SPECTRAL ANALYSIS OF OPERATOR POLYNOMIALS

by

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ABSTRACT

The main result of this thesis is a necessary and sufficient condition for an operator polynomial, $L(\lambda) = \sum_{i=o}^{\ell} A_i\lambda^i$, to have a Wiener-Hopf factorization with respect to a simple closed contour, γ, not intersecting the spectrum of L. A spectral approach to the particular case of stable factorization was developed by Gohberg, Lerer, and Rodman. This approach is developed further here, so that the general case can be considered. One obtains a spectral theory of operator polynomials which is an extension of the spectral theory of linear operators.

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RESUME

Le résultat principal de cette thèse est une condition nécessaire et suffisante pour qu'un polynome opérateur, $L(\lambda) = \sum_{i=o}^{\ell} A_i\lambda^i$, se factorise selon la méthode Wiener-Hopf par rapport à une courbe γ, simple et fermée, dont l'intersection avec le spectre de L est nulle. Une approche spectrale au cas particulier de factorisation stable a été développée par Gohberg, Lerer, et Rodman. Ici cette approche est élaborée davantage, pour que le cas général puisse être considéré. On obtient une théorie spectrale des polynomes opérateurs qui est une extension de la théorie spectrale des opérateurs linéaires.

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INTRODUCTION

Gohberg, Lancaster and Rodman [1,2] introduced a spectral theory for monic operator polynomials L in terms of standard triples (X,T,Y). If L is linear, i.e. $L(\lambda) = \lambda I - A$ for some operator A, then X = I, T = A, and Y = I defines a standard triple for L. More generally, suppose L is any monic operator polynomial and (X,T,Y) is a standard triple for L. If $e_1,\ldots,e_k$ is a Jordan chain for T then $Xe_1,\ldots,Xe_k$ is a Jordan (or Keldys) chain for L. In the finite dimensional case we can take T to be in Jordan canonical form and then the columns of X give a complete system of eigenvectors and generalized eigenvectors for L. Similarly, one can show that the rows of Y (in reverse order) give a complete system of eigenvectors and generalized eigenvectors for L^*.

Let L be an operator polynomial and γ a simple closed contour such that $\sigma(L) \cap \gamma = \emptyset$. It is always assumed γ is smooth and $\mathbb{C} - \gamma$ has two components : the interior F^+ and the exterior F^-. The aim of Chapter 1 is to give the definition of a γ-spectral triple (X_+,T_+,Y_+) for L and to show the significance of this definition. In particular, it is easy to show that $L^{-1}(\lambda) - X_+(\lambda I - T_+)^{-1}Y_+$ is holomorphic inside γ. If $\lambda_o \in \sigma(L)$ is isolated then this gives a natural way to compute the principal part of the Laurent expansion of $L^{-1}(\lambda)$ near λ_o, and leads to a simple proof, for operator polynomials, of a result in [10] that gives a formula for the resolvent in terms of Jordan chains for L and its transpose, L^*. Also a generalization of the resolvent form of [2] is given in section 1.2.

Section 1.3 discusses the connection between right divisors of L and invariant subspaces of T_+. The generality of the results here is new.

If L is a monic operator polynomial then $\sigma(L)$ is compact, and if γ contains $\sigma(L)$ in its interior then γ-spectral triples are the same as standard triples. In general, γ-spectral triples are really the same as the right and left γ-spectral pairs defined by Gohberg, Lerer, and Rodman [8]. The exact connection is discussed in section 1.1. The definition, given here seems to be simpler and a proof of the existence of γ-spectral triples is given in section 1.1, without using results on monic operator polynomials as was done in [8].

A continuous function $A: \gamma \to GL(X)$, where X is a Banach space, is said to admit a right factorization† relative to γ if, assuming $0 \in F^+$,

$$A(\lambda) = A_+(\lambda) \cdot \left(\sum_{i=1}^{\nu} \lambda^{\kappa_i} Q_i\right) \cdot A_-(\lambda)$$

where $A_\pm : \gamma \cup F^\pm \longrightarrow GL(X)$ are continuous functions holomorphic in $F^\pm$, $Q_1, \ldots, Q_\nu$ are mutually disjoint projectors such that $\sum_{i=1}^{\nu} Q_i = I$, and $\kappa_1 < \ldots < \kappa_\nu$ are integers, called the right partial indices. Interchanging the roles of A_+ and A_- one obtains the definition of a left factorization.

The main result of this thesis is that if L is an o.p. of degree $\leq \ell$ and γ is a simple closed contour such that $\sigma(L) \cap \gamma = \emptyset$ then L has a right factorization relative to γ if and only if

$$\frac{1}{2\pi i} \int_\gamma \begin{pmatrix} I \\ \vdots \\ \lambda^i I \end{pmatrix} L^{-1}(\lambda)\ (I \ \ldots\ \lambda^{\ell-1} I)\, d\lambda$$

---

† sometimes called spectral factorization or Wiener-Hopf factorization.

has a generalized inverse for $i = 0,1,\ldots,\ell-1$. to prove this, the results of Chapter 1 on spectral triples, as well as the concept of reducibility of operator polynomials introduced in Chapter 2, are used. For more details on the contents of Chapter 2, see the introduction there.

I would like to thank my thesis supervisor, B.Lawruk, for introducing me to the subject of this thesis, and for his interest and guidance. Finally, I am grateful to I.Gohberg for valuable remarks he made during the time he visited McGill University in November 1978.

CHAPTER 1

1.0 DEFINITIONS

Let X be a Banach space. $L(X)$ is the space of linear operators on and $GL(X)$ is the group of invertible linear operators. A map $L:\mathbb{C} \to L(X)$ is called an operator polynomial, abbreviated as o.p., if $L(\lambda) = \sum_{i=o}^{\ell} A_i\lambda^i$ for some $A_i \in L(X)$. If $A_\ell \neq 0$ then ℓ is called the degree of L. The spectrum of L is defined to be $\sigma(L) = \{\lambda \in C;\ L(\lambda)$ is not invertible$\}$. Notice that $\sigma(L)$ is closed, but not necessarily bounded. It is always assumed $\sigma(L) \neq \mathbb{C}$.

If L_1 and L are operator polynomials then L_1 is called a right divisor of L if there exists an operator polynomial L_2 such that $L = L_2L_1$. If γ is a simple closed contour such that $\sigma(L) \cap \gamma = \emptyset$ then L_1 is called a γ-spectral right divisor of L if $\sigma(L_1)$ is contained inside γ and $\sigma(L_2)$ is contained outside γ. In general, the part of $\sigma(L)$ inside γ will be denoted $\sigma_+(L)$, and that outside γ will be denoted $\sigma_-(L)$.

By a subspace of a Banach space, we will always mean a closed subspace. The sum of subspaces is denoted by + and the direct sum by $\oplus$. If $S_j \in L(X,X_j)$ $(j = 1,\ldots,n)$ are operators, where X_j and X are Banach spaces, then $\mathrm{col}(S_j)_{j=1}^n$ denotes the operator

$$\begin{pmatrix} S_1 \\ \vdots \\ S_n \end{pmatrix} \in L(X, \bigoplus_{j=1}^{n} X_j) .$$

If $T_j \varepsilon L(X_j,X)$ $(j = 1,\dots,m)$ then row $(T_j)_{j=1}^m$ denotes the operator

$$(T_1 \dots T_m) \varepsilon L(\bigoplus_{j=1}^m X_j, X).$$

The range of an operator $T \varepsilon L(X,Y)$ is denoted $R(T)$, and its kernel is $N(T)$. $S \varepsilon L(Y,X)$ is said to be a generalized inverse of T if STS = S and TST = T. It is easily seen that T has a generalized inverse if and only if $R(T)$ and $N(T)$ are complemented in Y and X, respectively.

In this chapter we will show how information about the spectral properties of an operator polynomial L can be concentrated into triples of operators (X,T,Y). It is convenient to introduce some terminology. A triple of operators (X,T,Y) is called an admissible triple if $X \varepsilon L(V,X)$, $T \varepsilon L(V)$, and $Y \varepsilon L(X,V)$ where X and V are Banach spaces; V is called the base space of (X,T,Y) and X is the target space. A pair of operators (X,T) is called a right admissible pair if $X \varepsilon L(V,X)$ and $T \varepsilon L(V)$ where X and V are Banach spaces, and a similar definition holds for left admissible pairs (T,Y). Often, the adjectives "right" and "left" will be omitted since the generic symbols (X,T) and (T,Y) will always be used and so there is no possibility of confusion.

The kernel of a right admissible pair (X,T) is defined to be

$$N(X,T) = \bigcap_{i=o}^{\infty} N(XT^i)$$

$$= \bigcap_{i=o}^{\infty} N(\mathrm{col}(XT^j)_{j=o}^i).$$

The least positive integer ℓ , if it exists, such that

$$N(\mathrm{col}(XT^j)_{j=o}^{\ell-1}) = N(\mathrm{col}(XT^j)_{j=o}^{\ell})$$

is called the index of stabilization of (X,T), and is denoted by $\mathrm{ind}(X,T)$. Otherwise, define $\mathrm{ind}(X,T) = \infty$. (X,T) is called an ℓ-independent admissible pair if

$$N(\mathrm{col}(XT^j)_{j=o}^{\ell-1}) = 0 .$$

This holds if and only if $N(X,T) = 0$ and $\ell \geq \mathrm{ind}(X,T)$. If (T,Y) is a left admissible pair then the range of (T,Y) is defined to be

$$R(T,Y) = \sum_{i=o}^{\infty} R(T^iY) .$$

We will say that (T,Y) is surjective if $R(T,Y)$ is equal to the base space of (T,Y). There are also analogous definitions of index of stabilization and ℓ-independence for left admissible pairs.

A right admissible pair (X',T') is said to be a restriction of the right admissible pair (X,T) if there exists a complemented T-invariant subspace W of the base space V of (X,T) and an invertible operator $S:V \longrightarrow W$ (where V' is the base space of (X',T') such that

$$X' = (X|_W)S , \qquad T' = S^{-1}(T|_W)S.$$

Also, (X,T) can be called an extension of (X',T'). If $W = V$ then (X,T) anc (X',T') are called similar.

1.1 SPECTRAL TRIPLES

1.1.1 <u>Definition</u>. Let $L(\lambda) = \sum_{i=o}^{\ell} A_i\lambda^i$ be an o.p., and let γ be a simple closed contour such that $\sigma(L) \cap \gamma = \phi$. An admissible triple (X_+,T_+,Y_+) is called a γ-spectral triple for L if $\sigma(T_+)$ is contained inside γ and

(i) $\quad \sum_{j=o}^{\ell} A_j X_+ T_+^j = 0;$

(ii) $\quad \mathrm{col}(X_+T_+^j)_{j=o}^{\ell-1}$ is injective;

(iii) $\quad X_+T_+^jY_+ = \frac{1}{2\pi i} \int_\gamma \lambda^j L^{-1}(\lambda)d\lambda$ for $j = 0,\ldots, \ell-1$.

A chain $x_1,\ldots,x_k \in X$ is called a Jordan chain for L at $\lambda_o \in \sigma(L)$ if

$$\sum_{p=o}^{j} \frac{1}{p!} L^{(p)}(\lambda_o)x_{j+1-p} = 0 , \quad (j = 0,\ldots,k-1).$$

The meaning of condition (i) is, basically, that it implies that if $e_1,\ldots,e_k$ is a Jordan chain for T_+ at λ_o then $Xe_1 ,\ldots,Xe_k$ is a Jordan chain for L at λ_o.

If the leading coefficient, A_ℓ, of L is invertible and γ is a simple closed contour containing $\sigma(L)$ then

$$\frac{1}{2\pi i} \int_\gamma \lambda^j L^{-1}(\lambda)d\lambda = \begin{cases} 0 & \text{for } j = 0,\ldots, \ell-2 \\ A_\ell^{-1} & \text{for } j = \ell-1 \end{cases} .$$

Thus a γ-spectral triple in this case is a standard triple in the sense of Gohberg, Lancaster, and Rodman [1]. For standard triples (X,T,Y), $\mathrm{col}(XT^j)_{j=o}^{\ell-1}$ is invertible.

We will also see shortly (1.1.7) that, in general, γ-spectral triples are an extension of γ-spectral pairs as defined in Gohberg, Lerer, and Rodman [8].

To establish the existence of γ-spectral triples, methods different from those in [8] will be used. The following easy proposition relating $L(\lambda)$ and $\lambda I - T$ is important. Define operator polynomials

$$L_j(\lambda) = A_j + A_{j+1}\lambda + \ldots + A_\ell \lambda^{\ell-j} \quad ,$$

for $j = 0,\ldots, \ell$. Notice that $L_o = L$.

1.1.2 <u>Proposition</u>. Let L be an o.p. and let (X,T) be a right admissible pair such that $\sum_{j=o}^{\ell} A_j XT^j = 0$. Then

$$L(\lambda)X = \left(\sum_{j=o}^{\ell-1} L_{j+1}(\lambda)XT^j\right)(\lambda I - T) \tag{1.1}$$

Similarly, if (T,Y) is a left admissible pair such that $\sum_{j=o}^{\ell} T^j YA_j = 0$ then

$$YL(\lambda) = (\lambda I - T)\left(\sum_{j=o}^{\ell} T^j Y L_{j+1}(\lambda)\right). \tag{1.2}$$

<u>Proof</u>. Since $\lambda L_{j+1}(\lambda) = L_j(\lambda) - A_j$, the right hand side of (1.1) equals

$$\sum_{j=o}^{\ell-1} (L_j(\lambda) - A_j)XT^j - \sum_{j=o}^{\ell-1} L_{j+1}(\lambda)XT^{j+1}$$

$$= L_o(\lambda)X - \sum_{j=o}^{\ell} A_j XT^j$$

$$= L(\lambda)X.$$

The proof of (1.2) is similar.

1.13 Theorem. Let L be an o.p. and γ a simple closed contour such that $\sigma(L) \cap \gamma = \emptyset$. There exists a γ-spectral triple for L and any two such triples are similar. Moreover, if (X_+,T_+,Y_+) is a γ-spectral triple for L then $\sigma(T_+) = \sigma_+(L)$ and

$$\text{(i)}' \quad \sum_{j=o}^{\ell} T_+^j T_+ A_j = 0,$$

$$\text{(ii)}' \quad \text{row } (T_+^j Y_+)_{j=o}^{\ell-1} \text{ is surjective,}$$

and 1.1.1(iii) holds for all $j = 0,1,2,\dots \ell$.

Proof. Let V_L^+ be the vector space of all $u \in C^\infty(\mathbb{R},\mathcal{X})$ such that, for some continuous $f:\overline{F^+} \to \mathcal{X}$ holomorphic in F^+,

$$u(t) = \frac{1}{2\pi i} \int_\gamma e^{t\lambda} L^{-1}(\lambda) f(\lambda) d\lambda \,. \tag{1.3}$$

Define $X_+' \in \mathcal{L}(V_L^+,\mathcal{X})$, $T_+' \in \mathcal{L}(V_L^+)$, and $Y_+' \in \mathcal{L}(\mathcal{X},V_L^+)$ as

$$X_+' u = u(0),$$

$$T_+' u = \frac{d}{dt} u \,,$$

$$(Y_+' x)(t) = \frac{1}{2\pi i} \int_\gamma e^{t\lambda} L^{-1}(\lambda) x \, d\lambda \,.$$

We will show that (X_+',T_+',Y_+') is a γ-spectral triple for L, called the natural γ-spectral triple for L. One can verify immediately that 1.1.1(i) and (iii) are true.

To show 1.1.1(ii) is true requires more effort.

V_L^+ has a natural topology induced from the usual Frechet topology on $C^\infty(\mathbb{R},X)$. It is not clear that V_L^+ is a Banach space in this topology but we will eventually show this. For the moment, however, V_L^+ is regarded simply as a vector space.

It is convenient to introduce the notation int $\gamma = F^+$ and ext $\gamma = F^-$. Let $\mu \in$ ext γ and define $S_\mu \in L(V_L^+)$ by

$$(S_\mu u)(t) = \frac{1}{2\pi i}\int_\gamma e^{t\lambda}L^{-1}(\lambda)f(\lambda)(\mu-\lambda)^{-1}d\lambda$$

where f determines u as in (1.3). S_μ is well-defined for if $\int_\gamma e^{t\lambda}L^{-1}(\lambda)f(\lambda)d\lambda = 0$ then $L^{-1}(\lambda)f(\lambda)$ is holomorphic inside γ. Thus so is $L^{-1}(\lambda)f(\lambda)(\mu-\lambda)^{-1}$, and hence $\int_\gamma e^{t\lambda}L^{-1}(\lambda)f(\lambda)(\mu-\lambda)^{-1}d\lambda = 0$. One can verify immediately that, for all $u \in V_L^+$,

$$(\mu I - T_+')S_\mu u = u$$

and

$$S_\mu(\mu I - T_+')u = u .$$

Now, there exists a closed contour γ' contained in the interior of γ such that the part of $\sigma(L)$ lying inside γ' is the same as that inside γ. We can replace γ by γ' in the above considerations and conclude that $\sigma(T_+') \subseteq$ int γ' and for $\mu \in$ ext γ'

$$(\mu I - T_+')^{-1}u = \frac{1}{2\pi i}\int_{\gamma'} e^{t\lambda}L^{-1}(\lambda)f(\lambda)(\mu-\lambda)^{-1}d\lambda \quad . \qquad (1.4)$$

Fix $u \in V_L^+$ and define

$$F(\mu) = \sum_{j=o}^{\ell-1} L_{j+1}(\mu)u^{(j)}(0) .$$

Then, for $\mu \in \gamma$, 1.1.2 and (1.4) imply

$$F(\mu) = X_+'(\mu I - T_+')^{-1}u$$

$$= \frac{1}{2\pi i} \int_{\gamma'} L^{-1}(\lambda)f(\lambda)(\mu-\lambda)^{-1}d\lambda \quad .$$

Now, Fubini's theorem and the Cauchy integral formula imply

$$\frac{1}{2\pi i} \int_{\gamma} e^{t\mu}F(\mu)d\mu = \frac{1}{2\pi i} \int_{\gamma'} \{\frac{1}{2\pi i} \int_{\gamma} e^{t\mu}(\mu-\lambda)^{-1}d\mu\} L^{-1}(\lambda)f(\lambda)d\lambda$$

$$= \frac{1}{2\pi i} \int_{\gamma'} e^{t\lambda}L^{-1}(\lambda)f(\lambda)d\lambda$$

$$= u(t).$$

In other words, for all $u \in V_L^+$,

$$u(t) = \frac{1}{2\pi i} \int_{\gamma} e^{t\lambda}L^{-1}(\lambda)\left(\sum_{j=o}^{\ell-1} L_{j+1}(\lambda)u^{(j)}(0)\right)d\lambda \quad . \qquad (1.5)$$

In particular, notice that 1.1.1 (ii) is true.

Now, define

$$P_\gamma = \frac{1}{2\pi i} \int_{\gamma} \begin{pmatrix} I \\ \vdots \\ \lambda^{\ell-1}I \end{pmatrix} L^{-1}(\lambda)(L_1(\lambda) \ \ldots \ L_\ell(\lambda))d\lambda \ . \qquad (1.6)$$

If $y \in X^\ell$ define $f_y(\lambda) = \sum_{j=o}^{\ell-1} L_{j+1}(\lambda)y_j$. When u is defined as in (1.3), but with f_y in place of f, then $u \in V_L^+$ and

$$\mathrm{col}(X_+'T_+'^j)_{j=o}^{\ell-1} u = P_\gamma y$$

Also, if $u \in V_L^+$ then (1.5) implies $P_\gamma y = y$, where $y = \mathrm{col}(X_+'T_+'^j)_{j=o}^{\ell-1} u$.

Hence $$R(\mathrm{col}(X_+'T_+'^j)_{j=o}^{\ell-1}) = R(P_\gamma) \qquad (1.7)$$

and $P_\gamma \in \mathcal{L}(X^\ell)$ is a projection.

A Banach space structure can be defined on V_L^+ by demanding that $\mathrm{col}(X_+'T_+'^j)_{j=0}^{\ell-1}$ be an isometry. The topology so defined on V_L^+ coincides with that inherited from $C^\infty(\mathbb{R},X)$. First of all, since $\mathrm{col}(X_+'T_+'^j)_{j=0}^{\ell-1}$ is continuous in the latter topology, it follows that this topology is finer than the Banach space topology. By the closed graph theorem, it therefore suffices to show that V_L^+ is a closed subspace of $C^\infty(\mathbb{R},X)$. Let $\{u_n\}_{n=1}^\infty$ be a sequence in V_L^+ such that $u_n \to u$ in $C^\infty(\mathbb{R},X)$. Then

$$u_n(t) = \frac{1}{2\pi i} \int_\gamma e^{t\lambda} L^{-1}(\lambda) f_{y_n}(\lambda) d\lambda \ ,$$

where
$$y_n = \mathrm{col}(X_+'T_+'^j)_{j=0}^{\ell-1} u_n \ .$$

Since $f_{y_n}(\lambda) \to f_y(\lambda)$ uniformly on γ, where $y = \mathrm{col}(u^{(j)}(0))_{j=0}^{\ell-1}$, we conclude that

$$u(t) = \frac{1}{2\pi i} \int_\gamma e^{t\lambda} L^{-1}(\lambda) f_y(\lambda) d\lambda \ .$$

Thus $u \in V_L^+$ and V_L^+ is a closed subspace of $C^\infty(\mathbb{R},X)$. This completes the proof of existence of a γ-spectral triple for L.

Let (X_+,T_+,Y_+) be a γ-spectral triple for L. Then

$$X_+ e^{tT_+} Y_+ = \frac{1}{2\pi i} \int_\gamma e^{t\lambda} L^{-1}(\lambda) d\lambda \quad . \tag{1.8}$$

Indeed, choose $x \in X$ and let $u(t)$ denote the difference between the right and left hand sides of (1.8) evaluated at x. Then 1.1.1(iii) implies $u^{(j)}(0) = 0$ for $j = 0,\dots, \ell-1$. Notice that

$$X_+e^{tT_+}Y_+x = \frac{1}{2\pi i} \int_\gamma e^{t\lambda} X_+(\lambda I - T_+)^{-1} Y_+ x \, d\lambda$$

$$= \frac{1}{2\pi i} \int_\gamma e^{t\lambda} L^{-1}(\lambda) f(\lambda) d\lambda$$

where $f(\lambda) = \sum_{j=o}^{\ell-1} L_{j+1}(\lambda)X_+T_+^jY_+x$. It follows that $u \varepsilon V_L^+$ and hence $u = 0$ by (1.5). Since x is arbitrary this proves (1.8). Differentiating both sides of (1.8) yields

$$X_+e^{tT_+}T_+^jY_+ = \frac{1}{2\pi i}\int_\gamma e^{t\lambda}\lambda^jL^{-1}(\lambda)d\lambda \qquad (1.9)$$

for all $j = 0,1,2,\ldots$. In particular, 1.1.1(iii) holds for all j.

