Multi-dimensional Lie-algebraic Operators

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Abstract

We investigate the geometric properties of multi-dimensional Lie-algebraic operators. Such operators are relevant to the study of quasi-exactly solvable, quantum mechanical systems. The present effort addresses several issues raised by the Q.E.S. research program. One such issue is the normalizability of an operator to Schrödinger form; this criterion is known as the operator closure conditions. We give a geometric, and a representation-theoretic reformulation of the closure conditions, and then use these techniques to obtain solutions for the case of linear SL(2) actions in the plane.

The study of multi-dimensional Lie-algebraic operators benefits from an intrinsic, geometrically based approach. We do this by taking as our setting the fibre bundle $\pi : \mathbf{G} \to \mathbf{M}$, where $\mathbf{G}$ is a Lie group, and the base is a homogeneous space. The symbol of a second-order Lie-algebraic operator induces a pseudo-Riemannian metric tensor, g, on the base; the symbol also induces a horizontal–vertical decomposition of the above bundle. Not surprisingly, the geometry of g is determined by this decomposition, and thus allows us to investigate g in terms of the horizontal and vertical vector fields associated to the decomposition.

Of particular interest is the class of flat geometries induced by Lie algebraic operators. One motivation for considering this class is furnished by Turbiner's Conjecture, which states that a Lie algebraic operator admits separation of variables if its symbol induces a flat metric. In the planar case we prove a global result to the effect that a flat, Riemannian manifold of the type described above, is isometric to the quotient of the Euclidean plane by reflections. We then use this result to give a proof of a limited form of Turbiner's conjecture.

Abstract

Nous étudions les propriétés géometriques des opérateurs induits par des algèbres de Lie en plusieurs dimensions. Ces opérateurs relèvent de la théorie des systèmes quasi-exactement résolubles en mécanique quantique. Ce travail traite de plusieurs questions soulevées par l'étude de ces opérateurs quasi-exactement résolubles. Une de ces questions est celle de l'équivalence d'un opérateur à la forme de Schrödinger; ce critère se formule en termes de conditions dites de fermeture. Nous donnons une formulation géométrique et enspirée par la théorie de représentations de groupes des conditions de fermeture. Ceci nous permet d'obtenir des solutions dans le cas des actions linéaires de SL(2) dans le plan.

L'étude des opérateurs induits par des algèbres de Lie en plusieurs dimensions se fait naturellement par une approche intrinsique, basée sur la géometrie sous-jacente. Le cadre est donc celui d'un fibré $\pi : \mathbf{G} \to \mathbf{M}$, où $\mathbf{G}$ est un groupe de Lie et $\mathbf{M}$ un espace homogène. Le symbole d'un opérateur du deuxième ordre induit par une algèbre de Lie definit un tenseur métrique pseudo-riemannien, g, sur la base; il définit également une décomposition horizontale–verticale du fibré. Il n'est pas etonnant que la géometrie de g soit déterminée par cette décomposition, et qu'elle permette donc l'étude de g en termes de champs de vecteurs horzontaux et verticaux.

Les géométries plates correspondant aux opérateurs induits par des algèbres de Lie sont d'un intérêt particulier. Une motivation pour l'étude de cette classe provient de la conjecture de Turbiner, selon laquelle l'opérateur doit être séparable si la métrique est plate. En dimension deux, nous démontrons un résultat global affirmant que une variété riemannienne plate du type décrit plus haut est isométrique au quotient du plan euclidien par un groupe de réflexions. Nous nous servons de ce résultat pour démontrer la conjecture de Turbiner sous une forme limitée.

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Finally, I would like to dedicate this thesis to my wife Dr. Susan Freter. This is really one case where I can't say enough, and so I will say nothing, save that it is good to have more than one reason to live.

Contents

Chapter 1

Introduction

Be patient toward all that is unresolved in your heart
And try to love the questions themselves.
– Rainer Maria Rilke

1.1 Overview

As per the title, the subject of the present dissertation is multi-dimensional, Lie-algebraic operators. Roughly speaking, these are differential operators that are defined on multi-dimensional manifolds, and that are generated from a finite dimensional Lie algebra of first-order operators. The primary impetus for the study of these mathematical objects comes from their application to quantum-dynamical spectral problems. The relevant concept here is the notion of quasi-exact solvability.

A quasi-exactly solvable operator is distinguished by the desirable property that a part of its spectrum can be computed using algebraic methods. This class of spectral problems was first defined in the research of Shifman, Turbiner, and Ushveridze [28] [31] [34]. There are a number of different ways to create quasi-exactly solvable operators. At present, the most comprehensive survey of the different methods is to be found in Ushveridze's book [35]. There are, as well, Turbiner's algebraic investigations of quasi-exactly solvable operators with polynomial eigenfunctions [32].

The Lie-algebraic approach to quasi-exact solvability was formulated independently by Kamran and Olver in [18], and also by Shifman and Turbiner in [28]. The idea behind this approach is simple; one must choose a Lie algebra of first-order operators that leaves invariant a finite dimension subspace of functions, and use a second-order Lie-algebraic operator constructed from this Lie algebra as the system Hamiltonian. The Lie-algebraic approach was taken up in the works of Gonzalez-Lopez, Kamran, and Olver. These authors completed a comprehensive analysis of the one-dimensional case [12], classified all quasi-exactly solvable Lie algebras in two dimensions [9], and created a formalism for the application of the Lie-algebraic approach to higher dimensions [13].

The article [13] has particular significance for the present work. Much of the research in the present dissertation was directly inspired by the questions raised in that paper, and can be understood as an attempt to address these question through the use of certain geometric and representation-theoretic techniques. The other driving force behind the present exposition is the relationship between quasi-exact solvability and the technique of separation of variables. One connection between these two notions is to be found in the works of Ushveridze [35], which detail techniques for creating Q.E.S. systems through the use of separation of variables. Turbiner has conjectured that there is also a connection in the opposite direction. He has put forward the conjecture (see p. 299 of [32]) that quasi-exactly solvable systems based on the Laplacian of flat, 2-dimensional space must always admit a separation of variables. The present work has something to say about the conjecture, as well as about Lie-algebraic Laplacians on flat space. In its full generality, however, the conjecture remains unresolved.

The rest of the introductory chapter is organized as follows. We begin with a brief description of the notion quasi-exact solvability, and follow with a discussion of the Lie-algebraic approach to Q.E.S. problems. We will not return to the Q.E.S. theme in the rest of the dissertation; it is presented in order to give the reader a sense of motivation and historical continuity. We will then give an overview of the issues that arise in the study of higher-dimensional Lie-algebraic operators. The end of the chapter is devoted to a summary of the research that is contained in the subsequent chapters.

1.2 Quasi-exact Solvability

Consider the time-independent Schrödinger equation for the quantum-mechanical harmonic oscillator:

$$-\psi_{xx} + ax^2\psi = E\psi, \qquad \text{where } a > 0.$$

One approach to this equation is to regard the differential operator $-\partial_{xx} + ax^2$ as a symmetric transformation of a certain linear subspace of the Hilbert space of square integrable functions. Solving the equation then amounts to finding an infinite orthonormal basis that diagonalizes $-\partial_{xx} + ax^2$. The quantum-mechanical harmonic oscillator is known as an exactly-solvable problem. This means that there is an algebraic procedure for diagonalizing $-\partial_{xx} + ax^2$.

One way to go about this is to employ the related operator:

$$\mathcal{H} = \partial_{xx} - 2\sqrt{a}x\partial_x.$$

The two operators are related by a change of scale:

$$\psi \mapsto e^{-\frac{\sqrt{a}}{2}x^2}\psi.$$

In other words,

$$e^{\frac{\sqrt{a}}{2}x^2}(-\partial_{xx} + ax^2)e^{-\frac{\sqrt{a}}{2}x^2} = -\mathcal{H} + \sqrt{a}.$$

This suggest that we first solve the spectral problem

$$\mathcal{H}\psi = E\psi,$$

and then relate the solutions to the original problem. The action of $\mathcal{H}$ on monomials is given by

$$x^n \mapsto -2\sqrt{a}nx^n + n(n-1)x^{n-2}.$$

Thus, using monomials as a function basis, we can express $\mathcal{H}$ as the following infinite matrix:

$$\begin{pmatrix} 0 & 0 & 2 & 0 & 0 & \dots \\ 0 & -2\sqrt{a} & 0 & 6 & 0 & \dots \\ 0 & 0 & -4\sqrt{a} & 0 & 12 & \dots \\ 0 & 0 & 0 & -6\sqrt{a} & 0 & \dots \\ 0 & 0 & 0 & 0 & -8\sqrt{a} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Diagonalizing the above matrix is a straight-forward algebraic procedure. The eigenvectors are polynomials,

$$\psi^{(0)} = 1, \quad \psi^{(1)} = x, \quad \psi^{(2)} = 2\sqrt{a}x^2 - 1, \quad \psi^{(3)} = 2\sqrt{a}x^3 - 3x, \quad \dots;$$

where the eigenvalue of $\psi^{(n)}$ is $-2n\sqrt{a}$. We are fortunate because $e^{-x^2/2}$ times any polynomial is square integrable. Therefore, the (unnormalized) state functions of the simple harmonic oscillator are given by $e^{-x^2/2}\psi^{(n)}$ with corresponding energy values $\sqrt{a}(2n+1)$.

The success of the above technique is, unfortunately, inextricably bound to the choice of a quadratic potential. Naively, one might expect that a system with a potential that was just a bit more complicated, say a quartic polynomial, would be amenable to a similar method; yet there is no known way to generate an exactly solvable model with a general quartic potential. Ushveridze in [35] points out an insurmountable analytic obstacle to the exact-solvability of a system with a general quadratic potential. The difficulty is revealed when one considers systems with Hamiltonian

$$-\partial_{xx} + x^2 + ax^4.$$

The parameter, a, determines the eigenvalues, $E(a)$, of operator. If one considers $a \mapsto E(a)$ as an analytic function, then there is a branch point type singularity at the value $a = 0$. This singularity was first studied by Bender and Wu [2]. They discovered that the function $E(a)$ can be analytically extended to a three-sheeted Riemann surface, where it has a complicated pattern of singular points. The point $a = 0$ turns out to be an accumulation point of the set of singularities. The exact solvability of the harmonic oscillator forced the the relation between energy levels and the parameter to be algebraic:

$$E^2 = (2n+1)^2 a.$$

The Bender/Wu pattern of singularities precludes such an algebraic relationship in the case of the quartic potential, and therefore makes it unlikely that there exists a method that gives exact solutions of such systems.

Interestingly enough, Turbiner and Ushveridze in [33] showed that the situation improves when one considers anharmonic oscillators with sextic potentials,

$$V(x) = Ax^6 + Bx^4 + Cx^2.$$

The resulting system is not exactly-solvable, but for certain values of the parameters A, B, and C it does become amenable to an algebraic treatment. Suppose that

$$A = b^2, \quad B = 2bc, \quad C = c^2 + (4n+3)b,$$

where b, c are arbitrary constants, and n is a natural number. For such parameter values the operator $-\partial_{xx} + V(x)$ turns out to stabilize a finite dimensional module of functions:

$$\exp\left(\frac{b}{4}x^4 + \frac{c}{2}x^2\right) x^k, \quad \text{where } k = 0 \ldots n.$$

When $b < 0$ these functions are square integrable, and therefore form a "finite block" within a matrix representation of the operator $-\partial_{xx}+V(x)$. This operator is symmetric, and therefore this block can be diagonalized by an algebraic procedure to produce n eigenvalues and eigenfunctions of the operator. For instance, when $n = 1$ the finite block of the operator is given by

$$\begin{pmatrix} 4b & -5c \\ -c & -2 \end{pmatrix}.$$

The eigenvalues (energy levels) are solutions of the equation

$$E^2 + (2 - 4b)E - 5c^2 - 8b = 0,$$

and the corresponding eigenfunctions are

$$\exp\left(\frac{b}{4}x^4 + \frac{c}{2}x^2\right)\left(x^2 + \frac{E + 5c}{4b}\right).$$

It is this property of leaving invariant a finite-dimensional subspace of functions that has come to be called quasi-exact solvability. As illustrated by the preceding example, if the invariant, n-dimensional subspace of a Q.E.S. operator consists of square integrable functions, then we can algebraically obtain n eigenfunctions and energy levels of the operator.

The next natural question is, how does one construct quasi-exactly solvable operators? The operator in the preceding example was not chosen by trial an error; it was constructed with the use of a certain method. One such method is based on the use of Lie-algebraic operators.

1.3 The Lie-algebraic approach

The authors of [18] and [28] observed that if one begins with a Lie algebra of first order operators that stabilizes a finite-dimensional subspace of functions, then any differential operator generated from this Lie algebra leaves the same subspace invariant. This simple, but important, observation forms the core of the Lie-algebraic approach to Q.E.S. operators.

On the line the most general quasi-exactly solvable Lie algebra of operators is given by

$$\partial_z, \quad z\partial_z - n/2, \quad z^2\partial_z - nz,$$

where n is a natural number. The corresponding module of invariant functions is the space of polynomials of degree less than or equal to n. Both the harmonic oscillator, and the system with a sextic potential described in the preceding section are generated from the above actions.

To get the anharmonic oscillator system with a sextic potential we begin with the operator

$$\mathcal{H} = \frac{1}{2}\left\{z\partial_z - \frac{n}{2}, \partial_z\right\} + 16b(z^2\partial_z - nz) + 4c\left(z\partial_z - \frac{n}{2}\right) + \frac{n}{2}\partial_z.$$

A change of coordinates, $z = x^2/4$, transforms the operator into

$$\mathcal{H} = \partial_{xx} + 2(bx^3 + cx)\partial_x - 4bnx^2 - 2nc.$$

Conjugating $\mathcal{H}$ by a change of scale with the inverse of the factor

$$\mu = \exp\left(\frac{b}{4}x^4 + \frac{c}{2}x^2\right),$$

gives the following:

$$\mu\mathcal{H}\mu^{-1} = \partial_{xx} - b^2x^6 - 2bcx^4 - (c^2 + b(3+4n))x^2 - 2c - 2nc.$$

Since the original operator was constructed so as to leave invariant the subspace generated by $\left\{x^k\right\}$, where $k = 0 \ldots n$, the operator

$$-\partial_{xx} + b^2x^6 + 2bcx^4 + (c^2 + b(3+4n))x^2$$

must leave invariant the subspace generated by $\left\{\mu x^k\right\}$.

The above discussion points the way to a general method of constructing quasi-exactly solvable operators. First, one needs a Lie algebra of first order operators,

$\{T_a\}$, that leaves invariant a finite-dimensional subspace of functions, $\mathcal{M}$. Next, one forms a second order differential operator,

$$\mathcal{H} = \sum_{ab} C^{ab} T_a T_b + \sum_a L^a T_a, \tag{1.1}$$

in such a way that $\mathcal{H}$ after a change of scale by a factor, μ, will give a Hamiltonian operator for a quantum-mechanical system. The required change of scale must also satisfy a normalizability constraint: the elements of $\mathcal{M}$ multiplied by μ must be square integrable. This program has been comprehensively carried out for 1-dimensional Q.E.S. operators by Gonzalez, Kamran, and Olver in [12]. One-dimensional Hamiltonians,

$$-\partial_{xx} + V(x)$$

are characterized by their potential, $V(x)$, and the afore-mentioned article tabulates all possible potentials that arise from the application of the Lie-algebraic method.

To generalize the method to higher dimensions one must resolve a number of issues that are not encountered in the 1-dimensional case. First, the second order part of an operator of the type shown in (1.1) will, in general, have the form

$$\sum_{ij} g^{ij} \partial_{ij},$$

where the coefficients g^{ij} depend on the constants C^{ab} and on the local coordinate expressions for the first order operators, T_a. The natural course is to interpret the g^{ij} as the components of a contravariant representation of a pseudo-Riemannian metric tensor, and to construct Schrödinger-type Hamiltonians of the form

$$\Delta + V, \tag{1.2}$$

where Δ is the corresponding Laplace-Beltrami operator and V is the system's potential.

Clearly, we must impose the constraint that g^{ij} be non-degenerate. The constituent first order operators are a sum of a vector field and a scalar, i.e.

$$T_a = V_a + \eta_a.$$

The vector field terms, V_a, must form a finite dimensional Lie algebra in their own right. In order for g^{ij} to be non-degenerate this Lie-algebra must act transitively, and this means that the setting for the system must be the subset of a homogeneous space. As to the scalar terms, η_a, they must satisfy two constraints: the operators T_a must be closed under the bracket operation, and the resulting Lie algebra of first order operators must be quasi-exactly solvable, i.e. it must stabilize some finite dimensional subspace of functions. A complete classification of quasi-exactly solvable Lie algebras of first-order operators in the complex plane has been carried out in [9]. An article by Milson [25] deals with the question of the scalar terms, η_a, and bracket closure. The quasi-exact solvability constraint for 2-dimensional operators is explored and illuminated in [8].

The next issue that must be resolved stems from the fact that in higher dimensions most second-order operators,

$$\mathcal{H} = \sum_{ij} g^{ij}\partial_{ij} + \sum_i h^i\partial_i + U,$$

cannot be transformed by a change of scale to an operator of the form (1.2). The difficulty is manifest if we write $\mathcal{H}$ in an invariant manner:

$$\mathcal{H} = \Delta + g^{ij}\omega_i\,\partial_j + U,$$

where $\omega = \omega_i\,dx^i$ is the so called magnetic 1-form associated with the operator. It is not hard to see that a change of scale results in the addition of an exact differential to ω, and therefore $\mathcal{H}$ is locally equivalent to a Schrödinger operator if and only if ω is closed. These closure conditions are automatically satisfied for 1-dimensional operators, because all 1-dimensional 1-forms are closed. In higher dimensions, choosing constants C^{al} and L^a so that the resulting operator satisfies the closure conditions is a formidable barrier to extending the Lie-algebraic approach to higher dimensions.

As in the 1-dimensional case, the final piece of the puzzle is the normalizability constraint: the change of scale that transforms an operator of the form (1.1) to an operator of the form (1.2) must make the functions in $\mathcal{M}$ square integrable. A number of systems that satisfy the normalizability constraint are exhibited in [28] and [13], but at this point there does not exist a more general analysis of this condition.

1.4 Summary of New Results

The dissertation opens with a chapter that defines and formalizes the notion of a Lie-algebraic operator. For reasons explained in the preceding section, we take the background setting to be a homogeneous space $\mathbf{M} = \mathbf{G}/\mathbf{H}$, and work with the associated representations of $\mathfrak{g}$ (the Lie algebra associated with $\mathbf{G}$) by first order differential operators. There are two original contributions in this chapter. First, we formally describe the set of Lie-algebraic operators associated with a given homogeneous space. We then show that the corresponding Lie group acts naturally on this set, and that the orbits of this action consist of equivalent operators. Second, we introduce the so called divergence cocycle. This is a 1-cocycle in the Lie algebra cohomology of $\mathfrak{g}$ with coefficients in $\mathcal{C}^\infty(\mathrm{M})$. This object is naturally associated with every second-order Lie-algebraic operator, and plays an important role in the theory of Lie-algebraic operators. In particular, the divergence cocycle is necessary for an invariant, coordinate-free formulation of this theory.

Chapter 3 is devoted to a discussion of representations of $\mathfrak{g}$ by non-homogeneous first order operators. The infinitesimal group actions of $\mathbf{G}$ on $\mathbf{M}$ engender a representation of $\mathfrak{g}$ by vector fields on $\mathbf{M}$. The additional data required for a representation by non-homogeneous first order operators turns out to be a cohomology class in $\mathrm{H}^1(\mathfrak{g}; \mathcal{C}^\infty(\mathrm{M}))$, and classes of inequivalent representations correspond to the classes of this cohomology. We present an isomorphism theorem for $\mathrm{H}^*(\mathfrak{g}; \mathcal{C}^\infty(\mathrm{M}))$ that, in particular, allows us to easily compute the dimension of H^1. The chapter also presents a method for explicitly computing representative cocycles for the classes in H^1. Some of the results in this chapter have appeared in a prior article by Milson [25]. The method of cocycle generation, however, is an improvement on the techniques presented in that earlier paper.

As mentioned above, the quadratic coefficients of a Lie-algebraic operator induce a pseudo-Riemannian geometry at the points of M where the corresponding metric tensor is non-degenerate. In chapter 4 we turn to the study of these Lie-algebraic metrics and spaces. The fundamental technique employed in this chapter is to lift the setting from M up to G. We show that a Lie-algebraic metric induces a vertical–horizontal decomposition of the tangent space of G, and that the geometry of the space below can be studied in terms of this decomposition. In particular, we show that the geodesic flows on M are given by the flow of horizontal vector fields on G. A particularly interesting phenomenon arises when G acts imprimitively on M. We will prove a theorem to the effect that in the imprimitive setting, if a geodesic and the invariant foliation are perpendicular at one point, then they must be perpendicular everywhere. A useful corollary is the following: if the invariant foliation has codimension 1 (such is the case when $\dim(M) = 2$), then the leaves of the perpendicular foliation are geodesic trajectories. We will also derive a number of formulas for standard objects like curvature and the Laplace-Beltrami operator in terms of the horizontal–vertical decomposition.

In Chapter 5 we return to the closure conditions that must be satisfied if a Lie-algebraic operator is to be of Schrödinger type, that is equivalent to a Schrödinger operator by a change of scale. We begin by giving an invariant reformulation of the closure conditions in terms of the horizontal–vertical decomposition defined in Chapter 4. We also list some conditions on Lie-algebraic operators that are sufficient, but not necessary for the operator to be of Schrödinger-type. The main thrust of the chapter is a further reformulation of the closure conditions as certain invariant equations on the group, G. Equivalently, this allows us to recast the closure conditions in terms of the representation theory of G. This approach is based on the fact that G acts invariantly on the set of Lie algebraic systems. Reduction by this invariant action was used in [12] to classify normalizable, 1-dimensional Lie algebraic potentials. We illustrate these ideas for the case of linear SL(2) action, and show how to classify Schrödinger-type operators by using the invariant group action to simplify the problem.

The final chapter takes up the study of flat Lie-algebraic metrics. We prove a fundamental theorem to the effect that a positive-definite, flat, Lie-algebraic metric that can be realized on a compact manifold admits a global cover by the Euclidean plane. This cover is an analytic mapping, but it is not everywhere invertible. The points of degeneracy correspond to k-fold branch points, and furthermore the locus of degeneracy forms a lattice of lines that tile the Euclidean plane into isometric cells. We apply this theorem to give the proof of a very limited form of Turbiner's conjecture. Specifically, we show that if **G** acts imprimitively on a compact **M**, then a flat Lie-algebraic operator on **M** can be separated in either flat or radial coordinates. We also exhibit a counter-example that illustrates that Turbiner's conjecture depends critically on the assumption that the metric be positive-definite. A discussion of the counter-example does not fit well into the context of Chapter 6, and so we relegate the details of the counter-example to an appendix.

Chapter 2

Lie Algebraic Operators

Such is the advantage of a well constructed
language that its simplified notation often
becomes the source of profound theories.
– Pierre-Simon de Laplace.

2.1 Preliminaries

The present section introduces the technical background material and notation that we will require in subsequent discussion. The goal here is to define some terms, to introduce some necessary notation, and to thereby eliminate ambiguity; there will be no attempt to be comprehensive. Most of our investigations will be set on a manifold M equipped with an action of a Lie group, G. We will be interested using the infinitesimal actions of G to construct a metric tensor on M, and therefore require that infinitesimally, at least, G act transitively on M. We will therefore suppose that M is a homogeneous space, or when working locally suppose that M is an open, contractible subset of one.

A group can act on itself with either a right or a left action. On a Lie group there are, correspondingly, two types of infinitesimal action; the infinitesimal left actions are given by so called right-invariant vector fields, and the infinitesimal right actions are given by the left-invariant vector fields. Even though the two types of action are formally equivalent, we cannot ignore the distinction, because in the settings considered below both types of action have a role and interact with one another.

With this in mind, let $\mathbf{G}$ be a Lie group and $\mathbf{H}$ a closed subgroup. We consider the homogeneous space, $\mathbf{M} = \mathbf{G}/\mathbf{H}$, to be the space of right cosets with a corresponding right $\mathbf{G}$ action. We define the associated Lie algebra, $\mathfrak{g}$, to be the tangent space of $\mathbf{G}$ at the identity point, e. The differential of the adjoint representation of the group $\mathrm{Ad} : \mathbf{G} \to \mathrm{Aut}(\mathfrak{g})$ induces the map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, and we specify that $[a, b] = \mathrm{ad}(a) \cdot b$, where $a, b \in \mathfrak{g}$.

For a $\mathfrak{g}$-valued function $f : \mathbf{G} \to \mathfrak{g}$, we define f^{L} to be the vector field

$$g \mapsto (\mathrm{L}_g)_* f_g, \quad \text{where } g \in \mathbf{G},$$

where $\mathrm{L}_g : \mathbf{G} \to \mathbf{G}$ denotes left multiplication by $g \in \mathbf{G}$. In particular, by regarding $a \in \mathfrak{g}$ as a constant function on $\mathbf{G}$, we can represent infinitesimal right multiplication by a as the left-invariant vector field a^{L}. More generally, if Φ is any tensor-valued function on $\mathbf{G}$, we will use Φ^{L} (respectively Φ^{R}) to denote the corresponding tensor field on $\mathbf{G}$ induced by right (respectively left) actions. If Φ is a constant, then Φ^{L} will be a left-invariant tensor field, and Φ^{R} a right invariant one.

Let $\pi : \mathbf{G} \to \mathbf{M}$ denote the canonical projection, and $o = \pi(e)$ the associated origin of the homogeneous space. It is important to realize that the various geometrical entities: functions, differential forms, transformations, and vector fields, that are associated with $\mathbf{M}$ are in correspondence with those $\mathbf{G}$-entities that are in some sense invariant with respect to left $\mathbf{H}$-actions. Right multiplication by $g \in \mathbf{G}$ is just such a diffeomorphism, and thereby induces a diffeomorphism of $\mathbf{M}$, which we will denote Υ_g. Smooth functions on the homogeneous space are naturally identified with the functions on $\mathbf{G}$ that are constant on the fibers of π, or equivalently those functions that are annihilated by the vector fields of $\mathfrak{h}^{\text{R}}$. A left-invariant, contravariant tensor field, Φ^{L}, is also $\mathfrak{h}^{\text{R}}$ invariant, and is thus projectable to a tensor field, $\Phi^{\pi} = \pi_*(\Phi^{\text{L}})$, on M. Two types of contravariant tensor fields will be of particular importance to us. Associated with the homogeneous space we have a representation of $\mathfrak{g}$ by vector fields $a^{\pi} = \pi_*(a^{\text{L}})$ on M. We will also be interested in type $\binom{0}{2}$ tensors $C \in \mathcal{S}^2\mathfrak{g}$, the corresponding left-invariant tensor field, C^{L}, and the projected tensor field, C^{π}. Working in terms of a basis, $a_1, \ldots, a_n$, the latter tensor field can be expressed as

$$C^{\pi} = \sum_{ij} C^{ij} a_i^{\pi} \otimes a_j^{\pi},$$

where C^{ij} is the symmetric $n \times n$ matrix of constants that determines C.

Some of the ideas that we will encounter are best described in terms of Lie algebra cohomology. We will briefly state the relevant definitions here; the reader is referred to Jacobson's book [16] for more complete information. Given a Lie algebra, $\mathfrak{g}$, and a $\mathfrak{g}$-module, $\mathcal{M}$, one defines a k-cochain with coefficients in $\mathcal{M}$ to be an alternating multi-linear map that takes k arguments from $\mathfrak{g}$ and returns values in $\mathcal{M}$. The space of all k-cochains is denoted by $\mathrm{C}^k(\mathfrak{g}; \mathcal{M})$. The corresponding cochain complex is defined to be the cochain spaces linked together by coboundary operators:

$$\mathrm{C}^0 \xrightarrow{\delta_0} \mathrm{C}^1 \xrightarrow{\delta_1} \mathrm{C}^2 \xrightarrow{\delta_2} \ldots \quad .$$

The k^{th} coboundary operator is defined by

$$(\delta_k\phi)(a_0, a_1, \ldots, a_k) = \sum_{i=0}^{k} (-1)^i a_i \phi(\ldots \widehat{a_i} \ldots) + \sum_{0 \le i < j \le k} (-1)^{i+j} \phi([a_i, a_j], \ldots \widehat{a_i} \ldots \widehat{a_j} \ldots).$$

The space of k-cocycles is defined to be the kernel of δ_k and is denoted by $\mathrm{Z}^k(\mathfrak{g}; \mathcal{M})$; the space of k-coboundaries is defined to be the image of δ_{k-1} and is denoted by $\mathrm{B}^k(\mathfrak{g}; \mathcal{M})$. The k^{th} cohomology space is defined to be the quotient $\mathrm{Z}^k/\mathrm{B}^k$, and is denoted by $\mathrm{H}^k(\mathfrak{g}; \mathcal{M})$.

2.2 The Components of a Lie-algebraic Operator System

In this section we define the basic objects of our study: Lie-algebraic operators, and the Lie-algebraic pseudo-Riemannian manifolds induced by these operators. A *Lie algebraic operator* is defined to be an operator that can be generated from the infinitesimal actions of a Lie group. In other words, it must be an element in the enveloping algebra of some Lie algebra of first order operators. We will call such a choice of a group action and an element in the enveloping algebra *the operator system.* Here we are interested exclusively in second order Lie algebraic operators, and in second order operator systems. Note: the distinction between Lie algebraic operators and operator systems is important because a given operator can be specified by several different systems.

One begins with a representation of a finite-dimensional Lie algebra, $\mathfrak{g}$, by first order operators,

$$a^{\pi} + \eta(a),$$

where $a \in \mathfrak{g}$, where a^{π} is a vector field on a background manifold, M, and η is a linear map from $\mathfrak{g}$ to $C^{\infty}(\mathrm{M})$. The vector field portion of the operators give a representation of $\mathfrak{g}$ by vector fields. Since we will want to construct non-degenerate metric tensors from these infinitesimal actions, we must assume that the vector fields a^{π} span the tangent space at every point of M. Thus, these vector fields define a local, transitive action of a Lie group, G, on M. We will fix a basepoint $o \in \mathrm{M}$, and make the technical assumption that M is just a contractible neighborhood of a global homogeneous space, $\mathrm{M} = \mathbf{G}/\mathbf{H}$, where $\mathbf{H}$ is the isotropy subgroup of o.

The operators $a^{\pi} + \eta(a)$ must be closed under the bracket operation, and hence:

$$\begin{aligned} [a^{\pi} + \eta(a), b^{\pi} + \eta(b)] &= [a^{\pi}, b^{\pi}] + a^{\pi}(\eta(b)) - b^{\pi}(\eta(a)) \\ &= [a, b]^{\pi} + \eta([a, b]). \end{aligned} \tag{2.1}$$

A linear map from $\mathfrak{g}$ to $C^{\infty}(\mathrm{M})$ is just a 1-cochain in the complex $\mathrm{C}^{\bullet}(\mathfrak{g}; C^{\infty}(\mathrm{M}))$.

Equation (2.1) is true if and only if $d\eta = 0$, i.e. if and only if η is a 1-cocycle.

A second order Lie algebraic operator is determined by the choice of the second and first order coefficients, and has the following form:

$$\mathcal{H} = \sum_{ij} C^{ij}(a_i^\pi + \eta_i)(a_j^\pi + \eta_j) + \sum_i L^i(a_i^\pi + \eta_i), \tag{2.2}$$

where $a_1, \dots, a_n$ is a basis of $\mathfrak{g}$, and where $\eta_i = \eta(a_i)$. In invariant terms, i.e. without a mention of a basis for $\mathfrak{g}$, this amounts to a choice of a symmetric, $\binom{0}{2}$ tensor $C \in \mathcal{S}^2\mathfrak{g}$, and an element of the Lie algebra (a type $\binom{0}{1}$ tensor), $L \in \mathfrak{g}$. We will call C, L, and η, respectively, the *second-order, linear and cohomology components* of the operator system. Later on we will consider a more intrinsic definition of $\mathcal{H}$. For now we must be content with the following simple tautology: the operator, $\mathcal{H}$, given by (2.2) is invariant under change of $\mathfrak{g}$-basis.

The symbol of an arbitrary second-order operator induces a contravariant tensor field. For a Lie-algebraic operator this tensor field is determined by the second-order component; i.e. the tensor field is given by

$$C^\pi = \sum_{ij} C^{ij} a_i^\pi \otimes a_j^\pi$$

In invariant terms, this is just $C^\pi = \pi_*(C^L)$, the projection of a left-invariant tensor field from G down to M. We will call C^π a *Lie-algebraic metric tensor*, and refer to M together with this metric tensor as a *Lie-algebraic pseudo-Riemannian manifold.*

The metric tensor, C^π, may have degenerate points. Since C^π is analytic with respect to the real-analytic structure of M, we know that the locus of degeneracy is either all of M, or an analytic hypersurface in M. We would like to assume that C^π is such that the first possibility does not hold. The following proposition gives a criterion for metric tensors, C^π, that are not identically degenerate. Two points of terminology used in the proposition must be explained. First, given an $x \in$ G/H, we use $\mathfrak{h}_x \subset \mathfrak{g}$ to denote the isotropy subalgebra of x, and $\mathfrak{h}_x^\perp \subset \mathfrak{g}^*$ to denote the 1-forms that annihilate that subalgebra. Second, it is legitimate to regard $C \in S^2\mathfrak{g}$ as a symmetric, bilinear form on $\mathfrak{g}^*$, or as a linear map from $\mathfrak{g}^*$ to $\mathfrak{g}$. As such, the restriction of the bilinear form, C, to $\mathfrak{h}_x^\perp$ is non-degenerate if and only if $\mathfrak{h}_x \oplus C(\mathfrak{h}_x^\perp) = \mathfrak{g}$.

Proposition 2.2.1 *The metric tensor field, C^π, is not identically degenerate if and only if there exists an $x \in \mathbf{G}/\mathbf{H}$, such that $\mathfrak{h}_x \oplus C(\mathfrak{h}_x^\perp) = \mathfrak{g}$.*

Proof: We use the canonical identification $\mathfrak{g}/\mathfrak{h}_x \cong \mathrm{T}_x\,\mathbf{G}/\mathbf{H}$, and the dual identification $\mathfrak{h}_x^\perp \cong \mathrm{T}_x^*\,\mathbf{G}/\mathbf{H}$. Thus, $\mathfrak{h}_x \oplus C(\mathfrak{h}_x^\perp) = \mathfrak{g}$ if and only if C^π is non-degenerate at x. □

At times it becomes necessary to consider cases where one or more of the components C, L, η is zero. To accommodate these possibilities we say that the operator system is *basic* if $\eta = 0$, and say that the operator system is *homogeneous* if $L = 0$.

In the next two sections we will see that different operator systems can give rise to equivalent operators. We will make precise this notion of equivalence, and distill the fundamental invariants that classify and distinguish Lie algebraic operators. The relevant data turns out to be the class of η in $\mathrm{H}^1(\mathfrak{g}; \mathcal{C}^\infty(\mathrm{M}))$, and the $\mathbf{G}$-orbit of (C, L) in $S^2\mathfrak{g} \oplus \mathfrak{g}$.

2.3 Change of Scale, Equivalence of Operators, and the Closure Conditions

Before we can discuss operator equivalence we must define and discuss the notion of a change of scale, also called a scaling (or a gauge) transformation. Every positive $\mu \in \mathcal{C}^\infty(\mathrm{M})$ gives rise to the scaling transformation

$$f \mapsto \mu f, \quad \text{where } f \in \mathcal{C}^\infty(\mathrm{M}).$$

A linear operator $\mathcal{H}$ of the original $C^\infty(\mathrm{M})$ corresponds to the conjugated operator $\mu\,\mathcal{H}\,\mu^{-1}$ of the scaled $C^\infty(\mathrm{M})$.

The effect of such a conjugation on the representation of $\mathfrak{g}$ by first order differential operators is the addition of a coboundary term, $\delta\lambda$, where $\lambda = \log(\mu)$. In other words,

$$\begin{aligned}\mu\,(a^\pi + \eta(a))\,\mu^{-1} &= a^\pi + \eta(a) - a^\pi(\lambda)\\ &= a^\pi + (\eta - \delta\lambda)(a), \qquad \text{where } a \in \mathfrak{g}.\end{aligned}$$

Scaling transformations impose an equivalence relation on the set of representations of $\mathfrak{g}$ by first order operators. Under this equivalence the distinguishing characteristic of a representation becomes the class of η in $\mathrm{H}^1(\mathfrak{g}; C^\infty(\mathrm{M}))$. The following self-evident proposition clarifies the relation between changes of scale and the choice of a cocycle η.

Proposition 2.3.1 *Let (C, L, η) be an operator system with corresponding Lie algebraic operator $\mathcal{H}$. Then, $\mu\,\mathcal{H}\,\mu^{-1}$ corresponds to the operator system $(C, L, \eta - \delta\lambda)$, where $\lambda = \log(\mu)$. In other words, a change of scale is equivalent to a change in the representative cocycle of a class in $\mathrm{H}^1(\mathfrak{g}; C^\infty(\mathrm{M}))$.*

We will say that two operators are *locally equivalent* if they can be related by the composition of a scaling transformation and a diffeomorphism. The theory of equivalence for second order differential operators was first worked out by É. Cotton [4]. In terms of local coordinates, a second order differential operator is given by

$$\mathcal{H} = g^{ij}\partial_{ij} + h^i\partial_i + U.$$

There is, however, a more intrinsic description of the operator. We will suppose that the symbol of $\mathcal{H}$, i.e. the matrix of second order coefficients, g^{ij}, is non-degenerate and thereby defines a pseudo-Riemannian metric in the ambient space, M. Working in terms of this metric, the operator can written as

$$\mathcal{H} = \Delta + L + U,$$

where Δ is the Laplace-Beltrami operator associated with the metric g^{ij}, and where $L = L^i\partial_i$ is a vector field determined by the coefficients g^{ij} and h^i. The local coordinate expression for the Laplacian is

$$\Delta = g^{ij}\partial_{ij} + \left(\partial_i(g^{ij})\right)\partial_j - \frac{g^{ij}\partial_i(\bar{g})}{2\bar{g}}\partial_j,$$

where $\bar{g}$ is the determinant of the g^{ij} matrix. This intrinsic description makes it clear that in order for two operators to be equivalent it is necessary that the pseudo-Riemannian metrics induced by their symbols be isometric. The question of equivalence is therefore reduced to the following. Given an operator

$$\widetilde{\mathcal{H}} = g^{ij}\partial_{ij} + \tilde{L} + \tilde{U},$$

when does there exist a scaling transformation that relates $\mathcal{H}$ to $\widetilde{\mathcal{H}}$? In order to answer this question, it is best to work in terms of the so-called magnetic 1-form, ω, obtained by lowering the indices on the linear term $L^i\partial_i$. In other words, the magnetic 1-form is given by

$$g^{ij}\omega_i = L^j.$$

Proposition 2.3.2 *The operators $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ with equal symbols are locally equivalent if and only if the following two conditions are satisfied. First, the difference of their magnetic 1-forms must be closed, i.e.*

$$\omega - \tilde{\omega} = 2\,\delta\lambda,$$

for some locally defined $\lambda \in C^\infty(\mathrm{M})$[1]*. Second, their scalar terms must be related by*

$$U = \tilde{U} + \Delta(\lambda) + \mathrm{grad}(\lambda)^2.$$

Proof: For a given positive $\mu \in C^\infty(\mathrm{M})$, we have

$$\mu^{-1}\widetilde{\mathcal{H}}\mu = \Delta + \tilde{L} + 2\,\mathrm{grad}(\lambda) + \tilde{U} + \Delta(\lambda) + \mathrm{grad}(\lambda)^2,$$

[1] The factor of 2 is there to simplify the expression of later formulas

where $\lambda = \log(\mu)/2$. Hence, in order for $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ to be locally equivalent it is necessary that

$$L - \tilde{L} = 2\,\mathrm{grad}(\lambda),$$

for some locally defined function λ. This gives the first condition regarding the magnetic 1-forms, and the preceding formula for the conjugated $\widetilde{\mathcal{H}}$ gives the second condition regarding the scalar terms. □

We are particularly interested in second order operators that can serve as Hamiltonians of quantum-dynamical systems. These operators have the form $\Delta + V$, where $V \in C^\infty(\mathrm{M})$; we will call them *Schrödinger operators.* It is therefore important to know which second order operators are locally equivalent to a Schröedinger operator. Proposition 2.3.2 tells us when that is the case.

