TOPOLOGICAL f-RINGS

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K.W. ARMSTRONG

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CHAPTER I

INTRODUCTION

In [1], Birkhoff and Pierce introduced the concept of f-ring. In [7], [9], and [10] it was shown that certain Archimedian f-rings can be represented as rings of continuous extended real valued functions on various topological spaces.

It was noted that the m-topology on rings of continuous real valued functions (see problem 2N of [6]) could be generalized to arbitrary f-rings provided they were convex in the sense of [5]. It was also shown that every f-ring can be embedded in a smallest convex f-ring. For this p-unit topology on a convex f-ring the ring becomes a topological ring. The concept of a C-ring was then introduced since these were the convex f-rings for which the p-unit topology is Hausdorff. By a slight generalization of the methods of topological ring theory it was established that the completion of a C-ring is a topological f-ring. The question then arose whether this completion is a C-ring and if its topology is the p-unit topology. It was then shown that a C-ring, R, could be represented as a subring of the ring of all continuous "sections" from the maximal ideal space of R (with the Stone topology) into the "bundle" space of R and that this ring of continuous sections with the p-unit topology is the completion of R. This answered the above question in the affirmative. It was then proved, as an application, that a bounded Archimedian f-ring can be represented as a subring of the ring of all continuous real valued functions on its maximal ℓ-ideal space, which is a special case of the results of [10]. By then using the methods of [5] it can be shown that an Archimedian f-ring with identity can be represented as a subring of the ring of continuous extended real valued functions on its maximal ℓ-ideal space.

The material in Chapter II is from [1] and [8]. The material in Chapter V is from [3] and [4]. The concept of convex f-ring in Chapter III is from [5]. The material in Chapter VI from 6-19 to 6-29 is a slight modification of the construction of the reals in [4].

CHAPTER II

PRELIMINARY RESULTS ON f-RINGS

Throughout by ring will be understood a commutative ring with identity, 1.

Definition 2-1

A partially ordered ring, $\langle R, \leq \rangle$, is a system such that R is a ring and $\leq$ is a partial order on R such that for all $a,b \in R$

1) $a \leq b \Rightarrow a+c \leq b+c$

2) $a,b \geq 0 \Rightarrow ab \geq 0$

Definition 2-2

An ℓ-ring is a partially ordered ring which is also a lattice. A totally ordered ring (t.o. ring) is a partially ordered ring such that $\leq$ is a total order on R.

Definition 2-3

An f-ring, R, is an ℓ-ring such that for all $a,b,c \in R$

$$a \wedge b = 0 \text{ and } c \geq 0 \Rightarrow (ca) \wedge b = 0$$

Proposition 2-4

Any t.o. ring is an f-ring.

In a ring R for $A,B \subseteq R$ define $A+B = \{a+b \mid a \in A, b \in B\}$ and similarly define $A \cdot B$ and $-A$.

In a partially ordered ring an element, r, is positive if $r \geqslant 0$ and negative if $r \leqslant 0$.

Proposition 2-5

If R is a partially ordered ring and P the subset of positive elements then $P+P \subseteq P$, $P \cdot P \subseteq P$, and $P \cap -P = \{0\}$. If R is a ring and Q a subset of R such that $Q+Q \subseteq Q$, $Q \cdot Q \subseteq Q$, and $Q \cap -Q = \{0\}$ then the relation $, \leqslant,$ defined by $a \leqslant b$ iff $b-a \in Q$ makes $\langle R, \leqslant \rangle$ a partially ordered ring with $Q = P$. R is a t.o. ring iff $P \cup -P = R$.

Definition 2-6

A mapping from an ℓ-ring, R, into an ℓ-ring, S, is an ℓ-homomorphism if it is both a ring and lattice homomorphism. Similarly define ℓ-monomorphism, ℓ-epimorphism and ℓ-isomorphism.

Definition 2-7

In any ℓ-ring define

1) $a^+ = a \vee 0$ 2) $a^- = (-a) \vee 0$ 3) $|a| = a \vee -a$.

Definition 2-8

In an ℓ-ring, R, an ℓ-ideal, I, is a ring ideal such that

$$b \in I \text{ and } |a| \leqslant |b| \Rightarrow a \in I.$$

Proposition 2-9

If I is an ℓ-ideal of the ℓ-ring, R, let r_I be the residue class of R/I containing r. Then R/I is an ℓ-ring under the definitions $r_I+s_I = (r+s)_I$, $-(r_I) = (-r)_I$, $r_I s_I = (rs)_I$, $r_I \vee s_I = (r \vee s)_I$, and $r_I \wedge s_I = (r \wedge s)_I$. It follows that $|r_I| = |r|_I$, $(r_I)^+ = (r^+)_I$, and $(r_I)^- = (r^-)_I$. Write R_I for the ℓ-ring R/I. Also $r \varepsilon I$ iff $r_I = 0_I$. The natural map, $r \rightarrow r_I$, is an ℓ-epimorphism of R onto R_I.

Definition 2-10

The cardinal product, $C(R_\alpha)$, of the partially ordered rings $\{R_\alpha | \alpha \varepsilon A\}$ is the cartesian product $\prod R_\alpha$ with operations defined as follows; $(a+b)_\alpha = a_\alpha + b_\alpha$, $(ab)_\alpha = a_\alpha b_\alpha$, $(-a)_\alpha = -a_\alpha$, $(1)_\alpha = 1$, and $(0)_\alpha = 0$ and also the relation, $\leq$, defined by $a \leq b$ iff for all $\alpha \varepsilon A$, $a_\alpha \leq b_\alpha$.

Proposition 2-11

The cardinal product of f-rings, R_α, is an f-ring. Also $(a \vee b)_\alpha = a_\alpha \vee b_\alpha$, $(a \wedge b)_\alpha = a_\alpha \wedge b_\alpha$, $|a|_\alpha = |a_\alpha|$, $(a^+)_\alpha = (a_\alpha)^+$, and $(a^-)_\alpha = (a_\alpha)^-$.

Proposition 2-12

For any f-ring, R, there exists a set of ℓ-ideals, $\{I_\alpha | \alpha \varepsilon A\}$, such that R_{I_α} is a t.o. ring and the mapping, π, defined by $(\pi(r))_\alpha = r_{I_\alpha}$ is an ℓ-monomorphism of R into $C(R_{I_\alpha})$.

The above proposition is the principal characterization of f-rings and is used to prove that many properties of t·o· rings extend to f-rings.

Definition 2-13

A sub-ℓ-ring of an ℓ-ring is a set which is both a subring and a sublattice.

Proposition 2-14

A sub-ℓ-ring of an f-ring is an f-ring.

Proposition 2-15

An f-field is totally ordered.

Proposition 2-16

Commutative f-rings with identity can be equationally defined in terms of a set, R, fixed elements $0,1 \in R$, the unary operation, $-$, and the binary operations $+$, $\cdot$, $\wedge$, $\vee$.

A regular element in a ring is one which is not a zero divisor. Thus all units are regular.

The following two propositions list the algebraic identities needed later.

Proposition 2-17

In any f-ring

1) $a \leq b$ and $c \geq 0 \Rightarrow ac \leq bc$

2) $a \leq b$ and $c \leq d \Rightarrow a+c \leq b+d$

3) $a \leq b \Rightarrow -b \leq -a$

4) $a \geq 0$ and $-a \geq 0 \Rightarrow a = 0$

5) $a \geq 0 \Rightarrow a(b \wedge c) = ab \wedge ac$ and $a(b \vee c) = ab \vee ac$

6) $a \leq b \Leftrightarrow a \wedge b = a$

7) $a \geq o \Leftrightarrow a = |a|$

8) $a \geq 0 \Leftrightarrow a = a^{+}$

9) $|a| \geq 0$

10) $a^{2} \geq 0$

11) $a^{2} = |a|^{2}$

12) $|b-c| \leq a \Leftrightarrow c-a \leq b \leq c+a$

13) $|a| = 0 \Leftrightarrow a = 0$

14) $|a+b| \leq |a| + |b|$

15) $||a| - |b|| \leq |a-b|$

16) $|ab| = |a||b|$

17) $|a^{+}-b^{+}| \leq |a-b|$

18) $|a| \leq |b| \Rightarrow a^{2} \leq b^{2}$

19) $a^{+}a^{-} = 0$

20) $a^{+} \wedge a^{-} = 0$

21) $1^{+} = 1$ and $(-1)^{-} = 1$

22) $a \vee b = a+(b-a)^{+}$

23) $a \wedge b = -(-a \vee -b)$

24) c regular and $c \geq 0 \Rightarrow c > 0$

25) c regular, $c \geq 0$ and $a < b \Rightarrow ac < bc$

26) $a, b \geq 1 \Rightarrow ab \geq 1$

27) $a = a^{+}-a^{-}$

28) $a \geq 1 \Rightarrow a$ regular

29) $1 > 0$

30) $|-a| = |a|$

A p-unit in a partially ordered ring is a positive unit.

Proposition 2-18

In any f-ring

1) v a unit $\Rightarrow \left|\frac{1}{v}\right| = \frac{1}{|v|}$

2) 2 a unit and $a < b \Rightarrow a < \frac{a+b}{2} < b$

3) u,v p-units and $u \leq v \Rightarrow 0 < \frac{1}{v} \leq \frac{1}{u}$

4) u,v p-units $\Rightarrow u \wedge v$ a p-unit

5) u,v p-units $\Rightarrow uv$ a p-unit

6) u a p-unit $\Rightarrow \frac{1}{u}$ a p-unit

7) 2 a unit $\Rightarrow a \vee b = \frac{1}{2}(a+b+|a-b|)$

Definition 2-19

In an ℓ-ring, R,

1) If $A \subseteq R$ then $\langle A \rangle$ is the smallest ℓ-ideal containing A, in particular, if $a \in R$ then $\langle a \rangle$ is the smallest ℓ-ideal containing a.

2) If A,B ℓ-ideals of R then $\langle AB \rangle$ is the smallest ℓ-ideal containing the ideal AB.

3) For a set of ℓ-ideals $\{A_i \mid i \in I\}$ of R, $\sum_{i \in I} A_i$ is the smallest ℓ-ideal containing every A_i. For two ideals A,B it is written as A+B.

4) A prime ℓ-ideal is an ℓ-ideal which is prime as a ring ideal.

Proposition 2-20

1) If R is a sub-ℓ-ring of the ℓ-ring, S, and

I an ℓ-ideal in S then $I \cap R$ is an ℓ-ideal in R.

2) For ℓ-ideals $\{A_i \mid i \varepsilon I\}$ of an ℓ-ring, R, $\sum_I A_i$ is the ordinary sum as ring ideals in R.

3) An ℓ-ideal P is prime iff for any two ℓ-ideals I and J,

$$I \nsubseteq P \text{ and } J \nsubseteq P \Rightarrow \langle IJ \rangle \nsubseteq P.$$

4) For every proper ℓ-ideal I there is a maximal ℓ-ideal M such that $I \subseteq M$. Also R_M is a totally ordered ring.

5) If R is an f-ring and $a,b \varepsilon R$ then

$$\langle\langle a\rangle \langle b\rangle\rangle = \langle ab\rangle .$$

6) A maximal ℓ-ideal is prime.

<u>Definition 2-21</u>

The J-radical of an ℓ-ring R, J(R), is the intersection of all the maximal ℓ-ideals in R.

<u>Definition 2-22</u>

In a partially ordered set, P, define for $a,b \varepsilon P$:

1) $(a,b) = \{x \varepsilon P \mid a < x < b\}$

2) $[a,b] = \{x \varepsilon P \mid a \leq x \leq b\}$

3) $(a,\infty) = \{x \varepsilon P \mid a < x\}$

4) $(-\infty, a) = \{x \varepsilon P \mid x < a\}$

5) $[a,\infty) = \{x \varepsilon P \mid a \leq x\}$

6) $(-\infty,a] = \{x \varepsilon P \mid x \leq a\}$

The intervals of types 1), 3), and 4) are called open intervals and the intervals of types 2), 5), and 6) are called closed intervals.

Proposition 2-23

If R is an ℓ-ring and $a \geqslant 0$ then

$$x \varepsilon [-a,a] \Leftrightarrow |x| \leqslant a.$$

Proposition 2-24

If I, J are ℓ-ideals of an f-ring, R, and $I \subseteq J$ then the mapping $\pi : R_I \to R_J$ defined by $\pi(r_I) = r_J$ is an ℓ-epimorphism.

Definition 2-25

An f-ring, R, is

1) bounded if for all $a \varepsilon R$ there exists a positive integer, n, such that $|a| \leqslant n1$,

2) Archimedian if $r, t \varepsilon R$ and for all positive integers, n, $nr \leqslant t$ then $r \leqslant 0$.

Proposition 2-26

1) The ℓ-homorphic image of a bounded f-ring is a bounded f-ring.

2) A bounded totally ordered field is ℓ-isomorphic to a sub-f-field of the reals.

3) If R is an Archimedian f-ring then $J(R) = \{0\}$.

CHAPTER III

THE CONVEX CLOSURE OF AN f-RING

Definition 3-1

A convex f-ring is an f-ring R such that

$r \in R$ and $r \geq 1 \Rightarrow r$ a unit

Proposition 3-2

In a convex f-ring all maximal ideals are l-ideals.

Proof

Let M be a maximal ideal in the convex f-ring R. Let $a, b \in R$ be such that $|a| \leq |b|$ and $a \notin M$. Since M maximal there exists $r \in R$ and $m \in M$ such that $ar+m = 1$ so $1-ar \in M$. Thus $(1-ar)(1+ar) = 1-a^2r^2 \in M$ so there exists $n \in M$ such that $n+a^2r^2 = 1$. By 2-17(18), $a^2 \leq b^2$ and so by 2-17(10,1,2), $1 = n+a^2r^2 \leq n+b^2r^2$. Since R convex, $n+b^2r^2$ is a unit and $n+b^2r^2 \notin M$ thus $b \notin M$ since otherwise $n+b^2r^2 \in M$. Thus if M a maximal ideal, $b \in M$, and $|a| \leq |b|$ then $a \in M$. Therefore by 2-8, M is an l-ideal.

Corollary 3-3

In a convex f-ring all maximal l-ideals are also maximal ideals.

Proof

Let M be a maximal l-ideal. Since M a proper ideal it is contained in a maximal ideal ,N. By 3-2, N is an l-ideal so $M = N$.

<u>Lemma 3-4</u> In an f-ring

$b \geq 0$, b regular, and $be \geq 0 \Rightarrow e \geq 0$

<u>Proof</u>

By 2-17(27), $b(e^+ - e^-) = be \geq 0$ and so by 2-17(2), $be^+ \geq be^-$ and $be^+ \wedge be^- = be^-$ by 2-17(6). Now $be^+ \wedge be^- = b(e^+ \wedge e^-) = b(0) = 0$ by 2-17(5,20). Thus $be^- = 0$ and since b regular then $e^- = 0$. By 2-17(27), $e = e^+$ and by 2-17(8), $e \geq 0$.

If R is a ring let Qc(R) be its classical ring of quotients. Let $\left[\frac{a}{b}\right]$ be the equivalence class of Qc(R) containing the fraction $\frac{a}{b}$.

<u>Lemma 3-5</u>

If R is an f-ring the set $P = \left\{\left[\frac{a}{b}\right] \varepsilon Qc(R) \middle| ab \geq 0\right\}$ is well defined.

<u>Proof</u>

Assume $\left[\frac{a}{b}\right] = \left[\frac{c}{d}\right]$ and $ab \geq 0$. Then $ad = cb$ and so $ad^2b = cb^2d$. Thus by 2-17(10,1), $cdb^2 \geq 0$. Now b regular so that b^2 is also and $b^2 \geq 0$. Therefore by 3-4, $cd \geq 0$.

<u>Lemma 3-6</u>

The set P satisfies the conditions of 2-5 and so defines a partial order of Qc(R), R an f-ring, which makes Qc(R) a partially ordered ring.

<u>Proof</u>

Assume $\left[\frac{a}{b}\right]$, $\left[\frac{c}{d}\right]$ εP so that $ab \geq 0$ and $cd \geq 0$. Then by 2-17(1), $acbd \geq 0$ and therefore $\left[\frac{a}{b}\right]\left[\frac{c}{d}\right] = \left[\frac{ac}{bd}\right] \varepsilon P$, that is $P \cdot P \subseteq P$. By 2-17(10,1,2), $0 \leq abd^2 + cdb^2 = (ad+cb)bd$ so that $\left[\frac{a}{b}\right] + \left[\frac{c}{d}\right] = \left[\frac{ad+cb}{bd}\right] \varepsilon P$, that is $P+P \subseteq P$.

Assume $\left[\frac{a}{b}\right] \varepsilon P \cap -P$ then there exists $\left[\frac{c}{d}\right] \varepsilon P$ such that $\left[\frac{a}{b}\right] = -\left[\frac{c}{d}\right] = \left[\frac{-c}{d}\right]$. Thus $cd \geqslant 0$ and $-cd \geqslant 0$ so by 2-17(4), $cd = 0$ but d is regular so $c = 0$. Therefore $\left[\frac{a}{b}\right] = 0$ and $P \cap -P = \{0\}$.

__Lemma 3-7__

If $\left[\frac{g}{h}\right]$, $\left[\frac{a}{b}\right] \varepsilon Qc(R)$ and $h,b \geqslant 0$ then $\left[\frac{g}{h}\right] \geqslant \left[\frac{a}{b}\right]$ iff $gb \geqslant ah$.

