EQUATIONS AND EQUATIONAL THEORIES

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ABSTRACT

We define a formula $\varphi(\vec{x};\vec{t})$ in a first-order language L, to be an equation in a category of L-structures $\mathbb{K}$ if,

for any H in $\mathbb{K}$, and set

$$p = \{\varphi(\vec{x};\vec{a}_i);\ i \in I,\ \vec{a}_i \in H\}$$

there is a finite set $I_0 \subset I$ such that for any $f:H \longrightarrow F$ in $\mathbb{K}$,

$$\cap_{i\in I_0}\varphi(F;f\vec{a}_i) = \cap_{i\in I}\varphi(F;f\vec{a}_i).$$

We say that a complete theory T is equational if any formula is equivalent in T to a boolean combination of equations in Mod(T), and we note that equational theories are stable.

Thus, we develop a theory of independence with respect to equations in general categories of structures, which is similar to the one introduced in stability (and actually identical to it in the case of equational theories) but which, in our context, have an algebraic character.

We then compare the concepts introduced in stability theory to corresponding notions in the context of equational theories.

RESUME

Nous disons qu'une formule $\varphi(\vec{x};\vec{t})$ dans un language du premier ordre L, est une équation dans une catégorie de L-structures $\mathbb{K}$ si,

pour tout H dans $\mathbb{K}$ et tout ensemble

$$p = \{\varphi(\vec{x};\vec{a}_i);\ i \in I,\ \vec{a}_i \in H\},$$

il existe un ensemble fini $I_0 \subset I$ tel que pour tout $f: H \longrightarrow F$ dans $\mathbb{K}$,

$$\cap_{i \in I_0} \varphi(F; f\vec{a}_i) = \cap_{i \in I} \varphi(F; f\vec{a}_i).$$

Une théorie complète est dite équationelle si toute formule est équivalente dans T à une combinaison booléénne d'équations dans $\mathrm{Mod}(T)$. Aussi nous notons qu'une théorie équationelle est stable.

Ainsi, nous developpons dans une catégorie de structures donnée, une théorie de l'indépendence qui est similaire à celle introduite en stabilité (et en fait, identique à celle-là dans le cas des théories équationelles) mais, qui a, dans notre contexte, un caractère algébrique.

Par la suite, nous comparons les concepts introduits en théorie de la stabilité à des concepts correspondants dans le contexte des théories équationelles.

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Introduction

The work presented in this thesis grew out of observations of the author made years ago to the effect that, in view of its applications, stability theory can/should be developed in an algebraic context.

The general idea would be to find and to investigate properties which confer to stable theories an algebraic character; a systematic study of such properties would be useful at the least for practical reasons if not for theoretical ones.

This thesis deals with chain condition properties in various categories of structures.

To make this more precise let us first recall a few points on stable theories.

A complete first order theory T is stable iff there is a cardinal λ such that there are exactly λ complete types over any model of T of cardinality λ.

This cardinality restriction on complete types is equivalent to a local combinatorial property on formulas which, in summary, means that we cannot define an order in T.

More precisely, T is stable iff any formula $\varphi(\vec{x};\vec{y})$ has the following (ladder) property:

for any model M of T there is no sequence $(\vec{a}_i)_{i<\omega}$ of tuples in M such that $\wedge_{i<n}\varphi(\vec{x};\vec{a}_i)\wedge\wedge_{i\geq n}\neg\varphi(\vec{x};\vec{a}_i)$ is consistent in M for any $n < \omega$.

Furthermore, satiblity permits the definition of canonical extensions of types, called non-forking extensions, in the following manner:

fix a large saturated model $\overline{M}$; all subsets of $\overline{M}$ considered below shall be subsets of cardinality strictly less than the cardinality of $\overline{M}$ and the types shall be types in tuples of variables of length strictly less than card$\overline{M}$.

Let $A \subset \overline{M}$ and let p be a complete type over A.

For q and r complete types over $\overline{M}$ extending p write $q \approx r$ if there is an automorphism σ of $\overline{M}$ which fixes A and such that $\sigma q = r$; of course $\approx$ is an equivalence realtion on the set of complete types over $\overline{M}$ extending p.

Now stability ensures the *existence and unicity* of an $\approx$-class C_p which has cardinality less or equal to $2^{(|T|+\aleph_0)}$. The elements of C_p are then called non-forking extensions of p to $\overline{M}$.

More generally, if $A \subset B \subset \overline{M}$, p a complete type over A and q a complete type over B, we say q is a non-forking extension of p to B or, that q does not fork over A, if there is an extension r of q to $\overline{M}$ which is a non-forking extension of p.

This non-forking notion is best understood in terms of a (ternary) relation of independence on the subsets of $\overline{M}$:

for $A \subset B, C \subset \overline{M}$, we write $B \underset{A}{\downarrow} C$ and say "B is independent from C over A" if $tp(B;C)$ is a non-forking extension of $tp(B;A)$ to C.

We intend here the word "independent" to convey the intuitive meaning attached to it. Thus, for instance, if T is the theory of infinite vector spaces over some fixed field then (T is stable and),

for $A \subset B, C \subset \overline{M}$, A, B, C subspaces of $\overline{M}$

$B \underset{A}{\downarrow} C$ iff the sum $B + C$ is direct over A i.e. $B \cap C = A$.

For a stable theory T, the following properties are true:

0. Given $A \subseteq \overline{M}$ and p, complete over A, there exists a unique $\approx$-class C_p as described above.

1. Monotonicity-transitivity: given $A \subset B \subset C \subset \overline{M}$ and $D \subset \overline{M}$,

$$D \underset{A}{\downarrow} C \text{ iff } D \underset{B}{\downarrow} C \text{ and } D \underset{A}{\downarrow} B.$$

2. Local-character: given $A, B, C \subseteq \overline{M}$

$B \underset{A}{\downarrow} C$ iff $\vec{b} \underset{A}{\downarrow} C$ for any finite tuple of elements in

B.

3. Symmetry: given $A,B,C \subset \bar{M}$

$$B \underset{A}{\downarrow} C \text{ iff } C \underset{A}{\downarrow} B.$$

4. If $A \subset \bar{M}$ and q is a complete type over $\bar{M}$ which does not fork over A then, for any formula $\varphi(\vec{x};\vec{t})$ in L, the type

$$q\restriction\varphi = \{\varphi(\vec{x};\vec{m})^{t} \in q;\ \vec{m} \subset \bar{M},\ t = 0,1\}$$

has finitely many distinct conjugates over A.

5. If q is a complete type over $B(\subset \bar{M})$, then there is $A \subset B$, $\mathrm{card}A \le \mathrm{card}T + \aleph_0$ such that q does not fork over A.

Note that properties 1. and 2. readily follow from 0. Property 5. can be seen as dual to property 0. and "stability", in the intuitive sense of the word as well as in its strict sense, is equivalent to the conjunct of 0. and 5.

Let us now describe (without any proof) the relation $\vec{a} \underset{M}{\downarrow} \vec{b}$, where $\vec{a}$, $\vec{b}$ are tuples of elements in $\bar{M}$ and M is an elementary submodel of $\bar{M}$, in two algebraic examples of stable theories:

1. For T the theory of algebraically closed fields in a

fixed characteristic (so that here $\overline{M}$ is a fixed large algebraically closed field and M is an algebraically closed subfield):

$$\vec{a} \underset{M}{\downarrow} \vec{b} \text{ iff } M\langle\vec{a}\rangle \text{ is linearly disjoint from } M\langle\vec{b}\rangle \text{ over } M,$$

where $M\langle\vec{a}\rangle$ is the field generated by M and $\vec{a}$.