Definition 2.3.3 We say that a second order operator,

$$\mathcal{H} = \Delta + L + U,$$

is of *Schrödinger type* if it satisfies the following equivalent *closure conditions*:

- $\mathcal{H}$ is locally equivalent to a Schrödinger operator, $\Delta + V$,
- locally, L is the gradient of a function,
- the corresponding magnetic 1-form is closed.

If $\mathcal{H}$ is of Schrödinger type, we will call V the *associated potential* of $\mathcal{H}$, and call $\Delta + V$ the *normalized form* of the operator.

Determining which Lie-algebraic operators satisfy the closure conditions is a difficult problem. Chapter 5 is devoted to the discussion of this question. At present, we will remark that the set of Schrödinger-type operators is invariant with respect to conjugation by a change of scale. Therefore, the closure conditions that determine whether or not a Lie-algebraic operator is of Schrödinger-type depend only on the cohomology class of the cocycle component, η. Moreover, the following is true.

Proposition 2.3.4 *The associated potential of a Schrödinger-type, Lie algebraic operator depends only on the cohomology class of the cocycle component, η.*

Proof: Let $\mathcal{H}$ be a Lie-algebraic operator that satisfies the closure conditions. Thus, we can conjugate $\mathcal{H}$ by some $\mu \in C^\infty(\mathrm{M})$ and obtain a Schrödinger operator, $\Delta + V$. Note that the above condition fixes μ up to a constant multiple. The reasons is that the gradient of a function is zero if and only if that function is a constant. This Schrödinger operator also happens to be a Lie-algebraic operator; the second-order and linear components do not change, but the cocycle component is $\eta - \delta \log(\mu)$. Now let us change the cocycle component by a coboundary, and obtain another Lie-algebraic operator, say $\mathcal{H}'$. Let's say that the normalized form of $\mathcal{H}'$ is $\Delta + V'$. Again, this normalized form is a Lie-algebraic operator; the cocycle component belongs to the same cohomology class as η, say it is $\eta - \delta \log(\mu')$. Therefore μ/μ' must be a constant, and therefore $V' = V$. □

2.4 Invariance with Respect to the Group Action.

There is a natural right-action of the group, G, on all the components of an operator system. The action of G on the tensor spaces of $\mathfrak{g}$ is derived from the adjoint representation. The right action of $g \in \mathrm{G}$ on $a \in \mathfrak{g}$ is given by

$$a \cdot g = \mathrm{Ad}(g)^{-1}(a).$$

Section 2.1, described how G acts on M via projected right multiplication diffeomorphisms, Υ_g, where $g \in \mathrm{G}$. These diffeomorphisms also induce a right G action on $C^\infty(\mathrm{M})$:

$$(\Upsilon_g)^*(f \cdot g) = f, \quad \text{where } f \in C^\infty(\mathrm{M}).$$

There is also a right action on $\mathrm{C}^*(\mathfrak{g}; C^\infty(\mathrm{M}))$:

$$(\Upsilon_g)^*((\eta \cdot g)(a \cdot g)) = \eta(a), \quad \text{where } \eta \in \mathrm{C}^1(\mathfrak{g}; C^\infty(\mathrm{M})),\ a \in \mathfrak{g}.$$

Thus, there is a well-defined **G** action on the set of operator systems. In this section we will show that two operator systems that are related by a **G**-action give rise to equivalent operators.

Proposition 2.4.1 *Let* (C, L, η) *be an operator system with corresponding Lie algebraic operator* $\mathcal{H}$. *For each* $g \in \mathbf{G}$ *the push forward,* $\widetilde{\mathcal{H}} = (\Upsilon_g)_* \mathcal{H}$, *is a Lie algebraic operator that corresponds to the operator system* $(C \cdot g, L \cdot g, \eta \cdot g)$.

Proof: Note that

$$(\Upsilon_g)_*(a^\pi + \eta(a)) = \tilde{a}^\pi + \tilde{\eta}(\tilde{a}), \qquad \text{where } a \in \mathfrak{g},\ \tilde{a} = a \cdot g,\ \tilde{\eta} = \eta \cdot g.$$

Hence, the push-forward of $\mathcal{H}$ is composed of first order Lie algebraic operators. In terms of a basis $a_1, \ldots, a_n$ of $\mathfrak{g}$, we have

$$\begin{aligned} \mathcal{H} &= \sum_{ij} C^{ij}(a_i^\pi + \eta(a_i))(a_j^\pi + \eta(a_j)) + \sum_i L^i(a_i^\pi + \eta(a_i)), \\ \widetilde{\mathcal{H}} &= \sum_{ij} C^{ij}(\tilde{a}_i^\pi + \tilde{\eta}(\tilde{a}_i))(\tilde{a}_j^\pi + \tilde{\eta}(\tilde{a}_j)) + \sum_i L^i(\tilde{a}_i^\pi + \tilde{\eta}(\tilde{a}_i)), \end{aligned}$$

where $\tilde{a}_i = a_i \cdot g$. We conclude by noting that the components of C and L with respect to the a_i basis are equal to the components of $C \cdot g$ and $L \cdot g$ with respect to the $\tilde{a}_i$ basis. □

The upshot of the above proposition is that if we modify the components of an operator system by a **G**-action, we will obtain a diffeomorphically equivalent operator. Therefore, all intrinsic properties of a Lie algebraic operator — quasi-exact solvability, the closure conditions, the curvature of the induced metric — are invariant under **G** actions.

It is natural to wonder what a **G** action does to the class of a cocycle $\eta \in \mathrm{Z}^1(\mathfrak{g}; \mathcal{C}^\infty(\mathrm{M}))$. It is not hard to show that **G** actions commute with the coboundary operator of $\mathrm{C}^\bullet(\mathfrak{g}; \mathcal{C}^\infty(\mathrm{M}))$. There is therefore a well-defined action of **G** on the cohomology groups.

Proposition 2.4.2 G *acts trivially on* $\mathrm{H}^*(\mathfrak{g}; \mathcal{C}^\infty(\mathrm{M}))$.

Proof: Since there is a G action on H^*, the latter must also be a $\mathfrak{g}$ module. These infinitesimal actions are given by Lie derivative operators with respect to the vector fields a^π, $a \in \mathfrak{g}$. Using the well-known homotopy formula, we see that if η is a cocycle then

$$\mathcal{L}_{a^\pi}(\eta) = i(a^\pi)d\eta + d(i(a^\pi)\eta) = d(i(a^\pi)\eta).$$

In other words $\mathfrak{g}$-actions take cocycles to coboundaries, and therefore the action on H^* must be trivial. □

In light of the above proposition we can summarize our results on equivalence of Lie-algebraic operators as follows.

Proposition 2.4.3 *Group actions and scaling transformations break up the set of operator systems into classes of equivalent operators. The set of these classes is given by*

$$[(\mathcal{S}^2\mathfrak{g} \oplus \mathfrak{g}) \mod \mathrm{G}] \times \mathrm{H}^1(\mathfrak{g}; \mathcal{C}^\infty(\mathrm{M})).$$

2.5 The Divergence Cocycle

In this section we will given an intrinsic, basis-free specification of the relation between an operator system (C, L, η) and the corresponding Lie algebraic operator. We will need this later in our study of the closure conditions. The key to this description is the divergence cocycle. This is an element of $\mathrm{Z}^1(\mathfrak{g}; \mathcal{C}^\infty(\mathrm{M}))$ that is naturally associated with the Lie algebraic metric induced by the second-order component $C \in \mathcal{S}^2\mathfrak{g}$.

For the moment, let g^{ij} be any pseudo-Riemannian metric on M, and put

$$\phi(a) = \mathrm{div}(a^\pi), \quad \text{where } a \in \mathfrak{g}.$$

Proposition 2.5.1 *$\delta\phi = 0$, i.e. ϕ is a cocycle.*

Proof: For A, B, vector fields on M, define

$$S(A,B) = A(\operatorname{div} B) - B(\operatorname{div} A) - \operatorname{div}[A,B].$$

By the standard properties of the divergence operator we see that for $f \in C^\infty(\mathrm{M})$ we have

$$S(fA,B) = S(A,fB) = fS(A,B)$$

i.e. S is a type $\binom{0}{2}$ tensor. Using local coordinates we have

$$\operatorname{div}\partial_i = \frac{-\partial_i \log(|g^{ij}|)}{2}$$

where $|g^{ij}| = \det(g^{ij})$. Therefore

$$S_{ij} = S(\partial_i, \partial_j) = -\partial_i\partial_j(|g^{ij}|) + \partial_j\partial_i(|g^{ij}|) = 0$$

The desired conclusion follows by taking A and B to be $\mathfrak{g}$ actions and recalling that $S(A,B)$ is just the definition of $(d\phi)(A,B)$. □

Proposition 2.5.2 *The class of ϕ in $\mathrm{H}^1(\mathfrak{g}; C^\infty(\mathrm{M}))$ is independent of the choice of pseudo-Riemannian metric, g^{ij}.*

Proof: With respect to a fixed set of local coordinates, the divergence of a vector field $X = \sum_i X^i\partial_i$ is given by

$$\sum_i \partial_i(X^i) - X(\log(|g^{ij}|))/2,$$

where $|g^{ij}|$ is the determinant of the matrix g^{ij}. The difference of divergence operators corresponding g^{ij} and to a different metric tensor, $\tilde{g}^{ij}$, is given by

$$\frac{1}{2}(\delta \log(|g^{ij}|) - \delta \log(|\tilde{g}^{ij}|)).$$

Therefore, a change in metric will alter the divergence of a^π, where $a \in \mathfrak{g}$, by a coboundary term. □

Definition 2.5.3 We call ϕ the *divergence cocycle* of the metric g. The divergence cocycle of an operator system is to defined as the divergence cocycle associated to the pseudo-Riemannian metric, C^π, induced by the second-order component, C, of that system.

Proposition 2.5.4 *Consider for the moment a basic, homogeneous operator system, $(C, 0, 0)$ with corresponding Lie-algebraic operator $\mathcal{H}$. An intrinsic description of $\mathcal{H}$ is given by*

$$\mathcal{H} = \Delta - (C\phi)^\pi,$$

where ϕ is the associated divergence cocycle, and where $(C\phi)^\pi$ is the invariant notation for

$$C^{ij} a_i^\pi \phi_j.$$

Proof: Let $f \in C^\infty(\mathrm{M})$ be given. In terms of a basis of $\mathfrak{g}$, we have

$$\begin{aligned} \Delta(f) &= \operatorname{div}(\operatorname{grad} f) \\ &= \operatorname{div}(C^{ij}\, a_i^\pi(f)\, a_j^\pi) \\ &= C^{ij} a_j^\pi(a_i^\pi(f)) + C^{ij} \operatorname{div}(a_j^\pi)\, a_i^\pi(f) \\ &= \mathcal{H}(f) + (C\phi)^\pi(f) \end{aligned}$$

□

Proposition 2.5.5 *More generally, consider an operator system (C, L, η). An intrinsic description of the corresponding $\mathcal{H}$ is given by*

$$\mathcal{H} = \Delta + (C(2\eta - \phi))^\pi + L^\pi + C(\eta, \eta) - C(\phi, \eta) + \operatorname{div}(C\eta)^\pi + \eta(L).$$

Proof: Again, working in terms of a basis of $\mathfrak{g}$, we have

$$\begin{aligned} \mathcal{H} &= C^{ij}(a_i^\pi + \eta_i)(a_j^\pi + \eta_j) + L^i(a_i^\pi + \eta_i) \\ &= C^{ij} a_i^\pi a_j^\pi + 2C^{ij}\eta_i a_j^\pi + C^{ij}\eta_i\eta_j + C^{ij} a_j^\pi(\eta_i) + L^\pi + \eta(L) \end{aligned} \tag{2.3}$$

The first term in the above expression is just the homogeneous, basic portion of $\mathcal{H}$, and hence, by the preceding proposition, is equal to $\Delta - C\phi^{\pi}$. The second and third terms are, respectively, equal to $2C\eta^{\pi}$ and $C(\eta, \eta)$. By the elementary properties of the divergence operator we have

$$\operatorname{div}(C^{ij}\eta_i a_j^{\pi}) = C^{ij}\eta_i\phi_j + C^{ij}a_j^{\pi}(\eta_i).$$

Hence, the fourth term of (2.3) is equal to $\operatorname{div}(C\eta)^{\pi} - C(\eta, \phi)$. □

Chapter 3

The Cohomology Component

Man muss immer generalisieren.
– Carl Gustav Jacobi

3.1 Isomorphism Theorem

As was mentioned in Section 2.2, every representation of a Lie algebra, $\mathfrak{g}$, by vector fields on a space, M, can be modified to a representation by non-homogeneous first order operators. The necessary ingredient is a linear function, $\eta : \mathfrak{g} \to \mathcal{C}^\infty(\mathrm{M})$ that satisfies

$$a^\pi(\eta(b)) - b^\pi(\eta(a)) - \eta([a^\pi, b^\pi]) = 0;$$

in other words, a cocycle of $\mathrm{H}^1(\mathfrak{g}; \mathcal{C}^\infty(\mathrm{M}))$. Conjugation by a scaling operator

$$f \mapsto \exp(\lambda) f, \quad \text{where } \lambda, f \in \mathcal{C}^\infty(\mathrm{M})$$

results in the addition of the coboundary term $a^\pi(\lambda)$, where $a \in \mathfrak{g}$. Thus, given a vector field representation, in order to classify the corresponding, non-homogeneous representations it is necessary to determine $\mathrm{H}^1(\mathfrak{g}; \mathcal{C}^\infty(\mathrm{M}))$. The cohomology in question has an infinite dimensional coefficient module, and one must wonder if the resulting cohomology will be finite dimensional. Furthermore, in order to obtain concrete examples of these representations, one must have techniques to explicitly determine cocycle representatives of the nontrivial cohomology classes.

These questions are addressed in Chapter 8 of Miller's book [24]. In that work we are presented with a method of calculating the dimension of H^1 and a technique for computing cocycle representatives for some special types of vector field realizations. Another relevant work is the by Gonzalez-Lopez, Kamran, and Olver [10]. This paper lists the cohomology dimensions and cocycle representatives for the 24 possible types of Lie algebras of planar vector fields. These questions were further taken up in [25], which presented a generalized isomorphism theorem for $H^*(\mathfrak{g}, \mathcal{C}^\infty(M))$.

We will begin with a discussion of the cochain complex $C^*(\mathfrak{g}; \mathcal{C}^\infty(M))$ and relate it to other simpler cochain complexes. In the end we will be able to show the resulting cohomology is finite dimensional, and to easily compute the dimension of H^1. The next step will be to describe some techniques for computing cocycle representatives, and illustrate these techniques with several examples.

Consider the cochain complex $C^*(\mathfrak{g}; \mathcal{C}^\infty(G)^L)$, where the notation $\mathcal{C}^\infty(G)^L$ indicates that $\mathfrak{g}$ acts via left-invariant vector fields. This complex can be naturally identified with the familiar deRham complex of differential forms on G. The cochains of the deRham complex take vector-field arguments and give back functions, while the cochains of $C^*(\mathfrak{g}; \mathcal{C}^\infty(G)^L)$ can be thought of as taking $\mathfrak{g}$-valued functions as arguments. Saying that $\mathfrak{g}$ acts via left-invariant vector fields, amounts to identifying a $\mathfrak{g}$-valued function f with the vector field f^L. In the projection $\pi : G \to M$, the right-invariant vector fields $\mathfrak{h}^R$ span the vertical directions. Thus, $\mathcal{C}^\infty(M)$ can be considered as the $\mathfrak{h}^R$-invariant submodule of $\mathcal{C}^\infty(G)$, and $C^*(\mathfrak{g}; \mathcal{C}^\infty(M))$ can be identified with the complex of $\mathfrak{h}^R$-invariant differential forms on G.

The local cohomology of the deRham complex is, of course, trivial, but the same cannot be said of the $\mathfrak{h}^R$-invariant cohomology. Certainly, every $\mathfrak{h}^R$-invariant, closed p-form can be locally integrated to a $p-1$ form, but it may be impossible to make the latter be $\mathfrak{h}^R$-invariant also. Such a p-form will then represent a non-trivial cohomology class. Consider, for instance, a closed, $\mathfrak{h}^R$-invariant 1-form, η. Up to a constant, η can be integrated to a function $f \in C^\infty(G)$, and $a^R(f)$ must be a constant for all $a \in \mathfrak{h}$ because

$$\mathcal{L}_{a^R}(df) = d(a^R \lrcorner\, df) = 0.$$

Let us denote this constant by $\tilde{\eta}(a)$. Note that

$$\tilde{\eta}([a,b]) = a^R b^R(f) - b^R a^R(f) = 0, \qquad \text{where } a, b \in \mathfrak{h},$$

and hence the map $\eta \mapsto \tilde{\eta}$ is actually a cocycle of $C^1(\mathfrak{h};1)$. Furthermore if η were trivial, i.e. if $\eta = df$ for some $f \in C^\infty(M)$, then $\tilde{\eta}$ would be zero. Thus $\eta \mapsto \tilde{\eta}$ factors to a map in cohomology.

Theorem 3.1.1 *The above cohomology map,*

$$H^1(\mathfrak{g}; C^\infty(M)) \to H^1(\mathfrak{h}; 1),$$

is an isomorphism

Proof: Suppose $\tilde{\eta} = 0$, or equivalently, $\eta(a^R) = 0$ for all $a \in \mathfrak{h}$. Since η is closed we can always integrate it locally to a function, f, which will be constant along the directions $\mathfrak{h}^R$. Since M is contractible we can perform an integration on all of $\pi^{-1}(M)$ to get an $f \in C^\infty(M)$ such that $df = \eta$. Hence, η is trivial, and therefore the cohomology map must be injective.

Now, let ρ be a cocycle of $H^1(\mathfrak{h};1)$. We identify ρ with the corresponding invariant 1-form on **H**. Since M is contractible we can choose a decomposition $\pi^{-1}(M) = M \times \mathbf{H}$ and pull ρ back along the second projection to get a $\eta \in \Omega^1(\pi^{-1}(M))$. It isn't hard to verify that $\rho = \tilde{\eta}$, and thus we have shown that the cohomology map is surjective as well. □

A related result appears as Theorem 8.4 of [24]. That particular version asserts that the dimension of $H^1(\mathfrak{g}; C^\infty(M))$ is equal to the codimension of $[\mathfrak{h},\mathfrak{h}]$ in $\mathfrak{h}$. In view of the above we can see why this is true; $H^1(\mathfrak{h};1)$ is really the same thing $[\mathfrak{h},\mathfrak{h}]^\perp \subset \mathfrak{h}^*$, the annihilators of the commutators of $\mathfrak{h}$.

Definition 3.1.2 We will call the linear form $\tilde{\eta} \in [\mathfrak{h},\mathfrak{h}]^\perp$ the *classifying form* of the cocycle η.

It is remarkable that a similar isomorphism theorem is true for the higher cohomology groups. For the sake of completeness we will consider this theorem, even though only H^1 will be relevant to the present discussion of operator systems. The basic technique in describing this isomorphism is to define a certain double complex with exact rows and columns and to show that this double complex computes both of the above cohomologies. This technique is quite similar to the bicomplex proof that the Cech and deRham cohomologies are isomorphic. A good reference for the Cech-deRham bicomplex is [3].

We will proceed by proving a general result about a bicomplex of Lie Algebras and then as an application generalize Theorem 3.1.1

Let $\mathfrak{g}_1$, $\mathfrak{g}_2$ be finite dimensional, real Lie algebras and M a $\mathfrak{g}_1$, $\mathfrak{g}_2$ bimodule. This means that M is both a $\mathfrak{g}_1$ and $\mathfrak{g}_2$ module, and that the $\mathfrak{g}_1$ and $\mathfrak{g}_2$ actions on M commute. Just as a Lie algebra and an associated module give rise to a cochain complex, a bimodule gives rise to a Lie algebra bicomplex. The cochain spaces of this bicomplex are defined to be

$$C^{p,q}(\mathfrak{g}_1,\mathfrak{g}_2;M) \cong \Lambda^p\mathfrak{g}_1^* \otimes \Lambda^q\mathfrak{g}_2^* \otimes M \quad \text{where } p,q \geq 0,$$

or $C^{p,q}$ for short. It is useful to think of $C^{p,q}$ as the space of linear forms with p anti-commuting arguments from $\mathfrak{g}_1$, q anti-commuting arguments from $\mathfrak{g}_2$ and values in M. The coboundary operators

$$\delta_1 : C^{p,q} \to C^{p+1,q}, \quad \delta_2 : C^{p,q} \to C^{p,q+1}$$

are defined as follows. δ_1 is defined just like the coboundary of $C^*(\mathfrak{g}_1; M)$ but with the $\mathfrak{g}_2$ arguments playing the role of parameters. More precisely, for $\omega \in C^{p,q}$, $a_i \in \mathfrak{g}_1$, $b_i \in \mathfrak{g}_2$ we put

$$\begin{aligned}(\delta_1\omega)(a_0,\ldots,a_p;b_1,\ldots,b_q) &= \sum_i(-1)^i a_i\omega(\ldots\widehat{a_i}\ldots;\vec{b}) \\ &+\sum_{i<j}(-1)^{i+j}\omega([a_i,a_j],\ldots\widehat{a_i}\ldots\widehat{a_j}\ldots;\vec{b}).\end{aligned}$$

δ_2 is defined analogously as a parametrized coboundary of $C^*(\mathfrak{g}_2; M)$ with the parameters coming from $\mathfrak{g}_1$. The coboundaries satisfy $\delta_1^2 = \delta_2^2 = 0$, and $\delta_1\delta_2 = \delta_2\delta_1$; the verification is trivial if not tedious.

To make use of the bicomplex we augment it by an extra row and an extra column of invariant cochains. We say that a cochain $\omega \in C^p(\mathfrak{g}_1; M)$ is $\mathfrak{g}_2$-*invariant* if

$$b\omega(a_1,\ldots,a_p) = 0$$

for all $a_i \in \mathfrak{g}_1$, $b \in \mathfrak{g}_2$; and denote the subspace of such cochains $C^{p,\mathrm{inv}}$. Note that $C^p(\mathfrak{g}_1; M) \cong C^{p,0}$ and thus the subspace of $\mathfrak{g}_2$-invariant cochains is precisely the kernel of $\delta_2 : C^{p,0} \to C^{p,1}$. Also note that δ_1 of a $\mathfrak{g}_2$-invariant cochain is invariant and that therefore $C^{*,\mathrm{inv}}$ is actually a subcomplex of $C^{*,0}$. We define $C^{\mathrm{inv},*}$, the $\mathfrak{g}_1$-invariant subcomplex of $C^{0,*}$, analogously. The augmented bicomplex is summarized by the following commutative diagram. The ι arrows label the inclusion maps of the invariant subcomplexes.

$$
\begin{array}{ccccccccc}
\uparrow & & \uparrow{\scriptstyle \delta_1} & & \uparrow & & \uparrow & & \\
C^{2,\mathrm{inv}} & \xrightarrow{\iota} & C^{2,0} & \xrightarrow{\delta_2} & C^{2,1} & \longrightarrow & C^{2,2} & \longrightarrow & \\
\uparrow & & \uparrow{\scriptstyle \delta_1} & & \uparrow & & \uparrow & & \\
C^{1,\mathrm{inv}} & \xrightarrow{\iota} & C^{1,0} & \xrightarrow{\delta_2} & C^{1,1} & \longrightarrow & C^{1,2} & \longrightarrow & \\
\uparrow & & \uparrow{\scriptstyle \delta_1} & & \uparrow & & \uparrow & & \\
C^{0,\mathrm{inv}} & \xrightarrow{\iota} & C^{0,0} & \xrightarrow{\delta_2} & C^{0,1} & \longrightarrow & C^{0,2} & \longrightarrow & \\
 & & \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \iota} & & \\
 & & C^{\mathrm{inv},0} & \longrightarrow & C^{\mathrm{inv},1} & \longrightarrow & C^{\mathrm{inv},2} & \longrightarrow &
\end{array}
$$

To avoid confusion let us agree to refer to the rows and columns of the main bicomplex by their row and column numbers. Thus row zero refers to the row $C^{0,*}$. The extra row of invariant cochains we will simply call the extra row or the invariant row, and likewise for the extra column.

The ultimate goal is to show that the bicomplex computes the cohomologies of the extra row and column. This is true if the rows and columns of the bicomplex are exact. Actually for the bicomplex under discussion a weaker assumption suffices.

Proposition 3.1.3 *In order for the rows and columns of the Lie algebra bicomplex to be exact it is sufficient that the* 0^{th} *row and column be exact, or equivalently, that* $\mathrm{H}^*(\mathfrak{g}_1; M)$ *and* $\mathrm{H}^*(\mathfrak{g}_2; M)$ *are trivial.*

Proof: Row p is obtained by tensoring $\Lambda^p\mathfrak{g}_1^*$ with row 0. Tensoring with a fixed vector space is an exact functor and hence if row 0 is exact so is row p. The same argument works for the columns. □

The justification for using the bicomplex is given by the following theorem.

Theorem 3.1.4 *If* $\mathrm{H}^*(\mathfrak{g}_1; M)$ *and* $\mathrm{H}^*(\mathfrak{g}_2; M)$ *are trivial then* $\mathrm{H}^{*,\mathrm{inv}}(\mathfrak{g}_1, \mathfrak{g}_2; \mathrm{M}) \cong \mathrm{H}^{\mathrm{inv},*}(\mathfrak{g}_1, \mathfrak{g}_2; \mathrm{M})$.

Proof: Note that by the preceding proposition we have the exactness of the rows and columns of the bicomplex. The standard proof of this theorem is accomplished by showing that the cohomology of the bicomplex is isomorphic to the cohomologies of the extra row and column. The reader is referred to [3] for the details. □

Let us now apply the preceding result to the computation of $\mathrm{H}^*(\mathfrak{g}; \mathcal{C}^\infty(\mathrm{M}))$. We do so by taking $\mathfrak{g}_1 = \mathfrak{g}$, $\mathfrak{g}_2 = \mathfrak{h}$, and $M = \mathcal{C}^\infty(\mathbf{G})^{\mathrm{LR}}$. The "LR" superscript means that $\mathfrak{g}$ acts with left-invariant vector fields, and while $\mathfrak{h}$ acts with right-invariant ones. The following propositions deal with the details of the resulting complex. First we determine the nature of the invariant forms and then prove the row and column exactness. It is also prudent to recall that we are working locally, and that at the moment **G**, **H**, **M** denote contractible open subsets of their corresponding global objects.

Proposition 3.1.5 $\mathrm{C}^{*,\mathrm{inv}}(\mathfrak{g}, \mathfrak{h}; \mathcal{C}^\infty(\mathrm{G})) \cong \mathrm{C}^*(\mathfrak{g}; \mathcal{C}^\infty(\mathrm{M}))$.

Proof: The values taken by a $\mathfrak{h}$ invariant cochain are functions that are annihilated by $\mathfrak{h}^{\mathrm{R}}$, i.e. they are elements of $\mathcal{C}^\infty(\mathrm{M})$. □

Proposition 3.1.6 $\mathrm{C}^{\mathrm{inv},*}(\mathfrak{g}, \mathfrak{h}; \mathcal{C}^\infty(\mathrm{G})) \cong \mathrm{C}^*(\mathfrak{h}; 1)$.

Proof: The values taken by a $\mathfrak{g}$ invariant cochain are annihilated by $\mathfrak{g}^{\mathrm{L}}$. The latter spans the tangent space at each point of **G** and hence the values of these invariant cochains are precisely the constant functions. □

Proposition 3.1.7 $\mathrm{H}^*(\mathfrak{g}; \mathcal{C}^\infty(\mathrm{G})^{\mathrm{L}}) = 0$.

Proof: As was mentioned earlier, $\mathrm{C}^*(\mathfrak{g}; \mathcal{C}^\infty(\mathrm{G})^{\mathrm{L}})$ is naturally isomorphic to the deRham complex of differential forms on **G**, and of course the latter has trivial local cohomology. □

Proposition 3.1.8 $H^*(\mathfrak{h}; \mathcal{C}^\infty(G)^R) = 0$.

Proof: First, let us choose a local decomposition $\mathbf{G} = \mathbf{M} \times \mathbf{H}$. With this decomposition $C^*(\mathfrak{h}; \mathcal{C}^\infty(G)^R)$ can be thought of as the local deRham complex of $\mathcal{H}$ parametrized by points of M. It will turn out that this parametrized complex is trivial because the cohomology at each value of the parameter, is itself trivial, and because **M** is contractible. Let us consider the details.

The cochain space of the parametrized complex in question is $\mathcal{C}^\infty(\mathrm{M}, \Omega^*(\mathbf{H}))$, where a map

$$\eta : \mathrm{M} \mapsto \Omega^k(\mathbf{H})$$

is considered smooth if for every choice of smooth vector fields $a_1, \ldots, a_k$ on **H**, the map

$$u \mapsto \eta(u)(a_1, \ldots, a_k)$$

is in $\mathcal{C}^\infty(\mathrm{M})$. The coboundary is given by

$$(d\eta)(x) = d(\eta(x)),$$

where $x \in \mathrm{M}$, and where the d on the right denotes the usual exterior derivative. We identify η with a cochain of $C^k(\mathfrak{h}; \mathcal{C}^\infty(G)^R)$ using the following formula

$$\eta(a_1, \ldots, a_k)(x, y) = \eta(x; b_1^R, \ldots, b_k^R)(y), \qquad \text{where } a_i \in \mathfrak{h},\ x \in \mathrm{M},\ y \in \mathbf{H}.$$

The triviality of $H(\mathcal{C}^\infty(\mathrm{M}, \Omega^*(\mathbf{H})))$ follows from the triviality of $\Omega^*(\mathbf{H})$); the proof requires a parametrized version of the Poincaré Lemma. We prove this lemma below and thereby conclude the present proof. □

Proposition 3.1.9 (Parametrized Poincaré Lemma) *Let B^n denote the unit open ball of $\mathbb{R}^n$. The cohomology of the parametrized complex, $\mathcal{C}^\infty(\mathrm{M}, \Omega^*(B^n))$, is trivial.*

Proof: Let us recall a proof of the fact that $\mathrm{H}^*_{\mathrm{DR}}(B^n) = 0$. We follow the ideas in [30]. It is sufficient to demonstrate the existence of a homotopy operator $K : \Omega^* \to \Omega^{*-1}$ such that $\pm Kd \pm Kd$ is the identity map on Ω^*. One such K is given by

$$(K\omega)(v) = \int_0^1 t^{k-1}(v \lrcorner \omega)(tv)\, dt, \qquad \text{where } \omega \in \Omega^k,\ v \in B^n,$$

The parametrized version of the above is

$$(K\omega)(x; v) = \int_0^1 t^{k-1}(v \lrcorner \omega(x))(tv)\, dt$$

where $\omega \in C^\infty(\mathrm{M}, \Omega^k(B^n))$, $v \in B^n$, $x \in \mathrm{M}$. □

As a consequence of the above propositions and Theorem 3.1.4 we have the following generalization of Theorem 3.1.1.

Theorem 3.1.10 $\mathrm{H}^*(\mathfrak{g}; C^\infty(\mathrm{M})) \cong \mathrm{H}^*(\mathfrak{h}; 1)$.

3.2 Determination of Cocycles

Now we turn to the problem of explicitly determining cocycle representatives for $\mathrm{H}^1(\mathfrak{g}; C^\infty(\mathrm{M}))$. Suppose we a have a representation of $\mathfrak{g}$ by vector fields $\{a_1^\pi, \ldots, a_n^\pi\}$ with structure constants c_{ij}^k. A cocycle $\eta \in \mathrm{Z}^1(\mathfrak{g}; C^\infty(\mathrm{M}))$ can be thought of as a solution to the following system of differential equations:

$$a_i^\pi(\eta_j) - a_j^\pi(\eta_i) - \sum_k c_{ij}^k \eta_k = 0, \qquad \text{where } i, j = 1, \ldots, \dim(\mathfrak{g}). \tag{3.1}$$

Work in the preceding sections allows us to compute the dimension of H^1. Here we are interested in methods of generating explicit cocycle representatives for every cohomology class.

Perhaps the simplest way to generate cocycles is to look for ones that have constant coefficients. The space of such cocycles is isomorphic to $[\mathfrak{g}, \mathfrak{g}]^\perp$, and thus can be easily computed.

By Theorem 3.1.1 we know that $H^1(\mathfrak{g}; C^\infty(M))$ is isomorphic to $[\mathfrak{h},\mathfrak{h}]^\perp$, but we need to consider the finer structure of the cocycle space $Z^1(\mathfrak{g}; C^\infty(M))$. Recall the classifying form map $\eta \mapsto \tilde{\eta}$ from Z^1 to $[\mathfrak{h},\mathfrak{h}]^\perp$ given by

$$\tilde{\eta}(a) = \eta(a^R), \qquad \text{where } a \in \mathfrak{h}.$$

As given, it is difficult to compute the classifying form explicitly, because the description of the cocyle η is in terms of the homogeneous space, M, and lacks explicit data about G. Fortunately, the classifying form has a more tractable description.

Proposition 3.2.1 *The classifying form, $\tilde{\eta}$, of a cocycle $\eta \in Z^1(\mathfrak{g}; C^\infty(M))$ is given by restricting η to $\mathfrak{h}$ and then evaluating at $o = \pi(e)$. In other words,*

$$\tilde{\eta}(a) = \eta(a)_o, \qquad \textit{where } a \in \mathfrak{h}.$$

Proof: In the original definition in Section 3.1, one identified η with a 1-form on G and put $\tilde{\eta}(a) = \eta(a^R)$; the latter was guaranteed to be a constant. But, a^R and a^L have the same value at $e \in G$, and with respect to the above mentioned identification, $\eta(a)_o = \eta(a^L)_e$. □

In light of the preceding proposition, the content of Theorem 3.1.1 is the assertion that the following sequence is exact. The d arrow is the coboundary operation, and the following arrow is the classifying form operation.

$$0 \longrightarrow \mathbb{R} \longrightarrow C^\infty(M) \xrightarrow{d} Z^1(\mathfrak{g}; C^\infty(M)) \longrightarrow [\mathfrak{h},\mathfrak{h}]^\perp \longrightarrow 0 \qquad (3.2)$$

A powerful technique for building a cocycle is to first define it on a subalgebra of $\mathfrak{g}$, and then to extend it to all of $\mathfrak{g}$. We therefore need a criterion for when such an extension is possible.

Theorem 3.2.2 *Let $\mathfrak{f} \subset \mathfrak{g}$ be a subalgebra whose action is transitive in a neighborhood of the basepoint, o. Let $\eta_0 \in Z^1(\mathfrak{f}; C^\infty(M))$ be a cocycle of $\mathfrak{f}$, and*

$$\tilde{\eta}_0 \in [\mathfrak{f}\cap\mathfrak{h}, \mathfrak{f}\cap\mathfrak{h}]^\perp \subset (\mathfrak{f}\cap\mathfrak{h})^*$$

the corresponding classifying form. One can extend η_0 to a cocycle

$$\eta \in Z^1(\mathfrak{g}; C^\infty(M))$$

if and only if $\tilde{\eta}_0$ can be extended to an element of $[\mathfrak{h},\mathfrak{h}]^\perp \subset \mathfrak{h}^*$, i.e. if and only if $\tilde{\eta}_0$ annihilates $[\mathfrak{h},\mathfrak{h}] \cap \mathfrak{f}$. The extension of the classifying form (if it exists), uniquely determines the extension of the cocycle.

Proof: The sequence (3.2) is a natural construction, and we therefore get the following commutative diagram.

$$\begin{array}{ccccccccc} 0 \longrightarrow \mathbb{R} \longrightarrow & C^\infty(M) & \longrightarrow & Z^1(\mathfrak{f}; C^\infty(M)) & \longrightarrow & [\mathfrak{f}\cap\mathfrak{h}, \mathfrak{f}\cap\mathfrak{h}]^\perp & \longrightarrow 0 \\ & \updownarrow & & \uparrow \iota_1 & & \uparrow \iota_2 & \\ 0 \longrightarrow \mathbb{R} \longrightarrow & C^\infty(M) & \longrightarrow & Z^1(\mathfrak{g}; C^\infty(M)) & \longrightarrow & [\mathfrak{h},\mathfrak{h}]^\perp & \longrightarrow 0 \end{array}$$

The first part of the theorem asserts that $\eta_0 \in Z^1(\mathfrak{f}; C^\infty(M))$ goes to the image of ι_2 if and only if η_0 is in the image of ι_1. One direction of the equivalence is true by the commutativity of the diagram. To prove the other direction, suppose that the classifying form of η_0 is in the image of ι_2. Then, there is an $\eta \in Z^1(\mathfrak{g}; C^\infty(M))$ such that $\iota_1(\eta)$ has the same classifying form as η_0. By the exactness of the horizontal sequences the difference $\eta_0 - \iota_1(\eta)$ is a coboundary, df, for some $f \in C^\infty(M)$. But, ι_1 takes $\eta + df$ to η_0, and therefore η_0 has an extension.

Now, let us consider the uniqueness part of the theorem. Suppose that $\eta_1, \eta_2 \in Z^1(\mathfrak{g}; C^\infty(M))$ are extensions of $\eta_0 \in Z^1(\mathfrak{g}; C^\infty(M))$, and that $\tilde{\eta}_1 = \tilde{\eta}_2$. The rows of the above diagram are exact, and hence η_1 and η_2 must differ by a coboundary. Hence, $\iota_1(\eta_1)$ and $\iota_1(\eta_2)$ differ by the same coboundary. But ι_1 takes both η_1 and η_2 to η_0. Therefore, the coboundary in question is zero, and therefore $\eta_1 = \eta_2$. □

To apply this theorem one executes the following steps. First we choose a subalgebra $\mathfrak{f}$, sufficiently large, so that $\mathfrak{f} + \mathfrak{h}$ is all of $\mathfrak{g}$. Let us put

$$n = \dim(\mathfrak{g}), \quad p = \dim(\mathfrak{f}), \quad q = \dim(\mathfrak{g}) - \dim(\mathfrak{h}).$$

Since a sufficiently large $\mathfrak{f}$ was chosen we have $p \geq q$, and thus we can choose an adapted basis $a_1, \ldots, a_n$ of $\mathfrak{g}$, such that $a_1, \ldots, a_p$ span $\mathfrak{f}$, and such that $a_{q+1}, \ldots, a_n$ span $\mathfrak{h}$. Next, choose a cocycle $\eta \in Z^1(\mathfrak{f}; C^\infty(M))$ with constant coefficients. This amounts to choosing constants $\eta_1, \ldots, \eta_p$ in such a way that the resulting form annihilates the commutators formed by $a_{q+1}, \ldots, a_p$, i.e. $[\mathfrak{f} \cap \mathfrak{h}, \mathfrak{f} \cap \mathfrak{h}]$. Finally, extend the classifying form of η by choosing initial values $\eta_{p+1}(o), \ldots, \eta_n(o)$, and then solve the overdetermined system of linear P.D.E.s,

$$a_i^\pi(\eta_j) = \sum_{k=1}^{n} c_{ij}^k \eta_k, \quad \text{where } i = 1 \ldots p,\ j = p+1 \ldots n,$$

for the unknowns $\eta_{p+1}, \ldots, \eta_n$. By Theorem 3.2.2, there will be a unique solution, and that solution will determine a cocycle of $\mathfrak{g}$.

