FLAT MODULES

by

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CHAPTER 0

1. The Tensor-Product and the Hom. Functors

Let R denote a ring, not necessarily commutative, with an identity element 1. Throughout we will concern ourselves with the Categories of right (left) R-modules, where R is a fixed ring.

$\emptyset_R$ will denote the category of right R-modules, whose typical objects will be written as A_R, B_R, E_R, etc. Similiarly, ${}_R\emptyset$ will denote the category of left R-modules, with objects denoted by ${}_RM, {}_RN$, etc.

Let $\emptyset_Z$ denote the category of Z-modules (abelian groups).

All maps will be R-homomorphisms.

We will not define the tensor-product formally, but merely recall its well known properties as a functor $T: \emptyset_R \times {}_R\emptyset \longrightarrow \emptyset_Z$. i.e. T is a covariant functor in both variables, additive and right exact. Let $E_R, {}_RM$ be arbitrary right and left R-modules respectively. We can characterize the tensor-product $E \otimes_R M$ by the universal property that $\forall$[1] middle-linear function $f: E \times M \longrightarrow G$, where G is a Z-module~~s~~, $\exists$ a unique Z-homomorphism g, $\ni$ the diagram

$$\begin{tikzcd} E \otimes_R M \arrow[dr, dashed, "g"] & \\ E \times M \arrow[u] \arrow[r, "f"] & G \end{tikzcd}$$

is commutative. i.e. $g(e \otimes m) = f(e,m)$, $e \in E, m \in M$.

For a given E_R, we will be particularily interested in the functor $T_E(.): {}_R\emptyset \longrightarrow \emptyset_Z$, defined by $T_E(M) = E \otimes_R M$, and if $f: M \longrightarrow N$,

1. We use the standard abbreviations $\forall, \exists, \ni$ which mean respectively: "for every", "there exists", and "such that".

then $T_E(f)=1_E \otimes f$.

We will require the following known results:

(0.1) The homomorphism $f: E \otimes M \longrightarrow E$, defined by $f(e \otimes r)=er$, is a Z-homomorphism. See [7]

(0.2) Let F_R be free, with basis $\langle x_i \rangle_{i \in I}$, and let ${}_R M$ be arbitrary. Then, each element of $F \otimes_R M$ has a unique representation in the form $\sum_i x_i \otimes m_i$, where $m_i \in M$ and $m_i = 0$ for all but finitely many i.

Since, in general, there is no ambiguity about the base ring, we will denote $T_E(M)=E \otimes_R M$ simply by $E \otimes M$.

Let ${}_R M, {}_R N$ be any two arbitrary R-modules, then $\mathrm{Hom}_R(MN)$ denotes the group of all R-homomorphisms $f: M \longrightarrow N$. Regarding Hom. as a functor $F: {}_R\mathcal{C} \times {}_R\mathcal{C} \longrightarrow \mathcal{C}_Z$, we know that it is additive, left exact, contravariant in M and covariant in N.

Again, we will be interested mainly, for a fixed ${}_R N$, with the functor $F_N(.): {}_R\mathcal{C} \longrightarrow \mathcal{C}_Z$, defined by $F_N(M)=\mathrm{Hom}_R(M,N)$. Thus, F_N is a contravariant functor in one variable. Let $f: M' \longrightarrow M$, then $F_N(f)=\mathrm{Hom}(f,1_N)=g \circ f$, where g is a variable element of $\mathrm{Hom}_R(M,N)$.

Recall that a module ${}_R N$ is <u>R-injective</u>, if given any diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & {}_R M & \xrightarrow{f} & M' \\ & & \downarrow g & & \\ & & N & & \end{array}$$

with the top row exact (i.e. f is a monomorphism), then there always exists a map $h: M' \longrightarrow N \ni h \circ f = g$.

<u>Proposition (0.3)</u>: F_N is an exact functor iff N is injective.

<u>Proof:</u>

Suppose F_N is an exact functor, and consider the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & {}_RM & \xrightarrow{f} & M' \\ & & \downarrow & & \\ & & N & & \end{array}$$

where the row is exact. Then,

$$F_N(M') \xrightarrow{F_N(f)} F_N(M) \longrightarrow 0 \text{ is exact.}$$

i.e. $\mathrm{Hom}_R(M',N) \longrightarrow \mathrm{Hom}_R(M,N)$ is an epimorphism.

Thus, given g, $\exists$ an $h \in \mathrm{Hom}_R(M',N)$, $\ni$ $h \circ f = g$

Hence, N is injective.

Conversly,

Let $0 \longrightarrow M' \xrightarrow{h} {}_RM \longrightarrow M'' \longrightarrow 0$ be exact.

Since F_N is an left exact functor, it suffices to show that

$$F_N(M) \xrightarrow{F_N(h)} F_N(M') \text{ is an epimorphism.}$$

But, if N is injective, then for any $f: M' \longrightarrow N$, $\exists$ $g: M \longrightarrow N$, $\ni$ $g \circ f = h$

i.e. $F_N(h)$ is an epimorphism.

<u>2. The Character-Group.</u>

Let Q denote the rationals, then Q/Z (the rationals modulo 1), is a divisible abelian group. But since a Z-module is divisible iff it is injective, (see [6]), Q/Z is injective. Of special interest is the functor $F'_{Q/Z}(\cdot): \mathcal{C}_R \longrightarrow \mathcal{C}_Z$, defined by $F'_{Q/Z}(E) = \mathrm{Hom}_Z(E, Q/Z)$. It is called the <u>character-group</u> of E, and is denoted by E^*. In view of the previous proposition, E^* is an exact functor. Furthermore, for any E_R, E^* may be given a structure of a left R-module, if we define for $f \in E^*$, the map rf by $rf(x)=f(xr)$, where $r \in R, x \in E$.

The following results will be of importance:

(0.4) There is a natural equivalence of the functors $(T_E)^* = F_{E^*}$.

i.e. the canonical map $f_M : (E \otimes M)^* \longrightarrow \mathrm{Hom}_R(M, E^*)$ is an isomorphism for every ${}_R M$. We note that both T_E^* and F_{E^*} are contravariant functors from ${}_R\phi \longrightarrow \phi_Z$.

To say that f_M is a natural isomorphism, we mean that given any map $g : M' \longrightarrow M$, then

$$\begin{array}{ccc} (E\otimes M)^* & \xrightarrow{\mathrm{Hom}(1_E \otimes g, 1)} & (E \otimes M')^* \\ \downarrow f_M & & \downarrow f_{M'} \\ \mathrm{Hom}_R(M, E^*) & \xrightarrow{\mathrm{Hom}(g,1)} & \mathrm{Hom}_R(M', E^*) \end{array}$$

is a commutative diagram.

(0.5) A module ${}_R N$ is injective iff for every left ideal ${}_R I$, and for every $f \in \mathrm{Hom}_R(I, N)$, $\exists\, x \in N \ni$ for all $r \in I$, $f(r) = rx$. (See [6])

Let $\prod_i E_i$ denote the direct-product of a family $\langle E_i \rangle$ of R-modules, and let $\sum_i E_i$ denote thier direct-sum. Given a finite family of R-modules, we will sometimes denote the direct sum by $E_1 \oplus E_2 \oplus \ldots \oplus E_n$.

(0.6) The direct-product of any family of injective (right) R-modules is injective iff each module is injective. (See [7])

(0.7) In any category $\not{C}_R$, $\mathrm{Hom}_R(\sum_i A_i, B) \cong \prod_i(\mathrm{Hom}(A_i,B))$ (See [3])

3. Direct-system of R-modules.

Let I be a directed set. i.e. $(I,<)$ is partially-ordered, and for every $i,j\in I$, $\exists$ $k\in I \ni i<k$ and $j<k$.

Let $< A_i, f_i^j : i<j\in I >$ be a collection of (right) R-modules and maps. Then it is called a direct-system if

(a) I is a directed set.

(b) For each pair $i<j$, $f_i^j : A_i \longrightarrow A_j$

(c) The maps f_i^j are consistent. i.e. if $i<j<k$, then $f_j^k \circ f_i^j = f_i^k$.

Given a direct system $< A_i, f_i^j >$, we can construct their direct-limit A, (denoted by $\varinjlim A_i$) as follows:

Let B=disjoint $\bigcup_{i\in I} A_i$, and define the following equivalence relation in B:

$$a_i \sim a_j \text{ iff } \exists \ k \ni i<k, j<k \text{ and } f_i^j(a_i)=f_j^k(a_j).$$

i.e. two element are equivalent iff they are eventually mapped into the same range. We denote by $[a_i]$, the equivalence class of the a_i, and then A is defined as the collection of these equivalence classes in B.

In order to give A a structure of a (right) R-module, we define:

1. $1_A=[1_i]$
2. $[a_i]^{-1}=[a_i^{-1}]$

3. $[a_i] + [a_j] = [f_i^k(a_i) + f_j^k(a_j)]$ $k>i,j$

4. $r[a_i] = [ra_i]$ $\forall$ $r\epsilon R$

These operations are well-defined, and the direct-limit A becomes a (right) R-module.

(0.8) The direct-limit preserves exactness of sequences, and commutes with the tensor-product.

i.e. $T_E(\underrightarrow{\lim} M_i) = \underrightarrow{\lim}(T_E(M_i))$ See [2]

4. <u>Resolutions of modules, and the Homology functor.</u>

Let (A): $A_2 \xrightarrow{f} A_1 \xrightarrow{g} A_0$ be an o-sequence of (right) R-modules. i.e. $Im(f) \subset Ker(g)$.

Define H(A) as Ker(g)/Im(f), then H(A) becomes a functor from the translation category of three-term o-sequences, into the category ϕ_R. H(A) is called the <u>Homology functor</u>, and it is both additive and R-linear.

As a consequence of the properties of the homology functor, we get the following results: (See [7])

(0.9) Let

$$\begin{array}{ccccccc} & & A & \longrightarrow & B_R & \longrightarrow & C & \longrightarrow & 0 \\ & f & \downarrow & g & \downarrow & h & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & & \end{array}$$

be a commutative diagram over R, with the rows exact. Then, we get the following canonical exact sequence:

$Ker(f) \longrightarrow Ker(g) \longrightarrow Ker(h) \longrightarrow Coker(f) \longrightarrow Coker(g)$

$\longrightarrow Coker(h)$.

For a given module A_R, we define a projective-resolution of A to be an exact sequence

$$\ldots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \ldots\ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0 \ ,$$

where the P_i are projective. We will similarily define free and flat resolutions of A. In particular, we know that every R-module has a free (and hence projective) resolutions.

$\mathrm{Tor}_n^R(E,M)$ is defined to be the left-derived functor of the functor $E\otimes M$. Since, in general, we are concerned mainly with the functor T_E, then $\mathrm{Tor}_n(E,M)$ becomes the left derived functor of T_E. In more detail, let $\ldots \longrightarrow P_n \longrightarrow \ldots\ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow {}_RM \longrightarrow 0$ be a projective resolution of a (left) R-module M, denoted by $\underline{P}$.
Then, we get the following o-sequence:
$\ldots \xrightarrow{f_{n+1}} T_E(P_n) \xrightarrow{f_n} \ldots \longrightarrow T_E(P_1) \xrightarrow{f_1} T_E(P_0) \xrightarrow{f_0} 0$, denoted by $E\otimes\underline{P}$.

