From M-Theory to Knot Theory via Topological Field Theory

Verónica Errasti Díez

Doctor of Philosophy

Department of Physics – Faculty of Science
McGill University, Montréal
Québec, Canada

March 27, 2018

A thesis submitted to McGill University
in partial fulfillment of the requirements of the degree of Doctor of Philosophy

Dedicada a mi madre, mi padre y mi hermano;
con amor infinito y en eterna gratitud.

Dedicated to my mother, my father and my brother;
with unbound love and in immense gratitude.

Después de todo, todo ha sido nada,
a pesar de que un día lo fue todo.
Después de nada, o después de todo
supe que todo no era más que nada.

Grito "¡Todo!", y el eco dice "¡Nada!".
Grito "¡Nada!", y el eco dice "¡Todo!".
Ahora sé que la nada lo era todo,
y todo era ceniza de la nada.

No queda nada de lo que fue nada.
(Era ilusión lo que creía todo
y que, en definitiva, era la nada.)

Qué más da que la nada fuera nada
si más nada será, después de todo,
después de tanto todo para nada.

– José Hierro

After all, all has been naught,
even though it once was it all.
After naught, or after all
I knew that all was but naught.

I shout "All!", and the echo says "Naught!".
I shout "Naught!", and the echo says "All!".
Now I know that the naught was it all,
and all was ash of the naught.

Naught is left of what was naught.
(It was delusion what I believed all
and that, ultimately, was the naught.)

What does it matter that the naught were naught
since more naught it shall be, after all,
after so much all for naught.

– José Hierro

Statement of Originality

The present thesis is based on original work done in collaboration with Professors Keshav Dasgupta, Pichai Ramadevi and Radu Tatar. The results here presented constitute original work that appeared in the following scientific articles:

- K. Dasgupta, **V. Errasti Díez**, P. Ramadevi and R. Tatar, *“Knot invariants and M-theory: Hitchin equations, Chern-Simons actions, and surface operators”*, Phys. Rev. D **95**, no. 2, 026010 (2017), doi:10.1103/PhysRevD.95.026010, [arXiv:1608.05128 [hep-th]].

- **V. Errasti Díez**, *“A companion to ”Knot invariants and M-theory I”: proofs and derivations”*, Phys. Rev. D **97**, no. 2, 026001 (2018) doi:10.1103/PhysRevD.97.026001 [arXiv:1702.07366 [hep-th]].

Regarding the first article above mentioned, I contributed to all aspects and at all stages of the project: definition of the problem, calculations, interpretation of the results and writing. On the other hand, I am the sole author of the second article.

In addition, the thesis contains two further chapters which represent related independent work by the author. This extra material has benefited from discussions with all the aforementioned Professors as well as with Doctors Rohit Jain and Gopala Krishna.

Acknowledgements

I am specially indebted to Professor Keshav Dasgupta, for his supervision and advice. His intuitive yet insightful approach to physics combines with his great communication skills and his amiable personality to yield an excellent supervisor. I am also thankful to Professors A. P. Balachandran, Rohini M. Godbole, Pichai Ramadevi, Andrés F. Reyes Lega, Aninda Sinha, Radu Tatar and Sachindeo Vaidya for the investigations we have carried out together. My gratitude extends to my collaborators Nirmalendu Acharyya and P. N. Bala Subramanian. For all the physics discussions together, I must thank Maxim Emelin, Rohit Jain, Gopala Krishna, Arnaud Lepage-Jutier, Evan McDonough, Mahul Pandey and Sam Selmani. I truly appreciate the technical help of Laure-Anne Douxchamps, Jatin Panwar and Julien Pinel. I am indebted to the Centre for High Energy Physics at the Indian Institute of Science for hospitality during my visit from September 2014 to December 2015.

No tengo palabras suficientemente hermosas para quienes me habéis dado la fuerza y el afecto necesarios para escribir esta tesis. Todo el amor del que soy capaz es para vosotras y vosotros. Mi madre merece elogios aparte, por la generosidad y ternura sin límites con las que me ha motivado a diario a lo largo de todo mi doctorado. Si fuera poeta, verdad y belleza requerirían que mis versos más dulces fueran para ella.

I do not have beautiful enough words for the people who gave me the strength and warmth I needed to write this thesis. All the love I am capable of is for them. My mother deserves extra praise, for the bottomless generosity and tenderness with which she has motivated me daily all through my PhD. If I were a poet, truth and beauty would require me to write the sweetest verses for her.

Abstract

We construct two M-Theory models and relate them to each other through a series of dualities. In doing so, we provide a unifying scheme of the supergravity proposals by Ooguri-Vafa and Witten to study knots and their invariants. Subsequently, we focus in the world-volume gauge theory following from one of the constructed models. This is a four-dimensional, $\mathcal{N} = 4$ Yang-Mills theory with generic gauge group $SU(N)$, in the presence of a boundary. We obtain its Hamiltonian and, for time-independent field configurations, we find that the equations of motion minimizing its energy are specific Hitchin integrable systems, along with certain "consistency conditions". All these results were first derived by Kapustin-Witten applying localization techniques to the path integral formulation of the gauge theory. Hence, our model provides a simplified scenario for calculations. Additionally, it allows for an interpretation of all the parameters in the theory in terms of supergravity quantities. We also derive the corresponding half-BPS boundary conditions. Upon a topological twist, we show that the boundary physics is governed by a complexified Chern-Simons action, thus providing a suitable subspace for the embedding of knots in our setup. Finally, we include knots in our model. At the M-Theoretical level, this is achieved by adding a given M2-brane state to the previously constructed model. In the bulk of the associated gauge theory, this M2-brane can be understood as a surface operator, whereas in the boundary it appears as a Wilson loop.

Abrégé

Nous construisons deux modèles de théorie M, que nous relions l'un á l'autre par une série de dualités. Ce faisant, nous fournissons un cadre unificateur aux supergravités proposées par Ooguri-Vafa et Witten pour étudier les noeuds et leurs invariants. Nous nous intéressons ensuite à la théorie de jauge dans le volume d'univers qui découle de l'un des modèles construits. Celle-ci est une théorie de Yang-Mills $\mathcal{N} = 4$ quadridimensionnelle possédant comme groupe de jauge $SU(N)$ en présence d'un bord. Nous obtenons son hamiltonien et, pour des configurations de champs indépendantes du temps, nous trouvons que les équations du mouvement qui minimisent l'énergie sont des systèmes intégrables définis de Hitchin, accompagnés de certaines " conditions de validité". Tous ces résultats ont été précédemment dérivés par Kapustin et Witten en appliquant des techniques de localisation à la formulation de la théorie de jauge en termes d'intégrale de chemin. Notre modèle apporte donc un scénario de simplification des calculs. De plus, il permet une interprétation de tous les paramètres de la théorie en termes de quantités de supergravité. Nous dérivons également les conditions au bord semi-BPS correspondantes. Par une torsion topologique, nous montrons que la physique au bord est régie par une action de Chern-Simons complexifiée, fournissant ainsi un sous-espace propice à l'insertion des noeuds dans notre cadre. Enfin, nous incluons les noeuds dans notre modèle. Au niveau de la théorie M, cela est réalisé par l'ajout d'un état de M2-brane donné au modèle construit précédemment. Dans l'espace intérieur associé à la théorie de jauge, cette M2-brane peut être vue comme un opérateur de surface, tandis qu'au bord elle apparaît comme une boucle de Wilson.

Contents

List of tables

List of figures

Preface

As already pointed out in the statement of originality, this thesis comprises results first obtained in [1,2]. In more detail, parts I and II are based on [2]. This in turn is a companion paper to [1] which complements, extends and generalizes its key results in the manners specified in section 1.2 of [2]. Both these works constitute what the authors refer to as "Part 1" in a series of papers to appear whose goal is to establish a direct connection between String Theory and Knot Theory via the construction of concrete models. We believe that having specific frameworks for computation hugely simplifies the approach to the complex topic of figuring out the many roles that knots possibly play in theoretical physics. In this context, part III contains new advancements to the aforementioned aim and, in due time, will probably be included in the paper(s) that shall form "Part 2" of the project.

Other works by the author, developed during the PhD period but not included in this thesis, are the following:

- N. Acharyya and **V. Errasti Díez**, *"Monopoles, Dirac operator, and index theory for fuzzy $SU(3)/(U(1)\times U(1))$"*, Phys. Rev. D **90**, no. 12, 125034 (2014), doi:10.1103/PhysRevD.90.125034, [arXiv:1411.3538 [hep-th]].
- **V. Errasti Díez**, R. M. Godbole and A. Sinha, *"Improvements to the Froissart bound from AdS/CFT"*, Phys. Lett. B **746**, 285 (2015), doi:10.1016/j.physletb.2015.05.016, [arXiv:1504.05754 [hep-ph]].
- N. Acharyya, A. P. Balachandran, **V. Errasti Díez**, P. N. Bala Subramanian and S. Vaidya, *"BRST Symmetry: Boundary Conditions and Edge States in QED"*, Phys. Rev. D **94**, no. 8, 085026 (2016), doi:10.1103/PhysRevD.94.085026, [arXiv:1604.03696 [hep-th]].

Together with M. Pandey, the author is also involved in the development of a supersymmetric extension of the matrix model for $SU(3)$ Yang-Mills theory first proposed in [3]. Additionally, the author is working with N. Acharyya, A. P. Balachandran and A. F. Reyes Lega in a book preliminary entitled "Constrained Hamiltonian Dynamics". The book intends to cover a wide variety of topics, ranging from a review of the classical dynamics of constrained systems to advanced topics related to the quantization of such systems, such as boundary value problems.

Chapter 1: Introduction

Knot Theory is the branch of Topology that studies knots. In this context, a knot is an embedding of a circle in a three-dimensional Euclidean space, or in its compact analogue: the three-sphere. Two such knots are said to be equivalent iff there exists an ambient isotopy transforming one to the other. This formal definition of equivalent knots is, unfortunately, insufficient in practice. To such a great extent that one of the main unresolved problems in Knot Theory consists in distinguishing knots. That is, determining when two knots are (or are not) equivalent. This is known as the *Classification Problem of Knots*. Very elaborate algorithms exist to this end, yet the problem persists.

Another approach to the knot differentiation puzzle involves knot invariants: numbers, polynomials or homologies defined for each knot which remain unchanged for equivalent knots. Interestingly, invariants such as Khovanov and Floer homologies are capable of telling apart the unknot from any other non-equivalent knot. Although this is a phenomenal achievement, there is still much to be accomplished. So much so that, at present, it is not known whether a knot invariant exists which is capable of distinguishing all inequivalent knots.

There are various ways to compute knot invariants. Mathematicians use recursive relations, known as skein relations, to compute the Conway [4, 5], Alexander [6] and Jones [7] polynomials, among others. The first physics understanding of knot invariants appeared much later, in the groundbreaking work [8]. In it, knot polynomials are obtained as expectation values of the holonomy of a Chern-Simons gauge field around a knot carrying a representation of the underlying (compact) gauge group. For instance, the Jones and HOMFLY-PT [5, 9] polynomials follow from considering the defining representations of $SU(2)$ and $SU(N)$, respectively.

Starting roughly at the same time and up to now, there have been a number of works that address the study of knot invariants from the point of view of four-dimensional physics: [10–16], to mention but a few. It is within this context that the present thesis attempts to provide a unifying and neat scheme of the results obtained so far and contribute new insights. Specifically, we will first establish a precise connection between the models in [14] and [11]. Then, we will reproduce the conclusions of [14] in the low energy supergravity description of a given M-Theory model. As we shall see, our approach leads to a strikingly simple analysis in the context of the usual classical Hamiltonian formalism. Last but not least, we will explain in details how knots are to be embedded in our model. It bears emphasizing that the appearance of knots in physical theories is generally not clarified. Instead, knot invariants are computed in setups that leave the reader wondering where the knot came from to begin with.

This thesis, together with [1, 2], constitute the first step in the path towards a clear and concrete derivation of knot invariants from M-Theory, compactified down to four dimensions. The simplest knot invariant, the so-called linking number, was computed in [1]. We leave the realization of more challenging invariants to the sequel(s).

Although the present thesis has knot invariants as its main (not yet achieved) motivation, it touches upon a wide range of topics in theoretical physics that have recently gathered plenty of attention. For example, we will discuss torsion classes, topological twists and surface operators. We will also briefly mention a connection to Morse Theory. An extension of our construction, currently under development, seems capable of bridging over to Seiberg-Witten Theory [17, 18] and certain Theories of Class S [19, 20]. All this points to knots as objects that play a plethora of vital roles in fundamental physics.

Beyond the theoretical realm of our interest, it should be noted that knots are not only abstract mathematical objects. Rather, they are existing, physical entities that have been observed in a wide variety of classical contexts. To mention some of the most relevant and surprising scenarios, one can create and then detect knots in condense matter systems like optical beams [21, 22] and nematic liquid crystals [23–25], but also in water [26] and even in DNA [27]! Additionally, knots have recently been discovered in a quantum framework involving Bose-Einstein condensates [28]. In short, knots are not only fascinating objects at an abstract level, but also a hot topic of research at an experimental level −even though this work shall not be concerned with the latter approach.

1.1 Organization of the thesis

As hinted by the title itself, the thesis is arranged in three parts. In part I, we construct two distinct M-Theory configurations that have all the necessary features to harbor knots. We refer to these as (M,1) and (M,5). Specifically, chapter 2 is devoted to the construction of (M,1), starting from the well-known D3-NS5 system in type IIB String Theory considered in [14]. The very same D3-NS5 system is also the basis for the construction of (M,5), presented in chapter 3. It is worth pointing out that (M,1) is dual to the model in [14], whereas (M,5) is dual to the resolved conifold in the presence of fluxes considered in [11].

Part II focuses on the study of the world-volume gauge theory that follows from appropriately compactifying model (M,1). In particular, chapter 4 deals with the derivation of its action. The corresponding Hamiltonian is obtained in chapter 5, where we also minimize its energy for static configurations of the fields. We thus find the BPS conditions of the gauge theory. After the energy minimization process, the Hamiltonian reduces to an action in the three-dimensional boundary subspace, as proved in chapter 6. Further, a careful analysis of the symmetries and physics of this boundary shows that knots can be consistently embedded in its Euclidean version, after a certain topological twist is performed.

At last, part III shows how knots can be consistently included in the model. This is achieved by entertaining a given M2-brane state in (M,1). In chapter 7 we consider a toy model M2-brane that allows for explicit computations but yields incorrect results. Chapter 8 suitably reorients the toy model M2-brane so that, in the world-volume gauge theory, it appears as a surface operator. The surface-operator-M2-brane sources a Wilson loop contribution to the boundary action. The thesis concludes in chapter 9 with a summary of the main results and a discussion of the challenging goals we intend to achieve through the model here developed.

Due to the considerable length of the computational details and arguments presented, we have included a graphical summary of the thesis. It works in the following manner. By looking at the fifteen figures (and their captions) here shown, the reader can quickly grasp the fundamental logic articulating each part and chapter. Additionally, most of the figures refer to equations in the text: these constitute our main results. Hence, the figures can be used to efficiently locate any particular information of interest within the text, as well as to gain a bird's eye view of the contents that follow. The graphical summary is further supplemented by a brief recapitulation paragraph in italic typeface at the end of chapters 2-8.

Part I
Two M-theory models to study knots: (M,1) and (M,5)

As the title suggests, in this first part we will construct two different M-Theory configurations that, as we shall show in due time, provide an appropriate framework for the study of knots. We will refer to these configurations as (M,1) and (M,5). Both of them will be directly obtained from the well-known type IIB system of a D3-brane ending on an NS5-brane considered in [14]. Chapter 2 contains the construction of (M,1) from the D3-NS5 system, while chapter 3 derives (M,5). As will be argued towards the end of this first part, in section 3.2.2, (M,5) is intimately related to the model in [11]. Consequently, this part lays the ground for an explicit connection between the two seemingly different supergravity approaches in [11,14] to study knots.

Before proceeding to the details, a word of warning: we will consider multiple intermediate type IIA, IIB and M-Theory configurations on our way to (M,1) and (M,5). Figure 1 provides a visual sketch of the overall logic in this part. Hence, the reader may find it clarifying to come back to this image while reading through chapters 2 and 3.

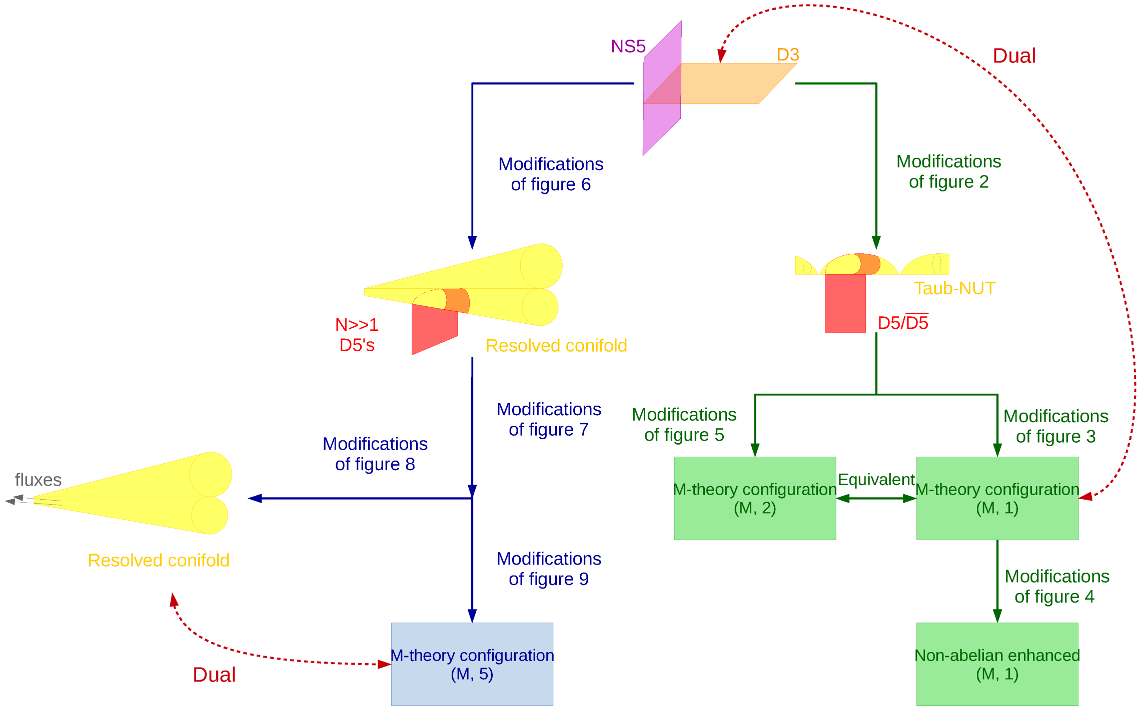

Figure 1: Graphical summary of part I. Starting from the type IIB D3-NS5 system of [14], we construct two different M-Theory configurations where knots and their invariants can be studied. We refer to these as (M,1) and (M,5). (The configuration (M,2) is equivalent to (M,1) for the purposes of our work, yet computationally tougher to handle. We will thus focus our efforts in the study of (M,1) only.) Note that (M,1) is dual to the supergravity model [14]. Similarly, (M,5) is dual to the resolved conifold in the presence of fluxes considered in [11]. The right-hand side of the figure, colored green, schematizes the contents of chapter 2. The left-hand side, in blue, depicts the discussion in chapter 3.

Chapter 2: Construction of (M,1) from the D3-NS5 system

As we just mentioned, the starting point of our analysis is the well-known type IIB String Theory configuration of a D3-brane ending on an NS5-brane. In more detail, we consider Minkowski spacetime $\mathbb{R}^{1,9}$, with mostly positive metric signature. We denote the coordinates as $(t,\, x_1,\, x_2,\, x_3, \theta_1,\, \phi_1,\, \psi,\, r,\, x_8,\, x_9)$. (The identifications $(x_4 \equiv \theta_1,\, x_5 \equiv \phi_1,\, x_6 \equiv \psi,\, x_7 \equiv r)$ will shortly become sensible.) We take the D3-brane to stretch along $(t,\, x_1,\, x_2,\, \psi)$ and the NS5-brane along $(t,\, x_1,\, x_2,\, x_3,\, x_8,\, x_9)$. The $U(1)$ gauge theory on the D3-brane has $\mathcal{N} = 4$ supersymmetry and the intersecting NS5-brane provides a half-BPS boundary condition. The world-volume gauge theory thus has $\mathcal{N} = 2$ supersymmetry as a whole. This is, essentially, the starting point of [14] as well. (The only difference is that, in [14], an axionic background $\mathcal{C}_0$ is switched on. We will elaborate on this point in section 2.2.)

Next, we do three modifications to the above setup. These are depicted schematically in figure 2 and discussed in the following.

- First, we introduce a second NS5-brane, parallel to the first one and which also intersects the D3-brane. This means that ψ, the direction of the D3-brane that is orthogonal to the NS5-branes, becomes a finite interval. The inclusion of the second NS5-brane halves the amount of supersymmetry of the gauge theory on the D3-brane. However, we consider the case when the ψ interval is very large (that is, the two NS5-branes are far from each other). Then, near the original NS5-brane, effectively no supersymmetry is lost in this step.

- Second, we do a T-duality to type IIA String Theory along x_3. As a result, we now have a D4-brane (instead of a D3-brane) between the same two NS5-branes of before.

- Third, we do a T-duality back to type IIB along ψ. The NS5-branes thus disappear and give rise to a warped Taub-NUT space in the $(\theta_1,\, \phi_1,\, \psi,\, r)$ directions. (This justifies the coordinate relabeling above.) As argued in [29], because ψ is a finite interval, the D4-brane converts to a D5/$\overline{\text{D5}}$ pair which wraps the ψ direction and stretches along the radial direction r.

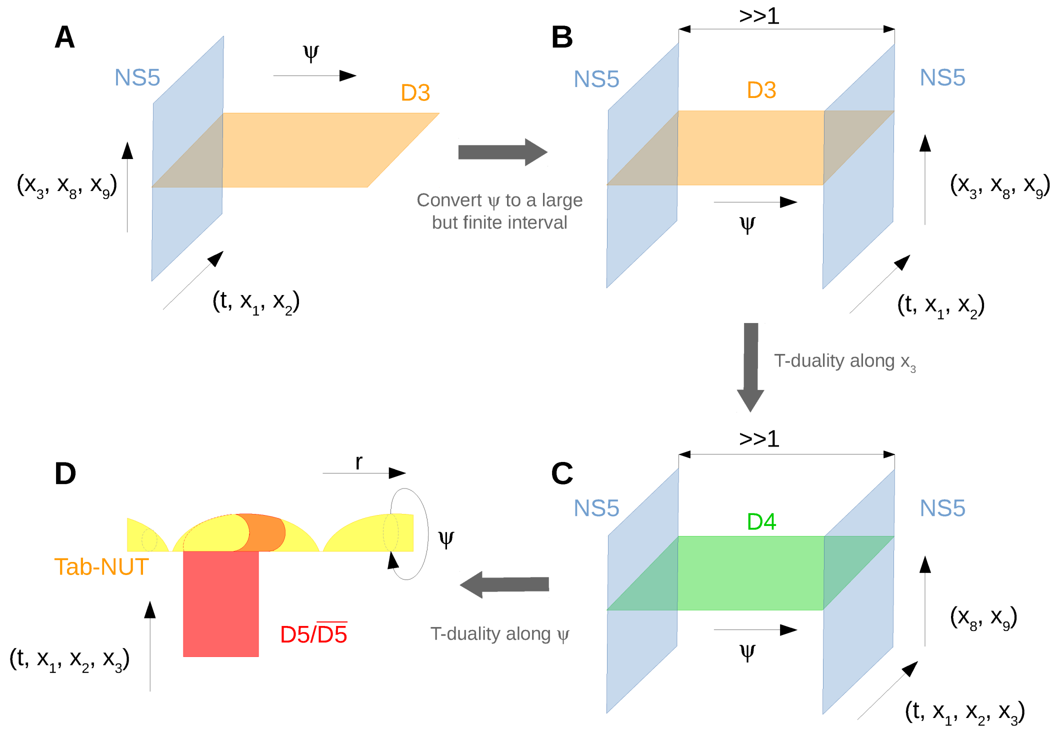

Figure 2: Caricature of the modifications to the D3-NS5 system described in chapter 2. This chain of dualities is done so that the corresponding metric can be written: the geometry of **D** is known. **A:** The type IIB D3-NS5 system. The corresponding world-volume gauge theory has $\mathcal{N} = 2$ supersymmetry. The D3-brane spans the $(t,\, x_1,\, x_2,\, \psi)$ directions and the NS5-brane the $(t,\, x_1,\, x_2,\, x_3,\, x_8,\, x_9)$ directions. The $(\theta_1,\, \phi_1,\, r)$ directions are suppressed. **B:** Introducing a second NS5-brane, parallel to the first one, converts the ψ direction into an interval. We take this interval to be large (but finite) in order to effectively retain the same amount of supersymmetry near the original NS5-brane. **C:** A T-duality along x_3 does not affect the parallel NS5-branes, but converts the D3-brane into a D4-brane. **D:** A T-duality along ψ converts the parallel NS5-branes to a warped Taub-NUT space along $(\theta_1,\, \phi_1,\, \psi,\, r)$. The D4-brane converts to a D5/$\overline{\text{D5}}$ pair that wraps the ψ direction and stretches along r. The $(\theta_1,\, \phi_1,\, x_8,\, x_9)$ directions are suppressed.

The geometry corresponding to this last configuration is known (in fact, the three modifications above were done only to be able to write the corresponding metric) and is given by

$$\begin{aligned} ds^2_{(B,1)} =& e^{-\phi}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + e^{\phi} F_4 (dx_8^2 + dx_9^2) \\ &+ e^{\phi}[F_1 dr^2 + F_2(d\psi + \cos\theta_1 d\phi_1)^2 + F_3(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2)], \end{aligned} \tag{2.1}$$

where $e^{-\phi}$ is the usual type IIB dilaton. (Since we will consider many metrics in the ongoing, we adopt the notation $ds^2_{(X,n)}$. Here $X = A$, B, M stands for type IIA, type IIB and M-Theory, respectively and $n \in \mathbb{N}$ is an index to label the different metrics that will occur.) We consider, for simplicity, the following dependence of the warp factors and dilaton[1]:

$$F_i = F_i(r), \qquad F_4 = F_4(r, x_8, x_9), \qquad \phi = \phi(\theta_1, r, x_8, x_9), \qquad i = 1, 2, 3. \tag{2.2}$$

The warped Taub-NUT space metric is, quite obviously, the second line in (2.1).

Let us temporarily move the $\overline{\text{D5}}$-brane far away along the (x_8, x_9) directions (the Coulomb branch) and consider only the D5-brane. This will simplify the flux discussion in the construction of the M-Theory configurations (M,1) and (M,2). Later on, in section 4.2, we will move this $\overline{\text{D5}}$-brane back to its original location and appropriately account for its effects. We will then see that the $\overline{\text{D5}}$-brane plays an important, non-trivial role in our investigations.

It has been known for quite some time now that D-branes carry Ramond-Ramond (RR) charges [30]. In this case that concerns us, the D5-brane sources an RR three-form flux $\mathcal{F}_3^{(B,1)}$ that can be computed as[2]

$$\mathcal{F}_3^{(B,1)} = e^{2\phi} * d\mathcal{J}_{(B,1)}, \tag{2.3}$$

where $\mathcal{J}_{(B,1)}$ stands for the fundamental form of the metric $e^{-\phi} ds^2_{(B,1)}$ along the Taub-NUT and Coulomb branch directions (θ_1, ϕ_1, ψ, r, x_8, x_9), which we call $ds^2_{(1)}$:

$$ds^2_{(1)} \equiv F_1 dr^2 + F_2(d\psi + \cos\theta_1 d\phi_1)^2 + F_3(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + F_4(dx_8^2 + dx_9^2). \tag{2.4}$$

Let us calculate $\mathcal{F}_3^{(B,1)}$ in details next.

[1] As made more precise in section 3.1, a definite choice of the warp factors and dilaton will in general not preserve the $\mathcal{N} = 2$ supersymmetry of the world-volume gauge theory. Consequently, any concrete choice one may wish to consider must be checked to indeed preserve the desired amount of supersymmetry.

[2] For a review on how fluxes can be determined, see [31].

We take the vielbeins of (2.4) to be

$$\begin{aligned} E_{\theta_1}^{(B,1)} &= \sqrt{F_3} e_{\theta_1}^{(B,1)} = \sqrt{F_3} d\theta_1, & E_{\phi_1}^{(B,1)} &= \sqrt{F_3} e_{\phi_1}^{(B,1)} = \sqrt{F_3} \sin\theta_1 d\phi_1, \\ E_{\psi}^{(B,1)} &= \sqrt{F_2} e_{\psi}^{(B,1)} = \sqrt{F_2}(d\psi + \cos\theta_1 d\phi_1), & E_r^{(B,1)} &= \sqrt{F_1} e_r^{(B,1)} = \sqrt{F_1} dr, \\ E_8^{(B,1)} &= \sqrt{F_4} e_8^{(B,1)} = \sqrt{F_4} dx_8, & E_9^{(B,1)} &= \sqrt{F_4} e_9^{(B,1)} = \sqrt{F_4} dx_9. \end{aligned} \tag{2.5}$$

These vielbeins can be used to compute the fundamental form $\mathcal{J}_{(B,1)}$:

$$\begin{aligned} \mathcal{J}_{(B,1)} \equiv& E_{\theta_1}^{(B,1)} \wedge E_{\phi_1}^{(B,1)} + E_{\psi}^{(B,1)} \wedge E_r^{(B,1)} + E_8^{(B,1)} \wedge E_9^{(B,1)} \\ =& F_3 \sin\theta_1 d\theta_1 \wedge d\phi_1 + \sqrt{F_1 F_2}(d\psi + \cos\theta_1 d\phi_1) \wedge dr + F_4 dx_8 \wedge dx_9. \end{aligned} \tag{2.6}$$

The exterior derivative of $\mathcal{J}_{(B,1)}$ is given by

$$d\mathcal{J}_{(B,1)} = (F_{3,r} - \sqrt{F_1 F_2}) \sin\theta_1 dr \wedge d\theta_1 \wedge d\phi_1 + F_{4,r} dr \wedge dx_8 \wedge dx_9, \tag{2.7}$$

where $(F_{3,r},\, F_{4,r})$ stand for the derivatives of $(F_3,\, F_4)$ with respect to r. The Hodge dual of the above, with respect to the metric (2.4), can be easily checked to yield

$$*d\mathcal{J}_{(B,1)} = e^{-2\phi} \left[k_2 (d\psi + \cos\theta_1 d\phi_1) \wedge dx_8 \wedge dx_9 + k_1 \sin\theta_1 d\psi \wedge d\theta_1 \wedge d\phi_1 \right], \tag{2.8}$$

where we have defined

$$k_1 \equiv e^{2\phi} \sqrt{\frac{F_2}{F_1}} \frac{F_3}{F_4} F_{4,r}, \qquad k_2 \equiv e^{2\phi} \sqrt{\frac{F_2}{F_1}} \frac{F_4}{F_3} (\sqrt{F_1 F_2} - F_{3,r}). \tag{2.9}$$

Further using the vielbeins (2.5), we obtain the desired result, the RR three-form flux $\mathcal{F}_3^{(B,1)}$:

$$\mathcal{F}_3^{(B,1)} = e_{\psi}^{(B,1)} \wedge \left(k_1 e_{\theta_1}^{(B,1)} \wedge e_{\phi_1}^{(B,1)} + k_2 e_8^{(B,1)} \wedge e_9^{(B,1)} \right). \tag{2.10}$$

It is important to note that this three-form is not closed: $d\mathcal{F}_3^{(B,1)} \neq 0$. This reflects the presence of the D5-brane in this configuration.

Summing up, the type IIB configuration shown in figure 2**D** can be obtained directly from the well-known D3-NS5 system. It has the metric (2.1), dilaton $e^{-\phi}$ and an RR three-form flux (2.10).

An essential ingredient that makes the study of knots using the D3-NS5 system possible is the presence of a Θ-term in the D3-brane gauge theory. In the case of [14], this term is sourced by an axionic background $\mathcal{C}_0$. In the following section, we will present an *alternative*, computationally simpler way to source the required Θ-term: by further modifying the above setup switching on a non-commutative deformation.

2.1 Sourcing the Θ-term: a non-commutative deformation

The starting point in this section is, of course, the just discussed type IIB geometry in (2.1). We will first T-dualize this to type IIA along ψ. (This means we will move from **D** to **C** in figure 2.) Here, we will do the non-commutative deformation, which will only affect the (x_3, ψ) directions: $(x_3, \psi) \to (\tilde{x}_3, \tilde{\psi})$. This will be followed by another T-duality along $\tilde{\psi}$. At this point, we will have a type IIB configuration capable of sourcing the required Θ-term in the $U(1)$ world-volume gauge theory. Then, we will T-dualize along ϕ_1 to type IIA. Finally, we will lift the resulting configuration to M-Theory. Along the way, we will also study the NS B-field, dilaton and fluxes associated to each geometry considered, which will in turn shed some light into the connection between the non-commutative deformation and the Θ-term. (The precise connection between these two will be shown early in section 5.2, see (5.82).) Figure 3 summarizes the just described chain of modifications and points out the most relevant equations in this section.

Let us go ahead and show in details the above outlined M-Theory construction. We start by rewriting the metric (2.1) in a more convenient way for our present purposes:

$$ds^2_{(B,1)} = ds^2_{(2)} + e^{-\phi}dx_3^2 + e^{\phi}F_2(d\psi + \cos\theta_1 d\phi_1)^2, \tag{2.11}$$

with $ds^2_{(2)}$ defined as

$$ds^2_{(2)} \equiv e^{-\phi}(-dt^2 + dx_1^2 + dx_2^2) + e^{\phi}[F_1 dr^2 + F_3(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + F_4(dx_8^2 + dx_9^2)]. \tag{2.12}$$

We recall that the dilaton here is

$$e^{\phi_{(B,1)}} = e^{-\phi} \tag{2.13}$$

and the RR three-form flux was given in (2.10).

T-dualizing along ψ[3], we get the metric

$$ds^2_{(A,1)} = ds^2_{(2)} + e^{-\phi}dx_3^2 + \frac{e^{-\phi}}{F_2}d\psi^2, \tag{2.14}$$

with associated NS B-field and dilaton

$$B_{(A,1)} = \cos\theta_1 d\psi \wedge d\phi_1, \qquad e^{\phi_{(A,1)}} = (e^{3\phi}F_2)^{-1/2}. \tag{2.15}$$

[3]For a practical compendium of formulae regarding how to perform T- and S-dualities and how to go from (to) type IIA to (from) M-Theory, see section 6.5 in [32].

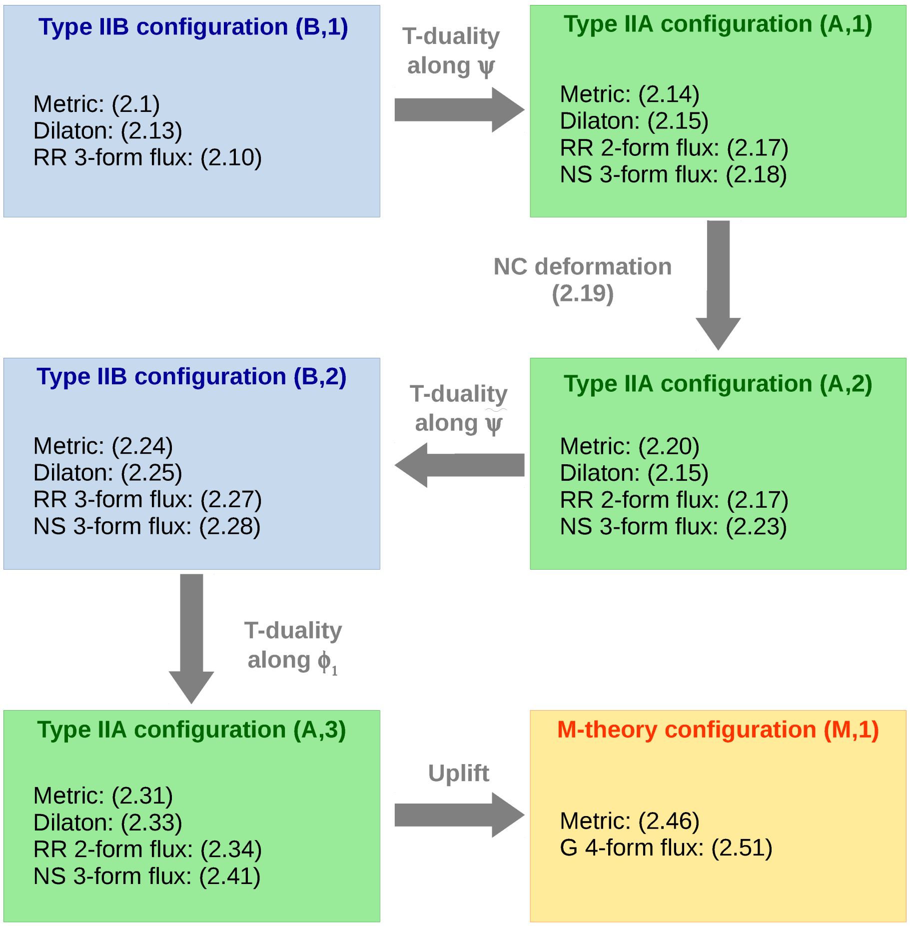

Figure 3: Graphical summary of section 2.1. To the type IIB configuration of figure 2**D** we do a series of modifications in order to source a Θ-term in the $U(1)$ world-volume gauge theory. This is achieved in going from the configuration (B,1) to (B,2). The presence of a Θ-term is essential to, later on, construct a three-dimensional space with the required features to allow for the realization of knots. The (B,2) configuration is then lifted to M-Theory. The configuration (M,1) (and its non-abelian enhanced version, studied in section 2.1.1) is the first M-Theory construction where knots can be studied.

We take the relevant vielbeins associated to $ds^2_{(A,1)}$ to be

$$\begin{array}{lll} e^{(A,1)}_{\theta_1} = d\theta_1, & e^{(A,1)}_{\phi_1} = \sin\theta_1 d\phi_1, & e^{(A,1)}_{\psi} = d\psi + \cos\theta_1 d\phi_1, \\ e^{(A,1)}_{r} = dr, & e^{(A,1)}_{8} = dx_8, & e^{(A,1)}_{9} = dx_9. \end{array} \tag{2.16}$$

As for the fluxes, the RR three-form flux in (2.10) now converts to an RR two-form flux:

$$\mathcal{F}^{(A,1)}_2 = k_1 e^{(A,1)}_{\theta_1} \wedge e^{(A,1)}_{\phi_1} + k_2 e^{(A,1)}_8 \wedge e^{(A,1)}_9. \tag{2.17}$$

Note that, for an arbitrary value of the warp factors and dilaton, the above flux is not closed: $d\mathcal{F}^{(A,1)}_2 \neq 0$. This is consistent with having a D4-brane as a source (see figure 2**C**). The NS three-form flux is given by

$$\mathcal{H}^{(A,1)}_3 = dB_{(A,1)} = -\sin\theta_1 d\theta_1 \wedge d\psi \wedge d\phi_1. \tag{2.18}$$

We will now deform the above type IIA configuration. The non-commutative deformation $(x_3,\, \psi) \to (\tilde{x}_3,\, \tilde{\psi})$ that we will consider is

$$\psi = \cos\theta_{nc}\tilde{\psi}, \qquad x_3 = \sec\theta_{nc}\tilde{x}_3 + \sin\theta_{nc}\tilde{\psi}, \tag{2.19}$$

where $\theta_{nc} \in [0,\, 2\pi)$ is the deformation parameter. Note that the $(x_3,\, \psi)$ directions in $ds^2_{(A,1)}$ form a square torus; that is, a geometry which is isometric to a square with opposite sides identified. Hence, the non-commutative deformation simply inclines the torus. This same deformation was considered in [33], albeit in a different context. Under this deformation, the above type IIA metric changes to

$$ds^2_{(A,2)} = ds^2_{(2)} + e^{-\phi}\left[\frac{\tilde{F}_2}{F_2}\sec^2\theta_{nc} d\tilde{x}_3^2 + \frac{\cos^2\theta_{nc}}{\tilde{F}_2}(d\tilde{\psi} + \tilde{F}_2 \sec^2\theta_{nc}\tan\theta_{nc} d\tilde{x}_3)^2\right], \tag{2.20}$$

where we have defined

$$\tilde{F}_2 \equiv \frac{F_2}{1 + F_2\tan^2\theta_{nc}} \tag{2.21}$$

and $ds^2_{(A,2)}$ has been written in a form suitable for the T-duality along $\tilde{\psi}$ that will soon follow. The NS B-field is also affected by the deformation and now takes the form

$$B_{(A,2)} = \cos\theta_{nc}\cos\theta_1 d\tilde{\psi} \wedge d\phi_1. \tag{2.22}$$

On the other hand, due to our simplifying choices in (2.2), the dilaton remains unchanged: $e^{\phi_{(A,2)}} = e^{\phi_{(A,1)}}$. The RR two-form flux (2.17) is also not affected by this deformation, namely

$\mathcal{F}_2^{(A,2)} = \mathcal{F}_2^{(A,1)}$, but the NS three-form flux in (2.18) changes to

$$\mathcal{H}_3^{(A,2)} = dB_{(A,2)} = -\cos\theta_{nc}\sin\theta_1 d\theta_1 \wedge d\tilde{\psi} \wedge d\phi_1. \tag{2.23}$$

T-dualizing the metric (2.20) along $\tilde{\psi}$, one obtains the type IIB metric

$$ds^2_{(B,2)} = ds^2_{(2)} + e^{-\phi}\frac{\tilde{F}_2}{F_2}\sec^2\theta_{nc}d\tilde{x}_3^2 + e^{\phi}\tilde{F}_2\left(\frac{d\tilde{\psi}}{\cos\theta_{nc}} + \cos\theta_1 d\phi_1\right)^2. \tag{2.24}$$

The NS B-field and dilaton associated to $ds^2_{(B,2)}$ are

$$B_{(B,2)} = \tilde{F}_2\sec^2\theta_{nc}\tan\theta_{nc}(d\tilde{\psi} + \cos\theta_{nc}\cos\theta_1 d\phi_1)\wedge d\tilde{x}_3, \qquad e^{\phi_{(B,2)}} = \sqrt{\tilde{F}_2/F_2}\sec\theta_{nc}e^{-\phi}, \tag{2.25}$$

respectively. To the $ds^2_{(B,2)}$ metric, we associate the following relevant vielbeins:

$$\begin{aligned} e_{\tilde{3}}^{(B,2)} &= d\tilde{x}_3, & e_{\theta_1}^{(B,2)} &= d\theta_1, & e_{\phi_1}^{(B,2)} &= \sin\theta_1 d\phi_1, \\ e_{\tilde{\psi}}^{(B,2)} &= d\tilde{\psi} + \cos\theta_{nc}\cos\theta_1 d\phi_1, & e_8^{(B,2)} &= dx_8, & e_9^{(B,2)} &= dx_9. \end{aligned} \tag{2.26}$$

In terms of these, it is not hard to see that the RR three-form flux $\mathcal{F}_3^{(B,2)}$ dual to $\mathcal{F}_2^{(A,2)}$ can be written as

$$\mathcal{F}_3^{(B,2)} = e_{\tilde{\psi}}^{(B,2)} \wedge \left(k_1 e_{\theta_1}^{(B,2)} \wedge e_{\phi_1}^{(B,2)} + k_2 e_8^{(B,2)} \wedge e_9^{(B,2)}\right). \tag{2.27}$$

Once again, it is important to note that the flux $\mathcal{F}_3^{(B,2)}$ is not closed: $d\mathcal{F}_3^{(B,2)} \neq 0$. This implies that indeed there is a D5-brane in this setup. Determining $\mathcal{H}_3^{(B,2)}$ is also not hard. Taking the exterior derivative of $B_{(B,2)}$ and using (2.21) and (2.26), we get

$$\mathcal{H}_3^{(B,2)} = \tilde{F}_2\sec\theta_{nc}\tan\theta_{nc}\Big(\frac{\tilde{F}_2 F_{2,r}}{F_2^2}\sec\theta_{nc}e_r^{(B,2)} \wedge e_{\tilde{\psi}}^{(B,2)} - e_{\theta_1}^{(B,2)} \wedge e_{\phi_1}^{(B,2)}\Big) \wedge e_{\tilde{3}}^{(B,2)}, \tag{2.28}$$

which is a closed form by definition.

So far, all we have done in this section boils down to introducing an NS B-field to the type IIB configuration that was our starting point (described in chapter 2 and depicted in figure 2**D**). This NS B-field, in turn, sources the NS three-form flux we just determined. In section 5.2, we will see how this NS flux sources the desired Θ-term in the world-volume gauge theory. For the time being, however, let us focus on the construction of the M-Theory configuration associated to this setup.

The following step in the duality chain outlined at the beginning of this section is to take the T-dual along ϕ_1 of (2.24). In order to make this step easy, we rewrite the aforementioned metric as

$$ds^2_{(B,2)} = ds^2_{(3)} + e^{\phi}(\tilde{F}_2 \cos^2\theta_1 + F_3 \sin^2\theta_1)\Big(d\phi_1 + \frac{\tilde{F}_2 \cos\theta_1 \sec\theta_{nc}}{\tilde{F}_2 \cos^2\theta_1 + F_3 \sin^2\theta_1} d\tilde{\psi}\Big)^2, \tag{2.29}$$

where we have defined

$$\begin{aligned} ds^2_{(3)} \equiv & e^{-\phi}(-dt^2 + dx_1^2 + dx_2^2 + \frac{\tilde{F}_2}{F_2}\sec^2\theta_{nc} d\tilde{x}_3^2) + e^{\phi}\frac{\tilde{F}_2 F_3 \sec^2\theta_{nc}\sin^2\theta_1}{\tilde{F}_2\cos^2\theta_1 + F_3\sin^2\theta_1} d\tilde{\psi}^2 \\ & + e^{\phi}[F_1 dr^2 + F_3 d\theta_1^2 + F_4(dx_8^2 + dx_9^2)]. \end{aligned} \tag{2.30}$$

T-dualizing along ϕ_1, we obtain the type IIA geometry

$$ds^2_{(A,3)} = ds^2_{(3)} + e^{-\phi}\frac{(d\phi_1 + \tilde{F}_2 \sec\theta_{nc}\tan\theta_{nc}\cos\theta_1 d\tilde{x}_3)^2}{\tilde{F}_2\cos^2\theta_1 + F_3\sin^2\theta_1}. \tag{2.31}$$

The NS B-field associated to the $ds^2_{(A,3)}$ metric is

$$B_{(A,3)} = \frac{\tilde{F}_2 \sec\theta_{nc}}{\tilde{F}_2\cos^2\theta_1 + F_3\sin^2\theta_1}(F_3 \sec\theta_{nc}\tan\theta_{nc}\sin^2\theta_1 d\tilde{\psi}\wedge d\tilde{x}_3 + \cos\theta_1 d\phi_1 \wedge d\tilde{\psi}). \tag{2.32}$$

The corresponding dilaton is given by

$$e^{\phi_{(A,3)}} = \sqrt{\frac{\tilde{F}_2}{F_2}}\frac{\sec\theta_{nc} e^{-3\phi/2}}{\sqrt{\tilde{F}_2\cos^2\theta_1 + F_3\sin^2\theta_1}}. \tag{2.33}$$

Coming to the fluxes, the type IIA two-form flux $\mathcal{F}_2^{(A,3)}$ dual to $\mathcal{F}_3^{(B,2)}$ in (2.27) can be easily seen to be

$$\mathcal{F}_2^{(A,3)} = k_1 \sin\theta_1 d\tilde{\psi}\wedge d\theta_1 + k_2\cos\theta_{nc}\cos\theta_1 dx_8 \wedge dx_9. \tag{2.34}$$

It is again important to note that, of course, this two-form flux is not closed: $d\mathcal{F}_2^{(A,3)} \neq 0$, which reflects the presence of a D6-brane (dual to the D5-brane in the previous type IIB configuration). Thus, if we denote as $\mathbf{A}_1$ the type IIA gauge field for this configuration, then it follows that $\mathcal{F}_2^{(A,3)}$ can be written as

$$\mathcal{F}_2^{(A,3)} = d\mathbf{A}_1 + \Delta, \qquad d\Delta = \text{sources}. \tag{2.35}$$

The explicit expression of the $d\Delta = d\mathcal{F}_2^{(A,3)}$ sources is

$$d\mathcal{F}_2^{(A,3)} = k_{1,a} \sin\theta_1 da \wedge d\tilde{\psi} \wedge d\theta_1 + (k_{2,a} \cos\theta_{nc} \cos\theta_1 da - k_2 \cos\theta_{nc} \sin\theta_1 d\theta_1) \wedge dx_8 \wedge dx_9. \tag{2.36}$$

We define $\mathbf{A}_1 = \mathbf{A}_1(\theta_1,\, x_8,\, x_9)$ as

$$\mathbf{A}_1 \equiv \mathbf{A}_{1\theta_1} d\theta_1 + \mathbf{A}_{18} dx_8 + \mathbf{A}_{19} dx_9, \tag{2.37}$$

We further define

$$\alpha_1 \equiv \frac{\partial \mathbf{A}_{19}}{\partial x_8} - \frac{\partial \mathbf{A}_{18}}{\partial x_9}, \qquad \alpha_2 \equiv \frac{\partial \mathbf{A}_{1\theta_1}}{\partial x_8} - \frac{\partial \mathbf{A}_{18}}{\partial \theta_1}, \qquad \alpha_3 \equiv \frac{\partial \mathbf{A}_{1\theta_1}}{\partial x_9} - \frac{\partial \mathbf{A}_{19}}{\partial \theta_1}. \tag{2.38}$$

Using the above quantities, the exterior derivative of $\mathbf{A}_1$ is

$$d\mathbf{A}_1 \equiv \alpha_1 dx_8 \wedge dx_9 + \alpha_2 dx_8 \wedge d\theta_1 + \alpha_3 dx_9 \wedge d\theta_1. \tag{2.39}$$

Since $d(d\mathbf{A}_1) = 0$, the α's just introduced are subject to the constraint

$$\frac{\partial \alpha_1}{\partial \theta_1} - \frac{\partial \alpha_2}{\partial x_9} + \frac{\partial \alpha_3}{\partial x_8} = 0. \tag{2.40}$$

The definition (2.37) will become sensible in the M-Theory uplift that follows. But first let us finish the flux discussion for this type IIA configuration. We note that the corresponding NS three-form flux is given by the exterior derivative of $B_{(A,3)}$. This is

$$\mathcal{H}_3^{(A,3)} = db \wedge (\hat{k}_{1,b} d\tilde{\psi} \wedge d\tilde{x}_3 + \hat{k}_{2,b} d\phi_1 \wedge d\tilde{\psi}), \tag{2.41}$$

where we have defined

$$\hat{k}_1 \equiv \frac{\tilde{F}_2 F_3 \sec^2\theta_{nc} \tan\theta_{nc} \sin^2\theta_1}{\tilde{F}_2 \cos^2\theta_1 + F_3 \sin^2\theta_1}, \qquad \hat{k}_2 \equiv \frac{\tilde{F}_2 F_3 \sec\theta_{nc} \cos\theta_1}{\tilde{F}_2 \cos^2\theta_1 + F_3 \sin^2\theta_1} \tag{2.42}$$

and $b \equiv (\theta_1,\, r)$ are the only coordinates on which the above two functions depend, as a consequence of our choices in (2.2).

Finally, we will uplift the above type IIA configuration to M-Theory. To this aim, we rewrite the metric $ds^2_{(A,3)}$ in (2.31) in a more convenient way. We first introduce the following quantities:

$$\begin{aligned} &H_1 \equiv (H_2 H_3)^{-1/3}, && H_2 \equiv (\cos^2\theta_{nc} + F_2 \sin^2\theta_{nc})^{-1}, \\ &H_3 \equiv (\tilde{F}_2 \cos^2\theta_1 + F_3 \sin^2\theta_1)^{-1}, && H_4 \equiv H_3 \tilde{F}_2 F_3 \sec^2\theta_{nc} \sin^2\theta_1, \\ &f_3 \equiv \tilde{F}_2 \sec\theta_{nc} \tan\theta_{nc} \cos\theta_1. \end{aligned} \tag{2.43}$$

In terms of these, the metric $ds^2_{(A,3)}$ can be written as

$$
\begin{aligned}
ds^2_{(A,3)} =&\frac{e^{-\phi}}{H_1}\left\{H_1[-dt^2 + dx_1^2 + dx_2^2 + H_2 d\tilde{x}_3^2 + H_3(d\phi_1 + f_3 d\tilde{x}_3)^2]\right. \\
&\left. + e^{2\phi}H_1\left[F_1 dr^2 + F_3 d\theta_1^2 + F_4(dx_8^2 + dx_9^2) + H_4 d\tilde{\psi}^2\right]\right\}. \qquad (2.44)
\end{aligned}
$$

It is essential to note that the M-Theory uplift will only be able to capture the dynamics of the type IIA theory in the strong coupling limit of the latter. For us, that means that we can only rely on the M-Theory description when $e^{\phi_{(A,3)}}$ is of order one or bigger. However, we will be interested in having a *finite* radius for the eleventh direction after we uplift. Therefore, we will be careful to avoid the infinite coupling limit where

$$
e^{\phi_{(A,3)}} \to \infty. \qquad (2.45)
$$

From (2.33) it follows that the above is true when $e^{-\phi} \to \infty$, for an arbitrary choice of $(F_2,\, F_3)$. Additionally, the infinite coupling limit also applies at two isolated points $(p_1,\, p_2)$ given by $p_1 = (\theta_1 = 0,\, r = r_1)$ and $p_2 = (\theta_1 = \pi/2,\, r = r_2)$ (for any value of the remaining coordinates), where $(r_1,\, r_2)$ are the values of the radial coordinate for which $F_2(r_1) = 0$ and $F_3(r_2) = 0$, respectively.

The M-Theory metric corresponding to (2.44) is

$$
\begin{aligned}
ds^2_{(M,1)} =&H_1[-dt^2 + dx_1^2 + dx_2^2 + H_2 d\tilde{x}_3^2 + H_3(d\phi_1 + f_3 d\tilde{x}_3)^2 + e^{2\phi}(F_1 dr^2 + H_4 d\tilde{\psi}^2)] \\
&+ e^{2\phi}H_1[F_3 d\theta_1^2 + F_4(dx_8^2 + dx_9^2)] + e^{-2\phi}H_1^{-2}(dx_{11} + \mathbf{A}_1)^2, \qquad (2.46)
\end{aligned}
$$

where $\mathbf{A}_1$ is the type IIA gauge field defined in (2.37). We note that, due to (2.2) and (2.37), for a fixed value of the radial coordinate, $r = r_0$, the second line above describes a warped Taub-NUT space in the $(\theta_1,\, x_8,\, x_9,\, x_{11})$ directions. (Indeed, this is what motivated the definition (2.37).) This is most easily seen by introducing

$$
G_1 \equiv e^{2\phi}H_1 F_3\Big|_{r=r_0}, \qquad G_2, G_3 \equiv e^{2\phi}H_1 F_4\Big|_{r=r_0}, \qquad G_4 \equiv e^{-2\phi}H_1^{-2}\Big|_{r=r_0} \qquad (2.47)
$$

and writing the warped Taub-NUT metric as

$$
ds^2_{TN_1} = G_1 d\theta_1^2 + G_2 dx_8^2 + G_3 dx_9^2 + G_4(dx_{11} + \mathbf{A}_1)^2. \qquad (2.48)
$$

Note that, as we just explained,

$$
G_i = G_i(\theta_1, x_8, x_9), \qquad i = 1, 2, 3, 4. \qquad (2.49)
$$

We take the vielbeins of (2.48) as

$$e_{\theta_1}^{(M,1)} = \sqrt{G_1}d\theta_1, \quad e_8^{(M,1)} = \sqrt{G_2}dx_8, \quad e_9^{(M,1)} = \sqrt{G_3}dx_9, \quad e_{11}^{(M,1)} = \sqrt{G_4}(dx_{11} + \mathbf{A}_1). \tag{2.50}$$

To better understand this Taub-NUT space, recall that, before the M-Theory uplift, we had a D6-brane in our type IIA configuration. The M-Theory uplift then converts this D6-brane to geometry. In particular, we obtain the metric (2.46), where (2.48) is a *single-centered* (warped) Taub-NUT space. In other words, in (2.48), $G_4^{-1} = 0$ occurs *once* and the coordinate singularity at this point is the location of the D6-brane in the dual type IIA picture. This is an important observation and essential to the G-flux computation that follows.

As we just hinted, the remaining of this section will be devoted to the determination of the G-flux corresponding to this M-Theory configuration. As is well-known, there exists a unique, normalizable (anti-)self-dual harmonic two-form ω associated to a single-centered (warped) Taub-NUT space [34]. Using which, the G-flux for our M-Theory configuration is given by

$$\mathcal{G}_4^{(M,1)} = \langle \mathcal{G}_4^{(M,1)} \rangle + \mathcal{F} \wedge \omega, \qquad \langle \mathcal{G}_4^{(M,1)} \rangle = \mathcal{H}_3^{(A,3)} \wedge dx_{11}, \qquad \mathcal{F} = d\mathcal{A}, \tag{2.51}$$

where $\langle \mathcal{G}_4^{(M,1)} \rangle$ is the background G-flux, $\mathcal{H}_3^{(A,3)}$ was determined in (2.41 and $\mathcal{A}$ is the seven-dimensional world-volume gauge field. Thus, in order to obtain the explicit form of $\mathcal{G}_4^{(M,1)}$, we have only one task left: ω must be computed. We do so in the following.

We start by making the following ansatz for ω:

$$\omega = d\zeta, \qquad \zeta = g(\theta_1, x_8, x_9)(dx_{11} + \mathbf{A}_1) \tag{2.52}$$

and then proceed to determine its precise value from the (anti-)self-duality requirement: $\omega = \pm * \omega$, where the Hodge dual is taken with respect to the metric (2.48). Let us see this in details. Using (2.39) and (2.50), ω can be written as

$$\begin{aligned}\omega =& g\Big(\frac{\alpha_1}{\sqrt{G_2G_3}}e_8^{(M,1)} \wedge e_9^{(M,1)} + \frac{\alpha_2}{\sqrt{G_1G_2}}e_8^{(M,1)} \wedge e_{\theta_1}^{(M,1)} + \frac{\alpha_3}{\sqrt{G_1G_3}}e_9^{(M,1)} \wedge e_{\theta_1}^{(M,1)}\Big)\\ &+ \frac{1}{\sqrt{G_4}}\Big(\frac{1}{\sqrt{G_1}}\frac{\partial g}{\partial\theta_1}e_{\theta_1}^{(M,1)} + \frac{1}{\sqrt{G_2}}\frac{\partial g}{\partial x_8}e_8^{(M,1)} + \frac{1}{\sqrt{G_3}}\frac{\partial g}{\partial x_9}e_9^{(M,1)}\Big) \wedge e_{11}^{(M,1)}.\end{aligned} \tag{2.53}$$

Rather straightforwardly it follows that its Hodge dual with respect to (2.48) is

$$\begin{aligned}*\omega =& \frac{1}{\sqrt{G_4}}\Big(\frac{1}{\sqrt{G_1}}\frac{\partial g}{\partial\theta_1}e_8^{(M,1)} \wedge e_9^{(M,1)} + \frac{1}{\sqrt{G_3}}\frac{\partial g}{\partial x_9}e_8^{(M,1)} \wedge e_{\theta_1}^{(M,1)} + \frac{1}{\sqrt{G_2}}\frac{\partial g}{\partial x_8}e_9^{(M,1)} \wedge e_{\theta_1}^{(M,1)}\Big)\\ &+ g\Big(\frac{\alpha_1}{\sqrt{G_2G_3}}e_{\theta_1}^{(M,1)} + \frac{\alpha_3}{\sqrt{G_1G_3}}e_8^{(M,1)} - \frac{\alpha_2}{\sqrt{G_1G_2}}e_9^{(M,1)}\Big) \wedge e_{11}^{(M,1)}.\end{aligned} \tag{2.54}$$

Imposing (anti-)self-duality of ω leads to three partial differential equations (PDEs):

$$\frac{1}{g}\frac{\partial g}{\partial\theta_1} = \pm\alpha_1\sqrt{\frac{G_1G_4}{G_2G_3}}, \qquad \frac{1}{g}\frac{\partial g}{\partial x_8} = \pm\alpha_3\sqrt{\frac{G_2G_4}{G_1G_3}}, \qquad \frac{1}{g}\frac{\partial g}{\partial x_9} = \mp\alpha_2\sqrt{\frac{G_3G_4}{G_1G_2}}. \tag{2.55}$$

Using (2.43) and (2.47) in the above, we can rewrite these equations in terms of the warp factors and dilaton as

$$\frac{1}{g}\frac{\partial g}{\partial\theta_1} = \pm\, e^{-2\phi}\frac{\alpha_1}{F_4}\sqrt{\frac{\tilde{F}_2F_3}{F_2}}\sec\theta_{nc}(\tilde{F}_2\cos^2\theta_1 + F_3\sin^2\theta_1)^{-1/2}\Bigg|_{r=r_0}, \tag{2.56}$$

$$\frac{1}{g}\frac{\partial g}{\partial x_8} = \pm\, e^{-2\phi}\alpha_3\sqrt{\frac{\tilde{F}_2}{F_2F_3}}\sec\theta_{nc}(\tilde{F}_2\cos^2\theta_1 + F_3\sin^2\theta_1)^{-1/2}\Bigg|_{r=r_0} = -\frac{\alpha_3}{\alpha_2}\frac{1}{g}\frac{\partial g}{\partial x_9}.$$

Solving the above set of PDEs generically is not easy. Consequently, we will do some more simplifying assumptions. To begin with, let us consider

$$\alpha_1 = 0, \qquad \alpha_2 = \beta_2(x_9)f(\theta_1, r, x_8, x_9)\Big|_{r=r_0}, \qquad \alpha_3 = \beta_3(x_8)f(\theta_1, r, x_8, x_9)\Big|_{r=r_0}, \tag{2.57}$$

where we have defined

$$f = f(\theta_1, r, x_8, x_9) \equiv e^{2\phi}\sqrt{\tilde{F}_2\cos^2\theta_1 + F_3\sin^2\theta_1}. \tag{2.58}$$

If we now choose the dilaton to be of the form

$$e^{2\phi} = \frac{e^{2\phi_0}Q(r, x_8, x_9)}{\sqrt{\tilde{F}_2\cos^2\theta_1 + F_3\sin^2\theta_1}}, \tag{2.59}$$

with ϕ_0 some constant, then (α_2, α_3) become independent of θ_1 (that is, functions of the coordinates (x_8, x_9) only). Recall that the α's were subject to the constraint (2.40). Hence, $Q = Q(r, x_8, x_9)$ above must satisfy

$$Q\left(\frac{d\beta_3}{dx_8} - \frac{d\beta_2}{dx_9}\right) + \beta_3\frac{\partial Q}{\partial x_8} - \beta_2\frac{\partial Q}{\partial x_9}\Bigg|_{r=r_0} = 0. \tag{2.60}$$

For notational convenience, we define

$$c_0 \equiv \sqrt{\frac{\tilde{F}_2}{F_2F_3}}\sec\theta_{nc}\Bigg|_{r=r_0}, \tag{2.61}$$

which is a constant that only depends on the deformation parameter θ_{nc}. Inserting all our choices and definitions in (2.56), these PDEs reduce to

$$\frac{1}{g}\frac{\partial g}{\partial x_8} = \pm c_0 \beta_3(x_8), \qquad \frac{1}{g}\frac{\partial g}{\partial x_9} = \mp c_0 \beta_2(x_9), \tag{2.62}$$

where g is now independent of θ_1 and thus $g = g(x_8, x_9)$. It is finally easy to use separation of variables to solve the above. Assuming $g = \tilde{g}_1(x_8)\tilde{g}_2(x_9)$, we obtain two ordinary differential equations,

$$\frac{d\tilde{g}_1}{\tilde{g}_1} = \pm c_0 \beta_3(x_8) dx_8, \qquad \frac{d\tilde{g}_2}{\tilde{g}_2} = \mp c_0 \beta_2(x_9) dx_9, \tag{2.63}$$

which can readily be solved to yield

$$g = g_0 \exp\left[\pm c_0 \left(\int_0^{x_8} \beta_3(x_8')dx_8' - \int_0^{x_9} \beta_2(x_9')dx_9'\right)\right], \tag{2.64}$$

with g_0 some integration constant. This completes the computation of ω in (2.52), which in turn gives us the explicit form of the G-flux in (2.51). Together with (2.46), the latter fully characterizes model (M,1).

2.1.1 Enhancing the world-volume gauge symmetry: tensionless M2-branes

It is an intrinsically interesting question to ask whether our first M-Theory construction above can be generalized to account for non-abelian world-volume gauge theories (and not just the particularly simple $U(1)$ case discussed so far). The answer is yes and the way to do so is discussed in [35]. Consequently, in this section we review and adapt the arguments in [35] to our case.

But before we jump into the details of non-abelian enhancement in M-Theory, it is instructive to recall the well-known equivalent discussion in type IIA String Theory [36]. Consider N parallel D6-branes ($N = 2, 3, 4, \ldots$). Consider there are open strings stretched between these D6-branes. In this case, the symmetry group of the corresponding world-volume gauge theory is

$$\underbrace{U(1) \times U(1) \times \ldots \times U(1)}_{N \text{ times}}. \tag{2.65}$$

In the limit when the open strings become tensionless, the D6-branes come on top of each other, leading to N coincident D6-branes. Then, the symmetry group of the corresponding world-volume gauge theory becomes $SU(N)$.

If we lift the above type IIA configuration to M-Theory, then the D6-branes convert to geometry and we obtain the metric (2.46)[4], with (2.48) a *multi-centered* (warped) Taub-NUT space. Indeed, $G_4^{-1} = 0$ now occurs N times in (2.48), the coordinate singularities at these points denoting the location of the D6-branes in the dual type IIA picture. As for the open strings, they convert to M2-branes wrapping the independent two-cycles in the Taub-NUT space (2.48). In the limit of tensionless M2-branes, the two-cycles vanish and the world-volume gauge theory symmetry group becomes $SU(N)$.

Let us see how the above discussion applies to our setup in details. The first step will be to construct the independent two-cycles in the space (2.48). In order to do so, let us start by rewriting the metric (2.48) in a more convenient way. Defining

$$U \equiv e^{2\phi} H_1^2\Big|_{r=r_0}, \qquad d\vec{\mathbf{x}}^2 \equiv H_1^{-1}[F_3 d\theta_1^2 + F_4(dx_8^2 + dx_9^2)]\Big|_{r=r_0}, \tag{2.66}$$

we can rewrite (2.48) as

$$ds^2_{TN_1} = U d\vec{\mathbf{x}}^2 + U^{-1}(dx_{11} + \mathbf{A}_1)^2. \tag{2.67}$$

Recall that now this warped Taub-NUT space is a multi-centered one. Using (2.43) and (2.59), U above can be written in terms of the warp factors and Q as

$$U = e^{2\phi_0} Q(\cos^2\theta_{nc} + F_2 \sin^2\theta_{nc})^{2/3}(\tilde{F}_2 \cos^2\theta_1 + F_3 \sin^2\theta_1)^{1/6}\Big|_{r=r_0}. \tag{2.68}$$

For simplicity, we will do two assumptions next: we will take the deformation parameter to be sufficiently small (that is, $\theta_{nc} << 1$) and we will consider

$$F_2\Big|_{r=r_0} = F_3\Big|_{r=r_0}. \tag{2.69}$$

Then, expanding to first order around $\theta_{nc} = 0$ and using (2.69), U becomes independent of θ_1:

$$\tilde{U} = \tilde{U}(x_8, x_9) \equiv \lim_{\theta_{nc}\to 0} U = e^{2\phi_0} Q(r, x_8, x_9) F_3^{1/6}\Big|_{r=r_0}. \tag{2.70}$$

$\tilde{U} = 0$ has N solutions, which we denote as $\vec{l}_i = (x_{8i}, x_{9i})$, with $i = 1, 2, \ldots, N$. Consider two such points $\vec{l}_i$ and $\vec{l}_j$ ($i \neq j$) and a geodesic $\mathcal{C}_g$ in the (x_8, x_9) space joining them. Attaching to each point in $\mathcal{C}_g$ a circle labeled by x_{11}, we obtain a minimal area two-cycle X_{ij}. We take $X_{k,k+1}$ (for $k = 1, 2, \ldots, N-1$) as the minimal area independent two-cycles.

[4]Since we never determined our warp factors and Q function in (2.59), we can absorb the changes in the geometry due to the inclusion of the D6-branes and open strings in these quantities.

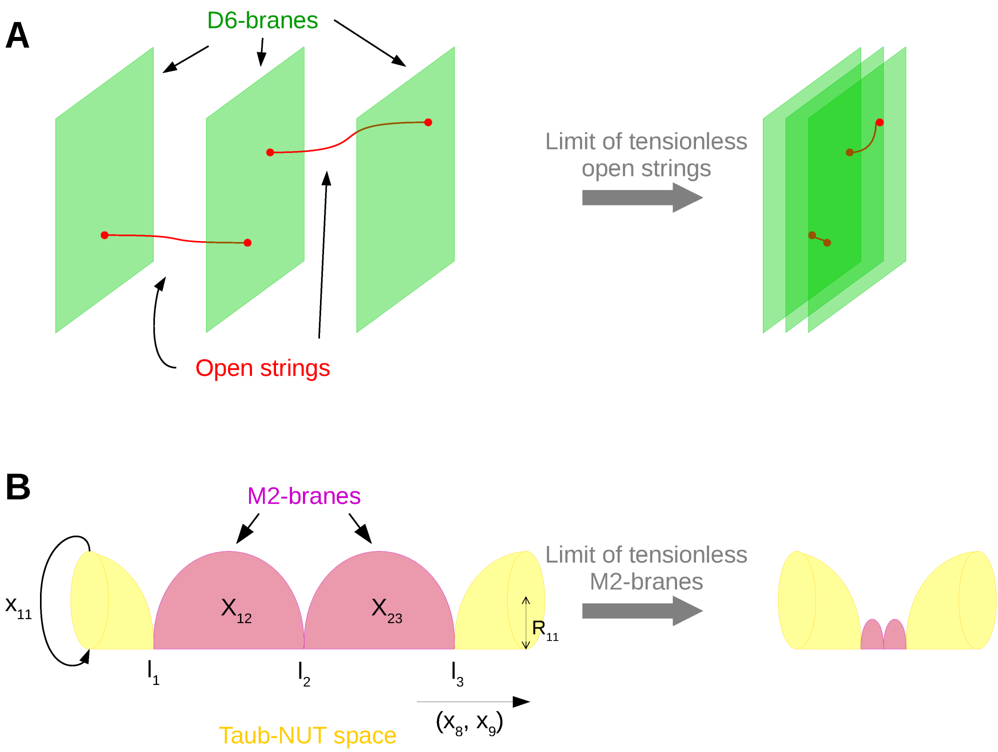

Figure 4: Schematics of the non-abelian enhancement of the world-volume gauge symmetry from $U(1) \times U(1) \times U(1)$ to $SU(3)$ in type IIA String Theory (top) and in M-Theory (bottom). **A:** Three parallel D6-branes in type IIA, with open strings stretching between them. The D6-branes span the $(t,\, x_1,\, x_2,\, \tilde{x}_3,\, \phi_1,\, \tilde{\psi},\, r)$ directions and the open strings are in the $(x_8,\, x_9)$ plane. The θ_1 direction is suppressed. When the open strings become tensionless, the D6-branes coincide. This produces the non-abelian enhancement. **B:** Uplift to M-Theory of the type IIA configurations in **A**. The D6-branes convert to geometry, giving rise to a multi-centered warped Taub-NUT space along $(\theta_1,\, x_8,\, x_9,\, x_{11})$, for a fixed value of the radial coordinate: $r = r_0$. R_{11} is the physical radius of the coordinate x_{11}. The $(t,\, x_1,\, x_2,\, \tilde{x}_3,\, \theta_1,\, \phi_1,\, \tilde{\psi},\, r)$ directions are suppressed in the figure. The singularities in the Taub-NUT space lie at $(\vec{l}_1,\, \vec{l}_2,\, \vec{l}_3)$: the position of the D6-branes in the dual type IIA configuration. The open strings become M2-branes wrapping the minimal area, independent two-cycles $(X_{12},\, X_{23})$ between the singularities. In the limit of tensionless M2-branes, these two-cycles vanish, leading to the desired non-abelian enhancement.

It is well-known that to each such two-cycle $X_{k,k+1}$, with k fixed, we can associate a unique, normalizable, (anti-)self-dual two-form ω_k. Obtaining the explicit form of ω_k is straightforward, in view of our earlier results. We only need to modify (2.52) to

$$\omega_k = d\zeta_k, \qquad \zeta_k = g_k(x_8, x_9)(dx_{11} + \mathbf{A}_1) \tag{2.71}$$

and restrict the integrals in (2.64) to the $X_{k,k+1}$ two-cycle:

$$g_k = \tilde{g}_0 \exp\left[\pm c_0 \int_{\vec{l}_k}^{\vec{l}_{k+1}} (\beta_3 - \beta_2)|d\vec{l}_{\mathcal{C}_g}|\right], \tag{2.72}$$

where $\tilde{g}_0$ is some integration constant and $d\vec{l}_{\mathcal{C}_g}$ denotes line element along the geodesic $\mathcal{C}_g$ joining $\vec{l}_k$ and $\vec{l}_{k+1}$.

Let us now compute the areas of the two-cycles $X_{k,k+1}$ and derive their intersection matrix. It will soon be clear why we do so. As measured in the Taub-NUT metric, the area of $X_{k,k+1}$ is given by

$$S_{k,k+1} = \int_{X_{k,k+1}} (\tilde{U}^{-1/2} dx_{11})\left(\tilde{U}^{1/2}\sqrt{F_4}\Big|_{r=r_0} |d\vec{l}_{\mathcal{C}_g}|\right) = \tilde{\beta} R_{11} \int_{\vec{l}_k}^{\vec{l}_{k+1}} \sqrt{F_4}\Big|_{r=r_0} |d\vec{l}_{\mathcal{C}_g}|, \tag{2.73}$$

with $\tilde{\beta}$ a constant that avoids possible conical singularities along $\mathcal{C}_g$ and R_{11} the physical radius of the x_{11} coordinate. It is easy to see that the self-intersection number for each $S_{k,k+1}$ is two: the $S_{k,k+1}$'s self-intersect at $\vec{l}_k$ and $\vec{l}_{k+1}$, with geodesics transversed in the same direction. $S_{k,k+1}$ intersects $S_{k-1,k}$ only at $\vec{l}_k$, their geodesics being transversed in opposite directions. No other two-cycles' areas intersect. Thus, the $(N-1) \times (N-1)$ intersection matrix of the areas of the two-cycles $X_{k,k+1}$ is

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}. \tag{2.74}$$

Or, written more compactly,

$$[S_{k,k+1}] \circ [S_{l,l+1}] = \begin{cases} 2\delta_{k,l} \\ -\delta_{l,k-1} \end{cases}. \tag{2.75}$$

This is, of course, the Cartan matrix of the A_{N-1} algebra.

Recall that there are M2-branes in this configuration. They wrap the $X_{k,k+1}$ two-cycles and thus their intersection matrix is (2.74). As previously explained, when the area of all these two-cycles tends to zero, the limit of tensionless M2-branes sets in. This corresponds to an A_{N-1} singularity, which in turn is responsible for enhancing the world-volume gauge symmetry to $SU(N)$, as shown in [37]. Figure 4 schematically depicts the above discussion for $N = 3$, both in the type IIA and M-Theory pictures.

To finish this section, we use all the above results to write the G-flux of this non-abelian enhanced M-Theory configuration as

$$\mathcal{G}_4^{(M,1)} = \langle \mathcal{G}_4^{(M,1)} \rangle + \sum_{k=1}^{N-1} \mathcal{F}_k \wedge \omega_k. \tag{2.76}$$

Here, $\mathcal{F}_k$'s are the Cartan algebra values of the world-volume field strength $\mathcal{F}$, the background G-flux $\langle \mathcal{G}_4^{(M,1)} \rangle$ is as earlier[5] in (2.51) and the two-forms ω_k were computed in (2.71).

2.2 Accounting for an axionic background: an additional RR B-field

Suppose we follow the prescription of [14] to source the Θ-term in the world-volume gauge theory. That is, suppose we consider the type IIB D3-NS5 system *with* an axionic background $\mathcal{C}_0$. How would that affect the results in the previous section 2.1, where $\mathcal{C}_0 = 0$?

Long story made short, we need to follow $\mathcal{C}_0$ along the modifications depicted in figure 2. We note that $\mathcal{C}_0$ would not be affected while going from **A** to **B** in figure 2. However, on going from **B** to **C**, $\mathcal{C}_0$ would dualize to a gauge field in the x_3 direction. Finally, on going from **C** to **D**, the gauge field would lead to an RR B-field in the (x_3, ψ) directions. Schematically,

$$\mathcal{C}_0 \xrightarrow[\text{but finite interval}]{\text{Convert } \psi \text{ to a large}} \mathcal{C}_0 \xrightarrow[\text{along } x_3]{\text{T-duality}} \mathcal{C}_1 = (\mathcal{C}_1)_3 dx_3 \xrightarrow[\text{along } \psi]{\text{T-duality}} \mathcal{C}_2 = (\mathcal{C}_2)_{3\psi} dx_3 \wedge d\psi. \tag{2.77}$$

Thus, in our construction, switching on an axionic background in the usual type IIB D3-NS5 system of [14], shown in figure 2**A**, amounts to adding an RR B-field in the (x_3, ψ) directions to the type IIB configuration shown in figure 2**D**.

However, here we will see a *different* way in which we can obtain such an RR B-field in the type IIB configuration before we uplift to M-Theory. This will involve another, distinct (although similar) chain of dualities and modifications to the type IIB configuration of figure 2**D** to that considered before, in section 2.1. In the following, we make precise this idea.

[5]Remember, however, that the warp factors and Q function introduced in (2.59) are different from those in the abelian case, due to the inclusion of the D6-branes and open strings in the dual type IIA theory.