(Recall that if $k \subset K,F$, are subfields of $\overline{M}$, we say K is linearly disjoint from F over k iff any sequence of elements in K which is linearly independent over k remains linearly independent over F.)

Equivalently,

$\vec{a} \underset{M}{\downarrow} \vec{b}$ iff for any fields $K \supset M$ and $F \supset M$, $K,F \subset \overline{M}$, and any field homomorphisms

$$f{:}M\langle\vec{a}\rangle \longrightarrow K \text{ and } g{:}M\langle\vec{b}\rangle \longrightarrow F$$

such that f and g are the identities on M, *there is* a homomorphism

$$h{:}M\langle\vec{a},\vec{b}\rangle \longrightarrow K\cdot F$$

such that $h{\restriction}M\langle\vec{a}\rangle = f$ and $h{\restriction}M\langle\vec{b}\rangle = g$.

And under slightly different terms,

$\vec{a} \underset{M}{\downarrow} \vec{b}$ iff the set of algebraic equations that the tuple $\vec{a}$ satisfy over M determines completely the set of algebraic equations that $\vec{a}$ satisfy over $M\langle\vec{b}\rangle$ i.e. $\vec{a} \underset{M}{\downarrow} \vec{b}$ iff the set of formulas

$$\{(P(\vec{x}) = 0);\ P(\vec{x}) \in M[\vec{x}],\ \overline{M} \models (P(\vec{a}) = 0)\}$$

is logically equivalent to the set

$$\{(P(\vec{x}) = 0),\ P(\vec{x}) \in M\langle\vec{b}\rangle[\vec{x}],\ \overline{M} \models (P(\vec{a}) = 0)\}.$$

2. For T the model completion of torsion-free abelian groups:

$\vec{a} \underset{M}{\downarrow} \vec{b}$ iff the sum $M\langle\vec{a}\rangle + M\langle\vec{b}\rangle$ is direct over M i.e. iff $M\langle\vec{a}\rangle \cap M\langle\vec{b}\rangle = M$. ($M\langle\vec{a}\rangle$, the group generated by M and $\vec{a}$).

Equivalently,

$\vec{a} \underset{M}{\downarrow} \vec{b}$ iff for any abelian groups $F \supset M$, $G \supset M$, $F,G \subset \overline{M}$, and group homomorphisms $f{:}M\langle\vec{a}\rangle \longrightarrow F$, $g{:}M\langle\vec{b}\rangle \longrightarrow G$ such that f and g are the identities on M *there is* a group homomorphism

$$h{:}M\langle\vec{a}\rangle + M\langle\vec{b}\rangle \longrightarrow F + G$$

such that $h \restriction M\langle\vec{a}\rangle = f$ and $h \restriction M\langle\vec{b}\rangle = g$.

In different words, we have

$\vec{a} \underset{M}{\downarrow} \vec{b}$ iff the set of atomic formulas that $\vec{a}$ satisfy over M determines completely the set of atomic formula that $\vec{a}$ satisfy over $M\langle\vec{b}\rangle$.

(We remind the reader that the two theories given above have elimination of quantifiers).

We notice in the examples above that the independence relation ($a \underset{M}{\downarrow} \vec{b}$) is stated *uniquely in terms of the atomic formulas*, without mention of negated atomic formulas; we could say that, intuitively, $\vec{a} \underset{M}{\downarrow} \vec{b}$ if $\vec{a}$ is independent from $\vec{b}$ over M with respect to the atomic formulas or that $\vec{a} \underset{M}{\downarrow} \vec{b}$ iff $\vec{a}$ and $\vec{b}$ do not satisfy any *atomic formula* over M which is not implied by the atomic formulas over M that $\vec{a}$ (or $\vec{b}$) satisfy on its own.

While, a priori, in arbitrary stable theories, the independence relation makes no distinction between e.g. atomic formulas and other formulas. For instance, if the theory has elimination of quantifiers we would have, in principle, that $\vec{a} \underset{M}{\downarrow} \vec{b}$ iff $\vec{a}$ and $\vec{b}$ do not satisfy any *quantifier-free* formula over M which is not "represented" by the *quantifier-free* formulas over M that $\vec{a}$ (or $\vec{b}$) satisfy on its own.

Note also that the ladder property given above of a formula φ does not establish any distinction between φ and $\neg\varphi$. Indeed, by compactness, the ladder property for a formula $\varphi(\vec{x};\vec{t})$ is equivalent to the following:

there is a natural number n for which, in any model M of T, there is no sequence $(\vec{a}_i)_{i<n}$ of tuples in M, such that the formulas

$$\wedge_{j<i}\varphi(\vec{x};\vec{a}_j)\wedge\wedge_{i\leq j<n}\neg\varphi(\vec{x};\vec{a}_j) \text{ for } i < n$$

are consistent in M;

and this last property is clearly symmetric in φ and $\neg\varphi$.

However, a crucial aspect of general algebraic theories is that some formulas are distinguished and considered "positive" while their negations are considered "negative"; also most of the basic notions defined in these theories, among which, notions of independence, mainly depend on these formulas.

For instance, in the theory of fields, the fundamental formulas are the algebraic equations; definitions or results in this theory are usually stated in terms of algebraic equations. Actually, in this case, we note that an important distinction between the algebraic equations and the inequations is the fact that the varieties (of some fixed dimension n) in a given field F are the basic closed sets of a noetherian compact topology on F^n.

Thus, in an attempt to abstract this notion of positiveness we say that a formula $\varphi(\vec{x};\vec{t})$ is an *equation* in T if in any model M of T, the intersection of a family of φ-definable subsets of M equals the intersection of a finite subfamily.

In other words, $\varphi(\vec{x};\vec{t})$ is an equation if the set of subsets of a model M which are definable by finite conjunctions of instances of φ, satisfy the descending chain condition.

Again equivalently, $\varphi(\vec{x};\vec{t})$ is an equation if, in any model M of T, there is no sequence $(\vec{a}_i)_{i<\omega}$ of tuples in M such that the formulas

$$\wedge_{i<n}\varphi(\vec{x};\vec{a}_i)\wedge\wedge_{i\geq n}\neg\varphi(x;\vec{a}_n) \text{ for } n < \omega,$$

are consistent in M;

one should here compare this last version of a formula being an equation to the ladder property: clearly the former implies the latter.

More generally, a set of formulas S is said *equational* if in any model M of T, the intersection of a family of S-definable subsets equals the intersection of a finite subfamily.

For instance, in the theory of algebraically closed fields in a fixed characteristic, the set of algebraic equations in some fixed sequence of variables $\vec{x}$ is equational. This follows immediately from the noetherian property, $F[\vec{x}]$ satisfies when F is a field.

Also, it is clear that equationality of a set of formulas S in the theory T makes the S-definbale subsets of a model M the basic closed sets of a noetherian compact topology on M, which, in the case of fields, with S the set of algebraic equations, is identified to the Zarisky topology.

We say that a complete theory T is *E-equational in* $\vec{x}$ for E a given set of equations if any formula $\Theta(\vec{x};\vec{t})$ is equivalent in T to a boolean combination of formulas $\varphi(\vec{x};\vec{t})$ in E;

T is equational in $\vec{x}$ if there is a set of equations E such that T is E-equational in $\vec{x}$.