__Proof__

If $gb \geqslant ah$ then by 2-17(1,2), $(gb-ah)hb \geqslant 0$ Therefore $\left[\frac{g}{h}\right] - \left[\frac{a}{b}\right] = \left[\frac{gb-ah}{hb}\right] \varepsilon P$ by 3-5 and by 2-5, $\left[\frac{g}{h}\right] \geqslant \left[\frac{a}{b}\right]$.

If $\left[\frac{g}{h}\right] \geqslant \left[\frac{a}{b}\right]$ then $\left[\frac{g}{h}\right] - \left[\frac{a}{b}\right] = \left[\frac{gb-ah}{hb}\right] \varepsilon P$ by 2-5 and by 3-5, $(gb-ah)hb \geqslant 0$. Since h and b are regular so is hb and $hb \geqslant 0$ by 2-17(1). Therefore by 3-4, $gb-ah \geqslant 0$ so by 2-17(2), $gb \geqslant ah$.

__Lemma 3-8__

For any $\left[\frac{a}{b}\right] \varepsilon Qc(R)$ there exists $c,d \varepsilon R$ such that $d \geqslant 0$ and $\left[\frac{a}{b}\right] = \left[\frac{c}{d}\right]$.

__Proof__

If b regular then b^2 is regular and $b^2 \geqslant 0$ by 2-17(10). Now $\left[\frac{a}{b}\right] = \left[\frac{ab}{b^2}\right]$.

__Lemma 3-9__

If $b,d \geqslant 0$ then $\left[\frac{a}{b}\right] \vee \left[\frac{c}{d}\right] = \left[\frac{ad \vee cb}{bd}\right]$ and $\left[\frac{a}{b}\right] \wedge \left[\frac{c}{d}\right] = \left[\frac{ad \wedge cb}{bd}\right]$.

Proof

Since b and d regular so bd is regular and $\left[\frac{ad \vee cb}{bd}\right] \in Qc(R)$. By 2-17(5), $b(ad \vee cb) = abd \vee cb^2 \geqslant abd$ so that by 3-7 $\left[\frac{ad \vee cb}{bd}\right] \geqslant \left[\frac{a}{b}\right]$. Similarly $\left[\frac{ad \vee cb}{bd}\right] \geqslant \left[\frac{c}{d}\right]$.

If $\left[\frac{g}{h}\right] \geqslant \left[\frac{a}{b}\right], \left[\frac{c}{d}\right]$ then by 3-8 it can be assumed $h \geqslant 0$. By 3-7, $gb \geqslant ah$ and $gd \geqslant ch$ so by 2-17(1,5), $g\ bd \geqslant ahd \vee chb = h(ad \vee cb)$. Thus by 3-7, $\left[\frac{g}{h}\right] \geqslant \left[\frac{ad \vee cb}{bd}\right]$ and therefore

$\left[\frac{a}{b}\right] \vee \left[\frac{c}{d}\right] = \left[\frac{ad \vee cb}{bd}\right]$ and similarly $\left[\frac{a}{b}\right] \wedge \left[\frac{c}{d}\right] = \left[\frac{ad \wedge cb}{bd}\right]$

Theorem 3-10

If R an f-ring then Qc(R) is an f-ring containing a sub-f-ring T l-isomorphic to R.

Proof

Qc(R) is a partially ordered ring by 3-6. If $\left[\frac{a}{b}\right], \left[\frac{c}{d}\right] \in Qc(R)$ then by 3-8 it may be assumed $b,d \geqslant 0$ so by 3-9, $\left[\frac{a}{b}\right] \wedge \left[\frac{c}{d}\right]$ and $\left[\frac{a}{b}\right] \vee \left[\frac{c}{d}\right]$ exist. Therefore Qc(R) is an l-ring.

If $\left[\frac{g}{h}\right] \geqslant 0$ and by 3-8 it is assumed $h \geqslant 0$ then by 3-5, $gh \geqslant 0$ and by 3-4, $g \geqslant 0$. If $\left[\frac{a}{b}\right] \wedge \left[\frac{c}{d}\right] = 0$ and by 3-8 it is assumed $b,d \geqslant 0$ then by 3-9 $ad \wedge cb = 0$ and by 2-3, $gad \wedge cbh = 0$. By 2-17(1), $hb \geqslant 0$ so by 3-9, $\left[\frac{g}{h}\right]\left[\frac{a}{b}\right] \wedge \left[\frac{c}{d}\right] = \left[\frac{ga}{hb}\right] \wedge \left[\frac{c}{d}\right] = \left[\frac{gad \wedge cbh}{hbd}\right] = 0$. Therefore by 2-3 Qc(R) is an f-ring.

Consider the subset $T = \left\{ \left[\frac{a}{1}\right] \mid a \in R \right\}$ of Qc(R). As is well known T is isomorphic to R as a ring under the mapping: $a \rightarrow \left[\frac{a}{1}\right]$. By 2-17(10), $1 \geqslant 0$ so by 3-9, $\left[\frac{a}{1}\right] \vee \left[\frac{b}{1}\right] = \left[\frac{a \vee b}{1}\right]$ and similarly $\left[\frac{a}{1}\right] \wedge \left[\frac{b}{1}\right] = \left[\frac{a \wedge b}{1}\right]$.

Thus T is l-isomorphic to R.

Proposition 3-11

The subset $\widehat{R} = \left\{ \left[\frac{a}{b}\right] \middle| \, b \geqslant 1 \right\}$ of Qc(R) is a convex f-ring containing a sub-f-ring ,T, l-isomorphic to R.

Proof

By 2-17(26) if $b,d \geqslant 1$ then $bd \geqslant 1$ so $\widehat{R}$ is a subring of Qc(R). Since $1 \geqslant 0$ then $bd \geqslant 0$ and by 3-9, $\widehat{R}$ is a sub-l-ring of Qc(R). By 2-14, $\widehat{R}$ is an f-ring. Now the set T of 3-10 is contained in $\widehat{R}$ and it l-isomorphic to R.

If $\left[\frac{a}{b}\right] \varepsilon \widehat{R}$ and $\left[\frac{a}{b}\right] \geqslant 1$ then $b \geqslant 1$ and by 3-7, $a \geqslant b$ so $a \geqslant 1$ and by 2-17(28), a is regular. Thus $\left[\frac{b}{a}\right] \varepsilon \widehat{R}$ so $\widehat{R}$ convex.

Lemma 3-12

If S is a convex f-ring containing a sub-f-ring T l-isomorphic to an f-ring R then S contains a sub-f-ring l-isomorphic to $\widehat{R}$.

Proof

Let π be the l-isomorphism of R onto T so if $b \varepsilon R$ and $b \geqslant 1$ then $\pi b \geqslant 1$. If $c,d \varepsilon T$ and $d \geqslant 1$ then $\frac{c}{d} \varepsilon S$ since S convex. Define a map η of $\widehat{R}$ into S by $\eta\left[\frac{a}{b}\right] = \frac{\pi a}{\pi b}$. It is easily verified by the techniques of 3-7, 3-8, and 3-9 that η is an l-monomorphism.

Definition 3-13

The convex closure of an f-ring R is the $\widehat{R}$ of 3-11.

As usual R can be considered as a sub-f-ring of $\widehat{R}$ and for any element $g \varepsilon \widehat{R}$ there exists $a,b \varepsilon R$ such that

$b \geqslant 1$ and $g = \frac{a}{b}$. Also by 2-17(28) if $a,b \varepsilon R$ and $b \geqslant 1$ then $\frac{a}{b} \varepsilon \widehat{R}$.

The relation between the l-ideals of R and of $\widehat{R}$ is now investigated.

<u>Lemma 3-14</u>

If R is an f-ring and I an l-ideal in R then $\widehat{I} = \left\{ \frac{a}{b} \varepsilon \widehat{R} \mid a \varepsilon I \right\}$ is an l-ideal in R.

<u>Proof</u>

If $\frac{a}{b}, \frac{c}{d} \varepsilon \widehat{I}$ then $\frac{a}{b}+\frac{c}{d} = \frac{ad+cb}{bd}$ and $ad+cb \varepsilon I$ so $\frac{a}{b}+\frac{c}{d} \varepsilon \widehat{I}$. Similarly if $\frac{a}{b} \varepsilon \widehat{I}$ and $\frac{c}{d} \varepsilon \widehat{R}$ then $\frac{a}{b}\,\frac{c}{d} \varepsilon \widehat{I}$. Therefore $\widehat{I}$ is an ideal in $\widehat{R}$.

If $\frac{a}{b} \varepsilon \widehat{I}$ and $\left|\frac{c}{d}\right| \leqslant \left|\frac{a}{b}\right|$ then by 2-17(16) and 2-18(1), $\frac{|c|}{|d|} \leqslant \frac{|a|}{|b|}$. Now by 2-17(9,1,16) $|bc| = |b|\,|c| \leqslant |a|\,|d| = |ad|$ and $ad \varepsilon I$ so by 2-8 $bc \varepsilon I$. Since $b \geqslant 1$ then $b \geqslant 0$ and by 2-17(7) $|b| = b \geqslant 1$. Now by 2-17(9,1,16), $|bc| = |b|\,|c| \geqslant |c|$ so by 2-8, $c \varepsilon I$ and $\frac{c}{d} \varepsilon \widehat{I}$. Thus $\widehat{I}$ is an l-ideal.

<u>Lemma 3-15</u>

If R is an f-ring, I an l-ideal in R, and J an l-ideal in $\widehat{R}$ then $\widehat{I} \cap R = I$ and $\widehat{J \cap R} = J$.

<u>Proof</u>

Since $\frac{a}{1} = a$, $I \subseteq \widehat{I}$ and so $I \subseteq \widehat{I} \cap R$. If $c \varepsilon \widehat{I} \cap R$ then there exists $\frac{a}{b} \varepsilon \widehat{I}$ such that $c = \frac{a}{b}$ and $a \varepsilon I$. Thus $a = bc$ so $bc \varepsilon I$ and $b \geqslant 1$. As in the proof of 3-14, $c \varepsilon I$ so $\widehat{I} \cap R \subseteq I$. Thus $\widehat{I} \cap R = I$.

By 2-20(1), $J \cap R$ is an l-ideal in R so that $\widehat{J \cap R}$ is an l-ideal of $\widehat{R}$ by 3-14. If $\frac{a}{b} \varepsilon \widehat{J \cap R}$ then $a \varepsilon J \cap R$ by 3-14 so $a \varepsilon J$. Now $\frac{1}{b} \varepsilon \widehat{R}$ so $(\frac{1}{b})a = \frac{a}{b} \varepsilon J$. Thus $\widehat{J \cap R} \subseteq J$. If

$\frac{a}{b} \varepsilon J$ then $b(\frac{a}{b}) = a \varepsilon J$ so $a \varepsilon J \cap R$ and by 3-14, $\frac{a}{b} \varepsilon \widehat{J \cap R}$. Thus $J \subsetneq \widehat{J \cap R}$. Therefore $J = \widehat{J \cap R}$.

If R an f-ring and $c \varepsilon R$ let $\langle c \rangle_{\hat{R}}$ be the l-ideal generated by c in $\hat{R}$ and let $\langle c \rangle_R$ be the l-ideal generated by c in R (see 2-19(1)).

<u>Lemma 3-16</u>

If R an f-ring and $c \varepsilon R$ then $\langle c \rangle_{\hat{R}} \cap R = \langle c \rangle_R$

<u>Proof</u>

By 2-20(1), $\langle c \rangle_{\hat{R}} \cap R$ is an l-ideal in R containing c. Now by 2-19(1), $\langle c \rangle_R \subseteq \langle c \rangle_{\hat{R}} \cap R$. Since $\widehat{\langle c \rangle_R}$ is an l-ideal in $\hat{R}$ by 3-14 then by 2-19(1), $\langle c \rangle_{\hat{R}} \subseteq \widehat{\langle c \rangle_R}$. Therefore $\langle c \rangle_{\hat{R}} \cap R \subseteq \widehat{\langle c \rangle_R} \cap R$ and by 3-15, $\langle c \rangle_R = \widehat{\langle c \rangle_R} \cap R$ so $\langle c \rangle_R \supseteq \langle c \rangle_{\hat{R}} \cap R$. Therefore $\langle c \rangle_R = \langle c \rangle_{\hat{R}} \cap R$.

<u>Lemma 3-17</u>

If M is a maximal l-ideal in R then $\hat{M}$ is a maximal l-ideal in $\hat{R}$.

<u>Proof</u>

If $\frac{a}{b} \not\varepsilon \hat{M}$ then $a \not\varepsilon M$ by 3-14 and $M + \langle a \rangle_R = R$ since M maximal l-ideal in R. By 2-20(2) there exists $m \varepsilon M$ and $x \varepsilon \langle a \rangle_R$ such that $m+x = 1$. Since $b(\frac{a}{b}) = a$ then $\langle a \rangle_{\hat{R}} \subseteq \langle \frac{a}{b} \rangle_{\hat{R}}$ by 2-19(1). By 3-15, $M \subsetneq \hat{M}$ and by 3-16 $\langle a \rangle_R \subseteq \langle a \rangle_{\hat{R}}$ so $1 = (m+x) \varepsilon \hat{M} + \langle \frac{a}{b} \rangle_{\hat{R}}$. Thus $\hat{M}$ is a maximal l-ideal in $\hat{R}$.

<u>Lemma 3-18</u>

If N a maximal l-ideal in $\hat{R}$ then $N \cap R$ is a maximal l-ideal in R.

Proof

By 2-20(1), $N \cap R$ is an l-ideal in R. If $c \varepsilon R$ and $c \notin N \cap R$ then $c \notin N$ so that $N + \langle c \rangle_{\hat{R}} = \hat{R}$. Thus there exists $\frac{a}{b} \varepsilon N$ and $\frac{e}{d} \varepsilon \langle c \rangle_{\hat{R}}$ such that $\frac{a}{b} + \frac{e}{d} = 1$. Since $b(\frac{a}{b}) = a$ and $d(\frac{e}{d}) = e$ then $a \varepsilon N \cap R$ and $e \varepsilon \langle c \rangle_{\hat{R}} \cap R = \langle c \rangle_R$ by 3-16. Now $bd = (ad+eb) \varepsilon (N \cap R) + \langle c \rangle_R$ and $bd \geqslant 1$ by 2-17(26). By 2-8 and 2-17(7), $1 \varepsilon (N \cap R) + \langle c \rangle_R$ so $R = (N \cap R) + \langle c \rangle_R$. Therefore $N \cap R$ is a maximal l-ideal in R.

Proposition 3-19

If $\mathcal{M}$ the set of maximal l-ideals of the f-ring R and $\mathcal{N}$ the set of maximal l-ideals of $\hat{R}$ and π is the mapping of $\mathcal{M}$ into $\mathcal{N}$ defined by $\pi(M) = \hat{M}$ then π is a one-to-one and onto mapping.

Proof

By 3-17 if M a maximal l-ideal in R then $\hat{M}$ is a maximal l-ideal in $\hat{R}$ so that π is a mapping of $\mathcal{M}$ into $\mathcal{N}$. Define a mapping η of $\mathcal{N}$ into $\mathcal{M}$ by $\eta(N) = N \cap R$. By 3-18, $N \cap R$ is a maximal l-ideal in R so η is a mapping of $\mathcal{N}$ into $\mathcal{M}$. Now $\eta\pi(M) = \eta(\hat{M}) = \hat{M} \cap R = M$ by 3-15 so $\eta\pi$ is the identity map on $\mathcal{M}$. Also $\pi\eta(N) = \pi(N \cap R) = \widehat{N \cap R} = N$ by 3-15 so $\pi\eta$ is the identity map on $\mathcal{N}$. Therefore π is a one-to-one mapping of $\mathcal{M}$ onto $\mathcal{N}$.

Lemma 3-20

For any f-ring R, $J(\hat{R}) = \widehat{J(R)}$.

Proof

By 3-19 for any $N \varepsilon \mathcal{N}$ there exists $M \varepsilon \mathcal{M}$ such that $N = \hat{M}$ so $J(\hat{R}) = \bigcap \{N \varepsilon \mathcal{N}\} = \bigcap \{\hat{M} \mid M \varepsilon \mathcal{M}\}$. Now

$J(\hat{R}) \cap R = \bigcap \{\hat{M} \cap R \mid M \varepsilon \mathcal{M}\} = \bigcap \{M \mid M \varepsilon \mathcal{M}\} = J(R)$ by 3-15.
So $\widehat{J(R)} = \widehat{J(\hat{R}) \cap R} = J(\hat{R})$ by 3-15.

Lemma 3-21

For any f-ring R, $J(R) = \{0\}$ iff $J(\hat{R}) = \{0\}$.

Proof

If $J(\hat{R}) = \{0\}$ then $J(R) \subseteq J(\hat{R})$ so $J(R) = \{0\}$. If $J(R) = \{0\}$ then $\{0\} = \widehat{J(R)} = J(\hat{R})$.

CHAPTER IV

MAXIMAL l-IDEAL SPACE OF AN f-RING

Throughout let $\mathcal{M}$ be the set of all maximal l-ideals of an f-ring R.

Definition 4-1

For any set $A \subseteq R$, let $S(A) = \{M \in \mathcal{M} \mid A \not\subseteq M\}$. In particular if $a \in R$ then $S(a) = \{M \in \mathcal{M} \mid a \notin M\}$.

Proposition 4-2

The collection of sets, $\{S(A) \mid A \subseteq R\}$, form a topology on $\mathcal{M}$. A basis for this topology is given by the sets $\{S(r) \mid r \in R\}$.