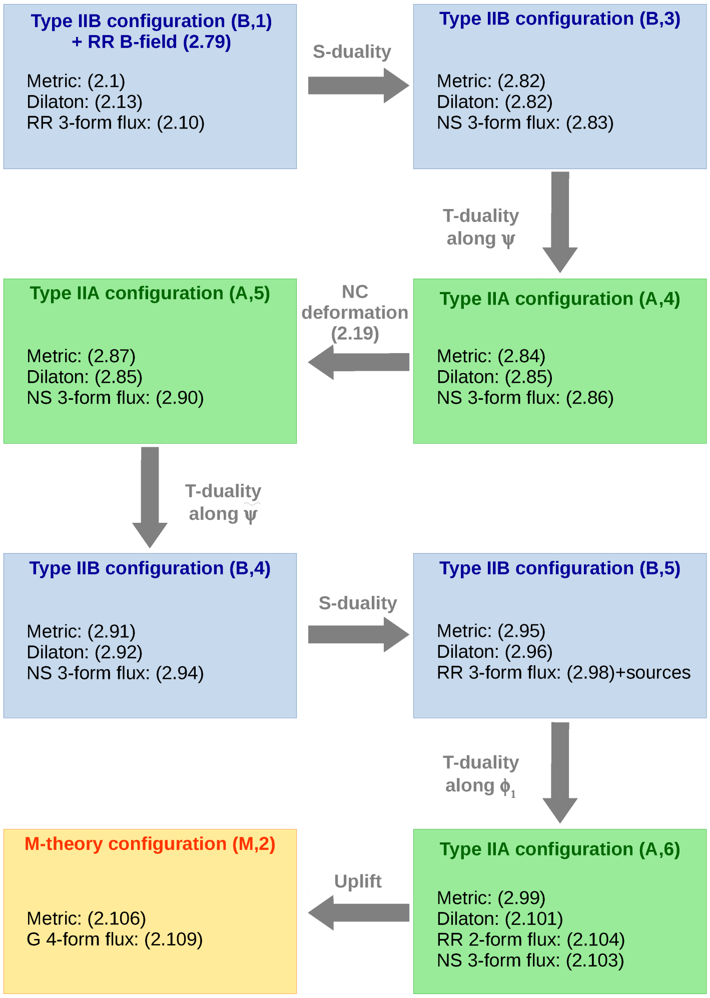

Figure 5: Graphical summary of section 2.2. To the type IIB configuration of figure 2**D** we associate an RR B-field and then proceed to do a series of modifications in order to account for the axionic background considered in [14]. This is achieved in going from the configuration (B,1), with the mentioned RR B-field added, to (B,5). The (B,5) configuration is then lifted to M-Theory. However, as argued in the text, it will suffice to study the M-Theory configuration (M,1) of figure 3.

The starting point here is the starting point of section 2.1 as well: the last configuration of section 2, schematically depicted in figure 2**D**. To this configuration we will associate an RR B-field. We will then do an S-duality. The next step will be a T-duality along ψ to type IIA, where we will do the same non-commutative deformation $(x_3, \psi) \to (\tilde{x}_3, \tilde{\psi})$ that was considered in section 2.1. Afterwards, we will consider a T-duality along $\tilde{\psi}$ back to type IIB, followed by an S-duality. At this point we will have a type IIB configuration with an RR B-field along $(\tilde{x}_3, \tilde{\psi})$. Thus, effectively we will have accounted for the axionic background, as we wished to do. The last T-duality will be along ϕ_1 to type IIA. The resulting configuration will then be lifted to M-Theory. As in section 2.1, the NS and RR B-fields, dilaton and fluxes of all the above geometries will be determined. Figure 5 serves as a summary of the chain of modifications just described and indicates the key equations in this section.

As just explained, we start by considering the type IIB geometry $ds^2_{(B,1)}$ in (2.1), which has a dilaton $e^{\phi_{(B,1)}}$ in (2.13) and an RR three-form flux $\mathcal{F}_3^{(B,1)}$ in (2.10). We will associate the following RR B-field $\mathcal{C}_2^{(B,1)}$ to this setup:

$$\mathcal{F}_3^{(B,1)} = d\mathcal{C}_2^{(B,1)} + \tilde{\Delta}, \qquad d\tilde{\Delta} = \text{sources} \neq 0. \tag{2.78}$$

Note that the sources above are required to keep consistent with the fact that $\mathcal{F}_3^{(B,1)}$ is not closed. These sources, of course, refer to the D5-brane present in this configuration. For concreteness and as a particularly simple case, we will assume that $\mathcal{C}_2^{(B,1)}$ is of the form

$$\mathcal{C}_2^{(B,1)} = b_{\theta_1\phi_1} d\theta_1 \wedge d\phi_1 + b_{89} dx_8 \wedge dx_9, \tag{2.79}$$

where $(b_{\theta_1\phi_1}, b_{89})$ are functions of only $a \equiv (\theta_1, r, x_8, x_9)$, in order to respect all isometries in (2.1). It follows then that its exterior derivative is

$$d\mathcal{C}_2^{(B,1)} = d\theta_1 \wedge d\phi_1 \wedge \Big(\sum_a \frac{\partial b_{\theta_1\phi_1}}{\partial a} da\Big) + \Big(\sum_a \frac{\partial b_{89}}{\partial a} da\Big) \wedge dx_8 \wedge dx_9. \tag{2.80}$$

Using (2.5), (2.10) and the above in (2.78), $\tilde{\Delta}$ can be easily checked to be

$$\begin{aligned}\tilde{\Delta} =& d\theta_1 \wedge d\phi_1 \wedge \Big(k_1 \sin\theta_1 d\psi - \sum_a \frac{\partial b_{\theta_1\phi_1}}{\partial a} da\Big) \\ &+ \Big(k_2 d\psi + k_2 \cos\theta_1 d\phi_1 - \sum_a \frac{\partial b_{89}}{\partial a} da\Big) \wedge dx_8 \wedge dx_9.\end{aligned} \tag{2.81}$$

S-dualizing the above, we obtain a type IIB configuration with metric, dipole and NS B-field given by

$$ds^2_{(B,3)} = e^{\phi} ds^2_{(B,1)}, \qquad e^{\phi_{(B,3)}} = e^{-\phi_{(B,1)}}, \qquad B_{(B,3)} = \mathcal{C}_2^{(B,1)}, \tag{2.82}$$

respectively. The corresponding NS three-form flux is the exterior derivative of $B_{(B,3)}$, plus sources coming from the NS5-brane (dual to the D5-brane before). Consequently, this is

$$\mathcal{H}_3^{(B,3)} = d\mathcal{C}_2^{(B,1)} + \tilde{\Delta} = \mathcal{F}_3^{(B,1)}, \tag{2.83}$$

not closed: $d\mathcal{H}_3^{(B,3)} \neq 0$. In other words, after the S-duality, the RR three-form flux becomes an NS one. This is of course very convenient (and the reason to take the S-dual to begin with): NS B-fields and fluxes are easier to deal with than RR ones.

A T-duality along ψ leads to the type IIA geometry

$$ds^2_{(A,4)} = e^{\phi} ds^2_{(2)} + dx_3^2 + \frac{e^{-2\phi}}{F_2} d\psi^2, \tag{2.84}$$

with $ds^2_{(2)}$ as in (2.12) and where the associated dilaton and NS B-field are given by

$$e^{\phi_{(A,4)}} = (F_2)^{-1/2}, \qquad B_{(A,4)} = \mathcal{C}_2^{(B,1)} + \cos\theta_1 d\psi \wedge d\phi_1. \tag{2.85}$$

The NS three-form flux is then

$$\mathcal{H}_3^{(A,4)} = dB_{(A,4)} = d\mathcal{C}_2^{(B,1)} - \sin\theta_1 d\theta_1 \wedge d\psi \wedge d\phi_1. \tag{2.86}$$

Note that this NS three-form flux *is* closed: $d\mathcal{H}_3^{(A,4)} = 0$. This is because, under the T-duality, the NS5-brane sources turn to geometry, as is well-known (see, for example, [38]).

Under the non-commutative deformation in (2.19), the type IIA metric changes to

$$ds^2_{(A,5)} = e^{\phi} ds^2_{(2)} + \frac{\hat{F}_2}{F_2} \sec^2\theta_{nc} d\tilde{x}_3^2 + \frac{e^{-2\phi}}{\hat{F}_2} \cos^2\theta_{nc} \big(d\tilde{\psi} + e^{2\phi} \hat{F}_2 \sec^2\theta_{nc} \tan\theta_{nc} d\tilde{x}_3\big)^2, \tag{2.87}$$

where we have defined

$$\hat{F}_2 \equiv \frac{F_2}{1 + e^{2\phi} F_2 \tan^2\theta_{nc}} \tag{2.88}$$

and $ds^2_{(A,5)}$ has been written in a form that anticipates the T-duality along $\tilde{\psi}$ that we will soon perform. Note the resemblance between $\hat{F}_2$ and $\tilde{F}_2$, defined in (2.21). Due to our choices in (2.2), the dilaton is not affected by the non-commutative deformation: $e^{\phi_{(A,5)}} = e^{\phi_{(A,4)}}$.

Similarly, our choice in (2.79) ensures that $\mathcal{C}_2^{(B,1)}$ remains unchanged too. The NS B-field, however, does change to

$$B_{(A,5)} = \mathcal{C}_2^{(B,1)} + \cos\theta_{nc}\cos\theta_1 d\tilde{\psi}\wedge d\phi_1, \tag{2.89}$$

which in turn induces the NS three-form flux to change accordingly:

$$\mathcal{H}_3^{(A,5)} = dB_{(A,5)} = d\mathcal{C}_2^{(B,1)} - \cos\theta_{nc}\sin\theta_1 d\theta_1\wedge d\tilde{\psi}\wedge d\phi_1. \tag{2.90}$$

Needless to say, this flux remains closed: $d\mathcal{H}_3^{(A,5)} = 0$.

Upon a T-duality along $\tilde{\psi}$, we obtain the type IIB geometry

$$ds^2_{(B,4)} = e^{\phi} ds^2_{(2)} + \frac{\hat{F}_2}{F_2}\sec^2\theta_{nc} d\tilde{x}_3^2 + e^{2\phi}\hat{F}_2\sec^2\theta_{nc}(d\tilde{\psi} + \cos\theta_{nc}\cos\theta_1 d\phi_1)^2, \tag{2.91}$$

with dilaton

$$e^{\phi_{(B,4)}} = \sqrt{\hat{F}_2/F_2}\sec\theta_{nc} e^{\phi}. \tag{2.92}$$

The NS B-field $B_{(A,5)}$ dualizes to

$$B_{(B,4)} = \mathcal{C}_2^{(B,1)} + e^{2\phi}\hat{F}_2\sec^2\theta_{nc}\tan\theta_{nc}(d\tilde{\psi} + \cos\theta_{nc}\cos\theta_1 d\phi_1)\wedge d\tilde{x}_3, \tag{2.93}$$

which contributes to the NS three-form flux

$$\mathcal{H}_3^{(B,4)} = \frac{\tan\theta_{nc}}{\cos\theta_{nc}}\Big[k_{3,a} da\wedge(\frac{d\tilde{\psi}}{\cos\theta_{nc}} + \cos\theta_1 d\phi_1) - k_3\sin\theta_1 d\theta_1\wedge d\phi_1\Big]\wedge d\tilde{x}_3 + \text{sources}, \tag{2.94}$$

where we have defined $k_3 \equiv e^{2\phi}\hat{F}_2$ and we recall that $a \equiv (\theta_1, r, x_8, x_9)$. Notice that these are the only coordinates on which k_3 depends, as a consequence of our choices in (2.2). The above flux is not closed, owing to the sources which denote the presence of an NS5-brane. We do not determine the precise form of the sources here, for reasons that will soon become clear.

Next, we do an S-duality. This changes the metric to

$$ds^2_{(B,5)} = \frac{e^{-\phi}}{\sec\theta_{nc}}\sqrt{\frac{F_2}{\hat{F}_2}}\Big[e^{\phi} ds^2_{(2)} + \frac{\hat{F}_2}{F_2}\sec^2\theta_{nc} d\tilde{x}_3^2 + k_3\sec^2\theta_{nc}(d\tilde{\psi} + \cos\theta_{nc}\cos\theta_1 d\phi_1)^2\Big]. \tag{2.95}$$

The corresponding dilaton is

$$e^{\phi_{(B,5)}} = \sqrt{F_2/\hat{F}_2}\cos\theta_{nc} e^{-\phi}. \tag{2.96}$$

The NS B-field now dualizes to an RR two-form flux given by

$$\mathcal{C}_2^{(B,5)} = -B_{(B,5)} = -\mathcal{C}_2^{(B,1)} + k_3 \sec^2\theta_{nc}\tan\theta_{nc} d\tilde{x}_3 \wedge (d\tilde{\psi} + \cos\theta_{nc}\cos\theta_1 d\phi_1). \tag{2.97}$$

The above contributes to an RR three-form flux as $\mathcal{F}_3^{(B,5)} = d\mathcal{C}_2^{(B,5)} +$ sources, where

$$d\mathcal{C}_2 = -d\mathcal{C}_2^{(B,1)} + \frac{\tan\theta_{nc}}{\cos\theta_{nc}}\left[k_3\cos\theta_1 d\theta_1 \wedge d\phi_1 - k_{3,a} da \wedge (\frac{d\tilde{\psi}}{\cos\theta_{nc}} + \cos\theta_1 d\phi_1)\right] \wedge d\tilde{x}_3 \tag{2.98}$$

and the sources reflect the presence of a D5-brane (S-dual to the previous NS5-brane), thus leading to $d\mathcal{F}_3^{(B,5)} \neq 0$.

All the modifications considered so far in this section have at this stage satisfied the desired goal: to source an RR 2-form flux along $(\tilde{x}_3, \tilde{\psi})$ in our type IIB configuration before the uplift to M-Theory. As we explained in the beginning of the section, this is equivalent to switching on an axionic background $\mathcal{C}_0$ in the usual D3-NS5 system. Having noted this important point, let us proceed with the remaining dualities to obtain the M-Theory uplift of the above configuration.

Upon a T-duality along ϕ_1, the type IIB configuration above leads to a type IIA geometry that can be conveniently expressed as

$$\begin{aligned} ds^2_{(A,6)} =& \frac{e^{-\phi}}{\tilde{H}_1\sqrt{\tilde{H}_2}}\left\{\tilde{H}_1\left(-dt^2 + dx_1^2 + dx_2^2 + \tilde{H}_2 d\tilde{x}_3^2 + \tilde{H}_3 d\phi_1^2\right)\right. \\ & \left. + e^{2\phi}\tilde{H}_1\left[F_1 dr^2 + F_3 d\theta_1^2 + F_4(dx_8^2 + dx_9^2) + \tilde{H}_4 d\tilde{\psi}^2\right]\right\}, \end{aligned} \tag{2.99}$$

where we have defined

$$\begin{aligned} &\tilde{H}_1 \equiv (F_3\sin^2\theta_1 + \hat{F}_2\cos^2\theta_1)^{1/3}\tilde{H}_2^{-1/3}, \qquad && \tilde{H}_2 \equiv \hat{F}_2 F_2^{-1}\sec^2\theta_{nc}, \\ &\tilde{H}_3 \equiv \tilde{H}_1^{-3}, && \tilde{H}_4 \equiv F_2 F_3 \sin^2\theta_1 \tilde{H}_1^{-3}. \end{aligned} \tag{2.100}$$

The type IIA dilaton in this case is

$$e^{\phi_{(A,6)}} = \left(\frac{F_2}{\hat{F}_2}\right)^{1/4}\left(\frac{e^{-3\phi}\sec\theta_{nc}}{F_3\sin^2\theta_1 + \hat{F}_2\cos^2\theta_1}\right)^{1/2}. \tag{2.101}$$

There is an NS B-field associated to this metric,

$$B_{(A,6)} = k_4 d\phi_1 \wedge d\tilde{\psi}, \qquad k_4 \equiv \frac{\hat{F}_2 \sec\theta_{nc}\cos\theta_1}{F_3\sin^2\theta_1 + \hat{F}_2\cos^2\theta_1}, \tag{2.102}$$

which gives rise to an NS three-form flux of the form

$$\mathcal{H}_3^{(A,6)} = dB_{(A,6)} = k_{4,a}da \wedge d\phi_1 \wedge d\tilde{\psi}. \tag{2.103}$$

Note that, as a consequence of our choices in (2.2) and because $\hat{F}_2$ depends on ϕ (see (2.88)), $k_4 = k_4(a)$ with $a \equiv (\theta_1, r, x_8, x_9)$. The RR three-form flux $\mathcal{F}_3^{(B,5)}$ dualizes to an RR two-form flux. Using (2.80), this can be written as

$$\mathcal{F}_2^{(A,6)} = d\theta_1 \wedge \Big(\sum_a \frac{\partial b_{\theta_1\phi_1}}{\partial a}da\Big) + \frac{\tan\theta_{nc}}{\cos\theta_{nc}}\left(k_{3,a}\cos\theta_1 da - k_3\sin\theta_1 d\theta_1\right)\wedge d\tilde{x}_3 + \text{sources} \tag{2.104}$$

and, of course, is not closed: $d\mathcal{F}_2^{(A,6)} \neq 0$, denoting a D6-brane source. This is dual to the D5-brane sourcing $\mathcal{F}_3^{(B,5)}$ before. In terms of the type IIA gauge field $\tilde{\mathbf{A}}_1$ of this configuration, we can further rewrite the above as

$$\mathcal{F}_2^{(A,6)} = d\tilde{\mathbf{A}}_1 + \Delta', \qquad d\Delta' = \text{sources}, \qquad \tilde{\mathbf{A}}_1 = b_{\theta_1\phi_1}d\theta_1 + k_3\frac{\tan\theta_{nc}}{\cos\theta_{nc}}\cos\theta_1 d\tilde{x}_3. \tag{2.105}$$

At last, we will uplift the just obtained type IIA configuration to M-Theory. Again it should be borne in mind that the M-Theory that follows only captures the dynamics of the above type IIA theory in the strong coupling limit where $e^{\phi_{(A,6)}}$ is, at least, of order one. Being once more interested in having a finite radius for the eleventh direction, we shall be careful to avoid the $e^{\phi_{(A,6)}} \to \infty$ limit. This limit applies in the same cases as discussed in (2.45) before. The corresponding M-Theory metric is

$$\begin{aligned} ds^2_{(M,2)} =& \tilde{H}_1\big[-dt^2 + dx_1^2 + dx_2^2 + \tilde{H}_2 d\tilde{x}_3^2 + \tilde{H}_3 d\phi_1^2 + e^{2\phi}(F_1 dr^2 + \tilde{H}_4 d\tilde{\psi}^2)\big] \\ &+ e^{2\phi}\tilde{H}_1[F_3 d\theta_1^2 + F_4(dx_8^2 + dx_9^2)] + \frac{e^{-2\phi}}{\tilde{H}_1^2\tilde{H}_2}(dx_{11} + \tilde{\mathbf{A}}_1)^2. \end{aligned} \tag{2.106}$$

In analogy to (2.47) earlier, fixing $r = r_0$ and defining

$$\tilde{G}_1 \equiv e^{2\phi}\tilde{H}_1 F_3\Big|_{r=r_0}, \qquad \tilde{G}_2, \tilde{G}_3 \equiv e^{2\phi}\tilde{H}_1 F_4\Big|_{r=r_0}, \qquad \tilde{G}_4 \equiv e^{-2\phi}\tilde{H}_1^{-2}\tilde{H}_2^{-1}\Big|_{r=r_0}, \tag{2.107}$$

the last line in (2.106) is easily seen to be a warped Taub-NUT space with metric

$$ds^2_{TN_2} = \tilde{G}_1 d\theta_1^2 + \tilde{G}_2 dx_8^2 + \tilde{G}_3 dx_9^2 + \tilde{G}_4 \Big(dx_{11} + \tilde{\mathbf{A}}_1 \Big|_{r=r_0} \Big)^2. \tag{2.108}$$

The G-flux corresponding to this second M-Theory construction is very similar to that in (2.51):

$$\mathcal{G}_4^{(M,2)} = \langle \mathcal{G}_4^{(M,2)} \rangle + \tilde{\mathcal{F}} \wedge \tilde{\omega}, \qquad \langle \mathcal{G}_4^{(M,2)} \rangle = \mathcal{H}_3^{(A,6)} \wedge dx_{11}, \tag{2.109}$$

where $\langle \mathcal{G}_4^{(M,2)} \rangle$ denotes the background G-flux, $\mathcal{H}_3^{(A,6)}$ is as in (2.103) and $\tilde{\omega}$ is the unique, normalizable (anti-)self-dual harmonic two-form associated to the single-centered (warped) Taub-NUT space in (2.108). Here, $\tilde{\mathcal{F}}$ stands for the seven-dimensional field strength of the world-volume gauge theory.

It would not be hard to adapt the computation of ω in section 2.1 to the present case and obtain the explicit form of $\tilde{\omega}$. In fact, we could adapt the discussion of section 2.1.1 too and thus obtain a non-abelian enhancement of the world-volume gauge theory in this setup. However, before doing any more computations, let us compare the two M-Theory metrics: (2.46) and (2.106). They are very similar. In fact, they just differ in the warp factors. It is important to note that both of them break the Lorentz invariance along the (t, x_1, x_2) and the $\tilde{x}_3$ directions. Moreover, both M-theories capture the dynamics of their dual type IIA configurations in the same limit, as we noted a bit earlier. Since the supergravity analysis that we will perform in parts II and III will only depend on the metric deformations, the above noted similarities are enough to consider that, for our purposes, both M-Theory configurations are equivalent. Nonetheless, it should be clear from our calculations so far that the first M-Theory configuration is computationally simpler to handle. Indeed, as we already anticipated, the non-commutative deformation by itself sources the required Θ-term in the world-volume theory and that is all we will really need. The present section explicitly has shown that (2.46) captures all the information needed from the type IIB configuration in [14] to embed knots and study their invariants. Consequently, we will drop any further study of the M-Theory configuration in (2.106) and instead carry all our investigations in the configuration with metric (2.46). That is, the first M-Theory construction to study knot invariants is (M,1) in figure 3. More precisely, we shall focus on its non-abelian enhancement, which was described in section 2.1.1.

It is important to bear in mind that the configuration (M,1) has been obtained from the D3-NS5 system of [14] using the well-defined chain of dualities depicted in figures 2 and 3 (along with figure 4, for the non-abelian enhanced case). Consequently, (M,1) is *dual* to the model in [14], by construction.

Part II will be devoted to the study of the physics following from (M,1). A special emphasize will be made on what and why this is a suitable framework for the realization of knots. Before proceeding in this direction, however, we shall first construct yet another M-Theory configuration, which we will refer to as (M,5). The configuration (M,5) also follows from [14], but is *not* dual to it, as we shall see. Instead, we will show that it is *dual* to the model in [11] and thus provides a second, independent ultraviolet-complete physical framework for the study of knots.

In chapter 2 we have constructed model (M,1). This is an M-theoretical model which has all required features to host knots. In particular, it has $\mathcal{N} = 4$ *supersymmetry, subject to half-BPS boundary conditions. Model (M,1) is dual to the non-abelian version of the D3-NS5 system proposed in [14] and is fully characterized by the eleven-dimensional metric (2.46) and the four-form G-flux (2.76).*

Chapter 3: Construction of (M,5) from the D3-NS5 system

As was the case in chapter 2 and as schematically shown in figure 1, the starting point of our analysis here too is the well-known type IIB String Theory configuration of a D3-brane ending on an NS5-brane considered in [14]. For the time being, we will not consider an axionic background: $\mathcal{C}_0 = 0$. The notation and orientation of the branes are exactly as before, but with the further identifications ($x_8 \equiv \theta_2$, $x_9 \equiv \phi_2$), which will soon become sensible.

Next, we do five modifications to the above setup. Figure 6 schematically depicts them. The modifications aim to ultimately make a precise connection between [14] and [11]. We will discuss such connection later on, in section 3.2.2. For the time being, let us just describe the modifications.

- First, we introduce a second NS5-brane, oriented along (t, x_1 x_2, x_3, θ_1, ϕ_1) and which intersects the D3-brane. In analogy to the first modification in chapter 2, this turns the direction ψ of the D3-brane, which is orthogonal to both NS5-branes, into a finite interval. The ψ interval in this case is taken to be not too large. Consequently, the $U(1)$ gauge theory on the D3-brane has only $\mathcal{N} = 1$ supersymmetry now.

- Second, we do a T-duality to type IIA String Theory along x_3, which results in the D3-brane converting to a D4-brane. The NS5-branes are not affected by this T-duality. This same duality was discussed at length in [39, 40].

- Third, we introduce a large number of coincident D4-branes, so that we have a stack of N (where $N \in \mathbb{N}$ and $N >> 1$) D4-branes between the two NS5-branes.

- Fourth, we do a T-duality along ψ back to type IIB. As a result, the NS5-branes disappear and give rise to a singular conifold in the (θ_1, ϕ_1, ψ, r, θ_2, ϕ_2) directions, which explains the coordinate relabeling above. The N D4-branes convert to N D5-branes, which wrap the vanishing two-cycle of the conifold. This T-duality has been carefully discussed in [29, 41]. Note that, unlike in chapter 2 (see figure 2**D**), there are no $\overline{\text{D5}}$-branes here. This is because there is no Coulomb branch in this setup (the associated world-volume gauge theory is an $\mathcal{N} = 1$ supersymmetric one).

- Finally, we blow up the two-cycle of the singular conifold and thus obtain a resolved conifold. The metric on the resolved conifold is a *non-Kähler* one, as succinctly pointed out in [41] and as discussed in details in [42].

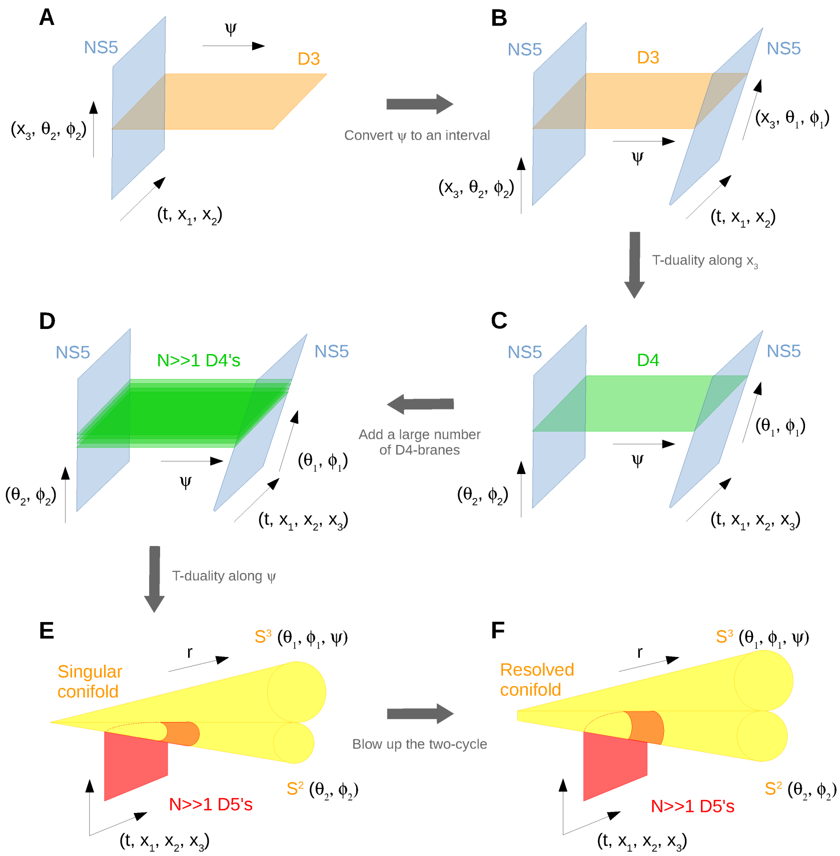

Figure 6: Caricature of the modifications to the D3-NS5 system described in chapter 3. The reason to consider this chain of dualities is twofold: to be able to write the corresponding metric (the geometry of **F** is known) and to ultimately connect [14] and [11]. **A:** The well-known type IIB D3-NS5 system. The D3-brane spans the $(t,\, x_1,\, x_2,\, \psi)$ directions and the NS5-brane the $(t,\, x_1,\, x_2,\, x_3,\, \theta_2,\, \phi_2)$ directions. The $(\theta_1,\, \phi_1,\, r)$ directions are suppressed. The gauge theory on the D3-brane has $\mathcal{N} = 2$ supersymmetry. **B:** Introducing a second NS5-brane, oriented along $(t,\, x_1,\, x_2,\, x_3,\, \theta_1,\, \phi_1)$ converts the ψ direction into an interval. This reduces the amount of supersymmetry of the gauge theory on the D3-brane from $\mathcal{N} = 2$ to $\mathcal{N} = 1$. The r direction is suppressed. **C:** A T-duality along x_3 does not affect the NS5-branes, but converts the D3-brane into a D4-brane. **D:** We add a large amount of coincident D4-branes to the previous configuration. The aim of this step is to later on establish a precise connection with the configuration studied in [11]. **E:** A T-duality along ψ converts the NS5-branes to a singular conifold along $(\theta_1,\, \phi_1,\, \psi,\, r,\, \theta_2,\, \phi_2)$. The D4-branes convert to as many D5-branes that wrap the vanishing two-cycle of the conifold. **F:** The blowing up of the two-cycle of the singular conifold leads to a resolved conifold. The D5-branes are not affected.

The geometry corresponding to this last configuration is known (which explains why the above modifications were done) and is given by

$$ds^2_{(B,7)} = e^{-\tilde{\phi}}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + e^{\tilde{\phi}} ds^2_{(4)}, \tag{3.1}$$

where the metric of warped internal six-dimensional manifold is

$$ds^2_{(4)} \equiv \mathscr{F}_1 dr^2 + \mathscr{F}_2 (d\psi + \sum_{i=1}^{2} \cos\theta_i d\phi_i)^2 + \sum_{i=1}^{2} \mathscr{F}_{2+i}(d\theta_i^2 + \sin^2\theta_i d\phi_i^2). \tag{3.2}$$

Here, $e^{-\tilde{\phi}}$ is the usual type IIB dilaton:

$$e^{\tilde{\phi}_{(B,7)}} = e^{-\tilde{\phi}}. \tag{3.3}$$

For simplicity, we assume that the warp factors and the dilaton only depend on the radial coordinate r:

$$\mathscr{F}_i = \mathscr{F}_i(r), \qquad \tilde{\phi} = \tilde{\phi}(r), \qquad i = 1, 2, 3, 4. \tag{3.4}$$

Under such assumption and for a fixed value of the radial coordinate, namely $r = r_0$, the second line in (3.1) is the resolved conifold metric. As was the case in chapter 2, the D5-branes in this configuration source an RR three-form flux $\mathcal{F}_3^{(B,7)}$ which can be computed as

$$\mathcal{F}_3^{(B,7)} = e^{2\tilde{\phi}} * d\mathcal{J}_{(B,7)}, \tag{3.5}$$

where $\mathcal{J}_{(B,7)}$ is the fundamental two-form associated to $ds^2_{(4)}$. We determine $\mathcal{F}_3^{(B,7)}$ in the following.

We start by defining the vielbeins associated to the internal metric (3.2) as

$$\begin{aligned} E_{\theta_i}^{(B,7)} &= \sqrt{\mathscr{F}_{2+i}} e_{\theta_i}^{(B,7)} = \sqrt{\mathscr{F}_{2+i}} d\theta_i, \qquad E_{\phi_i}^{(B,7)} = \sqrt{\mathscr{F}_{2+i}} e_{\phi_i}^{(B,7)} = \sqrt{\mathscr{F}_{2+i}} \sin\theta_i d\phi_i, \\ E_{\psi}^{(B,7)} &= \sqrt{\mathscr{F}_2} e_{\psi}^{(B,7)} = \sqrt{\mathscr{F}_2}(d\psi + \sum_{i=1}^{2} \cos\theta_i d\phi_i), \qquad E_r^{(B,7)} = \sqrt{\mathscr{F}_1} e_r^{(B,7)} = \sqrt{\mathscr{F}_1} dr, \end{aligned} \tag{3.6}$$

with $i = 1, 2$. Using these vielbeins, it is easy to write down the fundamental two-form of our interest:

$$\mathcal{J}_{(B,7)} \equiv \sum_{i=1}^{2} E_{\theta_i}^{(B,7)} \wedge E_{\phi_i}^{(B,7)} + E_{\psi}^{(B,7)} \wedge E_r^{(B,7)}$$
$$= \sum_{i=1}^{2} \mathscr{F}_{2+i} \sin\theta_i d\theta_i \wedge d\phi_i + \sqrt{\mathscr{F}_1 \mathscr{F}_2}(d\psi + \sum_{i=1}^{2} \cos\theta_i d\phi_i) \wedge dr. \tag{3.7}$$

The exterior derivative of the above is

$$d\mathcal{J}_{(B,7)} = \sum_{i=1}^{2} (\mathscr{F}_{2+i,r} - \sqrt{\mathscr{F}_1 \mathscr{F}_2}) \sin\theta_i dr \wedge d\theta_i \wedge d\phi_i, \tag{3.8}$$

where, quite obviously, $\mathscr{F}_{2+i,r}$ stands for the derivative with respect to r of $\mathscr{F}_{2+i}$ ($i = 1, 2$). The Hodge dual of the above, with respect to the metric (3.2), gives rise to the three-form flux (3.5) we were looking for:

$$\mathcal{F}_3^{(B,7)} = e^{2\tilde{\phi}} \sqrt{\frac{\mathscr{F}_2}{\mathscr{F}_1}} \sum_{\substack{i,j=1 \\ i \neq j}}^{2} \frac{\mathscr{F}_{2+j}}{\mathscr{F}_{2+i}} (\mathscr{F}_{2+i,r} - \sqrt{\mathscr{F}_1 \mathscr{F}_2}) \sin\theta_j (d\psi + \cos\theta_i d\phi_i) \wedge d\theta_j \wedge d\phi_j. \tag{3.9}$$

Note that, in good agreement with the previously pointed out presence of D5-branes in this configuration, the above flux is not closed: $d\mathcal{F}_3^{(B,7)} \neq 0$.

Later on, in section 3.2.1, we will be interested in making a fully precise choice of the warp factors and dilaton in (3.4). Accordingly, we note that not any such choice will eventually lead to a world-volume gauge theory with $\mathcal{N} = 1$ supersymmetry, as desired. The story is in fact a bit more involved: the warp factors and dilaton must satisfy a particular constraint equation so that we indeed have $\mathcal{N} = 1$ supersymmetry. In the following section, we derive this constraint equation.

3.1 Demanding $\mathcal{N} = 1$ supersymmetry: torsion classes

The aforementioned constraint equation relating the warp factors and dilaton in (3.4) that ensures $\mathcal{N} = 1$ supersymmetry in the associated world-volume gauge theory is most easily derived using the technique of torsion classes. A detailed yet concise review of the technique and its applications to String Theory can be found in [43]. A more mathematical approach to the same material is [44]. In this section, we review and adapt the results in these references to the present case and thus obtain the desired constraint equation. This is, essentially, the content of section 3.1 in [45] as well.

We start by noting that the type IIB configuration determined in the previous section has an internal six-dimensional manifold, whose (Riemannian) metric was given in (3.2). This manifold is equipped with a fundamental two-form, given in (3.7). In a more mathematical language, we say that this is a six-dimensional manifold with a $U(3)$ structure J. An $SU(3)$ structure is then determined by a real three-form Ω_+, which we will soon compute. There is an *intrinsic torsion* associated to each of these structures. For our purposes, only the intrinsic torsion τ_1 of the $SU(3)$ structure will be relevant. τ_1 belongs to a space which can be decomposed into five classes:

$$\tau_1 \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5, \tag{3.10}$$

according to its decomposition into the irreps of $SU(3)$

$$(\mathbf{1}+\mathbf{1}) + (\mathbf{8}+\mathbf{8}) + (\mathbf{6}+\bar{\mathbf{6}}) + (\mathbf{3}+\bar{\mathbf{3}}) + (\bar{\mathbf{3}}+\mathbf{3}). \tag{3.11}$$

We denote the component of τ_1 in $\mathcal{W}_i$ as W_i for all $i = 1, 2, 3, 4, 5$.

Before proceeding further, let us introduce the so-called contraction operator $\lrcorner$, which will immediately become useful to us. Let $(e_1, e_2, \ldots, e_i)$ be an orthonormal basis of the cotangent space T^*M of any i-dimensional manifold M. Given a j-form ω_1 and a k-form ω_2 in T^*M, with $i \geq j \geq k \geq 0$,

$$\omega_1 \equiv (\omega_1)_{12\ldots j} \prod_{l=1}^{j} e_l, \qquad \omega_2 \equiv (\omega_2)_{12\ldots k} \prod_{l=1}^{k} e_l, \tag{3.12}$$

the contraction operator $\lrcorner$ is a map from the pair (ω_1, ω_2) to a $(j-k)$-form given by

$$\omega_2 \lrcorner\, \omega_1 \equiv \frac{1}{j!} \begin{pmatrix} j \\ k \end{pmatrix} (\omega_1)^{12\ldots j} (\omega_2)_{12\ldots k} \prod_{l=k+1}^{j} e_l, \tag{3.13}$$

with the convention that $e_1 \wedge e_2 \lrcorner\, e_1 \wedge e_2 \wedge e_3 = e_3$, etc. Having introduced the contraction operator, we now have all the ingredients required to derive the desired constraint equation.

The necessary and sufficient conditions to ensure $\mathcal{N} = 1$ supersymmetry in the world-volume gauge theory corresponding to the geometry (3.1) have long been known [46][6]. These conditions were then reformulated in [43] in terms of the torsion classes we just introduced in (3.10). For the present case, they amount to demanding that the following should hold true:

$$2W_4 + W_5 = 0, \tag{3.14}$$

[6] The conditions in [46] are actually a bit too stringent. Later on, examples of $\mathcal{N} = 1$ supersymmetric theories which did not satisfy all these conditions were found (see, for example [33]). For our case, however, the list in [46] will suffice.

with $(W_4,\, W_5)$ defined as

$$W_4 \equiv \frac{1}{2} J \lrcorner\, dJ, \qquad W_5 \equiv \frac{1}{2} \Omega_+ \lrcorner\, d\Omega_+. \tag{3.15}$$

The remaining of this section is devoted to the calculation of (3.14) in terms of the warp factors and dilaton in (3.4).

In order to match the conventions in [45], where the interested reader can find an elaboration of the present discussion, we take the complex vielbeins of the internal six-manifold of (3.1) as in there:

$$\mathcal{E}_1^{(B,7)} = e^{\tilde{\phi}} (\sqrt{\mathscr{F}_1} e_r^{(B,7)} + i \sqrt{\mathscr{F}_2} e_\psi^{(B,7)}), \qquad \mathcal{E}_{1+i}^{(B,7)} = e^{\tilde{\phi} + i\psi/2} \sqrt{\mathscr{F}_{2+i}} (e_{\theta_i}^{(B,7)} + i e_{\phi_i}^{(B,7)}), \tag{3.16}$$

where the vielbeins $e^{(B,7)}$ where defined in (3.6) and $i = 1,\, 2$. In terms of these vielbeins, the $U(3)$ structure J of the internal space is given by

$$\begin{aligned} J &\equiv \overline{\mathcal{E}}_1^{(B,7)} \wedge \mathcal{E}_1^{(B,7)} + \sum_{i=1}^{2} \mathcal{E}_{1+i}^{(B,7)} \wedge \overline{\mathcal{E}}_{1+i}^{(B,7)} \\ &= 2ie^{2\tilde{\phi}} \Big(\sqrt{\mathscr{F}_1 \mathscr{F}_2} e_r^{(B,7)} \wedge e_\psi^{(B,7)} + \sum_{i=1}^{2} \mathscr{F}_{2+i} e_{\phi_i}^{(B,7)} \wedge e_{\theta_i}^{(B,7)} \Big), \end{aligned} \tag{3.17}$$

where the bar denotes complex conjugation. We also define the three-form Ω as

$$\Omega \equiv \mathcal{E}_1 \wedge \mathcal{E}_2 \wedge \mathcal{E}_3 = e^{3\tilde{\phi} + i\psi} \sqrt{\mathscr{F}_3 \mathscr{F}_4} \left(\sqrt{\mathscr{F}_1} e_r^{(B,7)} + i \sqrt{\mathscr{F}_2} e_\psi^{(B,7)} \right) \wedge \prod_{i=1}^{2} \left(e_{\theta_i}^{(B,7)} + i e_{\phi_i}^{(B,7)} \right). \tag{3.18}$$

The $SU(3)$ structure Ω_+ of the internal space is just the real part of the above three-form: $\Omega_+ \equiv \mathrm{Re}(\Omega)$. Using Euler's formula, it is not hard to show that

$$\begin{aligned} \Omega_+ = & e^{3\tilde{\phi}} \sqrt{\mathscr{F}_3 \mathscr{F}_4} \Big[\Big(\sqrt{\mathscr{F}_1} \cos\psi e_r^{(B,7)} - \sqrt{\mathscr{F}_2} \sin\psi e_\psi^{(B,7)} \Big) \wedge \Big(e_{\theta_1}^{(B,7)} \wedge e_{\theta_2}^{(B,7)} - e_{\phi_1}^{(B,7)} \wedge e_{\phi_2}^{(B,7)} \Big) \\ & - \Big(\sqrt{\mathscr{F}_1} \sin\psi e_r^{(B,7)} + \sqrt{\mathscr{F}_2} \cos\psi e_\psi^{(B,7)} \Big) \wedge \Big(e_{\theta_1}^{(B,7)} \wedge e_{\phi_2}^{(B,7)} + e_{\phi_1}^{(B,7)} \wedge e_{\theta_2}^{(B,7)} \Big) \Big]. \end{aligned} \tag{3.19}$$

In order to obtain the exterior derivative of the two structures of our interest, $(J,\, \Omega_+)$, it is necessary to use the explicit form of the vielbeins in (3.6). Rather tedious algebra yields

$$dJ = 2ie^{2\tilde{\phi}} \sum_{i=1}^{2} \left(\sqrt{\mathscr{F}_1 \mathscr{F}_2} - \mathscr{F}_{2+i,r} - 2\tilde{\phi}_r \mathscr{F}_{2+i} \right) e_r^{(B,7)} \wedge e_{\theta_i}^{(B,7)} \wedge e_{\phi_i}^{(B,7)}, \tag{3.20}$$

and the cumbersome looking four-form

$$\begin{aligned} d\Omega_+ = {} & k_1' e_r^{(B,7)} \wedge e_{\phi_1}^{(B,7)} \wedge e_{\phi_2}^{(B,7)} \wedge \sum_{i=1}^{2} \cot\theta_i e_{\theta_i}^{(B,7)} + k_2' e_r^{(B,7)} \wedge e_{\theta_1}^{(B,7)} \wedge e_{\theta_2}^{(B,7)} \wedge \sum_{i=1}^{2} \cot\theta_i e_{\phi_i}^{(B,7)} \\ & + d\psi \wedge e_r^{(B,7)} \wedge \big(k_1' e_{\theta_1}^{(B,7)} \wedge e_{\phi_2}^{(B,7)} + k_1' e_{\phi_1}^{(B,7)} \wedge e_{\theta_2}^{(B,7)} + k_2' e_{\theta_1}^{(B,7)} \wedge e_{\theta_2}^{(B,7)} - k_2' e_{\phi_1}^{(B,7)} \wedge e_{\phi_2}^{(B,7)} \big), \end{aligned} \tag{3.21}$$

where the subscript r, as before, denotes derivation with respect to the radial coordinate and we have defined

$$k_1' \equiv e^{3\tilde{\phi}} \sqrt{\mathscr{F}_2 \mathscr{F}_3 \mathscr{F}_4} \cos\psi \Big(3\tilde{\phi}_r - \sqrt{\frac{\mathscr{F}_1}{\mathscr{F}_2}} + \sum_{i=2}^{4} \frac{\mathscr{F}_{i,r}}{2\mathscr{F}_i} \Big), \qquad k_2' \equiv -\tan\psi k_1'. \tag{3.22}$$

Using (3.13) and all the above in (3.15), it is a matter of care and patience to obtain the relevant components of the intrinsic torsion of Ω_+ as

$$W_4 = \Big(\tilde{\phi}_r + \sum_{i=3}^{4} \frac{\mathscr{F}_{i,r} - \sqrt{\mathscr{F}_1 \mathscr{F}_2}}{4\mathscr{F}_i} \Big) e_r^{(B,7)}, \qquad W_5 = \frac{1}{2} \Big(\tilde{\phi}_r - \frac{1}{3} \sqrt{\frac{\mathscr{F}_1}{\mathscr{F}_2}} + \sum_{i=2}^{4} \frac{\mathscr{F}_{i,r}}{6\mathscr{F}_i} \Big) e_r^{(B,7)}. \tag{3.23}$$

Finally, inserting these values of (W_4, W_5) in (3.14), the desired constraint ensuring $\mathscr{N} = 1$ supersymmetry is obtained:

$$30\tilde{\phi}_r - 2\sqrt{\frac{\mathscr{F}_1}{\mathscr{F}_2}} + \frac{\mathscr{F}_{2,r}}{\mathscr{F}_2} + \sum_{i=3}^{4} \Big(7\frac{\mathscr{F}_{i,r}}{\mathscr{F}_i} - 6\frac{\sqrt{\mathscr{F}_1 \mathscr{F}_2}}{\mathscr{F}_i} \Big) = 0. \tag{3.24}$$

At this point one may wonder if similar constraints should not have been worked out for our configuration (M,1) with metric (2.1) in chapter 2 as well. Surely if $\mathscr{N} = 1$ supersymmetry constrains the choice of warp factors and dilaton in (3.4), $\mathscr{N} = 2$ supersymmetry should also constrain our choices in (2.2). The resolution to this issue is, unfortunately, beyond the scope of this work, as the powerful technique of torsion classes has not yet been generalized to the case of $\mathscr{N} = 2$ supersymmetry. Consequently, any specific choice for the warp factors in (2.2) and Q in (2.59) that one may want to consider will require an explicit verification that it indeed preserves the desired amount of supersymmetry[7].

To sum things up, so far we have obtained from the well-known D3-NS5 system (with no axion) of [14] the type IIB configuration with metric (3.1), dilaton $e^{-\tilde{\phi}}$ and an RR three-form flux (3.9). In order for this configuration to lead to a $\mathscr{N} = 1$ supersymmetric world-volume

[7]We will discuss how this is achieved in the gauge theory following from (M,1) in section 6.2 later on.

gauge theory, the constraint (3.24) should be satisfied. However, we would like to consider a type IIB configuration which, besides having an RR three-form flux, also has an NS three-form flux. This is, in principle, not an easy task. However, the series of dualities first presented in [41] and later on further studied in [42, 45], when applied to our above configuration, precisely serves this purpose. In the following section, we explain these dualities in details and obtain a type IIB configuration with both RR and NS fluxes. Such a generalization will then, in section 3.2.2, allow us to establish a direct connection with the model to study knots presented in [11].

3.2 Sourcing NS fluxes: a boost in M-Theory

We start this section considering the type IIB configuration described in chapter 3 and depicted in figure 6**F**. We will first perform three T-dualities, along (x_1, x_2, x_3), to type IIA. The resulting configuration will then be lifted to M-Theory, where we will perform a boost along the eleventh direction: $(t, x_{11}) \to (\tilde{t}, \tilde{x}_{11})$. This will be followed by a dimensional reduction to type IIA. The last step will be to T-dualize along (x_1, x_2, x_3) back to type IIB. Of course, we will work out the NS B-field, dilaton and RR and NS fluxes associated to each geometry considered along this chain of modifications. As we already pointed out, starting from a type IIB configuration which only has RR fluxes, we will thus obtain a type IIB configuration with RR and NS fluxes. As already said and as we shall show, the additional NS fluxes are required in order to precisely reproduce the model in [11]. Figure 7 outlines the just described chain of modifications and serves as a summary of the key results in the present section.

As just mentioned, to the type IIB configuration shown in figure 6**F** we do three T-dualities, along (x_1, x_2, x_3). It is rather straightforward to see that the metric then becomes

$$ds^2_{(A,7)} = -e^{-\tilde{\phi}}dt^2 + e^{\tilde{\phi}}(dx_1^2 + dx_2^2 + dx_3^2 + ds^2_{(4)}), \tag{3.25}$$

where $ds^2_{(4)}$ was defined in (3.2). Coming to the dilaton, its changes can be summarized as follows:

$$e^{\tilde{\phi}_{(B,7)}} = e^{-\tilde{\phi}} \xrightarrow[\text{along } x_1]{\text{T-duality}} e^{-\tilde{\phi}/2} \xrightarrow[\text{along } x_2]{\text{T-duality}} 1 \xrightarrow[\text{along } x_3]{\text{T-duality}} e^{\tilde{\phi}/2} = e^{\tilde{\phi}_{(A,7)}}. \tag{3.26}$$

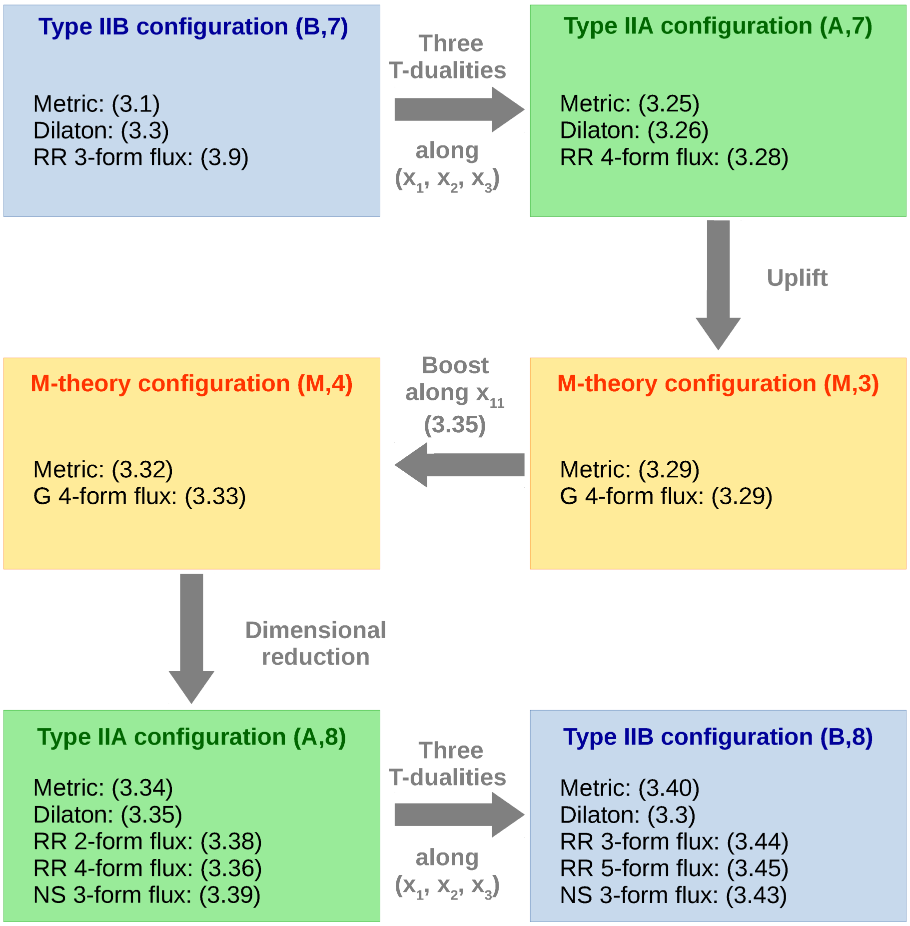

Figure 7: Graphical summary of section 3.2. To the type IIB configuration of figure 6**F** we do a series of modifications. In this manner, we obtain a type IIB configuration that, besides RR fluxes, has NS fluxes as well.

The $\mathcal{F}_3^{(B,7)}$ flux will give rise to an RR six-form flux. This is because each T-duality will add a leg to it along its corresponding Minkowskian direction (x_1, x_2, x_3):

$$\begin{aligned}\mathcal{F}_3^{(B,7)} \xrightarrow[\text{along } x_1]{\text{T-duality}} dx_1 \wedge \mathcal{F}_3^{(B,7)} \xrightarrow[\text{along } x_2]{\text{T-duality}} dx_2 \wedge dx_1 \wedge \mathcal{F}_3^{(B,7)} \\ \xrightarrow[\text{along } x_3]{\text{T-duality}} dx_3 \wedge dx_2 \wedge dx_1 \wedge \mathcal{F}_3^{(B,7)} = \mathcal{F}_6^{(A,7)}.\end{aligned} \tag{3.27}$$

This flux is not closed ($d\mathcal{F}_6^{(A,7)} \neq 0$), which is to be expected, since the three T-dualities convert the N coincident D5-branes of the previous type IIB configuration to N coincident D2-branes that source $\mathcal{F}_6^{(A,7)}$. The Hodge dual of this six-form flux then gives us the more convenient −from the point of view of the coming uplift− RR four-form flux of this type IIA configuration:

$$\mathcal{F}_4^{(A,7)} = *\mathcal{F}_6^{(A,7)} = *\mathcal{F}_3^{(B,7)} \wedge dt = \sum_{i=1}^{2} (\mathscr{F}_{2+i,r} - \sqrt{\mathscr{F}_1 \mathscr{F}_2}) \sin\theta_i dr \wedge d\theta_i \wedge d\phi_i \wedge dt, \tag{3.28}$$

where the first Hodge dual is with respect to the full ten-dimensional metric (3.25), whereas the second one is with respect to the internal metric in (3.2). The above result makes use of (3.5), (3.8) and (3.27).

Lifting this type IIA configuration to M-Theory is quite effortless. We get the following metric and G-flux:

$$ds^2_{(M,3)} = -e^{-4\tilde{\phi}/3} dt^2 + e^{2\tilde{\phi}/3}(dx_1^2 + dx_2^2 + dx_3^2 + ds^2_{(4)} + dx_{11}^2), \qquad \mathcal{G}_4^{(M,3)} = \mathcal{F}_4^{(A,7)}. \tag{3.29}$$

Note that the D2-branes now convert to N coincident M2-branes.

The key step in this chain of dualities comes next: we perform a boost in the eleventh direction. Explicitly,

$$x_{11} = \cosh\beta \tilde{x}_{11} - \sinh\beta \tilde{t}, \qquad t = -\sinh\beta \tilde{x}_{11} + \cosh\beta \tilde{t}, \tag{3.30}$$

with β the boost parameter. For brevity, we introduce the quantity

$$\Upsilon \equiv \sinh^2\beta (e^{2\tilde{\phi}/3} - e^{-4\tilde{\phi}/3}). \tag{3.31}$$

Using the above two equations in $ds^2_{(M,3)}$, it is a matter of simple algebra to check that the boosted M-Theory metric is given by

$$ds^2_{(M,4)} = e^{2\tilde{\phi}/3}(dx_1^2 + dx_2^2 + dx_3^2 + ds^2_{(4)}) - \frac{e^{-2\tilde{\phi}/3}}{\Upsilon + e^{2\tilde{\phi}/3}} d\tilde{t}^2$$
$$+ (\Upsilon + e^{2\tilde{\phi}/3})\big(d\tilde{x}_{11} - \frac{\Upsilon \coth\beta}{\Upsilon + e^{2\tilde{\phi}/3}} d\tilde{t}\big)^2. \tag{3.32}$$

Writing the metric as we did above makes it clear that the boost has now generated a gauge field in the M-Theory. On the other hand, the boosted G-flux can be easily seen to be

$$\mathcal{G}_4^{(M,4)} = d\mathcal{J}_{(B,7)} \wedge (\cosh\beta d\tilde{t} - \sinh\beta d\tilde{x}_{11}), \tag{3.33}$$

with $d\mathcal{J}_{(B,7)}$ as in (3.8).

The next step in the chain of dualities outlined in the beginning of this section is to dimensionally reduce the above to type IIA. The corresponding metric is

$$ds^2_{(A,8)} = -\frac{e^{-2\tilde{\phi}/3}}{\sqrt{\Upsilon + e^{2\tilde{\phi}/3}}} d\tilde{t}^2 + e^{2\tilde{\phi}/3}\sqrt{\Upsilon + e^{2\tilde{\phi}/3}}(dx_1^2 + dx_2^2 + dx_3^2 + ds^2_{(4)}) \tag{3.34}$$

and the associated dilaton is

$$e^{\tilde{\phi}_{(A,8)}} = \left(\Upsilon + e^{2\tilde{\phi}/3}\right)^{3/4}. \tag{3.35}$$

Coming now to the fluxes, we note that the M2-branes of the previous M-Theory setup now convert to D2-branes, which source an RR four-form flux given by

$$\mathcal{F}_4^{(A,8)} = \cosh\beta d\mathcal{J}_{(B,7)} \wedge d\tilde{t}. \tag{3.36}$$

The Hodge dual of the above will soon be useful. This is an RR six-form flux of the form

$$\mathcal{F}_6^{(A,8)} = *\mathcal{F}_4^{(A,8)} = \cosh\beta dx_1 \wedge dx_2 \wedge dx_3 \wedge \mathcal{F}_3^{(B,7)}, \tag{3.37}$$

. where $\mathcal{F}_3^{(B,7)}$ was given in (3.9) This flux is clearly not closed, $d\mathcal{F}_6^{(A,8)} \neq 0$, as expected. Additionally, the M-Theory gauge field generated by the boost (3.30), effectively converts to a "D0-charge". This D0-charge sources a closed RR two-form flux: the exterior derivative of the just mentioned gauge field. Explicitly,

$$\mathcal{F}_2^{(A,8)} = -d\Big(\frac{\Upsilon \coth\beta}{\Upsilon + e^{2\tilde{\phi}/3}} d\tilde{t}\Big) = \coth\beta \frac{d}{dr}\Big(\frac{\Upsilon}{\Upsilon + e^{2\tilde{\phi}/3}}\Big) d\tilde{t} \wedge dr, \tag{3.38}$$

where we have used the fact that, as a consequence of our choices in (3.4), the gauge field only depends on the radial coordinate r (and the boost parameter β). To finish this flux discussion, we note that the boost generates a closed NS three-form flux, just as we wanted:

$$\mathcal{H}_3^{(A,8)} = -\sinh\beta d\mathcal{J}_{(B,7)}. \tag{3.39}$$

To finish this section, the only remaining task is to perform three T-dualities, along (x_1, x_2, x_3), back to type IIB. From (3.34), it follows that the geometry corresponding to our final configuration is

$$ds^2_{(B,8)} = \frac{e^{-2\tilde{\phi}/3}}{\sqrt{\Upsilon + e^{2\tilde{\phi}/3}}}(-d\tilde{t}^2 + dx_1^2 + dx_2^2 + dx_3^2) + e^{2\tilde{\phi}/3}\sqrt{\Upsilon + e^{2\tilde{\phi}/3}}ds^2_{(4)}, \tag{3.40}$$

with $ds^2_{(4)}$ as in (3.2). The changes in the dilaton can be summarized as follows:

$$e^{\tilde{\phi}_{(A,8)}} \xrightarrow[\text{along } x_1]{\text{T-duality}} e^{-\tilde{\phi}/3}\big(\Upsilon + e^{2\tilde{\phi}/3}\big)^{1/2} \xrightarrow[\text{along } x_2]{\text{T-duality}} e^{-2\tilde{\phi}/3}\big(\Upsilon + e^{2\tilde{\phi}/3}\big)^{1/4} \xrightarrow[\text{along } x_3]{\text{T-duality}} e^{-\tilde{\phi}}. \tag{3.41}$$

Hence, the dilaton remains as in the beginning:

$$e^{\tilde{\phi}_{(B,8)}} = e^{\tilde{\phi}_{(B,7)}} = e^{-\tilde{\phi}}. \tag{3.42}$$

It is rather obvious that, since the dualities are along diagonal directions of the metric, the NS three-form flux will not be affected in this case:

$$\mathcal{H}_3^{(B,8)} = \mathcal{H}_3^{(A,8)} = -\sinh\beta d\mathcal{J}_{(B,7)}. \tag{3.43}$$

Regarding the $\mathcal{F}_6^{(A,8)}$ flux, we note that each T-duality will remove a leg to it along its corresponding Minkowskian direction (x_1, x_2, x_3). That is, we have the reverse process to that earlier in (3.27):

$$\begin{aligned}\mathcal{F}_6^{(A,8)} = \cosh\beta dx_1 \wedge dx_2 \wedge dx_3 \wedge \mathcal{F}_3^{(B,7)} &\xrightarrow[\text{along } x_1]{\text{T-duality}} \cosh\beta dx_2 \wedge dx_3 \wedge \mathcal{F}_3^{(B,7)} \\ \xrightarrow[\text{along } x_2]{\text{T-duality}} \cosh\beta dx_3 \wedge \mathcal{F}_3^{(B,7)} &\xrightarrow[\text{along } x_3]{\text{T-duality}} \cosh\beta\mathcal{F}_3^{(B,7)} = \mathcal{F}_3^{(B,8)}.\end{aligned} \tag{3.44}$$

We thus obtain a non-closed RR three-form flux, an indication of the N coincident D5-branes present in this configuration. Finally, the D0-charge previously sourcing $\mathcal{F}_2^{(A,8)}$ now converts to a D3-charge. The D3-charge then sources an RR five-form flux which, in analogy to (3.27), is given by $\mathcal{F}_2^{(A,8)} \wedge dx_1 \wedge dx_2 \wedge dx_3$, plus its Hodge dual (since the D3-charge is self-dual, the corresponding RR flux must be self-dual too). We get

$$\mathcal{F}_5^{(B,8)} = \coth\beta(1+*)\frac{d}{dr}\Big(\frac{\Upsilon}{\Upsilon+e^{2\tilde{\phi}/3}}\Big)d\tilde{t}\wedge dr\wedge dx_1\wedge dx_2\wedge dx_3, \tag{3.45}$$

where the Hodge dual is, of course, with respect to the metric (3.40). As a consistency check, one may verify that setting $\beta = 0$ (no boost), we recover the initial type IIB configuration with only dilaton and RR three-form flux:

$$\text{configuration } (B,8) \xrightarrow[\beta=0]{} \text{configuration } (B,7). \tag{3.46}$$

It is important to note that none of the modifications performed in this section affects the supersymmetry of the starting configuration (B,7). In other words, the previously derived constraint equation (3.24) is enough to ensure that the end configuration (B,8) is associated to an $\mathcal{N} = 1$ supersymmetric world-volume gauge theory too. We refer the interested reader to section 3.2 in [45] for an enlightening discussion on the difficulties to derive this constraint equation in the context of the configuration (B,8), where the internal 6-dimensional manifold is not complex, unlike in the configuration (B,7).

3.2.1 Exact results: a specific choice of the warp factors

At this stage, we would like to make our discussion fully precise. To this aim, we choose our warp factors as

$$\mathscr{F}_1 = \frac{e^{-\tilde{\phi}}}{2F}, \qquad \mathscr{F}_2 = \frac{r^2 e^{-\tilde{\phi}}F}{2}, \qquad \mathscr{F}_3 = \frac{r^2 e^{-\tilde{\phi}}}{4} + a^2, \qquad \mathscr{F}_4 = \frac{r^2 e^{-\tilde{\phi}}}{4}, \tag{3.47}$$

where, in good agreement with our previous choices in (3.4),

$$F = F(r), \qquad a^2 \equiv a_0^2 + \tilde{a}(r). \tag{3.48}$$

The constant a_0^2 is to be interpreted as the resolution parameter of the blown up two-cycle in the resolved conifold. (This choice was already studied in [45] and [47].) In this section, we work out three constraint equations that ultimately allow us to compute (F, $e^{\tilde{\phi}}$, a) above and thereby fully determine our type IIB configuration in this case. We will do so for a particularly simple case, as the most general scenario is computationally hard to handle.

The first constraint equation follows from demanding that the choice (3.47) leads to a world-volume gauge theory with $\mathcal{N} = 1$ supersymmetry. As we argued in section 3.1, this amounts to requiring that (3.24) holds true. Using (3.47) in (3.24), it is quite straightforward to show that the first constraint can be written as

$$\Big(15+88\frac{a^2e^{\tilde\phi}}{r^2}\Big)\tilde\phi_r+56e^{\tilde\phi}\frac{a}{r^2}a_r+\frac{2}{r}+\Big(\frac{4}{r}+\frac{1}{F}F_r-\frac{2}{rF}\Big)\Big(1+\frac{4a^2e^{\tilde\phi}}{r^2}\Big)=0, \tag{3.49}$$

where $(\tilde\phi_r,\, a_r,\, F_r)$ stand for the derivatives with respect to the radial coordinate r of $(\tilde\phi,\, a,\, F)$, respectively.

For the second constraint equation, we will demand quantization of the magnetic charge of the D5-branes in our configuration. Recall that, in spite of the duality chain of figure 7, our D5-branes remain as in figure 6**F**: oriented along $(t,\, x_1,\, x_2,\, x_3)$ and wrapping the two-cycle parametrized by $(\theta_2,\, \phi_2)$. As is well-known[8], the D5-branes' charge stems from the RR three-form flux $\mathcal{F}_3^{(B,8)}$. Accordingly, let us begin by giving the explicit form of this flux when the warp factors are chosen as just mentioned. This amounts to inserting (3.47) in (3.44) and further using (3.6) and (3.9). Rather easy and quick algebra then gives

$$\mathcal{F}_3^{(B,8)}=-\frac{e^{\tilde\phi}r^3F}{4}\cosh\beta\left(\tilde k_1 e_{\theta_1}^{(B,7)}\wedge e_{\phi_1}^{(B,7)}+\tilde k_2 e_{\theta_2}^{(B,7)}\wedge e_{\phi_2}^{(B,7)}\right)\wedge e_{\psi}^{(B,7)}, \tag{3.50}$$

where we have defined

$$\tilde k_1\equiv\tilde\phi_r\Big(1+\frac{4a^2e^{\tilde\phi}}{r^2}\Big),\qquad \tilde k_2\equiv\frac{r^2\tilde\phi_r-8aa_re^{\tilde\phi}}{r^2+4a^2e^{\tilde\phi}}. \tag{3.51}$$

Now, the magnetic charge of the D5-branes in our setup can be calculated as the integral of their RR three-form flux over the three cycle orthogonal to them:

$$q_m=\int_{S^3}\mathcal{F}_3^{(B,8)}, \tag{3.52}$$

with S^3 the three cycle labeled by $(\theta_1,\, \phi_1,\psi)$ and depicted in figure 6**F**. It is easy to see that only the first term in (3.50) will contribute to the magnetic charge. Normalizing the three cycle volume as

$$V_{S^3}\equiv\int_{S^3}e_{\theta_1}^{(B,7)}\wedge e_{\phi_1}^{(B,7)}\wedge e_{\psi}^{(B,7)}=1 \tag{3.53}$$

and demanding $q_m\in\mathbb{Z}$, we obtain the second constraint equation:

$$\tilde c_0\equiv\frac{\tilde k_1F}{4}e^{\tilde\phi}r^3\cosh\beta\in\mathbb{Z}. \tag{3.54}$$

[8]A succinct and clear review on charge quantization of D-branes can be found in [31].

The third and last constraint follows from $d^2\mathcal{F}_3^{(B,8)} = 0$. For simplicity, we will consider the limit when $(a,\, a_r)$ are of the same order and sufficiently small, $a \sim a_r << 1$. Under this assumption, we can expand $\tilde{k}_2$ around $a^2 = 0$ and obtain

$$\tilde{k}_2 = \tilde{\phi}_r\Big(1 - \frac{4a^2 e^{\tilde{\phi}}}{r^2}\Big) - \frac{8aa_r e^{\tilde{\phi}}}{r^2} + \mathcal{O}(a^3). \tag{3.55}$$

Further introducing the quantities

$$\eta_3 \equiv \Big(e_{\theta_1}^{(B,7)} \wedge e_{\phi_1}^{(B,7)} - e_{\theta_2}^{(B,7)} \wedge e_{\phi_2}^{(B,7)}\Big) \wedge e_{\psi}^{(B,7)}, \qquad G \equiv e^{2\tilde{\phi}} rF \cosh\beta\Big(2aa_r - \frac{e^{-\tilde{\phi}} r^2 \tilde{\phi}_r}{2}\Big), \tag{3.56}$$

it is not hard to convince oneself that $\mathcal{F}_3^{(B,8)}$ can be written in the very suggestive way

$$\mathcal{F}_3^{(B,8)} = -\tilde{c}_0\eta_3 + Ge_{\theta_2}^{(B,7)} \wedge e_{\phi_2}^{(B,7)} \wedge e_{\psi}^{(B,7)}, \tag{3.57}$$

where we have used our first constraint (3.54). Note that η_3 is a closed form ($d\eta_3 = 0$). Consequently, the exterior derivative of the above comes solely from the second term. Denoting as G_r the derivative of G with respect to r, we obtain

$$d\mathcal{F}_3^{(B,8)} = G_r e_r^{(B,7)} \wedge e_{\psi}^{(B,7)} \wedge e_{\theta_2}^{(B,7)} \wedge e_{\phi_2}^{(B,7)} - Ge_{\theta_1}^{(B,7)} \wedge e_{\phi_1}^{(B,7)} \wedge e_{\theta_2}^{(B,7)} \wedge e_{\phi_2}^{(B,7)}, \tag{3.58}$$

where we have made use of (3.6). Of course, the exterior derivative of the above must vanish and this leads to our third constraint equation:

$$0 = d^2\mathcal{F}_3^{(B,8)} = -G_r e_r^{(B,7)} \wedge e_{\theta_1}^{(B,7)} \wedge e_{\phi_1}^{(B,7)} \wedge e_{\theta_2}^{(B,7)} \wedge e_{\phi_2}^{(B,7)} \implies G_r = 0. \tag{3.59}$$

Having derived the three constraints of our interest, (3.49), (3.54) and (3.59), we will now solve them under the assumption $a \sim a_r << 1$, keeping only terms up to order $\mathcal{O}(a)$. (Other solutions to these equations are of course possible, but we will not attempt them here.) In this case, (3.49) reduces to

$$r\tilde{\phi}_r + \frac{r}{15F}F_r - \frac{2}{15F} + \frac{2}{5} + \mathcal{O}(a^2) = 0 \tag{3.60}$$

and (3.54) becomes

$$\tilde{c}_0 = \frac{e^{\tilde{\phi}} r^3 F}{4}\tilde{\phi}_r \cosh\beta + \mathcal{O}(a^2), \tag{3.61}$$

which immediately ensures that (3.59) is satisfied in the limit here considered. Defining

$Z \equiv e^{\tilde{\phi}}$ and $\hat{c}_0 \equiv \tilde{c}_0/\cosh\beta$, we can solve for F in the above

$$F = \frac{4\hat{c}_0}{r^3 Z_r} + \mathcal{O}(a^2). \tag{3.62}$$

Substitution in (3.60) then yields

$$rZ_{rr} - 3Z_r + r\Big(\frac{r^2}{2c_0} - \frac{15}{Z}\Big)Z_r^2 + \mathcal{O}(a^2) = 0, \tag{3.63}$$

with $Z_{rr} \equiv d^2Z/dr^2$. One may easily verify that a solution to (3.63) is given by $Z = 24\hat{c}_0 r^{-2}$. It follows then that

$$e^{\tilde{\phi}} = \frac{24\hat{c}_0}{r^2} + \mathcal{O}(a^2), \qquad F = -\frac{1}{12} + \mathcal{O}(a^2) \tag{3.64}$$

fully determines our choices in (3.47), up to order $\mathcal{O}(a^2)$. The explicit form of the type IIB configuration (B,8) in figure 7 can then be obtained by simply using (3.47) and (3.64) in (3.40) and in (3.42)-(3.45).

3.2.2 Connection to the model in [11]

The present section is devoted to sketching how the configuration (B,8) of figure 7 is related to the resolved conifold in the presence of fluxes considered by Ooguri and Vafa in [11]. Here, we will clearly point out the modifications needed to obtain the model in [11] from (B,8). These are depicted in figure 8, which serves as a graphical summary of the present section too. Nonetheless, unlike in previous sections, we will not present a thorough derivation of the geometries and fluxes for each intermediate configuration considered in the process. Such exhaustive study is beyond the scope of this thesis and is deferred to the sequel(s) of [1,2]. In such sequel(s), following [11], we also intend to explore knot invariants in the configuration (M,5), which follows from (B,8) and which is constructed in details in section 3.3. For the time being, we refer the interested reader to section 4.4 in [1] for a preliminary discussion of the physics stemming from (M,5) and the realization of knots in this setup.

As we just mentioned, our starting point in this section is the configuration (B,8) summarized in figure 7. Essentially, this is the same configuration as that drawn in figure 6**F**, but in the presence of both RR and NS fluxes. In figure 8, this is shown in the top, left corner. As can be seen, (B,8) consists of a large number N of D5-branes wrapping the two-cycle S^2 of a non-Kähler resolved conifold. Let us start by making an observation that will soon be relevant to us. From the orientation of the D5-branes shown in figure 6**F** it is clear that, upon a dimensional reduction, we expect to obtain an $SU(N)$ world-volume gauge theory along the spacetime directions ($\tilde{t}$, x_1, x_2, x_3).

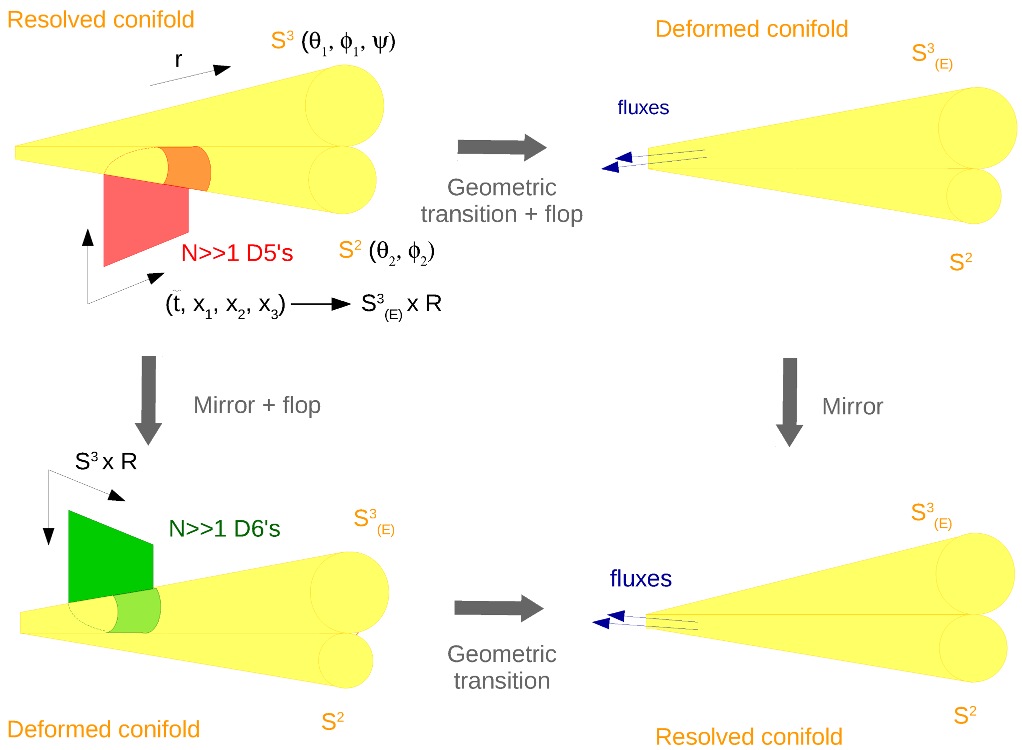

Figure 8: Depiction of the discussions in section 3.2.2. To the configuration (B,8) of figure 7 we do the following modifications: Euclideanize and compactify the $(\tilde{t},\, x_1,\, x_2)$ directions, go to the mirror picture, perform a flop operation and take the gravity dual. The resulting configuration is that of a resolved conifold in the presence of fluxes studied in [11]. Our configuration (B,8) is that on the top, left corner, whereas the most well-known realization of the model in [11] is drawn on the bottom, right corner. It should be noted that, as explained in the text, the mirror operations here shown are valid only in a certain energy range.

Next, recall that the metric corresponding to (B,8) was given in (3.40). Note in particular that the spacetime directions $(\tilde{t},\, x_1,\, x_2)$ in this geometry parametrize a three-dimensional Minkowski subspace. The first modification to (B,8) that one needs to consider in order to obtain the model in [11] consists in Euclideanizing and compactifying these directions, so that they parametrize a sphere: $(\tilde{t},\, x_1,\, x_2) \to S^3_{(E)}$. Then, the corresponding physical theory will lie in $S^3_{(E)} \times \mathbb{R}$, where $\mathbb{R}$ stands for the line labeled by the coordinate x_3.

Secondly, we must perform a series of T- and SYZ-dualities to the resulting configuration, which will take us to the so-called mirror picture. The required dualities are far from trivial, involving many subtleties. Nevertheless, the works [48–51] deal with all difficulties exhaustively and show that the mirror picture consists of N D6-branes wrapping the three-cycle S^3 of a non-Kähler deformed conifold. This is true only for energies higher than the inverse size of the two-cycle S^2 of the dual resolved conifold. As a consequence, we will restrict ourselves in the ongoing to this energy regime[9].

In the described mirror picture of our interest, the N D6-branes are oriented along the seven-dimensional subspace $S^3_{(E)} \times S^3 \times \mathbb{R}$. The third and last modification required to obtain the model in [11] is given by a flop operation, that exchanges $S^3_{(E)}$ and S^3: $S^3_{(E)} \leftrightarrow S^3$. Clearly, this does not affect the orientation of the D6-branes, yet it *transfers* the physics from $S^3_{(E)} \times \mathbb{R}$ to $S^3 \times \mathbb{R}$, thus yielding the D6-brane realization of the model in [11] depicted on the bottom, left corner of figure 8.

A more well-known realization of the setup in [11] is obtained by simply taking the large N dual (in other words, performing a geometric transition) of the above configuration. In this case, the deformed conifold becomes a resolved one. The D6-branes disappear in the dual picture, giving rise to fluxes. This configuration is precisely that shown on the bottom, right corner of figure 8.

Alternatively, one may take the large N dual of (B,8) first and consider the mirror picture afterwards. The result is the same: we obtain the deformed conifold with fluxes of [11]. This equivalent procedure is depicted on the top, right corner of figure 8.

At this stage, we have argued that our configuration (B,8) is related to the model in [11] by a simple chain of dualities. That is, (B,8) is *dual* to [11]. In the next section, we will build an M-Theory configuration (M,5) from (B,8). As we shall see, (B,8) is dual to (M,5) and so this will allow us to conclude that (M,5) is dual to [11] too.

3.3 Non-commutative deformation and M-Theory uplift

In this section we will obtain the second M-Theory construction where knot invariants can be studied: (M,5). Clearly, the starting point will be the configuration (B,8) in figure 7. We will first do a T-duality along ψ to type IIA, where we will perform the same non-commutative deformation we considered in section 2.1: $(x_3,\, \psi) \to (\tilde{x}_3,\, \tilde{\psi})$. As we argued in both sections 2.1 and 2.2, this deformation sources a Θ-term in the associated world-volume gauge theory, which is crucial to allow for the embedding of knots in our model. Finally, we will uplift the resulting configuration to M-Theory. As has been the case so far, the dilaton and fluxes for each geometry considered will be worked out here too. Figure 9 provides a graphical summary of this chain of modifications and indicates the main results in this section.

[9]As argued around (2.5) in [1], for energies lower than the size of S^2, the mirror picture will lead to D4-branes instead of D6-branes. Although such scenario may be interesting as well, it does not relate to the model in [11] and thus we are presently not concerned with it.