It will be easy to show that if T is equational in $\vec{x}$ then T is stable; we do not have an example of a stable theory which is not equational in some $\vec{x}$.

In the thesis, instead of a complete theory T, we consider an arbitrary category of L-structures for L a fixed first-order language. Furthermore, all formulas considered shall be formulas in a fixed set Δ e.g. the set of quantifier-free (or existential etc...) formulas; for emphasis we say $\mathbb{K}$ is a Δ-category.

We define the notion of an equation in $\mathbb{K}$ as follows: given $\varphi(\vec{x};\vec{a})$ and $\psi(\vec{x};\vec{b})$ formulas (in Δ) with parameters in $M \in \mathrm{Ob}(\mathbb{K})$, one has the notion

$$\varphi(\vec{x};\vec{a}) \vdash_H \psi(\vec{x};\vec{b}),$$

meaning that $F \models \forall\vec{x}(\varphi(\vec{x};f\vec{a}) \longrightarrow \psi(\vec{x};f\vec{b})$ for any morphism $f:H \longrightarrow F$ in $\mathbb{K}$.

Now, we say that $\varphi(\vec{x};\vec{t})$ is an equation in $\mathbb{K}$ if for any H in $\mathbb{K}$ and any set of the form

$$p = \{\varphi(\vec{x};\vec{a}_i);\ i \in I,\ \vec{a}_i \text{ in } H\}$$

there is a finite subset p_0 of p, such that

$$\wedge p_0 \vdash_H \varphi(\vec{x};\vec{a}_i) \text{ for any } i \in I;$$

similarly, we define the notion of an equational set of formulas, in $\mathbb{K}$.

Our first task then is to develop in this setting a theory of independence with respect to sets of equations which is identical to the existent one in stable complete theories.

We go about this task in the following manner:

given $A \subset H \in \mathbb{K}$ and a set p of formulas (in Δ) in the tuple of variables $\vec{x}$ with parameters in H, we say p is realized in $\mathbb{K}$ if there is $f:H \longrightarrow F$ in $\mathbb{K}$ such that the set

$$\{\varphi(\vec{x};f\vec{a});\ \varphi(\vec{x};\vec{a}) \in p\}$$

is realized in F; we say p is a type over A if p is consistent in $\mathbb{K}$ i.e. if any finite subset of p is realized in $\mathbb{K}$.

Let $A \subset B \subset H \in \mathbb{K}$ and p a type over A. Now, intuitively, as in the particular examples 1 and 2 described above, we would like an extension q of p to B, to be a "non-forking extension" of p to B with respect to a set of equations S if q does not imply any equation (meaning a formula in S) which is not induced by p.

Two points remain unclear in the definition above: first, what do we mean by "induced"? Second, are we to consider the equations that q should not imply among the formulas in S with parameters in B, among the formulas in S with parameters in H, or even among formulas in S with parameters in F when given some morphism $f:H \longrightarrow F$ in $\mathbb{K}$?

We get by the first point by saying that q is a "non-forking extension" of p to B with respect to S if q implies a *minimal* set of equations; and to investigate the second point we first relativize the search for non-forking extensions, to the structure H. The exact definition is as follows:

If r is a type over B let

$$r_H^S = \{\varphi(\vec{x};\vec{c});\ \varphi \in S,\ \vec{c} \in H,\ r \vdash_H \varphi(\vec{x};\vec{c})\}.$$

Let q be a complete type over B, $q \supset p$; then q is an S_H-*minimal* extension of p to B iff for any complete type r over B, $r \supset p$,

$$q_H^S \supset r_H^S \Longrightarrow q_H^S = r_H^S.$$

The S_H-minimal extensions play in our context the role that non-forking extensions play in stable theories.

We show then that S_H-minimal extensions of p to B exist; and, if S is equational, p has, up to S_H-equivalence finitely many S_H-minimal extensions to B i.e. there are S_H-minimal extensions, $q_0, \cdots, q_{n-1}$ of p to B such that for any S_H-minimal extension q of p to B there is $i < n$, $q_H^S \widetilde{H}$ $(q_i)_H^S$, (c.f. II.A.13). These two properties should be considered as weaker versions of properties 0 and 4 described above for non-forking extensions in stable theories.

Also we show (c.f. II.A.12) that the monotonicity-transitivity property for S_H-minimal extensions (see property 1. above) holds when considering types over "S_H-closed" subsets of H: for S an equational set, a subset A of H is S_H-closed in H if for any type q over A, q_H^S is definable over A, (note that for A arbitrary q_H^S is equivalent in **K** to a single formula in S with parameters *in* H).

As we pointed out above, S_H-minimal extensions are relative to the structure H; it is possible to have a morphism $f:H \longrightarrow F$ (say here an inclusion), $A \subset B \subset H$, p a type over A, q a type over B, such that q is an S_H-minimal extension of p to B but q is not an S_F-*minimal* extension of p to B. For that reason we define the notion of *S-full* types:

a complete type q over H is S-full if for any morphism

$$f{:}H \longrightarrow F, \ (fp)^S_F \underset{F}{\simeq} f[p^S_H].$$

And we show that, if p is over A and q is an S_H-minimal extension of p to H which is S-full, then for any $f{:}H \longrightarrow F$ (say here an inclusion) such that fq is consistent over F, q is an S_F-minimal extension of p to H (c.f. II.B.14); furthermore q has a unique S_F-minimal extension to F.

We say then that q^S_H is an S_H-component of p and we show (c.f. II.B.8) that, under some general assumptions on $\mathbb{K}$, as for instance that $\mathbb{K}$ is closed under unions of countable chains, there is $g{:}H \longrightarrow G$ such that for any S_G-minimal extension q of p to G, q^S_G is an S_G-component of p.

The results mentioned above hold in arbitrary Δ-categories. However, a most interesting example of a Δ-category is the case where Δ is the set of all formulas and $\mathbb{K}$ is the category of models of a first-order (not necessarily complete) theory with the Γ-elementary embeddings, for Γ some boolean-closed set of formulas, for morphisms.

We immediately check then that if H is a $\Sigma_1(\Gamma)$-closed structure in $\mathbb{K}$ then any complete type over H is S-full.

The essential property useful to us in such a category of structures is a definability lemma (c.f. III.7) which says that,

If $A \subset H$ and Θ is a formula with parameters in H such that for any morphisms $f_1, f_2: H \longrightarrow F$, $f_1\Theta \underset{F}{\approx} f_2\Theta$, *then* Θ is definable over A.

This lemma is just a generalization of the case when T is a complete theory, $A \subset H$, card$A$ < card$H$ and H is saturated; the lemma then says that if Θ is invariant under any automorphism of H over A, then Θ is definable over A.

It will follow from this lemma (c.f. III.10) that, for H a $\Sigma_1(\Gamma)$-closed structure, p a complete type over $A \subset H$, and S an equational set of formulas in Γ, the S_H-minimal extensions of p to H are conjugates of each other over A in the sense that, if q_1 and q_2 are such extensions, then there are $f_1, f_2: H \longrightarrow F$ such that

$$f_1[(q_1)^S_H] \underset{F}{\approx} f_2[(q_2)^S_H];$$

that every subset of H is S_H-closed in H so that the monotonicity-transivity property for S_H-minimal extensions holds when considering types over any subset of H;

and, that the local character property holds for S_H-minimal extensions.

The properties of S_H-minimal extension for S an arbitrary set of equations will then be immediately inferred from those of S_H-minimal extensions for S an equational set of formulas.