Proof

If $\langle A \rangle$ is the l-ideal generated by $A \subseteq R$ (see 2-19(1)) then $S(\langle A \rangle) = S(A)$ since $A \not\subseteq M$ iff $\langle A \rangle \not\subseteq M$ for any $M \in \mathcal{M}$. Now

$M \in \bigcup\{S(A_i) \mid i \in I\} \Leftrightarrow \exists i \in I,\ A_i \not\subseteq M \Leftrightarrow$
$M \in S(\sum\{\langle A_i \rangle \mid i \in I\})$. Therefore $\bigcup\{S(A_i) \mid i \in I\}$
$= S(\sum\{\langle A_i \rangle \mid i \in I\})$.

By 2-20(6,3), $M \in S(A) \cap S(B) \Leftrightarrow \langle A \rangle \not\subseteq M$ and $\langle B \rangle \not\subseteq M \Leftrightarrow \langle \langle A \rangle \ \langle B \rangle \rangle \not\subseteq M \Leftrightarrow M \in S(\langle \langle A \rangle \ \langle B \rangle \rangle) = S(\langle A \rangle \ \langle B \rangle)$. Therefore $S(A) \cap S(B) = S(\langle A \rangle \ \langle B \rangle)$.

Now $S(1) = \mathcal{M}$ and $S(0) = \emptyset$ so the sets $\{S(A) \mid A \subseteq R\}$ for a topology on $\mathcal{M}$.

Now $S(A) = S(\langle A \rangle) = S(\sum\{\langle a \rangle \mid a \in A\})$

$= \bigcup\{S(a) \mid a \in A\}$ so the sets $\{S(a) \mid a \in R\}$ form a basis for this topology.

<u>Lemma 4-3</u>

The space $\mathfrak{M}$ is compact.

<u>Proof</u>

Suppose $\{S(A_i) \mid i \in I\}$ is an open cover of $\mathfrak{M}$ then $\mathfrak{M} = \bigcup\{S(A_i) \mid i \in I\} = S(\sum\{\langle A_i\rangle \mid i \in I\})$ as in proof of 4-2 so for all $M \in \mathfrak{M}$, $\sum\{\langle A_i\rangle \mid i \in I\} \not\subseteq M$. Therefore by 2-20(4), $R = \sum\{\langle A_i\rangle \mid i \in I\}$, and so $1 \in \sum\{\langle A_i\rangle \mid i \in I\}$. By 2-20(2) there exists a finite subset F of I such that $1 \in \sum\{\langle A_i\rangle \mid i \in F\}$. Thus for all $M \in \mathfrak{M}$, $\sum\{\langle A_i\rangle \mid i \in F\} \notin M$ so that $\mathfrak{M} = S(\sum\{\langle A_i\rangle \mid i \in F\}) = \bigcup\{S(A_i) \mid i \in F\}$. Therefore $\{S(A_i) \mid i \in F\}$ is a finite subcover. Thus $\mathfrak{M}$ is compact.

<u>Lemma 4-4</u>

The space $\mathfrak{M}$ is Hausdorff.

<u>Proof</u>

Consider any $M, N \in \mathfrak{M}$ such that $M \neq N$. Then $M+N = R$ so by 2-20(2) there exists $r \in M$ and $s \in N$ such that $r+s = 1$. Let $a = r-s$. Now $a = r-s = r-(1-r) = r+r-1$ so by 2-6, $a_M = (r+r-1)_M = r_M+r_M-1_M = 0_M+0_M-1_M = -1_M$. Then by 2-6 and 2-17(21), $(a^-)_M = a_M{}^- = (-1_M)^- = 1_M \neq 0_M$ so that $a^- \notin M$ or $M \in S(a^-)$. Similarly $N \in S(a^+)$. Now as in proof of 4-2 and by 2-20(5), $S(a^+) \cap S(a^-) = S(\langle\langle a^+\rangle \langle a^-\rangle\rangle)$ $= S(\langle a^+a^-\rangle) = S(a^+a^-) = S(0) = \emptyset$ by 2-17(19). Thus

$M \varepsilon S(a^+)$ and $N \varepsilon S(a^-)$ and $S(a^-) \cap S(a^+) = \emptyset$ so $\mathcal{M}$ is Hausdorff.

Theorem 4-5

The maximal l-ideal space $\mathcal{M}$ of any f-ring is compact and Hausdorff.

Theorem 4-6

If $\mathcal{M}$ is the maximal l-ideal space of an f-ring R and $\mathcal{N}$ the maximal l-ideal space of $\hat{R}$ then $\mathcal{M}$ is homeomorphic to $\mathcal{N}$.

Proof

If $A \subseteq R$ let $S_{\mathcal{M}}(A) = \{ M \varepsilon \mathcal{M} | A \nsubseteq M \}$ and if $B \subseteq \hat{R}$ let $S_{\mathcal{N}}(B) = \{ N \varepsilon \mathcal{N} | B \nsubseteq N \}$.

Consider the mapping π of $\mathcal{M}$ into $\mathcal{N}$ defined by $\pi(M) = \hat{M}$. By 3-19, π is a one-to-one onto mapping. Let I be any l-ideal of R. If $I \subseteq M \varepsilon \mathcal{M}$ then $\hat{I} \subseteq \hat{M}$ and if $\hat{I} \subseteq \hat{M}$ then $I = \hat{I} \cap R \subseteq \hat{M} \cap R = M$ by 3-15. Thus $I \nsubseteq M \varepsilon \mathcal{M}$ iff $\hat{I} \nsubseteq \hat{M}$. Now if $M \varepsilon S_{\mathcal{M}}(I)$ then $\pi(M) = \hat{M} \varepsilon S_{\mathcal{N}}(\hat{I})$ so $\pi(S_{\mathcal{M}}(I)) \subseteq S_{\mathcal{N}}(\hat{I})$. If $N \varepsilon S_{\mathcal{N}}(\hat{I})$ then by 3-19 there exists $M \varepsilon \mathcal{M}$ such that $N = \hat{M}$ and so $\hat{I} \nsubseteq \hat{M}$. Then $I \nsubseteq M$ so $M \varepsilon S_{\mathcal{M}}(I)$ and $\pi M = \hat{M} = N \varepsilon \pi(S_{\mathcal{M}}(I))$. Thus $S_{\mathcal{N}}(\hat{I}) \subseteq \pi(S_{\mathcal{M}}(I))$ so $\pi(S_{\mathcal{M}}(I)) = S_{\mathcal{N}}(\hat{I})$.

Now all open sets of $\mathcal{M}$ are of the form $S_{\mathcal{M}}(A)$ for some $A \subseteq R$ and $S_{\mathcal{M}}(A) = S_{\mathcal{M}}(\langle A \rangle)$ as in the proof of 4-2. Thus all open sets of $\mathcal{M}$ are of the form $S_{\mathcal{M}}(I)$ for some

1-ideal I of R. Therefore by above π is a one-to-one onto open mapping of a Hausdorff space onto a compact space and so π is a homeomorphism.

CHAPTER V

PRELIMINARY RESULTS FROM THE THEORY OF TOPOLOGICAL RINGS

Definition 5-1

A family of subsets, $\mathcal{B}$, of a topological space, S, is a local base at $p \varepsilon S$ if every $B \varepsilon \mathcal{B}$ is a neighborhood of p and if V a neighborhood of p then there exists $W \varepsilon \mathcal{B}$ such that $W \subseteq V$.

Definition 5-2

In a topological space, S, a point $x \varepsilon S$ is an adherence point of the subset $A \subseteq S$ if all neighborhoods of x intersect A in a nonnull set.

Definition 5-3

A filter, $\mathcal{F}$, on a set, S, is a family of subsets of S such that

1) $F \varepsilon \mathcal{F}$ and $F \subseteq X \subseteq S \Rightarrow X \varepsilon \mathcal{F}$
2) $F_1, F_2 \varepsilon \mathcal{F} \Rightarrow F_1 \cap F_2 \varepsilon \mathcal{F}$
3) $S \varepsilon \mathcal{F}$
4) $\emptyset \not\varepsilon \mathcal{F}$

Definition 5-4

If $\mathcal{B}$ is a family of subsets of S such that the set $\mathcal{F} = \{X \subseteq S \mid \exists B \varepsilon \mathcal{B}, B \subseteq X\}$ is a filter on S then $\mathcal{B}$ is a filter base on S of $\mathcal{F}$ and $\mathcal{F}$ is the filter generated by $\mathcal{B}$.

Definition 5-5

If $\mathcal{B}$ a filter base on S then a point $p \varepsilon S$ is a limit point of $\mathcal{B}$ if for every neighborhood V of p there exists $B \varepsilon \mathcal{B}$ such that $B \subseteq V$. A filter base is convergent if it has a limit point.

Definition 5-6

If $\mathcal{B}$ a filter base on S then a point $p \varepsilon S$ is an adherence point of $\mathcal{B}$ if p adherent to every $B \varepsilon \mathcal{B}$.

Proposition 5-7

If f and g are two continuous mappings from a topological space, S, into a Hausdorff topological space, T, $A \subseteq S$ is dense in S and $f|A = g|A$ then $f = g$.

Proposition 5-8

If $A_i \subseteq S_i$ dense in the topological space, S_i, for each i then $A_1 X A_2 X \ldots X A_n$ is dense in $S_1 X S_2 X \ldots X S_n$.

Proposition 5-9

A set of subsets $\mathcal{B}$ of a set S is a filter base on S iff

1) $B_1, B_2 \varepsilon \mathcal{B} \Rightarrow \exists C \varepsilon \mathcal{B},\ C \subseteq B_1 \cap B_2$

2) $\mathcal{B} \neq \emptyset$ and $\emptyset \notin \mathcal{B}$.

Proposition 5-10

If s is a limit point of a filter, $\mathcal{H}$, in a topological space then s is an adherence point of $\mathcal{H}$.

Proposition 5-11

If S and T are topological spaces, f and g mappings from S into T, W open in S, $g(x) = f(x)$ for all $x \varepsilon W$, and g continuous at $s \varepsilon W$, then f is continuous at s.

Proposition 5-12

If S, T and V are topological spaces and the mapping, f, of $S \times T$ into V is continuous then all of the maps $g_s : T \to V$ and $h_t : S \to V$ defined by $g_s(t) = f(s,t)$ and $h_t(s) = f(s,t)$ are continuous.

Definition 5-13

A uniform structure on a set S is a family of subsets, $\mathcal{U}$, of $S \times S$ such that

1) $\mathcal{U}$ is a filter on $S \times S$
2) $U \varepsilon \mathcal{U} \Rightarrow \Delta \subseteq U$ where $\Delta = \{\langle x,x \rangle \mid x \varepsilon S\}$
3) $V \varepsilon \mathcal{U} \Rightarrow V^{-1} \varepsilon \mathcal{U}$ where $V^{-1} = \{\langle x,y \rangle \mid \langle y,x \rangle \varepsilon V\}$
4) $V \varepsilon \mathcal{U} \Rightarrow \exists W \varepsilon \mathcal{U}, W \circ W \subseteq V$ where

$$A \circ B = \{\langle x,z \rangle \mid \exists y \varepsilon S, \ \langle x,y \rangle \varepsilon A \text{ and } \langle y,z \rangle \varepsilon B\}.$$

A uniform space, $\langle S, \mathcal{U} \rangle$, is a set S with uniform structure, $\mathcal{U}$, on S. The sets of $\mathcal{U}$ are called entourages.

Definition 5-14

A base, $\mathcal{B}$, for a uniform structure, $\mathcal{U}$, on S is a family of subsets, $\mathcal{B}$, of $S \times S$ such that $\mathcal{B} \subseteq \mathcal{U}$ and

$$U \varepsilon \mathcal{U} \Rightarrow \exists B \varepsilon \mathcal{B}, \ B \subseteq U.$$

Definition 5-15

A mapping, f, from a uniform space, $\langle S, \mathcal{U} \rangle$, into a uniform space, $\langle T, \mathcal{W} \rangle$, is uniformly continuous if for all $W \varepsilon \mathcal{W}$ there exists $U \varepsilon \mathcal{U}$ such that

$$\langle x,y \rangle \varepsilon U \Rightarrow \langle f(x), f(y) \rangle \varepsilon W.$$

Definition 5-16

A uniform space, S, is (uniformly) isomorphic to a uniform space, T, if there exists a one-to-one onto mapping, f, of S onto T such that f and f^{-1} are both uniformly continuous.

Definition 5-17

If S_1 and S_2 are uniform spaces then the uniform structure on $S = S_1 X S_2$ is the smallest such that each of the maps $p_i : S \longrightarrow S_i$ is uniformly continuous where $p_i(\langle s_1, s_2 \rangle) = s_i$ for $i = 1,2$.

Definition 5-18

If $\langle S, \mathcal{U} \rangle$ is a uniform space and $A \subseteq S$ then the uniform structure, $\mathcal{W}$, induced on A by the uniform structure on S is the trace on AXA of the sets of the uniformity, $\mathcal{U}$. $\langle A, \mathcal{W} \rangle$ is called a uniform subspace of $\langle S, \mathcal{U} \rangle$.

Proposition 5-19

Let f, g_i be maps as follows;

$$g_i : S_1^i X S_2^i X \cdots X S_{m_i}^i \longrightarrow T_i \text{ and } f : T_1 X T_2 X \cdots X T_n \longrightarrow V$$

and also let h be the map

$$h : S_1^1 X \cdots X S_{m_1}^1 X S_1^2 X \cdots X S_{m_2}^2 X \cdots X S_1^n X \cdots X S_{m_n}^n \longrightarrow V$$

where $h(s_1^1, \cdots, s_{m_n}^n) = f(g_1(s_1^1, \cdots s_{m_1}^1), \cdots, g_n(s_1^n, \cdots, s_{m_n}^n))$ then

1) if f and all g_i are continuous then h is continuous,

2) if f and all g_i are uniformly continuous then

h is uniformly continuous.

Proposition 5-20

If S and T are uniform spaces, f a mapping from S into T, $\mathcal{B}$ a base for the uniformity of S, and $\mathcal{C}$ a base for the uniformity of T then f is uniformly continuous iff

$$\forall C \varepsilon \mathcal{C}, \exists B \varepsilon \mathcal{B}, \langle x,y \rangle \varepsilon \mathcal{B} \Rightarrow \langle f(x),f(y) \rangle \varepsilon C.$$

Proposition 5-21

If $\langle S, \mathcal{U} \rangle$ is a uniform space and $\mathcal{B}(x) = \{V(x) | V \varepsilon \mathcal{U}\}$ then there exists a unique topology on S such that for all $x \varepsilon S$, $\mathcal{B}(x)$ is the filter of neighborhoods of x. The topology induced on S by the uniform structure $\mathcal{U}$ is called the uniform topology for $\mathcal{U}$.

Proposition 5-22

If S and T are uniform spaces and $f:S \rightarrow T$ is a uniformly continuous mapping then f is continuous for the uniform topologies on S and T.

Definition 5-23

If $\langle S, \mathcal{U} \rangle$ is a uniform space then a set $A \subseteq S$ is of order $U \varepsilon \mathcal{U}$ if $A X A \subseteq U$.

Definition 5-24

If $\langle S, \mathcal{U} \rangle$ is a uniform space then a Cauchy filter, $\mathcal{F}$, is a filter on S such that for all $U \varepsilon \mathcal{U}$, there exists $F \varepsilon \mathcal{F}$ such that F is of order U.

A Cauchy filter base is a filter base whose generated filter is Cauchy.

Proposition 5-25

If $\mathcal{B}$ is a Cauchy filter base in a subspace, T, of a uniform space, S, then $\mathcal{B}$ is a Cauchy filter base in S.

Proposition 5-26

If S and T are uniform spaces, $f:S \rightarrow T$, a uniformly continuous mapping, and $\mathcal{B}$ a Cauchy filter base on S then the image of $\mathcal{B}$ is a Cauchy filter base on T.

Proposition 5-27

A complete space is a uniform space in which all Cauchy filters are convergent.

Proposition 5-28

In a complete Hausdorff uniform space every Cauchy filter base has a unique limit point.

Proposition 5-29

If S is a uniform space and T a dense subspace such that all Cauchy filter bases on T are convergent in S then S is complete.

Proposition 5-30

If S and T are uniform spaces, T Hausdorff and complete, A a dense subspace of S, and $f:A \rightarrow T$ is a uniformly continuous mapping, then there exists a unique mapping, $\overline{f}: S \to T$ such that $\overline{f}$ is uniformly continuous and $\overline{f}|A = f$.

Definition 5-31

A topological ring, $\langle R, \mathcal{J} \rangle$, is a ring, R, together with a topology, $\mathcal{J}$, on R such that

1) the mapping $x \to -x$ of R into R is continuous,

2) the mapping $\langle x,y \rangle \to x+y$ of $R \times R$ into R is continuous,

3) the mapping $\langle x,y \rangle \to xy$ of $R \times R$ into R is continuous.

Proposition 5-32

If A is a closed set in the topological ring, R, then $-A$ and $\{r\}+A$ are closed sets for any $r \in R$.

Proposition 5-33

If R, a commutative ring and $\mathcal{B}$, a filter base on R such that

1) $U \in \mathcal{B} \Rightarrow \exists V \in \mathcal{B},\ V+V \subseteq U$

2) $U \in \mathcal{B} \Rightarrow \exists V \in \mathcal{B},\ -V \subseteq U$

3) $U \in \mathcal{B} \Rightarrow \exists V \in \mathcal{B},\ V \cdot V \subseteq U$

4) $U \in \mathcal{B}$ and $r \in R \Rightarrow \exists V \in \mathcal{B}, \{r\} \cdot V \subseteq U$

then there exists a unique topology $\mathcal{J}$ on R which makes $\langle R, \mathcal{J} \rangle$, a topological ring and for which $\mathcal{B}$ is a local base at $0 \in R$ in the topology, $\mathcal{J}$. For the topology, $\mathcal{J}$, the set $\mathcal{B}+a = \{V+\{a\} \mid V \in \mathcal{B}\}$ is a local base at $a \in R$.