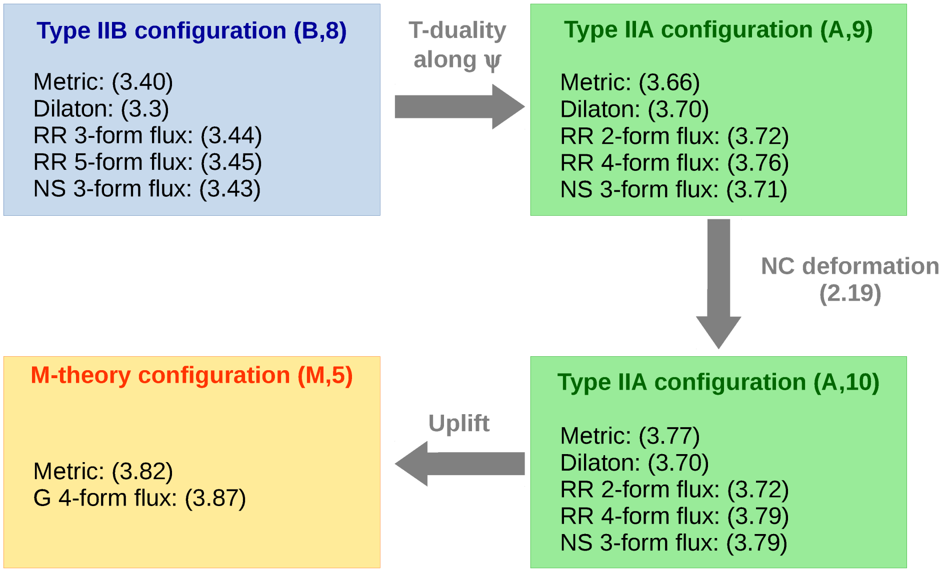

Figure 9: Graphical summary of section 3.3. To the configuration (B,8) of figure 7 we do a series of modifications, so as to source a Θ-term in the corresponding world-volume gauge theory. The resulting configuration is then lifted to M-Theory. The configuration (M,5) is the second M-Theory construction we propose for the study of knots.

In order to obtain the T-dual of the configuration (B,8), we first note that the geometry in (3.40) is associated to the following NS B-field:

$$B_{(B,8)} = \sinh\beta \sum_{i=1}^{2} \left(\sqrt{\mathcal{F}_1\mathcal{F}_2}\cos\theta_i dr - \mathcal{F}_{2+i}\sin\theta_i d\theta_i\right) \wedge d\phi_i, \tag{3.65}$$

as can be easily inferred from (3.43). T-dualizing along ψ the metric (3.40), we obtain the type IIA geometry

$$ds^2_{(A,9)} = \frac{1}{\sqrt{h}}\left(ds^2_{t12} + dx_3^2 + \frac{1}{\mathcal{F}_2}d\psi^2\right) + \sqrt{h}ds^2_{(5)}, \tag{3.66}$$

where we have defined

$$ds^2_{t12} \equiv -d\tilde{t}^2 + dx_1^2 + dx_2^2, \qquad ds^2_{(5)} \equiv \mathscr{F}_1 dr^2 + \sum_{i=1}^{2} \mathscr{F}_{2+i} \left(d\theta_i^2 + \sin^2\theta_i d\phi_i^2\right) \tag{3.67}$$

and, using Υ in (3.31), we have also introduced

$$h \equiv e^{4\tilde{\phi}/3}\left(\Upsilon + e^{2\tilde{\phi}/3}\right). \tag{3.68}$$

This has an associated NS B-field given by

$$B_{(A,9)} = B_{(B,8)} + \sum_{i=1}^{2} \cos\theta_i d\psi \wedge d\phi_i. \tag{3.69}$$

The dilaton for this type IIA configuration is, quite obviously,

$$e^{\tilde{\phi}_{(A,9)}} = h^{-1/4} \mathscr{F}_2^{-1/2} e^{-\tilde{\phi}}. \tag{3.70}$$

The NS three-form flux can be easily derived to be

$$\mathcal{H}_3^{(A,9)} = dB_{(A,9)} = \mathcal{H}_3^{(B,8)} + \sum_{i=1}^{2} \sin\theta_i d\theta_i \wedge d\phi_i \wedge d\psi, \tag{3.71}$$

with $\mathcal{H}_3^{(B,8)}$ as in (3.8) and (3.43). Coming to the RR fluxes now, we note that the T-duality converts the D5-branes that wrap the two-cycle of the resolved conifold in the configuration (B,8) to N coincident D6-branes that wrap the two-sphere parametrized by (θ_1, ϕ_1) in the dual type IIA picture[10]. Consequently, the RR three-form flux (3.44) that was sourced by the D5-branes now gives rise to the RR two-form flux

$$\mathcal{F}_2^{(A,9)} = e^{2\tilde{\phi}} \cosh\beta \sqrt{\frac{\mathscr{F}_2}{\mathscr{F}_1}} \sum_{\substack{i,j=1 \\ i\neq j}}^{2} \frac{\mathscr{F}_{2+j}}{\mathscr{F}_{2+i}} \left(\mathscr{F}_{2+i,r} - \sqrt{\mathscr{F}_1\mathscr{F}_2}\right) \sin\theta_j d\theta_j \wedge d\phi_j, \tag{3.72}$$

[10]Actually, this T-duality is more subtle and can also lead to D4-branes. We discussed this important point in section 3.2.2 already.

as well as to the RR four-form flux

$$\mathcal{F}_4^{(1)} = e^{2\tilde{\phi}} \cosh\beta \sqrt{\frac{\mathscr{F}_2}{\mathscr{F}_1}} \sum_{i,j=1}^{2} \frac{\mathscr{F}_{2+j}}{\mathscr{F}_{2+i}} (\mathscr{F}_{2+i,r} - \sqrt{\mathscr{F}_1\mathscr{F}_2}) \sin\theta_j \cos\theta_i d\psi \wedge d\phi_i \wedge d\theta_j \wedge d\phi_j. \tag{3.73}$$

Both are sourced by the dual D6-branes (and hence, $d\mathcal{F}_2^{(A,9)} \neq 0 \neq d\mathcal{F}_4^{(1)}$). On the other hand, the D3-charge that sourced the self-dual RR five-form flux in (3.45) converts to a D4-charge after the T-duality. They now source RR four- and six-form fluxes, which are Hodge dual to each other, with respect to the metric (3.66). Starting from (3.45) and using (3.68), it is clear that the RR six-form flux is

$$\mathcal{F}_6^{(A,9)} = \coth\beta \frac{d}{dr}\Big(\frac{e^{2\tilde{\phi}}}{h}\Big) d\tilde{t} \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge d\psi \wedge dr. \tag{3.74}$$

However, its Hodge-dual four-form will become more convenient once we perform the uplift to M-Theory, with views to computing the G-flux there. Since the metric (3.66) is diagonal, it is not hard to show that the flux of our interest is given by

$$\mathcal{F}_4^{(2)} = *\mathcal{F}_6^{(A,9)} = -\coth\beta \frac{d}{dr}\Big(\frac{e^{2\tilde{\phi}}}{h}\Big) h^2 \sqrt{\frac{\mathscr{F}_2}{\mathscr{F}_1}} \prod_{i=1}^{2} \mathscr{F}_{2+i} \sin\theta_i d\theta_i \wedge d\phi_i. \tag{3.75}$$

The total RR four-form flux for this configuration is thus

$$\mathcal{F}_4^{(A,9)} = \mathcal{F}_4^{(1)} + \mathcal{F}_4^{(2)}. \tag{3.76}$$

We will now apply the non-commutative deformation $(x_3, \psi) \to (\tilde{x}_3, \tilde{\psi})$ in (2.19) to the above type IIA configuration. The metric (3.66) then changes to

$$ds^2_{(A,10)} = (e^{\tilde{\phi}_{(A,9)}})^{2/3} \Big(\frac{\mathscr{F}_2 e^{2\tilde{\phi}}}{h}\Big)^{1/3} \Big[ds^2_{t12} + \big(\frac{d\tilde{x}_3}{\cos\theta_{nc}} + \sin\theta_{nc} d\tilde{\psi}\big)^2 + \frac{\cos^2\theta_{nc}}{\sqrt{h}\mathscr{F}_2} d\tilde{\psi}^2 + h ds^2_{(5)} \Big]. \tag{3.77}$$

The dilaton and RR two–form flux can be readily seen not to be affected by the deformation:

$$e^{\tilde{\phi}_{(A,10)}} = e^{\tilde{\phi}_{(A,9)}}, \qquad \mathcal{F}_2^{(A,10)} = \mathcal{F}_2^{(A,9)}. \tag{3.78}$$

However, the RR four-form flux and the NS three-form flux do change to

$$\mathcal{F}_4^{(A,10)} = e^{2\tilde{\phi}} \cosh\beta \cos\theta_{nc} \sqrt{\frac{\mathscr{F}_2}{\mathscr{F}_1}} d\tilde{\psi} \wedge \left(\hat{\hat{k}}_1 d\phi_1 \wedge d\theta_2 \wedge d\phi_2 + \hat{\hat{k}}_2 d\theta_1 \wedge d\phi_1 \wedge d\phi_2 \right) + \mathcal{F}_4^{(2)},$$

$$\mathcal{H}_3^{(A,10)} = \mathcal{H}_3^{(B,8)} + \cos\theta_{nc} \sum_{i=1}^{2} \sin\theta_i d\theta_i \wedge d\phi_i \wedge d\tilde{\psi}, \tag{3.79}$$

where we have defined

$$\hat{\hat{k}}_1 \equiv \frac{\mathscr{F}_4}{\mathscr{F}_3} (\mathscr{F}_{3,r} - \sqrt{\mathscr{F}_1 \mathscr{F}_2}) \sin\theta_2 \cos\theta_1, \qquad \hat{\hat{k}}_2 \equiv \frac{\mathscr{F}_3}{\mathscr{F}_4} (\mathscr{F}_{4,r} - \sqrt{\mathscr{F}_1 \mathscr{F}_2}) \sin\theta_1 \cos\theta_2. \tag{3.80}$$

Once more, the RR two-form flux not being closed, we can rewrite it in a similar fashion to what we did earlier in (2.35) and (2.105):

$$\mathcal{F}_2^{(A,10)} = d\hat{\mathbf{A}}_1 + \hat{\Delta}, \qquad \hat{\mathbf{A}}_1 \equiv \cosh\beta \sum_{i=1}^{2} \cos\theta_i d\phi_i, \qquad d\hat{\Delta} = \text{sources}, \tag{3.81}$$

with $\hat{\mathbf{A}}_1$ the type IIA gauge field for this configuration (A, 10). We will soon see that it is opportune to define $\hat{\mathbf{A}}_1$ as we just did. Before we proceed, let us make one last observation: the subsequent M-Theory uplift will only capture the dynamics of this type IIA theory when $e^{\tilde{\phi}_{(A,10)}}$ is of order one, or bigger.

The M-Theory metric corresponding to (3.77) is

$$ds^2_{(M,5)} = (e^{\tilde{\phi}_{(A,9)}})^{-2/3} ds^2_{(A,10)} + (h\mathscr{F}_2^2 e^{4\tilde{\phi}})^{-1/3} (dx_{11} + \hat{\mathbf{A}}_1)^2. \tag{3.82}$$

We note that, due to (3.4) and (3.81), for a fixed value of the ϕ_1 coordinate, namely $\phi_1 = \phi_1^*$, the metric along the directions $(r, \theta_2, \phi_2, x_{11})$ describes a warped Taub-NUT space. Introducing the quantities

$$\hat{G}_1 \equiv \mathscr{F}_1 (h^2 \mathscr{F}_2 e^{2\tilde{\phi}})^{1/3}, \qquad \hat{G}_2 \equiv \frac{\mathscr{F}_4}{\mathscr{F}_1} \hat{G}_1, \qquad \hat{G}_3 \equiv \sin^2\theta_2 \hat{G}_2, \qquad \hat{G}_4 \equiv (h\mathscr{F}_2^2 e^{4\tilde{\phi}})^{-1/3}, \tag{3.83}$$

which are only functions of the coordinates (r, θ_2) and the boost parameter β, we can write the metric for the Taub-NUT space as

$$ds^2_{TN_3} = \hat{G}_1 dr^2 + \hat{G}_2 d\theta_2^2 + \hat{G}_3 d\phi_2^2 + \hat{G}_4 (dx_{11} + \hat{\mathbf{A}}_1^*)^2, \tag{3.84}$$

where we have defined

$$\hat{\mathbf{A}}_1^* \equiv \hat{\mathbf{A}}_1\Big|_{\phi_1=\phi_1^*} = \cosh\beta\cos\theta_2 d\phi_2. \tag{3.85}$$

To the metric (3.84), we associate the following vielbeins:

$$e_r^{(M,5)} = \sqrt{\hat{G}_1}dr, \quad e_{\theta_2}^{(M,5)} = \sqrt{\hat{G}_2}d\theta_2, \quad e_{\phi_2}^{(M,5)} = \sqrt{\hat{G}_3}d\phi_2, \quad e_{11}^{(M,5)} = \sqrt{\hat{G}_4}(dx_{11} + \hat{\mathbf{A}}_1^*). \tag{3.86}$$

As was the case in section 2.1.1, this is a *multi-centered* warped Taub-NUT space. Recall that we had N D6-branes in the configuration (A, 10) prior to the uplift. Hence, $\hat{G}_4^{-1} = 0$ happens N times, leading to coordinate singularities that denote the location of the D6-branes in the dual type IIA picture. Further, the D6-branes in (A, 10) were coincident and consequently we are, by construction, at the non-abelian enhanced scenario discussed in 2.1.1: the symmetry group of the associated world-volume gauge theory is $SU(N)$. It follows then that the G-flux for this M-Theory configuration is of the same form as that in (2.76):

$$\mathcal{G}_4^{(M,5)} = \langle \mathcal{G}_4^{(M,5)} \rangle + \sum_{k=1}^{N-1} \hat{\mathcal{F}}_k \wedge \hat{\omega}_k, \tag{3.87}$$

where $\hat{\mathcal{F}}_k$'s are the Cartan algebra values of the seven-dimensional world-volume field strength $\hat{\mathcal{F}}$, the $\hat{\omega}_k$'s are the unique, normalizable, (anti-)self-dual two-forms associated to the minimal area independent two-cycles in the space (3.84) and the background G-flux is given by

$$\langle \mathcal{G}_4^{(M,5)} \rangle = \mathcal{F}_4^{(A,10)} + \mathcal{H}_3^{(A,10)} \wedge dx_{11}. \tag{3.88}$$

Explicitly,

$$\begin{aligned}\langle \mathcal{G}_4^{(M,5)} \rangle =& e^{2\tilde{\phi}} \cosh\beta\cos\theta_{nc}\sqrt{\frac{\mathscr{F}_2}{\mathscr{F}_1}}d\tilde{\psi} \wedge \left(\hat{\hat{k}}_1 d\phi_1 \wedge d\theta_2 \wedge d\phi_2 + \hat{\hat{k}}_2 d\theta_1 \wedge d\phi_1 \wedge d\phi_2\right) \\ &- \coth\beta \frac{d}{dr}\left(\frac{e^{2\tilde{\phi}}}{h}\right) h^2 \sqrt{\frac{\mathscr{F}_2}{\mathscr{F}_1}} \prod_{i=1}^{2} \mathscr{F}_{2+i} \sin\theta_i d\theta_i \wedge d\phi_i \\ &+ \sum_{i=1}^{2} \sin\theta_i d\theta_i \wedge d\phi_i \wedge dx_{11} \wedge \left[\sinh\beta(\mathscr{F}_{2+i,r} - \sqrt{\mathscr{F}_1\mathscr{F}_2})dr - \cos\theta_{nc} d\tilde{\psi}\right].\end{aligned} \tag{3.89}$$

It can be readily seen that the only quantities left to be computed are the $\hat{\omega}_k$'s. We do so in the following. The discussion is analogous to that in section 2.1.1, so we will be brief.

We begin the computation of the $\hat{\omega}_k$'s by constructing the minimal area independent two-cycles of (3.84) to which they are associated. Note that $\hat{G}_4 = \hat{G}_4(r)$. Thus, we can call the N solutions to $\hat{G}_4^{-1} = 0$ as $r_{(i)}$, where $i = 1, 2, \ldots, N$. Consider two such solutions, $r_{(i)}$ and $r_{(j)}$ (where $i \neq j$) and the straight line in the r direction connecting them, $\mathcal{C}_r$. Attaching to each point in $\mathcal{C}_r$ a circle labeled by x_{11}, we obtain the corresponding minimal area two-cycle X_{ij}. We take the set of all $X_{k,k+1}$, for $k = 1, 2, \ldots, N-1$, as the *independent* minimal area two-cycles where the $\hat{\omega}_k$'s are defined and consider the following ansatze for them:

$$\hat{\omega}_k = d\hat{\zeta}_k, \qquad \hat{\zeta}_k = \hat{g}_k(dx_{11} + \hat{\mathbf{A}}_1^*). \tag{3.90}$$

Easy algebra then yields

$$\begin{aligned} \hat{\omega}_k =& \frac{\hat{g}_{k,r}}{\sqrt{\hat{G}_1\hat{G}_4}} e_r^{(M,5)} \wedge e_{11}^{(M,5)} - \frac{\hat{g}_k}{\sqrt{\hat{G}_2\hat{G}_3}} \cosh\beta \sin\theta_2 e_{\theta_2}^{(M,5)} \wedge e_{\phi_2}^{(M,5)}, \\ *\hat{\omega}_k =& \frac{\hat{g}_{k,r}}{\sqrt{\hat{G}_1\hat{G}_4}} e_{\theta_2}^{(M,5)} \wedge e_{\phi_2}^{(M,5)} - \frac{\hat{g}_k}{\sqrt{\hat{G}_2\hat{G}_3}} \cosh\beta \sin\theta_2 e_r^{(M,5)} \wedge e_{11}^{(M,5)}, \end{aligned} \tag{3.91}$$

where, obviously, the Hodge dual is with respect to the metric (3.84) and $\hat{g}_{k,r}$ stands for the derivative of $\hat{g}_k$ with respect to the radial coordinate r. Using (3.83) and demanding (anti-)self-duality of $\hat{\omega}_k$ we obtain the ordinary differential equation

$$\frac{1}{\hat{g}_k}\frac{d\hat{g}_k}{dr} = \mp \cosh\beta \frac{e^{-\tilde{\phi}}}{\mathscr{F}_4}\sqrt{\frac{\mathscr{F}_1}{h\mathscr{F}_2}}, \tag{3.92}$$

which can be effortlessly solved to give

$$\hat{g}_k = \hat{g}_0 \exp\Big(\mp \int_{r_{(k)}}^{r_{(k+1)}} \frac{e^{-\tilde{\phi}}}{\mathscr{F}_4}\sqrt{\frac{\mathscr{F}_1}{h\mathscr{F}_2}} dr\Big), \tag{3.93}$$

with $\hat{g}_0$ some integration constant where we have absorbed the contribution of $\cosh\beta$. The above fully determines the G-flux in (3.87).

We remind the reader that all the discussion so far in this section is subject to the constraint (3.24) so as to ensure $\mathcal{N} = 1$ supersymmetry in the corresponding world-volume gauge theory.

The configuration (M,5) is the second and last model we construct for the study of knots and their invariants, the first one being (M,1). In the remaining of this work, we will only study the configuration (M,1). Indeed, in part II, we will understand in details the four-dimensional gauge theory stemming from (M,1). In doing so, we will argue how and why (M,1) provides a natural framework to realize knots. Part III will then exploit all the acquired knowledge to show how knots are to be incorporate in model (M,1). All investigations of the embedding of knots in (M,5) are deferred to the sequel(s) of [1, 2].

Before proceeding further, it is important to emphasize that, in constructing (M,1) and (M,5), we have already achieved a very major result in this thesis. Note that, as depicted in figure 1, the configuration (M,1) is dual to the D3-NS5 system of [14]. On the other hand, the configuration (M,5) follows from the very same D3-NS5 system and is dual to the resolved conifold in the presence of fluxes considered in [11]. Hence, we have made explicit the modifications that directly connect the seemingly very distinct models in [14] and [11]. In plain English, we have provided a *unifying picture* between the two existing approaches to computing knot invariants in String Theory. Comparing figures 2**B** and 6**B** we see that the Ooguri-Vafa and Witten proposals can be traced to a dual type IIB scenario of a D3 brane between two NS5-branes, where they only differ in the relative orientation between the NS5-branes.

In chapter 3 we have constructed model (M,5). As (M,1) in the previous chapter, (M,5) is an M-theoretical model that provides a suitable framework for the study of knots. Unlike (M,1), the (M,5) model has $\mathcal{N} = 1$ *supersymmetry and is dual to the resolved conifold in the presence of fluxes first suggested in [11]. It is important to note that in constructing (M,1) and (M,5) we have shown a direct connection between the seemingly distinct supergravity proposals of [11] and [14].*

Part II
The world-volume gauge theory of (M,1)

As hinted by the title itself, this second part focuses on the (non-abelian enhanced) M-Theory configuration (M,1) constructed in chapter 2. The fundamental purpose here will be to show that indeed (M,1) provides a suitable framework for the realization of knots. To this aim, we shall derive and investigate the four-dimensional world-volume gauge theory associated to (M,1). This is an $\mathcal{N} = 4$ supersymmetric theory with gauge group $SU(N)$ and we will subject it to half-BPS boundary conditions. Such study is presented in three main steps. In chapter 4, we obtain the action of the aforementioned gauge theory. Chapter 5 is devoted to the associated Hamiltonian and the minimization of its energy, which yields the BPS conditions for the theory. This analysis naturally leads to a three-dimensional boundary subspace X_3, whose action is the main object of interest in chapter 6. As we shall see, the physics in X_3 are governed by a complexified Chern-Simons action. Consequently, X_3 or, more precisely, its Euclideanization constitutes a suitable space where knots can be embedded.

Figure 10 provides a visual sketch of the overall logic and key results in this part. Given the considerable length of the calculations involved, the reader may find it useful to keep an eye in this image while reading through the following three chapters. In this way, the underlying principal flow of ideas shall hopefully not be lost during the presentation of the corresponding computational details.

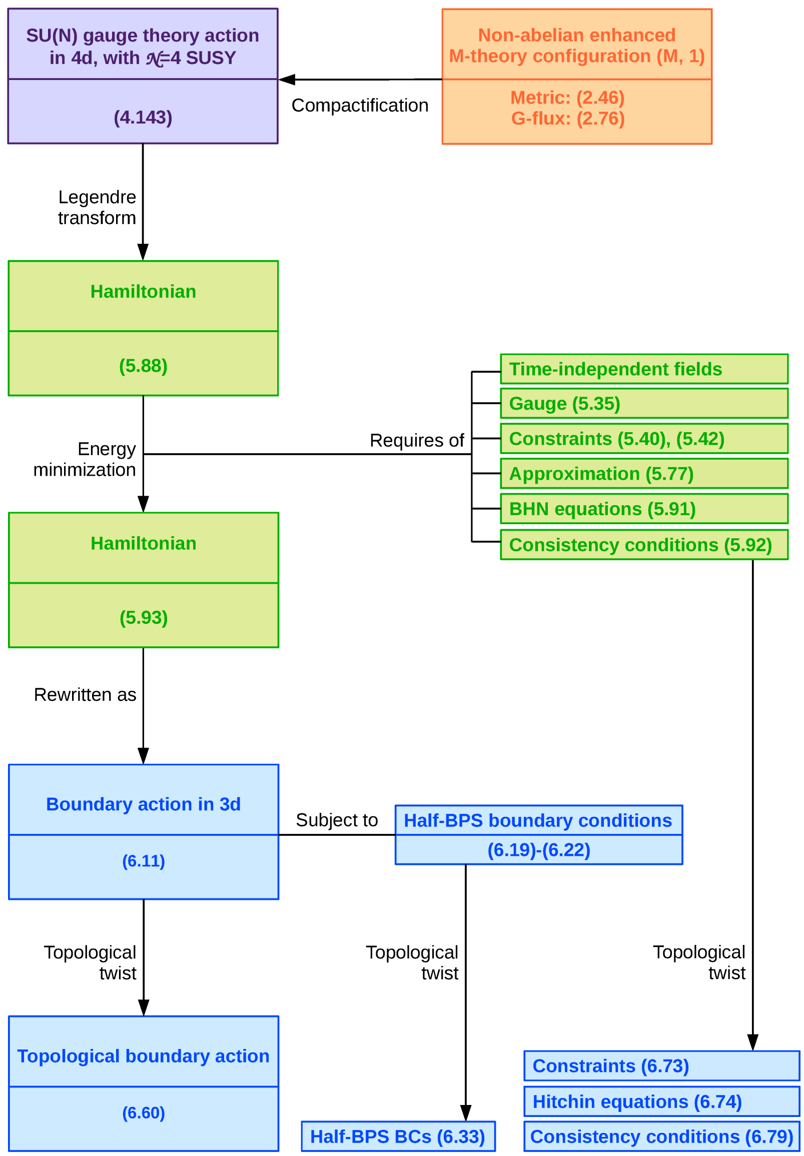

Figure 10: Graphical summary of part II. In orange, the starting point: the non-abelian enhanced M-Theory configuration (M,1) of chapter 2. In purple, the contents of chapter 4: the derivation of the four-dimensional gauge theory stemming from (M,1). Colored green, the derivation and minimization of the corresponding Hamiltonian, presented in chapter 5. Blue is associated to chapter 6, which focuses on the study of the action in the three-dimensional subspace X_3 where knots can be embedded.

Chapter 4: Bosonic action for the world-volume gauge theory

In accordance to the plan above outlined, in this chapter we argue what the bosonic action is for the $SU(N)$ world-volume gauge theory along $(t,\, x_1,\, x_2,\, \tilde{\psi})$ that follows from the non-abelian enhanced model (M,1). This gauge theory has $\mathcal{N} = 4$ supersymmetry by construction and we will impose maximally supersymmetric boundary conditions in due time. We will not be interested in doing so here, but obviously supersymmetry could be used to obtain the fermionic sector of the theory. In principle, one could explicitly write the eleven-dimensional M-Theory action and then work out the desired four-dimensional reduction[11]. However, this is more easily said than done. We will thus follow a different approach here: we will obtain the total action as the sum of three distinct contributions, providing ample motivation for each term.

The first two of these three terms directly stem from our construction of (M,1) in chapter 2 and are indeed initially written in terms of only quantities there defined. Rewriting these terms as functions of the fields in the $\mathcal{N} = 4$ vector multiplet is, however, far from trivial. In achieving this task, we further split the two terms in many parts.

The third and last term is, unluckily, hard to present in such a manner. Consequently, we start by directly writing it in terms of the aforementioned vector multiplet. Nonetheless, the length and complexity of the term lead us to further divide it into smaller pieces too.

The present chapter is heavy on computation. To help the reader make sense of the very many terms in the calculations that follow, we include figure 11. This figure provides a graphical summary of the entire chapter 4, pointing out all the different contributions to the total action and their origin.

A last important remark before jumping into computation. To avoid as much as possible dragging long prefactors, we set the Planck length to one right from the onset: $l_p \equiv 1$.

[11] For our case, compactification can be done via the G-flux (2.76) and metric (2.46), reduced over the normalizable internal harmonic forms. These two-forms are associated to the Taub-NUT subspace and were determined in (2.71).

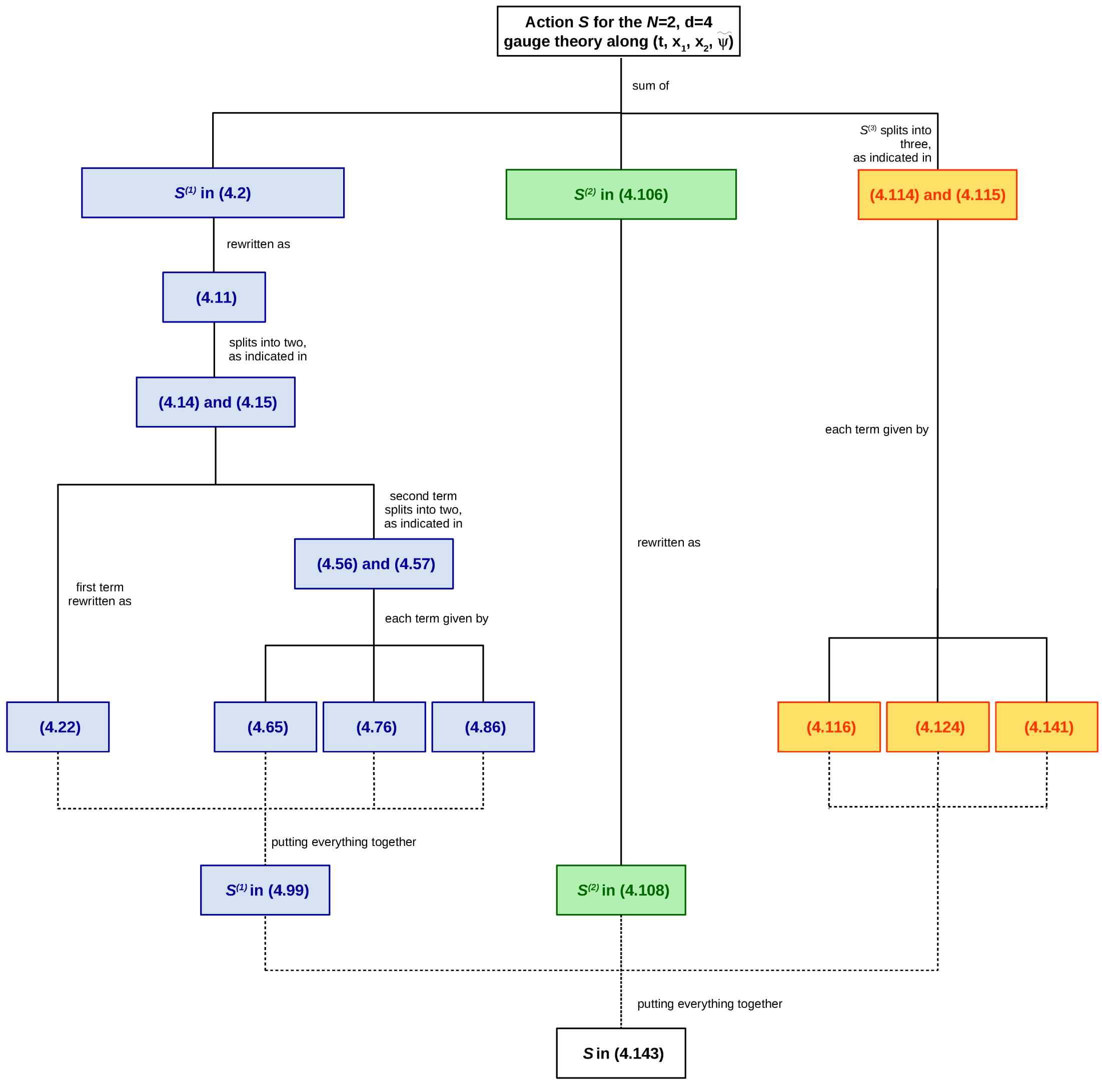

Figure 11: Graphical summary of chapter 4, where we obtain the bosonic action for the four-dimensional $SU(N)$ gauge theory following from the non-abelian M-Theory configuration (M,1) of part I. This figure sketches the connection between the very many terms whose addition gives the aforementioned action. The colors correspond to the subsections where the cited equations can be found: in blue results derived in section 4.1, in green those explained in section 4.2 and in yellow the terms worked out in section 4.3.

4.1 Kinetic term of the G-flux

The first contribution to the just described bosonic action we will consider is the kinetic term of the G-flux (2.76). Our approach will be to work out in details this term for the abelian configuration (M,1) of section 2.1 and then generalize the result to the non-abelian scenario of section 2.1.1. With this aim in mind, let us first rephrase the main features of both the abelian and non-abelian configurations (M,1) in a language suitable to such goals.

The geometry of the configuration (M,1) was given in (2.46), be it for the abelian or non-abelian case. By simple inspection, it can be readily seen that the eleven-dimensional manifold X_{11} on which this metric is defined naturally decomposes into three subspaces:

$$X_{11} = X_4 \otimes \Sigma_3 \otimes TN, \qquad X_4 = X_3 \otimes \mathbb{R}^+. \tag{4.1}$$

Here, X_4 is the four-dimensional subspace where we will define our gauge theory. This further decomposes into X_3 (the Minkowski-type three-dimensional subspace along $(t,\, x_1,\, x_2)$) and $\mathbb{R}^+$ (the half real line labeled by $\tilde{\psi}$). This second decomposition clearly denotes that there is no Lorentz invariance along $\tilde{\psi}$. On the other hand, Σ_3 is the three-cycle parametrized by $(\tilde{x}_3,\, \phi_1,\, r)$ and TN stands for the warped Taub NUT space spanning $(\theta_1,\, x_8,\, x_9,\, x_{11})$. For the abelian (M,1), this is a single-centered Taub NUT, whereas for the non-abelian (M,1) it is an N-times-centered one.

After the non-abelian enhancement, there are N coincident M2-branes oriented along $(x_8,\, x_9,\, x_{11})$ in the configuration (M,1), as depicted in figure 4**B**. Following the notation of section 2.1.1, we denote as $\vec{l}_1$ the location of these M2-branes in the $(x_8,\, x_9)$ plane. It is around this point $\vec{l}_1$ that we shall determine the action of the non-abelian world-volume gauge theory.

Coming to the fluxes, the G-flux for the non-abelian enhanced (M,1) was given in (2.76). This G-flux consists of two pieces: the *delocalized* background flux $\langle \mathcal{G}_4^{(M,1)} \rangle$ and the *localized* contribution of $\left(\sum_{k=1}^{N-1} \mathcal{F}_k \wedge \omega_k \right)$, sharply peaked around $\vec{l}_1$. As it is common practice in the literature, we will assume the delocalized piece is such that its contribution around $\vec{l}_1$ is negligible and restrict our attention to the localized piece only.

In the abelian case, the situation is essentially the same. The only difference being that the G-flux is now given by (2.51). The Taub-NUT space has a unique singularity, whose location we can denote as $\vec{l}_1$ as well. The G-flux again splits into delocalized and localized parts. We assume the delocalized part's contribution is inconsequential around $\vec{l}_1$.

We will now use all the above remarks to obtain the first term for the $U(1)$ world-volume gauge theory action:

$$S^{(1)} \equiv \int_{X_{11}} \left(\mathcal{G}_4^{(M,1)} \wedge * \mathcal{G}_4^{(M,1)} \right), \tag{4.2}$$

where the Hodge dual is with respect to the eleven-dimensional metric (2.46). Using (2.51) and because we are interested in the gauge theory around $\vec{l}_1$, where $\langle \mathcal{G}_4^{(M,1)} \rangle$ is negligible, the above reduces to

$$S^{(1)} = \int_{X_{11}} (\mathcal{F} \wedge \omega) \wedge * (\mathcal{F} \wedge \omega), \tag{4.3}$$

with $\mathcal{F}$ the seven-dimensional abelian field strength. By definition, ω is (anti-)self-dual and is restricted to the subspace TN. For concreteness, we take it to be self-dual in the ongoing. On the other hand, $\mathcal{F}$ spans $X_4 \otimes \Sigma_3$. Then, we can rewrite $S^{(1)}$ as

$$S^{(1)} = \int_{TN} \omega \wedge \omega \int_{X_4 \otimes \Sigma_3} \mathcal{F} \wedge * \mathcal{F}, \tag{4.4}$$

where the Hodge duals are taken with respect to the subspaces of (2.46) indicated by the corresponding integrals. This drastic simplification where the Taub-NUT completely decouples is not as trivial as we just made it sound. Hence, before proceeding further, let us carefully show how this can be made to happen consistently.

Naively, the decoupling happens if the following two conditions are satisfied:

- The integral over TN above only depends on the $(\theta_1, x_8, x_9, x_{11})$ coordinates.

- The integral over $X_4 \otimes \Sigma_3$ is independent of these very same coordinates.

The first condition can easily be seen to hold true. The two-form ω was defined in (2.52), with the gauge field $\mathbf{A}_1$ given by (2.37). It is clear from these expressions that the integrand $\omega \wedge \omega$ only depends on the Taub-NUT coordinates, as desired. The metric for the space TN was given in (2.48) and, as pointed out there, only depends on $(\theta_1, x_8, x_9, x_{11})$. This implies the measure for the integral over TN will have the desired coordinate dependence as well. The second condition, however, does not hold true a priori. An inspection of the metric (2.46) along the directions of X_4 and Σ_3 leads us to conclude that the measure of the second integral in (4.4) will depend on (θ_1, x_8, x_9) −recall our choices for the warp factors in (2.2) and for the dilaton in (2.59) to understand this last statement. Nevertheless, this desired decoupling can be *effectively* made to happen. Let us see how.

A careful inspection of (2.46) restricted to $X_4 \otimes \Sigma_3$ shows that the dependence of the second integral in (4.4) on (x_8, x_9) comes solely from the dilaton (2.59). We can therefore remove this dependence by assuming that the dilaton is given, to leading order, by its constant piece:

$$e^{2\phi} \approx e^{2\phi_0}. \tag{4.5}$$

Note that the above assumption is in excellent agreement to the strong coupling limit discussed around (2.45), required for our M-Theory configuration to be valid, as long as we consider $e^{2\phi_0}$ to be of order one. On the other hand, the θ_1 dependence of the second integral in (4.4) is not "removable". Let us thus turn to the θ_1 dependence of the first integral in (4.4).

We will refer to the first integral in (4.4) as

$$\frac{c_1}{v_3} \equiv \int_{TN} \omega \wedge \omega. \tag{4.6}$$

Using (2.37), (2.50), the first equation in (2.57) and (2.64) in (2.53), it is a matter of easy algebra to obtain the two-form ω as

$$\omega = \sum_{i=8}^{9} \frac{\partial g}{\partial x_i} dx_i \wedge (dx_{11} + \mathbf{A}_{1\theta_1} d\theta_1) + \big(\frac{\partial g}{\partial x_8}\mathbf{A}_{19} - \frac{\partial g}{\partial x_9}\mathbf{A}_{18}\big) dx_8 \wedge dx_9 \\ + g(\alpha_2 dx_8 + \alpha_3 dx_9) \wedge d\theta_1. \tag{4.7}$$

Then, (g, α_2, α_3) being all functions of only (x_8, x_9), it follows that (4.6) is actually independent of θ_1:

$$\omega \wedge \omega = 2g\big(\alpha_3 \frac{\partial g}{\partial x_8} - \alpha_2 \frac{\partial g}{\partial x_9}\big) d\theta_1 \wedge dx_8 \wedge dx_9 \wedge dx_{11}. \tag{4.8}$$

Consequently, choosing (4.5) and *transferring* the θ_1 integral to the second integral in (4.4) as an average, we can consistently decouple the contribution of the Taub-NUT space to $S^{(1)}$:

$$S^{(1)} = \frac{c_1}{v_3} \int_0^{\pi} \frac{d\theta_1}{2\pi} \int_{X_4 \otimes \Sigma_3} \mathcal{F} \wedge *\mathcal{F}, \tag{4.9}$$

where this prefactor should be understood, in this abelian case, as

$$\frac{c_1}{v_3} = \int_0^{R_8} dx_8 \int_0^{R_9} dx_9 \int_0^{R_{11}} dx_{11}\ 2g\left(\alpha_3 \frac{\partial g}{\partial x_8} - \alpha_2 \frac{\partial g}{\partial x_9}\right), \tag{4.10}$$

with R_i denoting the radius of the x_i direction (for i = 8, 9, 11). Note that (x_8, x_9) are non-compact directions, while x_{11} is compact.

At this point, it is easy to infer what the generalization of (4.9) is to the non-abelian case:

$$S^{(1)} = \frac{C_1}{V_3} I^{(1)}, \qquad I^{(1)} \equiv \int_0^{\pi} \frac{d\theta_1}{2\pi} \int_{X_4 \otimes \Sigma_3} \mathrm{Tr}(\mathcal{F} \wedge *\mathcal{F}), \tag{4.11}$$

where $\mathcal{F}$ is now the *non-abelian* seven-dimensional field strength and the trace is taken in the adjoint representation of $SU(N)$. There are just two subtleties in going from (4.9) to (4.11) that we better discuss.

The first one is regarding the prefactor (C_1/V_3). This prefactor is, of course, no longer given by (4.10). Instead, it depends on the two-forms $\{\omega_k\}$ in (2.71). Its explicit form is rather tedious to work out and we will not attempt to compute it here. For our purposes, it suffices to note that, by construction (see the details in section 2.1.1), we are guaranteed its independence on the θ_1 coordinate. So we can transfer the θ_1 integral to the subspace orthogonal to TN as an average and indeed obtain (4.11).

The second subtlety is regarding the appearance of the trace. (Note that the non-abelian G-flux in (2.76) only involves the Cartan algebra values of $\mathcal{F}$.) Let us try to shed some light to this point by first recalling how the non-abelian enhancement was achieved in section 2.1.1 −perhaps it suffices to take a second look at figure 4**B**. There, we wrapped M2-branes around the minimal area, independent two-cycles of the N-centered Taub-NUT space (2.48). The two-cycles were then shrunk to zero size, making the M2-branes tensionless. From this point of view, internal fluctuations of the Taub-NUT space are supposed to provide the Cartan values of the field strength. Fluctuations of the M2-branes along the Taub-NUT directions would then contribute the remaining roots and weights, thus leading to the full trace in (4.11). A more detailed version of this argument may be found in [35–37] and references therein. However, no rigorous proof of this conjecture exists. The argument between (3.91) and (3.98) in [1] in terms of a sigma model may well be the most solid evidence for this claim.

The fact that the trace should be in the adjoint representation has a simple enough heuristic explanation. Additionally, this very argument settles what the bosonic matter content is in our non-abelian world-volume gauge theory. Recall figure 2**B**. There, to the usual type IIB D3-NS5 system we added a second, parallel NS5-brane. The distance between the two NS5-branes being large enough then allows for effectively retaining $\mathcal{N} = 2$ supersymmetry near the original NS5-brane. By the same logic, deep in the bulk of the D3-brane, far away from both the NS5-branes, we expect $\mathcal{N} = 4$ supersymmetry effectively. As is well-known, any $\mathcal{N} = 4$ supersymmetric gauge theory has a vector multiplet consisting of four gauge fields and six real scalars, all of them in the adjoint representation. Certainly, this is the matter content we expect in the bosonic sector for our D3-brane gauge theory too, far from the NS5-branes. On the other hand, the bosonic matter content of any $\mathcal{N} = 2$ supersymmetric gauge theory is arranged in a vector multiplet of four gauge fields and two real scalars in the adjoint representation and a chiral multiplet containing four real scalars in any representation. Needless to say, this is the matter content we expect in the bosonic sector of our gauge theory nearby the NS5-branes. It then stands to reason that, if we are to reconcile these two limits in our setup, we require the four scalars of the $\mathcal{N} = 2$ chiral multiplet to be in the adjoint representation. Therefore, the bosonic matter content of our $SU(N)$ gauge theory is settled to that of the $\mathcal{N} = 4$ vector multiplet: four gauge fields and six real scalars, all of them in the adjoint representation.

Subtleties clarified, we take (4.11) as our starting point and devote the remaining of this section to writing $I^{(1)}$ in terms of the just discussed $\mathcal{N} = 4$ vector multiplet, which spans the spacetime directions $(t, x_1, x_2, \tilde{\psi})$. To begin with, we assume that the seven-dimensional non-abelian field strength $\mathcal{F}$ only depends on these coordinates:

$$\mathcal{F} = \mathcal{F}(t, x_1, x_2, \tilde{\psi}). \tag{4.12}$$

Secondly, and owing to the decomposition (4.1), we make a distinction between the components along X_4 and Σ_3:

$$\mathcal{F} = \mathcal{F}^{(X_4)} + \mathcal{F}^{(\Sigma_3)}. \tag{4.13}$$

Using such distinction in (4.11), we naturally split the first contribution to the non-abelian action into two pieces:

$$S^{(1)} = \frac{C_1}{V_3}\left(I^{(1,1)} + I^{(1,2)}\right), \tag{4.14}$$

with

$$I^{(1,1)} \equiv \int_0^{\pi} \frac{d\theta_1}{2\pi} \int_{X_4 \otimes \Sigma_3} \mathrm{Tr}(\mathcal{F}^{(X_4)} \wedge *\mathcal{F}^{(X_4)}), \quad I^{(1,2)} \equiv \int_0^{\pi} \frac{d\theta_1}{2\pi} \int_{X_4 \otimes \Sigma_3} \mathrm{Tr}(\mathcal{F}^{(\Sigma_3)} \wedge *\mathcal{F}^{(\Sigma_3)}). \tag{4.15}$$

Rather obviously, the Hodge dual in both $I^{(1,1)}$ and $I^{(1,2)}$ is with respect to the seven-dimensional metric of $X_4 \otimes \Sigma_3$.

Note that the crossed terms $(\mathcal{F}^{(X_4)} \wedge *\mathcal{F}^{(\Sigma_3)})$ and $(\mathcal{F}^{(\Sigma_3)} \wedge *\mathcal{F}^{(X_4)})$ are zero and thus have not been included in (4.14). The argument for the vanishing of the first such term is as follows. Each component of $\mathcal{F}^{(\Sigma_3)}$ spans two directions of Σ_3. Consequently, the corresponding term of $*\mathcal{F}^{(\Sigma_3)}$ is oriented along all four directions of Σ_4 and the remaining direction of Σ_3. As the components of $\mathcal{F}^{(X_4)}$ span two directions of Σ_4, the term $(\mathcal{F}^{(X_4)} \wedge *\mathcal{F}^{(\Sigma_3)})$ necessarily contains the wedge product of two same X_4 directions and thus yields zero. The argument for the vanishing of the second crossed term is similar.

At this stage, the only quantities left to be determined to explicitly write $S^{(1)}$ are $I^{(1,1)}$ and $I^{(1,2)}$, defined in (4.15). Their computation is quite long and involved. Consequently, we will do so in separate sections. In the end, we will put together in (4.14) the integrals $I^{(1,1)}$ and $I^{(1,2)}$ that we shall obtain.

4.1.1 Determining $I^{(1,1)}$: the contribution of gauge field strengths

As the title suggests, this section is devoted to the computation of $I^{(1,1)}$ in (4.15) in terms of the field strengths associated to the $\mathcal{N} = 4$ vector multiplet. But before jumping into the details of the calculation, let us introduce some quantities that will soon be useful.

We begin by taking a closer look at the seven-dimensional space $X_4 \otimes \Sigma_3$, where $I^{(1,1)}$ is defined. Its metric can be directly read from (2.46) to be

$$ds^2_{X_4 \otimes \Sigma_3} = H_1[-dt^2 + dx_1^2 + dx_2^2 + H_2 d\tilde{x}_3^2 + H_3(d\phi_1 + f_3 d\tilde{x}_3)^2 + e^{2\phi_0}(F_1 dr^2 + H_4 d\tilde{\psi}^2)], \tag{4.16}$$

where we have made use of our assumption (4.5). For later convenience, we denote as g_7 the determinant of the above metric:

$$g_7 \equiv \det(ds^2_{X_4 \otimes \Sigma_3}) = e^{4\phi_0} F_1 H_1^7 H_2 H_3 H_4 = e^{4\phi_0} F_1 H_1^4 H_4, \tag{4.17}$$

where in the last step we have used the fact that $H_1^3 H_2 H_3 = 1$, which follows from (2.43). It will also come in handy to write the metric along the subspace X_4, albeit in matrix form:

$$g_{ab} = H_1 \mathrm{diag}(-1, 1, 1), \qquad g_{\tilde{\psi}\tilde{\psi}} = e^{2\phi_0} H_1 H_4. \tag{4.18}$$

Here, the subscripts $(a,\, b)$ take values $(0,\, 1,\, 2)$ and stand for the Lorentz-invariant directions $(t,\, x_1,\, x_2)$. Being diagonal, it is straightforward to see that the inverse of the X_4 metric, in matrix form, is given by

$$g^{ab} = \frac{1}{H_1} \mathrm{diag}(-1, 1, 1), \qquad g^{\tilde{\psi}\tilde{\psi}} = \frac{e^{-2\phi_0}}{H_1 H_4}. \tag{4.19}$$

Calling g_4 the (absolute value of the) determinant of the X_4 metric, this is

$$g_4 \equiv \left|\det(ds^2_{X_4})\right| = e^{2\phi_0} H_1^4 H_4. \tag{4.20}$$

Having introduced our notation, we may now proceed to the determination of $I^{(1,1)}$. First of all, we explicitly write the wedge product $\left(\mathcal{F}^{(X_4)} \wedge *\mathcal{F}^{(X_4)}\right)$ as

$$\sqrt{g_7} \sum_{a,b,c,d=0}^{2} g^{ab} \left(g^{cd} \mathcal{F}_{ac} \mathcal{F}_{bd} + g^{\tilde{\psi}\tilde{\psi}} \mathcal{F}_{a\tilde{\psi}} \mathcal{F}_{b\tilde{\psi}} \right) = \sqrt{\frac{F_1}{H_4}} \left(e^{2\phi_0} H_4 \sum_{\substack{a,b=0 \\ a<b}}^{2} \mathcal{F}_{ab}^2 + \sum_{a=0}^{2} \mathcal{F}_{a\tilde{\psi}}^2 \right). \tag{4.21}$$

Using the above in (4.15), we have that

$$I^{(1,1)} = c_{11} \int d^4x \sum_{\substack{a,b=0 \\ a<b}}^{2} \mathrm{Tr}(\mathcal{F}_{ab}^2) + c_{12} \int d^4x \sum_{a=0}^{2} \mathrm{Tr}(\mathcal{F}_{a\tilde{\psi}}^2), \tag{4.22}$$

where the integration is with respect to the spacetime coordinates $(t,\, x_1,\, x_2,\, \tilde{\psi})$ and where we have defined the coefficients $(c_{11},\, c_{12})$ as

$$c_{11} \equiv e^{2\phi_0} \int d^4\tilde{\zeta} \sqrt{F_1 H_4}, \qquad c_{12} \equiv \int d^4\tilde{\zeta} \sqrt{\frac{F_1}{H_4}}. \tag{4.23}$$

As a short-hand notation that will keep appearing, we have introduced

$$\int d^4\tilde{\zeta} \equiv \int_0^{R_3} d\tilde{x}_3 \int_0^{2\pi} d\phi_1 \int_0^{\infty} dr \int_0^{\pi} \frac{d\theta_1}{2\pi} \tag{4.24}$$

too, with R_3 the radius of the non-compact direction $\tilde{x}_3$. Note that these coefficients have been taken out of the integral over the world-volume coordinates in (4.22) because F_1 and H_4 are only functions of the radial coordinate and θ_{nc}. To see this, recall our choices in (2.2) and the definitions in (2.21) and (2.43). For this same reason, we can right away perform the $(\tilde{x}_3,\, \phi_1)$ integrals above. Further using (2.43), we can express c_{11} and c_{12} as

$$c_{11} = 2R_3 e^{2\phi_0} \sec\theta_{nc} \int_0^{\infty} dr\, \sqrt{F_1 \tilde{F}_2 F_3}\, \mathcal{I}^{(1)}, \qquad c_{12} = 2R_3 \cos\theta_{nc} \int_0^{\infty} dr\, \sqrt{\frac{F_1}{\tilde{F}_2 F_3}}\, \mathcal{I}^{(2)}, \tag{4.25}$$

where we have defined

$$\mathcal{I}^{(1)} \equiv \int_0^{\pi/2} \frac{\sin\theta_1 d\theta_1}{\sqrt{\tilde{F}_2 \cos^2\theta_1 + F_3 \sin^2\theta_1}}, \qquad \mathcal{I}^{(2)} \equiv \int_0^{\pi/2} d\theta_1\, \csc\theta_1 \sqrt{\tilde{F}_2 \cos^2\theta_1 + F_3 \sin^2\theta_1}. \tag{4.26}$$

Since they will keep showing up, it is useful to introduce the functions

$$\chi(\theta_1) \equiv \sqrt{\tilde{F}_2 + F_3 + (\tilde{F}_2 - F_3)\cos 2\theta_1}, \qquad \tilde{\chi}(\theta_1) \equiv \sqrt{2(\tilde{F}_2 - F_3)}\cos\theta_1. \tag{4.27}$$

Using these, the first of these integrals can be readily performed to yield

$$\mathcal{I}^{(1)} = -\frac{1}{\sqrt{\tilde{F}_2 - F_3}} \ln|\chi(\theta_1) + \tilde{\chi}(\theta_1)| \Bigg|_{\theta_1=0}^{\theta_1=\pi/2} = \frac{\mathcal{J}_3}{2\sqrt{\tilde{F}_2 - F_3}}, \tag{4.28}$$

where we have defined

$$\mathcal{J}_3 \equiv \ln \left| \frac{\sqrt{\tilde{F}_2} + \sqrt{\tilde{F}_2 - F_3}}{\sqrt{\tilde{F}_2} - \sqrt{\tilde{F}_2 - F_3}} \right|, \tag{4.29}$$

a quantity which will appear in the present analysis very often. It is clear that the above will be real if and only if we require that $\tilde{F}_2 \geq F_3$, for all values of (r, θ_{nc}). Thus, we will demand this holds true in the ongoing. Using the above in (4.25), we obtain

$$c_{11} = R_3 e^{2\phi_0} \sec\theta_{nc} \int_0^\infty dr\ \mathcal{J}_3 \sqrt{\frac{F_1 \tilde{F}_2 F_3}{\tilde{F}_2 - F_3}}. \tag{4.30}$$

It is important to note that the above coefficient is just a number. The numerical value of c_{11} depends only on the choice of warp factors one would like to consider in (2.2). This choice is subject to the constraint $\tilde{F}_2 \geq F_3$ and should be checked to preserve the desired $\mathcal{N} = 4$ supersymmetry in the world-volume (later on reduced to $\mathcal{N} = 2$ supersymmetry via half-BPS boundary conditions).

Coming now to $\mathcal{I}^{(2)}$, we start by defining the soon to be useful three quantities

$$b_1 \equiv \sqrt{\frac{F_3}{\tilde{F}_2 - F_3}}, \quad b_2 \equiv \frac{1}{2}\sqrt{\frac{F_1(\tilde{F}_2 - F_3)}{\tilde{F}_2 F_3}}, \quad b_3 \equiv \frac{2}{b}\sqrt{\frac{F_3 + b^2(\tilde{F}_2 - F_3)}{\tilde{F}_2 - F_3}}, \quad b \in \mathbb{R}^+ - \{1\}. \tag{4.31}$$

We can use b_1 to rewrite the integral of our interest in the more convenient form

$$\mathcal{I}^{(2)} = \sqrt{\tilde{F}_2 - F_3} \int_0^{\pi/2} d\theta_1 \sqrt{\frac{b_1^2 + \cos^2\theta_1}{1 - \cos^2\theta_1}}. \tag{4.32}$$

Under the change of variables

$$\cos\theta_1 = z, \qquad d\theta_1 = -\frac{dz}{\sqrt{1 - z^2}}, \tag{4.33}$$

the above can be further rewritten as

$$\mathcal{I}^{(2)} = \frac{\sqrt{\tilde{F}_2 - F_3}}{2} \int_{-1}^{1} dz\, \frac{\sqrt{b_1^2 + z^2}}{b^2 - z^2}, \tag{4.34}$$

where b as defined in (4.31) is a regularization factor that we have introduced by hand in order to avoid the singularities of $\mathcal{I}^{(2)}$ at $z = \pm 1$. In the same spirit of $(\chi(\theta_1), \tilde{\chi}(\theta_1))$ before, let us introduce two more functions that will come in handy repeatedly:

$$\eta(z) \equiv \operatorname{arctanh}\Big(\frac{z}{b}\sqrt{\frac{b_1^2+b^2}{b_1^2+z^2}}\Big), \qquad \tilde{\eta}(z) \equiv \ln\left|z+\sqrt{b_1^2+z^2}\right|. \tag{4.35}$$

Finally, all the above can be used to integrate over z in (4.34) and obtain

$$\frac{2\mathcal{I}^{(2)}}{\sqrt{\tilde{F}_2-F_3}} = \frac{\sqrt{b_1^2+b^2}}{b}\eta(z)-\tilde{\eta}(z)\Bigg|_{z=-1}^{z=1} = b_3\mathcal{J}_4+\mathcal{J}_3^{-1}, \tag{4.36}$$

where we have defined the many times to occur quantity $\mathcal{J}_4$ as

$$\mathcal{J}_4 \equiv \operatorname{arctanh}\Big(\frac{1}{b}\sqrt{\frac{F_3+b^2(\tilde{F}_2-F_3)}{\tilde{F}_2}}\Big). \tag{4.37}$$

Plugging our result in (4.23), the coefficient c_{12} may be expressed as

$$c_{12} = 2R_3\cos\theta_{nc}\int_0^\infty dr\, b_2\left(b_3\mathcal{J}_4+\mathcal{J}_3^{-1}\right). \tag{4.38}$$

As was the case for c_{11} before, we want c_{12} to be a well-defined number for all choices of warp factors in (2.2) satisfying the constraint $\tilde{F}_2 \geq F_3$. It is not clear from our above result that this should be the case in the following two limits:

- $F_3 \to 0$. This limit also includes the case $(\tilde{F}_2, F_3) \to 0$ since, in order to be consistent with the constraint $\tilde{F}_2 \geq F_3$, we must demand that F_3 approaches zero *faster* than $\tilde{F}_2$. Hence, the case $(\tilde{F}_2, F_3) \to 0$ should be studied by first demanding $F_3 \to 0$ and afterwards considering the $\tilde{F}_2 \to 0$ limit of the resulting expression.

- $\tilde{F}_2 \to F_3 \nrightarrow 0$.

Let us thus study such subtle scenarios in details and show that c_{12} in (4.38) is indeed a finite number even then.

To consider the first case, namely $F_3 \to 0$, we start by rewriting the argument of the inverse hyperbolic tangent in (4.37) as

$$\frac{1}{b}\sqrt{\frac{F_3+b^2(\tilde{F}_2-F_3)}{\tilde{F}_2}} = \sqrt{1+\left(\frac{1-b^2}{b^2}\right)\frac{F_3}{\tilde{F}_2}}. \tag{4.39}$$

Next, we note that in the logarithmic term of (4.38), namely $\mathcal{J}_3$ in (4.29), only the numerator diverges as $F_3 \to 0$, while the denominator is well-defined in this limit. Hence, retaining only the divergent terms in the integrand of (4.38) and using (4.39), we focus on the study of

$$\lim_{F_3\to 0} c_{12} \sim \lim_{F_3\to 0}\left[b_2 b_3 \operatorname{arctanh}\left(\sqrt{1+\left(\frac{1-b^2}{b^2}\right)\frac{F_3}{\tilde{F}_2}}\right)+b_2\ln\left|\sqrt{\tilde{F}_2}-\sqrt{\tilde{F}_2-F_3}\right|\right]. \tag{4.40}$$

From our definitions in (4.31) it follows that

$$\lim_{F_3\to 0} b_2 = \lim_{F_3\to 0}\sqrt{\frac{F_1}{F_3}} = \lim_{F_3\to 0} b_2 b_3 \tag{4.41}$$

which, used in (4.40), gives

$$\lim_{F_3\to 0} c_{12} \sim \lim_{F_3\to 0}\sqrt{\frac{F_1}{F_3}}\left[\operatorname{arctanh}\left(\sqrt{1+\left(\frac{1-b^2}{b^2}\right)\frac{F_3}{\tilde{F}_2}}\right)+\ln\left|\sqrt{\tilde{F}_2}-\sqrt{\tilde{F}_2-F_3}\right|\right]. \tag{4.42}$$

Applying L'Hôpital's rule to the two terms above, it is easy to see that

$$\lim_{F_3\to 0}\sqrt{\frac{F_1}{F_3}}\operatorname{arctanh}\left(\sqrt{1+\left(\frac{1-b^2}{b^2}\right)\frac{F_3}{\tilde{F}_2}}\right) = -\lim_{F_3\to 0}\sqrt{\frac{F_1}{F_3}}\ln\left|\sqrt{\tilde{F}_2}-\sqrt{\tilde{F}_2-F_3}\right|. \tag{4.43}$$

That is, the divergent contribution to $\left(\lim_{F_3\to 0} c_{12}\right)$ is zero. This implies that c_{12} takes some finite numerical value when $F_3 \to 0$.

If we now turn our attention to the $(\tilde{F}_2, F_3) \to 0$ case, the above still holds true. However, the denominator the of logarithmic term of (4.38) is no longer well-defined and consequently, we must study it. As already argued, we first should consider the $F_3 \to 0$ limit of this term and then impose $\tilde{F}_2 \to 0$ there. Using (4.41) and applying L'Hôpital's rule, this additional divergent term can also be seen to vanish:

$$\lim_{\tilde{F}_2, F_3\to 0} c_{12} \sim \lim_{\tilde{F}_2, F_3\to 0}\frac{1}{\sqrt{F_3}}\ln\left|2\sqrt{\tilde{F}_2}\right| = \lim_{\tilde{F}_2, F_3\to 0} -\frac{F_3^{3/2}}{\tilde{F}_2} = 0. \tag{4.44}$$

Thus, $c_{12} = 0$ when $(\tilde{F}_2, F_3) \to 0$.

Finally, we study the limit $\tilde{F}_2 \to F_3 \nrightarrow 0$. From (4.31), it is not hard to work out the following two auxiliary limits:

$$\lim_{\tilde{F}_2\to F_3} b_2 = 0, \qquad \lim_{\tilde{F}_2\to F_3} b_2 b_3 = \frac{1}{b}\sqrt{\frac{F_1}{F_3}}. \tag{4.45}$$

Inserting the above in (4.38), we obtain

$$\lim_{\tilde{F}_2\to F_3} c_{12} = 2R_3\cos\theta_{nc}\int_0^\infty dr\frac{1}{b}\sqrt{\frac{F_1}{F_3}}\,\text{arctanh}\left(\frac{1}{b}\right) \sim \text{arctanh}\left(\frac{1}{b}\right), \tag{4.46}$$

which can be very large, yet is finite because the regularization factor satisfies $b \neq 1$ by definition. This proves that c_{12} is also just some number as $\tilde{F}_2 \to F_3$.

Summing up, $I^{(1,1)}$ is given by (4.22), with $(c_{11},\, c_{12})$ as in (4.30) and (4.38), respectively. Both of the coefficients are well-defined numbers for any choice of the warp factors one may want to consider, as long as the constraint $\tilde{F}_2 \geq F_3$ is respected.

4.1.2 Determining $I^{(1,2)}$: the contribution of three scalar fields

In this section we compute $I^{(1,2)}$ in (4.15) in terms of the $\mathcal{N} = 4$ vector multiplet's matter content. As in the previous section 4.1.1, it is convenient to first introduce certain quantities, which will be necessary in the subsequent calculation.

Let us begin by looking at the three-cycle Σ_3, parametrized by $(\tilde{x}_3,\, \phi_1,\, r)$. Its metric can be easily inferred from (4.16) to be

$$ds^2_{\Sigma_3} = H_1H_2d\tilde{x}_3^2 + H_1H_3(d\phi_1 + f_3d\tilde{x}_3)^2 + e^{2\phi_0}H_1F_1dr^2. \tag{4.47}$$

We take the vielbeins associated to the above metric as

$$e^{(\Sigma_3)}_{\tilde{3}} = \sqrt{H_1H_2}d\tilde{x}_3, \qquad e^{(\Sigma_3)}_{r} = e^{\phi_0}\sqrt{H_1F_1}dr, \qquad e^{(\Sigma_3)}_{\phi_1} = \sqrt{H_1H_3}(d\phi_1 + f_3dx_3). \tag{4.48}$$

It is not hard to see that these vielbeins satisfy

$$*e^{(\Sigma_3)}_{\tilde{3}} = e^{(\Sigma_3)}_{r}\wedge e^{(\Sigma_3)}_{\phi_1}, \qquad *e^{(\Sigma_3)}_{r} = e^{(\Sigma_3)}_{\phi_1}\wedge e^{(\Sigma_3)}_{\tilde{3}}, \qquad *e^{(\Sigma_3)}_{\phi_1} = e^{(\Sigma_3)}_{\tilde{3}}\wedge e^{(\Sigma_3)}_{r}, \tag{4.49}$$

where the Hodge duals are with respect to the metric (4.47).

Let us now focus on $\mathcal{F}^{(\Sigma_3)}$ in (4.15). This field strength is related to the corresponding three-dimensional non-abelian gauge field $\mathcal{A}^{(\Sigma_3)}$ in the usual manner

$$\mathcal{F}^{(\Sigma_3)} = \mathcal{D}\mathcal{A}^{(\Sigma_3)} + \mathcal{A}^{(\Sigma_3)}\wedge\mathcal{A}^{(\Sigma_3)}, \tag{4.50}$$

where the covariant derivative is defined as

$$\mathcal{D}_a \equiv \partial_a + i[\mathcal{A}_a,\quad], \qquad \mathcal{D}_{\tilde{\psi}} \equiv \partial_{\tilde{\psi}} + i[\mathcal{A}_{\tilde{\psi}},\quad], \tag{4.51}$$

with $a = (0,\, 1,\, 2)$ labeling the Lorentz-invariant directions $(t,\, x_1,\, x_2)$ and $(\mathcal{A}_a,\, \mathcal{A}_{\tilde{\psi}})$ standing for the world-volume gauge fields associated to the field strengths in (4.22). We define $\mathcal{A}^{(\Sigma_3)}$

as

$$\mathcal{A}^{(\Sigma_3)} \equiv \mathcal{A}_{\tilde{3}} d\tilde{x}_3 + \mathcal{A}_{\phi_1} d\phi_1 + \mathcal{A}_r dr = \hat{\alpha}_1 e_{\tilde{3}}^{(\Sigma_3)} + \hat{\alpha}_2 e_r^{(\Sigma_3)} + \hat{\alpha}_3 e_{\phi_1}^{(\Sigma_3)}. \tag{4.52}$$

In the last step above we have used (4.48) and the one-forms

$$\hat{\alpha}_1 \equiv \frac{\mathcal{A}_{\tilde{3}} - f_3 \mathcal{A}_{\phi_1}}{\sqrt{H_1 H_2}}, \qquad \hat{\alpha}_2 \equiv \frac{e^{-\phi_0} \mathcal{A}_r}{\sqrt{H_1 F_1}}, \qquad \hat{\alpha}_3 \equiv \frac{\mathcal{A}_{\phi_1}}{\sqrt{H_1 H_3}}. \tag{4.53}$$

Because of (4.12), $(\mathcal{A}_{\tilde{3}},\, \mathcal{A}_{\phi_1},\, \mathcal{A}_r)$ are functions of only $(t,\, x_1,\, x_2,\, \tilde{\psi})$. Note that this also explains our definitions in (4.51). On the other hand, from (2.2), (2.21) and (2.43), it is clear that the $\{\hat{\alpha}_i\}$'s, with $i = 1,\, 2,\, 3$, additionally depend on $(\theta_1,\, r)$. A vital remark follows: from the point of view of the four-dimensional gauge theory, $(\mathcal{A}_{\tilde{3}},\, \mathcal{A}_{\phi_1},\, \mathcal{A}_r)$ should be understood as three *real scalar fields*, in the adjoint representation of $SU(N)$.

Our above discussion settles the ground to determine $I^{(1,2)}$ in (4.15) in terms of the real scalar fields $(\mathcal{A}_{\tilde{3}},\, \mathcal{A}_{\phi_1},\, \mathcal{A}_r)$. The integrand there is of the form

$$\begin{aligned} \mathcal{F}^{(\Sigma_3)} \wedge *\mathcal{F}^{(\Sigma_3)} =& \mathcal{D}\mathcal{A}^{(\Sigma_3)} \wedge *\big(\mathcal{D}\mathcal{A}^{(\Sigma_3)}\big) + \mathcal{A}^{(\Sigma_3)} \wedge \mathcal{A}^{(\Sigma_3)} \wedge *\big(\mathcal{A}^{(\Sigma_3)} \wedge \mathcal{A}^{(\Sigma_3)}\big) \\ & + \mathcal{A}^{(\Sigma_3)} \wedge \mathcal{A}^{(\Sigma_3)} \wedge *\big(\mathcal{D}\mathcal{A}^{(\Sigma_3)}\big) + \mathcal{D}\mathcal{A}^{(\Sigma_3)} \wedge *\big(\mathcal{A}^{(\Sigma_3)} \wedge \mathcal{A}^{(\Sigma_3)}\big), \end{aligned} \tag{4.54}$$

where all the Hodge duals are with respect to the seven-dimensional metric (4.16) and we have made use of (4.50). Owing to the decomposition (4.1), it is easy to see that the last line above vanishes. The reason is analogous to that given around (4.15) for the vanishing of the there-called "crossed terms". For example, consider the first such term. The two-form $\mathcal{D}\mathcal{A}^{(\Sigma_3)}$ spans one direction in X_4 and another one in Σ_3. Consequently, its Hodge dual five-form is defined along the remaining three directions of X_4 and two directions of Σ_3. But, since $\mathcal{A}^{(\Sigma_3)} \wedge \mathcal{A}^{(\Sigma_3)}$ stretches along two directions of Σ_3, the wedge product of these two last forms will necessarily contain the wedge product of one of the directions of Σ_3 with itself. Anti-symmetry of the wedge product then implies zero value for this first term. A similar argument applies to the second term too. The decomposition (4.1) also allows for a drastic simplification of the two non-vanishing terms in the first line above. Indeed, we can decouple X_4 and Σ_3 completely and write

$$\begin{aligned} \mathcal{F}^{(\Sigma_3)} \wedge *\mathcal{F}^{(\Sigma_3)} =& \sqrt{g_4}\; d^4x \Big[\sum_{a=0}^{2} \mathcal{D}_a \mathcal{A}^{(\Sigma_3)} \wedge *\big(\mathcal{D}_a \mathcal{A}^{(\Sigma_3)}\big) + \mathcal{D}_{\tilde{\psi}} \mathcal{A}^{(\Sigma_3)} \wedge *\big(\mathcal{D}_{\tilde{\psi}} \mathcal{A}^{(\Sigma_3)}\big) \\ & + \mathcal{A}^{(\Sigma_3)} \wedge \mathcal{A}^{(\Sigma_3)} \wedge *\big(\mathcal{A}^{(\Sigma_3)} \wedge \mathcal{A}^{(\Sigma_3)}\big)\Big], \end{aligned} \tag{4.55}$$

where the Hodge dual on the left-hand side is with respect to the seven-dimensional metric (4.16), whereas the Hodge duals on the right-hand side are with respect to the three-dimensional metric (4.47). We remind the reader that g_4 was defined in (4.20) and that ($d^4x \equiv dt\, dx_1\, dx_2\, d\tilde{\psi}$), as in (4.22). Inserting the above in (4.15), we can split the computation of $I^{(1,2)}$ into three pieces:

$$I^{(1,2)} = \int d^4x \,\mathrm{Tr}\left(I^{(1,2,1)} + I^{(1,2,2)} + I^{(1,2,3)}\right), \tag{4.56}$$

where we have defined

$$\begin{aligned} I^{(1,2,1)} &\equiv \int_0^\pi \frac{d\theta_1}{2\pi} \int_{\Sigma_3} \sqrt{g_4}\; \mathcal{A}^{(\Sigma_3)} \wedge \mathcal{A}^{(\Sigma_3)} \wedge *\big(\mathcal{A}^{(\Sigma_3)} \wedge \mathcal{A}^{(\Sigma_3)}\big), \\ I^{(1,2,2)} &\equiv \int_0^\pi \frac{d\theta_1}{2\pi} \int_{\Sigma_3} \sqrt{g_4}\; \sum_{a=0}^{2} \mathcal{D}_a \mathcal{A}^{(\Sigma_3)} \wedge *\big(\mathcal{D}_a \mathcal{A}^{(\Sigma_3)}\big), \\ I^{(1,2,3)} &\equiv \int_0^\pi \frac{d\theta_1}{2\pi} \int_{\Sigma_3} \sqrt{g_4}\; \mathcal{D}_{\tilde{\psi}} \mathcal{A}^{(\Sigma_3)} \wedge *\big(\mathcal{D}_{\tilde{\psi}} \mathcal{A}^{(\Sigma_3)}\big), \end{aligned} \tag{4.57}$$

Clearly, the Hodge duals here are with respect to (4.47). In the following, we determine all these three terms separately.

Computation of $I^{(1,2,1)}$ in (4.57)

To begin with, we focus on $I^{(1,2,1)}$ in (4.57). Using (4.49) and (4.52), it is a matter or quick and easy algebra to obtain

$$\begin{aligned} \mathcal{A}^{(\Sigma_3)} \wedge \mathcal{A}^{(\Sigma_3)} &= [\hat{\alpha}_1, \hat{\alpha}_2] e_{\tilde{3}}^{(\Sigma_3)} \wedge e_r^{(\Sigma_3)} + [\hat{\alpha}_1, \hat{\alpha}_3] e_{\tilde{3}}^{(\Sigma_3)} \wedge e_{\phi_1}^{(\Sigma_3)} + [\hat{\alpha}_2, \hat{\alpha}_3] e_r^{(\Sigma_3)} \wedge e_{\phi_1}^{(\Sigma_3)}, \\ *\big(\mathcal{A}^{(\Sigma_3)} \wedge \mathcal{A}^{(\Sigma_3)}\big) &= [\hat{\alpha}_1, \hat{\alpha}_2] e_{\phi_1}^{(\Sigma_3)} - [\hat{\alpha}_1, \hat{\alpha}_3] e_r^{(\Sigma_3)} + [\hat{\alpha}_2, \hat{\alpha}_3] e_{\tilde{3}}^{(\Sigma_3)}. \end{aligned} \tag{4.58}$$

The wedge product of the above two quantities is then

$$\mathcal{A}^{(\Sigma_3)} \wedge \mathcal{A}^{(\Sigma_3)} \wedge *\big(\mathcal{A}^{(\Sigma_3)} \wedge \mathcal{A}^{(\Sigma_3)}\big) = \big([\hat{\alpha}_1, \hat{\alpha}_2]^2 + [\hat{\alpha}_1, \hat{\alpha}_3]^2 + [\hat{\alpha}_2, \hat{\alpha}_3]^2\big)\, e_{\tilde{3}}^{(\Sigma_3)} \wedge e_r^{(\Sigma_3)} \wedge e_{\phi_1}^{(\Sigma_3)}. \tag{4.59}$$

From the above, as well as our definitions in (4.20), (4.48) and (4.53), it follows without excessive algebraic effort that $I^{(1,2,1)}$ in (4.57) can be rewritten as

$$I^{(1,2,1)} = a_1 [\mathcal{A}_r, \mathcal{A}_{\phi_1} - \frac{a_3}{2a_1} \mathcal{A}_{\tilde{3}}]^2 + \frac{4a_1 a_2 - a_3^2}{4a_1} [\mathcal{A}_{\tilde{3}}, \mathcal{A}_r]^2 + a_4 [\mathcal{A}_{\tilde{3}}, \mathcal{A}_{\phi_1}]^2, \tag{4.60}$$

where we have defined, using (4.24),

$$a_1 \equiv \int d^4\tilde{\zeta}\sqrt{\frac{H_4}{F_1}}\left(\frac{1}{H_3}+\frac{f_3^2}{H_2}\right), \qquad a_2 \equiv \int d^4\tilde{\zeta}\sqrt{\frac{H_4}{F_1}}\frac{1}{H_2},$$
$$a_3 \equiv 2\int d^4\tilde{\zeta}\sqrt{\frac{H_4}{F_1}}\frac{f_3}{H_2}, \qquad a_4 \equiv e^{2\phi_0}\int d^4\tilde{\zeta}\frac{\sqrt{H_4F_1}}{H_2H_3}. \tag{4.61}$$

These coefficients can be easily written in terms of the warp factors using (2.43). Further, remember our warp factor choices in (2.2), the definition of $\tilde{F}_2$ in (2.21) and our assumption of constant dilaton in (4.5). Then, it is clear that all the above coefficients only depend on the $(r,\,\theta_1)$ coordinates and so the $(\tilde{x}_3,\,\phi_1)$ integrals in (4.24) are trivial and can be carried out right away. Altogether, we have that

$$a_1 = R_3\sec\theta_{nc}\int_0^\infty dr\sqrt{\frac{\tilde{F}_2F_3}{F_1}}\left(\mathcal{I}^{(3)}+\tilde{F}_2^2\tan^2\theta_{nc}(1+F_2\tan^2\theta_{nc})\mathcal{I}^{(4)}\right),$$
$$a_2 = 2R_3\sec\theta_{nc}\int_0^\infty dr\sqrt{\frac{\tilde{F}_2F_3}{F_1}}\mathcal{I}^{(1)}, \qquad a_3 \propto \mathcal{I}^{(5)}, \tag{4.62}$$
$$a_4 = e^{2\phi_0}R_3\sec\theta_{nc}\int_0^\infty dr\sqrt{F_1\tilde{F}_2F_3}(\cos^2\theta_{nc}+F_2\sin^2\theta_{nc})\mathcal{I}^{(3)},$$

where $\mathcal{I}^{(1)}$ was defined in (4.26) and where we have further defined

$$\mathcal{I}^{(3)} \equiv \int_0^\pi d\theta_1\;\sin\theta_1\sqrt{\tilde{F}_2\cos^2\theta_1+F_3\sin^2\theta_1}, \qquad \mathcal{I}^{(4)} \equiv \int_0^\pi \frac{\sin\theta_1\cos^2\theta_1 d\theta_1}{\sqrt{\tilde{F}_2\cos^2\theta_1+F_3\sin^2\theta_1}},$$
$$\mathcal{I}^{(5)} \equiv \int_0^\pi d\theta_1\frac{\sin\theta_1\cos\theta_1}{\sqrt{\tilde{F}_2\cos^2\theta_1+F_3\sin^2\theta_1}}. \tag{4.63}$$

It is most interesting to note that a_3 vanishes, since

$$\mathcal{I}^{(5)} \propto 1-\left.\sqrt{\frac{\tilde{F}_2+F_3+(\tilde{F}_2-F_3)\cos 2\theta_1}{\tilde{F}_2+F_3}}\right|_{\theta_1=0}^{\theta_1=\pi} = 0. \tag{4.64}$$

This greatly simplifies $I^{(1,2,1)}$ in (4.60). Specifically, (4.64) implies that there are no crossed terms for the interactions among the real scalars $(\mathcal{A}_{\tilde{3}},\,\mathcal{A}_{\phi_1},\,\mathcal{A}_r)$:

$$I^{(1,2,1)} = a_1[\mathcal{A}_r,\mathcal{A}_{\phi_1}]^2+a_2[\mathcal{A}_{\tilde{3}},\mathcal{A}_r]^2+a_4[\mathcal{A}_{\tilde{3}},\mathcal{A}_{\phi_1}]^2. \tag{4.65}$$

In the ongoing, we shall focus on the determination of the remaining coefficients in (4.62) and show that they are well-defined numbers for any choice of the warp factors one may wish to consider. With this aim in mind, we start by performing the integrals in (4.63).