Now, let V_+ denote the base space of (X_+,T_+,Y_+) and define $\varphi \varepsilon L(V_+,V_L^+)$ as $(\varphi x)(t) = X_+e^{tT_+}x$. Then 1.1.1(ii) implies φ is injective. Also, (1.5) and (1.9) imply that $\varphi \operatorname{row}(T_+^jY_+)_{j=o}^{\ell-1}$ is surjective, so φ is surjective. Hence φ is invertible (by the closed graph theorem) and (ii)' is true. One sees immediately that $X_+ = X_+'\varphi$ and $T_+'\varphi = \varphi T_+$, and also $\varphi Y_+ = Y_+'$ is simply a restatement of (1.8). Thus (X_+,T_+,Y_+) is similar to (X_+',T_+',Y_+') and any two γ-spectral pairs are similar. From (1.9) we see that $\varphi\sum_{j=o}^{\ell} T_+^jY_+A_j = 0$, which implies (i)' .

It remains to prove $\sigma(T_+) = \sigma_+(L)$. If $\lambda \notin \sigma(L)$ then 1.2.1 implies

$$\operatorname{col}(X_+T_+^i)_{i=o}^{\ell-1} = S(\lambda I - T_+) ,$$

where
$$S = \operatorname{col}\Big(L^{-1}(\lambda)\sum_{j=o}^{\ell-1} L_{j+1}(\lambda)X_+T_+^{j+i}\Big)_{i=o}^{\ell-1} ;$$

thus $\lambda I - T_+$ is injective. Similarly, for some S'

$$\operatorname{row}(T_+^iY_+)_{i=o}^{\ell-1} = (\lambda I - T_+)S' ;$$

thus (ii)' implies $\lambda I - T_+$ is surjective. Hence $\sigma(T_+) \subseteq \sigma(L)$.

To prove the reverse inclusion we start with the fact that $M(\lambda) = L^{-1}(\lambda) - X_+(\lambda I - T_+)^{-1}Y_+$ is holomorphic inside γ (more precisely, M has a holomorphic continuation there).

Indeed, for $j = 0,1,2,\ldots$

$$\frac{1}{2\pi i}\int_\gamma \lambda^j M(\lambda)d\lambda = \frac{1}{2\pi i}\int_\gamma \lambda^j L^{-1}(\lambda)d\lambda - X_+T_+^jY_+$$

$$= 0$$

which implies M is holomorphic inside γ. Define

$$N(\lambda) = M(\lambda) + X_+(\lambda I - T_+)^{-1}Y_+$$

for $\lambda \in \operatorname{int}\gamma\backslash\sigma(T_+)$. For $\lambda \in \operatorname{int}\gamma\backslash\sigma(L)$, $I = L(\lambda)N(\lambda)$, and then 1.1.2 implies

$$I = L(\lambda)M(\lambda) + \sum_{j=o}^{\ell-1} L_{j+1}(\lambda)X_+T_+^jY_+ . \tag{1.10}$$

But $\operatorname{int}\gamma$ is connected so (1.10) holds for all $\lambda \in \operatorname{int}\gamma$. Another application of 1.1.2 implies that $I = L(\lambda)N(\lambda)$ for all $\lambda \in \operatorname{int}\gamma\backslash\sigma(T_+)$. Similarly, $N(\lambda)L(\lambda) = I$ there. Hence $\sigma_+(L) \subseteq \sigma(T_+)$ and this completes the proof of the theorem.

1.1.4 Corollary. Let L be an o.p., and γ a simple closed contour such that $\sigma(L) \cap \gamma = \emptyset$. If (X_+,T_+,Y_+) is a γ-spectral triple for L then

$$L^{-1}(\lambda) - X_+(\lambda I - T_+)^{-1}Y_+$$

is holomorphic inside γ.

This was shown in the proof of 1.1.3.

1.1.5 Corollary. Let Γ and γ be simple closed contours not intersecting $\sigma(L)$, and suppose γ is contained in the interior of Γ. Let (X,T,Y) be a Γ-spectral triple for L and

$$P_\gamma = \frac{1}{2\pi i} \int_\gamma (\lambda I - T)^{-1} d\lambda \quad ,$$

the Riesz spectral projector. Then (X_+, T_+, Y_+) is a γ-spectral triple for L with base space $R(P_\gamma)$, where

$$X_+ = XP_\gamma \ ,$$

$$T_+ = P_\gamma T P_\gamma \quad ,$$

and
$$Y_+ = P_\gamma Y \ .$$

<u>Proof</u>. Only 1.1.1(iii) is not obvious for (X_+, T_+, Y_+).

But 1.1.4 implies

$$\begin{aligned} X_+ T_+^j Y_+ &= XT^j P_\gamma Y \\ &= XT^j \frac{1}{2\pi i} \int_\gamma (\lambda I - T)^{-1} d\lambda \ Y \\ &= \frac{1}{2\pi i} \int_\gamma \lambda^j X(\lambda I - T)^{-1} Y \, d\lambda \\ &= \frac{1}{2\pi i} \int_\gamma \lambda^j L^{-1}(\lambda) d\lambda \ . \end{aligned}$$

1.1.6 <u>Corollary</u>. Let L and γ be as in 1.13, and suppose (X_+, T_+, Y_+) is a γ-spectral triple for L. Let X^* denote the dual space to X and $L^*(\lambda) = \sum_{i=o}^{\ell} A_i^* \lambda^i$. Then $\sigma(L^*) = \sigma(L)$ and (Y_+^*, T_+^*, X_+^*) is a γ-spectral triple for L^*.

The proof is obvious.

If $L(\lambda) = \lambda I - A$ then P_γ , as defined in (1.6) , is the Riesz spectral projector for A with respect to the contour γ . In general, if L is any o.p. and (X_+, T_+, Y_+) is a γ-spectral triple for L then the range of

$\operatorname{col}(X_+T_+^i)_{i=0}^{\ell-1}$ is equal to the range of P_γ. Hence $R(\operatorname{col}(X_+T_+^i)_{i=0}^{\ell-1})$ is complemented in X^ℓ or, in other words, $\operatorname{col}(X_+T_+^i)_{i=0}^{\ell-1}$ is left invertible. Moreover, the range of $\operatorname{col}(X_+T_+^i)_{i=0}^{\ell-1}$ is equal to the set of all $y \in X^\ell$ such that, for some polynomial f of degree $\leq \ell-1$,

$$y = \operatorname{col}\Big(\int_\gamma \lambda^i L^{-1}(\lambda) f(\lambda) d\lambda\Big)_{i=0}^{\ell-1} .$$

Similarly, the kernel of $\operatorname{row}(T_+^iY_+)_{i=0}^{\ell-1}$ is equal to the kernel of the projector $\tilde{P}_\gamma \in L(X^\ell)$ defined by

$$\tilde{P}_\gamma = \frac{1}{2\pi i} \int_\gamma \begin{pmatrix} L_1(\lambda) \\ \vdots \\ L_\ell(\lambda) \end{pmatrix} L^{-1}(\lambda)\ (I \dots \lambda^{\ell-1} I) d\lambda .$$

Thus row $(T_+^iY_+)_{i=0}^{\ell-1}$ is right invertible. For the sake of completeness we also note that

$$P_\gamma = \operatorname{col}(X_+T_+^i)_{i=0}^{\ell-1} \cdot \operatorname{row}(T_+^iY_+)_{i=0}^{\ell-1} \cdot B$$

and

$$\tilde{P}_\gamma = B \cdot \operatorname{col}(X_+T_+^i)_{i=0}^{\ell-1} \cdot \operatorname{row}(T_+^iY_+)_{i=0}^{\ell-1}$$

where

$$B = \begin{pmatrix} A_1 & A_2 & \cdots & A_\ell \\ A_2 & & \cdot & \\ \vdots & \cdot & & \\ A_\ell & & & 0 \end{pmatrix} .$$

Also, in definition 1.1.1, ℓ can be any integer $\geq$ deg L.

The next proposition shows the connection between γ-spectral triples and the γ-spectral pairs defined in [8]. A right admissible pair (X,T) is called a right partial γ-spectral pair of L if $\sigma(T)$ is contained inside γ,

$\sum_{i=o}^{\ell} A_i X T^i = 0$, and $\mathrm{col}(XT^i)_{i=o}^{k-1}$ is left invertible for some integer $1 \leq k \leq \ell$.

1.1.7 Proposition. Let L be an o.p. and γ a simple closed contour not intersecting $\sigma(L)$. Let (X_+,T_+) be a right admissible pair with base V_+. The following statements are equivalent:

(i) There exists $Y_+ \varepsilon\, L(X,V_+)$ such that (X_+,T_+,Y_+) is a γ-spectral triple;

(ii) Every $u \;\varepsilon\; V_L^+$ has a representation of the form $u(t) = X_+ e^{tT_+} x$ for a unique $x \;\varepsilon\; V_+$, and conversely every u defined this way is in V_L^+;

(iii) (X_+,T_+) is a right partial γ-spectral pair for L and any other right partial γ-spectral pair for L is its restriction.

Proof. (i) $\Rightarrow$ (ii). If $u \;\varepsilon\; V_L^+$ then (1.5) and (1.9) imply

$$u(t) = X_+ e^{tT_+} x$$

where $\quad x = \mathrm{row}(T_+^i Y_+)_{i=o}^{\ell-1} \cdot B \cdot \mathrm{col}(u^{(i)}(0))_{i=o}^{\ell-1}$.

x is unique since $\mathrm{col}(u^{(i)}(0))_{i=o}^{\ell-1} = \mathrm{col}(X_+T_+^i)_{i=o}^{\ell-1}\, x$.

The converse statement in (ii) is clear.

(ii) $\Rightarrow$ (i) . Define the isomorphism $\varphi \;\varepsilon\; L(V_+,V_L^+)$ by $(\varphi x)(t) = X_+ e^{tT_+} x$. Let (X_+',T_+',Y_+') be the natural γ-spectral triple for L. One checks immediately that $X_+'\varphi = X_+$ and $T_+'\varphi = \varphi T_+$. Let $Y_+ = \varphi^{-1}Y_+'$, then (X_+,T_+,Y_+) is a γ-spectral triple for L.

(i) $\Rightarrow$ (iii) First of all, (X_+,T_+) is a right partial γ-spectral pair for L. Now we will assume, without loss of generality, that

$(X_+,T_+) = (X'_+,T'_+)$. If (X,T) is a right partial γ-spectral pair for L with base space V define $\varphi \in \mathcal{L}(V,V_L^+)$ as $(\varphi x)(t) = Xe^{tT}x$. Notice that $X = X'_+\varphi$ and $\varphi T = T'_+\varphi$; thus $R(\varphi)$ is invariant under T'_+. Let M be a left inverse of $\mathrm{col}(XT^i)_{i=o}^{k-1}$. Then $Q\varphi = \varphi$ where $Q = \varphi\, M\, \mathrm{col}(X'_+T'^{\,i}_+)_{i=o}^{k-1}$. Q is a projection of V_L^+ onto $R(\varphi)$ so $R(\varphi)$ is complemented in V_L^+. Hence (X,T) is a restriction of (X'_+,T'_+).

(iii) ⇒ (i). If (X_+,T_+) satisfies (iii) then (X'_+,T'_+) is a restriction of (X_+,T_+). Define the map φ as above but now for (X_+,T_+). Then $R(\varphi) = V_L^+$ so φ is invertible. Let $Y_+ = \varphi^{-1}Y'_+$, then (X_+,T_+,Y_+) is similar to (X'_+,T'_+,Y'_+) and is hence a γ-spectral triple for L.

1.1.8 <u>Definition.</u> Let L and γ be as in 1.1.9 . A right admissible pair (X_+,T_+) is called a right γ-spectral for L if there exists an operator Y_+ such that (X_+,T_+,Y_+) is a γ-spectral triple for L. The definition used in [8] is 1.1.9(iii). Similar definitions hold for left γ-spectral pairs.

1.2 THE RESOLVENT $L^{-1}(\lambda)$

Let L be an o.p. and $\lambda_o \in \sigma(L)$ isolated. In some neighbourhood of λ_o we have the Laurent expansion

$$L^{-1}(\lambda) = \sum_{j=-\infty}^{\infty} B_j(\lambda-\lambda_o)^j \tag{1.11}$$

The principal part of (1.11) is denoted $\Xi_{\lambda_o}(L^{-1})$.

Let (X_o,T_o,Y_o) be a spectral triple at λ_o for L, i.e. a γ-spectral triple for L where γ is a small circle about λ_o not containing any other points of $\sigma(L)$. Then 1.1.4 implies

$$\Xi_{\lambda_o}(L^{-1}) = X_o(\lambda I - T_o)^{-1}Y_o$$

$$= \sum_{j=o}^{\infty} X_o(T_o - \lambda_o I)^j Y_o\ (\lambda - \lambda_o)^{-(j+1)} . \qquad (1.12)$$

L^{-1} has a pole at λ_o if and only if there exists $\nu \geq 0$ such that $(T_o - \lambda_o I)^\nu = 0$, and if so then the order of the pole is equal to the minimal such ν. Indeed, if $(T_o - \lambda_o I)^\nu = 0$ then $B_j = 0$ for $j \geq \nu$. Conversely, if $B_j = 0$ for $j \geq \nu$, i.e. $X_o(T_o - \lambda_o I)^j Y_o = 0$ for $j \geq \nu$, then

$$\mathrm{col}(X_o T_o^i)_{i=o}^{\ell-1}\ (T_o - \lambda_o I)^\nu \cdot \mathrm{row}(T_o^j Y_o)_{j=o}^{\ell-1} = 0$$

and hence $(T_o - \lambda_o I)^\nu = 0$, by 1.1.1(ii) and 1.1.3(ii)'.

Similarly, one can show that B_j $(j = -1,-2,\ldots)$ have finite dimensional range if and only if the base space of (X_o, T_o, Y_o) is finite dimensional.

Now we prove the result mentioned in the introduction which is in Gohberg and Sigal [10].

1.2.1 Theorem. Let L be an o.p. and suppose $\lambda_o \in \sigma(L)$ is isolated. If the operators $B_j (j = -1,-2,\ldots)$ in the Laurent expansion (1.11) of L^{-1} have finite dimensional range then there exists a canonical system

$$x_1^i, \ldots, x_{r_i}^i \qquad (i = 1,\ldots,p)$$

of eigenvectors and generalized eigenvectors of L corresponding to λ_o and a canonical system

$$y_1^i, \ldots, y_{r_i}^i \qquad (i = 1,\ldots,p)$$

of eigenvectors and generalized eigenvectors of L^* corresponding to λ_o such that

$$\Xi_{\lambda_o}(L^{-1}) = \sum_{i=1}^{p} \sum_{j=o}^{r_i-1} (\lambda - \lambda_o)^{-(j+1)} \sum_{k=1}^{r_i-j} x^i_{r_i-j-k+1} \otimes y^i_k .$$

Proof. Let (X_o,T_o,Y_o) be a spectral triple at λ_o for L and recall (1.12). The base space of (X_o,T_o,Y_o) is finite dimensional. Hence we can assume:

(1) $T_o - \lambda_o I$ is in Jordan canonical form $\mathrm{diag}(J_i)^p_{i=1}$, where each J_i is an $r_i \times r_i$ nilpotent Jordan block;

(2) $X_o = \mathrm{row}(X_i)^p_{i=1}$, where $X_i = (x^i_j)^{r_i}_{j=1}$ and $x^i_j \in X$;

(3) $Y_o = \mathrm{col}(Y_i)^p_{i=1}$, where $Y_i = \mathrm{col}(y^i_{r_i-j})^{r_i-1}_{j=o}$ and $y^i_j \in X^*$.

Then

$$\Xi_{\lambda_o}(L^{-1}) = \sum_{i=1}^{p} \sum_{j=o}^{r_i-1} X_i J_i^j Y_i (\lambda - \lambda_o)^{-(j+1)}$$

and an easy computation completes the proof of the theorem.

We now consider the spectrum at ∞ for operator polynomials. This concept is due to Gohberg and Rodman [4,5]. If L is an o.p. of degree $\leq \ell$ and $\sigma(L)$ is compact then a spectral triple at ∞ for L, denoted $(X_\infty,T_\infty,T_\infty)$, is defined to be a spectral triple at 0 for the o.p. $\tilde{L}$, where $L(\lambda) = \lambda^\ell L(\lambda^{-1})$. More accurately, we should say ℓ-spectral triple at ∞, since increasing ℓ will increase the spectrum at ∞. For further explanation of this see [4, §3]. When using spectral triples at ∞ we will of course specify which ℓ is used.

A finite spectral triple (X,T,Y) for L is defined to be a Γ-spectral triple for L, where Γ is a simple closed contour containing $\sigma(L)$.

If L has invertible leading coefficient A_ℓ then there is no spectrum at ∞. The next theorem is a generalization of [2,theorem 13] where the resolvent is expressed in terms of a standard triple for L.

1.2.2 Theorem. Let L be an o.p. of degree $\leq \ell$ with $\sigma(L)$ compact. Let (X,T,Y) be a finite spectral triple for L and $(X_\infty,T_\infty,Y_\infty)$ a spectral triple at ∞ for L. Then,

$$L^{-1}(\lambda) = X(\lambda I - T)^{-1}Y + X_\infty T_\infty^{\ell-1}(I - \lambda T_\infty)^{-1}Y_\infty. \quad (1.13)$$

Proof. L^{-1} has a Laurent expansion in a neighbourhood of ∞ :

$$L^{-1}(\lambda) = \sum_{\infty}^{\infty} C_i \lambda^{-(i+1)} .$$

For $i = 0,1,2,\ldots$ $C_i = \frac{1}{2\pi i} \int_\Gamma \lambda^i L^{-1}(\lambda)d\lambda = XT^iY$.

Similarly, for $i \leq \ell-2$,

$$\begin{aligned} X_\infty T_\infty^{\ell-2-i} Y_\infty &= \frac{1}{2\pi i} \int_\gamma \lambda^{\ell-2-i} \tilde{L}^{-1}(\lambda)d\lambda \\ &= \frac{1}{2\pi i} \int_\gamma \lambda^{\ell-2-i} \lambda^{-\ell} L^{-1}(\lambda^{-1})d\lambda \\ &= \frac{1}{2\pi i} \int_\Gamma w^i L^{-1}(w)dw \\ &= C_i . \end{aligned}$$

Here we have made the substitution $w = \lambda^{-1}$, $dw = -\lambda^{-2}d\lambda$. Notice that the interior of γ is mapped to the exterior of Γ , where γ is a small contour about 0 and Γ is defined to be $\{\lambda^{-1};\ \lambda \in \gamma\}$. Combining the above facts, (1.13) follows easily.

We can use (1.13) to give a simple proof of the following result due, in the finite dimensional case, to [6].

1.2.3 <u>Theorem</u>. Let L be an o.p. of degree $\leq \ell$ with $\sigma(L)$ compact, and let $(X_\infty, T_\infty, T_\infty)$ be a spectral triple at ∞ for L. If $f \in C^\infty(\mathbb{R},X)$ then $L(\frac{d}{dt})u = f$ has the particular solution $u = u_o + v \in C^\infty(\mathbb{R},X)$, where

$$u_o(t) = \int_o^t \int_\Gamma e^{(t-s)\lambda} L^{-1}(\lambda)d\lambda \; f(s)ds$$

and

$$f(t) = \sum_{i=o}^{\infty} X_\infty T_\infty^{\ell-1+i} Y_\infty f^{(i)}(t) .$$

<u>Proof</u>. First of all, $u_o(t) = \int_o^t Xe^{(t-s)T}Y \; f(s)ds$ where (X,T,Y) is a finite spectral triple for L. An easy induction argument shows that

$$\frac{d^i u_o}{dt^i} = \sum_{j=o}^{i-1} XT^j Y \; f^{(i-1-j)}(t) + \int_o^t XT^i \; e^{(t-s)T}Y \; f(s)ds.$$

Hence

$$L(\frac{d}{dt})u_o = \sum_{i=o}^{\ell} A_i \frac{d^i u_o}{dt^i}$$

$$= \sum_{i=1}^{\ell} L_i(X,T)Y \; f^{(i-1)}(t) .$$

Now, by (1.1.2) and (1.13)

$$\sum_{i=1}^{\ell} L_i(X,T)Y \lambda^{i-1} = I - L(\lambda)X_\infty T_\infty^{\ell-1}(I - \lambda T_\infty)^{-1} Y_\infty .$$

Thus, $L(\frac{d}{dt})u_o = f - L(\frac{d}{dt})X_\infty T_\infty^{\ell-1}(I - \frac{d}{dt} T_\infty)^{-1} Y_\infty \; f$ and since the second term on the right hand side of this equation is

$$L(\frac{d}{dt})X_\infty T_\infty^{\ell-1}(I + \frac{d}{dt} T_\infty + \frac{d^2}{dt^2} T_\infty^2 + \dots)\ Y_\infty\ f,$$

which is $L(\frac{d}{dt})v$, the proof of the theorem is complete.

1.3 DIVISORS OF OPERATOR POLYNOMIALS

Let L be an o.p. with γ-spectral pair (X_+,T_+). In this section the connection between invariant subspaces of T_+ and right divisors of L is investigated. The next theorem and its corollary is due to Gohberg, Lerer, and Rodman [8] (except for (iii) of the theorem), but different proofs are given here.

1.3.1 Theorem. Let L and L_1 be o.p.'s and γ a simple closed contour such that $\sigma(L) \cap \gamma = \sigma(L_1) \cap \gamma = \emptyset$. Let (X_+,T_+) be a right γ-spectral pair for L. If LL_1^{-1} is holomorphic inside γ then there exists a unique invariant subspace $\mathcal{L}$ of T_+ such that $(X_+|_{\mathcal{L}}, T_+|_{\mathcal{L}})$ is a right γ-spectral pair of L_1. Moreover, the following statements are equivalent:

(i) LL_1^{-1} is holomorphic inside γ ;

(ii) any right γ-spectral pair for L_1 is a restriction of (X_+,T_+);

(iii) $V_{L_1}^+ \subseteq V_L^+$.

Proof. If $\mathcal{L}$ is an invariant subspace of T_+ such that $(X_+|_{\mathcal{L}}, T_+|_{\mathcal{L}})$ is a γ-spectral pair for L_1 then 1.1.7 implies

$$V_{L_1}^+ = \{u; u(t) = X_+|_{\mathcal{L}}\, e^{tT_+|_{\mathcal{L}}}x,\ x \in \mathcal{L}\}.$$

Hence $V^+_{L_1} = \varphi(L)$, where $\varphi \in L(V_+, V^+_L)$ is defined as in the proof of 1.1.3 : $(\varphi x)(t) = X_+ e^{tT_+} x$. This proves uniqueness of L in the statement of the theorem. Existence is a consequence of the results proved below. Without loss of generality (X_+, T_+) is the natural γ-spectral pair for L, since any two γ-spectral pairs for L are similar.

(i) ⇒ (iii). This is clear since $L_1^{-1} = L^{-1}(LL_1^{-1})$.

(iii) ⇒ (i) If $V^+_{L_1} \subseteq V^+_L$ then $L(\frac{d}{dt})(\int_\gamma e^{t\lambda} L_1^{-1}(\lambda)d\lambda) = 0$.

Differentiation under the integral sign yields

$$\int_\gamma e^{t\lambda} L(\lambda)L_1^{-1}(\lambda)d\lambda = 0,$$

which is equivalent to (i).

(ii) ⇒ (iii). This is clear from the first statement of the proof.

(iii) ⇒ (ii). If $V^+_{L_1} \subseteq V^+_L$ then $(X_+|V^+_{L_1}, T_+|V^+_{L_1})$ is the natural γ-spectral pair of L_1, and is a restriction of $(X_+, T_+)^1$ (Notice that $V^+_{L_1}$ is complemented in V^+_L since $\text{col}(X_+T_+^i)_{i=o}^{\ell-1}$ embeds $V^+_{L_1}$ and V^+_L into complemented subspaces of X^ℓ.). Since any two γ-spectral pairs for L_1 are similar, (ii) is proved.