Example 3.2.3 Consider the standard, projective line realization of $\mathfrak{g} = \mathfrak{sl}(2)$:

$$a_1^\pi = \partial_x, \quad a_2^\pi = x\partial_x, \quad a_3^\pi = x^2\partial_x.$$

Let us take $x = 0$ as the basepoint. With this choice the isotropy subalgebra is spanned by a_2 and a_3. The commutators of $\mathfrak{h}$ are spanned by a_3, and thus have codimension 1 in $\mathfrak{h}$. By Theorem 3.1.1 we can conclude that H^1 is one dimensional. By Proposition 3.2.1 we know that a cocycle must satisfy $\eta_3(0) = 0$, and that the class of the cocycle will be determined by the constant $c = \eta_2(0)$.

Let us begin by setting $\eta_1 = 0$ and then extend by solving

$$\begin{aligned} \partial_x(\eta_2) &= 0, & \eta_2(0) &= c; \\ \partial_x(\eta_3) &= 2\eta_2, & \eta_3(0) &= 0. \end{aligned}$$

The unique solution is $\eta_2 = c$, $\eta_3 = cx$.

Example 3.2.4 In this example we let $\mathfrak{g}$ be a certain subalgebra of linear endomorphisms of $\mathbb{R}^3$. The subalgebra in question will consist of those actions that stabilize a fixed direction in $\mathbb{R}^3$, say the direction spanned by the third unit vector.

Instead of choosing a single-index basis, we will use the more natural representation of $\mathfrak{gl}(3)$ and $\mathfrak{gl}(3)^*$ by 3×3 matrices. We take coordinates x^1, x^2, x^3 on $\mathbb{R}^3$, and thus the $\mathfrak{g}$ actions are given by

$$\begin{bmatrix} \partial_1 & \partial_2 & \partial_3 \end{bmatrix} \begin{bmatrix} \eta_{11} & \eta_{12} & 0 \\ \eta_{21} & \eta_{22} & 0 \\ \eta_{31} & \eta_{32} & \eta_{33} \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} = \sum_{ij} \eta_{ij} x^i \partial_j .$$

Taking $(1,0,0)$ as the basepoint, we see that the isotropy algebra, $\mathfrak{h}$, and its commutators are given by

$$\mathfrak{h} = \begin{bmatrix} 0 & * & 0 \\ 0 & * & 0 \\ 0 & * & * \end{bmatrix} \qquad [\mathfrak{h}, \mathfrak{h}] = \begin{bmatrix} 0 & * & 0 \\ 0 & 0 & 0 \\ 0 & * & 0 \end{bmatrix}$$

Since the codimension of $[\mathfrak{h}, \mathfrak{h}]$ in $\mathfrak{h}$ is two, we can conclude that H^1 is two-dimensional, and that the classifying forms are given by

$$\begin{bmatrix} & 0 & \\ & c_1 & \\ & 0 & c_2 \end{bmatrix} = c_1 \eta_{22} + c_2 \eta_{33},$$

where c_1, c_2 are arbitrary constants. Next, we find a cocycle that corresponds to the above classifying form. We begin with the following cocycle with constant coefficients.

$$\begin{bmatrix} 0 & & \\ 0 & c_1 & \\ 0 & & c_2 \end{bmatrix},$$

and extend by solving

$$\begin{aligned} x\,\partial_1 \eta_{12} &= -\eta_{12}, & x\,\partial_1 \eta_{32} &= 0, \\ x\,\partial_2 \eta_{12} &= -c_1, & x\,\partial_2 \eta_{32} &= 0, \\ x\,\partial_3 \eta_{12} &= -\eta_{32}, & x\,\partial_3 \eta_{32} &= 0, \\ \eta_{12}(1,0,0) &= 0 & \eta_{32}(1,0,0) &= 0 \end{aligned}$$

The resulting cocycle is

$$\begin{bmatrix} 0 & -c_1 y/x & \\ 0 & c_1 & \\ 0 & 0 & c_2 \end{bmatrix}.$$

We will consider one further technique that gives cocycles representatives of certain cohomology classes with virtually no effort. What are these cohomology classes? The choice of subalgebra $\mathfrak{h} \subset \mathfrak{g}$ singles out a certain element $\chi \in \mathrm{H}^1(\mathfrak{h})$, which we now describe. The adjoint action naturally makes $\mathfrak{g}/\mathfrak{h}$ into an $\mathfrak{h}$ module. We let $\chi \in \mathfrak{h}^*$ denote the character of this representation. Since χ kills all commutators of $\mathfrak{h}$, we can regard χ as an element of $\mathrm{H}^1(\mathfrak{h})$.

Proposition 3.2.5 *The classifying form of the divergence cocycle, η, is $-\chi$.*

Proof: Let's proceed by examining a slightly more general case. Let x be a point of a pseudo-Riemannian manifold, $X_1, \ldots, X_n$ a frame in a neighborhood of x, and $\theta^1, \ldots, \theta^n$ the dual 1-form coframe. For a vector field, X, that is zero at x we have

$$(\operatorname{div} X)_x = \sum_i \theta^i([X_i, X])_x.$$

Recall that $a^\pi_o = 0$ for $a \in \mathfrak{h}$. Now, let $X = a^\pi$ for an $a \in \mathfrak{h}$, and take $X_i = a^\pi$ where $\{a_i\}$ is a basis of some subspace of $\mathfrak{g}$ which is complementary to $\mathfrak{h}$. The preceding formula directly implies that

$$(\bar{\eta})(a) = (\operatorname{div} a^\pi)_o = -\chi(a).$$

□

Recall from Proposition 2.5.2 that the class of the divergence cocycle is independent of the choice of metric. To get a representative of the class one needs merely to use the most convenient available metric. As such, it is best to use the flat metric associated to the local coordinates that are being used to describe the $\mathfrak{g}$ action.

Example 3.2.6 Let us extend a preceding example by considering the action of $\mathfrak{sl}(3,\mathbb{R})$ on 2 dimensional projective space. Instead of using a single-index basis, we will use the more natural representation of $\mathfrak{sl}(3)$ by traceless 3×3 matrices. We take affine coordinates $[1, y^2, y^3]$ on $\mathbb{RP}^2$, and thus the $\mathfrak{sl}(3)$ actions are given by

$$\begin{bmatrix} -y^2\partial_2 - y^3\partial_3 & \partial_2 & \partial_3 \end{bmatrix} \begin{bmatrix} \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{31} & \eta_{32} & \eta_{33} \end{bmatrix} \begin{bmatrix} 1 \\ y^2 \\ y^3 \end{bmatrix}, \qquad \text{where } \eta_{11} + \eta_{22} + \eta_{33} = 0.$$

We take $y^2 = 0$, $y^3 = 0$ as the basepoint and hence $\mathfrak{h}$, the isotropy subalgebra, and its commutators are given by the following matrices

$$\mathfrak{h} = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \qquad [\mathfrak{h},\mathfrak{h}] = \begin{bmatrix} 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$$

Since the codimension of $[\mathfrak{h},\mathfrak{h}]$ in $\mathfrak{h}$ is one, we can conclude that H^1 is one-dimensional, and that the classifying form will be

$$\begin{bmatrix} 2c & 0 & 0 \\ & -c & 0 \\ & 0 & -c \end{bmatrix},$$

where c is an arbitrary constant. The classifying form is a multiple of the character, χ, and thus we can obtain a cocycle by simply taking the flat divergence of the $\mathfrak{g}$ actions with respect to the y^2, y^3 coordinates. The divergence cocycle is therefore

$$\begin{bmatrix} -2c & -3cy^2 & -3cy^3 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}.$$

Chapter 4

Geometric Aspects of a Lie Algebraic Operator System

Whoever ... proves his point and demonstrates
the prime truth geometrically should be believed
by all the world, for there we are captured.
– Albrecht Dürer

4.1 Preliminaries

To get at the intrinsic geometry induced by the second-order component of a Lie algebraic operator system it becomes necessary to expand our viewpoint from the quotient $\mathbf{G}/\mathbf{H}$ to all of $\mathbf{G}$. Unfortunately, a Lie-algebraic operator in its raw form does not determine $\mathbf{G}$ explicitly. Certainly, there is a list of vector fields in some system of local coordinates, and the structure constants that make that list into a Lie algebra, $\mathfrak{g}$; but $\mathbf{G}$, as such, is not given by the setup. Yet one knows that the group is there. In theory, it is possible to choose local coordinates of the group, and to explicitly specify the group action on $\mathbf{M}$ in terms of these coordinates. In certain simple cases this is actually a practical undertaking.

Perhaps, even more importantly, the Lie group setting possesses a rich, formal geometric theory, which can be used to describe and verify properties of the homogeneous space. We already encountered one such technique in Section 3.1. Every cocycle $\eta \in Z^1(\mathfrak{g}; C^\infty(M))$ can be locally integrated to a function $f \in C^\infty(G)$, and the derivatives of this function with respect to $\mathfrak{h}^R$ vector-fields are guaranteed to be a constant. The point is that we do not need to explicitly write down this function. Merely knowing that it exists tells us that $H^1(\mathfrak{g}; C^\infty(M))$ is isomorphic to $H^1(\mathfrak{h}; 1)$ (see Theorem 3.1.1).

The next few sections will proceed in the same vein. We will discuss properties of homogeneous space, in terms of objects defined on the group above.

4.2 Horizontal Vector Fields and Adapted Frames

The projection $\pi : G \to G/H$, gives a natural vertical distribution, $\mathfrak{h}^R$, on G. Dually, there is the natural cotangent sub-bundle of horizontal 1-forms. This sub-bundle is spanned by differential forms α^R, where $\alpha \in \mathfrak{g}^*$ annihilates $\mathfrak{h}$. We will denote it by $(\mathfrak{h}^\perp)^R$. Now, let us fix a $C \in S^2\mathfrak{g}$, and consider the corresponding left-invariant, type $\binom{0}{2}$ tensor field, C^L. In general, the extra information given by C allows us to decompose the tangent bundle of G, and allows us to speak of horizontal vectors and vertical 1-forms. The decomposition is given by

$$TG = \mathfrak{h}^R \oplus C^L(\mathfrak{h}^\perp)^R.$$

According to Proposition 2.2.1, the decomposition fails if and only if the projected metric tensor, C^π, is degenerate at $\pi(g)$. Thus, if $g \in G$ is not a point of degeneracy, the projection of the vector-fields $C^L(\mathfrak{h}^\perp)^R$ spans the tangent space of G/H at $\pi(g)$. Accordingly, we set M to be the subset of G/H where C^π is not degenerate. One further note on the abuse of notation: we will write the canonical projection as $\pi : \mathbf{G} \to \mathbf{M}$. It should be understood that we are excluding the fibers that lie above degenerate points of $\mathbf{G}/\mathbf{H}$.

Definition 4.2.1 For each $\alpha \in \mathfrak{h}^{\perp}$, we call $C^{\mathrm{L}}\alpha^{\mathrm{R}}$ a *horizontal vector field*, because such vector-fields span the horizontal distribution induced by C. For each $a \in \mathfrak{h}$, we call a^{R} a *vertical vector field*, because these vector fields span the vertical directions of the projection $\pi : \mathbf{G} \to \mathbf{M}$. We call this method of indexing horizontal vector fields by $\mathfrak{h}^{\perp}$ and vertical vector fields by $\mathfrak{h}$ the *adapted frame*. The restriction of this index to $\mathfrak{h}^{\perp}$ and the horizontal vector fields will be called the *horizontal subframe*.

It is helpful to have a description of the adapted frame in terms of a basis. Let

$$a_1, \ldots, a_{n-m}, a_{n-m+1}, \ldots, a_n$$

be an adapted basis of $\mathfrak{g}$, such that the last m entries give a basis of $\mathfrak{h}$. This way the first $n-m$ entries of the dual basis $\alpha^1, \ldots, \alpha^n$ will give a basis of $\mathfrak{h}^{\perp}$. In terms of such a basis we will give the adapted frame as $H^1, \ldots, H^{n-m}, V_{n-m+1}, \ldots, V_n$, where $H^i = C^{\mathrm{L}}\alpha^{i\mathrm{R}}$ and $V_j = a_j^{\mathrm{R}}$.

The structure equations of the adapted frame split into three classes: vertical–vertical, horizontal–vertical, and horizontal–horizontal. The first two classes do not depend on the geometry induced by C and are therefore of no interest to us.

Proposition 4.2.2 *The vertical vector fields form a Lie algebra isomorphic to $\mathfrak{h}$. The vertical–vertical structure equations are therefore equivalent to the structure equations of $\mathfrak{h}$. The Lie derivative of a vertical and a horizontal vector field is a horizontal vector field. In this way the space of horizontal vector fields is a module of the Lie algebra of vertical vector fields. The structure constants of this action are the same as the structure constants of the canonical, coadjoint action of $\mathfrak{h}$ on $\mathfrak{h}^{\perp}$.*

Proof: The assertion about the vertical–vertical structure equations is self-evident. For the vertical–horizontal case, since a horizontal vector field $C^{\mathrm{L}}\alpha^{\mathrm{R}}$ is a contraction involving a left-invariant tensor field, C^{L}, we have

$$[a^{\text{R}}, C^{\text{L}}\alpha^{\text{R}}] = C^{\text{L}}(\mathcal{L}_{a^{\text{R}}}(\alpha^{\text{R}})) = C^{\text{L}}(\text{ad}(a)^{*}(\alpha))^{\text{R}}.$$

□

To describe the structure equations of the adapted frame it will be convenient to switch to the indices-oriented approach. To that end, let us name the structure coefficients of the underlying Lie algebra $\mathfrak{g}$:

$$[a_i, a_j] = S^k_{ij}\, a_k, \quad i, j, k = 1 \dots n.$$

Recall that

$$[a^{\text{R}}, b^{\text{R}}] = -([a, b])^{\text{R}}, \quad a, b \in \mathfrak{g},$$

$$\mathcal{L}_{a^{\text{R}}}(\alpha^{\text{R}}) = (\text{ad}(a)^{*}(\alpha))^{\text{R}}, \quad a \in \mathfrak{g}, \alpha \in \mathfrak{g}^{*}.$$

Hence, the two types of structure equations for the adapted frame have the same coefficients as the structure equations for $\mathfrak{g}$. Namely,

$$[V_i, V_j] = -S^k_{ij} V_k, \quad [V_i, H^j] = S^j_{ik} H^k.$$

The horizontal–horizontal type of structure equations, however, have non-constant coefficients (the tilde above the A and B serves to remind us of this):

$$[H^i, H^j] = \sum_{k=1}^{n-m} 2\tilde{A}^{ij}_k H^k + \sum_{k=n-m+1}^{n} \tilde{B}^{ijk} V_k.$$

The factor of 2 in the above equation is there to simplify some later formulas.

The adapted frame of vertical and horizontal vector fields will be the primary tool for our investigation of the geometry of a Lie algebraic space. After two digressions devoted to technical issues we will resume with a discussion of the properties of adapted frames.

4.3 Tensor Fields of Mixed Type

A vector field like $C^{\mathrm{L}}\alpha^{\mathrm{R}}$ is a contraction of a left and a right-invariant tensor. We call the resulting object a tensor of mixed type. In this section we will develop calculation techniques to handle such tensors. First let us recall the following elementary facts.

Proposition 4.3.1 *For $a \in \mathfrak{g}$, $\alpha \in \mathfrak{g}^*$ we have*

$$a^{\mathrm{L}} = (\mathrm{Ad}\, a)^{\mathrm{R}}, \quad \alpha^{\mathrm{R}} = (\mathrm{Ad}^*\, \alpha)^{\mathrm{L}} = (\alpha\, \mathrm{Ad})^{\mathrm{L}},$$

where we regard Ad *as a function from* $\mathbf{G}$ *to* $\mathrm{End}(\mathfrak{g})$.

For ease of notation we will use $\tilde{\alpha}$ to denote the $\mathfrak{g}^*$-valued function $\mathrm{Ad}^*\,\alpha$. We thus have $\alpha^{\mathrm{R}} = \tilde{\alpha}^{\mathrm{L}}$. The identification of tensor fields with tensor valued functions allows us to define a left-invariant derivative on a Lie group. Let V be a vector space and $f : \mathbf{G} \to V$ a smooth function. We define $\mathrm{D}^{\mathrm{L}}(f) : \mathbf{G} \to \mathrm{Hom}(\mathfrak{g}, V)$ by

$$\mathrm{D}^{\mathrm{L}}(f)(a) = a^{\mathrm{L}} f, \quad a \in \mathfrak{g}.$$

We will need a formula for the Lie bracket of vector fields in this formalism.

Proposition 4.3.2 *Let f, g be $\mathfrak{g}$-valued functions on* $\mathbf{G}$. *Then,*

$$[f^{\mathrm{L}}, g^{\mathrm{L}}] = (\mathrm{ad}(f, g) + f\,\mathrm{D}^{\mathrm{L}}(g) - g\,\mathrm{D}^{\mathrm{L}}(f))^{\mathrm{L}}.$$

We also need a formula for the derivative of Ad.

Proposition 4.3.3 $\mathrm{D}^{\mathrm{L}}(\mathrm{Ad})_g(a) = \mathrm{Ad}_g\,\mathrm{ad}(a)$, *where $a \in \mathfrak{g}$, and $g \in \mathbf{G}$.*

Let us restate the above in a more convenient notation. Since C acts as an inner product on $\mathfrak{g}^*$ it also induces and inner product, on $\Lambda^2\mathfrak{g}^*$, which is given by

$$C(\alpha^i \wedge \alpha^j, \alpha^k \wedge \alpha^l) = \left\langle \alpha^i \wedge \alpha^j; C\alpha^k \wedge C\alpha^l \right\rangle = \begin{vmatrix} C^{ik} & C^{jk} \\ C^{il} & C^{jl} \end{vmatrix}$$

Thus, for $\alpha, \beta, \gamma \in \mathfrak{g}^*$ we have

$$\gamma(\mathrm{ad}(C\alpha, C\beta)) = -\delta\gamma(C\alpha, C\beta) = -C(\delta\gamma, \alpha \wedge \beta),$$

where $\delta : \mathfrak{g}^* \to \Lambda^2\mathfrak{g}^*$ is the usual coboundary operator. The preceding discussion and Propositions 4.3.2 and 4.3.3 combine to give the following formulas.

Proposition 4.3.4 *For $\alpha, \beta, \gamma \in \mathfrak{g}^*$ we have*

$$\begin{aligned}(C^{\mathrm{L}}\gamma^{\mathrm{R}})(C^{\mathrm{L}}(\alpha^{\mathrm{R}}, \beta^{\mathrm{R}})) &= -C(\delta\tilde{\alpha}, \tilde{\beta} \wedge \tilde{\gamma}) - C(\delta\tilde{\beta}, \tilde{\alpha} \wedge \tilde{\gamma}),\\ \gamma^{\mathrm{R}}([C^{\mathrm{L}}\alpha^{\mathrm{R}}, C^{\mathrm{L}}\beta^{\mathrm{R}}]) &= -C(\delta\tilde{\gamma}, \tilde{\alpha} \wedge \tilde{\beta}) + C(\delta\tilde{\beta}, \tilde{\alpha} \wedge \tilde{\gamma}) + C(\delta\tilde{\alpha}, \tilde{\beta} \wedge \tilde{\gamma}).\end{aligned}$$

At this point we will introduce a number of symbols that will help us with the calculations involving adapted frames. We work with an adapted basis as described in Definition 4.2.1. The indices i, j, k, will be presumed to range from 1 to $n - m$. We put

$$\tilde{C}^{ij} = C^{\mathrm{L}}(\alpha^{i\mathrm{R}}, \alpha^{j\mathrm{R}}) = C(\tilde{\alpha}^i, \tilde{\alpha}^j).$$

These symbols describe the inner product in terms of the horizontal vector fields. Thus,

$$H^i \cdot H^j = \tilde{C}^{ij}.$$

We define the symbols $\tilde{C}_{ij}$ (subscripts, rather than superscripts) as the entries of the matrix that is the inverse of $\tilde{C}^{ij}$. For a given $L \in \mathfrak{g}$ we put

$$\tilde{L}^i = \tilde{\alpha}^{i\mathrm{R}}(L).$$

Writing L as $L^j a_j$, the preceding symbol can also be defined as

$$\tilde{L}^i = \sum_{j=1}^{n} L^j \,\mathrm{Ad}^i_j.$$

Next, we define

$$\begin{aligned}\tilde{T}^{ijk} &= \tilde{\alpha}^k(\mathrm{ad}(C\tilde{\alpha}^i, C\tilde{\alpha}^j))\\ &= -C(\tilde{\alpha}^i \wedge \tilde{\alpha}^j, \delta\tilde{\alpha}^k)\\ &= \tilde{C}^{ir}\tilde{C}^{js}S^k_{rs}.\end{aligned}$$

Using these symbols Proposition 4.3.4 can be restated as

$$H^i(\tilde{C}^{jk}) = \tilde{T}^{ijk} + \tilde{T}^{ikj}, \tag{4.1}$$

$$(\alpha^k)^{\mathrm{R}}([H^i, H^j]) = \tilde{T}^{ijk} - \tilde{T}^{jki} - \tilde{T}^{kij} \tag{4.2}$$

4.4 A Generalized Covariant Derivative

In this section we fix a $C \in S^2\mathfrak{g}$, and work with the pseudo-Riemannian geometry induced by the metric tensor C^π on M. In particular we will show how to extend the covariant derivative operator, ∇, of the corresponding Levi-Civita connection to a more general operator that operates on sections of the bundle $\mathrm{TG}/\mathfrak{h}^{\mathrm{R}} \to \mathbf{G}$. We are particularly interested in the covariant derivatives of the horizontal vector fields, $C^{\mathrm{L}}\alpha^{\mathrm{R}}$, where $\alpha \in \mathfrak{h}^\perp$. These vector fields are not projectable, and that is why we need to implement a meaningful extension of the covariant derivative that can operate on them.

Consider a path, $\gamma : [-\epsilon, \epsilon] \to \mathbf{G}$. The essence of the present idea is to pull the parallel translation along the projected path, $\pi \circ \gamma$, back to γ. There is a complication; the parallel translation defined along γ does not operate on vectors of G, but on vectors of M. Note that

$$\mathrm{T}_x\mathrm{M} \cong \mathrm{T}_g\mathrm{G}/\mathfrak{h}^{\mathrm{R}}, \quad \text{where } g \in \mathbf{G},\ x = \pi(g),$$

and therefore, we must stipulate that the parallel translation along γ, and the corresponding covariant derivative operate on sections of the bundle $\mathrm{TG}/\mathfrak{h}^{\mathrm{R}} \to \mathbf{G}$.

With these preliminaries out of the way, we can derive the connection coefficients in terms of the adapted frame.

Proposition 4.4.1 *The parallel translation of a horizontal vector in a vertical direction is given by the flow of the corresponding vertical vector field. Thus, the covariant derivative of a horizontal vector field in a vertical direction is given by* *the Lie derivative of that vertical vector field. Formally,*

$$\nabla_{V_i} H^j = [V_i, H^j] = S^j_{ik} H^k.$$

Proof: Starting from a fixed point $g \in \mathbf{G}$, the flow of a vertical vector field, $V_i = a_i^{\mathrm{R}}$, projects down to an unmoving point $\pi(g) \in \mathbf{M}$. The vector H^j along that flow is not constant, however. It is given by $C^{\mathrm{L}}(\alpha(t))^{\mathrm{R}}$, where

$$\alpha(t) = \exp(ta_i)^* \alpha^j,$$

is the corresponding curve in $\mathfrak{h}^{\perp}$. The parallel translation along an constant path is just the identity automorphism of $\mathrm{T}_{\pi(g)}\mathbf{M}$, and therefore the covariant derivative of H^j at g is simply the derivative of the curve that $\pi_*(H^j \circ \gamma)$ makes in $\mathrm{T}_{\pi(g)}\mathbf{M}$. □

Proposition 4.4.2 *The covariant derivative of a horizontal vector-field in a horizontal direction is given by*

$$\begin{aligned} \nabla_{H^i} H^j &= 1/2\,[H^i, H^j] \quad \bmod\ \mathfrak{h}^{\mathrm{R}} \\ &= \tilde{A}^{ij}_k H^k. \end{aligned}$$

Proof: The standard formula for the covariant derivative of the Levi-Civita connection remains valid for non-projectable vector fields. The formula in question is

$$\begin{aligned} 2\nabla_{H^i} H^j \cdot H^k &= 2\,(\alpha^k)^{\mathrm{R}}(\nabla_{H^i} H^j) \\ &= H^i(H^j \cdot H^k) + H^j(H^i \cdot H^k) - H^k(H^i \cdot H^j) \\ &\quad -H^i \cdot [H^j, H^k] - H^j \cdot [H^i, H^k] + H^k \cdot [H^i, H^j] \end{aligned}$$

Combining the above with identities (4.1) and (4.2) gives

$$\begin{aligned} 2(\alpha^k)^{\mathrm{R}}(\nabla_{H^i} H^j) &= T^{ijk} - T^{jki} - T^{kij} \\ &= (\alpha^k)^{\mathrm{R}}([H^i, H^j]) \end{aligned}$$

By fixing i, j, and varying k, we see that $\nabla_{H^i} H^j$ must match the horizontal component of $\frac{1}{2}[H^i, H^j]$. □

4.5 Geometric Properties of the Adapted Frame

In this section we describe the metric geometry of (M,C^π) in terms of the adapted frame. We will obtain interesting formulas for the gradient, divergence and the Laplace-Beltrami operator, and also a formula for the divergence cocycle. The most striking result may well be the following.

Theorem 4.5.1 *The projections of the horizontal vector fields are auto-parallel. In other words, the flow of the horizontal vector fields projects down to geodesics on* M.

Proof: This theorem is a direct consequence of Proposition 4.4.2, which implies that $\nabla_{H^i} H^i = 0$. □

A word of caution is required at this point. A horizontal vector field is not, in general, projectable, and thus does not give a foliation of M by geodesic trajectories. The theorem merely states that if γ is a path in G such that $\dot{\gamma} = C^L \alpha^R$ for some fixed $\alpha \in \mathfrak{h}^\perp$, then the projection $\pi \circ \gamma$ is a geodesic down on M.

The following formulas give the gradient, divergence, and Laplace-Beltrami operators in terms of horizontal vector fields. At first glance these formulas do not appear to make sense, because they purport to equate projectable objects on the left hand side with non-projectable ones on the right. It must be understood that the proposition asserts that the end product of these formulas is a projectable object.

Proposition 4.5.2 *Let f be a function on* **M**, *and X a projectable vector field on* **G**. *We have*

$$\begin{aligned} \operatorname{grad}(f) &= \tilde{C}_{ij} H^i(f) H^j && (4.3) \\ \operatorname{div}(X) &= \tilde{C}_{ij} \nabla_{H^i}(X) \cdot H^j \\ \Delta(f) &= \tilde{C}_{ij} H^i(H^j(f)) && (4.4) \end{aligned}$$

Proof: In the sequel we identify f with its pullback to **G**, and X with its projection down to **M**. The formula for the gradient follows from the following:

$$\begin{aligned} H^i(f) &= H^i \cdot \operatorname{grad}(f), \\ \sum_{ij} \tilde{C}_{ij} H^i(f) H^j \cdot H^k &= H^k(f). \end{aligned}$$

Since X is a projectable vector field, the covariant derivative of X in a vertical direction is zero. Hence, $\nabla_{H^i} X$ at $g \in \mathbf{G}$ is equal to the conventional covariant derivative $\nabla_{\pi_*(H^i)} X$ at $\pi(g)$. Hence

$$(\tilde{C}_{ij} \nabla_{H^i} X \cdot H^j)_g = \operatorname{tr}(\nabla X)_{\pi(g)}.$$

The right hand side of the above expression is just the divergence of X at $\pi(g)$.

The formula for the Laplacian can be derived as follows. The covariant derivative is compatible with the metric inner product, and hence

$$\begin{aligned} H^i(H^j(f)) &= H^i(\operatorname{grad}(f) \cdot H^j) \\ &= (\nabla_{H^i} \operatorname{grad}(f)) \cdot H^j + \operatorname{grad}(f) \cdot \nabla_{H^i} H^j. \end{aligned}$$

After multiplying by $\tilde{C}_{ij}$, summing over i, j, and using the preceding formula for the divergence we obtain

$$\tilde{C}_{ij} H^i(H^j(f)) = \Delta(f) + \tilde{C}_{ij} \operatorname{grad}(f) \cdot \nabla_{H^i} H^j.$$

The desired formula follows from the observation that $\tilde{C}_{ij}$ is symmetric in i and j, whilest $\nabla_{H^i} H^j$ is anti-symmetric in these variables. Therefore, the second term of the right hand side of the above equation is zero. □

Next, we focus on the divergence cocycle, ϕ, described in Section 2.5. Since ϕ is a cocycle, it must (at least locally) be the coboundary of some function of **G**. We can describe this function in terms of the horizontal vector fields.

Proposition 4.5.3 *The divergence cocycle is equal to* $-\delta\lambda$, *where* $\lambda \in C^\infty(\mathbf{G})$ *is given by*

$$\begin{aligned}\lambda &= 1/2 \log(\det(\left\{H^i \cdot H^j\right\}) \\ &= 1/2 \log(\det\left\{\tilde{C}^{ij}\right\}).\end{aligned}$$

Proof: The volume of the **M** frame formed by the projections of $H^1, \ldots, H^{n-m}$ is given by

$$\sqrt{\det\left\{H^i \cdot H^j\right\}} = \exp(\lambda).$$

Hence, the pullback of the metric volume form to **G** is given by

$$\omega = \exp(-\lambda)\,(\alpha^1)^{\mathrm{R}} \wedge \ldots \wedge (\alpha^{n-m})^{\mathrm{R}},$$

where $\alpha^1, \ldots, \alpha^{n-m}$ is a basis of $\mathfrak{h}^\perp$. The Lie derivative commutes with the push-forward, and hence the divergence of a^π, where $a \in \mathfrak{g}$, is given by

$$\begin{aligned}\mathrm{div}(a^\pi)\omega &= \mathcal{L}_{a^{\mathrm{L}}}(\omega) \\ &= a^{\mathrm{L}}(\exp(-\lambda))\,(\alpha^1)^{\mathrm{R}} \wedge \ldots \wedge (\alpha^{n-m})^{\mathrm{R}} \\ &= -a^{\mathrm{L}}(\lambda)\,\omega.\end{aligned}$$

Hence, $\mathrm{div}(a^\pi) = -a^{\mathrm{L}}(\lambda)$, and therefore, the divergence cocycle is equal to $-\delta\lambda$. □

Definition 4.5.4 Because of its relation to the divergence cocycle, ϕ, we will call λ the *divergence function.*

Next, I would like to illustrate the formulas in Proposition 4.5.2 with a concrete example. Let us use the special orthogonal group, $\mathbf{G} = \mathrm{SO}(3)$, and the 2-sphere,

$\mathrm{M} = S^2 = \mathrm{SO}(3)/\mathrm{SO}(2)$. I will not base my calculations on explicit coordinates of the group and the sphere, such an approach can get quite messy, but rather use the matrix component functions of SO(3):

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}.$$

The projection $\pi : \mathrm{SO}(3) \to S^2$ is simply the operation of taking the first row of the orthogonal matrix. The functions x_{11}, x_{12}, x_{13} are constant on the fiber of this projection and are thus functions of the sphere. As a matter of convenience I will also label these three functions x, y, z, respectively.

First, I will compute the gradient and Laplacian of x. The sphere inherits its metric structure from the ambient Euclidean (x, y, z) space. The sphere gradient is therefore the orthogonal projection of the Euclidean gradient, $\frac{\partial}{\partial x}$, onto the tangent space of the sphere. A straightforward calculation gives

$$\begin{aligned} \mathrm{grad}(x) &= (y^2 + z^2)\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y} - xz\frac{\partial}{\partial z} && (4.5) \\ &= y(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}) + z(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}) && (4.6) \end{aligned}$$

The parenthesized vector fields are infinitesimal isometries, and thus have zero divergence. Hence,

$$\Delta(x) = \mathrm{div}\,\mathrm{grad}(x) = -2x. \tag{4.7}$$

Now we check the formulas in Proposition 4.5.2. against these givens. As the adapted basis of the Lie algebra let us take

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

The first of these span the isotropy algebra, and the second and third span the horizontal complement. The next step is to compute $H^i(x)$. Let θ^{R}_{ij} be the right-invariant Maurer-Cartan forms. From the identity

$$\begin{pmatrix} 0 & \theta^R_{12} & \theta^R_{13} \\ -\theta^R_{12} & 0 & \theta^R_{23} \\ -\theta^R_{13} & -\theta^R_{23} & 0 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} dx_{11} & dx_{12} & dx_{13} \\ dx_{21} & dx_{22} & dx_{23} \\ dx_{31} & dx_{32} & dx_{33} \end{pmatrix}$$

it follows that

$$\begin{aligned} dx_{11} &= x_{21}\theta^R_{12} + x_{31}\theta^R_{13} \\ dx_{21} &= -x_{11}\theta^R_{12} + x_{31}\theta^R_{23} \\ dx_{31} &= -x_{11}\theta^R_{13} - x_{21}\theta^R_{23} \end{aligned}$$

and hence

$$\begin{aligned} H^2(H^2(x)) &= H^2(x_{21}) = -x_{11} = -x \\ H^3(H^2(x)) &= H^3(x_{21}) = 0 \\ H^3(H^3(x)) &= H^3(x_{31}) = -x_{11} = -x \\ H^2(H^3(x)) &= H^2(x_{31}) = 0 \end{aligned}$$

In order to get the standard metric on the sphere we must use the negative of the Killing form on $\mathfrak{so}(3)$. The corresponding components are just

$$C^{ij} = \delta_{ij}.$$

According to (4.3) we must have

$$\text{grad}(x) = x_{21}H^2 + x_{31}H^3.$$

From the definition of the Maurer-Cartan form we get

$$dx_{1i}(H^2) = x_{2i}, \qquad dx_{1i}(H^3) = x_{3i},$$

and hence

$$\begin{aligned} dx(\text{grad}(x)) &= x_{21}^2 + x_{31}^2 = 1 - x_{11}^2 = y^2 + z^2 \\ dy(\text{grad}(x)) &= x_{21}x_{22} + x_{31}x_{32} = -x_{11}x_{12} = -xy \\ dz(\text{grad}(x)) &= x_{21}x_{23} + x_{31}x_{33} = -x_{11}x_{13} = -xz \end{aligned}$$

This agrees with our earlier result (4.5). As for the Laplacian, Proposition 4.5.2 predicts that $\Delta(x) = -x - x = -2x$, and this is in agreement with (4.7).

Similar calculations show agreement for x, y, and z. But notice that the formula for the gradient (4.3) defines a first order homogeneous differential operator and this operator agrees with grad on x, y, and z. The differentials of these functions span the cotangent space of M, and hence the gradient formula agrees with grad on all functions. Similar reasoning will show that the operator given by (4.4) must agree with Δ on all functions. We are first required, however, to demonstrate that the two operators agree on the quadratic as well as the linear functions. This can be accomplished by noting that the operator in (4.4) and Δ share the following identity:

$$\Delta(fg) = f\,\Delta(g) + 2\,\mathrm{grad}(f)\cdot\mathrm{grad}(g) + g\,\Delta(f)$$

Our example is also a good illustration of Proposition 4.5.3. Since the S^2 metric is both left and right invariant, we have $\tilde{C}^{ij} = C^{ij}$. Hence, according to Proposition 4.5.3 the divergence function, λ, is a constant, and the divergence cocycle is zero. This is in perfect agreement with the fact that the horizontal vector fields, H^i, are infinitesimal isometries, and hence have zero divergence.

4.6 The Bundle of Horizontal Frames and the Canonical Connection

We have already seen that to each $g \in \mathrm{G}$ corresponds the horizontal frame of $\mathrm{T}_{\pi(g)}\mathrm{M}$ given by $H^1, \ldots, H^{n-m}$.

Definition 4.6.1 We identify G with *the bundle of horizontal frames* of M, and define the *the canonical connection* on M as the connection given by the horizontal-vertical decomposition of the tangent space of G. This is in contrast to the Levi-Civita connection of C^π, or *the natural connection*, that we have already encountered.

The terms *natural connection* and *canonical connection* do not originate with us. Both of these connections arise in the theory of naturally reductive homogeneous spaces (see [19]). In a very real sense the objects at hand, Lie-algebraic spaces, extend the notion of a reductive homogeneous space, and these two types of connection come along with the extended theory.

To give an explicit description of the natural connection we again turn to the adapted frame and to the notion of the extended covariant derivative (see Section 4.4).

Proposition 4.6.2 *The extended covariant derivative associated with the natural connection obeys the following identities:*

$$\begin{aligned}\tilde{\nabla}_{V_i} H^j &= [V_i, H^j] \\ \tilde{\nabla}_{H^i} H^j &= 0\end{aligned}$$

Proof: The first of the above identities holds for reasons explained in Proposition 4.4.1. That proposition treats an analogous identity for the natural connection. Indeed, the identities involving the covariant derivative in a vertical direction are derived from the properties of the extended covariant derivative, rather than the specific nature of the connections in question.

The second identity is more or less a tautology based on the definition of the natural connection. Let γ be a **G**-path given by the flow of H^i. The definition of the extended covariant derivative states that $\nabla_{H^i} H^j$ is the ordinary covariant derivative of $\pi_*(H^j)$ along $\pi \circ \gamma$. This ordinary covariant derivative is calculated by lifting a vector field on **M** to an $\mathfrak{h}^\perp$-valued function on **G**, the bundle of horizontal frames; and then taking the derivative of this function along the horizontal lift of a curve on **M**. The identity follows because, by definition, γ is the horizontal lift of the projected path, $\pi \circ \gamma$, and because $\pi_*(H^j)$ lifts to a constant function. □

The natural and canonical connections are closely related. In fact, as the next result shows, they give rise to very similar geometries. The principal difference between the two connections is the fact that the natural connection is torsion free, while the canonical connection has, in general, non-zero torsion.

