The superstring and factorization algebras

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Contents

Abstract

In this predominantly expository thesis we describe the worldsheet Neveu-Schwarz-Ramond (NSR) superstring of high energy physics and factorization algebras. After providing some context for the aforementioned theory as well as the bc conformal field theory using common tools and constructions of theoretical physics, we explain how to implement the Becchi-Rouet-Stora-Tyutin (BRST) and Batalin-Vilkovisky (BV) formalisms to represent the ghosts and antifields, respectively, of any perturbative classical field theory described by an action functional and with a group of symmetries, actually forming a double complex (the BV-BRST complex) characterizing all of those fields and their symmetries and whose cohomology groups describe the observables of the theory and have the structure of a factorization algebra. Correspondingly, we explain how the equations of motion for the entire theory (fields, ghosts, antifields, and antighosts) can be portrayed in terms of an L-infinity algebra, which can be shown to be equivalent to a based formal moduli problem that encodes the perturbative behavior of the theory. This formal moduli problem forms a sheaf on the worldsheet, which implies that the algebra of functions over it has the structure of a factorization algebra. The majority of this thesis focuses on clarifying these relationships so as to abet the understanding of these perspectives on classical field theory.

Résumé

Dans cette thèse principalement explicative, nous décrivons la théorie des supercordes type Neveu-Schwarz-Ramond (NSR) sur une surface d'univers et les algèbres de factorisation. Nous commençons par fournir un molé de contexte de la théorie susmentionée ainsi que de la théorie classique conforme des champs bc. Dans ce but, nous utilisons des outils et constructions standards de la physique théorique. De plus, nous expliquons comment mettre en œuvre les formalismes de Becchi-Rouet-Stora-Tyutin (BRST) et de Batalin-Vilkovisky (BV). Le formalisme BRST décrit les champs fantômes de n'importe quelle théorie perturbative classique des champs possédant un groupe de symétrie et gouvernée par une fonctionnelle d'action. En outre, le formalisme BV précise les anti-champs d'une telle théorie. Les deux se réalisent en un double complexe différentiel (le complexe BV-BRST) qui encode tous ces champs et dont les groupes de cohomologie caractérisent les observables et présentent la structure d'une algèbre de factorisation. Parallèlement, nous illustrons comment les équations du mouvement de la théorie au complet (incluant tous les champs, champs fantômes, anti-champs, et anti-champs fantômes) peuvent être représentées comme une algèbre L-infinie. En particulier, on peut montrer que celle-ci est équivalente à un problème de modules formel avec point de base qui décrit le comportement perturbatif de la théorie. Ce problème de module formel constitue un faisceau, ce qui implique que l'algèbre des fonctions sur celui-ci forme une algèbre de factorisation. La majorité de cette thèse vise à clarifier ces relations afin d'encourager la compréhension des différentes perspectives de la théorie classique des champs.

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Chapter 1

Introduction

Modern high energy physics has and continues to provide a bounty of mathematical inspiration and examples. One particularly rich instance of this is string theory. Evolving throughout the 20^{th} century, the framework of string theory was shown during what is now called the *first superstring revolution*, to be able to describe not only the particles of the standard model, but also, hearkening back to its roots in S-matrix theory, their interactions. The theories that could encompass both bosonic and fermionic particles became known as *superstring theories*, and it is one of these, the *type II superstring* that shall be most discussed in this exposition.

The measurable quantities, or *observables* of a given physical theory are of particular interest to physicists and mathematicians alike: among other reasons, by checking the behavior of observables, the former may use them to determine how accurately a model is describing nature, and to the latter, measurable quantities offer an inroad into potential methods of formalization of perhaps then-hitherto ad-hoc constructions as well as inspiration for mathematical exploration in new directions. In this document, we will look at one algebraic construction, that of *factorization algebras* at the interface of several areas of mathematics and physics.

The journey to get from a classical field theory of physics to a description of its observables in terms of factorization algebras passes through several regions of mathematics, with one possible path as follows: given a particular classical field theory, one may apply the Batalin–Vilkovisky – Becchi–Rouet–Stora–Tyutin (BV-BRST) procedure to it to describe the theory in full, complete with ghosts, anti-fields, and anti-ghosts. This will produce a double complex, the total complex of which gives the sought-after cochain complex. The physical observables of the theory are then described by the cohomology groups of that complex, and these can be shown to form what looks like a cosheaf with respect to the tensor product instead of direct sum, i.e. a factorization algebra.

Let X be the space of fields of a classical field theory, and G denote its symmetry group. Here we assume that X has at least the structure of a vector space and G is finite-dimensional, though in many practical applications, one works with an infinite dimensional symmetry group. Looking infinitesimally, the space of fields can be described by the quotient $X/\mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra corresponding to G. One tractable way to examine this quotient is to look at it dually, i.e. to inspect the algebra of functions on $X/\mathfrak{g}$. In the cases that we are looking at, this space will be the derived critical locus of some action functional, and the algebraic dual to the homotopy quotient $X/\mathfrak{g}$ will be the BV-BRST double complex of the physical theory. It is the cohomology groups of this complex that then describe the physical observables, and also have the structure of factorization algebras.

This, however, is only one facet of the picture: a classical field theory can also be shown to correspond to a pointed elliptic formal moduli problem. This is illuminating as to the structure of the fields and observables of the theory since there is an equivalence of categories between formal moduli problems and L_∞ algebras, where the functor associating to an L_∞ algebra a formal moduli problem is the Maurer–Cartan functor. Modulo the choice of basepoint of the formal moduli problem, the Maurer–Cartan elements of the L_∞ algebra (here actually considered as a differential graded Lie algebra (dgLa) by that the brackets above degree 2 are set to zero) correspond directly to the solutions of the Euler-Lagrange equations, i.e. the physical fields. In fact, taking the symmetric algebra of this dgLa returns the BV–BRST complex, giving a way to algebraically interpret what could also be viewed as a map of simplicial sets. In the context of physics, it is possible to, in many cases, at least obtain the information of the BRST complex directly (we mention one strategy for this, the inversion of the Faddeev–Popov determinant, in the second chapter), so this whole picture is not used. At various points in this text, though, we will touch upon the examples of the *bc* conformal field theory (CFT) and the worldsheet NSR type II superstring, though not at every step of the way.

In this predominantly expository thesis, we aim to explain the type II superstring and factorization algebras. In the second chapter, we present some standard tools and computations from field theory, and become acquainted with the worldsheet NSR model for the type II superstring from the perspective of high energy physics. In the third chapter, the BV–BRST complex double complex for a classical field theory with a symmetry group and described by an action functional is introduced. In the fourth chapter, prefactorization and factorization algebras are introduced, and the path from formal moduli problems to factorization algebras is sketched, along with a means by which to describe a field theory governed by an action functional in this framework.

Chapter 2

Physics Concepts and Calculations

2.1 Introduction

The field theories in focus as examples in this exposition will be the closely-related $bc-$ and $\beta\gamma-$conformal field theories (CFTs) and the worldsheet Neveu–Schwarz Ramond (NSR) superstring, all free field theories over a one-complex-dimensional worldsheet, a complex manifold Σ. Physical models amenable to further mathematical description include such sigma-model field theories, so we examine those here from a physical perspective. We will begin the chapter with an introduction of some common methods in high energy physics, and subsequently apply those to the three example field theories.

All of the physical theories we consider here have transformations under which the field content of the theory is invariant. In this treatise, it is assumed that these symmetries form a group, the *gauge group*. Note that this is different than the notion of *global symmetry* found in physics: the former indicates that there is some notion of physical equivalence of all of the states in the same orbit of the gauge group, while the latter concerns symmetries on the space of physical states. Global symmetries are invariant under the parametrization of the theory, so no matter which action functional is chosen, these symmetries will be present.

The $bc-$CFT is a theory of two anticommuting tensor fields, b of weight $(\lambda, 0)$ and c of weight $(1-\lambda, 0)$ for $\lambda \in \mathbb{Z}_{\geq 0}$ governed by the complex worldsheet action

$$S_{bc} = \int_\Sigma dz d\bar{z} b \bar{\partial} c.$$

The $\beta\gamma$ theory is analogous, except that β and γ commute. While the $bc-$CFT can be treated as its own theory, it also naturally arises from the application of the BRST procedure to the bosonic string described by the Polyakov action as seen in [10, Equation 1.2.13],

$$S_P[X,g] = -\frac{1}{4\pi\alpha'}\int_\Sigma d\tau d\sigma(-g)^{\frac{1}{2}}g^{ab}\partial_a X^\mu \partial_b X_\mu. \tag{2.1}$$

Foreshadowing to the NSR worldsheet superstring, we may write this action, under a change of variables on a flat worldsheet,

$$(\sigma,\tau)\mapsto(\sigma^1,\sigma^2)\mapsto(z,\bar{z})$$

where the first transformation is relabeling and the second is $z=\sigma^1+i\sigma^2$, $\bar{z}=\sigma^1-i\sigma^2$, as

$$S_P = \frac{1}{2\pi\alpha'}\int_\Sigma d^2z\partial X^\mu\bar{\partial}X_\mu \tag{2.2}$$

for $X^\mu(z,\bar{z})$, often written without the variable dependence when that is presented by the context, a scalar field. The equation of motion for X^μ is $\partial\bar{\partial}X^\mu=0$. One may show, as is stated in [10, Chapter 1.2], that S_P is invariant under *Weyl transformations*, or local rescalings of the worldsheet metric γ_{ab}, i.e. transformations of the form

$$\gamma'_{ab} = exp(2\omega(\tau,\sigma))\gamma_{ab}(\tau,\sigma)$$

and *diffeomorphism transformations*, or worldsheet coordinate changes:

$$\frac{\partial\sigma'^c}{\partial\sigma^a}\frac{\partial\sigma'^d}{\partial\sigma^b}\gamma'_{cd}(\tau',\sigma') = \gamma_{ab}(\tau,\sigma),$$

where in the cases of both diff and Weyl transformations, $X^\mu(\tau,\sigma)$ does not change.

By studying these theories, in particular their symmetries, associated Noether conserved currents, and associated vertex operator algebras, we will gain much insight into the NSR superstring.

There are at least three ways to describe the superstring action: Neveu-Schwarz Ramond, Green-Schwarz, and Berkovits Pure-Spinor. In this exposition, we will be looking at the formermost. Without BRST-ghosts but with BV-antifields, this action is governed by the worldsheet action, Equation 10.1.5 of [11]:

$$S_{NSR} = \frac{1}{2\pi\alpha'}\int_\Sigma dzd\bar{z}\partial X^\mu\bar{\partial}X_\mu + \psi^\mu\bar{\partial}\psi_\mu + \tilde{\psi}^\mu\partial\tilde{\psi}_\mu$$

where the field content is as follows:

- X^μ is the sigma model map, i.e. a smooth map from the worldsheet to the target space.
- ψ^μ is a tensor of weight $(\frac{1}{2}, 0)$, and
- $\tilde{\psi}^\mu$ is a tensor of weight $(0, \frac{1}{2})$.

The fields with tildes throughout this exposition will denote the *antifields* of a theory. They arise as a consequence of the BV formalism, as will be explained in the subsequent chapter.

Formally, each summand of this action with the exception of the first one, which is the Polyakov action on a flat worldsheet (this free bosonic theory is essentially the subject of [10], and we direct the reader there for further information), (2.2), resembles a bc or $\beta\gamma$ model, so before delving into S_{NSR}, so our strategy will be to explore some properties of those free conformal field theories and then relate those findings to the NSR worldsheet superstring.

2.2 Physics Constructions

In the following sections, we will use some standard tricks, concepts, and computational tools familiar to many practitioners of physics. To promote clarity in that explanation, we introduce those here.

Most of these constructions are related by that all of the physical theories we consider here have symmetry groups. By varying the action by the symmetry transformations generating these groups, we may, by way of the *Noether procedure*, obtain the corresponding (conserved) currents. Alternatively, one can perform the *BRST procedure*, encoding some of these symmetries as additional fields, *ghost fields*, of the theory. The currents resulting from the Noether procedure, however, are themselves often informative, as the coefficients of their terms reveal information about the theory, such as its *central charge*. This central charge, which can be shown to commute with all operators in the theory, is a representative of the center of the symmetry group of the theory, and thus partially specifies the *vertex operator algebra* of the theory.

2.2.1 The Noether Procedure

Noether's theorem associates to any local and continuous symmetry a conserved current, which can be integrated to find a corresponding charge. Through the standard derivation of this theorem, one can read off what that current is explicitly for a given symmetry.

The Noether procedure works via the following steps:

1. Find a small local parameter (this will be denoted by ϵ here) and multiply this by a chosen field-wise symmetry transformation of the theory.

2. Vary the action with respect to ϵ: $S \mapsto \delta S$.

3. Insert all linear-order-in-ϵ behavior of the field a as δa in δS; integrate by parts to rearrange as necessary.

4. Note that if the transformation of a induced by the variation is a symmetry, the variation of the action will vanish, indicating that the term multiplying the $\bar{\partial}\epsilon$ term(s) in the action must also be equal to zero (while infinitesimal, ϵ is not itself 0).

5. Read off the term that must be 0 from 4. as the current corresponding to the symmetry.

In this procedure, it was not necessary to start with a symmetry of the action, but only by using that δS vanishes (by virtue of the transformation being a symmetry) is it possible to so read off a corresponding conserved current.

2.2.2 The BRST Procedure - Inverting the Faddeev-Popov Determinant

Given a field theory with gauge symmetries, it is in some sense possible to encode these symmetries as extra fields, *ghost fields*, whose behavior is described by a ghost action. The original action without gauge symmetries summed with the ghost action will describe the same theory as the original action replete with gauge symmetries. The formalism used to convert these symmetries into fields is called the *Becchi–Rouet–Stora–Tyutin (BRST) procedure*, and is the BRST portion of the method alluded to in the introduction. While we will explore this formalism mathematically to a further extent in the next chapter, we will provide here a way of explicitly computing the ghost fields given a classical gauge theory.

The BRST formalism is a way of discerning which fields in a space are physical and which encode symmetries, and then describing the latter as *ghosts* via a prescribed procedure well detailed in [10, Section 5.3].

In a field theory with a symmetry group, it is possible that multiple fields will be in the same orbit of the symmetry group, and in that sense equivalent. In the case of a gauge theory, this is what is known as "gauge equivalence". The physical fields of a theory are only those that are not directly related by symmetry transformation, i.e. in such a theory with symmetry group, a physical field is a representative of its equivalence class under the action of the symmetry group. The BRST procedure offers a way to identify each generator of this symmetry group with an extra field in the theory, expanding the field space, but reducing the redundancy caused by the identification of fields in the same orbit under the symmetry group's action.

In this section, we will mention some points of the BRST formalism, but refer to the source for more specifics. The mathematical interpretation and implementation of BRST will be further elaborated upon in the next chapter.

The path integrals corresponding to actions with any sorts of gauge symmetries are in a sense "ill-defined" because they sum over several physically-equivalent configurations, with this redundancy emanating from that two fields are considered equivalent, i.e. they describe the same physics, if they differ by a gauge transformation. One way to deal with this overcounting is by somehow summing only over the inequivalent configurations. Some intuition as to how to enact this is as follows:

1. Consider gauge-equivalent states as members of the same orbit of the gauge group. Then take a "slice" of the gauge group action that intersects each orbit only once. This is mathematically implemented via delta functions: for each gauge transformation η (and thus each orbit of the gauge group), insert a factor $\delta(p-\hat{p}^\eta)$ into the path-integral integrand for p some parameter (in bosonic string theory, p is usually the worldsheet metric g) and $\hat{p}^\eta$ a value for p gauge fixed using the gauge symmetry η (note that here $\hat{p}^\eta$ is some particular value: while we will range over η when considering all such

gauge-fixed p, in each δ only one single value of η is considered). It only will count the gauge orbit once, as the delta function vanishes unless $p = p^\eta$. One could then apply this to each gauge orbit and sum over the thereby constrained orbits to describe the slice.

2. When changing point of view to just look at this "slice" with respect to p there will be some sort of Jacobian. This is defined as

$$\Delta_{FP}[p] := \frac{1}{\int \mathscr{D}p\delta(p - \hat{p}^\eta)}$$

where integration is over the moduli space of p (in practice this could be, for instance, the moduli space of worldsheet metrics) and the measure of integration, $\mathscr{D}p$, is left- and right-invariant under the action of the gauge group. Δ_{FP} is called the *Faddeev-Popov determinant*, and it is this that will serve as the measure just on the slice, and will end up returning the ghosts.