Let $H_n(E\otimes\underline{P})$ denote the homology-modules of $E\otimes\underline{P}$:

$$\text{i.e.} \quad H_n(E\otimes\underline{P}) = \mathrm{Ker}(f_n)/\mathrm{Im}(f_{n+1}).$$

Then,

$$\mathrm{Tor}_n(E,M) = H_n(E\otimes\underline{P}), \text{the left-derived functor of } T_E(M).$$

(0.10) There exists a natural isomorphism $\mathrm{Tor}_0^R(E,M) \cong E\otimes_R M$ (See[7])

5. Rings and Modules of Quotients.

Let R be a commutative ring, S a multiplicatively-closed set with $1\in S$ and $0\notin S$. Let A be an R-module.

Let K_S be the collection of all formal quotints $\frac{a}{s}$ where $a\in A, s\in S$.
Define an equivalence relation in K_S as follows:

$$\frac{a_1}{s_1} \sim \frac{a_2}{s_2} \quad \text{if } \exists \ s \in S \ni s s_2 a_1 = s s_1 a_2$$

Let A_S be the totality of these equivalence classes (denoted by $\left[\frac{a}{s}\right]$), then A_S is an R-module if we define

1. $\left[\frac{a_1}{s_1}\right] + \left[\frac{a_2}{s_2}\right] = \left[\frac{a_1 s_2 + a_2 s_1}{s_1 s_2}\right]$

2. $r\left[\frac{a}{s}\right] = \left[\frac{ra}{s}\right] \quad r \in R$

As these operations are well defined, A_S is an R-module, called the <u>module of quotients</u>. There evidently exists a canonical homomorphism $A \longrightarrow A_S$, defined by $a \longrightarrow \left[\frac{a}{1}\right]$.

Considering R as a module over itself, we form the <u>ring of quotients</u> R_S, and A_S can be considered as an R_S-module if we define

$$\left[\frac{r}{s}\right] \cdot \left[\frac{a}{s_1}\right] = \left[\frac{ra}{ss_1}\right]$$

Furthermore, any R-homomorphism $f: A \longrightarrow A'$, determines an R_S-homomorphism $f_S: A_S \longrightarrow A'_S$ defined by $f_S\left[\frac{a}{s}\right] = \left[\frac{f(a)}{s}\right]$. In fact, we can consider the functor $S(.): \not{C}_R \longrightarrow \not{C}_{R_S}$, defined by $S(A) = A_S$ which is known to be covariant and exact.

If P is a prime ideal in R (in particular, a maximal ideal), then $S = R - P$ is always multiplicatively-closed, and we denote the corresponding module of quotients by A_P. In such a case, the ring of quotients is always <u>local</u> (i.e. with a unique maximal ideal).

(0.11). If E is an R_S-module considered as an R-module, then the canonical map $f: E \longrightarrow E_S$ is an R_S-isomorphism. (See [7])

(0.12) If A,B are R-modules, then for every $n>0$, there is an R_S-isomorphism $[\mathrm{Tor}_n^R(A,B)]_S \cong \mathrm{Tor}_n^{R_S}(A_S,B_S)$ See[7].

CHAPTER I

1. Definition and Properties of Flat Modules.

Let $0 \longrightarrow M' \longrightarrow_R M \longrightarrow M'' \longrightarrow 0$ be an exact sequence. We recall that for every object E_R in the category $\not{C}_R$, the functor T_E is a right-exact functor.

i.e. $T_E(M') \xrightarrow{T(f)} T_E(M) \xrightarrow{T(g)} T_E(M'') \longrightarrow 0$ is exact.

In general, $T(f)$ is <u>not</u> a monomorphism, and we can not idenfify $E\otimes M'$ as a subgroup of $E\otimes M$.

The following example will illustrate the above:

Let $E=Z_n$ – the integers mod. n $(n>1)$, considered as a right Z-module.

Let $M=nZ$ – the ideal generated by n, considered as a left Z-module.

Then, $0 \longrightarrow nZ \xrightarrow{i} Z$ is exact, but $1_E\otimes i : Z_n\otimes nZ \longrightarrow Z_n\otimes Z$ $(\cong Z_n)$, is not a monomorphism.

For let $z\in Z_n\otimes nZ$, hence $z=\sum_i x_i\otimes(ny_i)$ $\quad x_i, y_i \in Z \quad \forall\, i$

But $(1\otimes i)z = \sum_i x_i\otimes(ny_i) = \sum_i (x_i n)\otimes y_i = 0$ in Z_n.

This motivates the notion of flatness.

<u>DEFINITION</u>: A module E_R is <u>flat</u> iff the functor T_E is exact.

We define similarily a flat left R-module.

Before proceeding with the discussion of flat modules, we recall that $\forall$ E_R, $Hom_Z(E,Q/Z)$ $(=E^*)$ is exact and contravariant as a functor, and can be given a structure of a left R-module.

The following characterization of flat modules is very useful in proving many results concerning flatness.

PROPOSITION 1.1 E_R is flat iff ${}_RE^*$ is injective.

Proof:

E_R is flat iff the functor T_E is exact.i.e. given any exact sequence $0 \longrightarrow M' \xrightarrow{i} {}_RM$,

then $o \longrightarrow E\otimes M' \xrightarrow{1\otimes i} E\otimes M$ is exact.

Recalling the existance of a natural isomorphism $(E\otimes M)^* \cong Hom_R(M,E^*)$ we get the following commutative diagram:

$$\begin{array}{ccccc} (E\otimes M)^* & \longrightarrow & (E\otimes M')^* & \longrightarrow & 0 \\ \updownarrow & & \updownarrow & & \\ Hom_R(M,E^*) & \longrightarrow & Hom_R(M',E^*) & \longrightarrow & 0 \end{array},$$

and the exactness of one row certainly implies the exactness of the other. But this,in turn,implies the exactness of the functor F_{E^*},which by (0.3) is exact iff E^* is injective.

PROPOSITION 1.2 E_R is flat iff for every left ideal ${}_RI$,there exists a canonical isomorphism $E\otimes I \cong EI$.

Proof:

Suppose E_R is flat,then for any ideal ${}_RI$, $0 \longrightarrow E\otimes I \xrightarrow{1\otimes i} E\otimes R$ is exact. But under the natural isomorphism $g:E\otimes R \longrightarrow E$,the image of $1\otimes i$ is clearily EI.

Thus $E\otimes I \cong EI$.

Conversly,

if $E\otimes I \cong EI$,then $0 \longrightarrow E\otimes I \longrightarrow E$ is exact,hence $E^* \longrightarrow (E\otimes I)^* \longrightarrow 0$ is also exact.

But $(E\otimes I)^* \cong Hom_R(I,E^*)$, (by (0.4))

hence $E^* \longrightarrow Hom_R(I,E^*) \longrightarrow 0$ is exact for every ${}_RI$.

But the exactness of the sequence implies that for every ideal ${}_RI$, and every $f \in Hom_R(I,E^*)$, $\exists$ $x \in E^* \ni f(r)=rx$ $\forall$ $r \in I$, a property which characterizes injective module (0.5).

Hence E^* is injective, and by prop^n.1.1,

E is flat.

<u>COROLLARY 1.2</u> In the above proposition, it suffices to consider only the finitely-generated (f-g) left ideals.

<u>Proof:</u>

For suppose the result holds for every f-g ${}_RI$.

Let ${}_RJ$ be an arbitrary ideal and consider $1 \otimes i : E \otimes J \longrightarrow E \otimes R$.

Let $z \in E \otimes J$, $z = \sum_i x_i \otimes r_i$ where $x_i \in E$, $r_i \in J$ and $r_i = 0$ for all but finitely many i.

Suppose that $\sum_i x_i \otimes r_i = 0$ in $E \otimes R$, and let $I = \sum_i Rr_i$. Then, by hypothesis, $\sum_i x_i \otimes r_i = 0$ in $E \otimes I$.

But $\exists$ a canonical homomorphism $E \otimes I \longrightarrow E \otimes J$, thus

$$\sum_i x_i \otimes r_i = 0 \text{ in } E \otimes J.$$

i.e.

$1 \otimes i$ is a monomorphism (mono.) for every ${}_RI$.

<u>PROPOSITION 1.3</u> (a). Let $< E_i >_{i \in I}$ be an arbitrary family of right R-modules, and let $E = \sum_i E_i$. Then E is flat iff E_i is flat $\forall$ i.

(b). Let $< E_i, f_i^j >$ be a directed-system of flat right R-modules. Then $E = \varinjlim E_i$ is flat.

Proof:

(a). $\sum_i E_i$ is flat iff $(\sum_i E_i)^*$ is injective.

But,(0.7), $(\sum_i E_i)^* \cong \prod_i E_i^*$, and $\prod_i E_i^*$ is injective iff E_i^* is injective $\forall$ i. (0.6).

i.e. iff E_i is flat $\forall$ i.

(b). Recall that the direct-limit preserves exactness. (0.9)

i.e. as $0 \longrightarrow E_i \otimes I \xrightarrow{1 \otimes i} E_i \otimes R$ is exact $\forall$ ideal ${}_R I$, and $\forall$ i,

hence $0 \longrightarrow \underrightarrow{\lim}(E_i \otimes I) \longrightarrow \underrightarrow{\lim}(E_i \otimes R)$ is also exact.

Furthermore, the direct-limit commutes with the tensor-product.

i.e. $\underrightarrow{\lim}(E_i \otimes I) = (\underrightarrow{\lim} E_i) \otimes I \quad \forall i$.

Hence,

$$0 \longrightarrow E \otimes I \longrightarrow E \otimes R \text{ is exact } \forall {}_R I$$

i.e. Eis flat.

Some Examples of Flat Modules:

1. For any ring R, considered as a module over itself, is abviously flat. It follows, since every free module F is isomorphic to a direct-sum of copies of R, and by prop^n. 1.3a, that every free module is flat.

2. More generally, every projective module P_R is flat.

i.e. since every projective module is a direct-summand of some free module F_R, there exists a direct exact sequence

$$F \longrightarrow P_R \longrightarrow 0$$

hence, $0 \longrightarrow P^*_R \longrightarrow F^*_R$ is also direct exact.

i.e. P^*_R is a direct-factor of the injective module F^*_R, and hence itself injective (See[7]). But, by prop^n. 1.1, this implies that P_R is flat.

4. For any commutative ring R, we will show that the ring of quotients R_S is always a flat R-module.

<u>PROPOSITION 1.4</u> Let E_R be flat, then $\forall\, r \in R \ni r$ is not a right zero-divisor, the relation $xr=0$ where $x \in E$ implies that $x=0$.

<u>Proof:</u>

Consider $f: R \longrightarrow R$ defined by $f(t)=tr \ \forall\ t \in R$.