Proposition 5-34

A topological ring is Hausdorff iff the intersection of all neighborhoods of 0 is $\{0\}$.

Proposition 5-35

Let $\mathcal{U}$ be the collection of all subsets of RXR of the form $U = \{\langle x,y\rangle \mid x,y\varepsilon R \text{ and } x-y\varepsilon V\}$ where V is a neighborhood of 0 in the topological ring, R. Then $\mathcal{U}$ is the base of a uniform structure on R and the uniform structure generated by $\mathcal{U}$ is called the ring uniformity of R. The uniform topology for $\mathcal{U}$ is the original topology on R.

Proposition 5-36

If $\mathcal{B}$, a local base at 0 in the topological ring R then $\mathcal{U}$, the collection of all subsets of RXR of the form $U = \{\langle x,y\rangle \mid x,y\varepsilon R \text{ and } x-y\varepsilon B\varepsilon \mathcal{B}\}$ is a base for the ring uniformity of R.

Proposition 5-37

For a topological ring, R, the mapping $\langle x,y\rangle \rightarrow x+y$ of RXR into R and the mapping $x \rightarrow -x$ of R into R are uniformly continuous in the ring uniformity of R.

Definition 5-38

An isomorphism π of a topological ring, R, into a topological ring, S, is a mapping, π, of R onto S such that π is both a ring isomorphism and a homeomorphism of the topological spaces.

Proposition 5-39

If S_1 and S_2 are complete, Hausdorff topological rings and R_1,R_2 dense subrings of S_1 and S_2 respectively, and π, an isomorphism of R_1 onto R_2 then there exists a unique isomorphism, $\bar{\pi}$, of S_1 onto S_2 such that $\bar{\pi}|R = \pi$.

Proposition 5-40

For any commutative Hausdorff topological ring, R, with identity there exists a commutative Hausdorff topological ring, R^c, with identity such that

1) R is isomorphic to a dense subring of R^c

2) R^c is complete in its ring uniformity

3) if S is another commutative Hausdorff topological ring with identity satisfying (1) and (2) then S is isomorphic to R^c.

R^c is called the completion of R.

Proposition 5-41

If R is a Hausdorff topological ring with $\mathcal{B}$ as a local base at 0 then the collection $\{\bar{B} | B \in \mathcal{B}\}$ is a local base at 0 for the topology of R^c.

Definition 5-42

A topological field $\langle F, \mathcal{J} \rangle$ is a field F with a topology, $\mathcal{J}$, on F such that

1) $\langle F, \mathcal{J} \rangle$ is a topological ring

2) the mapping $x \to x^{-1}$ of F^* into F^* is continuous where F^* is the subspace, $F - \{0\}$, of F.

Proposition 5-43

The completion of a topological field, F, as a topological ring, is a topological field iff under the mapping $x \to x^{-1}$ of F^* into F^* any Cauchy filter in F^* which as the base of a Cauchy filter in F does not have 0 as an adherence point is mapped onto a Cauchy filter in F^*.

CHAPTER VI

O-RINGS AND THEIR TOPOLOGY

Definition 6-1

In an ℓ-ring, R, for u a p-unit in R define $I(u) = \{x \varepsilon R \mid |x| \leqslant u\}$.

Note that although I(u) could be defined for elements other than p-units, in the following it will be used for p-units only.

Theorem 6-2

If R is a convex f-ring there is a unique topology on R that makes R a topological ring and which has as a local base at 0 the collection $\mathcal{B} = \{I(u) \mid u \text{ a p-unit in } R\}$. The collection $x_o + \mathcal{B} = \{x_o + I(u) \mid I(u) \varepsilon \mathcal{B}\}$ is a local base at $x_o \varepsilon R$ in this topology.

Proof

By 2-18(4) if u,v p-units then $u \wedge v$ is a p-unit. Thus

$$x \varepsilon I(u) \cap I(v) \Leftrightarrow |x| \leqslant u \text{ and } |x| \leqslant v \Leftrightarrow |x| \leqslant u \wedge v$$
$$\Leftrightarrow x \varepsilon I(u \wedge v),$$

so $I(u) \cap I(v) = I(u \wedge v)$. Now $I(1) \varepsilon \mathcal{B}$ so $\mathcal{B} \neq \emptyset$ and $0 \varepsilon I(u)$ for all p-units so $\emptyset \notin \mathcal{B}$. Therefore by 5-9, $\mathcal{B}$ is a filter base on R.

Since $2 \geqslant 1$ and R convex then by 3-1, 2-17(29), 2-18(5,6), $\frac{u}{2}$ is a p-unit if u a p-unit. Thus $I(\frac{u}{2}) \varepsilon \mathcal{B}$ and by 2-17(14,2), $I(\frac{u}{2}) + I(\frac{u}{2}) \subseteq I(u)$. By

2-17(30), $-I(u) = I(u)$.

If $r \varepsilon R$ let $v = |r| \vee 1$ so $v \geqslant 1$ and since R convex then by 3-1 and 2-17(29), v is a p-unit. Now $|r| \leqslant v$ so by 2-18(6) and 2-17(1), $\frac{|r|}{v} \leqslant 1$. By 2-18(5,6) $I\left(\frac{u}{v}\right) \varepsilon \mathcal{B}$ for any p-unit u. Now

$x \varepsilon I\left(\frac{u}{v}\right) \Rightarrow |x| \leqslant \frac{u}{v} \Rightarrow |rx| = |r||x| \leqslant \frac{|r|u}{v} \leqslant u \Rightarrow rx \varepsilon I(u)$

by 2-17(16,9,1).

Thus $\{r\} \cdot I\left(\frac{u}{v}\right) \subseteq I(u)$.

If u a p-unit then by 2-18(4), $v = u \wedge 1$ is a p-unit and $v \leqslant 1$ so $v^2 \leqslant v \leqslant u$ by 2-17(1). Thus $I(v) \varepsilon \mathcal{B}$ and by 2-17(16,9,1), $I(v) \cdot I(v) \subseteq I(u)$.

Therefore by 5-33, there exists a unique topology $\mathcal{D}$ on R such that R is a topological ring and $\mathcal{B}$ is a local base at 0 in this topology. Also $x_0 + \mathcal{B}$ is a local base at x_0 in this topology.

From now on this topology $\mathcal{D}$ will be referred to as the p-unit topology on the convex f-ring R.

Definition 6-3

A C-ring, R, is a convex f-ring with $J(R) = \{0\}$.

A C-ring could be characterized as a sub-f-ring of a cardinal product of totally ordered fields.

Proposition 6-4

If R is an f-ring then $J(R) = \{0\}$ iff the convex closure, $\widehat{R}$, is a C-ring.

Proof

By 3-21, the proposition holds.

Lemma 6-5

If R a convex f-ring then $u \in R$ is a p-unit iff $u_M > 0$ for all $M \in \mathcal{M}$.

Proof

If u a p-unit then $u \notin M$ for all $M \in \mathcal{M}$ so $u_M \neq 0$ for all $M \in \mathcal{M}$. By 2-9, $u_M \geq 0$ for all $M \in \mathcal{M}$, so $u_M > 0$ for all $M \in \mathcal{M}$.

If $u_M > 0$ for all $M \in \mathcal{M}$ then for any ℓ-ideal I of R such that R_I is totally ordered let M be a maximal ℓ-ideal such that $I \subseteq M$ by 2-20(4). For any $r \in R$ if $r_I \leq 0$ then $r_M \leq 0$ by 2-24 and since R_I and R_M are totally ordered then if $r_M > 0$ then $r_I > 0$. Therefore $u_I > 0$ for all I such that R_I is totally ordered and by 2-12, $u \geq 0$. Now $u_M \neq 0$ for all $M \in \mathcal{M}$ so $u \notin M$ for all $M \in \mathcal{M}$ by 2-9. Therefore by 3-3, u is a unit and so u is a p-unit.

Lemma 6-6

If R a convex f-ring and $r \in R$, $M \in \mathcal{M}$ are such that $r_M > 0$ then there exists a p-unit $u \in R$ such that $r_M = u_M$.

Proof

For any $N \in \mathcal{M}$ such that $N \neq M$ there exists $s \in R$ such that $s \in S(N)$ and $s \notin S(M)$ by 4-4. For each $N \neq M$ choose such an s^N and let $\mathcal{C} = \{S(s^N) \mid N \neq M\} \cup S(r)$. Since $r_M \neq 0$ then $r \notin M$ or $M \in S(r)$ so $\mathcal{C}$ is an open cover of $\mathcal{M}$. Since by 4-3, $\mathcal{M}$ is compact then there exists $\mathcal{H} \subseteq \mathcal{C}$ such that $\mathcal{H}$ is a finite cover of $\mathcal{M}$. Since $M \notin S(s^N)$ for all $N \neq M$ then $S(r) \in \mathcal{H}$. Let $\mathcal{H}$ be determined by the elements $s_1, s_2, \cdots, s_n$ and r and let $u = |s_1| \vee \cdots \vee |s_n| \vee r$. Then for any $N \varepsilon \mathcal{M}$ either there exists s_i such that $N \varepsilon S(s_i)$ or $N \varepsilon S(r)$ since $\mathcal{H}$ is a cover of $\mathcal{M}$ so that either $(s_i)_N \neq 0$ or $r_N \neq 0$. By 2-9 and 2-17(9) $|s_i|_N = |(s_i)_N| > 0$ or $r_N > 0$. Therefore by 2-9, $u_N \geq |s_i|_N > 0$ or $u_N \geq r_N > 0$ so $u_N > 0$ for all $N \varepsilon \mathcal{M}$. By 6-5, u is a p-unit in R. Now all $s_i \varepsilon M$ so $(s_i)_M = 0$ and by 2-17(13), $|s_i|_M = 0$. Therefore $u_M = r_M$ by 2-9 and 2-17(9).

Lemma 6-7

If R a convex f-ring then $J(R) = \bigcap \{I(u) | u$ a p-unit in $R\}$.

Proof

If $r \varepsilon J(R)$ then $r_M = 0$ for all $M \varepsilon \mathcal{M}$ by 2-9 so $|r_M| = |r|_M = 0$ by 2-17(13). Thus by 6-5, $|r|_M < u_M$ for all $M \varepsilon \mathcal{M}$ and all p-units u. By 2-9, $(u - |r|)_M > 0$ for all $M \varepsilon \mathcal{M}$ so by 6-5, $u - |r| > 0$ and by 2-17(2), $u > |r|$. Therefore $r \varepsilon I(u)$ for all p-units u. Thus $J(R) \subseteq \bigcap \{I(u) | u$ a p-unit in $R\}$.

If v a p-unit then by 2-18(5,6), $\frac{v}{2}$ is a p-unit and by 2-18(3), $0 < \frac{1}{2} < 1$ so by 2-17(25), $0 < \frac{v}{2} < v$. If for some $M \varepsilon \mathcal{M}$, $(\frac{v}{2})_M = v_M$ then $0 = v_M - (\frac{v}{2})_M = (v - \frac{v}{2})_M = (\frac{v}{2})_M$ so by 2-9, $\frac{v}{2} \varepsilon M$. This is a contradiction since $\frac{v}{2}$ a unit so $(\frac{v}{2})_M \neq v_M$ for all $M \varepsilon \mathcal{M}$. By 2-9, $(\frac{v}{2})_M \leq v_M$ so

$(\frac{v}{2})_M < v_M$ for all $M \varepsilon \mathcal{M}$.

If $r \varepsilon \bigcap \{I(u) | u$ a p-unit$\}$ then $|r| \leq u$ for all p-units u so by 2-9 $|r|_M \leq u_M$ for all $M \varepsilon \mathcal{M}$ and all p-units u. If $r \notin J(R)$ then there exists $M \varepsilon \mathcal{M}$ such that $r \notin M$ or by 2-9, $r_M \neq 0$ so by 2-9 and 2-17(13,9), $|r_M| = |r|_M > 0$. By 6-6 there exists a p-unit v such that $v_M = |r|_M$. Then by above $(\frac{v}{2})_M < |r|_M$. This is a contradiction so $r \varepsilon J(R)$. Therefore $\bigcap \{I(u)\} \subseteq J(R)$ and so $J(R) = \bigcap \{I(u)\}$.

Proposition 6-8

A convex f-ring, R, is Hausdorff in the p-unit topology iff R is a C-ring.

Proof

Since the collection, $\{I(u) | u$ a p-unit$\}$, forms a local base at 0 the intersection of all neighborhoods of 0 by 5-1 is $\bigcap \{I(u) | u$ a p-unit$\} = J(R)$ by 6-7. By 5-34, R is Hausdorff iff $J(R) = \{0\}$. Thus R is Hausdorff iff R is a C-ring by 6-3.

The uniformity of the topological ring, R, where R is a convex f-ring with the p-unit topology will be referred to as the p-unit uniformity on R.

Lemma 6-9

If R a convex f-ring and u a p-unit let $V_u = \{\langle x,y \rangle \varepsilon R X R \,|\, |x-y| \leq u\}$. Then the collection $\{V_u | u$ a p-unit$\}$ is a base for the p-unit uniformity on R.

Proof

Since $x-y \in I(u)$ iff $|x-y| \leq u$ then $\{\langle x,y\rangle \in R \times R \mid x-y \in I(u)\} = V_u$. By 6-2 and 5-36 the collection, $\{V_u \mid u \text{ a p-unit}\}$, is a base for the p-unit uniformity on R.

Lemma 6-10

A function $f:R \to R$ on a convex f-ring, R, is uniformly continuous in the p-unit uniformity iff for all p-units u there exists a p-unit v such that $|x-y| \leq v$ implies $|f(x)-f(y)| \leq u$.

Proof

The result follows from 6-9 and 5-20.

Lemma 6-11

If R a convex f-ring then the mappings $\langle x,y\rangle \to x \vee y$ and $\langle x,y\rangle \to x \wedge y$ of $R \times R$ into R are uniformly continuous in the p-unit uniformity on R.

Proof

If u a p-unit in R and if $|x-y| \leq u$ then $|x^+-y^+| \leq u$ by 2-17(17) so by 6-10 the mapping $x \to x^+$ of R into R is uniformly continuous in the p-unit uniformity. By 2-17(22), 5-37, and 5-19 the mapping $\langle x,y\rangle \to x \vee y$ is uniformly continuous. By 2-17(23), 5-37, and 5-19 the mapping $\langle x,y\rangle \to x \wedge y$ is uniformly continuous.

Definition 6-12

A topological f-ring $\langle R, \mathcal{B} \rangle$ is an f-ring, R, with a topology, $\mathcal{B}$, on R such that

1) $\langle R, \mathcal{B} \rangle$ is a topological ring,

2) the mappings of $\langle x,y \rangle \to x \vee y$ and $\langle x,y \rangle \to x \wedge y$ of RXR into R are continuous.

Definition 6-13

An ℓ-isomorphism, π, of a topological f-ring, R, onto a topological f-ring, S, is a mapping, π, of R onto S such that π is an ℓ-isomorphism of R onto S and π is a homeomorphism of R onto S.

Proposition 6-14

A convex f-ring, R, is a topological f-ring in the p-unit topology.

Proof

By 6-2, R is a topological ring. By 6-11, 5-35, and 5-22, R is a topological f-ring.

Lemma 6-15

If S_1 and S_2 are complete Hausdorff topological f-rings and R_1, R_2 are dense sub-f-rings of S_1, S_2 and if π is an ℓ-isomorphism of R_1 onto R_2 then there is a unique ℓ-isomorphism $\overline{\pi}$ of S_1 onto S_2 such that $\overline{\pi}/R_1 = \pi$.

Proof

By 5-39 there is a unique isomorphism, $\overline{\pi}$, of S onto S as topological rings such that $\bar{\pi}|R_1 = \pi$. Now the mappings $\langle x,y\rangle \longrightarrow \bar{\pi}(x \wedge y)$ and $\langle x,y\rangle \longrightarrow \bar{\pi}(x) \wedge \bar{\pi}(y)$ of $S_1 X S_1$ into S_2 are continuous by 5-19, 6-12 and 6-13. Also $\bar{\pi}(x \wedge y) = \bar{\pi}(x) \wedge \bar{\pi}(y)$ for all $x,y \varepsilon R_1$ since $\bar{\pi}|R_1 = \pi$. So by 5-8 and 5-7, $\bar{\pi}(x \wedge y) = \bar{\pi}(x) \wedge \bar{\pi}(y)$ for all $x,y \varepsilon S_1$. Therefore $\bar{\pi}$ is an ℓ-isomorphism of S_1 onto S_2.

<u>Theorem 6-16</u>

If R is a C-ring with the p-unit topology then there exists a commutative Hausdorff topological f-ring, R^c, with identity such that

1) R is ℓ-isomorphic to a dense subring of R^c,
2) R^c is complete in its ring uniformity,
3) if S is another commutative Hausdorff topological f-ring with identity satisfying 1) and 2) then S is ℓ-isomorphic to R^c.

R^c is called the completion of R.

