in mind that $S^\times$ is only an accessory, and may have little relation with the diagram finally obtained in Section 4.

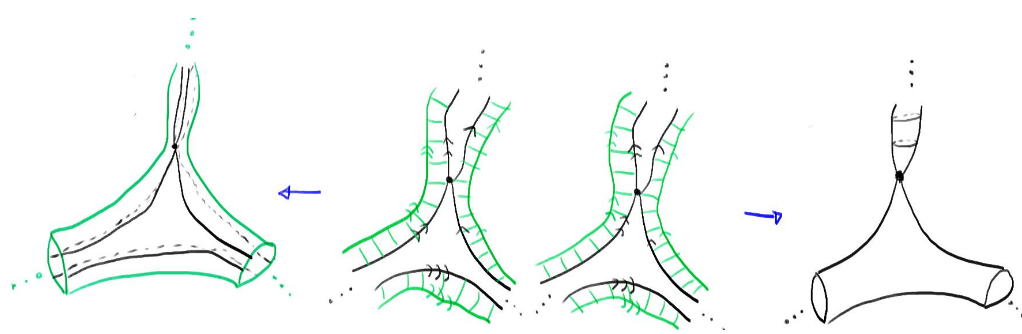

Figure 6. A scenario that requires a buffer as in Construction 3.16. The complex drawn in black is S^δ and the complex drawn in green in $S^\times$.

4. Jumps, horizontal paths, and bands

Let $\{\gamma_i\}$ be the immersed geodesics in S^δ along which the δ-ideal triangles $\{\widetilde{\Delta}_j\}$ have been identified to form S^δ.

For each i, let $\widehat{\Delta}_{\ell_i}$ and $\widehat{\Delta}_{r_i}$ be the ideal triangles meeting along γ_i. The *displacement path* α_i is the subpath $\alpha_i \subset \gamma_i$ whose endpoints are the vertices v, v' on γ_i of the intriangles $\Lambda_{\ell_i}, \Lambda_{r_i}$ of $\widehat{\Delta}_{\ell_i}, \widehat{\Delta}_{r_i}$.

We now describe "horizontal paths" that serve as combinatorial analogues to a "horocyclic flow". This will allow us to control the length of certain geodesic segments in S^δ and afterwards also in the "trimmed" genus 0 diagram Σ. Because geodesics are not unique in the combinatorial setting, it requires a bit of extra care to define these horizontal paths rigorously.

Definition 4.1 (Jumps). Let $\widehat{\Delta}' = \widehat{\Delta} - \Lambda$ where Λ is an open intriangle as in Definition 3.8. For each γ, let $t_r = \partial\widehat{\Delta}' - \gamma$, then for every $x \in \gamma$, and for every $y \in t_r$, there are geodesic arcs called *jumps* $J_x : [0,1] \to \widehat{\Delta}'$ with $J_x(0) = x$, $J_x(1) = y$ that pass through x, are parallel to a side of Λ, and have lengths $\leq K'$.

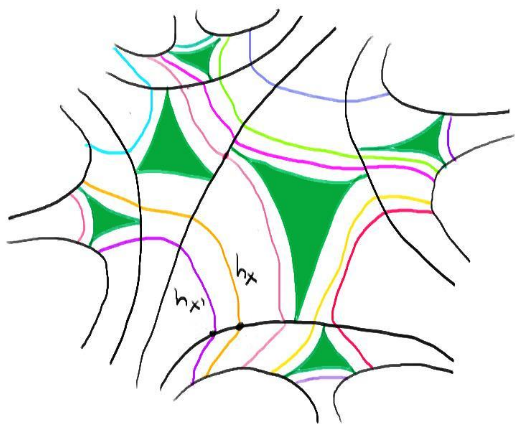

Figure 7. A collection of horizontal paths as described in Definition 4.2 drawn in the universal cover.

Definition 4.2 (Horizontal Paths). Two jumps J, J' are *opposite* if they lie in the same ideal triangle, and the terminal vertex of J is the initial vertex of J' (so the initial of J is the terminal of J'). We refer to Figure 8. A *horizontal path* h is a path in S that is a concatenation of jumps such that: Firstly, h has no *backtrack* consisting of a pair of consecutive jumps that are opposite. Secondly, no two jumps of h have the same initial point, and no two jumps of h have the same terminal point.

Horizontal paths can be finite, and also be infinite in one or both directions. Later we shall only consider finite horizontal paths.

Let $\iota(h)$ denote the set of initial points of jumps of h. Declare two horizontal paths h, g to be *equivalent* if $\iota(h) = \iota(g)$.

We shall often use the notation h_x to indicate a horizontal path having x as an initial or terminal vertex of some jump. The number of jumps of h_x shall be clear from the context.

Definition 4.3 (Returns). A finite horizontal path $J_1 J_2 \cdots J_n$ is an *i-return* if $J_1(0) \in \gamma_i$ and $J_n(1) \in \gamma_i$ but $J_b(0) \notin \gamma_i$ for $1 < b \leq n$. A horizontal path is a *return* if it is an i-return for some i. A *∂-return* is an i-return for some i for which there is a subarc $\lambda_i \subset \gamma_i$ and the concatenation $h\lambda_i$ is a cycle that separates S^δ into two components, and all intriangles of S^δ lie in the same component.