Then by hypothesis, f is mono., and as E is flat,

$$1_E \otimes f: E \otimes R \longrightarrow E \otimes R \text{ is also mono.}$$

But $E \otimes R \cong E$ by the map $x \otimes r \longrightarrow xr$ $x \in E$, hence $1_E \otimes f$ defines an endomorphism $x \longrightarrow xr$.

Thus $xr=0$ implies that $x=0$.

We recall that if R is an integral-domain, a module E is <u>torsion-free</u> if the relation $xr=0$ implies that $x=0$ or $r=0$. In view of the above prop^n. every flat module over an integral-domain is torsion-free.

<u>2. Flatness of Quotient-Modules</u>.

<u>PROPOSITION 1.4</u> E_R is flat iff for all exact sequences $0 \longrightarrow B \xrightarrow{f} C_R \xrightarrow{g} E \longrightarrow 0$, and for all left R-modules ${}_RM$, the sequence $0 \longrightarrow B \otimes M \xrightarrow{f \otimes 1} C \otimes M \xrightarrow{g \otimes 1} E \otimes M \longrightarrow 0$ is exact.

Proof:

Suppose E_R is flat and ${}_RM$ is arbitrary. Since every module is an epimorphic image of a free module, $\exists$ an exact sequence $0 \longrightarrow K \xrightarrow{i} {}_RF \xrightarrow{p} M \longrightarrow 0$, where F is free and K=Ker(p). Now consider the following diagram:

$$\begin{array}{ccccc}
T_B(K) & \xrightarrow{f\otimes 1_K} & T_C(K) & \xrightarrow{g\otimes 1_K} & T_E(K) \\
\downarrow{\scriptstyle T_B(i)} & & \downarrow{\scriptstyle T_C(i)} & & \downarrow{\scriptstyle T_E(i)} \\
T_B(F) & \xrightarrow{f\otimes 1_F} & T_C(F) & \xrightarrow{g\otimes 1_F} & T_E(F) \\
\downarrow{\scriptstyle T_B(p)} & & \downarrow{\scriptstyle T_C(p)} & & \downarrow{\scriptstyle T_E(p)} \\
T_B(M) & \xrightarrow{f\otimes 1_M} & T_C(M) & \xrightarrow{g\otimes 1_M} & T_E(M)
\end{array}$$

The diagram is abviously commutative, and all the rows and columns are exact.

Furthermore, as F is free (and hence flat), $f\otimes 1_F$ is mono., and as the tensor product is always right exact, $g\otimes 1_K$ is an epimorphism (epi.). But by (0.9), this implies that $\exists$ a map d $\ni$

$$\mathrm{Ker}(T_C(i)) \longrightarrow \mathrm{Ker}(T_E(i)) \xrightarrow{d} \mathrm{Coker}(T_B(i)) \longrightarrow \mathrm{Coker}(T_C(i)) \longrightarrow \mathrm{Coker}(T_E(i)) \text{ is exact.}$$

But as the map $T_B(p)$ is epi., $\mathrm{Coker}(T_B(i)) = T_B(M)$, and similarily for $T_C(p)$ and $T_E(p)$. However, the flatness of E implies that $\mathrm{Ker}(T_E(i))$ is null.

i.e. $0 \longrightarrow B\otimes M \longrightarrow C\otimes M \longrightarrow E\otimes M \longrightarrow 0$ is exact $\forall$ ${}_RM$.

Conversly,

consider any exact sequence of the form $0 \longrightarrow B \longrightarrow C \longrightarrow E_R \longrightarrow 0$ in which C is flat (say free).

Let ${}_RM = R/I$, where I is a f-g ideal. Then, $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$ is exact.

Subtituting $K=I, F=R$ and $M=R/I$ in the above diagram, then $\exists$ a map d $\ni$

$$Ker(1_C \otimes i) \longrightarrow Ker(1_E \otimes i) \xrightarrow{d} B \otimes M \xrightarrow{f \otimes 1_M} C \otimes M \text{ is exact.}$$

But, by hypothesis, $f \otimes 1_M$ is mono. and hence, $Im(d)=0$, and furthermore the flatness of C implies that $Ker(1_C \otimes i)=0$.

i.e. $0 \longrightarrow Ker(1_E \otimes i) \longrightarrow 0$ is exact, hence, $Ker(1_E \otimes i)=0$.

i.e. $1_E \otimes i : E \otimes I \longrightarrow E \otimes R$ is mono. for every f-g ideal ${}_RI$.

Thus, E_R is flat.

<u>PROPOSITION 1.5</u> Let $0 \longrightarrow E' \xrightarrow{f} E_R \xrightarrow{g} E'' \longrightarrow 0$ be exact where E" is flat. Then, E is flat iff E' is flat.

<u>Proof:</u>

Consider any monomorphism $u : M' \longrightarrow {}_RM$. Then we get the following commutative diagram:

$$\begin{array}{ccccc} E' \otimes M' & \xrightarrow{f \otimes 1_{M'}} & E \otimes M' & \xrightarrow{g \otimes 1_{M'}} & E'' \otimes M' \\ {\scriptstyle 1_{E'} \otimes u} \downarrow & & {\scriptstyle 1_E \otimes u} \downarrow & & {\scriptstyle 1_{E''} \otimes u} \downarrow \\ E' \otimes M & \xrightarrow{f \otimes 1_M} & E \otimes M & \xrightarrow{g \otimes 1_M} & E'' \otimes M \end{array}$$

Now, by $prop^n$.1.4, both $f \otimes 1_{M'}$ and $f \otimes 1_M$ are mono.

Furthermore, as E'' is flat, $1_{E''}\otimes u$ is mono..

If E is flat then $1_E\otimes u$ is mono., and hence, $(1_E\otimes u)\circ(f\otimes 1_M)$ is also mono.. But by commutativity of the diagram, this implies that $(f\otimes 1_M)\circ(1_{E'}\otimes u)$ is mono., and therefore, $1_{E'}\otimes u$ is mono. for every monomorphism u.

i.e. E' is flat.

If E' is flat, then the map $1_{E'}\otimes u$ is mono., and it follows directly from the diagram that $1_E\otimes u$ is mono..

i.e. E is flat.

From the above proposition, we can not conclude that a submodule of a flat module is flat. In fact, this is not generally true as can be seen by the following example:

Let $R=Z_4= \langle 0,1,2,3\rangle$ - the ring of integers mod4.
Now, R is abviously flat as an R-module, but we will show that the ideal $I=\langle 0,2\rangle$ is not flat.

By propositon 1.2, I is flat iff $I\otimes I \longrightarrow I\circ I$ is an isomorphism. As $I\circ I=0$ it suffices to show that $I\otimes I\neq 0$

Consider the map $f:I\times I\longrightarrow I$ defined by $f(x,y)=2$ if $x=y=2$, and $f(x,y)=0$ otherwise. Clearly, f is middle linear, and hence there must exist a Z-homomorphism $w:I\otimes I\longrightarrow I \ni w(x\otimes y)=f(x,y)$.

i.e. $I\otimes I$ can not be the trivial group, hence, I is not flat.

We may also note that in the above proposition, the flatness of both E' and E does not imply that E'' is flat.

For example, let R=Z and consider the exact sequence

$$0 \longrightarrow nZ \longrightarrow Z \longrightarrow Z/nZ \longrightarrow 0 \qquad n>1$$

It will be shown later that over Principal-ideal-domains (or more generally, over Prufer domains), a module is flat iff it is torsion-free. Hence, both Z and nZ are flat, but Z/nZ is not.

The following proposition gives both necessary and sufficient condition for the flatness of E".

PROPOSITION 1.6 Let E_R be an R-module, and E' a submodule of E.

(i) If E/E' is flat, then for all ideals $_RI$ of R, $E'I=E'\cap EI$ (*)

Conversly,

(ii) If E is flat, and for all (f-g) ideals $_RI$ (*) holds, then E/E' is flat.

Proof:

(i) Consider $0 \longrightarrow E' \xrightarrow{f} E \xrightarrow{g} E/E' \longrightarrow 0$ where E/E' is flat. Now, for every ideal $_RI$, $0 \longrightarrow {}_RI \xrightarrow{i} R$ is exact, hence, the diagram

$$\begin{tikzcd}
E'\otimes I \arrow[r, "f\otimes 1"] \arrow[d, "1_{E'}\otimes i"'] & E\otimes I \arrow[r, "g\otimes 1"] \arrow[d, dotted, "1_E\otimes i"'] & (E/E')\otimes I \arrow[d, dotted, "1_{E/E'}\otimes i"] \\
E'\otimes R \arrow[r, "f\otimes 1"] & E\otimes R \arrow[r, "g\otimes 1"] & (E/E')\otimes R
\end{tikzcd}$$

is commutative, and the rows are exact.

Since E/E' is flat, $1_{E/E'}\otimes i$ is mono., and by routine diagram-chasing, we can show that

$$\mathrm{Im}(1_E\otimes i)\cap \mathrm{Im}(f\otimes 1_I) = \mathrm{Im}[(f\otimes 1_R)\circ(1_{E'}\otimes i)]$$

$$\text{i.e. } EI \cap E' = E'I \quad \text{ideal } {}_RI.$$

(ii)Consider the exact sequence $0 \longrightarrow E' \xrightarrow{f} E \xrightarrow{g} E/E' \longrightarrow 0$ where E is flat. Let ${}_RI$ be a f-g ideal, and denote E/E' by E''. Hence,

$$E'\otimes I \xrightarrow{f\otimes 1} E\otimes I \xrightarrow{g\otimes 1} E''\otimes I \longrightarrow 0 \text{ is always exact.}$$

Flatness of E implies that $E\otimes I \cong EI$, and the image of $E'\otimes I$ under this canonical isomorphism is $E'I$,

$$\text{i.e. } E''\otimes I \cong EI/E'I.$$

Therefore, (by proposition 1.2), E'' is flat iff $E''I = EI/E'I$. Now, E'' consists of elements of the form $\sum_i g(x_i)r_i$ where $x_i \in E$, $r_i \in R$.

$$\text{But } \sum_i g(x_i)r_i = \sum_i g(x_ir_i) = \sum_i g(x_i') \text{ where } x_i' \in E$$

i.e. $E''I = g(EI) = EI/EI\cap E'$, hence, E'' is flat iff $EI/EI\cap E' = EI/E'I$.

$$\text{But } E'I \subset EI \cap E'$$

$$\text{i.e. } E'' \text{ is flat iff } E'I = EI \cap E'$$

The above proposition is now used to prove yet another result regarding flat modules, which will be very useful in future proofs. As preliminaries, let E_R be any module and consider the exact sequence $0 \longrightarrow K \longrightarrow F \xrightarrow{f} E \longrightarrow 0$, where F is free with basis $\langle f_i \rangle$, and $K = \mathrm{Ker}(f)$.

Each $x \in K$ can be written as $\sum_i f_ir_i$ where $r_i = 0$ for all but finitely many i.

Denote by I_x the ideal $\sum_i Rr_i$.

<u>COROLLARY 1.6</u> E is flat iff for every $x\in K$, $x\in KI_x$.

<u>Proof:</u>

Suppose E is flat, then (proposition 1.6) $x\in K\cap FI_x=KI_x$.