Using our definitions in (4.27), we obtain for $\mathcal{I}^{(3)}$

$$\mathcal{I}^{(3)} = -\frac{1}{4}\Big(\sqrt{2}\cos\theta_1\chi(\theta_1) + \frac{2F_3}{\sqrt{\tilde{F}_2 - F_3}}\ln|\chi(\theta_1) + \tilde{\chi}(\theta_1)|\Big)\Big|_{\theta_1=0}^{\theta_1=\pi} = \sqrt{\tilde{F}_2} + \frac{F_3\mathcal{J}_3}{2\sqrt{\tilde{F}_2 - F_3}}, \tag{4.66}$$

where $\mathcal{J}_3$ was defined in (4.29). Similarly, $\tilde{\mathcal{I}}^{(4)} \equiv (\tilde{F}_2 - F_3)\mathcal{I}^{(4)}$ yields

$$\tilde{\mathcal{I}}^{(4)} = \frac{1}{4}\Big(- \sqrt{2}\cos\theta_1\chi(\theta_1) + \frac{2F_3}{\sqrt{\tilde{F}_2 - F_3}}\ln|\chi(\theta_1) + \tilde{\chi}(\theta_1)|\Big)\Big|_{\theta_1=0}^{\theta_1=\pi} = \sqrt{\tilde{F}_2} - \frac{F_3\mathcal{J}_3}{2\sqrt{\tilde{F}_2 - F_3}} \tag{4.67}$$

after integration. We remind the reader that $\mathcal{I}^{(1)}$ was determined in (4.28) already. Then, substitution of these results in (4.62) immediately gives us the coefficients (a_1, a_2, a_4) in the desired form:

$$\begin{aligned} a_1 &= R_3 \sec\theta_{nc}\int_0^\infty dr\sqrt{\frac{\tilde{F}_2F_3}{F_1}}\Big(\tilde{a}_+\sqrt{\tilde{F}_2} + \frac{\tilde{a}_-F_3\mathcal{J}_3}{2\sqrt{\tilde{F}_2 - F_3}}\Big) \\ a_2 &= R_3\sec\theta_{nc}\int_0^\infty dr\ \tilde{a}_2\mathcal{J}_3, \qquad a_4 = R_3\sec\theta_{nc}\int_0^\infty dr\ \tilde{a}_4\Big(\sqrt{\tilde{F}_2} + \frac{F_3\mathcal{J}_3}{2\sqrt{\tilde{F}_2 - F_3}}\Big), \end{aligned} \tag{4.68}$$

where the new coefficients ($\tilde{a}_\pm$, $\tilde{a}_2$, $\tilde{a}_4$) appearing above are defined as

$$\begin{aligned} \tilde{a}_\pm &\equiv 1 \pm \frac{(\tilde{F}_2\tan\theta_{nc})^2}{\tilde{F}_2 - F_3}(1 + F_2\tan^2\theta_{nc}), \qquad \tilde{a}_2 \equiv (\cos^2\theta_{nc} + F_2\sin^2\theta_{nc})\sqrt{\frac{\tilde{F}_2F_3}{F_1(\tilde{F}_2 - F_3)}}, \\ \tilde{a}_4 &\equiv e^{2\phi_0}(\cos^2\theta_{nc} + F_2\sin^2\theta_{nc})\sqrt{F_1\tilde{F}_2F_3}. \end{aligned} \tag{4.69}$$

Upon a careful inspection of the coefficients in (4.68), it is not hard to convince oneself that these all are just numbers for any choice of the warp factors in (2.2). The only constraint is that $\tilde{F}_2 \geq F_3$ should hold true, as was the case for the other coefficients as well.

In short, $I^{(1,2,1)}$ is given by (4.65), with (a_1, a_2, a_3) in (4.68) well-defined numbers for any choice of warp factors satisfying $\tilde{F}_2 \geq F_3$.

Computation of $I^{(1,2,2)}$ in (4.57)

We now turn our attention to $I^{(1,2,2)}$ in (4.57). From (4.52), it is easy to obtain

$$\mathcal{D}_a\mathcal{A}^{(\Sigma_3)} = (\mathcal{D}_a\hat{\alpha}_1)e_{\tilde{3}}^{(\Sigma_3)} + (\mathcal{D}_a\hat{\alpha}_2)e_r^{(\Sigma_3)} + (\mathcal{D}_a\hat{\alpha}_3)e_{\phi_1}^{(\Sigma_3)}. \tag{4.70}$$

The Hodge dual of the above with respect to the metric (4.47) is straightforward in view of (4.49) and is given by

$$*\mathcal{D}_a\mathcal{A}^{(\Sigma_3)} = (\mathcal{D}_a\hat{\alpha}_1)e_r^{(\Sigma_3)} \wedge e_{\phi_1}^{(\Sigma_3)} - (\mathcal{D}_a\hat{\alpha}_2)e_{\tilde{3}}^{(\Sigma_3)} \wedge e_{\phi_1}^{(\Sigma_3)} + (\mathcal{D}_a\hat{\alpha}_3)e_{\tilde{3}}^{(\Sigma_3)} \wedge e_r^{(\Sigma_3)}. \tag{4.71}$$

The wedge product of the above two quantities is

$$(\mathcal{D}_a\mathcal{A}^{(\Sigma_3)}) \wedge *(\mathcal{D}_a\mathcal{A}^{(\Sigma_3)}) = \left[(\mathcal{D}_a\hat{\alpha}_1)^2 + (\mathcal{D}_a\hat{\alpha}_2)^2 + (\mathcal{D}_a\hat{\alpha}_3)^2\right] e_{\tilde{3}}^{(\Sigma_3)} \wedge e_r^{(\Sigma_3)} \wedge e_{\phi_1}^{(\Sigma_3)}. \tag{4.72}$$

Feeding the above to (4.57) and further using (4.20), (4.48) and (4.53), $I^{(1,2,2)}$ can be succinctly written as

$$I^{(1,2,2)} = \sum_{a=0}^{2}\left[c_{a\tilde{3}}(\mathcal{D}_a\mathcal{A}_{\tilde{3}} - \frac{\mu}{c_{a\tilde{3}}}\mathcal{D}_a\mathcal{A}_{\phi_1})^2 + c_{ar}(\mathcal{D}_a\mathcal{A}_r)^2 + c_{a\phi_1}(\mathcal{D}_a\mathcal{A}_{\phi_1})^2\right], \tag{4.73}$$

where, making use of (4.24), we have defined the coefficients

$$\begin{aligned} c_{a\tilde{3}} &\equiv e^{2\phi_0}\int d^4\tilde{\zeta}\ \frac{\sqrt{H_4F_1}}{H_2}, \qquad & \mu &\equiv e^{2\phi_0}\int d^4\tilde{\zeta}\ \sqrt{H_4F_1}\frac{f_3}{H_2}, \\ c_{ar} &\equiv \int d^4\tilde{\zeta}\ \sqrt{\frac{H_4}{F_1}}, \qquad & c_{a\phi_1} &\equiv e^{2\phi_0}\int d^4\tilde{\zeta}\ \frac{\sqrt{H_4F_1}}{H_3}. \end{aligned} \tag{4.74}$$

These coefficients can be written in terms of the warp factors using (2.43). Exactly as was the case before with the coefficients in (4.61), the $(\tilde{x}_3,\ \phi_1)$ integrals are trivial here too. Thus, we have that

$$\begin{aligned} c_{a\tilde{3}} &= 2R_3\sec\theta_{nc}\int_0^\infty dr\ \tilde{a}_4\mathcal{I}^{(1)}, \qquad & \mu &\propto \mathcal{I}^{(5)}, \\ c_{ar} &= 2R_3\sec\theta_{nc}\int_0^\infty dr\sqrt{\frac{\tilde{F}_2F_3}{F_1}}\mathcal{I}^{(1)}, \qquad & c_{a\phi_1} &= e^{2\phi_0}R_3\sec\theta_{nc}\int_0^\infty dr\ \sqrt{F_1\tilde{F}_2F_3}\mathcal{I}^{(3)}, \end{aligned} \tag{4.75}$$

where $(\mathcal{I}^{(1)},\ \mathcal{I}^{(3)},\ \mathcal{I}^{(5)},\ \tilde{a}_4)$ were defined in (4.26), (4.63) and (4.69), respectively.

In a similar fashion to what happened in the determination of $I^{(1,2,1)}$, the result in

(4.64) makes μ vanish. This implies that there are no crossed terms for the kinetic terms of $(\mathcal{A}_{\tilde{3}},\, \mathcal{A}_{\phi_1},\, \mathcal{A}_r)$. In other words, (4.73) reduces to

$$I^{(1,2,2)} = \sum_{a=0}^{2} \left[c_{a\tilde{3}} (\mathcal{D}_a \mathcal{A}_{\tilde{3}})^2 + c_{ar} (\mathcal{D}_a \mathcal{A}_r)^2 + \tilde{c}_{a\phi_1} (\mathcal{D}_a \mathcal{A}_{\phi_1})^2 \right], \tag{4.76}$$

with $\tilde{c}_{a\phi_1}$ defined as

$$\tilde{c}_{a\phi_1} \equiv c_{a\phi_1} + \frac{\mu^2}{c_{a\tilde{3}}}, \qquad \frac{\mu^2}{c_{a\tilde{3}}} = R_3 \sec^3\theta_{nc} \tan^2\theta_{nc} \int_0^\infty dr\ \tilde{a}_4 \tilde{F}_2 \mathcal{I}^{(4)}. \tag{4.77}$$

In writing the second equality above, we have made use of all (2.43), (4.24), (4.63), (4.69) and (4.74). At this point, we are left with the task of computing $(c_{a\tilde{3}},\, c_{ar},\, \tilde{c}_{a\phi_1})$ and showing that they are all well-defined real numbers for any choice of warp factors in (2.2).

The computation part is straightforward, in view of our earlier results in (4.28), (4.66) and (4.67). We thus obtain

$$\begin{aligned} c_{a\tilde{3}} &= R_3 \sec\theta_{nc} \int_0^\infty dr \frac{\tilde{a}_4 \mathcal{J}_3}{\sqrt{\tilde{F}_2 - F_3}}, \qquad c_{ar} = R_3 \sec\theta_{nc} \int_0^\infty dr\ \mathcal{J}_3 \sqrt{\frac{\tilde{F}_2 F_3}{F_1(\tilde{F}_2 - F_3)}}, \\ \tilde{c}_{a\phi_1} &= e^{2\phi_0} R_3 \sec\theta_{nc} \int_0^\infty dr \sqrt{F_1 \tilde{F}_2 F_3} \Big(\tilde{a}_+ \sqrt{\tilde{F}_2} + \frac{\tilde{a}_- F_3 \mathcal{J}_3}{2\sqrt{\tilde{F}_2 - F_3}} \Big), \end{aligned} \tag{4.78}$$

where $(\mathcal{J}_3,\, \tilde{a}_\pm,\, \tilde{a}_4)$ were defined in (4.29) and (4.69), respectively. On the other hand, the issue of proving that all three coefficients above are numbers is also simple enough. Once again, one must demand that $\tilde{F}_2 \geq F_3$ to prevent the "blowing up" of these quantities. However, any value of the warp factors in (2.2) satisfying this constraint can be readily seen to yield a finite, real result when used in (4.78).

Consequently, we conclude that $I^{(1,2,2)}$ is given by (4.76), with the coefficients $(c_{a\tilde{3}},\, c_{ar},\, \tilde{c}_{a\phi_1})$ there appearing given by (4.78). These are well-defined numbers as long as the warp factors are chosen such that $\tilde{F}_2 \geq F_3$ everywhere.

Computation of $I^{(1,2,3)}$ in (4.57)

At last, we consider $I^{(1,2,3)}$ in (4.57). Its computation is very similar to that of $I^{(1,2,2)}$, albeit algebraically more involved. In the following, we show all the relevant details. With the aid of (4.49) and (4.52), it is easy to see that

$$\mathcal{D}_{\tilde{\psi}} \mathcal{A}^{(\Sigma_3)} \wedge *\big(\mathcal{D}_{\tilde{\psi}} \mathcal{A}^{(\Sigma_3)}\big) = \big[\big(\mathcal{D}_{\tilde{\psi}} \hat{\alpha}_1\big)^2 + \big(\mathcal{D}_{\tilde{\psi}} \hat{\alpha}_2\big)^2 + \big(\mathcal{D}_{\tilde{\psi}} \hat{\alpha}_3\big)^2 \big] e_{\tilde{3}}^{(\Sigma_3)} \wedge e_r^{(\Sigma_3)} \wedge e_{\phi_1}^{(\Sigma_3)}. \tag{4.79}$$

With the above and the definitions (4.20), (4.48) and (4.53), one can write $I^{(1,2,3)}$ as

$$I^{(1,2,3)} = c_{\tilde{\psi}\tilde{3}}\big(\mathcal{D}_{\tilde{\psi}}\mathcal{A}_{\tilde{3}} - \frac{\nu}{c_{\tilde{\psi}\tilde{3}}}\mathcal{D}_{\tilde{\psi}}\mathcal{A}_{\phi_1}\big)^2 + c_{\tilde{\psi}r}\big(\mathcal{D}_{\tilde{\psi}}\mathcal{A}_r\big)^2 + c_{\tilde{\psi}\phi_1}\big(\mathcal{D}_{\tilde{\psi}}\mathcal{A}_{\phi_1}\big)^2, \tag{4.80}$$

where, making use of (4.24), we have defined

$$c_{\tilde{\psi}\tilde{3}} \equiv \int \frac{d^4\tilde{\zeta}}{H_2}\sqrt{\frac{F_1}{H_4}}, \quad \nu \equiv \int d^4\tilde{\zeta}\frac{f_3}{H_2}\sqrt{\frac{F_1}{H_4}}, \quad c_{\tilde{\psi}r} \equiv e^{-2\phi_0}\int \frac{d^4\tilde{\zeta}}{\sqrt{H_4 F_1}}, \quad c_{\tilde{\psi}\phi_1} \equiv \int \frac{d^4\tilde{\zeta}}{H_3}\sqrt{\frac{F_1}{H_4}}. \tag{4.81}$$

These coefficients can be expressed in terms of the warp factors in (2.2) by inserting (2.43) in the above. It is again the case that the $(\tilde{x}_3,\, \phi_1)$ integrals are trivial and so we obtain

$$\begin{aligned} c_{\tilde{\psi}\tilde{3}} &= 2R_3\cos\theta_{nc}\int_0^\infty dr\frac{\tilde{b}_2\mathcal{I}^{(2)}}{\sqrt{\tilde{F}_2 - F_3}}, & \nu &\propto \mathcal{I}^{(6)}, \\ c_{\tilde{\psi}r} &= 4e^{-2\phi_0}R_3\cos\theta_{nc}\int_0^\infty dr\frac{b_2\mathcal{I}^{(2)}}{F_1\sqrt{\tilde{F}_2 - F_3}}, & c_{\tilde{\psi}\phi_1} &= R_3\cos\theta_{nc}\int_0^\infty dr\sqrt{\frac{F_1}{\tilde{F}_2F_3}}\mathcal{I}^{(7)}, \end{aligned} \tag{4.82}$$

Here, we have defined $\tilde{b}_2$ as a slight variant of b_2 in (4.31):

$$\tilde{b}_2 \equiv (\cos^2\theta_{nc} + F_2\sin^2\theta_{nc})\sqrt{\frac{F_1(\tilde{F}_2 - F_3)}{\tilde{F}_2F_3}}, \tag{4.83}$$

$\mathcal{I}^{(2)}$ is as in (4.26) and the remaining integrals there appearing are defined as

$$\begin{aligned} \mathcal{I}^{(6)} &\equiv \int_0^\pi d\theta_1\;\cot\theta_1(\tilde{F}_2\cos^2\theta_1 + F_3\sin^2\theta_1)^{1/2}, \\ \mathcal{I}^{(7)} &\equiv \int_0^\pi d\theta_1\;\csc\theta_1(\tilde{F}_2\cos^2\theta_1 + F_3\sin^2\theta_1)^{3/2}. \end{aligned} \tag{4.84}$$

In view of our earlier results for $(a_3,\, \mu)$ in (4.62) and (4.75) respectively, it will come as no surprise that ν above vanishes. To see this, we simply need to use b_1 in (4.31) and the change of variables in (4.33). Then, after regularization, $\mathcal{I}^{(6)}$ vanishes by symmetry:

$$\mathcal{I}^{(6)} \propto \int_{-1}^1 dz\frac{z(b_1^2 + z^2)^{1/2}}{b^2 - z^2} = 0, \qquad b \in \mathbb{R}^+ - \{1\}. \tag{4.85}$$

Therefore, (4.80) simplifies considerably, leading to no crossed terms between the kinetic terms of $(\mathcal{A}_{\tilde{3}},\, \mathcal{A}_{\phi_1},\, \mathcal{A}_r)$:

$$I^{(1,2,3)} = c_{\tilde{\psi}\tilde{3}}\big(\mathcal{D}_{\tilde{\psi}}\mathcal{A}_{\tilde{3}}\big)^2 + c_{\tilde{\psi}r}\big(\mathcal{D}_{\tilde{\psi}}\mathcal{A}_r\big)^2 + \tilde{c}_{\tilde{\psi}\phi_1}\big(\mathcal{D}_{\tilde{\psi}}\mathcal{A}_{\phi_1}\big)^2, \tag{4.86}$$

with $\tilde{c}_{\tilde{\psi}\phi_1}$ defined as

$$\tilde{c}_{\tilde{\psi}\phi_1} \equiv c_{\tilde{\psi}\phi_1} + \frac{\nu^2}{c_{\tilde{\psi}\tilde{3}}}, \qquad \frac{\nu^2}{c_{\tilde{\psi}}} = R_3 \sec\theta_{nc}\tan^2\theta_{nc}\int_0^\infty dr\ \tilde{a}_2\sqrt{\tilde{F}_2 - F_3}\frac{F_1\tilde{F}_2}{F_3}\mathcal{I}^{(8)}. \tag{4.87}$$

In order to obtain the second equality above, the definitions in (2.43), (4.24), (4.69) and (4.81) have been used and we have further introduced

$$\mathcal{I}^{(8)} \equiv \int_0^\pi d\theta_1 \frac{\cos^2\theta_1}{\sin\theta_1}\sqrt{\tilde{F}_2\cos^2\theta_1 + F_3\sin^2\theta_1}. \tag{4.88}$$

At this stage, we are only left with the task of computing $(c_{\tilde{\psi}\tilde{3}},\, c_{\tilde{\psi}r},\, \tilde{c}_{\tilde{\psi}\phi_1})$.

To do so, we first recall $\mathcal{I}^{(2)}$ was already determined in (4.36) and so we still need to perform the integrals $(\mathcal{I}^{(7)},\, \mathcal{I}^{(8)})$. For $\mathcal{I}^{(7)}$, it is convenient to do the same set of transformations that we considered for $\mathcal{I}^{(2)}$ between (4.32) and (4.36) earlier on. Namely,

$$\begin{aligned}\frac{\mathcal{I}^{(7)}}{(\tilde{F}_2 - F_3)^{3/2}} &= \int_{-1}^{1} dz \frac{(b_1^2+z^2)^{3/2}}{b^2 - z^2} = \frac{(b_1^2+b^2)^{3/2}}{b}\eta(z) - \frac{3b_1^2+2b^2}{2}\tilde{\eta}(z) - \frac{z}{2}\sqrt{b_1^2+z^2}\Bigg|_{z=-1}^{z=1} \\ &= \frac{b^2}{4}b_3^3\mathcal{J}_4 - \frac{3b_1^2+2b^2}{2}\mathcal{J}_3 - \sqrt{\frac{\tilde{F}_2}{\tilde{F}_2 - F_3}},\end{aligned} \tag{4.89}$$

where $b \in \mathbb{R}^+ - \{1\}$ is a regularization factor, $(\eta(z),\, \tilde{\eta}(z))$ were defined in (4.35) and in the last step we have used (4.29), (4.31) and (4.37). In fact, we can do essentially the same for $\mathcal{I}^{(8)}$ and obtain

$$\begin{aligned}\frac{\mathcal{I}^{(8)}}{\sqrt{\tilde{F}_2 - F_3}} &= \int_{-1}^{1} dz\ z^2\frac{\sqrt{b_1^2+z^2}}{b^2 - z^2} = b\sqrt{b_1^2+b^2}\eta(z) - \frac{b_1^2+2b^2}{2}\tilde{\eta}(z) - \frac{z}{2}\sqrt{b_1^2+z^2}\Bigg|_{z=-1}^{z=1} \\ &= b^2 b_3\mathcal{J}_4 - \frac{b_1^2+2b^2}{2}\mathcal{J}_3 - \sqrt{\frac{\tilde{F}_2}{\tilde{F}_2 - F_3}}.\end{aligned} \tag{4.90}$$

With all these results at hand, it is now a matter of substitution and easy algebra to obtain the desired coefficients:

$$c_{\tilde{\psi}\tilde{3}} = R_3 \cos\theta_{nc} \int_0^\infty dr\ \tilde{b}_2 \left(b_3 \mathcal{J}_4 + \mathcal{J}_3^{-1}\right), \qquad \tilde{c}_{\tilde{\psi}\phi_1} = \int_0^\infty dr\ \left(a_{01}\mathcal{J}_4 + b_{01}\mathcal{J}_3^{-1} - c_{01}\right),$$
$$c_{\tilde{\psi}r} = 2e^{-2\phi_0} R_3 \cos\theta_{nc} \int_0^\infty dr\ \frac{b_2}{F_1}\left(b_3\mathcal{J}_4 + \mathcal{J}_3^{-1}\right). \tag{4.91}$$

Recall that $(\tilde{F}_2,\ b_2,\ b_3,\ \tilde{b}_2)$ were defined in (2.21), (4.31) and (4.83), respectively. The other factors in $\tilde{c}_{\tilde{\psi}\phi_1}$ are defined as

$$\begin{aligned} a_{01} \equiv & R_3 b^2 b_3 (\tilde{F}_2 - F_3)\Big(\cos\theta_{nc} b_3^2 \frac{\sqrt{F_1(\tilde{F}_2 - F_3)}}{4\tilde{F}_2 F_3} + \tilde{a}_2 \frac{\tan^2\theta_{nc}}{\cos\theta_{nc}} \frac{F_1 \tilde{F}_2}{F_3}\Big), \\ b_{01} \equiv & \frac{R_3}{2}\sqrt{\frac{F_1}{F_3}}\Big(\cos\theta_{nc} f^{(1)} \sqrt{\frac{\tilde{F}_2 - F_3}{\tilde{F}_2}} + \tilde{a}_2 \tilde{F}_2 f^{(2)} \frac{\tan^2\theta_{nc}}{\cos\theta_{nc}} \sqrt{\frac{F_1}{F_3}}\Big), \\ c_{01} \equiv & R_3(\tilde{F}_2 - F_3)\sqrt{\frac{F_1}{F_3}}\Big(\cos\theta_{nc} + \tilde{a}_2 \tilde{F}_2^2 \frac{\tan^2\theta_{nc}}{\cos\theta_{nc}} \sqrt{\frac{F_1(\tilde{F}_2 - F_3)}{\tilde{F}_2 F_3}}\Big), \end{aligned} \tag{4.92}$$

with $(f^{(1)},\ f^{(2)})$ given by

$$f^{(1)} \equiv 3F_3 + 2b^2(\tilde{F}_2 - F_3), \qquad f^{(2)} \equiv f^{(1)} - 2F_3. \tag{4.93}$$

In exactly the same way shown in the end of section 4.1.1 for c_{12}, it follows that $(c_{\tilde{\psi}\tilde{3}},\ c_{\tilde{\psi}r})$ are just numbers for any choice of the warp factors satisfying $\tilde{F}_2 \geq F_3$. The scenario is more subtle in the case of $\tilde{c}_{\tilde{\psi}\phi_1}$, for it is not clear at all that this coefficient is finite when:

- $F_3 \to 0$. As discussed after (4.38), this limit also includes the case $(\tilde{F}_2,\ F_3) \to 0$.
- $\tilde{F}_2 \to F_3 \nrightarrow 0$.

However, it turns out that

$$\lim_{F_3 \to 0} \tilde{c}_{\tilde{\psi}\phi_1} = 0, \tag{4.94}$$

the mathematical details precisely as in between (4.40) and (4.44) for c_{12} before. Consequently, we will just show that $\tilde{c}_{\tilde{\psi}\phi_1}$ is well-defined when $\tilde{F}_2 \to F_3$. To do this, we call $\epsilon^2 \equiv \tilde{F}_2 - F_3$ and take the $\epsilon \to 0$ limit. Plugged in $(b_3,\ \tilde{a}_4)$, as given by (4.31) and (4.69), we get

$$\lim_{\epsilon\to 0} b_3 \sim \lim_{\epsilon \to 0} \frac{1}{\epsilon} \sim \lim_{\epsilon \to 0} \tilde{a}_2. \tag{4.95}$$

Then, feeding the above to (4.92), we obtain

$$\lim_{\epsilon\to0} a_{01} \sim 1, \qquad \lim_{\epsilon\to0} b_{01} \sim \lim_{\epsilon\to0} \frac{1}{\epsilon}, \qquad \lim_{\epsilon\to0} c_{01} = 0. \tag{4.96}$$

We consider this very same limit for $(\mathcal{J}_3,\, \mathcal{J}_4)$ in (4.29) and (4.37):

$$\lim_{\epsilon\to0} \mathcal{J}_3 = \lim_{\epsilon\to0} \ln\left|\frac{1+\epsilon}{1-\epsilon}\right|, \qquad \lim_{\epsilon\to0} \mathcal{J}_4 = \operatorname{arctanh}\frac{1}{b}, \tag{4.97}$$

which is finite, since $b \neq 1$ by definition. All the above can be used in (4.91). Retaining only the divergent part, we have that

$$\lim_{\epsilon\to0} \tilde{c}_{\tilde{\psi}\phi_1} \sim \lim_{\epsilon\to0} \frac{1}{\epsilon} \ln\left|\frac{1+\epsilon}{1-\epsilon}\right| = \lim_{\epsilon\to0}\left(\frac{1}{1+\epsilon} + \frac{1}{1-\epsilon}\right) = 2, \tag{4.98}$$

where in the last step we have applied L'Hôpital's rule. In other words, the seemingly divergent part of $\tilde{c}_{\tilde{\psi}\phi_1}$ is actually finite. Consequently, there is no need to introduce an new constraint: $\tilde{c}_{\tilde{\psi}\phi_1}$ is a well-defined number for any warp factors one may wish to consider, as long as $\tilde{F}_2 \geq F_3$.

Quickly summing up, $I^{(1,2,3)}$ si given by (4.86) and the coefficients $(c_{\tilde{\psi}\tilde{3}},\, c_{\tilde{\psi}r},\, \tilde{c}_{\tilde{\psi}\phi_1})$ there appearing are all well-defined numbers if $\tilde{F}_2 \geq F_3$. Their explicit form is that in (4.91).

We can finally collect all our results so far into a quite simple form. First, we use (4.65), (4.76) and (4.86) in (4.56) and write $I^{(1,2)}$ accordingly. Next, inserting such $I^{(1,2)}$ and (4.22) in (4.14), the first term of the bosonic action for the $SU(N)$ world-volume gauge theory along $(t,\, x_1,\, x_2,\, \tilde{\psi})$ can be readily seen to be

$$\begin{aligned} S^{(1)} =& \frac{C_1 c_{11}}{V_3} \int d^4x \sum_{\substack{a,b=0\\a<b}}^{2} \operatorname{Tr}(\mathcal{F}_{ab}^2) + \frac{C_1 c_{12}}{V_3} \int d^4x \sum_{a=0}^{2} \operatorname{Tr}(\mathcal{F}_{a\tilde{\psi}}^2) \\ &+ \frac{C_1}{V_3} \int d^4x \operatorname{Tr}\Big\{a_1[\mathcal{A}_r, \mathcal{A}_{\phi_1}]^2 + a_2[\mathcal{A}_{\tilde{3}}, \mathcal{A}_r]^2 + a_4[\mathcal{A}_{\tilde{3}}, \mathcal{A}_{\phi_1}]^2 \\ &+ \sum_{a=0}^{2}\Big[c_{a\tilde{3}}\big(\mathcal{D}_a\mathcal{A}_{\tilde{3}}\big)^2 + c_{ar}\big(\mathcal{D}_a\mathcal{A}_r\big)^2 + \tilde{c}_{a\phi_1}\big(\mathcal{D}_a\mathcal{A}_{\phi_1}\big)^2\Big] \\ &+ c_{\tilde{\psi}\tilde{3}}\big(\mathcal{D}_{\tilde{\psi}}\mathcal{A}_{\tilde{3}}\big)^2 + c_{\tilde{\psi}r}\big(\mathcal{D}_{\tilde{\psi}}\mathcal{A}_r\big)^2 + \tilde{c}_{\tilde{\psi}\phi_1}\big(\mathcal{D}_{\tilde{\psi}}\mathcal{A}_{\phi_1}\big)^2\Big\}. \end{aligned} \tag{4.99}$$

It is important to bear in mind that all the coefficients appearing in this first term of the action have been shown to be real numbers for any choice of the warp factors satisfying $\tilde{F}_2 \geq F_3$. Without further delay, let us turn to the second term of the world-volume action.

4.2 Mass term of the G-flux

In order to obtain the second term for the bosonic action of the world-volume gauge theory associated to (M,1), we first need to brush up a bit the construction of this M-Theory model, which we presented in chapter 2. In particular, we need to recall how we moved far away along the Coulomb branch the $\overline{\text{D5}}$-brane of figure 2**D**. (Bear in mind that, as depicted, these branes stretch along the directions $(t,\, x_1,\, x_2,\, x_3,\, \psi,\, r)$.) In this manner, we managed to effectively ignore the presence of the $\overline{\text{D5}}$-brane in the configuration (B,1) of figure 3, thereby simplifying the starting point of our quantitative derivation of (M,1). It is now time to study the essential effects that the presence of this $\overline{\text{D5}}$-brane has for the world-volume gauge theory.

Let us begin by bringing back to its original position the $\overline{\text{D5}}$-brane. In other words, let us consider that the D5-brane in the configuration (B,1) has right next to it a parallel $\overline{\text{D5}}$-brane. To prevent the D5/$\overline{\text{D5}}$ pair from collapsing (thus giving rise to tachyons), we switch on a small NS B-field $\tilde{B}_2^{(B,1)}$ along the directions $(x_3,\, r)$ in both the D5- and $\overline{\text{D5}}$-branes. As carefully explained in [29], the D5/$\overline{\text{D5}}$ pair with such an NS B-field on it can alternatively be interpreted as two fractional D3-branes spanning $(t,\, x_1,\, x_2,\, \psi)$[12]. From this point of view, it is easy to infer that we must also switch on a small RR B-field $\tilde{\mathcal{C}}_2^{(B,1)}$ along the same directions $(x_3,\, r)$, so as to ensure the tadpole cancellation condition is satisfied[13]. As a particularly simple and consistent choice, we will consider both these fields to only depend on the $(\theta_1,\, r)$ coordinates:

$$\tilde{B}_2^{(B,1)} \equiv F^{(1)} dx_3 \wedge dr, \qquad \tilde{\mathcal{C}}_2^{(B,1)} \equiv F^{(2)} dx_3 \wedge dr, \qquad F^{(i)} = F^{(i)}(\theta_1, r), \qquad i = 1, 2. \tag{4.100}$$

With the goal of understanding how these new B-fields will affect the configuration (M,1), in the following we will subject them to the chain of modifications shown in figure 3.

For our present purposes, it turns out we need not do the whole analysis in details, as in part I before. Further, we need not worry about the NS B-field either. Rather, it suffices to note that, in going from (B,1) to (B,2), the above RR B-field will be affected by the non-commutative deformation in (2.19) and will also receive additional contributions along other directions. We shall not be interested in such additional terms, so we will consider simply

[12]Note that our choice of orientation of the NS B-field leads to the stretching of the fractional D3-branes along precisely the spacetime directions.

[13]The tadpole condition is, essentially, the statement that the charge of the fractional D3-branes should be conserved. It follows directly from the Bianchi identity and the equations of motion of the corresponding fluxes. A neat derivation of the tadpole condition can be found in section 4.2 of [52].

that

$$\tilde{\mathcal{C}}_2^{(B,2)} = \sec\theta_{nc} F^{(2)} d\tilde{x}_3 \wedge dr + \text{other terms}. \tag{4.101}$$

(The reader should not be worried at the drastic simplification in the analysis at this point, since it will shortly become clear why one can consistently do so.) Then, in T-dualizing along ϕ_1 to the configuration (A,3), we obtain an RR three-form potential of the form

$$\tilde{\mathcal{C}}_3^{(A,3)} = \sec\theta_{nc} F^{(2)} d\phi_1 \wedge d\tilde{x}_3 \wedge dr + \text{other terms}. \tag{4.102}$$

Without loss of generality, the relevant part of $\tilde{\mathcal{C}}_3^{(A,3)}$ will be assumed to be of the form

$$\tilde{\mathcal{C}}_3^{(A,3)} = \frac{N_r \sin 2\theta_{nc} \cos\theta_{nc} p(\theta_1) q(\theta_{nc})}{2(\cos^2\theta_{nc} + N\sin^2\theta_{nc})^2} dr \wedge d\tilde{x}_3 \wedge d\phi_1, \tag{4.103}$$

with $(p,\, q)$ periodic functions of $(\theta_1,\, \theta_{nc})$ with period $(\pi,\, 2\pi)$, respectively and $N = N(r,\, \theta_{nc})$ sufficiently small for all values of the radial coordinate and such that

$$\lim_{r\to 0} N = 0, \qquad \lim_{r\to\infty} N = 1. \tag{4.104}$$

Quite obviously, N_r above stands for the derivative of N with respect to r. Finally, in the uplift from (A,3) to (M,1), (4.103) will lead to the background G-flux in (2.51) receiving the additional contribution

$$\delta\langle\mathcal{G}_4^{(M,1)}\rangle = d\tilde{\mathcal{C}}_3^{(A,3)}. \tag{4.105}$$

For completeness, let us just mention that the NS B-field $\tilde{B}_2^{(B,1)}$ will also add to the background G-flux of (M,1), as roughly $d\tilde{B}_2^{(B,1)} \wedge dx_{11}$. This is, however, inconsequential from the point of view of the world-volume gauge theory.

Summing up, the inclusion of the $\overline{\text{D5}}$-brane in such a way that tachyons are avoided affects only the background G-flux of the abelian configuration (M,1). As already argued in section 4.1, the background G-flux does not contribute at all to the first term of the action (4.99). Consequently, the $\overline{\text{D5}}$-brane does *not* affect our results so far and so there is no need to make more precise the above analysis.

However, the particular contribution (4.103) to the RR three-form potential of the configuration (A,3) does play a key role. It sources a new term[14] for the gauge theory action,

[14] Actually, this second term for our bosonic action is well-known and usually referred to as "anomalous interaction term" in the literature. The interested reader can find a lucid review of its main features in section 4 of [53] and references therein.

which one may view as a mass term for the G-flux of (M,1):

$$S^{(2)} \equiv \int_{X_{11}} \tilde{\mathcal{C}}_3^{(A,3)} \wedge \mathcal{G}_4^{(M,1)} \wedge \mathcal{G}_4^{(M,1)}, \tag{4.106}$$

with $\mathcal{G}_4^{(M,1)}$ given by (2.51) in this abelian scenario and the eleven-dimensional manifold X_{11} was described around (4.1).

Moving on to the non-abelian enhanced case (constructed in section 2.1.1), our entire discussion hitherto straightforwardly goes through. The only two differences are that we have N number of D5/$\overline{\text{D5}}$ pairs instead of just one and that $\mathcal{G}_4^{(M,1)}$ in (4.106) is now the non-abelian G-flux in (2.76). Since the background G-flux in (2.76) is negligible and using the non-abelian generalization of (4.6), the second term of the action reduces to

$$S^{(2)} = \frac{C_1}{V_3}\int_0^{\pi}\frac{d\theta_1}{2\pi}\int_{X_4\otimes\Sigma_3} \text{Tr}\left(\tilde{\mathcal{C}}_3^{(A,3)} \wedge \mathcal{F} \wedge \mathcal{F}\right), \tag{4.107}$$

with $\mathcal{F}$ the non-abelian seven-dimensional field strength of (M,1). As was the case with the first term $S^{(1)}$ of the bosonic action, the trace is taken in the adjoint representation of the gauge group, in this case $SU(N)$. Also, note that we have transferred the θ_1 integral (as an average) to the $X_4 \otimes \Sigma_3$ subspace of X_{11}, to consistently decouple the contribution of the Taub-NUT space to $S^{(2)}$. Relevant comments regarding the appearance of this trace and the decoupling of the Taub-NUT subspace are as discussed before, between equations (4.4) and (4.12).

The $S^{(2)}$ term in (4.107) is actually very simple. Note that $\tilde{\mathcal{C}}_3^{(A,3)}$ spans all three directions of the three-cycle Σ_3. Recall also the decomposition of $\mathcal{F}$ in (4.13). It is clear that $\mathcal{F}^{(\Sigma_3)}$ cannot contribute to $S^{(2)}$, as it would then lead to a vanishing wedge product between two same directions of Σ_3. On the other hand, $\mathcal{F}^{(X_4)}$ does contribute, but is restricted to X_4 and does not depend on the θ_1 coordinate, both properties following by definition. Thus, the integral over $X_4 \otimes \Sigma_3$ naturally decomposes into independent integrals over X_4 and Σ_3 and (4.107) is in fact just given by

$$S^{(2)} = c_2 I^{(2)} \qquad I^{(2)} \equiv \int_{X_4} \text{Tr}\left(\mathcal{F}^{(X_4)} \wedge \mathcal{F}^{(X_4)}\right), \qquad c_2 \equiv \frac{C_1}{V_3}\int_0^{\pi}\frac{d\theta_1}{2\pi}\int_{\Sigma_3}\tilde{\mathcal{C}}_3^{(A,3)}. \tag{4.108}$$

It is important to highlight that this is Θ-term type of contribution to the action. For the moment, the above form of $I^{(2)}$ will suffice. We will work on further rewritings of this integral in due time, when the need arises. Consequently, let us focus on the only task left: the determination of the coefficient c_2.

This too turns out to be quite easy. Using (4.24) and (4.103), we can rewrite c_2 as

$$c_2 = \frac{C_1}{V_3} \int d^4\tilde{\zeta}\ \frac{N_r \sin 2\theta_{nc} \cos\theta_{nc} p(\theta_1) q(\theta_{nc})}{2(\cos^2\theta_{nc} + N \sin^2\theta_{nc})^2}. \tag{4.109}$$

Once more, the integrals over $(\tilde{x}_3, \phi_1)$ here are trivial. To simplify the notation a bit, we absorb the contribution of the θ_1 integral in the radius of the $\tilde{x}_3$ non-compact direction as

$$\tilde{R}_3 \equiv \frac{R_3}{2} \int_0^{\pi} d\theta_1\ p(\theta_1). \tag{4.110}$$

Then, c_2 can be checked to be

$$c_2 = \frac{C_1}{V_3} \tilde{R}_3 \sin 2\theta_{nc} \cos\theta_{nc} q(\theta_{nc}) \int_0^{\infty} \frac{N_r dr}{(\cos^2\theta_{nc} + N\sin^2\theta_{nc})^2} = 2\frac{C_1 \tilde{R}_3}{V_3} \sin\theta_{nc} q(\theta_{nc}), \tag{4.111}$$

where in the last step we have used the boundary values in (4.104). Our final expression for c_2 leaves no room for doubt: this coefficient is just some well-defined number. Without loss of generality, one may set $2\tilde{R}_3 = V_3$ and thus simply consider c_2 as

$$c_2 = C_1 \sin\theta_{nc} q(\theta_{nc}). \tag{4.112}$$

Written in this manner, C_1 accounts for the dependence of the c_2 coefficient on the non-abelian version of the M-Theory configuration (M,1) of chapter 2. The factor $\sin\theta_{nc}$ ensures that $\theta_{nc} = 0$ implies $c_2 = 0$. This is a most vital remark, once we recall that θ_{nc} was introduced to this aim precisely: sourcing a Θ-term in the world-volume gauge theory. Finally, $q(\theta_{nc})$ allows us to have as complex a dependence of c_2 on θ_{nc} as one may wish.

4.3 Completing the $\mathcal{N} = 4$ vector multiplet: third term for the action

In this section, we compute the third and last term $S^{(3)}$ that contributes to the bosonic action of the world-volume gauge theory. As we already pointed out in the beginning of the chapter 4, this third term is not easily derivable from the non-abelian M-Theory configuration (M,1). In fact, there is no rigorous derivation of this type of term in the literature till date. Nonetheless, all the knowledge we have gathered while deriving the first two terms, $S^{(1)}$ in (4.99) and $S^{(2)}$ in (4.108), will now pay off and allow us to obtain the remaining third term.

Let us begin by recalling that in the end of section 4.1 we argued that the bosonic matter content in the gauge theory must be exactly that in the $\mathcal{N} = 4$ vector multiplet. That is, in our action we must have four gauge fields and six real scalars, all of them in the adjoint representation of $SU(N)$. However, upon inspection of the already derived first two terms in the gauge theory action, we note that so far only the gauge fields ($\mathcal{A}_t$, $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_{\tilde{\psi}}$) and three real scalars ($\mathcal{A}_{\tilde{3}}$, $\mathcal{A}_{\phi_1}$, $\mathcal{A}_r$) have appeared in our analysis. Hence, what we are missing is the contribution of the other three real scalars in the $\mathcal{N} = 4$ vector multiplet. We will refer to these as (φ_1, φ_2, φ_3). Accordingly, $S^{(3)}$ will capture the dynamics of these scalar fields.

Let us next note that the terms $S^{(1)}$ and $S^{(2)}$ originate from the G-flux of the non-abelian configuration (M,1), which is given by (2.76). Further, these two terms exhaust all possible contributions of the G-flux to the action. This statement is most clearly seen by looking at the initial form of $S^{(1)}$ and $S^{(2)}$: that in (4.2) and (4.106), respectively. In consequence, $S^{(3)}$ must emerge purely from the geometry of (M,1). In other words, we expect the scalar fields φ_k (with $k = 1, 2, 3$) to stem from fluctuations of the eleven-dimensional supergravity Einstein term of (M,1). In terms of our non-abelian scenario of figure 4**B**, this means that the Taub-NUT space TN and the M2-branes wrapping its two-cycles fluctuate along $X_4 \otimes \Sigma_3$[15]. We will right away simplify the scenario and assume the fluctuations are restricted to X_4 only, so that

$$\varphi_k = \varphi_k(t, x_1, x_2, \tilde{\psi}) \qquad \forall k = 1, 2, 3. \tag{4.113}$$

We will further suppose that, in fluctuating along orthogonal directions of X_{11}, TN itself does not get back-reacted. Or, more accurately, that the back-reaction of TN is negligible compared to the change that the metric of $X_4 \otimes \Sigma_3$ experiences. This last key assumption allows us to write $S^{(3)}$ as an integral over $X_4 \otimes \Sigma_3$ only. In the same vein as for the previous two terms of the action, we will also average over the contribution of the θ_1 coordinate.

Having shed sufficient qualitative light into the nature and content of $S^{(3)}$, we are now ready to make this term in the action fully precise. Naturally, $S^{(3)}$ must contain the kinetic terms and the self-interaction terms of (φ_1, φ_2, φ_3), as well as their interaction terms with the other scalar fields ($\mathcal{A}_{\tilde{3}}$, $\mathcal{A}_{\phi_1}$, $\mathcal{A}_r$) in the theory. All in all, we have that

$$S^{(3)} = S^{(\varphi)}_{kin} + S^{(\varphi\varphi)}_{int} + S^{(\mathcal{A}\varphi)}_{int}, \tag{4.114}$$

which is written in a form that mimics the well-known $\mathcal{N} = 4$ vector multiplet's action. In the same spirit of (4.56)-(4.57), we can write the above as

$$S^{(\varphi)}_{kin} = \int_0^\pi \frac{d\theta_1}{2\pi} \int_{\mathcal{M}_7} \text{Tr} \sum_{k=1}^{3} \Big[\sum_{a=0}^{2} g^{aa} (\mathcal{D}_a \varphi_k)^2 + g^{\tilde{\psi}\tilde{\psi}} (\mathcal{D}_{\tilde{\psi}} \varphi_k)^2 \Big], \tag{4.115}$$

$$S^{(\varphi\varphi)}_{int} = \int_0^\pi \frac{d\theta_1}{2\pi} \int_{\mathcal{M}_7} \text{Tr} \sum_{k=1}^{3} [\varphi_k, \varphi_l]^2, \quad S^{(\mathcal{A}\varphi)}_{int} = \int_0^\pi \frac{d\theta_1}{2\pi} \int_{\mathcal{M}_7} \text{Tr} \sum_{k=1}^{3} \big([\mathcal{A}^{(\Sigma_3)}, \varphi_k] \wedge *[\mathcal{A}^{(\Sigma_3)}, \varphi_k] \big),$$

[15]As a reminder, the subspaces (TN, X_4, Σ_3) of the full eleven-dimensional manifold X_{11} were introduced and described around (4.1).

where we have introduced $\mathcal{M}_7 \equiv X_4 \otimes \Sigma_3$, $(g^{aa},\, g^{\tilde{\psi}\tilde{\psi}})$ are given by (4.19), the covariant derivatives were defined in (4.51), $\mathcal{A}^{(\Sigma_3)}$ stands for (4.52) and the Hodge dual is with respect to the three-dimensional metric of Σ_3 in (4.47). In the following, we shall determine these terms separately.

Computation of $S^{(\varphi)}_{kin}$ in (4.115)

This kinetic piece is rather unchallenging to work out. Simply writing out explicitly the integral over $X_4 \otimes \Sigma_3$ there appearing and using (4.17), (4.19) and (4.24), $S^{(\varphi)}_{kin}$ can be expressed as

$$S^{(\varphi)}_{kin} = \int d^4x \ \mathrm{Tr} \sum_{k=1}^{3} \Big[\sum_{a=0}^{2} b_{ak} (\mathcal{D}_a \varphi_k)^2 + b_{\tilde{\psi}k} (\mathcal{D}_{\tilde{\psi}} \varphi_k)^2 \Big] \tag{4.116}$$

where, once more, $d^4x \equiv dt dx_1 dx_2 d\tilde{\psi}$ and the coefficients $(b_{ak},\, b_{\tilde{\psi}k})$ are defined as

$$b_{ak} \equiv e^{2\phi_0} \int d^4\tilde{\zeta} H_1 \sqrt{F_1 H_4}, \qquad b_{\tilde{\psi}k} \equiv \int d^4\tilde{\zeta} H_1 \sqrt{\frac{F_1}{H_4}}. \tag{4.117}$$

Further introducing (2.43) in the above and noting that the integrands are independent of $(\tilde{x}_3,\, \phi_1)$, these coefficients considerably simplify to

$$\begin{aligned} b_{ak} = &\quad e^{2\phi_0} R_3 \sec\theta_{nc} \int_0^\infty dr \ (\cos^2\theta_{nc} + F_2 \sin^2\theta_{nc})^{1/3} F_3^{1/3} \sqrt{F_1 \tilde{F}_2}\, \mathcal{I}^{(9)}, \\ b_{\tilde{\psi}k} = &\quad R_3 \sec\theta_{nc} \int_0^\infty dr \ (\cos^2\theta_{nc} + F_2 \sin^2\theta_{nc})^{1/3} F_3^{1/3} \sqrt{\frac{F_1}{\tilde{F}_2}}\, \mathcal{I}^{(10)}, \end{aligned} \tag{4.118}$$

with the integrals there appearing defined as

$$\mathcal{I}^{(9)} \equiv \int_0^\pi d\theta_1 \, \frac{\sin\theta_1}{\hat{\chi}^{1/6}}, \qquad \mathcal{I}^{(10)} \equiv \int_0^\pi d\theta_1 \, \frac{\hat{\chi}^{5/6}}{\sin\theta_1}, \qquad \hat{\chi} = \hat{\chi}(\theta_1) \equiv 1 + \frac{\tilde{F}_2 - F_3}{F_3} \cos^2\theta_1. \tag{4.119}$$

These integrals are most easily performed after doing the by now familiar change of variables in (4.33). For $\mathcal{I}^{(9)}$ we obtain

$$\mathcal{I}^{(9)} = \int_{-1}^{1} dz \Big(1 + \frac{\tilde{F}_2 - F_3}{F_3} z^2 \Big)^{-1/6} = z \ {}_2F_1\Big(\frac{1}{6}, \frac{1}{2}, \frac{3}{2}; \frac{F_3 - \tilde{F}_2}{F_3} z^2 \Big) \Bigg|_{z=-1}^{z=1} = 2\Theta_{12}. \tag{4.120}$$

Similarly, using (4.33), introducing the regularization factor $b \in \mathbb{R}^+ - \{1\}$ in the same way as in (4.34) previously and further changing variables $z \to \hat{z} = z^2$, the integral $\mathcal{I}^{(10)}$ yields

$$\begin{aligned}\mathcal{I}^{(10)} =& 2\int_0^1 \frac{dz}{b^2 - z^2}\Big(1 + \frac{\tilde{F}_2 - F_3}{F_3} z^2\Big)^{5/6} = \int_0^1 \frac{d\hat{z}}{\sqrt{\hat{z}}}\frac{1}{b^2 - \hat{z}}\Big(1 + \frac{\tilde{F}_2 - F_3}{F_3}\hat{z}\Big)^{5/6} \\ =& \frac{2\sqrt{\hat{z}}}{b^2} F_1\Big(\frac{1}{2}, -\frac{5}{6}, 1, \frac{3}{2}; \frac{F_3 - \tilde{F}_2}{F_3}\hat{z}; \frac{\hat{z}}{b^2}\Big)\Bigg|_{\hat{z}=0}^{\hat{z}=1} = \frac{2}{b^2}\Theta_{34},\end{aligned} \tag{4.121}$$

where $(\Theta_{12}, \Theta_{34})$ above stand for the following hypergeometric functions:

$$\Theta_{12} \equiv {}_2F_1\Big(\frac{1}{6}, \frac{1}{2}, \frac{3}{2}; \frac{F_3 - \tilde{F}_2}{F_3}\Big), \qquad \Theta_{34} \equiv F_1\Big(\frac{1}{2}, -\frac{5}{6}, 1, \frac{3}{2}; \frac{F_3 - \tilde{F}_2}{F_3}; \frac{1}{b^2}\Big). \tag{4.122}$$

Putting everything together, we obtain

$$\begin{aligned} b_{ak} =& \quad 2e^{2\phi_0} R_3 \sec\theta_{nc} \int_0^\infty dr(\cos^2\theta_{nc} + F_2\sin^2\theta_{nc})^{1/3} F_3^{1/3}\sqrt{F_1\tilde{F}_2}\,\Theta_{12}, \\ b_{\tilde{\psi}k} =& \quad 2\frac{R_3}{b^2}\cos\theta_{nc}\int_0^\infty dr(\cos^2\theta_{nc} + F_2\sin^2\theta_{nc})^{1/3} F_3^{1/3}\sqrt{\frac{F_1}{\tilde{F}_2}}\,\Theta_{34}. \end{aligned} \tag{4.123}$$

Recalling the constraint $\tilde{F}_2 \geq F_3$ of section 4.1, the reader will not have a hard time of convincing herself/himself that the above two coefficients are well-defined numbers for any choice of warp factors in (2.2).

Computation of $S_{int}^{(\varphi\varphi)}$ in (4.115)

The determination of this self-interaction term is a simplified version of the computation we just presented for the kinetic term. As in there, all boils down to explicitly writing the integral over $X_4 \otimes \Sigma_3$ in (4.115) with the aid of (4.17) and (4.24):

$$S_{int}^{(\varphi\varphi)} = \int d^4x \ \mathrm{Tr}\sum_{k,l=1}^{3} d_{kl}[\varphi_k, \varphi_l]^2, \qquad d_{kl} \equiv e^{2\phi_0}\int d^4\tilde{\zeta} H_1^2\sqrt{F_1 H_4} \qquad \forall k, l = 1, 2, 3, \tag{4.124}$$

with $d^4x \equiv dtdx_1dx_2d\tilde{\psi}$. With regards to the determination of the d_{kl} coefficients, the first step is to use (2.43) and carry out the trivial $(\tilde{x}_3,\, \phi_1)$ integrals. We then find that

$$d_{kl} = e^{2\phi_0} R_3 \sec\theta_{nc} \int_0^\infty dr (\cos^2\theta_{nc} + F_2 \sin^2\theta_{nc})^{2/3} \sqrt{F_1 \tilde{F}_2 F_3}\,\mathcal{I}^{(11)} \qquad \forall k, l = 1, 2, 3, \tag{4.125}$$

where we have defined

$$\mathcal{I}^{(11)} \equiv F_3^{1/6} \int_0^\pi d\theta_1\ \sin\theta_1 \hat{\chi}^{1/6}, \tag{4.126}$$

and $\hat{\chi}$ is as in (4.119). Given the similarity between the above and $(\mathcal{I}^{(9)},\, \mathcal{I}^{(10)})$ before, the attentive reader may already have guessed that the easiest way to perform the above integral is by doing the change of variables in (4.33):

$$\begin{aligned} F_3^{-1/6}\mathcal{I}^{(11)} &= \int_{-1}^{1} dz \Big(1 + \frac{\tilde{F}_2 - F_3}{F_3} z^2\Big)^{1/6} \\ &= \frac{3z}{4}\Big(1 + \frac{\tilde{F}_2 - F_3}{F_3} z^2\Big)^{1/6} + \frac{z}{4} {}_2F_1\Big(\frac{1}{2}, \frac{5}{6}, \frac{3}{2}; \frac{F_3 - \tilde{F}_2}{F_3} z^2\Big)\Bigg|_{z=-1}^{z=1} = \frac{\Theta_{56}}{2F_3^{1/6}}, \end{aligned} \tag{4.127}$$

where Θ_{56} is

$$\Theta_{56} \equiv 3\tilde{F}_2^{1/6} + F_3^{1/6}\ {}_2F_1\Big(\frac{1}{2}, \frac{5}{6}, \frac{3}{2}; \frac{F_3 - \tilde{F}_2}{F_3}\Big). \tag{4.128}$$

As a result, we can write the d_{kl} coefficients as

$$d_{kl} = \frac{e^{2\phi_0}}{2} R_3 \sec\theta_{nc} \int_0^\infty dr \sqrt{F_1 \tilde{F}_2 F_3} (\cos^2\theta_{nc} + F_2 \sin^2\theta_{nc})^{2/3} \Theta_{56} \qquad \forall k, l = 1, 2, 3, \tag{4.129}$$

which are all just some real number whatever choice of warp factors one may wish to consider in (2.2).

Computation of $S_{int}^{(\mathcal{A}\varphi)}$ in (4.115)

The final term to be computed, namely the interaction term between the two sets of three real scalars $\mathcal{A}^{(\Sigma_3)}$ and φ_k (k = 1, 2, 3), is mathematically more involved than its previous two counterparts. Hence, let us first take a few preparatory steps. From (4.49) and (4.52) it follows that

$$\begin{aligned}[\mathcal{A}^{(\Sigma_3)},\varphi_k] =&[\hat{\alpha}_1,\varphi_k]\, e^{(\Sigma_3)}_{\tilde{3}} + [\hat{\alpha}_2,\varphi_k]\, e^{(\Sigma_3)}_{r} + [\hat{\alpha}_3,\varphi_k]\, e^{(\Sigma_3)}_{\phi_1},\\ *[\mathcal{A}^{(\Sigma_3)},\varphi_k] =&[\hat{\alpha}_1,\varphi_k]\, e^{(\Sigma_3)}_{r}\wedge e^{(\Sigma_3)}_{\phi_1} - [\hat{\alpha}_2,\varphi_k]\, e^{(\Sigma_3)}_{\tilde{3}}\wedge e^{(\Sigma_3)}_{\phi_1} + [\hat{\alpha}_3,\varphi_k]\, e^{(\Sigma_3)}_{\tilde{3}}\wedge e^{(\Sigma_3)}_{r},\end{aligned} \tag{4.130}$$

the Hodge dual having been taken with respect to (4.47). The wedge product between the above two quantities is then

$$[\mathcal{A}^{(\Sigma_3)},\varphi_k]\wedge *[\mathcal{A}^{(\Sigma_3)},\varphi_k] = ([\hat{\alpha}_1,\varphi_k]^2 + [\hat{\alpha}_2,\varphi_k]^2 + [\hat{\alpha}_3,\varphi_k]^2)\, e^{(\Sigma_3)}_{\tilde{3}}\wedge e^{(\Sigma_3)}_{r} e^{(\Sigma_3)}_{\phi_1}. \tag{4.131}$$

Since $H_1^3 H_2 H_3 = 1$, as a direct consequence of our definitions in (2.43), and reversing (4.48) and (4.52), the above can be rewritten in the more convenient form $\mathcal{K}_{\tilde{3}r\phi_1} d\tilde{x}_3\wedge dr\wedge d\phi_1$, with $\mathcal{K}_{\tilde{3}r\phi_1}$ given by

$$e^{\phi_0}\frac{\sqrt{F_1}}{H_1}\Big\{\frac{e^{-2\phi_0}}{F_1}[\mathcal{A}_r,\varphi_k]^2 + \frac{1}{H_2}[\mathcal{A}_{\tilde{3}},\varphi_k]^2 + \Big(\frac{f_3^2}{H_2}+\frac{1}{H_3}\Big)[\mathcal{A}_{\phi_1},\varphi_k]^2 - \frac{2f_3}{H_2}[\mathcal{A}_{\tilde{3}},\varphi_k][\mathcal{A}_{\phi_1},\varphi_k]\Big\}. \tag{4.132}$$

This is nothing but the integrand of $S^{(\mathcal{A}\varphi)}_{int}$ in (4.115). Once we substitute it there and after expanding the integral over $X_4\otimes\Sigma_3$ as well as using (4.20) and (4.24), we get the interaction term to be

$$S^{(\mathcal{A}\varphi)}_{int} = \int d^4x\ \mathrm{Tr}\sum_{k=1}^{3}\big(c_{rk}[\mathcal{A}_r,\varphi_k]^2 + c_{\tilde{3}k}[\mathcal{A}_{\tilde{3}},\varphi_k]^2 + c_{\phi_1 k}[\mathcal{A}_{\phi_1},\varphi_k]^2 - c_{kk}[\mathcal{A}_{\tilde{3}},\varphi_k][\mathcal{A}_{\phi_1},\varphi_k]\big)\,. \tag{4.133}$$

The four coefficients above (and these are the very last ones of their kind) are defined as

$$\begin{aligned} c_{rk} &\equiv \int d^4\tilde{\zeta}\ H_1\sqrt{\frac{H_4}{F_1}}, & c_{\tilde{3}k} &\equiv e^{2\phi_0}\int d^4\tilde{\zeta}\ \frac{H_1}{H_2}\sqrt{F_1 H_4},\\ c_{\phi_1 k} &\equiv e^{2\phi_0}\int d^4\tilde{\zeta}\ H_1\sqrt{F_1H_4}\left(\frac{f_3^2}{H_2}+\frac{1}{H_3}\right), & c_{kk} &\equiv 2e^{2\phi_0}\int d^4\tilde{\zeta}\ \frac{f_3}{H_2}\sqrt{F_1H_4}.\end{aligned} \tag{4.134}$$

Introducing (2.43) and carrying out the trivial ($\tilde{x}_3$, ϕ_1) integrals, these coefficients simplify notoriously to yield

$$
\begin{aligned}
c_{rk} =& 2R_3 \sec\theta_{nc} \int_0^\infty dr\ F_3^{1/3}\sqrt{\frac{\tilde{F}_2}{F_1}}(\cos^2\theta_{nc} + F_2\sin^2\theta_{nc})^{1/3}\Theta_{12}, \\
c_{\tilde{3}k} =& 2e^{2\phi_0} R_3 \sec\theta_{nc} \int_0^\infty dr\ F_3^{1/3}\sqrt{F_1\tilde{F}_2}(\cos^2\theta_{nc} + F_2\sin^2\theta_{nc})^{4/3}\Theta_{12}, \\
c_{\phi_1 k} =& e^{2\phi_0} R_3 \sec\theta_{nc} \int_0^\infty dr\ \sqrt{F_1\tilde{F}_2 F_3}(\cos^2\theta_{nc} + F_2\sin^2\theta_{nc})^{1/3}\Pi_{78}
\end{aligned}
\tag{4.135}
$$

and $c_{kk} \propto \mathcal{I}^{(5)}$, with $\mathcal{I}^{(5)}$ defined in (4.63). Note that in the case of $(c_{rk},\, c_{\tilde{3}k})$ we have also integrated over θ_1, using to this aim (4.119), (4.120) and (4.122). Additionally, we have defined Π_{78} as

$$
\Pi_{78} \equiv \hat{\Pi}_{78} + 3\sec^2\theta_{nc}\tan^2\theta_{nc}\tilde{F}_2^2(\cos^2\theta_{nc} + F_2\sin^2\theta_{nc})\tilde{\Pi}_{78}, \tag{4.136}
$$

with $(\hat{\Pi}_{78},\, \tilde{\Pi}_{78})$ depending on the $\hat{\chi}$ function in (4.119) as

$$
\hat{\Pi}_{78} \equiv F_3^{5/6}\int_0^\pi d\theta_1\ \sin\theta_1\hat{\chi}^{5/6}, \qquad \tilde{\Pi}_{78} \equiv \frac{1}{3F_3^{1/6}}\int_0^\pi d\theta_1\ \frac{\sin\theta_1\cos^2\theta_1}{\hat{\chi}^{1/6}}. \tag{4.137}
$$

Once more, these integrals are most easily carried out after doing the change of variables in (4.33). For $\hat{\Pi}_{78}$ we get

$$
\begin{aligned}
F_3^{-5/6}\hat{\Pi}_{78} =& \int_{-1}^1 dz\Big(1 + \frac{\tilde{F}_2 - F_3}{F_3}z^2\Big)^{5/6} = \frac{3z}{8}\Big(1 + \frac{\tilde{F}_2 - F_3}{F_3}z^2\Big)^{5/6} \\
&+ \frac{5z}{8}\,{}_2F_1\Big(\frac{1}{6},\frac{1}{2},\frac{3}{2};\frac{F_3 - \tilde{F}_2}{F_3}z^2\Big)\Big|_{z=-1}^{z=1} = \frac{3}{4}\big(\frac{\tilde{F}_2}{F_3}\big)^{5/6} + \frac{5}{4}\Theta_{12},
\end{aligned}
\tag{4.138}
$$

where in the last step we have made use of (4.122). Similarly, $\tilde{\Pi}_{78}$ gives

$$
\begin{aligned}
F_3^{-5/6}\tilde{\Pi}_{78} =& \frac{1}{3F_3}\int_{-1}^1 dz\ z^2\Big(1 + \frac{\tilde{F}_2 - F_3}{F_3}z^2\Big)^{-1/6} = \frac{z}{8(\tilde{F}_2 - F_3)}\Big[\Big(1 + \frac{\tilde{F}_2 - F_3}{F_3}z^2\Big)^{5/6} \\
&- {}_2F_1\Big(\frac{1}{6},\frac{1}{2},\frac{3}{2};\frac{F_3 - \tilde{F}_2}{F_3}z^2\Big)\Big]_{z=-1}^{z=1} = \frac{\Big(\tilde{F}_2/F_3\Big)^{5/6} - \Theta_{12}}{4(\tilde{F}_2 - F_3)}.
\end{aligned}
\tag{4.139}
$$

The above two results, when used in (4.136), allow us to write Π_{78} as

$$\Pi_{78} = \frac{3}{4}\tilde{F}_2^{5/6} + \frac{5}{4}F_3^{5/6}\Theta_{12} + \frac{3}{4}\Big(\frac{\tan\theta_{nc}}{\cos\theta_{nc}}\Big)^2(\cos^2\theta_{nc} + F_2\sin^2\theta_{nc})(\tilde{F}_2^{5/6} - F_3^{5/6}\Theta_{12})\frac{\tilde{F}_2^2}{\tilde{F}_2 - F_3}. \tag{4.140}$$

As we saw in (4.64), $\mathcal{I}^{(5)} = 0$ and so the coefficient c_{kk} vanishes. This reduces our interaction term in (4.133) to its final form:

$$S_{int}^{(\mathcal{A}\varphi)} = \int d^4x \,\mathrm{Tr}\sum_{k=1}^{3}\left(c_{rk}[\mathcal{A}_r,\varphi_k]^2 + c_{\tilde{3}k}[\mathcal{A}_{\tilde{3}},\varphi_k]^2 + c_{\phi_1 k}[\mathcal{A}_{\phi_1},\varphi_k]^2\right). \tag{4.141}$$

For the very last time, we observe that the coefficients appearing above are, as a simple inspection of their form in (4.135) suggests, well-defined numbers for any choice of the warp factors one may wish to consider in (2.2). Just to make the entire analysis transparent, we show that the only seemingly divergent term is actually finite. Defining $\epsilon \equiv (\tilde{F}_2 - F_3)$, we have that

$$\lim_{\tilde{F}_2\to F_3}\tilde{\Pi}_{78} = \lim_{\epsilon\to 0}\frac{(F_3+\epsilon)^{5/6} - F_3^{5/6}}{4\epsilon} \approx \frac{5}{24F_3^{1/6}}, \tag{4.142}$$

a finite result as we advanced a little ago. Recall that $F_3 \to 0$ cannot be considered in this case, as we explained after (4.38) earlier on.

It is now the time to collect all our results in this section. First, we introduce all (4.116), (4.124) and (4.141) in (4.114). We then obtain the third and last term $S^{(3)}$ for our gauge theory action. At last, adding such $S^{(3)}$ to $S^{(1)}$ in (4.99) and $S^{(2)}$ in (4.108), we obtain the total bosonic action for the world-volume gauge theory in (4.143).

To finish this section, we include table 1. This is a quick guide to finding the explicit form in terms of the warp factors in (2.2), the deformation parameter θ_{nc} in (2.19) and the constant dilaton in (4.5) of the abundant coefficients on which our above action depends. These will keep appearing all through the remaining of the thesis. Recall that we have explicitly shown that all these coefficients are well-defined numbers for any choice of the warp factors, as long as the constraint $\tilde{F}_2 \geq F_3$ is satisfied, with $\tilde{F}_2$ as in (2.21).

Before proceeding ahead in our analysis, it is worth noting that in the present work we do not study the four-dimensional bosonic action stemming from the configuration (M,2) of section 2.2. This is because (M,2) was shown to be equivalent to the configuration (M,1) of sections 2.1 and 2.1.1 (see figure 1), the latter being computationally easier to handle. However, this action is discussed in [1] and argued to be of the form (4.143), the only difference being that the coefficients of table 1 would in that case change. We refer the interested reader to [1] for the pertinent details.

$$
\begin{aligned}
S =& \frac{C_1 c_{11}}{V_3} \int d^4x \sum_{\substack{a,b=0 \\ a<b}}^{2} \text{Tr}(\mathcal{F}_{ab}^2) + \frac{C_1 c_{12}}{V_3} \int d^4x \sum_{a=0}^{2} \text{Tr}(\mathcal{F}_{a\tilde{\psi}}^2) + c_2 \int_{X_4} \text{Tr}\big(\mathcal{F}^{(X_4)} \wedge \mathcal{F}^{(X_4)}\big) \\
&+ \frac{C_1}{V_3} \int d^4x \, \text{Tr}\Big\{ a_1 [\mathcal{A}_r, \mathcal{A}_{\phi_1}]^2 + a_2 [\mathcal{A}_{\tilde{3}}, \mathcal{A}_r]^2 + a_4 [\mathcal{A}_{\tilde{3}}, \mathcal{A}_{\phi_1}]^2 + \sum_{a=0}^{2} \big[c_{a\tilde{3}} (\mathcal{D}_a \mathcal{A}_{\tilde{3}})^2 \\
&+ c_{ar} (\mathcal{D}_a \mathcal{A}_r)^2 + \tilde{c}_{a\phi_1} (\mathcal{D}_a \mathcal{A}_{\phi_1})^2 \big] + c_{\tilde{\psi}\tilde{3}} (\mathcal{D}_{\tilde{\psi}} \mathcal{A}_{\tilde{3}})^2 + c_{\tilde{\psi}r} (\mathcal{D}_{\tilde{\psi}} \mathcal{A}_r)^2 + \tilde{c}_{\tilde{\psi}\phi_1} (\mathcal{D}_{\tilde{\psi}} \mathcal{A}_{\phi_1})^2 \Big\} \\
&+ \int d^4x \, \text{Tr}\Big\{ \sum_{k=1}^{3} \Big[\sum_{a=0}^{2} b_{ak} (\mathcal{D}_a \varphi_k)^2 + b_{\tilde{\psi}k} (\mathcal{D}_{\tilde{\psi}} \varphi_k)^2 \Big] + \sum_{k,l=1}^{3} d_{kl} [\varphi_k, \varphi_l]^2 \\
&+ \sum_{k=1}^{3} \big(c_{rk} [\mathcal{A}_r, \varphi_k]^2 + c_{\tilde{3}k} [\mathcal{A}_{\tilde{3}}, \varphi_k]^2 + c_{\phi_1 k} [\mathcal{A}_{\phi_1}, \varphi_k]^2 \big) \Big\}. \qquad (4.143)
\end{aligned}
$$

(All the coefficients in blue depend on $\mathcal{J}_3$ in (4.29).)

Coefficient	Given in	
c_{11}	(4.30)	
c_{12}	(4.38)	$\longrightarrow$ Depends on (b, b_2, b_3) in (4.31) and $\mathcal{J}_4$ in (4.37).
a_1, a_2, a_4	(4.68)	$\longrightarrow$ Depend on ($\tilde{a}_\pm$, $\tilde{a}_2$, $\tilde{a}_4$) in (4.69).
$c_{a\tilde{3}}$, c_{ar}, $\tilde{c}_{a\phi_1}$	(4.78)	
$c_{\tilde{\psi}\tilde{3}}$, $c_{\tilde{\psi}r}$, $\tilde{c}_{\tilde{\psi}\phi_1}$	(4.91)	$\longrightarrow$ Depend on all the above via (a_{01}, b_{01}, c_{01}) in (4.92), as well as $\tilde{b}_2$ in (4.83) and ($f^{(1)}$, $f^{(2)}$) in (4.93).
c_2	(4.112)	
b_{ak}, $b_{\tilde{\psi}k}$	(4.123)	$\longrightarrow$ Depend on b in (4.31) and (Θ_{12}, Θ_{34}) in (4.122).
d_{kl}	(4.129)	$\longrightarrow$ Depends on Θ_{56} in (4.128).
$c_{\tilde{3}k}$, c_{rk}, $c_{\phi_1 k}$	(4.135)	$\longrightarrow$ Depend on (Θ_{12}, Π_{78}) in (4.122) and (4.140).

Table 1: List of coefficients appearing in the action (4.143), together with the equation numbers where they are expressed in terms of only the warp factors in (2.2) and (2.21), the deformation parameter in (2.19) and the leading constant term of the dilaton in (4.5). Note that we don't compute (C_1/V_3) explicitly. However, its abelian version (c_1/v_3) is given by (4.10). Note also that all the coefficients in blue require $\tilde{F}_2 \geq F_3$ to be finite. The colors in the table point to the origin of the coefficients: in blue those stemming from $S^{(1)}$ discussed in section 4.1, in green that related to $S^{(2)}$ in section 4.2 and in yellow the coefficients of $S^{(3)}$ in section 4.3.

In chapter 4 we have derived the four-dimensional physical action (4.143) following from the low energy limit of the M-theory model (M,1) constructed in chapter 2. We have shown that this action is well-defined everywhere under very mild constraints on the form of the warp factors and dilaton that characterize the metric (2.46) of (M,1). The action (4.143) depends on various coefficients, all of which can be expressed solely in terms of supergravity parameters, as detailed in table 1.

Chapter 5: The bulk theory

This chapter is devoted to the derivation of the BPS conditions for the $\mathcal{N} = 2$, four-dimensional gauge theory along $(t, x_1, x_2, \tilde{\psi})$ whose action we just obtained in (4.143). It goes without saying that the BPS conditions follow from minimizing the energy of the system with action (4.143), considering *static* configurations of the fields there. Hence, it is quite clear that the first step towards achieving our aim will be to obtain the Hamiltonian associated to (4.143). The second and last step will be to minimize this Hamiltonian, under the assumption that the gauge and scalar fields are all time-independent.

Yet once more, this is more easily said than done. Consequently, we will do the following. First, we shall determine and minimize the Hamiltonian following from (4.143) in a particularly simple limit: we will set $c_2 = 0$ there. That is to say, we will begin by performing the analysis when there is no Θ-term in the action. Then, we will use the insights thus gathered to generalize the results to the $c_2 \neq 0$ case we are really interested in.

This procedure is depicted in figure 12, where we also make reference to the main results in the present chapter. As such, the reader may find it useful to look at figure 12 as a guiding map: it captures the main logic behind the computational details shown in the following.

5.1 Analysis for the case $c_2 = 0$ in (4.143)

Obtaining the Hamiltonian associated to a given action is a well-defined problem in classical mechanics, which our readers surely know by heart. As such, after setting $c_2 = 0$ in (4.143), one could go ahead with the standard procedure: infer the conjugate momenta and write the Hamiltonian as the Legendre transform of the Lagrangian. However, in view of the length and complexity of the action (4.143), this procedure would be quite a long and tiresome mathematical exercise for us. Therefore, we will use a different approach to obtain the Hamiltonian: we will map our action to that in (2.1) in [54] and directly read off our Hamiltonian from (2.4) in the same reference. As we will point out when due, it shall pay off to develop such mapping for a number of other reasons as well.

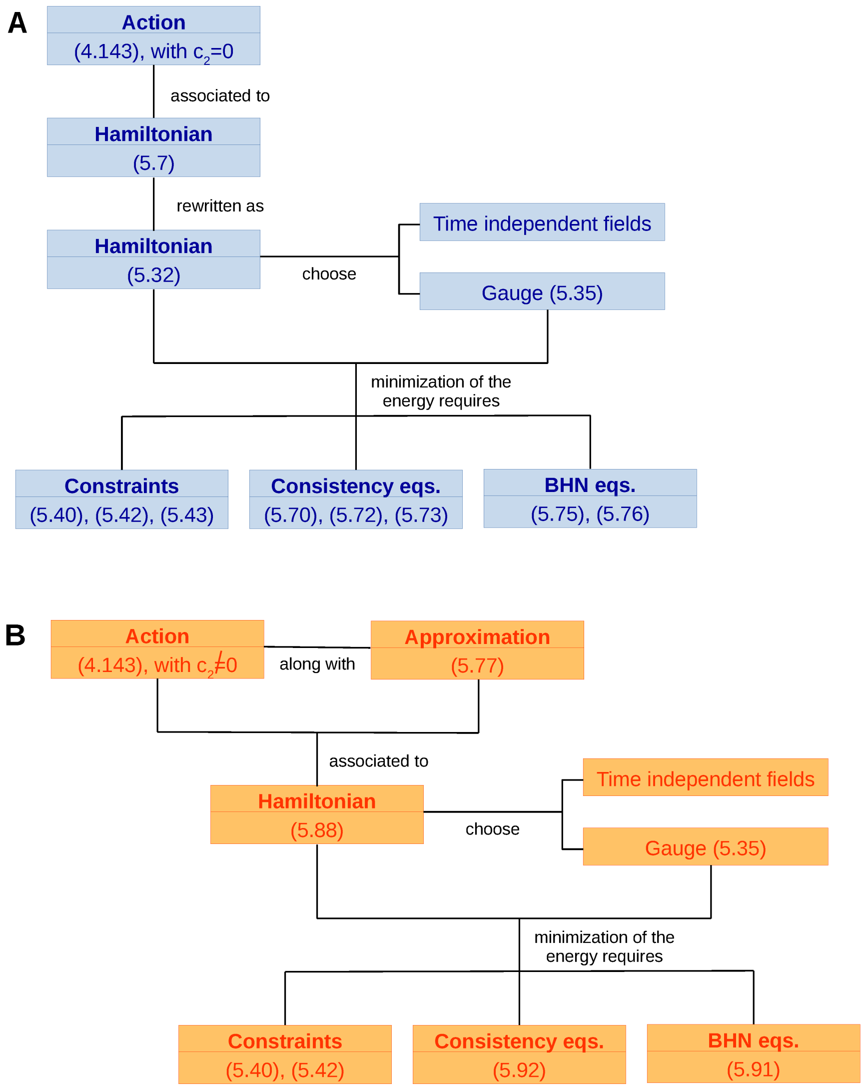

Figure 12: Sketch of the main results in chapter 5, where we obtain the Hamiltonian following from the gauge theory action (4.143) and minimize its energy. As a result, we obtain a set of equations the gauge and scalar fields in the theory must obey. The so-called BHN and consistency equations are particularly important, as they are related to knot invariants. **A:** Since the computation is a bit involved, in section 5.1 this is done in a particularly simple limit: setting $c_2 = 0$ in (4.143). **B:** The generalization to the case of interest, $c_2 \neq 0$ in (4.143), is done in section 5.2 and follows without much effort from the previous analysis.