Theorem 4.6.3 *The natural and the canonical connections have the same geodesics.*

Proof: Just like in Theorem 4.5.1 we have

$$\tilde{\nabla}_{H^i} H^i = 0,$$

and hence, the geodesics of both connections are given by the projected flows of the horizontal vector fields. □

The above result is related to a theorem due to H. Weyl (see Addendum 1. to Chapter 6 of [29]), which states that two connections, ∇ and $\tilde{\nabla}$, have the same geodesics if and only if their type $\binom{2}{1}$ difference tensor, $\nabla - \tilde{\nabla}$, is skew-symmetric in the contravariant arguments. This is clearly the case for the natural and the canonical connections, whose difference tensor is given by

$$\begin{aligned} \nabla_{H^i} H^j - \tilde{\nabla}_{H^i} H^j &= 1/2\,[H^i, H^j] \mod \mathfrak{h}^n \\ &= \tilde{A}^{ij}_k H^k. \end{aligned}$$

Another difference between the natural and canonical connections involves their respective structure groups. Like all affine connections, both of these connections can be described by using the bundle of linear frames of M, and then by restricting to a smaller sub-bundle. This process of reducing the structure group of a connection, has a fundamental limitation. If a frame F is in the restricted sub-bundle of frames, then all frames, F', obtained from the original by parallel translation must also be in the sub-bundle. Unfortunately, G, the bundle of horizontal frames does not, in general, contain a sufficient variety of frames to be able to describe parallel translation with respect to the natural, Levi-Civita connection. It is not, in general, possible to select a path through G along which the inner product coefficients, $H^i \cdot H^j$ remain constant. This is because the inner product matrix,

$H^i \cdot H^j$, has $(n-m)^2$ functional entries, but there are only n degrees of freedom for a path in $\mathbf{G}$.

4.7 Curvature

In this section we compute and compare the curvatures of the canonical and the natural connections on M. We will find that the formula for the curvature of the canonical connection is the simpler of the two, and that the curvature of the natural connection can be obtained by adding some corrective terms.

Proposition 4.7.1 *The curvature tensor, $\tilde{R}$, of the canonical connection is given by*

$$\begin{aligned} \tilde{R}(H^i, H^j)H^k &= \tilde{B}^{ija}[V_a, H^k] \\ &= \tilde{B}^{ija} S^k_{al} H^l \end{aligned}$$

Proof: The above is a direct consequence of the standard formula for the curvature tensor,

$$\tilde{R}(H^i, H^j)H^k = \tilde{\nabla}_{H^i}\tilde{\nabla}_{H^j}H^k - \tilde{\nabla}_{H^j}\tilde{\nabla}_{H^i}H^k - \tilde{\nabla}_{[H^i,H^j]}H^k,$$

and the fundamental relation for the covariant derivative of the canonical connection,

$$\tilde{\nabla}_{H^i}H^j = 0.$$

□

Working in terms of horizontal frames we will express the above result by writing

$$\tilde{R}^{ijk}{}_l = \tilde{B}^{ija} S^k_{al}.$$

The reader is cautioned not to confuse our symbol $\tilde{R}^{ijk}{}_l$ with the more traditional symbol for the Riemannian curvature tensor. The more traditional symbol is given with respect to a coordinate vector-field frame; that's why it has three subscripts and one superscript. We are working with respect to the horizontal frame, and we index the elements of this frame with a superscript. That is why our symbol for the Riemannian curvature tensor has three superscripts and one subscript.

Proposition 4.7.2 *In terms of the horizontal frame, the curvature tensor of the natural connection is given by*

$$R^{ijk}{}_l = H^i(\tilde{A}^{jk}_l) - H^j(\tilde{A}^{ik}_l) + \tilde{A}^{jk}_a \tilde{A}^{ia}_l - \tilde{A}^{ik}_a \tilde{A}^{i}_{ja} - 2\tilde{A}^{ij}_a \tilde{A}^{ak}_l - \tilde{R}^{ijk}{}_l.$$

Proof: The derivation is quite straight forward. It involves the usual formula for curvature as well as the fundamental relation for the covariant derivative of the natural connection, namely

$$\nabla_{H^i} H^j = \tilde{A}^{ij}_k H^k.$$

□

We will also give the formulas for the Ricci and scalar curvatures of the natural connection. These formulas have an intriguingly simple form, and may turn out to have some bearing on the relation between flatness and separation of variables (Turbiner's conjecture). First, we need to derive a formula that relates the divergence function (see Proposition 4.5.3) and the structure coefficients $\tilde{A}^{ij}_k$.

Proposition 4.7.3 *We have the following expression for the "pseudo-divergence" of a horizontal vector field.*[1]

$$-\operatorname{tr}(\nabla H^i) = \tilde{A}^{ij}_j = H^i(\lambda).$$

Proof: The standard formula for the derivative of a determinant gives

$$\begin{aligned} H^i(\det\left\{H^j \cdot H^k\right\}) &= H^i(\det\left\{\tilde{C}^{jk}\right\}) \\ &= \det\left\{\tilde{C}^{jk}\right\}(\tilde{C}_{jk} H^i(\tilde{C}^{jk})). \end{aligned}$$

[1] I say pseudo-divergence, because I do not see how to meaningfully define divergence for a non-projectable vector field.

In terms of λ the above reduces to

$$2H^i(\lambda) = \tilde{C}_{jk} H^i(\tilde{C}^{jk}).$$

According to identities (4.1) and (4.2),

$$\begin{aligned} H^i(\tilde{C}^{jk}) &= 1/2([H^i, H^j] \cdot H^k + [H^i, H^k] \cdot H^j) \\ &= (\tilde{A}^{ij}_a \tilde{C}^{ak} + \tilde{A}^{ik}_a \tilde{C}^{aj}). \end{aligned}$$

Combining the preceding two equations we obtain

$$2H^i(\lambda) = \tilde{A}^{ij}_a \delta^a_j + \tilde{A}^{ik}_a \delta^a_k = 2\tilde{A}^{ij}_j.$$

□

Proposition 4.7.4 *In terms of horizontal frames, the Ricci curvature of the natural connection is given by*

$$R^{jk} = H^j(H^k(\lambda)) - \mathrm{tr}(\nabla H^j \circ \nabla H^k) - \tilde{R}^{jk},$$

where

$$\tilde{R}^{jk} = \tilde{R}^{ajk}{}_a$$

is the Ricci-curvature of the canonical connection.[2]

Proof: The Ricci curvature tensor is given by $R^{jk} = R^{ijk}{}_i$, and is symmetric in the indices j, k. To get the desired expression we simply use the formula for $R^{ijk}{}_i$ given in Proposition 4.7.2; cancel all terms that are skew-symmetric in j, k; and use the identity

$$\tilde{A}^{ik}_i = -H^k(\lambda)$$

proven in Proposition 4.7.3. □

[2]Since the canonical connection will, in general, have non-zero torsion, $\tilde{R}^{jk}$ is not necessarily a symmetric tensor.

Next, we specialize to the gaussian curvature of a planar Lie algebraic space. In this setting we get an intriguing relationship between the curvature and the associated potential of the Lie algebraic operator. First, we need to make a few notational remarks. A tensor $C \in S^2\mathfrak{g}$ induces the metric tensor C^π on M, and the latter defines the gradient operation on functions of M. We can sensibly extend the gradient operation to functions of G by using the left-invariant tensor C^L:

$$\mathrm{grad}(f) = C^{ij} a_i^L(f) a_j^L, \quad f \in C^\infty(\mathrm{G}).$$

If f is a function of M, the above formula gives a vector field that projects down to the usual $\mathrm{grad}(f)$ on M. In Section 3.1 we encountered the rather special functions, $f \in C^\infty(\mathrm{G})$, whose coboundaries give cocycles with coefficients in $C^\infty(\mathrm{M})$, i.e. $a^L(f) \in C^\infty(\mathrm{M})$ for all $a \in \mathfrak{g}$. Such functions therefore enjoy the curious property of having their gradient be a projectable vector field. In particular, λ is such a function, because $-\delta\lambda$ is the divergence cocycle. Hence, $\mathrm{grad}(\lambda)$ is a projectable vector field. In a similar vein we have a meaningful definition of $\Delta(\lambda)$ as the divergence of $\mathrm{grad}(\lambda)$, and this definition gives the same result as the formula first introduced in Proposition 4.5.2:

$$\Delta(\lambda) = \tilde{C}_{ij} H^i(H^j(\lambda)).$$

Proposition 4.7.5 *Let $C \in S^2\mathfrak{g}$ be the second-order component of a basic, homogeneous operator system. If the corresponding Lie algebraic operator satisfies the closure conditions, then the associated potential is given by*

$$V = \Delta(\lambda/2) - \mathrm{grad}(\lambda/2)^2.$$

Proof: According to Proposition 2.5.5 the closure conditions are satisfied if there is a horizontal function, $f \in C^\infty(\mathrm{M})$, such that $\mathrm{grad}(\lambda) = \mathrm{grad}(f)$. Hence, in order to put the Lie algebraic operator, $\mathcal{H} = C^{ij} a_i^L a_j^L$, into Schrödinger form, we must conjugate $\mathcal{H}$ by the change of scale operator, $\exp(f/2)$. Formally, this is summarized by

$$\mathcal{H} \;=\; \Delta - \mathrm{grad}(f)$$

$$
\begin{aligned}
&= \exp(f/2)(\Delta + V)\exp(-f/2) \\
&= \Delta - \operatorname{grad}(f) - \Delta(f/2) + \operatorname{grad}(f/2)^2 + V.
\end{aligned}
$$

The last line of the above equation determines the form of the potential, V. □

Proposition 4.7.6 *Let $C \in S^2\mathfrak{g}$ be the second-order component of a basic, homogeneous operator system. If the corresponding Lie algebraic operator satisfies the closure conditions, then the curvature of the metric C^π is given by*

$$K = V - \operatorname{grad}(\lambda/2)^2 - 1/2\tilde{R},$$

where V is the associated potential and where

$$
\begin{aligned}
\tilde{R} &= \tilde{C}_{jk}\tilde{R}^{jk} \\
&= \frac{1}{(H^1 \cdot H^2)} \sum_{a=3}^{n} \tilde{B}^a_{12}([V_a, H^1] \cdot H^2 - [V_a, H^2] \cdot H^1)
\end{aligned}
$$

is the scalar curvature of the canonical connection. In particular, if the background metric is flat, we must have

$$-\exp(\lambda)\Delta(\exp(-\lambda)) = \Delta(\lambda) - \operatorname{grad}(\lambda)^2 = \tilde{R}.$$

Proof: On a surface there are only two horizontal vector fields: H^1, and H^2. Consequently, the formula for Ricci curvature given in Proposition 4.7.4 fields takes a particularly simple form:

$$R^{jk} = H^j(H^k(\lambda)) - H^j(\lambda)H^k(\lambda) - \tilde{R}^{jk}, \quad j,k = 1,2.$$

The gaussian curvature is given by $1/2\tilde{C}_{jk}R^{jk}$, and the desired conclusion follows when we use the potential formula in Proposition 4.7.6, and the formulas for the Laplacian and the gradient given in Proposition 4.5.2. □

4.8 Imprimitive Group Actions

Earlier we mentioned that the horizontal vector fields, H^i, are not, in general, projectable. This is a pity, because, otherwise we would get a foliation of M by geodesic trajectories. The purpose of the present section is to describe a condition that allows for something almost as good, the projectability of a portion of the horizontal distribution. The condition in question is the imprimitivity of the group action. More comprehensive information on primitive and imprimitive group actions is available in [26] [7] [20].

Definition 4.8.1 We say that the action of G on M = G/H is *imprimitive* if there exists a foliation of M that is invariant under the action of G. A *foliation* is a collection of immersed submanifolds of constant dimension (called the *leaves* of the foliation) such that a unique leaf passes through each point of M. A foliation can also be represented by its infinitesimal data: an integrable distribution of constant rank. A rank k *distribution* is an assignment of k-dimensional linear subspaces $\mathcal{D}_p \subset T_p M$ to all points of M. Integrability means that if vector fields X, Y are tangent to the distribution, then so is their Lie bracket, $[X, Y]$. Frobenius' theorem then tells us that if $\mathcal{D}$ is integrable, then through every $p \in$ M there passes a k-dimensional integral submanifold whose tangent space is $\mathcal{D}_p$. The foliation corresponding to $\mathcal{D}$ is the collection of the maximal integral submanifold engendered by $\mathcal{D}$. To rephrase our definition in terms of distributions, we can say that the G-action is imprimitive if there exists a constant rank, integrable, G-invariant distribution, $\mathcal{D}$. The G-invariance means that for each $g \in$ G and $p \in$ M we have

$$(T_g)_*(\mathcal{D}_p) = \mathcal{D}_{pg}.$$

The condition of imprimitivity was first described by Lie in his classification of low dimensional homogeneous spaces (see [22]). This concept is useful for the classification because an imprimitive group action can be projected to an action on a quotient of M. Consider as an example the following planar realization of the

Lie algebra $\mathfrak{gl}(2)$:

$$\partial_x, \quad x\partial_x, \quad y\partial_y, \quad x(x\partial_x - y\partial_y).$$

Notice how all the coefficients of ∂_x only involve functions of x. This means that the foliation $x = \text{const}$ is invariant under the local group action of GL(2), and hence we can project the group action down to a 1-dimensional quotient, where the infinitesimal actions will be

$$\partial_x, \quad x\partial x, \quad 0, \quad x^2\partial_x.$$

The above example illustrates that the imprimitive nature of a group action can be evident because of an appropriate choice of local coordinates. Fortunately, one does not have to play with local coordinates in order to test for imprimitivity; an invariant criterion based on abstract properties of the group action is available.

Theorem 4.8.2 *Consider a local, effective action of a Lie group,* **G***, on a homogeneous space,* $\mathbf{M} = \mathbf{G}/\mathbf{H}$*. Let* $\mathfrak{g}$*,* $\mathfrak{h}$ *be the Lie algebras of* **G** *and the isotropy subgroup,* **H***, respectively. The* **G***-action is imprimitive if and only if it is not a maximal subalgebra of* $\mathfrak{g}$*, i.e. if and only if there exists a Lie algebra* $\mathfrak{f}$ *that is properly intermediate between* $\mathfrak{h}$ *and* $\mathfrak{g}$*.*

Proof: Suppose that an intermediate $\mathfrak{f}$ exists. The desired invariant distribution can be obtained by projecting the invariant distribution $\mathfrak{f}^{\mathrm{R}}$ down to **M**. Now the vector fields a^{R}, where $a \in \mathfrak{f}$, are not themselves projectable, but we will show that $\mathfrak{f}^{\mathrm{R}}$ as a whole does project down. Let $a \in \mathfrak{f}$ be given and consider what happens to $\pi_*(a^{\mathrm{R}})_g$, as we move g within a single fibre of $\pi : \mathbf{G} \to \mathbf{M}$. It is straightforward to check that

$$\pi_*(a^{\mathrm{R}}_{hg}) = \pi_*(b^{\mathrm{R}})_g,$$

where $h \in \mathbf{H}$, and $b = \mathrm{Ad}(h)a$. Since $\mathfrak{f}$ is closed under the adjoint action of **H**, we do not get new vectors as we project $\mathfrak{f}^{\mathrm{R}}$ from different positions in the fibre. The foliation, $\pi_*(\mathfrak{f}^{\mathrm{R}})$ is **G** invariant, because $\mathfrak{f}^{\mathrm{R}}$ defines a right-invariant foliation on **G**, and because our convention is that the right **G** multiplications give the **G** action on M. The projected foliation , $\pi_*(\mathfrak{f}^R)$, is not trivial, i.e. the leaves are not just points of M, because by assumption $\mathfrak{f}$ is larger than $\mathfrak{h}$, and because the action of G on M is effective.

Conversely, suppose that we have a k-dimensional invariant foliation, $\mathcal{F}$, on M. For $g \in G$ let $\mathcal{D}_g \subset T_g G$ be the linear space of vectors that project down to the tangent distribution of $\mathcal{F}_p$, where $p = \pi(g)$. It is not hard to see that $\mathcal{D}_g$ is tangent to $\pi^{-1}(\mathcal{F})$, the preimages of the leaves of $\mathcal{F}$. Hence, $\mathcal{D}$ is a rank $k + \dim(\mathbf{H})$, integrable distribution. The G-invariance of $\mathcal{F}$ means that $\mathcal{D}$ is right invariant. Putting

$$\mathfrak{f} = \{a \in T_e G : \pi_*(a) \in T_o \mathcal{F}\},$$

we see that $\mathcal{D}$ must be $\mathfrak{f}^R$. The integrability of $\mathcal{D}$ means that $\mathfrak{f}$ is a subalgebra of $\mathfrak{g}$. Furthermore, since $\mathfrak{h}$ is in the kernel of $(\pi_*)_e$, it must be contained in $\mathfrak{f}$. On the other hand, if k is less than the dimension of M, then $\mathfrak{h}$ cannot be all of $\mathfrak{g}$. Therefore, $\mathfrak{h}$ is the desired intermediate subalgebra. ☐

For the rest of the section let us suppose that G acts imprimitively on M. We fix an intermediate subalgebra, $\mathfrak{f}$, and let $\mathcal{D} = \pi_*(\mathfrak{f}^R)$ be the G-invariant integrable distribution on M. We also fix a $C \in \mathcal{S}^2\mathfrak{g}$, and endow M with the pseudo-Riemannian metric C^π. Speaking in terms of the inner product of this geometry, let $\mathcal{D}^\perp$ denote the distribution of vectors that are perpendicular to $\mathcal{D}$.

Theorem 4.8.3 *The perpendicular distribution, $\mathcal{D}^\perp$ is generated by the projections of the horizontal vector fields $C^L\alpha^R$, where $\alpha \in \mathfrak{f}^\perp$. Furthermore, if $\mathcal{D}^\perp$ is tangent to a geodesic of* M *at one point, then it is tangent to that geodesic throughout.*

Proof: Let us fix an $\alpha \in \mathfrak{f}^\perp$, a $p \in$ M, and consider $(C^L\alpha^R)_g$ at various points in the fiber above p. In the proof of Theorem 4.8.2 we saw that

$$\pi_*(\mathfrak{f}^R_g) = \mathcal{D}_p$$

at all g above p. Since the inner product of $\pi_*((C^L\alpha^R)_g)$ and $\pi_*(u)$, where $u \in T_g G$, is just $\alpha^R(u)$, we can infer that $\pi_*((C^L\alpha^R)_g)$ is perpendicular to $\mathcal{D}_p$ for all g above

p. By the non-degeneracy assumption on C^π we know that $\dim(\mathcal{D}^\perp)$ is equal to the codimension of $\mathfrak{f}$ in $\mathfrak{g}$, and hence is equal to $\dim(\mathfrak{f}^\perp)$. Therefore, the projection of $C^{L}(\mathfrak{f}^\perp)^{R}$ generates all of $\mathcal{D}^\perp$.

The assertion about the geodesic follows directly from Theorem 4.5.1. For if a geodesic is tangent to $\mathcal{D}^\perp$ in one place, then it can be given by the projection of the flow of some horizontal vector field $C^{L}\alpha^{R}$, where $\alpha \in \mathfrak{f}^\perp$. Since the $\pi_*(C^{L}\alpha^{R})$ stays perpendicular to $\mathcal{D}$, so does the geodesic. □

Of course, the distribution $\mathcal{D}^\perp$ is not, in general, integrable and so we do not get a geodesic foliation of M. There is one special case when this does occur. We will encounter such special cases in our subsequent study of flat, planar, Lie algebraic metrics.

Corollary 4.8.4 *Suppose that* $\dim(\mathfrak{g}) = \dim(\mathfrak{f}) + 1$. *Then there exists a 1-dimensional foliation of* M *by geodesic trajectories. These geodesics are given by the flow of* $C^{L}\alpha^{R}$ *where* $\alpha \in \mathfrak{g}^*$ *is any non-zero annihilator of* $\mathfrak{f}$.

Proof: By the preceding theorem we get a rank 1 distribution on M, and this distribution is generated by $C^{L}\alpha^{R}$. The desired conclusion follows when we recall that a rank 1 distribution is always integrable. □

4.9 An example.

At this point it will be helpful to illustrate the concepts and formulas of the preceding sections with a concrete example. This example will be based on the two-dimensional linear representation of the $GL(2,\mathbb{R})$ group. This group is sufficiently "small" so as to permit concrete, manageable formulas.

Let us use group coordinates

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix}, \tag{4.8}$$

and the following basis for the lie algebra $\mathfrak{gl}(2)$:

$$a_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad a_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$a_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad a_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The homogeneous space, M, is $\mathbb{R}^2$ minus the origin, and the projection from the group to M will be the operation of taking the first row of the group matrix. As such, the group coordinates x, and y also furnish us with coordinates on M. This setup induces the following vector-field realization of $\mathfrak{gl}(2)$:

$$a_1^\pi = x\partial_x \quad a_2^\pi = x\partial_y$$

$$a_3^\pi = y\partial_x \quad a_4^\pi = y\partial_y$$

The natural basepoint of M is $x = 1$, $y = 0$. The isotropy algebra at this point is spanned by a_3 and a_4.

The Lie algebraic operator we will consider is given by

$$\mathcal{H} = a_1^2 + a_4^2 - \{a_2, a_3\} + \{a_1, a_3\} + \{a_2, a_4\},$$

where $\{a, b\}$ denote the anticommutator $ab + ba$. The background metric of this system is given by

$$C^\pi = \begin{pmatrix} x^2 + 2xy & -xy \\ -xy & 2xy + y^2 \end{pmatrix}.$$

This is a flat metric with flat coordinates $(\xi, \eta) \in \mathbb{R}^2$ given by

$$x = e^\xi \sin^2(\eta), \quad y = e^\xi \cos^2(\eta). \tag{4.9}$$

Next we will explicitly compute the adapted frame for this system. Since $GL(2, \mathbb{R})$ is an open subset of the Euclidean space of two-by-two matrices, we can represent the tangent vectors of the group with matrices, and conveniently describe vector fields as matrices with entries that are functions of x, y, z, w. Thus, to get a left-invariant vector field we simply right multiply a constant matrix by the generic group element (4.8). The vertical vector fields of the adapted frame are therefore given by,

$$V_3 = a_3^{\mathrm{R}} = \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix}, \quad V_4 = a_4^{\mathrm{R}} = \begin{pmatrix} 0 & 0 \\ z & w \end{pmatrix}.$$

To describe the horizontal vector fields we need to be able to express the contractions of right-invariant vector fields and left-invariant 1-forms (see Section 4.3). To this end we use the formula

$$\alpha^{\mathrm{R}}(a^{\mathrm{L}})_g = \alpha(\mathrm{Ad}_g(a)), \qquad \text{where } a \in \mathfrak{g}, \alpha \in \mathfrak{g}^*, g \in \mathbf{G}.$$

The adjoint representation matrix is

$$\frac{1}{xw - yz}\begin{pmatrix} xw & -xz & yw & -yz \\ -xy & x^2 & -y^2 & xy \\ wz & -z^2 & w^2 & -wz \\ -yz & xz & -yw & xw \end{pmatrix}.$$

The horizontal vector fields, $H^i = C^{\mathrm{L}}(\alpha^i)^{\mathrm{R}}$, where $i = 1, 2$, are therefore given by

$$H^i = (\mathrm{Ad}_1^i + \mathrm{Ad}_3^i)a_1^{\mathrm{L}} + (\mathrm{Ad}_4^i - \mathrm{Ad}_3^i)a_2^{\mathrm{L}} + (\mathrm{Ad}_1^i - \mathrm{Ad}_2^i)a_1^{\mathrm{L}} + (\mathrm{Ad}_4^i + \mathrm{Ad}_2^i)a_4^{\mathrm{L}}.$$

In matrix form this is

$$\begin{aligned} H^1 &= \frac{1}{xw - yz}\begin{pmatrix} x(2yw + yz + xw) & -y(xw + 2xz + yz) \\ w(xw + 2xz + yz) & -z(2yw + yz + xw) \end{pmatrix} \\ H^2 &= \frac{1}{xw - yz}\begin{pmatrix} -2xy(x + y) & 2xy(x + y) \\ -(x + y)(yz + xw) & (x + y)(yz + xw) \end{pmatrix} \end{aligned}$$

The structure equation for the frame H^1, H^2, V_3, V_4 are

$$\begin{array}{ll} [H^1, H^2] = 0 & [V_3, V_4] = V_3 \\ [V_3, H^1] = -H^2 & [V_3, H^2] = 0 \\ [V_4, H^1] = 0 & [V_4, H^2] = -H^2 \end{array}$$

Next, let us compute the divergence function and the divergence cocycle. According to Proposition 4.5.3, the divergence function is given by

$$\lambda = 1/2\,\log(\tilde{C}^{11}\tilde{C}^{22} - (\tilde{C}^{12})^2),$$

where

$$\tilde{C}^{ij} = C^{\mathrm{L}}((\alpha^i)^{\mathrm{R}}, (\alpha^j)^{\mathrm{R}}) = C^{ab}\,\mathrm{Ad}^i_a\,\mathrm{Ad}^j_b.$$

The necessary calculation is tedious, but the result is surprisingly simple:

$$\lambda = \log(x) + \log(y) + 2\log(x+y) - 2\log(xw - yz).$$

Let us check that $-\delta\lambda$ really is the divergence cocycle, ϕ. Recall that $\phi_i = \mathrm{div}(a_i^{\pi})$. The determinant of the metric matrix in (4.9) is $2xy(x+y)^2$, and hence,

$$\begin{aligned}
\phi_1 &= \mathrm{div}(x\partial_x)\\
&= 1 - x/2\,\partial_x(\log(xy(x+y)^2)\\
&= 1 - \frac{1}{2}(1 + \frac{x}{y} + \frac{2x}{x+y})\\
&= \frac{-x+y}{2(x+y)}.
\end{aligned}$$

On the other hand we have,

$$a_1^{\mathrm{L}}(\lambda) = (x\partial_x + z\partial_z)(\lambda) = \frac{x-y}{2(x+y)},$$

confirming that $\phi = -\delta\lambda$. The remaining components of the divergence cocycle are

$$\phi_2 = \frac{-(x+3y)x}{2(x+y)y}, \quad \phi_3 = \frac{-(3x+y)y}{2(x+y)x}, \quad \phi_4 = \frac{x-y}{2(x+y)}.$$

Another calculation shows that

$$H^1(\lambda) = 0, \quad H^2(\lambda) = 0.$$

This is in accordance with Proposition 4.7.3 and the fact that the bracket, $[H^1, H^2]$ is zero. Thus, in this example λ is a purely vertical function. This means that

$$C^{\mathrm{L}}((\alpha^i)^{\mathrm{R}}, \delta\lambda) = 0, \quad i = 1, 2,$$

or to put it another way, $(C\phi)^\pi = 0$. We can now confirm the formula for the Laplacian given in Proposition 2.5.4. Using that formula and the fact that $(C\phi)^\pi = 0$ we have

$$\Delta = \mathcal{H} + (C\phi)^\pi = \mathcal{H}.$$

Let's do a computation to confirm this formula. We have

$$\begin{aligned} \mathcal{H} &= (x\partial_x)^2 + (y\partial_y)^2 - \{x\partial_y, y\partial_x\} + \{x\partial_x, y\partial_x\} + \{x\partial_y, y\partial_y\} \\ &= (x^2 + 2xy)\partial_{xx} - 2xy\partial_{xy} + (2xy + y^2)\partial_{yy} + y\partial_x + x\partial_y. \end{aligned}$$

The standard formula for the Laplacian corresponding to a metric g^{ij} is

$$\Delta = g^{ij}\partial_{ij} + \partial_i(g^{ij})\partial_j - \frac{g^{ij}\partial_i(|g^{ij}|)}{2|g^{ij}|}\partial_j,$$

where $|g^{ij}|$ is the determinant of the g^{ij} matrix. Accordingly, for our example we have

$$\begin{aligned} \Delta &= (x^2 + 2xy)\partial_{xx} - 2xy\partial_{xy} + (2xy + y^2)\partial_{yy} \\ &\quad +2(x+y)\partial_x - y\partial_x - x\partial_y + 2(x+y)\partial_y - (x+y)(\partial_x + \partial_y) \\ &= (x^2 + 2xy)\partial_{xx} - 2xy\partial_{xy} + (2xy + y^2)\partial_{yy} + y\partial_x + x\partial_y. \end{aligned}$$

Thus $\Delta = \mathcal{H}$, as expected.

These computations also serve to illustrate Proposition 4.7.5. According to that proposition the associated potential of the basic, homogeneous operator is

$$\Delta(\lambda/2) + \mathrm{grad}(\lambda/2)^2.$$

Since λ is annihilated by horizontal vector fields, Proposition 4.5.2 tells us that the above expression must be zero. This is in agreement with the fact that $\mathcal{H} = \Delta$. The latter equation means that the normalized Schrödinger operator does not require a scale change, and that the associated potential is zero.

Next, let us illustrate Theorem 4.5.1 by integrating the horizontal vector fields and showing, explicitly, that their flows project to straight lines on M. We proceed by finding constants of motion for these two vector fields. One constant we already know; it is λ. It will be more convenient to use the exponential of λ,

$$\kappa_1 = \frac{xy(x+y)^2}{(xw - yz)^2}.$$

Another constant of motion can be obtained by careful inspection of the matrix expressions for H^1 and H^2. The second constant of motion is

$$\kappa_2 = \frac{x+y}{z+w}.$$

With the help of κ_1, and κ_2 we see that the flow of H^1 is given by

$$\frac{d}{dt}(x) = x + 2\kappa\sqrt{xy}, \quad \frac{d}{dt}(y) = y - 2\kappa\sqrt{xy},$$

where $\kappa = \sqrt{\kappa_1}/\kappa_2$. These equations can be solved by rewriting them as

$$\frac{d}{dt}(x+y) = x+y, \quad \frac{d}{dt}\left(\sqrt{\frac{x}{y}}\right) = 2\kappa\left(\frac{x}{y}+1\right).$$

The solutions in terms of the flat coordinates given in (4.9) are

$$\eta = 2\kappa\xi + \text{const.}$$

The flow of H^2 is given by

$$\frac{d}{dt}(x) = -2\kappa_1\sqrt{xy}, \quad \frac{d}{dt}(y) = 2\kappa_1\sqrt{xy}.$$

The solutions are simply

$$\xi = \text{const.}$$

Thus we see that the flows of H^1 and H^2 project down to straight lines.

In section 4.8 we considered an imprimitive planar GL(2) action. That action is equivalent to the group action under present discussion; one merely makes the change of coordinates:

$$x' = x/y, \quad y' = y.$$

In terms of the coordinates used here, the invariant foliation is given by the radial lines, $y = \kappa x$. In terms of the flat coordinates, ξ and η, this foliation is given by $\eta =$ const. According to Theorem 4.8.2 this foliation must correspond to the subalgebra spanned by

$$(a_1 + a_4)^\pi = x\partial_x + y\partial_y, \quad a_3^\pi = y\partial_x, \quad a_4^\pi = y\partial_y.$$

The annihilators of this subalgebra are generated by α^2. Thus, according to Corollary 4.8.4, H^2 must project to a a foliation of geodesic trajectories which are perpendicular to the invariant foliation. This is in accordance with the above calculations, which tell us that the projection of the flow of H^2, namely $\xi =$ const, is perpendicular to the invariant foliation, namely $\eta =$ const.

Chapter 5

The Closure Conditions

What is the difference between method and device?
A method is a device which you used twice.
- George Polyá, How to Solve It.

5.1 A Reformulation of the Closure Conditions

At this point we return to the closure conditions, which we first defined in Section 2.3. The goal is to place this notion into a setting based on the group, $\mathbf{G}$, and the decomposition of the group's tangent space into vertical and horizontal directions.

Let a Lie algebraic operator, $\mathcal{H}$, be given. Recall from Section 2.2 that this entails a choice of an operator system (C, L, η) where $C \in S^2\mathfrak{g}$, $L \in \mathfrak{g}$, and $\eta \in \mathrm{Z}^1(\mathfrak{g}; \mathcal{C}^\infty(\mathrm{M}))$. The choice of C also gives us the vertical-horizontal decomposition described in Section 4.2:

$$\mathbf{TG} = \mathfrak{h}^{\mathrm{R}} \oplus C^{\mathrm{L}}(\mathfrak{h}^{\perp})^{\mathrm{R}}.$$

Note that a vertical vector field can be given as v^{R}, where v is a $\mathfrak{h}$-valued function on $\mathbf{G}$. Similarly, every horizontal vector field can be given as $C^{\mathrm{L}}\psi$, where ψ is a 1-form that annihilates $\mathfrak{h}^{\mathrm{R}}$. Also recall Proposition 2.5.5, which tells us that

$\mathcal{H}$ decomposes into three terms: the Laplacian, a linear vector field term, and a scalar term. The proposition also tells us that the linear part is the projection of the vector field $C^{L}(2\eta-\phi)+L^{L}$ down to M (Recall that ϕ is the divergence cocycle described in Section 2.5). We are now ready to restate the closure conditions.

Proposition 5.1.1 *Let ψ be the $\mathfrak{h}^{R}$ annihilating 1-form such that $C^{L}\psi$ is the horizontal part of $C^{L}(2\eta-\phi)+L^{L}$. The operator $\mathcal{H}$ satisfies the closure conditions if and only if ψ is closed.*

Proof: If ψ is closed then there exists a local function, $f \in C^{\infty}(\mathrm{G})$ such that $2\,df = \psi$. Since ψ annihilates vertical vector fields, f must actually be a function of M. Hence,

$$C(2\eta-\phi)^{\pi}+L^{\pi}=2\,\mathrm{grad}(f). \tag{5.1}$$

It follows that a change of scale by $\exp(-f)$ will change $\mathcal{H}$ into a Schröedinger operator.

Conversely, if $\mathcal{H}$ satisfies the closure conditions, then there exists an $f \in C^{\infty}(\mathrm{M})$ such that (5.1) holds. But that means that $\psi = 2\,df$. □

A closer look at equation (5.1) reveals that there are essentially two components to the closure conditions. For homogeneous operator systems the closure conditions reduce to the following question: for which cocycles, η, is the expression $C(2\eta-\phi)$ a gradient of some function of M? We will call this the *homogeneous closure conditions.* On the other hand if we take the cocycle component, η to be equal to one-half times the divergence cocycle, ϕ, then the closure conditions reduce to the following criterion: for which $C \in \mathcal{S}^2\mathfrak{g}$ and $L \in \mathfrak{g}$ does there exists an $f \in C^{\infty}(\mathrm{M})$ such that L^{π} is equal to grad(f)? If this criterion is satisfied we will say that C and L are *compatible.*

It isn't difficult to find all compatible $L \in \mathfrak{g}$ for a single, fixed $C \in \mathcal{S}^2\mathfrak{g}$. One merely has to check which 1-forms $(C^{\pi})^{-1}(L)$ are closed. This turns out to be a straight-forward, linear condition on L. In general, a fixed C will not admit any compatible $L \in \mathfrak{g}$. A fundamental problem is therefore to determine those C that do admit compatible linear terms.

Let us now reformulate the closure conditions in terms of the adapted frame (see Section 4.2). We will look at the homogeneous closure conditions and at the compatibility criterion separately.

Proposition 5.1.2 *Let $(C, 0, \eta)$ be a homogeneous operator system with corresponding Lie algebraic operator $\mathcal{H}$. Let $\rho \in \mathfrak{h}^*$ be the classifying form of $2\eta - \phi$ (see Section 3.1). Then, $\mathcal{H}$ satisfies the closure conditions if and only if for horizontal indices, $i, j = 1, \ldots, n-m$ we have*

$$\tilde{B}^{ijk}\rho_k = 0,$$

where $\tilde{B}^{ijk}$ are the vertical structure coefficients of the horizontal frame (see Section 4.2).

Proof: According to Proposition 5.1.1 the homogeneous closure conditions are satisfied if and only if the horizontal component of $2\eta - \phi$ is closed. Since both η and ϕ are closed this is equivalent to the requirement that the vertical component, let us call it ψ, of $2\eta - \phi$ is closed. By Theorem 3.1.1 the vertical component is given by

$$\psi(H^i) = 0, \quad \psi(V_i) = \rho_i,$$

where $H^1, \ldots, H^{n-m}, V_{n-m+1}, \ldots, V_n$ is the adapted frame. The next step is to evaluate $\delta\psi$ with three types of vector combinations: vertical–vertical, vertical–horizontal, horizontal–horizontal. In what follows we are relying on the coboundary formula for 1-cochains:

$$\delta\psi(a, b) = a\psi(b) - b\psi(a) - \psi([a, b]);$$

and on the fact that ψ evaluated on any vertical vector field is a constant. Since a classifying form is a cocycle of $\mathrm{H}^1(\mathfrak{h}; 1)$, it annihilates all commutators of $\mathfrak{h}$, and hence

$$\delta\psi(V_i, V_j) = -\rho([V_i, V_j]) = 0.$$

Since the bracket of a horizontal with a vertical vector field is a horizontal vector field we have

$$\delta\psi(H^i, V_j) = -\rho([H^i, V_j]) = 0.$$

Thus, we get no conditions on ψ from these first two types of combinations. The horizontal-horizontal combination, however, yields

$$\delta\psi(H^i, H^j) = -\psi(\tilde{A}^{ij}_k H^k + \tilde{B}^{ijk} V_k) = -\tilde{B}^{ijk}\rho_k.$$

Therefore, all such expressions must be zero in order for ψ to be closed. □

The next proposition concerns the compatibility criterion. The results are formulated in terms of tensor fields of mixed type. The relevant concepts and notation are described in Section 4.3.

Proposition 5.1.3 *The compatibility of $C \in S^2\mathfrak{g}$ and $L \in \mathfrak{g}$ is expressed by the following equations:*

$$d\tilde{\alpha}^i(C\tilde{\alpha}^j, L) - d\tilde{\alpha}^j(C\tilde{\alpha}^i, L) - 2A^{ij}_k\tilde{\alpha}^k(L) = 0,$$

where i, j, k range from 1 to $\dim(\mathfrak{h}^\perp)$, i.e. they are indices of horizontal vector fields in the adapted frame.

Proof: Let ψ be the $\mathfrak{h}^R$ annihilating 1-form such that $C^L\psi$ is the horizontal component of L^L. In other words

$$\begin{aligned} \psi(H^i) &= (\alpha^i)^R(L^L) = \tilde{\alpha}^i(L), \\ \psi(V_j) &= 0. \end{aligned}$$

By Proposition 5.1.1, C and L are compatible if and only if $\delta\psi = 0$. As in the preceding proposition we must evaluate $\delta\psi$ on the three types of horizontal–vertical vector combinations. It's not hard to see that $\delta\psi$ is zero when both of its arguments are vertical vector fields.

If one vector field is horizontal, say $C^{\mathrm{L}}\alpha^{\mathrm{R}}$, where $\alpha \in \mathfrak{h}^{\perp}$, and the other is vertical, say a^{R}, where $a \in \mathfrak{h}$, we get

$$\delta\psi(a^{\mathrm{R}}, C^{\mathrm{L}}\alpha^{\mathrm{R}}) = a^{\mathrm{R}}(\alpha^{\mathrm{R}}(L^{\mathrm{L}})) - \psi([a^{\mathrm{R}}, C^{\mathrm{L}}\alpha^{\mathrm{R}}]).$$

Since $\mathcal{L}_{a^{\mathrm{R}}}(\alpha^{\mathrm{R}}) = (\mathrm{ad}(a)^{*}(\alpha))^{\mathrm{R}}$ both of the right hand side terms of the above equations are equal to

$$(\mathrm{ad}(a)^{*}(\alpha))^{\mathrm{R}}(L^{\mathrm{L}}),$$

and therefore $\delta\psi(a^{\mathrm{R}}, C^{\mathrm{L}}\alpha^{\mathrm{R}}) = 0$.