Conversly,

Let ${}_RI$ be any ideal, and suppose that $x\in K\cap FI$.

Clearly, $I_x\subset I$ for every $x\in K\cap FI$, and hence, $K\cap FI=KI$.(*)

As F is free, it is flat, and since (*) is satisfied, E is also flat (proposition 1.6).

<u>REMARK:</u> It follows from the corollary that if $x\in K$, $x=\sum_i f_i r_i$, then $x=\sum_i k_i r_i$ where $k_i\in K$.

<u>3. Tensor Product of Flat Modules</u>

Recall that, in general, $E\otimes_R M$ is in the category of Z-modules, but if R is a commutative ring then $E\otimes_R M$ can be considered as an R-module, if we define $r(x\otimes y)=(xr)\otimes y$ $x\in E$, $y\in M$ and $r\in R$.

Suppose that E and M are both flat. Can we, in the commutative case, conclude that $E\otimes M$ is a flat R-module ? The answer is in the affirmative. In fact, we prove a more general result.

Let R_1, R_2 be two rings, E_{R_1} an R_1-module and ${}_{R_1}M_{R_2}$ a (R_1-R_2) bimodule. We can consider $E\otimes_{R_1}M$ as a right R_2-module, if we define $(x\otimes y)r=x\otimes(yr)$ where $x\in E$, $y\in M$, $r\in R_2$.

In such a case, we have the following:

PROPOSITION 1.7 Suppose that E_{R_1} is a flat R_1-module, and ${}_{R_1}M_{R_2}$ is a flat R_2-module, then $E \otimes_{R_1} M$ is a flat (right) R_2-module.

Proof:

Consider any exact sequence $0 \longrightarrow N' \longrightarrow {}_{R_2}N$.

As M_{R_2} is flat, $0 \longrightarrow M \otimes_{R_2} N' \longrightarrow M \otimes_{R_2} N$ is exact.

But, we can consider $M \otimes_{R_2} N$ as a left R_1-module,

hence, $E \otimes_{R_1} (M \otimes_{R_2} N') \longrightarrow E \otimes_{R_1} (M \otimes_{R_2} N)$ is a monomorphism.

But there is a trivial identifaction of $E \otimes_{R_1} (M \otimes_{R_2} N)$ with $(E \otimes_{R_1} M) \otimes_{R_2} N$.

i.e. $(E \otimes_{R_1} M) \otimes_{R_2} N' \longrightarrow (E \otimes_{R_1} M) \otimes_{R_2} N$ is mono.

Thus, $E \otimes_{R_1} M$ is a flat R_2-module.

COROLLARY 1.7 Let R be commutative, E and M flat R-modules. Then, $E \otimes M$ is flat.

Proof:

It is a direct consequence of the above proposition.

As a final characterization of flatness in this chapter, we show that if E_R is flat, then each linear relation in E is a consequence of linear relations in R.

PROPOSITION 1.8 E_R is flat iff the following hold: If $\langle e_i\rangle_{i\in I}$ and $\langle t_i\rangle_{i\in I}$ are two finite families of elements in E and R respectively, such that $\sum_i e_i t_i = 0$, then $\exists$ a finite family $\langle x_j\rangle_{j\in J}$ in E and a family $\langle a_{ji}\rangle_{j\in J, i\in I}$ of elements in R, such that $\sum_i a_{ji} t_i = 0$ $\forall j\in J$, and $e_i = \sum_j x_j a_{ji}$ $\forall i\in I$.

Proof:

Suppose that E_R is flat and let ${}_RI = \sum_i Rt_i$.

Now, $E\otimes I = = EI$, and hence, $\sum_i e_i t_i = 0$ in EI implies that $\sum_i e_i \otimes t_i = 0$ in $E\otimes I$.

Let ${}_RF$ be free with basis $\langle u_i\rangle_{i\in I}$, then

$0 \longrightarrow K \xrightarrow{i} F \xrightarrow{f} I \longrightarrow 0$ is exact, where $f(u_i) = t_i \ \forall i$.

Hence, $0 \longrightarrow E\otimes K \xrightarrow{1\otimes i} E\otimes F \xrightarrow{1\otimes f} E\otimes I \longrightarrow 0$ is exact.

The $\sum_i e_i \otimes t_i = 0$ in $E\otimes I$ implies that $\sum_i e_i \otimes u_i \in \mathrm{Ker}(1\otimes f) = \mathrm{Im}(1\otimes i)$.

i.e. $\sum_i e_i \otimes u_i = \sum_j x_j \otimes i(r_j)$ for some $x_j \in E$, $r_j \in R$, where $j\in J$ is a finite family.

But as $i(r_j)\in F$, $i(r_j) = \sum_i a_{ji} u_i$ for some $a_{ji}\in R$ $\forall j\in J$.

As $f\circ i(r_j) = 0$, this implies that $\sum_i a_{ji} t_i = 0$ $\forall j\in J$.

Furthermore, $\sum_i e_i \otimes u_i = \sum_j x_j \otimes (\sum_i a_{ji} u_i) = \sum_i (\sum_j x_j a_{ji}) \otimes u_i$.

But, as F is free with basis u_i, the representation of $E\otimes F$ in terms of the $\sum_i e_i \otimes u_i$ is unique (0.2).

hence, $e_i = \sum_j x_j a_{ji}$ $\forall i \in I$

Conversly,

let $_R I$ be any f-g ideal, and suppose that $y = \sum_i e_i \otimes t_i$ is in $Ker(\emptyset)$, where $\emptyset : E \otimes I ---> EI$.

Then, by hypothesis, $\exists$ families $\langle x_j \rangle_{j \in J}$, $\langle a_{ji} \rangle_{j \in J, i \in I}$ and

$$y = \sum_i (\sum_j x_j a_{ji}) \otimes t_i = \sum_j x_j \otimes (\sum_i a_{ji} t_i)$$

$$= \sum_j x_j \otimes 0 = 0$$

i.e. $Ker(\emptyset) = 0$, and hence $E \otimes I \cong EI$, which implies that E is flat.

CHAPTER 2

1. Preliminaries

Given any ring R, we have seen that free===>projective===>flat, and if R is an integral-domain, flat===>torsion-free. [1]

In this chapter, we will deal with characterizations of classes of rings, for which some of the above implications are reversed. For example, what rings are characterized by the property that every flat (right) module is projective?

Finally, we look at the class of rings over which every module is flat.

Before attempting to look at these questions, we need some further results regarding flatness.

PROPOSITION 2.1 Let $o \longrightarrow K \longrightarrow F_R \longrightarrow E \longrightarrow 0$ be an exact sequence. Then, the following are equivalent:

(a). E is flat.

(b) Given any $u \in K$, $\exists$ a homomorphism $\phi: F \longrightarrow K \ni \phi(u)=u$

(c) Given any finite set $\langle u_i \rangle_{i=1,..,n}$ $u_i \in K$, $\exists$ a homomorphism $\phi: F \longrightarrow K$ with $\phi(u_i)=u_i$ $\forall i$.

Proof:

(a)==>(b). Let $\langle x_j \rangle_{j \in J}$ be a base for F, then for $u \in K$, $u=\sum_{i=1}^{k} x_{j_i} r_i$ where $r_i \in R$.

Let $I_u=\sum_{i=1}^{k} Rr_i$, then as E is flat, $u \in KI_u$.

i.e. $u=\sum_{i=1}^{k} k_i r_i$, where $k_i \in K$. (proposition 1.6)

1. We use ===> to denote "implies".

Define,

$$\phi: F \longrightarrow K \text{ by } \begin{cases} \phi(x_{j_i}) = k_i & i=1,\ldots,k \\ \phi(x_j) = 0 & j \neq j_1,\ldots,j_k \end{cases}$$

Then, clearly, ϕ satisfies the required properties.

(b)==>(a).

Suppose that $u = \sum_{i=1}^{k} x_{j_i} r_i \in K$, and that $\exists$ a $\phi: F \longrightarrow K \ni$ $\phi(u) = u$. Then

$$u = \sum_{i=1}^{k} \phi(x_{j_i}) r_i \in KI_u \quad \forall u \in K.$$

Thus, E is flat. (corollary 1.6)

To complete the proof, it suffices to show that (b)==>(c). We proceed by induction on n.

If n=1, then the existance of ϕ is given by (b). Suppose, that there exists a map ϕ for all $m < n$, $n > 1$.

Consider $u_1,\ldots,u_n$ $u_i \in K$, and let $\phi_n: F \longrightarrow K$ be such that $\phi_n(u_n) = u_n$.

Define $v_i = u_i - \phi_n(u_i)$ for $i=1,\ldots,n-1$.

Then, by the induction hypothesis, $\exists$ $\phi': F \longrightarrow K$, where $\phi'(v_i) = v_i$ $i=1,\ldots,n-1$. Now, define $\phi: F \longrightarrow K$ by

$$\phi = \phi' \cdot \phi_n - \phi' - \phi_n .$$

It is easily verified that ϕ has the desired properties.

We recall that the direct-limit of flat module is flat. We now show that, in fact, every flat module is a direct-limit of projective, and even free modules. With these results, the problem of over what rings is flat==>projective, is reduced to finding the class of rings over which the direct-limit of projective modules is projective, and similarily for free modules.

PROPOSITION 2.2A E_R is flat iff it is the direct-limit of projective modules.

Proof:

Over any ring R, projective==>flat, hence the direct-limit of projectives is abviously flat.

Conversly,

suppose that E_R is flat, and consider the exact sequence $0 \longrightarrow K \longrightarrow F_R \longrightarrow E \longrightarrow 0$ where F is free.

Let $\langle K_i, f_i^j \rangle$ be the direct-system of all f-g submodules K_i of K, ordered in the natural way, where f_i^j are the canonical monomorphisms.

K is abviously the $\underset{\longrightarrow}{\lim} K_i$.

Suppose that each K_i is generated by $\langle u_{i1}, u_{i2}, \ldots, u_{is_i} \rangle$.

But as E is flat, $\forall\, i$, $\exists\, \phi_i : F \longrightarrow K_i \ni \phi_i(u_{i_j}) = u_{i_j}$ $j=1,\ldots,s_i$.

(proposition 2.1).

i.e. $\forall i$, K_i is a direct-summand of F,

$\Longrightarrow K_i$ is projective $\forall\, i$.

Hence, $F = K_i \oplus R_i \ \forall i$, where $R_i \cong F/K_i$ and projective.
Therefore,

$< F/K_i, g_i^j >$ is a direct-system of projective modules,

where $F/K_i < F/K_j$ if $K_i \subset K_j$, and g_i^j are the natural homomorphisms.

But $\underset{\longrightarrow}{\lim} F/K_i = F/K \cong E$

i.e. E is a direct-limit of projective modules.

<u>PROPOSITION 2.2B</u> E_R is flat iff it is the direct-limit of free modules.

<u>Proof:</u>

By proposition 2.2A, E is a direct-limit of $< P_i, f_i^j >$, where P_i are projectives.