An *augmented ∂-return* is a concatenation $h\lambda_i$ as above.

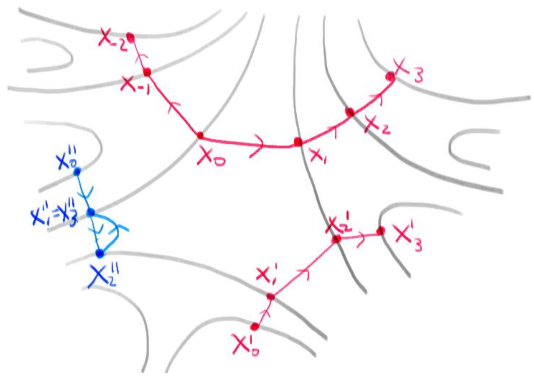

Figure 8. Two horizontal paths and a path with a backtrack.

Lemma 4.4. *Let h be a ∂-return. Then $|h| \leq 2(n-2) \cdot K'$.*

Proof. Let h be a ∂-return. Then h is a concatenation $h_1 \cdots h_k$ where each jump h_i has length $\leq K'$ and is parallel to the side of Λ_j facing a cusp $\widehat{C}_i$ (i.e., the side of Λ_j that separates $\widehat{C}_i$ from the other two sides of Λ_j within Δ_j). Moreover, h has no backtracks and no repetitions in the sense that no two subpaths are parallel to each other. Indeed, if they were, the endpoints of h would lie on distinct $\gamma_i, \gamma_{i'}$ or h could be subdivided into two or more returns h', h'', which contradicts that h was a return to begin with. Consequently, the subpaths that compose h are in bijective correspondence with the sides of Λ_j's facing $\widehat{C}_i$. Thus $k \leq 2(n-2)$. $\square$

Construction 4.5 (Trimming S^δ). We now describe how to obtain Σ from S^δ. Observe that S^δ contains a subsurface $\bar{S}^\delta$ with boundary such that $S^\delta - \bar{S}^\delta$ consists of n open cylinders $\sqcup \widehat{C}_i$, which we loosely refer to as the "cusps" of S^δ.

For each cusp $\widehat{C}_j$ of S^δ choose ∂-returns $h_{x_1}, \ldots, h_{x_n}$ satisfying the following:

(1) each $x_j \in \gamma_j$ is a vertex of an intriangle in S^δ,
(2) h_{x_j} separates $\cup_\ell \Lambda_\ell - x_j$ from $\widehat{C}_i$,
(3) $y_j \in h_{x_j} \cap \gamma_j$ is such that $h_{x_j}(t) = y_j$ and $y_j \notin \alpha_j$.

Let $\lambda_{x_jy_j}$ be the subarc of γ_j with endpoints x_j and y_j. Let ρ_j be the closed path that is the concatenation $h_{x_j}\lambda_{x_jy_j}$. We emphasize that each ρ_j corresponds to a cusp $\widehat{C}_j$, and ρ_j bounds an infinite annulus $A_j \subset \widehat{S}$.

Let $\Sigma = S^\delta - (\cup_i A_i)$. Then $\Sigma \hookrightarrow S^\delta$ is a compact genus 0 surface with n boundary components $\mathcal{C}_1 = \rho_1, \ldots, \mathcal{C}_n = \rho_n$. To have control over the length of the $\mathcal{C}_i$, we will need the following:

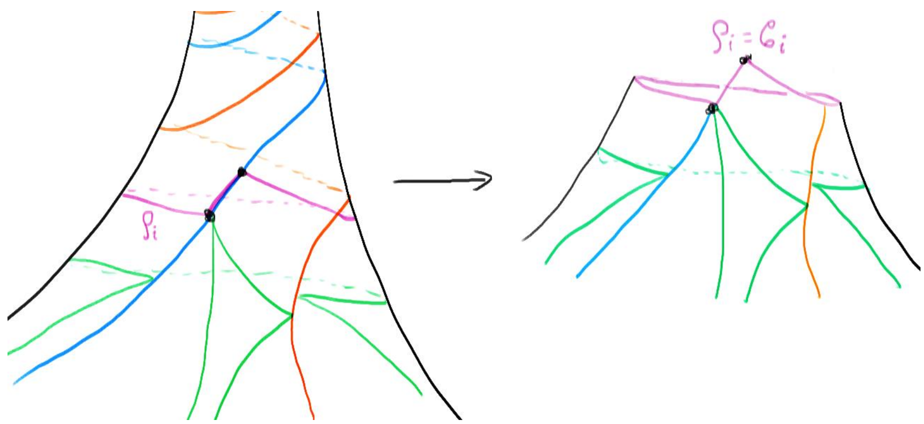

FIGURE 9. Trimming along an augmented ∂-return to obtain Σ.