3. To calculate Δ_{FP}, one uses the following trick, based on the definition of Δ_{FP} from 2. Since

$$1 = \Delta_{FP} \int \mathscr{D}p\delta(p - \hat{p}^\eta),$$

one may insert this right-hand-side quantity multiplicatively into any path integral without any immediate consequence. This procedure is local, so expanding p in a small neighborhood allows for the calculation of Δ_{FP}^{-1} perturbatively. In terms of conformal symmetries and the Polyakov action, (2.1), with $p = g_{\alpha\beta}$, the worldsheet metric, this looks like

$$g_{\alpha,\beta} \mapsto \delta g_{\alpha\beta} = 2\delta\omega g_{\alpha,\beta} - \nabla_\alpha \sigma_\beta - \nabla_\beta \sigma_\alpha$$

for ω parametrizing a Weyl transformation and σ a diffeomorphism transformation. We may rewrite this transformation as [10, Equation 3.3.16].

$$\delta g_{\alpha,\beta} = (2\delta\omega - \nabla_\gamma \delta\sigma^\gamma) g_{\alpha\beta} - 2(P\delta\sigma)_{\alpha\beta}$$

where ([10, Equation 3.3.17]) P is the following operator from vectors to symmetric, traceless 2−tensors,

$$(P\delta\sigma)_{\alpha\beta} = \frac{1}{2}\Bigg(\nabla_\alpha \delta\sigma_\beta + \nabla_\beta \delta\sigma_\alpha - g_{\alpha\beta}\nabla_\gamma \delta\sigma^\gamma\Bigg).$$

This allows one calculate that the Jacobian inverse of the Faddeev-Popov determinant is

$$\Delta^{-1}_{FP}[\hat{g}] = \int [d\beta' d\delta\sigma] exp\left(4\pi i \int d^2\sigma \hat{g}^{\frac{1}{2}} \beta'^{\alpha\beta}(\hat{P}\delta\sigma)_{\alpha\beta}\right),$$

which is [10, Equation 3.3.18] for β a symmetric traceless 2−tensor.

4. The Jacobian for the coordinate transformation has been found, and it remains to find the Faddeev-Popov determinant, which is by definition the inverse of this Jacobian. One way to invert a path integral is to switch the even variables with Grassmannian ones. In the example of 3., the coordinate transformations are

$$\beta'_{\alpha\beta} \leftrightarrow b_{\alpha\beta}$$

and

$$\delta\sigma^\gamma \leftrightarrow c^\gamma$$

so the Faddeev-Popov determinant can be written as

$$\begin{aligned}\Delta_{FP}[g] &= \int [dbdc] exp\left(4\pi i \int d^2\sigma \hat{g}^{\frac{1}{2}} b^{\alpha\beta}(\hat{P}c)_{\alpha\beta}\right) \\ &= \int [dbdc] exp[-S_{gh}],\end{aligned}$$

where S_{gh} is the ghost action. In summary, by avoiding overcounting via looking only at the "physical" modes, through integrating over one point only in each gauge orbit, one implements a change of frame, requiring a coordinate-change Jacobian. The inverse of this quantity, necessary in the calculation of the Jacobian (from the insertion of 1), causes the appearance of a ghost fields (arising from the inversion of the path integral by replacing even variables with Grassmann ones). In the case of the bosonic string, and indeed that of the superstring, the ghost action wrapping them is the bc action, or some linear combination of $bc-$action factors, to be elaborated upon in the forthcoming.

2.2.3 Vertex (Operator) Algebras and Operator Product Expansions

The objective of an operator product expansion (OPE) is to explain the behavior of two operators (fields) in the vicinity of one another over a complex worldsheet. Computationally, this consists of expanding the operators around some point between them, in what amounts to a Laurent expansion. Since these expansions are formal power series, the OPE is valid both in the perturbative and non-perturbative regimes, though only the former will be considered in this document. Given a two-dimensional CFT [1], there are multiple common axiomatic approaches to the OPE. One is to derive it from the vertex operator algebra of the theory, and another is to take it as an axiom of the theory. This latter option is frequently what is done in practice, though the formalism for the former does exist in several cases. In this section, we will introduce vertex operator algebras following [2] and explain from the definition how they encode the state-operator correspondence and also induce the operator product expansion. To begin, we overload the term "field".

Definition 2.2.1. *[2, Definition 5.0.1.1] For V a vector space, a formal power series $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n} \in EndV[[z, z^{-1}]]$ is called a* field *if for each $v \in V$, there exists some $N \in \mathbb{N}$ such that $a_i v = 0$ for all $i > N$.*

The notion of field defined directly above does not usually correspond with that of classical field theory, where the fields are often sections of some bundle(s) over the worldsheet or target space of the theory. As explained in [2], one way of looking at fields in this context (i.e. that of vertex operator algebras) is as observables supported at points, or as operators.

Definition 2.2.2. *[2, Definition 5.0.1.2] A* vertex algebra *consists of the following data*

- *The* state space*, a $\mathbb{C}-$vector space V*
- *a* vacuum vector*, a nonzero vector $|0\rangle \in V$;*
- *a* shift operator*, a linear map $T : V \to V$; and*

[1] The machinery of the OPE exists in some cases for higher-dimensional conformal field theories, but the relationship between the OPE and the vertex operator algebra is not yet as clear there.

- *the* vertex operation*, a linear map* $Y(-,z) : V \to EndV[[z,z^{-1}]]$ *associating to each vector a field.*

satisfying the following axioms.

- *The* vacuum axiom*:* $Y(|0\rangle, z) = id_V$ *and* $Y(v,z)|0\rangle \in v + zV[[z]]$ *for all* $v \in V$*.*

- *The* translation axiom*:* $[T, Y(v,z)] = \partial_z Y(v,z)$ *for each* $v \in V$ *and* $T|0\rangle = 0$*.*

- *The* locality axiom*: For any pair of vectors* $v_1, v_2 \in V$*, there exists* $N \in \mathbb{N} \cup \{0\}$ *such that* $(z-w)^N [Y(v_1,z), Y(v_2,w)] = 0$ *as elements of* $EndV[[z^{\pm 1}, w^{\pm 1}]]$*.*

Let us clarify what some of these requirements and axioms mean and imply in the world of conformal field theories. The *state space* is taken to be the (Hilbert) space of (physical) states associated to the theory. It would also be possible to view the non-physical states in this way. The *vacuum vector* corresponds to the vacuum state, or the lowest-energy physical state possible for the theory (the bottom rung of the ladder for the classical harmonic oscillator, or $|0\rangle$, for instance). The *shift operator* is a technical tool, allowing navigation between states within the state space. The *vertex operation* assigns a field to each vector. Now, if one views these fields as operators, this can be interpreted as an encoding of a state-operator correspondence, which is a principle of field theory stating that it is possible to describe all physical states of a theory in terms of local operators, such as the power series living in $EndV[[z,z^{-1}]]$. Frequently, computations can take place either in the state space or the operator space, so this vertex operation offers a practical convenience in providing a way to navigate between those spaces. It is the axioms that give us a way to calculate the operator product expansion for a particular theory. We will give a definition of the operator product expansion and then explain how it is related to these three axiomata.

Definition 2.2.3. *Let* z*,* z_1*,* $z_2 \in \mathbb{C}$ *be points on the complex plane,* $U \backslash \{0\}$ *be an open neighborhood of* z_1*, and* F_i*,* $F_j \in EndV[[z,z^{-1}]]$ *fields in the context of vertex algebras. Then, the* operator product expansion*, locally, inside of* U*, is the sum*

$$F_i(z_1)F_j(z_2) := \sum_{j \in \mathbb{Z}} f_{ij}^k (z_1 - z_2) F_k(z_2) \tag{2.3}$$

for f_k *analytic functions and* F_k *an operator determined by* F_i *and* F_j*.*

This is a local expression depicting what happens when two fields approach each other in a small neighborhood. One may show that while 2.3 is convergent for all CFTs, there are many other field theories with OPEs that have infinite radii of convergence. Notice that the OPE is of the form of a sum of terms like $Y(v, z)$ from the definition of a vertex operator. The *translation axiom* will tell us how to calculate the OPE of fields that are described in terms of partial derivatives, with the partial derivative corresponding in the vertex algebra picture to a change in location in the vector space of physical states, i.e. the shift operator generates translations in z. The *locality axiom* says that the commutator of operator product expansions in this context terminates. In the body of this chapter, we will see examples of the OPE calculation, and one may also visit chapter 2 of [10] for another perspective on this material.

Computation of the OPE

Given a particular classical field theory, the calculation of the operator product expansion can be done with the assistance of the Green's functions (or *correlation functions*) of the theory. Physically the correlation functions give S-matrix elements in the form of vacuum expectation values for time-ordered products of fields. Green's functions seen mathematically, each as a in some sense piecewise solution to a differential equation, found by setting that equation equal to a delta function for each possible point and then solving pointwise, define the Feynman propagators for any fields. There is some subtlety here regarding the form of the solutions to the differential equation, since the background geometry of the space upon which the solutions are being sought can contribute.

The procedure of obtaining the operator product expansion given a particular field theory works as follows.

1. Obtain the equations of motion from the Lagrangian of the theory.

2. Use the Euler–Lagrange equations as the linear differential operator L and solve the following equation

$$LG = \delta$$

for G, the Green's function corresponding to the Euler–Lagrange equation. Do this for each equation of motion of the theory.

3. Substitute the Green's functions for their corresponding fields (i.e. the ones from whose equations of motion they are derived) in the expression to obtain OPE. If there is an additional differential operator present in the problem, apply this to the Green's function (for instance, if one has found the Green's function for c in the bc−CFT, but is faced with a term of the form $\bar{\partial}c$, apply $\bar{\partial}$ to the Green's function with which c is replaced).

4. Using some convention for normal ordering, calculate the operator product expansions.

In the subsequent sections of this chapter, we will come across instances where this procedure was used to obtain correlation functions and then the operator product expansion.

2.2.4 Central Charge and Worldsheet Energy-Momentum Tensor

If the symmetries of a classical field theory form a group, G, it may be possible to form central extensions thereof. Fix a classical field theory and a central extension. The generators of the central extension will be entities in the center of G, and correspond, via the afore-described Noether procedure, to a conserved current. The integral of this current returns a "charge", the *central charge* of the theory. In practice, such a current is the operator product expansion of the worldsheet stress-energy tensor of the theory with itself, and the central charge will generally be the coefficient of the $\frac{1}{z^2}$ term in that expansion.

2.3 The bc and $\beta\gamma$ CFTs

Now that we have working apparatus for exploring physical theories, we will examine the bc and $\beta\gamma$ conformal field theories over a Riemann surface worldsheet without boundary Σ that are integral to the superstring action. The bc and $\beta\gamma$ field theories are formally nearly alike, differing in the commutativity of the fields: b and c anticommute, whereas β and γ commute. Due to these extreme similarities, we will work through the details for the bc

theory and make mentions along the way of how the analogous results look for the $\beta\gamma$ one. We essentially follow the presentation in [10, Section 2.5].

Let $(z, \bar{z})$ be (anti)holomorphic coordinates on Σ. The fields of the $bc-$theory are anticommuting tensor fields $b(z, \bar{z})$ and $c(z, \bar{z})$ (frequently just written as b and c, with the variable dependence assumed) of weights $(\lambda, 0)$ and $(1-\lambda, 0)$ respectively. These tensors are "fields" in the context of classical field theory, and when we henceforward refer to fields, we will mean the term in this sense unless otherwise specified. The action of the bc theory is

$$S = \frac{1}{2\pi\alpha'} \int_{\Sigma} dz d\bar{z} b\bar{\partial}c$$

An anti-version of this can be developed, but it is formally the same as this theory, with anticommuting tensor fields $\tilde{b}(z, \bar{z})$ and $\tilde{c}(z, \bar{z})$ of weights $(0, \lambda)$ and $(0, 1-\lambda)$ respectively. In the $\tilde{b}\tilde{c}-$action, due to the weights of the fields as tensors, ∂ replaces $\bar{\partial}$. Here α' is a constant called the *string tension*, and, frequently, $dzd\bar{z}$ will be denoted by d^2z. In this section we will focus on the bc theory since under the analogy just above, computations in $\tilde{b}\tilde{c}$ CFT follow using the same methods. We first vary the action and set that variation to zero to obtain the equations of motion for b and c.

$$0 = \delta S = \frac{1}{2\pi\alpha'} \int_{\Sigma} d^2z(\underbrace{\delta b\bar{\partial}c}_{A} + \underbrace{b\bar{\partial}\delta c}_{B}) \tag{2.4}$$

Now integrating summand B by parts (with respect to $\bar{\partial}$):

$$\begin{aligned} B &= \frac{1}{2\pi\alpha'}\left(\int_{\partial\Sigma} d^2z b\delta c - \int_{\Sigma} d^2z\delta c\bar{\partial}b\right) \\ &= \frac{1}{2\pi\alpha'}\int_{\Sigma} d^2z\delta c\bar{\partial}b \end{aligned}$$

since Σ has no boundary and b and c anticommute. Looking at A and what remains of B together then gives

$$\delta S = \int_{\Sigma} d^2z(\delta b\bar{\partial}c + \delta c\bar{\partial}b)$$

Inside the integral composing δS there are two variations such that

$$\delta S = \frac{\delta S}{\delta b}\delta b + \frac{\delta S}{\delta c}\delta c.$$

Since the variation was set to zero at the outset, this means that both $\frac{\delta S}{\delta b}$ and $\frac{\delta S}{\delta c}$ must be equal to zero, and the means of achieving this gives the equations of motion:

$$\bar{\partial} b = 0$$

and

$$\bar{\partial} c = 0.$$

With the action and equations of motion in hand, a next step is to calculate the energy-momentum tensor(s) T_B (and $\tilde{T}_B$, though we will only actually calculate the former, and appeal again to analogy to obtain the latter) and central charge(s) $\aleph$ (and $\tilde{\aleph}$) of the bc theory. We will continue only to work with the bc theory, but the arguments all apply for the $\tilde{b}\tilde{c}$ theory, as well.

To calculate the energy-momentum tensor of the bc theory, we use the Noether procedure with conformal transformations. Our first step is to ascertain how b and c transform under a conformal transformation, $z \mapsto z + \epsilon(z)$ for $\epsilon(z)$ holomorphic and small:

$$b(z) \mapsto \left(1 + \frac{\partial \epsilon}{\partial z}\right)^{\lambda} b(z + \epsilon(z))$$

$$= b(z) + \lambda \partial \epsilon(z) b(z) + \epsilon(z) \partial(b(z)) + \mathscr{O}((\epsilon(z))^2),$$

where we have only written the $z-$dependence of $b(z, \bar{z})$ since it is there that this coordinate transfomation is implemented. Since ϵ is an infinitesimal parameter, the terms in the above transformation above linear order shall be ignored. The transformation of $c(z, \bar{z})$ is up to this order the same, except for that the multiplicative factor of λ is replaced by one of $(1 - \lambda)$, reflecting the (tensorial) weight of the field c.

In this case, the current from using the conformal transformations of b and c as indicated above will be the energy-momentum tensor, T_B, of the bc CFT. In the language of the introductory sections on the OPE and energy-momentum tensor, calculating this current and taking the operator product of it with itself will describe the central extension of the symmetry group generated by the above conformal transformation. Recall Equation (2.4), the variation of the bc action. Now, varying with respect to conformal transformations, we obtain

$$\delta S = \frac{1}{2\pi\alpha'}\int_\Sigma d^2z([\lambda\partial\epsilon(z)b(z) + \epsilon(z)\partial b(z)]\bar{\partial}c(z) + b\bar{\partial}[(1-\lambda)\partial\epsilon(z)c(z) + \epsilon(z)\partial c(z)])$$
$$= \frac{1}{2\pi\alpha'}\int_\Sigma d^2z(1-\lambda)b(z)\bar{\partial}\partial\epsilon(z)c(z) + b(z)\bar{\partial}\epsilon(z)\partial c(z).$$

Integrating by parts again, this time in terms of ∂, and only on the first term delivers

$$= \frac{1}{2\pi\alpha'}\left((1-\lambda)\left(\int_{\partial\Sigma} d^2zb(z)c(z)\bar{\partial}\epsilon(z) - \int_\Sigma d^2z(\bar{\partial}\epsilon(z)\partial(b(z)c(z))\right) + \int_\Sigma d^2zb(z)\bar{\partial}\epsilon(z)\partial c(z)\right)$$

where again the boundary integral vanishes since M is assumed to have no boundary. Now arranging the material inside the integrals by factoring out $\bar{\partial}\epsilon(z)$:

$$\delta S = \frac{1}{2\pi\alpha'}\left(\int_\Sigma d^2z((1-\lambda)\bar{\partial}\epsilon\partial(bc) + b\bar{\partial}\epsilon\partial c)\right)$$
$$= \frac{1}{2\pi\alpha'}\left(\int_\Sigma d^2z\bar{\partial}\epsilon((-(1-\lambda)(\partial(bc))) + b\partial c)\right)$$

The sought-after current is then given by the terms multiplying $\bar{\partial}\epsilon$:

$$\begin{aligned}(-(1-\lambda)(\partial(bc))) + b\partial c &= (\lambda-1)(\partial(bc) + b\partial c \\ &= (\lambda-1)(\partial bc + b\partial c) + b\partial c \\ &= (\lambda-1)(\partial bc) + \lambda(b\partial c).\end{aligned}$$

In free field theories, normal ordering conventions are established to determine how products of operators are computed, and to agree with what is often seen in the literature (see for example the :: normal ordering of [10]), one may insert normal ordering symbols (here reflecting that same :: normal ordering) into the above expression to obtain

$$T(z) = -(1-\lambda){:}\partial bc{:} + \lambda{:}b\partial c{:}. \tag{2.5}$$

One may also calculate the "antiholomorphic" energy-momentum tensor, $\tilde{T}_B$, but in the case of a theory such as this one where all of the variations take place on the $z-$coordinate, all terms vanish, leaving

$$\tilde{T}_B(\bar{z}) = 0.$$

Notice that if the variations of b and c were functions of $\bar{z}$ and not of z, the antiholomorphic $\tilde{T}_B$ would have the form the holomorphic one does here and vice versa. This is important for the anti-fields later to be introduced: they will essentially be the antiholomorphic versions of the fields to which they correspond.