Finally the symmetry property appears to be a global property, in the sense that it needs consideration of the set of *all* equations in $\mathbb{K}$. It will be best discussed in chapter IV and chapter V.

Let us only state here (c.f. V.A.6) that the symmetry property will hold for S_H-minimal extensions when Γ is the set of all formulas and S is the set of all equations in $\mathbb{K}$.

Of course, all the results above apply to the case when $\mathbb{K}$ is the category of models of a complete theory T with the elementary embeddings for morphisms. In particular, when T is E-equational, whence stable, we will show that E_H-minimal extensions of types (H a model of T) as introduced above identify with non-forking extensions as introduced in stability theory.

Our concern throughout this thesis, has been to show that the notion of an equation well translates the idea of positiveness we mentioned above, and to show that this notion relates in a natural way to algebra.

Our general aim would be to attach to this notion, properties and definitions which are familiar to (standard) equations in algebra.

It will be easily seen for instance that taking an S_H-minimal extension of a type p over $A \subset H$ (for S equational), is similar to taking an "irreducible component" (with respect to S) of p_H^S.

More precisely, we will show (c.f. II.B.6) that $p_0, \cdots, p_{n-1}$ are the S_H-minimal extensions of p to H iff $p_H^S \underset{H}{\simeq} \vee_{i<n} (p_i)_H^S$, and for any $i < n$, $(p_i)_H^S \not\vdash_H (p_j)_H^S$.

Note that $(p_i)_H^S$ is S_H-irreducible, in the sense that for any φ_1, φ_2 in S with parameters in H,

$$(p_i)_H^S \vdash_H \varphi_1 \vee \varphi_2 \Longrightarrow (p_i)_H^S \vdash_H \varphi_\ell, \text{ for } \ell = 1 \text{ or } 2,$$

(this follows immediately from the fact that p_i is complete).

In chapter 0, we fix some notations, make precise the setting in which we want to work and define the notion of an equational set of formulas.

In chapter I, we compare the notion of equationality to some natural variants as for instance a formula having finite height. We investigate the basic properties of equational formulas and, as an application, prove that the set of differential equations in the category of differential fields with characteristic 0, is equational.

In chapter II, section A, we investigate S_H-minimal extensions of a type, show that such extensions always exist (c.f. A.2), that they satisfy the monotonicity-transitivity property when considering types over S_H-closed subsets of H and that, for S equational, a type p over $A \subset B \subset H$ has, up to S_H-equivalence, finitely many S_H-minimal extensions to B.

In section B, we define what S-irreducible and S-full types are, as well as S-irreducible and S-full structures. We also define what an S_H-component of a type is. We show (c.f. 10) that if S is equational and $\mathbb{K}$ is ω-conservative then, given p over H there is a morphism $f:H \longrightarrow F$ in $\mathbb{K}$ such that fp is consistent in $\mathbb{K}$ and for any S_F-minimal extension q of p to f, q_F^S is an S_f-component of p. Finally, we show that if $\mathbb{K}$ is inductive then for any H in $\mathbb{K}$ there is $f:H \longrightarrow F$ in $\mathbb{K}$ such that F is S-full.

In section C, we observe that the theory of Sections A and B goes through in a very general abstract context; we then describe such a context.

In chapter III we study the case of a Δ-category $\mathbb{K}$ of models of a first-order theory T with the Γ-elementary embeddings for morphisms, where we assume Δ is the set of all formulas in L, Γ is a boolean-closed set of formulas, $S \subset \Gamma$, $S = cl^+(S)$ and $\mathbb{K}$ reflects S. We show then, for S a set of equations, that any $\Sigma_1(\Gamma)$-closed structure H in $\mathbb{K}$ is S-full and that any subset of such a structure is S_H-closed in H; furthermore we prove the local-character property for S_H-minimal extensions of types over subsets of H.

In chapter IV, we introduce the notion of S-minimal amalgam and relate it to S_H-minimal extensions. We also define the notion of a full set of formulas in a structure H and prove the symmetry property for S_H-minimal extensions of types when S is full in H. Then, we discuss the existence of full sets of formulas. We prove in particular that the set of all equations, in the category of models of a complete theory with the elementary embeddings for morphisms, is full in any given model. (For more details as to the existence of full sets of formulas in categories such as in chapter III, see [S.2]).

In chapter V, we consider the case of the category of models of a complete first-order theory T with the elementary embeddings for morphisms. We consider the case T stable and show that if p is a complete type over $A \subset B$ and q is a non-forking extension of p to B then q is an S_H-minimal extension of p to B whenever S is a set of equations in T. We define what an equational theory is. We then classify equational theories in terms of equational theories having the d.c.c and equational theories having the d.c.c on irreducible types. We show that equational theories with the d.c.c (resp. with the d.c.c on irreducible types) are totally-transcendental (resp. superstable) and give (in both cases) criterions for the two notions to be equivalent. We also characterize the fundamental order in equational theories in terms of equations.

We have not investigated in the thesis the theory of dimension introduced in stability theory (c.f. [M] or [Sh]). This, as well as applications of the results in this thesis to some algebraic theories, will be undertaken in [S.2].

Although the motivation for this work comes largely from stability theory, the theory in this thesis is, from a logical point of view, independent of stability theory. In particular the technical work in the thesis is self-contained. However, frequent references are made, for motivational purposes, to stability theory. Although efforts have been made to explain the relevant parts of stability theory when the comparisons are made, some familiarity with stability theory is necessary to fully appreciate these connections.

Useful introductory references to stability theory are [M], [P.1], [L,P], [B].

Unless expressly stated to the contrary, all the results and concepts in this thesis are due to the author. The notions of equation and equational theory have first been defined in [S.1].

I must express my gratitude to Professor M. Makkai for directing me in this work and for allowing me to benefit through my years of graduate studies from his experience and professional views of mathematics. I am also indebted to Professor A. Pillay for sharing with me his knowledge of stability theory.

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CHAPTER O

Preliminaries

In this chapter we fix the setting in which we want to work and define the notions of equational and strongly equational sets of formulas. We also give some examples.

Preliminaries.

a) We fix once and for all a first-order language L; the rules of formation for formulas and terms in L are the usual ones.

$\varphi,\psi,\chi\cdots$ denote formulas in L; $H,H',H\cdots$ denote L-structures; $\vec{a},\vec{b},\vec{c},\cdots$ denote finite tuples of elements in given L-structures.

We do not distinguish between L-structures and their underlying sets.

We divide all variables in L in two classes X and T, and call the variables in X type variables, the variables in T parameter variables.

Unless stated otherwise, $\vec{x},\vec{y},\vec{x}_1,\cdots$ denote finite tuples of type variables;

$t,\vec{u},\vec{t}_1,\cdots$ denote finite tuples of parameter variables.

A formula φ is a formula in $\vec{x}$ if, whenever x is a type variable, x occurs freely in φ iff x is mentioned in $\vec{x}$; if in addition $\vec{t}$ is the finite tuple of free parameter variable in φ we write φ under the form $\varphi(\vec{x},\vec{t})$.

If S is a set of formulas in L, we let $S^{\vec{x}}$ denote the set of formulas in $\vec{x}$ which are in S.