1.3.2 <u>Corollary.</u> Let L, L_1, and γ be as in the theorem. LL_1^{-1} is holomorphic and invertible inside γ if and only if the right γ-spectral pairs of L and L_1 coincide, or if and only if $V^+_L = V^+_{L_1}$.

1.3.3 <u>Remark</u>. The above corollary applies in the case L_1 is a γ-spectral right divisor of L.

If LL_1^{-1} is holomorphic inside γ then the subspace L in 1.3.1 will be called the invariant subspace of T_+ corresponding to L_1. Notice that L is a complemented subspace of the base space of (X_+, T_+).

1.3.4 Remark. There are two other ways to characterize $\mathcal{L}$:

1) $\quad (\mathrm{col}\,(X_+T_+^i)_{i=o}^{\ell-1}|_{\mathcal{L}}) = \{\int_\gamma \lambda^i L_1^{-1}(\lambda)f(\lambda)d\lambda \;;\; f \text{ a polynomial}\}.$

Since $\mathrm{col}(X_+T_+^i)_{i=o}^{\ell-1}$ is left invertible, this uniquely determines $\mathcal{L}$;

2) Define $\varphi \in \mathcal{L}(V_+,V_L^+)$ as usual, then $\mathcal{L} = \varphi^{-1}(V_{L_1^+})$.

If LL_1^{-1} is holomorphic inside γ and $\sigma(L_1)$ is contained inside then LL_1^{-1} is entire and in the finite dimensional case it is a polynomial. This is no longer true in infinite dimensions (an entire function is a polynomial if and only if it has a pole at ∞). In [8], the following example is given. If A is an operator such that $\sigma(A) = \{0\}$ and $A^n \neq 0$ $(n = 1,2,...)$ let $L(\lambda) = I$ and $L_1(\lambda) = I + \lambda A$, then $L(\lambda)\, L_1^{-1}(\lambda) = (I + \lambda A)^{-1}$ is entire but is not a polynomial. The next theorem gives a necessary and sufficient condition for the quotient to be a polynomial.

1.3.5 Theorem. Let L be an o.p. of degree $\leq \ell$ and L_1 an o.p. of degree $\leq k$. Suppose $\sigma(L_1)$ is compact and $M = LL_1^{-1}$ is entire. Let (X_1,T_1,Y_1) be a finite spectral triple for L_1 and $(X_\infty,T_\infty,Y_\infty)$ a spectral triple at for L_1. Then an explicit formula for M can be given in terms of the coefficients of L and the spectral triples for L_1:

$$M(\lambda) = \sum_{j=o}^{\ell-1} \{ \sum_{i=j+1}^{\ell} A_i X_1 T_1^{i-(j+1)} Y_1 + \sum_{i=o}^{j} A_i X_\infty T_\infty^{j+k-i-1} Y_\infty \}\lambda^j$$

$$+ \sum_{j=\ell}^{\infty} \{ \sum_{i=o}^{\ell} A_i X_\infty T_\infty^{j+k-i-1} Y_\infty \}\lambda^j \qquad (1.14)$$

M is polynomial if and only if there exists a non-negative integer μ such that

$$\sum_{i=o}^{\ell} A_{\ell-i} X_\infty T_\infty^{i+\mu} = 0 \quad . \tag{1.15}$$

If such a μ exists then the degree of M is $\leq \max(\mu + \ell-k, \ell-1)$.

Proof. 1.2.3 implies that

$$L_1^{-1}(\lambda) = X_1(\lambda I - T_1)^{-1} Y_1 + X_\infty T_\infty^{k-1}(I - \lambda T_\infty)^{-1} Y_\infty \tag{1.16}$$

The fact that M is entire implies

$$\sum_{i=o}^{\ell} A_i X_1 T_1^i = 0 \quad .$$

Indeed, let γ be a simple closed contour containing $\sigma(L_1)$ and use 1.1.7:

$$\sum_{i=o}^{\ell} A_i \int_\gamma \lambda^i L_1^{-1}(\lambda) f(\lambda) d\lambda = \int_\gamma L(\lambda) L_1^{-1}(\lambda) f(\lambda) d\lambda$$

$$= 0 \ .$$

Thus we can apply 1.1.2 and (1.16) to obtain:

$$M(\lambda) = \sum_{j=o}^{\ell-1} L_{j+1}(\lambda) X_1 T_1^j Y_1 + L(\lambda) X_\infty T_\infty^{k-1} (I - \lambda T_\infty)^{-1} Y_\infty \tag{1.17}$$

Notice that, since $\sigma(T_\infty) = \{0\}$, $I - \lambda T_\infty$ is entire,

$$(I - \lambda T_\infty)^{-1} = I + \lambda T_\infty + \lambda^2 T_\infty^2 + \dots , \tag{1.18}$$

and $\lim_{n\to\infty} \|T_\infty^n\|^{1/n} = 0$. This shows that there is no problem of convergence in (1.14). Using (1.18) one easily sees that the second term in (1.17) can be rewritten as

$$\sum_{j=o}^{\ell-1} \{ \sum_{i=o}^{j} A_i X_\infty T_\infty^{j+k-1-i} Y_\infty \} \lambda^j + \sum_{j=\ell}^{\infty} \{ \sum_{i=o}^{\ell} A_i X_\infty T_\infty^{j+k-1-i} Y_\infty \} \lambda^j \ .$$

Collecting the various terms together one obtains (1.14). Finally, rewrite the last term in (1.14) in the form

$$\sum_{j=\ell}^{\infty} \left(\sum_{i=o}^{\ell} A_{\ell-i} X_\infty T_\infty^i \right) T_\infty^{j+k-\ell-1} Y_\infty \quad \lambda^j \quad .$$

M is a polynomial if and only if this series terminates, and since $(Y_\infty \ \ldots \ T_\infty^{\ell-1} Y_\infty)$ is surjective this is equivalent to (1.15). The last statement of the theorem is now easily verified.

The next corollary gives a formula for the quotient and remainder in a generalized Euclidean algorithm. Actually, it is difficult to attach any significance to this result in general, but if L_1 has invertible leading coefficient then we obtain theorem 6 of [1], which was a crucial part of that paper. An important generalization of this case is when L_1 is reducible in the sense of Chapter 2. With respect to the decomposition determined by L_1 we have $R(\lambda) = (r_{ij}(\lambda))_{i,j=o}^{\ell}$ and deg $r_{ij} < j$. Thus 1.3.6 gives a formula for the (unique) remainder satisfying this condition, and also a formulae for the quotient.

1.3.6 Corollary. Let L be an o.p. of degree $\leq \ell$ and L_1 an o.p. of degree $\leq k$, and suppose $\sigma(L_1)$ is compact. Let (X_1, T_1, Y_1) be a finite spectral triple for L_1 and $(X_\infty, T_\infty, Y_\infty)$ a spectral triple at ∞ for L_1. Write $L_1(\lambda) = \sum_{j=o}^{k} B_j \lambda^j$ and $L(\lambda) = \sum_{j=o}^{\ell} A_j \lambda^j$. Then

$$L(\lambda) = M(\lambda) L_1(\lambda) + R(\lambda) \quad ,$$

where

$$R(\lambda) = \sum_{j=o}^{k-1} R_j \lambda^j \quad \text{and}$$

$$R_j = \left(\sum_{i=o}^{\ell} A_i X_1 T_1^i \right) \left(\sum_{m=o}^{\ell-j} T_1^m Y_1 B_{j+m} \right) ;$$

M is defined as in (1.14) except with A_i replaced by

$$A_i' = \begin{cases} A_i - R_i & i = 0,\dots,k-1 \\ A_i & i \geq k \end{cases}$$

<u>Proof</u>. We claim that $LL_1^{-1} - RL_1^{-1}$ is entire. Indeed, by 1.1.2

$$R(\lambda)L_1^{-1}(\lambda) = L(X_1,T_1)\left(\sum_{j=o}^{k-1} T_1^j Y_1 L_{1,j+1}(\lambda)\right)L_1^{-1}(\lambda)$$

$$= L(X_1,T_1)(\lambda I - T_1)^{-1}Y_1$$

where $L(X_1,T_1) = \sum_{i=o}^{\ell} A_i X_1 T_1^i$. Hence if γ is a simple closed contour containing $\sigma(L_1)$ then for all $j = 0,1,2,\dots$

$$\int_\gamma \lambda^j R(\lambda)L_1^{-1}(\lambda)d\lambda = L(X_1,T_1)T_1^j Y_1$$

$$= \int_\gamma \lambda^j L(\lambda)L_1^{-1}(\lambda)d\lambda ,$$

and the claim follows. Apply 1.3.5 to $M = L'L_1^{-1}$ where $L' = L-R$ and the corollary is proved.

Suppose $L = L_2L_1$. The next theorem shows how a left γ-spectral pair for L_2 can be constructed from a left γ-spectral pair (T_+,Y_+) for L and the invariant subspace of T_+ corresponding to L_1.

1.3.7 <u>Theorem</u> Let L, L_1, and L_2 be o.p.'s such that $L = L_2L_1$, and γ a simple closed contour such that $\sigma(L) \cap \gamma = \sigma(L_1) \cap \gamma = \emptyset$.

Let (T_+, Y_+) be a left γ-spectral pair of L, and $\mathcal{L}$ the invariant subspace of T_+ corresponding to L_1. If $\overline{T}_+ \in \mathcal{L}(V_+/\mathcal{L})$ is the operator induced from T_+ and $\overline{Y}_+ = \pi Y_+$, where $\pi \in \mathcal{L}(V_+, V_+/\mathcal{L})$ is the natural projection, then $(\overline{T}_+, \overline{Y}_+)$ is a left γ-spectral pair for L_2.

Proof. Without loss of generality, (T_+, Y_+) is the natural left γ-spectral pair for L with base space V_L^+, and then $\mathcal{L} = V_{L_1}^+$. Also, let (T_2^+, Y_2^+) be the natural left γ-spectral pair for L_2 with base space $V_{L_2}^+$. In particular, recall that

$$Y_+(t) = \frac{1}{2\pi i} \int_\gamma e^{t\lambda} L^{-1}(\lambda) d\lambda \quad \text{and} \quad Y_2^+(t) = \frac{1}{2\pi i} \int_\gamma e^{t\lambda} L_2^{-1}(\lambda) d\lambda \quad .$$

Define $\psi \in \mathcal{L}(V_L^+, V_{L_2}^+)$ as $\psi u = L_1(\frac{d}{dt})u$. Then ψ is surjective, for if $v \in V_{L_2}^+$ there is a polynomial $f(\lambda)$ such that $v(t) = \int_\gamma e^{t\lambda} L_2^{-1}(\lambda) f(\lambda) d\lambda$ and thus $\psi u = v$, where $u(t) = \int_\gamma e^{t\lambda} L^{-1}(\lambda) f(\lambda) d\lambda$. Moreover, $N(\psi) = V_{L_1}^+$. Indeed, it is clear that $V_{L_1}^+ \subseteq N(\psi)$ and if $u \in N(\psi)$ then $u(t) = \int_\gamma e^{t\lambda} L^{-1}(\lambda) f(\lambda) d\lambda$ for some polynomial $f(\lambda)$. Since $L_1(\frac{d}{dt})u = 0$, i.e. $\int_\gamma e^{t\lambda} L_2^{-1}(\lambda) f(\lambda) d\lambda = 0$, it follows that $g(\lambda) = L_2^{-1}(\lambda) f(\lambda)$ is holomorphic inside γ and $u(t) = \int_\gamma e^{t\lambda} L_1^{-1}(\lambda) g(\lambda) d\lambda$, which implies $u \in V_{L_1}^+$. Thus ψ induces an isomorphism $\overline{\psi}$ between $V_L^+/V_{L_1}^+$ and $V_{L_2}^+$, by the closed graph theorem.

Now, notice that $\psi T_+ = T_2^+ \psi$. Hence $T_2^+ \overline{\psi} = \overline{\psi}\, \overline{T}_+$. Also, $\psi Y_+ = Y_2^+$ and so $\overline{\psi}\, \overline{Y}_+ = Y_2^+$. This shows that the admissible pair $(\overline{T}_+, \overline{Y}_+)$ is similar to (T_2^+, Y_2^+) and is hence a γ-spectral pair for L_2 .

1.3.8 Corollary. (cf. [1,§6]) Let the hypotheses be as in the theorem. Then (Y_+^*, T_+^*) is a right γ-spectral pair for L^*, L_2^* is a right divisor of L^*, and the invariant subspace of T_+^* corresponding to L_2^* is $\mathcal{L}^\perp$.

Proof. The first claim has already been proved in 1.1.6, and the second is trivial. Recall that $T_+(\mathcal{L}) \subseteq \mathcal{L}$ implies $T_+^*(\mathcal{L}^\perp) \subseteq \mathcal{L}^\perp$. We must show that $(Y_+^*|\mathcal{L}^\perp, T_+^*|\mathcal{L}^\perp)$ is a γ-spectral pair for L_2^*. The theorem states that $(\overline{T}_+, \overline{Y}_+)$ is a left γ-spectral pair for L_2 and hence $(\overline{Y}_+^*, \overline{T}_+^*)$ is a right γ-spectral pair for L_2^*. But $\pi^* \in \mathcal{L}((V_+|\mathcal{L})^*, V_+^*)$ defines an isomorphism between $(V_+/\mathcal{L})^*$ and $\mathcal{L}^\perp$, and $Y_+^* \pi^* = \overline{Y}_+^*$ and $\pi^* T_+^* = T_+^* \pi^*$. It follows that $(Y_+^*|\mathcal{L}^\perp, T_+^*|\mathcal{L}^\perp)$ is similar to $(\overline{Y}_+^*, \overline{T}_+^*)$. This completes the proof of the corollary.

1.3.9 Remark. Suppose $L = L_2 L_1$, and (X_+, T_+) and (X_2^+, T_2^+) are right γ-spectral pairs for L and L_2 with base spaces V_+ and V_2^+, respectively. The proof of 1.3.7 shows that there is a natural surjection $\psi \in \mathcal{L}(V_+, V_2^+)$ with kernel $\mathcal{L}$, the invariant subspace of T_+ corresponding to L_1, uniquely determined by

$$X_2^+(T_2^+)^j \psi = \frac{1}{2\pi i} \int_\gamma \lambda^j L_1(\lambda) X_+(\lambda I - T_+)^{-1}\, d\lambda \qquad (j=0,\ldots,\ell_2-1).$$

Indeed, when (X_+, T_+) and (X_2^+, T_2^+) are the natural γ-spectral pairs of L and L_2, respectively, this is just the ψ defined in 1.3.7. In the general case one can argue by similarity. We also note that ψ has a natural section $\chi \in \mathcal{L}(V_2^+, V_+)$ uniquely determined by

$$X_+ T_+^j \chi = \frac{1}{2\pi i} \int_\gamma \lambda^j L_1^{-1}(\lambda) X_2^+(\lambda I - T_2^+)^{-1} d\lambda \qquad (j = 0, \ldots, \ell-1).$$

Again, in the case of natural spectral pairs, for $v \in V_{L_2}^+$

$$(\chi v)(t) = \frac{1}{2\pi i} \int_\gamma e^{t\lambda} L_1^{-1}(\lambda) X_2^+(\lambda I - T_2^+)^{-1} v \; d\lambda .$$

Then $(\psi X v)(t) = L_1(\frac{d}{dt})Xv = \frac{1}{2\pi i} \int_\gamma e^{t\lambda} X_2^+(\lambda I - T_2^+)^{-1} v \; d\lambda$

$$= v(t) \quad .$$

Hence $\psi \chi v = v$ and X is a section of ψ.

This remark is used in the proof of the next theorem, which gives a formula for a γ-spectral triple for $L = L_2 L_1$ in terms of γ-spectral triples for L_1 and L_2. This is a generalization of [2,theorem 17], which is for standard triples of monic operator polynomials.

1.3.10 Theorem. Suppose $L = L_2 L_1$ and $\sigma(L_i) \cap \gamma = \emptyset$ $(i = 1,2)$. Let (X_i^+, T_i^+, Y_i^+) be a γ-spectral triple for L_i with base space V_i^+ $(i = 1,2)$. Let $V_+ = V_1^+ \times V_2^+$ and define $X_+ \in \mathcal{L}(V_+, X)$, $T_+ \in \mathcal{L}(V_+)$, and $Y_+ \in \mathcal{L}(X, V_+)$ as

$$X_+ = (X_1^+ A)$$

$$T_+ = \begin{pmatrix} T_1^+ & Y_1^+ X_2^+ \\ 0 & T_2^+ \end{pmatrix} ,$$

$$Y_+ = \begin{pmatrix} B \\ Y_2^+ \end{pmatrix} ,$$

where
$$A = \frac{1}{2\pi i} \int_\gamma L_1^{-1}(\lambda) X_2^+ (\lambda I - T_2^+)^{-1} \, d\lambda ,$$

$$B = \frac{1}{2\pi i} \int_\gamma (\lambda I - T_1^+)^{-1} Y_1^+ \{L_2^{-1}(\lambda) - X_2^+(\lambda I - T_2^+)^{-1} Y_2^+ \} \, d\lambda .$$

Then (X_+, T_+, Y_+) is a γ-spectral triple for L.

Proof. Let (X_+', T_+', Y_+') be a γ-spectral triple for L. Without loss of generality $(X_1^+, T_1^+) = (X_+'|_{\mathcal{L}}, T_+'|_{\mathcal{L}})$ and $V_1^+ = \mathcal{L}$ where $\mathcal{L}$ is the invariant subspace of T_+' corresponding to L_1. Let $\psi \in \mathcal{L}(V_+', V_2^+)$ be defined as in the previous remark. Recall that $N(\psi) = \mathcal{L}$ and there is a natural section $\chi \varepsilon\ L(V_2^+,V_+)$. Thus $V_+' = L \oplus \chi(V_2^+)$, and there is the isomorphism $\varphi \varepsilon\ L(V_1^+ \times V_2^+, V_+)$ defined by $\varphi\binom{u}{v} = u + \chi v$. We claim that φ defines a similarity between (X_+,T_+,Y_+) and (X_+',T_+',Y_+'). Now,

$$\varphi T_+\binom{u}{v} = \varphi\begin{pmatrix} T_1^+ u + Y_1^+ X_2^+ v \\ T_2^+ v \end{pmatrix}$$

$$= T_+' u + Y_1^+ X_2^+ v + \chi T_2^+ v$$

and $Y_1^+X_2^+ + \chi T_2^+ = T_+'\,\psi$, since for $j = 0,1,\ldots$

$$X_+'T_+'^j Y_1^+ X_2^+ + X_+'T_+'^j\ \chi T_2^+ = \frac{1}{2\pi i}\int_\gamma \lambda^j L_1^{-1}(\lambda) X_2^+ \{I + T_2^+(\lambda I - T_2^+)^{-1}\}\, d\lambda$$

$$= \frac{1}{2\pi i}\int_\gamma \lambda^{j+1} L_1^{-1}(\lambda) X_2^+(\lambda I - T_2^+)^{-1} d\lambda$$

$$= X_+'T_+'^j T_+'\chi \quad .$$

Hence

$$\varphi T_+\binom{u}{v} = T_+'(u + \chi v)$$

$$= T_+'\ \varphi\binom{u}{v}$$

so $\varphi T_+ = T_+'\varphi$. Also

$$(X_+'\varphi)\binom{u}{v} = X_+'(u + \chi v)$$

$$= X_1^+ u + Av$$

$$= X_+\binom{u}{v}$$

so $X_+'\varphi = X_+$. Finally,

$$\varphi^{-1}Y'_+ = \begin{pmatrix} Y'_+ - \chi Y_2^+ \\ Y_2^+ \end{pmatrix}$$

and we claim that $Y'_+ - \chi Y_2^+ = B$. Indeed, for $j = 0,1,\ldots$

$$X'_+Y'^{\,j}_+Y'_+ - X'_+Y'^{\,j}_+\chi Y_2^+ = \frac{1}{2\pi i}\int_\gamma \lambda^j L_1^{-1}(\lambda)\{\, L_2^{-1}(\lambda) - X_2^+(\lambda I - T_2^+)^{-1}Y_2^+\}\, d\lambda$$

$$= \frac{1}{2\pi i}\int_\gamma \lambda^j X_1^+(\lambda I - T_1^+)^{-1}Y_1^+\{L_2^{-1}(\lambda) - X_2^+(\lambda I - T_2^+)^{-1}Y_2^+\}d\lambda$$

$$= X'_+T'^{\,j}_+B\ .$$

Hence $\varphi^{-1}Y'_+ = Y_+$ and the proof of the theorem is complete.

1.3.11 Remark. If $\sigma(L_i)$ is contained inside γ (so finite spectral triples are considered) then one can give other formulas for A and B. Let $(X_{i\infty}, T_{i\infty}, Y_{i\infty})$ be a spectral triple at ∞ for L_i for some chosen $\ell_i \geq \deg L_i$ $(i = 1,2)$.

Then
$$A = \sum_{j=o}^{\infty} (X_{1\infty}T_{1\infty}^{\ell_1-1+j}Y_{1\infty})X_2^+(T_2^+)^j$$

$$B = \sum_{j=o}^{\infty} (T_1^+)^j Y_1^+(X_{2\infty}T_{2\infty}^{\ell_2-1+j}Y_{2\infty})\ .$$

Indeed, enlarge γ to be a circle with large radius and then

$$A = \frac{1}{2\pi i}\int_\gamma L_1^{-1}(\lambda)X_2^+(\lambda I - T_2^+)^{-1}d\lambda$$

$$= \frac{1}{2\pi i}\sum_{j=o}^{\infty}\left(\int_\gamma L_1^{-1}(\lambda)\ \lambda^{-(j+1)}d\lambda\right)X_2^+(T_2^+)^j$$

which gives the required result. For B one uses

$$L_2^{-1}(\lambda) - X_2^{+}(\lambda I - T_2^{+})^{-1}Y_2^{+} = X_{2\infty}T_{2\infty}^{\ell_2 - 1}(I - \lambda T_{2\infty})^{-1}Y_{2\infty}.$$

1.4 TRANSFORMATIONS OF OPERATOR POLYNOMIALS

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C})$ let φ be the fractional linear transformation $\frac{a\lambda + b}{c\lambda + d}$. Recall that φ is bijective with inverse $\frac{d\lambda - b}{-c\lambda + a}$. If L is an o.p. of degree $\leq \ell$ define the transformation of L under φ as

$$L_{\varphi}(\lambda) = (-c\lambda + a)^{\ell} L(\varphi^{-1}(\lambda)),$$

which is also an o.p. of degree $\leq \ell$.