Finally, let us consider the case of two horizontal arguments. Using the notation and results of Section 4.3, and the structure coefficients introduced in Section 4.2 we get

$$\begin{aligned}\delta\psi(H^{i}, H^{j}) &= H^{i}(\tilde{\alpha}^{j}(L)) - H^{j}(\tilde{\alpha}^{i}(L)) - \psi([H^{i}, H^{j}]) \\ &= \tilde{\alpha}^{j}(\mathrm{ad}(C\tilde{\alpha}^{i}, L)) - \tilde{\alpha}^{i}(\mathrm{ad}(C\tilde{\alpha}^{j}, L)) - 2\tilde{A}^{ij}_{k}\tilde{\alpha}^{k}(L)\end{aligned}$$

The last line is equivalent to the desired formula. □

5.2 Simplified Closure Conditions

Formula (5.1) suggests a number of ways to simplify the closure conditions. The idea of working with simplified closure conditions was introduced in [13]. Some similar ideas are also mentioned in [28]. The most basic approach is to take η to be $\phi/2$, and $L = 0$. Then, the closure conditions are automatically satisfied regardless of the choice of $C \in S^2\mathfrak{g}$. Actually, there is no need to have η exactly equal to $\phi/2$.

Proposition 5.2.1 *In order for an operator, $\mathcal{H}$, to satisfy the closure conditions it suffices for $\mathcal{H}$ to be engendered by a homogeneous operator system, $(L, 0, \eta)$, where η has the same cohomology class as $\phi/2$.*

Proof: Recalling the discussion in Section 2.3 we see that if these two cocycles have the same cohomology class in $\mathrm{H}^1(\mathfrak{g}; \mathcal{C}^\infty(\mathrm{M}))$, then their difference is df, for some $f \in \mathcal{C}^\infty(\mathrm{M})$, and hence

$$C(2\eta - \phi)^\pi = \mathrm{grad}(f).$$

This means (see formula (5.1)) that the closure conditions are satisfied if we take $L = 0$. □

Next, let us consider basic operators systems $(C, L, 0)$. The zero cocycle component means that the corresponding Lie-algebraic operator, $\mathcal{H}$, is generated by pure vector fields.

Definition 5.2.2 We say that the action of G on M is unimodular if there exists a volume form, ω, on M which is invariant with respect to the G-actions.

Proposition 5.2.3 *The* G*-action is unimodular if and only if* χ*, the character of the representation of* $\mathfrak{h}$ *on* $\mathfrak{g}/\mathfrak{h}$ *(see Section 3.2), is zero.*

Proof: Suppose that $\chi = 0$. Let $\alpha^1, \ldots, \alpha^m$ be a basis of $\mathfrak{h}^\perp$, and set

$$\omega = (\alpha^1)^{\mathrm{R}} \wedge \ldots \wedge (\alpha^m)^{\mathrm{R}}.$$

An easy calculation shows that the Lie derivative of ω with respect to a^{R}, where $a \in \mathfrak{h}$, is zero. Hence, ω is a pullback of a volume form on M. But ω is right-invariant and hence invariant under the G action on M.

Conversely, suppose that ω is a G-invariant volume form on M. Consider the divergence cocycle, ϕ, with respect to ω. This cocycle is given by

$$\mathcal{L}_{a^{\mathrm{L}}}(\omega) = \phi(a)\omega, \qquad \text{where } a \in \mathfrak{g},$$

and so must be zero by the assumption of invariance. By a slightly modified version of Proposition 3.2.5 we know that $-\chi$ is the classifying form of ϕ, and therefore χ must be zero. □

Proposition 5.2.4 *The following three conditions suffice for the action of* $\mathbf{G}$ *to be unimodular:*

- $\mathfrak{h}$ *is semi-simple,*
- $\mathfrak{g}$ *is compact,*
- *both* $\mathfrak{g}$ *and* $\mathfrak{h}$ *are reductive.* [1]

Proof: A semi-simple Lie algebra can only be represented by trace-free matrices. Hence, if $\mathfrak{h}$ is semi-simple, χ must be zero.

If $\mathfrak{g}$ is compact, then there exists an Ad-invariant, positive–definite inner-product on $\mathfrak{g}$. In particular, this means that the representation of $\mathfrak{h}$ on $\mathfrak{g}/\mathfrak{h}$ can be given by skew-symmetric, and hence trace-free, matrices.

The adjoint character of a reductive Lie algebra must be zero. Thus, if both $\mathfrak{g}$ and $\mathfrak{h}$ are reductive we have

$$\chi_{\mathfrak{g}/\mathfrak{h}}(a) = \chi_{\mathfrak{g}}(a) - \chi_{\mathfrak{h}}(a) = 0, \qquad \text{where } a \in \mathfrak{h}.$$

□

The preceding two propositions combine to give the following.

Proposition 5.2.5 *If one of the three conditions listed in Proposition 5.2.4 is satisfied, then every Lie algebraic operator engendered by a basic, homogeneous operator system will satisfy the closure conditions.*

We now turn to another method to simplify the closure conditions. This technique is useful whenever a cohomology class of $\mathrm{H}^1(\mathfrak{g}; C^\infty(\mathrm{M}))$ has a representative cocycle with constant coefficients. Suppose that $\rho \in \mathrm{Z}^1(\mathfrak{g}; 1)$ is such a cocycle. Then $C\rho$ will actually be an element of $\mathfrak{g}$.

[1] Recall that a reductive Lie algebra is a direct sum of simple and abelian components. As such, the character of the adjoint representation of a reductive Lie algebra is always zero. See [16] and [6] for the background material used in this proposition.

Proposition 5.2.6 *A Lie algebraic operator, $\mathcal{H}$, engendered by an operator system (C, L, η) will automatically satisfy the closure conditions if $2\eta - \phi$ has the same cohomology class as a cocycle with constant coefficients, ρ, and if $L = -C\rho$.*

Proof: If the premise of the proposition holds, then

$$2\eta - \phi = \rho + df,$$

for some $f \in \mathcal{C}^\infty(\mathrm{M})$. By Proposition 2.5.5 we have

$$\begin{aligned}\mathcal{H} &= \Delta + C(2\eta - \phi)^\pi + L^\pi + \text{scalar} \\ &= \Delta + \text{grad}(f) + \text{scalar},\end{aligned}$$

and hence the closure conditions are satisfied. □

Let us illustrate the above technique with an example. We use the following planar realization of $\mathfrak{gl}(2) \ltimes \mathbb{R}^{r+1}$:

$$x\partial_x,\ y\partial_y,\ x^2\partial_x + rxy\partial_y,\ \partial_x,\ \partial_y,\ x\partial_y,\ x^2\partial_y, \ldots,\ x^r\partial_y,$$

where r is some natural number. Let us call these vector fields, $a_1, \ldots, a_{r+5}$, according to the order in the above sequence. Taking the basepoint to be $(0, 0)$, the isotropy algebra, $\mathfrak{h}$, is spanned by a_1, a_2, a_3, $a_6, \ldots$, a_{r+5}. The commutator ideal, $[\mathfrak{h}, \mathfrak{h}]$, is spanned by a_3, $a_6, \ldots$, a_{r+5}. Hence, by Theorem 3.1.1, $\mathrm{H}^1(\mathfrak{g}; \mathcal{C}^\infty(\mathrm{M}))$ has dimension 2. Representative cocycles can be obtained by using the techniques of Section 3.2. We will use

$$\eta = c_1\alpha^1 + c_2\alpha^2,$$

where α^i, with $i = 1..r+5$, is the dual basis of $\mathfrak{g}^*$, and c_1, c_2 are arbitrary constants. By Proposition 3.2.5 the divergence cocycle is represented by $\alpha^1 + \alpha^2$.

This example is so convenient precisely because all cohomology classes can be represented by cocycles with constant coefficients. For instance, Proposition 5.2.6 tells us that for any $C \in S^2\mathfrak{g}$, and η the closure conditions will be satisfied if we take

$$L = C((1 - 2c_1)\alpha^1 + (1 - 2c_2)\alpha^2),$$

where c_1, c_2 are the cohomology parameters that determine the class of η.

5.3 The Representation Theory Perspective

Fix an $\alpha \in \mathfrak{g}^*$ and consider the equation

$$C^{\mathrm{L}}(\alpha^{\mathrm{R}}, \alpha^{\mathrm{R}}) = 0, \tag{5.2}$$

where $C \in \mathcal{S}^2\mathfrak{g}$ is the variable. In Section 4.3 we showed that the above is equivalent to the equation

$$C(\tilde{\alpha}, \tilde{\alpha}) = 0,$$

where $\tilde{\alpha}_g = \mathrm{Ad}^*_g(\alpha)$, and $g \in \mathbf{G}$. In other words, this equation demands that C annihilate the entire $\mathbf{G}$-invariant subspace of $\mathcal{S}^2\mathfrak{g}^*$ that is generated by $\alpha \otimes \alpha$. Solving this equation is a feasible undertaking if $\mathbf{G}$ is semi-simple, because of the well developed representation theory for semi-simple Lie groups and algebras. Indeed, if $\mathbf{G}$ were semi-simple then $\mathcal{S}^2\mathfrak{g}^*$ would be the direct sum of certain irreducible submodules, and the entire task would reduce to computing this decomposition, as well as determining which components of this decomposition are generated by the single element $\alpha \otimes \alpha$. We can summarize by saying that solutions to (5.2) are given by a certain set of *invariant equations*. These equations are just the submodule of $\mathcal{S}^2\mathfrak{g}^*$ generated by $\alpha \otimes \alpha$.

The above simplistic example is meant to be a guiding analogy for the fundamental nature of the closure conditions. In Section 2.4 we established that the closure conditions are invariant under the action of the underlying group. To put it another way, if $C \in \mathcal{S}^2\mathfrak{g}$ and $L \in \mathfrak{g}$ are compatible, then every element of the $\mathbf{G}$-orbit that is generated by (C, L) gives another compatible pair. Unlike the simplistic example above, however, the solution orbits cannot be specified by a linear criterion. Actually, as was shown in Section 5.1, the closure conditions are linear in the L variable. As we are about to show, however, the invariant equations for the closure conditions are polynomial in the C variable, and this is the major source of difficulty in obtaining general solutions.

From now on we will specialize to the case where $\mathbf{M}$ is 2-dimensional. This way the formulas will simpler, but the essential features of the approach can still be illustrated. In the 2-dimensional case there are only 2 horizontal vector fields: H^1, and H^2. From equation (4.2) we get

$$\alpha^{in}([H^1, H^2]) = 2\tilde{T}^{12i} - \tilde{T}^{2i1} - \tilde{T}^{i12}, \quad i = 1 \dots n$$

The structure coefficients are therefore determined by the following equations

$$\begin{pmatrix} \tilde{C}^{11} & \tilde{C}^{12} \\ \tilde{C}^{12} & \tilde{C}^{22} \end{pmatrix} \begin{pmatrix} \tilde{A}_1^{12} \\ \tilde{A}_2^{12} \end{pmatrix} = \begin{pmatrix} \tilde{T}^{121} \\ \tilde{T}^{122} \end{pmatrix}, \tag{5.3}$$

$$\tilde{T}^{12i} - \tilde{T}^{2i1} - \tilde{T}^{i12} = 2\tilde{A}_1^{12}\tilde{C}^{1i} + 2\tilde{A}_2^{12}\tilde{C}^{2i} + \tilde{B}^{12i} \tag{5.4}$$

Recall from Section 4.3 that $\tilde{C}^{ij}$ is linear in the C variable, and that $\tilde{T}^{ijk}$ is quadratic. Hence $\tilde{A}_i^{12}$ is a ratio of polynomials in the C variable; the numerator has degree 3, and the denominator degree 2. Hence $\tilde{B}^{12i}$ is also a rational expression in C whose numerator has degree 4 and whose denominator has degree 2. By proposition 5.1.2 the invariant equations for the homogeneous closure conditions are homogeneous fourth degree polynomials in C. By proposition 5.1.3 the invariant equations for the compatibility of C and L are linear in L and third degree, homogeneous polynomials in C.

Next, let us compute the invariant equations for an uncomplicated, 2-dimensional example. We will use the linear representation of $\mathfrak{sl}(2)$:

$$x\partial_y, \quad x\partial_x - y\partial_y, \quad y\partial_x.$$

As per the usual we will use the above sequence as the basis a_1, a_2, a_3 of $\mathfrak{g} = \mathfrak{sl}(2)$, and take α^1, α^2, α^3 as the dual basis. With basepoint $x = 1$, $y = 0$, the isotropy subalgebra, $\mathfrak{h}$, is spanned by a_3. By theorem 3.1.1 the dimension of $\mathrm{H}^1(\mathfrak{g}; C^\infty(\mathrm{M}))$ is 1, and the cohomology classes can be represented by

$$\eta_1 = 0, \quad \eta_2 = 0, \quad \eta_3 = \frac{P}{x^2},$$

where P is the cohomology class parameter. The classifying form of η, as given above, is $\rho = P\alpha^3$. It will also be convenient to write out the structure equations for $\mathfrak{sl}(2)$:

$$d\alpha^1 = 2\alpha^{12}, \quad d\alpha^2 = -\alpha^{13}, \quad d\alpha^3 = 2\alpha^{23},$$

where we use α^{ij} as an abbreviation for $\alpha^i \wedge \alpha^j$. Let us also introduce the abbreviation

$$\alpha^{ij} \cdot \alpha^{kl} = C^{\wedge 2}(\alpha^{ij}, \alpha^{kl}) = C^{ik}C^{jl} - C^{il}C^{jk}, \tag{5.5}$$

From the structure equations of $\mathfrak{sl}(2)$, and from equations (5.3) (5.4) it follows that

$$\begin{aligned}
\tilde{T}^{121} &= -2\tilde{\alpha}^{12} \cdot \tilde{\alpha}^{12} \\
\tilde{T}^{122} &= \tilde{\alpha}^{13} \cdot \tilde{\alpha}^{12} \\
\tilde{A}^{12}_1 &= \frac{\tilde{C}^{22}(-2\tilde{\alpha}^{12} \cdot \tilde{\alpha}^{12}) - \tilde{C}^{12}(\tilde{\alpha}^{13} \cdot \tilde{\alpha}^{12})}{\tilde{\alpha}^{12} \cdot \tilde{\alpha}^{12}} \\
\tilde{A}^{12}_2 &= \frac{-\tilde{C}^{12}(-2\tilde{\alpha}^{12} \cdot \tilde{\alpha}^{12}) + \tilde{C}^{11}(\tilde{\alpha}^{13} \cdot \tilde{\alpha}^{12})}{\tilde{\alpha}^{12} \cdot \tilde{\alpha}^{12}} \\
\tilde{B}^{123} &= -2\tilde{\alpha}^{23} \cdot \tilde{\alpha}^{12} + 2\tilde{\alpha}^{12} \cdot \tilde{\alpha}^{23} + \tilde{\alpha}^{13} \cdot \tilde{\alpha}^{13} - 2\tilde{A}^{12}_1\tilde{C}^{13} - 2\tilde{A}^{12}_2\tilde{C}^{23} \\
&= \frac{(\tilde{\alpha}^{12} \cdot \tilde{\alpha}^{12})(\tilde{\alpha}^{13} \cdot \tilde{\alpha}^{13} - 4\tilde{\alpha}^{12} \cdot \tilde{\alpha}^{23}) - 2(\tilde{\alpha}^{12} \cdot \tilde{\alpha}^{13})^2}{\tilde{\alpha}^{12} \cdot \tilde{\alpha}^{12}}
\end{aligned}$$

From the above identities and from Proposition 5.1.2 we see that the invariant equations for the homogeneous closure conditions are generated by

$$\mu_{\text{hom}} = (\alpha^{12} \odot \alpha^{12}) \odot (\alpha^{13} \odot \alpha^{13} - 4\alpha^{12} \odot \alpha^{23}) - 2(\alpha^{12} \odot \alpha^{13}) \odot (\alpha^{12} \odot \alpha^{13}), \tag{5.6}$$

where the circled dot is a symbol for the symmetric tensor product. Ostensibly, the space of fourth-degree polynomials with arguments in $C \in \mathcal{S}^2\mathfrak{g}$ is the tensor space $\mathcal{S}^4(\mathcal{S}^2\mathfrak{g}^*)$. So why does μ_{hom} seem to belong to the tensor space $\mathcal{S}^2\mathcal{S}^2(\Lambda^2\mathfrak{g}^*)$? The explanation is that the latter tensor space be canonically mapped to the space of fourth order polynomials via formula (5.5).

Proposition 5.1.3 gives the formula for the compatibility condition. Evaluating the first two terms in that formula we obtain:

$$d\tilde{\alpha}^1(C\tilde{\alpha}^2, L) = 2\tilde{\alpha}^{12}(C\tilde{\alpha}^2, L)$$
$$d\tilde{\alpha}^2(C\tilde{\alpha}^1, L) = -\tilde{\alpha}^{13}(C\tilde{\alpha}^1, L)$$

The formulas for the $\tilde{A}_k^{ij}$ are given above. Putting all of these formulas together, we see that the invariant equations for compatibility are generated by

$$\mu_{\text{comp}} = (\alpha^{12} \odot \alpha^{12}) \otimes (\alpha^{13} \otimes \alpha^1 - 2\alpha^{12} \otimes \alpha^2) - 2(\alpha^{12} \odot \alpha^{13}) \otimes (\alpha^{12} \otimes \alpha^1). \quad (5.7)$$

The above tensor is an element of

$$\mathcal{S}^2(\Lambda^2\mathfrak{g}^*) \otimes (\Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}^*),$$

but we interpret μ_{comp} as form that is homogeneous of degree 3 in $C \in \mathcal{S}^2\mathfrak{g}$ and is linear in $L \in \mathfrak{g}$, i.e. an element of

$$\mathcal{S}^3(\mathcal{S}^2\mathfrak{g}^*) \otimes \mathfrak{g}^*.$$

This interpretation is accomplished by using the following canonical maps:

$$\begin{array}{ll} \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathcal{S}^2\mathfrak{g}^* \otimes \mathfrak{g}^*, & \alpha^{ij} \otimes \alpha^k \mapsto (\alpha^i \odot \alpha^k) \otimes \alpha^j - (\alpha^j \odot \alpha^k) \otimes \alpha^i; \\ \mathcal{S}^2\Lambda^2\mathfrak{g}^* \to \mathcal{S}^2(\mathcal{S}^2\mathfrak{g}^*), & \text{given by formula (5.5).} \end{array}$$

Before we can derive the full set of invariant equations generated by μ_{hom} and μ_{comp} we need to summarize the representation theory of $\mathfrak{sl}(2)$. A good reference for this subject as and other interesting aspects of Lie representation theory is [6]. The standard presentation of $\mathfrak{sl}(2)$ is given in terms of the raising operator J^+, the lowering operator J^-, the weight operator J^0, and the following relations:

$$[J^+, J^-] = J^0, \quad [J^0, J^+] = 2J^+, \quad [J^0, J^-] = -2J^-.$$

Every finite dimensional $\mathfrak{sl}(2)$ module decomposes as a direct sum of irreducible components. The irreducible modules, $\mathcal{U}_n$, are indexed by natural numbers, and the index, n, is the dimension of the respective module. We will describe $\mathcal{U}_n$ in terms of the following basis and relations

$$u^{(n)}_{-n+1}, \quad u^{(n)}_{-n+3}, \quad u^{(n)}_{-n+5}, \quad \ldots, u^{(n)}_{n-3}, \quad u^{(n)}_{n-1},$$

$$J^0 u_k^{(n)} = k u_k^{(n)}, \quad J^+ u_k^{(n)} = \frac{n+k+1}{2} u_{k+2}^{(n)}, \quad J^- u_k^{(n)} = \frac{n-k+1}{2} u_{k-2}^{(n)} \quad (5.8)$$

Working with any $\mathfrak{sl}(2)$ module, we will call the eigenvectors, u, of J^0 homogeneous elements; the weight, $w(u)$, of these elements is their J^0 eigenvalue. A lowest (respectively highest) weight element is one that is annihilated by J^- (respectively J^+). A lowest weight element, u, generates an irreducible submodule of dimension equal to $-w(u)+1$. Having fixed a lowest weight element, u, we will call the sequence of elements,

$$J^{+(k)}(u)/k!, \quad k = 0, \ldots, w(u)+1,$$

the adapted basis of that submodule. This basis will then obey the standard relations given in (5.8).

Speaking in terms of the $\mathfrak{sl}(2)$ basis described at the beginning of this section, let us take a_1 as the raising operator, a_3 as the lowering operator, a_2 as the weight operator, and consider the decomposition of some tensor spaces constructed from the adjoint representation of $\mathfrak{g} = \mathfrak{sl}(2)$. Recall that the action on $\mathfrak{g}^*$ is given by the negative transpose of the adjoint actions. Thus, taking α^1 as the lowest weight element of $\mathfrak{g}^*$ we obtain the following adapted basis:

$$\alpha^1, \quad 2\alpha^2, \quad -\alpha^3.$$

Turning to $\mathcal{S}^2\Lambda^2\mathfrak{g}^*$, a good choice for the lowest weight element is $\alpha^{12} \odot \alpha^{12}$. Let us label the resulting adapted basis as

$$\begin{aligned}
u_{-4} &= \alpha^{12} \odot \alpha^{12}, \\
u_{-2} &= -2\alpha^{12} \odot \alpha^{13}, \\
u_0 &= \alpha^{13} \odot \alpha^{13} - 2\alpha^{12} \odot \alpha^{23}, \\
u_2 &= 2\alpha^{13} \odot \alpha^{23}, \\
u_4 &= \alpha^{23} \odot \alpha^{23}.
\end{aligned}$$

Since $\mathcal{S}^2\Lambda^2\mathfrak{g}^*$ has dimension 6, and the above submodule has dimension 5, there must also be a 1-dimensional submodule. It is generated by the following invariant tensor:

$$u_0' = \alpha^{13} \odot \alpha^{13} + 4\alpha^{12} \odot \alpha^{23}.$$

We can now write μ_{hom} quite simply as

$$u_{-4}\tilde{u}_0 - 1/2u_{-2}u_{-2},$$

where we omit the $\odot$ for the sake of brevity, and where

$$\tilde{u}_0 = \frac{4}{3}u_0 - \frac{1}{3}u_0' = \alpha^{13} \odot \alpha^{13} - 4\alpha^{12} \odot \alpha^{23}.$$

Presented in this form, it is not hard to verify that $J^-(\mu_{\text{hom}}) = 0$, i.e. μ_{hom} is an element of lowest weight. Since μ_{hom} has weight -4, it generates a 5-dimensional module. The basis of this module constitutes the invariant equations for the homogeneous closure conditions; this basis is given below:

$$\begin{gathered} u_{-4}\tilde{u}_0 - \tfrac{1}{2}u_{-2}u_{-2}, \\ u_{-2}\tilde{u}_0 + 4u_{-4}u_2 - 2u_{-2}v_0, \\ u_0\tilde{u}_0 + u_{-2}u_2 + 8u_{-4}u_4 - 2u_0u_0, \\ u_2\tilde{u}_0 - 2u_0u_2 + 4u_{-2}u_4 \\ u_4\tilde{u}_0 - \tfrac{1}{2}u_2u_2. \end{gathered} \tag{5.9}$$

Now, let us consider μ_{comp}. Some of the factors come from $S^2\Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}^*$. The most relevant one is $\alpha^{12} \otimes \alpha^1$. This is a weight -4 element of lowest weight, and hence generates a 5 dimensional submodule. We name the adapted basis as follows:

$$\begin{aligned} v_{-4} &= \alpha^{12} \otimes \alpha^1, \\ v_{-2} &= -\alpha^{13} \otimes \alpha^1 + 2\alpha^{12} \otimes \alpha^2, \\ v_0 &= -\alpha^{23} \otimes \alpha^1 - 2\alpha^{13} \otimes \alpha^2 - \alpha^{12} \otimes \alpha^3, \\ v_2 &= \alpha^{13} \otimes \alpha^3 - 2\alpha^{23} \otimes \alpha^2, \\ v_4 &= \alpha^{23} \otimes \alpha^3. \end{aligned}$$

We can therefore give μ_{comp} simply as

$$u_{-2}v_{-4} - u_{-4}v_{-2},$$

where again we can omit the ⊗ sign without fear of ambiguity. We can now see that μ_{comp} is an element of lowest weight, and hence generates the following 7-dimensional module. These are the invariant equations for the compatibility conditions.

$$\begin{gathered}
-u_{-4}v_{-2} + u_{-2}v_{-4}, \\
-2u_{-4}v_0 + 2u_0v_{-4}, \\
-3u_{-4}v_2 - u_{-2}v_0 + u_0v_{-2} + 3u_2v_{-4}, \\
-4u_{-4}v_4 - 2u_{-2}v_2 + 2u_2v_{-2} + 4u_4v_{-4}, \\
-3u_{-2}v_4 - u_0v_2 + u_2v_0 + 3u_4v_{-2}, \\
-2u_0v_4 + 2u_4v_0, \\
-u_2v_4 + u_4v_2
\end{gathered} \tag{5.10}$$

The above invariant equations for μ_{hom} and μ_{comp} tell us something important about solutions to the general closure conditions, i.e. Lie algebraic operators such that $2\eta - \phi$ is non-trivial, and such that $L \neq 0$. The invariant equations for the general closure conditions are generated by $\mu_{\text{hom}}+\mu_{\text{comp}}$. But now we know that the two terms in questions generate non-isomorphic irreducible modules, and therefore the module generated by $\mu_{\text{hom}} + \mu_{\text{comp}}$ is just the direct sum of the 5-dimensional module generated by μ_{hom} and the 7-dimensional module generated by μ_{comp}. In other words, the general closure conditions are satisfied if and only if both the homogeneous closure condition and the compatibility condition are satisfied.

5.4 Using the Group Action to Solve the Closure Conditions.

The invariant equations derived in the preceding section tell us that a Lie algebraic operator satisfies the homogeneous closure conditions if it is engendered by an operator system whose second-order component, $C \in S^2\mathfrak{g}$, is the simultaneous zero of 5 fourth degree polynomials. Likewise, now we know that $C \in \mathcal{S}^2\mathfrak{g}$ and $L \in \mathfrak{g}$ are compatible if (C, L) is the simultaneous zero of 7 polynomials of degree $(3, 1)$. Unfortunately, this knowledge only serves to reformulate the problem, and does not provide an effective computational tool. [2] Indeed, we could also obtain polynomial equations for the $\mathfrak{sl}(2)$ closure conditions in terms of local coordinates.

A truly useful bit of knowledge is the fact that the closure conditions are invariant under the action of the underlying group (see Section 2.4). To exploit this fact we need to analyze the orbit structure of $\mathcal{S}^2\mathfrak{g}$, to find canonical representatives for each orbit, and to test the closure conditions on these representatives. The idea of using an invariant group action to simplify a given problem is a veritable mathematical leitmotif. In the context of Lie-algebraic operator research this idea has been mentioned in [12] and in [27].

In the present section we will compute the orbit structure of $\mathcal{S}^2\mathfrak{g}$ for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ and use this information to obtain the solutions to the closure conditions for the homogeneous space described in the last section. Fortunately, it is not difficult to describe the orbit structure of the irreducible modules of $\mathfrak{sl}(2)$, once we realize that $\mathcal{U}_n$ is isomorphic to the module of homogeneous, degree $(n-1)$ polynomials in two variables, say x and y. The group action for the polynomial modules is given by the change of variables represented by each matrix in SL(2). Such polynomials factor into a number of linear and quadratic components, and this factorization is stable under the SL(2) action. Therefore, the multiplicities of the irreducible factors and the number of irreducible quadratics are two fundamental invariants of the group action.

We are particularly interested in $\mathcal{U}_5$, the module of fourth degree polynomials. Such a polynomial is specified by 5 parameters, and so the group action should give us the freedom to eliminate 3 of them. Furthermore, if we are willing to consider polynomials up to linear scaling, we should be able to cut down to just 1 parameter. This is just an upper bound on the number of required parameters; indeed there

[2] This is not quite true. See the last paragraph of Section 5.3

are certain types of degenerate orbit types that will require no parameters at all. The following basic propositions will allow us to choose canonical representatives for the various orbits.

Proposition 5.4.1 SL(2) *is doubly transitive on the space of linear factors; in other words, given two distinct linear factors there exists a group action that takes one factor to a multiple of* x*, and the other to a multiple of* y*. Furthermore, given three distinct linear factors, there exists a group action that takes two of the factors to multiples of* x *and* y*, respectively, and takes the third factor to a multiple of* $x+y$*.*

Proposition 5.4.2 *Given two distinct irreducible quadratics, there exists an* $\mathrm{SL}(2,\mathbb{R})$ *action that takes the first one to a multiple of* x^2+y^2*, and takes the second one to a multiple of* $x^2+(ky)^2$*, where* k *is an invariant of the pair.*

Proposition 5.4.3 *Given an irreducible quadratic and a linear factor, there exists an* $\mathrm{SL}(2,\mathbb{R})$ *action that takes the quadratic to a multiple of* x^2+y^2*, and takes the linear factor to a multiple of* x*.*

The following is a list of all possible root multiplicities, for degree 4 polynomials as well as a canonical representative for each such possibility (see the book by Gurevich [15] for a systematic treatment). The parameter, R, where it appears is an invariant that serves to parametrize the given orbits. We also list the corresponding elements of $\mathcal{S}^2\mathfrak{g}$ by using the following identifications:

$$x^4 \cong 6a_3^2, \quad 4x^3y \cong 12a_2a_3, \quad 6x^2y^2 \cong 6a_2^2 - 12a_1a_3, \quad 4xy^3 \cong -12a_1a_2, \quad y^4 \cong 6a_1^2.$$

Since $\mathcal{S}^2\mathfrak{g}$ is the direct sum of $\mathcal{U}_5$ and $\mathcal{U}_1$ we must also add a multiple of the invariant tensor

$$C_{\mathrm{inv}} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

in order to get representatives for all orbits of $\mathcal{S}^2\mathfrak{g}$. This is the role served by the parameter S. There is also the orbit of the invariant tensor, but we won't include in our list because it generates a degenerate metric.

1. $xy(x^2 + 2Rxy + y^2)$. Four distinct linear factors ($|R| > 1$); three linear factors, one of them with double multiplicity ($|R| = 1$); two distinct linear factors and a quadratic factor ($|R| < 1$).

$$C_{(1)} = \begin{pmatrix} 0 & -3/2 & -2R+2S \\ -3/2 & 2R+S & 3/2 \\ -2R+2S & 3/2 & 0 \end{pmatrix}$$

2. x^2y^2. Two linear factors, both with double multiplicty.

$$C_{(2)} = \begin{pmatrix} 0 & 0 & -1+2S \\ 0 & 1+S & 0 \\ -1+2S & 0 & 0 \end{pmatrix}$$

3. x^3y. Two linear factors, one of them with a triple multiplicity.

$$C_{(3)} = \begin{pmatrix} 0 & 0 & 2S \\ 0 & S & 3/2 \\ 2S & 3/2 & 0 \end{pmatrix}$$

4. x^4. A single linear factor of quadruple multiplicity.

$$C_{(4)} = \begin{pmatrix} 0 & 0 & 2S \\ 0 & S & 0 \\ 2S & 0 & 6 \end{pmatrix}$$

5. $(x^2 + y^2)(x^2 + (Ry)^2)$. Two distinct quadratic factors ($R \neq 0, \pm 1$); a double quadratic factor ($R = \pm 1$); a double linear factor, and a quadratic factor ($R = 0$).

$$C_{(5)} = \begin{pmatrix} 6R^2 & 0 & -R^2-1+2S \\ 0 & R^2+1+S & 0 \\ -R^2-1+2S & 0 & 6 \end{pmatrix}$$

Now let us plug these orbital representatives in the invariant equations and find out which, if any, parameters give solutions to the closure conditions. To keep things reasonably brief we will work out the answer in detail for one of the above representatives, say $C_{(1)}$, and list the results for the others in the appendix. To obtain solutions for the homogeneous closure conditions, we evaluate the invariant equations (5.9) on the above tensor. The relevant calculations for the first equation are shown below:

$$\begin{aligned}
u_{-4} &= C^{11}C^{22} - (C^{12})^2 \\
&= -9/4 \\
\tilde{u}_0 &= C^{11}C^{33} - (C^{13})^2 - 4(C^{12}C^{23} - C^{13}C^{22}) \\
&= 9 + 8(S-R)(2R+S) - 4(S-R)^2 \\
u_{-2} &= -2C^{11}C^{23} + 2C^{13}C^{23} \\
&= -6R + 6S \\
u_{-4}\tilde{u}_0 - \frac{1}{2}u_{-2}u_{-2} &= -27(R^2 - S^2 - \frac{3}{4})
\end{aligned} \tag{5.11}$$

The second equations in (5.9) expands as follows:

$$u_{-2}\tilde{u}_0 + 4u_{-4}u_2 - 2u_{-2}u_0 = -3(S-R)(2S+3-2R)(2S-2R-3) \tag{5.12}$$

Therefore, the only solutions to equations (5.11) and (5.12) are

$$R = 0, S = 1/2, \qquad R = 1, S = -1/2.$$

We won't bother expanding the rest of the equations; the upshot is that both of the above solutions satisfy the other 3 polynomials in (5.11). The corresponding solution for C is given by a multiple of

$$\begin{pmatrix} 0 & -1 & \pm 2 \\ -1 & \mp 1 & 1 \\ \pm 2 & 1 & 0 \end{pmatrix} \tag{5.13}$$

Let us now consider which $L \in \mathfrak{g}$, if any, are compatible with a type $C_{(1)}$ quadratic component. One has to evaluate the 7 expressions given in (5.10) with $C = C_{(1)}$, and then check for which values of the parameters R, S there exists a non-zero solution for L. This is best done with a symbolic calculation package, so we won't bother considering the intermediate computations here. The interesting result is that again, the (5.13) gives the only solutions for C. The solutions for L must be a multiple of $a_1 + a_3$. The most general Lie-algebraic operator corresponding to these solutions is

$$\begin{aligned} \mathcal{H} &= -\{x\partial_y, x\partial_x - y\partial_y\} + 2\left\{x\partial_y, y\partial_x + P/x^2\right\} + \\ &\quad +(x\partial_x - y\partial_y)^2 + \left\{y\partial_x + P/x^2, x\partial_x - y\partial_y\right\} + Q(x\partial_y + y\partial_x + P/x^2) \\ &= \Delta + 2\,\mathrm{grad}(\log(\mu)) - \frac{P(Q-2)}{x^2} \end{aligned}$$

where the Laplacian and the gradient are determined by the following contravariant metric tensor:

$$g^{ij} = \begin{pmatrix} x(x+2y) & -(x+y)^2 - xy \\ -(x+y)^2 - xy & y(2x+y) \end{pmatrix},$$

and where

$$\mu = e^{\frac{-P}{x(x+y)}}(x+y)^{1-\frac{Q}{2}}.$$

The above metric has hyperbolic signature and a constant positive curvature: $K = 4$. After a change of scale by μ^{-1} this corresponds to the Schrödinger operator

$$\Delta + \frac{Q^2}{4} - 1.$$

Chapter 6

Flat Lie Algebraic Spaces

Say what you know, do what you must, come what may.
– Sonja Kovalevskaia

6.1 Turbiner's Conjecture

Certain, rare values of $C \in \mathcal{S}^2\mathfrak{g}$ induce a flat background metric. Turbiner [32] has conjectured that a 2-dimensional Q.E.S. system that (i) satisfies the closure conditions, and (ii) has a flat background metric, must admit separation of variables in some suitable system of coordinates. The work in this section derives from the impetus to resolve this conjecture.

We will begin by considering some examples of flat Lie-algebraic metrics. These examples will illustrate Turbiner's separation phenomenon, and also reveal two interesting properties possessed by such metrics.

The first property is a local one. We make two assumptions: the coefficients of the contravariant metric tensor are non-singular analytic functions, and the curvature vanishes identically. These facts determine the behaviour of the metric at the locus of the metric tensor's degeneracy. One aspect of this behaviour is the fact that the flow of a gradient vector field can never cross the locus of degeneracy; the flow is trapped in the non-degenerate region. This is proved in Corollary 6.4.2.

The second property is global in nature, and requires the additional assumption that the analytic, contravariant metric tensor be defined on a compact manifold. The compactness hypothesis implies that there exists a global analytic map from the Euclidean plane onto the region where the metric is positive definite. The locus of degeneracy is pulled back by this covering to a collection of straight lines, and these lines tile the plane into isometric sectors. This tiling result is proved in Theorem 6.5.12.

At the present we are not aware of a comprehensive proof of Turbiner's conjecture. However, it is our belief that the conjecture is true. The evidence for this assertion is two-fold.

First, the conjecture holds for all examples of flat Lie-algebraic metrics known to us. We will illustrate this point with two examples. In the first example the separation takes place in both flat and polar coordinates, in the second example the separation takes place in a parabolic coordinate system.

Second, we will give the proof of a limited form of the conjecture under the additional hypothesis that the action of the underlying group is imprimitive [1] When the group acts imprimitively the geometry of the Lie algebraic systems has some important properties (see Section 4.8 for a discussion). One consequence of these properties is that a Lie algebraic operator that satisfies the closure conditions will separate in either flat or radial coordinates.

We should also note that Turbiner's conjecture is critically dependent on the assumption that the signature of the underlying flat metric be positive-definite. The relevant counter-example will be presented in an appendix.

[1] The current proof also requires a certain compactness condition.

6.2 Two Flat Examples

Let us begin our discussion of flat systems with two clarifying examples. The first of these examples will illustrate the separability that arises from an imprimitive group action. The second example shows that separation can also occur in a coordinate system that is neither flat nor polar. The second example will also serve to illustrate the relationship between the closure conditions and separability.

Our first example is based on the following two dimensional $\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)$ actions:

$$\partial_x, \quad x\partial_x, \quad x^2\partial_x, \quad \partial_y, \quad y\partial_y, \quad y^2\partial_y.$$

Our Lie algebraic operator will be given by

$$\begin{aligned}\mathcal{H} &= \partial_x^2 + \{2x\partial_x + 2y\partial_y, \partial_y\} + 2K_1x\partial_x + 4K_1y\partial_y + 4K_2\partial_y \\ &= \Delta - 2\partial_y + 2K_1x\partial_x + 4K_1y\partial_y + 4K_2\partial_y \\ &= \Delta + \operatorname{grad}\left(K_1y + (K_2 - 1/2)\log(y - x^2)\right),\end{aligned}$$

where the Laplacian and gradient are taken with respect to the induced metric,

$$\begin{pmatrix} 1 & 2x \\ 2x & 4y \end{pmatrix}. \tag{6.1}$$

The curvature of this metric is zero. Flat coordinates are given by

$$x = \xi, \quad y = \xi^2 + \eta^2. \tag{6.2}$$

The $\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)$ actions admits two invariant foliations:

$$x = \text{const}, \quad \text{and} \quad y = \text{const}.$$

As a consequence, $\mathcal{H}(x)$ is a function of x and $\mathcal{H}(y)$ is a function of y. As predicted by Corollary 4.8.4, the perpendicular distributions,

$$\{\partial_\xi\}, \quad \{\xi\partial_\xi + \eta\partial_\eta\},$$

run in straight lines in the flat (ξ, η) coordinates. These two invariant distributions give rise to two sets of coordinates in which the equation

$$\mathcal{H}\Psi = E\Psi \tag{6.3}$$

can be separated. We will give a general explanation of why invariant foliations induce separation of variables later, in Theorem 6.6.2. For now, let us illustrate the phenomenon with the example at hand. Switching to the flat coordinates we see that the first invariant foliation is given by $\xi = $ const; the leaves are straight lines. As we already noted, this implies that $\mathcal{H}(\xi)$ must be a function of ξ, and thereby forces the linear part of the operator to separate into a sum of a gradient of a ξ-function and a gradient of a η-function:

$$\mathcal{H} = \Delta + \text{grad}\left(K_1\xi^2\right) + \text{grad}\left(K_1\eta^2 + (K_2 - 1/2)\log(\eta^2)\right).$$

We can therefore separate (6.3) into

$$\begin{aligned} (\partial_{\xi\xi} + 2K_1\xi\partial_\xi - E)\Psi_1(\xi) &= \lambda\Psi_1(\xi), \\ (\partial_{\eta\eta} + 2K_1\eta\partial_\eta + \frac{2K_2 - 1}{\eta}\partial_\eta - E)\Psi_2(\eta) &= -\lambda\Psi_2(\eta), \end{aligned}$$

where λ is a constant of separation.