For $s:\vec{t} \longrightarrow \vec{a}$ an evaluation function of $\vec{t}$ into a structure H, $\varphi(\vec{x},\vec{a})$ denotes as usual the formula obtained from $\varphi(\vec{x};\vec{t})$ by substituting $\vec{a}$ for $\vec{t}$; if $\vec{a}$ has its elements in $A \subset H$, we say $\varphi(\vec{x};\vec{a})$ is a formula with parameters in A. $\varphi(\vec{b};\vec{a})$ and $\varphi(\vec{b};\vec{t})$ are obtained from $\varphi(\vec{x};\vec{t})$ in a similar manner. We frequently write φ or $\varphi(\vec{x})$ for $\varphi(\vec{x};\vec{t})$ or $\varphi(\vec{x};\vec{a})$, that is when the context makes it clear which one is meant.

As always, given an L-structure H and a formula $\varphi(\vec{x})$ with parameters in H, we write "$H \models \varphi(\vec{a})$" to signify that the sentence $\varphi(\vec{a})$ in the language L with new individual constants naming the elements of $\vec{a}$ is true in $\langle H,\vec{a}\rangle$;

$\varphi(H) = \{\vec{a} \in H^n; H \models \varphi(\vec{a})\}$. ($n$ = length$(\vec{x})$).

A set of formulas p in $\vec{x}$ with parameters in H is realized in H if there is a finite tuple $\vec{a}$ of elements in H such that

$H \models \varphi(\vec{a})$ whenever $\varphi(\vec{x})$ belongs to p.

b) We fix a category $\mathbb{K}$: the objects of $\mathbb{K}$ are L-strucutres and the morphisms of $\mathbb{K}$ are maps between the underlying sets of objects in $\mathbb{K}$; composition of morphisms is then the composition of maps, and, for H in $\mathbb{K}$, the identity morphism on H is the identity map from H into H.

Later on we shall consider additional assumptions on $\mathbb{K}$ as for instance that $\mathbb{K}$ is the category of models of a first order theory with embeddings or elementary embeddings for morphisms.

To simplify the presentation we extend $\mathbb{K}$ to the category $\hat{\mathbb{K}}$ which includes the subsets of structures in $\mathbb{K}$ as objects and the inclusion maps between subsets of a structure in $\mathbb{K}$ as morphisms:

- $\text{Object}(\hat{\mathbb{K}}) = \{(A,H) ; H \in \mathbb{K} \text{ and } A \subset H\}$

- A morphism $f:(A,H) \longrightarrow (B,F)$ in $\hat{\mathbb{K}}$ is formally defined as a morphism, denoted again f, $f:H \longrightarrow F$ in $\mathbb{K}$ such that $\text{range}(f \upharpoonright A) \subseteq B$. [We mean by that, that $f:(A,H) \longrightarrow (B,F)$ is identified to the triple $\langle f:H \longrightarrow F ; (A,H) ; (B,F) \rangle$].

Thus, the identity morphism $\text{id}:(A,H) \longrightarrow (A,H)$ is formally defined as the identity morphism on H.

If $f:(A,H) \longrightarrow (B,F)$ and $g:(B,F) \longrightarrow (C,G)$ are in $\hat{\mathbb{K}}$ so that $f:H \longrightarrow F$ and $g:F \longrightarrow G$ are in $\mathbb{K}$ then

$g\cdot f:(A,H)\longrightarrow(C,G)$ is formally defined as the morphism $g\cdot f:H\longrightarrow G$.

For $f:(A,H)\longrightarrow(B,F)$ we let $f[(A,H)]=(f(A),F)$. Thus if $f:H\longrightarrow H$ is the identity on H and $A\subset B\subset H$ then $f:(A,H)\longrightarrow(B,H)$ is a morphism in $\hat{\mathbb{K}}$ and $f[(A,H)] = (A,H)$; in that case, we refer to f as an inclusion map.

Note that two morphisms $f,g:(A,H)\longrightarrow(B,F)$ in $\hat{\mathbb{K}}$ are identified if the morphisms $f,g:H\longrightarrow F$ are equal and not just if f and g take the same values on A.

When there is no ambiguity, we write A instead of (A,H) and $f:A\longrightarrow B$ instead of $f:(A,H)\longrightarrow(B,F)$.

If $\varphi=\varphi(\vec{x},\vec{a})$ is a formula in $\vec{x}$ with parameters in A ($A\in\hat{\mathbb{K}}$) and $f:A\longrightarrow B$ is a morphism in $\hat{\mathbb{K}}$ we let $f\varphi=\varphi(\vec{x},f\vec{a})$ (where $f\vec{a}=\langle fa_1,\cdots,fa_n\rangle$ when $\vec{a}=\langle a_1,\cdots,a_n\rangle$). Of course $f\varphi$ is a formula in $\vec{x}$ with parameters in B.

If p is a set of formulas in $\vec{x}$ with parameters in A, we let $fp=\{f\varphi;\varphi\in p\}$;

We say p is *realized in* $\mathbb{K}$ *over* (A,H) if there is a morphism $f:(A,H)\longrightarrow F$ such that fp is realized in F; p is *consistent in* $\mathbb{K}$ *over* (A,H) if every finite subset of p is realized in $\mathbb{K}$ over (A,H); p is inconsistent in $\mathbb{K}$ if p is not consistent in $\mathbb{K}$. When no confusion arises we just say p is realized (resp. consistent) in $\mathbb{K}$ instead of p is realized (resp. consistent) in $\mathbb{K}$ over (A,H).

For $A=(A,H)$ in $\mathbb{K}$, $\varphi=\varphi(\vec{x})$ and $\psi=\psi(\vec{x})$ formulas in $\vec{x}$ with parameters in A, we write $\varphi \vdash_{\overline{A}} \psi$ if for any morphism $f:H\longrightarrow F$, $(f\varphi)(F) \subset (f\psi)(F)$. Write $\varphi \underset{A}{\sim} \psi$ iff $\varphi \vdash_{\overline{A}} \psi$ and $\psi \vdash_{\overline{A}} \varphi$.

If p and q are sets of formulas in $\vec{x}$ with parameters in A, φ as above, we write $p \vdash_{\overline{A}} \varphi(\vec{x})$ if there is a finite subset p_0 of p such that $\wedge p_0 \vdash_{\overline{A}} \varphi$; $p \vdash_{\overline{A}} q$ if $p \vdash_{\overline{A}} \Theta$ for every $\Theta \in q$; $p \underset{A}{\sim} q$ if $p \vdash_{\overline{A}} q$ and $q \vdash_{\overline{A}} p$.

If p and q are sets of formulas in $\vec{x}$ with parameters in A, or just single formulas, and $p \underset{A}{\sim} q$ we say that p is equivalent to q in $\mathbb{K}$.

More generally, for φ, ψ, formulas with parameters in $A \subset H$, $g:(A,H)\longrightarrow(B,G)$ a morphism in $\hat{\mathbb{K}}$ we write $\varphi \vdash_{\overline{g}} \psi$ if for any morphism $f:G\longrightarrow F$, $f\cdot g\varphi(F) \subset f\cdot g\psi(F)$; $\varphi \underset{g}{\sim} \psi$ if $\varphi \vdash_{\overline{g}} \psi$ and $\psi \vdash_{\overline{g}} \varphi$.

In other words $\varphi \vdash_{\overline{g}} \psi$ if $g\varphi \vdash_{\overline{G}} g\psi$, and $\varphi \underset{g}{\sim} \psi$ if $g\varphi \underset{G}{\sim} g\psi$.

Note also that $\varphi \vdash_{\overline{H}} \psi$ iff $\varphi \vdash_{\overline{id_H}} \psi$.