The Lagrangian density $\mathcal{L}$ of our theory can be directly inferred from the action (4.143), since

$$S = \int d^4x \; \mathcal{L}. \tag{5.1}$$

With $c_2 = 0$, $\mathcal{L}$ in (4.143) is precisely of the form of the Lagrangian (2.1) in [54], up to relative factors and under the following identifications:

$$x_M \to (t, x_1, x_2, \tilde{\psi}), \qquad \phi_A \to (\mathcal{A}_{\tilde{3}}, \mathcal{A}_{\phi_1}, \mathcal{A}_r, \varphi_1, \varphi_2, \varphi_3), \qquad \phi_5 \to \mathcal{A}_{\tilde{3}}. \tag{5.2}$$

Note that our definitions for the covariant derivatives in (4.51) differ from the covariant derivatives in [54]. This mismatch is rectified by replacing factors of (i) there by $(-i)$ in our case. Properly accounting for the additional prefactors in our theory as well, it is rather simple to see that the different terms that compose the Hamiltonian (2.4) in [54] are, in the language of the present paper, given by

$$\begin{gathered} \sum_a (F_{a0} - D_a\phi_5)^2 \to \mathcal{T}_1, \qquad \sum_a (D_0\phi_a + i[\phi_5, \phi_a])^2 \to \mathcal{T}_2, \qquad (D_0\phi_5)^2 \to \mathcal{T}_3, \\ \frac{1}{2}\sum_{a\neq b}(F_{ab} - \epsilon_{abcd}D_c\phi_d + i[\phi_a, \phi_b])^2 \to \mathcal{T}_4, \qquad \Big(\sum_a D_a\phi_a\Big)^2 \to 0, \end{gathered} \tag{5.3}$$

where we have defined $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ as

$$\begin{aligned} \mathcal{T}_1 &\equiv \frac{C_1}{V_3}\sum_{\alpha=1}^{2}(\sqrt{c_{11}}\mathcal{F}_{\alpha 0} - \sqrt{c_{\alpha\tilde{3}}}\mathcal{D}_\alpha\mathcal{A}_{\tilde{3}})^2 + \frac{C_1}{V_3}(\sqrt{c_{12}}\mathcal{F}_{\tilde{\psi}0} - \sqrt{c_{\tilde{\psi}\tilde{3}}}\mathcal{D}_{\tilde{\psi}}\mathcal{A}_{\tilde{3}})^2, \\ \mathcal{T}_2 &\equiv \frac{C_1}{V_3}(\sqrt{c_{0r}}\mathcal{D}_0\mathcal{A}_r - i\sqrt{a_2}[\mathcal{A}_{\tilde{3}}, \mathcal{A}_r])^2 + \frac{C_1}{V_3}(\sqrt{\tilde{c}_{0\phi_1}}\mathcal{D}_0\mathcal{A}_{\phi_1} - i\sqrt{a_4}[\mathcal{A}_{\tilde{3}}, \mathcal{A}_{\phi_1}])^2 \\ &\quad + \sum_{k=1}^{3}(\sqrt{b_{0k}}\mathcal{D}_0\varphi_k - i\sqrt{c_{\tilde{3}k}}[\mathcal{A}_{\tilde{3}}, \varphi_k])^2, \qquad \mathcal{T}_3 \equiv \frac{C_1}{V_3}c_{0\tilde{3}}(\mathcal{D}_0\mathcal{A}_{\tilde{3}})^2 \end{aligned} \tag{5.4}$$

and where $\mathcal{T}_4$ naturally splits into two, $\mathcal{T}_4 = \mathcal{T}_4^{(1)} + \mathcal{T}_4^{(2)}$, due to the decomposition of the subspace X_4 explained in (4.1):

$$\mathcal{T}_4^{(1)} = \frac{1}{2}\sum_{\alpha,\beta=1}^{2}\Big(\sqrt{\frac{C_1}{V_3}}\tau^{(1)} + \tau^{(2)}\Big)^2, \qquad \mathcal{T}_4^{(2)} = \frac{1}{2}\sum_{\alpha=1}^{2}\Big(\sqrt{\frac{C_1}{V_3}}\tau^{(3)} + \tau^{(2)}\Big)^2, \tag{5.5}$$

with ($\tau^{(1)}$, $\tau^{(2)}$, $\tau^{(3)}$) standing for

$$\begin{aligned}
\tau^{(1)} \equiv& \sqrt{c_{11}}\mathcal{F}_{\alpha\beta} - \sqrt{c_{\tilde{\psi}r}}\epsilon_{\alpha\beta\tilde{\psi}r}\mathcal{D}_{\tilde{\psi}}\mathcal{A}_r - \sqrt{\tilde{c}_{\tilde{\psi}\phi_1}}\epsilon_{\alpha\beta\tilde{\psi}\phi_1}\mathcal{D}_{\tilde{\psi}}\mathcal{A}_{\phi_1} - \sqrt{\frac{V_3}{C_1}}\sum_{k=1}^{3}\sqrt{b_{\tilde{\psi}k}}\epsilon_{\alpha\beta\tilde{\psi}k}\mathcal{D}_{\tilde{\psi}}\varphi_k, \\
i\tau^{(2)} \equiv& \sqrt{\frac{C_1 a_1}{V_3}}[\mathcal{A}_r, \mathcal{A}_{\phi_1}] + \sum_{k,l=1}^{3}\left(\sqrt{c_{rk}}[\mathcal{A}_r, \varphi_k] + \sqrt{c_{\phi_1 k}}[\mathcal{A}_{\phi_1}, \varphi_k] + \sqrt{d_{kl}}[\varphi_k, \varphi_l]\right), \qquad (5.6) \\
\tau^{(3)} \equiv& \sqrt{c_{12}}\mathcal{F}_{\alpha\psi} - \sqrt{c_{\beta r}}\epsilon_{\alpha\tilde{\psi}\beta r}\mathcal{D}_\beta\mathcal{A}_r - \sqrt{\tilde{c}_{\beta\phi_1}}\epsilon_{\alpha\tilde{\psi}\beta\phi_1}\mathcal{D}_\beta\mathcal{A}_{\phi_1} - \sqrt{\frac{V_3}{C_1}}\sum_{k=1}^{3}\sqrt{b_{\beta k}}\epsilon_{\alpha\tilde{\psi}\beta k}\mathcal{D}_\beta\varphi_k.
\end{aligned}$$

Putting everything together as in (2.4) in [54], we obtain the Hamiltonian associated to the action (4.143) for $c_2 = 0$:

$$H = \int d^4x \,\mathrm{Tr}\left\{\sum_{i=1}^{3}\mathcal{T}_i + \frac{1}{2}\sum_{\alpha,\beta=1}^{2}\left(\sqrt{\frac{C_1}{V_3}}\tau^{(1)} + \tau^{(2)}\right)^2 + \frac{1}{2}\sum_{\alpha=1}^{2}\left(\sqrt{\frac{C_1}{V_3}}\tau^{(3)} + \tau^{(2)}\right)^2\right\} + Q_{EM}, \qquad (5.7)$$

where Q_{EM} denotes the sum of electric and magnetic charges in the theory. As is well-known (for instance, see (2.5) in [54]), these charges are boundary terms. We will study these boundary terms in exquisite detail in section 6.1, for the case where $c_2 \neq 0$ in (4.143). Hence, for the time being, we shall not make them precise and instead we focus on the bulk terms. Also, it should be noted that the above Hamiltonian incorporates the Gauss law in it, as explained in [54]. Consequently, there are no constraints on the gauge and scalar fields of our theory imposed by the Gauss law[16].

According to the plan of action described in the beginning of this section, having obtained the Hamiltonian for our gauge theory, we should now proceed to minimize it. It turns out, however, that the minimization process simplifies considerably if we first rewrite (5.7) in a certain manner. This is the first side-benefit of having mapped our setup to that in section 2 of [54]. Further, in section 5.1.2 we shall obtain important results from this rewriting! Thus, we will presently simply rewrite the Hamiltonian (5.7) in a more convenient form and postpone the minimization problem to section 5.1.1.

The rewriting we will carry out consists in introducing new, arbitrary coefficients in some of the terms inside the sums of squares of (5.7) and, at the same time, adding new terms to the Hamiltonian so that there is no change in its quadratic components. We shall not yet make precise the additional crossed terms produced in this manner. But the reader should not worry, the crossed terms will be determined meticulously in section 5.1.2. In fact, their

[16]The skeptical reader can alternatively be convinced of this last statement by the combination of (5.2) and our later gauge choice in (5.35) and (5.40).

study leads to the important results we were anticipating a little before. Perhaps a toy model will make the rewriting we intend to perform most transparent. Consider the Hamiltonian

$$H^{(1)} = (\mathbb{A} + \mathbb{B})^2 + \mathbb{C}. \tag{5.8}$$

Introducing the arbitrary parameters $(\hat{x},\, \hat{y})$, the above can be rewritten as

$$H^{(1)} = (\mathbb{A} + \hat{x}\mathbb{B})^2 + \hat{y}\mathbb{B}^2 + \tilde{\mathbb{C}}, \tag{5.9}$$

as long as the constraints

$$\hat{x}^2 + \hat{y} = 1, \qquad \tilde{\mathbb{C}} = \mathbb{C} + 2\mathbb{A}\mathbb{B}(1 - \hat{x}), \tag{5.10}$$

are enforced. Written in this language, our earlier statement of ignoring the "additional crossed terms" simply means that the second constraint above shall not be studied presently, but is rather postponed to section 5.1.2.

Actually, we shall only rewrite the term $\mathcal{T}_4$ and leave $(\mathcal{T}_1,\, \mathcal{T}_2,\, \mathcal{T}_3)$ as they are. We do so piecewise and first focus on the first three terms of $\mathcal{T}_4^{(1)}$ in (5.5):

$$\frac{1}{2}\sum_{\alpha,\beta=1}^{2}\Big(\sqrt{\frac{C_1 c_{11}}{V_3}}\mathcal{F}_{\alpha\beta} - \sqrt{\frac{C_1 c_{\tilde{\psi}r}}{V_3}}\epsilon_{\alpha\beta\tilde{\psi}r}\mathcal{D}_{\tilde{\psi}}\mathcal{A}_r - \sqrt{\frac{C_1 \tilde{c}_{\tilde{\psi}\phi_1}}{V_3}}\epsilon_{\alpha\beta\tilde{\psi}\phi_1}\mathcal{D}_{\tilde{\psi}}\mathcal{A}_{\phi_1} + \dots\Big)^2. \tag{5.11}$$

In the above, we introduce arbitrary coefficients in the second and third terms, which depend on $(\alpha,\, \beta)$. Clearly, these must be antisymmetric in the mentioned indices, so as not to yield zero due to the Levi-Civita symbols. We absorb the minus signs in the coefficients and also transfer the factor of $(1/2)$ inside the square. All in all, we rewrite the above as

$$\begin{aligned}\sum_{\alpha,\beta=1}^{2}&\Big(\sqrt{\frac{C_1 c_{11}}{2V_3}}\mathcal{F}_{\alpha\beta} + \sqrt{\frac{C_1 c_{\tilde{\psi}r}}{V_3}}s^{(1)}_{\alpha\beta}\epsilon_{\alpha\beta\tilde{\psi}r}\mathcal{D}_{\tilde{\psi}}\mathcal{A}_r + \sqrt{\frac{C_1 \tilde{c}_{\tilde{\psi}\phi_1}}{V_3}}s^{(2)}_{\alpha\beta}\epsilon_{\alpha\beta\tilde{\psi}\phi_1}\mathcal{D}_{\tilde{\psi}}\mathcal{A}_{\phi_1} + \dots\Big)^2\\ &+ \frac{C_1 c_{\tilde{\psi}r}}{V_3}s^{(1)}(\mathcal{D}_{\tilde{\psi}}\mathcal{A}_r)^2 + \frac{C_1 \tilde{c}_{\tilde{\psi}\phi_1}}{V_3}s^{(2)}(\mathcal{D}_{\tilde{\psi}}\mathcal{A}_{\phi_1})^2 + \chi_s,\end{aligned} \tag{5.12}$$

where χ_s contains the additional crossed terms created by the inclusion of the $(s^{(1)}_{\alpha\beta},\, s^{(2)}_{\alpha\beta})$ coefficients and we demand the constraints

$$2(s^{(i)}_{12})^2 + s^{(i)} = 1, \qquad \forall i = 1, 2 \tag{5.13}$$

hold true, so as to ensure the quadratic pieces remain the same. In exactly the same way, the first three terms of $\mathcal{T}_4^{(2)}$ in (5.5), namely

$$\frac{1}{2}\sum_{\alpha=1}^{2}\Big(\sqrt{\frac{C_1c_{12}}{V_3}}\mathcal{F}_{\alpha\tilde{\psi}}-\sqrt{\frac{C_1c_{\beta r}}{V_3}}\epsilon_{\alpha\tilde{\psi}\beta r}\mathcal{D}_\beta\mathcal{A}_r-\sqrt{\frac{C_1\tilde{c}_{\beta\phi_1}}{V_3}}\epsilon_{\alpha\tilde{\psi}\beta\phi_1}\mathcal{D}_\beta\mathcal{A}_{\phi_1}+\dots\Big)^2, \tag{5.14}$$

can be rewritten as

$$\sum_{\alpha=1}^{2}\Big(\sqrt{\frac{C_1c_{12}}{2V_3}}\mathcal{F}_{\alpha\tilde{\psi}}+\sqrt{\frac{C_1c_{\beta r}}{V_3}}t_\alpha^{(1)}\epsilon_{\alpha\tilde{\psi}\beta r}\mathcal{D}_\beta\mathcal{A}_r+\sqrt{\frac{C_1\tilde{c}_{\beta\phi_1}}{V_3}}t_\alpha^{(2)}\epsilon_{\alpha\tilde{\psi}\beta\phi_1}\mathcal{D}_\beta\mathcal{A}_{\phi_1}+\dots\Big)^2$$
$$+\frac{C_1c_{\beta r}}{V_3}t^{(1)}(\mathcal{D}_\beta\mathcal{A}_r)^2+\frac{C_1\tilde{c}_{\beta\phi_1}}{V_3}t^{(2)}(\mathcal{D}_\beta\mathcal{A}_{\phi_1})^2+\chi_t, \tag{5.15}$$

where χ_t takes into account the additional crossed terms created by the inclusion of $(t_\alpha^{(1)}, t_\alpha^{(2)})$ and we impose the constraints

$$\sum_{\alpha=1}^{2}(t_\alpha^{(i)})^2+t^{(i)}=1, \qquad \forall i=1,2, \tag{5.16}$$

which guarantee the squared terms are not affected in the rewriting.

With the very same idea in mind, we look at the fifth terms in both $\mathcal{T}_4^{(1)}$ and $\mathcal{T}_4^{(2)}$ next:

$$\frac{1}{2}\sum_{\alpha,\beta=1}^{2}\Big(\dots-i\sqrt{\frac{C_1a_1}{V_3}}[\mathcal{A}_r,\mathcal{A}_{\phi_1}]+\dots\Big)^2+\frac{1}{2}\sum_{\alpha=1}^{2}\Big(\dots-i\sqrt{\frac{C_1a_1}{V_3}}[\mathcal{A}_r,\mathcal{A}_{\phi_1}]+\dots\Big)^2. \tag{5.17}$$

We introduce antisymmetric (in their indices) coefficients in both terms, add squared terms that make sure we do not alter that part and encompass the new crossed terms in χ_4, which we do not presently determine. We also pull in the factor of (1/2), as before. Explicitly, the above becomes

$$\sum_{\alpha,\beta=1}^{2}\Big(\dots-ig_{\alpha\beta}^{(4)}\sqrt{\frac{C_1a_1}{V_3}}[\mathcal{A}_r,\mathcal{A}_{\phi_1}]+\dots\Big)^2+\sum_{\alpha=1}^{2}\Big(\dots-ih_{\alpha\tilde{\psi}}^{(4)}\sqrt{\frac{C_1a_1}{V_3}}[\mathcal{A}_r,\mathcal{A}_{\phi_1}]+\dots\Big)^2$$
$$+\frac{C_1a_1}{V_3}q^{(4)}[\mathcal{A}_r,\mathcal{A}_{\phi_1}]^2+\chi_4, \tag{5.18}$$

where we require that the following must be satisfied:

$$2(g_{12}^{(4)})^2+\sum_{\alpha=1}^{2}(h_{\alpha\tilde{\psi}}^{(4)})^2-q^{(4)}=1. \tag{5.19}$$

The relative difference in signs between (5.19) and the previous constraints (5.13) and (5.16) is a consequence of the overall factors of $(-i)$ in the terms of the action presently being considered. Similarly, the last terms in $\mathcal{T}_4^{(1)}$ and $\mathcal{T}_4^{(2)}$,

$$\frac{1}{2}\sum_{\alpha,\beta=1}^{2}\Big(\ldots - i\sum_{k,l=1}^{3}\sqrt{d_{kl}}[\varphi_k,\varphi_l]\Big)^2 + \frac{1}{2}\sum_{\alpha=1}^{2}\Big(\ldots - i\sum_{k,l=1}^{3}\sqrt{d_{kl}}[\varphi_k,\varphi_l]\Big)^2, \tag{5.20}$$

are rewritten in the form

$$\sum_{\alpha,\beta=1}^{2}\Big(\ldots - i\sum_{k,l=1}^{3}g^{(1)}_{\alpha\beta kl}\sqrt{d_{kl}}[\varphi_k,\varphi_l]\Big)^2 + \sum_{\alpha=1}^{2}\Big(\ldots - i\sum_{k,l=1}^{3}h^{(1)}_{\alpha\tilde{\psi}kl}\sqrt{d_{kl}}[\varphi_k,\varphi_l]\Big)^2$$
$$+\sum_{k,l=1}^{3}q^{(1)}_{kl}d_{kl}[\varphi_k,\varphi_l]^2 + \chi_1, \tag{5.21}$$

with the constraint

$$2(g^{(1)}_{12kl})^2 + \sum_{\alpha=1}^{2}(h^{(1)}_{\alpha\tilde{\psi}kl})^2 - q^{(1)}_{kl} = 1, \qquad \forall k,l = 1,2,3, \tag{5.22}$$

where $g^{(1)}_{\alpha\beta kl}$ has been defined to be antisymmetric in $(\alpha,\,\beta)$ and in $(k,\,l)$. Analogously, we say that $h^{(1)}_{\alpha\tilde{\psi}kl}$ is antisymmetric in $(\alpha,\,\tilde{\psi})$ and in $(k,\,l)$ by definition. We do an identical rewriting of the sixth and seventh terms of $\mathcal{T}_4^{(1)}$ and $\mathcal{T}_4^{(2)}$ too. That is, we rewrite the aforementioned terms,whose original form can be directly read from (5.5) and (5.6) or even simply inferred from the subsequent equation, as

$$\sum_{\alpha,\beta=1}^{2}\Big(\ldots - i\sum_{k=1}^{3}g^{(1)}_{\alpha\beta k}\sqrt{c_{rk}}[\mathcal{A}_r,\varphi_k] - i\sum_{k=1}^{3}g^{(2)}_{\alpha\beta k}\sqrt{c_{\phi_1 k}}[\mathcal{A}_{\phi_1},\varphi_k] + \ldots\Big)^2$$
$$+\sum_{\alpha=1}^{2}\Big(\ldots - i\sum_{k=1}^{3}h^{(1)}_{\alpha\tilde{\psi}k}\sqrt{c_{rk}}[\mathcal{A}_r,\varphi_k] - i\sum_{k=1}^{3}h^{(2)}_{\alpha\tilde{\psi}k}\sqrt{c_{\phi_1 k}}[\mathcal{A}_{\phi_1},\varphi_k] + \ldots\Big)^2$$
$$+\sum_{k=1}^{3}q^{(1)}_k c_{rk}[\mathcal{A}_r,\varphi_k]^2 + \sum_{k=1}^{3}q^{(2)}_k c_{\phi_1 k}[\mathcal{A}_{\phi_1},\varphi_k]^2 + \chi_2 + \chi_3. \tag{5.23}$$

We also demand the following constraints

$$2(g_{12k}^{(i)})^2 + \sum_{\alpha=1}^{2}(h_{\alpha\tilde{\psi}k}^{(i)})^2 - q_k^{(i)} = 1, \qquad \forall i = 1,2, \qquad \forall k = 1,2,3. \tag{5.24}$$

Here, $g_{\alpha\beta k}^{(i)}$ has been defined to be antisymmetric in $(\alpha,\, \beta)$ and $h_{\alpha\tilde{\psi}k}^{(i)}$ in $(\alpha,\, \tilde{\psi})$, for both $i = 1,\, 2$.

The only two terms left, namely the fourth terms of $\mathcal{T}_4^{(1)}$ and $\mathcal{T}_4^{(2)}$ in (5.5), will be rewritten in a slightly trickier way. Essentially, we will first "mix" them and then multiply those mixed terms with new coefficients. Again, we will make sure that the squared terms are not affected in the rewriting by subjecting the coefficients introduced to constraint equations. For the time being, we will not determine the additional crossed terms thus produced. To make the idea more precise, let us first consider a toy model to illustrate how we will proceed. Consider the Hamiltonian

$$H^{(2)} = \frac{1}{2}(\hat{\mathbb{A}} + \hat{\mathbb{B}})^2 + \frac{1}{2}(\hat{\mathbb{C}} + \hat{\mathbb{D}})^2 = \frac{1}{2}(\hat{\mathbb{A}}^2 + \hat{\mathbb{B}}^2 + \hat{\mathbb{C}}^2 + \hat{\mathbb{D}}^2) + \text{ crossed terms.} \tag{5.25}$$

We will "mix" the terms $(\hat{\mathbb{B}},\, \hat{\mathbb{D}})$ in the above. To this aim, we define $\hat{\mathbb{E}} \equiv \hat{\mathbb{B}} + \hat{\mathbb{D}}$. Next, we insert inside the squares the factors of $(1/2)$ and introduce the arbitrary coefficients $(\hat{u},\, \hat{v})$. All these changes allow us to rewrite the toy Hamiltonian as

$$H^{(2)} = (\frac{\hat{\mathbb{A}}}{\sqrt{2}} + \hat{u}\hat{\mathbb{E}})^2 + (\frac{\hat{\mathbb{C}}}{\sqrt{2}} + \hat{v}\hat{\mathbb{E}})^2 = \frac{1}{2}(\hat{\mathbb{A}}^2 + \hat{\mathbb{C}})^2 + (\hat{u}^2 + \hat{v}^2)(\hat{\mathbb{B}}^2 + \hat{\mathbb{D}}^2) + \text{ crossed terms.} \tag{5.26}$$

If we demand that the squared terms in (5.25) and (5.26) match, then it is clear that $(\hat{u},\, \hat{v})$ must satisfy the following constraint:

$$\hat{u}^2 + \hat{v}^2 = \frac{1}{2}. \tag{5.27}$$

Coming back to the fourth terms in $\mathcal{T}_4^{(1)}$ and $\mathcal{T}_4^{(2)}$ that motivated the just explained toy model, these are given by

$$\frac{1}{2}\sum_{\alpha,\beta=1}^{2}\Big(\ldots - \sum_{k=1}^{3}\sqrt{b_{\tilde{\psi}k}}\epsilon_{\alpha\beta\tilde{\psi}k}\mathcal{D}_{\tilde{\psi}}\varphi_k + \ldots\Big)^2 + \frac{1}{2}\sum_{\alpha=1}^{2}\Big(\ldots - \sum_{k=1}^{3}\sqrt{b_{\beta k}}\epsilon_{\alpha\tilde{\psi}\beta k}\mathcal{D}_{\beta}\varphi_k + \ldots\Big)^2. \tag{5.28}$$

Following the logic above exposed, we introduce $\delta \equiv (\alpha,\, \tilde{\psi})$ and rewrite (5.28) as

$$\sum_{\alpha,\beta=1}^{2}\Big(\ldots+\sum_{\delta,k=1}^{3}\sqrt{b_{\delta k}}\epsilon_{\alpha\beta}\cdot m^{(1)}_{\delta k}\mathcal{D}_\delta\varphi_k+\ldots\Big)^2+\sum_{\alpha=1}^{2}\Big(\ldots+\sum_{\delta,k=1}^{3}\sqrt{b_{\delta k}}\epsilon_{\alpha\tilde{\psi}}\cdot m^{(2)}_{\delta k}\mathcal{D}_\delta\varphi_k+\ldots\Big)^2, \tag{5.29}$$

plus some extra crossed terms which we shall refer to symbolically as χ_m. The dot products appearing above will be made precise soon enough, in section 5.1.1. The new coefficients above must satisfy

$$\sum_{i=1}^{2}(m^{(i)}_{\delta k})^2=\frac{1}{2}, \qquad \forall\delta, k=1,2,3, \tag{5.30}$$

which makes sure the quadratic terms have not been changed during the rewriting. Note that there is *no* antisymmetry relating the indices of these coefficients, unlike in all previous cases.

We are now ready to collect results and present the Hamiltonian following from the action (4.143), with $c_2=0$, in the most convenient form for our subsequent investigations. Appropriately summing (5.12), (5.15), (5.18), (5.21), (5.23) and (5.29) we obtain the desired rewriting of $\mathcal{T}_4$ in (5.5). As a short-hand notation, let us introduce $(y_2,\, y_3)\equiv(r,\, \phi_1)$ and

$$\chi_T\equiv\chi_s+\chi_t+\chi_4+\chi_1+\chi_2+\chi_3+\chi_m. \tag{5.31}$$

That is, χ_T accounts for all crossed terms produced when rewriting $\mathcal{T}_4$ as just explained. χ_T will be the main object of study of section 5.1.2, but presently we shall not shed light into it. If to the rewritten $\mathcal{T}_4$ we further add $(\mathcal{T}_1,\, \mathcal{T}_2,\, \mathcal{T}_3)$ as given by (5.4), then the Hamiltonian in (5.7) can be expressed as (5.32).

We remind the reader that most of the notation used in (5.7) was introduced in chapter 4. In particular, table 1 provides a quick guide to find the explicit form of the prefactors that have a supergravity interpretation in terms of the warp factors in (2.2) and (2.21), the deformation parameter θ_{nc} in (2.19) and the leading term of the dilaton in (4.5). For clarity and completeness, we include table 2, which summarizes the form and properties of the new coefficients introduced in going from (4.143) to (5.32). Note that these coefficients do *not* admit a supergravity interpretation. Instead, the constraint relations we demanded they satisfy should be regarded as their *defining* equations. These are (5.13), (5.16), (5.19), (5.22), (5.24) and (5.30). Nearly all coefficients with indices in table 2 fulfill antisymmetry properties. Nonetheless, note that the $\{m^{(i)}\}$'s are *not* constrained by any such requirement.

$$
\begin{aligned}
H = \quad & Q_{EM} + \int d^4x \ \mathrm{Tr}\Bigg\{ \frac{C_1}{V_3}\Bigg[\sum_{\alpha=1}^{2} (\sqrt{c_{11}}\mathcal{F}_{\alpha 0} - \sqrt{c_{\alpha\tilde{3}}}\mathcal{D}_\alpha \mathcal{A}_{\tilde{3}})^2 + (\sqrt{c_{12}}\mathcal{F}_{\tilde{\psi}0} - \sqrt{c_{\tilde{\psi}\tilde{3}}}\mathcal{D}_{\tilde{\psi}}\mathcal{A}_{\tilde{3}})^2 \\
& + (\sqrt{c_{0r}}\mathcal{D}_0\mathcal{A}_r - i\sqrt{a_2}[\mathcal{A}_{\tilde{3}}, \mathcal{A}_r])^2 + (\sqrt{\tilde{c}_{0\phi_1}}\mathcal{D}_0\mathcal{A}_{\phi_1} - i\sqrt{a_4}[\mathcal{A}_{\tilde{3}}, \mathcal{A}_{\phi_1}])^2 + c_{0\tilde{3}}(\mathcal{D}_0\mathcal{A}_{\tilde{3}})^2 \Bigg] \\
& + \sum_{k,l=1}^{3} \Bigg[(\sqrt{b_{0k}}\mathcal{D}_0\varphi_k - i\sqrt{c_{\tilde{3}k}}[\mathcal{A}_{\tilde{3}}, \varphi_k])^2 + q_{kl}^{(1)} d_{kl}[\varphi_k, \varphi_l]^2 + \sum_{\gamma=2}^{3} q_k^{(\gamma-1)} c_{y_\gamma k}[\mathcal{A}_{y_\gamma}, \varphi_k]^2 \Bigg] \\
& + \sum_{\alpha,\beta=1}^{2} \Bigg(\sqrt{\frac{C_1 c_{11}}{2V_3}}\mathcal{F}_{\alpha\beta} + \sqrt{\frac{C_1 c_{\tilde{\psi}r}}{V_3}} s_{\alpha\beta}^{(1)} \epsilon_{\alpha\beta\tilde{\psi}r}\mathcal{D}_{\tilde{\psi}}\mathcal{A}_r + \sqrt{\frac{C_1 \tilde{c}_{\tilde{\psi}\phi_1}}{V_3}} s_{\alpha\beta}^{(2)} \epsilon_{\alpha\beta\tilde{\psi}\phi_1}\mathcal{D}_{\tilde{\psi}}\mathcal{A}_{\phi_1} \\
& - i g_{\alpha\beta}^{(4)} \sqrt{\frac{C_1 a_1}{V_3}}[\mathcal{A}_r, \mathcal{A}_{\phi_1}] - i \sum_{k,l=1}^{3} g_{\alpha\beta kl}^{(1)} \sqrt{d_{kl}}[\varphi_k, \varphi_l] - i \sum_{k=1}^{3}\sum_{\gamma=2}^{3} g_{\alpha\beta k}^{(\gamma-1)} \sqrt{c_{y_\gamma k}}[\mathcal{A}_{y_\gamma}, \varphi_k] \\
& + \sum_{\delta,k=1}^{3} \sqrt{b_{\delta k}}\epsilon_{\alpha\beta} \cdot m_{\delta k}^{(1)} \mathcal{D}_\delta \varphi_k \Bigg)^2 + \sum_{\alpha=1}^{2} \Bigg(\sqrt{\frac{C_1 c_{12}}{2V_3}}\mathcal{F}_{\alpha\tilde{\psi}} + \sqrt{\frac{C_1 c_{\beta r}}{V_3}} t_\alpha^{(1)} \epsilon_{\alpha\tilde{\psi}\beta r}\mathcal{D}_\beta \mathcal{A}_r \\
& + \sqrt{\frac{C_1 \tilde{c}_{\beta\phi_1}}{V_3}} t_\alpha^{(2)} \epsilon_{\alpha\tilde{\psi}\beta\phi_1}\mathcal{D}_\beta\mathcal{A}_{\phi_1} - i h_{\alpha\tilde{\psi}}^{(4)} \sqrt{\frac{C_1 a_1}{V_3}}[\mathcal{A}_r, \mathcal{A}_{\phi_1}] - i \sum_{k,l=1}^{3} h_{\alpha\tilde{\psi}kl}^{(1)} \sqrt{d_{kl}}[\varphi_k, \varphi_l] \\
& - i \sum_{k=1}^{3}\sum_{\gamma=2}^{3} h_{\alpha\tilde{\psi}k}^{(\gamma-1)} \sqrt{c_{y_\gamma k}}[\mathcal{A}_{y_\gamma}, \varphi_k] + \sum_{\delta,k=1}^{3} \sqrt{b_{\delta k}}\epsilon_{\alpha\tilde{\psi}} \cdot m_{\delta k}^{(2)} \mathcal{D}_\delta\varphi_k \Bigg)^2 + \frac{C_1}{V_3}\Bigg[c_{\tilde{\psi}r} s^{(1)} (\mathcal{D}_{\tilde{\psi}}\mathcal{A}_r)^2 \\
& + \tilde{c}_{\tilde{\psi}\phi_1} s^{(2)} (\mathcal{D}_{\tilde{\psi}}\mathcal{A}_{\phi_1})^2 + c_{\beta r} t^{(1)} (\mathcal{D}_\beta \mathcal{A}_r)^2 + \tilde{c}_{\beta\phi_1} t^{(2)} (\mathcal{D}_\beta\mathcal{A}_{\phi_1})^2 + a_1 q^{(4)} [\mathcal{A}_r, \mathcal{A}_{\phi_1}]^2 \Bigg] + \chi_T \Bigg\}.
\end{aligned}
\tag{5.32}
$$

Coefficient	Given in
$s_{\alpha\beta}^{(i)}$, $s^{(i)}$	(5.13)
$t_\alpha^{(i)}$, $t^{(i)}$	(5.16)
$g_{\alpha\beta}^{(4)}$, $h_{\alpha\tilde{\psi}}^{(4)}$, $q^{(4)}$	(5.19)
$g_{\alpha\beta kl}^{(1)}$, $h_{\alpha\tilde{\psi}kl}^{(1)}$, $q_{kl}^{(1)}$	(5.22)
$g_{\alpha\beta k}^{(i)}$, $h_{\alpha\tilde{\psi}k}^{(i)}$, $q_k^{(i)}$	(5.24)
$m_{\delta k}^{(i)}$	(5.30)

with: α, β, $i = 1, 2$ and k, l, $\delta = 1, 2, 3$,

$(s_{\alpha\beta}^{(i)}, g_{\alpha\beta}^{(4)}, g_{\alpha\beta kl}^{(1)}, g_{\alpha\beta k}^{(i)})$ antisymmetric in (α, β),

$(h_{\alpha\tilde{\psi}}^{(4)}, h_{\alpha\tilde{\psi}kl}^{(1)}, h_{\alpha\tilde{\psi}k}^{(i)})$ antisymmetric in $(\alpha, \tilde{\psi})$ and

$(g_{\alpha\beta kl}^{(1)}, h_{\alpha\tilde{\psi}kl}^{(1)}, q_{kl}^{(1)})$ antisymmetric in (k, l).

Table 2: List of coefficients appearing in the Hamiltonians (5.32) and (5.88) that do not have a supergravity interpretation, their defining relations and their antisymmetry properties.

5.1.1 Minimization of the Hamiltonian

Having written the Hamiltonian of our theory as (5.32), we now make the following simple yet crucial observation: this is a sum of squared terms, plus boundary terms Q_{EM} and "crossed terms" χ_T. Ignoring momentarily $(Q_{EM},\, \chi_T)$, it is clear that in order to minimize the energy of the system each such squared term must vanish separately. In this section, we enforce the just described minimization and thus obtain the bulk equations of motion for the world-volume gauge theory.

Let us start by setting to zero the first six squared terms in (5.32). These are the terms stemming from $(\mathcal{T}_1,\, \mathcal{T}_2,\, \mathcal{T}_3)$ in (5.4) and that experienced no change in the previous section. Since we wish our discussion to be as general as possible, we assume that the coefficients C_1/V_3 and $c_{0\tilde{3}}$ do not vanish. Then, we obtain the following:

$$\begin{aligned}
&(\sqrt{c_{11}}\mathcal{F}_{\alpha 0} - \sqrt{c_{\alpha\tilde{3}}}\mathcal{D}_\alpha \mathcal{A}_{\tilde{3}})^2 = 0, && (\sqrt{c_{12}}\mathcal{F}_{\tilde{\psi}0} - \sqrt{c_{\tilde{\psi}\tilde{3}}}\mathcal{D}_{\tilde{\psi}}\mathcal{A}_{\tilde{3}})^2 = 0,\\
&(\sqrt{c_{0r}}\mathcal{D}_0\mathcal{A}_r - i\sqrt{a_2}[\mathcal{A}_{\tilde{3}}, \mathcal{A}_r])^2 = 0, && (\sqrt{\tilde{c}_{0\phi_1}}\mathcal{D}_0\mathcal{A}_{\phi_1} - i\sqrt{a_4}[\mathcal{A}_{\tilde{3}}, \mathcal{A}_{\phi_1}])^2 = 0,\\
&(\sqrt{b_{0k}}\mathcal{D}_0\varphi_k - i\sqrt{c_{\tilde{3}k}}[\mathcal{A}_{\tilde{3}}, \varphi_k])^2 = 0, && \mathcal{D}_0\mathcal{A}_{\tilde{3}} = 0,
\end{aligned} \tag{5.33}$$

which should hold true $\forall \alpha = 1,\, 2$ and $\forall k = 1,\, 2,\, 3$. Recall now that both the gauge fields $(\mathcal{A}_a,\, \mathcal{A}_{\tilde{\psi}})$ (with $a = 0,\, 1,\, 2$) and the real scalars $(\mathcal{A}_{\tilde{3}},\, \mathcal{A}_{\phi_1},\, \mathcal{A}_r)$ (in the adjoint representation of $SU(N)$) depend only on the coordinates $(t,\, x_1,\, x_2,\, \tilde{\psi})$. As we pointed out in the beginning of chapter 5, not only are we interested in obtaining the minimum energy configuration for the aforementioned fields, but we also want them to satisfy BPS conditions. Hence, we search for static solutions to (5.33). This implies we will consider in the ongoing that the fields only depend on $(x_1,\, x_2,\, \tilde{\psi})$ and thus, using (4.51), the above reduces to

$$\begin{aligned}
&(\sqrt{c_{11}}\mathcal{D}_\alpha\mathcal{A}_0 - \sqrt{c_{\alpha\tilde{3}}}\mathcal{D}_\alpha \mathcal{A}_{\tilde{3}})^2 = 0, && (\sqrt{c_{12}}\mathcal{D}_{\tilde{\psi}}\mathcal{A}_0 - \sqrt{c_{\tilde{\psi}\tilde{3}}}\mathcal{D}_{\tilde{\psi}}\mathcal{A}_{\tilde{3}})^2 = 0,\\
&(\sqrt{c_{0r}}[\mathcal{A}_0, \mathcal{A}_r] - \sqrt{a_2}[\mathcal{A}_{\tilde{3}}, \mathcal{A}_r])^2 = 0, && (\sqrt{\tilde{c}_{0\phi_1}}[\mathcal{A}_0, \mathcal{A}_{\phi_1}] - \sqrt{a_4}[\mathcal{A}_{\tilde{3}}, \mathcal{A}_{\phi_1}])^2 = 0,\\
&(\sqrt{b_{0k}}[\mathcal{A}_0, \varphi_k] - \sqrt{c_{\tilde{3}k}}[\mathcal{A}_{\tilde{3}}, \varphi_k])^2 = 0, && [\mathcal{A}_0, \mathcal{A}_{\tilde{3}}] = 0,
\end{aligned} \tag{5.34}$$

valid again $\forall \alpha = 1,\, 2$ and $\forall k = 1,\, 2,\, 3$.

To proceed further, we need to choose a gauge. We shall make the following gauge choice:

$$\mathcal{A}_0 = \mathcal{A}_{\tilde{3}}. \tag{5.35}$$

This follows from our earlier identifications in (5.2), where the scalar field $\mathcal{A}_{\tilde{3}}$ was singled out from the other two scalars $(\mathcal{A}_{\phi_1},\, \mathcal{A}_r)$. One could certainly single out $\mathcal{A}_{\phi_1}$ or $\mathcal{A}_r$ instead and appropriately modify the above gauge choice. We will not entertain these options in the present work, as they do not lead to further physical insight. However, the interested reader can find enough detail on the $\mathcal{A}_0 = \mathcal{A}_r$ gauge choice in (3.178)-(3.182) in [1]. Essentially, all our conclusions will be valid in such a gauge as well, but $\mathcal{A}_{\tilde{3}}$ and $\mathcal{A}_r$ would then exchange roles. Although this is not a proof that our conclusions are gauge-independent, it strongly points towards such a possibility. We regard this as yet another advantage of relating our model to that in [54]. Anyhow, for our choice (5.35), the set of equations in (5.34) reduces to

$$\begin{array}{ll} (\sqrt{c_{11}} - \sqrt{c_{\alpha\tilde{3}}})^2 (\mathcal{D}_\alpha \mathcal{A}_{\tilde{3}})^2 = 0, & (\sqrt{c_{12}} - \sqrt{c_{\tilde{\psi}\tilde{3}}})^2 (\mathcal{D}_{\tilde{\psi}} \mathcal{A}_{\tilde{3}})^2 = 0, \\ (\sqrt{c_{0r}} - \sqrt{a_2})^2 [\mathcal{A}_{\tilde{3}}, \mathcal{A}_r]^2 = 0, & (\sqrt{\tilde{c}_{0\phi_1}} - \sqrt{a_4})^2 [\mathcal{A}_{\tilde{3}}, \mathcal{A}_{\phi_1}]^2 = 0, \\ (\sqrt{b_{0k}} - \sqrt{c_{\tilde{3}k}})^2 [\mathcal{A}_{\tilde{3}}, \varphi_k]^2 = 0, & \forall \alpha = 1,2, \quad \forall k = 1,2,3. \end{array} \tag{5.36}$$

Note that the last equation in (5.34) does not appear here, since it is trivially satisfied by our gauge choice.

All the equations in (5.36) admit the trivial solution $\mathcal{A}_{\tilde{3}} = 0$. Another possible solution would be to simultaneously have that

$$c_{11} = c_{\alpha\tilde{3}}, \quad c_{12} = c_{\tilde{\psi}\tilde{3}}, \quad c_{0r} = a_2, \quad \tilde{c}_{0\phi_1} = a_4, \quad b_{0k} = c_{\tilde{3}k}, \quad \forall \alpha = 1,2, \ \forall k = 1,2,3. \tag{5.37}$$

Let us explore this option by using the explicit form of the above coefficients, summarized in table 1. From (4.30), (4.69) and (4.78), we immediately see that the first equation will be satisfied iff

$$\cos^2\theta_{nc} + F_2 \sin^2\theta_{nc} = 1. \tag{5.38}$$

Similarly, using (4.31), (4.38), (4.83) and (4.91) in the second equation, one can right away conclude (5.38) is required so that $c_{12} = c_{\tilde{\psi}\tilde{3}}$. The same deduction follows from introducing (4.68), (4.69) and (4.78) in $c_{0r} = a_2$. On the other hand, using these same results in $\tilde{c}_{0\phi_1} = a_4$, one finds that, besides (5.38), it is also necessary to impose

$$\frac{(\tilde{F}_2 \tan\theta_{nc})^2}{\tilde{F}_2 - F_3} (1 + F_2 \tan^2\theta_{nc}) = 0. \tag{5.39}$$

Finally, from (4.123) and (4.91) it follows that $b_{0k} = c_{\tilde{3}k}$ iff we demand (5.38). Summing up, to ensure (5.37) we must enforce both (5.38) and (5.39). But in doing so, we do not wish to constraint our setup by choosing a particular form for the warp factors. On the contrary, we want to keep our M-Theory configuration (M,1) of part I as general as possible. Hence, we conclude that the second possible solution to (5.36) is given by $\theta_{nc} = 0$.

Between $\mathcal{A}_{\tilde{3}} = 0$ and $\theta_{nc} = 0$, there is a preferred solution to (5.36). Recall section 2.1: θ_{nc} was introduced as an alternative and computationally simpler way to account for the axionic background of [14], which was there shown to be an essential ingredient to study knots using the D3-NS5 system. As we will show in section 6.3, in our approach too θ_{nc} shall play a key role and allow us to construct a three-dimensional space capable of supporting knots. Accordingly, we set to zero the first six squared terms in the Hamiltonian (5.32) via

$$\mathcal{A}_{\tilde{3}} = 0, \tag{5.40}$$

along with the gauge choice in (5.35)[17]. Also, bear in mind all fields are time-independent in the ongoing.

Let us next turn our attention to the final five terms, as well as the last two terms in the third line of the Hamiltonian (5.32). These are the squared terms we introduced to make sure that while rewriting the Hamiltonian (5.7) as (5.32) all quadratic terms remain unaffected. Minimization of the energy requires them all to vanish which, for $(C_1/V_3) \neq 0$, means that

$$\begin{aligned}
&s^{(1)}(\mathcal{D}_{\tilde{\psi}}\mathcal{A}_r)^2 = 0, \qquad s^{(2)}(\mathcal{D}_{\tilde{\psi}}\mathcal{A}_{\phi_1})^2 = 0, \qquad t^{(1)}(\mathcal{D}_\beta\mathcal{A}_r)^2 = 0, \qquad t^{(2)}(\mathcal{D}_\beta\mathcal{A}_{\phi_1})^2 = 0,\\
&a_1 q^{(4)}[\mathcal{A}_r, \mathcal{A}_{\phi_1}]^2 = 0, \quad q^{(1)}_{kl} d_{kl}[\varphi_k, \varphi_l]^2 = 0, \quad q^{(\gamma-1)}_k c_{y_\gamma k}[\mathcal{A}_{y_\gamma}, \varphi_k]^2 = 0,
\end{aligned} \tag{5.41}$$

for all $\beta = 1,\, 2$, all $k,\, l = 1,\, 2,\, 3$ and all $\gamma = 2,\, 3$. If we consider that, generically, all the coefficients appearing above are not zero, then satisfying (5.41) implies

$$\mathcal{D}_\delta\mathcal{A}_r = \mathcal{D}_\delta\mathcal{A}_{\phi_1} = [\mathcal{A}_r, \mathcal{A}_{\phi_1}] = [\mathcal{A}_r, \varphi_k] = [\mathcal{A}_{\phi_1}, \varphi_k] = 0, \tag{5.42}$$

$\forall \delta = 1, 2, \tilde{\psi}$ and $\forall k = 1, 2, 3$. On the other hand, if we do not wish to trivialize the system, we cannot conclude that most generically all $q^{(1)}_{kl}$'s are non-zero. Observe that this would imply $[\varphi_k, \varphi_l] = 0$ for all $(k,\, l)$. Hence, as the simplest non-trivial case, we will consider only one such (independent) coefficient vanishes. We choose $q^{(1)}_{12} = 0$. Then, we impose

$$[\varphi_1, \varphi_2] \neq 0, \qquad [\varphi_1, \varphi_3] = [\varphi_2, \varphi_3] = 0. \tag{5.43}$$

In this manner, we have enforced (5.41).

In our minimization of the Hamiltonian (5.32), we next focus on the squared term between the fourth and sixth lines and demand its vanishing:

$$\begin{aligned}
&\sqrt{\frac{C_1}{V_3}}\Big(\sqrt{\frac{c_{11}}{2}}\mathcal{F}_{\alpha\beta} + \sqrt{c_{\tilde{\psi}r}}\, s^{(1)}_{\alpha\beta}\epsilon_{\alpha\beta\tilde{\psi}r}\mathcal{D}_{\tilde{\psi}}\mathcal{A}_r + \sqrt{\tilde{c}_{\tilde{\psi}\phi_1}}\, s^{(2)}_{\alpha\beta}\epsilon_{\alpha\beta\tilde{\psi}\phi_1}\mathcal{D}_{\tilde{\psi}}\mathcal{A}_{\phi_1} - ig^{(4)}_{\alpha\beta}\sqrt{a_1}[\mathcal{A}_r, \mathcal{A}_{\phi_1}]\Big)\\
&- i\sum_{\delta,k,l=1}^{3}\Big(g^{(1)}_{\alpha\beta kl}\sqrt{d_{kl}}[\varphi_k, \varphi_l] + \sum_{\gamma=2}^{3} g^{(\gamma-1)}_{\alpha\beta k}\sqrt{c_{y_\gamma k}}[\mathcal{A}_{y_\gamma}, \varphi_k] + i\sqrt{b_{\delta k}}\epsilon_{\alpha\beta}\cdot m^{(1)}_{\delta k}\mathcal{D}_\delta\varphi_k\Big) = 0,
\end{aligned} \tag{5.44}$$

[17] This implies $\mathcal{A}_0 = 0$, known as the Weyl gauge or also as the axial gauge.

which should be true for all α, $\beta = 1$, 2. Needless to say, minimization of the energy requires all squared terms to vanish simultaneously. This implies the choices previously made to set to zero other squared terms must now be enforced as well. Thus, inserting (5.42) and (5.43) in the above, our equations reduce to

$$\sqrt{\frac{C_1 c_{11}}{2V_3}}\mathcal{F}_{\alpha\beta} - 2ig^{(1)}_{\alpha\beta 12}\sqrt{d_{12}}[\varphi_1, \varphi_2] + \sum_{\delta,k=1}^{3} \sqrt{b_{\delta k}}\epsilon_{\alpha\beta} \cdot m^{(1)}_{\delta k}\mathcal{D}_\delta\varphi_k = 0, \tag{5.45}$$

again true $\forall\alpha, \beta = 1, 2$ and where we have used the fact that $g^{(1)}_{\alpha\beta 12} = -g^{(1)}_{\alpha\beta 21}$ by definition and $d_{12} = d_{21}$, as can be seen from (4.129). Since (5.45) is antisymmetric in (α, β), we can restrict our attention to the case where $(\alpha = 1, \beta = 2)$. Following the normalization convention that $\epsilon_{12} = 1$, noting that (4.123) tells us that $b_{12} = b_{21}$ and choosing

$$g^{(1)}_{1212} = m^{(1)}_{\tilde{\psi}3} = m^{(1)}_{12} = -m^{(1)}_{21} = \frac{1}{\sqrt{2}}, \qquad m^{(1)}_{11} = m^{(1)}_{22} = m^{(1)}_{13} = m^{(1)}_{23} = m^{(1)}_{\tilde{\psi}1} = m^{(1)}_{\tilde{\psi}2} = 0, \tag{5.46}$$

it is a matter of minor algebra to obtain

$$\mathcal{F}_{12} + \sqrt{\frac{V_3}{C_1 c_{11}}}\Big(- 2i\sqrt{d_{12}}[\varphi_1, \varphi_2] + \sqrt{b_{12}}(\mathcal{D}_1\varphi_2 - \mathcal{D}_2\varphi_1) + \sqrt{b_{\tilde{\psi}3}}\mathcal{D}_{\tilde{\psi}}\varphi_3\Big) = 0. \tag{5.47}$$

Note that the dot product in (5.45) has been interpreted as a usual scalar product in this case.

The above is the first BPS equation following from the minimization of the energy of the Hamiltonian (5.32). Notice that, schematically, our BPS condition is of the form

$$\mathcal{F} + \mathcal{D}\varphi + [\varphi, \varphi] = 0. \tag{5.48}$$

The well-versed reader will of course be familiar with the Bogomolny [55], Hitchin [56] and Nahm [57] equations, which we can sketch as follows:

$$\text{Bogomolny:}\;\; \mathcal{F} + \mathcal{D}\varphi = 0, \qquad \text{Hitchin:}\;\; \mathcal{F} + [\varphi, \varphi] = 0, \qquad \text{Nahm:}\;\; \mathcal{D}\varphi + [\varphi, \varphi] = 0. \tag{5.49}$$

Written in this manner, it is evident that our BPS condition is just some linear combination of all Bogomolny, Hitchin and Nahm equations. Collecting initials into an acronym, we will refer to (5.47) as the first *BHN equation*.

Before proceeding ahead, let us pause for a moment and study what are the consequences following from the choices of coefficients made so far. These choices are $q^{(1)}_{12} = 0$ and (5.46). As can be checked in table 2, these coefficients are required to satisfy the constraint equations

(5.22) and (5.30). So, combining our choices and the constraints, we are led to conclude that

$$2\left(g^{(1)}_{12kl}\right)^2+\sum_{\alpha=1}^{2}\left(h^{(1)}_{\alpha\tilde{\psi}kl}\right)^2-q^{(1)}_{kl}=1\quad\forall k,l=2,3,\qquad h^{(1)}_{\alpha\tilde{\psi}12}=-h^{(1)}_{\alpha\tilde{\psi}21}=0\quad\forall\alpha=1,2,$$
$$m^{(2)}_{\psi 3}=m^{(2)}_{12}=m^{(2)}_{21}=0,\qquad m^{(2)}_{11},m^{(2)}_{22},m^{(2)}_{13},m^{(2)}_{23},m^{(2)}_{\tilde{\psi}1},m^{(2)}_{\tilde{\psi}2}=\pm\frac{1}{\sqrt{2}}\tag{5.50}$$

must hold true in the following.

The last step in the minimization of the energy of our system with Hamiltonian (5.32) is to demand the vanishing of the squared term between its sixth and eighth lines. This must be done in a consistent manner to all previous choices made in this section. The necessary vanishing we just mentioned is

$$\sqrt{\frac{C_1}{V_3}}\Big(\sqrt{\frac{c_{12}}{2}}\mathcal{F}_{\alpha\tilde{\psi}}+\sqrt{c_{\beta r}}t^{(1)}_{\alpha}\epsilon_{\alpha\tilde{\psi}\beta r}\mathcal{D}_\beta\mathcal{A}_r+\sqrt{\tilde{c}_{\beta\phi_1}}t^{(2)}_{\alpha}\epsilon_{\alpha\tilde{\psi}\beta\phi_1}\mathcal{D}_\beta\mathcal{A}_{\phi_1}-ih^{(4)}_{\alpha\tilde{\psi}}\sqrt{a_1}[\mathcal{A}_r,\mathcal{A}_{\phi_1}]\Big)$$
$$-i\sum_{\delta,k,l=1}^{3}\Big(h^{(1)}_{\alpha\tilde{\psi}kl}\sqrt{d_{kl}}[\varphi_k,\varphi_l]+\sum_{\gamma=2}^{3}h^{(\gamma-1)}_{\alpha\tilde{\psi}k}\sqrt{c_{y_\gamma k}}[\mathcal{A}_{y_\gamma},\varphi_k]+i\sqrt{b_{\delta k}}\epsilon_{\alpha\tilde{\psi}}\cdot m^{(2)}_{\delta k}\mathcal{D}_\delta\varphi_k\Big)=0,\tag{5.51}$$

for all α, $\beta=1$, 2. Using (5.42), (5.43) and (5.50) in the above, we have that

$$\sqrt{\frac{C_1c_{12}}{2V_3}}\mathcal{F}_{\alpha\tilde{\psi}}+\sum_{\delta,k=1}^{3}\sqrt{b_{\delta k}}\epsilon_{\alpha\tilde{\psi}}\cdot m^{(2)}_{\delta k}\mathcal{D}_\delta\varphi_k=0\qquad\forall\alpha=1,2.\tag{5.52}$$

Here, $\delta=3$ should be understood as making reference to the bulk direction $\tilde{\psi}$. Without loss of generality, we take the definition of the dot product above to be

$$\sum_{\delta,k=1}^{3}\sqrt{b_{\delta k}}\epsilon_{\alpha\tilde{\psi}}\cdot m^{(2)}_{\delta k}\mathcal{D}_\delta\varphi_k\equiv-6\sum_{\delta,k=1}^{3}\sqrt{b_{\delta k}}\epsilon_{[\alpha\tilde{\psi}}m^{(2)}_{\delta k]}\mathcal{D}_\delta\varphi_k+\sqrt{b_{\tilde{\psi}\alpha}}\epsilon_{\alpha\tilde{\psi}}m^{(2)}_{\tilde{\psi}\alpha}\mathcal{D}_{\tilde{\psi}}\varphi_\alpha,\tag{5.53}$$

with the indices of the first term on the right-hand side necessarily different from each other. This seemingly involved term is not so complicated and, using the antisymmetry of the epsilon tensors, is explicitly given by

$$-\frac{1}{2}\sum_{\delta,k=1}^{3}\sqrt{b_{\delta k}}\Big[\epsilon_{\alpha\tilde{\psi}}(m^{(2)}_{\delta k}-m^{(2)}_{k\delta})+\epsilon_{\delta\tilde{\psi}}(m^{(2)}_{\alpha k}-m^{(2)}_{k\alpha})+\epsilon_{\delta k}(m^{(2)}_{\alpha 3}-m^{(2)}_{\tilde{\psi}\alpha})$$
$$+\epsilon_{\alpha k}(m^{(2)}_{\delta 3}-m^{(2)}_{\tilde{\psi}\delta})+\epsilon_{\alpha\delta}(m^{(2)}_{k3}-m^{(2)}_{\tilde{\psi}k})+\epsilon_{k\tilde{\psi}}(m^{(2)}_{\alpha\delta}-m^{(2)}_{\delta\alpha})\Big]\mathcal{D}_\delta\varphi_k.\tag{5.54}$$

In good agreement with (5.50), we now implement the second line there, choosing the plus sign for all the $m^{(2)}$ coefficients in the last equality. In this manner, the above reduces considerably to

$$-\frac{1}{2}\sum_{\delta,k=1}^{3}\sqrt{b_{\delta k}}[\epsilon_{\delta k}m^{(2)}_{\tilde{\psi}\alpha}+\epsilon_{\alpha k}m^{(2)}_{\tilde{\psi}\delta}+\epsilon_{\alpha\delta}m^{(2)}_{\tilde{\psi}k}]\mathcal{D}_{\delta}\varphi_{k}. \tag{5.55}$$

As we said, the dot product is taken by definition such that all indices in this term should be different from each other. In other words, $\delta = 1(2)$ if $\alpha = 2(1)$ and $k = 3$. This leads to

$$-\frac{1}{2}\sqrt{b_{\beta 3}}[\epsilon_{\beta\tilde{\psi}}m^{(2)}_{\tilde{\psi}\alpha}+\epsilon_{\alpha\tilde{\psi}}m^{(2)}_{\tilde{\psi}\beta}+\epsilon_{\alpha\beta}m^{(2)}_{\tilde{\psi}3}]\mathcal{D}_{\beta}\varphi_{3}=\begin{cases}\sqrt{\frac{b_{23}}{2}}\mathcal{D}_2\varphi_3 & \text{if } \alpha=1,\ \beta=2,\\ \sqrt{\frac{b_{13}}{2}}\mathcal{D}_1\varphi_3 & \text{if } \alpha=2,\ \beta=1,\end{cases} \tag{5.56}$$

where the normalization convention used is $\epsilon_{1\tilde{\psi}} = \epsilon_{2\tilde{\psi}} = 1$. Finally, plugging the above in (5.52), it is a matter of minor algebra to obtain the remaining two BHN equations as

$$\mathcal{F}_{\alpha\tilde{\psi}}+\sqrt{\frac{V_3}{C_1c_{12}}}\big(\sqrt{b_{\tilde{\psi}\alpha}}\mathcal{D}_{\tilde{\psi}}\varphi_\alpha+\sqrt{b_{\alpha 3}}\mathcal{D}_\alpha\varphi_3\big)=0, \qquad \forall\alpha=1,2. \tag{5.57}$$

Collecting thoughts, in this section we have shown that the vanishing of the different squared terms in the Hamiltonian (5.32) for static configurations leads to the BHN equations (5.47) and (5.57). The name BHN simply denotes that these are a combination of the well-known Bogomolny, Hitchin and Nahm equations. In obtaining such BHN equations, we chose the gauge (5.35) and further demand that the gauge and scalar fields in the bosonic sector of the theory satisfy (5.40), (5.42) and (5.43). Additionally, we made the coefficient choices $q^{(1)}_{12} = 0$, (5.46) and (5.50) −selecting the plus sign in the last equality there. One can easily check that all our choices respect the defining equations of the coefficients, summarized previously in table 2. However, this analysis completely ignored the (Q_{EM}, χ_T) terms in (5.32). In the next section, we start to shed light in this direction by studying χ_T.

5.1.2 Consistency requirements and advantage of rewriting (5.7) as (5.32)

We already pointed out the crucial fact that the electric and magnetic charges Q_{EM} in the Hamiltonian (5.32) are not yet specified boundary terms. That is, the Hamiltonian as a whole is defined in the X_4 space (the bulk) but the terms Q_{EM} are defined solely in X_3 (the boundary). We remind the reader that the spaces X_4 and X_3 were defined in (4.1). The goal in this section is to ensure that χ_T in (5.32) does *not* contribute to the boundary terms Q_{EM}. Further, we want to ensure that χ_T is in good agreement with the bulk energy minimization performed in the previous section. Anticipating events, we will see that such consistency leads to new constraints on the scalar fields of our gauge theory. In this manner, we shall be able to focus on the study of the boundary theory only, since the bulk theory will by then be set to zero by requiring that the fields satisfy (5.40), (5.42) and (5.43), together with the BHN equations (5.47) and (5.57) and the new constraints we shall presently find.

But let us take a step back first: what is χ_T to begin with? In order to determine χ_T precisely we will compare the Hamiltonians (5.7) and (5.32), i.e. the Hamiltonians before and after the inclusion of the coefficients in table 2. By definition, χ_T is simply the collection of all crossed terms produced during this rewriting. To make our task computationally easier, we will make use of all the equations above mentioned, which guarantee that the bulk theory is minimized. That is, our analysis will be valid on shell.

Explicitly, using (5.40), (5.42) and (5.43) in (5.7), the on shell Hamiltonian before the rewriting is given by

$$
\begin{aligned}
H = \int d^4x \ \mathrm{Tr}\Big[\frac{1}{2}\sum_{\alpha,\beta=1}^{2}\Big(\sqrt{\frac{C_1 c_{11}}{V_3}}\mathcal{F}_{\alpha\beta} - \sum_{k=1}^{3}\sqrt{b_{\tilde{\psi}k}}\epsilon_{\alpha\beta\tilde{\psi}k}\mathcal{D}_{\tilde{\psi}}\varphi_k - i\sum_{k,l=1}^{2}\sqrt{d_{kl}}[\varphi_k,\varphi_l]\Big)^2 \\
+ \frac{1}{2}\sum_{\alpha=1}^{2}\Big(\sqrt{\frac{C_1 c_{12}}{V_3}}\mathcal{F}_{\alpha\tilde{\psi}} - \sum_{k=1}^{3}\sqrt{b_{\beta k}}\epsilon_{\alpha\tilde{\psi}\beta k}\mathcal{D}_{\beta}\varphi_k - i\sum_{k,l=1}^{2}\sqrt{d_{kl}}[\varphi_k,\varphi_l]\Big)^2\Big] + Q_{EM}. \quad (5.58)
\end{aligned}
$$

Let us for the time being ignore Q_{EM}. We already said and it can be clearly seen from (4.129), that $d_{12} = d_{21}$. However, $[\varphi_1, \varphi_2] = -[\varphi_2, \varphi_1]$. Hence, when summing over k, $l = 1$, 2 in the pertinent terms above, these will vanish unless they are squared. In other words, the non-zero crossed terms in our Hamiltonian (5.58) are just two:

$$
\begin{aligned}
\zeta_1 &\equiv -\frac{1}{2}\sqrt{\frac{C_1 c_{11}}{V_3}}\sum_{\alpha,\beta=1}^{2}\sum_{k=1}^{3}\sqrt{b_{\tilde{\psi}k}}\epsilon_{\alpha\beta\tilde{\psi}k}\mathrm{Tr}\{\mathcal{F}_{\alpha\beta}, \mathcal{D}_{\tilde{\psi}}\varphi_k\}, \\
\zeta_2 &\equiv -\frac{1}{2}\sqrt{\frac{C_1 c_{12}}{V_3}}\sum_{\alpha=1}^{2}\sum_{k=1}^{3}\sqrt{b_{\beta k}}\epsilon_{\alpha\tilde{\psi}\beta k}\mathrm{Tr}\{\mathcal{F}_{\alpha\tilde{\psi}}, \mathcal{D}_{\beta}\varphi_k\}.
\end{aligned} \quad (5.59)
$$

Simply carrying out the sums above and noting that (4.123) implies that $b_{\tilde{\psi}k}$ and b_{ak} are the same for all values of $a = 1$, 2 and $k = 1$, 2, 3, we get

$$
\begin{aligned}
\zeta_1 &= \sqrt{\frac{C_1 c_{11} b_{\tilde{\psi}3}}{V_3}} \ \mathrm{Tr}\left\{\mathcal{F}_{12}, \mathcal{D}_{\tilde{\psi}}(\varphi_1+\varphi_2+\varphi_3)\right\}, \\
\zeta_2 &= -\frac{1}{2}\sqrt{\frac{C_1 c_{12} b_{12}}{V_3}}\mathrm{Tr}\left[\left\{\mathcal{F}_{2\tilde{\psi}}, \mathcal{D}_1(\varphi_1+\varphi_2+\varphi_3)\right\} - \left\{\mathcal{F}_{1\tilde{\psi}}, \mathcal{D}_2(\varphi_1+\varphi_2+\varphi_3)\right\}\right],
\end{aligned} \quad (5.60)
$$

with the normalization convention $\epsilon_{12k\tilde{\psi}} = 1$ for all $k = 1, 2, 3$. On the other hand, using (5.40), (5.42), (5.43) and the choices $q^{(1)}_{12}$, $h^{(1)}_{\alpha\tilde{\psi}12} = 0$ (for all $\alpha = 1, 2$) in (5.32), the on shell Hamiltonian after the rewriting is

$$H = \int d^4x \ \mathrm{Tr}\Big[\sum_{\alpha,\beta=1}^{2} \Big(\sqrt{\frac{C_1 c_{11}}{2V_3}} \mathcal{F}_{\alpha\beta} + \sum_{\delta,k=1}^{3} \sqrt{b_{\delta k}} \epsilon_{\alpha\beta} \cdot m^{(1)}_{\delta k} \mathcal{D}_\delta \varphi_k - i \sum_{k,l=1}^{2} g^{(1)}_{\alpha\beta kl} \sqrt{d_{kl}} [\varphi_k, \varphi_l] \Big)^2$$
$$+ \sum_{\alpha=1}^{2} \Big(\sqrt{\frac{C_1 c_{12}}{2V_3}} \mathcal{F}_{\alpha\tilde{\psi}} + \sum_{\delta,k=1}^{3} \sqrt{b_{\delta k}} \epsilon_{\alpha\tilde{\psi}} \cdot m^{(2)}_{\delta k} \mathcal{D}_\delta \varphi_k \Big)^2 + \chi_T \Big] + Q_{EM}. \tag{5.61}$$

We know that the squared terms of this and the previous Hamiltonian are the same, provided the coefficients above satisfy the constraints in table 2, as already discussed in the previous section. Hence, let us just focus on the crossed terms. There are four of them:

$$\begin{aligned}
\zeta'_1 &\equiv -i \sqrt{\frac{2C_1 c_{11} d_{12}}{V_3}} \sum_{\alpha,\beta=1}^{2} g^{(1)}_{\alpha\beta 12} \ \mathrm{Tr}\{\mathcal{F}_{\alpha\beta}, [\varphi_1, \varphi_2]\}, \\
\zeta'_2 &\equiv \sqrt{\frac{C_1 c_{11}}{2V_3}} \sum_{\alpha,\beta=1}^{2} \sum_{\delta,k=1}^{3} \sqrt{b_{\delta k}} \epsilon_{\alpha\beta} \cdot m^{(1)}_{\delta k} \ \mathrm{Tr}\{\mathcal{F}_{\alpha\beta}, \mathcal{D}_\delta \varphi_k\}, \\
\zeta'_3 &\equiv -2i \sqrt{d_{12}} \sum_{\alpha,\beta=1}^{2} \sum_{\delta,m=1}^{3} g^{(1)}_{\alpha\beta 12} \sqrt{b_{\delta m}} \epsilon_{\alpha\beta} \cdot m^{(1)}_{\delta m} \ \mathrm{Tr}\{[\varphi_1, \varphi_2], \mathcal{D}_\delta \varphi_m\}, \\
\zeta'_4 &\equiv \sqrt{\frac{C_1 c_{12}}{2V_3}} \sum_{\alpha=1}^{2} \sum_{\delta,k=1}^{3} \sqrt{b_{\delta k}} \epsilon_{\alpha\tilde{\psi}} \cdot m^{(2)}_{\delta k} \ \mathrm{Tr}\{\mathcal{F}_{\alpha\tilde{\psi}}, \mathcal{D}_\delta \varphi_k\},
\end{aligned} \tag{5.62}$$

where we have used the (anti)symmetry properties $d_{12} = d_{21}$ and $g^{(1)}_{\alpha\beta 12} = -g^{(1)}_{\alpha\beta 21}$ to carry out the sums over k, l in the first and third terms. In this language, χ_T is

$$\chi_T = \sum_{i=1}^{2} \zeta_i - \sum_{i=1}^{4} \zeta'_i. \tag{5.63}$$

On our way to determine χ_T, let us first focus on ζ'_4. Using the coefficient choices in (5.50) for the plus sign in all cases, the dot product definition in (5.53), the result (5.56) and further summing over α, it is easy to see that

$$\zeta'_4 = \frac{1}{2} \sqrt{\frac{C_1 c_{12}}{V_3}} \ \mathrm{Tr} \Big(\{\mathcal{F}_{1\tilde{\psi}}, \sqrt{b_{\tilde{\psi}1}} \mathcal{D}_{\tilde{\psi}} \varphi_1 + \sqrt{b_{23}} \mathcal{D}_2 \varphi_3\} + \{\mathcal{F}_{2\tilde{\psi}}, \sqrt{b_{\tilde{\psi}2}} \mathcal{D}_{\tilde{\psi}} \varphi_2 + \sqrt{b_{13}} \mathcal{D}_1 \varphi_3\} \Big), \tag{5.64}$$

where the normalization convention employed is once again $\epsilon_{13} = \epsilon_{23} = 1$. With the aid of the BHN equations in (5.57), ζ_4' can be seen to be a squared term, not a crossed term:

$$\zeta_4' = -\frac{C_1 c_{12}}{2V_3} \sum_{\alpha=1}^{2} \mathrm{Tr}(\mathcal{F}_{\alpha\tilde{\psi}})^2. \tag{5.65}$$

The conclusion that ζ_4' is not a crossed term of course implies that it does not contribute to Q_{EM}, as we wished in the first place. Further, since ζ_4' is a squared term, it can be absorbed by an appropriate relabeling of the coefficients in table 2, where the defining equations remain unaltered. Consequently, ζ_4' does not contribute to χ_T and we need not worry over it in the ongoing.

We turn our attention to ζ_1', ζ_2' and ζ_3' next. As before, we interpret the dot product in ζ_2' and ζ_3' as a regular scalar product, we use our coefficient choices in (5.46) and sum over α, β in (5.62). In the process, one must not forget the antisymmetric properties of the coefficients summarized in table 2. The described computation is not hard and yields

$$\begin{aligned}
\zeta_1' &= 2i\sqrt{\frac{C_1 c_{11} d_{12}}{V_3}}\ \mathrm{Tr}\Big\{\mathcal{F}_{12}, [\varphi_1, \varphi_2]\Big\}, \\
\zeta_2' &= \sqrt{\frac{C_1 c_{11}}{V_3}}\ \mathrm{Tr}\Big\{\mathcal{F}_{12}, \sqrt{b_{12}}(\mathcal{D}_1\varphi_2 - \mathcal{D}_2\varphi_1) + \sqrt{b_{\tilde{\psi}3}}\mathcal{D}_{\tilde{\psi}}\varphi_3\Big\}, \\
\zeta_3' &= 2i\sqrt{d_{12}}\ \mathrm{Tr}\{[\varphi_1, \varphi_2], \sqrt{b_{12}}(\mathcal{D}_1\varphi_2 - \mathcal{D}_2\varphi_1) + \sqrt{b_{\tilde{\psi}3}}\mathcal{D}_{\tilde{\psi}}\varphi_3\}.
\end{aligned} \tag{5.66}$$

It can be easily checked that, further introducing the first BHN equation (5.47) in the above, the following holds true:

$$\zeta_1' + \zeta_2' = -\frac{2C_1 c_{11}}{V_3}\ \mathrm{Tr}(\mathcal{F}_{12})^2, \qquad \zeta_3' = 8d_{12}\ \mathrm{Tr}[\varphi_1, \varphi_2]^2 - 2i\sqrt{\frac{C_1 c_{11} d_{12}}{V_3}}\ \mathrm{Tr}\{[\varphi_1, \varphi_2], \mathcal{F}_{12}\}. \tag{5.67}$$

The same observation we made for ζ_4' should be invoked presently too: the squared terms can be absorbed by a relabeling of the coefficients in table 2. They do not contribute to Q_{EM} and do not affect the bulk minimization of section 5.1.1. In other words, we can consistently conclude that they do not contribute to χ_T and simply ignore them in the following. The only term which contributes to χ_T from the above is

$$\zeta_3' = -2i\sqrt{\frac{C_1 c_{11} d_{12}}{V_3}}\ \mathrm{Tr}\{[\varphi_1, \varphi_2], \mathcal{F}_{12}\}. \tag{5.68}$$

Putting everything together, we say that

$$\chi_T = \zeta_1 + \zeta_2 - \zeta_3', \tag{5.69}$$

which must either be set to zero or reduced to a sum of squared terms −which would then be accounted for by an inconsequential redefinition of the coefficients in table 2. In this manner, the Hamiltonian (5.32) will lead to a boundary theory determined by Q_{EM} solely, while a consistent bulk energy minimization is ensured via BHN and other constraining equations on the gauge and scalar fields. What is more, it is evident that $\zeta_1 - \zeta_3'$ and ζ_2 will have to satisfy this condition separately, as the BHN equations (5.47) and (5.57) do not mix $\mathcal{F}_{12}$ with $(\mathcal{F}_{1\tilde{\psi}}, \mathcal{F}_{2\tilde{\psi}})$. For this very same reason, we must demand right away

$$\mathcal{D}_{\tilde{\psi}}\varphi_1 = \mathcal{D}_{\tilde{\psi}}\varphi_2 = \mathcal{D}_1\varphi_3 = \mathcal{D}_2\varphi_3 = 0. \tag{5.70}$$

We will refer to these as the first set of consistency requirements we mentioned in the title of the present section. Implementing the above and using (5.47), ζ_1 in (5.60) and ζ_3' in (5.68), we get

$$\zeta_1 - \zeta_3' = -\frac{2C_1c_{11}}{V_3}\,\mathrm{Tr}(\mathcal{F}_{12})^2 - \sqrt{\frac{C_1c_{11}b_{12}}{V_3}}\,\mathrm{Tr}\{\mathcal{F}_{12}, \mathcal{D}_1\varphi_2 - \mathcal{D}_2\varphi_1\}. \tag{5.71}$$

It goes without saying that the first term on the right-hand side above is squared and thus does not contribute to χ_T. That is not the case with the second term, though. To make it vanish, we will demand

$$\mathcal{D}_1\varphi_2 - \mathcal{D}_2\varphi_1 = 0, \tag{5.72}$$

another consistency requirement. The attentive reader won't take long staring at ζ_2 in (5.60) in combination with the two relevant BHN equations in (5.57) to realize that yet one last consistency requirement is needed:

$$\mathcal{D}_1\varphi_1 + \mathcal{D}_2\varphi_2 = 0. \tag{5.73}$$

Then, ζ_2 simplifies to

$$\zeta_2 = \frac{1}{2}\sqrt{\frac{C_1c_{12}b_{12}}{V_3}}\,\mathrm{Tr}\left[\left\{\mathcal{F}_{2\tilde{\psi}}, \mathcal{D}_1(\varphi_1 + \varphi_2)\right\} + \left\{\mathcal{F}_{1\tilde{\psi}}, \mathcal{D}_1(\varphi_1 - \varphi_2)\right\}\right]. \tag{5.74}$$

We cannot make squares of the above, so it better vanish. Fortunately, this is indeed zero, as can be seen from the combination of the requirements (5.70) and the BHN equations (5.57), which lead to

$$\mathcal{F}_{1\tilde{\psi}} = \mathcal{F}_{2\tilde{\psi}} = 0. \tag{5.75}$$

The other BHN equation, namely (5.47), also reduces in view of our consistency requirements and is now given by

$$\mathcal{F}_{12} + \sqrt{\frac{V_3}{C_1 c_{11}}}\left(2i\sqrt{d_{12}}[\varphi_1, \varphi_2] + \sqrt{b_{\tilde{\psi}3}}\mathcal{D}_{\tilde{\psi}}\varphi_3\right) = 0. \tag{5.76}$$

Finally, we note that χ_T has by now been converted to some sum of squared terms which does not affect our analysis and definitely does not contribute to Q_{EM}, as was our goal in the beginning of this section.

In conclusion, for the gauge choice (5.35), the energy of the Hamiltonian (5.32) is minimized when all (5.40), (5.42), (5.43), (5.70), (5.72) and (5.73) are satisfied, together with the BHN equations (5.75) and (5.76). In this case, χ_T is zero. More precisely, χ_T is absorbed by an immaterial redefinition of coefficients, as already explained. Then, we are only left with the boundary terms Q_{EM} to be considered.

To finish this section, let us clarify what is the advantage of rewriting the Hamiltonian (5.7) as (5.32). The so-called consistency requirements (5.70), (5.72) and (5.73) that we obtained in this section to ensure no crossed terms were produced in the aforementioned rewriting are actually vital results in our analysis. They simplify the BHN equations and, together with them, are known to be directly related to knot invariants (for example, see section 3.2 in [14]). We will discuss such relation at length in part III. For the time being, we will devote the coming section to the generalization of all the results so far in chapter 5 to the case that really concerns us: $c_2 \neq 0$ in (4.143). This will in turn directly lead us to the study of the corresponding boundary theory in chapter 6.

5.2 Generalization to the case where $c_2 \neq 0$ in (4.143)

We have by now gained considerable insight into the bulk physics of the theory with action (4.143) but with no topological term (i.e. $c_2 = 0$ there). The inclusion of this topological term is, however, far from trivial, both conceptually and computationally. To relax a bit the computational difficulties, we will begin this section by doing the following approximation: we will in the ongoing consider that

$$c_{11} = c_{12} \tag{5.77}$$

in (4.143). Looking at the definitions of these coefficients in (4.23), we see that this amounts to requiring that $e^{2\phi_0}H_4 = 1$. Further using (2.43), our simplification reduces to a constraint equation on the so far completely arbitrary warp factors (2.2) and (2.21) and constant leading value of the dilaton in (4.5):

$$\frac{e^{2\phi_0}\tilde{F}_2 F_3 \sec^2\theta_{nc}\sin^2\theta_1}{\tilde{F}_2\cos^2\theta_1 + F_3\sin^2\theta_1} = 1. \tag{5.78}$$

Clearly, this is not too stringent a constraint, as there is ample freedom of choice to satisfy it. For a physical interpretation of our assumption, one should look at the metric of the M-Theory configuration (M,1) in (2.46). We then see that (5.77) implies that $(t,\, x_1,\, x_2,\, \tilde{\psi})$ are now Lorentz invariant directions. In other words, our approximation leads to a restoration of the Lorentz symmetry along $\tilde{\psi}$ in the spacetime X_4, defined in (4.1).

Having made this simplification, we proceed to show an intermediate result, which will immediately prove useful in deriving the Hamiltonian following from the action (4.143) with $c_2 \neq 0$. This consists in working out a convenient component form of the integrand of this topological term in the action:

$$\mathcal{F}^{(X_4)} \wedge \mathcal{F}^{(X_4)} \equiv \sum_{\mu<\nu,\, \rho<\lambda} \mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\lambda} dx_\mu \wedge dx_\nu \wedge dx_\rho \wedge dx_\lambda = d^4x \sum_{\mu<\nu} \mathcal{F}_{\mu\nu} * \mathcal{F}^{\mu\nu}, \tag{5.79}$$

where, as usual, the Hodge dual of the field strength is defined as

$$*\mathcal{F}^{\mu\nu} \equiv \frac{1}{2}\sum_{\rho,\lambda} \epsilon^{\mu\nu\rho\lambda}\mathcal{F}_{\rho\lambda}, \tag{5.80}$$

d^4x is the volume element of the now Minkowskian spacetime X_4 and x_μ refers collectively to its coordinates $(t,\, x_1,\, x_2,\, \tilde{\psi})$.

Using the approximation (5.77), (5.79) and recalling (4.112), we are ready to write the first line in the action (4.143) of our theory, which we denote as S_{L1}, in the following suitable manner:

$$S_{L1} = \int d^4x \ \mathrm{Tr} \sum_{\mu<\nu} \left(\frac{C_1 c_{11}}{V_3} \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + C_1 \sin\theta_{nc} q(\theta_{nc}) \mathcal{F}_{\mu\nu} * \mathcal{F}^{\mu\nu} \right). \tag{5.81}$$

The reader will of course right away notice that S_{L1} is precisely Maxwell's action with a Θ-term (see, for example, in (2.1) in [58]). The correlation becomes fully apparent once we identify our coefficients, which only depend on supergravity variables, with the Yang-Mills coupling and gauge theory Θ-parameter as

$$\frac{C_1 c_{11}}{V_3} \equiv \frac{4\pi}{g_{YM}^2}, \qquad C_1 \sin\theta_{nc} q(\theta_{nc}) \equiv \frac{\Theta}{2\pi}. \tag{5.82}$$

The above makes concrete the long standing promise of section 2.1. There, we claimed that introducing the non-commutative deformation labeled by the parameter θ_{nc} would lead to a Θ-term in the four-dimensional gauge theory associated to the M-Theory configuration (M,1). From (5.82) it is clear that $\theta_{nc} = 0$ would lead to no Θ-term in the gauge theory, so the deformation is indeed successful in replacing the axionic background of [14] to source this topological term. Later on, in section 6.3, we shall see that this topological term is a fundamental ingredient to convert the boundary X_3 of X_4 into a suitable space for the embedding of knots. This is because such term allows us to define a topological theory in X_3. But let us not get ahead of ourselves. It is standard to combine the Yang-Mills coupling and the Θ-parameter into a single *complex* coupling constant τ as

$$\tau \equiv \frac{\Theta}{2\pi} + i\frac{4\pi}{g_{YM}^2} = C_1\big(\sin\theta_{nc} q(\theta_{nc}) + i\frac{c_{11}}{V_3}\big), \tag{5.83}$$

where the last equality follows from our prior identification (5.82).