By taking the operator product expansion of this energy-momentum tensor with itself, one may obtain the central charge of the bc theory. Here this calculation will be demonstrated for the bc theory, still with unspecified weights, and in the next section the result, calculated in the same manner, will be documented for the commuting $\beta\gamma$ theory. First to note is that the energy-momentum tensor from before can be rewritten as follows:

$$\begin{aligned} T(z) &= -(1-\lambda):\partial bc: + \lambda :b\partial c: \\ &= -:\partial bc: + \lambda :\partial bc: + \lambda :b\partial c: \\ &= \lambda\partial :bc: - :\partial bc: \end{aligned}$$

This agrees with [10, Equation 2.5.11a] up to a sign. Now calculating the OPE:

$$\begin{aligned} T(z_1)T(z_2) &= (\lambda\partial :b(z_1)c(z_1): - :\partial b(z_1)c(z_1):)(\lambda\partial :b(z_2)c(z_2): - :\partial b(z_2)c(z_2):) \\ &= \underbrace{\lambda^2\partial :b(z_1)c(z_1):\partial :b(z_2)c(z_2):}_{A} - \underbrace{\lambda\partial :b(z_1)c(z_1):\partial b(z_2)c(z_2):}_{B} \\ &\quad - \underbrace{\lambda :\partial b(z_1)c(z_1):\partial :b(z_2)c(z_2):}_{C} + \underbrace{:\partial b(z_1)c(z_1):\partial b(z_2)c(z_2):}_{D} \end{aligned}$$

Dealing with each term individually, and where $\sim$ denotes "has terms that are the most singular proportional to" let

$$\begin{aligned} A &= \lambda^2\partial_{z_1}\partial_{z_2}(b(z_1)c(z_1)b(z_2)c(z_2)) \sim \partial_{z_1}\partial_{z_2}\left(\frac{\lambda^2}{(z_1-z_2)^2}\right) \sim \frac{-6\lambda^2}{(z_1-z_2)^4} \\ B &= -\lambda(\partial(b(z_1)c(z_1))\partial b(z_2)c(z_2)) \sim -\lambda\partial_{z_1}\left(\left(\partial_{z_2}\left(\frac{1}{z_1-z_2}\right)\right)\left(\frac{1}{z_1-z_2}\right)\right) \sim \frac{3\lambda}{(z_1-z_2)^4} \\ C &= -\lambda(\partial b(z_1)c(z_1)\partial(b(z_2)c(z_2))) \sim -\lambda\partial_{z_2}\left(\left(\partial_{z_1}\left(\frac{1}{z_1-z_2}\right)\right)\left(\frac{1}{z_1-z_2}\right)\right) \sim \frac{3\lambda}{(z_1-z_2)^4} \\ D &= \partial_{z_1}b(z_1)c(z_1)\partial_{z_2}b(z_2)c(z_2) \sim \partial_{z_1}\left(\frac{1}{z_1-z_2}\right)\partial_{z_2}\left(\frac{1}{z_1-z_2}\right) \sim \frac{-1}{(z_1-z_2)^4}, \end{aligned}$$

where we note that the singularities in these OPEs come from the replacement of b and c by their corresponding Green's functions, as described in the prior section on computing the OPE. Combining numerators of A, B, C, and D, we obtain, where the central charge, $\aleph$ is the coefficient of the $\frac{1}{2z^4}$ term in this case,

$$\aleph = 2(-6\lambda^2 + 6\lambda - 1) = -12\lambda^2 + 12\lambda - 6.$$

Similarly, since $T(\bar{z}) = 0$, the corresponding central charge is

$$\tilde{\aleph} = 0.$$

This calculation looks only at singular behavior, so the results for $\aleph$ and $\tilde{\aleph}$ would be reversed in the case that the variation was on the antiholomorphic coordinate.

Symmetries of the bc system

The bc system is conformally invariant, hence the name "$bc-$CFT", as well as invariant under a symmetry colloquially known as *ghost number symmetry*, which will be significant in the BRST formalism, as it tracks the degree of a function in the enlarged phase space coming from the turning of symmetries into ghost fields via the BRST procedure. The worldsheet models for the superstring partially consist of bc and $\beta\gamma$ systems of different particular weights, emphasizing the importance and convenience of these symmetries to the particular case in focus. The number symmetry has not yet been explored, but it is generated by the following transformations:

$$b(z,\bar{z}) \mapsto -i\epsilon(z)b(z,\bar{z})$$
$$c(z,\bar{z}) \mapsto i\epsilon(z)c(z,\bar{z})$$

for $\epsilon(z)$ an infinitesimal holomorphic function. Then, one shows via varying that the bc action is invariant under this symmetry. Indeed, we have

$$\begin{aligned}\delta(b\bar{\partial}c) &= \delta b\bar{\partial}c + b\bar{\partial}\delta c\\ &= (-i\epsilon b)\bar{\partial}c + b\bar{\partial}(i\epsilon c)\\ &= -i\epsilon b\bar{\partial}c + ib\bar{\partial}\epsilon c + ib\bar{\partial}c\epsilon\\ &= 0,\end{aligned}$$

with the last equality following from the holomorphicity of ϵ.

Since this is a symmetry under which the action is invariant, there corresponds to it, by Noether's theorem, a conserved current described by

$$j^\mu(c) = \frac{\partial L}{\partial c}(i\epsilon c) = bi\epsilon c$$

and

$$j^\mu(b) = \frac{\partial L}{\partial b}(-i\epsilon b) = 0.$$

Summary of the $\beta\gamma$ Theory

As mentioned thus far, the $\beta\gamma$ theory is set up in the same way as the bc theory, except for that β and γ are communting fields with weights $h_\beta = h_b = (\lambda, 0)$ and $h_\gamma = h_c = (1-\lambda, 0)$. The $\beta\gamma$ action is

$$S_{\beta\gamma} = \frac{1}{2\pi\alpha'} \int_\Sigma d^2 z \beta\bar{\partial}\gamma,$$

and it is invariant under conformal and ghost number symmetries of the form of those above, with b and β, likewise c and γ substituted for one another. It has a corresponding antiholomorphic theory, where the fields are $\bar{\beta}(z,\bar{z})$ and $\bar{\gamma}(z,\bar{z})$, which are tensors of weights $h_{\bar{\beta}} = h_{\bar{b}} = (0, \lambda)$ and $h_{\bar{\gamma}} = h_{\bar{c}} = (0, 1-\lambda)$, respectively. The energy-momentum tensor is again formally identical to that of the bc theory:

$$T(z) = \lambda\partial : \beta\gamma : - : \partial\beta\gamma :$$

$$T(\bar{z}) = 0$$

where for the anti-$\beta\gamma$ theory, $T(\bar{z}) = \tilde{T}$ is of the form of $T(z)$ of this theory, and vice versa. Since the central charge is calculated by the TT OPE, it is

$$\aleph = 12\lambda^2 - 12\lambda + 1$$

$$\tilde{\aleph} = 0$$

and vice versa for the anti-theory, with the difference in sign for these theories coming from the commutativity of the fields: instead of as in the bc theory, where

$$b(z_1)c(z_2) \sim \frac{1}{z_1 - z_2}$$

and

$$c(z_1)b(z_2) \sim \frac{1}{z_1 - z_2},$$

in the $\beta\gamma$ theory, due to the commutativity of the fields, switching b with c and z_1 with z_2 generates only a single minus sign (from $z_1 \leftrightarrow z_2$) instead of two, making the products resemble

$$\beta(z_1)\gamma(z_2) \sim \frac{1}{z_1 - z_2}$$

and

$$\gamma(z_1)\beta(z_2) \sim -\frac{1}{z_1 - z_2}.$$

Again, the products and central charges for the anti-theory are of the same form as those of the theory.

Relation to NSR Superstring

All of the terms in the worldsheet action of the NSR superstring (including after BRST is applied), with the exception of $\partial X^\mu \bar{\partial} X_\mu$, are of the form of some theory from one of the previous two subsections. Making this correspondence explicit, we have the following formal similarities:

$$(\psi^\mu(z,\bar{z}), \psi^\mu(z,\bar{z}) \leftrightarrow (b(z,\bar{z}), c(z,\bar{z}))$$

with $h_b = h_c = \left(\frac{1}{2}, 0\right)$.

$$(\tilde{\psi}^\mu(z,\bar{z}), \tilde{\psi}^\mu(z,\bar{z})) \leftrightarrow (\tilde{b}(z,\bar{z}), \tilde{c}(z,\bar{z}))$$

with weights $h_{\bar{b}} = h_{\bar{c}} = \left(0, \frac{1}{2}\right)$.

There will also be BRST ghost terms in this superconformal CFT, to be explored in the next subsection. They also look formally like bc and $\beta\gamma$ CFTs, as we shall see.

2.3.1 the BC-CFT

The ghost structure of the NSR superstring will be very similar to that of an amalgamated $bc-$CFT and $\beta\gamma-$CFT. We will refer to this resulting CFT as the $BC-$CFT. Following [11, Section 10.1] we give a brief summary of this, noting that all of the computations are done in the same manner as those for the $bc-$CFT.

The action of the $BC-$CFT, [11, Equation 10.1.17] is

$$S_{BC} = \frac{1}{2\pi} \int_\Sigma d^2 z (b\bar{\partial}c + \beta\bar{\partial}\gamma)$$

where b and c are anticommuting tensors of weights $h_b = \lambda$ and $h_c = 1 - \lambda$, as before, and β and γ are commuting tensors with weights $h_\beta = \lambda - \frac{1}{2}$ and $h_\gamma = \frac{3}{2} - \lambda$. One may also calculate the energy-momentum tensors of these theories as

$$T_B = (\partial b)c - \lambda\partial(bc) + (\partial\beta)\gamma - \frac{1}{2}(2\lambda - 1)\partial(\beta\gamma)$$

and

$$T_F = -\frac{1}{2}(\partial\beta)c + \frac{2\lambda - 1}{2}\partial(\beta c) - 2b\gamma,$$

which are respectively Equations 10.1.18 and 10.1.19 of [11]. Recall that each of these currents corresponds to a symmetry under Noether's theorem. T_B will come from worldsheet conformal invariance, and T_F from a supersymmetry transformation mixing bosons and fermions (as is there indicated by the terms consisting of both commuting and anticommuting fields). There is naturally an anti-theory to this one, as was the case with the $bc-$ and $\beta\gamma-$CFTs.

2.4 The Worldsheet NSR Superstring

Recall from the introduction that the worldsheet action of the NSR superstring, [11, Equation 10.1.5], is

$$S_{NSR} = \frac{1}{2\pi\alpha'}\int_\Sigma dzd\bar{z}\,\underbrace{\partial X^\mu \bar{\partial} X_\mu}_{A} + \underbrace{\psi^\mu \bar{\partial}\psi_\mu}_{B} + \underbrace{\tilde{\psi}^\mu \partial \tilde{\psi}_\mu}_{C} \tag{2.6}$$

where the ψ terms represent bc systems with $\lambda = \frac{1}{2}$, and the $\tilde{\psi}$ terms stand for the corresponding $\tilde{b}\tilde{c}$ systems (also, therefore, with $\lambda = \frac{1}{2}$). One subtlety here worth noticing is that the antifields are already included in the action, though the ghosts are not. The antifields are a result of the BV portion of the Lagrangian BV-BRST formalisn, and will be further explained in a subsequent chapter. Here, though, it is taken for granted that the BV resolution has already been implemented. The ghost insertions will come from the BRST part of the procedure.

After BRST, a systematic way of encoding symmetries of a physical theory in terms of extra fields, the BRST *ghosts*, is performed, one obtains an action of the form

$$S_{NSR}^{BRST} = \frac{1}{2\pi\alpha'}\int_\Sigma d^2z \partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial}\psi_\mu + \tilde{\psi}^\mu \partial\tilde{\psi}_\mu + \underbrace{b\bar{\partial}c}_{D} + \underbrace{B\bar{\partial}C}_{E} + \underbrace{\tilde{B}\partial\tilde{C}}_{F}$$

where the first three summands are as in (2.6), D is the bosonic ghosts, a $bc-$CFT with $\lambda = 2$ and E and F encode the superconformal ghosts and anti-ghosts, which each comprise a $BC-$CFT with $\lambda = 2$.

The superstring action presented above is *worldsheet supersymmetric*, meaning it is invariant under particular transformations switching the bosonic and fermionic fields. We will take the currents as given and calculate the corresponding supersymmetry transformations. [11, Equation 10.1.8] states the *worldsheet supercurrents*, which will, up to a (noncommuting) parameter, end up corresponding with the sought-after symmetries. The holomorphic and antiholomorphic worldsheet supercurrents are

$$T_F(z) = i\left(\frac{2}{\alpha'}\right)^{\frac{1}{2}}\psi^\mu(z)\partial X_\mu(z)$$

$$T_{\tilde{F}}(\bar{z}) = i\left(\frac{2}{\alpha'}\right)^{\frac{1}{2}}\tilde{\psi}^\mu(\bar{z})\bar{\partial} X_\mu(\bar{z}).$$

The currents under inspection here are then [11, Equation 10.1.9]

$$j^\mu(z) = \nu(z)T_F(z)$$

and

$$\tilde{j}^\mu(\bar{z}) = \nu(\bar{z})^*\tilde{T}_{\tilde{F}}(\bar{z}).$$

By calculating the OPE of these currents with each field, one may find out how each field transforms under the symmetry corresponding to the chosen current. With this in mind,

$$T_F(z)X^\mu(0) = i\left(\frac{2}{\alpha'}\right)^{\frac{1}{2}} :\psi^\mu(z)\partial X_\mu(z) :: X^\nu(0):$$

$$= i\left(\frac{2}{\alpha'}\right)^{\frac{1}{2}}\left(\overbrace{\psi^\mu(z)X^\nu(0)} :\partial X_\mu(0): + \overbrace{\partial X_\mu(z)X^\nu(0)} :\psi^\mu(z):\right)$$

where the brackets above the fields indicate that they are being contracted (i.e. as above, all of the fields outside of the normal ordering symbols are pairwise contracted: due to this, the extra notation is not necessary unless there are more than two fields outside of the colons). We will calculate out this OPE in some detail to give an example of the above-introduced procedure.

The Green's functions associated to the contractions $X^\mu(z_1, \bar{z}_1)X^\nu(z_2, \bar{z}_2)$ are

$$X^\mu(z_1, \bar{z}_1)X^\nu(0, 0) \sim -\frac{\alpha'}{2}\eta^{\mu\nu}ln|z_1|^2$$

for $\eta^{\mu\nu}$ the worldsheet metric, and where $\sim$ denotes "the singular part of the left-hand side is proportional to".