<u>Proof</u>

By 6-8 and 6-14, R is a commutative Hausdorff topological f-ring with identity in the p-unit topology. By 5-40 there exists a commutative Hausdorff topological f-ring R^c with identity such that R is isomorphic to a dense subring, T, of R^c and R^c is complete in its uniformity. Define lattice operations on T by means of the isomorphic mapping of R onto T and T is a C-ring for these operations. Since the mapping is a homeomorphism the topology induced on T by R^c must be the p-unit topology on T. By 6-11, 5-8 and 5-30 the binary operations $\vee$ and $\wedge$ on T can be extended to R^c. By 5-35 and 5-22, $\vee$ and $\wedge$ are continuous operations from $R^c \times R^c$ into R^c. Then by 5-19, 5-12 and 5-7 any equation in terms of $0, 1, -, +, \cdot, \vee$, and $\wedge$ holding in R also hold in R^c. Therefore by 2-16, R^c is a commutative f-ring with identity. Also by 6-12, R^c is a topological f-ring.

If S is another commutative Hausdorff topological f-ring with identity satisfying 1) and 2) then by 6-15, S is ℓ-isomorphic to R^c.

The question immediately occurs, "Is R^c a C-ring and is its topology the p-unit topology." In the following an affirmative answer is given by representing R^c as a ring of functions.

<u>Definition 6-17</u>

A topological f-field is an f-field which is both a topological f-ring and a topological field (see 5-42).

<u>Lemma 6-18</u>

If R is a convex f-ring with the p-unit topology then for u a p-unit the set $\{x\varepsilon R \mid |x-a| \leq u\}$ is a neighborhood of $a\varepsilon R$.

Proof

By 6-2, $a+I(u)$ is a neighborhood of $a\varepsilon R$. Now

$$y\varepsilon a+I(u) \Longleftrightarrow (y-a)\varepsilon I(u) \Longleftrightarrow |y-a| \leqslant u$$

so
$$a+I(u) = \{y\varepsilon R \mid |y-a| \leqslant u\}.$$

Lemma 6-19

If F is an f-field then F is a topological f-field in the p-unit topology.

Proof

Now F is automatically a C-ring so F is a topological f-ring in the p-unit topology by 6-14. If $a\varepsilon F^*$ then by 2-17(13,9), $|a| > 0$ so if u a p-unit then by 2-18(5,6,2) $\frac{|a|}{2} \wedge \frac{|a|^2 u}{2}$ is a p-unit. By 6-18, $W = \{x\varepsilon F \mid |x-a| \leqslant \frac{|a|}{2} \wedge \frac{|a|^2 u}{2}\}$ is a neighborhood of a in F. Now $|0-a| \not\leqslant \frac{|a|}{2}$ by 2-18(3) and 2-17(25) so $0 \not\varepsilon W$ and W is a neighborhood of a in F^*. If $|x-a| \leqslant \frac{|a|}{2}$ then $\big||x| - |a|\big| \leqslant \frac{|a|}{2}$ by 2-17(15) so by 2-17 (12) $|x| \geqslant |a| - \frac{|a|}{2} = \frac{|a|}{2}$. Therefore if $x\varepsilon W$ then $|x| \geqslant \frac{|a|}{2}$ so by 2-18(1) and 2-17(9,1) $\frac{1}{|x|} \leqslant \frac{2}{|a|}$. If $x\varepsilon W$ then $\left|\frac{1}{x} - \frac{1}{a}\right| = \frac{1}{|x|\,|a|}|x-a| \leqslant \frac{|a|\,u}{2|x|} \leqslant u$ by 2-18(1) and 2-17(16,1,9). Thus if f is the mapping $x \to \frac{1}{x}$ of F^* into F then $f(W) \subseteq \{y\varepsilon R \mid |y-a| \leqslant u\}$. Therefore by 6-18, f is continuous for all $a\varepsilon F^*$. Now the range of f is F^* so the mapping $x \longrightarrow \frac{1}{x}$ of F^* into F^* is continuous. By 5-42, F is a topological field so by 6-17, F is a topological f-field in the p-unit topology.

Lemma 6-20

If F is an f-field with the p-unit topology then every Cauchy filter in F^*, which as the base of a Cauchy filter in F does not have 0 as an adherence point, under the mapping $x \longrightarrow \frac{1}{x}$ of F^* into F^* is mapped onto a Cauchy filter in F^*.

Proof

Let $\mathcal{C}$ be a Cauchy filter in F^* such as that the base of a Cauchy filter in F does not have 0 as an adherence point. Let $\mathcal{B}$ be the image of $\mathcal{C}$ under the mapping $x \longrightarrow \frac{1}{x}$ of F^* into F^* so $\mathcal{B}$ is a filter on F^*. Since 0 is not an adherence point of $\mathcal{C}$ then there exists a p-unit, v, such that I(v) is disjoint from some $V \varepsilon \mathcal{C}$. Thus for all $x \varepsilon V$, $|x| \nleq v$ so by 2-15, for all $x \varepsilon V$, $|x| > v$.

If u a p-unit then by 2-18(5), uv^2 is a p-unit so by 6-9 and 5-24 there exists $W \varepsilon \mathcal{C}$ such that for all $x,y \varepsilon W$, $|x-y| \leq uv^2$. Now $W \cap V \varepsilon \mathcal{C}$ and for all $x,y \varepsilon W \cap V$ it holds that

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \frac{1}{|x||y|}\,|x-y| \leq \frac{1}{v^2}uv^2 = u$$

by 2-17(16,1) and 2-18(1,3). Thus the image of $W \cap V$ under the mapping $x \rightarrow \frac{1}{x}$ is a set of order V_u (see 6-9 and 5-23) and so $\mathcal{B}$ contains a set of order V_u for any p-unit, u, and by 5-24, $\mathcal{B}$ is a Cauchy filter.

<u>Proposition 6-21</u>

If F is an f-field with the p-unit topology then F^c is a topological f-field.

<u>Proof</u>

Now F is a C-ring so by 6-16, F^c exists and is a topological f-ring. By 5-43, 6-19, and 6-20, F^c is a topological field so by 6-17, F^c is a topological f-field.

As usual F is henceforth considered as a subfield of its completion F^c.

<u>Lemma 6-22</u>

If F is an f-field with the p-unit topology and F^c its completion let P be the positive elements of F and P^c the positive elements of F^c then $\overline{P} = P^c$.

<u>Proof</u>

Since the mapping $x \rightarrow -x$ of F^c into F^c is continuous then $-(\overline{P}) \subseteq \overline{(-P)}$. By 5-32 $-(\overline{P})$ is closed in F^c and $-P \subseteq -(\overline{P})$ so $\overline{(-P)} \subseteq -(\overline{P})$. Therefore $-(\overline{P}) = \overline{(-P)}$. Since $x^+ = x$ for all $x \epsilon P$ by 2-17(8) then by 5-7, 6-8, and 6-11, $x^+ = x$ for all $x \epsilon \overline{P}$. Thus by 2-17(8) $x \geqslant 0$ for all $x \epsilon \overline{P}$ so $\overline{P} \subseteq P^c$.

Now $\overline{P} \cup -(\overline{P}) = \overline{P} \cup \overline{(-P)} = \overline{P \cup -P} = \overline{F} = F^c$ by 6-16, 2-15 and 2-5.

Also $-(\overline{P})\cap P^c \subseteq -P^c \cap P^c = \{0\}$ by 2-5 so $-(\overline{P})\cap P^c = \{0\}$. If $x\varepsilon P^c$ and $x \not\varepsilon \overline{P}$ then $x\varepsilon-(\overline{P})$ so $x\varepsilon-(\overline{P})\cap P^c$, that is $x = 0$. This is a contradiction since $0\varepsilon\overline{P}$. Therefore $\overline{P} = P^c$.

In the following let $[\ \ ]$, $(\ \ )$, etc. indicate intervals in F and $[\ \ ]^c$, $(\ \ )^c$, etc. indicate intervals in F^c (see 2-22).

<u>Lemma 6-23</u>

All closed (open) intervals in F^c are closed (open) sets in the topology of F^c.

<u>Proof</u>

By 6-22, $\overline{[0,\infty)} = [0,\infty)^c$ so $[0,\infty)^c$ is closed in F^c. By 2-17(2), $a+[0,\infty)^c = [a,\infty)^c$ so that by 5-32, $[a,\infty)^c$ is closed in F^c. By 2-17(3), $-[-a,\infty)^c = (-\infty,a]^c$ so by 5-32, $(-\infty,a]^c$ is closed in F^c. By 2-15, 6-21 the complement of $[a,\infty)^c$ is $(-\infty,a)^c$ and that of $(-\infty,a]^c$ is $(a,\infty)^c$ so $(a,\infty)^c$ and $(-\infty,a)^c$ are open in F^c. Now $(a,b)^c = (a,\infty)^c \cap (-\infty,b)^c$ and $[a,b] = [a,\infty)^c \cap (-\infty,b]^c$ so $(a,b)^c$ is open in F^c and $[a,b]^c$ is closed in F^c.

<u>Lemma 6-24</u>

If F an f-field with the p-unit topology and $a,b\varepsilon F$ then $\overline{[a,b]} = [a,b]^c$.

<u>Proof</u>

Now $[a,b] \subseteq [a,b]^c$ and by 6-23, $[a,b]^c$ is closed in F^c so $\overline{[a,b]} \subseteq [a,b]^c$. If $x \varepsilon (a,b)^c$ and V any neighborhood of x in F^c then $W = V \cap (a,b)^c$ is a neighborhood of x in F^c since $(a,b)^c$ open by 6-23. By 6-16, F is dense in F^c so there exists $s \varepsilon F$ such that $s \varepsilon W$. Now $s \varepsilon (a,b)^c$ and $s \varepsilon F$ so $s \varepsilon (a,b)$, that is every neighborhood of x contains a point of (a,b). Thus x is an adherence point of (a,b) and so x is an adherence point of $[a,b]$. Therefore $(a,b)^c \subseteq \overline{[a,b]}$ and so $[a,b]^c = (a,b)^c \cup \{a,b\} \subseteq \overline{[a,b]}$. Thus $\overline{[a,b]} = [a,b]^c$.

<u>Lemma 6-25</u>

If F is an f-field then for any $x,y \varepsilon F^c$ such that $x < y$ there exists $r \varepsilon F$ such that $x < r < y$, that is F is order dense in F^c.

<u>Proof</u>

By 2-18(2) and since F^c a field, $x < \frac{x+y}{2} < y$ so $(x,y)^c \neq \emptyset$. By 6-23, $(x,y)^c$ is open in F^c and by 6-16, F is dense in F^c so there exists $r \varepsilon F$ such that $r \varepsilon (x,y)^c$, that is $x < r < y$.

<u>Proposition 6-26</u>

If F is an f-field then the topology of F^c is the p-unit topology.

<u>Proof</u>

By 5-41 the collection $\{\overline{I(v)} \mid v \text{ a p-unit in } F\}$ is a local base at 0 for the topology of F^c. For v a p-unit in F let $I(v) = \{x\varepsilon F \mid |x| \leq v\}$ and for u a p-unit in F^c let $I^c(u) = \{x\varepsilon F^c \mid |x| \leq u\}$. By 2-23, $I(v) = [-v,v]$ and $I^c(u) = [-u,u]^c$, so by 6-24, $\overline{I(v)} = I^c(v)$. Thus the collection $\{I^c(v) \mid v$ a p-unit in $F\}$ is a local base at 0 for the topology of F^c. By 6-25 for all p-units u in F^c there exists a p-unit v in F such that $0 < v < u$ and so $I^c(v) \subseteq I^c(u)$. Thus for all p-units u in F^c, $I^c(u)$ is a neighborhood of 0 and for every neighborhood, V, of 0 there exists a p-unit v in F and so in F^c such that $I^c(v) \subseteq V$. Thus the collection $\{I^c(u) \mid u$ a p-unit in $F^c\}$ is a local base at 0 for the topology of F^c. By 6-2 the topology of F^c is the p-unit topology.

Thus the question asked after theorem 6-16 has been answered in the affirmative for f-fields.

Theorem 6-27

For every f-field, F, there exists an f-field, F^c, such that if F^c has the p-unit topology then

1) F is a dense sub-f-field of F^c
2) F^c induces the p-unit topology on F
3) F is order dense in F^c
4) F^c is complete in the p-unit uniformity
5) F^c is unique up to ℓ-isomorphism.

Proof

The theorem follows from 6-16, 6-21, 6-25 and 6-26.

Definition 6-28

The interval topology on a totally ordered set is the topology with the open intervals as a base. (See 2-22).

Note that by 2-15 the f-fields are the same as the totally ordered fields.

Proposition 6-29

The interval topology on an f-field, F, is the p-unit topology.

Proof

Let F have the interval topology. If V is a neighborhood of 0 then there exists an open interval, I, such that $0 \in I \subseteq V$. If $I = (a,b)$ let $u = |a| \wedge |b|$, if $I = (c, \infty)$ or $I = (-\infty, c)$ let $u = |c|$. By 2-17 (13,9), $|a|, |b|, |c| > 0$ so they are all p-units since F a field and by 2-18(4), u is a p-unit in all cases. By 2-17(25) and 2-18(3) $\frac{u}{2} < u$ so $\left[-\frac{u}{2}, \frac{u}{2}\right] \subseteq I$ since F is totally ordered by 2-15. For any $v > 0$, $0 \in (-v,v) \subseteq [-v,v]$ so $[-v,v]$ is a neighborhood of 0. By 2-23, $I(v) = [-v,v]$ so that the collection $\{I(v) \mid v \text{ a p-unit in } F\}$ is a local base at 0 for the interval topology of F. Therefore by 6-2, the interval topology is the p-unit topology.

CHAPTER VII

THE TOPOLOGY OF THE BUNDLE SPACE

In this chapter let R be a C-ring and $\mathcal{M}$ the maximal ℓ-ideal space of R with the topology of 4-2.

Lemma 7-1

For any $M \varepsilon \mathcal{M}$, R_M is an f-field and R_M^c is an f-field containing R_M.

Proof

By 3-3, R_M is an f-field. By 6-21, R_M^c is an f-field.

Definition 7-2

For a C-ring, R, the bundle space is $\mathcal{B} = \dot{\bigcup}\{R_M^c \mid M \varepsilon \mathcal{M}\}$, that is the disjoint union.

Definition 7-3

A section, g, is a mapping from $\mathcal{M}$ into $\mathcal{B}$ such that $g(M) \varepsilon R_M^c$.

Lemma 7-4

The set of all sections, S, under the definitions

$$(g+h)(M) = g(M)+h(M),\ (-g)(M) = -g(M),$$

$$(gh)(M) = g(M)h(M),\ (g \vee h)(M) = g(M) \vee h(M),$$

$$(g \wedge h)(M) = g(M) \wedge h(M),\ 1(M) = 1_M,\ 0(M) = 0_M,$$

form an f-ring.

Proof

S is the cardinal product of the f-fields, R_M^c,

$M \varepsilon \mathcal{M}$ by 2-10. So by 2-11, S is an f-ring.

Defintion 7-5

For each $r \varepsilon R$ define a section, $\hat{r}$, by $\hat{r}(M) = r_M \varepsilon R_M^c$.

Proposition 7-6

The mapping, $r \longrightarrow \hat{r}$ is an ℓ-monomorphism of R into S.

Proof

Now $\widehat{(r+s)}(M) = (r+s)_M = r_M + s_M = \hat{r}(M) + \hat{s}(M)$ for all $M \varepsilon \mathcal{M}$ by 2-9 so $\widehat{(r+s)} = \hat{r} + \hat{s}$. Similarly, $\widehat{rs} = \hat{r}\hat{s}$, $\widehat{(r \vee s)} = \hat{r} \vee \hat{s}$, $\widehat{(r \wedge s)} = \hat{r} \wedge \hat{s}$, and $\hat{1}(M) = 1_M = 1(M)$ so $\hat{1} = 1$. Thus by 2-6, the mapping $r \longrightarrow \hat{r}$ is an ℓ-homomorphism. If $\hat{r} = 0$ then $\hat{r}(M) = r_M = 0_M$ for all $M \varepsilon \mathcal{M}$ so by 2-9, $r \varepsilon M$ for all $M \varepsilon \mathcal{M}$. Thus $r \varepsilon \bigcap \{M | M \varepsilon \mathcal{M}\} = J(R)$ so by 6-3, $r = 0$. There the kernel of the mapping is $\{0\}$ and so the mapping is an ℓ-monomorphism.

Definition 7-7

Let $R = \{\hat{r} | r \varepsilon R\}$. The sub-f-ring, $\hat{R}$, of S is called the Gelfand representation of R.

A topology for $\mathcal{B}$ is desired such that $\mathcal{B}$ induces the p-unit topology on each R_M^c and if each R_M is a subfield of the reals then a section is continuous iff it is continuous as a real valued function on $\mathcal{M}$.

Definition 7-8

For V open in $\mathcal{M}$ and $r, s \varepsilon R$ such that $r_M < s_M$ for all $M\varepsilon V$ define the subset of $\mathcal{B}$,

$$\langle r,s\ ;\ V\rangle = \{\alpha\varepsilon R_M^c \mid M\varepsilon V \text{ and } r_M<\alpha<s_M\}\ .$$

<u>Lemma 7-9</u>

For $r,s\varepsilon R$, the set $\{M \mid r_M< s_M\}$ is open in $\mathcal{M}$.