Now, $\forall i$, $\exists$ a free right R-module F_i, $\ni F_i = P_i \oplus Q_i$.
We can thus, represent every P_i as a direct-limit of free modules as follows:

$$F_i \xrightarrow{g_{i1}} F_i \xrightarrow{g_{i2}} F_i \xrightarrow{g_{i3}} \ldots\ldots \quad \text{where } \operatorname{Ker}(g_{ik}) = Q_i \ \forall k.$$

i.e.

P_i is a direct-limit of $< F_{ik}, g_{ik}^{k+1} >$ where $F_{ik} = F_i$, $g_{ik}^{k+1} = g_i \ \forall i$.

Consider the $< F_{ik} >$, ordered in the natural way. i.e. $F_{ik} < F_{jl}$ if $P_i \subset P_j$.

Define the maps h_{jl} to be the canonical monomorphisms of P_j into F_{jl}.

i.e.

h_{jl} are defined $\ni$ the diagram
$$\begin{tikzcd} 0 \arrow[r, dashed] & P_j \arrow[r, dashed, "h_{jl}"] \arrow[dr, "1_j"'] & F_{jl} \arrow[d, dashed, "g_{jl}"] \\ & & P_j \end{tikzcd}$$

is commutative $\forall j$.

Define the homomorphisms $\phi_{ik}^{jl}: F_{ik} \dashrightarrow F_{jl}$, by the composition map

$$\phi_{ik}^{jl} = h_{jl} \cdot f_i^j \cdot g_{ik}.$$

i.e.

$$F_{ik} \xrightarrow{g_{ik}} P_i \xrightarrow{f_i^j} P_j \xrightarrow{h_{jl}} F_{jl}.$$

Then,

$\langle F_{ik}, \phi_{ik}^{jl} \rangle$ is a direct-system, as the maps are consistent.

i.e.

$$\phi_{jl}^{rs} \cdot \phi_{ik}^{jl} = h_{rs} \cdot f_j^r \cdot \overbrace{g_{jl} \cdot h_{jl}}^{1} \cdot f_i^j \cdot g_{ik} .$$

$$= h_{rs} \cdot f_j^r \cdot f_i^j \cdot g_{ik} = h_{rs} \cdot f_i^r \cdot g_{ik}.$$

$$= \phi_{ik}^{rs}$$

It is also evident that the direct-limit of the F_{ik} is E.

We can now proceed to the discuss the questions which were raised in the beginning of the chapter.

2. Domains over which Flat = Torsion-free.

We recall that a ring R is a (right) semi-hereditary if every f-g ideal I_R is projective.

DEFINITION: A ring R is called a Prufer-domain iff R is a semi-hereditary integral-domain.

We may note that, over integral domains, a non-zero ideal is projective iff it is invertible. We will now quote the following, well known characterization of Prufer domains. (See[3])

Corollary 2.3: R is a Prufer domain iff every f-g, torsion-free R-module is projective.

We can now give, a complete classification, of the domains over which flat=torsion-free.

THEOREM 2.4: Every torsion-free module is flat iff R is a Prufer domain.

Proof:

Suppose R is Prufer.

Then, by the above corollary, every f-g, torsion-free R-module is projective.

But, every torsion-free module is a direct-limit of f-g torsion-free modules, and hence, a direct-limit of projectives. Since, a direct-limit of projectives is flat (proposition 2.2A), hence, every torsion-free module is flat.

Conversly,

suppose that torsion-free==>flat.

Hence, any f-g ideal I_R is flat, since it is torsion-free. We will prove later, that any f-g flat module over an integral-domain is projective. i.e. every f-g ideal I_R is projective,

$\Longrightarrow$ R is a Prufer domain.

---

2. Rings over which Flat=Projective, and Flat=Free.

We have already seen, that it suffices to look at the class of rings for which the direct-limit of projective (or free) modules is projective (or free).

Given a ring R, let J_R denote its Jacobson radical. i.e. J_R is the intersection of all maximal right ideals in R.

DEFINITIONS: (1). An ideal I in a ring R is (right) T-nilpotent, if given any sequence $\langle a_i \rangle_i$ in I, $\exists$ n, $\ni a_n a_{n-1} \cdots a_1 = 0$.

(2). A ring R is semi-simple if $J_R = 0$.

H. Baas (See[1]), in his paper on Finistic Dimension, proved the following:

THEOREM 2.5 The following are equivalent:

(a). The direct-limit of projective right R-modules is projective.

(b). J_R is right T-nilpotent and R/J is semi-simple and Artinian.

(c). R satisfies the descending chain condition on principal left ideals.

A ring R, satisfying any of the above equivalences, is called (right) <u>Perfect</u>. We will attempt here to reproduce the proof of the above result, but proceed to discuss a subclass of the perfect rings, namely, those rings over which flat=free.

The following results are due mainly to Gomorov (See[5]).

<u>THEOREM 2.6</u> In order that all flat right modules are free, it is necessary and sufficient that R is a local ring with a right T-nilpotent maximal ideal.

<u>Proof:</u>

Assume flat==>free.

<u>Lemma 1.</u> Every element $a \in R$ with a right inverse, has a left inverse.

Suppose $ab=1$, and consider the descending chain $Ra \supset Ra^2 \supset Ra^3 \supset \ldots$.

Since flat==>free (and hence flat==>projective), by theorem 2.6, R satisfies the descending chain condition on principal left ideals.

$$\text{i.e. } a^n \subset Ra^{n+1} \text{ for some } n.$$

$$\Longrightarrow \quad a^n = ra^{n+1} \text{ for some } r \in R.$$

hence,

$$(ra-1)a^n=0$$

but then,

$$(ra-1)a^n b^n=0=(ra-1)(ab)^n=ra-1$$

i.e. $ra=1$, hence a has a left inverse (and $r=b$).

<u>Lemma 2</u> The non-invertible elements of R form a right T-nilpotent ideal, if every non-invertible element is nilpotent.

For suppose that every non-invertible is nilpotent.

Let I be the set of all non-invertible elements.

Clearly,

$$\text{if } a\in I, \text{ then } ar\in I \ \forall r\in R.$$

Suppose that $a,b\in I$, then we show that $a+b\in I$. For assume the contrary,

then $\exists\ c\in R \ni (a+b)c=1$

but as a is nilpotent, $\exists$ a minimum $n \ni a^n=0$

$$\implies a^{n-1}(a+b)c=a^{n-1}$$

$$\implies a^{n-1}bc=a^{n-1}$$

$$\implies (1-bc)a^{n-1}=0$$

but as $bc\in I$, it follows that $1-bc$ is invertible

hence, $a^{n-1}=0$, which contradicts minimality of n.

Therefore, I_R, the set of all non-invertible elements, is an ideal in R, and hence, automatically the unique maximal ideal.

i.e. $I_R=J_R$ the Jacobson radical.

By theorem 2.5, I_R is T-nilpotent.

We have reduced the proof ~~n~~ow, to proving that every non-invertible element is nilpotent.

Let <u>a</u> be any non-invertible element, and assume that it is not nilpotent. Let F_R be a free module with basis $\langle x_i\rangle_{i\in I}$, where I is arbitrary.

Let G be the free submodule of F generated by $z_i = x_i - x_{i+1}a$. Then, G is a proper submodule of F. For assume that $F=G$,

$$\Longrightarrow x_1 = \sum_{i=1}^{n} z_i r_i = x_1 r_1 + x_2(r_2 - ar_1) + \ldots + x_n(r_n - ar_{n-1}) + x_{n+1}ar_n .$$

But, as x_i form a base,

$$\Longrightarrow r_1 = 1 \text{ and } r_i = a^{i-1} \quad i = 1, \ldots, n$$

Also,

$$ar_n = 0 \Longrightarrow aa^{n-1} = a^n = 0$$

i.e. a is nilpotent.

Contradiction.

Let G_n denote the free module generated by $(z_1, \ldots, z_n)$. Then,

F/G_n is free, with basis $\bar{x}_{n+1}, \bar{x}_{n+2}, \ldots$, where $\bar{x}_i$ are the images of the x_i, under the epimorphism $F \longrightarrow F/G_n$.

But,

$$F/G = \varinjlim F/G_n .$$

i.e. F/G is a direct-limit of free module , and by hypothesis, is flat, and hence, free.

Now, as $F/G \neq 0$, let $v_1, v_2, \ldots$ be a basis.

Let t_i be the images of the x_i under the map $F \longrightarrow F/G$. Then, $v_1 = \sum_i t_i r_i$.

but $t_i = t_{i+1}a$, $\Longrightarrow \exists\, k \ni v_1 = t_{k+1}(\sum_i a^{k-i} r_i)$.

Let $b = \sum_i a^{k-i} r_i$.

now, $t_{k+1} = \sum_j v_j w_j$, where $w_j \epsilon R$.

hence, $v_1 = (\sum_j v_j w_j)b \Longrightarrow 1 = w_1 b$, and by lemma 1 ,

$$\Longrightarrow w_1 b = 1 = b w_1$$

Furthermore,

$$v_1 w_1 = (t_{k+1} b) w_1 = t_{k+1}(b w_1) = t_{k+1} = t_{k+2} a \ .$$

but,

$$t_{k+2} = \sum_j v_j k_j \text{ where } k_j \epsilon R.$$

$$\Longrightarrow v_1 w_1 = (\sum_j v_j k_j) a$$

hence,

$$w_1 = k_1 a$$

but

$$b(k_1 a) = (b k_1) a = b w_1 = 1$$

$$\Longrightarrow a \text{ is invertible}$$

Contradiction.

i.e. every non-invertible element is nilpotent, hence, by lemma 2, we have the necessary condition.

Conversly,

suppose that R is a local ring with a right T-nilpotent maximal ideal. Then (theorem 2.5), every flat right module is projective. But over a local ring, by the well-known result of I. Kaplanski, every projective module is free.

i.e. every flat right module is free.

---

4. Rings over which the Direct-Product of Flat Modules is Flat.

Let us recall that if $< E_i >_{i \in I}$ is an arbitrary family of flat right R-modules, then $E = \sum_i E_i$ is also flat. This immediately suggests another problem, i.e., For what class of rings is $\prod_i E_i$ always flat?. This question was answered by S. Chase (See[4]), and we reproduce here some of its main results.

<u>DEFINITION:</u> A module A_R is <u>finitely-related</u> if $\exists$ an exact sequence $0 \longrightarrow K \longrightarrow F_R \longrightarrow A \longrightarrow 0$, where F is free, and both K and F are finitely-generated,

<u>COROLLARY 2.7</u> Every finitely-related flat module is projective.

<u>Proof:</u>

Suppose E_R is finitely-related, then $\exists$ an exact sequence

$0 \longrightarrow K \longrightarrow F \longrightarrow E \longrightarrow 0$, where K is f-g.

Let $\langle u_i \rangle_{i=1,\ldots,n}$ generate K. Since E is flat, $\exists$ a map $\phi : F \longrightarrow K$,

$\ni \quad \phi(u_i) = u_i \ \forall i.$ (proposition 2.1)

i.e. the above sequence is split-exact,

$\Longrightarrow$ E is a direct-summand of F, and hence, projective

---

We come now, to the main theorem.