The third step in the above procedure is to calculate the OPEs from this. Recall that they will be of the form of Equation 2.3. Then,

$$T_F(z)X^\mu(0) \sim i\left(\frac{2}{\alpha'}\right)^{\frac{1}{2}} \frac{2ln(z)}{z}\psi^\mu(z).$$

Similarly,

$$T_{\tilde{F}}(\bar{z})X^\mu(0) \sim i\left(\frac{2}{\alpha'}\right)^{\frac{1}{2}} \frac{2ln(\bar{z})}{\bar{z}}\psi^\mu(\bar{z}).$$

Next looking at the OPEs with the fermionic fields,

$$T_F(z)\psi^\mu(0) = i\left(\frac{2}{\alpha'}\right)^{\frac{1}{2}}\Big(:\psi^\mu(z)\partial X_\mu(z) :: \psi^\mu(0): \Big)$$

$$= i\left(\frac{2}{\alpha'}\right)^{\frac{1}{2}}\Big(\psi^\mu(z)\psi^\mu(0) : \partial X_\mu(z) : +\partial X_\mu(z)\psi^\mu(0) : \psi^\mu(0): \Big)$$

$$\sim i\left(\frac{2}{\alpha'}\right)^{\frac{1}{2}} \frac{\partial X^\mu(0)}{z}.$$

The anti-holomorphic fermion-to-boson transformation has the same form, i.e.

$$T_{\tilde{F}}(\bar{z})\tilde{\psi}^\mu(0) \sim i\left(\frac{2}{\alpha'}\right)^{\frac{1}{2}} \frac{\partial X^\mu(0)}{\bar{z}}\tilde{\psi}^\mu(\bar{z}).$$

Also, the holomorphic and anti-holomorphic fields do not switch, which is evidenced by that

$$T_F(z)\tilde{\psi}^\mu(0) = T_{\tilde{F}}(\bar{z})\psi^\mu(0) = 0.$$

The residues, i.e. the coefficients of the $\frac{1}{z}$ term of each OPE, describe how the field with which the worldsheet supercurrent OPE (energy-momentum tensor) was taken (here these fields are X^μ, ψ^μ, and $\tilde{\psi}^m u$) transforms . Thus,

$$X^\mu(z,\bar{z}) \mapsto i\left(\frac{2}{\alpha'}\right)^{\frac{1}{2}}\Big(2ln(z)\psi^\mu(z) + 2ln(\bar{z})\psi^\mu(\bar{z})\Big),$$

$$\psi^\mu(z) \mapsto i\left(\frac{2}{\alpha'}\right)^{\frac{1}{2}}\partial X^\mu(0),$$

and

$$\tilde{\psi}^\mu(\bar{z}) \mapsto i\left(\frac{2}{\alpha'}\right)^{\frac{1}{2}}\bar{\partial} X^\mu(0),$$

which are in accord with transformations [11, 10.1.10a-10.1.10c].

Besides this supersymmetry, the NSR worldsheet superstring is conformally invariant. The generators of conformal transformations on the worldsheet can be shown to correspond to the "bosonic" energy-momentum tensor, which is itself often referred to as a "generator of the conformal algebra". The currents T_B and T_F together generate the *superconformal algebra* of the superstring, encoding the previously-discussed worldsheet supersymmetry as well as conformal invariance. Since this algebra encapsulates the symmetries of the theory, it characterizes the physical states: they are the states that are annihilated by this algebra. As we saw earlier in this chapter, Noether's theorem allows us to calculate the conserved current associated with a symmetry of an action or vice versa, so we will hereby use this technique to calculate T_B. As per usual, $\tilde{T}_B$, which is generated by a conformal transformation on $\bar{z}$, could be computed analogously.

T_B will correspond to worldsheet translations on the bosonic variables X^μ and conformal transformations on the fermionic variables, which we note is the same symmetry that was used to calculate the $bc-$CFT energy-momentum tensor above. Here, T_B turns out to be the energy-momentum tensors for each summand in the worldsheet NSR superstring action pasted together. To make this expressly clear, we will calculate each term separately.

For A of (2.6), we let $\delta X^\mu = -\epsilon \partial X^\mu$ parametrize worldsheet translations. Varying this term in the action with respect to this, one obtains

$$\begin{aligned}\delta A &= \frac{1}{4\pi\alpha'}\int d^2z(\delta(\partial X^\mu)\bar{\partial}X_\mu + \partial X^\mu \delta\bar{\partial}X_\mu)\\ &= \frac{1}{4\pi\alpha'}\int d^2z(\underbrace{\partial(-\epsilon\partial X^\mu)\bar{\partial}X_\mu}_{A'} + \underbrace{\partial X^\mu\bar{\partial}(-\epsilon\partial X^\mu)}_{A''})\end{aligned}$$

Working with each term individually, we may re-write A' as

$$A' = \frac{1}{4\pi\alpha'}\int_\Sigma d^2z(\underbrace{-\partial\epsilon\partial X^\mu\bar{\partial}X_\mu}_{A'_1} - \underbrace{\epsilon\partial\partial X^\mu\bar{\partial}X_\mu}_{A'_2})$$

Integrating A'_1 by parts with respect to ∂, we find

$$\begin{aligned}A'_1 &= \frac{1}{4\pi\alpha'}\left(\int_{\partial\Sigma} d^2z(-\partial X^\mu\bar{\partial}X_\mu\epsilon) + \int_\Sigma d^2z\partial((\partial X^\mu)\bar{\partial}X_\mu\right)\epsilon\\ &= \frac{1}{4\pi\alpha'}\left(\int_\Sigma d^2z(\partial\partial X^\mu\bar{\partial}X^\mu + \partial X^\mu\partial\bar{\partial}X_m u\right)\epsilon\\ &= \frac{1}{4\pi\alpha'}\int_\Sigma d^2z\partial\partial X^\mu\bar{\partial}X^\mu\epsilon\end{aligned}$$

where in this integration by parts and the subsequent ones of this calculation, all integrals over the boundary $\partial\Sigma$ of Σ vanish, since Σ is taken to be without boundary. The second line here also used the equation of motion for X^μ, namely $\partial\bar{\partial}X^\mu = 0$. Now integrating A_1' by parts with respect to $\bar{\partial}$:

$$\begin{aligned}
A_1' &= \frac{1}{4\pi\alpha'}\left(\int_{\partial\Sigma} d^2z\partial\partial X^\mu \epsilon X^\mu - \int_\Sigma d^2z X^\mu \bar{\partial}(\partial\partial X^\mu \epsilon)\right) \\
&= -\frac{1}{4\pi\alpha'}\int_\Sigma (d^2z X^\mu \bar{\partial}\partial\partial X^\mu \epsilon + X^\mu \partial\partial X^\mu \bar{\partial}\epsilon) \\
&= -\frac{1}{4\pi\alpha'}\int_\Sigma d^2z X^\mu \partial\partial X^\mu \bar{\partial}\epsilon
\end{aligned}$$

where again the X^μ equation of motion was used. The term multiplying $\bar{\partial}\epsilon$ will be A_1's contribution to the energy-momentum tensor here.

Integrating A_2' by parts with respect to $\bar{\partial}$ we obtain

$$\begin{aligned}
A_2' &= -\frac{1}{4\pi\alpha'}\left(\int_{\partial\Sigma} d^2z\epsilon\partial\partial X^\mu X_\mu - \int_\Sigma d^2z X^\mu \bar{\partial}(\epsilon\partial\partial X^\mu)\right) \\
&= \frac{1}{4\pi\alpha'}\int_\Sigma d^2z(X_\mu \bar{\partial}\epsilon\partial\partial X^\mu + X^\mu \epsilon\bar{\partial}\partial\partial X^\mu) \\
&= \frac{1}{4\pi\alpha'}\int_\Sigma d^2z X_\mu \bar{\partial}\epsilon\partial\partial X^\mu.
\end{aligned}$$

Since all the fields in A commute, the contributions to the current from A_1' and A_2' cancel. Now looking at A'', calculate

$$\begin{aligned}
A'' &= -\frac{1}{4\pi\alpha'}\int_\Sigma (\partial X^\mu \bar{\partial}\epsilon\partial X_\mu + \partial X^\mu \epsilon\bar{\partial}\partial X_\mu) \\
&= -\frac{1}{4\pi\alpha'}\int_\Sigma \partial X^\mu \bar{\partial}\epsilon\partial X_\mu.
\end{aligned}$$

This indicates that the entirety of the contribution to T_B from the bosonic part of the NSR worldsheet superstring is $\frac{1}{4\pi\alpha'}\partial X^\mu \partial X_\mu$ and comes from A''.

Term B contributes to T_B, but C will not, since though it is formally the same as B, the anti-fields $\tilde{\psi}^\mu$ are present in lieu of the fields ψ^μ. This means that the only remaining contribution to T_B should come from B. We have, however, already calculated the worldsheet energy-momentum tensor for the $bc-$CFT, and according to [10, Chapter 2.5] and [11, Chapter 10.1], the ψ, $\bar{\psi}$ CFT present in (2.6) is a $bc-$CFT with $\lambda = \frac{1}{2}$, and ψ playing the role of b and $\tilde{\psi}$ that of c. This CFT can be split into two pieces preserving the conformal invariance such that [10, Equation 2.5.18a]

$$\psi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2)$$
$$\tilde{\psi} = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2).$$

Starting from (2.5), we may, using the substitutions just suggested, obtain the energy-momentum tensor for the ψ^μ, and analogously, for the $\tilde{\psi}^\mu$ summands in worldsheet NSR superstring action. Rewriting $T(z)$ in terms of this substitution data, we obtain

$$\begin{aligned} T(z) &= (1-\lambda)\partial\psi^\mu \partial\tilde{\psi}^\mu - \lambda\psi^\mu\tilde{\psi}^\mu \\ &= \frac{1}{2}\Big(\underbrace{\partial\psi^\mu\tilde{\psi}^\mu}_{A} - \underbrace{\psi^\mu\partial\tilde{\psi}^\mu}_{B}\Big) \end{aligned}$$

Treating A and B separately, we proceed.

$$\begin{aligned} A &= \frac{1}{2}\left(\partial\left(\frac{1}{\sqrt{2}}(\psi_1 + i\psi_2)\right)\frac{1}{\sqrt{2}}(\psi_1 - i\psi_2)\right) \\ &= \frac{1}{4}(\partial\psi_1 + i\partial\psi_2)(\psi_1 - i\psi_2) \\ &= \frac{1}{4}(\partial\psi_1\psi_1 - i\partial\psi_2\psi_2 + i\partial\psi_2\psi_1 + \partial\psi_2\psi_2). \end{aligned}$$

Similarly, for B,

$$\begin{aligned} B &= -\frac{1}{4}(\psi_1 + i\psi_2)\partial(\psi_1 - i\psi_2) \\ &= -\frac{1}{4}(\psi_1 + i\psi_2)(\partial\psi_1 - i\partial\psi_2) \\ &= -\frac{1}{4}(\psi_1\partial\psi_1 - i\psi_1\partial\psi_2 + i\psi_2\partial\psi_1 + \psi_2\partial\psi_2). \end{aligned}$$

Summing these terms and noting that the fields ψ_1 and ψ_2 anticommute, we obtain

$$\begin{aligned} A + B &= \frac{1}{4}(\partial\psi_1\psi_1 - i\partial\psi_2\psi_2 + i\partial\psi_2\psi_1 + \partial\psi_2\psi_2) - (\psi_1\partial\psi_1 - i\psi_1\partial\psi_2 + i\psi_2\partial\psi_1 + \psi_2\partial\psi_2) \\ &= \frac{1}{4}(-\psi_1\partial\psi_1 + i\psi_2\partial\psi_2 - i\psi_1\partial\psi_2 - \psi_2\partial\psi_2 - \psi_1\partial\psi_1 + i\psi_1\partial\psi_2 - i\psi_2\partial\psi_1 - \psi_2\partial\psi_2) \\ &= \frac{1}{4}(-2\psi_1\partial\psi_1 - 2\psi_2\partial\psi_2) \\ &= -\frac{1}{2}(\psi_1\partial\psi_1 + \psi_2\partial\psi_2) \\ &= -\frac{1}{2}(\psi^\mu\partial\psi_\mu) = T_\psi(z, \bar{z}) \end{aligned}$$

agreeing with [10, Equation 2.5.18c] as expected. The contribution of term C, encoding the role of the fermions in the theory, to the antiholomorphic energy-momentum tensor of the worldsheet NSR superstring is calculated analogously from the standard $bc-$CFT energy-momentum tensor and is

$$T_{\tilde{\psi}} = -\frac{1}{2}(\tilde{\psi}^{\mu}\partial\tilde{\psi}_{\mu}).$$

Thus, since $T_B = T_X + T_{\psi}$,

$$T_B = \frac{1}{4\pi\alpha'}(\partial X^{\mu}\partial X^{\mu}) - \frac{1}{2}(\psi^{\mu}\partial\psi_{\mu}),$$

and, in the same manner, since $\tilde{T}_B = T_X + T_{\tilde{\psi}}$,

$$\tilde{T}_B = \frac{1}{4\pi\alpha'}(\partial X^{\mu}\partial X^{\mu}) - \frac{1}{2}(\tilde{\psi}^{\mu}\partial\tilde{\psi}_{\mu}).$$

One may, using the Noether procedure with the appropriate symmetries, calculate any currents corresponding to symmetries, including the BRST current.

Chapter 3

The BV-BRST Formalism

3.1 Introduction

In the previous chapter, we saw a sketch of how the BRST procedure, a method of viewing the symmetries of a theory by adding terms replete with symmetry-encoding *ghost fields* to the action, can be implemented. We also assumed that all of the fermionic fields in the theories we were examining had anti-holomorphic partners, the *antifields*, which arise from the Batalin–Vilkovisky (henceforth BV) procedure. In this chapter, we will outline a mathematical formalism for the combined BV-BRST procedure.

Frequently when presented with a gauge theory or any physical theory with some collection of symmetries, it is informative to analyze the behavior of the fields (and hence observables) modulo the action of these symmetries. An example to keep in mind is spacetime translation and rotation invariance (invariance under the action of the restricted Lorentz group): essentially the behavior of a theory invariant under these symmetries will be the same no matter when and where it is located in spacetime. In order to look at the fields up to gauge invariance, mathematically one needs to take a quotient of the field space by the symmetry group(oid), or infinitesimally, its corresponding Lie algebra(/oid). Doing this directly amounts to taking a homotopy quotient of the space of fields by the action Lie algebroid, which is the entity encoding the infinitesimal versions of the symmetries of the theory, but the BRST formalism allows us work in the algebraic dual: instead of quotienting out by the gauge symmetries, we can construct a differential graded algebra describing dually the algebra of functions on the quotient, such that the zeroth cohomology group of the complex this forms is the space of functions invariant under the action of the gauge symmetries.

The BV formalism introduces antifields into the action to accommodate a mathematical nuance: it is possible that the intersection between the submanifolds comprising the critical locus of the action are not transverse. To see how the non-transversality could arise, let us consider the physical fields. In a given Lagrangian field theory, the physical fields will be the ones that satisfy the equations of motion (the Euler–Lagrange equations). Viewing the action S of such a field theory as a function on the space of fields, this set of physical fields is the critical locus of S, i.e. the vanishing locus of the differential dS. Let X denote the field space, so that its cotangent bundle is written as T^*X with zero section identifiable with a copy of X, and $gr(dS)$, the graph of the one-form dS. Then, the critical locus is

$$S_{crit} := gr(dS) \cap X.$$

In some situations, this will not be transverse, and to track this lack of transversality, it will be useful to consider a "dual" of this derived intersection, its ring of functions, which is a differential graded (dg) algebra denoted by

$$\mathscr{O}(S_{Crit}) = \mathscr{O}(gr(dS)) \otimes^{\mathbb{L}}_{\mathscr{O}(T^*X)} \mathscr{O}(X) \tag{3.1}$$

where $\mathbb{L}$ designates the derived tensor product. The lack of transversality is there chronicled by the Tor groups of the derived tensor product. The dg algebra $\mathscr{O}(S_{Crit})$ can be described using the tools of homological algebra: this module is quasi-isomorphic to the cochain complex coming from resolving the zero-section of the cotangent bundle with a Koszul-Tate resolution. This by construction provides another chain complex (this one with differential provided by contraction of vector fields), and we will show that this form of BV complex in tandem with the aforementioned BRST complex form a double complex, encoding all of the fields, antifields, ghosts, and anti-ghosts of the theory, the cohomology of which represents that theory's observables.

In this chapter, we shall construct the BV and BRST complexes and the double complex they form. This will be done in a way general enough to be suitable to describe any sigma-model physical theory with a collection of symmetries (such that its corresponding Lie algebra is finite dimensional[1]). This machinery may be applied to any thus-quantified field theory, including the superstring.