1.4.1. <u>Theorem</u>. Let L be an o.p. of degree $\leq \ell$ and γ a simple closed contour such that $\sigma(L) \cap \gamma = \phi$. Let φ be a fractional linear transformation, suppose $\frac{-d}{c}$ is in the exterior of γ, and set $\tilde{\gamma} = \varphi(\gamma)$. If (X_+, T_+, Y_+) is a γ-spectral triple for L then $(\tilde{X}_+, \tilde{T}_+, \tilde{Y}_+)$ is a $\tilde{\gamma}$-spectral triple for L_{φ}, where

$$\tilde{X}_+ = X_+,$$

$$\tilde{T}_+ = \varphi(T_+),$$

and

$$\tilde{Y}_+ = (\det A)^{-(\ell-1)}(dI + cT_+)^{\ell-2}Y_+ .$$

<u>Proof.</u> Notice that φ maps the interior of γ to the interior of $\tilde{\gamma}$ and by the spectral mapping theorem $\sigma(\tilde{T}_+) = \varphi(\sigma(\tilde{T}_+))$ is contained inside $\tilde{\gamma}$. Now,

$$L_{\varphi}(\lambda) = \sum_{i=0}^{\ell} A_i (d\lambda - b)^i (-c\lambda + a)^{\ell - i}$$

and, by lemma 1.4.4, $(d\lambda - b)^i(-c\lambda + a)^{\ell-i} = \sum_{j=0}^{\ell} u_{ij}\lambda^j$, where $(u_{ij})_{i,j=o}^{\ell} = (\det A)^{\ell} u_{A^{-1}}$ is invertible. Then $L_\varphi(\lambda) = \sum_{i=o}^{\ell} \tilde{A}_i \lambda^i$ where $(\tilde{A}_o \ldots \tilde{A}_\ell) = (A_o \ldots A_\ell)U$ and $U = (u_{ij}\ I)_{i,j=o}^{\ell} \in L(X^\ell)$ is invertible. Also, one checks easily that

$$(d\tilde{T}_+ - bI)^i(-c\tilde{T}_+ + aI)^{\ell-i} = T_+^i(cT_+ + dI)^{-\ell}\ (\det A)^\ell$$

for $i = 0,\ldots,\ell$ and thus

$$U\ \mathrm{col}(\tilde{X}_+\tilde{T}_+^i)_{i=o}^{\ell} = \mathrm{col}(X_+T_+^i)_{i=o}^{\ell}\ (cT_+ + dI)^{-\ell}\ (\det A)^\ell \quad .$$

Hence $\mathrm{col}(\tilde{X}_+\tilde{T}_+^i)_{i=o}^{\ell}$ is injective, and

$$\begin{aligned} \sum_{i=o}^{\ell} \tilde{A}_i\tilde{X}_+\tilde{T}_+^i &= (\tilde{A}_o \ldots \tilde{A}_\ell)\ \mathrm{col}(\tilde{X}_+\tilde{T}_+^i)_{i=o}^{\ell} \\ &= (A_o \ldots A_\ell)\ \mathrm{col}(X_+T_+^i)_{i=o}^{\ell}\ (cT_+ + dI)^{-\ell}(\det A)^\ell \\ &= 0 . \end{aligned}$$

It remains to show (c) of definition 1.1.1:

$$\begin{aligned} \frac{1}{2\pi i}\int_{\tilde{\gamma}} \lambda^i\ L_\varphi^{-1}(\lambda)d\lambda &= \frac{1}{2\pi i}\int_{\tilde{\gamma}} \lambda^i(-c\lambda + a)^{-\ell}\ L^{-1}(\varphi^{-1}(\lambda))d\lambda \\ &= \frac{1}{2\pi i}\int_{\gamma} (\varphi(w))^i(\det A\ \ (cw+d)^{-1})^{-\ell}\ L^{-1}(w)\ \frac{\det A}{(cw+d)^2}\ dw \\ &= \tilde{X}_+\tilde{T}_+^i\tilde{Y}_+ . \end{aligned}$$

In the second equality the substitution $\lambda = \varphi(w)$ was made.

1.4.2 Corollary. Let L be an o.p. of degree $\leq \ell$ such that $\sigma(L)$ is compact and let (X,T,Y) and $(X_\infty,T_\infty,Y_\infty)$ be finite and infinite spectral triples for L, respectively. Then, $(X,T-aI,Y)$ and $(X_\infty,T_\infty(I-aT_\infty)^{-1},(I-aT_\infty)^{\ell-2}Y_\infty)$ are finite and infinite spectral triples for $L_a(\lambda) = L(\lambda+a)$, respectively.

Proof. For the case of finite spectral triples take $\varphi(\lambda) = \lambda-a$, and let γ be a large contour containing $\sigma(L)$. For the case of infinite spectral triples note that $\tilde{L}(\lambda) = \lambda^\ell L(\lambda^{-1}) = \sum_{i=0}^{\ell} \tilde{A}_i\lambda^i$ where $\tilde{A}_i = A_{\ell-i}$. Then,

$$\begin{aligned}\tilde{L}_a(\lambda) &= \lambda^\ell L_a(\lambda^{-1}) \\ &= \sum_{i=0}^{\ell} A_i\lambda^i(1+a\lambda)^{\ell-i} .\end{aligned}$$

Let γ be a small contour about 0 not containing any other points of $\sigma(\tilde{L})$, take $\varphi(\lambda) = \dfrac{\lambda}{1-a\lambda}$, and apply the theorem to $\tilde{L}$ and φ.

1.4.3 Corollary. Let L be an o.p. of degree $\leq \ell$ such that $\sigma(L)$ is compact and $0 \notin \sigma(L)$, and let (X,T,Y) and $(X_\infty,T_\infty,Y_\infty)$ be finite and infinite spectral triples for L, respectively. Define

$$\tilde{X} = (X\ X_\infty),$$

$$\tilde{T} = \begin{pmatrix} T^{-1} & 0 \\ 0 & T_\infty \end{pmatrix},$$

and

$$\tilde{Y} = \begin{pmatrix} -T^{\ell-2}Y \\ Y_\infty \end{pmatrix},$$

then, $(\tilde{X},\tilde{T},\tilde{Y})$ is a finite spectral triple for $\tilde{L}$.

Proof. Choose a simple closed contour γ containing $\sigma(L)$ but not 0, and let λ_o be a small circle about 0 not containing any points of $\sigma(L)\setminus 0$.

Then $\Gamma = \tilde{\gamma} + \gamma_o$ contains $\sigma(\tilde{L})$ in its interior, where $\tilde{\gamma} = \{\lambda^{-1};\lambda \in \gamma\}$. The theorem implies that $(X, T^{-1}, T^{\ell-2}Y)$ is a $\tilde{\gamma}$-spectral triple for $\tilde{L}$ (take $\varphi(\lambda) = \lambda^{-1}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$). Also, by definition, $(T_\infty, T_\infty, Y_\infty)$ is a γ_o-spectral triple for $\tilde{L}$. Thus 1.1.5 implies that $(\tilde{X}, \tilde{T}, \tilde{Y})$ is a $\tilde{\gamma}$-spectral triple for $\tilde{L}$.

1.4.4 Lemma. Define the map $u: M_2(\mathbb{C}) \to M_\ell(\mathbb{C})$ as follows : for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $u(A) = u_A$ is defined via

$$\mathrm{col}\{(a\lambda + b)^i (c\lambda + d)^{\ell - i}\}_{i=o}^{\ell} = u_A\, \mathrm{col}(\lambda^j)_{i=0}^{\ell}.$$

Then u is a homomorphism, $u_{\alpha A} = \alpha^\ell u_A$, and $\det u_A = (\det A)^{\ell(\ell+1)/2}$.

Proof. The defining property for u_A can be written

$$(c\lambda + d)^\ell\, \mathrm{col}\{(\varphi_A(\lambda))^i\}_{i=o}^{\ell} = u_A \mathrm{col}(\lambda^i)_{i=o}^{\ell}$$

where $\varphi_A(\lambda) = \dfrac{a\lambda + b}{c\lambda + d}$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, and $AB = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}$.

Then

$$\begin{aligned} u_{AB}\mathrm{col}(\lambda^i)_{i=o}^{\ell} &= (\tilde{c}\lambda + \tilde{d})^\ell\, \mathrm{col}\{(\varphi_{AB}(\lambda))^i\}_{i=o}^{\ell} \\ &= (\tilde{c}\lambda + \tilde{d})\quad \mathrm{col}\{(\varphi_A(\varphi_B(\lambda))^i\}_{i=o}^{\ell} \\ &= (\tilde{c}\lambda + \tilde{d})^\ell\, (c\varphi_B(\lambda) + d)^{-\ell}\, u_A (c'\lambda + d')^{-\ell}\, u_B \mathrm{col}(\lambda^i)_{i=o}^{\ell} \\ &= u_A u_B\, \mathrm{col}(\lambda^i)_{i=o}^{\ell}. \end{aligned}$$

Hence $u_{AB} = u_A u_B$, and also $u_I = I$, so u is a homomorphism. Thus to prove $\det u_A = (\det A)^{\ell(\ell+1)/2}$ it suffices to consider the five cases

$$A = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$$

since any $A \in M_2(\mathbb{C})$ can be decomposed as a finite product of such elementary matrices. The proof is then completely elementary and the details won't be provided here. Finally, $u_{\alpha A} = \alpha^{\ell} u_A$ for $\alpha \in \mathbb{C}$ is also obvious.

1.5 DEPENDENCE ON PARAMETERS

Let M be a C^∞ manifold and let $L_m(\lambda) = \sum_{i=0}^{\ell} A_i(m)\lambda^i$ be an o.p. with coefficients depending smoothly on $m \in M$. This means that $A_i : M \to L(X)$ is smooth for $i = 0,\dots,\ell$.

If V_1 and V_2 are vector bundles over M, denote by $L(V_1,V_2)$ the usual space of homomorphisms of vector bundles: $f \in L(V_1,V_2)$ if f is a smooth fibre-preserving map from V_1 to V_2 and is linear on each fibre.

1.5.1. <u>Theorem.</u> Let M and L_m be as above, and let $\{\gamma_m\}_{m \in M}$ be a family of simple closed contours such that $\sigma(L_m) \cap \gamma_m = \emptyset$ and depending smoothly on $m \in M$ in the following sense: for each $m_o \in M$ there exists a neighbourhood U of m_o such that for all $m \in U$ the portion of $\sigma(L_m)$ inside γ_m is contained inside γ_{m_o}. Then there exists a vector bundle V_+ over M and $X_+ \in L(V_+,X)$, $T_+ \in L(V_+)$, $Y_+ \in L(X,V_+)$ such that, for all $m \in M$, $((X_+)_m, (T_+)_m, (Y_+)_m)$ is a γ_m-spectral triple for L_m.

<u>Proof</u>. Define the projections $P_m = P_{\gamma_m} \in L(X^{\ell})$ as in the proof of 1.1.3:

$$P_m = \frac{1}{2\pi i} \int_{\gamma_m} \begin{pmatrix} I \\ \vdots \\ \lambda^{\ell-1} I \end{pmatrix} L_m^{-1}(\lambda)(L_{m,1}(\lambda) \cdots L_{m,\ell}(\lambda))d\lambda.$$

Choose $m_o \in M$. The hypotheses on γ_m imply that for m in a neighbourhood of m_o we can replace the contour integral over γ_m in the definition of P_m by that over γ_{m_o}. Thus it is clear that P_m depends smoothly on $m \in M$. This defines a vector bundle $R(P) = \{R(P_m)\}_{m \in M}$ over M. Now, let $((X_+)_m, (T_+)_m, (Y_+)_m)$ be the natural γ_m-spectral triple for L_m with base space $V^+_{L_m}$ as in the proof of 1.1.3. Also, let $Q = \text{col}(X_+T_+^i)_{i=o}^{\ell-1}$. Since Q_m defines an isometry between $V^+_{L_m}$ and $R(P_m)$ it follows that $V^+_L = \{V^+_{L_m}\}_{m \in M}$ has a natural vector bundle structure over M. Also, $Q_m^{-1} \in \mathcal{L}(R(P_m), V^+_{L_m})$ is given by

$$(Q_m^{-1}x)(t) = \frac{1}{2\pi i} \int_{\gamma_m} e^{t\lambda} L_m^{-1}(\lambda) \left(\sum_{j=o}^{\ell-1} L_{m,j+1}(\lambda) x_j \right) d\lambda .$$

Then, one computes easily that

$$(X_+Q^{-1})_m = \frac{1}{2\pi i} \int_{\gamma_m} L_m^{-1}(\lambda) \; (L_{m,1}(\lambda) \ldots L_{m,\ell}(\lambda))|_{R(P_m)} \; d\lambda ,$$

$$(QY_+)_m = \frac{1}{2\pi i} \int_{\gamma_m} \begin{pmatrix} I \\ \vdots \\ \lambda^{\ell-1} I \end{pmatrix} L_m^{-1}(\lambda)\, d\lambda ,$$

$$(QT_+Q^{-1})_m = \frac{1}{2\pi i} \int_{\gamma_m} \begin{pmatrix} \lambda I \\ \vdots \\ \lambda^{\ell} I \end{pmatrix} L_m^{-1}(\lambda)(L_{m,1}(\lambda) \ldots L_{m,\ell}(\lambda))|_{R(P_m)} \; d\lambda .$$

Now, since R(P) is an embedded submanifold of $M \times \mathcal{X}^{\ell}$ it follows that $X_+Q^{-1} \in \mathcal{L}(R(P),\mathcal{X})$, $QT_+Q^{-1} \in \mathcal{L}(R(P))$, and $QY_+ \in \mathcal{L}(\mathcal{X},R(P))$. This completes the proof of the theorem.

CHAPTER 2

The purpose of this chapter is to introduce "reducibility" of admissible pairs and of operator polynomials and to show how these concepts are natural for considering spectral factorization (see section 2.3).

In section 2.1, reducibility of admissible pairs is defined and a sufficient condition for an admissible pair to be reducible is given. Reducibility of operator polynomials is defined in 2.2 and necessary and sufficient conditions for an operator polynomial to be reducible are given in terms of its coefficients and also in terms of the spectrum at infinity. These results make it clear that reducibility of operator polynomials is a generalization of operator polynomials simply behaved at infinity as defined in [8].

Given a reducible admissible pair (X,T) there is a natural way to construct a reducible operator polynomial having (X,T) as finite spectral pair. This is very closely related to a construction via special left inverses introduced in [5,6] (see section 2.5). Also, a reducible operator polynomial has a natural finite spectral triple, which we call the companion triple in analogy with the case of invertible leading coefficient discussed in [1,2].

In section 2.4 some simple applications to initial value problems for ordinary differential equations are obtained, and we give a generalization of Lopatinskii's theorem on Υ-spectral right divisors. In section 2.5, inverse problems for spectral pairs and triples are considered.

2.1 REDUCIBILITY OF ADMISSIBLE PAIRS

2.1.1 <u>Definition.</u> Let (X,T) be a right admissible pair with base space V and let $\mathcal{D}$ denote a decomposition of its target space: $\mathcal{X} = \mathcal{X}_o \oplus \dots \oplus \mathcal{X}_\ell$. $P_i \in L(\mathcal{X})$ will denote the projection of $\mathcal{X}$ onto $\mathcal{X}_i$. Write $X = \mathrm{col}(X_i)_{i=o}^{\ell}$, where $X_i = P_i X$, and define

$$Q_{\mathcal{D}}(X,T) = \mathrm{col}(X_i T^j)_{j=o,\,i=1}^{i-1\quad \ell} .$$

If $Q_{\mathcal{D}}(X,T)$ is invertible then (X,T) is said to be reducible with respect to $\mathcal{D}$. The possibility that some of the $\mathcal{X}_i$ are zero is not excluded.

An admissible pair (X,T) is said to be reducible if it is reducible with respect to some decomposition of its target space. The next proposition gives a partial answer as to when an admissible pair is reducible. In the finite dimensional case it gives a complete answer : (X,T) is reducible if and only if N(X,T) = 0.

2.1.2 <u>Proposition</u>. Let (X,T) be a right admissible pair. If (X,T) is reducible then N(X,T) = 0 and ind(X,T) < ∞ . Conversely, if N(X,T) = 0, ind(X,T) < ∞ , and R(X) and $R(XT^i \,|N(\mathrm{col}(XT^j)_{j=o}^{i-1}))$ for $i = 1,\dots,\ell$ are complemented in $\mathcal{X}$ then (X,T) is reducible. Indeed, let $\ell \geq \mathrm{ind}(X,T)$ and choose subspaces $\mathcal{X}_o,\dots,\mathcal{X}_\ell$ of $\mathcal{X}$ such that for $i = 1,\dots,\ell-1$

$$\mathcal{X}_{i+1} \oplus \dots \oplus \mathcal{X}_\ell = R(XT^i \,|N(\mathrm{col}(XT^j)_{j=o}^{i-1})),$$

$$\mathcal{X}_1 \oplus \dots \oplus \mathcal{X}_\ell = R(X),$$

$$\mathcal{X}_0 \oplus R(X) = \mathcal{X} .$$

Then $\mathcal{X} = \mathcal{X}_o \oplus \dots \oplus \mathcal{X}_\ell$ and this decomposition reduces (X,T).

Proof. If (X,T) is reduced by a decomposition $X = X_o \oplus \dots \oplus X_\ell$ then $N(X,T) = 0$ and $\mathrm{ind}(X,T) \leq \ell$. Thus the first assertion is clear. Notice also that $X_\ell \neq 0$ if and only if $\mathrm{ind}(X,T) = \ell$. For the converse, choose subspaces X_i as above (see lemma 2.1.3). Let Q_i denote the projection of X onto $X_{i+1} \oplus \dots \oplus X_\ell$ along $X_o \oplus \dots \oplus X_i$, and let $S = \mathrm{col}(Q_i XT^i)_{i=o}^{\ell-1}$. Notice that S is obtained from $Q_D(X,T)$ by a permutation of rows. Thus it suffices to show that S is bijective.

One can easily prove by induction on i that for $i = 0,1,\dots,\ell-1$

$$N(\mathrm{col}(XT^j)_{j=o}^{i}) = N(\mathrm{col}(Q_j XT^j)_{j=o}^{i}) .$$

Letting $i = \ell-1$ shows that S is injective.

To prove surjectivity of S, let $y_i \in R(Q_i)$ $(i = 0,1,\dots,\ell-1)$ be given. Choose $x_o \in X$ such that $y_o = Xx_o$ and then choose successively $x_i \in N(\mathrm{col}(XT^j)_{j=o}^{i-1})$ for $i = 1,\dots,\ell$ such that

$$XT^i x_j = y_i - \sum_{j=o}^{i-1} Q_i XT^i x_j .$$

Let $x = \sum_{j=o}^{\ell} x_j$, then for $i = 0,1,\dots,\ell-1$

$$Q_i XT^i x = \sum_{j=o}^{i} Q_i XT^i x_j = y_i .$$

Hence, $Sx = \mathrm{col}(y_i)_{i=o}^{\ell-1}$ and the proof is complete.

2.1.3 Lemma. Let E and F be closed subspaces of a Banach space G such that $E \subseteq F$. If E is complemented in G then E is complemented in F.

Proof. $E \oplus M = G$ implies $E \oplus (M \cap F) = F$.

2.1.4 <u>Remark</u>. If (X,T) is a right admissible pair such that $\mathrm{col}(XT^j)_{j=o}^{i-1}$ has a generalized inverse for some $i \geq 1$, then $R(XT^i|N(\mathrm{col}(XT^j)_{j=o}^{i-1}))$ is complemented in X^{i+1} if and only if $R(\mathrm{col}(XT^j)_{j=o}^{i})$ is complemented in X. Indeed, let W denote the former subspace and let S be a generalized inverse of $\mathrm{col}(XT^j)_{j=o}^{i-1}$. If $y \in R(\mathrm{col}(XT^j)_{j=o}^{i})$ with $y = \begin{pmatrix} y' \\ y_{i+1} \end{pmatrix}$ where $y' \in X^i$ and $y_{i+1} \in X$, then

$$y = \begin{pmatrix} y' \\ XT^iSy' \end{pmatrix} + \begin{pmatrix} 0 \\ y_{i+1}-XT^iSy' \end{pmatrix}$$

and it is easily seen that

$$R(\mathrm{col}(XT^j)_{j=o}^{i}) = \{\begin{pmatrix} y \\ XT^iSy \end{pmatrix} ; \; y \in R(\mathrm{col}(XT^j)_{j=o}^{i-1})\} \oplus \binom{0}{W} .$$

If W is closed in X then $R(\mathrm{col}(XT^j)_{j=o}^{i})$ is closed in X^{i+1}, and if W is complemented then $R(\mathrm{col}(XT^j)_{j=o}^{i})$ is complemented. Conversely, if the latter subspace is complemented in X^{i+1} then $\binom{0}{W}$ is complemented in X^{i+1}, and 2.1.3 implies W is complemented in X.

There is an analogous definition of reducibility for left admissible pairs (T,Y): given a decomposition D of X write $Y = \mathrm{row}(Y_i)_{i=o}^{\ell}$ and define

$$R_D(T,Y) = \mathrm{row}(T^jY_i)_{j=o,i=1}^{i-1\ \ \ell} ,$$

then (T,Y) is said to be reducible with respect to D if $R_D(T,Y)$ is invertible.

2.1.5 <u>Proposition</u>. Let (T,Y) be a left admissible pair. If (T,Y) is reducible then (T,Y) is surjective and $\mathrm{ind}(T,Y) < \infty$. Conversely, if (T,Y) is surjective, $\mathrm{ind}(T,Y) < \infty$, and $N(Y)$ and $(T^iY)^{-1}(R(\mathrm{row}(T^jY)_{j=o}^{i-1}))$ for $i = 1,\ldots\ell$ are complemented in X then (T,Y) is reducible. Indeed, let $\ell \geq \mathrm{ind}(T,Y)$ and choose subspaces $X_o,\ldots,X_\ell$ of X such that for $i = 1,\ldots,\ell$

$$\mathcal{X}_o \oplus \ldots \oplus \mathcal{X}_i = (T^iY)^{-1}(R(\mathrm{row}(T^jY)_{j=o}^{i-1})) \quad \text{and} \quad \mathcal{X}_o = N(Y).$$

Then $\mathcal{X} = \mathcal{X}_o \oplus \ldots \oplus \mathcal{X}_\ell$ and this decomposition reduces (T,Y).

The proof of this proposition is similar to that of 2.1.2.

Notice that

$$[R(XT^i | N(\mathrm{col}(XT^j)_{j=o}^{i-1}]^o = (T^{*i}X^*)^{-1}(R(\mathrm{row}(T^{*j}X^*)_{j=o}^{i-1})). \quad (2.1)$$

For future reference notice that if subspaces $\mathcal{X}_i$ of $\mathcal{X}$ are chosen as in 2.1.5 then for $i = 0,1,\ldots, \ell-1$

$$R(\mathrm{row}(T^jY)_{j=o}^{i}) = R(\mathrm{row}(T^jY|_{R(Q_j)})_{j=o}^{i}).$$

Also, (T,Y) is reduced by the decomposition $\mathcal{X} = \mathcal{X}_o \oplus \ldots \oplus \mathcal{X}_\ell$ if and only if row $(T^jY|_{R(Q_j)})_{j=o}^{i}$ is ivertible. Here Q_j are the projectors defined in the proof of 2.1.2.