The associated potential of the normalized Schrödinger operator is given by

$$V = -K_1^2\xi^2 - K_2^2\eta^2 - \frac{(K_2 - 1/2)(K_2 - 3/2)}{\eta^2} - K_1 - 2K_1K_2.$$

Here we have an illustration of another interesting phenomenon: a coordinate system that separates (6.3) also separates the normalized equation,

$$(\Delta + V)\Psi = E\Psi. \tag{6.4}$$

The second invariant foliation corresponds to the level lines of the radius function, $r = \sqrt{\xi^2 + \eta^2}$. Again, this means that $\mathcal{H}(r)$ is a function of r, and therefore by switching to polar coordinates,

$$\xi = r\cos(\theta), \quad \eta = r\sin(\theta),$$

the linear portion of $\mathcal{H}$ will separate:

$$\mathcal{H} = \Delta + \operatorname{grad}\left(K_1 r^2 + (2K_2 - 1)\log(r)\right) + \operatorname{grad}\left((2K_2 - 1)\log(\sin(\theta))\right).$$

We can therefore separate (6.3) into

$$\begin{aligned} \left(\partial_{rr} + \partial_r + 2K_1 r\partial_r + \frac{2K_2 - 1}{r}\partial_r - E\right)\Psi_1(r) &= \frac{\lambda}{r^2}\Psi_1(\theta) \\ \left(\partial_{\theta\theta} + (2K_2 - 1)\cot(\theta)\partial_\theta\right)\Psi_2(\theta) &= -\lambda\Psi_2(\theta). \end{aligned}$$

Writing the associated potential in polar coordinates we obtain

$$V = -K_1^2 r^2 - \frac{1}{r^2}\frac{(K_2 - 1/2)(K_2 - 3/2)}{\sin^2(\theta)} - K_1 - 2K_1K_2.$$

Thus the normalized equation (6.4) also separates in polar coordinates into

$$\begin{aligned} \left(\partial_{rr} + \partial_r - K_1^2 r^2 - E - K_1 - 2K_1K_2\right)\Psi_1(r) &= \frac{\lambda}{r^2}\Psi_1(r), \\ \left(\partial_{\theta\theta} - \frac{(K_2 - 1/2)(K_2 - 3/4)}{\sin^2(\theta)}\right)\Psi_2(\theta) &= -\lambda\Psi_2(\theta). \end{aligned}$$

The next example of a Lie algebraic operator will be generated from the following realization of $\mathfrak{sl}(3)$ actions:

$$\partial_x, \quad \partial_y, \quad x\partial_x, \quad x\partial_y, \quad y\partial_x, \quad y\partial_y, \quad x^2\partial_x + xy\partial_y, \quad xy\partial_x + y^2\partial_y.$$

The operator itself is given by:

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}\{x\partial_x, \partial_x\} + \partial_y^2 + K_1\partial_x + K_2\partial_y + K_3 x\partial_x + K_4 y\partial_y + K_5(x\partial_y - xy\partial_x) \\ &= \Delta + \operatorname{grad}\left(K_1\frac{\log(x)}{2} + K_2\frac{y}{2} + K_3\frac{x}{2} + K_4\frac{y^2}{4} + K_5\left(\frac{xy}{2} + \frac{y^3}{6}\right)\right), \end{aligned}$$

where the Laplacian and gradient operators are given with respect the induced metric,

$$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}.$$

This metric has zero curvature, with flat coordinates given by:

$$x = \xi^2/4, \quad y = \eta.$$

The basis of linear terms $L \in \mathfrak{g}$ that are compatible with the given second-order component, $C \in \mathcal{S}^2\mathfrak{g}$, is

$$\partial_x, \quad \partial_y, \quad x\partial_x, \quad y\partial_y, \quad x\partial_y - xy\partial_x - y^2\partial_y.$$

Thus, the given $\mathcal{H}$ is the most general operator with the given second-order component that satisfies the closure conditions. The interesting feature of this operator is that for all choices of parameters $K_1, \ldots, K_5$, equation (6.3) separates, although *the choice of coordinates in which separation takes place depends on the value of the parameters.* This suggests that there is some hidden connection between the closure conditions and separability, and since the $\mathfrak{sl}(3)$ action is not imprimitive, this connection goes deeper than the imprimitivity phenomenon discussed in the preceding example.

When $K_5 = 0$, the equation separates in the flat coordinates, (ξ, η). When $K_5 \neq 0$, equation (6.3) separates in parabolic coordinates:

$$\xi = 2uv, \quad \eta = u^2 - v^2 + \frac{K_3 - K_4}{K_5}.$$

In the parabolic coordinates equation (6.3) becomes

$$\begin{aligned} &\partial_{uu}\Psi + \left(K_5u^5 + (2K_3 - K_4)u^3 + \frac{K_3(K_3 - K_4)}{K_5}u + K_2u + \frac{K_1}{u}\right)\partial_u\Psi \\ &+\partial_{vv}\Psi - \left(K_5v^5 + (2K_3 - K_4)v^3 - \frac{K_3(K_3 - K_4)}{K_5}v - K_2v + \frac{K_1}{v}\right)\partial_v\Psi \\ &\qquad = E(u^2 + v^2)\Psi. \end{aligned} \tag{6.5}$$

It is obvious how to separate the above equation.

The metric tensor in the separation coordinates is

$$\frac{1}{4(u^2 + v^2)}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The above equation makes clear that

$$\mathcal{H} = \Delta + \mathrm{grad}(f(u) + g(v)),$$

where $f(u)$ and $g(v)$ are rational functions given by (6.5). These two observations show that the normalized Schrödinger operator obtained from $\mathcal{H}$ by a change of scale, will separate in the same coordinates as $\mathcal{H}$. This is another indication of a deep connection between the closure conditions and separability.

6.3 Contravariant Metric Tensor and Curvature

The goal of the present section is to derive a criterion for flatness based on the contravariant form of the Riemannian metric tensor. The formula in question is presented in Proposition 6.3.3. This formula is the key to unlocking the striking properties of flat metrics with non-singular contravariant tensors that we will be discussing in the next section.

Again, our starting data is the contravariant form of a Riemannian metric tensor, g^{ij}. One could invert the matrix g^{ij} and then apply the usual formula for sectional curvature, but the result would have a cumbersome form, and therefore a limited usefulness. Instead, we will base our approach on the moving frame of coordinate function gradients. Suppose then that we are given a pseudo-Riemannian, contravariant metric tensor, g^{ij}, and that the coefficients of the metric are analytic functions of local coordinates $x^1, \ldots, x^n$. It will be useful to use the abbreviations

$$H^i = \mathrm{grad}(x^i) = g^{ij}\partial_j.$$

Note that $H^i \cdot H^j = g^{ij}$, i.e. the functions g^{ij} are the metric coefficients of the coordinate gradients' frame. It will also be useful to define the following two types of symbols:

$$\begin{aligned} \alpha^{ijk} &= H^i(g^{jk}) \\ \beta^{ijk} &= \alpha^{ijk} - \alpha^{jik} \end{aligned}$$

These symbol are useful for describing the bracket of coordinate gradients:

$$[H^i, H^j] = \beta^{ijk}\partial_k.$$

They can also be used to describe the coefficients of the corresponding Levi-Civita connection,

$$(\nabla_{H^i} H^j) \cdot H^k,$$

relative to the coordinate gradients frame. We will abbreviate these coefficients as γ^{ijk}.

Proposition 6.3.1 *The connection coefficients, γ^{ijk}, of the coordinate gradients frame are given by*

$$\begin{aligned}\gamma^{ijk} &= \frac{1}{2}\left(\alpha^{ijk} - \alpha^{jik} + \alpha^{kij}\right) \\ &= \frac{1}{2}\left(\beta^{ijk} + \alpha^{kij}\right) \\ &= \frac{1}{2}\left(\alpha^{ijk} - \beta^{jki}\right).\end{aligned}$$

Proof: The standard derivation of the covariant derivative of a Levi-Civita connection gives

$$\begin{aligned}2(\nabla_{H^i}H^j)\cdot H^k &= H^i(H^j\cdot H^k) + H^j(H^i\cdot H^k) - H^k(H^i\cdot H^j) \\ &\quad - [H^i,H^k]\cdot H^j - [H^j,H^k]\cdot H^i + [H^i,H^j]\cdot H^k.\end{aligned}$$

Expanding the right hand side in terms of the α^{ijk} type symbols we obtain

$$2(\nabla_{H^i}H^j)\cdot H^k = \alpha^{ijk} - \alpha^{jik} + \alpha^{kij}.$$

□

We are now ready to derive a contravariant version of the formula for the Riemannian curvature. In what follows, let R denote the Riemannian curvature tensor and R^{ijkl} the tensor's components relative to the coordinate gradients frame, i.e. $R(H^i,H^j)H^k\cdot H^l$.

Proposition 6.3.2 *The curvature tensor is given by*

$$\begin{aligned}R^{ijij} = &\quad \frac{1}{2}\left(H^i(\beta^{jij}) + H^j(\beta^{iji})\right) \\ &+ \frac{1}{4}\left(\mathrm{grad}(g^{ii})\cdot\mathrm{grad}(g^{jj}) - \mathrm{grad}(g^{ij})^{\cdot 2}\right) \\ &+ \frac{3}{4}\,[H^i,H^j]^{\cdot 2}.\end{aligned} \tag{6.6}$$

Proof: By definition, the curvature tensor is determined by

$$R^{ijij} = \nabla_{H^i}\nabla_{H^j}H^i \cdot H^j - \nabla_{H^j}\nabla_{H^i}H^i \cdot H^j - \nabla_{[H^i,H^j]}H^i \cdot H^j.$$

The following identities serve to re-express the three terms in the right hand side of the above expression.

$$\begin{aligned} H^i(\gamma^{jij}) &= \nabla_{H^i}\nabla_{H^j}H^i \cdot H^j + g_{ab}\,\gamma^{jia}\gamma^{ijb} \\ H^j(\gamma^{iij}) &= \nabla_{H^j}\nabla_{H^i}H^i \cdot H^j + g_{ab}\,\gamma^{iia}\gamma^{jjb} \\ \nabla_{[H^i,H^j]}H^i \cdot H^j &= g_{ab}\beta^{ija}\gamma^{bij} \end{aligned}$$

Thanks to these three identities we can reformulate the formula for the curvature tensor as

$$\begin{aligned} R^{ijij} = \quad & H^i(\gamma^{jij}) - H^j(\gamma^{iij}) \\ + \quad & g_{ab}\left(-\gamma^{jia}\gamma^{ijb} + \gamma^{iia}\gamma^{jjb} - \beta^{ija}\gamma^{bij}\right) \end{aligned} \tag{6.7}$$

The curvature tensor is symmetric with respect to a switch of the i and j indices, and consequently we can symmetrize these indices without affecting the right hand side's value. Upon symmetrizing the first two right hand side terms of (6.7) and using the identity

$$\gamma^{jij} - \gamma^{jji} = \beta^{jij},$$

we obtain

$$\begin{aligned} H^i(\gamma^{jij}) - H^j(\gamma^{iij}) &= \frac{1}{2}\,H^i\left(\gamma^{jij} - \gamma^{jji}\right) + \frac{1}{2}\,H^j\left(\gamma^{iji} - \gamma^{iij}\right) \\ &= \frac{1}{2}H^i(\beta^{jij}) + \frac{1}{2}H^j(\beta^{iji}) \end{aligned} \tag{6.8}$$

Next we use the identity $g_{ab}\,\alpha^{aij} = \partial_b g^{ij}$ and the formula in Proposition 6.3.1 to transform the remaining terms in the right hand side of (6.7) as follows.

$$\begin{aligned} -g_{ab}\,\gamma^{jia}\gamma^{ijb} &= -\frac{1}{4}\,g_{ab}\,(\beta^{jia} + \alpha^{aij})(\beta^{ijb} + \alpha^{bij}) \\ &= -\frac{1}{4}\left(g_{ab}\,\beta^{jia}\beta^{ijb} + \partial_b g^{ij}\beta^{ijb} + \partial_a g^{ij}\beta^{jia} + g_{ab}\partial_a g^{ij}\partial_b g^{ij}\right) \\ &= \frac{1}{4}\,g_{ab}\,\beta^{ija}\beta^{ijb} - \frac{1}{4}\,g^{ab}\partial_a g^{ij}\partial_b g^{ij}. \end{aligned} \tag{6.9}$$

The final transformation is justified by the fact that β^{ijb} is skew-symmetric in i and j while g^{ij} is symmetric, and therefore the second and third terms in the next to last line vanish after we symmetrize with respect to i and j. We also have

$$g_{ab}\,\gamma^{jja}\gamma^{iib} = \frac{1}{4}\,g^{ab}\partial_a g^{ii}\partial_b g^{jj}. \tag{6.10}$$

Also, by the formula in Proposition 6.3.1 we have

$$\begin{aligned} -g_{ab}\,\beta^{ija}\gamma^{bij} &= -\frac{1}{2}\,g_{ab}\,\beta^{ija}\left(\alpha^{bij}+\beta^{ijb}\right) \\ &= -\frac{1}{2}\,\partial_a g^{ij}\beta^{ija} + \frac{1}{2}\,g_{ab}\,\beta^{ija}\beta^{ijb} \\ &= \frac{1}{2}\,g_{ab}\,\beta^{ija}\beta^{ijb}. \end{aligned} \tag{6.11}$$

As above, the first term in the next to last line vanished because we symmetrized in i and j. Putting (6.8) (6.9) (6.10) (6.11) together we obtain

$$R^{ijij} = \frac{1}{2}\left(H^i(\beta^{jij}) + H^j(\beta^{iji})\right) + \frac{1}{4}\,g^{ab}\left(\partial_a g^{ii}\partial_b g^{jj} - \partial_a g^{ij}\partial_b g^{ij}\right) + \frac{3}{4}\,g_{ab}\,\beta^{ija}\beta^{ijb},$$

and this is equivalent to the formula given in the body of the current proposition. □

Our next step is to specialize the above curvature formula to the two dimensional case, and to obtain a certain criterion for metric's flatness. In what follows it will be convenient to denote the two coordinate variables as $x^1 = x$, $x^2 = y$, and to give the contravariant metric tensor, g^{ij} as

$$\begin{pmatrix} p & q \\ q & r \end{pmatrix}.$$

We also introduce the following abbreviations. For the determinant of the contravariant metric tensor we write

$$|g^{ij}| = pr - q^2.$$

Setting

$$\Omega = dx \wedge dy,$$

we can express the metric's volume form as

$$\omega = |g^{ij}|^{-\frac{1}{2}}\Omega.$$

Given a vector field $V = a\partial_x + b\partial_y$, we will express the divergence of V relative to Ω as

$$\mathrm{Div}(V) = a_x + b_y.$$

The divergence relative to the metric g and to ω is given by

$$\mathrm{div}(V) = \mathrm{Div}(V) - \frac{V(|g^{ij}|)}{2|g^{ij}|}.$$

Given any function, f, we will write

$$f_x = \partial_x f, \quad f_y = \partial_y f,$$
$$f_1 = H^1(f), \quad f_2 = H^2(f).$$

We also put

$$\begin{aligned} \beta^1 &= \beta^{121} = q_1 - p_2, \\ \beta^2 &= \beta^{122} = r_1 - q_2. \end{aligned} \tag{6.12}$$

The bracket of the coordinate gradients can therefore be given as

$$[H^1, H^2] = \beta^1\partial_x + \beta^2\partial_y.$$

Before proceeding further we also need to gather in one place some basic identities that will be required later on. The first identity is the formula for the area of a parallelogram relative to the metric, g:

$$U^2V^2 - (U\cdot V)^2 = (\omega(U,V))^2 = |g^{ij}|^{-1}\Omega(U,V)^2, \tag{6.13}$$

where U and V are two-dimensional vectors. The polarized version of this identity is

$$U^2(V\cdot W) - (U\cdot V)(U\cdot W) = |g^{ij}|^{-1}\Omega(U,V)\Omega(U,W), \tag{6.14}$$

where U, V, and W are vectors. We will also need the following divergence identity:

$$\begin{aligned} V(|g^{ij}|) &= V(\Omega(H^1, H^2)) \\ &= \mathrm{Div}(V)|g^{ij}| + \Omega([V, H^1], H^2) + \Omega(H^1, [V, H^2]). \end{aligned} \tag{6.15}$$

Another divergence identity is the following. For any function, f, we have

$$\begin{aligned} &\mathrm{Div}(\mathrm{grad}(f))|g^{ij}| = (\mathcal{L}(\mathrm{grad}(f))\Omega)(H^1, H^2) \\ &\quad = d(\mathrm{grad}(f) \lrcorner \Omega)(H^1, H^2) \\ &\quad = H^1\left(\Omega(\mathrm{grad}(f), H^2)\right) - H^2\left(\Omega(\mathrm{grad}(f), H^1)\right) - \Omega(\mathrm{grad}(f), [H^1, H^2]) \\ &\quad = H^1(f_x|g^{ij}|) - H^2(f_y|g^{ij}|) - \Omega(\mathrm{grad}(f), [H^1, H^2]) \end{aligned} \tag{6.16}$$

Let f, h be smooth functions. The obvious identities,

$$\partial_x \cdot \mathrm{grad}(f) = f_x, \qquad \partial_y \cdot \mathrm{grad}(f) = f_y,$$

immediately imply

$$rf_1 - qf_2 = |g^{ij}|f_x, \qquad pf_2 - qf_1 = |g^{ij}|f_y. \tag{6.17}$$

The obvious identities,

$$\begin{aligned} \Omega(\mathrm{grad}(f), \mathrm{grad}(h)) &= \Omega(f_x H^1 + f_y H^2, h_x H^1 + h_y H^2) \\ &= |g^{ij}|(f_x h_y - f_y h_x) \\ &= \Omega(f_1\partial_x + f_2\partial_y, h_1\partial_x + h_2\partial_y) \\ &= f_1 h_2 - f_2 h_1, \end{aligned}$$

imply the following identity, which we will need later:

$$f_1 h_2 - f_2 h_1 = |g^{ij}|(f_x h_y - f_y h_x) \tag{6.18}$$

Proposition 6.3.3 *The Gaussian curvature is given by the following formula:*

$$\begin{aligned} -4p|g^{ij}|^2 K = &\; 3H^1(|g^{ij}|)^2 - 2|g^{ij}|\, H^1(H^1(|g^{ij}|)) \\ &+ |g^{ij}|\left(-p_x H^1(|g^{ij}|) - 4q_y H^1(|g^{ij}|) + 3p_y H^2(|g^{ij}|)\right) \\ &+ |g^{ij}|^2\left(2p_x q_y - 2p_y q_x + 4q_y q_y - 4r_y p_y + 2pp_{xx} - 2rp_{yy} + 4qq_{yy} + 4pq_{xy}\right) \end{aligned} \tag{6.19}$$

Proof: Using (6.13) with $U = H^1$, $V = [H^1, H^2]$ we obtain

$$p|g^{ij}|\,([H^1, H^2])^2 - |g^{ij}|\,(\beta^1)^2 = (p\beta^2 - q\beta^1)^2 \tag{6.20}$$

Making a linear change of coordinates, if necessary, we assume without loss of generality that $p \neq 0$. Recall that K, the Gaussian curvature, is related to the Riemannian curvature tensor by

$$-|g^{ij}|K = R^{1212}.$$

Multiplying both sides of (6.6) by $4p|g^{ij}|$, and using (6.20) we obtain:

$$\begin{aligned} -4p|g^{ij}|^2K &= p|g^{ij}|\,(2H^2(\beta^1) - 2H^1(\beta^2)) \\ &\quad + p|g^{ij}|\,(\text{grad}(p)\cdot\text{grad}(r) - \text{grad}(q)\cdot\text{grad}(q)) \\ &\quad + 3|g^{ij}|\,(\beta^1)^2 + 3(p\beta^2 - q\beta^1)^2. \end{aligned} \tag{6.21}$$

Next, using (6.17) we obtain

$$\begin{aligned} H^1(p\beta^2 - q\beta^1) &= pH^1(\beta^2) - qH^1(\beta^1) + p_1\beta^2 - q_1\beta^1 \\ &= pH^1(\beta^2) - pH^2(\beta^1) + |g^{ij}|\,\partial_y\beta^1 + p_1\beta^2 - q_1\beta^1. \end{aligned} \tag{6.22}$$

Using (6.14) with $U = H^1$, $V = \text{grad}(p)$, $W = \text{grad}(r)$ we get

$$\begin{aligned} p\ \text{grad}(p)\cdot\text{grad}(r) - p_1r_1 &= \omega(H^1, p_xH^1 + p_y H^2)\,\omega(H^1, r_xH^1 + r_yH^2) \\ &= |g^{ij}|p_yr_y. \end{aligned} \tag{6.23}$$

Similarly, we obtain

$$p\ \text{grad}(q)^2 - q_1^2 = |g^{ij}|q_y^2. \tag{6.24}$$

By substituting (6.22), (6.23), and (6.24) into (6.21) we transform the curvature formula into

$$\begin{aligned} -4|g^{ij}|^2K &= |g^{ij}|\left[-2H^1(p\beta^2 - q\beta^1) + 2(p_1\beta^2 - q_1\beta^1) + p_1r_1 - q_1^2 + 3(\beta^1)^2\right] + \\ &\quad + |g^{ij}|^2\left[2\partial_y\beta^1 + (p_yr_y - q_y^2)\right] + \\ &\quad + (p\beta^2 - q\beta^1)^2. \end{aligned} \tag{6.25}$$

Next, using (6.12), and (6.18) with $f = p$ and $h = \beta^1$ we obtain

$$\begin{aligned} p_1\beta^2 - q_1\beta^1 &= p_1r_1 - p_1q_2 - q_1^2 + q_1p_2 \\ &= p_1r_1 - q_1^2 + |g^{ij}|(q_xp_y - p_xq_y). \end{aligned} \tag{6.26}$$

The following is also true,

$$\begin{aligned} p_1\beta^2 - q_1\beta^1 + (\beta^1)^2 &= p_1\beta^2 - q_1\beta^1 + \beta^1(q_1 - p_2) \\ &= p_1\beta^2 - p_2\beta^1 \\ &= \Omega(\mathrm{grad}(p), [H^1, H^2]) \end{aligned} \tag{6.27}$$

Substituting (6.26) and (6.27) into (6.25) we obtain

$$\begin{aligned} -4|g^{ij}|^2K &= |g^{ij}|\left[-2H^1(p\beta^2 - q\beta^1) + 3\,\Omega\left(\mathrm{grad}(p), [H^1, H^2]\right)\right] + && (6.28) \\ &+|g^{ij}|^2\left[(2\,\partial_y\beta^1 + p_xq_y - p_yq_x + p_yr_y - q_y^2)\right] && (6.29) \\ &+3(p\beta^2 - q\beta^1)^2 \end{aligned}$$

Next, we substitute (6.16) with $f = p$ into (6.28) to obtain

$$\begin{aligned} -4|g^{ij}|^2K &= |g^{ij}|\left[-2H^1(p\beta^2 - q\beta^1) + 3H^1(p_x|g^{ij}|) + 3H^2(p_y|g^{ij}|)\right] \\ &+|g^{ij}|^2\left[(\mathrm{Div}(\mathrm{grad}(p)) + 2\,\partial_y\beta^1 + p_xq_y - p_yq_x + p_yr_y - q_y^2)\right] \\ &+3(p\beta^2 - q\beta^1)^2 \end{aligned}$$

The penultimate step is to use (6.15) to derive

$$\begin{aligned} H^1(|g^{ij}|) &= \mathrm{Div}(H^1)|g^{ij}| + \Omega(H^1, [H^1, H^2]) \\ &= (p_x + q_y)|g^{ij}| + p\beta^2 - q\beta^1, \end{aligned}$$

and then to substitute this identity into (6.27) to obtain

$$\begin{aligned} -4|g^{ij}|^2K &= 3H^1(|g^{ij}|)^2 - 2|g^{ij}|H^1(H^1(|g^{ij}|)) + \\ &+|g^{ij}|\left[-4(p_x + q_y)H^1(|g^{ij}|) + 3H^1(p_x|g^{ij}|) + 3H^2(p_y|g^{ij}|)\right] + \\ &+|g^{ij}|^2\left[(\mathrm{Div}(\mathrm{grad}(p)) + 2\,\partial_y\beta^1 + p_xq_y - p_yq_x + p_yr_y - q_y^2 + 3(p_x + q_y)^2)\right] \end{aligned}$$

Finally, some straight-forward expansion and simplification yields (6.19). □

6.4 The Trapping Theorem

The aim of the present section is to present an important local property of flat Riemannian metrics whose metric coefficients are analytic functions.

Let us first illustrate this phenomenon with an example. Consider again the flat metric given in (6.1). The metric matrix divides the plane into two regions according to whether the determinant, $y-x^2$, is positive or negative. The boundary between these regions is the locus of the metric's degeneracy, the curve $y = x^2$. The flat coordinates given by (6.2) are in fact a cover of the positive definite region by the full Euclidean plane:

$$x = \xi, \quad y = \xi^2 + \eta^2,$$

where (ξ, η) are the flat coordinates. It is as if we endowed the paraboloid $y = x^2 + z^2$ with the flat metric structure from the projection to the (x, z) plane, and then projected the paraboloid to the (x, y) plane. This covering is irregular. There are two points in the (ξ, η) plane above every point in the (x, y) plane; the exception is the boundary curve, where the relationship is one to one. Thus, the boundary curve is a "crease" formed by the projection; it is precisely at the boundary that the rank of the Jacobian drops, and where the covering ceases to be a diffeomorphism.

It turns out that there are two types of degenerate points. There are the unreachable points; the distance between these and non-degenerate points is infinite. We will see examples of such degeneracies later. The reachable degeneracies, on the other hand correspond to places where the Euclidean plane "bends back on itself". More precisely, near a reachable, degenerate point the Lie-algebraic space is analytically covered by Euclidean space, and the locus of degeneracy corresponds to the points where the covering map is degenerate and the degree of the covering drops. This behaviour is described in Theorem 6.4.1

This "crease" analogy becomes even more marked when we trace out the geodesics of the flat metric. In the (ξ, η) coordinates the geodesics are given by

$$a\xi + b\eta = c, \quad \text{where } a,\ b,\ c \text{ are constants;}$$

and in the (x, y) coordinates by

$$(a^2 + b^2)x^2 - 2acx - b^2y + c^2 = 0.$$

Thus, in the (x, y) plane, the geodesics are represented by a family of parabolas that are all "trapped" in the range of the projection, the region $\{y - x^2 > 0\}$.

The behaviour of the geodesics at the boundary is particularly interesting. The geodesic parabolas never cross the boundary. As they get close to the boundary curve, the geodesics become tangent to it, and are then "reflected" back into the region $\{y - x^2 > 0\}$.

What can be said about the curves that are not trapped by the boundary? Consider, for instance, the curve

$$x = k, \quad y = t,$$

where k is a constant and t is the parameter of motion. The square of the curve's velocity with respect to (6.1) is $\frac{1}{4(y-x^2)}$, i.e. the curve's velocity is singular as it crosses the boundary. In contrast, a path with finite velocity must be tangent to the boundary when the two meet. This condition has a more analytical description: the derivative of $\det(g^{ij})$ along a path with finite velocity must be zero whenever the determinant is zero. Consider for instance the gradient of an analytic function, $f(x, y)$. Clearly, the square of this function's gradient will not have any singularities, and so we would predict that

$$\text{grad}(f)(y - x^2) = 0, \quad \text{whenever } y - x^2 = 0.$$

Let us verify this. We have

$$\begin{aligned} \text{grad}(f) &= f_x(\partial_x + 2x\partial_y) + f_y(2x\partial_x + 4y\partial_y), \\ \text{grad}(f)(y - x^2) &= 4f_y(y - x^2), \end{aligned}$$

and so, our prediction is confirmed.

The following Theorem and its Corollary serve to formalize the above discussion. We continue to use the notation established in Section 6.3

Theorem 6.4.1 *Let g^{ij} be a contravariant, planar metric tensor with non-singular, analytic coefficients p, q, and r. If the curvature of the corresponding metric is identically zero, then there exist locally defined, analytic functions, μ_1 and μ_2 such that*

$$H^1(|g^{ij}|) = \mu_1 |g^{ij}|, \quad \text{and} \quad H^2(|g^{ij}|) = \mu_2 |g^{ij}|.$$

Proof: The theorem is obviously true for non-singular points of the metric. So suppose without loss of generality that $|g^{ij}|$ is zero at the origin. It is a well known fact that the ring of convergent power series with complex coefficients is a unique factorization domain (see for instance the book by Gunning and Rossi, [14]). This means that up to multiplication by invertible functions, $|g^{ij}|$ factors uniquely into a product of irreducible, complex-valued, analytic functions that are 0 at the origin. Let f be one such irreducible factor, and let k be the multiplicity with which f occurs in the factorization of $|g^{ij}|$. We therefore have

$$|g^{ij}| = f^k \sigma,$$

where σ and f are relatively prime. Now suppose that $H^1(f)$ and f are relatively prime. Hence,

$$\begin{aligned} H^1(|g^{ij}|) &= k\sigma H^1(f) f^{k-1} + \rho_1 f^k \\ H^1(H^1(|g^{ij}|)) &= k(k-1)\sigma (H^1(f))^2 f^{k-2} + \rho_2 f^{k-1}, \end{aligned} \tag{6.30}$$

where ρ_1 and ρ_2 are some analytic functions. Hence we can apply Proposition 6.3.3 to conclude that

$$3H^1(|g^{ij}|)^2 - 2|g^{ij}|\, H^1(H^1(|g^{ij}|))$$

is divisible by f^{2k-1}. Using (6.30) to expand the above we obtain

$$(3k^2 - 2k(k-1))\sigma^2 (H^1(f))^2 f^{2k-2} + \rho_3 f^{2k-1},$$

where ρ_3 is some analytic function. Hence, $k(k+2)\sigma^2(H^1(f))^2$ must be divisible by f. But this cannot be because both σ and $H^1(f)$ are assumed to be relatively prime to f, and because $k(k+2)$ is non-zero for all positive k. Therefore $H^1(f)$ is divisible by f. This must be true for all irreducible factors of $|g^{ij}|$, and therefore $H^1(|g^{ij}|)$ is divisible by $|g^{ij}|$. □

Corollary 6.4.2 (The Trapping Theorem) *Let g^{ij} be as in the preceding theorem, and let f be an analytic function. Then, the flow of* grad(f) *can never cross the locus of degeneracy. More precisely, this means that the trajectories of the flow of* grad(f) *are either contained in the locus of degeneracy of g^{ij}, or they never intersect it.*

Proof: Note that $\text{grad}(f) = f_xH^1 + f_yH^2$. By Theorem 6.4.1 $\text{grad}(f)(|g^{ij}|)$ is divisible by $|g^{ij}|$, and hence $\text{grad}(f)^{(k)}(f)$ is divisible by $|g^{ij}|$ for any positive integer, k. Consider an analytically parametrized curve, $\phi(t)$, whose derivative is equal to $\text{grad}(f)$. Since an analytic vector field integrates to an analytically parametrized curve, $|g^{ij}| \circ \phi$ must be an analytic function of t. If $|g^{ij}| = 0$ at one point of $\phi(t)$, then all orders of the derivative of $|g^{ij}|$ along this curve will also be zero. Therefore, there are exactly two possibilities: either $|g^{ij}|$ is never zero, or $|g^{ij}| \circ \phi$ is identically zero. □

Having defined the multipliers μ_1, and μ_2 we will use them to give yet another formula for Gaussian curvature. We will need this formula later. We put

$$\begin{aligned} P &= \mu_2 - r_y, & Q &= \mu_1 - 2q_y \\ R &= \mu_1 - p_x, & S &= \mu_2 - 2q_x. \end{aligned}$$

Another way to express the nature of μ_1 and μ_2 is to write

$$r\mu_1 - q\mu_2 = |g^{ij}|_x, \qquad -q\mu_1 + p\mu_2 = |g^{ij}|_y. \tag{6.31}$$

With these definitions equations (6.31) can be restated as the following relations:

$$rR - qS \;=\; pr_x \tag{6.32}$$

$$pP - qQ \;=\; rp_y \tag{6.33}$$

Proposition 6.4.3 *The Gaussian curvature is given by*

$$-4pK = QR + p_yS - 2p\,(P_y + Q_x - p_{xx}) \tag{6.34}$$
$$-4qK = RP - p_yr_x - 2q(Q_x + P_y - p_{xx}) \tag{6.35}$$

Proof: Recall that

$$H^i(|g^{ij}|) = \mu_i|g^{ij}|, \quad i = 1, 2.$$

We can therefore rewrite the formula in Proposition 6.3.3 as

$$\begin{aligned} 4pK &= 3\mu_1^2 - 2\mu_1^2 - 2H^1(\mu_1) - (p_x + 4q_y)\mu_1 + 3p_y\mu_2 \\ &\quad +2p_xq_y - 2p_yq_x + 4q_yq_y - 4r_yp_y + 2pp_{xx} - 2rp_{yy} + 4qq_{yy} + 4pq_{xy} \end{aligned}$$

Using the y-derivative of the relation in (6.31) we obtain

$$4pK = \mu_1^2 - 2q_y\mu_1 - p_x\mu_1 + py\mu_2 - 2p(\mu_{1x} + \mu_{2y}) - 2p_yq_x + 2p_xq_y + 4pq_{xy} + 2pp_{xx} + 2r_{yy},$$

which can be abbreviated as equation (6.34). To obtain equation (6.35) we use the following equational relation

$$p(6.35) - q(6.34) = R(6.32) + p_y(6.33).$$

□

We will conclude this section with a restatement of the above results in terms of Newton-Puiseux series. Such a series is an expansion of a function of one variable in terms of fractional powers of that variable. More precisely, a Newton-Puiseux series is an expansion of the form $\phi(x^{\frac{1}{k}})$ where k is a fixed positive integer and $\phi(X)$ is a convergent power series in the dummy variable X. The fundamental result about such power series is the Newton-Puiseux Theorem. The proof is widely available; see for instance Lecture 12 of Abhyankar's book [1]. We also need to recall the notion of a Weierstrass polynomial and the Weierstrass Preparation Theorem. These are also well known topics; a discussion can be found in Lecture 16 of Abhyankar's book [1].

We use the standard notations $\mathbb{C}[[x]]$ to denote complex power series in x, and $\mathbb{C}[[x]][y]$ to denote the ring of polynomials in y whose coefficients are power series in x. A *Weierstrass polynomial of degree n in the variable y* is defined to be an n^{th} degree monic polynomial in $\mathbb{C}[[x]][y]$ whose coefficients are non-invertible power series in x. In other words, such a polynomial, $h(x,y)$, has the form

$$h(x,y) = y^n + h_{n-1}(x)y^{n-1} + \ldots + h_1(x)y + h_0(x),$$

where the $h_i(x) \in \mathbb{C}[[x]]$, and $h_i(0) = 0$.

Theorem 6.4.4 (Newton–Puiseux Theorem) *Suppose that $f(x,y) \in \mathbb{C}[[x]][y]$ is irreducible and convergent as a power series in x and y. Also suppose that $f(0,0) = 0$ and that f is a monic polynomial in the y variable. Then, the equation*

$$f(x,y) = 0,$$

can be solved for y in terms of a Newton-Puiseux series of x. More precisely, there exists a convergent fractional power series, $\phi\left(x^{\frac{1}{k}}\right)$ such that

$$f(x,y) = \prod_{\omega} \left(y - \phi\left(\omega x^{\frac{1}{k}}\right)\right),$$

where the product is taken over the primitive k^{th} roots of unity.

Theorem 6.4.5 (Weierstrass Preparation Theorem) *Suppose that $f \in \mathbb{C}[[x,y]]$ has order n in the y variable, i.e. n is the smallest integer such that $f_{0n} \neq 0$. Then, f can be written uniquely as a product of power series $h(x,y)$ and $u(x,y)$, such that h is an n^{th} degree Weierstrass polynomial in y and such that $u(0,0) \neq 0$.*

Going back to the discussion of flat analytic metrics, let us suppose that the metric tensor is degenerate at the origin, and let $f(x,y)$ be an irreducible analytic factor of $|g^{ij}| = \det(g^{ij})$. By the Weierstrass Preparation Theorem we can without loss of generality assume that f is a Weierstrass polynomial in y. Let $y = \phi\left(x^{\frac{1}{k}}\right)$ be a Newton-Puiseux series solution of

$$f(x, y) = 0.$$

The curve $y = \phi\left(x^{\frac{1}{k}}\right)$ is a branch of the locus of degeneracy of the metric. The following proposition is yet another way of saying that gradient vector fields flow along the locus of the metric's degeneracy.

Proposition 6.4.6 *Let g^{ij} be as in Theorem 6.4.1. Then, the following relations hold:*

$$q(x, \phi) = p(x, \phi)\, \phi', \qquad r(x, \phi) = q(x, \phi)\, \phi'.$$

Proof: By Theorem 6.4.1 we have

$$pf_x + qf_y \equiv 0 \ (\mathrm{mod}\ f). \tag{6.36}$$

Let us write f as follows:

$$f(x, y) = \left(y - \phi\left(x^{\frac{1}{k}}\right)\right)\sigma(x, y),$$

where σ is the product of factors involving roots of unity different from 1. Rewriting (6.36) we have

$$(p(x, \phi)\, \phi' - q(x, \phi))\sigma(x, \phi) = 0.$$

It is clear that $y - \phi(x)$ and $\sigma(x, y)$ are relatively prime, and therefore

$$p(x, \phi)\, \phi' - q(x, \phi) = 0.$$

The relation involving r and q follows analogously. □

6.5 The Tiling Theorem

In the present section we use the trapping theorem of the preceding section to derive a crucial global property of flat analytic metrics on a compact space. In Theorem 6.5.12 we will prove that positive-definite regions of a flat, two-dimensional, Riemannian manifold, M, with a non-singular, but possibly degenerate, analytic contravariant metric tensor are isometric to the Euclidean plane modulo a discrete group generated by reflections. The fixed points of these reflections form a lattice that tiles the plane into isometric regions. Speaking intuitively, the isometry from $\mathbb{R}^2$ to M is a process of folding the plane along the tiling lattice onto a positive-definite region of M.

We will begin with five examples to illustrate the salient features of this tiling theorem. First, let us return to the flat metric given in (6.1). Note that the region $\{y \geq x^2\}$ is isometric to the quotient of the (ξ, η) plane by a single reflection isometry. Let us search for some apriori reasons why this should be.