3.2 The BRST Complex

Let us consider a field theory with symmetries and denote its space of fields by X and symmetry group by G (we are avoiding the use of the terminology *gauge theory*, similarly *gauge group*, since though all the theories we look at have symmetries, they are not all called "gauge theories", with the notable instance of this being the type-II superstring). We will also assume that G is a finite-dimensional Lie group with corresponding Lie algebra $\mathfrak{g}$. The action of $\mathfrak{g}$ on X produces a cochain complex

$$C^\infty(X) \otimes \mathfrak{g}^* \xrightarrow{d_{BRST}} C^\infty(X) \otimes \bigwedge^2 \mathfrak{g}^* \xrightarrow{d_{BRST}} C^\infty(X) \otimes \bigwedge^3 \mathfrak{g}^* \xrightarrow{d_{BRST}} \cdots \tag{3.2}$$

where the differential on Lie algebra elements comes from the dual Lie bracket and on elements of $C^\infty(X)$ is the map

$$d_{BRST}\Big|_{C^\infty(X)} = \rho(-) \otimes z^\alpha \tag{3.3}$$

for $z^\alpha \in \mathfrak{g}^*$ basis elements of $\mathfrak{g}^*$, with

$$\begin{aligned} \rho : C^\infty(X) \otimes \mathfrak{g} &\to C^\infty(X) \\ z_\alpha(-) &\mapsto \rho^i_\alpha \frac{\partial(-)}{\partial x^i}, \end{aligned}$$

taking as input elements of $C^\infty(X)$, where the x_i are coordinates on X.

This complex is an example of a *Chevalley–Eilenberg complex* of a Lie algebra.

Definition 3.2.1. *[17, Section 7.1] Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$, and $U\mathfrak{g}$ its universal enveloping algebra. The* Chevalley–Eilenberg chain complex *of $\mathfrak{g}$ is the chain complex*

$$CE_*(\mathfrak{g}) = U\mathfrak{g} \otimes_{\mathbb{K}} \wedge^p \mathfrak{g} = \wedge^\bullet \mathfrak{g}^* \tag{3.4}$$

[1]It is possible to work with the case of infinite-dimensional Lie algebras, similarly algebroids, but one must take technical care when dualizing twice: it is not a given that in these cases that the bi-dual space is equal or isomorphic to the original space. There are some ways to work around this, so in general this is not at all an insurmountable problem.

with, for $u \in U\mathfrak{g}$ and $x_1 \wedge \cdots \wedge x_k \in \wedge^k \mathfrak{g}$, k^{th} differential

$$\begin{aligned} d(u \otimes x_1 \wedge \cdots \wedge x_k) = & \sum_{i=1}^{k} (-1)^{i+1} u x_i \otimes x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_k \\ & + \sum_{i<j} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_k \end{aligned}$$

One may show that the following alternative description of the Chevalley–Eilenberg complex corresponding to a particular Lie algebra $\mathfrak{g}$ in terms of a cochain complex is dual to the previous definition. This is seen by applying $Hom_{U\mathfrak{g}}(-, \mathbb{K})$ to the previous chain complex.

Definition 3.2.2. *Let $\mathfrak{g}$ be a Lie algebra and M a $\mathfrak{g}-$module. Then, the corresponding* Chevalley–Eilenberg cochain complex *is defined by*

$$CE^*(\mathfrak{g}, M) = Hom_{\mathbb{K}}(\wedge^\bullet \mathfrak{g}, M)$$

with differential

$$\begin{aligned} (d_{CE}\omega)(x_1, \cdots, x_k) = & \sum_{i=1}^{k} (-1)^{i+1} x_i \cdot \omega(x_1, \cdots, \hat{x}_i, \cdots, x_k) \\ & + \sum_{i<j} (-1)^{i+1} \omega([x_i, x_j], x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots x_k) \end{aligned}$$

for $x_1, \cdots, x_k \in \mathfrak{g}$ and $\omega \in \wedge^k \mathfrak{g}^* \otimes M$, for any $\mathfrak{g}-$module M.

One may show that Chevalley–Eilenberg complexes are functorial with respect to module morphisms.

Note that in the cases that $\mathfrak{g}$ is, as we have assumed here, finitely generated, due to $Hom-$tensor adjunction,

$$CE^*(\mathfrak{g}, M) = Hom_{\mathbb{K}}(\wedge^\bullet \mathfrak{g}, M) = \wedge^\bullet \mathfrak{g} \otimes_{\mathbb{K}} M.$$

If M is the ground field $\mathbb{K}$, the previous definition simplifies to

$$CE^*(\mathfrak{g}) = Sym(\mathfrak{g}^*[-1])$$

with differential on dual Lie algebra elements z^α

$$d := [,]^* : \mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*$$

the dual Lie bracket. The shift here puts $\mathfrak{g}^*$ in degree one, which leaves $Hom_{U\mathfrak{g}}(\mathbb{K}, \mathbb{K}) = \mathbb{K}$, the ground field, in degree zero.

In the case of the BRST complex, we are looking at the Chevalley–Eilenberg cochain complex of $\mathfrak{g}$, the finite-dimensional Lie algebra corresponding to the symmetry group of whichever classical field theory is being considered, tensored with $C^\infty(X)$, the ring of functions over the field space encoding the physical fields. Concretely, choosing $\{z^\alpha\}_{\alpha \in J}$ as the basis elements of $\mathfrak{g}^*$ and $\{f^i\}_{i \in I}$ as generating elements of $C^\infty(X)$, with J and I as indexing sets, one may compute in coordinates that (3.2) is a cochain complex with d_{BRST} as the differential. Using the dual Lie bracket and (3.3) one obtains that on dual Lie algebra elements such as z^α,

$$d_{BRST} z^\alpha = c^\alpha_{\beta\gamma} z^\beta \wedge z^\gamma$$

for $c^\alpha_{\beta\gamma}$ the structure constants of $\mathfrak{g}^*$, and for $f_i \in C^\infty(X)$,

$$d_{BRST}(f_i) = \rho^i_\alpha \frac{\partial f}{\partial x^i} z^\alpha$$

for ρ as in (3.3).

Remark 3.2.1. *The Chevalley–Eilenberg cochain complex of $\mathfrak{g}$ computes its Lie algebra cohomology, so the second cohomology group determines equivalence classes of extensions of $\mathfrak{g}$ by the module M. Hearkening back to the central charge introduced in Chapter 2, this indicates that the central charge will be the same for any representative of a particular equivalence class of central extensions by a given M (any representative of the cohomology class $H^2_{CE}(\mathfrak{g}, M)$), but that it is possible that there will be central extensions by M that do not have equal central charges. In these cases, one must specify both the module M and the equivalence class of central extensions under consideration.*

3.3 The BV Complex

The BV complex is formed by choosing a projective resolution of $\mathscr{O}(S_{Crit})$ as an $\mathscr{O}(T^*X)$-module, so as to compute the derived tensor product (3.1). We use here the Koszul–Tate resolution, which provides a way of describing a quotient ring consisting of a local complete intersection in terms of a differential graded algebra (see [15] for this construction).

In the case that $gr(dS) \cap S$ is transverse, i.e. the codimension of the intersection is equal to the sum of the codimension of each of $gr(dS)$ and X, the Tor groups, $Tor_\bullet^{\mathscr{O}(T^*X)}(\mathscr{O}(X), \mathscr{O}(gr(dS)))$, will be trivial for all non-zero cohomological degrees, but if it is not, these Tor groups will track the lack of transversality of the intersection.

The complex arising from the Koszul-Tate resolution is of the form

$$\Gamma\Big(\bigwedge\nolimits_{n\in\mathbb{N}}^{\bullet} TX\Big) = C^\infty(X) \xrightarrow{d_{BV}} \Gamma(TX) \xrightarrow{d_{BV}} \Gamma \bigwedge\nolimits^2 (TX) \xrightarrow{d_{BV}} \cdots$$

with differential as the contraction of differential forms with the one-form dS, i.e.

$$d_{BV} = \iota_{dS}.$$

Definition 3.3.1. *The* interior product *or* contraction *of polyvector fields is a map*

$$\iota_\alpha : \Gamma(\wedge^n TM) \to \Gamma(\wedge^{n-1} TM)$$

for $\alpha \in \Gamma(T^*M) := \Omega^1(M)$ *a one-form. Let* $X \in \Gamma(\wedge^n TM)$*. Then,*

$$\iota_\alpha X(\alpha_1, \cdots, \alpha_{n-1}) := X(\alpha, \alpha_1, \cdots, \alpha_{n-1})$$

where for $1 \leq i \leq n-1$*,* $\alpha_i \in \Gamma(T^*M)$ *are arbitrary one-forms.*

In other words, ι_α contracts the polyvector field of order $(n+1)$ with one one-form to make an $n-$polyvector field. Note that an $n-$polyvector field is dual to an n-form field. The contraction operator satisfies the identity that

$$\iota_\alpha \iota_\beta X = -\iota_\beta \iota_\alpha X \tag{3.5}$$

for any polyvector X and any one-forms α and β, and it is this property that shows that ι squares to zero. The contraction operator is by definition a degree -1 derivation on $\Gamma(\bigwedge^\bullet TM)$, which makes $\Gamma(\bigwedge^\bullet TM)$ a dg algebra. To show that ι_α squares to zero, making $\Gamma(\bigwedge^\bullet TM)$ into a complex with ι_α as the differential, we use its skew-symmetry property 3.5. Along those lines,

$$\begin{aligned} \iota_\alpha \iota_\alpha X(\alpha_1, \cdots \alpha_{n-2}) &= \iota_\alpha X(\alpha, \alpha_1, \cdots, \alpha_{n-2}) \\ &= X(\alpha, \alpha, \alpha_1, \cdots, \alpha_{n-2}) \\ &= 0 \end{aligned}$$

with the last equality following from the antisymmetry of X under the switching of α (in slot 1) with α (in slot 2).

3.4 The BV-BRST Double Complex

It is possible to describe the derived critical locus and higher gauge symmetries all in one, since the BRST and BV complexes can be shown to unite into a single double complex, the following, where $C^\infty(X)$ is situated by design in bidegree $(deg_{BV}, deg_{BRST}) = (0,0)$ and an application of d_{BRST} increases deg_{BRST} by one, leaving fixed deg_{BV} whereas applying d_{BV} increases deg_{BV} by one and leaves deg_{BRST} fixed.

$$\begin{tikzcd}
C^\infty(X) \arrow[r, "d_{BRST}"] & C^\infty(X)\otimes\mathfrak{g}^* \arrow[r, "d_{BRST}"] & C^\infty(X)\otimes\mathfrak{g}^*\wedge\mathfrak{g}^* \arrow[r, "d_{BRST}"] & C^\infty(X)\otimes\mathfrak{g}^*\wedge\mathfrak{g}^*\wedge\mathfrak{g}^* \arrow[r, "d_{BRST}"] & \cdots \\
\Gamma(TX) \arrow[u, "d_{BV}"] \arrow[r, "d_{BRST}"] & \Gamma(TX)\otimes\mathfrak{g}^* \arrow[u, "d_{BV}"] \arrow[r, "d_{BRST}"] & \Gamma(TX)\otimes\mathfrak{g}^*\wedge\mathfrak{g}^* \arrow[u, "d_{BV}"] \arrow[r, "d_{BRST}"] & \Gamma(TX)\otimes\mathfrak{g}^*\wedge\mathfrak{g}^*\wedge\mathfrak{g}^* \arrow[u, "d_{BV}"] \arrow[r, "d_{BRST}"] & \cdots \\
\Gamma(\bigwedge^2 TX) \arrow[u, "d_{BV}"] \arrow[r, "d_{BRST}"] & \Gamma(\bigwedge^2 TX)\otimes\mathfrak{g}^* \arrow[u, "d_{BV}"] \arrow[r, "d_{BRST}"] & \Gamma(\bigwedge^2 TX)\otimes\mathfrak{g}^*\wedge\mathfrak{g}^* \arrow[u, "d_{BV}"] \arrow[r, "d_{BRST}"] & \Gamma(\bigwedge^2 TX)\otimes\mathfrak{g}^*\wedge\mathfrak{g}^*\wedge\mathfrak{g}^* \arrow[u, "d_{BV}"] \arrow[r, "d_{BRST}"] & \cdots \\
\cdots \arrow[u, "d_{BV}"] & \cdots \arrow[u, "d_{BV}"] & \cdots \arrow[u, "d_{BV}"] & \cdots \arrow[u, "d_{BV}"] &
\end{tikzcd}$$

with differentials

$$d_{BV} = \iota_{dS} \otimes \mathrm{id}_{\wedge^\bullet \mathfrak{g}^*}$$

and d_{BRST} as the Chevalley–Eilenberg differential for the $\mathfrak{g}-$module. The total complex of this complex gives an algebraic model of the quotient $S_{crit}/\mathfrak{g}$ tracking (in the BV-direction) the lack of transversality of the intersection $gr(dS)\cap X$ and (in the BRST-direction) the *ext* groups limning the sheaf cohomology on $X/\mathfrak{g}$. The zeroth cohomology of this complex is the space of invariant functions on the field space, i.e. the space of physical fields.

In order to show that this is indeed a double complex, we must show that the differential squares to zero, i.e. $(d_{BRST} + d_{BV})^2 = 0$. Expanded out, this is

$$(d_{BRST} + d_{BV})^2 = d_{BRST}^2 + d_{BRST}d_{BV} + d_{BV}d_{BRST} + d_{BV}^2.$$

It follows from the two sections just prior to this one that $d_{BRST}^2 = 0$ since d_{BRST} here is still a Chevalley–Eilenberg differential, and that $d_{BV}^2 = 0$ since in this double complex, d_{BV}

is modified only by tensoring the contraction operator with the identity. What remains then is to show that the other two terms contribute nothing: they will sum to zero. This arises since the action S is invariant under the action of $\mathfrak{g}$ by design, which makes the interior product a map of $\mathfrak{g}-$modules so that $\iota_{dS} \otimes \mathrm{id}_{\wedge^\bullet \mathfrak{g}^*}$ gives a morphism between the corresponding Chevalley–Eilenberg complexes by functoriality of Chevalley–Eilenberg complexes with respect to module morphisms. A similar computation will be demonstrated in the next section for a Chevalley–Eilenberg complex isomorphic to the one presented here.

3.5 The Cotangent Action Lie Algebroid

In a field theory, the fields are not some arbitrary collection of sections, rather they are the solutions to the (here derived) equations of motion for that theory. In order to incorporate that into a description of the BV-BRST complex, it suffices to look at the *cotangent Lie algebroid*, which besides the original BV-BRST generators, also has vector field generators corresponding to "taking derivatives" of the usual BV-BRST generators. Since this also describes a double complex, it has differentials. The BV-differentials are still contraction of vector fields, but the BRST differentials, while each still a derivation on some Chevalley–Eilenberg complex, are "one-order higher" than those of the original BV-BRST complex: they come from bracketing the original differentials with the action. The algebraic structure used here to formulate these infinitesimals is that of action Lie algebroids.

Definition 3.5.1. *[12] Let M be a manifold. A* Lie algebroid *consists of $E \xrightarrow{\pi} M$ a vector bundle over M equipped with*

- *A vector bundle homomorphism $E \xrightarrow{\rho} TM$, the* anchor map *of the Lie algebroid.*
- *A Lie bracket on $\Gamma(E)$*

satisfying the following axioms:

1. *Let α, $\beta \in \Gamma(E)$. Then,*
$$\rho([\alpha, \beta]) = [\rho(\alpha), \rho(\beta)],$$
where the bracket on the left is the Lie bracket on E and that on the right is the Lie bracket on vector fields, i.e. ρ gives a homomorphism on sections of E.

2. $\rho(\alpha) \in \Gamma(TM)$ *is a vector field satisfying the Leibniz rule*

$$[\alpha, f \cdot \beta] = \rho(\alpha)(f) \cdot \beta + f[\alpha, \beta]$$

for $f \in C^\infty(M)$*.*

Definition 3.5.2. *An* action Lie algebroid *is a Lie algebroid such that the base manifold* M *comes equipped with the action of a Lie algebra,* $\mathfrak{g}$ *and*

- $E = \mathfrak{g} \times M$ *is the vector bundle over* M*.*
- $\rho : \mathfrak{g} \times M \to TM$ *is the Lie algebra homomorphism induced by the infinitesimal action of* $\mathfrak{g}$ *on* M *by sending each constant section of* $\mathfrak{g} \times M$ *to the vector field encoding that action.*
- *The Lie bracket comes from the Lie bracket on* $\mathfrak{g}$ *and extends from constant sections to all sections by the Leibniz rule.*

One may dually describe a Lie algebroid in terms of its Chevalley–Eilenberg algebra.

Definition 3.5.3. *The* Chevalley–Eilenberg algebra $CE(A)$ corresponding to a given Lie algebroid A *is the complex*

$$CE(A) := (\wedge_{C^\infty(M)}\Gamma(E)^*, d)$$

where the differential in degree zero is given by

$$(d_{CE}f)(X) = \rho(X)(f)$$

and in degree one is

$$d_{CE} = \rho(X_1)\omega(X_2) - \rho(X_2)\omega(X_1) + \omega([X_1, X_2])$$

for ρ *the anchor map of* A*,* $\omega \in \Gamma(E^*)$*, and* X_1*,* $X_2 \in \Gamma(E)$*, and is extended to other degrees by the Leibniz rule.*

Notice that this dual description of a Lie algebroid in terms of a Chevalley–Eilenberg algebra is just the first row of what was used to form the rows of the BV-BRST complex as introduced earlier, and that over a point, i.e. with $M = *$, this CE algebra returns that of an ordinary Lie algebra.