2.1.6. Remark If (T,Y) is a left admissible pair such that $\mathrm{row}(T^jY)_{j=o}^{i-1}$ has a generalized inverse for some $i \geq 1$, then $(T^iY)^{-1}(R(\mathrm{row}(T^jY)_{j=o}^{i-1}))$ is complemented in $\mathcal{X}$ if and only if $N(\mathrm{row}(T^jY)_{j=o}^{i})$ is complemented in $\mathcal{X}^{i+1}$. Indeed, let W denote the former subspace and let S be a generalized inverse of $\mathrm{row}(T^jY)_{j=o}^{i-1}$. If $x \in N(\mathrm{row}(T^jY)_{j=o}^{i})$ write $x = \binom{x'}{x_{i+1}}$ where $x' \in \mathcal{X}^i$ and $x_{i+1} \in \mathcal{X}$. Then

$$x = \begin{pmatrix} -ST^iY\, x_{i+1} \\ x_{i+1} \end{pmatrix} + \begin{pmatrix} x' + ST^iY\, x_{i+1} \\ 0 \end{pmatrix}$$

and it is easily seen that

$$N(\mathrm{row}(T^jY)_{j=o}^{i}) = A \oplus \begin{pmatrix} N(\mathrm{row}(T^jY)_{j=o}^{i-1}) \\ 0 \end{pmatrix},$$

where $$A = \{ \binom{-ST^iY\,x}{x} \; ; \quad x \in W \; \}$$

If W is a complemented subspace of $\mathcal{X}$ then clearly $N(\mathrm{row}(T^jY)_{j=o}^{i})$ is complemented in $\mathcal{X}^{i+1}$. Conversely, suppose the latter subspace is complemented in $\mathcal{X}^{i+1}$. Then, by 2.1.3, it is complemented in

$$\tilde{A} = \{\begin{pmatrix} -ST^iYx \\ x \end{pmatrix} ; x \in \mathcal{X}\} \oplus \begin{pmatrix} N(\mathrm{row}(T^jY)_{j=o}^{i-1}) \\ 0 \end{pmatrix},$$

i.e.

$$\tilde{A} = N(\mathrm{row}(T^jY)_{j=o}^{i}) \oplus B,$$

for some subspace B. Let C be a complement of $N(\mathrm{row}(T^jY)_{j=o}^{i-1})$ in $\mathcal{X}^i$ then

$$\tilde{A} \oplus \binom{C}{0} = \mathcal{X}^{i+1} .$$

Hence $A \oplus M = \mathcal{X}^{i+1}$, where $M = B \oplus \binom{\mathcal{X}^i}{0}$. M is closed in $\mathcal{X}^{i+1}$, since there is a continuous projection P of $\mathcal{X}^{i+1}$ onto A such that $M = N(P)$. Let $\tilde{M}$ be the projection of M onto the $(i+1)^{st}$ coordinate, then $\tilde{M}$ is closed since $M \supseteq \binom{\mathcal{X}^i}{0}$. Now, we can show that

$$W \oplus \tilde{M} = \mathcal{X} .$$

Indeed, $W + \tilde{M} = \mathcal{X}$ is clear and if $x \in W \cap \tilde{M}$ then $\binom{x'}{x} \in M$ for some $x' \in \mathcal{X}^i$. Hence

$$\binom{x'}{x} = \begin{pmatrix} -ST^iY\bar{x} \\ \bar{x} \end{pmatrix} + \binom{y}{0} ,$$

where the first term is in B.

Then $\bar{x} = x \in W$ and so

$$\begin{pmatrix} -ST^iYx \\ x \end{pmatrix} \in N(\mathrm{row}(T^jY)_{j=o}^{i}) \cap B = 0 .$$

Thus $x = 0$. This completes the proof that W is complemented in $\mathcal{X}$.

2.2 REDUCIBILITY OF OPERATOR POLYNOMIALS

The purpose of this section is to show that given a reducible admissible pair (X,T) there is a natural way to construct an operator polynomial having (X,T) as finite spectral pair.

4.2.1 Definition. An o.p. L is said to be right reducible if there is a decomposition $X = X_o \oplus \dots \oplus X_\ell$ such that if we write $L(\lambda) = (a_{ij}(\lambda))_{i,j=o}^{\ell}$, where $a_{ij}(\lambda)$ is an o.p. with coefficients in $L(X_j,X_i)$, then

(i) $a_{jj}(\lambda)$ has degree j with invertible leading coefficient;

(ii) $\deg a_{ij}(\lambda) < j$ for $i \neq j$.

As in 2.1, $X_i = 0$ is allowed and we regard (i) and (ii) as holding vacuously in that case.

The next theorem gives a necessary and sufficient condition for L to be right reducible, in terms of its coefficients. Another characterization of reducibility in terms of the spectrum at ∞ will be given in 2.2.6. Notice that the easy part of 2.2.6 implies that if L is right reducible then $\sigma(L)$ is compact. This is needed in the proof of 2.2.2.

2.2.2 Theorem Let $L(\lambda) = \sum_{i=o}^{\ell} A_i\lambda^i$ be an o.p. . For $i = -1,0,\dots,\ell-1$ define $W_i = \bigcap_{j=i+1}^{\ell} N(A_j)$ and also set $W_\ell = X$. L is right reducible if and only if

(i) $W_{-1} = 0$ and W_i is an invariant subspace of A_i for all i;

(ii) for all i, $R(A_i|W_i)$ is closed and

$$R(A_i|W_i) \oplus N(A_i|W_i) = W_i .$$

Moreover, if L is right reducible then there is a unique decomposition of $\mathcal{X}$ that reduces L (which will be called the canonical decomposition of $\mathcal{V}$ determined by L) given by

$$\mathcal{X}_i = R(A_i|W_i) \qquad (i = 0,\dots, \ell) .$$

<u>Proof</u>. First of all, notice that $N(A_i|W_i) = W_{i-1}$ for $i = 0,\dots, \ell$ and that $W_{-1} = 0$ if $L(\lambda)$ is 1-1 for some $\lambda \in \mathbb{C}$.

Now, suppose the decomposition $\mathcal{X} = \bigoplus_{j=o}^{\ell} \mathcal{X}_j$ reduces L. One can easily show by induction on ℓ that $W_i = \bigoplus_{j=o}^{i} \mathcal{X}_j$ and $\mathcal{X}_i = R(A_i|W_i)$ for $i = 0,\dots, \ell$. This proves uniqueness of the decomposition reducing L. Also the validity of (i) and (ii) follows immediately.

Consider the following statement: if L is an o.p. of degree $\leq \ell$ satisfying (i) and (ii) then L is right reducible. We will prove this statement is true by induction on ℓ (which proves the converse to the theorem). For $\ell = 0$ there is no problem because then $L(\lambda) = A_o$ is invertible. Suppose the statement is true for o.p.'s of degree $\leq \ell-1$. If L is an o.p. of degree $\leq \ell$ satisfying (i) and (ii) then $X = N(A_\ell) \oplus X_\ell$ where $X_\ell = R(A_\ell)$. Corresponding to this decomposition of X,

$$A_\ell = \begin{pmatrix} 0 & 0 \\ 0 & (I-P)A_\ell(I-P) \end{pmatrix}$$

and
$$L(\lambda) = \begin{pmatrix} P\,L(\lambda)P & PL(\lambda)(I-P) \\ (I-P)L(\lambda)P & (I-P)L(\lambda)(I-P) \end{pmatrix}$$

where P is the projection on $N(A_\ell)$ along $\mathcal{X}_\ell$. Notice that $(I-P)A_\ell(I-P) \in \mathcal{L}(R(A_\ell))$ is invertible - it is bijective continuous, hence invertible by the closed graph theorem. Now, $\bar{L}(\lambda):=PL(\lambda)P$ is an o.p. of degree $\leq \ell-1$ with coefficients in $L(N(A_\ell))$. The corresponding conditions (i) and (ii) for $\bar{L}$ are a subset of those for L. The induction hypothesis implies that $\bar{L}$ is reduced by a decomposition $N(A_\ell) = \bigoplus_{j=o}^{\ell-1} X_j$. If $i \leq j \leq \ell$ then $X_i \subseteq W_j$ so $A_j(X_i) \subseteq W_j$ and $(I-P)A_j|X_i = 0$. Hence

$$\deg(I-P)L(\lambda)P|X_i \leq i-1 \quad .$$

Referring to (2.2) it is now easily seen that the decomposition $X = \bigoplus_{j=o}^{\ell} X_j$ reduces L. This completes the proof of the theorem.

Let L be right reducible. From now on we assume that the leading coefficients of a_{ii} $(i = 0,\dots,\ell)$ are monic, i.e. $a^j_{ij} = \delta_{ij}I_{X_j}$. Then, $a_{ij}(\lambda) = \sum_{k=o}^{\ell} a^k_{ij}\lambda^k$ for $k > j$. A finite spectral pair for L will now be constructed from its coefficients. When L is monic (i.e. $X_\ell = X$) this is simply the companion pair defined in [1].

2.2.3 Definitions. Let $V_c = \bigoplus_{i=1}^{\ell} (X_i)^i$. Elements of V_c will be denoted by $x = \mathrm{col}(x_1,\dots,x_\ell)$, where $x_i = \mathrm{col}(x^1_i,\dots,x^i_i) \in (X_i)^i$, and $x^k_i \in X_i$ will be called the (i,k) coordinate of x. Let $P^k_i \in L(V_c,X_i)$ denote the projection onto the (i,k) component. Let $J^k_i \in L(X_i,V_c)$ denote the inclusion of X_i into the (i,k) component of V_c. Both the latter definitions are for $i = 1,\dots,\ell$ and $k = 1,\dots,i$. Define, for $i = 0,\dots,\ell$, $X_i \in L(V_c,X_i)$ and $Y_i \in L(X_i,V_c)$, and, for $i = 1,\dots,\ell$, $T_i \in L(V_c,(X_i)^i)$ as follows:

$$X_o = -(a_{01}^o | a_{02}^o \; a_{02}^1 \; | \; \cdots \; | a_{0\ell}^o \; a_{0\ell}^1 \; \cdots \; a_{0\ell}^{\ell-1}),$$

$$X_i = P_i^1 \quad (i = 1,\ldots,\ell),$$

$$Y_o = 0,$$

$$Y_i = J_i^i \quad (i = 1,\ldots \ell),$$

$$T_i|(X_i)^i = \begin{pmatrix} 0 & I_{X_i} & & 0 \\ \cdot & & & \\ \cdot & & & I_{X_i} \\ \cdot & & & \\ -a_{ii}^o & & \cdots & -a_{ii}^{i-1} \end{pmatrix} \text{ is the}$$

companion map for $a_{ii}(\lambda)$, and

$$T_i|(X_j)^j = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & & 0 \\ -a_{ij}^o & & -a_{ij}^{j-1} \end{pmatrix} \quad \text{for } j \neq i .$$

Finally, define $X_c \in L(V_c,X)$, $T_c \in L(V_c)$, and $Y_c \in L(X,V_c)$

such that $X_c = \operatorname{col}(X_i)_{i=o}^{\ell}$

$$T_c = \operatorname{col}(T_i)_{i=1}^{\ell} ,$$

and $Y_c = \operatorname{row}(Y_i)_{i=1}^{\ell}$.

The admissible triple (X_c,T_c,Y_c) is called the companion triple for L .

The next theorem and its proof give the motivation for these definitions.

2.2.4 Theorem. If L is a right reducible o.p. then the companion triple is a finite spectral triple for L.

Proof. First of all, we note an important property of T_c:

$$P_i^k T_c = P_i^{k+1} \quad (k = 1,\dots,i-1). \quad \text{It follows}$$

$$P_i^1 T_c^k = P_i^{k+1} \quad (k = 0,\dots,i-1) \quad \text{and also}$$

$$P_i^1 T_c^i = P_i^i T_c = -(a_{i1}^o \mid a_{i2}^o a_{i2}^1 \mid \dots \mid a_{i\ell}^o \dots a_{i\ell}^{\ell-1}). \tag{2.3}$$

The fact that (X_c, T_c) is a finite spectral pair for L is due to 1.1.7 and the usual linearization procedure for ODE's. Indeed, if $L(\frac{d}{dt})u = 0$ (where $u \in C^\infty(R,X)$) then for $i = 0,\dots,\ell$

$$\left(\frac{d}{dt}\right)^i u_j = -\sum_{j=1}^{\ell} \sum_{k=o}^{j-1} a_{ij}^k \left(\frac{d}{dt}\right)^k u_j, \quad \text{and}$$

hence for $i = 1,\dots,\ell$

$$\left(\frac{d}{dt}\right) v_j^i = -\sum_{j=1}^{\ell} \sum_{k=1}^{j} a_{ij}^{k-1} v_j^k,$$

where $v = \mathrm{col}(v_1,\dots, v_\ell)$

and $v_i = \mathrm{col}(u_i,\dots, (\frac{d}{dt})^{i-1} u_i)$.

Thus every solution of $L(\frac{d}{dt})u = 0$ has a (unique) representation of the form $u(t) = X_c e^{tTc} v_o$ for some $v_o \in V_c$, and as usual the converse is true. Therefore (X_c, T_c) is a finite spectral pair for L.

It remains to show that for $k = 0,\dots, \ell-1$

$$\int_\Gamma \lambda^k L^{-1}(\lambda) d\lambda = X_c T_c^k Y_c \tag{2.4}$$

where Γ is a large contour containing $\sigma(L)$.

From the remarks at the beginning of the proof:

$$P_i(X_c T_c^k Y_c)P_j \;=\; X_i T_c^k Y_i$$

$$= \begin{cases} 0 & i \neq j \\ 0 & i = j \quad k = 0,\dots,i-2, i \geq 2 \\ I_{X_i} & i = j \quad k = i-1,\ i \geq 1 \\ 0 & i = 0\ . \end{cases}$$

Also, we can write $L(\lambda) = M(\lambda^{-1}) \left(\sum_{m=o}^{\ell} \lambda^m P_m \right)$ where $M(\lambda)$ is an o.p. such that $M(0) = I$.

Hence $$P_i L^{-1}(\lambda) P_j \;=\; P_i \left(\sum_{m=o}^{\ell} \lambda^{-m} P_m \right) M(\lambda^{-1}) P_j$$

$$= P_i \left(\sum_{i=o}^{\ell} \lambda^{-m} P_m \right) P_j + \varepsilon(\lambda)$$

$$= \delta_{ij}\ \lambda^{-i} P_i + \varepsilon(\lambda),$$

where $\varepsilon(\lambda) \to 0$ as $\lambda \to \infty$.

Thus for $k = 0,1,\dots,i-1$

$$P_i \int_\Gamma \lambda^k L^{-1}(\lambda) d\lambda\ P_j = P_i X_c T_c^k Y_c P_j \quad .$$

We claim that this implies (2.4). Indeed, since (X_c, T_c) is a finite spectral pair for we know (1.1.7) that there exists $\tilde{Y}_c \in L(X,V)$ such that for all $k = 0,1,\dots$

$$\int_\Gamma \lambda^k L^{-1}(\lambda) d\lambda \;=\; X_c T_c^k Y_c \quad .$$

Thus $0 = Q_D(X_c, T_c)(Y_c - \tilde{Y}_c) = Y_c - \tilde{Y}_c$, where D is the canonical decomposition of L, so $Y_c = \tilde{Y}_c$ and the theorem is proved.

We now show that given a decomposition of X reducing an admissible pair (X,T) there is a natural way to construct a reducible o.p. with finite spectral pair (X,T).

2.2.5 <u>Theorem.</u>

(i) Let L be a right reducible o.p. An admissible pair (X,T) is a finite spectral pair for L if and only if

$$\sum_{i=o}^{\ell} A_i X T^i = 0$$

and the canonical decomposition of X determined by L reduces (X,T)

(ii) Conversely, let (X,T) be an admissible pair reduced by a decomposition D of its target space X. There is a unique right reducible o.p. L with canonical decomposition D and finite spectral pair (X,T). Explicitly, if D is the decomposition $X = \bigoplus_{j=o}^{\ell} X_j$ then

$$L(\lambda) = (a_{ij}(\lambda))_{i,j=o}^{\ell} ,$$

$$a_{ij}(\lambda) = \delta_{ij}\, \lambda^j I_{X_j} + \sum_{k=o}^{j-1} a_{ij}^k\, \lambda^k ,$$

$$a_{ij}^k = - X_i T^i V_j^{k+1} , \qquad (2.5)$$

where $V_j^k \in L(X_j, V)$ for $j = 1,\ldots, \ell$ and $k = 1,\ldots,j$ are defined via

$$(Q_D(X,T))^{-1} = \operatorname{row}(V_j^k)_{k=1,j=1}^{j\ \ \ \ell} .$$

Also (X,T,Y) is a finite spectral triple for L where

$$Y = \operatorname{row}(Y_j)_{j=o}^{\ell} ,$$

$$Y_o = 0,$$

and $$Y_j = V_j^j \qquad (j = 1,\ldots,\ell) .$$

Proof. (i) Let (X,T) be a finite spectral pair for L. By definition, $\sum_{i=o}^{\ell} A_i XT^i = 0$. Also, (X,T) is similar to the comparison pair (X_c,T_c) of L so it is clear that $\mathcal{D}$, the canonical decomposition of $\mathcal{X}$ determined by L, reduces (X,T).

Conversely, suppose (X,T) is an admissible pair such that $\sum_{i=o}^{\ell} A_i XT^i = 0$ and $\mathcal{D}$ reduces (X,T). Let

$$S = Q_{\mathcal{D}}(X_c,T_c)^{-1} Q_{\mathcal{D}}(X,T) .$$

$S \in \mathcal{L}(V,V_c)$ is invertible and I claim that $X = X_c S$ and $T = S^{-1}T_c S$. This can be calculated directly but we use a trick to simplify the proof. Let $\tilde{X} = X_c S$ and $\tilde{T} = S^{-1}T_c S$. If $x \in V$ define $u \in V_L$ by $u(t) = Xe^{tT}x$. There is a unique $y \in V_c$ such that $Xe^{tT}x = X_c e^{tT_c} y$. Differentiating both sides of this equality several times with respect to t yields

$$Q_{\mathcal{D}}(X_c,T_c)y = Q_{\mathcal{D}}(X,T)x .$$

Hence, $Xe^{tT} = X_c e^{tT_c} S = \tilde{X}e^{t\tilde{T}}$.

Again, differentiation with respect to t yields

$$Q_{\mathcal{D}}(X,T)T = Q_{\mathcal{D}}(\tilde{X},\tilde{T})\tilde{T} = Q_{\mathcal{D}}(X,T)\tilde{T} .$$

Hence, $T = \tilde{T}$ and $X = \tilde{X}$. This proves that (X,T) is similar to (X_c,T_c), showing that it is a finite spectral pair for L.

(ii) If $L(\lambda) = (a_{ij}(\lambda))_{i,j=o}^{\ell}$ is a right reducible o.p. with canonical decomposition $\mathcal{D}$ and finite spectral pair (X,T) then one can show that (2.5) holds, thus proving uniqueness. Indeed, without loss of generality $(X,T) = (X_c,T_c)$, the companion pair of L. Since $Q_{\mathcal{D}}(X_c,T_c) = I$, then $V_j^k = J_j^k$ and (2.5) follows from (2.3) for $i > 0$ and for $i = 0$ from the definition of X_o. Now, given any admissible pair (X,T) reduced by $\mathcal{D}$ simply define L via (2.5). For $i = 0,\ldots,\ell$

$$P_i \sum_{k=o}^{\ell} A_k XT^k = \sum_{k=o}^{\ell} \sum_{j=o}^{\ell} a_{ij}^k X_j T^k$$

$$= \sum_{j=o}^{\ell} \delta_{ij} \lambda^j XT^j - X_i T^i \sum_{j=1}^{\ell} \sum_{k=o}^{j-1} V_j^{k+1} X_j T^k$$

$$= 0 .$$

Hence $\sum_{k=o}^{\ell} A_k XT^k = 0$ and (i) implies that (X,T) is a finite spectral pair for L. Finally, 2.2.4 proves that the companion triple (X_c,T_c,Y_c) is a finite spectral triple for L. It follows that (X,T,Y) is a finite spectral triple for L since $X = X_c S$, $T = S^{-1} T_c S$, and $Y = S^{-1} Y_c$ where $S = Q_{\mathcal{D}}(X,T)$.

The next theorem gives a characterization of right reducibility in terms of the spectrum at ∞ .

2.2.6 <u>Theorem.</u> Let L be an o.p. of degree $\leq \ell$.

The following statement are equivalent:

(i) there exists an invertible $C \in \mathcal{L}(X)$ such that $CL(\lambda)$ is right reducible;

(ii) there exist mutually disjoint projectors $P_i \in L(X)$ $(i = 0,\dots,\ell)$ such that $I = \sum_{i=o}^{\ell} P_i$ and $L(\lambda)\left(\sum_{i=o}^{\ell} \lambda^{\ell-i} P_i\right)$ is an o.p. of degree ℓ with invertible leading coefficient;

(iii) $\sigma(L)$ is compact and if (X_∞, T_∞) is a spectral pair at ∞ then $R(X_\infty T_\infty^i)$ is complemented in X for all $i = 0,1, \dots$ and for $i = 1,2,\dots$

$$R(X_\infty T_\infty^i) = R(X_\infty T_\infty^i|N(\mathrm{col}(X_\infty T_\infty^j)_{j=o}^{i-1})) \tag{2.6}$$

<u>Proof</u>. The equivalence of (i) and (ii) is obvious.

Now, let $\widetilde{X}_j = X_{\ell-j}$ and let $\widetilde{P}_i^k \in L(\bigoplus_{j=o}^{\ell} (\widetilde{X}_j)^j, \widetilde{X}_i)$ be defined as in 2.2.3.

(ii) → (iii). If (ii) holds then there is an o.p. $M(\lambda)$ with $M(0)$ invertible such that $L(\lambda) = M(\lambda^{-1})\left(\sum_{i=o}^{\ell} \lambda^i P_i\right)$.

Then,
$$\widetilde{L}(\lambda) = \lambda^\ell L(\lambda^{-1})$$
$$= M(\lambda)\left(\sum_{i=o}^{\ell} \lambda^i P_{\ell-i}\right).$$

Thus for (X_∞, T_∞) we can take (in block operator form)

$$V_\infty = \bigoplus_{i=o}^{\ell} (\widetilde{X}_i)^i$$

$$X_\infty = \mathrm{col}(\widetilde{X}_i)_{i=o}^{\ell}$$

and
$$T_\infty = \mathrm{diag}(\widetilde{T}_i)_{i=o}^{\ell}$$

where $\tilde{X}_i \in L(V_\infty, \tilde{X}_i)$ and $\tilde{T}_i \in L((\tilde{X}_i)^i)$ are defined by

$$\tilde{X}_o = 0$$

$$\tilde{X}_i = P_i^1 \qquad (i = 1,\ldots,\ell)$$

and

$$\tilde{T}_i = \begin{pmatrix} 0 & I_{\tilde{X}_i} & & \\ & & \ddots & \\ & & & I_{\tilde{X}_i} \\ 0 & & & 0 \end{pmatrix} \qquad (i = 1,\ldots \ell)$$

One easily computes that for $i = 0,1,2,\ldots, \ell-1$

$$X_\infty T_\infty^i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ P_{i+1}^{i+1} \\ \vdots \\ P_\ell^{i+1} \end{pmatrix}.$$

and also $T_\infty^\ell = 0$. It is now clear that for $i = 0,\ldots, \ell-1$

$$R(X_\infty T_\infty^i) = \tilde{X}_{i+1} \oplus \ldots \oplus \tilde{X}_\ell$$

and

$$R(X_\infty T_\infty^i) = R(X_\infty T_\infty^i|N\begin{pmatrix} X_\infty \\ \vdots \\ X_\infty T_\infty^{i-1} \end{pmatrix}) .$$

This proves (iii) .