LEMMA 4.6. *Let $\mathcal{C}_j$ be a boundary circle of Σ. Then the universal cover $\widetilde{\mathcal{C}}_j$ maps to an (a,b)-quasigeodesic in $\widetilde{X}$, where a, b depend only on n and δ.*

PROOF. As described in Construction 3.10, we have $\mathcal{C}_j = \rho_j = h_{x_j}\lambda_{x_jy_j}$. We will apply Theorem 2.10, where the lifts of h_{x_j} play the roles of the various σ_i and the lifts of $\lambda_{x_jy_j}$ play the roles of the various λ_i. First we show there is $L' > 0$ such that for any two consecutive lifts $\tilde{\lambda}_{x_jy_j}$ and $\tilde{\lambda}'_{x_jy_j}$, the intersection $\tilde{\lambda}_{x_jy_j} \cap \mathcal{N}_{3\delta}(\tilde{\lambda}'_{x_jy_j})$ has diameter at most L'. Let $\tilde{\gamma}_j, \tilde{\gamma}'_j$ be the lifts of γ_j with $\tilde{\lambda}_{x_jy_j} \subset \tilde{\gamma}_j$ and $\tilde{\lambda}'_{x_jy_j} \subset \tilde{\gamma}'_j$. Let $\bar{h}_{x_j}$ be the lift of h_{x_j} connecting $\tilde{\lambda}_{x_jy_j}$ to $\tilde{\lambda}'_{x_jy_j}$. By Theorem 2.6, it suffices to show that the subrays $\bar{\gamma}_j : [0,\infty) \to \Sigma$ and $\bar{\gamma}'_j : [0,\infty) \to \Sigma$ of $\tilde{\gamma}_j$ and $\tilde{\gamma}'_j$ having an endpoint on $\bar{h}_{x_j}$ and containing respectively $\tilde{\lambda}_{x_jy_j}$ and $\tilde{\lambda}'_{x_jy_j}$, do not represent the same point on $\partial\widetilde{X}$.

Notice that the lift $\bar{h}$ of the horizontal path containing h_{x_j} is two-sided in the sense that $\bar{h}$ separates $\widetilde{S}^\delta$, and $\bar{\gamma}_j$ and $\bar{\gamma}'_j$ lie on opposite sides of $\bar{h}$. Arguing by contradiction, suppose $\bar{\gamma}_j$ and $\bar{\gamma}'_j$ represent the same point on $\partial\widetilde{X}$. Since $\bar{\gamma}_j$ and $\bar{\gamma}'_j$ both have an endpoint on $\bar{h}_{x_j}$, it follows that $\tilde{\lambda}_{x_jy_j}(1)$ and $\tilde{\lambda}'_{x_jy_j}(0)$ project to the same point in the quotient, and a neighbourhood of $\tilde{h}_{x_j}$ connecting the lifts

$\tilde{\gamma}_j$ and $\tilde{\gamma}_j'$ as in Figure 10 produces a Möbius strip in Σ, which is impossible. Hence, $\bar{\gamma}_j$ and $\bar{\gamma}_j{}'$ have different endpoints on $\partial\widetilde{X}$.

Since $\bar{\gamma}_j(1) = h_{x_j}(0)$ and $h_{x_j}(1) = \bar{\gamma}_j{}'(0)$, and $\mathsf{d}(\bar{\gamma}_j(0), \bar{\gamma}_j{}'(0)) = |h_{x_j}| \leq 2K'(n-2)$ by Lemma 4.4, it follows from Corollary 2.9 that there exists $L' > 0$ for which $\bar{\gamma}_j, \bar{\gamma}_j{}'$ do not lie in the 3δ-neighbourhood of each other.

Since $|h_{x_j}| \leq 2K'(n-2)$, choosing $L'' > 2K'(n-2)$ ensures that diam $\big(\tilde{\lambda}_{x_jy_j} \cap \mathcal{N}_{3\delta}(\tilde{h}_{x_j}\tilde{\lambda}'_{x_jy_j})\big) \leq L''$, so letting $L = max\{L', L''\}$ yields the desired result, provided that $\frac{1}{2}|\tilde{\lambda}_{x_jy_j}| \geq 6(L+\delta)$. If $\frac{1}{2}|\tilde{\lambda}_{x_jy_j}| < 6(L+\delta)$ then $\frac{1}{2}|\lambda_{x_jy_j}| < 6(L+\delta)$, and so $\tilde{\rho}_j$ is still a uniform quasigeodesic, since it projects to an essential path in Σ of length uniformly bounded by $6(L+\delta) + 2K'(n-2)$ and there are only finitely many such combinatorial paths. □

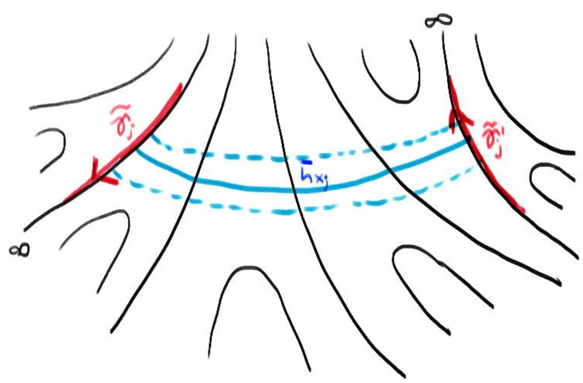

FIGURE 10. Möbius strip.