Definition 3.5.4. *[16, Definition 10.22] A* connected Lie-∞ algebroid *over a manifold X with algebra of smooth functions $C^\infty(X)$ consists of the following data:*

- *A graded module $\mathfrak{a}_\bullet = (\mathfrak{a}_k)_{k\in\mathbb{N}}$ where each of the $\mathfrak{a}_k$ is a free $C^\infty(X)-$module of finite rank.*

- *A degree 1 differential d_{CE} such that*

$$\left(Sym_{C^\infty(X)}(\mathfrak{a}^*_\bullet), d_{CE}\right) =: CE(\mathfrak{a}),$$

  is a cochain differential graded commutative $\mathbb{R}-$algebra, the Chevalley–Eilenberg algebra *of $\mathfrak{a}$.*

A connected $Lie_\infty-$algebroid is called a *derived* connected $Lie_\infty-$algebroid if $\mathfrak{a}_\bullet$ has nonzero modules in negative as well as non-negative degrees.

To describe the entire BV-BRST complex as seen before, which incorporates the infinitesimal perturbations of the solutions to the Euler–Lagrange equations used to explore the critical locus of a perturbative field theory in the BV-BRST perspective, we will require a sort of cotangent space to the action Lie algebroid, the the corresponding *infinitesimal cotangent action Lie algebroid.*

Definition 3.5.5. *[16, Definition 11.2] Let $\mathfrak{a}$ be a connected Lie$-\infty$ algebroid. The corresponding* infinitesimal cotangent Lie algebroid $(T^*_{inf}\mathfrak{a})^*_\cdot$ *consists of the following data:*

- *The graded module $(T^*_{inf}\mathfrak{a})^*_\cdot = \mathfrak{a}^* \oplus Der(CE(\mathfrak{a}))$ where $Der(CE(\mathfrak{a}))$ denotes the graded derivations of the graded algebra upon which $CE(\mathfrak{a})$ is based.*

- *The degree-one differential making*

$$\left(Sym_{C^\infty(X)}((T^*_{inf}\mathfrak{a})^*), d_{CE(T^*_{inf})}\right) = \left(Sym_{C^\infty(X)}(\mathfrak{a}^* \oplus Der(CE(\mathfrak{a})), d_{CE(T^*_{inf})}\right)$$

  into the Chevalley–Eilenberg algebra of $(T^*_{inf}\mathfrak{a})*$ where on each component this differential acts as follows:

$$d_{CE(T^*_{inf}\mathfrak{a})}\Big|_{\mathfrak{a}^*} = d_{CE(\mathfrak{a})}$$
$$d_{CE(T^*_{inf}\mathfrak{a})}\Big|_{Der(CE(\mathfrak{a}))} = [d_{CE(\mathfrak{a})}, -]$$

Note the parallel with the usual cotangent bundle T^*M of a manifold M: the zero section of T^*M is identifiable with a copy of M itself. Somewhat dually, the first term of the Chevalley–Eilenberg algebra corresponding to the infinitesimal cotangent Lie algebroid is the Chevalley–Eilenberg algebra of the underlying Lie-∞ algebroid $\mathfrak{a}$.

To specify the infinitesimal cotangent Lie algebroid of an action Lie algebroid, in light of the above defintion, requires determining the generators of the derivations on the Chevalley–Eilenberg algebra dual to $X/\mathfrak{g}$, the Lie algebroid encoding the action of the infinitesimal symmetries of the symmetry group G on the field space X. The generators of the first summand are the same as those in the underlying Lie$-\infty$ algebroid. In an arbitrary theory, we will have vector field generators in two degrees, corresponding to the cohomological-degree zero generators of the physical fields in the Chevalley–Eilenberg algebra, and the cohomological-degree one generators of the ghosts (elements of the Lie algebra of symmetries). If we represent a ghost element by z^α and field by $f(x^i)$, then the associated coordinate-basis elements will be denoted respectively by $\frac{\partial}{\partial z^\alpha}$ in cohomological degree negative-one and $\frac{\partial}{\partial x^i}$ in cohomological degree zero, where this construction can be done for any $\alpha \in J$, $i \in I$ respectively, with as prior, the z^α providing a basis for the dual Lie algebra $\mathfrak{g}^*$ of the gauge group G, and the $\{x^i\}_{i\in I}$ a coordinate basis on the worldsheet of the theory. The differential is also defined here to be the Schouten bracket of the element with the action S:

$$d_{BV} = [S, -] = \iota_{dS} Y$$

for $Y \in \bigwedge^n TX$ a (poly)vector field. Note that since the Schouten bracket satisfies the Jacobi identity, the differential it defines squares to zero (this is analogous to how d_{BRST} on Lie algebra elements vanishes).

In the case, such as here, that we are working with a field theory described by an action functional S, the BV-BRST complex as we have noted dually describes the combined derived intersection $gr(S) \cap X$ and homotopy quotient $X/\mathfrak{g}$. As seen in [16] and from the definitions of this section, the Chevalley–Eilenberg complex of the infinitesimal cotangent Lie algebroid of the action Lie algebroid corresponding to the theory has the same generators as those needed to describe the BV-BRST complex. We inspect this in terms of coordinate bases. Let $\{z^\alpha\}_{\alpha\in I}$ and $\{f(x^j)\}_{x\in J}$ respectively describe the generators of $\mathfrak{g}^*$ and X of the action

Lie algebroid. By construction, the associated BV-BRST complex has these two sets of generators, as well as the corresponding vector fields. Of these vector fields, there are also two sets: $\left\{\frac{\partial}{\partial z^\alpha}\right\}_{\alpha\in I}$, which will be in cohomological degree negative-one since the z^α have been set to be in cohomological degree one, and $\left\{\frac{\partial}{\partial x^j}\right\}_{j\in J}$ in cohomological degree zero. In the perspective of the infinitesimal cotangent Lie algebroid, the vector field generators fulfill the roles of the generators of the derivations over $X/\mathfrak{g}$, up to a shift in degree, which can be viewed as follows: the differential on the Chevalley–Eilenberg complex on the derivation factor of the infinitesimal cotangent Lie algebroid comes from the bracket of the Chevalley–Eilenberg differential, here on $CE(X/\mathfrak{g})$ with each generator, the degree of each of the vector fields (the derivation-generators) will be lowered by one for consistency. Since up to the shift in degree, the generators of the BV-BRST complex of a theory and those of its infinitesimal cotangent Lie algebroid are the same, those two constructions are isomorphic.

Slightly more explicitly, since the BV-BRST complex as defined in the previous section has rows defined by the Chevalley–Eilenberg algebra of some module and columns coming from the Koszul–Tate resolution, it directly encompasses the structure provided by the infinitesimal cotangent Lie algebroid: the rows are still of the form of a Chevalley–Eilenberg algebra, but now incorporating the extra vector field generators and the columns are as before, so that the total complex is $CE(T^*_{inf}(X/\mathfrak{g})^*)$. Since each term $CE(T^*_{inf}(X/\mathfrak{g})^*)$ has two factors, the differentials will be written correspondingly. In this case,

$$\begin{aligned} d_{BV}\Big|_{T^*_{inf(X/\mathfrak{g})^*}} &= d_{BV}\otimes 1 \\ d_{BRST}\Big|_{T^*_{inf}(X/\mathfrak{g})^*} &= d_{CE}\Big|_{T^*_{inf}(X/\mathfrak{g})^*}. \end{aligned}$$

In this setting, as well, $(d_{BV})^2 = 0$, since the only modification to it is tensoring by the identity, and because the first component of d_{BV}, the Schouten bracket with the action, satisfies a Jacobi identity and thus squares to zero, the entire tensor product does too. d_{BRST} squares to zero since it is still a Chevalley–Eilenberg differential. What remains, then, to show is that

$$d_{BV}\Big|_{T^*_{inf(X/\mathfrak{g})^*}}\circ d_{BRST}\Big|_{T^*_{inf}(X/\mathfrak{g})^*} = -d_{BRST}\Big|_{T^*_{inf}(X/\mathfrak{g})^*}\circ d_{BV}\Big|_{T^*_{inf(X/\mathfrak{g})^*}}.$$

To that end, recall that $\Gamma(TX)$ has a Lie bracket and define for $z \in \mathfrak{g}$ and $Y \in \Gamma(TX)$ a Lie algebra homomorphism

$$\rho : \mathfrak{g} \to \Gamma(TX)$$

acting on a (poly)vector field $Y \in \Gamma(\wedge^n TX)$ as

$$z \cdot Y = [\rho(z), Y]_{Sch}$$

where the bracket on the right side is the Schouten bracket, and $\rho(z)$ encodes the Lie algebra action of z. In our setup, the action will be represented by a function $S \in \Gamma(\wedge^0 TX)$ invariant under the action of the symmetries of G, or the action of $\mathfrak{g}$.

$$z \cdot \iota_{dS} Y = \iota_{dS} \cdot z(Y).$$

The left side of this equation may be re-written using the Jacobi identity on the Schouten bracket as

$$\begin{aligned}
z \cdot \iota_{dS} Y &= [\rho(z), [S, Y]_{Sch}]_{Sch} \\
&= [S, [Y, \rho(z)]_{Sch}]_{Sch} + [Y, [\rho(z), S]_{Sch}]_{Sch} \\
&= -[S, [\rho(z), Y]_{Sch}]_{Sch} + [Y, [\rho(z), S]_{Sch}]_{Sch} \\
&= -[S, [\rho(z), Y]_{Sch}]_{Sch}
\end{aligned}$$

where the last equivalence comes from that the action S was assumed to be invariant under the action of the Lie algebra elements. Thus, since the right side is equal to $-\iota_{dS} \cdot z(Y)$, Equation 3.5 holds.

To interpret the cohomology of the BV-BRST complex considering the physical action S, notice that the underlying structure is as before: the BV columns still track the lack of transversality of the intersection defining the derived critical locus, and the rows are the Chevalley–Eilenberg complexes, with the first, as previously, encoding the on-shell behavior as the Chevalley–Eilenberg complex of the infinitesimal cotangent action Lie algebroid.

Chapter 4

Factorization Algebras

4.1 Introduction

Thus far, we have seen that the fields of a physical theory can be organized into a double complex, the BV-BRST complex (explained in Chapter 3), and by taking the cohomology of the total complex thereof, one obtains the classical observables of the theory. It is upon these that there is the structure of a factorization algebra. This is only one side of the story, however. The BV-BRST complex itself consists of the symmetric algebra of some differential graded Lie algebra, which, under an equivalence of infinity-categories (to be addressed later) itself is associated to a formal moduli problem. In this chapter we will introduce the notions and associations needed to gain some understanding of these correspondences.

4.2 Prefactorization Algebras and Factorization Algebras

A prefactorization algebra is in rough analogy with a precosheaf, except for with the direct sum replaced by the direct product. There are multiple ways of describing such entities more formally, and we shall include two here. Before presenting these definitions, though, we will introduce some additional underlying machinery.

Definition 4.2.1. *[9, Definition 1.1] An* operad*, O over a symmetric monoidal category C, is a collection of objects $J(n)$ indexed by the natural numbers, with the following extra structure:*

1. *Composition morphisms*

$$\gamma : J(k) \times J(m_1) \times \cdots \times J(m_k) \to J(m)$$

   where $m = \sum_{i=0}^{k} m_i$ fulfill the following associativity relation: for all $c \in J(k)$, $d_i \in J(m_i)$, $e_t \in J(l_t)$,

$$\gamma(\gamma(c; d_1, \cdots d_k); e_1, \cdots, e_m) = \gamma(c; f_1, \cdots, f_k)$$

   where $f_i = \gamma(d_i; e_{\sum_{i=0}^{m-1} i+1}, \cdots e_{j_{\sum_{i=0}^{m} j_i}})$.

2. *An identity element, $e \in J(1)$, such that*
   - *for $d \in J(m)$, $\gamma(1; d) = d$ and*
   - *for $c \in J(k)$ and $1^k = (1, \cdots, 1) \in J(1)^k$, $\gamma(c, 1^k) = c$.*

3. *A right operation of the symmetric group S_m on $J(m)$ for each m such that for all $c \in J(k)$, $d_i \in J(m_i)$, $\sigma \in S_k$, and $\tau_i \in S_{m_i}$, such that:*
   - *$\gamma(c\sigma; d_1, \cdots, d_k) = \gamma(c, d_{\sigma^{-1}(1)}, \cdots, d_{\sigma^{-1}(k)})\sigma(m_1, \cdots m_k)$ for $\sigma(m_1, \cdots, m_k)$ the permutation of m letters permuting the k blocks of letters given by the particular partition of m as σ permutes k letters, and*
   - *$\gamma(c; d_1\tau_1, \cdots, d_k\tau_k) = \gamma(c; d_1, \cdots d_k)(\tau_1 \oplus \cdots \oplus \tau_k)$ for $\tau_1 \oplus \cdots \oplus \tau_k$ the image of the tuple $(\tau_1, \cdots, \tau_k)$ under the inclusion $S_{m_1} \times \cdots \times S_{m_k} \hookrightarrow S_m$.*

Definition 4.2.2. *Let V be a symmetric monoidal category. A* colored operad *or* multicategory *$\mathscr{M}$, over a symmetric monoidal category $(C, \boxtimes)$, is a symmetric multicategory enriched over V.*

One may view a colored operad in a rough sense as the horizontal categorification of an operad, an *operadoid*, meaning the generalization of the idea of an operad to a situation with multiple objects (in analogy to groups and groupoids).

In the construction of Costello and Gwilliam in [2], a particular colored operad, $Disj_M$, is fundamental.

Definition 4.2.3. *[2, Definition 3.7.2.1] Let M be a topological space. $Disj_M$ is a colored operad consisting of the following structure.*

1. *The objects of $Disj_M$, $Ob(Disj_M)$, are all connected open subsets of M.*

2. *A set of maps, denoted $Disj_M(\{U_\alpha\}_{\alpha\in A}|V)$ from each finite collection of open sets $\{U_\alpha\}_{\alpha\in A} \in M$ to each open set $V \in M$. If the U_α are pairwise-disjoint and $U_\alpha \in V$, $\forall\alpha \in A$, the set of maps is a single point. The set of maps is empty in any other case.*

3. *Composition of maps corresponds to operadic composition.*

Property 2 in the above definition tells us that the set of maps between two disjoint U_{α_i}, $i \in \{1,2\}$ is empty.

The definition of a prefactorization algebra is then as follows:

Definition 4.2.4. *[2, Section 3.1.2] A* prefactorization algebra *on M, a topological space, valued in C, a multicategory, is a functor*

$$P : Disj_M \to C.$$

One may construct a prefactorization algebra more explicitly, in a sort of analog to a pre-cosheaf, but with respect to the tensor product versus the direct sum. This gives the following alternative definition.

Definition 4.2.5. *[2, Section 3.1.1]A* prefactorization algebra *on a topological space M taking values in some linear multicategory C is a rule that assigns an object in $\mathscr{F}(U) \in C$ to each open set $U \in M$ along with, corresponding to each finite set of pairwise disjoint subsets $U_i \in V$ a linear map*

$$m_V^{U_1,\cdots,U_n} : \mathscr{F}(U_1) \otimes \cdots \otimes \mathscr{F}(U_n) \to \mathscr{F}(V).$$

This inclusion rule satisfies the property that for $U_{i,1} \sqcup \cdots \sqcup U_{i,n_i} \subseteq V_i$ and $V_1 \sqcup \cdots \sqcup V_k \subseteq W$,

$$\begin{tikzcd}
\bigotimes_{i=1}^k \bigotimes_{j=1}^{n_i} \mathscr{F}(U_j) \arrow[rr] \arrow[dr] & & \bigotimes_{j=1}^k \mathscr{F}(V_i) \arrow[dl] \\
& \mathscr{F}(W) &
\end{tikzcd}$$

commutes.

Usually C in practice is a linear multicategory such as that of vector spaces or cochain complexes (taking values in an additive category). This allows for a notion of addition of the values of the outputs, and also a linear version of composition.

The generalization from prefactorization algebra to *factorization algebra* requires gluing conditions preserving the multiplicative structure as the (co)sheaf axiom does the additive structure of a pre(co)sheaf. To achieve this, one may look at a particularly generous sort of cover of M.