(iii) → (ii). Define a decomposition of X as follows:

$$\tilde{X}_{i+1} \oplus \dots \oplus \tilde{X}_{\ell} = R(X_\infty T_\infty^i)$$

for $i = 0,\dots, \ell-1$ and

$$\tilde{X}_o \oplus R(X_\infty) = X \ .$$

The hypotheses of (iii) and proposition 2.1.2 imply that this decomposition reduces (X_∞,T_∞). Thus, by 2.2.5, there is a right reducible o.p. $L_1(\lambda) = (b_{ij}(\lambda))_{i,j=o}^{\ell}$ with finite spectral pair (X_∞,T_∞). Here

$$b_{ij}(\lambda) = \delta_{ij}\,\lambda^j I_{\tilde{X}_j} + \sum_{k=o} b_{ij}^k\,\lambda^k$$

and $$b_{ij}^k = -\tilde{X}_i T_\infty^i \tilde{V}_j^{k+1} \quad \text{for } k = 0,\dots,j-1 \text{ and } j \geq 1,$$

where $$X_\infty = \operatorname{col}(\tilde{X}_i)_{i=o}^{\ell}$$

and the $\tilde{V}_j^k \in L(\tilde{X}_j,V_\infty)$ are defined as in 2.2.5. We want to show that all $b_{ij}^k = 0$, i.e. $\tilde{X}_i T_\infty^i = 0$ for $i = 0,\dots, \ell$. Notice that the hypotheses of (iii) imply $T_\infty^\ell = 0$, so $\tilde{X}_\ell T_\infty^\ell = 0$. Also, for $i = 1,\dots, \ell-1$,

$$\begin{aligned}\tilde{X}_i T_\infty^i &= P_{\tilde{X}_i} X_\infty T_\infty^i \\ &= 0\end{aligned}$$

and $\tilde{X}_o = 0$. Hence

$$L_1(\lambda) = \sum_{i=o}^{\ell} \lambda^i P_{\tilde{X}_i} \ .$$

Now, since $T_\infty^\ell = 0$, 1.3.5 implies that

$$L(\lambda) = M(\lambda)\, L_1(\lambda)$$

for some o.p. $M(\lambda)$ with $M(\lambda)$ invertible. Thus if we let $P_i = P\widetilde{X}_{\ell-i}$ for $i = 0,\dots,\ell$ then $I = \sum_{i=o}^{\ell} P_i$ and $L(\lambda)\left(\sum_{i=o}^{\ell} \lambda^{\ell-i} P_i\right)$ is an o.p. of degree ℓ with invertible leading coefficient.

This completes the proof of the theorem.

There are definitions and results dual to those above. An o.p. L is said to be left reducible if there is a decomposition $X = X_o \oplus \dots \oplus X_\ell$ such that if we write $L(\lambda) = (a_{ij}(\lambda))_{i,j=o}^{\ell}$, where $a_{ij}(\lambda)$ is an o.p. with coefficients in $L(X_j, X_i)$, then

(i) $a_{ij}(\lambda)$ has degree j with invertible leading coefficient;

(ii) $\deg a_{ij}(\lambda) < i$ for $i \neq j$.

If L is left reducible then the spectrum of L is compact. One can define a companion triple (X_c, T_c, Y_c) for L in the obvious way (for the sake of brevity this definition is omitted). A quick proof that this triple is a finite spectral triple for L can be given by duality. Indeed, L^* is a right reducible o.p. and (Y_c^*, T_c^*, X_c^*) is the comparison triple for L^*. Hence, by 2.2.4, (Y_c^*, T_c^*, Y_c^*) is a finite spectral triple for L^*. Take the transpose of the conditions satisfied by (Y_c^*, T_c^*, X_c^*) and restrict to X, then it follows that (X_c, T_c, Y_c) is a finite spectral triple for L.

2.2.7. <u>Theorem.</u> Let $L(\lambda) = \sum_{i=o}^{\ell} A_i \lambda^i$ be an o.p. For $i = -1, 0, \dots, \ell-1$ define $\widetilde{W}_i = \underset{j=i+1}{\overset{\ell}{+}} R(A_j)$ and $\widetilde{W}_\ell = 0$. L is left reducible if and only if

(i) $\widetilde{W}_{-1} = X$ and $\widetilde{W}_i$ is an invariant subspace of A_i for all i ;

(ii) $$(A_i)^{-1}(\widetilde{W}_i) \cap \widetilde{W}_{i-1} = \widetilde{W}_i \quad \text{for } i = 0,\dots,\ell \tag{2.7}$$
$$(A_i)^{-1}(\widetilde{W}_i) + \widetilde{W}_{i-1} = X \quad \text{for } i = 1,\dots,\ell \tag{2.8}$$

__Proof__ Suppose L is reduced by the decomposition $X = \bigoplus_{j=o}^{\ell} X_j$. Then $\sigma(L)$ is compact which implies $\tilde{W}_{-1} = X$. Also, one can show by induction that

$$\tilde{W}_i = \bigoplus_{j=i+1}^{\ell} X_j$$

and $\bigoplus_{j\neq i} X_j = (A_i)^{-1}(W_i)$ for $i = 0,\ldots, \ell$.

Hence (i) and (ii) follow immediately.

Consider the following statement: if L is an o.p. of degree $\leq \ell$ satisfying (i) and (ii) then L is left reducible. We will prove this statement is true by induction on ℓ (which proves the converse to the theorem). For $\ell = 0$ there is no problem because then $L(\lambda) = A_o$ is invertible. Suppose the statement is true for o.p.'s of degree $\leq \ell-1$. If L is an o.p. of degree $\leq \ell$ satisfying (i) and (ii) then $X = N(A_\ell) \oplus X_\ell$ where $X_\ell = R(A_\ell)$. Corresponding to this decomposition of X ,

$$A_\ell = \begin{pmatrix} 0 & 0 \\ 0 & (I-P)A_\ell(I-P) \end{pmatrix}$$

and

$$L(\lambda) = \begin{pmatrix} PL(\lambda)P & PL(\lambda)(I-P) \\ (I-P)L(\lambda)P & (I-P)L(\lambda)(I-P) \end{pmatrix} \tag{2.9}$$

where P is the projection on $N(A_\ell)$ along X_ℓ .

Now, $\tilde{L}(\lambda): = PL(\lambda)P$ is an o.p. of degree $\leq \ell-1$ with coefficients in $L(\overline{X})$ where $\overline{X} = PX$ and we will show that (i) and (ii) hold for this o.p. and then apply the induction hypothesis. First of all, write

$\bar{L}(\lambda) = \sum_{i=o}^{\ell-1} \bar{A}_i\lambda^i$, where $\bar{A}_i = PA_iP$, and set $\bar{W}_i = \dotplus_{j=i+1}^{\ell-1} R(\bar{A}_i)$ for $i = 0,\ldots, \ell-2$, and $\bar{W}_{\ell-1} = 0$. I claim that $\bar{W}_i = P\tilde{W}_i$ for $i \le \ell-1$. Indeed, $\bar{W}_i \subseteq P\tilde{W}_i$ is clear for all i and if $\bar{W}_i = P\tilde{W}_i$ for some $i \le \ell-1$ then since $\tilde{W}_{i-1} = R(A_i) + \tilde{W}_i$ we have

$$\begin{aligned}
P\tilde{W}_{i-1} &\subseteq R(PA_i) + P\tilde{W}_i \\
&\subseteq R(PA_iP) + R(PA_i(I-P)) + \bar{W}_i \\
&\subseteq R(\bar{A}_i) + \bar{W}_i \\
&\subseteq \bar{W}_{i-1}
\end{aligned}$$

(notice that $X_\ell \subseteq \tilde{W}_i$ so $A_i(X_\ell) \subseteq \tilde{W}_i$ for $i \le \ell-1$). Hence $P\tilde{W}_{i-1} = \bar{W}_{i-1}$ and thus the claim is proved by induction on i, the case $i = \ell-1$ being obvious. Now, one can show that (i) and (ii) hold for $\bar{L}$:

(i) First of all, $\bar{W}_{-1} = P\tilde{W}_{-1} = PX = \bar{X}$.

Since
$$\begin{aligned}
P\tilde{W}_i &\subseteq \tilde{W}_i + (I-P)\tilde{W}_i \\
&\subseteq \tilde{W}_i + X_\ell \\
&\subseteq \tilde{W}_i ,
\end{aligned}$$

we deduce that $\bar{W}_i \subseteq \tilde{W}_i$. Hence $P|\tilde{W}_i$ is a projector in $\tilde{W}_i$ with kernel X_ℓ and range $\bar{W}_i$. It follows that $\bar{W}_i$ is closed and $\bar{W}_i \oplus X_\ell = \tilde{W}_i$ for $i \le \ell-1$. Also, $\bar{A}_i(\bar{W}_i) = PA_i(\bar{W}_i) \subseteq P\tilde{W}_i = \bar{W}_i$ so $\bar{W}_i$ is an invariant subspace of $\bar{A}_i$;

(ii) The fact that $\bar{W}_i$ is an invariant subspace of $\bar{A}_i$ is equivalent to $\bar{W}_i \subseteq (\bar{A}_i)^{-1}(\bar{W}_i) \cap \bar{W}_{i-1}$, since $\bar{W}_i \subseteq \bar{W}_{i-1}$. Now, if $x \in (\bar{A}_i)^{-1}(\bar{W}_i) \cap \bar{W}_{i-1}$ then

$x \in W_{i-1}$ and

$$A_i x = PA_i Px + PA_i(I-P)x + (I-P)A_i x$$

$$\in \bar{W}_i + \bar{W}_i + X_\ell$$

$$\subseteq \tilde{W}_i \quad .$$

Hence $x \in \tilde{W}_i$ so $x = Px \in \bar{W}_i$. Thus (2.7) holds for $\bar{L}$. Also $X = (A_i)^{-1}(\tilde{W}_i) + \tilde{W}_{i-1}$ implies

$$\bar{X} = P(A_i)^{-1}(\tilde{W}_i) + P\tilde{W}_{i-1}$$

$$\subseteq (\bar{A}_i)^{-1}(\bar{W}_i) + \bar{W}_{i-1} \quad .$$

for $i = 1,\ldots, \ell-1$ and thus (2.8) holds for $\bar{L}$. Thus the induction hypothesis implies that $\bar{L}$ is reducible, i.e. there exists a decomposition $\bar{X} = \bigoplus_{j=o}^{\ell-1} X_j$ that reduces $\bar{L}$. By the first part of this proof

$$\bar{W}_i = \bigoplus_{j=i+1}^{\ell-1} X_j$$

and hence $$\tilde{W}_i = \bigoplus_{j=i+1}^{\ell} X_j \quad .$$

If we notice that $A_j(X_\ell) \subseteq \tilde{W}_j$, i.e.

$$P_{X_i} PA_j(I-P) = 0 \qquad (i \leq j \leq \ell\ )$$

or

$$\deg P_{X_i} PL(\lambda)(I-P) \leq i-1,$$

and refer to (2.9) it is easily seen that the decomposition $X = \bigoplus_{j=o}^{\ell} X_j$ reduces L. This completes the proof of the theorem.

2.2.8 Remark By induction on ℓ, starting with the fact that $X_\ell = R(A_\ell)$, one can also prove that if L is left reducible then there is a unique decomposition of X reducing L.

The next theorem is the analogue of 2.2.6.

2.2.9 Theorem. Let L be an o.p. of degree $\leq \ell$. The following statements are equivalent:

(i) there exists an invertible $C \in L(X)$ such that $L(\lambda)C$ is left reducible;

(ii) there exist mutually disjoint projectors $P_i \in L(X)$ $(i = 1,\ldots,\ell)$ such that $I = \sum_{i=o}^{\ell} P_i$ and $(\sum_{i=o}^{\ell} \lambda^{\ell-i} P_i)L(\lambda)$ is an o.p. of degree ℓ with invertible leading coefficient;

(iii) $\sigma(L)$ is compact, and if (T_∞, Y_∞) is a left spectral pair at ∞ for L then $N(T_\infty^i Y_\infty)$ is complemented in X for $i = 0,1,2,\ldots$ and for $i = 1,2,\ldots$

$$N(T_\infty^i Y_\infty) = (T_\infty^i Y_\infty)^{-1}(R(\mathrm{row}(T_\infty^j Y_\infty)_{j=o}^{i-1})) \quad .$$

2.3 SPECTRAL FACTORIZATION.

Let L be an o.p. and γ a simple closed contour not intersecting $\sigma(L)$. The purpose of this section is to give a necessary and sufficient condition for L to have a right factorization with respect to γ. Gohberg, Lerer, and Rodman [7,8] considered the case of canonical and quasi-canonical factorization. First we need some preliminary results, which are of independent interest also.

2.3.1 Theorem. A left admissible pair (T,Y) is the left finite spectral pair of a right reducible o.p. of degree $\leq \ell$ if and only if $\mathrm{row}(T^j Y)_{j=o}^{i}$

has a generalized inverse for $i = 0,1,\ldots, \ell-2$, and is right invertible for $i = \ell-1$.

Proof. We first establish some notation. Given a decomposition $\mathcal{X} = \bigoplus_{j=o}^{\ell} \mathcal{X}_j$ set $W_i = \bigoplus_{j=o}^{i} \mathcal{X}_j$ for $i=0,\ldots,\ell$ and $\tilde{W}_i = \bigoplus_{j=i+1}^{\ell} \mathcal{X}_j$ for $i = 0,\ldots, \ell-1$. Also, let $W_{-1} = 0$ and $\tilde{W}_\ell = 0$. Define $P_{ij} \in \mathcal{L}(\tilde{W}_{j-1},\tilde{W}_{i-1})$ as the natural pojection, and notice that $P_{i+1,i}\, P_{ij} = P_{i+1,j}$.

Now, if (T,Y) is the left finite spectral pair for a right reducible o.p. L of degree $\leq \ell$, then without loss of generality $(Y,T) = (Y_c,T_c)$, the left companion pair for L. After a moment's reflection it is clear that there is a permutation of blocks $\sigma \in \mathcal{L}(V_c,\tilde{V})$ such that if we let $\tilde{Y} = \sigma Y_c$ and $\tilde{T} = \sigma T_c\, \sigma^{-1}$ then

$$\tilde{Y} = \begin{pmatrix} P_{10} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and

$$\tilde{T} = \begin{pmatrix} M_1 & M_2 & \cdots & M_{\ell-1} & M_\ell \\ P_{21} & 0 & \cdots & 0 & 0 \\ & P_{32} & & & \vdots \\ & & \ddots & 0 & \\ 0 & & & P_{\ell,\ell-1} & 0 \end{pmatrix} \tag{2.10}$$

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where $M_j \in \mathcal{L}(\tilde{W}_{j-1},\tilde{W}_o)$ are given by

$$M_j = -\begin{pmatrix} a^o_{1j} & a^1_{1,j+1} & \cdots & a^{\ell-j}_{1\ell} \\ a^o_{2j} & a^1_{2,j+1} & \cdots & a^{\ell-j}_{2\ell} \\ \vdots & \vdots & & \vdots \\ a^o_{\ell j} & a^1_{\ell,j+1} & \cdots & a^{\ell-j}_{\ell\ell} \end{pmatrix}. \tag{2.11}$$

Now, for simplicity of notation, and without loss of generality, we assume $(Y,T) = (\tilde{Y},\tilde{T})$ and $V = \tilde{V}$. A direct computation shows that for $i = 0,\ldots, \ell-1$

$$(Y \ldots T^iY) = \begin{pmatrix} P_{10} & N_1^1 & N_1^2 & \cdots & N_1^i \\ 0 & P_{20} & N_2^2 & \cdots & N_2^i \\ 0 & 0 & P_{30} & & \vdots \\ \vdots & \vdots & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & N_i^i \\ & & \vdots & & P_{i+1,0} \\ 0 & 0 & 0 & & 0 \\ \vdots & \vdots & \vdots & \cdot & \vdots \\ 0 & 0 & 0 & & 0 \end{pmatrix}$$

for some $N_j^k \in L(X,\tilde{W}_{j-1})$. Notice that $N(P_{j+1,0}) = W_j$ and $P_{j+1,0}|\tilde{W}_j = I_{\tilde{W}_j}$; thus

$$R(Y \ldots T^{i-1}Y) = \begin{pmatrix} \mathrm{col}(\tilde{W}_j)_{j=o}^{i-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

is complemented in V and

$$N(Y\ldots T^{i-1}Y) \oplus \mathrm{col}(\tilde{W}_j)_{j=o}^{i-1} = X^i .$$

Hence row $(T^jY)_{j=o}^{i-1}$ has a generalized inverse for $i = 1,\ldots, \ell$ and is of course surjective for $i = \ell$. As an aside, notice that $W_i = (T^iY)^{-1}(\mathrm{row}(T^jY)_{j=o}^{i-1})$ and $X_o = N(Y)$.

Conversely, suppose row $(T^jY)_{j=o}^{i}$ has a generalized inverse for $i = 0,\ldots, \ell-1$. Due to remark 2.1.6, there exists a decomposition $X = \bigoplus_{j=o}^{\ell} X_j$, chosen as in 2.1.5, that reduces (T,Y) . For $i = 1,\dots,\ell$ let

$$\psi_i = \text{row}\,(T^j Y|\widetilde{W}_j)_{j=o}^{i-1} .$$

Then ψ_ℓ is invertible, ψ_i is left invertible ($i = 1,\dots,\ \ell-1$) and, for some $M_{ji} \in L(X_i,\widetilde{W}_{j-1})$ ($j = 1,\dots,i$), we have

$$T^i Y|_{X_i} = \psi_i\ \text{col}(M_{ji})_{j=1}^{i} .$$

Then

$$\psi_\ell^{-1} T\psi_\ell = \begin{pmatrix} M_{11} & 0 & M_{12} & 0 & 0 & \dots & M_{1,\ell-1} & 0 & M_{1\ell} \\ 0 & I_{\widetilde{W}_1} & M_{22} & 0 & & \dots & M_{2,\ell-1} & 0 & M_{2\ell} \\ & & 0 & I_{\widetilde{W}_2} & & \ddots & & & \\ & & & & & & M_{\ell-1,\ell-1} & 0 & \\ & & & & & & 0 & I_{\widetilde{W}_{\ell-1}} & M_{\ell\ell} \end{pmatrix} \quad (2.12)$$

In other words, if M denotes the right hand side of (2.12) we have to show

$\psi_\ell M = T\psi_\ell$.

Now, for $j = 1,\dots\ell$

$$T\psi_\ell|\widetilde{W}_{j-1} = TT^{j-1}Y|\widetilde{W}_{j-1}$$

$$= (T^j Y|X_j\ \ T^j Y|_{\widetilde{W}_j})$$

(recall $X_j \oplus \widetilde{W}_j = \widetilde{W}_{j-1}$) and

$$\psi_\ell M|_{\tilde{W}_{j-1}} = \psi_\ell \begin{pmatrix} M_{1j} & 0 \\ \vdots & \vdots \\ M_{jj} & 0 \\ 0 & I_{\tilde{W}_j} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}$$

$$= (\psi_j \cdot \operatorname{col}(M_{ij})_{i=1}^{j} \qquad T^j Y|_{\tilde{W}_j}).$$

Hence $T\psi_\ell|_{W_{j-1}} = \psi_\ell \, M|_{W_{j-1}}$ for all j, so $T\psi_\ell = \psi_\ell \, M$.

Also, $\psi_\ell \begin{pmatrix} 0 & I_{\tilde{W}_o} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} = (0 \quad Y|\tilde{W}_o) = Y$

and thus $\psi_\ell^{-1} Y = \begin{pmatrix} P_{10} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

At this point we apply 2.3.3 with $V_j = \tilde{W}_j$ to obtain $\psi \in L(\bigoplus_{i=o}^{\ell-1} \tilde{W}_i)$ and $M_j \in L(\tilde{W}_{j-1}, \tilde{W}_o)$ as in the lemma. Let $\varphi = \psi_\ell \, \psi^{-1}$, then $\varphi^{-1} T\varphi = \psi(\psi_\ell^{-1} T\psi_\ell)\psi^{-1}$ and is equal to the same expression as in (2.10).

Also, $\varphi^{-1} Y = \psi \begin{pmatrix} P_{10} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} P_{10} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

Now, defining a_{ij}^k $(i,j = 1,\ldots, \ell,\ k = 0,\ldots,j-1)$ so that (2.11) holds and choosing a_{0j}^k $(j = 1,\ldots, \ell\ ,\ k = 0,\ldots,j-1)$ arbitrarily in $L(X_j, X_o)$ we obtain a right reducible o.p. L with (T,Y) as left finite spectral pair if we define

$$L(\lambda) = (a_{ij}(\lambda))_{i,j=o}^{\ell}$$

where $$a_{ij}(\lambda) = \delta_{ij}\, \lambda^j I_{X_j} + \sum_{k=o}^{j-1} a_{ij}^k \lambda^k .$$

2.3.2. Theorem. A right admissible pair (X,T) is the right finite spectral pair for a left reducible o.p. of degree $\leq \ell$ if and only if $\mathrm{col}(XT^j)_{j=o}^{i}$ has a generalized inverse for $i = 0,1,\ldots, \ell-2$, and is left invertible for $i = \ell-1$.

Proof. If (X,T) is the right finite spectral pair for a left reducible o.p. L then, as in 2.3.1, using the explicit form of the right companion pair for L, one can show that $\mathrm{col}(XT^j)_{j=o}^{i}$ has a generalized inverse for $i = 0,1,\ldots, \ell-1$, and is of course injective for $i = \ell-1$.

Conversely, suppose $\mathrm{col}(XT^j)_{j=o}^{i}$ has a generalized inverse for $i = 0,1,\ldots, \ell-2$ and is left invertible for $i = \ell-1$. Due to remark 2.1.4, there exists a decomposition $X = \bigoplus_{j=o}^{\ell} X_j$, chosen as in 2.1.2, that reduces (X,T). For $i = 1,\ldots, \ell$ let

$$\psi_i = \mathrm{col}(Q_j XT^j)_{j=o}^{i-1} ,$$

where the projectors Q_j are defined as in 2.1.2 with $R(Q_j) = \widetilde{W}_j$. Then ψ_ℓ is invertible, ψ_i is right invertible $(i = 1,\ldots, \ell-1)$, and, for some $M_{ij} \in L(\widetilde{W}_{j-1}, X_i)$ $(j = 1,\ldots,i)$, we have

$$P_i XT^i = \mathrm{row}(M_{ij})_{j=1}^{i}\, \psi_i .$$

Here P_i is the projection of X onto X_i. The construction of a left reducible o.p. L having (X,T) as right finite spectral pair is now completely analogous to 2.3.1 and the details are omitted.

2.3.3. Lemma. Given a decreasing sequence of subspaces $V_o \supseteq V_1 \supseteq \cdots \supseteq V_{\ell-1} \supseteq V_\ell = 0$, and subspaces X_i $(i = 1,\ldots,\ell)$ such that $X_i \oplus V_i = V_{i-1}$ and operators $M_{ij} \in L(X_j, V_{i-1})$, $i \leq j$, define the operator $M \in L(\bigoplus_{i=o}^{\ell-1} V_i)$ as in the right hand side of (2.12). There exist $\psi \in L(\bigoplus_{i=o}^{\ell-1} V_i)$ and $M_j \in L(V_{j-1}, V_o)$ $(j = 1,\ldots,\ell)$ such that

$$\psi M = \begin{pmatrix} M_1 & M_2 & \cdots & & M_\ell \\ P_{21} & 0 & & & 0 \\ & P_{32} & & & \vdots \\ & & \ddots & & \vdots \\ & & & P_{\ell,\ell-1} & 0 \end{pmatrix} \psi, \tag{2.13}$$

and in block matrix form ψ is upper triangular with identity along the diagonal.