Similarly, for p and q as above define $p \vdash_{\overline{g}} q$ if $gp \vdash_{\overline{G}} gq$ and $p \underset{g}{\sim} q$ if $gp \underset{G}{\sim} gq$.

Remark. For p a set of formulas in $\vec{x}$ with parameters in A

$(A \subset H)$ p is inconsistent in $\mathbb{K}$ iff some finite subset of p is inconsistent in $\mathbb{K}$ iff some finite subset of p is not realized in $\mathbb{K}$ iff there is a finite set $\{\varphi_I ; i \in I\}$, (I finite) of formulas in p such that $(\vec{x} = \vec{x}) \models_{A} \bigvee_{i \in I} \neg\varphi_i$.

Examples.

1. Let $\mathbb{K}$ be the category of fields with field embeddings for morphisms $(L = \{+,\cdot,0,1\})$; F a field. If $\varphi(\vec{x}) = (P(\vec{x}) = 0)$ and $\psi(\vec{x}) = (Q(\vec{x}) = 0)$ are two algebraic equations in $\vec{x}$ with coefficients in F then it is easily seen that $\varphi(\vec{x}) \models_{F} \psi(\vec{x})$ iff $\varphi(\hat{F}) \subset \psi(\hat{F})$, where $\hat{F}$ is the algebraic closure of F. It is clear also that a quantifier free formula $\theta(\vec{x})$ with parameters in F is consistent in $\mathbb{K}$ iff $\theta(\vec{x})$ has a solution in $\hat{F}$.

2. The situation in 1. above can be generalized in the following manner: let $\mathbb{K}$ be the category of models of a first-order theory T with embeddings for morphisms; E an existentially-closed structure in $\mathbb{K}$. If $\varphi(\vec{x})$ and $\psi(\vec{x})$ are quantifier-free formulas with parameters in E, then $\varphi(\vec{x}) \models_{E} \psi(\vec{x})$ iff $\varphi(E) \subset \psi(E)$.

   Indeed, if $\varphi(\vec{x}) \models_{E} \psi(\vec{x})$ then obviously $\varphi(E) \subset \psi(E)$.

   Conversely, suppose $\varphi(E) \subset \psi(E)$. $\varphi(\vec{x}) \not\models_{E} \psi(\vec{x})$ means there is a morphism, i.e. an embedding, $f: E \longrightarrow G$ in $\mathbb{K}$, such that $\varphi(\vec{x}) \wedge \neg\psi(\vec{x})$ is realized in G. Since E is existentially-closed it follows that $\varphi(\vec{x}) \wedge \neg\psi(\vec{x})$ is realized in E ✠. We conclude that $\varphi(\vec{x}) \models_E \psi(\vec{x})$.

Suppose now that T has the amalgamation property and is closed under unions of increasing chains of structures. Let F be a structure in $\mathbb{K}$, $\varphi(\vec{x})$ and $\psi(\vec{x})$ quanitifier-free formulas with parameters in F.

Then, $\varphi(\vec{x}) \models_F \psi(\vec{x})$ iff for some existentially-closed structure E in $\mathbb{K}$ and morphism $e:F \longrightarrow E$ we have $e\varphi(E) \subset e\psi(E)$: clearly if $\varphi(\vec{x}) \models_F \psi(\vec{x})$ then for any morphism $e:F \longrightarrow E$, $e\varphi(E) \subset e\psi(E)$; so it suffices to choose e with E existentially-closed.

Conversely, suppose there is a morphism $e:F \longrightarrow E$ with E existentially-closed such that $e\varphi(E) \subset e\psi(E)$; we want to show that $\varphi(\vec{x}) \models_F \psi(\vec{x})$ i.e. for any embedding $g:F \longrightarrow G$ $(g\varphi)(G) \subset (g\psi)(G)$

Let $g:F \longrightarrow G$ be a morphism in $\mathbb{K}$. We know from what preceded that $e\varphi \models_E e\psi$. By the amalgamation property, there are morphisms $h_1:E \longrightarrow H$ and $h_2:G \longrightarrow H$ such that $h_1e = h_2g$. Since $e\varphi \models_E e\psi$, $h_1e\varphi(H) \subset h_1e\psi(H)$ i.e. $h_2g\varphi(H) \subset h_2g\psi(H)$. It easily follows that $(g\varphi)(G) \subset (g\psi)(G)$, which is what we wanted.

The definitions below formalize some of the properties used in the examples above.

1. **Definition.** Let Δ be a set of formulas in L, $f: H \longrightarrow F$ a morphism in $\mathbb{K}$. Then,

(i) f is Δ-elementary if for any formula $\varphi(\vec{x})$ in Δ with parameters in H and $\vec{a}$ a tuple of elements in H,

$$H \models \varphi(\vec{a}) \iff F \models f\varphi(f\vec{a}).$$

(ii) f reflects Δ if for $\varphi(\vec{x})$ and $\psi(\vec{x})$ in Δ with parameters in H,

$$f\varphi \models_F f\psi \text{ (resp. } f\varphi \models_F (\vec{x} \neq \vec{x});\ (\vec{x} = \vec{x}) \models_F f\varphi)$$

$$\Longrightarrow \varphi \models_H \psi \text{ (resp. } \varphi \models_H (\vec{x} \neq \vec{x});\ (\vec{x} = \vec{x}) \models_H \varphi).$$

(iii) $\mathbb{K}$ is Δ-elementary (resp. reflects Δ) if every morphism in $\mathbb{K}$ is Δ-elementary (resp. reflects Δ).

(iv) $\mathbb{K}$ has the amalgamation property (A.P. for short) if for any morphisms $f_1: H \longrightarrow F_1$ and $f_2: H \longrightarrow F_2$ in $\mathbb{K}$, there are morphisms $g_1: F_1 \longrightarrow G$ and $g_2: F_2 \longrightarrow G$ in $\mathbb{K}$ such that

$g_1 \cdot f_1 = g_2 \cdot f_2$.

Example. Suppose $\mathbb{K}$ is the category of models of a first-order theory T with embeddings for morphisms and Δ is the set of quantifier-free formulas. Then $\mathbb{K}$ reflects Δ iff T has the amalgamation property.

Proof. Suppose $\mathbb{K}$ reflects Δ and $f_1:H \longrightarrow F_1$ $f_2:H \longrightarrow F_2$ are embeddings in $\mathbb{K}$. We want to show the existence of embeddings $g_1:F_1 \longrightarrow G$ and $g_2:F_2 \longrightarrow G$ in $\mathbb{K}$ such that $g_1 \cdot f_1 = g_2 \cdot f_2$. To simplify notation , write $f_1(h) = h$ and $f_2(h) = h$ for $h \in H$.

Let $\{c_a;a \in F_1\}$ and $\{d_a;a \in F_2\}$ be sets of individual constants (not occuring in L), with $c_a = d_b$ iff $a = b \in H$. Let $D(F_1)$ be the set of quantifier-free sentences in $L \cup \{c_a;a \in F_1\}$ satisfied in F_1 when we interpret the constant c_a by a in F_1. Similarly, define $D(F_2)$.

Clearly then, to amalgamate f_1 and f_2 in $\mathbb{K}$ it suffices to show the consistency of $T \cup D(F_1) \cup D(F_2)$.