As it stands, the background manifold of this metric tensor is $\mathbb{R}^2$, a non-compact space. For reasons that are about to become clear, we would like to compactify our setting, i.e. to extend $\mathbb{R}^2$ to a compact manifold such that the metric tensor extends in a non-singular fashion to the enlarged space. One such compactification is given by embedding $\mathbb{R}^2$ into $\mathbb{RP}^1 \times \mathbb{RP}^1$. To describe the enlarged space we introduce extra coordinates,

$$\tilde{x} = \frac{1}{x}, \quad \tilde{y} = \frac{1}{y}.$$

The whole of $\mathbb{RP}^1 \times \mathbb{RP}^1$ is covered by the following four coordinate systems: (x, y), $(x, \tilde{y})$, $(\tilde{x}, y)$, $(\tilde{y}, \tilde{x})$. In each of these coordinates the metric (6.1) has non-singular coefficients. For instance, in the $(\tilde{x}, \tilde{y})$ coordinates the metric matrix is

$$\begin{pmatrix} \tilde{x}^4 & 2\tilde{x}\tilde{y}^2 \\ 2\tilde{x}\tilde{y}^2 & 4\tilde{y}^3 \end{pmatrix}.$$

In this compactification the locus of the metric's degeneracy is the closed curve

$$\left\{y = x^2\right\} \cup \{\tilde{x} = 0\} \cup \{\tilde{y} = 0\}.$$

We can now begin to see an apriori explanation for why the region $\{y \geq x^2\}$ is isometric to a finite quotient of the plane. Let us choose a base-point in the $\{y - x^2 > 0\}$ region, and extend out geodesic paths away from this point. Since we are in a compact setting, a geodesic will either travel away from the base-point forever, or it will come in contact with the locus of degeneracy. At that point the geodesic will for an instant match directions with the boundary curve and then reflect back into the region where the metric is positive definite. In summary, we can indefinitely extend a geodesic trajectory in any direction. Speaking in a more formal language, we are asserting that the exponential map from the tangent space of a base-point has as its domain the full tangent space.

It is important to note that the property in question differs essentially from the usual notion of geodesic completeness. The usual setting for geodesic completeness is a Riemannian manifold with a *non-degenerate metric tensor.* We, on the other hand, wish to investigate the geometry of a space whose contravariant metric tensor has non-singular coefficients, but possesses degeneracies.

The next example is based on the following flat metric:

$$\begin{pmatrix} 1-x^2 & 0 \\ 0 & 1-y^2 \end{pmatrix}$$

Flat coordinates are given by

$$x = \sin(\xi), \quad y = \sin(\eta).$$

The locus of degeneracy is the union of the lines, $x = \pm 1$ and $y = \pm 1$, which divide the (x, y) plane into a 3 by 3 grid. The given flat coordinates cover the central region of the grid with an infinite-fold covering. The pullback of the degeneracy locus to the complete, (ξ, η), plane gives an infinite, bi-directional grid, which tiles that plane into infinitely many isometric squares. It is therefore clear that the central region of the (x, y) plane is isometric to the (ξ, η) plane modulo two vertical and two horizontal reflections. In the preceding example the Euclidean plane was tiled into two isometric regions; in the present example we obtain infinitely many; in the next example we will show that a single region is also possible.

The metric in question is given by:

$$\begin{pmatrix} x^2+y^2 & 0 \\ 0 & x^2+y^2 \end{pmatrix}. \tag{6.37}$$

Flat coordinates are given by

$$x = e^\xi \cos(\eta), \quad y = e^\xi \sin(\eta).$$

In this example the origin is the only degenerate point. The given flat coordinates cover the punctured plane with an infinite-fold covering. Note that the origin of the (x, y) plane is an *unreachable point*, i.e. the length of any curve from the origin to any other point is infinite. This example is meant to illustrate the dichotomy of points on the locus of degeneracy; such points are either unreachable points, or points where the Euclidean space folds back on itself.

The next example illustrates a more complicated tiling pattern. In this example the preimage of the locus of degeneracy divides the Euclidean plane into $2k$ equal sectors (k is any positive integer) radiating from a common center. The metric in question is given by

$$\begin{pmatrix} 4x & 2ky \\ 2ky & k^2x^{k-1} \end{pmatrix}. \tag{6.38}$$

Flat coordinates are given by

$$x = \xi^2 + \eta^2, \qquad y = \Re(\xi + i\eta)^k,$$

where the symbol $\Re$ denotes the real part of a complex number. Let us verify that with this change of coordinates, the above metric really is equivalent to

$$\partial_\xi \otimes \partial_\xi + \partial_\eta \otimes \partial_\eta.$$

It is clear that

$$dx \cdot dx = x_\xi^2 + x_\eta^2 = 4(\xi^2 + \eta^2) = 4x.$$

Writing $\zeta = \xi + i\eta$ we have

$$dy = k\Re(\zeta^{k-1})d\xi - k\Im(\zeta^{k-1})d\eta,$$

and hence

$$dx \cdot dy = 2k\left(\Re(\zeta^{k-1})\xi - \Im(\zeta^{k-1})\eta\right) = 2ky.$$

Finally we have

$$dy \cdot dy = k^2\Re(\zeta^{k-1})^2 + k^2\Im(\zeta^{k-1})^2 = k^2x^{k-1}.$$

Finally, let us examine the pullback of the locus of degeneracy to the (ξ,η) Euclidean plane. The determinant of the matrix in (6.38) is $4k^2(x^k - y^2)$. Hence, the locus of degeneracy in the flat coordinates is given by

$$(\zeta\bar{\zeta})^k - \left(\frac{\zeta^k + \bar{\zeta}^k}{2}\right)^2 = -[\Im(\zeta^k)]^2.$$

It is clear that the locus of

$$\Im(\zeta^k) = 0,$$

consists of k straight lines that divide the plane into $2k$ equal sectors. We therefore see that the positive-definite region $\left\{x^k - y^2 \geq 0\right\}$ is isometric to the (ξ,η) Euclidean plane modulo k centrally based reflections.

The final example is meant to illustrate the necessity for the compact setting. Consider a contravariant metric tensor with matrix

$$\begin{pmatrix} x^4 & 0 \\ 0 & y^4 \end{pmatrix}$$

The reasonable choice for flat coordinates is

$$x = 1/\xi, \quad y = 1/\eta.$$

These coordinates show that there exist finite length paths such as

$$x = t+1,\ y = 1, \quad t \geq 0$$

that do not converge to a limit point. To put it another way $\mathbb{R}^2$ with the given metric does not have enough points to form a complete Euclidean space. The difficulty disappears as soon as we add some points at infinity by stipulating that the metric is given on $\mathbb{RP}^1 \times \mathbb{RP}^1$. We therefore have ξ and η available as coordinates, and with these coordinates the metric tensor assumes the standard form,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We now generalize the above examples into theorems. The setting will be a compact, two-dimensional, real-analytic manifold, M^2, and a pseudo-Riemannian metric, g^{ij}, with analytic coefficients. The locus of degeneracy of g^{ij} divides M into connected open regions. The signature of the metric may change from region to region; we assume that one such region, R, has been fixed, and that the metric in that region is positive-definite. Finally, and most importantly, we assume that the curvature of g is identically zero.

Before proving the main result we will need some definitions and lemmas. Let us call a point on the boundary of R an *unreachable point* if all smooth curves that end in that point have infinite length. Let us write the power series expansion of g about the origin as

$$g^{(0)} + g^{(1)} + g^{(2)} + \dots,$$

where $g^{(k)}$ is a two-by-two symmetric matrix whose coefficients are homogeneous k^{th}-degree polynomial in two variables. Our analysis of boundary points will be based on this expansion. Clearly, the expansion about a degenerate point must have a degenerate $g^{(0)}$. In what follows we will without loss of generality treat generic points on the locus of degeneracy as if they were the origin. Unless stated otherwise, we will write the contravariant tensor matrix, g^{ij} as

$$\begin{pmatrix} p & q \\ q & r \end{pmatrix},$$

and use $|g^{ij}|$ to abbreviate the determinant of this matrix. Much of our discussion will be based on the notion of order of an analytic function, $f(x, y)$. We define this as the smallest total degree, $i+j$, of all monomials, $x^i y^j$ with a non-zero coefficient in the expansion of f, and denote it as $\operatorname{ord}(f)$.

Our first result is a basic criterion for the unreachability of a degenerate point.

Proposition 6.5.1 *Suppose that the power series expansions of all three metric coefficients, p, q, and r, have order 2 or higher. Then, the origin is unreachable. This statement continues to hold even if the coefficients of g are real-valued Newton-Puiseux series.*

Proof: The main idea of the proof will be to compare g with the metric given in (6.37). The eigenvalues, λ, of g are given by

$$\lambda = \frac{p + r \pm \sqrt{(p-r)^2 + 4q^2}}{2}.$$

Let us put

$$\varrho = x^2 + y^2.$$

Because the expansion of g begins with second degree terms, if the coefficients of g are convergent power series, then we have

$$p = \varrho^2 \tilde{p}, \quad q = \varrho^2 \tilde{q}, \quad r = \varrho^2 \tilde{r},$$

where $\tilde{p}$, $\tilde{q}$, $\tilde{r}$ are non-singular analytic functions in polar coordinates. If the coefficients of g are Puiseux series, then we can be assured that $\tilde{p}$, $\tilde{q}$, and $\tilde{r}$ do not have a pole at the origin. In fact all three of these functions must tend to zero near the origin. The formula for eigenvalues can now be written as

$$\lambda = \varrho^2 \left(\frac{\tilde{p} + \tilde{r} \pm \sqrt{(\tilde{p} - \tilde{r})^2 + 4\tilde{q}^2}}{2} \right).$$

Since the parenthetical factor tends to zero near the origin, we can choose a $K > 0$ such that for $\varrho \leq 1$ we will have $\lambda \leq K\varrho^2$. This implies that the length of any path measured with respect to g is greater or equal to $1/K$ times the length of that path measured with respect to the metric in (6.37). But the origin is unreachable with respect to (6.37), and therefore, a fortiori, it is unreachable with respect to g. □

The next proposition classifies degenerate points where the metric has a non-zero constant term.

Proposition 6.5.2 *Suppose that* $\operatorname{ord}(|g^{ij}|) > 0$*, but* $g^{(0)} \neq 0$*. Then, after a suitable change of coordinates* $|g^{ij}| = y^k h(x,y)$*, where* h *is invertible.*

Proof: Using a linear change of coordinates we can always change g so that the constant term has the form

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

i.e. so that p is an invertible function. Let $y - \phi(x)$ be Puiseux series solution of

$$|g^{ij}|(x,y) = 0.$$

By Proposition 6.4.6 we must have

$$p(x,\phi)\phi' = q(x,\phi). \tag{6.39}$$

Since p is invertible, the order of the left-hand side is controlled by ϕ'. Also note that $q(x,\phi)$ is $q(x,0)$ plus a multiple of ϕ. It is therefore impossible for ϕ to have any fractional powers. Let us see why. Suppose ϕ has terms with non-integral exponents. Let d be the smallest *non-integral* rational number such that the coefficient of x^d in $\phi(x)$ is non-zero. Hence, the left hand side will have a non-zero x^{d-1} term. However, all terms with fractional powers in the right hand side must have degree d or higher. We have reached a contradiction, and can thereby conclude that $\phi(x)$ is actually a convergent power series.

We can now take x and $y = y - \phi(x)$ as new coordinates. We may therefore assume, without loss of generality, that y is a factor of $|g^{ij}|$. As we saw in the proof to Proposition 6.5.3, the preceding assumption means that q and r are divisible by y. Let us now show that $|g^{ij}|$ has no other factors other than y. Again, let $y = \phi(x)$ be a Puiseux series equation for the locus of degeneracy, and again (6.39) must be true. Our present assumptions mean that $q(x,0) = 0$, and hence the right hand side of (6.39) is a multiple of $\phi(x)$. The order on the left hand side, however, is equal to the order of $\phi'(x)$. This can only be possible if $\phi(x) = 0$. □

Much of our work will be based on one fundamental tool: *the quadratic fold map.* We define this to be an analytic map from $\mathbb{R}^2$ to $\mathbb{R}^2$ that under a suitable choice of local coordinates can be expressed as

$$x = \xi, \quad y = \eta^2 \tag{6.40}$$

The name "quadratic fold" reflects the fact that the (ξ, η) plane generically covers the (x, y) plane in a two-to-one relationship. The exception are the points on the x-axis (the line of the fold) where the relationship is one-to-one. One fundamental use of the quadratic fold is to resolve a first order degeneracy in the contravariant metric tensor.

Proposition 6.5.3 *Suppose that* $\operatorname{ord}(|g^{ij}|) = 1$. *Then, there exists a non-degenerate contravariant metric tensor* $\tilde{g}^{ij}$ *with analytic coefficients defined on some neighborhood,* $N \subset \mathbb{R}^2$, *and a quadratic fold map* $\phi : N \to R$, *such that* $\phi_*(\tilde{g}) = g$.

Proof: Our assumptions about $|g^{ij}|$ amount to the fact that $|g^{ij}|$ can be used as a coordinate function. We can therefore choose a coordinate system, (x, y), with the property that the determinant of g^{ij} relative to these coordinates is y times an invertible analytic function. The locus of degeneracy is therefore the x-axis, and without loss of generality we assume that near the origin, g is positive definite in the upper half plane. Proposition 6.4.6 tells us that both $q(x, 0)$ and $r(x, 0)$ must be zero. This directly implies that y is a factor of both q and r; let us say that $q = 2y\tilde{q}$ and that $r = 4y\tilde{r}$. Next we define a quadratic fold map as per (6.40). We take N, the domain of this map, to be a neighborhood of the origin, sufficiently small so that the image of the map is contained in the domain of definition of the x and y coordinate functions. Contravariant tensors in the two planes are related by

$$\begin{aligned} \partial_\xi \partial_\xi &= \partial_x \partial_x, \\ \eta\, \partial_\xi \partial_\eta &= 2y\, \partial_x \partial_y, \\ \partial_\eta \partial_\eta &= 4y\, \partial_y \partial_y. \end{aligned}$$

Therefore g is the pushforward of the following non-singular metric:

$$\begin{pmatrix} p & \eta\tilde{q} \\ \eta\tilde{q} & \tilde{r} \end{pmatrix}.$$

Also note that the determinant of the Jacobian of the quadratic fold is $4\eta^2 = 4y$, and hence the determinant of $\tilde{g}$ is equal to $|g^{ij}|/4y$. The result is a non-zer), analytic function in the ξ, η variables, i.e. $\tilde{g}$ is non-singular and non-degenerate.

□

The next proposition shows that quadratic factors of $|g^{ij}|$ can be put into normal form.

Proposition 6.5.4 *Let $f(x,y)$ be a second order analytic factor of $|g^{ij}|$. Then, after a suitable change of coordinates $f(x,y)$ can be expressed as the product of an invertible function, and of one of the following 3 canonical forms: y^2, $y^2 - x^k$, where $k > 0$, or $y^2 + x^k$, where k is even.*

Proof: Using the Weierstrass Preparation theorem we can factor $f(x,y)$ into a unit and into a second order Weierstrass polynomial. We can therefore, without loss of generality assume that

$$f(x,y) = y^2 + f_1(x)y + f_0(x).$$

Note that

$$f(x,y) = \left(y + \frac{f_1(x)}{2}\right)^2 + f_0(x) - \frac{f_1(x)^2}{4}.$$

The first case in our classification occurs when

$$f_0(x) - \frac{f_1(x)^2}{4} = 0.$$

To obtain the canonical form for f we take x and $y + f_1(x)/2$ as new coordinates.

Now, suppose that

$$f_0(x) - \frac{f_1(x)^2}{4} \neq 0,$$

and let us say that

$$f_0(x) - \frac{f_1(x)^2}{4} = x^k\,\phi(x),$$

where $\phi(x)$ is an invertible power series. Making the analytic change of coordinates

$$y = y + \frac{f_1(x)}{2}, \quad x = x\,(\pm\phi(x))^{\frac{1}{k}},$$

we can without loss of generality assume that

$$f(x,y) = y^2 + x^k, \quad \text{or} \quad f(x,y) = y^2 - x^k.$$

In the former case, if k is odd, we can do a $x \mapsto -x$ change of coordinate, and thereby obtain a polynomial of the $y^2 - x^k$ form. □

The second and third cases in the above classification are instances where the degeneracy of the metric can be resolved by using a certain class of analytic maps. The quadratic fold map can be generalized to the notion of a *k-fold map*. These are analytic maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ of the type presented in example (6.38), i.e. under a suitable choice of local coordinates (ξ,η) of the domain and (x,y) of the range, the k-fold map is given by:

$$x = \zeta\bar{\zeta}, \qquad y = \Re(\zeta^k), \tag{6.41}$$

where $\zeta = \xi + i\eta$. These objects are important because they resolve second order degeneracies in the metric tensor. Also note that in the case $k = 1$ the above map is equivalent to a quadratic fold. We also need to introduce a hyperbolic variant of the above k-fold map:

$$x = \xi\eta, \qquad y = \frac{1}{2}(\xi^k - \eta^k). \tag{6.42}$$

This map is quite similar to the standard k-fold given in (6.41), but differs in that it is an isometry between flat pseudo-Riemannian manifolds with a hyperbolic signature.

The following two propositions are necessary for the subsequent discussion about k-fold maps.

Proposition 6.5.5 *Suppose that $x^k - y^2$ is an analytic factor of $|g^{ij}|$. Then g^{ij} is equal to the push-forward of some analytic metric tensor, $\tilde{g}$, via the k-fold map given in (6.41). Furthermore, if the order of $|g^{ij}|$ is 2 then $\tilde{g}$ is non-degenerate. If the order of $|g^{ij}|$ is greater than 2, then the origin is an unreachable point.*

Proof: Let us write the metric tensor as follows:

$$\begin{pmatrix} p & q \\ q & r \end{pmatrix} = \begin{pmatrix} p_0(x) + p_1(x)y & q_0(x) + q_1(x)y \\ q_0(x) + q_1(x)y & r_0(x) + r_1(x)y \end{pmatrix} + (y^2 - x^k) \begin{pmatrix} \tilde{p} & \tilde{q} \\ \tilde{q} & \tilde{r} \end{pmatrix},$$

where the various functions of x are convergent power series. The Puiseux series solutions of

$$y^2 - x^k = 0$$

are given by

$$y = x^{k/2}.$$

Thus, we can use Proposition 6.4.6 to obtain the following relations:

$$\begin{aligned} q_0 + x^{k/2}\, q1 &= \frac{k}{2} x^{k/2-1} (p_0 + p_1 x^{k/2}) \\ r_0 + x^{k/2}\, r1 &= \frac{k}{2} x^{k/2-1} (q_0 + q_1 x^{k/2}). \end{aligned}$$

Separating the terms with half-powers from those with whole powers we further obtain the following:

$$\begin{array}{llll} p_0 &= \frac{2}{k}\, x\, q_1, & q_0 &= \frac{k}{2}\, x^{k-1}\, p_1, \\ r_0 &= \frac{k}{2}\, x^{k-1}\, q_1, & r_1 &= \frac{k^2}{4}\, x^{k-2}\, p_1. \end{array} \tag{6.43}$$

Let us set

$$A = \begin{pmatrix} 4x & 2ky \\ 2ky & k^2 x^{k-1} \end{pmatrix}, \quad B = \begin{pmatrix} 4y & 2kx^{k-1} \\ 2kx^{k-1} & k^2 x^{k-2} y \end{pmatrix}, \quad C = \begin{pmatrix} \tilde{p} & \tilde{q} \\ \tilde{q} & \tilde{r} \end{pmatrix}.$$

Using (6.43) and the above definitions we can write the metric tensor as

$$\frac{q_1}{2k} A + \frac{p_1}{4} B + (y^2 - x^k)\, C. \tag{6.44}$$

Next, let us see if the above terms can be given as push-forwards of non-singular tensors via the standard k-fold map given in (6.41). In the discussion accompanying example (6.38), we saw that A is the push-forward of the identity matrix. We also saw that the square of the Jacobian of the k-fold is equal to $4k^2(y^2 - x^k)$. This implies that $y^2 - x^k$ times any analytic metric tensor in the (x, y) space is the push-forward of some non-singular metric tensor in the (ξ, η) space.

We have dealt with the first and the third terms in (6.44); but what about the middle term? Note that

$$xB = yA + 2k(x^k - y^2)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

A bit of calculation shows that the right hand side is equal to

$$\begin{pmatrix} \xi\Re(\zeta^{k-1}) + \eta\Im(\zeta^{k-1}) & \eta\Re(\zeta^{k-1}) - \xi\Im(\zeta^{k-1}) \\ \eta\Re(\zeta^{k-1}) - \xi\Im(\zeta^{k-1}) & -\xi\Re(\zeta^{k-1}) - \eta\Im(\zeta^{k-1}) \end{pmatrix}.$$

Using the self-evident identities

$$\begin{aligned} \zeta\bar{\zeta}\Re(\zeta^{k-2}) &= \xi\Re(\zeta^{k-1}) + \eta\Im(\zeta^{k-1}) \\ -\zeta\bar{\zeta}\Im(\zeta^{k-2}) &= \eta\Re(\zeta^{k-1}) - \xi\Im(\zeta^{k-1}) \end{aligned}$$

we can conclude that B is the push-forward of

$$\begin{pmatrix} \Re(\zeta^{k-2}) & -\Im(\zeta^{k-2}) \\ -\Im(\zeta^{k-2}) & -\Re(\zeta^{k-2}) \end{pmatrix}.$$

To recap, we have shown that all three terms of (6.44) are push-forwards of non-singular metric tensors in the (ξ, η) space, and therefore the same can be said of our flat metric tensor, g.

To finish the proof, let us suppose that the order of $|g^{ij}|$ is greater than 2. From equation (6.44) we see that this implies that the order of $q_1(x)$ is greater than 0. But this, in turn implies that all coefficients of g have order 2 or higher, and therefore by Proposition 6.5.1 the origin is unreachable. If, on the other hand the order of $|g^{ij}|$ is 2, then $\tilde{g}$ must be non-degenerate at the origin because the determinant of $\tilde{g}$ is the determinant of g divided by the square of the determinant of the Jacobian, □

Proposition 6.5.6 *Suppose that $x^k + y^2$, where k is even, is an analytic factor of $|g^{ij}|$. Then g^{ij} is equal to the push-forward of some analytic metric tensor via the k-fold map given in (6.42). Furthermore, the order of the series expansion of $|g^{ij}|$ must be greater than 2, and the origin is an unreachable point.*

Proof: Our proof proceeds analogously to the one for Proposition 6.5.5. Again, we write the metric tensor as

$$\begin{pmatrix} p & q \\ q & r \end{pmatrix} = \begin{pmatrix} p_0(x) + p_1(x)y & q_0(x) + q_1(x)y \\ q_0(x) + q_1(x)y & r_0(x) + r_1(x)y \end{pmatrix} + (y^2 + x^k) \begin{pmatrix} \tilde{p} & \tilde{q} \\ \tilde{q} & \tilde{r} \end{pmatrix}.$$

The Puiseux series solutions of

$$y^2 + x^k = 0$$

are given by

$$y = i\, x^{k/2}.$$

As before we use Proposition 6.4.6 to obtain the following relations:

$$\begin{aligned} p_0 &= \tfrac{2}{k}\, x\, q_1, & q_0 &= -\tfrac{k}{2}\, x^{k-1}\, p_1, \\ r_0 &= -\tfrac{k}{2}\, x^{k-1}\, q_1, & r_1 &= -\tfrac{k^2}{4}\, x^{k-2}\, p_1. \end{aligned} \tag{6.45}$$

Let us set

$$A = \begin{pmatrix} 4x & 2ky \\ 2ky & -k^2 x^{k-1} \end{pmatrix}, \quad B = \begin{pmatrix} 4y & -2kx^{k-1} \\ -2kx^{k-1} & k^2 x^{k-2} y \end{pmatrix}, \quad C = \begin{pmatrix} \tilde{p} & \tilde{q} \\ \tilde{q} & \tilde{r} \end{pmatrix}.$$

Using (6.45) and the above definitions we can write the metric tensor as

$$\frac{q_1}{2k}\, A + \frac{p_1}{4}\, B + (y^2 + x^k)\, C. \tag{6.46}$$

Next, let us show that A is the push-forward of $2\partial_\xi\partial_\eta$. Computing the push-forward of the latter, we obtain:

$$\begin{aligned}
dx &= \xi d\eta + \eta d\xi, \\
dy &= \frac{k}{2}\left(\xi^{k-1}d\xi - \eta^{k-1}d\eta\right), \\
dx^2 &= 4\xi\eta = 4x, \\
dxdy &= k(\xi^k - \eta^k) = 2ky, \\
dy^2 &= -k^2\xi^{k-1}\eta^{k-1} = -k^2x^{k-1}.
\end{aligned}$$

This is in agreement with the multiplication induced by A. A similar calculation shows that B is the push-forward of the following:

$$\begin{pmatrix} -2\eta^{k-2} & 0 \\ 0 & 2\xi^{k-2} \end{pmatrix}.$$

Again, since the determinant of the Jacobian is equal to $-k(y^2+x^k)$, every multiple of $y^2 + x^k$ is the push-forward of some non-singular tensor. We can therefore conclude that our flat tensor, g, is the push-forward of some non-singular tensor, $\tilde{g}$.

Now let us show that the order of $|g^{ij}|$ must be greater than 2. Let us suppose the opposite. From this we will presently deduce that g must have hyperbolic signature in a neighborhood of the origin. This conclusion is, of course, incompatible with our overall, initial assumption that the origin is on the boundary of a region where g is positive definite. Since the determinant of $\tilde{g}$ is the determinant of g divided by the square of the Jacobian, we can deduce that $\tilde{g}$ is non-degenerate at the origin. Clearly, whatever our choice for an open domain of the hyperbolic k-fold (6.42), if that domain includes the origin of the (ξ, η) space, then the range must necessarily cover some neighborhood of the origin in the (x, y) space (this is a consequence of the assumption that k is even). Thus, the signature of g around the origin is identically hyperbolic, and so we have our contradiction.

Now that we know that the order of $|g^{ij}|$ is greater than 2 we can use equation (6.46) to deduce that the order of $q_1(x)$ must be greater than 0. But this implies that all the coefficients of the metric tensor have order 2 or higher, and therefore by Proposition 6.5.1 the origin is an unreachable point. □

As a matter of fact, Propositions 6.5.3 and 6.5.5 exhaust the possibilities for reachable points on the locus of degeneracy. We will prove this by systematically showing that all other types of degenerate points must be unreachable.

Proposition 6.5.7 *Let $f(x, y)$ be a non-trivial, analytic factor of $|g^{ij}|$. If* $\operatorname{ord}(f) = 1$, *and if the multiplicity of f in the factorization of $|g^{ij}|$ is greater than 1, then the origin is unreachable.*

Proof: The conditions on f imply that it can be a coordinate function. Thus, without loss of generality let us assume that $f = y$. By Theorem 6.4.6 both $q(x, 0)$ and $r(x, 0)$ must be zero. As we did in the proof of Proposition 6.5.3 we can write $q = y\tilde{q}$ and $r = y\tilde{r}$, where $\tilde{q}$ and $\tilde{r}$ are non-singular. We again employ a quadratic fold map as per (6.40), to conclude that g is the push-forward of the following metric tensor:

$$(\tilde{g}^{ij}) = \begin{pmatrix} p & \eta\tilde{q} \\ \eta\tilde{q} & \tilde{r} \end{pmatrix}.$$

Let k be the multiplicity of y in $|g^{ij}|$. The square of the determinant of the Jacobian is $4\eta^2 = 4y$, and hence the multiplicity of η in the determinant of $\tilde{g}$ must be $2k - 2$. The assumption that $k > 1$ implies that η continues to be a factor of the metric tensor's determinant, For reasons we have already seen, η must be a factor $\tilde{r}$, and this can only be possible if y^2 is a factor of r.

We may therefore write the contravariant metric tensor of g as

$$\begin{pmatrix} p & y\tilde{q} \\ y\tilde{q} & y^2\hat{r} \end{pmatrix},$$

where $\tilde{q}$, and $\hat{r}$ are non-singular. The next step is to compare g to a metric with the following contravariant tensor:

$$(\hat{g}^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & y^2 \end{pmatrix}. \tag{6.47}$$

This is also a flat metric, with flat coordinates ξ, η given by

$$x = \xi, \qquad y = e^{\eta}.$$

Let ρ be the type $\binom{1}{1}$ tensor field given by $g\hat{g}^{-1}$. In the x, y coordinates ρ is represented by the matrix

$$\begin{pmatrix} p & \tilde{q}/y \\ y\tilde{q} & \hat{r} \end{pmatrix}.$$

The eigenvalues, λ, of ρ are given by

$$\lambda = \frac{p + \hat{r} \pm \sqrt{(p - \hat{r})^2 + 4\tilde{q}^2}}{2}.$$

The above formula makes it clear that in a fixed neighborhood, N, of the origin we can find an upper bound, $K > 0$, for the eigenvalues, λ. The eigenvalues of $\hat{g}$ are never negative. Hence, in the region, $R \cap N$, where g is positive definite, the eigenvalues of ρ are non-negative, and in that region K^{-1} serves as a lower bound on the eigenvalues of ρ^{-1}. Hence, given a tangent vector, $\mathbf{v}$, based at a point in $R \cap N$, we must have

$$K^{-1}\,\hat{g}^{-1}(\mathbf{v},\mathbf{v}) \leq \hat{g}^{-1}(\rho^{-1}(\mathbf{v}),\mathbf{v}) = g^{-1}(\mathbf{v},\mathbf{v}).$$

This in turns implies that the length functional on curves engendered by g is bounded below by K^{-1} times the length functional engendered by $\hat{g}$. The origin is unreachable with respect to the latter metric, and therefore, a fortiori, it is unreachable with respect to g. □

Proposition 6.5.8 *Suppose that* $\mathrm{ord}(|g^{ij}|) \geq 3$, *but that the order of at least one of the metric tensor's coefficients is 1. Then after a linear change of coordinates the linear term of g can be put into one of the following two canonical forms:*

$$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}, \quad or \quad \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof: Let $p^{(1)}$, $q^{(1)}$, $r^{(1)}$ be the linear terms in the expansion of the metric tensor's coefficients. We are assuming that $p^{(1)}r^{(1)}-(q^{(1)})^2=0$, and hence $p^{(1)}$ and $r^{(1)}$ must be proportional to $q^{(1)}$. We can therefore perform a linear change of coordinates so that the linear term in the expansion of g has the form

$$\begin{pmatrix} ax+by & 0 \\ 0 & 0 \end{pmatrix},$$

where a and b are constants. If $a \neq 0$, then a change of coordinates

$$\tilde{x} = \frac{x}{a} + \frac{by}{a^2}, \quad \tilde{y} = y,$$

will result in a tensor of the form $\tilde{x}\partial_{\tilde{x}} \otimes \partial_{\tilde{x}}$. If $a=0$, then a change of coordinates

$$\tilde{x} = \frac{x}{b}, \quad \tilde{y} = y,$$

will result in a tensor of the form $\tilde{y}\partial_{\tilde{x}} \otimes \partial_{\tilde{x}}$. □

Proposition 6.5.9 *Suppose that* $\mathrm{ord}(|g^{ij}|) \geq 3$, *and that the linear term of g has the form*

$$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}.$$

Then, $|g^{ij}|$ is the product of first order factors, and this implies that the origin is an unreachable point.

Proof: First, consider the case where $|g^{ij}|$ factors into a product of first order factors. Since $\mathrm{ord}(|g^{ij}|) \geq 3$ there must be at least 3 such factors. If two of them are equal then the origin is unreachable by Proposition 6.5.7. If two of the factors are not equal then by Proposition 6.5.4 their product can be put into the form $y^2 \pm x^k$, and hence by Propositions 6.5.5 and 6.5.6 the origin is unreachable.

Let us therefore suppose that $|g^{ij}|$ has an irreducible factor, $f(x,y)$, whose order is greater than 1. Let $y=\phi(x)$ be a Puiseux series solution of $f(x,y)=0$. By Proposition 6.4.6 we must have

$$(x+p_1(x)\phi+p_2(x)\phi^2+\ldots)\phi' = (q_0(x)+q_1(x)\phi+q_2(x)\phi^2+\ldots.$$

Since f is irreducible, ϕ must have terms of non-integral degree. Let d be the smallest non-integral degree in the expansion of $\phi(x)$. Since the left-hand side contains the term $x\phi'$, the smallest non-integral degree of the left hand side of the above equation must also be d. But since we are assuming that the order of $q_1(x)$ is 1 or more, the smallest possible non-integral degree of the right-hand side is $d+1$. This is impossible, i.e. $|g^{ij}|$ must be a product of first order factors, and as we noted above, this implies that the origin is unreachable. □

Proposition 6.5.10 *Suppose that* $\mathrm{ord}(|g^{ij}|) \geq 3$ *and that the linear term of* g *has the form*

$$\begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}.$$

Then, the origin is an unreachable point.

Proof: Since $\mathrm{ord}(p) = 1$ we can take x and $y = p$ as new coordinates. Doing so changes the products $dx \cdot dy$ and $dy \cdot dy$; the product $dx \cdot dx$ remains unchanged. Therefore, the g^{11} coefficient of the resulting metric is equal to p, which is y in the new coordinates. Thus, we can without loss of generality assume that the metric tensor is such that $p = y$.

The constraint of zero curvature will allow us to deduce crucial information about the order of the expansions of the metric tensor's coefficients. This information will prove sufficient to prove unreachability. First, let us write the flatness equation by simplifying the curvature formula in (6.34) with the assumption that $p = y$:

$$QR + S - y(P_y + Q_x) = 0, \tag{6.48}$$

$$RP - r_x - 2q(P_y + Q_x) = 0. \tag{6.49}$$

Recall that

$$R = Q + 2q_y, \qquad S = P + r_y - 2q_x.$$

From (6.33) we also have

$$r = yP - qQ. \tag{6.50}$$

Our hypotheses imply that the orders of q and r are at least 2. Hence, the action of $H^2 = q\partial_x + r\partial_y$ on any analytic function raises the order of that function by at least one. Recall that $P = \mu_2 - r_y$ where μ_2 is an eigenfunction of the H^2 action. Hence, the orders of P and S are at least 1. Hence, $\text{ord}(Q) \geq 1$, for otherwise the left hand side of (6.48) would have order 0. But now we know that the first and third terms of (6.49) have order at least 2, and hence, $\text{ord}(r_x) \geq 2$. From (6.50) we deduce that $\text{ord}(P_x) \geq 1$. Taking an x-derivative of (6.48) we obtain

$$(QR)_x + S_x - y(P_{xy} + Q_{xx}) = 0.$$

Hence,

$$\text{ord}(S_x) = \text{ord}(P_x + r_{xy} - 2q_{xx})$$

is greater than or equal to 1. We have already deduced that $\text{ord}(P_x)$ and $\text{ord}(r_{xy})$ are at least 1, and hence $\text{ord}(q_{xx}) \geq 1$. Let us write the metric tensor coefficients as power series in y:

$$\begin{aligned} q &= q_0(x) + q_1(x)y + q_2(x)y^2 + \ldots \\ r &= r_0(x) + r_1(x)y + r_2(x)y^2 + \ldots, \end{aligned}$$

and summarize what we know about the order of these coefficients:

$$\text{ord}(q_0) \geq 3, \qquad \text{ord}(q_1) \geq 1, \qquad \text{ord}(r_0) \geq 3, \qquad \text{ord}(r_1) \geq 2. \tag{6.51}$$

Again, let $y = \phi(x)$ be a Puiseux series solution of $f(x, y) = 0$, and put $d = \text{ord}(\phi)$. By Proposition 6.4.6 we have

$$q(x, \phi) = q_0 + q_1\phi + q_2\phi^2 + q_3\phi^3 + \ldots = \phi\phi'. \tag{6.52}$$

Hence,

$$2d - 1 = \text{ord}(\phi\phi') \geq \min(\text{ord}(q_0), \text{ord}(q_1) + d),$$

and this implies that $d \geq 2$.

Our next step is to make the change of coordinates $\tilde{y} = y - \phi(x)$. Since ϕ may have terms of fractional degree, this may also be true of the coefficients of the transformed metric tensor. Fortunately, we will not need the assumption of strict analyticity in order to conclude our proof, but we do have to explain why this procedure is valid. Note that $|g^{ij}|(x,0) \leq 0$, because $p = y$. Since we are assuming that the origin is a boundary point of a region where g is positive definite, at least one real-valued, irreducible factor, say $f(x,y)$, of $|g^{ij}|$ must take both positive and negative values near the origin. By Proposition 6.4.1, $yf_x + qf_y$ is divisible by f, and hence $f \neq x$. We will demand that $y - \phi(x)$ be a factor of such an f. Since $f(0,0) = 0$, the branch $y = \phi(x)$ does not intersect either the x or y axis near the origin. Therefore, we can assume without loss of generality that $y = \phi(x)$ forms a part of the boundary of R (the chosen region of positive definiteness), and that in some neighborhood of the origin this curves lies in the upper, right quadrant. This, in turn implies that $\phi(x) = \hat{\phi}\left(x^{\frac{1}{k}}\right)$, where $\hat{\phi}(X)$ is a convergent power series with real coefficients.

Now, let us compute the coefficients of the metric tensor after the above change of coordinates. We have

$$\begin{aligned} dx \cdot dx &= y = \tilde{y} + \phi, \\ dx \cdot d\tilde{y} &= q(x,y) - y\phi'(x). \\ d\tilde{y} \cdot d\tilde{y} &= r(x,y) - q(x,y)\phi' - \phi'(q(x,y) - y\phi'). \end{aligned}$$

Using (6.52) we see that

$$\begin{aligned} q(x,y) - y\phi' &= (q_0 + q_1 y + q_2 y \ldots) - y\phi' \\ &= (q_0 + q_1 y + q_2 y \ldots) - (q_0 + q_1\phi + q_2\phi^2 \ldots) - \phi'(y - \phi) \\ &= (q_1 - \phi')(y - \phi) + q_2(y^2 - \phi^2) + \ldots \end{aligned}$$

Similarly,

$$r(x,y) - q(x,y)\phi' = (r_1 - q_1\phi')(y - \phi) + (r_2 - q_2\phi')(y^2 - \phi^2) + \ldots$$

Therefore we can write the transformed metric as

$$\begin{pmatrix} \tilde{y}+\phi & 2\tilde{y}\tilde{q} \\ 2\tilde{y}\tilde{q} & r\tilde{y}\tilde{r} \end{pmatrix},$$

where $\tilde{q}$ and $\tilde{r}$ may have fractional x-exponents. Nonetheless, from (6.51) and from the fact that $\mathrm{ord}(\phi) \geq 2$ we can deduce that

$$\mathrm{ord}(\tilde{q}_1) \geq 1 \qquad \mathrm{ord}(\tilde{r}_1) \geq 2.$$

Next, we use a quadratic fold map: $y = \eta^2$. The image of this mapping is the upper half-plane, and earlier we able to assume that R is contained in the upper half-plane. The resulting metric tensor has the form

$$\begin{pmatrix} \eta^2+\phi(x) & \eta\tilde{q}(x,\eta^2) \\ \eta\tilde{q}(x,\eta^2) & \tilde{r}(x,\eta^2) \end{pmatrix};$$

for the reason see the proof of Proposition 6.5.3. Since all the coefficients have order 2 or more, by Proposition 6.5.7 the origin is unreachable. □

Metrics with a nilpotent linear part are not just a theoretical possibility. There exist a truly large number of such metrics; our preliminary research indicates that their cardinality is at least as great as the set of all convergent power series! Due to lack of time and space we can only indulge the reader with a couple of examples:

$$\begin{pmatrix} y & xy+x^3 \\ xy+x^3 & 4y^2+yx^2-x^4 \end{pmatrix} \qquad \begin{pmatrix} y & -y^2 \\ -y^2 & 2y^3 \end{pmatrix}.$$

Much work remains to be done in regard to these metrics. Some natural questions are

- How many such metrics are there? Is there some natural way to index them?
- Clearly, such metrics are widely related by changes of coordinates. How many inequivalent classes of such metrics are there?