<u>Proof</u>

If $r_M<s_M$ then $|s_M - r_M| = s_M - r_M \neq 0_M$ so $(s_M - r_M) + |s_M - r_M| = 2(s_M - r_M) \neq 0_M$. Thus by 2-9, $(s-r)+|s-r| \not\varepsilon M$ so by 4-1, $M\varepsilon S[(s-r)+|s-r|]$. If $r_M \not< s_M$ then by 7-1 and 2-15, $s_M \leqslant r_M$ so $|s_M - r_M| = r_M - s_M$ and $(s_M - r_M) + |s_M - r_M| = 0_M$. Thus by 2-9, $(s-R)+|s-r|\varepsilon M$ so by 4-1, $M\not\varepsilon S\ [(s-r)+|s-r|]$. Therefore $\{M \mid r_M<s_M\} = S\ [(s-r)+|s-r|]$ and by 4-2, it is open in $\mathcal{M}$.

<u>Lemma 7-10</u>

For any $r,s,t,u\varepsilon R$ and any open sets U,V in $\mathcal{M}$,

$$\langle r,s;U\rangle \cap \langle t,u;V\rangle = \langle r\vee t, s\wedge u;W\rangle$$

where $W = U\cap V\cap\{M \mid (r\vee t)_M < (s\wedge u)_M\}$.

<u>Proof</u>

Now,

$$\alpha\varepsilon \langle r,s;U\rangle \cap \langle t,u;V\rangle \Longleftrightarrow$$

$$\alpha\varepsilon R_M^c,\ M\varepsilon U\cap V,\ r_M<\alpha<s_M \text{ and } t_M<\alpha<u_M$$

$$\Longleftrightarrow M\varepsilon U\cap V,\ (r\vee t)_M<\alpha<(s\wedge u)_M,\ \alpha\varepsilon R_M^c$$

$$\Longleftrightarrow \alpha\varepsilon R_M^c\ ,\ M\varepsilon W, \text{ and } (r\vee t)_M<\alpha<(s\wedge u)_M$$

$$\Longleftrightarrow \alpha\varepsilon \langle r\vee t,\ s\wedge u;\ W\rangle .$$

By 7-9, W is open in $\mathcal{M}$ so

$$\langle r,s;U\rangle \cap\langle t,u;V\rangle = \langle r\vee t,\ s\wedge u;W\rangle .$$

Theorem 7-11

The collection $\{\langle r,s;V\rangle \mid r,s\in R,\ V \text{ open in } \mathfrak{M}\}$ is a base for a topology on $\mathcal{B}$.

Proof

If $\alpha\in\mathcal{B}$ then for some $M\in\mathfrak{M}$, $\alpha\in R_M^c$ and so by 6-27(3) there exists $r,s\in R$ such that $\alpha-1_M<r_M<\alpha<s_M<\alpha+1_M$. Let $V=\{M \mid r_M<s_M\}$ so V is open in $\mathfrak{M}$ by 7-9 and $\alpha\in\langle r,s;V\rangle$. Thus by 7-10 the collection $\{\langle r,s;V\rangle \mid r,s\in R \text{ and } V \text{ open in } \mathfrak{M}\}$ forms a base for a topology on $\mathcal{B}$.

Henceforth the bundle space, $\mathcal{B}$, will be assumed to have this topology.

Lemma 7-12

If F an f-field then a base for the interval topology on F^c is given by the collection $\{(r,s)^c \mid r,s\in F\}$.

Proof

If $\alpha\in F^c$ and V a neighborhood of α in the interval topology then there exists an open interval, I, such that $\alpha\in I\subseteq V$. Whether I is bounded or unbounded there exists $\beta,\gamma\in F^c$ such that $\alpha\in(\beta,\gamma)^c\subseteq I$. By 6-27(3) there exists $r,s\in F$ such that $\beta<r<\alpha<s<\gamma$ so $\alpha\in(r,s)^c\subseteq(\beta,\gamma)^c\subseteq I\subseteq V$. Thus the collection $\{(r,s)^c \mid r,s\in F\}$ is a base for the interval topology on F^c.

Proposition 7-13

The topology of $\mathcal{B}$ induces the p-unit topology on R_M^c.

Proof

If $M\varepsilon V$ then $\langle r,s;V\rangle \cap R_M^c = (r_M,s_M)^c$ and if $M\notin V$ then $\langle r,s;V\rangle \cap R_M^c = \emptyset$. Now consider $(r_M,s_M)^c$ in R_M^c for $r,s\varepsilon R$. Let $V = \{N|r_N < s_N\}$ so by 7-9, V is open and $M\varepsilon V$. Then $\langle r,s;V\rangle \cap R_M^c = (r_M,s_M)^c$. Thus the topology on $\mathcal{B}$ induces a topology on R_M^c with base the collection $\{(r_M,s_M)^c | r,s\varepsilon R\}$. By 7-12 the induced topology on R_M^c is the interval topology. By 6-29 the induced topology on R_M^c is the p-unit topology.

Proposition 7-14

If F, G are f-fields, F a sub-f-field of G and F order dense in G then F^c is ℓ-isomorphic to G^c.

Proof

If $\alpha,\beta\varepsilon G^c$ and $\alpha<\beta$ then by 2-18(2) and 6-25 there exists $g,h\varepsilon G$ such that $\alpha<g<\frac{\alpha+\beta}{2}<h<\beta$. Now there exists $r\varepsilon F$ such that $\alpha<g<r<h<\beta$ since F order dense in G so F order dense in G^c. As in 7-12, F is dense in G^c in the interval topology so by 6-29, F is dense in G^c in the p-unit topology. By 6-16(3), G^c is ℓ-isomorphic to F^c.

Corollary 7-15

If F, G are f-fields such that $F\subseteq G\subseteq F^c$ then F^c is ℓ-isomorphic to G^c.

Proof

By 6-27(3), F is order dense in F^c so F is order dense in G so by 7-14, G^c is ℓ-isomorphic to F^c.

For the rationals, Q, define Q^c to be the reals, $\mathbb{R}$. If F a sub-f-field of $\mathbb{R}$ then since $Q \subseteq F$ by 7-15, F^c is ℓ-isomorphic to $\mathbb{R}$. Thus for any sub-f-field, F, of the reals, $\mathbb{R}$, take F^c to be $\mathbb{R}$.

Lemma 7-16

If for R each R_M is a sub-f-field of the reals, $\mathbb{R}$, then the collection $\{\langle m,n;V\rangle \mid m,n\varepsilon Q \text{ and } V \text{ open in } \mathcal{M}\}$ is a base for the topology of $\mathcal{B}$.

Proof

Since R convex, Q is a sub-f-ring of R. Also Q is order dense in each R_M^c. Consider any $\langle r,s;V\rangle$ and any $\alpha\varepsilon\langle r,s;V\rangle$. Then there exists $M\varepsilon V$ such that $\alpha\varepsilon R_M^c$ and $r_M < \alpha < s_M$ and also there exists $m,n\varepsilon Q$ such that $r_M < m < \alpha < n < s_M$. Let $W = \{N \mid r_N < m\} \cap \{N \mid n < s_N\} \cap V$ so by 7-9, W is open in $\mathcal{M}$ and $M\varepsilon W$ so $\alpha\varepsilon\langle m,n;W\rangle$. If $\beta\varepsilon\langle m,n;W\rangle$ then there exists $N\varepsilon W$ such that $\beta\varepsilon R_N^c$ and $r_N < m < \beta < n < s_N$ so that $\beta\varepsilon\langle r,s;V\rangle$. Thus $\alpha\varepsilon\langle m,n;W\rangle \subseteq \langle r,s;V\rangle$ so the collection $\{\langle m,n;V\rangle \mid m,n\varepsilon Q \text{ and } V \text{ open in } \mathcal{M}\}$ is a base for the topology of $\mathcal{B}$.

Definition 7-17

If for R each R_M is a sub-f-field of the reals, $\mathbb{R}$, then for each section, $f: \mathcal{M} \to \mathcal{B}$ define $f_{\mathbb{R}}: \mathcal{M} \to \mathbb{R}$ by $f_{\mathbb{R}}(M) = f(M)$ where $\mathbb{R}$ has the usual topology of the reals.

Proposition 7-18

If for R each R_M is a sub-f-field of the reals, $\mathbb{R}$, then a section $f: \mathcal{M} \to \mathcal{B}$ is continuous iff $f_{\mathbb{R}}$ is continuous.

Proof

Now $M \varepsilon f^{-1}(\langle m,n;V\rangle) \Leftrightarrow M \varepsilon V$ and $m < f(M) < n$ $\Leftrightarrow M \varepsilon V$ and $M \varepsilon f_{\mathbb{R}}^{-1}(m,n) \Leftrightarrow M \varepsilon V \cap f_{\mathbb{R}}^{-1}(m,n)$. Thus $f^{-1}(\langle m,n;V\rangle) = f_{\mathbb{R}}^{-1}(m,n) \cap V$.

If $f_{\mathbb{R}}$ is continuous then consider a basic open set $\langle m,n;V\rangle$ in $\mathcal{B}$. Now V is open in $\mathcal{M}$ and since $f_{\mathbb{R}}$ is continuous, $f_{\mathbb{R}}^{-1}(m,n)$ is open in $\mathcal{M}$ so $f^{-1}(\langle m,n;V\rangle)$ $= f_{\mathbb{R}}^{-1}(m,n) \cap V$ is open in $\mathcal{M}$. Thus f is continuous.

If f is continuous then $f^{-1}(\langle m,n;\mathcal{M}\rangle)$ $= f_{\mathbb{R}}^{-1}(m,n) \cap \mathcal{M} = f_{\mathbb{R}}^{-1}(m,n)$ is open in $\mathcal{M}$. Thus $f_{\mathbb{R}}$ is continuous.

CHAPTER VIII

THE RING OF CONTINUOUS SECTIONS

In this chapter let R be a C-ring, $\mathcal{M}$ the maximal ℓ-ideal space of R with the topology of 4-2, and $\mathcal{B}$ the bundle space of R with the topology of 7-11.

Definition 8-1

Let F be the set of all continuous sections from $\mathcal{M}$ into $\mathcal{B}$.

Proposition 8-2

For any $r \in R$, $\hat{r}: \mathcal{M} \to \mathcal{B}$ is continuous, that is $\hat{R} \subseteq F$.

Proof

Now, $M \in \hat{r}^{-1}(\langle s,t;V\rangle) \Leftrightarrow \hat{r}(M) \in \langle s,t;V\rangle \Leftrightarrow$ $M \in V$ and $s_M < r_M < t_M \Leftrightarrow M \in \{N \mid r_N < t_N < s_N\} \cap V$. Thus $\hat{r}^{-1}(\langle s,t;V\rangle) = V \cap \{M \mid r_M < t_M < s_M\} =$ $V \cap \{M \mid r_M < t_M\} \cap \{M \mid t_M < s_M\}$ so by 7-9 $\hat{r}^{-1}(\langle s,t;V\rangle)$ is open in $\mathcal{M}$. Therefore $\hat{r}$ is continuous.

Lemma 8-3

If $f \in F$ then $-f \in F$.

<u>Proof</u>

Now by 7-4, 2-9, and 2-17(3)

$M \varepsilon (-f)^{-1} (\langle r,s;V \rangle) \Leftrightarrow (-f)(M) \varepsilon \langle r,s;V \rangle$

$\Leftrightarrow M \varepsilon V$ and $r_M < (-f)(M) < s_M$

$\Leftrightarrow M \varepsilon V$ and $(-s)_M < f(M) < (-r)_M$

$\Leftrightarrow M \varepsilon f^{-1} (\langle -s,-r;V \rangle)$.

Thus $(-f)^{-1} (\langle r,s;V \rangle) = f^{-1} (\langle -s,-r;V \rangle)$ which is open in $\mathcal{M}$ since $f \varepsilon F$. Therefore $-f \varepsilon F$.

<u>Lemma 8-4</u>

If $f \varepsilon F$ then $\{M \varepsilon \mathcal{M} | f(M) > 0\}$ is open in $\mathcal{M}$.

<u>Proof</u>

Let $A = \{\alpha \varepsilon R_M^c | \alpha > 0$ and $M \varepsilon \mathcal{M}\}$. If $\beta \varepsilon A$ then there exists $M \varepsilon \mathcal{M}$ and $r \varepsilon R$ such that $\beta \varepsilon R_M^c$ and $0 < \beta < r_M < \beta + 1$ by 6-27(3). Let $V = \{N \varepsilon \mathcal{M} | r_N > 0\}$ then by 7-9, V is open in $\mathcal{M}$ and $M \varepsilon V$. Therefore $\beta \varepsilon \langle 0,r;V \rangle \subsetneq A$ so A is open in $\mathcal{B}$. Now $\{M \varepsilon \mathcal{M} | f(M) > 0\} = f^{-1}(A)$ which is open in $\mathcal{M}$ since $f \varepsilon F$.

<u>Lemma 8-5</u>

If $f \varepsilon F$ and $S(f) = \{M \varepsilon \mathcal{M} | f(M) \neq 0\}$ then S(f) is open in $\mathcal{M}$.

<u>Proof</u>

Since each R is totally ordered then

$\{M \varepsilon \mathcal{M} | f(M) \neq 0\} = \{M \varepsilon \mathcal{M} | f(M) > 0\} \cup \{M \varepsilon \mathcal{M} | (-f)(M) > 0\}$.

By 8-3, $-f \varepsilon F$ so by 8-4, S(f) is open in $\mathcal{M}$.

Lemma 8-6

If $f \varepsilon F$ and $t \varepsilon R$ then $f+\hat{t} \varepsilon F$.

Proof

Now by 2-9 and 7-4

$M \varepsilon (f+\hat{t})^{-1} \langle r,s;V\rangle \Leftrightarrow M \varepsilon V$ and $r_M < (f+\hat{t})(M) < s_M$

$\Leftrightarrow M \varepsilon V$ and $(r-t)_M < f(M) < (s-t)_M$

$\Leftrightarrow M \varepsilon f^{-1} \langle (r-t),(s-t);V\rangle$.

Thus $(f+\hat{t})^{-1} \langle r,s;V\rangle = f^{-1} \langle (r-t),(s-t);V\rangle$ is open in $\mathcal{M}$ since $f \varepsilon F$. Therefore $f+\hat{t} \varepsilon F$.

Lemma 8-7

If $f,g \varepsilon F$, W open in $\mathcal{M}$, and for all $N \varepsilon W, f(N) < g(N)$ then $\langle f,g;W\rangle =$ $\{\alpha \varepsilon R_M^c \mid M \varepsilon W$ and $f(M) < \alpha < g(M)\}$ is open in $\mathcal{B}$.

Proof

If $\alpha \varepsilon \langle f,g;W\rangle$ then there exists $M \varepsilon W$ such that $\alpha \varepsilon R_M^c$ and since R_M dense in R_M^c by 6-25 there exists $r,s \varepsilon R$ such that $f(M) < r_M < \alpha < s_M < g(M)$. Let $V_1 = \{N \mid \hat{s}(N) < g(N)\}$, $V_2 = \{N \mid f(N) < \hat{r}(N)\}$, and $V_3 = \{N \mid \hat{r}(N) < \hat{s}(N)\}$. Since $V_1 = \{N \mid (g-\hat{s})(N) > 0\}$ then by 8-5 and 8-4, V_1 is open in $\mathcal{M}$ and similarly by 8-3, 8-5 and 8-4, V_2 is open in $\mathcal{M}$. Now by 7-9, V_3 is open in $\mathcal{M}$ so $V = V_1 \cap V_2 \cap V_3 \cap W$ is open in $\mathcal{M}$. Then $M \varepsilon V$ and $\alpha \varepsilon \langle r,s;V\rangle \subseteq \langle f,g;W\rangle$ so that $\langle f,g;W\rangle$ is open in $\mathcal{B}$.

Lemma 8-8

If $f,g \varepsilon F$ then $f+g \varepsilon F$.

Proof

Now by 7-4

$M\varepsilon(f+g)^{-1}\ \langle r,s;V\rangle \Leftrightarrow M\varepsilon V$ and $\hat{r}(M) < (f+g)(M) < \hat{s}(M)$

$\Leftrightarrow M\varepsilon V$ and $(\hat{r}-g)(M) < f(M) < (\hat{s}-g)(M)$

$\Leftrightarrow M\varepsilon \langle(\hat{r}-g),(\hat{s}-g);V\rangle$.

Thus $(f+g)^{-1}\ \langle r,s;V\rangle = f^{-1}\ \langle(\hat{r}-g),(\hat{s}-g);V\rangle$ which is open in $\mathfrak{M}$ by 8-3, 8-6 and 8-7. Therefore $f+g\varepsilon F$.

Lemma 8-9

If $f\varepsilon F$ then $|f|\varepsilon F$.

Proof

If $f(M) > 0$ then $M\varepsilon W = \{N | f(N) > 0\}$ and W is open by 8-4. Since each R_M^c is totally ordered then $f(N) = |f(N)| = |f|(N)$ for all $N\varepsilon W$. So by 5-11, $|f|$ is continuous at M.

If $f(M) < 0$ then $M\varepsilon W = \{N | (-f)(N) > 0\}$ and W is open by 8-3 and 8-4. Now $(-f)(N) = |f(N)| = |f|(N)$ for all $N\varepsilon W$ so by 8-3 and 5-11, $|f|$ is continuous at M.

If $f(M) = 0$ then $|f|(M) = |f(M)| = 0$. If $|f|(M)\varepsilon \langle r,s;V\rangle$ then $r_M < 0 < s_M$. Let $V_1 = f^{-1}\langle r,s;V\rangle$, $V_2 = (-f)^{-1}\langle r,s;V\rangle$ and $V_3 = V_1 \cap V_2$ so by 8-3, V_3 is open. Since $f(M) = (-f)(M) = 0$ then $M\varepsilon V_3$. Now if $f(N) \geq 0$ then $|f|(N) = f(N)$ so if $N\varepsilon V_1$ then $r_N < |f|(N) < s_N$. Similarly if $f(N) < 0$ then $|f|(N) = (-f)N$ so if $N\varepsilon V_2$ then $r_N < |f|(N) < s_N$. Thus $|f|(V_3) \subseteq \langle r,s;V\rangle$ so $|f|$ is continuous at M.