COROLLARY 4.7. *In the setting of Construction 4.5, each boundary component* $\mathcal{C}_i$ *of* Σ *is homotopic in* X *to a boundary component* C_i *of* S, *and* $\partial\Sigma = V \cup H$ *where* $V = \cup\lambda_{x_jy_j}$ *and* $H = \cup h_{x_j}$. *Moreover, there is a constant* $\mathcal{K} > 0$ *with* $|\mathcal{C}_i| \leq \mathcal{K}|C_i|$. *See Figure 9.*

COROLLARY 4.8. *There is a genus 0 diagram* Σ' *homotopic to* Σ *and such that* $\partial\Sigma' = \partial S$. *Moreover,* $\mathsf{Area}(\Sigma') \leq \mathsf{Area}(\Sigma) + nM|\partial\Sigma|$.

PROOF. This follows from Proposition 2.5 and Corollary 4.7. □

REMARK 4.9. Henceforth we restrict our attention to genus 0 diagrams Σ obtained via Construction 4.5. By Corollary 4.8 it suffices to bound $\mathsf{Area}(\Sigma)$ to prove Theorem 3.1.

DEFINITION 4.10 (Bands). Two finite horizontal paths h and h' are *parallel* if their sequences of initial points of jumps $p_1, \ldots, p_m$ and $p_1', \ldots, p_m'$ have the same length and moreover, for each i the points p_i and p_i' lie on the same geodesic γ but are not separated by either vertex of an intriangle on γ.

Equivalent horizontal paths are obviously parallel, and parallelism is an equivalence relation.

A *band* is an entire parallelism class.

Since Σ is compact, each band B has representatives h, h' whose initial points x, x' are farthest from each other in the sense that $\mathsf{d}_\gamma(x, x')$ is maximal, where γ is the geodesic containing these initial points. The *thickness* of B is defined to be $\mathsf{d}_\gamma(x, x')$. We say B is *bounded* by h and h'.

Let y denote the terminal point of h, and let y' denote the terminal point of h'. Let $\iota(B)$ be the subarc of γ bounded by x, x', and likewise let $\tau(B)$ be the subarc of γ bounded by y, y'.

A band is *linear* if $\iota(B) \cap \tau(B)$ is empty or a singleton. Otherwise it is *annular* (this includes the extreme case where $\iota(B) = \tau(B)$). See Figure 11.

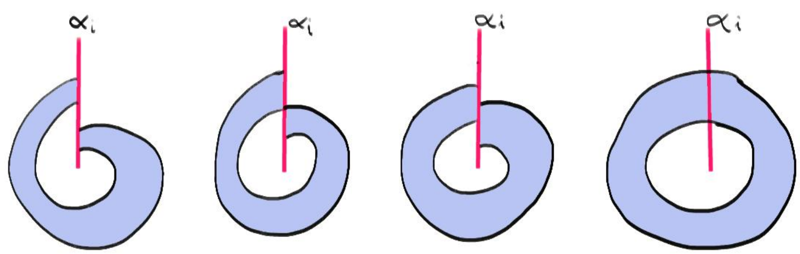

FIGURE 11. Possible scenarios for linear (left and centre left) and annular (centre right and right) bands.

DEFINITION 4.11 (Crossings and multiplicity). Let B be a band bounded by h and h'. Let $h_1, h_2, \ldots, h_m$ be the initial point of h followed by the sequence of terminal points of jumps of h. Let $h'_1, h'_2, \ldots, h'_m$ be the analogous points of h'. For each i, let ν_i be the subpath of the geodesic bounded by h_i, h'_i. We refer to $\nu_1, \ldots, \nu_m$ as the sequence of *verticals arcs* of B. Note that $\nu_1 = \iota(B)$ and $\nu_m = \tau(B)$.

The *multiplicity* of the band B is the number m of vertical arcs above. We use the notation $m(B)$ for the multiplicity of B.

LEMMA 4.12. *There are at most* $6n - 9$ *parallelism classes of returns.*

PROOF. Each i-return separates (together with a subarc of α_i) its complement in Σ into two connected components. The parallelism class of this i-return is uniquely determined by the resulting partition of intriangles and boundary circles. As horizontal paths do not cross each other and do not self-cross, the number of parallelism classes of returns is bounded above by the maximal number of disjoint non-parallel simple closed curves in a $(3n-2)$-punctured sphere, where n is the number of boundary circles of Σ and $2(n-2)$ is the number of intriangles. By [**JMM96**, Lem. 3.2], the maximal number of such curves in a planar surface is bounded above by $6n-9$. $\square$

DEFINITION 4.13. A *generalised backtrack* is a horizontal path whose first jump is parallel to the opposite of its last jump. In particular, the first and last jumps of a generalised backtrack lie in the same δ-ideal triangle. See Figure 12.