THEOREM 2.7 The direct-product of any family of flat right R-modules is flat iff every f-g left ideal ${}_RI$ is finitely-related.

Proof:

Suppose that the direct-product of flat modules is flat.

Let ${}_RI$ be any f-g ideal. i.e. $I=\sum_{i=1}^{r} Ru_i \quad u_i \in R$.

Let ${}_RF$ be free with basis $(x_1,\ldots,x_r)$, and consider the exact sequence

$$0 \longrightarrow K \longrightarrow {}_RF \xrightarrow{f} I \longrightarrow 0, \text{ where } f(x_i)=u_i,\ K=\mathrm{Ker}(f).$$

Now,

$k\epsilon K$, let $R^{(k)}$ be a copy of the ring R.

$$\text{Let } A_R=\prod_{k\epsilon K} R^{(k)}$$

hence,

if $k=a_1(k)x_1+a_2(k)x_2+\ldots+a_r(k) \in K$, then

$f(k)=\sum_{i=1}^{r} a_i(k)u_i=0$

Let $a_j=\langle a_j(k)\rangle_{k\epsilon K}$, then $\sum_{i=1}^{r} a_iu_i=0$

By hypothesis, A is flat,

hence (proposition 1.8), $\exists\ b_1,\ldots,b_n \epsilon A$, $\langle m_{ij}\rangle_{\substack{i=1,\ldots,n\\ j=1,\ldots,r}}$ in R and

$$\sum_{j=1}^{r} m_{ij}a_j=0, \quad a_j=\sum_{i=1}^{n} b_im_{ij} \qquad (*)$$

Now,

let $z_i=\sum_{j=1}^{r} m_{ij}x_j \in F \quad i\leq n$

then, as $f(z_i)=0$, $\Longrightarrow z_1, z_2, \ldots, z_n \in K$.

Let $b_i = \langle b_i(k)\rangle_{k\in K}$, $b_i(k)\in R$. Then, by (*),

$$a_j(k) = \sum_{i=1}^{n} b_i(k) m_{ij} \quad j=1,\ldots,r$$

But,

$$k = \sum_{j=1}^{r} a_j(k)x_j = \sum_{j=1}^{r}\left(\sum_{i=1}^{n} b_i(k)m_{ij}\right)x_j$$

$$= \sum_{i=1}^{n} b_i(k)\left(\sum_{j=1}^{r} m_{ij}x_j\right)$$

$$= \sum_{i=1}^{n} b_i(k)z_i$$

i.e. $z_1, z_2, \ldots, z_n$ generate K.

$\Longrightarrow {}_R I$ is finitely-related.

Conversly,

suppose that every f-g left ideal is finitely-related.

We quote the following lemma without proof (See[2]) :

<u>Lemma:</u> Let ${}_R M$ be a finitely-related module, $\langle E_i\rangle_{i\in I}$, an arbitrary family of right R-modules. Then, the canonical homomorphism

$$\left(\prod_{i\in I} E_i\right)\otimes M \longrightarrow \prod_{i\in I}(E_i\otimes M)$$

is an isomorphism.

Assuming the lemma, let E_i be flat $\forall\, i\in I$, and hence, for every f-g ideal ${}_R J$,

$$E_i\otimes J \xrightarrow{1_{E_i}\otimes j} E_i\otimes R \text{ is mono.}$$

hence, $\prod_i (E_i \otimes J) \xrightarrow{f} \prod_i (E_i \otimes R)$ is mono.

But, by hypothesis, J is finitely-related. Thus, we have

$$\begin{array}{ccc} \prod_i (E_i \otimes J) & \xrightarrow{f} & \prod_i (E_i \otimes R) \\ \updownarrow & & \updownarrow \\ (\prod_i E_i) \otimes J & \xrightarrow{g} & (\prod_i E_i) \otimes R \end{array}$$

Evidently, g is also mono. for every f-g ideal ${}_RI$,

$\Longrightarrow$ $\prod_i E_i$ is flat. (proposition 1.2)

The class of rings over which every f-g (left) ideal is finitely related, can be completely given by an ideal-theoritic characterization. In fact, S.Chase proved the following:

<u>THEOREM 2.8</u> R is a ring $\ni$ every f-g left ideal is finitely-related iff $\forall$ $a \in R$, the ideal of all left zero-divisors of a is f-g, and the intersection of any f-g ideals is again f-g.

We will not give a proof here, but give few examples of rings which satisfy the above. We note that in view of the theorem, if R is an integral-domain, then the direct-product of any family of flat right R-modules is flat iff the intersection of any two f-g left ideals is again f-g.

EXAMPLES:

(a). Left-Noetherian rings, as clearly, every f-g left module is finitely-related.

(b). Left semi-hereditary rings.

For let ${}_RI$ be any f-g ideal, and hence, projective.
Thus, $\exists$ an exact sequence

$$0 \longrightarrow K \longrightarrow {}_RF \longrightarrow I \longrightarrow 0,$$ where F is free and f-g.

But, $F \cong K \oplus I$

$\Longrightarrow$ K is f-g.

i.e. ${}_RI$ is finitely-related.

(c). $R = K[X_1, X_2, \ldots]$ - the ring of polynomials over a field.

(i). If the number of indeterminants is finite, then R is Noetherian.

(ii). If the number is infinite, then R is an integral-domain, with the property that the intersection of any two f-g ideals is also f-g.

5. Rings over which every Module is Flat.

DEFINITION: A ring R is regular if $\forall\, r \in R$, $\exists\, a \in R \ni rar = r$.

PROPOSITION 2.9 Every right R-module is flat iff R is a regular ring.

Note: As the concept of regularity is symmetrical, we can conclude that all left R-modules are also flat.

Proof:

Suppose that every right R-module is flat.

consider the sequence

$$0 \longrightarrow rR \longrightarrow R \longrightarrow R/rR \longrightarrow 0.$$

Now, as R/rR is flat

$\Longrightarrow rR \cap I = rI$ for any left ideal ${}_R I$ (proposition 1.8).

Let ${}_R I = Rr$, then

$$rR \cap Rr = rRr$$

but, $r \in rR \cap Rr$,

$\Longrightarrow r = rar$ for some a .

i.e. R is regular.

Conversly,

suppose that R is regular.

It is well-known, that over regular rings, every f-g ideal is principal . Let A_R be any module.

consider $0 \longrightarrow K \longrightarrow F_R \longrightarrow A \longrightarrow 0$

It suffices to show that $\forall\, u \in K$, $u \in KI_u$ (proposition 1.6)

since I_u is f-g $\forall\, u \in K$, it is principal. i.e. $I_u = Rr$ for some r.

now, $u \in K \Longrightarrow u = fr$ for some $f \in F$

but, $r = rar$ for some $a \in R$.

i.e. $u = frar = uar \in KI_u$

$\Longrightarrow A$ is flat.

<u>REMARK:</u> WE will show later, that R is regular iff every cyclic module is flat.

CHAPTER 3

1. Faithfully-Flat Modules

DEFINITION: A module E_R is faithfully-flat (f-flat) if E is flat, and for all $_RM$, $E \otimes M = 0 \implies M = 0$.

PROPOSITION 3.1 E_R is f-flat iff for all maximal left ideals $_RM$, $E \neq EM$.

Proof:

Suppose that E is f-flat, then as $R/M \neq 0 \implies E \otimes R/M \neq 0$.

but $E \otimes R/M \cong E/EM$

i.e. $E \neq EM$.

Conversly,

every ideal $_RI \neq R$, is contained in some maximal ideal $_RM$.

hence, $E \neq EM \implies E \neq EI$.

i.e. $E \otimes R/I \ (\cong E/EI) \neq 0$

i,e. for all cyclic modules $R/I \neq 0$, $E \otimes R/I \neq 0$.

Now, if $_RN$ is an arbitrary, non-trivial module, it contains a cyclic submodule N'. But as E is flat, we can identify $E \otimes N'$ as a subgroup of $E \otimes N$.

hence, as $E \otimes N' \neq 0 \implies E \otimes N \neq 0$.

Remarks:

(a). It follows from the above proposition, that the direct-sum of f-flat modules is also f-flat. In particular, as R is evidently f-flat as a module over itself, it follows that every free module is f-flat. We will show now, that the same is not true for projective modules. In fact, if every projective module over R is f-flat then R contains no non-trivial idempotent.

i.e. suppose that $e \neq 0,1$ is idempotent, then Re being a direct-summand of R, is projective.

But $Re \otimes (1-e)R = 0$ as $xe \otimes (1-e)y = xe^2 \otimes (1-e)y$

$$= xe \otimes e(1-e)y = 0$$

(b). If R is a Principal-Ideal-Domain, it follows that E_R is f-flat iff it is torsion-free, and $E \neq Ep$ for any prime element p.

In particular, Q is not f-flat Z-module. However, we can show this directly, for consider

$$z \in Q \otimes Z_m \qquad m > 1$$

$$\text{then, } z = \sum_i x_i \otimes y_i \quad , \; x_i \in Q \;\; y_i \in Z_m \; .$$

$$\text{but, } x_i \otimes y_i = \frac{mx_i}{m} \otimes y_i = \frac{x_i}{m} \otimes my_i = 0 \;\; \forall i.$$

$$\text{i.e. } Q \otimes Z_m = 0.$$

(c), If R is a local ring, then every f-g flat module is f-flat. This follows from the proposition, for if M is the unique maximal ideal, and E is f-g, then EM=E ==> E=0.

<u>PROPOSITION 3.2</u> Let $0 \longrightarrow E' \longrightarrow E_R \longrightarrow E'' \longrightarrow 0$ be exact. Then if E' and E" are flat, and one of them is f-flat, then E is also f-flat.

<u>Proof:</u>

by proposition 1.5, E is flat.

As E" is flat, then for any ${}_RM$, $0 \longrightarrow E'\otimes M \longrightarrow E\otimes M \longrightarrow E"\otimes M \longrightarrow 0$ is exact (proposition 1.4).

Now, if $E\otimes M=0$, then $E'\otimes M$ and $E"\otimes M$ are both the trivial groups. but, as one of E' and E" is f-flat, $\Longrightarrow M=0$

i.e. E is f-flat.

---

2. Flatness in terms of the Tor. Functor, and Homological Dimension.

Let R be a fixed ring. Recall, that the functor $Tor_n^R(E_R; {}_RM)$ is the left-deived functor of $E\otimes_R M$. In particular, keeping E_R fixed, then $Tor_n(E,.)$ is the left-derived functor of $T_E(.): {}_R\mathcal{C} \longrightarrow \mathcal{C}_Z$ defined by $T_E(M)=E\otimes M$.

We also recall that $\exists$ a natural isomorphism $Tor_0(E,M)\cong E\otimes M$, hence given any exact sequence $0 \longrightarrow M' \longrightarrow {}_RM \longrightarrow M" \longrightarrow 0$, we get the following derived, exact sequence:

$$\ldots \longrightarrow Tor_2(E,M") \longrightarrow Tor_1(E,M') \longrightarrow Tor_1(E,M) \longrightarrow Tor_1(E,M")$$
$$\longrightarrow E\otimes M' \longrightarrow E\otimes M \longrightarrow E\otimes M" \longrightarrow 0.$$

We can now relate the flatness of E with the Tor. functor.