Proof. We need to find ψ in the form $\psi = (S_{ij})_{i,j=1}^{\ell}$, where $S_{ij} \in L(V_{j-1}, V_{i-1})$, $S_{ij} = 0$ for $i > j$, and $S_{ii} = I_{V_{i-1}}$. Now, direct computation yields

$$\psi M = \begin{pmatrix} N_{11} & S_{12} & N_{12} & S_{13} & \cdots & & N_{1,\ell-1} & S_{1\ell} & N_{1\ell} \\ 0 & I_{V_1} & N_{22} & S_{23} & \cdots & & N_{2,\ell-1} & S_{2\ell} & N_{2\ell} \\ & & 0 & I_{V_2} & & & \vdots & \vdots & \vdots \\ & & & & \ddots & & & & \\ & & & & 0 & I_{V_{\ell-2}} & N_{\ell-1,\ell-1} & S_{\ell-1,\ell} & N_{\ell-1,\ell} \\ & & & & & & 0 & I_{V_{\ell-1}} & N_{\ell\ell} \end{pmatrix}$$

where $N_{ij} = \sum_{m=i}^{j} S_{im} M_{mj}$, $j \geq i$, $i = 1,\ldots,\ell$.

Also, the right hand side of (2.13) is

$$\begin{pmatrix} \tilde{M}_1 & \tilde{M}_2 & \cdot & \cdot & \cdot & \tilde{M}_{\ell-1} & \tilde{M}_\ell \\ P_{21} & P_{21}S_{12} & \cdot & \cdot & \cdot & P_{21}S_{1,\ell-1} & P_{21}S_{1\ell} \\ & P_{32} & P_{32}S_{23} & & & P_{32}S_{2,\ell-1} & P_{32}S_{2\ell} \\ & & \cdot & \cdot & & \vdots & \vdots \\ & & & \cdot & \cdot & & \\ & 0 & & & P_{\ell-1,\ell-2} & P_{\ell-1,\ell-2}S_{\ell-2,\ell-1} & P_{\ell-1,\ell-2}S_{\ell-2,\ell} \\ & & & & & P_{\ell,\ell-1} & P_{\ell,\ell-1}S_{\ell-1,\ell} \end{pmatrix}$$

where $\tilde{M}_j = \sum_{m=1}^{j-1} M_m S_{mj} + M_j \quad (j = 1,\ldots,\ell\,)$.

Let $P_{ij} \in L(V_{j-1},V_{i-1})$, $j \leq i$, be the natural projections. Now, choose $S_{\ell-1,\ell} \in L(V_{\ell-1},V_{\ell-2})$ such that $P_{\ell,\ell-1}\, S_{\ell-1,\ell} = N_{\ell\ell}$.

Having chosen $S_{ij} \in L(V_{j-1},V_{i-1})$ for $i = k,k+1,\ldots,\ell-1$, $j = i+1,\ldots,\ell$ choose $S_{k-1,j} \in L(V_{j-1},V_{k-2})$ for $j = k,k+1,\ldots,\ell$ such that

$$P_{k,k-1}S_{k-1,j} = \begin{cases} (N_{kk}S_{k,j+1}) & j = k,k+1,\ldots,\ell-1 \\ N_{k\ell} & j = \ell \end{cases} .$$

This defines the components of ψ by induction. Finally, M_j $(j = 1,\ldots,\ell\,)$ are defined successively such that

$$M_j + \sum_{m=1}^{j-1} M_m S_{mj} = \begin{cases} (N_{1j} \; S_{1,j+1}) & j = 1,\dots, \ell-1 \\ N_{1\ell} & j = \ell \end{cases}$$

There is an analogous lemma needed for the proof of 2.3.2, but for the sake of brevity this is omitted.

For the main theorem on spectral factorization we need the following simple lemma. The proof is standard.

2.3.4 Lemma. Let $C \in L(X,Y)$ and $B \in L(Y,Z)$ where X,Y,Z are Banach spaces, and set $A = BC$. If C is right invertible then A has a generalized inverse if and only if B has a generalized inverse.

2.3.5 Theorem. Let L be an operator polynomial of degree $\leq \ell$ and γ a simple closed contour not intersecting $\sigma(L)$. Then L has a right factorization with respect to γ if and only if

$$\frac{1}{2\pi i} \int_\gamma \begin{pmatrix} I \\ \vdots \\ \lambda^i I \end{pmatrix} L^{-1}(\lambda) \; (I \; \dots \; \lambda^{\ell-1} I) d\lambda \tag{2.14}$$

has a generalized inverse for $i = 0,1,\dots, \ell-1$.

Proof. First of all, the statement of the theorem can be expressed in terms of spectral pairs. Let (X_+, T_+, Y_+) be a γ -spectral triple for L. Since (2.14) is equal to

$$\mathrm{col}(X_+ T_+^j)_{j=o}^{i} \cdot \mathrm{row}(T_+^j Y_+)_{j=o}^{\ell-1} ,$$

and the second factor is right invertible, lemma 2.3.4 shows that (2.14) has a generalized inverse if and only if $\mathrm{col}(X_+ T_+^j)_{j=o}^{i}$ has a generalized inverse.

Without loss of generality, 0 is in the interior of γ. Suppose L has a right factorization with respect to γ:

$$L(\lambda) = L_+(\lambda) \left(\sum_{i=1}^{\nu} \lambda^{\kappa_i} Q_i \right) L_-(\lambda) ,$$

where $L_\pm : \gamma \cup F^\pm \to GL(X)$ are continuous, and holomorphic in $F^\pm$. Let $L_1(\lambda) = \left(\sum_{i=1}^{\nu} \lambda^{\kappa_i} Q_i \right) L_-(\lambda)$. Without loss of generality, $L_-(\infty) = I$. I claim that L_+ is an o.p. and L_1 is a left reducible o.p.. Indeed,

$$\left(\sum_{i=1}^{\nu} \lambda^{-\kappa_i} Q_i \right) L_+^{-1}(\lambda)\, L(\lambda) = L_-(\lambda) .$$

The right hand side is holomorphic in F^- and the left hand side is holomorphic in $F^+ \setminus 0$ with at most a pole at 0. Hence L_- is a polynomial in λ^{-1} and, in fact, $Q_i L_-$ is a polynomial in λ^{-1} of degree $\leq \kappa_i$. Thus $\kappa_i \geq 0$ because otherwise $Q_i L_- = 0$ which implies L_- is not surjective for $\lambda \in \gamma$, a contradiction. It is now easy to see that L_1 is a left reducible o.p. Similarly,

$$L(\lambda) L_-^{-1}(\lambda) \left(\sum_{i=1}^{\nu} \lambda^{-\kappa_i} Q_i \right) = L_+(\lambda)$$

and it follows that L_+ is an o.p. and $\kappa_i \leq \ell$ for $i = 1, \dots, \nu$. Now, since $L = L_+ L_-$ is a γ-spectral factorization, 1.3.2 implies (X_+, T_+) is a right γ-spectral pair for L_1. But $\sigma(L_1) \subseteq F^+$ so 2.3.2 implies $\mathrm{col}(X_+ T_+^j)_{j=0}^{i}$ has a generalized inverse for $i = 0, 1, \dots, \ell-1$.

Conversely, suppose $\mathrm{col}(X_+ T_+^j)_{j=0}^{i}$ has a generalized inverse for $i = 0, 1, \dots, \ell-1$. Then 2.3.2 implies that there is a left reducible o.p. L_1 of degree $\leq \ell$ with right finite spectral pair (X_+, T_+). If (X_∞, T_∞) is a right spectral pair at ∞ for L_1 we know that $T_\infty^\ell = 0$ and so 1.3.5 implies L_1 is a right divisor of L, i.e.

$$L(\lambda) = L_+(\lambda)L_1(\lambda)$$

for some o.p. L_+ which is in fact invertible in $\gamma \cup F^+$ by 1.3.4 . Also,

$$L_1(\lambda) = \left(\sum_{i=o}^{\ell} \lambda^i P_i\right) L_-(\lambda)$$

for some L_-, an o.p. in λ^{-1} of degree $\leq \ell$, which is invertible in $\gamma \cup F^-$; here P_i is the projection of X onto the component X_i in the decomposition of X that reduces L_1. Thus L has a right factorization with respect to γ. This completes the proof of the theorem.

Similarly, we have

2.3.6. Theorem. Let L be an operator polynomial of degree $\leq \ell$ and γ a simple closed contour not intersecting $\sigma(L)$. Then L has a left factorization with respect to γ if and only if

$$\frac{1}{2\pi i} \int_\gamma \begin{pmatrix} I \\ \vdots \\ \lambda^{\ell-1} I \end{pmatrix} L^{-1}(\lambda) \; (I \; \ldots \; \lambda^i I) d\lambda$$

has a generalized inverse for $i = 0,1,\ldots \; \ell-1$.

2.3.7. Example. Let $X = X_o \oplus X_2$, $A \in L(X_2, X_o)$, and

$$L(\lambda) = \begin{pmatrix} I_{X_o} & -A\lambda \\ 0 & \lambda^2 I_{X_2} \end{pmatrix}$$

L is a right reducible o.p. with $\sigma(L) = \{0\}$ and the companion triple for L is

$$X = \begin{pmatrix} 0 & A \\ I_{X_2} & 0 \end{pmatrix},$$

$$T = \begin{pmatrix} 0 & I_{X_2} \\ 0 & 0 \end{pmatrix},$$

$$T = \begin{pmatrix} 0 & 0 \\ 0 & I_{X_2} \end{pmatrix}.$$

Now, $R(X) = R(A) \oplus X_2$ and $N(X) = N(A)$.. Since the pair (X,T) is reducible (whatever the choice of A) the hypotheses of the second part of 2.1.2 are not necessary. But, L has a right factorization relative to γ, where γ is a simple closed contour containing 0, if and only if A has a generalized inverse. Notice that L always has a left factorization relative to γ (cf. [11]).

The partial indices (right or left) are always unqiue; this is proved by a standard argument using Liouville's theorem. The next two theorems show how to determine these indices.

2.3.8. Theorem. Let L be an operator polynomial of degree $\leq \ell$ and γ a simple closed contour not intersecting $\sigma(L)$. Suppose L has a right factorization relative to γ. For $i = 0,1,\ldots,\ell$ let

$$M_i = \{\int_\gamma \lambda^i L^{-1}(\lambda) f(\lambda) d\lambda;\ f \text{ is a polynomial of degree } \leq \ell-1$$
$$\text{such that } \int_\gamma \lambda^j L^{-1}(\lambda) f(\lambda) d\lambda = 0 \text{ for } j = 0,1,\ldots,i-1\}.$$

Then $M_o, M_1, \ldots, M_\ell$ is a decreasing sequence of complemented subspaces of X and i is one of the right partial indices for L relative to γ if and only if $M_i \subsetneq M_{i-1}$.

Proof. Let (X_+, T_+, Y_+) be a γ-spectral triple for L. One can show easily, since $\mathrm{row}(T_+^j Y_+)_{j=o}^{\ell-1}$ is surjective, that

$$M_i = R(X_+ T_+^i | N(\mathrm{col}(X_+ T_+^j)_{j=o}^{i-1})) .$$

It follows that M is a complemented subspace of X and there is a decomposition $X = \bigoplus_{j=o}^{\ell} X_j$ such that, for $j = 0, \ldots, \ell-1$,

$$X_{i+1} \oplus \ldots \oplus X_\ell = M_i ,$$

$$X_o \oplus M_o = X$$

(see the proof of 2.3.1 and 2.3.2). Now, i is one of the right partial indices for L relative to γ if and only if $X_i \neq 0$, which clearly holds if and only if $M_i \subsetneq M_{i-1}$.

2.3.9 Corollary. Let the hypotheses be as in the theorem, and let k be a positive integer. The right partial indices of L relative to γ are $\geq$ k if and only if

$$\int_\gamma \begin{pmatrix} I \\ \vdots \\ \lambda^{k-1} I \end{pmatrix} L^{-1}(\lambda)\ (I \ \ldots \ \lambda^{\ell-1} I)\, d\lambda$$

is surjective, and are $\leq$ k if and only if

$$N\Big(\int_\gamma \begin{pmatrix} I \\ \vdots \\ \lambda^{k-1} I \end{pmatrix} L^{-1}(\lambda)(I \ldots \lambda^{\ell-1} I)\, d\lambda \Big) \subseteq N\Big(\int_\gamma \lambda^k L^{-1}(\lambda)\ (I \ \ldots \ \lambda^{\ell-1} I)\, d\lambda \Big).$$

Proof. From the proof of the theorem one sees that the right partial indices are $\geq$ k if and only if $M_{k-1} = X$, and are $\leq$ k if and only if $M_k = 0$, from which the corollary follows easily.

It is customary in finite dimensions to use a basis in X to write the middle term in a factorization for L in the form $\operatorname{diag}(\lambda^{\kappa_i})_{i=1}^n$, where $n = \dim X$ and $\kappa_1 \leq \ldots \leq \kappa_n$. Now, let

$$q_i = \operatorname{rank} \int_\gamma \begin{pmatrix} I \\ \vdots \\ \lambda^{i-1} I \end{pmatrix} L^{-1}(\lambda)\ (I \ \ldots \ \lambda^{\ell-1} I)\, d\lambda$$

$(i = 1, \ldots, \ell)$ and let $q_o = 0$, then $\dim M_i = q_{i+1} - q_i$. A simple counting argument based on the proof of 2.3.8 shows that

$$\kappa_j = \operatorname{card}\{i;\ q_{i+1} - q_i \geq n-j+1\}$$

for $j = 1, \ldots, n$, which is the formula of [9]. Furthermore, 2.3.9 can be regarded as a generalization of Lopatinskii's theorem [2, theorem 21].

2.3.10. Theorem. Let L be an operator polynomial of degree $\leq \ell$ and γ a simple closed contour not intersecting $\sigma(L)$. Suppose L has a left factorization relative to γ . For $i = 0,1,\ldots, \ell$ let

$$\widetilde{M}_i = \{x;\ \int_\gamma \begin{pmatrix} I \\ \vdots \\ \lambda^{\ell-1}I \end{pmatrix} \lambda^i L^{-1}(\lambda)d\lambda\ x \in R(\int_\gamma \begin{pmatrix} I \\ \vdots \\ \lambda^{\ell-1}I \end{pmatrix} L^{-1}(\lambda)(I \ldots \lambda^{i-1})d\lambda)\}.$$

Then $\widetilde{M}_o, \widetilde{M}_1, \ldots, \widetilde{M}_\ell$ is an increasing sequence of complemented subspaces of X and i is one of the left partial indices for L relative to γ if and only if $\widetilde{M}_{i-1} \subsetneqq \widetilde{M}_i$.

Proof. Let (X_+,T_+,Y_+) be a γ-spectral triple for L. Since $\mathrm{col}(X_+T_+^j)_{j=0}^{\ell-1}$ is injective, one can show that

$$\widetilde{M}_i = (T_+^i Y_+)^{-1}(R(\mathrm{row}(T_+^j Y_+)_{j=o}^{i-1})) .$$

Hence $\widetilde{M}_i$ is a complemented subspace of X and there is a decomposition $X = \bigoplus_{j=o}^{\ell} X_j$ such that, for $i = 0,1,\ldots, \ell$,

$$X_o \oplus \ldots \oplus X_i = \widetilde{M}_i .$$

Now, i is one of the left partial indices if and only if $X_i \neq 0$, which holds if and only if $\widetilde{M}_{i-1} \subsetneqq \widetilde{M}_i$.

2.3.11 Corollary. Let the hypotheses be as in the theorem, and let k be a positive integer. The left partial indices of L relative to γ are $\geq$ k if and only if

$$\int_\gamma \begin{pmatrix} I \\ \vdots \\ \lambda^{\ell-1}I \end{pmatrix} L^{-1}(\lambda)(I \ \ldots \ \lambda^{k-1}I)d\lambda$$

is injective, and are $\leq$ k if and only if

$$R\left(\int_\gamma \begin{pmatrix} I \\ \vdots \\ \lambda^{\ell-1}I \end{pmatrix} \lambda^{k} L^{-1}(\lambda)d\lambda\right) \subseteq R\left(\int_\gamma \begin{pmatrix} I \\ \vdots \\ \lambda^{\ell-1}I \end{pmatrix} L^{-1}(\lambda)(I \ldots \lambda^{k-1}I)d\lambda\right)$$

Proof. The right partial indices are $\geq k$ if and only if $\tilde{M}_{k-1} = 0$, and are $\leq k$ if and only if $\tilde{M}_k = X$, from which the corollary follows easily.

Let us return to example 2.3.7. We can determine the partial indices of L for various choices of $A \in L(X_2, X_o)$. The left partial indices L (relative to a simple closed contour containing 0) are always 0 and 2. For right factorization, assuming A has a generalized inverese, we have the following cases:

	right partial indices
A invertible	1
$N(A) = 0; R(A) \neq X_o$	0,1
$N(A) \neq 0, X_2; R(A) = X_o$	1,2
$N(A) \neq 0, X_2; R(A) \neq X_o$	0,1,2
$A = 0$	0,2

2.4 ORDINARY DIFFERENTIAL EQUATIONS

2.4.1 Proposition. Let L be an o.p. and γ a simple closed contour such that $\sigma(L) \cap \gamma = \emptyset$. Let $B(\lambda) = \sum_{j=o}^{m} B_j \lambda^j$ be an o.p. with coefficients $B_j \in L(X,Y)$, where Y is a Banach space. Then the following statements are equivalent:

(i) For any $y \in Y$ there is a unique $u \in V_L^+$ such that $B(\frac{d}{dt})u|_{t=o} = y$;

(ii) If (X_+, T_+) is a γ-spectral pair for L then $\sum_{j=o}^{m} B_j X_+ T_+^j$ is invertible.

<u>Proof.</u> V_L^+ is the set of all $u \in C^\infty(\mathbb{R},X)$ such that $u(t) = X_+e^{tT_+}x$ for some x. Also, if $u(t) = X_+e^{tT_+}x$ then $B(\frac{d}{dt})u|_{t=o} = (\sum_{j=o}^{m} B_jX_+T_+^j)x$. The proof of the proposition is now obvious.

For the rest of this section we will assume X is finite dimensional. First, a result that generalizes theorem 21 of [2] is given and then we prove an extension of 2.4.1.

2.4.2 <u>Theorem.</u> Let L be an o.p. of degree $\leq \ell$ and γ a simple closed contour such that $\sigma(L) \cap \gamma = \emptyset$. Let (X_+,T_+) be a γ-spectral pair of L and let D be a decomposition of X. Then the following statements are equivalent:

(i) There is a γ-spectral right divisor L^+ of L that is reducible with canonical decomposition D ;

(ii) D reduces (X_+,T_+) ;

(iii) $\int_\gamma B_D(\lambda) L^{-1}(\lambda) (I \dots \lambda^{\ell-1}I)d\lambda$ is surjective, where B_D is the o.p. with coefficients in $L(X, \overset{k}{\underset{j=1}{\oplus}} (X_j)^j)$ (here D is the decomposition $X = \overset{k}{\underset{j=o}{\oplus}} X_j$) defined by $B_D(\lambda)|X_o = 0$ and for $i = 1,\dots,k$

$$B_D(\lambda)|_{X_i} = \mathrm{col}(\lambda^j I_{X_i})_{j=o}^{i-1} \in (X_i)^i \subseteq \overset{k}{\underset{j=1}{\oplus}} (X_j)^j .$$

<u>Proof.</u> The equivalence of (i) and (ii) is true in infinite dimensions and is a consequence of 1.3.2 and 2.2.5. Also, D reduces (X_+,T_+) if and only if $\sum_{j=o}^{m} B_jX_+T_+^j$ is invertible, where $B_D(\lambda) = \sum_{j=o}^{m} B_j\lambda^j$. Thus the equivalence of (ii) and (iii) follows from 1.1.3 (ii)' and the fact that

$$\int_\gamma B_D(\lambda)L^{-1}(\lambda)(I\dots\lambda^{\ell-1}I)d\lambda = (\sum_{j=o}^{m} B_jX_+T_+^j)(Y_+\dots T_+^{\ell-1}Y_+).$$

2.4.3 Theorem. Let L be an o.p. and γ a simple closed contour such that $\sigma(L) \cap \gamma = \emptyset$. Let p^+ denote the number of roots of $M: = \det L$ inside γ (counted according to multiplicity), and let $B(\lambda) = \sum_{j=o}^{m} B_j \lambda^j$ be an o.p. with coefficients B_j in $L(X, \mathbb{C}^{p^+})$. Take $Y = \mathbb{C}^{p^+}$ in 2.4.1, then (i) and (ii) of 2.4.1 are also equivalent to the following statements:

(iii) $\int_\gamma B(\lambda) L^{-1}(\lambda)\, (I \ldots \lambda^{\ell-1} I) d\lambda$ is surjective, i.e. has rank p^+;

(iv) Factor M as $M^- M^+$ where M^+ contains all the roots of M inside γ. Let L^c denote the o.p. such that

$$L(\lambda) L^c(\lambda) = L^c(\lambda) L(\lambda) = M(\lambda) I .$$

Then the rows of $B(\lambda)\, L^c(\lambda)$ are linearly independent modulo $M^+(\lambda)$.

Proof. The equivalence of (i) and (ii) is proved as before. Notice that $\dim V_L^+ = p^+$ so existence is equivalent to uniqueness in (i). A similar statement holds for (ii). The operator in (iii) is equal to $(\sum_{j=o}^{m} B_j X_+ T_+^m)(Y_+ \ldots T_+^{\ell-1} Y_+)$. Using 1.1.3 (ii)' the equivalence of (ii) and (iii) follows easily.

(iii) $\Rightarrow$ (iv). If $x \in \mathbb{C}^{p+}$ such that $xB(\lambda) L^c(\lambda) = P(\lambda) M^+(\lambda)$ for some X-valued polynomial P, then $xB(\lambda)L^{-1}(\lambda)$ is holomorphic inside γ and so

$$x \int_\gamma B(\lambda) L^{-1}(\lambda)(I \ldots \lambda^{\ell-1} I) d\lambda = 0.$$

Thus (iii) implies $x = 0$, and (iv) holds.

(iv) $\Rightarrow$ (iii). There exists a reducible γ-spectral right divisor L^+ of L, by 2.4.2 and 2.1.2. Also (iii), and hence (iv), holds if B is replaced by B_D, where D is the canonical decomposition of X determined by L^+. Write

$$B = QL^+ + R$$

where Q and R are operator polynomials and, with respect to the decomposition D, $R = (r_{ij})_{i,j=o}^{k}$ with $\deg r_{ij} < j$. Then there is a unique $p^+ \times p^+$ matrix S such that $R(\lambda) = SB_D(\lambda)$. Now,

$$\int_\gamma B(\lambda)L^{-1}(\lambda)(I\ldots\lambda^{\ell-1}I)d\lambda = S\int_\gamma B_D(\lambda)L^{-1}(\lambda)(I\ldots\lambda^{\ell-1}I)d\lambda$$

is surjective, which implies L is invertible. Since $L = L^-L^+$ for some o.p. L^- then $L^c = (L^+)^c(L^-)^c$ and

$$\begin{aligned} BL^c &= QL^+L^c + SB_DL^c \\ &\equiv S\,B_DL^c \text{ modulo } M^+ \quad . \end{aligned}$$

Hence BL^c is linearly independent modulo M^+.