If $T \cup D(F_1) \cup D(F_2)$ is inconsistent, there is a formula $\varphi(\vec{c})$ in $D(F_2)$ such that $T \cup D(F_1) \cup \varphi(\vec{c})$ is inconsistent, i.e. there is a quantifier-free formula $\varphi(\vec{x};\vec{a})$ with parameters in H such that $\varphi(\vec{x};\vec{a})$ is realized in F_2 and $\varphi(\vec{x};\vec{a})$ is inconsistent in $\mathbb{K}$ over F_1. This means, $\varphi(\vec{x},\vec{a}) \vdash_{F_1} (\vec{x} \neq \vec{x})$, from which we deduce, using reflection,

$$\varphi(\vec{x};\vec{a}) \vdash_{H} (\vec{x} \neq \vec{x}).$$

But $\varphi(\vec{x};\vec{a})$ is realized in F_2 ⌘.

That proves one direction of the claim. The converse easily follows from lemma 2. below. ◻

2. Lemma. If $\mathbb{K}$ has the amalgamation property and Δ is a set of formulas in L such that $\mathbb{K}$ is Δ-elementary, then $\mathbb{K}$ reflects Δ.

Proof. Suppose $\mathbb{K}$ has the A.P. and $\mathbb{K}$ is Δ-elementary. Let $f_1:H \longrightarrow F_1$ be a morphism in $\mathbb{K}$, $\varphi(\vec{x})$, $\psi(\vec{x})$ formulas in $\Delta \cup \{(\vec{x} = \vec{x}), (\vec{x} \neq \vec{x})\}$ with parameters in H such that $f\varphi \models_{F_1} f\psi$; we want to show $\varphi \models_H \psi$, that is: $f_2\varphi(F_2) \subset f_2\psi(F_2)$ whenever $f_2:H \longrightarrow F_2$ is a morphism in $\mathbb{K}$.

So let $f_2:H \longrightarrow F_2$ be a morphism in $\mathbb{K}$ and let $g_1:F_1 \longrightarrow G$ and $g_2:F_2 \longrightarrow G$ be morphisms such that $g_1f_1 = g_2f_2$. (g_1, g_2 exists by A.P.). From $f_1\varphi \models_{F_1} f_1\psi$, we deduce that $g_1f_1\varphi(G) \subset g_1f_1\psi(G)$ or equivalently $g_2f_2\varphi(G) \subset g_2f_2\psi(G)$. Since g_2 is Δ-elementary, it follows $f_2\varphi(F_2) \subset f_2\psi(F_2)$. □

3. Definition. Let S be a set of formulas in L, n a natural number,

(i) S is equational (resp. n-strongly-equational) if for any H, $\vec{x}$ and any set p of formulas in $S^{\vec{x}}$ with parameters in H, there is $p_0 \subset p$, p_0 finite (resp. card$(p_0) \leq n$) and $p_0 \models_H p$.

S is strongly equational if there is a natural number m such that S is m-strongly-equational.

(ii) $\varphi(\vec{x};\vec{t})$ is an equation (resp. n-strong-equation, strong equation) if $\{\varphi(\vec{x};\vec{t})\}$ is equational (resp. n-strongly-equational, strongly-equational).

4. Examples.

(i) Let $\mathbb{F}$ be the category of fields with field embeddings for morphisms; $L = \{+,\cdot,0,1\}$. Let S be the set of atomic formulas in L.

Claim. S is equational.

Proof. Suppose $H \in \mathbb{K}$ and $p = \{\varphi_i(\vec{x};\vec{a}_i);\ i \in I,\ \varphi_i \in S.\ \vec{a}_i \in H\}$. Each $\varphi_i(\vec{x};\vec{a}_i)$ is equivalent in $\mathbb{K}$ to an algebraic equation $(P_i(\vec{x};\vec{a}_i) = 0)$, where P_i is a polynomial in the variables $\vec{x}$ with coefficients in H.

Since the ring of polynomials $H[\vec{x}]$ is noetherian, there is a finite set $J \subset I$ such that for any $i \in I$, P_i is a linear combination of P_j's for $j \in J$. It follows that for any morphism f in $\mathbb{K}$, fP_i is a linear combination of fP_j's for $j \in J$. Clearly then, if $p_0 = \{\varphi_j; j \in J\}$, $p_0 \vdash_H (P_i(\vec{x};\vec{a}_i) = 0)$ for every $i \in I$, i.e. $p_0 \vdash_H p$ which is what we wanted. □

(ii) Let R be a fixed ring and let L be the standard language of R-modules.

A homomorphism of modules $f: H \longrightarrow F$, is pure if for any positive primitive formula (p.p.f. in short) $\varphi(\vec{x})$ and $\vec{a} \in H$,

$$H \models \varphi(\vec{a}) \iff F \models \varphi(f\vec{a}).$$

In other words f is pure if f is Δ-elementary with Δ the set of p.p.f.

Let $\mathbb{M}_R$ be the category of modules with pure embeddings for morphisms.

Claim. Every positive primitive formula $\varphi(\vec{x};\vec{t})$ is a strong equation in $\mathbb{M}_R$.

Proof. For H an R-module, $\varphi(H;\vec{o})$ ($\vec{o} = \langle o, \cdots o \rangle$) is an additive subgroup of H^n (n = length $\vec{x}$) and $\varphi(H;\vec{a})$ ($\vec{a} \in H$), if not empty, is a coset of $\varphi(H;\vec{o})$ in H^n. It follows that for $\vec{a}, \vec{b} \in H$, $\varphi(\vec{x};\vec{a})$ and $\varphi(\vec{x};\vec{b})$ are either equivalent *in* H or contradictory *in* H.

Since the morphisms in $\mathbb{K}$ are pure, in fact, either $\varphi(\vec{x};\vec{a})$ and $\varphi(\vec{x};\vec{b})$ are equivalent *in* $\mathbb{K}$ or $\{\varphi(\vec{x};\vec{a}), \varphi(\vec{x};\vec{b})\}$ is inconsistent in $\mathbb{K}$. The claim follows immediately. □

(iii) If $\mathbb{K}$ is the category of models of a first-order theory T with elementary embeddings for morphisms and $E(\vec{x};\vec{t})$ is a formula which defines an equivalence relation in models of T then $E(\vec{x};\vec{t})$ is a 2-strong-equation.

5. Remark. Clearly, if S is equational and p is a set of formulas in $S^{\vec{x}}$ with parameters in H then p is consistent in $\mathbb{K}$ iff p is realized in $\mathbb{K}$. Thus the property of equationality induces a compactness property.

Now, we could have defined the property of equationality only for those sets of formulas in $S^{\vec{x}}$ (with parameters) which are realized in $\mathbb{K}$. We would have then that for p as above, p is equivalent to a finite subset once p is realized in $\mathbb{K}$.

This property is however in general too weak for what follows, but it is worthwhile bearing it in mind and checking at different stages what additional conditions on $\mathbb{K}$ make this property sufficient to obtain analagous results.

Example. Consider the ring of integers $\mathbb{Z}$. Let $\mathbb{K}$ be the category with single object $\mathbb{Z}$ and single morphism the identity on $\mathbb{Z}$; $L = \{+,\cdot,0,1\}$.

Let $\varphi(x,t) = \exists s(x = s\cdot t)$. φ says "x is a multiple of t".

φ is not an equation: take $p = \{\varphi(x;k); k \in \mathbb{Z}\}$; p says

"x is a multiple of all integers", and clearly there is no finite set $\mathbb{Z}_0$ of integers such that "x is a multiple of all integers" iff "x is a multiple of all integers in $\mathbb{Z}_0$".