Since each R_M^c is totally ordered then $|f|$ is continuous for all $M \varepsilon \mathcal{M}$ and so $|f| \varepsilon F$.

Lemma 8-10

If $r \varepsilon R$ and $r_M > 0$ then there exists V open in $\mathcal{M}$ and a p-unit u in R such that $M \varepsilon V$ and $r_N = u_N$ for all $N \varepsilon V$.

Proof

Since R convex then $\frac{r}{2} \varepsilon R$ and $0 < \left(\frac{r}{2}\right)_M < r_M$. By 6-6 there exists a p-unit v in R such that $v_M = \left(\frac{r}{2}\right)_M$. Let $u = v \vee r$ and $V = \{N | v_N < r_N\}$ so by 7-9, V is open in $\mathcal{M}$ and $M \varepsilon V$. Now $u_N = v_N \vee r_N = r_N$ for all $N \varepsilon V$. Since $u \geqslant v$ then $u_N \geqslant v_N > 0$ for all $N \varepsilon \mathcal{M}$ by 2-9 and 6-5. Thus by 6-5, u is a p-unit in R.

Lemma 8-11

If $f \varepsilon F$ and u a p-unit in R then $\hat{u} f \varepsilon F$.

Proof

Since u a p-unit in R then by 2-18(6), $\frac{1}{u}$ a p-unit in R and $\left(\frac{1}{u}\right)_M = \frac{1}{u_M} > 0$. If $r_M < s_M$ for all $M \varepsilon V$ then by 2-17(25) and 2-9, $\left(\frac{r}{u}\right)_M < \left(\frac{s}{u}\right)_M$ for all $M \varepsilon V$. Now by 2-17(25), 2-9 and 7-4

$$M \varepsilon (\hat{u} f)^{-1} \langle r, s; V \rangle \Leftrightarrow M \varepsilon V \text{ and } r_M < \hat{u}(M) f(M) < s_M$$
$$\Leftrightarrow M \varepsilon V \text{ and } \frac{r_M}{u_M} < f(M) < \frac{s_M}{u_M} \Leftrightarrow$$
$$M \varepsilon V \text{ and } \left(\frac{r}{u}\right)_M < f(M) < \left(\frac{s}{u}\right)_M \Leftrightarrow M \varepsilon f^{-1} \left\langle \frac{r}{u}, \frac{s}{u}; V \right\rangle .$$

Thus $(\hat{u} f)^{-1} \langle r, s; V \rangle = f^{-1} \left\langle \frac{r}{u}, \frac{s}{u}; V \right\rangle$ which is open in $\mathcal{M}$ since $f \varepsilon F$ so $\hat{u} f \varepsilon F$.

<u>Lemma 8-12</u>

If $f, g \varepsilon F$ then $\{M \varepsilon \mathcal{M} | f(M) < g(M)\}$ is open in $\mathcal{M}$.

<u>Proof</u>

Now $\{M \varepsilon \mathcal{M} | f(M) < g(M)\} = \{M \varepsilon \mathcal{M} | (g-f)(M) > 0\}$. By 8-3 and 8-8, $g-f \varepsilon F$ so by 8-4, $\{M \varepsilon \mathcal{M} | (g-f)(M) > 0\}$ is open in $\mathcal{M}$.

<u>Lemma 8-13</u>

If $f, g \varepsilon F$ and $g(M) = 0$ then fg is continuous at M.

<u>Proof</u>

By 6-25 there exists $t \varepsilon R$ such that $0 \leqslant |f|(M) < t_M < (|f|+1)(M)$. By 6-6 there exists a p-unit u in R such that $u_M = t_M$. Let $V = \{N | |f|(N) < \hat{u}(N)\}$ then by 8-2, 8-9 and 8-12, V is open in $\mathcal{M}$. If $fg(M) \varepsilon \langle r, s; W \rangle$ then $M \varepsilon W$ and $r_M < 0 < s_M$. Let $X = \{N | |g|(N) < \widehat{\left(\frac{s \wedge -r}{u}\right)}(N)\}$ then by 8-2, 8-9 and 8-12, X is open in $\mathcal{M}$. Now $s_M > 0$ and $(-r)_M > 0$ so $\widehat{\left(\frac{s \wedge -r}{u}\right)}(M) > 0$ since R_M^c is totally ordered and by 2-18(6) and by 2-17(25). Now $|g|(M) = |g(M)| = 0$ so $M \varepsilon X$. Let $Z = V \cap W \cap X$ then Z open and $M \varepsilon Z$. If $N \varepsilon Z$ then

$|fg|(N) = |fg(N)| = |f(N)| \, |g(N)|$

$= |f|(N) |g|(N) \leqslant \hat{u}(N) |g|(N) < \hat{s}(N) \wedge (\widehat{-r})(N)$

by 2-17(1,25) so that by 2-23 and 2-17(23) $\hat{r}(N) < fg(N) < \hat{s}(N)$. Thus $fg(Z) \subseteq \langle r, s; W \rangle$ and $M \varepsilon Z$ so that fg is continuous at M.

Lemma 8-14

If $f\epsilon F$ and $r\epsilon R$ then $\hat{r}f\epsilon F$.

Proof

If $\hat{r}(M) = 0$ then by 8-2 and 8-13, $\hat{r}f$ is continuous at M. If $\hat{r}(M) > 0$ then by 8-10 there exists V open in $\mathcal{M}$ and u a p-unit in R such that $M\epsilon V$ and $\hat{u}(N) = \hat{r}(N)$ for all $N\epsilon V$. Thus $(\hat{u}f)(N) = (\hat{r}f)(N)$ for all $N\epsilon V$ so by 8-11 and 5-11, $\hat{r}f$ is continuous at M. If $\hat{r}(M) < 0$ then $\widehat{(-r)}(M) > 0$ so by above and 8-3, $\widehat{(-r)}(-f) = \hat{r}f$ is continuous at M. Since each R_M^c is totally ordered then $\hat{r}f$ is continuous for all $M\epsilon\mathcal{M}$ so $\hat{r}f\epsilon F$.

Lemma 8-15

If $f,g\epsilon F$ then $f\vee g$, $f\wedge g\epsilon F$.

Proof

Since R convex $\frac{1}{2}\epsilon R$. Now S is an f-ring by 7-4 so $f\vee g = \widehat{\left(\frac{1}{2}\right)}(f+g+|f-g|)$ by 2-18(7). So by 8-3, 8-8, 8-9 and 8-14, $f\vee g\epsilon F$. Also by 2-17(23) $f\wedge g = -(-f\vee -g)$ so by 8-3, $f\wedge g\epsilon F$.

Definition 8-16

If $u\epsilon S$ and $u(M) \neq 0$ for all $M\epsilon\mathcal{M}$ then define $\frac{1}{u}(M) = \frac{1}{u(M)}$.

Note that $\frac{1}{u}\epsilon S$ since $\frac{1}{u(M)}\epsilon R_M^c$.

Lemma 8-17

$u\epsilon S$ is a unit in S iff $u(M) \neq 0$ for all $M\epsilon\mathcal{M}$.

Proof

If $u(M) \neq 0$ then $(u)\left(\frac{1}{u}\right)(M) = u(M)\frac{1}{u(M)} = 1$ so u is a unit in S. If $u\varepsilon S$ is a unit then there exists $v\varepsilon S$ such that $uv = 1$. Thus for all $M\varepsilon \mathcal{M}$, $u(M)v(M) = 1$ so $u(M) \neq 0$ for all $M\varepsilon \mathcal{M}$.

Lemma 8-18

If $u\varepsilon F$ then u is a p-unit in F iff $u(M) > 0$ for all $M\varepsilon \mathcal{M}$.

Proof

If u a p-unit in F then $u(M) \geqslant 0$ for all $M\varepsilon \mathcal{M}$ by 2-9. By 8-17, $u(M) \neq 0$ since u a unit in S so $u(M) > 0$ for all $M\varepsilon \mathcal{M}$.

If $u(M) > 0$ for all $M\varepsilon \mathcal{M}$ then $\frac{1}{u}\varepsilon S$ by 8-16. If $\frac{1}{u}(M)\varepsilon \langle r,s;V\rangle$ then $r_M < \frac{1}{u}(M) < s_M$ and $M\varepsilon V$. Since $u(M) > 0$ then by 2-18(6), $\frac{1}{u}(M) > 0$ so by 2-9, $\frac{1}{u}(M) > \widehat{(r \vee 0)}(M)$ and since R^c_M is totally ordered. By 6-25, there exists $t\varepsilon R$, $(r\vee 0)_M < t_M < \frac{1}{u}(M)$. Let $W = \{N \mid (r\vee 0)_N < t_N < s_N\} \cap V$ then $M\varepsilon W$ and W is open by 7-9. Therefore $\frac{1}{u}(M)\varepsilon \langle t,s;W\rangle \subseteq \langle r,s;V\rangle$. Now by 8-10 there exists open sets U_1 and U_2 and p-units v,w in R such that $v_N = t_N$ for all $N\varepsilon U_1$ and $w_N = s_N$ for all $N\varepsilon U_2$ and also $M\varepsilon U_1 \cap U_2$. Let $X = U_1 \cap U_2 \cap W$ so $M\varepsilon X$ and $\frac{1}{u}(M)\varepsilon \langle v,w;X\rangle \subseteq \langle t,s;W\rangle$. By 2-18(3), $\left(\frac{1}{w}\right)_N < \left(\frac{1}{v}\right)_N$ for all $N\varepsilon X$ and X is open so let $Y = u^{-1}\left\langle \frac{1}{w}, \frac{1}{v};X\right\rangle$.

Now $\left(\frac{1}{w}\right)_M < u(M) < \left(\frac{1}{v}\right)_M$ by 2-18(3) so $M\varepsilon Y$. If $N\varepsilon Y$ then $N\varepsilon X$ and $\left(\frac{1}{w}\right)_N < u(N) < \left(\frac{1}{v}\right)_N$ so by 2-18(3), $v_N < \frac{1}{u}(N) < w_N$. Thus $\frac{1}{u}(Y) \subseteq \langle v,w;X\rangle \subseteq \langle r,s;V\rangle$ and $M\varepsilon Y$. Thus $\frac{1}{u}$ is continuous for all $M\varepsilon \mathcal{M}$ so $\frac{1}{u}\varepsilon F$. Since $u(M)\frac{1}{u}(M) = 1$ for all $M\varepsilon \mathcal{M}$ then $u\,\frac{1}{u} = 1$ so u is a p-unit in F.

Lemma 8-19

If $f\varepsilon F$ and $f(M) > 0$ then there exists V open in $\mathcal{M}$ and u a p-unit in F such that $M\varepsilon V$ and $f(N) = u(N)$ for all $N\varepsilon V$.

Proof

By 6-25 there exists $r\varepsilon R$ such that $0 < r_M < f(M)$. By 6-6 there exists a p-unit v in R such that $v_M = r_M$. Let $u = \hat{v} \vee f$ so by 8-15, $u\varepsilon F$. Now $u(N) \geqslant \hat{v}(N) > 0$ for all $N\varepsilon \mathcal{M}$ by 6-5 so by 8-18, u is a p-unit in F. Let $V = \{N \mid \hat{v}(N) < f(N)\}$ so by 8-2 and 8-12, V is open in $\mathcal{M}$ and $M\varepsilon V$. Since each R_N^c is totally ordered then for all $N\varepsilon V$, $u(N) = f(N) \vee \hat{v}(N) = f(N)$.

Lemma 8-20

If $f\varepsilon F$ and u a p-unit in F then $uf\varepsilon F$.

Proof

Now $u(M) > 0$ for all $M\varepsilon \mathcal{M}$ by 8-18 so by 2-18(6) and 2-17(25), $\hat{r}(M) < \hat{s}(M)$ iff $\frac{\hat{r}(M)}{u(M)} < \frac{\hat{s}(M)}{u(M)}$ for any $r,s\varepsilon R$. Also by 8-14, $\frac{\hat{r}}{u}, \frac{\hat{s}}{u} \varepsilon F$. Now by 8-18, 2-18(6) and 2-17(25)

$M\varepsilon(uf)^{-1}\langle r,s;V\rangle \Leftrightarrow \hat{r}(M) < (uf)(M) < \hat{s}(M)$ and $M\varepsilon V \Leftrightarrow \frac{\hat{r}}{u}(M) < f(M) < \frac{\hat{s}}{u}(M)$ and $M\varepsilon V \Leftrightarrow M\varepsilon f^{-1}\langle \frac{\hat{r}}{u},\frac{\hat{s}}{u};V\rangle$. Therefore $(uf)^{-1}\langle r,s;V\rangle = f^{-1}\langle \frac{\hat{r}}{u},\frac{\hat{s}}{u};V\rangle$ which is open in $\mathcal{M}$ by 8-7 and thus $uf\varepsilon F$.

<u>Lemma 8-21</u>

If $f,g\varepsilon F$ then $fg\varepsilon F$.

<u>Proof</u>

Let $M\varepsilon\mathcal{M}$ and consider $(fg)(M) = f(M)g(M)$. If $g(M) = 0$ then by 8-13, fg is continuous at M. If $g(M) > 0$ then by 8-19 there exists V open in $\mathcal{M}$ and a p-unit u in F such that $M\varepsilon V$ and $u(N) = g(N)$ for all $N\varepsilon V$. Thus $(fg)(N) = (fu)(N)$ for all $N\varepsilon V$ so by 5-11, fg is continuous at M. If $g(M) < 0$ then $(-g)(M) > 0$ by 2-17(3) and 7-4 so $fg = (-f)(-g)$ by 7-4 and by above fg is continuous at M. Therefore fg is continuous for all $M\varepsilon\mathcal{M}$ so $fg\varepsilon F$.

<u>Proposition 8-22</u>

F is a convex f-ring.

<u>Proof</u>

Now S is an f-ring and by 8-3, 8-8, 8-15 and 8-21, F is a sub-ℓ-ring of S so by 2-14, F is an f-ring. If $f\varepsilon F$ and $f \geqslant 1$ then by 7-4, $f(M) \geqslant 1 > 0$ for all $M\varepsilon\mathcal{M}$ so by 8-18, f is a unit in F. Therefore by 3-1, F is convex.

Lemma 8-23

If for $f\varepsilon F$ and u a p-unit in F for each $M_1, M_2 \varepsilon \mathcal{M}$ there exists $s\varepsilon R$ such that $|\hat{s}(M_i)-f(M_i)| < u(M_i)$ for $i = 1,2$ then there exists $r\varepsilon R$ such that $|\hat{r}-f| \leqslant u$.

Proof

For a given $M\varepsilon \mathcal{M}$ let $s^N \varepsilon R$ be such that $|\widehat{s^N}(N)-f(N)| < u(N)$ and $|\widehat{s^N}(M)-f(M)| < u(M)$. Let $V_N = \{Q | \widehat{s^N}(Q)-f(Q) < u(Q)\}$ then by 8-3, 8-8 and 8-12, V_N is open in $\mathcal{M}$ and by 2-23, $N\varepsilon V_N$ so $\{V_N | N\varepsilon \mathcal{M}\}$ is an open cover of $\mathcal{M}$. By 4-3, $\mathcal{M}$ is compact so a finite subcover of $\{V_N | N\varepsilon \mathcal{M}\}$ exists, say for $N_1, N_2, \cdots, N_p$. Let $t^M = s^{N_1} \wedge \cdots \wedge s^{N_p}$ so $t^M \varepsilon R$ and $\widehat{t^M}(Q) \leqslant \widehat{s^{N_i}}(Q)$ for all $Q\varepsilon \mathcal{M}$. For any $Q\varepsilon \mathcal{M}$ there exists V_{N_i} such that $Q\varepsilon V_{N_i}$ so $\widehat{s^{N_i}}(Q)-f(Q) < u(Q)$. Therefore $\widehat{t^M}(Q)-f(Q) \leqslant \widehat{s^{N_i}}(Q)-f(Q) < u(Q)$ by 2-17(2) so by 2-17(2), $\widehat{t^M}(Q) < f(Q)+u(Q)$ for all $Q\varepsilon \mathcal{M}$. Since each R_Q^c is totally ordered there exists N_i such that $\widehat{t^M}(Q) = \widehat{s^{N_i}}(Q)$. Let $W_M = \bigcap_{i=1}^{p} \{Q | \widehat{s^{N_i}}(Q) > f(Q)-u(Q)\}$ then as above W_M is open in $\mathcal{M}$ and by 2-23 and 2-17(2), $M\varepsilon W_M$. Now $\widehat{t^M}(Q) = \widehat{s^{N_i}}(Q) > f(Q)-u(Q)$ for all $Q\varepsilon W_M$. Then $\{W_M | M\varepsilon \mathcal{M}\}$ is an open cover of $\mathcal{M}$ so a finite subcover exists say for $M_1, M_2, \cdots, M_q$. Let $r = t^{M_1} \vee \cdots \vee t^{M_q}$ so $r\varepsilon R$ and $\hat{r}(Q) \geqslant \widehat{t^{M_i}}(Q)$ for all $Q\varepsilon \mathcal{M}$. Now for any $Q\varepsilon \mathcal{M}$ there exists W_{M_i} such that $Q\varepsilon W_{M_i}$ so that $\hat{r}(Q) \geqslant \widehat{t^{M_i}}(Q) > f(Q)-u(Q)$. Thus for all $Q\varepsilon \mathcal{M}$, $r(Q) > f(Q)-u(Q)$ so by 7-4, $\hat{r} \geqslant f-u$. Since each

R_Q^c is totally ordered then there exists M_i such that $\hat{r}(Q) = \hat{t}^{M_i}(Q)$ and so $\hat{r}(Q) < f(Q)+u(Q)$ for all $Q \varepsilon \mathcal{M}$. Thus by 7-4, $\hat{r} \leqslant f+u$ and so by 2-17(2) and 2-23, $|\hat{r}-f| \leqslant u$.