<u>PROPOSITION 3.3</u> The following are equivalent:

(a). E_R is flat.

(b). for all left R-module M, $\forall$ $n>0$, $Tor_n(E,M)=0$.

(c). for all left R-module M, $Tor_1(E;M)=0$

(d). for all f-g left ideals I of R, $Tor_1(E,R/I)=0$

Proof:

(a)==>(b). Let ${}_RM$ be arbitrary, and consider the free resolution of M, $\ldots \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \ldots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$.

since E is flat,

$(\underline{F})\quad \ldots \longrightarrow E\otimes F_n \longrightarrow E\otimes F_{n-1} \longrightarrow \ldots \longrightarrow E\otimes F_0 \longrightarrow E\otimes M \longrightarrow 0$ is exact.

but, $Tor_n(E,M)$ is isomorphic to the homology modules of the complex $(\underline{F})$, and as $(\underline{F})$ is an exact sequence, $H_n(E\otimes\underline{F})=0\ \forall\, n>0$

i.e. $Tor_n(E,M)=0$ for $n>0$.

(b)==>(c)==>(d) Trivial.

(d)==>(a). Consider any f-g ideal ${}_RI$,

then,

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0 \text{ is exact.}$$

hence,

$$Tor_1(E,R/I) \longrightarrow E\otimes I \longrightarrow E\otimes R \text{ is exact.}$$

but $Tor_1(E,R/I)=0$,

i.e. $E\otimes I \longrightarrow E\otimes R$ is mono. $\forall$ f-g ${}_RI$

$\Longrightarrow$ E is flat.

DEFINITIONS: (1). A_R has projective dimension = n, and denoted by $\dim A_R=n$, if $\exists$ a projective resolution for A of length n, where n is minimum.

Otherwise, we denote $\dim A_R=\infty$

(2). A_R has weak-dimension = n, and denoted by

$w.dimA_R=n$, if $\exists$ a flat resolution for A of length n, where n is minimum.

(3). Define the <u>right global dimension of R</u> ($g.dimR_R$) as the $\sup_{A_R} dimA_R$.

(4). Define the <u>weak right global dimension of R</u> ($w.g.dimR_R$) as $\sup_{A_R} w.dimA_R$

As customary, the (weak) dimension of the trivial module is denoted by -1.

Before we proceed, we need the following relavent result, which we only state as a theorem.

<u>THEOREM 3.4</u> $Tor_n(E,M) \cong H_n(E \otimes \underline{P})$, where $\underline{P}$ is a flat resolution of ${}_RM$.

i.e. this theorem states that, in effect, to compute $Tor_n(E,M)$ it suffices to consider ~~only~~ the flat resolutions of M, rather then the projective ones. With this result, we are now able to prove the following:

<u>PROPOSITION 3.5</u> The following are equivalent:

(a). $w.dimA_R \leqslant n$

(b). $Tor_k(A,M)=0 \ \forall \ {}_RM$ and $\forall \ k>n$.

(c). Given any exact sequence $0 \longrightarrow X \longrightarrow E_{n-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow A \longrightarrow 0$, where E_i $i=0,1,\ldots,n-1$ are flat, then X is necessarily flat.

<u>Proof:</u>

(a)==>(b). Let $\underline{P}$ be a flat resolution of A_R. Then,

$$\mathrm{Tor}_n(A,M) = H_n(\underline{P}\otimes M) \qquad \text{(theorem 3.4)}$$

But, as $w.\dim A_R \leqslant n$, we can choose $\underline{P} \ni P_k=0\ \forall\ k>n$.

i.e. $\mathrm{Tor}_k(A,M)=0\ \forall\ k>n$.

(b)==>(c). Let $\underline{P}$ be a flat resolution of A, and put $A_i=\mathrm{Im}(E_i \longrightarrow E_{i-1})$. We then get the following exact sequences:

$$0 \longrightarrow A_1 \longrightarrow E_0 \longrightarrow A \longrightarrow 0$$

$$0 \longrightarrow A_2 \longrightarrow E_1 \longrightarrow A_1 \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow A_{n-1} \longrightarrow E_{n-2} \longrightarrow A_{n-2} \longrightarrow 0$$

$$0 \longrightarrow X \longrightarrow E_{n-1} \longrightarrow A_{n-1} \longrightarrow 0$$

this, in term, gives the following exact sequences:

$$0=\mathrm{Tor}_{n+1}(E_0,M) \longrightarrow \mathrm{Tor}_{n+1}(A,M) \longrightarrow \mathrm{Tor}_n(A_1,M) \longrightarrow \mathrm{Tor}_n(E_0,M)=0$$

$$0 \longrightarrow \mathrm{Tor}_n(A_1,M) \longrightarrow \mathrm{Tor}_{n-1}(A_2,M) \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow \mathrm{Tor}_3(A_{n-2},M) \longrightarrow \mathrm{Tor}_2(A_{n-1},M) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Tor}_2(A_{n-1},M) \longrightarrow \mathrm{Tor}_1(X,M) \longrightarrow 0$$

hence,

$$\mathrm{Tor}_1(X,M) \cong \mathrm{Tor}_2(A_{n-1},M) \cong \ldots \cong \mathrm{Tor}_n(A_1,M) \cong \mathrm{Tor}_{n+1}(A,M)=0$$

i.e. $\mathrm{Tor}_1(X,M)=0$

==> X is flat.

(c)==>(a). Trivial

PROPOSITION 3.6 Let $0 \longrightarrow E' \longrightarrow E_R \longrightarrow E'' \longrightarrow 0$ be exact, where E is flat. Then, w.dim$E'' \leq 1+$ w.dimE', Furthermore, equality holds if E" is not flat.

Proof:

Suppose E" is not flat. i.e. w.dim$E''>0$.

Now, for $n>0$ and any ${}_RM$, we have

$$0 \longrightarrow \mathrm{Tor}_{n+1}(E'',M) \longrightarrow \mathrm{Tor}_n(E',M) \longrightarrow 0 \text{ is exact.}$$

$$\text{i.e. } \mathrm{Tor}_{n+1}(E'',M) \cong \mathrm{Tor}_n(E',M)$$

hence, by the previous proposition,

$$\Longrightarrow \text{w.dim}E'' = 1+\text{w.dim}E'$$

Furthermore, if E" is flat, then E' is also flat, and the inequality is trivially satisfied.

It is well-known (See[7]), that g.dim $R_R=$ sup dim C_R, where C_R ranges over all cyclic modules. As the main result of this section, we will prove that the same is true for the weak global dimension of R.

i.e. w.g.dim $R_R = \sup\limits_{A_R}$ w.dim A_R, where the A_R are cyclic.

The theorem is the consequence of the following lemma:

<u>LEMMA 3.7:</u> Let $< A_i >_{i\in I}$ be a family of submodules of a module E_R, where $(I,\leqslant)$ is a well-ordered set, and the A_i satisfy the following:

(1). $A_1=0$, and $A_i \subset A_j$ for $i\leqslant j$.

(2). If j is a limit-ordinal, then $A_j=\bigcup_{i<j} A_i$.

(3). $w.\dim(A_{i+1}/A_i)\leqslant n \ \forall i\in I$.

Then, if $A_R=\bigcup_{i\in I} A_i$, $\Longrightarrow$ $w.\dim A \leqslant n$.

<u>Proof:</u>

We proceed by induction on n.

Suppose $n=0$.

We show that A_i is flat $\forall\, i\in I$.

Now, A_1 is flat, and as $w.\dim(A_2/A_1)=0, \Longrightarrow A_2$ is flat.

suppose that A_i is flat $\forall\, i< j$.

(i). if j is not a limit-ordinal, then

$$0 \longrightarrow A_{j-1} \longrightarrow A_j \longrightarrow A_j/A_{j-1} \longrightarrow 0 \text{ is exact.}$$

but, both A_{j-1} and A_j/A_{j-1} are flat,

$$\Longrightarrow A_j \text{ is flat.}$$

(ii). if j is a limit-ordinal, then

$$A_j= \varinjlim_{i<j} A_i, \text{ and hence, } A_j \text{ is flat.}$$

i.e. A_i is flat $\forall\, i\in I$.

but $A=\bigcup_{i\in I} A_i = \varinjlim A_i$

hence, A is flat, and $w.\dim A = 0$

Proceeding with the induction hypothesis, assume that the lemma is true for all $m < n$, $n > 0$.

Let F_R be the free module generated by the non-zero elements of A.

Let F_i be the free modules generated by the non-zero elements of A_i.

$$\text{Then, } F_i \subset F_{i+1} \subset F \quad \forall\, i \in I.$$

consider the epimorphism $f: F \longrightarrow A$ defined in the natural way. Let $K = \mathrm{Ker}(f)$, then , by restricting the domain of f, we get maps $F_i \longrightarrow A_i$ with kernels $K_i = F_i \cap K$.

Hence, we get the following commutative diagram :

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_i \arrow[r] \arrow[d] & F_i \arrow[r] \arrow[d] & A_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_{i+1} \arrow[r] \arrow[d] & F_{i+1} \arrow[r] \arrow[d] & A_{i+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_{i+1}/K_i \arrow[r] \arrow[d] & F_{i+1}/F_i \arrow[r] \arrow[d] & A_{i+1}/A_i \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The columns are exact, the upper two rows are exact, and the bottom row is an 0-sequence. However, by routine diagram-chasing, we conclude that $0 \longrightarrow K_{i+1}/K_i \longrightarrow F_{i+1}/F_i \longrightarrow A_{i+1}/A_i \longrightarrow 0$ is also exact $\forall\, i \in I$.

Now, F_{i+1}/F_i is free, thus flat. It follows (proposition 3.6) that $w.\dim(K_{i+1}/K_i) \leqslant n-1$.

This implies that we can use the induction hypothesis on $\bigcup_i K_i$, as clearly, $< K_i >_{i \in I}$ satisfy the postulates of the lemma.

But, $\bigcup_i K_i = \bigcup_i (F_i \cap K) = (\bigcup_i F_i) \cap K = F \cap K = K$

$$\Longrightarrow w.\dim A < n-1.$$

Furthermore,

$$0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0 \text{ is exact.}$$

Reapplying proposition 3.6,

$$\begin{aligned} w.\dim A &\leqslant 1 + w.\dim K \\ &\leqslant 1 + (n-1) \\ &\leqslant n \end{aligned}$$

<u>THEOREM 3.7</u> $w.g.\dim R_R = \sup_A w.\dim A_R$ where A ranges over all cyclic right R-modules.

<u>Proof:</u>

Trivially, $\sup_A w.\dim A_R \leqslant w.g.\dim R_R$

Assume now, that $\forall$ cyclic module A, $w.\dim A \leqslant n \quad n \geqslant 0$.

We will show that for an arbitrary E_R, $w.\dim E \leqslant n$

let $\langle x_i \rangle_{i \in I}$ generate E, and well-order the set I.

Define the submodule A_j as the one generated by $\langle x_i \rangle$ $i < j$.