With the assumptions as in 2.4.3, an o.p. B with coefficients in $L(X,\mathbb{C}^{p+})$ is said to cover (or complement) L with respect to γ if (iv) is satisfied. This is the condition used by Agmon, Douglis, Nirenberg in their paper: "Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II", Comm. Pure. Appl. Math. 17 (1964), 35-92. The condition (iii) was that used by Lopatinskii in: "A method of reducing boundary problems for a system of differential equations of elliptic type to regular elliptic equations", Ukrain. Mat. Z. 5 (1953), 123-151. A direct proof of the equivalence of these two conditions does not seem to exist in the literature.

2.5 OTHER RESULTS

Let (X,T) be an ℓ-independent admissible pair. Let $X = \bigoplus_{j=o}^{\ell} X_j$ be a decomposition D of X and define $W_i = \bigoplus_{j=o}^{i} X_j$ for $i = 0,\ldots,\ell$. Then it is easy to see that (X,T) is reduced by D if and only if

$$\mathrm{col}(W_j)_{j=o}^{\ell-1} \oplus R(\mathrm{col}(XT^j)_{j=o}^{\ell-1}) = X^\ell . \qquad (2.15)$$

Indeed, let Q_i denote the projection of X onto $X_{i+1} \oplus \ldots \oplus X_\ell$ along $X_o \oplus \ldots \oplus X_i$, as in 2.1.2. Then $W_i = R(I-Q_i)$ and (X,T) is reduced by D if and only if $\mathrm{col}(Q_i\, XT^i)_{i=o}^{\ell-1}$ is invertible, or if and only if (2.15) holds.

This shows that reducibility of admissible pairs is very closely related to the special left inverses of $\mathrm{col}(XT^j)_{j=o}^{\ell-1}$ used in [5,6], except that here it is not necessary for T to be invertible. Given a reducible admissible pair (X,T), 2.2.5 showed how to construct a reducible o.p. with (X,T) as finite spectral triple. The next proposition shows that this construction is really the same as that in [5,6].

2.5.1 <u>Proposition.</u> Let L be a right reducible o.p. such that $0 \notin \sigma(L)$, and with canonical decomposition $X = \bigoplus_{j=o}^{\ell} X_j$. Let (X,T) be a finite spectral pair for L and define V_j^i for $1 \le j \le \ell$ and $1 \le i \le j$ as in 2.2.5. Also, let $V_j^i = 0$ for $j < i$, and set $V^i = (V_o^i \ldots V_\ell^i)$. Then $(V^1 \ldots V^\ell)$ is a left inverse of $\mathrm{col}(XT^j)_{j=o}^{\ell-1}$ and

$$L(\lambda) = L(0)\,\{\, I - XT^{-\ell}(V^\ell\lambda^\ell + \ldots + V^1\lambda)\} . \qquad (2.16)$$

Also, $(V_1 \ldots V_\ell) = T^{\ell-1}(V^\ell \ldots V^1)$ is a left inverse to $\mathrm{col}(XT^{-j})_{j=o}^{\ell-1}$ and

$$L(\lambda) = L(0)\,\{I - XT^{-\ell}(V_1\lambda^\ell + \ldots + V_\ell\lambda\ )\}. \qquad (2.17)$$

<u>Proof.</u> Since $\sum_{j=1}^{\ell} \sum_{i=1}^{\ell} V_j^i X_j T^{i-1} = I$ it follows that

$$I = \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} V_j^i X_j T^{i-1}$$

$$= \sum_{i=1}^{\ell} V^i XT^{i-1} ,$$

which proves $(V^1 \ldots V^\ell)$ is a left inverse of $\mathrm{col}(XT^j)_{j=o}^{\ell-1}$. Now, $A_oX + A_1XT + \ldots + A_\ell XT^\ell = 0$ so

$$A_oXT^{-1} = -(A_1 \ldots A_\ell)\ \mathrm{col}(XT^j)_{j=o}^{\ell-1} .$$

But L is reducible so $A_i|W_{i-1} = 0$ and

$$N(V^1 \ldots V^\ell) = \mathrm{col}(W_i)_{i=o}^{\ell-1}$$

$$\subseteq N(A^1 \ldots A^\ell) .$$

Hence, $$-A_oXT^{-1}(V^1 \ldots V^\ell) = (A_1 \ldots A_\ell).$$

Now, (2.16) and (2.17) follow immediately.

Let L be an o.p. of degree $\leq \ell$, and for $j = 1,\ldots,\ell$ define $L_j(T,Y) = \sum_{i\ o}^{\ell-j} T^iYA_{i+j}$. It is a consequence of (1.5) that $\mathrm{row}(L_j(T,Y))_{j=1}^{\ell}$ is a left inverse of $\mathrm{col}(XT^j)_{j=o}^{\ell-1}$. In the case that L is a right reducible o.p. we have $V^j = L_j(T,Y)$. The next theorem can thus be regarded as an extension of 2.5.1.

2.5.2 <u>Theorem.</u> Admissible pairs (X,T) and (X_∞,T_∞), with $\sigma(T_\infty) = 0$, are finite and infinite spectral pairs, respectively, for an o.p. L of degree $\leq \ell$ such that $\sigma(L)$ is compact if and only if

$$\begin{pmatrix} X & X_\infty T_\infty^{\ell-1} \\ \vdots & \vdots \\ XT^{\ell-1} & X_\infty \end{pmatrix} \qquad (2.18)$$

is invertible. Moreover, if (X,T,Y) and $(X_\infty,T_\infty,Y_\infty)$ are finite and infinite spectral triples for an o.p. of degree $\leq \ell$ such that $\sigma(L)$ is compact, then (2.18) has inverse

$$\begin{pmatrix} L_1(T,Y) & \ldots\ldots & L_\ell(T,Y) \\ \tilde{L}_\ell(T_\infty,Y_\infty)\ldots & & \tilde{L}_1(T_\infty,Y_\infty) \end{pmatrix} \tag{2.19}$$

where $\tilde{L}_{\ell-j}(T_\infty,Y_\infty) = \sum_{k=o}^{j} T_\infty^{j-k} Y_\infty A_k$ for $j = 0,\ldots, \ell-1$.

<u>Proof.</u> We prove the last statement of the theorem first. Define

$$Q = \mathrm{col}(XT^j)_{j=o}^{\ell-1}, \quad Q_\infty = \mathrm{col}(X_\infty T_\infty^{\ell-1-j})_{j=o}^{\ell-1}, \quad W = \mathrm{row}(L_j(T,Y))_{j=1}^{\ell},$$

and $W_\infty = \mathrm{row}(\tilde{L}_{\ell-j}(T_\infty,Y_\infty))_{j=o}^{\ell-1}$.

Now, if Γ is a simple closed contour containing $\sigma(L)$ then

$$XT^i L_{j+1}(T,Y) = \sum_{k=j+1}^{\ell} XT^{k+i-j-1} Y A_k$$

$$= \frac{1}{2\pi i} \int_\gamma \lambda^{i-j-1} L^{-1}(\lambda) \left(\sum_{k=j+1}^{\ell} A_k \lambda^k \right) d\lambda$$

and also,

$$X_\infty T_\infty^{\ell-1-i} \tilde{L}_{\ell-j}(T_\infty,Y_\infty) = \sum_{k=o}^{j} X_\infty T_\infty^{\ell-1-i+j-k} Y_\infty A_k$$

$$= \frac{1}{2\pi i} \int_\gamma \lambda^{i-j-1} L^{-1}(\lambda) \left(\sum_{k=o}^{j} A_k \lambda^k \right) d\lambda .$$

Then, for $i,j = 0,1,\ldots, \ell-1$

$$XT^i L_{j+1}(T,Y) + X_\infty T_\infty^{\ell-1-i} \tilde{L}_{\ell-j}(T_\infty,Y_\infty) = \delta_{ij} I$$

and hence $QW + Q_\infty W_\infty = I_{X^\ell}$. From (1.5) it follows that $WQ = I_V$ and, similarly, $W_\infty Q_\infty = I_{V_\infty}$. To complete the proof that (2.18) is invertible with inverse (2.19) it is left to show that $W_\infty Q = 0$ and $WQ_\infty = 0$, and for this lemma 2.5.3 is used.

$$WQ_\infty = (Y \ \dots \ T^{\ell-1}Y)B\begin{pmatrix} X_\infty T_\infty^{\ell-1} \\ \vdots \\ X_\infty \end{pmatrix}$$

$$= -(Y \ \dots \ T^{\ell-1})\sigma \, \widetilde{B}\begin{pmatrix} X_\infty \\ \vdots \\ X_\infty T_\infty^{\ell-1} \end{pmatrix} \quad T_\infty^{\ell}$$

$$= - \ (T^{\ell-1}Y \ \dots \ Y)\widetilde{B}\begin{pmatrix} X_\infty \\ \vdots \\ X_\infty T_\infty^{\ell-1} \end{pmatrix} \quad T_\infty^{\ell}$$

$$= \ T^{\ell}(Y \ \dots \ T^{\ell-1}Y)B\begin{pmatrix} X_\infty T_\infty^{\ell-1} \\ \vdots \\ X_\infty \end{pmatrix} \quad T_\infty^{\ell}$$

$$= \ T^{\ell}(WQ_\infty)T_\infty^{\ell} \quad .$$

Iteration yields, for all $i = 1,2,\dots$

$$\|WQ_\infty\|^{1/\ell i} \leq \text{const.}\|T\| \ \|T_\infty^{\ell}\|^{1/\ell i} \quad .$$

Since the right hand side tends to zero as $i \to \infty$, this implies $WQ_\infty = 0$. Similarly, $W_\infty Q = 0$.

Conversely, suppose (2.18) is invertible. Choose $a \notin \sigma(T)$ and replace T by $T-aI$ and T_∞ by $T_\infty(I-aT_\infty)^{-1}$ in (2.18). One can show that the resulting operator is still invertible.

Hence $\mathrm{col}(\widetilde{X}\widetilde{T}^{i})_{i=o}^{\ell-1}$ is invertible, where

$$\widetilde{X} = (X \ X_\infty)$$

and

$$\widetilde{T} = \begin{pmatrix} (T-aI)^{-1} & 0 \\ 0 & T_\infty(T-aT_\infty)^{-1} \end{pmatrix} \quad .$$

Proposition 2.2.5 implies that $(\tilde{X},\tilde{T})$ is a finite spectral pair of a monic o.p. $\tilde{L}$. Define

$$L(\lambda) = (\lambda - a)^{\ell}\,\tilde{L}((\lambda - a)^{-1}) ,$$

then 1.4.1 implies that L has (X,T) as finite spectral pair and (X_∞,T_∞) as spectral pair at ∞ . This completes the proof of the theorem.

2.5.3 <u>Lemma</u>. Let L be an o.p. of degree $\leq \ell$. Define B as in 1.1.9, and $\tilde{B}$ with A_j replaced by $\tilde{A}_j = A_{\ell-j}$. Let σ be the permutation of blocks in X^{ℓ} that maps the j^{th} block to the $(\ell-j)^{th}$ block. If (X,T) is an admissible pair such that $\sum_{i=o}^{\ell} A_i XT^i = 0$ then

$$\tilde{B}\begin{pmatrix} XT^{\ell-1} \\ \vdots \\ X \end{pmatrix} = -\,\sigma B \begin{pmatrix} X \\ \vdots \\ XT^{\ell-1} \end{pmatrix} T^{\ell} \tag{2.20}$$

Similarly, if (T,Y) is an admissible pair such that $\sum_{i=o}^{\ell} T^i Y A_i = 0$ then

$$(T^{\ell-1}Y \;\ldots\; Y)\tilde{B} = -T^{\ell}(Y \;\ldots\; T^{\ell-1}Y)B\sigma .$$

<u>Proof.</u> To establish (2.20) we must show that for $j = 0,\ldots, \ell-1$

$$A_{\ell-1-j}XT^{\ell-1} +\ldots + A_o XT^j = -(A_{\ell-j}X +\ldots+ A_{\ell}XT^j)T^{\ell} .$$

But, this is an immediate consequence of $(\sum_{i=o}^{\ell} A_i XT^i)T^j = 0.$

The proof for (T,Y) is similar.

The next proposition considers the inverse problem for spectral triples.

2.5.4 Proposition. Admissible triples (X,T,Y) and $(X_\infty,T_\infty,Y_\infty)$, with $\sigma(T_\infty) = 0$, are finite and infinite spectral triples for an o.p. L of degree $\leq \ell$ such that $\sigma(L)$ is compact if and only if

(i) (2.18) is invertible;

(ii) $X_\infty T_\infty^i Y_\infty = XT^{\ell-2-i}Y$ for $i = 0,1,\ldots \ell-2$;

(iii) $X(aI-T)^{-1}Y + X_\infty T_\infty^{\ell-1}(I-aT_\infty)^{-1}Y_\infty$ is invertible for some $a \in \mathbb{C}$.

Proof. If (X,T,Y) and $(X_\infty,T_\infty,Y_\infty)$ are finite and infinite spectral triples for L then 2.5.2 implies (i), and 1.2.2 implies (ii) and (iii) .

Conversely, suppose (i) - (iii) hold for admissible triples (X,T,Y) and $(X_\infty,T_\infty,Y_\infty)$, with $\sigma(T_\infty) = 0$. Choose $a \notin \sigma(T)$ so that (iii) holds and define $\tilde{X},\tilde{T}$, and $\tilde{L}$ as in the second part of the proof of 2.5.2. Also, define

$$\tilde{Y} = \begin{pmatrix} -(T-aI)^{\ell-2}Y \\ (I-aT_\infty)^{\ell-2}Y \end{pmatrix} ,$$

then

$$\tilde{X}\tilde{T}^i\tilde{Y} = -X(T-aI)^{\ell-2-i}Y + X_\infty T_\infty^i(I-aT_\infty)^{\ell-2-i}Y_\infty$$

$$= 0 \quad \text{for } i = 0,\ldots, \ell-2$$

and is invertible for $i = \ell-1$. It follows that $(\tilde{X},\tilde{T},\tilde{Y})$ is a finite spectral triple for $C\tilde{L}$ for some invertible $C \in L(X)$. Define $L(\lambda) = (\lambda-a)^\ell\, C\tilde{L}((\lambda-a)^{-1})$, then 1.4.1 implies L has (X,T,Y) as finite spectral triple and $(X_\infty,T_\infty,Y_\infty)$ as spectral triple at ∞ .

2.5.5. Remark. We can rewrite the condition in (iii) as: there exists $a \in \mathbb{C}$ such that

$$\sum_{i=o}^{\infty} (XT^iY)a^{-(i+1)} + \sum_{i=o}^{\infty} (X_\infty T_\infty^{\ell-1+i} Y_\infty)a^i$$

is invertible.

2.5.6. Remark . The proof of 2.5.4 shows that (ii) and (iii) imply

$$\mathrm{col}(\tilde{X}\,\tilde{T}^i)_{i=o}^{\ell-1} \cdot \mathrm{row}(\tilde{T}^i\tilde{Y})_{i=o}^{\ell-1}$$

is invertible. Thus if X has finite dimension n one can replace (i) by

(i)' $\dim V + \dim V_\infty = n\ell$

where V and V_∞ are the base spaces of (X,T,Y) and $(X_\infty, T_\infty, Y_\infty)$, respectively.

Assume now that X has finite dimension n. The next theorem shows what sequences of operators can be the Fourier coefficients of the inverse of an o.p.. This result partially generalizes one in F. Gantmacher, The Theory of Matrices, p.207, where rational functions for the case n = 1 are considered.

Given $c_i \in L(X)$ $(i = 0,1,2,\dots)$, let $H(c_i)_{i=o}^{\infty}$ denote the block "Toeplitz" or "Hankel" operator

$$\begin{pmatrix} c_o & c_1 & c_2 & \cdots \\ c_1 & c_2 & \cdots & \\ c_2 & \cdots & & \\ \vdots & & & \end{pmatrix} .$$

We say that $H(c_i)_{i=o}^{\infty}$ has degree ℓ if

$$\mathrm{rank}\; H(c_i)_{i=o}^{\infty} = \mathrm{rank} \begin{pmatrix} c_o & c_1 & c_2 & \cdots & c_{\ell-1} \\ c_1 & c_2 & \cdot & \cdot & c_\ell \\ c_2 & & \cdot & & \vdots \\ \vdots & \cdot & & & \vdots \\ c_{\ell-1} & c_\ell & & \cdots & c_{2\ell-2} \end{pmatrix} ,$$

and if ℓ is the smallest positive integer having this property. In the case there is no such ℓ we say that the degree is ∞. It is easy to verify that $H(c_i)_{i=0}^{\infty}$ has finite degree if and only if it has finite rank. If $c_i \in L(X)$ $(i = 0, \pm 1, \pm 2, \ldots)$ we say that $\deg(c_i)_{i=-\infty}^{\infty} = \ell$ if

$$\deg H(c_i)_{i=0}^{\infty} \le \ell \quad \text{and} \quad \deg H(c_{\ell-2-i})_{i=0} \le \ell \quad ,$$

and ℓ is the smallest positive integer having this property.

2.4.7 Theorem. Let $c_i \in L(X)$ $(i = 0, \pm 1, \pm 2, \ldots)$, with $c_{-i} = 0$ for all but finitely many positive integers i. Then there is an o.p. L of degree $\le \ell$ such that for $i = 0, \pm 1, \ldots$

$$c_i = \frac{1}{2\pi i} \int_{\Gamma} \lambda^i L^{-1}(\lambda)\, d\lambda \,, \tag{2.21}$$

where Γ is a simple closed contour containing $\sigma(L)$, if and only if

(i) $\quad \operatorname{rank} H(c_i)_{i=0}^{\ell} + \operatorname{rank} H(c_{\ell-2-i})_{i=0}^{\infty} = n\ell \quad ;$

(ii) $\quad \ell \ge \deg(c_i)_{i=-\infty}^{\infty} \quad ;$

(iii) $\quad \sum_{-\infty}^{\infty} c_i\, a^{-(i+1)}$ converges and is invertible for some $a \in C$.

Proof. If L is an o.p. of degree $\le \ell$ with the Fourier coefficients c_i let (X,T,Y) and $(X_\infty, T_\infty, Y_\infty)$ be finite and infinite spectral triples for L, respectively. Then

$$H(c_i)_{i=0}^{\infty} = \operatorname{col}(XT^i)_{i=0}^{\infty} \cdot \operatorname{row}(T^i Y)_{i=0}^{\infty} \tag{2.22}$$

$$H(c_{\ell-2-i})_{i=0}^{\infty} = \operatorname{col}(X_\infty T_\infty^i)_{i=0}^{\infty} \cdot \operatorname{row}(T_\infty^i Y_\infty)_{i=0}^{\infty} \,.$$

Since $\mathrm{col}(XT^i)_{i=o}^{\ell-1}$ is injective and $\mathrm{row}(T^iY)_{i=o}^{\ell-1}$ is surjective, (ii) follows. Also, rank $H(c_i)_{i=o}^{\infty} = \dim V$ and rank $H(c_{\ell-2-i})_{i=o}^{\infty} = \dim V_\infty$ so (i) is true. Finally, (iii) holds since, for a large,

$L^{-1}(a) = \sum_{-\infty}^{\infty} c_i \, a^{-(i+1)}$.

Notice that if (X,T,Y) is the natural finite spectral triple for L then

$$Y(t) = \frac{1}{2\pi i} \int_\Gamma e^{t\lambda} L^{-1}(\lambda) d\lambda$$

$$= \sum_{i=o}^{\infty} c_i \frac{t^i}{i!} \tag{2.23}$$

Similarly, if $(X_\infty, T_\infty, Y_\infty)$ is the natural spectral triple at ∞ for L then

$$Y_\infty(t) = \sum_{i=o}^{\infty} c_{\ell-2-i} \frac{t^i}{i!} \quad . \tag{2.24}$$

Now, suppose the operators c_i satisfy (i) - (iii). Condition (iii) implies, in particular, that there is an $\alpha > 0$ such that $\| c_i \alpha^i \| \leq$ constant for all $i = 0,1,2,\ldots,$ and thus

$$\sum_{o}^{\infty} \| c_i \frac{t^i}{i!} \| \leq \text{constant} \cdot \sum_{o}^{\infty} \frac{1}{i!} \left(\frac{|t|}{\alpha}\right)^i < \infty \quad .$$

Hence if we define $Y(t)$ by the expression (2.23) then Y is an analytic function $\mathbb{R} \to L(X)$, which can be regarded as a map $X \to C^\infty(\mathbb{R},X)$. Then, define the subspace $V = R(Y \ldots (\frac{d}{dt})^{\ell-1} Y)$ of $C^\infty(\mathbb{R},X)$. I claim that V is invariant under $\frac{d}{dt}$. Indeed, if $u \in V$ then for some $x \in X^\ell$

$$u = (Y \ \ldots \ (\frac{d}{dt})^{\ell-1} Y) x$$

$$= \sum_{i=o}^{\infty} \frac{t^i}{i!} r_i x \ ,$$

where $r_i = (c_i \ \ldots \ c_{i+\ell-1})$ for $i = 0,1,2, \ \ldots$.

Since $\ell \geq \deg H(c_i)_{i=o}^{\infty}$ there are operators $S_j \in L(X)$ $(j = 0,\ldots, \ell-1)$ such that

$$c_{i+\ell} = \sum_{j=o}^{\ell-1} c_{i+j} S_j \; .$$

Then

$$\frac{du}{dt} = \sum_{i=o}^{\infty} \frac{t^i}{i!} r_{i+1} x$$

$$= \sum_{i=o}^{\infty} \frac{t^i}{i!} r_i x' \quad ,$$

where $x' \in X^{\ell}$ is defined by $x'_o = S_o x_\ell$ and $x'_j = x_{j-1} + S_j x_\ell$ for $j = 1,\ldots \ell-1$. Thus $\frac{du}{dt} \in V$, and V is invariant under $\frac{d}{dt}$. Let $T \in L(V)$ denote the restriction of $\frac{d}{dt}$ to V and define $X \in L(V,X)$ as $Xu = u(0)$. By definition $(Y \ldots T^{\ell-1}Y)$ is surjective and I claim that $\operatorname{col}(XT^j)_{j=o}^{\ell-1}$ is injective. Indeed, if $u = \sum_{i=o}^{\infty} \frac{t^i}{i!} r_i x$ such that

$$0 = \operatorname{col}(XT^j)_{j=o}^{\ell-1} x$$

$$= \operatorname{col}(r_i)_{i=o}^{\ell-1} x$$

then the fact that $\ell \geq \deg H(c_i)_{i=o}^{\infty}$ implies $r_i x = 0$ for all i and hence $u = 0$. Now. (2.22) holds and it follows that $\dim V = \operatorname{rank} H(c_i)_{i=o}^{\infty}$. Analogous definitions for Y_∞, starting with (2.24), and for V yields an admissible triple $(X_\infty, T_\infty, Y_\infty)$ such that $\dim V = \operatorname{rank} H(c_{\ell-2-i})_{i=o}^{\infty}$. Notice that T_∞ is nilpotent since $c_{-i} = 0$ for i large. Also, $XT^iY = c_i$ and $X_\infty T_\infty^i Y_\infty = c_{\ell-2-i}$ for $i = 0,1,2,\ldots$. It follows from 2.5.4, and remark 2.5.5 and 2.5.6, that (X,T,Y) and $(X_\infty, T_\infty, Y_\infty)$ are finite and infinite spectral triples for an o.p. of degree $\leq \ell$. Then (2.21) holds, and the proof of the theorem is complete.

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