Lemma 8-24

For any $f \varepsilon F$, u a p-unit in F and any $M_1, M_2 \varepsilon \mathcal{M}$ there exists $s \varepsilon R$ such that $|f(M_i)-\hat{s}(M_i)| < u(M_i)$ for $i = 1,2$.

Proof

By 6-27(1), R_{M_i} is dense in $R_{M_i}^c$ for the p-unit topology and by 2-18(5), $\frac{u}{2}(M_i)$ is a p-unit in $R_{M_i}^c$. Thus by 6-18, there exists $s_1, s_2 \varepsilon R$ such that $|\hat{s}_i(M_i)-f(M_i)| \leqslant \frac{u}{2}(M_i) < u(M_i)$ by 2-17(25). If $M_1 = M_2$ let $s = s_1$. If $M_1 \neq M_2$ then by 4-4 there exists $r_1, r_2 \varepsilon R$ such that $M_i \varepsilon S(r_i)$ for $i = 1,2$ and $S(r_1) \cap S(r_2) = \emptyset$. Thus $\hat{r}_1(M_1) \neq 0 \neq \hat{r}_2(M_2)$ and $\hat{r}_1(M_2) = \hat{r}_2(M_1) = 0$. Since R_{M_1} is a field by 7-1 then there exists $t_1 \varepsilon R$ such that $\hat{r}_1(M_1)\hat{t}_1(M_1) = \hat{s}(M_1)$ and similarly there exists $t_2 \varepsilon R$ such that $\hat{r}_2(M_2)\hat{t}_2(M_2) = \hat{s}_2(M_2)$. Let $s = r_1 t_1 + r_2 t_2$ then $s \varepsilon R$ and $\hat{s}(M_1) = \hat{r}_1(M_1)\hat{t}_1(M_1)+\hat{r}_2(M_1)\hat{t}_2(M_1) = \hat{s}_1(M_1)$ and similarly $\hat{s}(M_2) = \hat{s}_2(M_2)$. Therefore $|\hat{s}(M_i)-f(M_i)| < u(M_i)$ for $i = 1,2$.

Note that as F is a convex f-ring the p-unit topology can be defined on it.

Proposition 8-25

If F has the p-unit topology then $\hat{R}$ is dense in F.

Proof

By 8-23 and 8-24 for any $f \varepsilon F$ and any p-unit u in F there exists $r \varepsilon R$ such that $|f-\hat{r}| \leqslant u$. Thus by 6-18 and 6-2 every neighborhood of f contains an element of $\hat{R}$ so $\hat{R}$ is dense in F.

Lemma 8-26

If u a p-unit in F then there exists a p-unit $v \varepsilon R$ such that $\hat{v} \leqslant u$.

Proof

Since F is convex then 4 and 8 are p-units in F so by 2-18(5,6) $\frac{u}{4}$ and $\frac{u}{8}$ are p-units in F. By 8-25 there exists $v \varepsilon R$ such that $|\hat{v}-\frac{3u}{8}| \leqslant \frac{u}{4}$ so by 2-23 and 2-17(2) $0 < \frac{u}{8} = \left(\frac{3u}{8} - \frac{u}{4}\right) \leqslant \hat{v} \leqslant \left(\frac{u}{4} + \frac{3u}{8}\right) = \frac{5u}{8} < u$. Thus $0 < \hat{v} < u$ and $\hat{v}(M) \geqslant \frac{u}{8}(M) > 0$ by 7-4 so by 6-5 v is a p-unit in R.

Proposition 8-27

The p-unit topology on F induces the p-unit topology on $\hat{R}$.

Proof

Let $I^F(u) = \{f \varepsilon F \,|\, |f| \leqslant u\}$ for u a p-unit in F and $I^{\hat{R}}(\hat{v}) = \{\hat{r} \varepsilon \hat{R} \,|\, |\hat{r}| \leqslant \hat{v}\}$ for $\hat{v}$ a p-unit in $\hat{R}$. By 8-26 for any $I^F(u)$ there exists a p-unit v in R such that $I^F(\hat{v}) \subseteq I^F(u)$. Thus by 6-2 the collection $\{I^F(\hat{v}) \,|\, \hat{v}$ a p-unit in $\hat{R}\}$ is a local base at 0 for the p-unit topology of F. Now $I^F(\hat{v}) \cap \hat{R} = I^{\hat{R}}(\hat{v})$ so that the collection $\{I^{\hat{R}}(\hat{v}) \,|\, \hat{v}$ a p-unit in $\hat{R}\}$ is a local base at 0 in the topology induced on $\hat{R}$ by the p-unit topology on F. Thus by 6-2, the topology induced on $\hat{R}$ is the p-unit topology.

<u>Definition 8-28</u>

A set V in a convex f-ring R is u-small where u a p-unit if for all $x,y \in V$ $|x-y| \leq u$.

<u>Lemma 8-29</u>

If R is a convex f-ring with the p-unit uniformity then a filter, $\mathcal{C}$, on R is Cauchy iff for all p-units, u, in R there exists $V \in \mathcal{C}$ such that V is u-small.

<u>Proof</u>

The result follows from 5-14, 5-23, 5-24, 6-9 and 8-28.

<u>Lemma 8-30</u>

If R is a convex f-ring with the p-unit uniformity then a Cauchy filter, $\mathcal{C}$, on R converges to $s \in R$ iff for every u-small set $V \in \mathcal{C}$, $|v-s| \leq u$ for all $v \in V$.

<u>Proof</u>

Let $\mathcal{C}$ be a Cauchy filter on R and $s \in R$ such that for any u-small set $V \in \mathcal{C}$, $|v-s| \leq u$ for all $V \in V$. Let $W_u = \{r \in R \mid |r-s| \leq u\}$ then by 6-2 and 6-18 the collection $\{W_u \mid u \text{ a p-unit in } R\}$ is a local base at s in the p-unit topology. By 8-29 there exists

$V \varepsilon \mathcal{C}$ such that V is u-small so by condition $V \subseteq W_u$. By 5-5 and 5-1, $\mathcal{C}$ converges to s.

Let $\mathcal{C}$ be a Cauchy filter on R converging to $s \varepsilon R$ and let $V \varepsilon \mathcal{C}$ be a u-small set. If there exists $v \varepsilon V$ such that $|v-s| \nleq u$ then there exists $M \varepsilon \mathcal{M}$ such that $|v-s|_M \nleq u_M$ by 7-4, 7-5 and 7-6. Since each R_M totally ordered then $|v-s|_M > u_M$ so by 2-17(2,25) and 2-18(5,6) $\left(\frac{|v-s|-u}{2}\right)_M > 0$. By 6-6 there exists w, a p-unit in R such that $w_M = \left(\frac{|v-s|-u}{2}\right)_M$. Now if W_w is as above and if $r \varepsilon V \cap W_w$ then $|r-s| \leqslant w$ and $|v-r| \leqslant u$ by 8-28 so by 2-17(14,2) $|v-s| \leqslant |v-r| + |r-s| \leqslant u+w$. Then by 2-17(2) and 2-18(2)

$|v-s|_M \leqslant u_M + w_M \leqslant \left(\frac{|v-s|+u}{2}\right)_M < |v-s|_M$. This is a contradiction so $V \cap W_w = \emptyset$ and thus by 5-2 and 5-6 s is not an adherence point of $\mathcal{C}$. So by 5-10, s is not a limit point of $\mathcal{C}$ which is a contradiction. Therefore for all $v \varepsilon V$, $|v-s| \leqslant u$.

Lemma 8-31

If $f \varepsilon S$ such that for all p-units $u \varepsilon R$ there exists $r \varepsilon R$ such that $|f-\hat{r}| \leqslant \hat{u}$ then $f \varepsilon F$.

Proof

If $f(M) \varepsilon \langle s,t;V \rangle$ then $M \varepsilon V$ and $s_M < f(M) < t_M$ and since R^c_M is totally ordered then $(f(M)-s_M) \wedge (t_M - f(M)) > 0$ by 2-17(2). Since R_M dense in R_M^c there exists $v \varepsilon R$ such that $0 < v_M < (f(M) - s_M) \wedge (t_M - f(M))$ and by 6-6 there exists a p-unit $u \varepsilon R$ such that $u_M = v_M$. Let $W = \{N \mid (s + \frac{u}{2})_N < (t - \frac{u}{2})_N\} \cap V$ so by 7-9, W is open in $\mathcal{M}$ and by 6-5 and 2-17(2) $\langle (s+\frac{u}{2}), (t-\frac{u}{2}); W \rangle \subseteq \langle s, t; V \rangle$. Then by 6-5, 2-17(2) and 2-18(5,6) $(s+\frac{u}{2})_M < (s+u)_M < f(M) < (t-u)_M < (t-\frac{u}{2})_M$ and $M \varepsilon V$ so $M \varepsilon W$. By assumption there exists $r \varepsilon R$ such that $|f - \hat{r}| \leq (\widehat{\frac{u}{2}})$ so let $X = \hat{r}^{-1} \langle (s+\frac{u}{2}), (t-\frac{u}{2}); W \rangle$. If $N \varepsilon X$ then $r_N < (t-\frac{u}{2})_N$ so by 2-17(12,2) and 6-5, $f(N) \leq (r+\frac{u}{2})_N < t_N$. Similarly $f(N) > s_N$ and $N \varepsilon W$ so $f(X) \subseteq \langle s, t; V \rangle$. Also by 2-17(12), $r_M < f(M) + (\frac{u}{2})_M$ so by 6-5 and 2-17(2), $r + (\frac{u}{2})_M \leq f(M) + u_M < t_M$ and so $r_M < t_M - (\frac{u}{2})_M$. Similarly $s_M + (\frac{u}{2})_M < r_M$ and $M \varepsilon W$ so $M \varepsilon X$. Now X is open in $\mathcal{M}$ since $\hat{r} \varepsilon F$ so f continuous at M. Thus f continuous at all $M \varepsilon \mathcal{M}$ so $f \varepsilon F$.

Lemma 8-32

Under the natural mapping of R onto R_M let the image of $V \subseteq R$ be V_M and the image of a filter, $\mathcal{C}$, be $\mathcal{C}_M$. If V is a u-small set in R then V_M is a u_M-small set in R_M. If $\mathcal{C}$ is a Cauchy filter in R then $\mathcal{C}_M$ is a Cauchy filter in R_M.

Proof

If $\beta, \gamma \varepsilon V_M$ then there exists $r, s \varepsilon V$ such that

$\beta = r_M$ and $\gamma = s_M$ and since by 8-28, $|r-s| \leq u$ then by 2-9, $|r_M - s_M| \leq u_M$.

If α a p-unit in R_M then by 6-6 there exists a p-unit $v \varepsilon R$ such that $v_M = \alpha$. Thus if $|r-s| \leq v$ then by 2-9, $|r_M - s_M| \leq v_M = \alpha$ so by 6-10 the natural mapping is uniformly continuous. Thus by 5-26, $\mathcal{C}_M$ is a Cauchy filter on R_M.

Lemma 8-33

If $\mathcal{C}$ a Cauchy filter in R then $\mathcal{C}_M$ is a Cauchy filter base in R_M^c and has a unique limit point in R_M^c. Let lim $\mathcal{C}_M$ be this unique limit point.

Proof

By 8-32, $\mathcal{C}_M$ is a Cauchy filter on R_M so by 5-25, $\mathcal{C}_M$ is a Cauchy filter base on R_M^c. Now R_M is a C-ring so by 6-16, R_M^c is a Hausdorff and complete space in the p-unit uniformity. By 5-28, $\mathcal{C}_M$ has a unique limit point in R_M^c.

Lemma 8-34

If $\mathcal{C}$ a Cauchy filter on $\hat{R}$ define $f \varepsilon S$ by $f(M) = \lim \mathcal{C}_M$ then $f \varepsilon F$ and $\mathcal{C}$ converges in F to f.

Proof

If $V \varepsilon \mathcal{C}$ is a $\hat{u}$-small set, $\hat{u}$ a p-unit in $\hat{R}$, then V_M is u_M-small in R_M^c by 8-32. Since $\mathcal{C}_M$ converges to f(M) then by 8-33 and 8-30, $|\hat{v}(M) - f(M)| \leq \hat{u}(M)$ for all $\hat{v}\varepsilon V$. Thus by 7-4 and 7-6 $|\hat{v}-f| \leqslant \hat{u}$ for all $\hat{v}\varepsilon V$. Now by 8-29 there exists $V\varepsilon \mathcal{C}$ such that V is $\hat{u}$-small for any p-unit $\hat{u}\varepsilon\hat{R}$ so that for any p-unit $\hat{u}\varepsilon\hat{R}$ there exists $\hat{r}\varepsilon\hat{R}$ such that $|\hat{r}-f| \leqslant \hat{u}$. Therefore by 8-31, $f\varepsilon F$.

Let $\mathcal{D}$ be the Cauchy filter on F generated by $\mathcal{C}$ and let $W\varepsilon\mathcal{D}$ be a v-small set, where v a p-unit in F. Let $\mathcal{D}_M$ be the image of $\mathcal{D}$ in R^c_M. Now $\mathcal{D}_M$ and $\mathcal{C}_M$ are bases for the same filter on R^c_M so $\mathcal{D}_M$ converges in R^c_M to f(M). Now $|w_1-w_2| \leqslant v$ for all $w_1, w_2 \varepsilon W$ so by 7-4, $|w_1(M)-w_2(M)| \leqslant v(M)$. Therefore by 8-30 $|w(M)-f(M)| \leqslant v(M)$ for all $w\varepsilon W$ and thus by 7-4, $|w-f| \leqslant v$ for all $w\varepsilon W$. Since $f\varepsilon F$ then by 8-30, $\mathcal{D}$ converges in F to f.

Proposition 8-35

F is complete in the p-unit uniformity.

Proof

The result follows from 8-25, 8-27, 8-34 and 5-29.

Proposition 8-36

F is a C-ring.

Proof

If $f\varepsilon F$ and $f \neq 0$ then there exists $M\varepsilon\mathcal{M}$ such that $f(M) \neq 0$ so by 7-4 and 2-17(9,13), $|f|(M)>0$. Now by 8-19 there exists a p-unit $u\varepsilon F$ such that $u(M) = |f|(M)$

so by 2-18(5,6) and 2-17(25), $\left(\frac{u}{2}\right)(M) < |f|(M)$. Thus $f \notin I\left(\frac{u}{2}\right)$ and so $\bigcap \{I(v) \mid v$ a p-unit in $F\} = \{0\}$. Therefore by 8-22, 6-7 and 6-3, F is a C-ring.

Theorem 8-37

F is the completion of R.

Proof

By 8-36 and 6-7, F is Hausdorff in the p-unit topology and by 8-22 F is a topological f-ring in the p-unit topology. F is complete in its ring uniformity by 8-35. By 8-25, $\hat{R}$ is dense in F and the topology induced on $\hat{R}$ is the p-unit topology by 8-27. By 7-6, R is ℓ-isomorphic to $\hat{R}$ and R is homeomorphic to $\hat{R}$ since both have the p-unit topology. Therefore by 6-16(3), F is the completion of R.

The following result answers the questions raised after 6-16.

Corollary 8-38

If R is C-ring with the p-unit topology then its completion, R^c, is a C-ring with the p-unit topology.

Proof

The result follows from 8-36 and 8-37.

As an application of this theorem consider Archimedian f-rings.

Proposition 8-39

A bounded convex Archimedian f-ring, R, is

ℓ-isomorphic to a sub-f-ring of $C(\mathcal{M})$, the ring of continuous real valued functions on the maximal ideal space of R, and $C(\mathcal{M})$ is the completion of R.

Proof

By 7-1 and 2-26(1,2) for all $M \epsilon \mathcal{M}$, R_M is a sub-f-field of the reals. By 2-26(3), R is a C-ring so by 7-18, F is ℓ-isomorphic to $C(\mathcal{M})$. Thus $C(\mathcal{M})$ is the completion of R by 8-37.

Corollary 8-40

A bounded Archimedian f-ring, R, is ℓ-isomorphic to a sub-f-ring of $C(\mathcal{M})$.

Proof

If $\frac{a}{b}, \frac{c}{d} \epsilon \widehat{R}$ and $n\left(\frac{a}{b}\right) \leqslant \frac{c}{d}$ for all positive integers, n, then by 2-17(1), $n(ad) \leqslant bc$ and by 2-25(2), $ad \leqslant 0$. Thus by 2-17(3), $-(ad) \geqslant 0$ so by 3-4, $-a \geqslant 0$. Then by 2-18(6) and 2-17(1), $\frac{-a}{b} \geqslant 0$ so that by 2-17(3), $\frac{a}{b} \leqslant 0$. Therefore $\widehat{R}$ is Archimedian.

Since R a sub-f-ring of $\widehat{R}$ then by 8-39 and 4-6, R is ℓ-isomorphic to a sub-f-ring of $C(\mathcal{M})$.

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