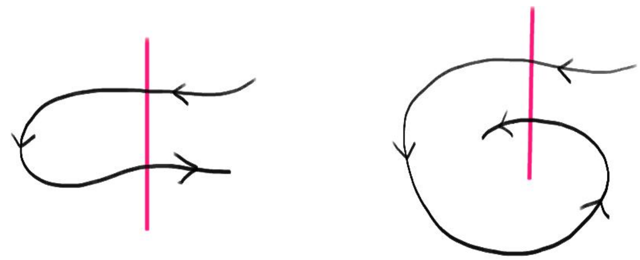

FIGURE 12. On the left: a horizontal path that contains a generalised backtrack. On the right: a horizontal path that does not contain a generalised backtrack.

LEMMA 4.14. *No subpath of a horizontal path is a generalised backtrack.*

PROOF. See Figure 13. Let h be a combinatorially minimal counterexample, so h has no subpath that is a counterexample. Let γ be the geodesic containing the initial and terminal points of h, then h contains at least 2 jumps, whose endpoint v, w not on γ are distinct. So there exist geodesics γ', γ'' (possibly identical) that intersect h at v, w, thus providing at least one subpath g that is a generalised backtrack, contradicting our initial choice of h. $\square$

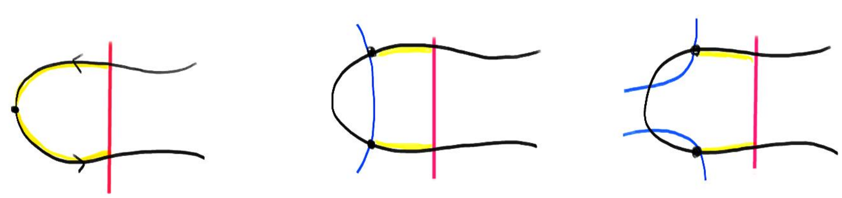

FIGURE 13. Finding an innermost counterexample. The situation on the left implies the situations on the centre or on the right, since jumps in horizontal paths are parallel to sides of intriangles.

COROLLARY 4.15. *For every i and for every $j \neq i$, an i-return h_i intersects γ_j at most twice.*

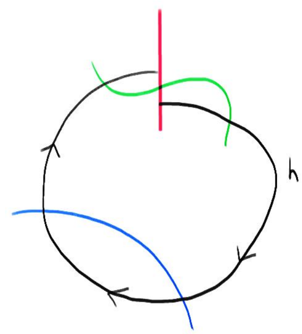

FIGURE 14. The two possible ways in which (a priori) a geodesic could double-cross an i-return.

PROOF. Let h be an i-return, let a be the subpath of γ_i bounded by the endpoints of h and let γ_j be a geodesic intersecting h more than twice. Then either γ_j crosses a, which is impossible, since horizontal paths do not cross, or one of the subpaths of h with endpoints in γ_j is a generalised backtrack, thus contradicting Lemma 4.14. See Figure 14. □

DEFINITION 4.16. Let $h = J_1 \cdots J_N$ be a horizontal path, then the *jump length* of h is equal to N.

COROLLARY 4.17. *Let h be an i-return for some i, then the jump length of h is bounded above by* $6(n-2)$.

PROOF. By Corollary 4.15, an i-return crosses each $\gamma_j \neq \gamma_i$ at most twice. Since $\{\gamma_i\}_i$ has cardinality $3(n-2)$, there are at most $6(n-2)$ intersections between h_i and $\{\gamma_i\}$. □

REMARK 4.18. Since δ-ideal triangles are K'-slim, it follows from Corollary 4.17 that the length of a return is bounded above by $K'6(n-2)$.

LEMMA 4.19. *Let $\{d_1, \ldots, d_k\}$ be the set of vertical arcs of linear bands. Then* $\sum_j |d_j| \leq \frac{|\partial\Sigma|}{2}(6(n-2))$.

PROOF. For each B_j, there are right and left projections $\pi_r^j : d_j \to V \subset \partial\Sigma$, $\pi_\ell^j : d_j \to V \subset \partial\Sigma$ which are distance preserving, since the thickness of each linear band B_j is constant. Therefore $|d_j| = |\pi_r^j(d_j)| = |\pi_\ell^j(d_j)|$. It remains to bound the number of vertical arcs having the same image under the left and right projections. That is, we must bound the multiplicity of each linear band. There are two cases to consider:

(1) Each band B that is not a return with respect to any of the γ_i crosses each displacement path at most once, so there are at most $3(n-2)$ vertical arcs in B.

(2) Each band B that is a return with respect to a γ_i crosses at most every displacement path at most twice. Hence at most $6(n-2)$ vertical arcs lie in B.