- Based on the examples of such metrics obtained by us, it would appear that the determinants of these metrics always factor into first order factors, i.e. the locus of degeneracy consists of curves that are non-degenerate at the origin. To date we have not been able to prove this fact, and would like to leave it as a conjecture.

We have now accumulated a sufficient number of results so as to give a classification of degenerate points of g.

Proposition 6.5.11 *A degenerate point of g is either an unreachable point, or there exists a contravariant,* non-degenerate *metric tensor $\bar{g}^{ij}$ with analytic coefficients defined on some neighborhood, $N \subset \mathbb{R}^2$, and an analytic $\phi : N \to R$, such that $\phi_*(\bar{g}) = g$. Furthermore, such a map, if it exists, is either a quadratic fold (Proposition 6.5.3), or it is a k-fold map (Proposition 6.5.5).*

Proof: As usual, without loss of generality we assume that degenerate point in question is the origin. Suppose then that the origin is not unreachable. Thus, by Proposition 6.5.1, $g^{(0)} + g^{(1)} \neq 0$. If $g^{(0)} \neq 0$, then by Proposition 6.5.3, $\mathrm{ord}(|g^{ij}|)$ must be 1, i.e. the locus of degeneracy is a non-singular curve. This Proposition also tells shows that a suitable ϕ exists and that this map is a quadratic fold.

For the remainder of the proof assume that $g^{(0)} = 0$. If $g^{(1)}$ is degenerate, then by Proposition 6.5.8 there exists a linear change of coordinates that transforms $g^{(1)}$ into a normal form which is within the scope of hypotheses' of Propositions 6.5.9 and 6.5.10. In either case, we can be assured that the origin is unreachable.

The only case left to consider is a metric such that $\mathrm{ord}(|g^{ij}|) = 2$. By Proposition 6.5.4, $|g^{ij}|$ up to an invertible multiple is of the form y^2, $y^2 + x^k$, or $y^2 - x^k$. By Propositions 6.5.7 and 6.5.6 we can exclude the first two possibilities. In the y^2 case the origin is unreachable. The $y^2 + x^k$ case, where k is even, cannot occur, because we restrict ourselves to the boundary points of a region where the metric tensor is positive definite. The $y^2 - x^k$ case is covered by Proposition 6.5.5, which shows that ϕ exists and is a k-fold map. □

We now come to the fundamental theorem of this section. In what follows, we will regard $\mathbb{R}^2$ as the Euclidean plane with the standard Euclidean metric g_E. Our goal is to show that the union of R and its reachable boundary points is isometric to the quotient of the Euclidean plane by a certain group of isometries. Once we build a map

$$\psi : \mathbb{R}^2 \to \mathrm{M},$$

let us say that an *isometric symmetry of* ψ is an isometry, σ of the plane such that $\psi\sigma = \psi$. We will employ this notion to isolate the isometries in the quotient group.

Theorem 6.5.12 (Tiling Theorem) *There exists a globally defined, real-analytic map* $\psi : \mathbb{R}^2 \to \mathrm{M}$ *such that* $\psi_*(g_E) = g$, *and such that* ψ *that covers all of* R *plus the reachable portions of its boundary. Furthermore, the preimage of the locus of degeneracy,* $|g^{ij}| = 0$, *under this map, if it is non-empty, consists of lines that tile* $\mathbb{R}^2$ *into isometric cells. These cells are related by the group of isometric symmetries of* ψ*; indeed the union of* R *and the reachable points is isometric to the quotient of* $\mathbb{R}^2$ *by this group.*

Proof: Since the curvature of g is identically zero, there exists an isometry from an open neighborhood, O, of $\mathbb{R}^2$ to an open neighborhood of M. We fix this germ of an isometry, and try to extend its domain to all of $\mathbb{R}^2$. We proceed by a process akin to analytic continuation, although in our case it should more properly be called isometric continuation, because it is based on the rigidity of isometries, rather than the rigidity of analytic maps. Both types of continuation operate on the same principle: information about a sufficiently high order of derivatives of a mapping will locally determine that mapping. In this sense, isometries are even more rigid than analytic mappings; the former are completely determined by first order information, where as the latter require information about all orders of derivatives.

Our goal is to build up an atlas, $\mathcal{A}$, of compatible analytic isometries

$$\psi_\alpha : O_\alpha \to \mathrm{M},$$

where compatibility means that the mappings agree on overlaps. The range of the O_α may include points where the metric on M is degenerate, and so we should remark that in such instances the term "isometry" simply means that the push-forward of the Euclidean metric via ψ_α is equal to g. By Zorn's lemma there exists a maximal such atlas, $\mathcal{A}$. Let us show that $\mathcal{A}$ must cover all of $\mathbb{R}^2$. Suppose not. Then, there exists a straight path

$$\gamma : [0, 1] \to \mathbb{R}^2,$$

such that with the exception of one endpoint, say 1, the image of γ lies entirely in some O_α. Since M is compact $\psi_\alpha(\gamma)$ must have a limit point, say $\mathbf{x} \in$ M. Now, either $\mathbf{x}$ lies in R, or it lies on the locus of degeneracy of g. Let us show that in the latter case, $\mathbf{x}$ must be a reachable point.

Let ℓ be the length functional on paths in R that is induced by the metric g, and let $\tilde{\ell}$ be the length functional induced by the metric $dx^2 + dy^2$, where x, y is some choice of local coordinates around $\mathbf{x}$. Now g^{ij} is degenerate at $\mathbf{x}$, but it is not singular there, and therefore in some neighborhood of $\mathbf{x}$ the eigenvalues of g^{ij} are bounded from above, say by B. Hence, the eigenvalues of g_{ij} are bounded from below by $1/B$. This directly implies that $\tilde{\ell} \leq \frac{1}{B}\ell$. Hence,

$$\lim_{t\to 1} \tilde{\ell}(\gamma(t, 1)) \leq \lim_{t\to 1} \ell(\gamma(t, 1)) = 0.$$

Since we are assuming that $\mathbf{x}$ is a limit point of $\psi_\alpha(\gamma)$ in the coordinate chart topology, the above implies that $\mathbf{x}$ is actually the unique limit of this path. Since $\ell(\gamma)$ is finite, $\mathbf{x}$ must be a reachable point.

Thus, by Proposition 6.5.11 even if $\mathbf{x}$ is a degenerate point, there exists a non-degenerate metric tensor $\tilde{g}^{ij}$ defined on some $N \subset \mathbb{R}^2$, and an analytic map $\phi : N \to$ M such that $\phi_*(\tilde{g}) = g$. Note that $\phi(N) \cap \psi_\alpha(O_\alpha) \neq \emptyset$, because $\mathbf{x}$ is on the boundary of $\psi_\alpha(O_\alpha)$. We therefore have an isometry

$$\psi_\alpha^{-1} \circ \phi : \ \phi^{-1}(\phi(N) \cap \psi_\alpha(O_\alpha)) \to \mathbb{R}^2.$$

Since $\tilde{g}$ is non-degenerate on N we can extend this to an isometry

$$\tilde{\phi} : N \to \mathbb{R}^2.$$

But then, $\phi \circ \tilde{\phi}^{-1}$ with domain $\tilde{\phi}(N)$ extends $\mathcal{A}$, implying that the atlas isn't maximal, and causing a contradiction.

Since the topology of $\mathbb{R}^2$ is trivial, the maps ψ_α piece together to give a global analytic mapping $\psi : \mathbb{R}^2 \to \mathrm{M}$ such that g^{ij} is the push-forward of g_E. In turn, this implies that the locus of degeneracy, $|g^{ij}| = 0$, is the image of the locus of degeneracy of ψ, i.e. those points where $J(\psi)$, the Jacobian is degenerate. The mapping ψ therefore tiles $\mathbb{R}^2$ into connected open cells, C_i that are the preimages of R. The boundary between these cells is the locus $|J(\psi)| = 0$.

Next, suppose that there is more than one such cell, C_i. Since the images of all the C_i are isometric to R, the C_i must be isometric to one another. Let σ be an isometry that relates two of these cells, say $\sigma(C_1) = C_2$. Hence, $\psi \circ \sigma$ and ψ agree on C_1. Above we saw that the germ of ψ completely determines ψ, and therefore

$$\psi \circ \sigma = \psi.$$

By the classification of degenerate points in Proposition 6.5.11, we know that at a degenerate point ψ is equivalent to either a quadratic fold or to a k-fold.

Let ξ, η and x, y be coordinates on $\mathbb{R}^2$ and on M, respectively such that the quadratic fold in question has the normal form

$$\psi : x = \xi, \quad y = \eta^2.$$

It is clear that this mapping possesses a symmetry:

$$\sigma : (\xi, \eta) \mapsto (\xi, -\eta),$$

i.e. locally, at least, $\psi \circ \sigma = \psi$. But this means that the image of σ is isometric to its domain, and therefore σ is an isometry of $\mathbb{R}^2$ that fixes the curve $\eta = 0$. The range of Euclidean isometries is not large; the only one that has such behaviour is a reflection. The locus $\eta = 0$ is therefore a straight line.

Let us now consider the case of a k-fold. Again, let us choose normal form coordinates on $\mathbb{R}^2$ and on M such that the action of ψ is given by

$$x = \xi^2 + \eta^2, \quad y = \Re(\zeta^k),$$

where $\zeta = \xi + i\eta$. The symmetries of this mapping are generated by

$$\sigma : \zeta \mapsto \omega\bar{\zeta},$$

where ω is a primitive k^{th} root of unity. Since σ fixes the curve $\Im(\omega^{\frac{1}{2}}\zeta) = 0$, it must again be a reflection of $\mathbb{R}^2$. Furthermore, note that from the perspective of the points

$$\left\{\zeta : \Im(\omega^{\frac{1}{2}}\zeta) = 0, \quad \zeta \neq 0\right\}$$

this σ is a quadratic fold. We have therefore shown that the locus $|J(\psi)| = 0$ consists of straight lines, and that these straight lines are the fixed points of the quadratic fold symmetries of ψ.

The actions of quadratic fold symmetries are transitive on the set of cells, C_i; any two adjacent cells can be related by such a symmetry. Furthermore, since a symmetry of ψ that acts as the identity on a cell must be the global identity (isometric rigidity), the quadratic fold symmetries generate the group of isometric symmetries of ψ. This shows that R is isometric to the quotient of $\mathbb{R}^2$ by a group generated by reflections. □

To conclude, let us just remark that the group of isometric symmetries is a subgroup of the symmetry group of the tiling $|J(\psi)| = 0$. It is the largest such subgroup with the property that for all symmetries σ, and cells C, if $\sigma(C) = C$, then σ is the identity. This follows directly from the requirement that $\psi\sigma = \psi$.

6.6 Flat Metrics Arising From Imprimitive Actions

In this section we will explore the relation between flat Lie-algebraic metrics, and separation of variables. The main theorem is a partial affirmation of the Turbiner conjecture in 2-dimensions. We will shows that the conjecture is true if the underlying group acts imprimitively, and if this action can be realized on a compact manifold where the action is regular. The present result has another limitation; we will only consider basic Lie algebraic operators, that is operators with a trivial cohomology parameter, η.

Note that the class of manifold with regular **G**-action is larger than the class of homogeneous spaces of **G**, because we are not requiring that the **G** action be transitive. In particular, almost all two dimensional homogeneous spaces are either compact, or can be extended to a compact manifold with **G**-action.

The result that we will prove has wider applicability than might appear at first glance. In the case of 2-dimensional local group actions one has the advantage of a classification of all such actions. The original classification for the complex plane was done by Lie in [22]. This classification was extended to the real plane in [11], and enriched Lie's list with some new classes of group actions that are inequivalent under a change of real coordinates.

The classification reveals that there exist exactly 5 types of maximal local group actions in the real plane. The following table list these maximal actions. The table is an excerpt of the classification list in [11]; the ID column gives the identification labels used in that particular work. The significance of this list lies in the fact that every conceivable Lie algebraic operator in the plane can be formed by using one of the 5 maximal entries. Therefore, in order to prove Turbiner's conjecture in the plane it suffices to verify it for each of the 5 types of maximal actions.

ID	Generators	Structure
7.	$p,\ q,\ xp+yq,\ yp-xq,\ (x^2-y^2)p+2xyq,\ 2xyp+(y^2-x^2)q$	$\mathfrak{so}(3,1)$

8.	$p,\ q,\ xp,\ yp,\ xq,\ yq,\ x^2p+xyq,\ xyp+y^2q$	$\mathfrak{sl}(3)$
16.	$p,\ q,\ xp,\ yq,\ x^2p,\ y^2q$	$\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)$
23.	$p,\ yq,\ \eta_1(x)q,\ \ldots,\ \eta_r(x)q, \quad (r\geq 1)$	$\mathbb{R}^2 \ltimes \mathbb{R}^{r+1}$
28.	$p,\ q,\ xp,\ xq,\ yq,\ x^2p+rxyq,\ x^2,\ \ldots,\ x^rq, \quad (r\geq 1)$	$\mathfrak{gl}(2)\ltimes\mathbb{R}^{r+1}$

Entries 7 and 8 describe primitive actions, and the group action in entry 23 does not have a compact global model. Entries 16 and 18, however, fit the imprimitivity and compactness requirements. A compact manifold on which the actions of entry 16 are realized is just $\mathbb{RP}^1\times\mathbb{RP}^1$. Compact manifolds with actions of the type in entry 28 are given by the Hirzerbruch surfaces. We won't give an explicit description of these compact global models here; for us it suffices to know that they exist. The reader is referred to [8] for some helpful remarks on compact global models of 2-dimensional group actions.

To proof of Turbiner's conjecture for the cases at hand rests on the properties of imprimitive Lie algebraic systems described in Section 4.8, and in Theorem 6.5.12. We need one more ancillary result before we can give the proof for the conjecture.

Proposition 6.6.1 *Let $f(x,y)$ be a real-analytic function defined on all of $\mathbb{R}^2$ with the property that the vector field*

$$\mathrm{grad}(f)=f_x\partial_x+f_y\partial_y$$

flows in straight lines. Then, the level lines of $f(x,y)$ are either mutually parallel straight lines, or concentric circles about a common center. To put it another way, either

$$f=f(ax+by)$$

for some constants a, b; or

$$f=f((x-x_0)^2+(y-y_0)^2)$$

for some x_0, and y_0.

Proof: The condition that grad(f) flows in straight lines is expressed more analytically as

$$\mathrm{grad}(f)(f_y/f_x) = 0,$$

or equivalently as the condition that the vector $f_x\partial_x + f_y\partial_y$ is an eigenvector of the matrix

$$D^2 f = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}.$$

Note that the vector field

$$T = -f_y\partial_x + f_x\partial_y$$

annihilates f, or equivalently that T is tangent to the level lines of f. Also note that

$$\begin{aligned} T(f_x^2 + f_y^2) &= 2(-f_y, f_x)\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}\begin{pmatrix} f_x \\ f_y \end{pmatrix} \\ &= 2T \cdot D^2 f(\mathrm{grad}(f)) \\ &= 0. \end{aligned}$$

This tells us that the magnitude of grad(f) is constant along the level lines of f, and that if $df \neq 0$ at some $p \in \mathbb{R}^2$, then df is non-zero everywhere along the same level line as f. We can therefore conclude that level lines where $df \neq 0$ are closed 1-dimensional submanifolds of $\mathbb{R}^2$.

We may without loss of generality assume that $df \neq 0$ at the origin, and that the flow of grad(f) at the origin is along the y-axis. Hence, $f_x(0, y) = 0$ for some interval, $-\epsilon < y < \epsilon$. Since we have assumed f to be analytic everywhere, we can conclude that $f_x(0, y) = 0$ for all y. To put this point more geometrically, at any given point on the y-axis grad(f) is either 0 or a vector parallel to the y-axis. Furthermore, since $df \neq 0$ at the origin, the zeroes of df on the y-axis are isolated from one another.

Now consider the level-line submanifold,

$$L = \{(x, y) : f(x, y) = f(0, 0)\},$$

that runs through the origin. At this point two possibilities arise. If L is the x-axis, then $\mathrm{grad}(f)_{(x,0)}$ is a constant vector in the ∂_y direction, and hence $\mathrm{grad}(f)$ is either zero or parallel to ∂_y everywhere. In this case, f is a function of y only.

Let us turn to the other possibility, and assume that L is not equal to the x-axis. At each point $(x_0, y_0) \in L$, we know that $\mathrm{grad}(f)$ is normal to L. The normal line has equation

$$-f_y(x_0, y_0)(x - x_0) + f_x(x_0, y_0)(y - y_0) = 0,$$

and on the points of the normal line $\mathrm{grad}(f)$ is either zero or parallel to the normal. Since L is not the x-axis, some normal line of L must intersect the y-axis, and the point of this intersection varies continuously with $(x_0, y_0) \in L$. At the points of intersection $\mathrm{grad}(f)$ is simultaneously parallel to the y-axis and to the normal line through (x_0, y_0), and hence is zero. Since the points along the y-axis where $\mathrm{grad}(f) = 0$ are isolated, the point of intersection must be independent of the choice of $(x_0, y_0) \in L$. This is only possible if L is a circle. Therefore, $\mathrm{grad}(f)$ flows along the normals of this circle, and the level lines of f are the other circles with the same center as L. □

We can now state and prove the main theorem. The reader is well-advised to keep in mind the first example of Section 6.2 while studying this theorem. The setting for the theorem is a Lie algebraic operator $\mathcal{H}$ with quadratic component, $C \in \mathcal{S}^2\mathfrak{g}$, linear component $L \in \mathfrak{g}$, but with a zero cohomology component, $\eta \in \mathrm{H}^1(\mathfrak{g}; \mathcal{C}^\infty(\mathrm{M}))$. We furthermore assume that $\mathbf{G}$ acts imprimitively and that the $\mathbf{G}$ action is realized on a compact manifold. An important note: the compactness assumption does not mean that the homogeneous space, $\mathbf{M} = \mathbf{G}/\mathbf{H}$, must be compact. The case of the Hirzebruch surfaces bears this out. The group action on these compact spaces breaks up into 2 orbits; one of the orbits is a non-compact 2-dimensional homogeneous space, and the other orbit is a 1-dimensional homogeneous space.

Theorem 6.6.2 *If the background metric induced by C is flat, and if $\mathcal{H}$ satisfies the closure conditions (i.e. C and L are compatible), then both the unnormalized* *equation*

$$\mathcal{H}\Psi = E\Psi,$$

and the normalized Schrödinger equation

$$(\Delta + V)\Psi = E\Psi,$$

where V is the associated potential, admit a separation of variables in either a flat or a radial coordinate system.

Proof: Since the action of $\mathbf{G}$ is imprimitive we can choose a locally defined real-analytic function f of $\mathbf{M}$ such that the level-lines of f give the invariant foliation. By Corollary 4.8.4, grad(f) flows along geodesic trajectories. By Theorem 6.5.12 there exists a real analytic map $\Phi : \mathbb{R}^2 \to \mathbf{M}$, such that the Jacobian of Φ is degenerate along a certain lattice of straight lines, but at those points where the Jacobian is non-degenerate the map is a local isometry. Hence, $\Phi^*(f)$ is a real-analytic function with the property that grad(f) flows in straight lines. At this point we would like to apply Proposition 6.6.1 to conclude that $\Phi^*(f)$ is either a function of a flat or of a radial coordinate. However, a remark is in order at this point. The hypothesis of Proposition 6.6.1 speaks of a globally defined function, whereas we have a patchwork of locally defined functions that nonetheless "piece together" in the sense that they all have the same level lines. A closer inspection of the proof of Proposition 6.6.1 will show that the result continues to hold even with the more general, patchwork data, and therefore the pullback of the invariant foliation to $\mathbb{R}^2$ are the level lines of either a flat or a radial function.

For the rest of the proof we will move the setting to $\mathbb{R}^2$. There is still the local action of the group, $\mathbf{G}$, but this action is non-degenerate only wherever the Jacobian of Φ is not degenerate. Separation of variables is a local phenomenon, so for the present purposes we can safely ignore the points of degeneracy. Let us consider the case where the invariant foliation is given by a flat function. Let us use x and y as the flat, orthogonal coordinates on $\mathbb{R}^2$, and suppose that the invariant foliation is given by the level lines $x = \text{const}$. Since $\mathcal{H}$ satisfies the closure conditions, there exists a function $\phi(x,y)$ such that

$$\mathcal{H} = \Delta - 2\,\mathrm{grad}(\phi).$$

Since the foliation is invariant, $\mathcal{H}(x)$ must continue to be a function of x, and this is possible if and only if ϕ is of the form $\xi(x) + \eta(y)$, for some single-variable functions ξ and η. It is now obvious that the equation

$$\mathcal{H}\Psi = E\Psi \tag{6.53}$$

separates in the x, y coordinates. The associated potential of the associated Schrödinger operator is given by

$$\begin{aligned} V &= \Delta(\phi) + \mathrm{grad}(\phi)^2 \\ &= \xi_{xx} + \eta_{yy} + \xi_x^2 + \eta_y^2, \end{aligned}$$

and therefore the equation

$$(\Delta + V)\Psi = E\Psi \tag{6.54}$$

separates in the x, y coordinates as well.

Now let us treat the case where the invariant foliation is given by the level lines of a radial function. Without loss of generality we will assume that this radial function is just $r = \sqrt{x^2+y^2}$. The claim is that the differential equations in question separate in the standard radial coordinates r and θ. The contravariant form of the flat metric tensor in these coordinates is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix},$$

and the Laplacian by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

As above, $\mathcal{H}$ is a sum of the Laplacian and a gradient, say $-2\,\mathrm{grad}(\phi)$. Since the level lines of r are invariant with respect to the G actions, $\mathcal{H}(r)$ must be a function of r, and hence $\phi(r,\theta)$ must be of the form $\rho(r)+\sigma(\theta)$. Hence, equation (6.53) separates into

$$\begin{aligned}(\partial_{rr}+\partial_r+\rho_r\partial_r-E)\,\Psi_1(r) &= \frac{\lambda}{r^2}\Psi_1(r)\\ (\partial_{\theta\theta}+\sigma_\theta\partial_\theta)\,\Psi_2(\theta) &= -\lambda\Psi_2(\theta),\end{aligned}$$

where λ is a constant of separation. The associated potential of the normalized Schrödinger operator is given by

$$V=\rho_{rr}+\rho_r^2+\frac{\sigma_{\theta\theta}+\sigma_\theta^2}{r^2},$$

and therefore equation (6.54) also separates in polar coordinates. □

Appendix A

Closure Condition Solutions for the Linear SL(2) Action

I wish to God these calculations had been executed by steam.
– Charles Babbage

In this appendix we will list the solutions to the closure conditions obtained by checking the invariant equations (5.9) and (5.10) against the 5 types of $\mathcal{S}^2\mathfrak{g}$ orbits given in Section 5.4 For each solution we will present the operator system (C, L, η), the potential, V, of the normalized Schrödinger operator, the gauge factor, μ, the contravariant metric tensor, g^{ij}, induced by C, and the curvature, K, of that metric.

The $C_{(1)}$ type: there is one class of general solutions.

$$C = \begin{pmatrix} 0 & -1 & -2 \\ -1 & 1 & 1 \\ -2 & 1 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} Q \\ 0 \\ Q \end{pmatrix} \qquad \eta = \begin{pmatrix} 0 \\ 0 \\ P/x^2 \end{pmatrix}.$$

$$V = \frac{Q^2}{4} - 1 \qquad \mu = (x+y)^{1-Q/2} e^{\frac{-P}{x(x+y)}}.$$

$$g^{ij} = \begin{pmatrix} x^2+2xy & -x^2-3xy-y^2 \\ -x^2-3xy-y^2 & 2xy+y^2 \end{pmatrix} \qquad K=4.$$

The $C_{(2)}$ type: there are no homogeneous solutions, and 1 class of basic solutions.

$$C = \begin{pmatrix} 0 & 0 & -1+2S \\ 0 & 1+S & 0 \\ -1+2S & 0 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} Q \\ 0 \\ Q \end{pmatrix} \qquad \eta = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$V = -S + \frac{1}{2} - \frac{Q^2}{6} \qquad \mu = x^{\frac{1}{2}+\frac{Q^2}{6}} y^{\frac{1}{2}-\frac{Q^2}{6}}$$

$$g^{ij} = \begin{pmatrix} x^2(1+S) & (S-2)xy \\ (S-2)xy & (1+S)y^2 \end{pmatrix} \qquad K=0$$

The $C_{(3)}$ type: there is one class of general solutions, and another of basic solutions.

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 3/2 \\ 0 & 3/2 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 0 \\ 0 \\ Q \end{pmatrix} \qquad \eta = \begin{pmatrix} 0 \\ 0 \\ P/x^2 \end{pmatrix}$$

$$V = 0 \qquad \mu = e^{\frac{-P}{xy}} y^{1-\frac{Q}{3}}$$

$$g^{ij} = \begin{pmatrix} 3xy & -3/2y^2 \\ -3/2y^2 & 0 \end{pmatrix} \qquad K=0.$$

$$C = \begin{pmatrix} 0 & 0 & 2S \\ 0 & S & 3/2 \\ 2S & 3/2 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 0 \\ 4/3QS \\ Q \end{pmatrix} \qquad \eta = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$V = -\frac{512S^3Q^2x^3 + 192S^2(9+Q^2)x^2y - 72(27+5Q^2)Sxy^2 + 81(Q^2-9)y^3}{27y(8Sx-3y)^2}$$

$$\mu = e^{\frac{2SQx}{9y}} (8Sx-3y)^{\frac{1}{4}} y^{\frac{-Q}{3}+\frac{3}{4}}.$$

$$g^{ij} = \begin{pmatrix} Sx^2+3xy & Sxy-3/2y^2 \\ Sxy-3/2y^2 & Sy^2 \end{pmatrix} \qquad K = \frac{-108Sy^2}{(8Sx-3y)^2}.$$

The $C_{(4)}$ type: there is one class of basic solutions.

$$C = \begin{pmatrix} 0 & 0 & 2S \\ 0 & S & 0 \\ 2S & 0 & 6 \end{pmatrix} \qquad L = \begin{pmatrix} 0 \\ Q_1 \\ Q_2 \end{pmatrix} \qquad \eta = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$V = -\frac{4SQ_1^2x^2 + 4SQ_1Q_2xy + (SQ_2^2 + 24S^2 + Q_1^2)y^2}{24Sy^2}$$

$$\mu = y^{\frac{2S-Q_1}{2S}} e^{\frac{x(Q_1x+Q_2y)}{12y^2}}.$$

$$g^{ij} = \begin{pmatrix} Sx^2 + 6y^2 & Sxy \\ Sxy & Sy^2 \end{pmatrix} \qquad K = 0.$$

The $C_{(5)}$ type: there is one general solution, and one class of basic solutions.

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 6 \end{pmatrix} \qquad L = \begin{pmatrix} 0 \\ Q \\ 0 \end{pmatrix} \qquad \eta = \begin{pmatrix} 0 \\ 0 \\ P/x^2 \end{pmatrix}$$

$$V = \frac{3}{2} - \frac{Q^2}{6}, \qquad \mu = y^{1-Q/3}.$$

$$g^{ij} = \begin{pmatrix} 3/2x^2 + 6y^2 & -3/2xy \\ -3/2xy & 3/2y^2 \end{pmatrix}, \qquad K = -6$$

$$C = \begin{pmatrix} 6 & 0 & -2+2S \\ 0 & 2+S & 0 \\ -2+2S & 0 & 6 \end{pmatrix} \qquad L = \begin{pmatrix} Q \\ 0 \\ -Q \end{pmatrix} \qquad \eta = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$V = -S - \frac{Q^2}{24} - 2, \qquad \mu = \sqrt{x^2+y^2}e^{-\frac{Q}{12}\arctan(x/y)}.$$

$$g^{ij} = \begin{pmatrix} (2+S)x^2 + 6y^2 & (S-4)xy \\ (S-4)xy & 6x^2 + (2+S)y^2 \end{pmatrix}, \qquad K = 0.$$

Appendix B

Hyperbolic Signature Counter-example

We think in generalities, but we live in details.
- Alfred North Whitehead

In the present appendix we will show that Turbiner's conjecture cannot be true without the assumption that the Lie-algebraic metric in question is positive-definite as well as flat. We will exhibit a Lie-algebraic operator that engenders a flat, hyperbolic metric, but does not have a separable potential. We will organize our discussion into three parts. First we will introduce the Lie-algebraic operator in question. Second, we will define and discuss the notion of "separation of variables" for this operator. Finally, we will demonstrate that there does not exist a coordinate system that separates our operator.

The focus of our attention will be the Lie-algebraic operator

$$\mathcal{H} = \{\partial_v, -u\partial_u + v\partial_v\} + u\partial_u - 2v\partial_v + u\partial_v.$$

There any number of Lie algebras that will generate this operator — projective SL(3) actions, for instance — and there is no need to fix one specific algebra. The corresponding contravariant metric tensor is given by

$$\begin{pmatrix} 0 & -u \\ -u & 2v \end{pmatrix}.$$

Flat coordinates, (x, y), are given by

$$x = \frac{1}{u}, \quad y = uv.$$

The metric tensor in these coordinates is given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Writing our operator in the (x, y) coordinates we obtain

$$\begin{aligned} \mathcal{H} &= \left\{\frac{1}{x}\partial_y, x\partial_x\right\} - x\partial_x - y\partial_y + \frac{1}{x^2}\partial_y \\ &= \Delta - \text{grad}\left(\frac{1}{x} + \log(x) + xy\right) \end{aligned}$$

A change of scale transforms the above into the following Schrödinger operator:

$$\Delta - \frac{xy}{2} + \frac{1}{2x} + \frac{1}{2}. \tag{B.1}$$

The theoretical underpinnings for our treatment of separation of variables come from articles by Miller [23] and Koornwinder [21]. We will say that a second-order partial differential equation

$$H(x, u, u_i, u_{ij}; E) = 0, \tag{B.2}$$

with a parameter, E, additively separates in the coordinate system $x_1, \ldots, x_n$, if there exist n single-variable ordinary differential equations,

$$\frac{d^2u^{(i)}}{dx_i^2} + f_i\left(x_i, u^{(i)}, \frac{du^{(i)}}{dx_i}; E, \lambda_1, \ldots, \lambda_{n-1}\right) = 0, \quad \text{where } i = 1 \ldots n,$$

depending on E and $n-1$ other parameters, such that for all values of the parameters the function

$$u(x_1, \ldots, x_n) = \sum_i u^{(i)}(x_i)$$

is a solution of (B.2). The definition of multiplicative separability is similar; for all values of the parameters the function

$$u(x_1, \ldots, x_n) = \prod_i u^{(i)}(x_i)$$

must be a solution of (B.2). The above definition of additive separability is equivalent to the notion of regular separability introduced in [23]. This article gives an equivalent criterion based on H and its derivatives:

$$\begin{aligned} &H_{u_{ii}} H_{u_{jj}} (\tilde{D}_i \tilde{D}_j H) + H_{u_{ii} u_{jj}} (\tilde{D}_i H)(\tilde{D}_j H) \\ &\quad = H_{u_{jj}} (\tilde{D}_i H)(\tilde{D}_j H_{u_{ii}}) + H_{u_{ii}} (\tilde{D}_j H)(\tilde{D}_i H_{u_{jj}}), \quad i \neq j, \end{aligned} \tag{B.3}$$

where $\tilde{D}_i$ is the total derivative operator $\partial_{x_i} + u_i \partial_u + u_{ii} \partial_{u_i}$.

We are interested in the multiplicative separability of Schrödinger equations:

$$(\Delta + V)\Psi = E\Psi.$$

This can be converted into a question of additive separability by introducing the related dependent variable $u = \log(\Psi)$. Writing the above in terms of u we have:

$$\Delta(u) + \mathrm{grad}(u)^2 + V = E. \tag{B.4}$$

Applying the criterion in (B.3) we obtain the following conditions (see [23] and [21] for the proofs and further references):

- The coordinates $x_1, \ldots, x_n$ must be orthogonal. In other words, the metric tensor must satisfy

$$g^{ij} = 0, \quad \text{whenever } i \neq j.$$

- The metric tensor must satisfy the Levi-Civita separability conditions:

$$\partial_{ij}G^a + \partial_i G^a \partial_j G^a - \partial_i G^a \partial_j G^i - \partial_j G^a \partial_i G^j = 0, \qquad \text{for all } a \text{ and } i \neq j,$$

  and where $G^i = \log(g^{ii})$. This condition is equivalent to the requirement that the metric tensor be in Stäckel form. In other words, there exists a so-called Stäckel matrix, $\{s_{ij}\}$, such that the i^{th} row depends only on the variable x_i, and such that the first row of the matrix's inverse is equal to $(g^{11}, \ldots, g^{nn})$.

- The metric tensor must satisfy the Robertson condition:

$$\partial_{ij}\left(G^i - \sum_{k \neq i} G^k\right) = 0, \qquad \text{for all } i \neq j.$$

  This, in turn, is equivalent to the condition that

$$\partial_i \left(G^i - \sum_{k \neq i} G^k\right) = f_i(x_i),$$

  for some single variable function $f_i(x_i)$. Eisenhart proved in [5] that the Robertson condition is equivalent to the requirement that the Ricci tensor in the given coordinates is in diagonal form, i.e. that $R_{ij} = 0$, if $i \neq j$. Therefore, the Robertson condition is automatically satisfied whenever the metric g^{ij} is flat.

- The potential must satisfy

$$\partial_{ij}V = \partial_i G^j \partial_j V + \partial_j G^i \partial_i V.$$

  It can be shown that this is equivalent to the condition that the potential has the form

$$V = \sum_i h_i(x^i) g^{ii},$$

  for some list of single variable functions $h_i(x_i)$.

If the above conditions are satisfied, then (B.4) takes the form

$$\sum_i g^{ii}(u_{ii} + u_i^2 + f_i u_i + h_i - E s_{1i}) = 0,$$

and therefore separates into the following equations:

$$\frac{d^2 u^{(i)}}{dx_i^2} + \left(\frac{du^{(i)}}{dx_i}\right)^2 + f_i(x_i)\frac{du^{(i)}}{dx_i} + h_i(x_i) + \sum_{j=1}^{n} \lambda_j s_{1i}(x_i) = 0,$$

where $\lambda_1 = -E$, and $\lambda_2, \ldots, \lambda_n$ are the other constants of separation.

The preceding discussion tells us that the Lie algebraic operator (B.1) will separate in coordinates (ξ, η) if and only if the following two conditions are satisfied. First, these coordinates must give multiplicative separation of the following hyperbolic form of Laplace's equation,

$$\Psi_{xy} = E\Psi.$$

In other words, the metric in the (ξ, η) coordinates must be in Stäckel form. Note that the Robertson conditions are automatically satisfied, because the metric in question is flat. Second, the potential must have the form

$$V = \frac{f(\xi)}{p} + \frac{g(\eta)}{q},$$

where $p\,d\xi^2 - q\,d\eta^2$ is the expression for the metric in the (ξ, η) coordinates.

The separation of the planar, hyperbolic Laplace equation has been studied by Kalnins in [17]. This article classifies all coordinate systems that allow separation of variables for this equation. These coordinate systems are presented in the following table. Note that we will reserve the symbols $\tilde{x}$ and $\tilde{y}$ for the orthogonal Cartesian coordinates.

	Coordinate System		Metric	Description
1.	$x = \tilde{x} + \tilde{y}$	$y = \tilde{x} - \tilde{y}$	$d\tilde{x}^2 - d\tilde{y}^2$	Cartesian
2.	$\tilde{x} = \xi \cosh \eta$	$\tilde{y} = \xi \sinh \eta$	$d\xi^2 - \xi^2 d\eta^2$	Polar
3.	$\tilde{x} = \frac{1}{2}(\xi^2 + \eta^2)$	$\tilde{y} = \xi\eta$	$(\xi^2 - \eta^2)(d\xi^2 - d\eta^2)$	Parabolic I
4.	$x = -(\xi - \eta)^2$	$y = 2(\xi + \eta)$	$\frac{\xi-\eta}{4}(d\xi^2 - d\eta^2)$	Parabolic II
5.	$x = \cosh \frac{1}{2}(\xi - \eta)$	$y = \sinh \frac{1}{2}(\xi + \eta)$	$(\sinh \xi - \sinh \eta)(d\xi^2 - d\eta^2)$	Hyperbolic I
6.	$x = \sinh(\xi - \eta)$	$y = \exp(\xi + \eta)$	$(\exp(2\xi) + \exp(2\eta))(d\xi^2 - d\eta^2)$	Hyperbolic II
7.	$x = \cosh(\xi - \eta)$	$y = \exp(\xi + \eta)$	$(\exp(2\xi) - \exp(2\eta))(d\xi^2 - d\eta^2)$	Hyperbolic III
8.	$\tilde{x} = \sinh \xi \cosh \eta$	$\tilde{y} = \cosh \xi \sinh \eta$	$(\cosh^2 \xi + \sinh^2 \eta)(d\xi^2 - d\eta^2)$	Elliptic I
9.	$\tilde{x} = \cosh \xi \cosh \eta$	$\tilde{y} = \sinh \xi \sinh \eta$	$(\sinh^2 \xi + \sinh^2 \eta)(d\xi^2 - d\eta^2)$	Elliptic II
10.	$\tilde{x} = \cos \xi \cos \eta$	$\tilde{y} = \sin \xi \sin \eta$	$(\sin^2 \xi + \sin^2 \eta)(d\xi^2 - d\eta^2)$	Elliptic IIa

Strictly speaking, the classification is complete up to conjugation by elements of the pseudo-Euclidean group,

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} \cosh\theta & \sinh\theta & s_1 \\ \sinh\theta & \cosh\theta & s_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ 1 \end{pmatrix},$$

the reflection transformation,

$$(\tilde{x}, \tilde{y}) \mapsto (-\tilde{x}, \tilde{y}),$$

and the permutation transformation,

$$(\tilde{x}, \tilde{y}) \mapsto (\tilde{y}, \tilde{x}).$$

Thus, in order to show that the potential given in (B.1) does not separate, it suffices to check each of the above coordinate systems for the property that an expression of the form

$$V = xy + ax + by + \frac{1}{cx + d}, \quad \text{or} \quad V = xy + ax + by + \frac{1}{cy + d}$$

cannot be reexpressed as

$$\frac{f(\xi)}{p} + \frac{g(\eta)}{q},$$

where $pd\xi^2 + qd\eta^2$ is the corresponding form of the metric.

Indeed, none of the above coordinate systems has this property, and therefore (B.1) does not separate. The verification is done by straight-forward inspection. Consider, as an example, the Parabolic II system. We must check that $V(\xi - \eta)$ cannot have the form $f(\xi) + g(\eta)$. Let us focus on the $\frac{1}{cx+d}$ term. Writing in terms of ξ and η we have

$$\frac{\xi - \eta}{-c(\xi - \eta)^2 + d} = \frac{-1}{c(\xi - \eta) + \frac{d}{\xi - \eta}}.$$

There is no way to add choose c and d so that the result will have the form $f(\xi) + g(\eta)$. The same conclusion holds for terms of the form $\frac{1}{cy+d}$.

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