Clearly, $< A_i >_{i \in I}$ the conditions of the lemma, and as A_{i+1}/A_i is cyclic (generated by the natural image of x_i), then by hypothesis

$$w.\dim(A_{i+1}/A_i) \leqslant n \ \forall i.$$

But, $E=\bigcup_i A_i$, hence, by the lemma,

$$w.\dim E \leqslant n$$

<u>COROLLARY 3.7</u> A ring R is regular iff every cyclic right (left) module is flat.

<u>3. Weak Dimensions of Noetherian Rings, and Rings of Quotients</u>.

Before proceeding these two important examples, we make the following observation regarding the weak global dimension of R, which is a consequence of the symmetric properties of the Tor. functor.

Recall that $w.\dim A_R \leqslant n$ iff $\mathrm{Tor}_{n+1}(A,M)=0$ for all ${}_RM$. Similarily, $w.\dim_R M \leqslant n$ iff $\mathrm{Tor}_{n+1}(A,M)=0$ for all A_R.

Now, let both A and M vary. Suppose that $\sup w.\dim A_R \leqslant n$, then Tor_{n+1} takes only the null values, and conversly. Hence, by the above symmetry of Tor_{n+1}, we conclude that $\sup w.\dim_R M \leqslant n$.

i.e. left weak global dimension = right weak global dimension, and it is denoted simply by $w.g.\dim R$.

As a trivial consequence of the definition, we note that over any ring R, $w.g.\dim R \leqslant g.\dim R_R$

$$\leqslant g.\dim {}_RR.$$

(A). Let R be a right <u>Noetherian</u> ring.

<u>PROPOSITION 3.8</u> For any f-g module A_R, w.dim A = dim A.

<u>Proof:</u>

It suffices to show that dim A $\leqslant$ w.dim A, as the reverse inequality is abviously true.

Let A_R be f-g, and assume that w.dim A $\leqslant$ n. Then, $\exists$ a flat resolution $0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \ldots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$, where the P_i are f-g and flat.

But, as R is right noetherian, every f-g right module is finitely related, and every finitely-related flat module is projective (corollary 2.7).

i.e. P_i $i=0,1,\ldots,n$ are projectives.

$\Longrightarrow$ dim A $\leqslant$ w.dim A.

<u>COROLLARY 3.8</u> If R is right-noetherian, then w.g.dim R = g.dimR_R.

<u>Proof:</u>

Again, it is sufficient to prove that g.dim $R_R \leqslant$ w.g.dim R. suppose that A_R is cyclic, then

dim A = w.dim A $\leqslant$ w.g.dim R (proposition 3.8)

hence,

$\sup_A$ dim A $\leqslant$ w.g.dim R, where A_R ranges over all cyclic modules.

i.e. g.dim $R_R \leqslant$ w.g.dim R

(B). Let R be a <u>commutative</u> ring.

We recall that for every multiplicatively-closed set S, the functor $S(.):\mathcal{C}_R \dashrightarrow \mathcal{C}_{R_S}$ defined by $S(A)=A_S$ is an exact functor into the category of R_S-modules, where R_S denotes the ring of quotients. Furthermore, $\exists$ an R_S-isomorphism

$$\mathrm{Tor}_{n+1}^{R_S}(A_S,B_S)\cong[\mathrm{Tor}_{n+1}^{R}(A,B)]_S .$$

<u>PROPOSITION 3.9</u> Over any commutative ring R, w.g.dim R_S $\leqslant$ w.g.dim R.

<u>Proof:</u>

Assume that w.g.dim R = n $n \geqslant 0$.

Let A,B be arbitrary R_S-modules. It suffices to show that $\mathrm{Tor}_{n+1}^{R_S}(A,B)=0$.

A can be regarded as an R-module if we define $ra=\left[\frac{r}{1}\right]a \ \forall r\in R,\ a\in A$, and similarily for B.

$$\text{hence, } \mathrm{Tor}_{n+1}^{R}(A,B)=0$$

but,

$$\mathrm{Tor}_{n+1}^{R_S}(A_S,B_S)\cong[\mathrm{Tor}_{n+1}^{R}(A,B)]_S \qquad \text{(by 0.12)}$$

$$\text{i.e. } \mathrm{Tor}_{n+1}^{R_S}(A_S,B_S)=0$$

But as an R_S-module, $A\cong A_S$, and similarily for B (by 0.11)

$$\text{thus, } \mathrm{Tor}_{n+1}^{R_S}(A,B)=0, \implies \text{w.g.dim } R_S \leqslant n$$

PROPOSITION 3.10 Let A be an R-module, then

$$w.\dim(A_S)_{R_S} \leqslant w.\dim A_R.$$

Proof:

Assume $w.\dim A_R = n$, then we show that for any R_S-module B,

$$\mathrm{Tor}_{n+1}^{R_S}(A_S,B)=0.$$

Now, B can be considered as an R-module, hence,

$$\mathrm{Tor}_{n+1}^{R_S}(A_S,B_S)\cong[\mathrm{Tor}_{n+1}^{R}(A,B)]_S = 0.$$

But B and B_S are isomorphic as R_S-modules,

$$\text{i.e } \mathrm{Tor}_{n+1}^{R_S}(A_S,B_S)=0$$

COROLLARY 3.10 If E is a flat R-module, then E_S is a flat R_S-module.

Proof:

E flat $\implies w.\dim E_R \leqslant 0$

$\implies w.\dim(E_S)_{R_S} \leqslant 0$ (proposition 3.10)

$\implies E_S$ is R_S-flat

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4. Finitely-Generated Flat Modules over Integral-Domains.

As a final application, we will prove that every f-g flat module over an integral-domain is projective. We first prove the following result, which is interesting in itself:

Let R be a local ring and M its unique maximal ideal, then

THEOREM 3.11 Every f-g flat module over a local ring R is free[1].

Proof:

Let E_R be f-g flat module, and let $(u_1, u_2, \ldots, u_n)$ be a minimal set of generators for E. Consider the exact sequence

$$0 \longrightarrow K \longrightarrow F_R \longrightarrow E \longrightarrow 0,$$

where F is free with basis $(x_1, \ldots, x_n)$, $f(x_i)=u_i \ \forall i$, $K=\mathrm{Ker}(f)$.

Now, let $k \in K$, then $k = \sum_{i=1}^{m} x_i a_i \quad a_i \in R$.

but since $f(k)=0$, $\Longrightarrow \sum_{i=1}^{m} u_i a_i = 0$.

Hence, $a_i \in M \ \forall i$. For suppose that $a_{i_0} \notin M$ for some i_0. Then, a_{i_0} is invertible, which implies that we can express a_{i_0} in term of $(a_i)_{i \neq i_0}$, which contradicts the minimality of $(a_i)_i$.

hence, $K \subset FM$.

As before, we denote by $I_k = \sum_{i=1}^{m} Ra_i$, thus I_k is f-g $\forall k \in K$.

Now, as E is flat, then $\forall k \in K$, $k \in KI_k$ (proposition 1.6)

1. I have not seen in the literature a proof of the above result, which is as short and direct as the one presented here.

$$\Longrightarrow k=\sum_{i=1}^{m} k_i a_i \text{ for some } k_i \in K.$$

but as $K \subset FM$,

$$\Longrightarrow k_i = \sum_j x_j m_{ij}$$

hence,

$$k=\sum_i\left(\sum_j x_j m_{ij}\right)a_i = \sum_j x_j\left(\sum_i m_{ij} a_i\right)$$

but $k=\sum_{j=1}^{m} x_j a_j$, and since x_j form a basis, $\Longrightarrow a_j=\sum_i m_{ij} a_i \ \forall j$.

i.e. $I_k \subset I_k M \ \forall k \in K$.

but, as I_k is finitely-generated, $\Longrightarrow I_k=0 \ \forall k \in K$.

Hence, $K=0$, and $E \cong F$

i.e. E is a free R-module.

Let R be now an integral-domain. Denote by $E_f=E_S$ the module of quotients, where $S=\langle f^n/n>0, f\in R\rangle$. Let R_f be the corresponding ring of quotients.

We note that for every torsion-free module E, $E\subset E_S$, and hence, a set of elements is linearily-indepedent in E iff they are linear-indepedent in E_S.

<u>LEMMA 3.12</u> Let E be f-g and flat over an integral-domain R. Then, $\forall$ maximal ideal M, $\exists$ an $f\in R-M$, $\ni$ E_f is R_f-free.

<u>Proof:</u>

$\forall M$, E_M is a f-g flat R_M-module (proposition 3.10)

But, R_M is always local, hence, E_M is R_M-free (theorem 3.11), with basis, say, $(\frac{e_1}{s_1}, \frac{e_2}{s_2}, \ldots, \frac{e_n}{s_n})$ where $e_i \in E$, $s_i \in R-M$. But, as s_i are invertible, $\Longrightarrow e_1, e_2, \ldots, e_n$ also form a base for E_M.

Extent the base to a set $e_1, \ldots, e_n, \ldots, e_r$ of generators for E over R. As E is flat, and hence, torsion-free,

$$\forall\, i>n,\ e_i = \Sigma \frac{r_{ij}}{s_{ij}} e_j \qquad r_{ij},\ s_{ij} \in R,\ s_{ij} \notin M$$

let $f = \prod_{i,j} s_{ij}$, then it follows that E_f is R_f-free, with basis $(e_1, e_2, \ldots, e_n)$

THEOREM 3.12 Every f-g flat module over an integral-domain R is projective.

Proof:

Let $P^* = \langle f / E_f \text{ is } R_f\text{-free} \rangle$. Then, by the above lemma, P^* is not contained in any maximal ideal, and hence, the ideal generated by P^* is R.

$$\Longrightarrow \exists \text{ a finite set } \ni R = f_1 R + \ldots + f_n R.$$

Let $B = \prod_i R_{f_i}$, then B is faithfully-flat R-module, as evidently, $BM \neq B$ for any maximal ideal M (proposition 3.1)

As $R \subset B$, and B is f-flat, hence, E is finitely-related R-module if $B \otimes_R E$ is a finitely-related B-module (See[2]).

But, the map $f: B \otimes_R E \longrightarrow \prod_{i=1}^{n} E_{f_i}$ defined by

$$\left(\frac{r_1}{f_1^{k_1}}, \ldots, \frac{r_n}{f_n^{k_n}}\right) \otimes x \longrightarrow \left(\frac{r_1}{f_1^{k_1}} x, \ldots, \frac{r_n}{f_n^{k_n}} x\right)$$

is a B-isomorphism. Furthermore, each E_{f_i} is f-g, free R_{f_i}-module, and hence, projective

$$\Longrightarrow \prod_{i=1}^{n} E_{f_i} \text{ is f-g, projective over } \prod_{i=1}^{n} R_{f_i}.$$

But every f-g, projective module is finitely-related. i.e.

$\prod_{i=1}^{n} E_{f_i} \cong B \otimes_R E$ is a finitely-related B-module,

$$\Longrightarrow E \text{ is finitely-related } R\text{-module}.$$

But, every finitely-related flat module is projective (proposition 2.7).

i.e. E is a projective R-module.

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