Definition 4.2.6. *A* Weiss cover *of an open set $U \subset M$ is a collection of open sets, $\mathfrak{U} = \{U_i | i \in I\}$ such that for any finite collection of points $\{x_1, \ldots, x_k\} \subset U$ there exists an open set $U_i \in \mathfrak{U}$ such that $\{x_1, \ldots, x_k\} \subset U_i$.*

A trivial example is the case that $I = \{1\}$ and $U = U_i = \mathfrak{U}$, but one sees that rapidly this sort of covering has no dearth of open sets: in fact, for a space with point-separation, it is like a superset of all coverings, in that it contains all possible ones, from the coarsest through the finest. As noted in [2], a Weiss cover defines a Grothendieck topology (see [1] for this and related notions) on the set of open sets of a topological space M, $Opens(M)$.

Now we have all of the ingredients necessary to define a factorization algebra from a prefactorization algebra.

Definition 4.2.7. *(Like [2, Definition 6.1.3.1].) Let M be a topological space and let C be a linear multicategory. A* factorization algebra *on M with values in C is a prefactorization algebra on M with values in C that is a cosheaf with respect to the Weiss topology. In other words: a factorization algebra is a prefactorization algebra with the additional property that for every open subset $U \in M$ and every Weiss cover $\{U_i\}_{I \in I}$, the following sequence is exact:*

$$\bigoplus_{i,j} \mathscr{F}(U_i \cap U_j) \to \bigoplus \mathscr{F}(U_k) \to \mathscr{F}(U) \to 0.$$

Definition 4.2.8. *[2, Section 6.3.1] A* multiplicative factorization algebra *is a factorization algebra such that for all pairs of disjoint open subsets (V, W) in M, the structure map*

$$m^{V,W}_{V \sqcup W} : \mathscr{F}(V) \otimes \mathscr{F}(W) \to \mathscr{F}(V \sqcup W)$$

is an isomorphism.

The target category here will usually be CCh, the category of cochain complexes. CCh can actually be equipped with (actually several different choices of) a model structure, which means that its morphisms will be divided into three classes: fibrations, cofibrations, and weak equivalences. In the setting of factorization algebras, however, only the existence of a notion of weak equivalence plays a role: this indicates that a more nuanced version of equivalence would have a place in the context of factorization algebras. In order to build up a such variant of factorization algebras, where the analog of the precosheaf axiom holds only up to weak equivalence (in the CCh, one may consider weak equivalences as quasi-isomorphisms of chain complexes), we require some more concepts.

Construction 4.2.1. *[2, Section 7.1.4] Let M be a topological space and $\mathfrak{U} = \{U_i\}_{i \in I}$ be a cover of some open subset $U$of M (where U can be any open subset, the importance being that one is chosen). Define Φ to be a precosheaf on M valued in cochain complexes in an additive category C.*

One may construct a precosheaf valued in double cochain complexes where the degree $(k, \bullet)$ term looks like

$$\bigoplus_J \Phi(U_{j_o} \cap \cdots \cap U_{j_k})$$

where the sum is over tuples $J = (j_0, \cdots, j_k)$ with $j_0, \ldots, j_k \in J$ pairwise distinct. One differential of these double complexes is the differential on the complex Φ and the other is the alternating sum of the structure maps of Φ coming from the inclusion of one term in the internal sum into the next.

*One may build a $\check{C}$ech complex out of the totalization of the double complex these terms comprise. To this end, define the $\check{C}$*ech complex of $\mathfrak{U}$ with values in Φ *as*

$$\check{C}(\mathfrak{U}, \Phi) = \bigoplus_{k=0}^{\infty} \left(\bigoplus_J \Phi(U_{j_0} \cap \cdots \cap U_{j_k})[k] \right).$$

where $[k]$ is a shift in degree of k in the positive direction.

For any term in this sum that is of the form $\Phi(U_{j_i})$, one may show that the structure map from that Φ is a precosheaf provides the natural map

$$H : \check{C}(\mathfrak{U}, \Phi) \to \Phi(U).$$

This can be constructed for any such term, and any U, so long as $U_{j_i} \subseteq U$, and thus gives a map from the Čech complex on any smaller open to the precosheaf on a larger one containing it.

Definition 4.2.9. *[2, Section 6.1.4] Φ as defined above is a* homotopy cosheaf *if H is a quasi-isomorphism for U any open set of M and $\{U_i\}_{i\in I}$ any open cover of U.*

Now enough information has been presented to understand the aforementioned generalization of factorization algebras.

Let C be a multicategory over a Grothendieck abelian category, and M a topological space. Let $CCh(C)$ denote the category formed by cochain complexes taking values in C. One may form a multicategory over $CCh(C)$, where the higher-arity operations are multilinear maps between these cochain complexes. Note that since $CCh(C)$ consists of cochain complexes, it comes with a notion of weak equivalence: quasi-isomorphism of cochain complexes.

Definition 4.2.10. *[2, Definition 6.1.4.1] Let $\mathscr{F}$ be a prefactorization algebra on M valued in $CCh(C)$. $\mathscr{F}$ is called a* homotopy factorization algebra *if for every $U \in M$ an open set, and every Weiß cover $\mathfrak{U}$ of U, the map H from Construction 4.2.1 is a quasi-isomorphism.*

As stated in [2], and seen from Construction 4.2.1, this indicates that if $\mathscr{F}$ is a homotopy factorization algebra, it is by definition a homotopy cosheaf on the topology on U defined by $\mathscr{U}$, and, in particular, on M itself.

It is also worth mentioning that not only does a notion of weak equivalence in the target category allow for the generalization to homotopy factorization algebras, but it also provides a way of recognizing different homotopy factorization algebras themselves as weakly equivalent. One may show (as can be found in [2, Section 6.1.5.1]), that for F and G homotopy factorization algebras, if for each open set U, $F(U) \to G(U)$ is a weak equivalence in the target category, then F and G are themselves weakly equivalent as factorization algebras. This may be checked explicitly by looking at particular refinements of U.

4.3 Formal Moduli Problems

Historically, that which became referred to as "formal moduli problems" was explored as a way to encode order-wise deformations. In the context of partial differential equations (PDEs), which is particularly relevant to field theories governed by action functionals since the Euler–Lagrange equations are PDEs, this means taking a perturbation expansion of the equation under inspection and trying to find order-wise solutions. The space of all solutions at all orders, as well as the relationships between those deformations, forms the formal moduli problem associated with that equation. Now there is a way to phrase formal moduli problems (FMPs) in terms of infinity categories, and we will hop between the two, first introducing FMPs in that generality, but referencing the importance of order-wise solutions to the equations of motion in a given field theory to the understanding of the structure of its space of solutions.

We will here introduce the notion of FMPs in some generality, as found in [7], and then specialize to the case seen in [3] and most relevant to factorization algebras. Before stating these main definitions, however, we require some background.

Definition 4.3.1. *Let $\mathbb{K}$ be a field of characteristic zero and A a commutative $\mathbb{K}-$algebra. A is* local Artinian *if it satisfies the following two conditions.*

1. *A is a finite-dimensional $\mathbb{K}$ vector space.*

2. *A has a unique maximal ideal $\mathfrak{m}_A$, and moreover $\mathfrak{m}_A^N = 0$ for some $N \in \mathbb{N}$ large enough.*

Similarly in the differential-graded setting,

Definition 4.3.2. *(Similar to [3, Definition 3.1.0.1].) Let $\mathbb{K}$ be a field of characteristic zero and A a commutative differential graded $\mathbb{K}-$algebra only in degrees ≤ 0. A is an* Artinian dg algebra *if it fulfills the following two conditions.*

1. *With respect to the $\mathbb{N}-$grading on A, each graded component A^i is finite-dimensional, and there exists a minimal i, i.e. for i small enough, $A^i = 0$.*

2. *A has a unique maximal (differential) ideal $\mathfrak{m}_A$ where $\mathfrak{m}_A^N = 0$ for some $N \in \mathbb{R}$ large enough, and $A/\mathfrak{m}_A = \mathbb{K}$.*

Definition 4.3.3. *[7, Definition 0.0.6] Let $\mathscr{C}$ be an $\infty-$category and $N(Ring_{\mathbb{K}}^{art})$ the nerve of the category of local Artinian $\mathbb{K}-$algebras. A $\mathscr{C}-$*valued formal moduli problem *is a functor*

$$FMP : N(Ring_{\mathbb{K}}^{art}) \to \mathscr{C}.$$

The case explored in [3] is a specification on this theme.

Definition 4.3.4. *[3, Definition 3.1.0.2] Let $\mathbb{K}$ be a field of characteristic zero. A* formal pointed moduli problem *over $\mathbb{K}$ is a functor of simplicially enriched categories*

$$F : Art_{\mathbb{K}}^{dg} \to sSets,$$

where $sSets$ denotes the category of simplicial sets and $Art_{\mathbb{K}}^{dg}$ differential graded Artinian algebras over a characteristic-zero field $\mathbb{K}$, satisfying the following properties.

1. *$F(\mathbb{K})$ is contractible.*
2. *F takes surjective maps of dg Artinian algebras to fibrations of simplicial sets.*
3. *For $B \xrightarrow{f} A$, $C \xrightarrow{g} B$ $\in Art_{\mathbb{K}}^{dg}$, and $B \times_A C$ the fiber product of B and C, the natural induced map*

$$F(B \times_A C) \to F(B) \times_{F(A)} F(C)$$

is a weak homotopy equivalence.

The objective here is to, in defining a formal moduli problem, construct a functor from one of these categories to an arbitrary infinity category, and one way to do this practically is to pass to simplicial sets on both the source and the target of the functor. Recall that

Definition 4.3.5. *An $(\infty, 1)-$category is a simplicial set satisfying the Kan condition, i.e. such that each horn has a filler under Kan extension.*

We know that Joyal's quasi-categories ($(\infty, 1)-$categories as in the previous definition) provide a model for any infinity category, essentially breaking it down into simplicial sets, and will choose to look in terms of this model here. It follows that for the target we will have no problem. For the source, one must show how a simplicial structure arises on the category of local Artinian algebras or Artinian dg algebras.

In the former case, notice that the source of the formal moduli problem functor is the nerve of $Ring_{\mathbb{K}}^{art}$, and by construction, this interprets a category in terms of the data of simplicial sets. This works as follows:

Construction 4.3.1. *(The Nerve of a Category) Let C be a category. We will show how this construction works for C and mention how it specifies to the case of interest here, that of $Ring_{\mathbb{K}}^{art}$. The construction in any setting can be accomplished in three stages:*

1. *Associate to each $n-$morphism in $\mathscr{C}$ a $n-$simplex. Since C is an ordinary category, this means that we associate a point to each object, a $1-$simplex to each morphism, and a $2-$simplex to every composition of morphisms.*

2. *Form the degeneracy maps*

$$d_k : \Delta_k \to \Delta_{k-1}$$

by deleting the i^{th} entry in a sequence of k objects, to obtain a sequence of $k-1$ objects. Viewed in terms of simplicial sets, this entails deleting a single $0-$simplex, and connecting the two $1-$simplices that would have joined it to the neighboring two $0-$simplicies. This connection can itself be seen, as a composition of morphisms, as a $2-$simplex.

3. *Establish the face maps,*

$$s_k : \Delta_k \to \Delta_{k+1},$$

by adding an identity morphism from one $k-$morphism to itself. In terms of simplicial sets, this "adds a side" to the simplices, making them simplices of one dimension higher.

In general, the nerve offers a way to view any ordinary category as an $(\infty, 1)-$category. In the case of $Ring_{\mathbb{K}}^{art}$, the $0-$morphisms are the locally Artinian algebras, and the $1-$morphisms the structure-preserving algebra homomorphisms between them. The simplicial sets are then constructed by these in the same manner as above.

Since the target category $\mathscr{C}$ in the context of formal moduli problems is already an $\infty-$category (meaning here and everywhere else in this document, $(\infty, 1)-$ category) in the general case, it can be realized in terms of simplicial sets, and a formal moduli problem can be viewed as mapping between simplicial sets.

In the case of $Art_{\mathbb{K}}^{dg}$, since it is a small category, we may take its nerve, and by construction this describes (at least up to isomorphism) the contents of that category. Thus, in order to get a map of infinity categories in this case as well, we may pass to the nerve and look at, as in the above case, the corresponding map of simplicial sets.

Construction 4.3.2. *(Simplicial enrichment of $Art_{\mathbb{K}}^{dg}$, similar to [3, Section 3.1]). It is possible to describe a simplicial enrichment of $Art_{\mathbb{K}}^{dg}$ (providing a model of that category as an $(\infty, 1)-$category) in terms of differential forms on simplices.*

Differential forms on a simplex can be constructed in rough analogy to differential forms over a coordinate chart $\{U, x_I\}$ on a topological space M: locally, on a coordinate chart, a differential form $\alpha \in \Omega^*(U)$ can be written as

$$\alpha = \sum_I f_I \wedge dx^I$$

where each f is a function on U and I is a multi-index encoding lists of integers of increasing length. For the analogy, one may view a simplex as a particular choice of chart (they locally do look like subsets of $\mathbb{R}^n$), as

$$\Omega^n(\Delta^n) \simeq \frac{\mathbb{R}[x_1, \cdots, x_n, dx^i, \cdots, dx^n]}{\sum_{i=1}^n x_i = 1, \sum_{i=1}^n dx^i = 0},$$

with the relations in the denominator those coming from the rules of simplices. This defines a commutative algebra of differential forms over each $k-$simplex. This construction for all k produces a cochain complex of differential forms, with differential the de Rham differential.

Now, let us consider the setting of dg$-$Artin algebras. Let $\mathfrak{m}_A$ be the maximal ideal of $A \in Art_{\mathbb{K}}^{dg}$, and $\mathfrak{m}_B$ the maximal ideal of $B \in Art_{\mathbb{K}}^{dg}$. A morphism in $Art_{\mathbb{K}}^{dg}$,

$$f : A \to B,$$

must be such that $f(\mathfrak{m}_A) = \mathfrak{m}_B$ (since the morphisms are structure-preserving). The simplicial structure on $Art_{\mathbb{K}}^{dg}$ then is built as follows:

For any level $n = k$, the $k-$simplices are maximal-ideal-preserving maps

$$\mathfrak{m}_A \to \mathfrak{m}_B \otimes \Omega^*(\Delta^k)$$

where $\Omega^*(\Delta^k)$ is essentially the de Rham complex of differential forms on the $k-$simplex as prior within this construction.

This construction gives us a more concrete way of seeing how the formal moduli problem functor is indeed between $\infty-$categories, and hints at how one might explicitly compute the correspondence.

Remark 4.3.1. *There is a classical version of this theory, wherein the functor constructed is from Artinian algebras to sets. This means that one should have the following commutative diagram:*

$$\begin{tikzcd} Art_{\mathbb{K}}^{dg} \arrow[r] \arrow[d, "H^0"] & SSet \arrow[d, "\pi_0"] \\ Art_{\mathbb{K}} \arrow[r] & Set \end{tikzcd}$$

The top row describes a formal pointed moduli problem, and the bottom row a classical moduli problem. To get from the dg case to the classical case, as indicated on the downward arrows, one takes the connected components of $SSet$ and zeroth homology group of $Art_{\mathbb{K}}^{dg}$. It is also possible to get back from the bottow row to the top row. On the right, that amounts to taking the constant simplicial set, and on the left, that is inclusion of regular Artinian algbras into the degree-zero part of dg Artinian algebras. Note that this does not really capture the graded nature of $Art_{\mathbb{K}}^{dg}$, since all information is concentrated in degree zero.

In the setting of Definition 4.3.4, and in this exposition, we would like to be able to explain the formal moduli problem associated to a classical field theory described by an action functional. For intuition into this, one may consider again the historical motivation of formal moduli problems in the context of differential equations: to find and describe the order-wise solutions thereto. As we will later see, the solutions to the Euler–Lagrange equations to our theories will describe a local $L_\infty-$algebra, the Maurer-Cartan equations of which will correspond directly to those equations of motion. To see how this contributes to the structure of a formal moduli problem defined at the beginning of this subsection, we will examine the connection between these and $L_\infty-$algebras.

4.3.1 Connection of Formal Moduli Problems to L_∞ Algebras

It is possible to construct a formal moduli problem given an L_∞ algebra. The formal moduli problem will consist of solutions to the Maurer–Cartan equations of that algebra. This is explained in great detail in [6] and [5], and referenced and partially elaborated upon in, respectively, [7] and [3]. Here most of the time it will be adequate to work with differential graded Lie algebras rather than $L_\infty-$algebras, so to begin we will make the relationship between the two explicit.