The Hamiltonian associated to S_{L1} can be directly read from (2.2) in [58]. Note however that we must do an overall sign change, since we work in the opposite Minkowski signature convention. We must also account for the different overall normalization too. All in all,

$$\begin{aligned} H_{L1} = & \int d^4x \ \mathrm{Tr}\Big(\frac{2i}{\tau-\bar\tau}\Pi^i\Pi_i + i\frac{\tau+\bar\tau}{\tau-\bar\tau}\Pi^i B_i + \frac{i}{2}\frac{\tau\bar\tau}{\tau-\bar\tau}B^i B_i\Big) \\ = & \frac{2i}{\tau-\bar\tau}\int d^4x \ \mathrm{Tr}\Big(\Pi^i + \frac{\tau}{2}B^i\Big)\Big(\Pi_i + \frac{\bar\tau}{2}B_i\Big), \end{aligned} \tag{5.84}$$

where $i = x_1, x_2, \tilde\psi$ spans the spatial coordinates of X_4 and the canonical momenta and magnetic field in our case are given by

$$\Pi^i = \frac{C_1 c_{11}}{V_3}\mathcal{F}^{0i}, \qquad B^i = 2\epsilon^{ijk}\mathcal{F}_{jk}. \tag{5.85}$$

That is, we get the Hamiltonian

$$H_{L1} = \frac{2i}{\tau-\bar\tau}\int d^4x \ \mathrm{Tr}\Big(\frac{C_1 c_{11}}{V_3}\mathcal{F}^{0i} + \tau\epsilon^{ijk}\mathcal{F}_{jk}\Big)\Big(\frac{C_1 c_{11}}{V_3}\mathcal{F}_{0i} + \bar\tau\epsilon_{ilm}\mathcal{F}^{lm}\Big), \tag{5.86}$$

where $\bar\tau$ denotes the complex conjugate of τ. An uncomplicated yet very useful rewriting of this Hamiltonian is

$$H_{L1} = \int d^4x\ \mathrm{Tr}\Big[\frac{\tau-\bar{\tau}}{2i}\sum_{i=1}^{3}(\mathcal{F}_{0i}\mathcal{F}^{0i}) + \frac{4i|\tau|^2}{\tau-\bar{\tau}}\sum_{\alpha,\beta=1}^{2}(\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta}) + \frac{8i|\tau|^2}{\tau-\bar{\tau}}\sum_{\alpha=1}^{2}(\mathcal{F}_{\alpha\tilde{\psi}}\mathcal{F}^{\alpha\tilde{\psi}})$$
$$+ (\tau+\bar{\tau})\sum_{i,j,k=1}^{3}\epsilon_{0ijk}(\mathcal{F}^{0i}\mathcal{F}^{jk})\Big], \tag{5.87}$$

which the reader may verify quite effortlessly.

At this point, we are ready to write the full Hamiltonian following from (4.143), topological piece included:

$$\begin{aligned}
H = \int d^4x\ \mathrm{Tr}\Big\{ &\sum_{\alpha=1}^{2}\Big(\sqrt{\frac{\tau-\bar{\tau}}{2i}}\mathcal{F}_{\alpha 0} - \sqrt{\frac{C_1 c_{\alpha\tilde{3}}}{V_3}}\mathcal{D}_\alpha\mathcal{A}_{\tilde{3}}\Big)^2 + \Big(\sqrt{\frac{\tau-\bar{\tau}}{2i}}\mathcal{F}_{\tilde{\psi}0} - \sqrt{\frac{C_1 c_{\tilde{\psi}\tilde{3}}}{V_3}}\mathcal{D}_{\tilde{\psi}}\mathcal{A}_{\tilde{3}}\Big)^2 \\
&+\frac{C_1}{V_3}\Big[(\sqrt{c_{0r}}\mathcal{D}_0\mathcal{A}_r - i\sqrt{a_2}[\mathcal{A}_{\tilde{3}},\mathcal{A}_r])^2 + (\sqrt{\tilde{c}_{0\phi_1}}\mathcal{D}_0\mathcal{A}_{\phi_1} - i\sqrt{a_4}[\mathcal{A}_{\tilde{3}},\mathcal{A}_{\phi_1}])^2 + c_{0\tilde{3}}(\mathcal{D}_0\mathcal{A}_{\tilde{3}})^2\Big] \\
&+\sum_{k,l=1}^{3}\Big[(\sqrt{b_{0k}}\mathcal{D}_0\varphi_k - i\sqrt{c_{\tilde{3}k}}[\mathcal{A}_{\tilde{3}},\varphi_k])^2 + q_{kl}^{(1)}d_{kl}[\varphi_k,\varphi_l]^2 + \sum_{\gamma=2}^{3}q_k^{(\gamma-1)}c_{y_\gamma k}[\mathcal{A}_{y_\gamma},\varphi_k]^2\Big] \\
&+\sum_{\alpha,\beta=1}^{2}\Big(\sqrt{\frac{2i|\tau|^2}{\tau-\bar{\tau}}}\mathcal{F}_{\alpha\beta} + \sqrt{\frac{C_1 c_{\tilde{\psi}r}}{V_3}}s_{\alpha\beta}^{(1)}\epsilon_{\alpha\beta\tilde{\psi}r}\mathcal{D}_{\tilde{\psi}}\mathcal{A}_r + \sqrt{\frac{C_1\tilde{c}_{\tilde{\psi}\phi_1}}{V_3}}s_{\alpha\beta}^{(2)}\epsilon_{\alpha\beta\tilde{\psi}\phi_1}\mathcal{D}_{\tilde{\psi}}\mathcal{A}_{\phi_1} \\
&-ig_{\alpha\beta}^{(4)}\sqrt{\frac{C_1 a_1}{V_3}}[\mathcal{A}_r,\mathcal{A}_{\phi_1}] - i\sum_{k,l=1}^{3}g_{\alpha\beta kl}^{(1)}\sqrt{d_{kl}}[\varphi_k,\varphi_l] - i\sum_{k=1}^{3}\sum_{\gamma=2}^{3}g_{\alpha\beta k}^{(\gamma-1)}\sqrt{c_{y_\gamma k}}[\mathcal{A}_{y_\gamma},\varphi_k] \\
&+\sum_{\delta,k=1}^{3}\sqrt{b_{\delta k}}\epsilon_{\alpha\beta}\cdot m_{\delta k}^{(1)}\mathcal{D}_\delta\varphi_k\Big)^2 + \sum_{\alpha=1}^{2}\Big(\sqrt{\frac{4i|\tau|^2}{\tau-\bar{\tau}}}\mathcal{F}_{\alpha\tilde{\psi}} + \sqrt{\frac{C_1 c_{\beta r}}{V_3}}t_\alpha^{(1)}\epsilon_{\alpha\tilde{\psi}\beta r}\mathcal{D}_\beta\mathcal{A}_r \\
&+\sqrt{\frac{C_1\tilde{c}_{\beta\phi_1}}{V_3}}t_\alpha^{(2)}\epsilon_{\alpha\tilde{\psi}\beta\phi_1}\mathcal{D}_\beta\mathcal{A}_{\phi_1} - ih_{\alpha\tilde{\psi}}^{(4)}\sqrt{\frac{C_1 a_1}{V_3}}[\mathcal{A}_r,\mathcal{A}_{\phi_1}] - i\sum_{k,l=1}^{3}h_{\alpha\tilde{\psi}kl}^{(1)}\sqrt{d_{kl}}[\varphi_k,\varphi_l] \\
&-i\sum_{k=1}^{3}\sum_{\gamma=2}^{3}h_{\alpha\tilde{\psi}k}^{(\gamma-1)}\sqrt{c_{y_\gamma k}}[\mathcal{A}_{y_\gamma},\varphi_k] + \sum_{\delta,k=1}^{3}\sqrt{b_{\delta k}}\epsilon_{\alpha\tilde{\psi}}\cdot m_{\delta k}^{(2)}\mathcal{D}_\delta\varphi_k\Big)^2 + \frac{C_1}{V_3}\Big[c_{\tilde{\psi}r}s^{(1)}(\mathcal{D}_{\tilde{\psi}}\mathcal{A}_r)^2 \\
&+\tilde{c}_{\tilde{\psi}\phi_1}s^{(2)}(\mathcal{D}_{\tilde{\psi}}\mathcal{A}_{\phi_1})^2 + c_{\beta r}t^{(1)}(\mathcal{D}_\beta\mathcal{A}_r)^2 + \tilde{c}_{\beta\phi_1}t^{(2)}(\mathcal{D}_\beta\mathcal{A}_{\phi_1})^2 + a_1 q^{(4)}[\mathcal{A}_r,\mathcal{A}_{\phi_1}]^2\Big] + \tilde{\chi}_T \\
&+(\tau-\bar{\tau})\sum_{i,j,k=1}^{3}\epsilon_{0ijk}\mathcal{F}^{0i}\mathcal{F}^{jk}\Big\} + \tilde{Q}_{EM}.
\end{aligned} \tag{5.88}$$

All that we have done in writing the above is to couple the Hamiltonian (5.87) to the real scalar fields ($\mathcal{A}_r$, $\mathcal{A}_{\phi_1}$, $\mathcal{A}_{\tilde{3}}$) and $\{\varphi_k\}$'s, with $k = 1, 2, 3$. Our prior meticulous analysis of the $c_2 = 0$ case made this task almost trivial. In details, keeping the last term in (5.87) separate, we coupled the scalar fields as in (5.32). The only difference is that, in the present case, the prefactors for the terms involving field strengths were different, matching the ones in (5.87). Of course, the coefficients that do not have a supergravity interpretation remain constrained as summarized in table 2. Note that the terms ($\tilde{\chi}_T$, $\tilde{Q}_{EM}$) are now written with a tilde to denote they are not the same as those appearing in (5.32), although they still stand for the crossed terms related to the coefficients of table 2 and the electric and magnetic charges in the theory, respectively. Note the close resemblance between the above and the Hamiltonian for the $c_2 = 0$ case in (5.32). Essentially, they are the same up to prefactors in the terms containing field strengths, but there is an all important additional term now: that in the last line of (5.88).

The similarity between the $c_2 = 0$ Hamiltonian and the $c_2 \neq 0$ one allows us to easily generalize the results in section 5.1 to the present and relevant case. In particular, it is remarkably simple to minimize the energy of (5.88) for static configurations. That is, to find the BPS conditions for our gauge and scalar fields. Let us nevertheless show a few steps in the process for clarity, since we will not minimize the energy in exactly the same way as before.

Again, we choose to work in the gauge (5.35) and demand that (5.40) and (5.42) hold true. This time, instead of ensuring the vanishing of the seventh squared term via (5.43), we will choose

$$q^{(1)}_{kl} = 0, \qquad \forall k, l = 1, 2, 3. \tag{5.89}$$

This choice leads to a richer dynamics of the scalar fields $\{\varphi_k\}$'s than that considered in the $c_2 = 0$ case). As we shall see, the above will play an important role in the study of the boundary theory in section 6.3. For the time being, the mentioned choices reduce the Hamiltonian to

$$\begin{aligned} H = \int d^4x \ \mathrm{Tr}\Big\{ &\sum_{\alpha=1}^{2} \Big(\sqrt{\frac{4i|\tau|^2}{\tau - \bar{\tau}}} \mathcal{F}_{\alpha\tilde{\psi}} - i \sum_{k,l=1}^{3} h^{(1)}_{\alpha\tilde{\psi}kl} \sqrt{d_{kl}} [\varphi_k, \varphi_l] + \sum_{\delta,k=1}^{3} \sqrt{b_{\delta k}} \epsilon_{\alpha\tilde{\psi}} \cdot m^{(2)}_{\delta k} \mathcal{D}_\delta \varphi_k \Big)^2 \\ &+ \sum_{\alpha,\beta=1}^{2} \Big(\sqrt{\frac{2i|\tau|^2}{\tau - \bar{\tau}}} \mathcal{F}_{\alpha\beta} - i \sum_{k,l=1}^{3} g^{(1)}_{\alpha\beta kl} \sqrt{d_{kl}} [\varphi_k, \varphi_l] + \sum_{\delta,k=1}^{3} \sqrt{b_{\delta k}} \epsilon_{\alpha\beta} \cdot m^{(1)}_{\delta k} \mathcal{D}_\delta \varphi_k \Big)^2 \\ &+ (\tau + \bar{\tau}) \sum_{i,j,k=1}^{3} \epsilon_{0ijk} \mathcal{F}^{0i} \mathcal{F}^{jk} + \tilde{\chi}_T \Big\} + \tilde{Q}_{EM}. \end{aligned} \tag{5.90}$$

In section 5.1, we did many coefficient choices to simplify the computation as much as possible. On this occasion, we wish to keep our coefficients arbitrary for as long as possible, because this freedom of choice will be beneficial once we look at the boundary theory. Consequently, we will take as our BHN equations the following:

$$\begin{aligned}\sqrt{\frac{4i|\tau|^2}{\tau-\bar{\tau}}}\mathcal{F}_{\alpha\tilde{\psi}} - i\sum_{k,l=1}^{3} h^{(1)}_{\alpha\tilde{\psi}kl}\sqrt{d_{kl}}[\varphi_k,\varphi_l] + \sum_{\delta,k=1}^{3}\sqrt{b_{\delta k}}\epsilon_{\alpha\tilde{\psi}}\cdot m^{(2)}_{\delta k}\mathcal{D}_\delta\varphi_k = 0,\\ \sqrt{\frac{2i|\tau|^2}{\tau-\bar{\tau}}}\mathcal{F}_{\alpha\beta} - i\sum_{k,l=1}^{3} g^{(1)}_{\alpha\beta kl}\sqrt{d_{kl}}[\varphi_k,\varphi_l] + \sum_{\delta,k=1}^{3}\sqrt{b_{\delta k}}\epsilon_{\alpha\beta}\cdot m^{(1)}_{\delta k}\mathcal{D}_\delta\varphi_k = 0,\end{aligned} \tag{5.91}$$

for all α, $\beta = 1, 2$. In view of the detailed computation in section 5.1.2, it is not hard to infer that on this occasion too we will be able to absorb $\tilde{X}_T$ through a meaningless renaming of coefficients by imposing certain consistency requirements to our scalar fields $\{\varphi_k\}$'s. The conditions there derived, namely (5.70), (5.72) and (5.73), are completely independent of the prefactors in the various terms of the Hamiltonian. Hence, the only alteration needed in that calculation consists in accommodating the choice (5.89) instead of (5.43). The attentive reader will surely be easily convinced that the consistency requirements generalize to

$$\mathcal{D}_1\varphi_2 - \mathcal{D}_2\varphi_1 = \mathcal{D}_1\varphi_3 - \mathcal{D}_{\tilde{\psi}}\varphi_1 = \mathcal{D}_2\varphi_3 - \mathcal{D}_{\tilde{\psi}}\varphi_2 = \mathcal{D}_1\varphi_1 + \mathcal{D}_2\varphi_2 + \mathcal{D}_{\tilde{\psi}}\varphi_3 = 0 \tag{5.92}$$

in the present case. Once the energy has thus been minimized, the Hamiltonian reduces to

$$H = (\tau+\bar{\tau})\int d^4x \sum_{i,j,k=1}^{3}\epsilon_{0ijk}\mathrm{Tr}(\mathcal{F}^{0i}\mathcal{F}^{jk}) + \tilde{Q}_{EM}. \tag{5.93}$$

In the following chapter, we will devote quite some effort to the study of the above Hamiltonian. But before jumping into the pertinent details, let us briefly review the main contents of the present chapter.

We have shown that the action (4.143) is associated to the Hamiltonian (5.88). Both of them are defined in the spacetime X_4. A consistent minimization of the energy of (5.88) for static configurations of the fields, working in the gauge (5.35), is obtained by imposing the constraints (5.40), (5.42) and (5.92). We also require that the BHN equations in (5.91) be satisfied. In this energy minimization process, the coefficients of table 2 remain mostly arbitrary. The only choice made is that in (5.89). The on shell Hamiltonian then reduces to (5.93), which is defined in the boundary $X_3 \subset X_4$.

In chapter 5 we have obtained the four-dimensional Hamiltonian (5.88) associated to the action (4.143) derived in chapter 4. This Hamiltonian depends on two kinds of coefficients: some that admit a supergravity interpretation and others that have no physical meaning. The first set already appeared in the action (4.143) and was summarized in table 1. The second, *new set of coefficients is succinctly defined in table 2. Note that the Hamiltonian is of the form sum of squared terms, plus a three-dimensional boundary contribution. We have then obtained a consistent minimization of the energy of (5.88), for static configurations of the fields and working in the axial gauge (5.35), by setting to zero each squared term independently. This leads to the constraints (5.40), (5.42) and (5.92). We also require that the BPS equations in (5.91), which are generalized monopole equations, be satisfied. The Hamiltonian then reduces to the boundary piece (5.93).*

Chapter 6: The boundary theory

As we just mentioned, the minimization of the energy of the Hamiltonian stemming from the M-Theory configuration (M,1) presented in section 5.2 leads to (5.93). In the present section, we will first show that (5.93) is defined only in X_3, the boundary of X_4.

This realization then requires us to find suitable boundary conditions for all the fields in the gauge theory. Of course, we are referring to half-BPS boundary conditions: ones that break the $\mathcal{N} = 4$ supersymmetry of the theory to $\mathcal{N} = 2$. Although so far we have insisted that by construction the configuration (M,1) is $\mathcal{N} = 2$ supersymmetric, it is only at this stage that we shall be able to make this claim fully precise. As we shall see, this desired amount of supersymmetry requires of no constraint on the parameters that characterize (M,1) summarized in table 1 but is enforced by appropriate boundary conditions instead.

Finally, we shall note that, if the configuration (M,1) is to be useful for the study of knots, the theory in X_3 better be topological. In this manner, it will be possible to embed the knots −which are topological objects− in X_3 consistently. To this aim, we will present the notion of topological twist and show that, upon twisting, our gauge theory indeed becomes a suitable framework for the realization of knots.

A graphical summary of the main results of chapter 6 is as shown in blue in figure 10. From this schematic point of view, section 6.1 can be understood as the derivation of the boundary action (6.11). Similarly, section 6.2 contains the details on the half-BPS boundary conditions (6.19)-(6.22) and sections 6.3 and 6.3.1 deal with the technicalities involved in topologically twisting all previously cited results.

6.1 First steps towards determining the boundary theory

In this section, we have one very concrete goal: to rewrite the Hamiltonian of our gauge theory after its energy has been minimized, i.e. (5.93), as an integral over X_3 instead of X_4. Once more, we remind the reader that these spaces were defined and described around (4.1). In other words, we want to show that, for the gauge choice (5.35) and after imposing the BPS conditions (5.40), (5.42), (5.91) and (5.92), the total Hamiltonian (5.88) reduces to a boundary Hamiltonian. As a matter of fact, this does not involve any conceptual hurdle, so let us jump into computation right away.

After having left the electric and magnetic charges $\tilde{Q}_{EM}$ unspecified for the whole of chapter 5, we finally take it upon us to specify them. As we already hinted previously, we will do so by comparing our Hamiltonian (5.88) to that in (2.4) in [54] and then inferring $\tilde{Q}_{EM}$ from (2.5) in that same reference. Obviously, one could do the computation explicitly. However, this won't give us any further insight into our theory and so we do not attempt such approach here. From our identifications in (5.2) and our choice (5.40), it is clear that the electric charge vanishes in our case:

$$\tilde{Q}_{EM} \equiv \tilde{Q}_E + \tilde{Q}_M, \qquad \tilde{Q}_E = 0. \tag{6.1}$$

It is also easy to see that the magnetic charge is of the form

$$\tilde{Q}_M = \int d^4x \; \partial_{\tilde{\psi}} q_M = \int d^3x \; q_M, \qquad d^3x \equiv dt dx_1 dx_2, \tag{6.2}$$

where we have ignored terms which are total derivatives along the unbounded directions $(t,\, x_1,\, x_2)$, since they do not affect the physics of our theory and where we have rewritten $\tilde{Q}_M$ as a boundary term, defined in X_3 instead of the whole of X_4. Of course, this comes as no surprise: we have long been anticipating that the electric and magnetic charges would be restricted to X_3 only. Further using (5.2) and noting that (5.88) is exactly (2.4) in [54] up to prefactors, it is clear that q_M is given by

$$q_M = \sum_{k,l,m=1}^{3} \mathrm{Tr}\left[\sum_{\alpha,\beta=1}^{2} d_1 \epsilon_{k\alpha\beta} \varphi_k \mathcal{F}_{\alpha\beta} + \epsilon_{klm} \left(\frac{i d_2}{3} \varphi_k [\varphi_l, \varphi_m] + d_3 \varphi_k \mathcal{D}_l \varphi_m \right) \right], \tag{6.3}$$

where $(d_1,\, d_2,\, d_3)$ are coefficients that account for the difference of prefactors between our Hamiltonian and that in [54]. Their determination is not straightforward, so let us work them out in details.

Simply looking at our Hamiltonian (5.88), it is evident that the field strength $\mathcal{F}_{\alpha\beta}$ picks up the additional prefactor $\sqrt{2i|\tau|^2/(\tau-\bar{\tau})}$ for all $\alpha,\, \beta = 1,\, 2$, as compared to [54]. Similarly, for fixed values of $(l,\, m)$, it follows that to $\mathcal{D}_l \varphi_m$ we must associate the prefactor $\sqrt{b_{lm}} m^{(1)}_{lm}$[18]. Actually, the only non-trivial prefactors are those that we should attach to $[\varphi_l, \varphi_m]$. To establish what they are, we first note that

$$\sum_{\alpha,\beta=1}^{2} \sum_{k,l=1}^{3} g^{(1)}_{\alpha\beta kl} \sqrt{d_{kl}} [\varphi_k, \varphi_l] = 4\sqrt{d_{12}} \left(g^{(1)}_{1212} [\varphi_1, \varphi_2] + g^{(1)}_{1213} [\varphi_1, \varphi_3] + g^{(1)}_{1223} [\varphi_2, \varphi_3] \right), \tag{6.4}$$

where we have used that $g^{(1)}_{\alpha\beta kl}$ is antisymmetric in $(\alpha,\, \beta)$ and in $(k,\, l)$ by definition (see

[18]To fully understand this prefactor, the reader may find it useful to recall that the dot product appearing in the relevant term of the Hamiltonian was taken to be the usual scalar product around (5.47).

table 2) and d_{kl} is independent of $(k,\, l)$ due to (4.129). From the above it follows that to the $[\varphi_l, \varphi_m]$ term we must associate the factor $4\sqrt{d_{lm}}g^{(1)}_{12lm}$. The next subtlety consists in how to relate the $(\sqrt{2i|\tau|^2/(\tau-\bar{\tau})},\, \sqrt{b_{lm}}m^{(1)}_{lm},\, 4\sqrt{d_{lm}}g^{(1)}_{12lm})$ prefactors to $(d_1,\, d_2,\, d_3)$. Let us address this issue next.

From (2.4) and (2.5) in [54] it can be readily seen that the magnetic charge simply ensures the vanishing of the crossed terms within the BHN-like equations squared of the bulk Hamiltonian. Schematically, if after gauge fixing the bulk Hamiltonian is of the form

$$\int_{X_4} \big(\mathcal{F} + \mathcal{D}\varphi + [\varphi, \varphi]\big)^2, \tag{6.5}$$

then the magnetic charge will be roughly as

$$\int_{X_3} \big(\varphi\mathcal{F} + \varphi[\varphi, \varphi] + \varphi\mathcal{D}\varphi\big). \tag{6.6}$$

Here, the first two terms follow from the $(\mathcal{F}\cdot\mathcal{D}\varphi)$ and $(\mathcal{D}\varphi\cdot[\varphi,\varphi])$ type of crossed terms after integration by parts using the covariant derivative $\mathcal{D}$. For its part, the last term follows from the $(\mathcal{F}\cdot[\varphi,\varphi])$-like crossed term after expressing the field strength as a commutator of covariant derivatives and using these to integrate by parts. Our discussion implies

$$d_1 = \sqrt{\frac{2i|\tau|^2}{\tau-\bar{\tau}}}\sqrt{b_{\tilde{\psi}k}}m^{(1)}_{\tilde{\psi}k}, \qquad d_2 = 4\sqrt{b_{\tilde{\psi}k}d_{lm}}g^{(1)}_{12lm}m^{(1)}_{\tilde{\psi}k}, \qquad d_3 = 4\sqrt{\frac{2i|\tau|^2}{\tau-\bar{\tau}}}\sqrt{d_{lm}}g^{(1)}_{12lm}, \tag{6.7}$$

which fully specifies the magnetic charge in our theory. Note that the indices of these coefficients are to be contracted with the appropriate terms in (6.3).

Once we have the explicit form of $\tilde{Q}_{EM}$ in (5.93), we can focus on the only other term in this Hamiltonian, namely

$$H_{top} \equiv (\tau+\bar{\tau})\int d^4x \sum_{i,j,k=1}^{3} \epsilon_{0ijk}\mathrm{Tr}(\mathcal{F}^{0i}\mathcal{F}^{jk}). \tag{6.8}$$

Recall that $(i,\, j,\, k)$ stand for the spatial directions of X_4: $(x_1,\, x_2,\, \tilde{\psi})$. Recall also that, after our simplifying assumption in (5.77), X_4 is now a Lorentz-invariant space. A quick exercise of opening indices in both (5.79) and the above allows us to rewrite H_{top} as

$$H_{top} = (\tau+\bar{\tau})\int_{X_4} \mathrm{Tr}\big(\mathcal{F}^{(X_4)} \wedge \mathcal{F}^{(X_4)}\big). \tag{6.9}$$

It is well-known that the above can be rewritten as a Chern-Simons type of boundary integral,

$$S_{top} = (\tau + \bar{\tau}) \int_{X_3} \mathrm{Tr}\big(\mathcal{A} \wedge d\mathcal{A} + \frac{2i}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\big), \tag{6.10}$$

which is gauge-invariant iff $(\tau + \bar{\tau})$ is an integer multiple of 2π. We will discuss this subtlety shortly, in section 6.3. For the time being, however, we will just collect our results so far. Using (6.2) and H_{top} in (5.93), we can indeed write the Hamiltonian of our theory, after its bulk energy has been minimized, as a boundary action, the way we wanted:

$$S_{bnd} \equiv \tilde{Q}_M + S_{top} = \int d^3x \; q_M + (\tau + \bar{\tau}) \int_{X_3} \mathrm{Tr}\big(\mathcal{A} \wedge d\mathcal{A} + \frac{2i}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\big), \tag{6.11}$$

with q_M as in (6.3) and the gauge and scalar fields in the theory satisfying the constraint and BHN equations mentioned at the end of the previous chapter.

At this stage, we have been able to minimize the energy of the four-dimensional gauge theory defined in X_4 that follows from the M-Theory configuration (M,1) of part I. By construction, this bulk theory has $\mathcal{N} = 4$ supersymmetry. After such minimization, we have just found out that we are left with a theory whose action is given by (6.11). That is, we have a theory defined on the three-dimensional boundary X_3 of X_4. All through parts I and II, we have insisted that the presence of this boundary provides a half-BPS condition to the full four-dimensional theory, thus reducing the amount of supersymmetry to $\mathcal{N} = 2$. But, of course, this does not happen naturally: in general, arbitrary boundary conditions on the fields break all supersymmetry. In the next section, we derive the constraints required to ensure the desired maximally supersymmetric boundary conditions. In this way, we will finally make precise what we mean when we say that the warp factors in (2.2) and (2.21) and the dilaton in (4.5) should be chosen such that $\mathcal{N} = 2$ supersymmetry is ensured[19].

6.2 Ensuring maximally supersymmetric boundary conditions

Whether boundary conditions that preserve some amount of supersymmetry are possible in a four-dimensional, $\mathcal{N} = 4$ Yang-Mills theory coupled to matter and, if so, what these look like are fundamental questions that were answered in [59]. In this section, we review the relevant results in this work and adapt them to our own model. As we shall see, ensuring that the boundary theory (6.11) previously derived has $\mathcal{N} = 2$ supersymmetry is indeed possible and only requires a mild constraint be satisfied by our supergravity parameters.

[19] We remind the reader that, presently, such supergravity parameters' choice is solely constrained by (5.78), owing to our simplifying assumption in (5.77).

As a first step towards obtaining the much desired $\mathcal{N}=2$ boundary conditions, we must first understand the symmetries of our M-Theory configuration (M,1). As was explained in chapter 2 and as sketched in figure 1, (M,1) is dual to the D3-NS5 system in type IIB. The non-abelian enhanced scenario amounts to considering N number of superposed D3-branes, as argued in section 2.1.1. In the following, we will use this duality to our advantage and discuss the spacetime symmetries of (M,1), in its non-abelian version, in the simpler scenario of the multiple D3's ending on an NS5 system. We remind the reader that the underlying metric and orientations of both the multiple D3-branes and the single NS5-brane in this setup were introduced right at the beginning of chapter 2 and are graphically summarized in figure 2**A**. It is also worth bearing in mind that, upon dimensional reduction, the four-dimensional gauge theory in the world-volume of the D3-branes has $SU(N)$ as its gauge group and $\mathcal{N}=4$ supersymmetry. Having refreshed a bit our memory, it is easy enough to argue what symmetries are present in the D3-NS5 system.

Consider the usual type IIB String Theory. This is defined in $\mathbb{R}^{1,9}$. We will label the corresponding coordinates as x_I, with $I = 0, 1, \ldots, 9$. The associated metric is simply $\eta_{IJ} = \text{diag}(-1, 1, \ldots, 1)$. Hence, the spacetime symmetry group is $SO(1,9)$. As is well-known, $SO(1,9)$ is generated by Gamma matrices Γ_I, which satisfy the usual Clifford algebra

$$\{\Gamma_I, \Gamma_J\} = 2\eta_{IJ}, \tag{6.12}$$

and has **16** as is its irrep. Here, we consider a ten-dimensional gauge field and Majorana-Weyl fermion, related to each other by their supersymmetry transformations. We denote as ε the supersymmetry generator. This is a Majorana-Weyl spinor that satisfies

$$\bar{\Gamma}\varepsilon = \varepsilon, \qquad \bar{\Gamma} \equiv \Gamma_0\Gamma_1 \ldots \Gamma_9 \tag{6.13}$$

and consequently it transforms in the **16** of $SO(1,9)$. Here, $\Gamma_0\Gamma_1 \ldots \Gamma_9$ stands for the anti-symmetrized product of $(\Gamma_0, \Gamma_1, \ldots, \Gamma_9)$.

The inclusion of multiple, coincident D3-branes breaks $SO(1,9)$ to $SO(1,3) \times SO(6)$, the $SO(1,3)$ oriented along the same directions as the D3's. The NS5-brane further breaks the symmetry group to

$$\mathcal{U} \equiv SO(1,2) \times SO(3) \times SO(3). \tag{6.14}$$

This is most easily understood in two steps. First, the NS5-brane restricts one of the spatial coordinates of the D3-branes to take only non-negative values. In our notation: $\psi \geq 0$, as can be seen in figure 2**A**. Demanding that Lorentz transformations leave the boundary $\psi = 0$ invariant, $SO(1,3)$ breaks down to $SO(1,2)$. On the other hand, the NS5-brane also breaks $SO(6)$ to $SO(3) \times SO(3)$. One of these $SO(3)$'s acts on the three-dimensional subspace spanned by the NS5-brane which is orthogonal to the directions shared with the D3's. In the language of figure 2**A**, it acts on (x_3, x_8, x_9). The other $SO(3)$ then acts on the remaining spacetime directions. These are the directions labeled by $(\theta_1,\, \phi_1,\, r)$ and suppressed in figure 2**A**. We denote as $\boldsymbol{V_8}$ the irrep of $\mathcal{U}$: the $(\mathbf{2},\mathbf{2},\mathbf{2})$ tensor product.

Having established $\mathcal{U}$ in (6.14) as the symmetry group of the D3-NS5 system, it follows that $\mathcal{U}$ is the symmetry of the configuration (M,1) too. However, caution is needed: some of the dualities required to obtain (M,1) from the D3-NS5 system are non-trivial. For example, consider the T-duality relating figure 2**C** to 2**D**. Consequently, for our coming analysis to hold true, any specific choice of the warp factors in (2.2) and (2.21) and constant dilaton in (4.5) should be checked to not only enforce the constraint (5.78), but should also be $\mathcal{U}$-invariant.

Focusing on the case where (M,1) is indeed $\mathcal{U}$-invariant, we can precisely reproduce the results in [14]. Let us see how. As we saw in chapter 4, the scalar fields associated to the directions on which the $SO(3)$'s of $\mathcal{U}$ act are $(\mathcal{A}_{\bar{3}},\, \varphi_1,\, \varphi_2)$ and $(\varphi_3,\, \mathcal{A}_{\phi_1},\, \mathcal{A}_r)$, respectively. In the language of [14, 59], these are collectively referred to as $\vec{X}$ and $\vec{Y}$:

$$\vec{X} \equiv (\mathcal{A}_{\bar{3}}, \varphi_1, \varphi_2), \qquad \vec{Y} \equiv (\varphi_3, \mathcal{A}_{\phi_1}, \mathcal{A}_r), \tag{6.15}$$

an identification that will soon prove useful to us.

Let us make yet one more observation before we determine the desired half-BPS boundary conditions. We note that the **16** of $SO(1,9)$ naturally decomposes as

$$\mathbf{16} = \boldsymbol{V_8} \otimes \boldsymbol{V_2}, \tag{6.16}$$

where $\boldsymbol{V_2}$ is a 2-dimensional real vector space. The natural elements that act on $\boldsymbol{V_2}$ are the even elements of the $SO(1,9)$ Clifford algebra that commute with $\mathcal{U}$. It follows then that the supersymmetry generator ε can be decomposed as

$$\varepsilon = \varepsilon_8 \otimes \varepsilon_2, \qquad \varepsilon_8 \in \boldsymbol{V_8}, \qquad \varepsilon_2 \in \boldsymbol{V_2}. \tag{6.17}$$

In order for ε to be $\mathcal{U}$-invariant, ε_2 must be a non-zero, fixed element of $\boldsymbol{V_2}$. On the other hand, ε_8 is just some arbitrary element of $\boldsymbol{V_8}$. Again following [14, 59], we choose

$$\varepsilon_2 = \begin{pmatrix} -a \\ 1 \end{pmatrix}, \tag{6.18}$$

with a a real parameter. The above is precisely the last ingredient we needed to finally discuss half-BPS boundary conditions in the four-dimensional gauge theory following from (M,1).

It is well-established (for example, see [60]) that boundary conditions preserve some degree of supersymmetry iff they ensure that the normal (to the boundary) component of the corresponding supercurrent vanishes. This in turn constrains the associated supersymmetry generator too. Thanks to the above discussion and, in particular, to our identifications (6.15), we can directly read off from [14, 59] the boundary conditions and constraint on ε_2 thus obtained. We refer the interested reader to [59] for a detailed derivation of the results we now quote. The boundary conditions on the fields are as follows. The scalar fields (φ_3, $\mathcal{A}_{\phi_1}$, $\mathcal{A}_r$) must all vanish at $\tilde{\psi} = 0$:

$$\varphi_3 = \mathcal{A}_{\phi_1} = \mathcal{A}_r = 0. \tag{6.19}$$

The remaining scalar fields must satisfy

$$\mathcal{D}_{\tilde{\psi}}\mathcal{A}_{\bar{3}} - \frac{2a}{1+a^2}[\varphi_1, \varphi_2] = 0, \quad \mathcal{D}_{\tilde{\psi}}\varphi_1 - \frac{2a}{1+a^2}[\varphi_2, \mathcal{A}_{\bar{3}}] = 0, \quad \mathcal{D}_{\tilde{\psi}}\varphi_2 - \frac{2a}{1+a^2}[\mathcal{A}_{\bar{3}}, \varphi_1] = 0 \tag{6.20}$$

at the boundary. Due to our choice (5.40), the above further simplifies to

$$[\varphi_1, \varphi_2] = \mathcal{D}_{\tilde{\psi}}\varphi_1 = \mathcal{D}_{\tilde{\psi}}\varphi_2 = 0, \tag{6.21}$$

for a general value of the parameter a. At $\tilde{\psi} = 0$, the gauge fields are required to obey

$$\mathcal{F}_{\tilde{\psi}\mu} + \frac{a}{1-a^2}\epsilon_{\mu\nu\lambda}\mathcal{F}^{\nu\lambda} = 0, \qquad \forall\mu, \tag{6.22}$$

where (μ, ν, λ) label the spacetime directions (t, x_1, x_2, $\tilde{\psi}$). As for the constraint on the supersymmetry generator, it relates the parameter a in (6.18) to the Yang-Mills coupling and gauge theory Θ-parameter as

$$\frac{\Theta/(2\pi)}{4\pi/g_{YM}^2} = \frac{2a}{1-a^2}. \tag{6.23}$$

Owing to our prior identifications in (5.82), we can give a supergravity interpretation to a:

$$\frac{V_3 \sin\theta_{nc} q(\theta_{nc})}{c_{11}} = \frac{2a}{1-a^2} \quad \to \quad a = \sqrt{1 + \left(\frac{c_{11}}{V_3 \sin\theta_{nc} q(\theta_{nc})}\right)^2} - \frac{c_{11}}{V_3 \sin\theta_{nc} q(\theta_{nc})}. \tag{6.24}$$

Yet another way to express the same relation follows from using (4.112) and (5.82) in (6.23):

$$c_2 = \frac{4\pi}{g_{YM}^2}\frac{2a}{1-a^2}. \tag{6.25}$$

Now that our boundary theory in (6.11) is $\mathcal{N} = 2$-supersymmetric, we still need to overcome one more difficulty. If our M-Theory configuration (M,1) and the four-dimensional gauge theory stemming from it through dimensional reduction are to be of use in the study of knots and their invariants: what is the three-dimensional space where knots should be realized? Undoubtedly, X_3 spanned by (t, x_2, x_2). Or more precisely, its Euclidean version. Now, since knots are topological objects, it is clear that the theory in X_3 ought to be topological too. At least, this should be the case for our construction to be an appropriate framework to support knots. However, a quick look at our action (6.11) immediately tells us that this is not the case in our setup. The second, Chern-Simons term in the boundary action is indeed topological, but the presence of the magnetic charge adds a non-topological contribution that naively seems undesirable from our point of view. The resolution to this puzzle was first worked out in the well-known work [61] and it consists in performing a so-called topological twist to our four-dimensional gauge theory. In the following, we summarize the basics of this technique and apply it to our own model.

6.3 Obtaining a Chern-Simons boundary action: topological twist

We begin this section by introducing the concept of topological twist. Following which, we shall show that topologically twisting our gauge theory, its corresponding boundary action is Chern-Simons-like.

If we momentarily ignore the fact that $\tilde{\psi} \geq 0$, then the symmetry of our M-Theory configuration (M,1) is as in (6.14), but with $SO(1,2)$ replaced by $SO(1,3)$. In this case, the topological twist consists in extending the Lorentz symmetry $SO(1,3)$ acting along $(t,\, x_1,\, x_2,\, \tilde{\psi})$ to a new symmetry S'. S' rotates the $(t,\, x_1,\, x_2,\, \tilde{\psi})$ subspace and, simultaneously, the $(\tilde{x}_3,\, \theta_1,\, x_8,\, x_9)$ subspace too. It is not hard to see that this new symmetry necessarily leads to the reinterpretation of the scalar fields $(\mathcal{A}_{\tilde{3}},\, \varphi_1,\, \varphi_2,\, \varphi_3)$ associated to the new rotation directions as a one-form:

$$\Phi = \sum_\mu \Phi_\mu dx^\mu, \qquad (\Phi_0, \Phi_1, \Phi_2, \Phi_3) = i(\varphi_3, \varphi_1, \varphi_2, \mathcal{A}_{\tilde{3}}). \tag{6.26}$$

There should be no confusion regarding notation. As introduced in (5.79) and used through all the previous section, x_μ refers to the spacetime coordinates $(t,\, x_1,\, x_2,\, \tilde{\psi})$. The precise identification between the components of this one-form and our scalars suggested above is such that we match the notation in [14]. However, other identifications could also be entertained. In fact, we will do so later on, in section 6.3.1.

As a short aside, it will soon prove useful to introduce some notation. Following [14], we combine the real scalar fields $(\mathcal{A}_{\phi_1},\, \mathcal{A}_r)$ associated to the directions $(\phi_1,\, r)$ not affected by S' into a complex scalar field:

$$\sigma \equiv \mathcal{A}_r + i\mathcal{A}_{\phi_1}, \qquad \bar{\sigma} = \mathcal{A}_r - i\mathcal{A}_{\phi_1}. \tag{6.27}$$

In the same spirit, we shall topologically twist our gauge fields to

$$A = \sum_{\mu} A_\mu dx^\mu, \qquad A_\mu = i\mathcal{A}_\mu, \qquad \forall \mu. \tag{6.28}$$

The corresponding field strengths are then

$$F = dA + A \wedge A = \sum_{\mu,\nu} F_{\mu\nu} dx^\mu \wedge dx^\nu, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{6.29}$$

Clearly, this demands that we define new covariant derivatives, which happen to match the ones used so far and introduced earlier in (4.51):

$$D_\mu \equiv \partial_\mu + [A_\mu, \quad] = \partial_\mu + i[\mathcal{A}_\mu, \quad] \equiv \mathcal{D}_\mu, \qquad \forall \mu. \tag{6.30}$$

Of course, the above topological twist must be made compatible with the fact that $\tilde{\psi} \geq 0$ in our setup, before we can apply it to our four-dimensional gauge theory. What is more, it must also be made compatible with having $\mathcal{N} = 2$ supersymmetric boundary conditions on the fields. In other words, before proceeding further, all the results in section 6.2 must be extended to the case where the gauge theory is twisted. Such generalization was first done in [14, 61], where the reader may find all the computational details. In the following, we simply review the main pertinent results in these works, while adapting them to our present construction.

We begin by making the supersymmetry generator ε in (6.13) compatible with the new symmetry S'. That is, we demand

$$(\Gamma_{\mu\nu} + \Gamma_{\tilde{\mu}\tilde{\nu}})\varepsilon = 0, \qquad \forall \mu, \nu = t, x_1, x_2, \tilde{\psi}, \qquad \forall \tilde{\mu}, \tilde{\nu} = \tilde{x}_3,\, \theta_1,\, x_8,\, x_9, \tag{6.31}$$

so that ε is S'-invariant. This condition has a two-dimensional space of solutions. If we denote as $(\varepsilon_l,\, \varepsilon_r)$ the basis of solutions, then the supersymmetry generator can be written as a linear combination of them both:

$$\varepsilon = \varepsilon_l + \hat{t}\varepsilon_r, \qquad \hat{t} \in \mathbb{C}, \tag{6.32}$$

where the hat on $\hat{t}$ is meant to differentiate the above complex variable from the time coordinate t. At this point, one repeats the same procedure as in the previous section: one requires that the component of the supercurrent associated to ε above that is normal to the $\tilde{\psi} = 0$ boundary vanishes. In this manner, we reproduce the same boundary conditions (6.19)-(6.22) as before, but in the twisted case:

$$\sigma = \bar{\sigma} = \Phi_0 = [\Phi_1, \Phi_2] = D_{\tilde{\psi}}\Phi_1 = D_{\tilde{\psi}}\Phi_2 = F_{\tilde{\psi}\mu} - \frac{i}{2}\frac{\hat{t}^2 + 1}{\hat{t}^2 - 1}\epsilon_{\mu\nu\lambda}F^{\nu\lambda} = 0, \qquad \forall \mu. \tag{6.33}$$

Comparing the last boundary condition above with its untwisted counterpart in (6.22), it follows that the parameters a and $\hat{t}$ are related to each other. Since a is additionally related to the gauge theory parameters (g^2_{YM}, Θ), so must $\hat{t}$ be. These relationships also follow from studying the constraint imposed on the supersymmetry generator by demanding the vanishing of the normal component of its supercurrent. In this latter approach, as shown in [14], the constraint that ε in (6.32) must satisfy turns out to be the exact same constraint that ε_2 in (6.18) has to satisfy in the untwisted case, which then led us to (6.23). Either of the two approaches yields

$$\hat{t} = -i\frac{1+ia}{1-ia}. \tag{6.34}$$

The above can be rewritten in many suggestive ways. For example, using (6.23), we can write $\hat{t}$ as a function of the Yang-Mills coupling and Θ-parameter of our gauge theory: $\hat{t} = \hat{t}(g^2_{YM}, \Theta)$. Further using (5.82), we can express $\hat{t}$ in terms of supergravity parameters of our M-Theory configuration (M,1): $\hat{t} = \hat{t}(c_{11}, V_3, \theta_{nc})$. A particularly neat result follows from considering (5.83) as well:

$$\hat{t} = \pm\frac{|\tau|}{\tau}, \tag{6.35}$$

which the reader can verify without excessive algebraic effort. The above is interesting because it is not obvious a priori that the two complex parameters $(\tau, \hat{t})$ that characterize the twisted gauge theory should be related to one another. Additionally, it is surprising that they should have such a mathematically simple relation.

Having introduced the topological twist and verified its consistency with all the (super)symmetries in our setup, we can proceed to twist the boundary action (6.11). As anticipated, this will give rise to a topological theory in X_3. Let us see how exactly.

Using (6.26)-(6.30) in (6.11), we verify that the boundary theory after twisting becomes

$$S^{(t)}_{bnd} = -\int d^3x\ q^{(t)}_M - (\tau+\bar{\tau})\int_{X_3}\mathrm{Tr}\big(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\big). \tag{6.36}$$

From (6.3), the twisted magnetic charge density $q^{(t)}_M$ can be easily seen to be

$$q^{(t)}_M = \sum_{a,b,c=0}^{2}\mathrm{Tr}\Big[\sum_{\alpha,\beta=1}^{2} d_1\epsilon_{a\alpha\beta}\Phi_a F_{\alpha\beta} + \epsilon_{abc}\Big(\frac{d_2}{3}\Phi_a[\Phi_b,\Phi_c] + d_3\Phi_a D_b\Phi_c\Big)\Big], \tag{6.37}$$

with (d_1, d_2, d_3) as in (6.7), albeit the indices there need to be appropriately reinterpreted. As we will soon open up all indices and make explicit their meaning, the reader should not worry too much over notation at this stage. It is perhaps worth mentioning that, in the last term,

$D_{\tilde{\psi}}$ does not appear, unlike in the untwisted case (6.3). This is simply because the boundary conditions (6.33) guarantee no such contribution occurs. On the other hand, although (5.35) and (5.40) also force $D_0\Phi = 0$, we shall carry these vanishing terms around because they will make the coming derivation of the topological boundary action more transparent. It goes without saying that one can do the same calculation without them too.

It turns out, however, that (6.36) is not quite the correct twisted boundary theory. One more term, proportional to the Chern-Simons term in (6.36), must be added to the above:

$$S^{(t)}_{bnd,tot} = S^{(t)}_{bnd} + b_2 \int_{X_3} \mathrm{Tr}\big(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\big), \qquad b_2 \in \mathbb{C}. \tag{6.38}$$

This additional term is required to ensure that all observables and states on the twisted gauge theory are invariant under the supersymmetry generated by ε in (6.32). Upon including such term, one more striking observation can be made: not only are τ and $\hat{t}$ related to each other, but also all physics of the twisted theory depends solely on a particular combination of the two parameters:

$$\Psi \equiv \frac{\tau + \bar{\tau}}{2} + \frac{\tau - \bar{\tau}}{2} \frac{\hat{t} - \hat{t}^{-1}}{\hat{t} + \hat{t}^{-1}}. \tag{6.39}$$

Ψ is usually referred to as *the canonical parameter* and it appears in the correct boundary theory as

$$S^{(t)}_{bnd,tot} = -\int d^3x \; q^{(t)}_M + i\Psi \int_{X_3} \mathrm{Tr}\big(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\big). \tag{6.40}$$

Note that this allows us to determine the value of b_2, the coefficient of the required extra piece in the boundary action, since

$$-(\tau + \bar{\tau}) + b_2 = i\Psi \quad \Longrightarrow \quad b_2 = \frac{\tau + \bar{\tau}}{2}(2 + i) + i \frac{\tau - \bar{\tau}}{2} \frac{\hat{t} - \hat{t}^{-1}}{\hat{t} + \hat{t}^{-1}}. \tag{6.41}$$

Of course, none of the statements in the above paragraph are obvious. Their proofs were worked out in exquisite detail in sections 3.4 and 3.5 of [61]. Unfortunately, a review of these derivations is beyond the scope of the thesis. Nonetheless, the reader should find no difficulty going through the cited reference, as we have carefully made our notation coincident with the one employed there.

Having established (6.40) as the twisted boundary action, showing its topological nature amounts to appropriately rewriting it. We will do so in a few steps, the first consisting in expressing the twisted magnetic charge density $q^{(t)}_M$ in differential geometry language. To this aim, let us first introduce the exterior covariant derivative of the twisted scalar fields (6.26):

$$d_A\Phi \equiv d\Phi + [A, \Phi]. \tag{6.42}$$

If we restrict $d_A\Phi$ to X_3 (where $\tilde{\psi} = 0$ and thus $d\tilde{\psi} = 0$ too) and since $\Phi_3 = 0$ due to (5.40) and (6.26), the above can be explicitly written as

$$\begin{aligned} d_A\Phi &= \sum_{a,b=0}^{2} \Big(\frac{\partial \Phi_b}{\partial x^a} dx^a \wedge dx^b + [A_a dx^a, \Phi_b dx^b]\Big) \\ &= (D_0\Phi_1 - D_1\Phi_0) dt \wedge dx_1 + (D_0\Phi_2 - D_2\Phi_0) dt \wedge dx_2 + (D_1\Phi_2 - D_2\Phi_1) dx_1 \wedge dx_2. \end{aligned} \tag{6.43}$$

Then, we can use (6.43) to introduce three more quantities, defined in X_3, that will soon become relevant to us:

$$\begin{aligned} \Phi \wedge F &= \Big(\sum_{a=0}^{2} \Phi_a dx^a\Big) \wedge \Big(\sum_{\alpha,\beta=1}^{2} F_{\alpha\beta} dx^\alpha \wedge dx^\beta\Big) = 2\Phi_0 F_{12} d^3x, \\ \Phi \wedge \Phi \wedge \Phi &= (\Phi_0[\Phi_1, \Phi_2] - \Phi_1[\Phi_0, \Phi_2] + \Phi_2[\Phi_0, \Phi_1]) d^3x, \\ \Phi \wedge d_A\Phi &= [\Phi_0(D_1\Phi_2 - D_2\Phi_1) - \Phi_1(D_0\phi_2 - D_2\Phi_0) + \Phi_2(D_0\Phi_1 - D_1\Phi_0)] d^3x. \end{aligned} \tag{6.44}$$

We remind the reader that $d^3x = dt \wedge dx_1 \wedge dx_2$ is the normalized volume element of X_3. Note that, in the above, we did not take into account the whole twisted field strength introduced in (6.29). The reasons are similar to those which led us to (6.43). Specifically, $F_{0\mu} = 0$ for all μ, due to the constraint (5.35) and our gauge choice (5.40). Also, $\tilde{\psi} = 0$ at the three-dimensional boundary X_3 of our spacetime X_4, implying $d\tilde{\psi} = 0$ there and thus no field strength stretches along this direction.

To appreciate the benefit of having calculated (6.44), let us now carry out the sums in (6.37). In doing so, we shall use (6.7) and, through explicit computation, clear any doubt regarding index notation, as previously promised. The first sum can be easily seen to yield

$$\sum_{a,b,c=0}^{2} \sum_{\alpha,\beta=1}^{2} d_1 \epsilon_{a\alpha\beta} \Phi_a F_{\alpha\beta} = 2\sqrt{\frac{2i|\tau|^2}{\tau - \bar{\tau}}} \sqrt{b_{\tilde{\psi}3}}\, m^{(1)}_{\tilde{\psi}3} \Phi_0 F_{12}, \tag{6.45}$$

with the normalization convention $\epsilon_{012} = 1$. The second sum gives

$$\sum_{a,b,c=0}^{2} d_2 \epsilon_{abc} \Phi_a [\Phi_b, \Phi_c] = 8\sqrt{b_{\tilde{\psi}3} d_{12}}\, m^{(1)}_{\tilde{\psi}3} (g^{(1)}_{1212} \Phi_0[\Phi_1, \Phi_2] - g^{(1)}_{1232} \Phi_1[\Phi_0, \Phi_2] + g^{(1)}_{1231} \Phi_2[\Phi_0, \Phi_1]), \tag{6.46}$$

where we have used the fact that d_{kl} in (4.129) is independent of $(k,\, l)$ to take d_{12} as common factor. The third and last sum appearing in the twisted magnetic charge density is

$$\sum_{a,b,c=0}^{2} \epsilon_{abc} d_3 \Phi_a D_b \Phi_c = 4\sqrt{\frac{2i|\tau|^2}{\tau-\bar{\tau}}}\sqrt{d_{12}}\Big[g^{(1)}_{1212}\Phi_0\big(D_1\Phi_2 - D_2\Phi_1\big) - g^{(1)}_{1232}\Phi_1\big(D_0\Phi_2 - D_2\Phi_0\big) + g^{(1)}_{1231}\Phi_2\big(D_0\Phi_1 - D_1\Phi_0\big)\Big]. \tag{6.47}$$

Recall that, so far, we have only made the choice of coefficients in (5.89). We shall now make further choices. In particular, we want to impose

$$g^{(1)}_{1212} = g^{(1)}_{1232} = g^{(1)}_{1231}. \tag{6.48}$$

It is important to note that our choices are in good agreement with the defining relation (5.22), since we have the full spectrum of $\{h^{(1)}_{\alpha\tilde{\psi}kl}\}$'s unfixed to satisfy those equalities. Now, comparing our prior auxiliary quantities in (6.44) with the sums (6.45)-(6.47), it follows that $q^{(t)}_M$ in (6.37) can be written in the very convenient form

$$\int d^3x\; q^{(t)}_M = -\int_{X_3} \mathrm{Tr}\big(2D_1\Phi\wedge F + \frac{2}{3}D_2\Phi\wedge\Phi\wedge\Phi + D_3\Phi\wedge d_A\Phi\big), \tag{6.49}$$

where we have defined the coefficients $(D_1,\, D_2,\, D_3)$ as

$$D_1 \equiv -\frac{1}{2}\sqrt{\frac{2i|\tau|^2}{\tau-\bar{\tau}}}\sqrt{b_{\tilde{\psi}3}}m^{(1)}_{\tilde{\psi}3}, \quad D_2 \equiv -4\sqrt{b_{\tilde{\psi}3}d_{12}}m^{(1)}_{\tilde{\psi}3}g^{(1)}_{1212}, \quad D_3 \equiv -4\sqrt{\frac{2i|\tau|^2}{\tau-\bar{\tau}}}\sqrt{d_{12}}g^{(1)}_{1212}. \tag{6.50}$$

Using the above in our boundary action (6.40), we obtain

$$S^{(t)}_{bnd,tot} = \int_{X_3} \mathrm{Tr}\big(2D_1\Phi\wedge F + \frac{2}{3}D_2\Phi\wedge\Phi\wedge\Phi + D_3\Phi\wedge d_A\Phi\big) + i\Psi\int_{X_3}\mathrm{Tr}\big(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\big). \tag{6.51}$$

The second step required to rewrite (6.51) as a topological action consists in suitably fixing $(D_2,\, D_3)$. Specifically, we require

$$D_2 = \frac{D_1^3}{(i\Psi)^2}, \qquad D_3 = \frac{D_1^2}{i\Psi}. \tag{6.52}$$

From (6.50) it follows that, in terms of the coefficients of tables 1 and 2 (the first ones having a supergravity interpretation and the second ones lacking it), the above constraints are

$$1=\frac{b_{\tilde{\psi}3}}{32\sqrt{d_{12}}g^{(1)}_{1212}}\left(\frac{m^{(1)}_{\tilde{\psi}3}}{i\Psi}\right)^2\left(\frac{2i|\tau|^2}{\tau-\bar{\tau}}\right)^{3/2},\qquad 1=-\frac{b_{\tilde{\psi}3}}{16\sqrt{d_{12}}g^{(1)}_{1212}}\frac{\left(m^{(1)}_{\tilde{\psi}3}\right)^2}{i\Psi}\sqrt{\frac{2i|\tau|^2}{\tau-\bar{\tau}}}. \tag{6.53}$$

Of course, we would rather not constrain our model (M,1) by demanding that the warp factors (2.2) and constant dilaton (4.5) appearing in its metric (2.46) are obliged to satisfy the above. Instead, we prefer to choose our coefficients of table 2 in such a way that (6.53) is true. Clearly, there is just enough freedom of choice for us to do so: we thus fix $(m^{(1)}_{\tilde{\psi}3}, g^{(1)}_{1212})$. Note that these coefficients must fulfill (5.22) and (5.30), where we have already chosen (5.89) and (6.48). This is possible and simply fixes $(m^{(2)}_{\tilde{\psi}3}, h^{(1)}_{1\tilde{\psi}12})$ as well. In this case and after some easy algebra, we obtain

$$\begin{aligned} S^{(t)}_{bnd,tot}=&i\Psi\int_{X_3}\mathrm{Tr}\Big(A\wedge dA+\frac{2}{3}A\wedge A\wedge A+2\Phi'\wedge dA+2\Phi'\wedge A\wedge A\\ &+\frac{2}{3}\Phi'\wedge\Phi'\wedge\Phi'+\Phi'\wedge d\Phi'+\Phi'\wedge[A,\Phi']\Big), \end{aligned}\tag{6.54}$$

where we have used (6.29) and(6.42) and where Φ' is just the one-form Φ in (6.26) rescaled in the following manner:

$$\Phi'\equiv\frac{D_1}{i\Psi}\Phi. \tag{6.55}$$

A couple of trace identities come in handy at this stage. These shall allow us to further rewrite the boundary theory in what will soon become a particularly enlightening form. The identities in question are

$$\mathrm{Tr}(\Phi'\wedge[A,\Phi'])=2\mathrm{Tr}(\Phi'\wedge A\wedge\Phi'),\qquad \mathrm{Tr}(A\wedge d\Phi')=\mathrm{Tr}(\Phi'\wedge dA), \tag{6.56}$$

which the reader may easily verify through explicit computation with the aid of (5.35), (5.40), (6.26), (6.28) and (6.29). The second identity holds up to a total derivative only. However, since these terms are defined in X_3, the three-dimensional space labeled by the unbound directions $(t,\, x_1,\, x_2)$, the total derivative term does not affect the physics following from $S^{(t)}_{bnd,tot}$ and so we ignore it in the ongoing. Combining (6.54) and (6.56), we obtain

$$\begin{aligned} S^{(t)}_{bnd,tot}=&i\Psi\int_{X_3}\mathrm{Tr}\Big(A\wedge dA+\frac{2}{3}A\wedge A\wedge A+2A\wedge d\Phi'+2\Phi'\wedge A\wedge A\\ &+\frac{2}{3}\Phi'\wedge\Phi'\wedge\Phi'+\Phi'\wedge d\Phi'+2\Phi'\wedge A\wedge\Phi'\Big). \end{aligned}\tag{6.57}$$

The third and last step on our way towards a topological boundary theory consists in defining a *modified* gauge field, which is a linear combination of the twisted gauge and scalar fields (6.26) and (6.28):

$$A_D \equiv A + \Phi'. \tag{6.58}$$

In fact, A_D is the *complexification* of A. It is a matter of simple algebra to check that

$$\begin{aligned} A_D \wedge dA_D =& A \wedge dA + \Phi' \wedge dA + A \wedge d\Phi' + \Phi' \wedge d\Phi', \\ A_D \wedge A_D \wedge A_D =& A \wedge A \wedge A + A \wedge \Phi' \wedge \Phi' + \Phi' \wedge A \wedge A + \Phi' \wedge \Phi' \wedge \Phi' \\ &+ A \wedge A \wedge \Phi' + A \wedge \Phi' \wedge A + \Phi' \wedge A \wedge \Phi' + \Phi' \wedge \Phi' \wedge A. \end{aligned} \tag{6.59}$$

Since the trace of a product is invariant under cyclic permutations of the terms in that product and also due to (6.56), it is easy to see that, as promised, indeed (6.57) defines a topological field theory in X_3, albeit in terms of the just introduced modified gauge field A_D:

$$S^{(t)}_{bnd,tot} = i\Psi \int_{X_3} \mathrm{Tr}(A_D \wedge dA_D + \frac{2}{3} A_D \wedge A_D \wedge A_D). \tag{6.60}$$

The above *complexified* Chern-Simons satisfies the goal stated at the beginning of the present section. Yet, before proceeding ahead, there are a couple of issues worth mentioning.

First, we note that in (6.60) there is still one free parameter: D_1. Recall that Ψ is given by (6.39). Hence, it depends only on $(\tau, \hat{t})$. These two parameters have an interpretation in terms of our supergravity parameters, i.e. the warp factors and dilaton of the M-Theory configuration (M,1). As such, they are fixed when a specific model (M,1) is considered. It turns out that D_1 can also be determined. As argued in [14], supersymmetric Wilson loop operators can be associated to the boundary theory with action (6.60) iff the Chern-Simons gauge field A_D is invariant under the supersymmetry generated by ε in (6.32). Schematically, we can express this as

$$\delta A_D = \delta(A + \Phi') = \delta(A + \frac{D_1}{i\Psi}\Phi) = 0, \tag{6.61}$$

where we have made use of (6.55) and (6.58). As our notation is now such that it precisely matches the one used in [14], the interested reader should have no difficulty in following the discussion in section 2.2.4 of that same reference. In it, the reader shall find the proof that the above constraint sets the value of D_1 to

$$D_1 = i\Psi \frac{t - t^{-1}}{2} = \frac{i}{4}(t - t^{-1})\Big[\tau + \bar{\tau} + (\tau - \bar{\tau})\frac{\hat{t} - \hat{t}^{-1}}{\hat{t} + \hat{t}^{-1}}\Big], \tag{6.62}$$

where the second equality follows from (6.39). As we just said, $(\tau, \hat{t})$ are fixed for a given model (M,1). However, from (6.50), we see that D_1 depends on various coefficients: (τ, $b_{\tilde{\psi}3}$, $m^{(1)}_{\tilde{\psi}3}$). As given by (4.123), $b_{\tilde{\psi}3}$ is also fixed once a particular model (M,1) is chosen via warp factors and constant dilaton. We remind the reader that $m^{(1)}_{\tilde{\psi}3}$ was already chosen to take the value that renders the first equality in (6.53) true. Consequently, on this occasion we have no way out but to constrain the model (M,1) by demanding that from the very beginning it is defined in such a way that D_1, understood as in (6.50), satisfies (6.62).

Second, we must refer to the point already mentioned in passing in section 6.1. Namely, the fact that the non-abelian Chern-Simons theory (6.60) is gauge-invariant iff ($i\Psi$) is an integer multiple of 2π[20]. In other words, a path integral formalism associated to the action (6.60) is only well-defined when

$$(-2\pi i)^{-1})\Psi \in \mathbb{Z}. \tag{6.63}$$

From its very definition in (6.39), we see that Ψ does not necessarily satisfy such a property. Perhaps this observation is even more evident from (5.83) and (6.35), expressing Ψ only in terms of coefficients with a supergravity interpretation, which depend only on the specific choice of M-Theory model (M,1):

$$\Psi = C_1 \sin\theta_{nc} q(\theta_{nc}) - \frac{C_1 c_{11}^2}{V_3 \sin\theta_{nc} q(\theta_{nc})} \frac{V_3 \sin\theta_{nc} q(\theta_{nc}) - ic_{11}}{V_3 \sin\theta_{nc} q(\theta_{nc}) + ic_{11}}. \tag{6.64}$$

The conclusion from both perspectives is one and the same: we must impose some constraints on the warp factors (2.2) and (2.21) dilaton in (4.5) if our topological boundary is to have a path integral representation. (See table 1 for a guide to the equations linking the coefficients in (6.64) and the just mentioned warp factors and dilaton.) Given that in the present work we wish not study a concrete model (M,1), we will not elaborate on the required constraints here. However, it should be noted that our present analysis is only valid for the subset of M-Theory configurations (M,1) that satisfy (6.63).

The reader should not get disheartened, though. These are, hopefully, passing shortcomings. Once we consider the truly relevant scenario in part III (that is, the scenario that allows us to discuss knots embedded in a ultraviolet-complete physical model), we shall be able to fulfill (6.62) without constraining model (M,1). We shall adopt an optimist attitude with respect to (6.63): once topological M-Theory is formulated, we should have no need to impose this constraint to begin with!

[20] As the lucid work [62] shows, an appropriate analytical continuation of (6.60) would allow for a path integral formalism in case that the requirement (6.63) is not met. This is hard to realize in our M-Theory construction of model (M,1), since it would require a (to date) nonexistent formalism: *topological M-Theory*. Needless to say, a careful study of such scenario is beyond the scope of the thesis and we shall not proceed in this direction. The interested reader can gain more insights from the discussion around (3.346)-(3.350) in [1].

6.3.1 Twisting the bulk

In order to understand our next goal, let us briefly refresh our memory. In part I, we constructed the M-Theory model (M,1). In the present part II, we derived the Hamiltonian (5.88), defined in the bulk X_4 and associated to (M,1). Then, a consistent minimization of its energy, for static configurations of the fields, led to the Hamiltonian (5.93). We further rewrote this as the action (6.11), which is defined in X_3: the boundary of X_4. Upon topologically twisting (6.11), we obtained the Chern-Simons action (6.60): a suitable framework for the realization of knots in our setup. Quite evidently, our analysis shall be consistent only when we also topologically twist the bulk energy minimization equations that allowed us to obtain (6.11) to begin with. Doing so is the aim of the present section.

The set of energy minimization equations we must twist are (5.40), (5.42), (5.91) and (5.92), as already pointed out at the very end of section 5.2. Before twisting, however, we make the following observation: the various coefficient choices made so far in order to obtain a topological boundary theory considerably simplify the BHN equations (5.91).

To be precise, consider the third term in the second BHN equation for ($\alpha = 1$, $\beta = 2$) and interpret the dot product there appearing as a usual scalar product, in the same spirit as we did earlier in (5.45). Once more, we work with the normalization convention that $\epsilon_{12} = 1$. Then, this term can be written as

$$\sum_{\delta,k=1}^{3} \sqrt{b_{\delta k}}\, m^{(1)}_{\delta k}\mathcal{D}_\delta \varphi_k = \sqrt{b_{12}}\Big(\sum_{\alpha=1}^{2}\sum_{k=1}^{3} m^{(1)}_{\alpha k}\mathcal{D}_\alpha\varphi_k\Big) + \sqrt{b_{\tilde\psi 1}}\Big(\sum_{k=1}^{3} m^{(1)}_{\tilde\psi k}\mathcal{D}_{\tilde\psi}\varphi_k\Big), \tag{6.65}$$

where we have used the fact that $b_{1k} = b_{2k}$ for all $k = 1, 2, 3$ and the same is true for $b_{\tilde\psi k}$, as can be seen from (4.123). We will now fix the coefficients ($m^{(1)}_{13}$, $m^{(1)}_{21}$, $m^{(1)}_{23}$, $m^{(1)}_{\tilde\psi 1}$, $m^{(1)}_{\tilde\psi 2}$) to

$$\frac{g^{(1)}_{1212}}{g^{(1)}_{1213}} m^{(1)}_{13} = -m^{(1)}_{21} = \frac{g^{(1)}_{1212}}{g^{(1)}_{1223}} m^{(1)}_{23} = -\left|\sqrt{\frac{b_{\tilde\psi 1}}{b_{12}}}\right| \frac{g^{(1)}_{1212}}{g^{(1)}_{1213}} m^{(1)}_{\tilde\psi 1} = -\left|\sqrt{\frac{b_{\tilde\psi 1}}{b_{12}}}\right| \frac{g^{(1)}_{1212}}{g^{(1)}_{1223}} m^{(1)}_{\tilde\psi 2} = m^{(1)}_{12}, \tag{6.66}$$

with $m^{(1)}_{12}$ not yet fixed to any particular value. Further, we shall set the till now arbitrary parameters ($m^{(1)}_{11}$, $m^{(1)}_{22}$, $m^{(1)}_{12}$) to

$$m^{(1)}_{11} = m^{(1)}_{22} = m^{(1)}_{12} = m^{(1)}_{\tilde\psi 3}\sqrt{\frac{b_{\tilde\psi 1}}{b_{12}}}, \tag{6.67}$$

with $m^{(1)}_{\tilde{\psi}3}$ such that the left-hand equation in (6.53) is true. Then, we obtain

$$\sum_{\delta,k=1}^{3} \sqrt{b_{\delta k}} m^{(1)}_{\delta k} \mathcal{D}_\delta \varphi_k = \sqrt{b_{12}} m^{(1)}_{12} \Big[\big(\mathcal{D}_1\varphi_1 + \mathcal{D}_2\varphi_2 + \mathcal{D}_{\tilde{\psi}}\varphi_3\big) + \big(\mathcal{D}_1\varphi_2 - \mathcal{D}_2\varphi_1\big) \\ + \frac{g^{(1)}_{1213}}{g^{(1)}_{1212}} \big(\mathcal{D}_1\varphi_3 - \mathcal{D}_{\tilde{\psi}}\varphi_1\big) + \frac{g^{(1)}_{1223}}{g^{(1)}_{1212}} \big(\mathcal{D}_2\varphi_3 - \mathcal{D}_{\tilde{\psi}}\varphi_2\big) \Big]. \tag{6.68}$$

Written in this manner, it is straightforward to see that the consistency requirements (5.92) set to zero each term between brackets on the right-hand side above. Further, since the BHN equation of which this term is part of is antisymmetric under the exchange of (α, β), the above holds true for all allowed values of these indices. That is,

$$\sum_{\delta,k=1}^{3} \sqrt{b_{\delta k}} \epsilon_{\alpha\beta} \cdot m^{(1)}_{\delta k} \mathcal{D}_\delta \varphi_k = 0, \qquad \forall \alpha, \beta = 1, 2. \tag{6.69}$$

In much the same way, one can show that the third term in the first BHN equation (5.91) also vanishes:

$$\sum_{\delta,k=1}^{3} \sqrt{b_{\delta k}} \epsilon_{\alpha\tilde{\psi}} \cdot m^{(2)}_{\delta k} \mathcal{D}_\delta \varphi_k = 0, \qquad \forall \alpha = 1, 2. \tag{6.70}$$

If one interprets the dot product above as the usual scalar product, the proof is exactly as before. In more details, one must obtain the values of the $m^{(2)}$ coefficients from (5.30), (6.53), (6.66) and (6.67). Also, one must realize that $b_{12} = b_{\tilde{\psi}1}$ owing to our approximation (5.77), which implies $e^{2\phi_0} H_4 = 1$ in (4.117). However, if one would like to consider the more general scenario where (5.77) is not imposed, (6.70) can still be enforced by simply entertaining more elaborated interpretations of the dot product, in the vein of (5.53) earlier on.

All in all, the conclusion is that our choices of the coefficients in table 2 reduce the BHN equations in (5.91) to

$$\sqrt{\frac{4i|\tau|^2}{\tau - \bar{\tau}}} \mathcal{F}_{\alpha\tilde{\psi}} - i \sum_{k,l=1}^{3} h^{(1)}_{\alpha\tilde{\psi}kl} \sqrt{d_{kl}} [\varphi_k, \varphi_l] = 0, \quad \sqrt{\frac{2i|\tau|^2}{\tau - \bar{\tau}}} \mathcal{F}_{\alpha\beta} - i \sum_{k,l=1}^{3} g^{(1)}_{\alpha\beta kl} \sqrt{d_{kl}} [\varphi_k, \varphi_l] = 0, \tag{6.71}$$

for all α, $\beta = 1$, 2. As explained around (5.49), the above are just Hitchin equations! This is a remarkable result: in our setup, the BHN equations *naturally* decouple to Hitchin equations, together with and a set of consistency conditions on the scalar fields there appearing. Such result becomes even more relevant in view that Hitchin equations are precisely the starting point in the study of knots and their invariants in [13]. The very same Hitchin and consistency equations are also related to a number of other interesting topics, such as the Geometric Langlands Program [63–67] or Ngô's fundamental lemma for Lie algebras [68,69].

However exciting these directions may be, let us get back on track: currently, our aim is to twist all energy minimization equations. To this aim and as already anticipated in section 6.3, it is convenient to consider a different mapping between our scalar fields and their twisted one-form counterpart. In particular, instead of the boundary map (6.26), we would like to consider the bulk identification

$$\Lambda = \sum_{\mu} \Lambda_\mu dx^\mu, \qquad (\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_{\tilde{\psi}}) = i(\mathcal{A}_{\tilde{3}}, \varphi_1, \varphi_2, \varphi_3). \tag{6.72}$$

All other twisted fields remain as previously explained in (6.27)-(6.30). In this manner, the twisted version of (5.40) and (5.42) is

$$\Lambda_0 = D_\eta \sigma = D_\eta \bar{\sigma} = [\sigma, \bar{\sigma}] = [\sigma, \Lambda_k] = [\bar{\sigma}, \Lambda_k] = 0, \qquad \forall \eta = x_1, x_2, \tilde{\psi}, \quad \forall k = 1, 2, 3. \tag{6.73}$$

Similarly, the twisted version of the Hitchin equations in (6.71) is given by

$$F_{\alpha\tilde{\psi}} - \frac{\aleph}{\sqrt{2}} \sum_{k,l=1}^{3} h^{(1)}_{\alpha\tilde{\psi}kl} [\Lambda_k, \Lambda_l] = 0, \qquad F_{\alpha\beta} - \aleph \sum_{k,l=1}^{3} g^{(1)}_{\alpha\beta kl} [\Lambda_k, \Lambda_l] = 0, \qquad \forall \alpha, \beta = 1, 2. \tag{6.74}$$

where we have defined $\aleph$ as the following constant:

$$\aleph \equiv \sqrt{\frac{d_{12}(\tau - \bar{\tau})}{2i|\tau|^2}}. \tag{6.75}$$

The above definition uses the fact that, as can be seen from (4.129), all d_{kl} coefficients have the same value. Note that, from (5.83) and the equations mentioned in table 1, it follows that $\aleph$ depends entirely on supergravity parameters only. That is, parameters that characterize the M-Theory model (M,1).

At this stage, the only equations left to be twisted are those in (5.92). These become

$$D_1\Lambda_2 - D_2\Lambda_1 = D_1\Lambda_{\tilde{\psi}} - D_{\tilde{\psi}}\Lambda_1 = D_2\Lambda_{\tilde{\psi}} - D_{\tilde{\psi}}\Lambda_2 = D_1\Lambda_1 + D_2\Lambda_2 + D_{\tilde{\psi}}\Lambda_{\tilde{\psi}} = 0. \tag{6.76}$$

Our identifications (6.72) allow us to further rewrite the above in a very concise manner in the language of differential geometry. To do so, we first compute a few auxiliary quantities. We begin with the Hodge dual of Λ. Since (6.73) sets the time component of this one-form to zero, we can carry out this computation in the three-dimensional subspace spanned by $(x_1, x_2, \tilde{\psi})$. As we already explained, the simplifying assumption (5.77) converts this to a

Euclidean space. Consequently, the calculation is trivial and yields

$$*\Lambda = \Lambda_1 dx_2 \wedge d\tilde{\psi} - \Lambda_2 dx_1 \wedge d\tilde{\psi} + \Lambda_{\tilde{\psi}} dx_1 \wedge dx_2. \tag{6.77}$$

Making use of the exterior covariant derivative introduced in (6.42) and in much the same way as earlier in (6.43), it is easy to see that

$$\begin{aligned} d_A\Lambda &= (D_1\Lambda_2 - D_2\Lambda_1)dx_1 \wedge dx_2 + \sum_{\alpha=1}^{2}(D_\alpha\Lambda_{\tilde{\psi}} - D_{\tilde{\psi}}\Lambda_\alpha)dx_\alpha \wedge d\tilde{\psi}, \\ d_A * \Lambda &= (D_1\Lambda_1 + D_2\Lambda_2 + D_{\tilde{\psi}}\Lambda_{\tilde{\psi}})dx_1 \wedge dx_2 \wedge d\tilde{\psi}. \end{aligned} \tag{6.78}$$

Upon comparing the above with (6.76), it is clear that this last set of constraint equations can be succinctly written as

$$d_A\Lambda = 0 = d_A * \Lambda, \tag{6.79}$$

which completes the twisting of all energy minimization equations in X_4.

Hereupon, we have gathered a good amount of knowledge about the four-dimensional gauge theory following from the M-Theory configuration (M,1), dual to the model in [14]. In the following, we rephrase our findings in such a way that their merit is made most visible.

Appropriately compactifying (M,1), we have obtained its associated four-dimensional action (4.143), defined in the space X_4. Then, we have derived the corresponding Hamiltonian and written it in the particularly convenient form (5.32). By construction, the coefficients appearing in the Hamiltonian can be expressed only in terms of the supergravity parameters of (M,1). Minimization of the energy of this Hamiltonian for static configurations of the fields led to a series of BPS and consistency conditions on these gauge and scalar fields. For the gauge choice (5.35), they are given by (5.40), (5.42), (5.91) and (5.92). It turns out that all these are the same equations mentioned in [14] and derived using localization techniques for path integrals in [61]. Consequently, we have reproduced the results of [14], but we have done so in the well-known, conceptually easy classical Hamiltonian formalism. In the process, we have established a precise mapping between the usual gauge theory parameters (g_{YM}, Θ, τ) and the parameters that characterize model (M,1): (5.82) and (5.83). In other words, we have given a concrete, *simple* procedure to reproduce [14] and simultaneously provided a *supergravity interpretation* for it.

After the minimization process above described, the non-vanishing part of the Hamiltonian was rewritten as the action in (6.11). This is defined in the three-dimensional space X_3, the boundary of X_4. Of course, if our construction is to be a suitable framework for the study of knots, these should be embedded in X_3. Hence, the boundary action should be topological for our goals to be achievable. Upon a topological twist, this was proven to indeed be the case: (6.11) converts to the Chern-Simons action (6.60). Note that the Chern-Simons gauge field is a linear combination of the twisted gauge and scalar fields, as given by (6.58). Further, $\mathcal{N} = 2$ supersymmetry was made compatible with this construction, requiring only appropriate boundary conditions for the twisted fields, stated in (6.33).

The careful analysis of the theory in X_3 showed that it has all required features to host knots. What is more, additional support to this claim followed from this very same analysis in the following manner. Overall coherence required us to twist the energy minimization conditions in the bulk if we were to focus on the twisted boundary theory. We then noted that, in obtaining (6.60), we were forced to make certain choices for the coefficients summarized in table 2. Aptly translating such choices to our BPS conditions revealed that these were simplified to precisely the set of equations that are the starting point for the study of knots and their invariants in [13]! Note that [13] is just a symbolic reference here, one that helps us make our point. In part III, we will cite many more works that elaborate on the connection between our bulk equations and knots. For completeness, we remind the reader that the twisted BPS bulk equations are those in (6.73), (6.74) and (6.79).