Moreover, the images of π_r^j, $\pi_r^{j'}$ are disjoint subarcs of V whenever d_j, $d_{j'}$ belong to different bands, and the same holds for π_ℓ^j, $\pi_\ell^{j'}$. Therefore $|\cup_j \pi_r^j(d_j)| \leq \frac{1}{2}|\partial\Sigma|$ and $|\cup_j \pi_\ell^j(d_j)| \leq \frac{1}{2}|\partial\Sigma|$, hence $\sum_j |d_j| \leq \frac{|\partial\Sigma|}{2}(6(n-2))$. □

LEMMA 4.20. *There exists $\kappa > 0$ depending only on δ, and a genus 0 diagram Σ' with $\partial\Sigma' = \partial\Sigma$ and such that every annular band in Σ' has thickness at most κ.*

PROOF. For each annular band A_i, the subarc $\iota(A_i) \cup \tau(A_i)$ can be divided into three subarcs u, v, w where $v = \iota(A_i) \cap \tau(A_i)$ and u, w are the connected components of $\iota(A_i) \cup \tau(A_i) - v$. Let A_i be bounded by the two horizontal paths β and β', so $\partial A_i = \vartheta \sqcup \vartheta'$ where $\vartheta = u\beta$ and $\vartheta' = w\beta'$. We claim that $\tilde{\vartheta}$ and $\tilde{\vartheta}'$ are uniform quasigeodesics in $\widetilde{X}$.

We prove the claim for $\tilde{\vartheta}$, and the proof for $\tilde{\vartheta}'$ is analogous. The proof is an application of Theorem 2.10, and is identical to that of Lemma 4.6 with the lifts of u playing the role of the λ_i, the lifts of β playing the role of the σ_i, and $L'' > K'6(n-2)$. Hence, $\tilde{\vartheta}$ and $\tilde{\vartheta}'$ are uniform quasigeodesics, and therefore $\tilde{\vartheta}$ lies in the κ-neighbourhood of $\tilde{\vartheta}'$ in $\widetilde{X}$ for some $\kappa > 0$. It follows that ϑ, ϑ' bound an annulus A_i' that has thickness at most κ.

Replacing Σ by the genus 0 diagram Σ' containing A_i' for each i yields the desired result. Note that, since annular bands are disjoint from each other, this procedure can be performed simultaneously for all annular bands. □

COROLLARY 4.21. *Let $\{d'_{j'}\}$ be the set of vertical arcs of annular bands. Then $\sum_j |d_j| \leq \kappa 12(n-2)6(n-9)$.*

PROOF. Since there are at most $6(n-9)$ parallelism classes of returns, there are at most $6(n-9)$ annular bands, each annular band intersects $\cup\alpha_i$ at most in $2 \cdot 6(n-2)$ subpaths. Combining this information with Lemma 4.20, we see that $\sum_{j'} |d'_{j'}| \leq \kappa 12(n-2)6(n-9)$. □

THEOREM 4.22. *There exists a genus 0 diagram $\Sigma^\flat$ with $\partial\Sigma^\flat = \partial\Sigma$ and a constant k_n such that $\mathsf{Area}(\Sigma^\flat) \leq k_n|\partial S|$.*

PROOF. We will find a forest F in Σ composed of displacement paths and sides of intriangles where

(1) $|F|$ is bounded
(2) F cuts Σ into a disc.

The length of a side of an intriangle is $\leq K'$ by construction. To bound $|\cup\alpha_i|$, we let $\cup\alpha_i = (\cup_j d_j) \cup (\cup_{j'} d'_{j'})$ where $d_1, \ldots, d_k$ are the subpaths of $\cup\alpha_i$

intersecting linear bands and $d'_1, \ldots, d'_{k'}$ are the subpaths of $\cup\alpha_i$ intersecting annular bands. By Lemma 4.19

$$\sum_j |d_j| \leq \frac{|\partial\Sigma|}{2}(6(n-2)),$$

and by Corollary 4.21

$$\sum_{j'} |d'_{j'}| \leq \kappa 12(n-2)6(n-9).$$

Let $F = (\cup_i^{3(n-2)}\alpha_i) \cup (\cup_j^{3(n-2)-1} s_j)$, where each s_j is a side of an intriangle connecting distinct α_i's. Then cutting Σ along F yields a (possibly singular) disc diagram D. By Theorem 2.4, there exists a constant $k^\flat$ depending only on δ and n, and a disc diagram $D^\flat$ with $\partial D = \partial D^\flat$ and $\mathsf{Area}(D^\flat) \leq k^\flat |\partial D^\flat|$. By the previous observations and Corollary 4.7,

$$|F| \leq \sum_j |d_j| + \sum_{j'} |d'_{j'}| + (3(n-2)-1)K' \leq$$
$$\frac{\mathcal{K}|\partial S|}{2}(6(n-2)) + \kappa 12(n-2)6(n-9) + (3(n-2)-1)K'$$

and $|D| = |\partial\Sigma| + 2|F|$. Hence, setting $k^\# = \frac{\mathcal{K}}{2}(6(n-2)) + \kappa 12(n-2)6(n-9) + (3(n-2)-1)K'$, then

$$area(D^\flat) \leq k^\flat|\partial D^\flat| \leq k^\flat(|\partial\Sigma| + 2|f|) \leq k^\flat(\mathcal{K}|\partial S| + 2k^\#|\partial S|).$$

Letting $\Sigma^\flat$ be the genus 0 diagram obtained from $D^\flat$ by re-gluing the sides of $\partial D^\flat$ corresponding to F, and noting that $\mathsf{Area}(\Sigma^\flat) = \mathsf{Area}(D^\flat)$ finishes the proof. □

Remark 4.23. Since it does not affect the strength of the main result, and to clarify the exposition, we made no attempt to obtain sharp constants, particularly for Theorem 4.22. We believe that the constants could be improved by a more exacting analysis of the combinatorics that govern the behaviour of horizontal paths.