Definition 4.3.6. *A* differential graded Lie algebra *is a graded vector space $V = \bigoplus_i V_i$ (over a field of characteristic zero) equipped with two maps:*

1. *A bilinear bracket:*

$$[\cdot,\cdot] : V_i \times V_j \to V_{i+j}$$

fulfilling

* *skew-anti-symmetry: for $x, y \in V$ homogeneous*

$$[x, y] = (-1)^{|x||y|+1}[y, x]$$

and

* *The graded Jacobi identity: for $x, y, z \in V$ homogeneous:*

$$(-1)^{|x||z|}[x,[y,z]] + (-1)^{|y||x|}[y,[z,x]] + (-1)^{|z||y|}[z,[x,y]] = 0$$

2. *A differential of homological grading*

$$d : V_i \to V_{i-1}$$

or cohomological grading:

$$d : V^i \to V^{i+1}$$

(where in the latter case, the notation of the grading in superscripts instead of subscripts is to emphasize that the differential raises the degree) that is a derivation of the bracket, i.e. for homogeneous $x, y \in V$,

$$d[x, y] = [dx, y] + (-1)^{|x|}[x, dy].$$

Note that the choice of homological versus cohomological grading on a vector space is easy to switch: starting with homological grading, set $V_i = V^{-i}$ for all i to obtain cohomological grading.

There is a direct generalization of the contents of this definition into that of the next.

Definition 4.3.7. *An $L_\infty-$*algebra *or* homotopy Lie algebra *is a graded vector space $V = \bigoplus_i V_i$ (over a field of characteristic zero) equipped with level-wise bracket operations: for each $n \in \mathbb{N}$ there is a degree $n-2$ multilinear map, the* n-ary bracket*,*

$$b_n := [\cdot, \cdots, \cdot]_n : V \times \cdots \times V \to V$$

from n copies of V to V such that these maps fulfill

1. *graded skew-symmetry: for $(v_1, \cdots v_n)$, a n-tuple of homogeneous elements in V, $\sigma \in S_n$ any permutation, and χ the graded signature of the permutation,*

$$b_n(v_{\sigma(1)}, \cdots, v_{\sigma(n)}) = \chi(\sigma, v_1, \cdots, v_n) b_n(v_1, \cdots v_n)$$

   and

2. *the strong homotopy Jacobi identity: for $n \in \mathbb{N}$ and any nuple $v_1, \cdots, v_n$ of homogeneous elements in V (i.e. all in a single V_i),*

$$\sum_{\substack{i,j \in \mathbb{N} \\ i+j=n+1}} \sum_{\sigma \in US(i,j-1)} \chi(\sigma, v_1, \cdots, v_n)(-1)^{i(j-1)} b_j(b_i(v_{\sigma(1)}, \cdots, v_{\sigma(i)}), v_{\sigma(i+1)}, \cdots, v_{\sigma(n)}) = 0$$

where US stands for the unshuffle permutations*.*

Definition 4.3.8. *A permutation $\sigma \in S_n$ is called an $(k, n-k)$* unshuffle *if $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(n)$.*

Observe that for $n \leq 2$, the definition of an $L_\infty-$algebra corresponds with that of a dg-Lie algebra, so the latter can in a sense be viewed as a truncation of the former. This can be seen in that the strong homotopy Jacobi identity in this case simplifies to the graded Jacobi identity and that the unary bracket, b_1, acts as d, and is Leibniz-compatible with the bracket, or b_2.

As mentioned at the end of the previous section and explained in [3, Chapter 3.1.1], the Maurer-Cartan elements of a $L_\infty-$algebra can be associated with a formal moduli problem. This works as follows.

Construction 4.3.3. *(Formal moduli problems from Maurer–Cartan elements of $L_\infty-$algebras) Let $\mathfrak{g}$ be an $L_\infty-$algebra. It then has levelwise brackets, and correspondingly its Maurer–Cartan equation on elements of cohomological degree* 1 *is defined to be*

$$d + \sum_{n>1} \frac{1}{n!} b_n(-, \cdots, -) = 0 \tag{4.1}$$

where the b_n terms stand for the $n^{th}-$order Lie brackets, and take n inputs. The Maurer–Cartan functor*, for $\mathfrak{g}$ a $L_\infty-$algebra, and $A \in Art^{dg}_{\mathbb{K}}$,*

$$MC_{\mathfrak{g}} : Art^{dg}_{\mathbb{K}} \to sSet$$

$$MC_{\mathfrak{g}}(A) = MC(\mathfrak{g} \otimes m_A \otimes \Omega(\Delta^n)),$$

takes as input a dg Artin algebra and returns the Maurer-Cartan elements of the $L_\infty-$algebra formed by $L \otimes \mathfrak{m}_A$ (one may show that the tensor product of a $L_\infty-$algebra with a commutative dga is again an $L_\infty-$algebra). In Construction 4.3.4, we saw that the form of the simplicial sets associated to a dg Artinian algebra are of the form

$$m_A \otimes \Omega^*(\Delta^n)$$

where $\Omega^*(\Delta^n)$ is again the de Rham complex of differential forms on the simplicial set Δ^n. To determine the entire simplicial structure, one looks at all values of n. This time, however, we are looking at the simplicial structure of the elements of the tensor algebra, so elements of the form

$$\alpha \in \mathfrak{g} \otimes m_A \otimes \Omega^*(\Delta^n).$$

As was the case in Construction 4.3.2, these are associated to a formal moduli problem.

To make contact with [3], the formal moduli problem associated to $\mathfrak{g}$ in this way is in that source known as $B\mathfrak{g}$.

This association provides one very direct way of working with formal moduli problems: via the manipulations of the corresponding $L_\infty-$algebra. The next subsection will state an underlying form of this connection.

4.3.2 Fundamental Equivalence

One motivation for introducing formal moduli problems here at all is so that they can be used to encode the "physical fields" of a classical field theory; the solutions to the equations of motion. Since the formalism of factorization algebras works predominantly with L_∞ algebras (or frequently the special case of $dg-$Lie algebras), one needs to establish a way to navigate between these and formal moduli problems, and that is encoded in the following theorem seen in [7] (not stated in its most general version, but instead rather colloquially).

Theorem 4.3.1. *Let $\mathbb{K}$ be a field of characteristic zero. Then there is an equivalence of $\infty-$categories between formal moduli problems and dg Lie algebras over this field.*

We will not prove this here, and shall only note that it holds for the case relevant to factorization algebras, i.e. when looking at formal pointed moduli problems, as well as more generally.

To give some intuition into this equivalence, we will peek at the functors of which this equivalence of categories is comprised.

Let FMP denote the $(\infty)-$category of formal moduli problems and $dgLa$ that of differential graded Lie algebras, both with the morphisms as structure-preserving maps. Recall that since there is a classical $1-$categorical formulation of this relation, it makes sense to consider both equivalence of classical categories, and that of $(\infty, 1)-$categories.

As we saw above, the functor from dgLa to FMP is the Maurer–Cartan functor. This functor has an inverse, coming from taking the tangent complex of FMP, which we are not discussing here, and the two together form an equivalence of $\infty-$categories.

4.4 From Formal Moduli Problems to Factorization Algebras

Now we have all the content necessary in place to at least trace a route from a given formal moduli problem to the structure of a factorization algebra on the cohomology groups of the corresponding BV-BRST complex (the observables of the associated physical theory).

From the correspondence in Theorem 4.3.1, any given formal moduli problem may be described as the Maurer–Cartan functor of a corresponding dgLa. In the case of a field theory, one may, over each open neighborhood U on the worldsheet, examine the solutions to the Euler–Lagrange equations. This collection of fields forms the field space of the theory over each U. If one picks some solution in this field space and expands around it, this chosen field may serve as a "basepoint" for the formal moduli problem associated to this moduli space. The data of this FMP is determined, as previously discussed, by the order-wise (with respect to this perturbation expansion) solutions to the equations of motion. Taking the Chevalley–Eilenberg complex of this will produce the BV–BRST complex of the theory (with respect to this choice of basepoint), and the cohomology groups of this Chevalley–Eilenberg complex describe the observables of the theory. One may show that these cohomomlogy groups have the structure of a factorization algebra.

Schematically, this chain of relationships looks like the following:

$$FMPs \xleftarrow[MCs]{} dgLas \xrightarrow{Sym} BV - BRST \xrightarrow{H^\bullet} PreFAs \xrightarrow{Gluing} FAs.$$

where the maps are from left to right the Maurer–Cartan equations, taking the symmetric algebra, calculating cohomology of the total complex of the BV-BRST double complex, and a gluing condition analogous to the pre-sheaf-to-sheaf gluing axiom. In this section, we will further explain these correspondences.

4.4.1 Between Maurer–Cartan Equations and Physical Fields

In this subsection, we present how to construct a pointed formal moduli problem of fields given a classical field theory. An essential point here is, as illustrated in [3], that the moduli of the solutions to the equations of motion of a specified theory can be described by based formal moduli problems.

All of the field theories we work with here are perturbative, so to probe the space of solutions to the equations of motion, one may expand about some particular such solution, ϕ_0, to search for others in some small ϕ_0-surrounding neighborhood. Following the notation of [3], one may construct an expansion around ϕ_0 in terms of some small parameter ϵ. This returns

$$\phi_\epsilon = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \cdots ,$$

and one may check if such an expansion is in the space of physical fields by applying relevant equations of motion to it orderwise, with the answer affirmative if this expansion satisfies the Euler–Lagrange equation. In the case here, ϵ is taken to be nilpotent, meaning that for some n, $\epsilon^n = 0$.

One will note the similarity between the previous statement and the condition that a (dg) Artinian algebra have a maximal ideal. This is not an accidental formal analog, rather an integral piece of the formalism in use: in the language of formal deformation theory, in this search for physical fields described by terminating expansions as written above, we may look for deformations parametrized by the Artin ring $[\epsilon]/\epsilon^n$.

Consider the space of fields as a derived stack (recall that this is actually what $X/\mathfrak{g}$ is, it is just computationally direct to deal with it in terms of its BV-BRST complex, so that is what is done, then without explanation, in the previous chapters). To every open neighborhood $U \in X$, assign a formal moduli problem, denoted here by $M(U)$, containing some solution, ϕ_0, as introduced above. Again as above, check for other solutions in the region by perturbing this ϕ_0. The space containing (here terminating) expansions around some particular solution is some completion $\hat{M}(U)$ of $M(U)$. Recall as well that a formal moduli problem is defined to be a functor

$$F : Art_{\mathbb{K}}^{dg} \to Sset$$

satisfying several properties listed above. Let A be a dg Artin algebra with maximal ideal $\mathfrak{m}_A$. Since by definition, $A/\mathfrak{m}_A$ is $\mathbb{K}$ and contractible, it may be viewed as a point. Then, the formal moduli problem functor can be viewed as mapping this point to the solution around which the perturbative expansion takes place.

Using the aforestated equivalence of $\infty-$categories between formal moduli problems and dgLas, it is possible, under the Maurer-Cartan functor and its counterpart (from FMPs to dgLas) in the equivalence of categories, to consider this formal moduli problem data in terms of a specific dgLa. The Maurer-Cartan elements corresponding to the nested ideals of the chain condition on Artinian algebras correspond to the coefficients of the higher-order terms in the expansion of ϕ_0. The value of N capping the chain of ideals is the same as the exponent of the highest-order term in the perturbation expansion of ϕ_0.

Note that while this chain of viewpoints is a nice way of obtaining some information about the original derived stack of fields, it does not offer a complete picture of that entity, for this sequence causes a significant loss of data: we go from describing the entirety of the field space to just looking at locally at solutions near one particular fixed one (the basepoint of the formal moduli problem, which is chosen). This is an acceptable setting for perturbative quantum field theory, but to understand other field theories, one may require the preservation of more information from the derived stack itself.

We shall now sketch out how the solutions to these "levels" of equations of motion correspond to the Maurer–Cartan elements of some dgLa, and hence to a formal moduli problem.

Recall that the Maurer–Cartan equations of an $L_\infty-$algebra are of the form of Equation 4.1. For $n = 1$, the Maurer–Cartan bracket is defined in terms of the "one bracket", or differential, so that for ϕ an element of the $L_\infty-$algebra, the equation

$$d\phi = 0$$

means that ϕ is a level one Maurer–Cartan element. Similarly, the level-two Maurer–Cartan equation spelled out is

$$d(-) + \frac{1}{2}[-,-] = 0,$$

the level-three one is

$$d(-) + \frac{1}{2}[-,-] + \frac{1}{6}l_3(-,-,-) = 0,$$

et cetera. In Chapter 2 when we calculated the Euler–Lagrange equations of the $bc-$CFT, we suggestively saw that they were of the form

$$\partial b = \partial c = 0.$$

This inspection of the $bc-$CFT reveals a lack of higher bracket terms arising from the equations of motion, terminating the analogy rather early. Note that in an interacting theory, such as that of the holomorphic bosonic string in [4], one will obtain terms containing b_n for $n \geq 2$.

It is possible to find solutions to the Euler–Lagrange equations on any open submanifold of Σ a differentiable manifold (the worldsheet of the theory), and those restricted Euler–Lagrange equations also trace out the elements of a dgLa and hence a formal moduli problem. This gives a formal moduli problem, $EL(U)$, corresponding to each open set $U \subset M$. The relationships between these formal moduli problems will induce the prefactorization algebra structure on the observables.

4.4.2 The Prefactorization Algebra Structure on Physical Observables

As is described in the first section of [3], the formal moduli problems $EL(U)$ can be patched together so that the $\mathscr{O}(EL(U))$ satisfy the axioms of a prefactorization algebra. As mentioned there and earlier in this text, there is a direct parallel with sheaf-like structures: a prefactorization algebra is axiomatically the same as a precosheaf.

4.4.3 The Factorization Algebra Structure on Physical Observables

The factorization algebra structure on the observables can be induced by the sheaf structure of EL. This is roughly because showing that EL is a sheaf indicates that the space of formal moduli problems corresponding to the theory, one associated to each $U \subset X$ forms a sheaf, and a "dual" of that gives a cosheaf consisting of the algebras of functions $\mathscr{O}(EL(U))$ for all $U \subset M$ open. A factorization algebra is, recall, essentially a cosheaf with respect to the tensor product, rather than the direct sum. This tells us that since for two disjoint open sets U and V of X, $EL(U \sqcup V) = EL(U) \times EL(V)$, if we take the factorization algebra $\mathcal{F}$ to be $\mathscr{O}(EL(U \sqcup V))$, it follows that this is equal to $\mathscr{O}(EL(U)) \otimes \mathscr{O}(EL(V))$.

Bibliography

[1] *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos.* Lecture Notes in Mathematics, Vol. 269. Springer-Verlag, Berlin-New York, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

[2] Kevin Costello and Owen Gwilliam. *Factorization Algebras in Quantum Field Theory*, volume 1 of *New Mathematical Monographs*. Cambridge University Press, 2016.

[3] Kevin Costello and Owen Gwilliam. *Factorization Algebras in Quantum Field Theory*, volume 2 of *New Mathematical Monographs*. Cambridge University Press, 2021.

[4] Owen Gwilliam and Brian Williams. The holomorphic bosonic string, 2017.

[5] V. Hinich. Dg coalgebras as formal stacks, 1998.

[6] Maxim Kontsevich and Yan Soibelman. Deformations of algebras over operads and the Deligne conjecture. In *Conférence Moshé Flato 1999, Vol. I (Dijon)*, volume 21 of *Math. Phys. Stud.*, pages 255–307. Kluwer Acad. Publ., Dordrecht, 2000.

[7] Jacob Lurie. *Derived Algebraic Geometry X: Formal Moduli Problems.* 2011.

[8] K. Mackenzie. *Lie groupoids and Lie algebroids in differential geometry*, volume 124 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1987.

[9] J. P. May. *The geometry of iterated loop spaces.* Lecture Notes in Mathematics, Vol. 271. Springer-Verlag, Berlin-New York, 1972.

[10] Joseph Polchinski. *String Theory*, volume 1 of *Cambridge Monographs on Mathematical Physics*. Cambridge University Press, 1998.

[11] Joseph Polchinski. *String Theory*, volume 2 of *Cambridge Monographs on Mathematical Physics*. Cambridge University Press, 1998.

[12] Jean Pradines. Théorie de Lie pour les groupoïdes différentiables. Calcul différenetiel dans la catégorie des groupoïdes infinitésimaux. *C. R. Acad. Sci. Paris Sér. A-B*, 264:A245–A248, 1967.

[13] Daniel G. Quillen. *Homotopical algebra*. Springer, 1967.

[14] M. Schottenloher. *A mathematical introduction to conformal field theory*. Lecture notes in physics, 759. Springer, 2008.

[15] John Tate. Homology of Noetherian rings and local rings. *Illinois J. Math.*, 1:14–27, 1957.

[16] Urs Schreiber. Geometry of physics – perturbative quantum field theory - mathematical quantum field theory - lecture notes, winter term 2017. `http://ncatlab.org/nlab/show/geometry%20of%20physics%20--%20perturbative%20quantum%20field%20theory`, June 2022. Revision 198.

[17] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.