In chapter 6 we have studied the three-dimensional boundary contribution (5.93) to the four-dimensional Hamiltonian (5.88) obtained in chapter 5. In particular, we have shown that, upon performing a topological twist to the gauge theory, the boundary physics is captured by a complexified Chern-Simons action. Consequently, this boundary constitutes a suitable space for the embedding of knots. On the other hand, the twist simplifies the four-dimensional BPS equations (5.91) into a Hitchin integrable system given by (6.74) and (6.76).

Part III
Inclusion of knots in (M,1)

In this third and last part, we will render useful to knot theory the contents of the previous two parts. In other words and as the title advances, we will embed knots in (M,1). We will do so by entertaining certain M2-brane states in (M,1). Because our goal is easier to postulate than to achieve, we will proceed in two steps. First, we will consider a toy model M2-brane in chapter 7. This hugely simplifies the analysis because it allows for explicit computations. Once sufficient insight has been gained through the toy model, we will study the knot-embedding M2-brane of interest in chapter 8. In this latter case and at the world-volume gauge theory level, knots will appear as Wilson loops localized at the three-dimensional boundary X_3 and as surface operators in the spacetime X_4 as a whole.

For the convenience of the reader, we have included a graphical summary of the contents in this part. Figure 13 points out the main equations in the discussion that follows and depicts the logic that connects them to one another.

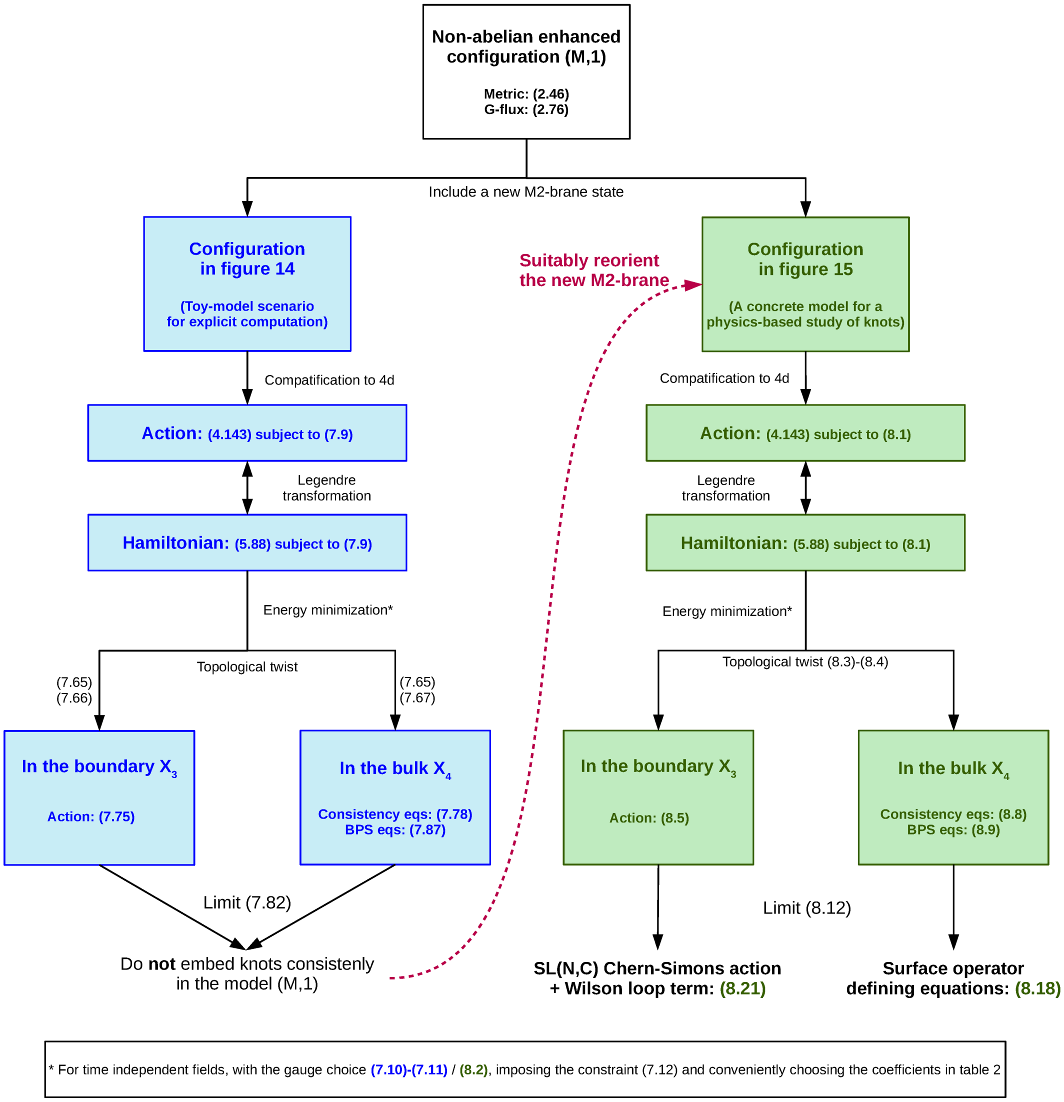

Figure 13: Graphical summary of part III. To the configuration (M,1) of part I we now add a new M2-brane so as to consistently embed knots in our setup. After that, we adapt the contents of part II to this new configuration, compactifying to four dimensions so as to study the associated world-volume gauge theory. In blue, we show the contents of chapter 7, where we consider a particularly simple new M2-brane that allows for explicit computations. The correct embedding of knots is then discussed in chapter 8, here depicted in green. We find that the new M2-brane of chapter 8 correctly incorporates any given knot at the boundary via a Wilson loop, whereas in the bulk it can be understood as a surface operator.

Chapter 7: A wrong but instructive realization of knots

Let us start by considering once more the configuration (M,1). We remind the reader that, for the non-abelian enhanced scenario, (M,1) was fully characterized by its metric (2.46) and its associated G-flux (2.76). It is also worth bearing in mind that the eleven-dimensional manifold X_{11} on which (2.46) is defined naturally decomposes into three subspaces: the physical spacetime X_4 −with a three-dimensional boundary X_3−, a three cycle Σ_3 and a multi-centered, warped Taub-NUT space TN, as described around (4.1). To this setup we will now add an M2-brane, oriented along the $(t,\, x_1,\, \tilde{\psi})$ directions of X_4. This additional M2-brane should not be confused with the M2-branes wrapping the two-cycles of the Taub-NUT subspace that we included in section 2.1.1 with the goal of enhancing the gauge symmetry group underlying the configuration (M,1). The just described setup is schematically depicted in figure 14. As we shall show in this section, the new M2-brane clarifies how one should embed any arbitrary knot in the model (M,1) and thus sets the ground to address the interesting and challenging question of understanding knots in a theoretical physics context.

7.1 Elucidating the new action

In this section, we will argue what the four-dimensional action associated to the M-Theoretical configuration in figure 14 looks like. In other words, we shall review the contents of chapters 2 and 4 in the light of the newly added M2-brane state. We shall begin our discussion at the M-Theoretical level and then dimensionally reduce to the usual four-dimensional physical scenario. In particular, we shall first discuss how the novel M2-brane affects the inherent symmetries of the (M,1) model by explaining why we have chosen it to stretch along precisely $(t,\, x_1,\, \tilde{\psi})$.

The choice of orientation of the new M2-brane will become increasingly sensible as we proceed. However, a fundamental observation should be highlighted already: the orientation is such that this M2-brane does not break any of the supersymmetries in our setup. For simplicity, let us explain this claim in the context of a type IIA dual picture. For just a moment, consider the scenario with no new M2-brane. In particular, consider the configuration (A,3) summarized in figure 3. As a quick reminder, (A,3) is the dimensional reduction on the

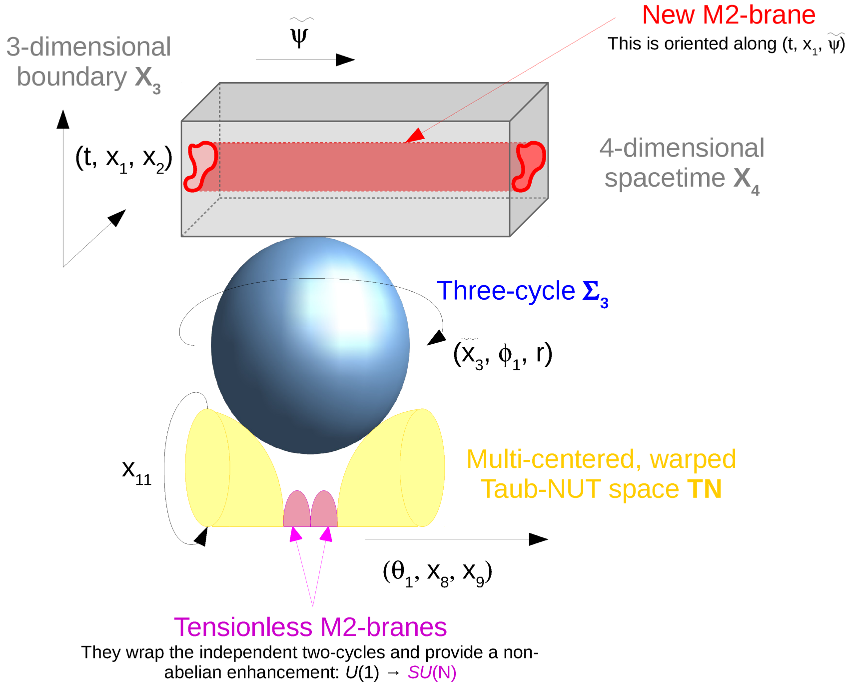

Figure 14: To the configuration (M,1) we add a new M2-brane, oriented along the $(t,\, x_1,\, \tilde{\psi})$ spacetime directions. As we will show, this additional M2-brane state serves as a toy model that clarifies how knots should be incorporated in (M,1).

circle labeled by x_{11} of the low-energy limit of (M,1). It consists of N coincident D6-branes oriented along $X_4 \otimes \Sigma_3$, with tensionless open strings between them and in the presence of both NS and RR fluxes. This was illustrated in figure 4**A**. For later convenience, we note that

$$X_4 \otimes \Sigma_3 = Y_3 \otimes \Sigma_4, \qquad Y_3 \equiv span\{t, x_1, \tilde{\psi}\}, \qquad \Sigma_4 \equiv span\{x_2, \tilde{x}_3, \phi_1, r\}. \tag{7.1}$$

Now, add the new M2-brane of our interest in (M,1). Since this M2-brane extends along Y_3 (that is, it is placed at a fixed point of the compactification circle parametrized by x_{11}), in the configuration (A,3) it converts to a D2-brane along $Y_3 \in X_4$. Consequently, the novel D2-brane is embedded in the aforementioned N coincident D6-branes. In other words, the D2-brane forms a bound state with the D6-branes and thus all supersymmetry is indeed preserved. Hence, our prior analyses in chapter 6 regarding boundary conditions and topological twist remain the same even in the presence of the new M2-brane.

Let us now understand how the new M2-brane affects the action of the configuration (M,1). Once again momentarily placing ourselves in the configuration (A,3), we note that the new D2-brane there is, of course, a carrier of RR charge and so it naturally couples to an RR three-form potential oriented along its world-volume Y_3. As explained in section 4.2, (A,3) already has an associated RR three-form potential (4.103). In order to not confuse both RR potentials, we shall refer to the former as

$$\Delta\tilde{\mathcal{C}}_3^{(A,3)} = (\Delta\tilde{\mathcal{C}}_3^{(A,3)})_{t1\tilde{\psi}}\, dt \wedge dx_1 \wedge d\tilde{\psi}. \tag{7.2}$$

In the most general possible case, this RR potential will depend on various parameters. Firstly, it will depend on the non-commutative deformation parameter θ_{nc} that we introduced in (2.19)[21]. Secondly, it will depend on all the non-compact coordinates of the D6-branes' world-volume where it is embedded. Finally, it will also depend on θ_1, the only coordinate transverse to the D6-branes that does not specify their location −recall the discussion in the beginning of section 4.1 to this respect. All in all, we have that

$$\Delta\tilde{\mathcal{C}}_3^{(A,3)} = \Delta\tilde{\mathcal{C}}_3^{(A,3)}(\theta_{nc}, t, x_1, x_2, \tilde{\psi}, r, \theta_1). \tag{7.3}$$

Next, observe that the presence of this D2-brane makes the N coincident D6-branes in which it is embedded develop a world-volume field strength expectation value $\langle\mathcal{F}\rangle$, oriented along the orthogonal directions to the D2-brane: $\langle\mathcal{F}\rangle \in \Sigma_4$. Further, since the D2-{D6} system is translation invariant along Y_3, it follows that $\langle\mathcal{F}\rangle = \langle\mathcal{F}\rangle(x_2,\, \tilde{x}_3,\, \phi_1,\, r)$[22]. Lastly, recall that after an uplift to the M-Theory model (M,1) and in its supergravity limit, (4.103) led to the contribution (4.107) to the bosonic four-dimensional gauge theory action. Similarly, the newly added brane sources the following novel term to the world-volume action:

$$S^{(4)} = \frac{C_1}{V_3}\int_0^{\pi}\frac{d\theta_1}{2\pi}\int_{X_4\otimes\Sigma_3}\mathrm{Tr}\left(\Delta\tilde{\mathcal{C}}_3^{(A,3)} \wedge \langle\mathcal{F}\rangle \wedge \langle\mathcal{F}\rangle\right). \tag{7.4}$$

Before proceeding ahead, it is perhaps worth refreshing the memory of the reader by pointing out where exactly all subtleties involved in deriving (7.4) can be found. Let us first note that (C_1/V_3) is the (θ_1-independent) non-abelian generalization of (4.6), as discussed below (4.11). Since (7.4) is a topological term (i.e. no measure appears in this integral), it is clear that the average over θ_1 only affects the new RR potential $\Delta\tilde{\mathcal{C}}_3^{(A,3)}$. The trace is taken in the adjoint representation of $SU(N)$, for the reasons given between (4.11) and (4.12).

[21]It should be borne in mind that we require $\theta_{nc} \neq 0$, so that the usual topological Θ-term is present chapter 6, this topological piece is in turn essential so that a complexified Chern-Simons action governs the physics in X_3, where ultimately we shall place knots.

[22] The situation is a higher dimensional analogue to the well-known electromagnetic setup where an infinitely long line of electric charge sources an orthogonal electric field that only depends on transverse coordinates.

Last but not least, in posing (7.4) as an analogue to (4.107), we have omitted comparison to the latter's root equation (4.106). Such comparison allows us to understand the new M2-brane as additional contributions $\Delta\mathcal{G}_4^{(M,1)}$ to the already present G-fluxes (2.76) in (M,1). Mathematically,

$$S^{(4)} = \int_{X_{11}} \mathrm{Tr}\left(\Delta\tilde{\mathcal{C}}_3^{(A,3)} \wedge \Delta\mathcal{G}_4^{(M,1)} \wedge \Delta\mathcal{G}_4^{(M,1)}\right). \tag{7.5}$$

As argued in section 4.1, the background G-flux $\langle\mathcal{G}_4^{(M,1)}\rangle$ part of (2.76) is delocalized and so we assume that it does not affect the corresponding four-dimensional physics. For identical reasons, we can neglect the new M2-brane's contribution to $\langle\mathcal{G}_4^{(M,1)}\rangle$ and understand it simply as a novel *localized* G-flux in (M,1).

Recapitulating, to the non-abelian enhanced M-Theory model (M,1) we add an M2-brane along Y_3. This does not break the amount of supersymmetry of (M,1), which remains to be $\mathcal{N} = 4$ with half-BPS boundary conditions. However, the new M2-brane sources a novel contribution, that in (7.4), to the action (4.143) following from (M,1). Our next goal is to explain what the total action, (4.143) plus (7.4), looks like. We shall do so in a simplified scenario following quite stringent assumptions. The reader should not fear loss of generality, though. This is just for the sake of clarity, so that we can explicitly write down intermediate formulae as we argue. The total action we shall ultimately write holds true in the unsimplified case of interest too.

In order to understand the total action (4.143) plus (7.4), we begin by looking only at its new contribution (7.4). As warned, let us momentarily postulate unnecessary simplifying assumptions. In particular, let us postulate the following separations of variables:

$$\begin{aligned}\Delta\tilde{\mathcal{C}}_3^{(A,3)} &= f_a(\theta_{nc}, t, x_1, x_2, \tilde{\psi})\, f_b(\theta_1)\, f_c(r)\, dt \wedge dx_1 \wedge d\tilde{\psi}, \\ \langle\mathcal{F}\rangle \wedge \langle\mathcal{F}\rangle &= g_a(x_2)\, g_b(\tilde{x}_3)\, g_c(\phi_1)\, g_d(r)\, dx_2 \wedge d\tilde{x}_3 \wedge d\phi_1 \wedge dr,\end{aligned} \tag{7.6}$$

with $\{f,\, g\}$ arbitrary smooth functions. In this case, (7.4) reduces to

$$S^{(4)} = \frac{C_1}{V_3} \int d^4x\, \mathrm{Tr}\left(\varkappa\, f_a \cdot g_a\right), \tag{7.7}$$

where $d^4x \equiv dt\, dx_1\, dx_2\, d\tilde{\psi}$ as in previous occasions, $\varkappa$ is a constant defined as

$$\varkappa \equiv -\left(\int_0^{R_3} d\tilde{x}_3\, g_b\right)\left(\int_0^{\pi} \frac{d\theta_1}{2\pi} f_b\right)\left(\int_0^{2\pi} d\phi_1\, g_c\right)\left(\int_0^{\infty} dr\, f_c \cdot g_d\right) \tag{7.8}$$

and we have suppressed the arguments of the $\{f,\, g\}$ functions for brevity. In the spirit of our detailed computations in chapter 4, the above will only be a sensible term in the total action iff $\varkappa$ is well-defined everywhere. This means that we must demand that the integral of $(g_b,\, f_c \cdot g_d)$ does not diverge at either $(R_3,\, r \to \infty)$ nor $(R_3,\, r = 0)$, respectively. Further, the functions $(f_b,\, g_c)$ should not be $(\pi,\, 2\pi)$-periodic, since this would make $\varkappa$ vanish and we would end up in the perplexing, if not forthwith nonsensical, scenario where the new M2-brane in (M,1) plays no physical role whatsoever! Supposing all these conditions are satisfied and hence $\varkappa \in \mathbb{R} - \{0\}$, let us turn our attention to the $(f_a \cdot g_a)$ piece in (7.7). By definition, g_a is some function of $(\langle \mathcal{A}_2 \rangle,\, \langle \mathcal{A}_{\tilde{3}} \rangle,\, \langle \mathcal{A}_{\phi_1} \rangle,\, \langle \mathcal{A}_r \rangle)$ that only depends on the spacetime coordinate x_2. Then, f_a provides an additional dependence over the remaining spacetime coordinates $(t,\, x_1,\, \tilde{\psi})$[23]. Without loss of generality, we can further absorb $C_1 \varkappa / V_3$ in $(f_a \cdot g_a)$, which simply rescales this integrand. Consequently and from the point of view of the matter content in the world-volume gauge theory studied in part II, we see that the effect of the new M2-brane amounts to sourcing a non-zero expectation value for the $\mathcal{A}_2$ component of the gauge field as well as for the $(\mathcal{A}_{\tilde{3}},\, \mathcal{A}_{\phi_1},\, \mathcal{A}_r)$ real scalars in the adjoint representation of $SU(N)$. Effectively, we thus conclude that the total action (4.143) plus (7.7) is of the form (4.143), but with the following replacements:

$$\mathcal{A}_x \to \mathcal{A}_x^{(tot)} \equiv \langle \mathcal{A}_x \rangle + \mathcal{A}_x, \qquad \forall x = 2, \tilde{3}, \phi_1, r. \tag{7.9}$$

It should be rather obvious that, as we advanced a little ago, our conclusion holds true for the total action (4.143) plus (7.4). That is, when we do not postulate (7.6) with appropriate constraints on the $\{f,\, g\}$ functions, the total action remains exactly as in (4.143), but with the replacements in (7.9). Explicit computations that prove this claim are not possible because we do not know the exact form of the RR potential $\Delta \tilde{\mathcal{C}}_3^{(A,3)}$. However, it is in principle always possible to carry out the integrals along $(\tilde{x}_3,\, \theta_1,\, \phi_1,\, r)$ and, under the conjecture that these integrals neither vanish nor blow up, we would end up inferring that the net effect of (7.4) is precisely to switch on the expectation values suggested in (7.9).[24]

7.2 The bulk theory revisited

Having established (4.143) subject to the replacements in (7.9) as the physical action corresponding to the configuration in figure 14, it is now time to show how the contents of chapter 5 vary in the presence of the new M2-brane. In other words, in this section we will explore how the BHN equations (5.91) change in the present context. Before starting this task, maybe a few caution words are due. The present section amounts to a computationally pretty strenuous exercise, with little supplementary physical insight compared to chapter 5

[23]It provides dependence over the parameter θ_{nc} as well, but this is not important at the moment.

[24]Any reader who remains skeptical to our conclusion may find it reassuring to know that, sooner than later, we shall be able to reproduce equations known to encode numerous knot invariants from precisely the just argued total effective action. This result may well be taken as confirmation of our conjecture.

before. Consequently, readers wishing to skip technical details are welcome to jump to the last paragraph in the section, where the results here obtained are summarized. Nevertheless, this is an important section from the point of view of maintaining a mathematically rigorous analysis throughout the text and its omission would make it hard to reproduce the concluded equations. Therefore, in the following, we shall advance steadily and reliably on our way towards understanding knots in a physical context.

Acceptance of (4.143) modified by (7.9) as the action for (M,1) in the presence of the new M2-brane automatically implies that the Hamiltonian for this configuration is given by (5.88), as long as we carry out the replacements in (7.9). Let us recall that in posing such total effective Hamiltonian there are a few underlying assumptions. On the one hand, we considered the warp factors appearing in the metric of (M,1) to be as in (2.2), we further simplified to a constant dilaton in (4.5), later on we required the warp factors to satisfy $\tilde{F}_2 \geq F_3$ (with $\tilde{F}_2$ defined in (2.21)) and finally a Minkowski-like spacetime was assumed, which means the warp factors are constrained to also satisfy (5.78). On the other hand, tables 1 and 2 provide all information about the coefficients in the total effective Hamiltonian of interest. The only coefficient not accounted for in the tables is τ, which was defined in (5.83).

We are now at the point where we can start minimizing (5.88) subject to (7.9). First of all, let us fix a gauge. Specifically, let us modify our previous gauge choice in (5.35) to

$$\mathcal{A}_0 = \mathcal{A}_{\tilde{3}}^{(tot)}. \tag{7.10}$$

Let us also restrict ourselves to time-independent fields, so that we can study BPS configurations. As shown earlier in section 5.1.1, the only sensible way to make the first six squared terms in the Hamiltonian vanish in this case consists in setting

$$\mathcal{A}_{\tilde{3}}^{(tot)} = 0. \tag{7.11}$$

In order to allow for richer dynamics of all the fields in the gauge theory, instead of an apt modification of (5.42) to the present case, this time we will relax our simplifying assumptions to

$$\mathcal{D}_{\tilde{\psi}} \mathcal{A}_{\phi_1}^{(tot)} = \mathcal{D}_{\tilde{\psi}} \mathcal{A}_r^{(tot)} = 0. \tag{7.12}$$

The above in turn forces us to a more constraining initial choice on some of the coefficients summarized in table 2. Specifically, we choose

$$q_{kl}^{(1)} = q_k^{(i)} = t^{(i)} = q^{(4)} = 0 \qquad \forall k, l = 1, 2, 3, \qquad \forall i = 1, 2. \tag{7.13}$$

All these choices reduce our total effective Hamiltonian to

$$\hat{H} = \int d^4x \,\mathrm{Tr}\Big\{ \sum_{\alpha,\beta=1}^{2} \Big(\sqrt{\frac{2i|\tau|^2}{\tau-\bar{\tau}}}\mathcal{F}^{(tot)}_{\alpha\beta} - ig^{(4)}_{\alpha\beta}\sqrt{\frac{C_1 a_1}{V_3}}[\mathcal{A}^{(tot)}_r, \mathcal{A}^{(tot)}_{\phi_1}] - i\sum_{k,l=1}^{3} g^{(1)}_{\alpha\beta kl}\sqrt{d_{kl}}[\varphi_k, \varphi_l]$$
$$-i\sum_{k=1}^{3}\sum_{\gamma=2}^{3} g^{(\gamma-1)}_{\alpha\beta k}\sqrt{c_{y_\gamma k}}[\mathcal{A}^{(tot)}_{y_\gamma}, \varphi_k] + \sum_{\delta,k=1}^{3}\sqrt{b_{\delta k}}\epsilon_{\alpha\beta}\cdot m^{(1)}_{\delta k}\mathcal{D}^{(tot)}_{\delta}\varphi_k\Big)^2 + \sum_{\alpha=1}^{2}\Big(\sqrt{\frac{4i|\tau|^2}{\tau-\bar{\tau}}}\mathcal{F}^{(tot)}_{\alpha\tilde{\psi}}$$
$$+\sqrt{\frac{C_1 c_{\beta r}}{V_3}}t^{(1)}_{\alpha}\epsilon_{\alpha\tilde{\psi}\beta r}\mathcal{D}^{(tot)}_{\beta}\mathcal{A}^{(tot)}_{r} + \sqrt{\frac{C_1 \tilde{c}_{\beta\phi_1}}{V_3}}t^{(2)}_{\alpha}\epsilon_{\alpha\tilde{\psi}\beta\phi_1}\mathcal{D}^{(tot)}_{\beta}\mathcal{A}^{(tot)}_{\phi_1}$$
$$-ih^{(4)}_{\alpha\tilde{\psi}}\sqrt{\frac{C_1 a_1}{V_3}}[\mathcal{A}^{(tot)}_r, \mathcal{A}^{(tot)}_{\phi_1}] - i\sum_{k,l=1}^{3}h^{(1)}_{\alpha\tilde{\psi}kl}\sqrt{d_{kl}}[\varphi_k,\varphi_l] - i\sum_{k=1}^{3}\sum_{\gamma=2}^{3}h^{(\gamma-1)}_{\alpha\tilde{\psi}k}\sqrt{c_{y_\gamma k}}[\mathcal{A}^{(tot)}_{y_\gamma}, \varphi_k]$$
$$+\sum_{\delta,k=1}^{3}\sqrt{b_{\delta k}}\epsilon_{\alpha\tilde{\psi}}\cdot m^{(2)}_{\delta k}\mathcal{D}^{(tot)}_{\delta}\varphi_k\Big)^2 + \hat{\chi}_T + (\tau-\bar{\tau})\sum_{i,j,k=1}^{3}\epsilon^{0ijk}\mathcal{F}^{(tot)}_{0i}\mathcal{F}^{(tot)}_{jk}\Big\} + \hat{Q}_{EM}, \tag{7.14}$$

where $(\hat{\chi}_T,\, \hat{Q}_{EM})$ are the soon to be written explicitly appropriate modifications of their tilde counterparts in (5.88), which were described below the aforementioned equation. Here, $\mathcal{F}^{(tot)}_{\alpha\beta}$ stands for the field strength associated to the $(\mathcal{A}_1,\, \mathcal{A}^{(tot)}_2)$ gauge fields, in a similar fashion to (4.50) before. Likewise, $\mathcal{D}^{(tot)}_{\alpha}$ is related to these same gauge fields as in (4.51). We remind the reader that $(y_2,\, y_3) \equiv (r,\, \phi_1)$ is simply a short-hand notation.

At this point, our goal is to see how the BHN equations in (5.91) get modified due to the presence of the new M2-brane [25]. These modified BHN equations shall follow from requiring that the squared terms in (7.14) vanish. To the extent that it is possible, we shall choose the coefficients of table 2 in such a way that we retain the form of (5.91). Let us start with the BHN equation associated to the $(\mathcal{F}^{(tot)}_{12})$ component of the field strength. Following our just explained strategy, we choose

$$g^{(\gamma-1)}_{\alpha\beta k} = g^{(4)}_{\alpha\beta} = 0 \qquad \forall\alpha,\beta = 1,2, \qquad \forall k = 1,2,3, \qquad \forall\gamma = 2,3. \tag{7.15}$$

Then, our first BHN equation is precisely of the desired form:

$$\sqrt{\frac{2i|\tau|^2}{\tau-\bar{\tau}}}\mathcal{F}^{(tot)}_{\alpha\beta} - i\sum_{k,l=1}^{3} g^{(1)}_{\alpha\beta kl}\sqrt{d_{kl}}[\varphi_k,\varphi_l] + \sum_{\delta,k=1}^{3}\sqrt{b_{\delta k}}\epsilon_{\alpha\beta}\cdot m^{(1)}_{\delta k}\mathcal{D}^{(tot)}_{\delta}\varphi_k = 0, \tag{7.16}$$

[25] It is perhaps convenient to refresh the memory of the reader a bit: we understand as a BHN equation any relation of the form suggested in (5.48). Besides, it will soon be useful to recognize a Hitchin equation as indicated in (5.49).

where α, $\beta = 1, 2$. However, the above result is only possible due to a drastic reduction of the parameter space for the coefficients of table 2. Due to (7.12), we need not worry about satisfying the defining relation (5.13). But (5.16), (5.19), (5.22) and (5.24) get constrained to

$$\sum_{\alpha=1}^{2}\left(t_{\alpha}^{(i)}\right)^2=\sum_{\alpha=1}^{2}\left(h_{\alpha\tilde{\psi}}^{(4)}\right)^2=2\left(g_{12kl}^{(1)}\right)^2+\sum_{\alpha=1}^{2}\left(h_{\alpha\tilde{\psi}kl}^{(1)}\right)^2=\sum_{\alpha=1}^{2}\left(h_{\alpha\tilde{\psi}k}^{(i)}\right)^2=1, \tag{7.17}$$

for all k, $l = 1, 2, 3$ and $i = 1, 2$. None of our choices so far affect the relation (5.30). Nevertheless, it is most convenient to further choose all the $m^{(1)}$ coefficients −except for $m_{12}^{(1)}$, which we will presently take to be the arbitrary coefficient in terms of which all the rest get fixed− as in (6.66) and (6.67). As shown around (6.69), these choices together with the enforcement of the consistency requirements (5.92)[26] make the last term in (7.16) vanish and so, the ($\mathcal{F}_{12}^{(tot)}$)-related BHN equation decouples to a Hitchin equation:

$$\sqrt{\frac{2i|\tau|^2}{\tau-\bar{\tau}}}\mathcal{F}_{\alpha\beta}^{(tot)}-i\sum_{k,l=1}^{3}g_{\alpha\beta kl}^{(1)}\sqrt{d_{kl}}[\varphi_k,\varphi_l]=0, \qquad \forall\alpha,\beta=1,2. \tag{7.18}$$

The defining relation (5.30) can then be satisfied with an appropriate choice of the relevant $m^{(2)}$ coefficients. Additionally, the enforcement of (5.92) sets $\hat{\chi}_T = 0$ in our total effective Hamiltonian (7.14). Section 5.1.2 proves this last claim, as the heedful reader may recall.

Turning our attention to the ($\mathcal{F}_{2\tilde{\psi}}^{(tot)}$)-related BHN equation, we will first do two coefficient replacements in (7.14):

$$t_{\alpha}^{(1)}\epsilon_{\alpha\tilde{\psi}\beta r}\rightarrow t_{\beta r}^{(1)}\epsilon_{\alpha\tilde{\psi}}, \qquad t_{\alpha}^{(2)}\epsilon_{\alpha\tilde{\psi}\beta\phi_1}\rightarrow t_{\beta\phi_1}^{(2)}\epsilon_{\alpha\tilde{\psi}}. \tag{7.19}$$

The advantage of such replacements consists in increasing the number of different values the index β can take in the above terms: for $\alpha = 2$, now β can be equal to either 1 or 2, whereas before antisymmetry of the Levi-Civita symbol forced us to consider only $\beta = 1$. In exchange, we must modify the first constraint in (7.17) as follows:

$$\sum_{\alpha=1}^{2}\left(t_{\alpha}^{(i)}\right)^2=1\rightarrow\sum_{\beta=1}^{2}\left(t_{\beta r}^{(1)}\right)^2=\sum_{\beta=1}^{2}\left(t_{\beta\phi_1}^{(2)}\right)^2=1. \tag{7.20}$$

No other changes apply. Hence, we can go ahead and do a few convenient coefficient choices:

$$t_{1r}^{(1)}=t_{1\phi_1}^{(2)}=h_{2\tilde{\psi}12}^{(1)}=h_{2\tilde{\psi}13}^{(1)}=h_{2\tilde{\psi}1}^{(1)}=h_{2\tilde{\psi}1}^{(2)}=0, \qquad t_{2r}^{(1)}=t_{2\phi_1}^{(2)}=1. \tag{7.21}$$

[26] Strictly speaking, we consider (5.92) after imposing the replacement $\mathcal{D}_2\rightarrow\mathcal{D}_2^{(tot)}$.

Working in the $\epsilon_{2\tilde{\psi}} = 1$ normalization convention and interpreting the dot product as the usual scalar product, the second squared term in the Hamiltonian (7.14), for $\alpha = 2$, gives rise to the following BHN equation:

$$\sqrt{\frac{4i|\tau|^2}{\tau-\bar{\tau}}}\mathcal{F}^{(tot)}_{2\tilde{\psi}} + \sqrt{\frac{C_1}{V_3}}\left(\sqrt{c_{2r}}\mathcal{D}^{(tot)}_2\mathcal{A}^{(tot)}_r + \sqrt{\tilde{c}_{2\phi_1}}\mathcal{D}^{(tot)}_2\mathcal{A}^{(tot)}_{\phi_1}\right) - ih^{(4)}_{2\tilde{\psi}}\sqrt{\frac{C_1 a_1}{V_3}}[\mathcal{A}^{(tot)}_r, \mathcal{A}^{(tot)}_{\phi_1}]$$
$$- 2ih^{(1)}_{2\tilde{\psi}23}\sqrt{d_{23}}[\varphi_2,\varphi_3] - i\sum_{k,\gamma=2}^{3} h^{(\gamma-1)}_{2\tilde{\psi}k}\sqrt{c_{y_\gamma k}}[\mathcal{A}^{(tot)}_{y_\gamma},\varphi_k] + \sum_{\delta,k=1}^{3}\sqrt{b_{\delta k}}m^{(2)}_{\delta k}\mathcal{D}^{(tot)}_\delta\varphi_k = 0, \tag{7.22}$$

where we have used the fact that $d_{23} = d_{32}$, as can be seen from (4.129). What is more, the last term above has already been proven to vanish: see (6.70). As there shown, such vanishing only requires us to choose all $m^{(2)}$ coefficients $-$except $m^{(2)}_{12}-$ in a manner compatible with both our earlier $m^{(1)}$ choices and with the defining relation (5.30). The proof also uses the consistency conditions (5.92), suitably adapted to the present case where there is an additional M2-brane in the configuration (M,1). The above may not look too enlightening right away, but as it turns out, we can bring it into a most suggestive form with just some uncomplicated algebra. Let us see how in details.

In appropriate generalization of (6.27) before, let us introduce

$$\sigma^{(tot)} \equiv \mathcal{A}^{(tot)}_r + i\mathcal{A}^{(tot)}_{\phi_1}, \qquad \bar{\sigma}^{(tot)} = \mathcal{A}^{(tot)}_r - i\mathcal{A}^{(tot)}_{\phi_1}. \tag{7.23}$$

In terms of the above complex scalar fields, the BHN equation (7.22) takes a far simpler form. In details, one can easily check that

$$\begin{aligned}
&\sum_{k,\gamma=2}^{3} h^{(\gamma-1)}_{2\tilde{\psi}k}\sqrt{c_{y_\gamma k}}[\mathcal{A}^{(tot)}_{y_\gamma},\varphi_k] = 2\sqrt{\frac{4i|\tau|^2}{\tau-\bar{\tau}}}\mathrm{Re}\big(\gamma_1[\bar{\sigma}^{(tot)},\varphi_2] + \gamma_2[\bar{\sigma}^{(tot)},\varphi_3]\big),\\
&ih^{(4)}_{2\tilde{\psi}}\sqrt{a_1}[\mathcal{A}^{(tot)}_r,\mathcal{A}^{(tot)}_{\phi_1}] = \sqrt{\frac{4i|\tau|^2}{\tau-\bar{\tau}}}\sqrt{\frac{V_3}{C_1}}\gamma_3[\bar{\sigma}^{(tot)},\sigma^{(tot)}],\\
&\sqrt{c_{2r}}\mathcal{D}^{(tot)}_2\mathcal{A}^{(tot)}_r + \sqrt{\tilde{c}_{2\phi_1}}\mathcal{D}^{(tot)}_2\mathcal{A}^{(tot)}_{\phi_1} = 2\sqrt{\frac{4i|\tau|^2}{\tau-\bar{\tau}}}\sqrt{\frac{V_3}{C_1}}\mathrm{Re}\big(\gamma_5\mathcal{D}^{(tot)}_2\sigma^{(tot)}\big),
\end{aligned} \tag{7.24}$$

where $\{\gamma\}$ are a set of new constants, defined as certain linear combinations of the constants we are already familiar with:

$$\gamma_i \equiv \frac{1}{2}\sqrt{\frac{\tau-\bar{\tau}}{4i|\tau|^2}}\Big(h^{(1)}_{2\tilde{\psi}(i+1)}\sqrt{c_{r(i+1)}}+ih^{(2)}_{2\tilde{\psi}(i+1)}\sqrt{c_{\phi_1(i+1)}}\Big), \qquad \gamma_3 \equiv \frac{1}{2}\sqrt{\frac{\tau-\bar{\tau}}{4i|\tau|^2}}\sqrt{\frac{C_1 a_1}{V_3}}h^{(4)}_{2\tilde{\psi}},$$
$$\gamma_4 \equiv \frac{1}{2}\sqrt{\frac{d_{23}(\tau-\bar{\tau})}{4i|\tau|^2}h^{(1)}_{2\tilde{\psi}23}}, \qquad \gamma_5 \equiv \frac{1}{2}\sqrt{\frac{\tau-\bar{\tau}}{4i|\tau|^2}}\sqrt{\frac{C_1}{V_3}}\left(\sqrt{c_{2r}}-i\sqrt{\tilde{c}_{2\phi_1}}\right), \tag{7.25}$$

with $i = 1,\,2$. The cumbersome looking prefactors, along with the still unused γ_4 constant, are about to pay off: using (7.24) in (7.22), the $(\mathcal{F}^{(tot)}_{2\tilde{\psi}})$-related BHN equation becomes

$$\mathcal{F}^{(tot)}_{2\tilde{\psi}} + 2\mathrm{Re}\big(\gamma_5\mathcal{D}^{(tot)}_2\sigma^{(tot)}\big) = i\gamma_4[\varphi_2,\varphi_3] + 2i\mathrm{Re}\Big(\sum_{i=1}^{2}\gamma_i[\bar{\sigma}^{(tot)},\varphi_{(i+1)}]\Big) + \gamma_3[\bar{\sigma}^{(tot)},\sigma^{(tot)}]. \tag{7.26}$$

But we can do better. Let

$$\hat{\varphi}_2 \equiv \varphi_2 + 2\mathrm{Re}\Big(\frac{\gamma_2\bar{\sigma}^{(tot)}}{\gamma_4}\Big), \qquad \hat{\varphi}_3 \equiv \varphi_3 - 2\mathrm{Re}\Big(\frac{\gamma_1\bar{\sigma}^{(tot)}}{\gamma_4}\Big). \tag{7.27}$$

Then, it does not take much effort to show that the commutator of the above two fields yields

$$[\hat{\varphi}_2,\hat{\varphi}_3] = [\varphi_2,\varphi_3] + 2\mathrm{Re}\Big(\sum_{i=1}^{2}\frac{\gamma_i}{\gamma_4}[\bar{\sigma}^{(tot)},\varphi_{(i+1)}]\Big) + 2i\mathrm{Im}\Big(\frac{\gamma_1\bar{\gamma}_2}{\gamma_4^2}\Big)[\bar{\sigma}^{(tot)},\sigma^{(tot)}], \tag{7.28}$$

where $\bar{\gamma}$ denotes the complex conjugate of γ. In other words, this commutator is proportional (with proportionality constant $(i\gamma_4)^{-1}$) to the right-hand side of the BHN equation (7.26), as long as we demand that

$$\gamma_3 = -2\mathrm{Im}\Big(\frac{\gamma_1\bar{\gamma}_2}{\gamma_4}\Big) \tag{7.29}$$

holds true. If we use the equalities $c_{r2} = c_{r3}$ and $c_{\phi_1 2} = c_{\phi_1 3}$ following from (4.135), together with the $\{\gamma\}$'s definition in (7.25), the above can be rewritten exclusively in terms of the coefficients in tables 1 and 2:

$$\frac{h^{(4)}_{2\tilde{\psi}}h^{(1)}_{2\tilde{\psi}23}}{h^{(1)}_{2\tilde{\psi}3}h^{(2)}_{2\tilde{\psi}2} - h^{(1)}_{2\tilde{\psi}2}h^{(2)}_{2\tilde{\psi}3}} = -2\sqrt{\frac{c_{\phi_1 2}c_{r2}V_3}{a_1 d_{23} C_1}}. \tag{7.30}$$

Here, we have arranged the result so that the coefficients with (without) supergravity interpretation appear on the right-hand (left-hand) side of the equation. Since we wish to constrain the M-Theory model (M,1) as less as possible, we shall choose to satisfy the above by suitably fixing the $h^{(1)}_{2\tilde{\psi}3}$ coefficient. In this case, the $(\mathcal{F}^{(tot)}_{2\tilde{\psi}})$-related BHN equation reads

$$\mathcal{F}^{(tot)}_{2\tilde{\psi}} - i\gamma_4[\hat{\varphi}_2, \hat{\varphi}_3] = -2\mathrm{Re}\big(\gamma_5 \mathcal{D}^{(tot)}_2 \sigma^{(tot)}\big), \tag{7.31}$$

which we shall consider to be its final form for now. Note that, as was the case with the $\big(\mathcal{F}^{(tot)}_{12}\big)$-related BHN equation in (7.18), the above is a Hitchin equation too. However, this time the Hitchin equation is *sourced*.

To finish our discussion regarding this second BHN equation, let us collect all coefficient choices made in deriving it and write down the constraints that the still undetermined coefficients are subjected to. To begin with, we no longer need to worry about the constraints in (7.20), since the coefficient choices in (7.21) satisfy them leaving no degree of freedom. On the other hand, those same choices used in (7.17) lead to

$$\sum_{\alpha=1}^{2}\big(h^{(4)}_{\alpha\tilde{\psi}}\big)^2 = 2\big(g^{(1)}_{1223}\big)^2 + \sum_{\alpha=1}^{2}\big(h^{(1)}_{\alpha\tilde{\psi}23}\big)^2 = \big(h^{(i)}_{1\tilde{\psi}1}\big)^2 = \sum_{\alpha=1}^{2}\big(h^{(i)}_{\alpha\tilde{\psi}2}\big)^2 = \sum_{\alpha=1}^{2}\big(h^{(2)}_{\alpha\tilde{\psi}3}\big)^2 = 1, \tag{7.32}$$

valid for all $i = 1,\, 2$, together with the constraint

$$2\big(g^{(1)}_{12kl}\big)^2 + \big(h^{(1)}_{1\tilde{\psi}kl}\big)^2 = 1, \tag{7.33}$$

which applies when $(k,\, l) = \{(1,\, 2),\, (1,\, 3)\}$. Note that we have already taken into account the enforcement of (7.30) too. Lastly and as explained in the text, in order to set to zero the last terms in (7.16) and (7.22) we have almost exhausted the defining relation (5.30), which now is simply given by

$$\big(m^{(1)}_{12}\big)^2 + \big(m^{(2)}_{12}\big)^2 = \frac{1}{2}. \tag{7.34}$$

We consider the $(\mathcal{F}^{(tot)}_{1\tilde{\psi}})$-related BHN equation next. This follows from the vanishing of the second squared term in the Hamiltonian (7.14), for $\alpha = 1$. Self-consistency of our analysis requires us to implement the coefficient replacements in (7.19), together with the relevant choices in (7.21). Additionally, we already argued that the result in (6.70) applies to the present case. If we further consider that

$$h^{(1)}_{1\tilde{\psi}12} = h^{(1)}_{1\tilde{\psi}23} = h^{(1)}_{1\tilde{\psi}2} = h^{(2)}_{1\tilde{\psi}2} = 0 \tag{7.35}$$

and working in the $\epsilon_{1\tilde{\psi}} = 1$ normalization convention, the BHN equation of interest is

$$\sqrt{\frac{4i|\tau|^2}{\tau-\bar{\tau}}}\mathcal{F}^{(tot)}_{1\tilde{\psi}} + \sqrt{\frac{C_1}{V_3}}\Big(\sqrt{c_{2r}}\mathcal{D}^{(tot)}_2\mathcal{A}^{(tot)}_r + \sqrt{\tilde{c}_{2\phi_1}}\mathcal{D}^{(tot)}_2\mathcal{A}^{(tot)}_{\phi_1}\Big) - ih^{(4)}_{1\tilde{\psi}}\sqrt{\frac{C_1 a_1}{V_3}}[\mathcal{A}^{(tot)}_r, \mathcal{A}^{(tot)}_{\phi_1}]$$
$$- 2ih^{(1)}_{1\tilde{\psi}13}\sqrt{d_{13}}[\varphi_1,\varphi_3] - i\sum_{k=1,3}\sum_{\gamma=2}^{3} h^{(\gamma-1)}_{1\tilde{\psi}k}\sqrt{c_{y_\gamma k}}[\mathcal{A}^{(tot)}_{y_\gamma},\varphi_k] = 0. \tag{7.36}$$

In much the same way as before, rewriting the above in terms of the $(\sigma^{(tot)}, \bar{\sigma}^{(tot)})$ complex scalar fields introduced in (7.23) is convenient. The result this time looks as follows:

$$\mathcal{F}^{(tot)}_{1\tilde{\psi}} + 2\mathrm{Re}\big(\gamma_5\mathcal{D}^{(tot)}_2\sigma^{(tot)}\big) = i\tilde{\gamma}_4[\varphi_1,\varphi_3] + 2i\mathrm{Re}\big(\tilde{\gamma}_1[\bar{\sigma}^{(tot)},\varphi_1] + \tilde{\gamma}_2[\bar{\sigma}^{(tot)},\varphi_3]\big)$$
$$+ \tilde{\gamma}_3[\bar{\sigma}^{(tot)},\sigma^{(tot)}], \tag{7.37}$$

where γ_5 is as in (7.25) and we have defined

$$\tilde{\gamma}_1 \equiv \frac{1}{2}\sqrt{\frac{\tau-\bar{\tau}}{4i|\tau|^2}}\Big(h^{(1)}_{1\tilde{\psi}1}\sqrt{c_{r1}} + ih^{(2)}_{1\tilde{\psi}1}\sqrt{c_{\phi_1 1}}\Big), \qquad \tilde{\gamma}_3 \equiv \frac{1}{2}\sqrt{\frac{\tau-\bar{\tau}}{4i|\tau|^2}}\sqrt{\frac{C_1 a_1}{V_3}}h^{(4)}_{1\tilde{\psi}},$$
$$\tilde{\gamma}_2 \equiv \frac{1}{2}\sqrt{\frac{\tau-\bar{\tau}}{4i|\tau|^2}}\Big(h^{(1)}_{1\tilde{\psi}3}\sqrt{c_{r3}} + ih^{(2)}_{1\tilde{\psi}3}\sqrt{c_{\phi_1 3}}\Big), \qquad \tilde{\gamma}_4 \equiv \frac{1}{2}\sqrt{\frac{d_{13}(\tau-\bar{\tau})}{4i|\tau|^2}}h^{(1)}_{1\tilde{\psi}13}. \tag{7.38}$$

At this stage, it should not come as a surprise that the above BHN equation can be brought into a more suggestive form. To this aim, we now define

$$\hat{\varphi}_1 \equiv \varphi_1 + 2\mathrm{Re}\Big(\frac{\tilde{\gamma}_2\bar{\sigma}^{(tot)}}{\tilde{\gamma}_4}\Big) \tag{7.39}$$

and note that the commutator between this field and $\hat{\varphi}_3$ in (7.27), namely

$$[\hat{\varphi}_1,\hat{\varphi}_3] = [\varphi_1,\varphi_3] + 2\mathrm{Re}\Big(\frac{\gamma_1}{\gamma_4}[\bar{\sigma}^{(tot)},\varphi_1] + \frac{\tilde{\gamma}_2}{\tilde{\gamma}_4}[\bar{\sigma}^{(tot)},\varphi_3]\Big) + 2i\mathrm{Im}\Big(\frac{\gamma_1\overline{\tilde{\gamma}_2}}{\gamma_4\tilde{\gamma}_4}\Big)[\bar{\sigma}^{(tot)},\sigma^{(tot)}], \tag{7.40}$$

is exactly $(i\tilde{\gamma}_4)^{-1}$ times the right-hand side of (7.37) iff the following two equalities are imposed:

$$\tilde{\gamma}_1 = \frac{\gamma_1\tilde{\gamma}_4}{\gamma_4}, \qquad \tilde{\gamma}_3 = -2\mathrm{Im}\Big(\frac{\gamma_1\overline{\tilde{\gamma}_2}}{\gamma_4}\Big). \tag{7.41}$$

In view of (7.25) and (7.38) and considering separately the real and imaginary parts of the first constraint above, we can write these equalities in terms of the coefficients of tables 1 and 2 as

$$\frac{h^{(1)}_{2\tilde{\psi}2}h^{(1)}_{1\tilde{\psi}13}}{h^{(1)}_{1\tilde{\psi}1}h^{(1)}_{2\tilde{\psi}23}} = 1 = \frac{h^{(2)}_{2\tilde{\psi}2}h^{(1)}_{1\tilde{\psi}13}}{h^{(2)}_{1\tilde{\psi}1}h^{(1)}_{2\tilde{\psi}23}}, \qquad \frac{h^{(4)}_{1\tilde{\psi}}h^{(1)}_{2\tilde{\psi}23}}{h^{(1)}_{1\tilde{\psi}3}h^{(2)}_{2\tilde{\psi}2} - h^{(1)}_{1\tilde{\psi}2}h^{(2)}_{1\tilde{\psi}3}} = -2\sqrt{\frac{c_{r2}c_{\phi_1 2}V_3}{a_1 d_{23} C_1}}, \tag{7.42}$$

where we have also used $d_{12} = d_{23}$ and $(c_{r1} = c_{r2},\, c_{\phi_1 1} = c_{\phi_1 2})$ following from (4.129) and (4.135), respectively. We shall satisfy the rightmost equality by suitably fixing $h^{(4)}_{1\tilde{\psi}}$. The implementation of the other two equalities will be done in an a priori strange fashion, which will turn out to be most useful not much later on:

$$h^{(1)}_{1\tilde{\psi}1} = h^{(1)}_{2\tilde{\psi}2} = h^{(2)}_{1\tilde{\psi}1} = h^{(2)}_{2\tilde{\psi}2} = 1, \qquad h^{(1)}_{1\tilde{\psi}13} = h^{(1)}_{2\tilde{\psi}23}. \tag{7.43}$$

Putting everything together, we find that the $(\mathcal{F}^{(tot)}_{1\tilde{\psi}})$-related BHN equation decouples to the following *sourced* Hitchin equation:

$$\mathcal{F}^{(tot)}_{1\tilde{\psi}} - i\gamma_4[\hat{\varphi}_1, \hat{\varphi}_3] = -2\mathrm{Re}\big(\gamma_5 \mathcal{D}^{(tot)}_2 \sigma^{(tot)}\big). \tag{7.44}$$

Notice that here we have used the fact that $\tilde{\gamma_4} = \gamma_4$ after choosing (7.43), as can be checked from their definitions in (7.25) and (7.38). This equality combined with (7.41) also implies that $\tilde{\gamma_1} = \gamma_1$, which will soon come in handy. It is pretty straightforward to check that the coefficient choices made in deriving this Hitchin equation reduce the possible values that the still arbitrary parameters in (7.32) and (7.33) can take. Specifically, they are now required to satisfy the very stringent conditions

$$2\big(g^{(1)}_{1223}\big)^2 = 2\big(g^{(1)}_{1213}\big)^2 = 1 - \big(h^{(1)}_{1\tilde{\psi}13}\big)^2, \qquad 2\big(g^{(1)}_{1212}\big)^2 = 1 = \sum_{\alpha=1}^{2}\big(h^{(2)}_{\alpha\tilde{\psi}3}\big)^2. \tag{7.45}$$

Conversely, the constraint (7.34) is not affected.

It will certainly not look any nicer, but in anticipation to the boundary analysis to come, we must rewrite our $(\mathcal{F}^{(tot)}_{12})$-related Hitchin equation (7.18) in terms of the hatted complex scalar fields $(\hat{\varphi}_1,\, \hat{\varphi}_2,\, \hat{\varphi}_3)$ defined in (7.27) and (7.39). This amounts to a straightforward yet tedious algebraic exercise. Hence, let us show a few intermediate results before stating the end result:

$$\sqrt{\frac{2i|\tau|^2}{\tau - \bar{\tau}}}\mathcal{F}^{(tot)}_{\alpha\beta} - i\sum_{k,l=1}^{3} g^{(1)}_{\alpha\beta kl}\sqrt{d_{kl}}[\hat{\varphi}_k, \hat{\varphi}_l] = -\sum_{k=1}^{3}\mathrm{Re}\big(\kappa_k[\bar{\sigma}^{(tot)}, \varphi_k]\big) - \kappa_4[\bar{\sigma}^{(tot)}, \sigma^{(tot)}], \tag{7.46}$$

where α, $\beta = 1$, 2 and we have defined the $\{\kappa\}$ constants as

$$\begin{aligned} \kappa_1 &\equiv -\frac{4i\sqrt{d_{12}}}{\gamma_4}(\gamma_2 g^{(1)}_{\alpha\beta 12} - \gamma_1 g^{(1)}_{\alpha\beta 13}), & \kappa_2 &\equiv \frac{-4i\sqrt{d_{12}}}{\gamma_4}(\tilde{\gamma}_2 g^{(1)}_{\alpha\beta 12} - \gamma_1 g^{(1)}_{\alpha\beta 23}), \\ \kappa_3 &\equiv \frac{4i\sqrt{d_{12}}}{\gamma_4}(\tilde{\gamma}_2 g^{(1)}_{\alpha\beta 13} + \gamma_2 g^{(1)}_{\alpha\beta 23}), & \kappa_4 &\equiv 4i\sqrt{d_{12}}\Big(\frac{\bar{\gamma}_2\tilde{\gamma}_2 - \bar{\tilde{\gamma}}_2\gamma_2}{\gamma_4^2}\Big) g^{(1)}_{\alpha\beta 12}. \end{aligned} \tag{7.47}$$

Here, we have taken into account that d_{kl} in (4.129) is the same for all (k, l). The last subtleties employed when defining the above $\{\kappa\}$ constants amount to implementing $\tilde{\gamma}_i = \gamma_i$ for $i = 1$, 4, as just explained. Once again making use of the definitions in (7.27) and (7.39), one can verify that

$$\sum_{k=1}^{3} \mathrm{Re}\big(\kappa_k[\bar{\sigma}^{(tot)}, \varphi_k]\big) = \sum_{k=1}^{3} \mathrm{Re}\big(\kappa_k[\bar{\sigma}^{(tot)}, \hat{\varphi}_k]\big) - \frac{i}{\gamma_4}\mathrm{Re}\big(\kappa_1\bar{\tilde{\gamma}}_2 + \kappa_2\gamma_2 + \kappa_3\gamma_1\big)[\bar{\sigma}^{(tot)}, \sigma^{(tot)}]. \tag{7.48}$$

Consequently, we can introduce a last κ constant,

$$\kappa_5 \equiv -\frac{i}{\gamma_4}\mathrm{Re}\big(\kappa_1\bar{\tilde{\gamma}}_2 + \kappa_2\gamma_2 + \kappa_3\gamma_1\big) + \kappa_4, \tag{7.49}$$

and rewrite the Hitchin equation (7.18), for $(\alpha = 1, \beta = 2)$, in what shall soon be seen to be its most convenient form:

$$\sqrt{\frac{2i|\tau|^2}{\tau - \bar{\tau}}}\mathcal{F}^{(tot)}_{12} - i\sum_{k,l=1}^{3} g^{(1)}_{12kl}\sqrt{d_{kl}}[\hat{\varphi}_k, \hat{\varphi}_l] = -\sum_{k=1}^{3}\mathrm{Re}\big(\kappa_k[\bar{\sigma}^{(tot)}, \hat{\varphi}_k]\big) - \kappa_5[\bar{\sigma}^{(tot)}, \sigma^{(tot)}]. \tag{7.50}$$

Summing up, in the case where we add an M2-brane oriented along $(t, x_1, \tilde{\psi})$ to the configuration (M,1), the energy associated to the corresponding world-volume Hamiltonian is minimized when the Hitchin equations (7.31), (7.44) and (7.50) are satisfied[27]. It is noteworthy to emphasize that these all are *sourced* Hitchin equations. This is true for the gauge choice (7.10)-(7.11), for time-independent field configurations and when the consistency requirements (5.92) are met −after the replacement $\mathcal{D}_2 \to \mathcal{D}_2^{(tot)}$. Moreover, the energy minimization process restricts enormously the values that the coefficients of table 2, which appear in the Hamiltonian, can take. Among those coefficients, the ones that have not yet been appropriately fixed must fulfill both (7.34) and (7.45).

[27]To fully make sense of these Hitchin equations in terms of variables introduced prior to this section, the reader will need to see the definitions in (7.25), (7.27), (7.38), (7.39), (7.47) and (7.49), as well as set $\tilde{\gamma}_i = \gamma_i$ for $i = 1$, 4.

7.3 The boundary theory revisited

Now that we know the BPS equations that govern the four-dimensional gauge theory following from the configuration depicted in figure 14, it is time to investigate what the boundary action is. That is, the present section is devoted to the adaptation of the contents in sections 6.1 and 6.2 to the present case, where an M2-brane along $(t,\, x_1,\, \tilde{\psi})$ is added to the model (M,1). By the end of this section, we will be ready to topologically twist the theory. As was the case before in part II, this twist will hugely clarify the situation and (finally!) open the possibility for an exciting physical interpretation of the new M2-brane. However, for the time being, the reader must remain patient: the computations in this section are the last ones required to reach such a climactic point in our discussion.

The first thing we will do in this section is work out the generalization of (6.11) to the case where we include the new M2-brane of figure 14. To this aim, we will proceed in the short-cut manner used before: by comparison to [54]. However, the comparison is only possible after we rewrite the Hamiltonian (7.14) using the sourced Hitchin equations (7.31), (7.44) and (7.50). In doing so, we must take into account that these Hitchin equations imply $\hat{\chi}_T = 0$, as explained below (7.18). All in all, the total effective Hamiltonian can be written as

$$\begin{aligned}\hat{H} = \; & \frac{4i|\tau|^2}{\tau-\bar{\tau}} \int d^4x \, \mathrm{Tr}\Big\{ \sum_{\alpha=1}^{2} \Big(\mathcal{F}^{(tot)}_{\alpha\tilde{\psi}} - i\gamma_4 [\hat{\varphi}_\alpha, \hat{\varphi}_3] + 2\mathrm{Re}\big(\gamma_5 \mathcal{D}^{(tot)}_2 \sigma^{(tot)}\big)\Big)^2 \\ & + \Big(\mathcal{F}^{(tot)}_{12} - i\sqrt{\frac{\tau-\bar{\tau}}{2i|\tau|^2}} \sum_{k,l=1}^{3} g^{(1)}_{12kl} \sqrt{d_{kl}} [\hat{\varphi}_k, \hat{\varphi}_l] + \sum_{k=1}^{3} \mathrm{Re}\big(\tilde{\kappa}_k [\bar{\sigma}, \hat{\varphi}_k]\big) + \tilde{\kappa}_5 [\bar{\sigma}, \sigma]\Big)^2 \Big\} \\ & + \int d^4x \, \mathrm{Tr}\Big\{ (\tau-\bar{\tau}) \sum_{i,j,k=1}^{3} \epsilon^{0ijk} \mathcal{F}^{(tot)}_{0i} \mathcal{F}^{(tot)}_{jk} \Big\} + \hat{Q}_{EM}. \end{aligned} \tag{7.51}$$

It is perhaps worth mentioning a few details regarding the overall prefactor in the bulk Hamiltonian part, i.e. the first two lines above. The first squared term follows from (7.31) for $\alpha = 2$ and (7.44) for $\alpha = 1$ in a slightly tricky way. On the one hand, we must remember that we suppressed the above appearing overall prefactor (actually, its square root) when going from (7.22)/(7.36) to (7.26)/(7.37) before obtaining (7.31)/(7.44), respectively. On the other hand, the seemingly arbitrary choices in (7.43) here pay off again: they set $\gamma_4 = \tilde{\gamma}_4$, as can be seen from their definitions in (7.25) and (7.38) after noting that $d_{13} = d_{23}$ due to (4.129). Compared to (7.50), the second squared term above is the result of summing over $\alpha,\, \beta = 1,\, 2$ and exploiting antisymmetry over these indices before taking the common factor out of the bulk integral. The $\{\tilde{\kappa}\}$ constants are therefore just a rescaled version of the $\{\kappa\}$'s in (7.47) and (7.49), for ($\alpha = 1$, $\beta = 2$):

$$\tilde{\kappa}_x = \sqrt{\frac{\tau - \bar{\tau}}{2i|\tau|^2}}\,\kappa_x\bigg|_{\alpha=1,\beta=2} \qquad \forall x = 1,2,3,5. \tag{7.52}$$

With these subtleties in mind, the suggested Hamiltonian should make perfect sense. Additionally, it is written in a manner that allows us to directly compare it to (2.4) in [54].

Note that, by definition, we should understand the last term $\hat{Q}_{EM}$ in the Hamiltonian (7.51) as an apt modification of (6.1) and (6.2). Namely, our gauge choice (7.10)-(7.11) together with the time-independence of all fields in the theory set the electric charge to zero. At the same time, the magnetic charge contributes to the theory at the boundary X_3 of the spacetime X_4, since total derivatives along the unbound directions (t, x_1, x_2) do not affect the physics under study. Mathematically,

$$\hat{Q}_{EM} \equiv \hat{Q}_E + \hat{Q}_M, \qquad \hat{Q}_E = 0, \qquad \hat{Q}_M = \frac{4i|\tau|^2}{\tau - \bar{\tau}} \int d^4x\; \partial_{\tilde{\psi}} \hat{q}_M = \frac{4i|\tau|^2}{\tau - \bar{\tau}} \int d^3x\; \hat{q}_M, \tag{7.53}$$

where, quite evidently, $d^4x \equiv dt dx_1 dx_2 d\tilde{\psi}$ while $d^3x \equiv dt dx_1 dx_2$. Let us turn towards the precise form of $\hat{q}_M$ next. It is necessary for the reader to recall the paragraph containing equations (6.5) and (6.6) before. In view of what is there explained, it should be easy to recognize that the second line in the Hamiltonian (7.51) does not contribute to the magnetic charge $\hat{Q}_M$. Briefly, there are only ($\mathcal{F}$, $[\varphi,\varphi]$) type of terms in this squared BHN equation, so they will contribute ($\varphi\mathcal{D}\varphi$)-like terms to the magnetic charge. However, since this field strength is associated to only the unbound directions (x_1, x_2), the integration by parts will render such contributions to the boundary physics negligible: we can set them to zero by the usual convention that all fields vanish at spatial infinity. Contrarily, the first line in the Hamiltonian (7.51) does contribute to $\hat{Q}_M$: both the ($\mathcal{F}\cdot\mathcal{D}\varphi$) and the ($\mathcal{F}\cdot[\varphi,\varphi]$) crossed terms lead to non-trivial terms in the magnetic charge. Note that this is not true for the ($\mathcal{D}\varphi\cdot[\varphi,\varphi]$) crossed term, since the covariant derivative is along the unbound direction x_2 in this case. Putting everything together while following [54], we can write $\hat{q}_M$ as

$$\hat{q}_M \equiv \sum_{\alpha=1}^{2} \mathrm{Tr}\left[2\mathcal{A}_\alpha^{(tot)}\mathrm{Re}(\gamma_5 \mathcal{D}_2^{(tot)}\sigma^{(tot)}) - i\gamma_4(\hat{\varphi}_\alpha \mathcal{D}_\alpha^{(tot)}\hat{\varphi}_3 - \hat{\varphi}_3\mathcal{D}_\alpha^{(tot)}\hat{\varphi}_2)\right]. \tag{7.54}$$

This fully specifies the magnetic charge in the theory. What is more, the Hamiltonian (7.51) is now completely described as well.

In precisely the same way as that pointed out around (6.8)-(6.10) earlier on, we can convert the first term in the last line of the Hamiltonian (7.51) to a boundary action. This means that, once on shell −i.e. the first two lines in (7.51) are set to zero−, the generalization of the boundary action (6.11) we were looking for is given by

$$\hat{S}_{bnd} = \frac{4i|\tau|^2}{\tau - \bar{\tau}} \int d^3x \; \hat{q}_M + (\tau + \bar{\tau}) \int_{X_3} \mathrm{Tr}\big(\mathcal{A}^{(tot)} \wedge d\mathcal{A}^{(tot)} + \frac{2i}{3} \mathcal{A}^{(tot)} \wedge \mathcal{A}^{(tot)} \wedge \mathcal{A}^{(tot)}\big), \tag{7.55}$$

with $\hat{q}_M$ as defined in (7.54). What we will do next is "massage" the above until we bring it into its most convenient form for our later purposes.

As we argued at the very beginning of section 7.1, the new M2-brane does not alter the supersymmetry of model (M,1). Consequently and as long as the replacements in (7.9) are carried out, the boundary conditions previously derived are still valid when the M2-brane is present. In particular, we want to focus on the following appropriate modifications of the boundary conditions in (6.19) and (6.21):

$$\varphi_3 = \mathcal{A}^{(tot)}_{\phi_1} = \mathcal{A}^{(tot)}_{r} = [\varphi_1, \varphi_2] = \mathcal{D}_{\tilde{\psi}}\varphi_1 = \mathcal{D}_{\tilde{\psi}}\varphi_2 = 0 \qquad \text{at } \tilde{\psi} = 0. \tag{7.56}$$

These boundary conditions, when taken into account simultaneously with the suitably modified ($\mathcal{D}_2 \to \mathcal{D}_2^{(tot)}$) consistency requirements in (5.92) and with the definitions (7.23), (7.27) and (7.39) imply that

$$\mathcal{D}_2^{(tot)}\varphi_3 = \mathcal{D}_{\tilde{\psi}}\varphi_2, \qquad \mathcal{D}_1\varphi_3 = \mathcal{D}_{\tilde{\psi}}\varphi_1, \qquad \hat{\varphi}_k = \varphi_k \qquad \forall k = 1,2,3 \text{ and at } \tilde{\psi} = 0. \tag{7.57}$$

Such observations are pertinent because they are just now going to help us rewrite $\hat{q}_M$ in (7.54) in the form of q_M in (6.3) and in this manner exploit previous computations. The idea is to add and subtract terms that are identically zero until $\hat{q}_M \sim q_M$. More concretely, because $\hat{\varphi}_3 = \varphi_3 = 0$ at X_3, we can simply go ahead and add

$$\gamma_6 \mathrm{Tr}\big(\sum_{\alpha,\beta=1}^{2} \hat{\varphi}_3 \mathcal{F}^{(tot)}_{\alpha\beta}\big), \qquad \gamma_7 \sum_{k,l,m=1}^{3} \epsilon_{klm} \mathrm{Tr}\big(\hat{\varphi}_k[\hat{\varphi}_l, \hat{\varphi}_m]\big), \qquad -i\gamma_4 \mathrm{Tr}\big(\hat{\varphi}_3 \mathcal{D}_2^{(tot)} \hat{\varphi}_1\big) \tag{7.58}$$

to $\hat{q}_M$ without any repercussion. Here, (γ_6, γ_7) are some arbitrary real constants while γ_4 was defined in (7.25). Similarly, because of (7.57), it is also possible to add the following terms to $\hat{q}_M$ without the slightest consequence:

$$-i\gamma_4 \mathrm{Tr}\big(\hat{\varphi}_\alpha \mathcal{D}_\beta^{(tot)} \hat{\varphi}_3\big), \qquad -i\gamma_4 \mathrm{Tr}\big(\hat{\varphi}_\alpha \mathcal{D}_{\tilde{\psi}} \hat{\varphi}_\beta\big) \qquad \forall \alpha, \beta = 1,2 \text{ with } \alpha \neq \beta. \tag{7.59}$$

The last of these type of manipulations we wish to do amounts to subtracting

$$-i\gamma_4 \mathrm{Tr}\big(\hat{\varphi}_\alpha \mathcal{D}_\alpha^{(tot)} \hat{\varphi}_3\big), \qquad i\gamma_4 \mathrm{Tr}\big(\hat{\varphi}_3 \mathcal{D}_2^{(tot)} \hat{\varphi}_2\big) \qquad \forall \alpha = 1,2 \tag{7.60}$$

from $\hat{q}_M$. In conclusion, the most beneficial form for the boundary action remains as in

(7.55), but with the magnetic charge density $\hat{q}_M$ written as

$$\hat{q}_M = \sum_{\alpha,\beta=1}^{2} \sum_{k,l,m=1}^{3} \mathrm{Tr}\big[\gamma_6 \epsilon_{k\alpha\beta} \hat{\varphi}_k \mathcal{F}^{(tot)}_{\alpha\beta} + \epsilon_{klm}\big(\gamma_7 \hat{\varphi}_k [\hat{\varphi}_l, \hat{\varphi}_m] - i\gamma_4 \hat{\varphi}_k \mathcal{D}^{(tot)}_l \hat{\varphi}_m\big) + 2\mathcal{A}^{(tot)}_\alpha \mathrm{Re}\big(\gamma_5 \mathcal{D}^{(tot)}_2 \sigma^{(tot)}\big)\big] \tag{7.61}$$

instead of as in its defining form in (7.54).

Even tough (7.55) and (7.61) constitute the results we were seeking in this section, it is worth pausing for a moment before continuing our way to the promised topological twist. Indeed, it is quite enriching to reflect upon the differences between the two magnetic charge densities considered so far: (6.3) in the absence of the new M2-brane of figure 14 and (7.61) in its presence. We can divide their dissimilarities into three classes, from less to more important:

1. Differences in the coefficients. Although the magnetic charges take a similar form in both cases, all the terms are separately rescaled. In more detail, we find that the following coefficient changes are required to try to establish a biconditional statement $Q_M \leftrightarrow \hat{Q}_M$:

$$d_1 \leftrightarrow \frac{4i|\tau|^2}{\tau - \bar{\tau}}\gamma_6, \qquad \frac{id_2}{3} \leftrightarrow \frac{4i|\tau|^2}{\tau - \bar{\tau}}\gamma_7, \qquad d_3 \leftrightarrow -i\frac{4i|\tau|^2}{\tau - \bar{\tau}}\gamma_4, \tag{7.62}$$

   where we have taken into account the global prefactor difference following from comparing (6.2) and (7.53). We remind the reader that τ was defined in (5.83), (d_1, d_2, d_3) in (6.7), γ_4 in (7.25) and (γ_6, γ_7) are arbitrary real parameters. All (τ, d_1, d_2, d_3, γ_4) admit a supergravity interpretation in terms of the warp factors and constant dilaton that characterize the M-Theory model (M,1).

2. Differences in the fields. As already discussed, the new M2-brane forces us to consider new contributions to the gauge and scalar fields ($\mathcal{A}_2$, σ, $\bar{\sigma}$), after gauge fixing −see (7.9). These changes do not alter the form of the terms to which the above rescaling applies, but they compel us to consider

$$\mathcal{F}_{\alpha\beta} \leftrightarrow \mathcal{F}^{(tot)}_{\alpha\beta}, \qquad \mathcal{D}_2 \leftrightarrow \mathcal{D}^{(tot)}_2, \qquad \varphi_k \leftrightarrow \hat{\varphi}_k \qquad \forall \alpha, \beta = 1, 2 \text{ and } \forall k = 1, 2, 3 \tag{7.63}$$

   when mapping $Q_M \leftrightarrow \hat{Q}_M$. It is perhaps worth reminding the reader that $\{\hat{\varphi}\}$ are certain linear combinations of the φ and (σ, $\bar{\sigma}$) scalar fields in the gauge theory, as can be seen from their definitions in (7.27) and (7.39).

3. Appearance of a new term. There is a new term in $\hat{Q}_M$ that is not present in Q_M and that spoils an easy relation of the form $Q_M \leftrightarrow \hat{Q}_M$:

$$2\frac{4i|\tau|^2}{\tau-\bar{\tau}}\sum_{\alpha,\beta=1}^{2}\mathrm{Tr}\Big[\mathcal{A}_{\alpha}^{(tot)}\mathrm{Re}\big(\gamma_5\mathcal{D}_2^{(tot)}\sigma^{(tot)}\big)\Big], \tag{7.64}$$

with γ_5 a constant given by (7.25). This term will soon become crucial to understand the essential role of the new M2-brane in realizing knots in (M,1). For the time being, it is important to note that this term is proportional to $(\mathcal{A}_1,\,\mathcal{A}_2^{(tot)})$. That is, proportional to those components of the gauge field that are oriented along the boundary X_3 and that are not set to zero by our gauge choice (7.10)-(7.11).

7.4 The topological twist revisited: making sense of the new M2-brane

We are finally ready to perform the topological twist described in section 6.3, but applied to this present scenario where we have a novel M2-brane in (M,1). From the point of view of the bosonic matter content in the world-volume gauge theory following from the configuration in figure 14, the topological twist we wish to consider amounts to a suitable adaptation of equations (6.26)-(6.30) and (6.72) to the current case. Specifically, we must entertain a certain rescaling of the gauge fields:

$$\hat{A}=\sum_{\mu}\hat{A}_{\mu}dx^{\mu},\qquad \hat{A}_{\mu}=i\mathcal{A}_{\mu}^{(tot)}\qquad \forall\mu, \tag{7.65}$$

with x^{μ} labeling the spacetime directions $(t,\,x_1,\,x_2,\,\tilde{\psi})$. We will refer to the twisted field strength and covariant derivative associated to the above twisted gauge field as $\hat{F}$ and $\hat{D}_{\mu}$, where $\hat{F}=-i\mathcal{F}^{(tot)}$ and $\hat{D}_{\mu}=\mathcal{D}_{\mu}^{(tot)}$ as a direct consequence of the above. In sections 6.3 and 6.3.1, we learnt that the topological twist affects some of the scalar fields in the theory via a certain rescaling and rearrangement into a one-form. We also learnt that it is convenient to consider such rearrangement differently for boundary and bulk. Consequently, for the boundary we will consider an appropriate modification of (6.26):

$$\hat{\Phi}=\sum_{\mu}\hat{\Phi}_{\mu}dx^{\mu},\qquad (\hat{\Phi}_0,\hat{\Phi}_1,\hat{\Phi}_2,\hat{\Phi}_3)=i(\hat{\varphi}_3,\hat{\varphi}_1,\hat{\varphi}_2,\mathcal{A}_{\tilde{3}}^{(tot)}), \tag{7.66}$$

whereas for the bulk we will generalize (6.72):

$$\hat{\Lambda} = \sum_{\mu} \hat{\Lambda}_\mu dx^\mu, \qquad (\hat{\Lambda}_0, \hat{\Lambda}_1, \hat{\Lambda}_2, \hat{\Lambda}_3) = i(\mathcal{A}_{\bar{3}}^{(tot)}, \hat{\varphi}_1, \hat{\varphi}_2, \hat{\varphi}_3). \tag{7.67}$$

It should be noted that the complex scalar fields ($\sigma^{(tot)}$, $\bar{\sigma}^{(tot)}$) are not affected by the twisting process when they appear by themselves. However, when they are part of the $\{\hat{\varphi}\}$ scalar fields, they do "feel" the twist.

In this section, we will start by implementing the above twist in the boundary action (7.55), with $\hat{q}_M$ as in (7.61). Then, we will twist the bulk BPS equations (7.31), (7.44) and (7.50), together with the consistency requirements (5.92), subject to the replacements in (7.9). For both boundary and bulk, we will ponder over the thus obtained results so as to fully grasp the importance and connection to Knot Theory of the new M2-brane in figure 14.

Twisting the boundary

The first of the just described tasks, namely twisting (7.55) and (7.61), is straightforward in view of the above exposed. The result is

$$\hat{S}^{(t)}_{bnd} = -\frac{4i|\tau|^2}{\tau - \bar{\tau}} \int d^3x \hat{q}^{(t)}_M - (\tau - \bar{\tau}) \int_{X_3} \mathrm{Tr}\big(\hat{A} \wedge d\hat{A} + \frac{2}{3} \hat{A} \wedge \hat{A} \wedge \hat{A}\big), \tag{7.68}$$

with the twisted magnetic charge density $\hat{q}^{(t)}_M$ given by

$$\hat{q}^{(t)}_M = \sum_{\alpha,\beta=1}^{2} \sum_{a,b,c=0}^{2} \mathrm{Tr}\Big[\gamma_6 \epsilon_{a\alpha\beta} \hat{\Phi}_a \hat{F}_{\alpha\beta} - i\epsilon_{abc}\big(\gamma_7 \hat{\Phi}_a [\hat{\Phi}_b, \hat{\Phi}_c] + \gamma_4 \hat{\Phi}_a \hat{D}_b \hat{\Phi}_c\big) + 2i\hat{A}_\alpha \mathrm{Re}\big(\gamma_5 \hat{D}_2 \sigma^{(tot)}\big)\Big]. \tag{7.69}$$

Another way to obtain this twisted boundary action is by comparison with the previously performed topological twist, where (6.11) and (6.3) transformed to (6.36) and (6.37), respectively. The comparison is possible thanks to the enumeration that concluded the preceding section, with special emphasis on equations (7.62) and (7.63). As already happened with (6.36) before, the above is not quite the correct twisted boundary action because it is not compatible with the amount of supersymmetry we require: $\mathcal{N} = 4$ with half-BPS boundary conditions. The proof for this claim and an elegant way out can be found in the brilliant work [61], sections 3.4 and 3.5. The interested reader may also find worth browsing through [70], which contains an illuminating discussion on how subtle and difficult it is to correct this shortcoming, as well as a different yet equivalent procedure to fix the situation.

Here, we shall limit ourselves to the postulation of the correct twisted boundary action:

$$\hat{S}^{(t)}_{bnd,tot} = -\frac{4i|\tau|^2}{\tau - \bar{\tau}} \int d^3x \hat{q}^{(t)}_M + i\Psi \int_{X_3} \mathrm{Tr}\big(\hat{A} \wedge d\hat{A} + \frac{2}{3}\hat{A} \wedge \hat{A} \wedge \hat{A}\big), \tag{7.70}$$

where $\hat{q}^{(t)}_M$ remains as in (7.69) and the so-called canonical parameter Ψ was defined in (6.39). Note that all the comments on Ψ previously made remain true now as well, most significantly equation (6.41). Also well worth mentioning, the above is the counterpart to (6.40) before.