Remark 4.24. Similar results to those obtained in the previous sections should hold for an arbitrary surface diagram (i.e., a compact combinatorial 2-complex that embeds into a surface with punctures and such that the surface deformation retracts to the 2-complex), following a more-or-less identical line of thought as that adopted here, and keeping in mind the relevant considerations regarding separating curves and Euler characteristic.

5. A special case

This section proves a linear isoperimetric function for 2-complexes that satisfy the negative weight test [**Pri88, Ger87**]. This was first explained for disc diagrams by Gersten. We recall the Combinatorial Gauss Bonnet Theorem and its associated formulas, and refer to Gersten and Pride as above for proofs, or to [**MW02**] for the slight generalization we use.

The curvatures of vertices and 2-cells are defined as follows:
$\kappa(v) = 2\pi - \sum_{c\in \text{Corners}(v)} \sphericalangle(c) - \pi\chi(\text{link}(v))$
$\kappa(f) = 2\pi - \sum_{c\in \text{Corners}(f)} \mathsf{def}(c)$, where $\mathsf{def}(c) = \pi - \sphericalangle(c)$

THEOREM 5.1 (Combinatorial Gauss-Bonnet). *Let Y be a compact 2-complex with an angle assigned at each corner of each 2-cell, then*

$$2\pi\chi(Y) = \sum_{v\in \mathit{Vertices}(Y)} \kappa(v) + \sum_{f\in \mathit{2\text{-}cells}(Y)} \kappa(f)$$

We have in mind the case where Y is a (possibly singular) surface.

DEFINITION 5.2. An *angled 2-complex* X is a 2-complex with an angle assigned to each corner of each 2-cell. (More precisely, an angle is assigned to each edge in the link of each 0-cell.) An angled 2-complex X has *coarse negative curvature* if the following holds:

(1) Each 2-cell of X has nonpositive curvature.
(2) For each 0-cell $v \in X^0$, each cycle in link(v) has length $\geq 2\pi$.
(3) There exists $R \in \mathbb{N}$ such that for each near-immersed disc diagram $D \to X$ and $p \in D$, there exists v with $\mathsf{d}(p, v) < R$ and either $v \in \partial D$ or $\kappa(v) < 0$.

A map $Y \to X$ between 2-complexes is a *near-immersion* if it is a local-injection outside Y^0. The angles of X are pulled back to Y, so Y is itself an angled 2-complex.

For a genus 0 diagram S with boundary circles $\{C_i\}$ we let $|\partial S| = \sum |C_i|$.

PROPOSITION 5.3. *Let X be a compact angled 2-complex with coarse negative curvature. There exists $K \geq 0$ with the following property:*

Let S be a genus 0 diagram with n boundary circles. Then $\mathsf{Area}(S) \leq K|\partial S|$ *for any near-immersion.*

We first explain things assuming negative curvature at each vertex:

PROOF ASSUMING GENUINE NEGATIVE CURVATURE.
By Theorem 5.1 we have:

$$2\pi(2-n) = 2\pi\chi(S) \;=\; \sum \kappa(v) + \sum \kappa(f)$$

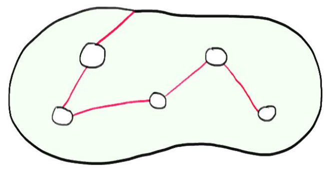

FIGURE 15. A cut-forest for a genus 0 diagram with 6 boundary circles.

so

$$2\pi(2-n) \;\leq\; \nu|\partial S| - \xi\mathsf{Area}(S)$$

where ν is an upperbound on positive curvature at a boundary vertex and $-\xi < 0$ is an upperbound on the negative curvature of a 2-cell (i.e., $|\xi|$ is the maximum in absolute value). Since X is compact, such ν and ξ always exist. Hence

$$\mathsf{Area}(S) \;\leq\; \frac{\nu}{\xi}|\partial S| + 2\pi(n-2) \;\leq\; |\partial S|(\frac{\nu}{\xi} + 2\pi)$$

as $n \leq |\partial S|$, since each boundary circle of S has at least one edge. □