At this stage, we have fulfilled the promise of establishing the twisted action governing the physics at the boundary X_3. However, (7.70) and (7.69) are not a priori enlightening and some algebraic effort is needed to bring these into a physically meaningful form. In fact, the effort is minimal, since we can exploit previous computations to our advantage. There is a "but". As already advanced in the end of the previous section, the term $\sim (\hat{A}\hat{D}_2\sigma^{(tot)})$ in the magnetic charge is somewhat special and does not admit a clear mapping to the simpler scenario of parts I and II where no M2-brane was present in the spacetime directions. We will, for the time being, keep this term singled out and only get back to it in the very end of the section, when it will fit in place perfectly. Proceeding in this cautious manner, all we need to do is carry out the sums in (7.69). This is an easy enough task, specially in view of the earlier sums in (6.45)-(6.47). Indeed, a substitution $(A, \Phi) \to (\hat{A}, \hat{\Phi})$ in (6.42)-(6.44) immediately allows us to express the twisted magnetic charge in the language of differential geometry as

$$\begin{aligned}\frac{4i|\tau|^2}{\tau - \bar{\tau}} \int d^3x \; \hat{q}^{(t)}_M = &- \int_{X_3} \mathrm{Tr}\Big(2\tilde{D}_1\hat{\Phi} \wedge \hat{F} + \frac{2}{3}\tilde{D}_2\hat{\Phi} \wedge \hat{\Phi} \wedge \hat{\Phi} + \tilde{D}_3\hat{\Phi} \wedge d_{\hat{A}}\hat{\Phi}\Big) \\ &- \frac{8|\tau|^2}{\tau - \bar{\tau}} \int d^3x \; \mathrm{Tr}\Big[\hat{A}_\alpha \mathrm{Re}\big(\gamma_5 \hat{D}_2 \sigma^{(tot)}\big)\Big],\end{aligned} \tag{7.71}$$

where we have defined

$$\tilde{D}_1 \equiv -\frac{1}{2}\frac{4i|\tau|^2}{\tau - \bar{\tau}}\gamma_6, \qquad \tilde{D}_2 \equiv 3i\frac{4i|\tau|^2}{\tau - \bar{\tau}}\gamma_7, \qquad \tilde{D}_3 \equiv i\frac{4i|\tau|^2}{\tau - \bar{\tau}}\gamma_4. \tag{7.72}$$

Following the same logic as that between (6.51)-(6.60), we demand

$$\tilde{D}_2 = \frac{\tilde{D}_1^3}{(i\Psi)^2}, \qquad \tilde{D}_3 = \frac{\tilde{D}_1^2}{i\Psi} \tag{7.73}$$

and define a new gauge field: a linear combination of the original twisted gauge field and the one-form in (7.66), given by

$$\hat{A}_D \equiv \hat{A} + \frac{\tilde{D}_1}{i\Psi}\hat{\Phi}. \tag{7.74}$$

Then, the twisted action in X_3 following from the configuration depicted in figure 14 can be written as a *topological*, complexified Chern-Simons action plus an additional contribution that we will discuss in the end of the section:

$$\hat{S}^{(t)}_{bnd,tot} = i\Psi \int_{X_3} \mathrm{Tr}(\hat{A}_D \wedge d\hat{A}_D + \frac{2}{3}\hat{A}_D \wedge \hat{A}_D \wedge \hat{A}_D) + \frac{8|\tau|^2}{\tau - \bar{\tau}} \mathrm{Tr} \sum_{\alpha=1}^{2} \int d^3x \left[\hat{A}_\alpha \mathrm{Re}\big(\gamma_5 \hat{D}_2 \sigma^{(tot)}\big)\right]. \tag{7.75}$$

A few more comments are due before we move from the boundary X_3 to the bulk X_4. To begin with, note that the constraints in (7.73) actually don't constrain us at all! We can satisfy them by conveniently choosing the arbitrary constants (γ_6, γ_7). On the other hand, there are two more constraints we did not mention yet: the apt modification of (6.62) given by $\tilde{D}_1 = i\Psi(\hat{t} - \hat{t}^{-1})/2$, with $\hat{t}$ introduced in (6.32), and the very same (6.63). As previously explained, these constraints follow form supersymmetry and gauge invariance requirements, respectively. Let us see how we go about satisfying those:

- Because of the constraint (7.73), $\tilde{D}_1$ is related to $\tilde{D}_3$. Then, looking at the definition of $\tilde{D}_3$ in (7.72), we note that this parameter depends on γ_4, which in turn was defined in (7.25). γ_4 depends one of the coefficients, $h^{(1)}_{2\tilde{\psi}23}$, that is listed in table 2 and that does not have any physical meaning. As can be seen from (7.43) and (7.45), $h^{(1)}_{2\tilde{\psi}23}$ has not yet been fixed. Consequently, we can choose its value such that the supersymmetry constraint is satisfied without imposing any condition on our model (M,1). For concreteness, the just described calculation yields the following choice:

$$h^{(1)}_{2\tilde{\psi}23} = \frac{\Psi}{2}\frac{\hat{t} - \hat{t}^{-1}}{\sqrt{d_{23}}}\sqrt{\frac{\tau - \bar{\tau}}{4i|\tau|^2}}. \tag{7.76}$$

  Notice that this means fixing $(h^{(1)}_{1\tilde{\psi}23}, g^{(1)}_{1223}, g^{(1)}_{1213})$ as well, up to an irrelevant sign. But even more importantly, notice that this is in sharp contrast to what happened in section 6.3 before, where we (unfortunately) needed to narrow down the family of M-Theoretical configurations (M,1) to fulfill (6.62)!

- Against this background, (6.63) still requires us to impose conditions on the model (M,1) to hold true. However, as pointed out in both footnote 20 and at the end of section 6.3, this is more of a structural trouble than a limitation of our own construction.

Twisting the bulk

We will now generalize the contents of section 6.3.1 to the case where a new M2-brane along $(t,\, x_1,\, \tilde{\psi})$ is added to the model (M,1). After doing so we will be able to give a most interesting four-dimensional interpretation to this new M2-brane. Just to get the ball rolling, let us advance that such interpretation will finally land us in the exciting discipline of Knot Theory. On the not-so-bright side, it will be an "emergency landing", where things will not fully work out. Like the phoenix, we shall profit from the ashes of the present section to resurrect in the next chapter and, then yes, correct course to land with the elegance that is due in the fascinating discipline of Knot Theory.

We will start with the consistency requirements in (5.92), subject to $\mathcal{D}_2 \to \mathcal{D}_2^{(tot)}$. First of all, we note that these are written in terms of the unhatted scalar fields $(\varphi_1,\, \varphi_2,\, \varphi_3)$, unlike the boundary action we just discussed and the Hitchin equations we shall turn to after. We fix this situation by using (7.27) and (7.39) to rewrite the consistency requirements as

$$\begin{aligned} \mathcal{D}_1\hat{\varphi}_2 - \mathcal{D}_2^{(tot)}\hat{\varphi}_1 \quad &= \frac{2}{\gamma_4}\mathrm{Re}\big(\gamma_2\mathcal{D}_1\bar{\sigma}^{(tot)} - \tilde{\gamma}_2\mathcal{D}_2^{(tot)}\bar{\sigma}^{(tot)}\big), \\ \mathcal{D}_\alpha^{(tot)}\hat{\varphi}_3 - \mathcal{D}_{\tilde{\psi}}\hat{\varphi}_\alpha \quad &= -\frac{2}{\gamma_4}\mathrm{Re}\big(\gamma_1\mathcal{D}_\alpha^{(tot)}\bar{\sigma}^{(tot)}\big) \qquad\qquad \forall\alpha = 1, 2, \\ \mathcal{D}_1\hat{\varphi}_1 + \mathcal{D}_2^{(tot)}\hat{\varphi}_2 + \mathcal{D}_{\tilde{\psi}}\hat{\varphi}_3 \quad &= \frac{2}{\gamma_4}\mathrm{Re}\big(\tilde{\gamma}_2\mathcal{D}_1\bar{\sigma}^{(tot)} + \gamma_2\mathcal{D}_2^{(tot)}\bar{\sigma}^{(tot)}\big), \end{aligned} \tag{7.77}$$

where we have used $\tilde{\gamma}_4 = \gamma_4 \in \mathbb{R}$, as explained below (7.44), together with $\mathcal{D}_{\tilde{\psi}}\sigma^{(tot)} = 0 = \mathcal{D}_{\tilde{\psi}}\bar{\sigma}^{(tot)}$, which readily follow from considering both (7.12) and (7.23). We are now ready to topologically twist the consistency requirements. It is not hard to see that, under (7.65) and (7.67), the above equations read

$$\begin{aligned} \hat{D}_1\hat{\Lambda}_2 - \hat{D}_2\hat{\Lambda}_1 \quad &= \frac{2i}{\gamma_4}\mathrm{Re}\big(\gamma_2\hat{D}_1\bar{\sigma}^{(tot)} - \tilde{\gamma}_2\hat{D}_2\bar{\sigma}^{(tot)}\big), \\ \hat{D}_\alpha\hat{\Lambda}_3 - \hat{D}_{\tilde{\psi}}\hat{\Lambda}_\alpha \quad &= -\frac{2i}{\gamma_4}\mathrm{Re}\big(\gamma_1\hat{D}_\alpha\bar{\sigma}^{(tot)}\big) \qquad\qquad \forall\alpha = 1, 2, \\ \hat{D}_1\hat{\Lambda}_1 + \hat{D}_2\hat{\Lambda}_2 + \hat{D}_{\tilde{\psi}}\hat{\Lambda}_3 \quad &= \frac{2i}{\gamma_4}\mathrm{Re}\big(\tilde{\gamma}_2\hat{D}_1\bar{\sigma}^{(tot)} + \gamma_2\hat{D}_2\bar{\sigma}^{(tot)}\big). \end{aligned} \tag{7.78}$$

Consider the replacements $\Lambda \to \hat{\Lambda}$ and $D \to \hat{D}$ in (6.78). It is then easy to see that the left-hand side of the first two lines above amounts to the different components of $d_{\hat{A}}\hat{\Lambda}$. Similarly, the left-hand side of the last equation above is the only component of $d_{\hat{A}} * \hat{\Lambda}$. What we are hinting at here is that perhaps a generalization of (6.79) can accommodate the new M2-brane. But to write the consistency equations in such a concise manner, we need to put some effort into the right-hand sides. This is our next goal.

Uncomplicated calculations allow us to rewrite the right-hand sides of these consistency equations as $(2i/\gamma_4)$ times

$$\begin{aligned}
&\mathrm{Re}(\gamma_2)\mathrm{Re}\big(\hat{D}_1\bar{\sigma}^{(tot)}\big) - \mathrm{Im}(\gamma_2)\mathrm{Im}\big(\hat{D}_1\bar{\sigma}^{(tot)}\big) - \mathrm{Re}(\tilde{\gamma}_2)\mathrm{Re}\big(\hat{D}_2\bar{\sigma}^{(tot)}\big) + \mathrm{Im}(\tilde{\gamma}_2)\mathrm{Im}\big(\hat{D}_2\bar{\sigma}^{(tot)}\big),\\
&\mathrm{Im}(\gamma_1)\mathrm{Im}\big(\hat{D}_\alpha\bar{\sigma}^{(tot)}\big) - \mathrm{Re}(\gamma_1)\mathrm{Re}\big(\hat{D}_\alpha\bar{\sigma}^{(tot)}\big) \qquad \forall\alpha = 1, 2,\\
&\mathrm{Re}(\tilde{\gamma}_2)\mathrm{Re}\big(\hat{D}_1\bar{\sigma}^{(tot)}\big) - \mathrm{Im}(\tilde{\gamma}_2)\mathrm{Im}\big(\hat{D}_1\bar{\sigma}^{(tot)}\big) + \mathrm{Re}(\gamma_2)\mathrm{Re}\big(\hat{D}_2\bar{\sigma}^{(tot)}\big) - \mathrm{Im}(\gamma_2)\mathrm{Im}\big(\hat{D}_2\bar{\sigma}^{(tot)}\big),
\end{aligned} \tag{7.79}$$

respectively. Be warned: the next equation is crucial, so let us motivate it meticulously. We observe that the right-hand sides of all these consistency equations depend solely on the dynamics of the complex scalar fields $(\sigma^{(tot)}, \bar{\sigma}^{(tot)})$. When no new M2-brane was present, these same fields were denoted $(\sigma, \bar{\sigma})$. By definition, the difference between these two sets of fields is that, in the former, we have switched on expectation values for the fields −recall (7.9) and (7.23). Further refreshing our memory, $(\sigma, \bar{\sigma})$ were really "useless" all along part II, in the absence of this new M2-brane. So much so, that we completely trivialized their dynamics in (5.42)[28]. Consequently, when the M2-brane is present, it is perfectly possible to reinterpret $(\sigma^{(tot)}, \bar{\sigma}^{(tot)})$ as their expectation values $(\langle\sigma\rangle, \langle\bar{\sigma}\rangle)$, after demanding that (5.42) holds true. This does not lead to any hurdle. For concreteness, the reader may choose to do so, for example, at the level of the Hamiltonian (7.14). The take-home message is this: we can identify the new M2-brane of figure 14 as the one and only source for the dynamics of $(\sigma^{(tot)}, \bar{\sigma}^{(tot)})$. Such being the case, these scalar fields must be oriented orthogonal to the M2-brane, which in this case means that they are oriented along the spatial direction x_2. In other words, $(\sigma^{(tot)}, \bar{\sigma}^{(tot)})$ are *localized* along x_2 if we impose (5.42). Yet another way to put it is to say that these scalar fields behave as delta functions in the direction orthogonal to the new M2-brane state:

$$\sigma^{(tot)}, \bar{\sigma}^{(tot)} \to \langle\sigma^{(tot)}\rangle, \langle\bar{\sigma}^{(tot)}\rangle \sim \delta_{M2}. \tag{7.80}$$

Having clearly established the key identification (7.80), our job of rewriting (7.78) as some sort of extension of (6.79) hugely simplifies. We start by noting that, in this case,

$$[\bar{\sigma}^{(tot)}, \sigma^{(tot)}] = 0, \qquad \hat{D}_\alpha\sigma^{(tot)}, \hat{D}_\alpha\bar{\sigma}^{(tot)} \sim \delta_{M2} \qquad \forall\alpha = 1, 2. \tag{7.81}$$

In the limit when

$$\hat{D}_\alpha\bar{\sigma}^{(tot)} \to (1+i)\delta_{M2} \qquad \forall\alpha = 1, 2 \tag{7.82}$$

[28]The attentive reader will surely note that this equation was posed when we were working in the oversimplified scenario where no topological piece was present in the action of the four-dimensional gauge theory, i.e. when $c_2 = 0$ in (4.143). However, (5.42) was chosen in the $c_2 \neq 0$ case of interest too, as pointed out above (5.89).

in (7.79), we have that the right-hand sides of (7.78) considerably reduce to yield $(2i\delta_{M2}/\gamma_4)$ times

$$\mathrm{Re}(\gamma_2-\tilde{\gamma}_2)-\mathrm{Im}(\gamma_2-\tilde{\gamma}_2),\qquad \mathrm{Im}(\gamma_1)-\mathrm{Re}(\gamma_1),\qquad \mathrm{Re}(\gamma_2+\tilde{\gamma}_2)-\mathrm{Im}(\gamma_2+\tilde{\gamma}_2),\tag{7.83}$$

respectively. We can then demand that the first two quantities equal each other, namely

$$\mathrm{Re}(\gamma_2-\tilde{\gamma}_2)-\mathrm{Im}(\gamma_2-\tilde{\gamma}_2)=\mathrm{Im}(\gamma_1)-\mathrm{Re}(\gamma_1).\tag{7.84}$$

Using (7.25) and (7.38), the above can be rewritten in terms of the coefficients listed in tables 1 and 2 as

$$\sqrt{c_{r2}}\big(h^{(1)}_{2\tilde{\psi}3}-h^{(1)}_{1\tilde{\psi}3}+1\big)+\sqrt{c_{\phi_1 3}}\big(h^{(2)}_{1\tilde{\psi}3}-h^{(2)}_{2\tilde{\psi}3}-1\big)=0,\tag{7.85}$$

where we have used $c_{r2}=c_{r3}$ and $c_{\phi_1 2}=c_{\phi_1 3}$, as can be seen from (4.135). The above can be satisfied by setting $h^{(2)}_{1\tilde{\psi}3}$ to whatever value makes it hold true. Then, by virtue of the last constraint in (7.45), the parameter $h^{(2)}_{2\tilde{\psi}3}$ becomes fixed too[29]. Putting everything together, we find that the consistency requirements in (7.78) can be brought to the concise form we sought:

$$\begin{aligned}
&d_{\hat{A}}\hat{\Lambda}\Big|_{\| M2}=0,\qquad\qquad d_{\hat{A}}\hat{\Lambda}\Big|_{\perp M2}=\frac{2i}{\gamma_4}\big[\mathrm{Re}(\gamma_2-\tilde{\gamma}_2)-\mathrm{Im}(\gamma_2-\tilde{\gamma}_2)\big]\delta_{M2},\\
&d_{\hat{A}}*\hat{\Lambda}=\frac{2i}{\gamma_4}\big[\mathrm{Re}(\gamma_2+\tilde{\gamma}_2)-\mathrm{Im}(\gamma_2+\tilde{\gamma}_2)\big]dv\wedge\delta_{M2},
\end{aligned}\tag{7.86}$$

where dv stands for a three-form defined in the directions spanned by the new M2-brane and measures its volume element. When evaluated orthogonal (parallel) to the M2-brane, $d_{\hat{A}}\hat{\Lambda}$ consist of all the terms on the left-hand sides of (7.78) that contain (exclude) $\langle\mathcal{A}_2\rangle$, since this is orthogonal to the M2-brane itself: $\langle\mathcal{A}_2\rangle\sim\delta_{M2}$, as explained below (7.3) and in (7.9). As an initial naive comment, we can note that this is a very fortunate form for the equations. It reminds us of what happened to the (still not twisted) Hitchin equations: they developed sources in the presence of the new M2-brane. Similarly, the consistency requirements for no M2-brane in (6.79) now acquire sources due to the M2-brane. As a second, more down to earth, comment: the left-hand sides of the last two lines above are a two- and a three-form

[29] At this stage, the reader may appreciate an update on the status of the coefficients of table 2 that have not yet been fixed. We have that the $\{g^{(1)}\}$'s must satisfy

$$g^{(1)}_{1212}=\pm\frac{1}{\sqrt{2}}\quad\text{ and }\quad g^{(1)}_{1223}=\pm g^{(1)}_{1213},\quad\text{ with } |g^{(1)}_{1213}|=\text{fixed}.$$

Besides, (7.34) should be true.

respectively whereas the right-hand sides correspond to a one- and a four-form respectively. This makes all alarms ring: we are for the first time noting the wrongness of our approach. Anyway, this should not be too shocking, since the title of the chapter already warned us. The reader will need to bear with these nonsensical equations a bit longer: we still have more to learn from them.

Let us now turn to the BPS equations to finish our analysis of the twisted bulk in the presence of the new M2-brane. Topologically wisting the sourced Hitchin equations in (7.31), (7.44) and (7.50) according to the prescription in (7.65) and (7.67) we get

$$\begin{aligned} \hat{F}_{\alpha\tilde{\psi}} - \gamma_4[\hat{\Lambda}_\alpha, \hat{\Lambda}_3] &= -2i\text{Re}\big(\gamma_5 \hat{D}_2 \sigma^{(tot)}\big) \qquad \forall \alpha = 1,2, \\ \sqrt{\frac{2i|\tau|^2}{\tau-\bar{\tau}}}\hat{F}_{12} - \sum_{k,l=1}^{3} g^{(1)}_{12kl}[\hat{\Lambda}_k, \hat{\Lambda}_l] &= i\sum_{k=1}^{3}\text{Im}\big(\kappa_k[\bar{\sigma}^{(tot)}, \hat{\Lambda}_k]\big) - i\kappa_5[\bar{\sigma}^{(tot)}, \sigma^{(tot)}]. \end{aligned} \tag{7.87}$$

As a reminder, all the coefficients appearing above can be expressed solely in terms of the coefficients of tables 1 and 2. We did so for τ in (5.83) and for (γ_4, γ_5) in (7.25). The reader who is eager for more computation can use all (7.25), (7.38), (7.47) and (7.49) to express all $\{\kappa\}$'s above in this manner. However, what we want to do is consider the limit (7.82) in these twisted, sourced Hitchin equations. To this aim, we first work out the two non-vanishing terms on the right-hand sides. It is easy to see that the first one is

$$\text{Re}\big(\gamma_5\hat{D}_2\sigma^{(tot)}\big) = \text{Re}(\gamma_5)\text{Re}\big(\hat{D}_2\sigma^{(tot)}\big) - \text{Im}(\gamma_5)\text{Im}\big(\hat{D}_2\sigma^{(tot)}\big) \to [\text{Re}(\gamma_5) - \text{Im}(\gamma_5)]\delta_{M2}. \tag{7.88}$$

On the other hand, we have that

$$[\bar{\sigma}^{(tot)}, \hat{\Lambda}_k] \to [(1+i)\delta_{M2}, \hat{\Lambda}_k] = i(1+i)\sum_{a,b,c=1}^{N^2-1} f_{abc}\delta^a_{M2}\hat{\Lambda}^b_k T^c \equiv \mathcal{K}_k\delta_{M2} \qquad \forall k = 1,2,3, \tag{7.89}$$

with T^c and f_{abc} standing for the generators and structure constants of the gauge group $SU(N)$, respectively. Consequently, the second and last non-zero term is

$$\begin{aligned} \text{Im}\big(\kappa_k[\bar{\sigma}^{(tot)}, \hat{\Lambda}_k]\big) &= \text{Re}(\kappa_k)\text{Im}\big([\bar{\sigma}^{(tot)}, \hat{\Lambda}_k]\big) + \text{Im}(\kappa_k)\text{Re}\big([\bar{\sigma}^{(tot)}, \hat{\Lambda}_k]\big) \\ &\to [\text{Re}(\kappa_k)\text{Im}(\mathcal{K}_k) + \text{Im}(\kappa_k)\text{Re}(\mathcal{K}_k)]\delta_{M2}. \end{aligned} \tag{7.90}$$

Unfortunately, the left-hand sides are not to work out smoothly in the wrong approach of this chapter. Ideally, we would like to recognize them as the components of the differential geometrical quantity $(\hat{F} - \gamma_4\hat{\Lambda} \wedge \hat{\Lambda})$, but there are two difficulties that prevent us from establishing such an identification. The first one is that the prefactors of the $\hat{\Lambda}$ commutator terms do not match between the $(\alpha\tilde{\psi})$- and the $(\alpha\beta)$-oriented equations. The second obstacle amounts to the appearance of the terms $[\hat{\Lambda}_\alpha, \hat{\Lambda}_3]$ in the $(\alpha\beta)$-oriented equation. The resolution to this situation is presented in the next chapter. For the time being and just momentarily, let us ignore the aforementioned complications and claim that, when (7.82) is true, (7.87) takes the form

$$\hat{F} - \gamma_4 \hat{\Lambda} \wedge \hat{\Lambda}\Big|_{\parallel M2} = 0, \qquad \hat{F} - \gamma_4 \hat{\Lambda} \wedge \hat{\Lambda}\Big|_{\perp M2} \propto \delta_{M2}. \tag{7.91}$$

Now comes the time to make some sense of the new M2-brane in figure 14. The fact is that our bulk equations orthogonal to the M2-brane in (7.86) and (7.91) strongly resemble the defining equations for a *surface operator*, see for example [13,14,65,71,72]. For concreteness, let us reproduce (6.8) in [14]:

$$F - \phi \wedge \phi = 2\pi\alpha\delta_K, \qquad d_A * \phi = 2\pi\beta ds \wedge \delta_K, \qquad d_A\phi = 2\pi\gamma\delta_K. \tag{7.92}$$

Here, K stands for an arbitrary knot, $(\alpha,\, \beta,\, \gamma)$ are parameters that characterize K, ds is a line element along K and δ_K is a delta function two-form that is Poincaré dual to K. On the other hand, $(F,\, \phi)$ are the field strength and the twisted scalar fields of a four-dimensional, time-independent $\mathcal{N} = 4$ Yang-Mills theory. It thus follows that simply identifying $(\hat{A},\, \hat{F},\, \hat{\Lambda})$ in the left-hand sides of (7.86) and (7.91) with $(A,\, F,\, \phi)$ in [14], we can reproduce the left-hand sides above. The right-hand sides are clearly impossible to match, but the crux of the matter is clear by now: we want to interpret the new M2-brane as a surface operator from the point of view of the four-dimensional gauge theory. Further, we want this surface operator to correctly implement any arbitrary knot in our setup. In the next chapter, we will discuss surface operators and make the stated wishes come true by a suitable reorientation of the new M2-brane. But, before doing so, it is instructive to quickly observe what the limit (7.82) −the very same limit that brought our bulk equations so close to the defining equations for surface operators− yields when implemented in the twisted boundary action (7.75).

Back to the boundary

The twisted boundary action that we derived in (7.75) contains two terms: a complexified Chern-Simons term and an additional term. The limit (7.82) in which we wish to consider the action (7.75) clearly leaves the Chern-Simons piece unaltered. However, it affects the integrand of the extra piece in the following manner:

$$\mathrm{Re}\big(\gamma_5 \hat{D}_2 \sigma^{(tot)}\big) = \mathrm{Re}(\gamma_5)\mathrm{Re}\big(\hat{D}_2 \sigma^{(tot)}\big) - \mathrm{Im}(\gamma_5)\mathrm{Im}\big(\hat{D}_2 \sigma^{(tot)}\big) \to [\mathrm{Re}(\gamma_5) - \mathrm{Im}(\gamma_5)]\delta_{M2}. \tag{7.93}$$

Plugging the above in (7.75) and integrating over x_2, we see that in this case the topological boundary theory is supplemented by

$$\frac{8|\tau|^2}{\tau - \bar{\tau}} [\mathrm{Re}(\gamma_5) - \mathrm{Im}(\gamma_5)] \mathrm{Tr} \sum_{\alpha=1}^{2} \int dt dx_1 \ \hat{A}_\alpha \Big|_{\|\delta_{M2}} . \tag{7.94}$$

Let us verbalize the above, omitting prefactors. Consider the twisted connection $\hat{A}$. Evaluate it parallel to the M2-brane and integrate it in the corresponding plane. Finally, take its trace in the adjoint representation. Surely enough, this rings a bell: this sounds exactly like a two-dimensional generalization of a *Wilson loop* action, given by the holonomy of the connection $\hat{A}$ around a knot K. Our suspicions seem to be further encouraged by the mismatch between δ_{M2} and δ_K that prevented us from mapping our bulk equations to the surface operator definition (7.92). Indeed, if we could somehow replace $\delta_{M2} \to \delta_K$ in (7.93), we would have obtained the boundary action to be exactly Chern-Simons in the presence of a Wilson loop. We owe the connection between such an action and knot theory to [8]. There, it was shown that, for the gauge group $SU(2)$, the vacuum expectation value of the holonomy of the gauge field around a knot K traced in the fundamental representation of $SU(2)$ yields the Jones polynomial. Other knot invariants follow from considering different ranks and irreps of the gauge group.

So close yet so far! In the next chapter, we will see how one can easily fix the situation and reach the picture we are but scratching with our fingers here.

In chapter 7 we have worked out in details a toy-model inclusion of knots in the M-theoretical model (M,1) of chapter 2. Our incorrect approach helps build intuition on the delicate issue of knot embedding, since we work in a simplified (albeit unrealistic) scenario where explicit calculations are possible.

Chapter 8: The correct realization of knots

As we advanced just a moment ago, our primary task in this chapter will be to make suitable changes to the new M2-brane of figure 14 so as to be able to interpret it as a surface operator from the four-dimensional gauge theory point of view. This should fix all the troubles we were facing before, namely:

- The degree of the differential forms in the consistency conditions should match for left- and right-hand sides.
- The sourced Hitchin equations should admit a rewriting of the form (7.91), for some constant γ_4.
- All the above bulk equations should admit a unique and sensible map to the surface operator defining equations (7.92), in an appropriate limit. A most interesting consequence of establishing such a map would then be to obtain a supergravity interpretation of the $(\alpha,\, \beta,\, \gamma)$ parameters that label any knot K.
- At the three-dimensional boundary, the surface-operator-M2-brane should then contribute a Wilson loop term to the complexified Chern-Simons action.

It turns out that the task is not particularly thorny and simply amounts to a careful reorientation of the new M2-brane in the previous chapter. Instead of along the spacetime directions $(t,\, x_1,\, \tilde{\psi})$, an M2-brane that admits a surface-operator interpretation in X_4 must be oriented along either $(K,\, \tilde{\psi},\, \tilde{x}_3)$ or $(K,\, \tilde{\psi},\, r)$. We shall here choose the first option. This is schematically depicted in figure 15.

We shall begin our discussion in section 8.1 with a lightning overview of surface operators. We shall then argue why an additional M2-brane in (M,1) along either $(K,\, \tilde{\psi},\, \tilde{x}_3)$ or $(K,\, \tilde{\psi},\, r)$ constitutes a surface operator from a spacetime perspective. Afterwards, we will explain where the two possibilities originate from and why the first one is a better choice in our case. Once these basic aspects have been clarified, in section 8.2 we will resolve all the four items listed above and bring this thesis to an end.

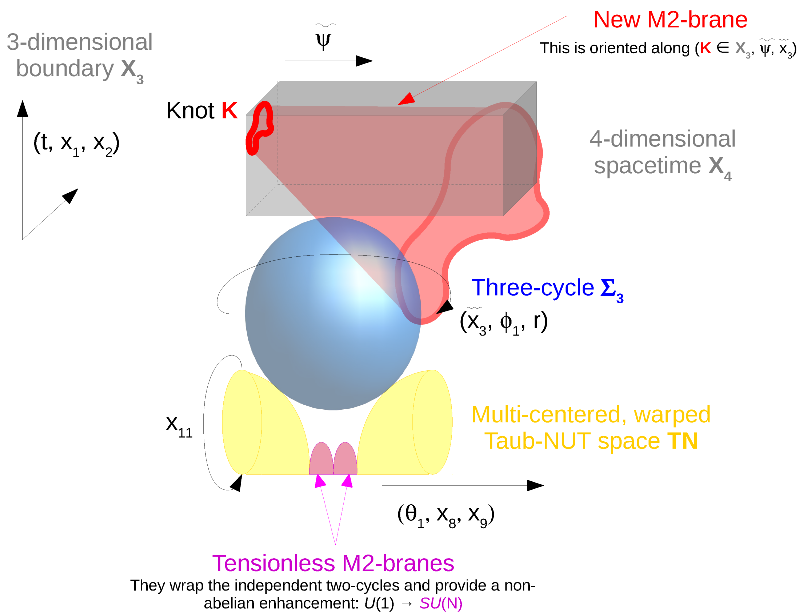

Figure 15: To the configuration (M,1) we add a different new M2-brane. This time the new M2-brane is oriented along $(K, \tilde{\psi}, \tilde{x}_3)$, where K stands for a one-dimensional knot in the X_3 boundary. As we will show, this M2-brane corresponds to a surface operator in the world-volume gauge theory defined in X_4 and it appears as a Wilson loop in the boundary action of X_3.

8.1 What went wrong in chapter 7?: Understanding surface operators

In general, field theories admit the definition of operators, that is, maps from one space of physical states to another (or the same) such space. This is true for classical as well as for quantum field theories. There are different ways to classify operators. Here, we want to focus on four-dimensional gauge theories and introduce two classifications. On the one hand, one can distinguish operators according to the co-dimension of their support. Co-dimension four operators are the well-known local operators supported on a point that we usually consider in field theory, starting at the textbook level. Co-dimension three operators are line operators, such as the familiar Wilson and t'Hooft operators. Surface operators, also called surface defects, are *co-dimension two* operators and are the relevant ones presently. Finally, it is also possible to have co-dimension one operators. An example that has recently gained plenty of attention is that of domain walls. On the other hand, attending to their positive description, it is customary to differentiate between electric and magnetic type of operators. Electric operators are constructed directly from the fields, as an additional term in the action of the field theory. Let the previously mentioned Wilson operator stand as an example. On their part, magnetic operators require a modification on the space of fields in the theory for their construction. For instance, consider t'Hooft operators, which are typically defined via a change in the measure of the path integral formulation of the field theory. In general, surface operators are *magnetic*, in the sense that they are introduced by requiring that the fields present prescribed singularities on the support of these operators.

Just in the light of the above shallow overview, we can right away discard the M2-brane of figure 14 as a surface operator in X_4: this is a co-dimension one object from the point of view of the world-volume gauge theory! If we insist on converting it into one, we must begin by at the very least "pushing" one of its "legs" out of X_4 and into the internal directions. As pointed out in the beginning of section 7.1, in order to not break the supersymmetry of the M-Theory model (M,1), the internal leg must be along either of the directions of the three-cycle Σ_3: ($\tilde{x}_3$, ϕ_1, r). Of course, ϕ_1 is a compact direction and hence is not suited to host the surface-operator-M2-brane's internal leg. We are thus left with the unbound, radial directions ($\tilde{x}_3$, r) as feasible choices.

Exceptionally memorious and attentive readers may right away understand these two possibilities from a four-dimensional perspective as well, even applaud our particular selection of $\tilde{x}_3$ over r. In general, however, it bears repeating at this juncture where such freedom of choice comes from after compactification. Briefly, we must recall the discussion after (5.35). In more detail, in chapter 4 we derived the four-dimensional Lagrangian (4.143) associated to (M,1). In chapter 5 we inferred the corresponding Hamiltonian by comparison to [54]. To this aim, we singled out the scalar field $\mathcal{A}_{\tilde{3}}$ −see (5.2). We could have equally well singled out $\mathcal{A}_r$. Then, instead of (5.35) and (5.40), the most convenient gauge choice would have been $\mathcal{A}_0 = \mathcal{A}_r = 0$. As a matter of fact, this latter gauge choice was explored in [1] and seen to be equivalent to the one here considered, under an exchange of the roles played by the scalar fields ($\mathcal{A}_{\tilde{3}}$, $\mathcal{A}_r$). At present and in order to benefit from the explicit computations in the previous chapters, it stands to reason that the internal leg for the surface-operator-M2-brane is chosen along $\tilde{x}_3$.

Coming now to the other two legs, we note that these are fixed if the aim of including the new M2-brane is to realize knots in the configuration (M,1). By definition, a knot is an embedding of a circle in three-dimensional Euclidean space. After a Wick rotation[30], the boundary X_3 governed by a complex topological action is the suitable space to host the knot K. Hence, we use one of the M2-brane's legs as a knot $K \in X_3$. Then, the other leg must be along the bulk direction $\tilde{\psi}$. This explains the new M2-brane's orientation in figure 15.

[30]In our time-independent gauge theory, this is a formal step rather than a practical one.

It is important to note that such M2-brane is a co-dimension two object in X_4, as well as in X_3. Further, if we can indeed bring our BPS and consistency equations to the form (7.92), this will provide an *electric* description of the object in X_4. This is very interesting because, as noted a moment ago, surface operators are generally introduced magnetically in the literature (for example, see [13, 14]), which obscures their origin. In fact, we shall regard this as one of the main results in the thesis: in the next and final section, we provide a detailed account of the supergravity emergence of the surface operator in the world-volume gauge theory.

8.2 Solving it all: the new M2-brane as a knot embedding surface operator

In this section, we will be able to use all the computations in chapter 7 as a basis from which we can quickly infer the action, Hamiltonian, bulk equations and boundary action associated to the M-theory configuration of figure 15. It is important to highlight why the present section is the punch-line of the thesis. That in figure 15 is a concrete model for the study of knots in an ultraviolet-complete physical context. The equations we shall state in this section lay the foundations for future research in this direction, since they pave the way for the explicit computation of knot invariants in a physics language. In a field as subtle and complex as the intersection of knot and gauge theories, the advantages of having a concrete scheme for calculations are not to be discarded too quickly. Note for example that our construction is valid for a general gauge group $SU(N)$, whereas most of the literature discusses only $SU(2)$ and little is known about the highly non-trivial generalization to a higher rank. In addition, a process of quantization is natural in our approach and can potentially shed light into the notion of quantum knots at the fundamental theoretical level.

Following the very same logic as that in section 7.1 before, it is easy to see that the four-dimensional action associated to (M,1) in the presence of a new M2-brane along (K, $\tilde{\psi}$, $\tilde{x}_3$) is effectively given by (4.143) under the replacements

$$\mathcal{A}_x \to \mathcal{A}_x^{(tot)} \equiv \langle \mathcal{A}_x \rangle + \mathcal{A}_x, \qquad \forall x = 0, 1, 2, \phi_1, r, \tag{8.1}$$

where all fluctuations $\langle \mathcal{A}_x \rangle$'s depend on only spacetime coordinates and are oriented orthogonal to K in two-dimensional slices X_2 which are parallel to X_3 for all fixed values of the bulk direction $\tilde{\psi} = \tilde{\psi}_0 \geq 0$. That is, all $\langle \mathcal{A}_x \rangle$'s are supported on two-dimensional subspaces $X_2 \subset X_4$. The Hamiltonian is then (5.88) subject to the above replacements. Clearly, the most convenient gauge choice is a suitable modification of (7.10) and (7.11) to the present scenario:

$$\mathcal{A}_0^{(tot)} = \mathcal{A}_{\bar{3}} = 0. \tag{8.2}$$

To be able to focus on BPS configurations, we demand all fields be time-independent in the following. We shall choose the coefficients of table 2 exactly as in chapter 7. In this case, the difference between the configurations in figures 14 and 15 consists in considering (7.9) or (8.1), respectively. Regarding the topological twist, we will adjust (7.65)-(7.67) to[31]

$$\tilde{\chi} = \sum_\mu \tilde{\chi}_\mu dx^\mu, \qquad \forall \chi = A, \Phi, \Lambda, \tag{8.3}$$

where x^μ labels the spacetime directions and where

$$\tilde{A}_\mu = i\mathcal{A}_\mu^{(tot)}, \quad (\tilde{\Phi}_0, \tilde{\Phi}_1, \tilde{\Phi}_2, \tilde{\Phi}_3) = i(\hat{\varphi}_3, \hat{\varphi}_1, \hat{\varphi}_2, \mathcal{A}_{\bar{3}}), \quad (\tilde{\Lambda}_0, \tilde{\Lambda}_1, \tilde{\Lambda}_2, \tilde{\Lambda}_3) = i(\mathcal{A}_{\bar{3}}, \hat{\varphi}_1, \hat{\varphi}_2, \hat{\varphi}_3), \tag{8.4}$$

with $\mathcal{A}^{(tot)}$ as in (8.1). In view of the detailed computations in chapter 7 −in particular (7.75)−, it can be readily inferred that the twisted boundary action for the configuration in figure 15 is

$$\tilde{S}^{(t)}_{bnd,tot} = i\Psi\, CS_D + \frac{8|\tau|^2}{\tau - \bar{\tau}} \mathrm{Tr} \sum_{\alpha=1}^{2} \int d^3x \left[\tilde{A}_\alpha \mathrm{Re}\big(\gamma_5 \tilde{D}_2 \sigma^{(tot)}\big) \right], \tag{8.5}$$

where for brevity we have defined

$$CS_D \equiv \int_{X_3} \mathrm{Tr}(\tilde{A}_D \wedge d\tilde{A}_D + \frac{2}{3} \tilde{A}_D \wedge \tilde{A}_D \wedge \tilde{A}_D). \tag{8.6}$$

Here, the traces are taken in the adjoint representation of the gauge group $SU(N)$, the constants (τ, Ψ, γ_5) were defined in (5.83), (6.39) and (7.25), respectively and the field $\sigma^{(tot)}$ is as in (7.23). Regarding the Chern-Simons gauge field $\tilde{A}_D$ and in analogy to section 6.3, this is defined as a complexification of the usual gauge field:

$$\tilde{A}_D \equiv \tilde{A} + \frac{D_1}{i\Psi} \tilde{\Phi}, \tag{8.7}$$

with the constant D_1 fixed to make $\tilde{A}_D$ compatible with $\mathcal{N} = 4$ supersymmetry with half-BPS boundary conditions. Turning to the bulk X_4, the twisted consistency conditions in

[31] This is mostly for the convenience of the reader: a change from hatted to tilde fields is meant as a reminder that we are presently dealing with a different, novel configuration that should not be confused with any prior model.

(7.78) now become

$$\begin{aligned}\tilde{D}_1\tilde{\Lambda}_2 - \tilde{D}_2\tilde{\Lambda}_1 &= \frac{2i}{\gamma_4}\mathrm{Re}\big(\gamma_2\tilde{D}_1\bar{\sigma}^{(tot)} - \tilde{\gamma}_2\tilde{D}_2\bar{\sigma}^{(tot)}\big),\\ \tilde{D}_\alpha\tilde{\Lambda}_3 - \tilde{D}_{\tilde{\psi}}\tilde{\Lambda}_\alpha &= -\frac{2i}{\gamma_4}\mathrm{Re}\big(\gamma_1\tilde{D}_\alpha\bar{\sigma}^{(tot)}\big) \qquad \forall\alpha = 1,2,\\ \tilde{D}_1\tilde{\Lambda}_1 + \tilde{D}_2\tilde{\Lambda}_2 + \tilde{D}_{\tilde{\psi}}\tilde{\Lambda}_3 &= \frac{2i}{\gamma_4}\mathrm{Re}\big(\tilde{\gamma}_2\tilde{D}_1\bar{\sigma}^{(tot)} + \gamma_2\tilde{D}_2\bar{\sigma}^{(tot)}\big),\end{aligned} \tag{8.8}$$

where the constants $\{\gamma, \tilde{\gamma}\}$ were defined in (7.25) and (7.38), respectively. Similarly, the twisted BPS equations are given by a sourced Hitchin integrable system which is nothing but an apt modification of (7.87):

$$\begin{aligned}\tilde{F}_{\alpha\tilde{\psi}} - \gamma_4[\tilde{\Lambda}_\alpha, \tilde{\Lambda}_3] =& \ -2i\mathrm{Re}\big(\gamma_5\tilde{D}_2\sigma^{(tot)}\big) \qquad \forall\alpha = 1,2,\\ \sqrt{\frac{2i|\tau|^2}{\tau-\bar{\tau}}}\tilde{F}_{12} - \sum_{k,l=1}^{3} g^{(1)}_{12kl}[\tilde{\Lambda}_k, \tilde{\Lambda}_l] =& \ i\sum_{k=1}^{3}\mathrm{Im}\big(\kappa_k[\bar{\sigma}^{(tot)}, \tilde{\Lambda}_k]\big) - i\kappa_5[\bar{\sigma}^{(tot)}, \sigma^{(tot)}],\end{aligned} \tag{8.9}$$

with the constants $\{\kappa\}$ given by (7.47) and (7.49).

In the following, we will bring all the above results to an enlightening form. That is, to a differential geometrical form that makes explicit the connection of the configuration in figure 15 to Knot Theory. The easiest way to do so is to start with the twisted consistency requirements (8.8). As was the case with (7.78) before, the left-hand sides of the first two lines above capture the different components of $d_{\tilde{A}}\tilde{\Lambda}$, while the left-hand side in the last line corresponds to the only component of $d_{\tilde{A}} * \tilde{\Lambda}$, with the Hodge dual taken with respect to the Euclidean space[32] labeled by $(x_1, x_2, \tilde{\psi})$. As for the corresponding right-hand sides, the observation we made after (7.79) remains true: $(\sigma^{(tot)}, \bar{\sigma}^{(tot)})$ can be regarded as *localized* fields, oriented within X_4 but orthogonal to the new M2-brane in figure 15. That is, these scalar fields behave as delta functions in any plane X_2 labeled by (t, x_1, x_2) and orthogonal to K. Note that there exists one such plane for every fixed value $\tilde{\psi}_0$ of the bulk coordinate $\tilde{\psi}$. Mathematically, we can say that

$$\sigma^{(tot)}, \bar{\sigma}^{(tot)} \to \langle\sigma^{(tot)}\rangle, \langle\bar{\sigma}^{(tot)}\rangle \sim \delta_K \qquad \forall\,\tilde{\psi} = \tilde{\psi}_0 \geq 0. \tag{8.10}$$

Note that δ_K is in this case a two-form supported on the two-dimensional *knot complement* X_2, not a one-form as in (7.80). It follows immediately that

$$[\bar{\sigma}^{(tot)}, \sigma^{(tot)}] \to 0, \qquad \tilde{D}_\alpha\sigma^{(tot)}, \tilde{D}_\alpha\bar{\sigma}^{(tot)} \sim \delta_K \qquad \forall\alpha = 1,2. \tag{8.11}$$

[32]The spacetime X_4 is Minkowskian since the approximation (5.77).

In particular, we wish to consider the very special limit when

$$\tilde{D}_\alpha \sigma^{(tot)}, \tilde{D}_\alpha \bar{\sigma}^{(tot)} \to (1+i)\delta_K \qquad \forall \alpha = 1, 2. \tag{8.12}$$

This limit acts like a *restriction map* in the sense that it is such that δ_K takes values in the Lie algebra t of the maximal torus $\mathbb{T}$ of the gauge group:

$$\delta_K = \sum_a \delta_K^a U^a, \qquad \delta_K^a \in t, \qquad U^a \in \mathbb{T} \subseteq SU(N). \tag{8.13}$$

As can be inferred from our computations in (7.79)-(7.85), the above limit allows us to rewrite (8.8) in precisely the form we were wishing for:

$$\begin{gathered} d_{\tilde{A}}\tilde{\Lambda}\Big|_{\parallel K} = 0, \qquad d_{\tilde{A}}\tilde{\Lambda}\Big|_{\perp K} = \frac{2i}{\gamma_4}\big[\mathrm{Re}(\gamma_2 - \tilde{\gamma}_2) - \mathrm{Im}(\gamma_2 - \tilde{\gamma}_2)\big]\delta_K, \\ d_{\tilde{A}} * \tilde{\Lambda} = \frac{2i}{\gamma_4}\big[\mathrm{Re}(\gamma_2 + \tilde{\gamma}_2) - \mathrm{Im}(\gamma_2 + \tilde{\gamma}_2)\big] dl \wedge \delta_K, \end{gathered} \tag{8.14}$$

where dl stands for a one-form along K that measures its line element. Let us clarify notation, since these are some of the most relevant equations in the entire thesis. Evaluation orthogonal to K means we only consider terms proportional to either $\langle \tilde{A}_1 \rangle$ or $\langle \tilde{A}_2 \rangle$, both defined in X_2. This is because such fluctuations are orthogonal to the new M2-brane of figure 15 in exactly the same sense as that in (8.10) for the case of the scalar fields ($\bar{\sigma}^{(tot)}$, $\sigma^{(tot)}$). Conversely, when we evaluate parallel to the knot we are left with all the terms that do not involve neither $\langle \tilde{A}_1 \rangle$ nor $\langle \tilde{A}_2 \rangle$. Notice that powers of these fluctuations vanish, so these two options exhaust all the terms in the consistency conditions. It is worth mentioning that the degrees of all differential forms above are perfectly coherent, which implies we have just fulfilled the first of the objectives stated at the beginning of this chapter 8.

This is certainly encouraging, but not enough to allow us to identify the reoriented M2-brane as a knot embedding surface operator in our gauge theory. Hence, let us proceed towards establishing this fact. We shall now focus on the limit (8.12) of the BPS Hitchin equations in (8.9). First of all, we should note that this limit only applies for fixed values of the bulk coordinate: $\psi = \psi_0 \geq 0$. This means that, for all $\alpha = 1$, 2, the $(\alpha\tilde{\psi})$-oriented equations in (8.9) become trivially satisfied in this case (i.e. they yield $0 = 0$). Conversely, their (12)-oriented counterpart does admit the consideration of the limit (8.12). In differential geometric language, (8.3) and (8.4) tell us $\tilde{\Lambda}_3$ is associated to the differential $d\tilde{\psi}$. Consequently, there is no contribution from this component to the limit we seek. No other subtlety applies and, in view of our earlier calculations in (7.88)-(7.90), it is easy to infer the following result:

$$\tilde{F} - \hat{\mathcal{K}}\tilde{\Lambda} \wedge \tilde{\Lambda}\Big|_{\perp K} = i\sqrt{\frac{\tau - \bar{\tau}}{2i|\tau|^2}} \sum_{k=1}^{2} [\mathrm{Re}(\kappa_k)\mathrm{Im}(\mathcal{K}_k) + \mathrm{Im}(\kappa_k)\mathrm{Re}(\mathcal{K}_k)]\delta_K, \tag{8.15}$$

where $\mathcal{K}_k$ was defined in (7.89) for all $k = 1, 2$ and we have introduced the new constant

$$\hat{\mathcal{K}} \equiv 2g^{(1)}_{1212}\sqrt{\frac{\tau-\bar{\tau}}{2i|\tau|^2}}. \tag{8.16}$$

The above is of the desired form in (7.91) and consequently we can consider achieved the second of the objectives stated at the beginning of this chapter 8.

The third objective we set for ourselves was to be able to determine a meaningful single map from our bulk equations (8.14) and (8.15) to the surface operator defining equations (7.92). This is not too complicated and essentially boils down to establishing the coefficient identifications

$$\alpha^a \equiv \frac{i}{2\pi}\sqrt{\frac{\tau-\bar{\tau}}{2i|\tau|^2}}\sum_{k=1}^{2}[\mathrm{Re}(\kappa_k)\mathrm{Im}(\mathcal{K}_k)+\mathrm{Im}(\kappa_k)\mathrm{Re}(\mathcal{K}_k)]U^a, \tag{8.17}$$

$$\beta^a \equiv \frac{i\sqrt{\hat{\mathcal{K}}}}{\pi\gamma_4}\big[\mathrm{Re}(\gamma_2+\tilde{\gamma}_2)-\mathrm{Im}(\gamma_2+\tilde{\gamma}_2)\big]U^a, \qquad \gamma^a \equiv \frac{i\sqrt{\hat{\mathcal{K}}}}{\pi\gamma_4}\big[\mathrm{Re}(\gamma_2-\tilde{\gamma}_2)-\mathrm{Im}(\gamma_2-\tilde{\gamma}_2)\big]U^a,$$

so that our twisted fields $\big(\tilde{A},\, \Lambda' \equiv \sqrt{\hat{\mathcal{K}}}\tilde{\Lambda}\big)$ can be recognized as $(A,\, \phi)$ in (7.92). Explicitly,

$$\tilde{F}-\Lambda'\wedge\Lambda'\Big|_{\perp K} = 2\pi\alpha\delta_K, \qquad d_{\tilde{A}} * \Lambda' = 2\pi\beta dl\wedge\delta_K, \qquad d_{\tilde{A}}\Lambda'\Big|_{\perp K} = 2\pi\gamma\delta_K, \tag{8.18}$$

where we have omitted the color indices. These equations are truly important in the context of Knot Theory, since their moduli space is known to encode multiple knot invariants. For example, see [73, 74] for mathematical references and [14, 75] for more physics oriented ones. Perhaps at this point we should highlight the work of A. Shende as a whole, which makes him one of the leading figures equating algebra-geometric invariants associated to equations like (8.18) with topological invariants of knots. At this stage, we can make further sense of (8.13). Viewing the moduli space of (8.18) −away from the singularities generated by K− as the space of *flat* complex connections, the requirement (8.13) ensures that we consider the maximal compact and connected component of such connections. It is precisely in this subspace that the knot invariants are defined [62].

It bears emphasizing that (8.17) provides a supergravity interpretation to the Lie-algebra valued parameters $(\alpha^a,\, \beta^a,\, \gamma^a)$ labeling the surface operator. This is because all the right-hand sides there can be explicitly traced to the parameters appearing in the Hamiltonian (5.88) and summarized in table 1. As there noted, all these coefficients are defined as certain integrals of the warp factors (2.2) and constant dilaton (4.5) that characterize the M-Theoretical model (M,1).

At this stage, the bulk equations have demonstrated that we can indeed regard the new M2-brane of figure 15 as a surface operator after compactification from eleven to four dimensions. It is important to call the reader's attention to the fact that, contrary to the overwhelming majority of the literature (starting with the pioneering mathematical reference [76] and all the way to the most recent physical works such as [65] or [72]), we have here introduced surface operators in an *electric* formulation, thus exploiting the Hamiltonian formalism developed in parts I and II. The equivalent magnetic description would consist of considering the bulk equations of part II, when no surface-operator-M2-brane was present in model (M,1) and *postulating* the twisted fields present singularities of the form

$$\hat{A} = \alpha d\theta + \ldots, \qquad \hat{\Lambda} = \beta \frac{dr}{r} - \gamma d\theta + \ldots \tag{8.19}$$

in the planes X_2 orthogonal to the surface operator, which are here being parametrized by polar coordinates (r, θ). The ellipses stand for other possible, less singular terms. Although correct, the magnetic description would not have shed any light on the origin of the singularity-producing surface operator. It would also not have allowed for a rewriting of its defining parameters (α, β, γ) in terms of supergravity quantities as in (8.17). In our opinion, this is one of the most appealing features of our construction: it allows for an easy to grasp, almost tangible manner to introduce surface operators in the Hitchin system that minimizes the energy of $\mathcal{N} = 4$ Yang-Mills BPS states.

The only task left is to demonstrate that the "knot-leg" of the surface-operator-M2-brane in figure 15 correctly incorporates any knot K in the boundary X_3, as expected. In other words, we must check the fourth and last goal in the beginning of chapter 8. To this aim, we must consider the action (8.5) and show that, in the limit (8.12), the second term is nothing but the trace of the twisted gauge field transported along K. Based on our earlier calculations (7.93) and (7.94), this is rather straightforward. Rather unsurprisingly at this stage, the result is given by the desired integral:

$$\frac{8|\tau|^2}{\tau - \bar{\tau}}[\mathrm{Re}(\gamma_5) - \mathrm{Im}(\gamma_5)]\mathrm{Tr}\sum_{\alpha=1}^{2}\int d^3x\ \tilde{A}_\alpha \delta_K \propto \mathrm{Tr}\sum_{\alpha=1}^{2}\int d^3x\ \tilde{A}_\alpha\bigg|_{\|K} = \mathrm{Tr}\oint_K \tilde{A}. \tag{8.20}$$

Hence, the twisted boundary action is

$$\tilde{S}^{(t)}_{bnd,tot} = i\Psi\, CS_D + \frac{8|\tau|^2}{\tau - \bar{\tau}}[\mathrm{Re}(\gamma_5) - \mathrm{Im}(\gamma_5)]\mathrm{Tr}\oint_K \tilde{A}, \tag{8.21}$$

with CS_D defined in (8.6).

Much like the bulk equations (8.18) before, the above boundary action is also source to diverse knot invariants. In fact, bulk and boundary are intimately related, beyond stemming from the same Hamiltonian in our particular construction. The last issue we would like to call the reader's attention to is precisely this bulk-boundary interrelation. In the case where no new M2-brane is present in (M,1), it is well-known that the (bulk) Hitchin equations can be rewritten in the language of Morse Theory. In this way, their relation to the complexified Chern-Simons (boundary) action becomes apparent [15]. The interested reader can find a concise yet clear review of this claim in section 5.3.2 of [14], for instance. More concretely, regarding the complexified Chern-Simons functional as a Morse function, its Morse flow equation is equivalent to the Hitchin integrable system, for a suitable choice of metric. Consequently, we expect the sourced Hitchin equations we obtained in (8.18) to be expressible as the Morse flow equation associated to regarding the boundary functional (8.21) as a Morse function, for some appropriate metric.

In chapter 8 we have shown how knots are to be embedded in the M-theoretical model (M,1) of chapter 2: one must include a carefully oriented M2-brane state in (M,1), with one of its legs in the form of a knot. We have then repeated the analysis performed for (M,1) in chapters 4-6, but taking into account the effect of the knot-embedding M2-brane. Namely, we have obtained the eleven-dimensional low energy limit supergravity action for this M-theory configuration, reduced it to four dimensions, Legendre transformed it into a Hamiltonian, minimized the energy of the Hamiltonian for static configurations of the fields in the axial gauge thereby obtaining a three-dimensional boundary action and we have performed a topological twist. In the four-dimensional spacetime, we thus obtain BPS equations that are given by a sourced Hitchin integrable system (8.18) that defines surface operators and that is known to encode multiple knot invariants (for example, see [72]). In the three-dimensional spacetime, we recover the topological Chern-Simons action, together with a Wilson loop term. As shown in the seminal work [8], such action is also known to capture various knot invariants.

Chapter 9: Summary, conclusions and outlook

In the first part I of the thesis, we have constructed two M-Theory models: (M,1) and (M,5). They have both been obtained from the D3-NS5 system in type IIB String Theory considered in [14] by means of well-defined chains of dualities and modifications. As depicted in figure 1, (M,1) has been proven to be dual to the aforementioned system in [14], while (M,5) has been argued to be dual to the resolved conifold with fluxes in [11]. An apparent indication of the seeming unrelatedness between our models −and hence between [14] and [11]− is their supersymmetry: $\mathcal{N} = 2$ for (M,1) and $\mathcal{N} = 1$ in the case of (M,5). However, we have been able to trace all dissimilarities to a difference in the orientation of an NS5-brane in a dual type IIB picture: compare figures 2**B** and 3**B**. We have thus showed that, although distinct, [14] and [11] are intimately related, much more than one would suspect a priori.

In the second part II, we have derived and studied in depth the world-volume gauge theory of (M,1). This gauge theory is defined in the four-dimensional spacetime X_4. In chapter 4, we have obtained its action. In chapter 5 we have written the corresponding Hamiltonian in a particularly enlightening form: as a sum of squared terms, plus contributions from the three-dimensional boundary $X_3 \subset X_4$. We have then minimized the bulk energy by setting each squared term to zero independently. For static configurations of the fields, we have thus found BPS conditions that match the "localization equations" of [14, 15, 61], which were obtained via elaborate techniques of localization of certain path integrals. This correspondence implies that our approach reproduces all the results in [14], but in a much simpler formalism. Further, our construction has enabled us to map all the parameters in [14] to variables of the low energy limit of (M,1). In this manner, we have been able to give a supergravity interpretation to all the findings in [14].

In the last chapter 6 of part II, we have focused on the boundary theory. We have shown that, upon a topological twist, a complexified Chern-Simons action captures the physics in X_3. Additionally, we have obtained the appropriate half-BPS boundary conditions for all the fields, which ensure that the theory in X_4 as a whole is indeed $\mathcal{N} = 2$ supersymmetric. It follows that the space X_3 has all required features to host knots. In other words, after Euclideanization, knots can consistently be embedded in X_3 and studied in the framework of the previously described world-volume gauge theory.

The last part III of the thesis has been devoted to precisely the incorporation of an arbitrary knot K in X_3. Specifically, we have included a carefully oriented, knot embedding

M2-brane state in (M,1). The M2-brane has led to the presence of sources in the localization equations in X_4, while it has contributed a novel term to the topological action governing X_3. After twisting and in a certain, well-defined limit, the bulk equations have been shown to reproduce the surface operator defining relations, supported on a two-dimensional subspace $X_2 \subset X_4$ orthogonal to the M2-brane state. In the very same limit, we have been able to understand the extra term in the boundary action as a Wilson loop around K. Both the surface operator bulk equations and the complexified Chern-Simons action in the presence of a Wilson loop in the boundary are known to encode a wide variety of knot invariants. Additionally, it is most reasonable to conjecture that we should be able to relate them to one another using Morse Theory. A particularly noteworthy merit of our model in this part is having provided an electric description of the surface-operator-M2-brane.

There are many interesting future directions. In fact and as pointed out in the preface, the two references on which the present thesis is based, namely [1, 2], form the first volume in a series of papers to appear that will attempt to cover a good deal of them. We are particularly captivated by explicit computations of knot invariants. On the one hand, turning our attention to model (M,5), we see that most of the analysis is pending. Most notoriously, the details on its connection to [11] through a flop transition, the derivation of its pertinent four-dimensional gauge theory and the suitable embedding of knots in it. Once this is done, a wide range of possibilities unfolds. Two such are the computation of HOMFLY-PT polynomials, along the lines of [77] and the study of A-polynomials, as in [16].

On the other hand, we have not yet exploited most of the immense potential of model (M,1) and its world-volume gauge theory. Although it was not included here, section 3.3 in [1] computes the linking number of any arbitrary knot in the abelian version of the configuration in figure 15. In chapter 8 of the thesis, we have worked out the corresponding non-abelian extension, so we are ready to attempt the derivation of more challenging knot invariants from (M,1). The first one in mind is the A-polynomial. What is more, we conjecture that in our model the zero locus of the surface operator equations (8.18) is itself the A-polynomial of whatever knot K one embeds in the boundary X_3. An easy way to motivate our conjecture is, for example, by comparison to the Simple Harmonic Oscillator toy model discussed in [78]. Remarkably, we are exceptionally well-equipped for this goal, since we work in a framework with a generic gauge group $SU(N)$, while the vast majority of the physics oriented literature restricts attention to the $SU(2)$ case.

Given a Hamiltonian and its classical equations of motion, any physicist will surely think of quantization as a logical next-step. We are no exception and quantization is one of the crucial topics we would like explore in the sequel(s) to [1, 2]. In the quantum realm, the connection to knot invariants becomes even more appealing. For concreteness, let us mention our favorite first candidate, Khovanov homology. Khovanov homology arises naturally from a four-dimensional gauge theory in the presence of surface operators, just like ours. It is particularly interesting to note that Khovanov's invariants categorify the all-famous Jones polynomial and are stronger than it when addressing the knot classification problem [79]. In fact, the puzzle of why the coefficients of the Jones and related polynomials should be integers was resolved in the pioneering work [80], in terms of Khovanov homology. What is more, following indications in [12], Witten argues [14] that Khovanov homology should appear as observables in a four-dimensional Topological Quantum Field Theory, in a higher dimensional analogue to the Jones polynomial case in three-dimensional Chern-Simons theory. It would certainly be a most significant result to confirm this hypothesis in our model.

Besides the captivating but demanding goal of calculating knot invariants, the model presented in this thesis offers other lush possibilities. One on which we have already made some progress consists in further exploiting the brane configurations in part I. We are presently trying to figure out the exact modifications one would need at the String Theory level so as to extend our construction and include a direct connection to Seiberg-Witten theory [17, 18] and to Theories of Class S [19, 20], at least to the subset of such theories that admits a Lagrangian description. On the other hand, we have shown that the bosonic sector of the world-volume gauge theory explored in part II captures a wide range of mathematical and physical results in a unifying, simple and ultraviolet-complete formalism. It is then natural to speculate that a careful investigation of its fermionic sector is likely to yield interesting results as well. Finally and as already mentioned at the end of part III, we would like to better understand the relation between the boundary action and the bulk equations of the configuration in figure 15. That is, we would like to establish the precise Morse flow equation relating them both, following the prescription in [15].

References

[1] K. Dasgupta, V. Errasti Díez, P. Ramadevi and R. Tatar, "Knot invariants and M-Theory: Hitchin equations, Chern-Simons actions, and surface operators," Phys. Rev. D **95**, no. 2, 026010 (2017) doi:10.1103/PhysRevD.95.026010 [arXiv:1608.05128 [hep-th]].

[2] V. Errasti Díez, "Knot invariants and M-theory: Proofs and derivations," Phys. Rev. D **97**, no. 2, 026001 (2018) doi:10.1103/PhysRevD.97.026001 [arXiv:1702.07366 [hep-th]].

[3] A. P. Balachandran, A. de Queiroz and S. Vaidya, "A matrix model for QCD: QCD color is mixed," Int. J. Mod. Phys. A **30**, no. 09, 1550064 (2015) doi:10.1142/S0217751X15500645 [arXiv:1407.8352 [hep-th]].

[4] J. H. Conway, "An enumeration of knots and links and some of their algebraic properties; Computational problems in Abstract Algebra," Pergamon Press, New York, 329 (1970).

[5] J. H. Przytycki and K. P. Traczyk, "Invariants of links of Conway Type," Kobe J. Math. **4**, 115 (1987).

[6] J. W. Alexander, "Topological invariants of knots and links," Trans. Amer. Math. Soc. **30**, 275 (1928) doi:10.2307/1989123.

[7] V. F. R. Jones, "A polynomial invariant for knots via von Neumann algebras," Bull. Amer. Math. Soc. **12**, 103 (1985).

[8] E. Witten, "Quantum Field Theory and the Jones Polynomial," Commun. Math. Phys. **121**, 351 (1989) doi:10.1007/BF01217730.

[9] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millet and A. Ocneanu, "A new polynomial invariant of knots and links," Bull. Amer. Math. Soc. **12**, 239 (1985).

[10] R. Gopakumar and C. Vafa, "On the gauge theory / geometry correspondence," Adv. Theor. Math. Phys. **3**, 1415 (1999) [hep-th/9811131].

[11] H. Ooguri and C. Vafa, "Knot invariants and topological strings," Nucl. Phys. B **577**, 419 (2000) doi:10.1016/S0550-3213(00)00118-8 [hep-th/9912123].

[12] S. Gukov, A. S. Schwarz and C. Vafa, "Khovanov-Rozansky homology and topological strings," Lett. Math. Phys. **74**, 53 (2005) doi:10.1007/s11005-005-0008-8 [hep-th/0412243].

[13] S. Gukov, "Gauge theory and knot homologies," Fortsch. Phys. **55**, 473 (2007) doi:10.1002/prop.200610385 [arXiv:0706.2369 [hep-th]].

[14] E. Witten, "Fivebranes and Knots," arXiv:1101.3216 [hep-th].

[15] D. Gaiotto and E. Witten, "Knot Invariants from Four-Dimensional Gauge Theory," Adv. Theor. Math. Phys. **16**, no. 3, 935 (2012) doi:10.4310/ATMP.2012.v16.n3.a5 [arXiv:1106.4789 [hep-th]].

[16] M. Aganagic and C. Vafa, "Large N Duality, Mirror Symmetry, and a Q-deformed A-polynomial for Knots," arXiv:1204.4709 [hep-th].

[17] N. Seiberg and E. Witten, "Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory," Nucl. Phys. B **426**, 19 (1994) Erratum: [Nucl. Phys. B **430**, 485 (1994)] doi:10.1016/0550-3213(94)90124-4, 10.1016/0550-3213(94)00449-8 [hep-th/9407087].

[18] N. Seiberg and E. Witten, "Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD," Nucl. Phys. B **431**, 484 (1994) doi:10.1016/0550-3213(94)90214-3 [hep-th/9408099].

[19] D. Gaiotto, "N=2 dualities," JHEP **1208**, 034 (2012) doi:10.1007/JHEP08(2012)034 [arXiv:0904.2715 [hep-th]].

[20] D. Gaiotto, G. W. Moore and A. Neitzke, "Wall-crossing, Hitchin Systems, and the WKB Approximation," arXiv:0907.3987 [hep-th].

[21] J. Leach, M. R. Dennis, J. Courtial and M. J. Padgett, "Vortex knots in light," New J. Phys. **7**, 55 (2005).

[22] M. R. Dennis, R. P. King, B. Jack, K. O'Holleran and M. J. Padgett, "Isolated optical vortex knots," Nature Phys. **6**, 118 (2010).

[23] I. I. Smalyukh, Y. Lansac, N. A. Clark and R. P. Trivedi, "Three-dimensional structure and multistable optical switching of triple-twisted particle-like excitations in anisotropic fluids," Nature Mater. **9**, 139 (2009).

[24] U. Tkalec, M. Ravnik, S. Copar, S. Zumer, I. Musevic, "Reconfigurable knots and links in chiral nematic colloids," Science **333**, 62 (2011).

[25] D. Seč, S. Čopar and S. Žumer, "Topological zoo of free-standing knots in confined chiral nematic fluids," Nat. Commun. **5**, 1 (2014).

[26] D. Kleckner and W. T. M. Irvine, "Creation and dynamics of knotted vortices," Nature Phys. **9**, 253 (2013).

[27] D. Han, S. Pal, Y. Liu and H. Yan, "Folding and cutting DNA into reconfigurable topological nanostructures," Nature Nanotech. **5**, 712 (2010).

[28] D. S. Hall, M. W. Ray, K. Tiurev, E. Ruokokoski, A. H. Gheorghe and M. Möttönen, "Tying Quantum Knots," Nature Physics **12**, 478 (2016) doi:10.1038/nphys3624 [arXiv:1512.08981 [cond-mat.quant-gas]].

[29] K. Dasgupta and S. Mukhi, "Brane constructions, fractional branes and Anti-de Sitter domain walls," JHEP **9907**, 008 (1999) doi:10.1088/1126-6708/1999/07/008 [hep-th/9904131].

[30] J. Polchinski, "Dirichlet Branes and Ramond-Ramond charges," Phys. Rev. Lett. **75**, 4724 (1995) doi:10.1103/PhysRevLett.75.4724 [hep-th/9510017].

[31] R. Blumenhagen, D. Lüst and S. Theisen, "Basic concepts of String Theory," Series in Theoretical and Mathematical Physics, Springer, Berlin (2013).

[32] http://people.brandeis.edu/ headrick/HeadrickCompendium.pdf.

[33] S. Chakravarty, K. Dasgupta, O. J. Ganor and G. Rajesh, "Pinned branes and new nonLorentz invariant theories," Nucl. Phys. B **587**, 228 (2000) doi:10.1016/S0550-3213(00)00354-0 [hep-th/0002175].

[34] P. J. Ruback, "The Motion of Kaluza-Klein Monopoles," Commun. Math. Phys. **107**, 93 (1986).

[35] A. Sen, "A Note on enhanced gauge symmetries in M and String Theory," JHEP **9709**, 001 (1997) doi:10.1088/1126-6708/1997/09/001 [hep-th/9707123].

[36] E. Witten, "Bound states of strings and p-branes," Nucl. Phys. B **460**, 335 (1996) doi:10.1016/0550-3213(95)00610-9 [hep-th/9510135].

[37] A. Sen, "Dynamics of multiple Kaluza-Klein monopoles in M and String Theory," Adv. Theor. Math. Phys. **1**, 115 (1998) [hep-th/9707042].

[38] D. Tong, "NS5-branes, T duality and world sheet instantons," JHEP **0207**, 013 (2002) doi:10.1088/1126-6708/2002/07/013 [hep-th/0204186].

[39] K. Dasgupta, K. Oh and R. Tatar, "Geometric transition, large N dualities and MQCD dynamics," Nucl. Phys. B **610**, 331 (2001) doi:10.1016/S0550-3213(01)00296-6 [hep-th/0105066].

[40] K. Dasgupta, K. Oh and R. Tatar, "Open / closed string dualities and Seiberg duality from geometric transitions in M theory," JHEP **0208**, 026 (2002) doi:10.1088/1126-6708/2002/08/026 [hep-th/0106040].

[41] J. Maldacena and D. Martelli, “The Unwarped, resolved, deformed conifold: Fivebranes and the baryonic branch of the Klebanov-Strassler theory,” JHEP **1001**, 104 (2010) doi:10.1007/JHEP01(2010)104 [arXiv:0906.0591 [hep-th]].

[42] F. Chen, K. Dasgupta, P. Franche, S. Katz and R. Tatar, “Supersymmetric Configurations, Geometric Transitions and New Non-Kähler Manifolds,” Nucl. Phys. B **852**, 553 (2011) doi:10.1016/j.nuclphysb.2011.07.013 [arXiv:1007.5316 [hep-th]].

[43] G. Lopes Cardoso, G. Curio, G. Dall'Agata, D. Lust, P. Manousselis and G. Zoupanos, “NonKähler string backgrounds and their five torsion classes,” Nucl. Phys. B **652**, 5 (2003) doi:10.1016/S0550-3213(03)00049-X [hep-th/0211118].

[44] S. Chiossi and S. Salamon, “The intrinsic torsion of $SU(3)$ and G_2 structures,” [arXiv:math/0202282 [math.DG]].

[45] K. Dasgupta, M. Emelin and E. McDonough, “Non-Kähler resolved conifold, localized fluxes in M-Theory and supersymmetry,” JHEP **1502**, 179 (2015) doi:10.1007/JHEP02(2015)179 [arXiv:1412.3123 [hep-th]].

[46] A. Strominger, “Superstrings with torsion,” Nucl. Phys. B **274**, 253 (1986).

[47] F. Chen, K. Dasgupta, J. M. Lapan, J. Seo and R. Tatar, “Gauge/Gravity Duality in Heterotic String Theory,” Phys. Rev. D **88**, 066003 (2013) doi:10.1103/PhysRevD.88.066003 [arXiv:1303.4750 [hep-th]].

[48] A. Strominger, S. T. Yau and E. Zaslow, “Mirror symmetry is T duality,” Nucl. Phys. B **479**, 243 (1996) doi:10.1016/0550-3213(96)00434-8 [hep-th/9606040].

[49] M. Becker, K. Dasgupta, A. Knauf and R. Tatar, “Geometric transitions, flops and nonKähler manifolds. I.,” Nucl. Phys. B **702**, 207 (2004) doi:10.1016/j.nuclphysb.2004.09.020 [hep-th/0403288].

[50] S. Alexander, K. Becker, M. Becker, K. Dasgupta, A. Knauf and R. Tatar, “In the realm of the geometric transitions,” Nucl. Phys. B **704**, 231 (2005) doi:10.1016/j.nuclphysb.2004.10.036 [hep-th/0408192].

[51] M. Becker, K. Dasgupta, S. H. Katz, A. Knauf and R. Tatar, “Geometric transitions, flops and non-Kähler manifolds. II.,” Nucl. Phys. B **738**, 124 (2006) doi:10.1016/j.nuclphysb.2005.12.023 [hep-th/0511099].

[52] M. Grana, “Flux compactifications in String Theory: A Comprehensive review,” Phys. Rept. **423**, 91 (2006) doi:10.1016/j.physrep.2005.10.008 [hep-th/0509003].

[53] A. Sen, “Strong coupling dynamics of branes from M theory,” JHEP **9710**, 002 (1997) doi:10.1088/1126-6708/1997/10/002 [hep-th/9708002].

[54] K. M. Lee and H. U. Yee, "BPS String Webs in the 6-dim (2,0) Theories," JHEP **0703**, 057 (2007) doi:10.1088/1126-6708/2007/03/057 [hep-th/0606150].

[55] E. B. Bogomolny, "Stability of Classical Solutions," Sov. J. Nucl. Phys. Yad. Fiz **24**, 449 (1976).

[56] N. Hitchin, "Stable bundles and integrable systems," Duke Math. J. **54**, no.1, 91 (1987) doi:10.1215/S0012-7094-87-05408-1.

[57] W. Nahm, "All Selfdual Multi - Monopoles For Arbitrary Gauge Groups," CERN-TH-3172.

[58] C. R. Leao and V. O. Rivelles, "Off-shell duality in Born-Infeld theory," Nucl. Phys. B **602**, 514 (2001) doi:10.1016/S0550-3213(01)00105-5 [hep-th/0101031].

[59] D. Gaiotto and E. Witten, "Supersymmetric Boundary Conditions in N=4 Super Yang-Mills Theory," J. Statist. Phys. **135**, 789 (2009) doi:10.1007/s10955-009-9687-3 [arXiv:0804.2902 [hep-th]].

[60] L. Brink, J. H. Schwarz and J. Scherk, "Supersymmetric Yang-Mills Theories," Nucl. Phys. B **121**, 77 (1977).

[61] A. Kapustin and E. Witten, "Electric-Magnetic Duality And The Geometric Langlands Program," Commun. Num. Theor. Phys. **1**, 1 (2007) doi:10.4310/CNTP.2007.v1.n1.a1 [hep-th/0604151].

[62] E. Witten, "Analytic Continuation Of Chern-Simons Theory," AMS/IP Stud. Adv. Math. **50**, 347 (2011) [arXiv:1001.2933 [hep-th]].

[63] R. P. Langlands, "Problems in the theory of automorphic forms," Lect. Notes in Math. **170**, 18, Springer Verlag (1970).

[64] T. Hausel and M. Thaddeus, "Mirror symmetry, Langlands duality, and the Hitchin system," Invent. Math. **153**, 197 (2003) doi:10.1007/s00222-003-0286-7 [math/0205236 [math-ag]].

[65] S. Gukov and E. Witten, "Gauge Theory, Ramification, And The Geometric Langlands Program," hep-th/0612073.

[66] E. Witten, "Mirror Symmetry, Hitchin's Equations, And Langlands Duality," doi:10.1093/acprof:oso/9780199534920.003.0007 arXiv:0802.0999 [math.RT].

[67] E. Witten, "Geometric Langlands And The Equations Of Nahm And Bogomolny," arXiv:0905.4795 [hep-th].

[68] B. C. Ngô, "Fibration de Hitchin et endoscopie," Invent. Math. **164**, 399 (2006).

[69] B. C. Ngô, "Le lemme fondamental pour les algèbres de Lie," Publ. Math. I.H.E.S. **111**, 1 (2010).

[70] N. Marcus, "The Other topological twisting of N=4 Yang-Mills," Nucl. Phys. B **452**, 331 (1995) doi:10.1016/0550-3213(95)00389-A [hep-th/9506002].

[71] S. Gukov and E. Witten, "Rigid Surface Operators," Adv. Theor. Math. Phys. **14**, no. 1, 87 (2010) doi:10.4310/ATMP.2010.v14.n1.a3 [arXiv:0804.1561 [hep-th]].

[72] S. Gukov, "Surface Operators," doi:10.1007/978–3-319-18769-3_8 arXiv:1412.7127 [hep-th].

[73] V. Shende, "Hilbert schemes of points on a locally planar curve and the Severi strata of its versal deformation," Compositio Mathematica **148.2**, 531 (2012) doi:10.1112/S0010437X11007378 arXiv:1009.0914 [math.AG].

[74] A. Oblomkov and Z. Yun, "Geometric representations of graded and rational Cherednik algebras," Advances in Mathematics **292** (2014) doi:10.1016/j.aim.2016.01.015 arXiv:1407.5685 [math.RT].

[75] T. Dimofte and S. Gukov, "Refined, Motivic, and Quantum," Lett. Math. Phys. **91**, 1 (2010) doi:10.1007/s11005-009-0357-9 [arXiv:0904.1420 [hep-th]].

[76] C. T. Simpson, "Nonabelian Hodge theory," Proceedings of the International Congress of Mathematicians, vols. I, II (Kyoto, 1990), pp. 747–756. Math. Soc. Jap., Tokyo (1991).

[77] J. M. F. Labastida and M. Marino, "Polynomial invariants for torus knots and topological strings," Commun. Math. Phys. **217**, 423 (2001) doi:10.1007/s002200100374 [hep-th/0004196].

[78] S. Gukov and I. Saberi, "Lectures on Knot Homology and Quantum Curves," doi:10.1090/conm/613/12235 arXiv:1211.6075 [hep-th].

[79] D. Bar-Natan, "On Khovanov's Categorification Of The Jones Polynomial," Alg. Geom. Topology **2**, 337 (2002) [arXiv:math/0201043].

[80] M. Khovanov, "A Categorification Of The Jones Polynomial," Duke. Math. J. **101**, 359 (2000) [arXiv:math/9908171].