PROOF OF PROPOSITION 5.3. By Lemma 5.6, there exists a cut-forest F for S with $|F| \leq M|\partial S|$ where $M = 2R(\frac{1}{\mu}\big(\xi + 2\pi\big) + 1)$. Let D be the disc diagram obtained from S by cutting along F. By Theorem 2.4 there exists $K > 0$ with $\mathsf{Area}(D) \leq K|\partial D|$. Since every edge of F appears twice in ∂D the following holds with $K' \geq K + 2MK$:

$$\mathsf{Area}(S) \;=\; \mathsf{Area}(D) \leq K|\partial D| \;\leq\; K(|\partial S| + 2|F|) \;\leq\; K'|\partial S| \qquad \square$$

DEFINITION 5.4. A *cut-forest* F in a genus 0 diagram S is a finite, properly embedded acyclic graph with

1. vertices $V = \{p_i\} \sqcup \{q_j\}$ with $\{p_i\} \subset \partial S$ and $\{q_j\} \subset \mathrm{Int}(S)$,
2. paths $\{\sigma_k\}$ with endpoints in V and disjoint interiors,
3. and with $S - F$ contractible.

The *length* of F is the sum: $|F| := \sum_k |\sigma_k|$.

REMARK 5.5. A cut-forest F exists for each genus 0 diagram S. Indeed, it suffices to choose paths $\{\sigma_k\}_{k=1}^{n-1}$ connecting each boundary component C_k to C_{k+1} (mod n) within $S - \cup_{i<k}\sigma_i$. See Figure 15.

LEMMA 5.6. *Let X be a compact angled 2-complex with coarse negative curvature. There exists $M > 0$ such that the following holds:*

Let $S \to X$ be a nearly-immersed genus 0 diagram. Then S contains a cut-forest F with $|F| \leq M|\partial S|$.

PROOF. Let $\xi > 0$ be an upperbound on curvatures of boundary vertices. Let $-\mu < 0$ be an upperbound on curvatures of negatively curved vertices.

Let R be as in Definition 5.2. Let $V = \{v_1, \ldots, v_p\}$ be the negatively curved vertices in $\mathrm{Int}(S)$. By Theorem 5.1:

$$2\pi(2-n) \;=\; 2\pi\chi(S) \;=\; \sum \kappa(v) + \sum \kappa(f) \;\leq\; \xi|\partial S| - p\mu.$$

Consequently

$$p\mu \;\leq\; 2\pi(n-2) + \xi|\partial S| \;<\; (\xi + 2\pi)|\partial S|$$

and so

$$p \;<\; \frac{1}{\mu}\big(\xi + 2\pi\big)|\partial S|.$$

Let $\Theta_1 = C_1$ and for each $m > 0$, let σ_m be a shortest geodesic in S from C_{m+1} to Θ_m where

$$\Theta_m = C_1 \cup \cdots \cup C_m \cup \sigma_1 \cup \cdots \cup \sigma_{m-1},$$

(and where we allow subscripts posterior to m to be changed to minimise $|\sigma_m|$ by possibly choosing a different C_{m+1}.)

Let σ'_{m+1} be the subpath obtained by omitting the initial and terminal subpaths of σ of length R. Observe that σ'_{m+1} is disjoint from $N_R(\Theta_m \cup \partial S)$ for otherwise, there would be a shorter choice of σ_{m+1} – perhaps ending at a different C'_m (after reordering subscripts) or starting in a different place on Θ_m. More precisely, assume σ'_{m+1} is not disjoint from $N_R(\Theta_m \cup \partial S)$, let $x \in \sigma'_{m+1}$ be the vertex closest to C_m that lies in $N_R(\Theta_m \cup \partial S)$, and let σ''_{m+1} be the subpath of σ_{m+1} starting at C_m and ending at x. Then there exists a geodesic path β with endpoints x, y where $y \in \Theta_m \cup \partial S$ and $\mathsf{d}(x,y)$ is minimal, then $\sigma''_{m+1} \cup \beta$ is shorter than σ_{m+1} and connects C_m to Θ_m.

Similarly, for any points $p, q \in \sigma'_{m+1}$ with $\mathsf{d}_{\sigma'_{m+1}}(p,q) > 2R$ then $\mathsf{d}_S(p,q) > R$, for otherwise σ'_{m+1} would not have been a geodesic to begin with. It follows that:

$$\sum_{m=1}^{n-1} |\sigma_m - 2R| = \sum 2R|\sigma'_m| < 2R|V|$$

since we can consider points at distance $2R$ from each other within σ'_m, and each is uniquely associated to a vertex of V. Hence $\sum |\sigma_m| < 2R(|V|) + 2R(n-1)$.

Finally, $F = \cup \sigma_i$ is a forest, since at each stage, if $\cup_{i=1}^{m} \sigma_i$ is not a forest, then we could have chosen σ_m shorter by removing edges from its cycles. □

Remark 5.7. As in Theorem 3.1, we avoid estimating $\mathsf{Area}(S)$ directly. Instead the argument focuses on constructing a short cut-forest for S, which we can then use to cut S into a disc diagram D with $|\partial_{\mathsf{p}} D|$ proportional to $|\partial S|$. We then apply Theorem 2.4 to replace D by D', and then glue D' along the cut-forest paths to obtain a genus 0 diagram S' with area bounded linearly by $|\partial S|$.

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