However, if $p = \{\varphi(x;a_i);\ i \in I,\ a_i \in \mathbb{Z}\}$ is realized in $\mathbb{Z}$, then p is equivalent to a finite subset: for let k realize p. Then a_i divides k for any $i \in I$. Since k has finitely many divisors it follows that there are at most finitely many distinct a_i's. The assertion has now become obvious.

Note here that the formula $\varphi(t;x) = \exists s(t = sx)$ is an equation.

CHAPTER 1

Basic Properties

We define the notions of an ℓ-equational set and the height of a set of formulas, and compare these notions to equational sets.

Then, we work out some basic properties of equational sets and equational formulas. We show for instance that S is equational iff the closure of S under positive boolean combinations is equational (c.f. 7), and, a formula $\varphi(\vec{x};\vec{t})$ has a finite ℓ-height in $\vec{x}$ iff $\varphi(\vec{x};\vec{t})$ has finite ℓ-height in $\vec{t}$ (c.f. 8).

Finally we introduce some terminology on types and complete types and give a criterion for equationality using complete types (c.f. 13). We then apply this criterion to prove that in the category of differentially closed fields of characteristic 0, the set of differential equations is equational (c.f. application 2 after proposition 13).

Throughout this chapter S denotes a set of formulas in L closed under substitution of parameter variables.

If Δ is a set of formulas in L we let $cl^{+}(\Delta)$ denote the closure of Δ under finite conjunctions, finite disjunctions and substitution of parameter variables; $cl(\Delta)$

denotes the closure of Δ under boolean combinations and substitution of parameter variables.

Definitions. n, a positive integer

(i) S is ℓ-equational (resp. ℓ-n-strongly-equational) in $\mathbb{K}$ (ℓ for local) if for any H in $\mathbb{K}$, $\vec{x}$ and p, a set of formulas in $S^{\vec{x}}$ with parameters in H, p is logically equivalent *in* H to a finite subset (resp. to a finite subset of cardinality less or equal to n).

(ii) S has ℓ-height (resp. height) less than n if there is no structure H in $\mathbb{K}$, $\vec{x}$ and sequence $(\varphi_i)_{i<n}$ of formulas in $S^{\vec{x}}$ with parameters in H such that the formulas $\wedge_{i<k}\varphi_i \wedge \neg\varphi_k$ for $0 < k < n$ as well as the formula $\wedge_{i<n}\varphi_i$ are consistent *in* H (resp. in $\mathbb{K}$)
write ℓ-height$(S) = n$ (resp. height$(S) = n$) if S has ℓ-height (resp. height) less than $n + 1$ but not less than n.

S has finite ℓ-height (resp. height) if there is a natural number m such that S has ℓ-height (resp. height) less than m; S has infinite ℓ-height (resp, height) otherwise.

Thus, S has ℓ-height less than 1 if for any H in $\mathbb{K}$ and φ in S with parameters in H, φ is inconsistent in H.

1. Proposition.

(i) S is not $\mathbb{I}$-equational (resp. equational) in $\mathbb{K}$ iff there is a structure H in $\mathbb{K}$, and a countable sequence $(\varphi_n)_{n<\omega}$ of formulas in $S^{\vec{x}}$ with parameters in H such that for any $k < \omega$, $\wedge_{i\leq k}\varphi_i \wedge \neg\varphi_{k+1}$ is consistent *in* H (resp. in $\mathbb{K}$)

(ii) S is not $\mathbb{I}$-n-strongly-equational (resp. n-strongly-equational) iff either S is not $\mathbb{I}$-equational (resp. equational) or there is a structure H in $\mathbb{K}$, $\vec{x}$ and a finite set q of formulas in $S^{\vec{x}}$ with parameters in H such that q has cardinality $n + 1$ and q is not logically equivalent in H (resp. in $\mathbb{K}$) to any proper subset.

Proof.

(i) Suppose S is not $\mathbb{I}$-equational (resp. equational). Then there are $H \in \mathbb{K}$, $\vec{x}$ and p, a set of formulas in $S^{\vec{x}}$ with parameters in H, such that p is not logically equivalent *in* H (resp. in $\mathbb{K}$) to any finite subset. We construct a sequence $(\varphi_n)_{n<\omega}$ of elements in p by induction on n in this way: assume $\varphi_0,\cdots,\varphi_n$ has been chosen. Since $\wedge_{i\leq n}\varphi_i$ does not imply p in H (resp. in $\mathbb{K}$) there is a formula $\psi \in p$ such that $\wedge_{i\leq n}\varphi_i \wedge \neg\psi$ is consistent in H (resp. in $\mathbb{K}$). Let $\varphi_{n+1} = \psi$.

$(\varphi_n)_{n<\omega}$ thus constructed satisfies our condition.

The other direction of the claim is obvious.

(ii) Suppose S is not ℓ-n-strongly-equational (resp. n-strongly-equational) but ℓ-equational (resp. equational). Then there are $H \in \mathbb{K}$, $\vec{x}$ and p, a set of formulas in $S^{\vec{x}}$ with parameters in H such that p is not logically equivalent in H (resp. in $\mathbb{K}$) to any finite subset of cardinality less than $n + 1$. But, by ℓ-equationality (resp. equationality) p is equivalent in H (resp. in $\mathbb{K}$) to a finite subset; let p_0 be such a subset of least cardinality. Evidently, $\mathrm{card}(p_0) > n$. Let q be a subset of p_0 of cardinality $n + 1$. If q is equivalent in H (resp. in $\mathbb{K}$) to a proper subset q_0, it would follow that $p_0 (= q \cup (p_0 \setminus q))$ is equivalent in H (resp. in $\mathbb{K}$) to a proper subset $q_0 \cup (p_0 \setminus q)$, contradicting the minimality of p_0. So q satisfies the conditions of the claim.

The other direction of the claim is clear. ∎

2. **Corollary.** We have the following diagram of implications:

```
                          1                                 3
S has finite height───→S is strongly-equational───→S is
           |                          |              equational
           |                          |                   |
           |                          |                   |
           v              2           v             4     v
S has finite ß-height→S is ß-strongly-equational→S is
                                                   ß-equational
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Proof. The vertical implications as well as 3 and 4 follow immediately from the definitions.

Proof of 1. Suppose height(S) $= n$ but S is not $n+1$-strongly-equational. By lemma 1.(i) S is equational (since S not equational is easily seen to contradict height(S) finite). By 1.(ii), if follows there are H in $\mathbb{K}$, $\vec{x}$, and a set q of formulas in $S^{\vec{x}}$ with parameters in H, q of cardinality $n + 2$ such that q is not equivalent in $\mathbb{K}$ to any proper subset. Let $q = \{\varphi_i; i < n + 2\}$. Our assertion on q clearly implies that the formulas $\wedge_{i<k}\varphi_i \wedge \neg\varphi_k$ for any k, $0 < k < n + 1$ as well as $\wedge_{i<n+1}\varphi_i$ are consistent in $\mathbb{K}$. But that means height$(S) \geq n + 1$. Contradiction.

The proof of 2 is similar. ∎

Remark. The implications given in the diagram above are all the possible implications that exist between the different terms of the diagram.

More specifically, if we let L consist of one binary relation $R(x;t)$, we can easily construct categories in which R is, for instance, $\aleph$-equational but not equational, or equational but not strongly-equational, etc... We consider some of the cases below.

a) Let $|H|$ be an infinite set; $(A_i)_{i<\omega}$ a sequence of subsets of $|H|$; $(a_i)_{i<\omega}$ a sequence of two by two distinct elements of $|H|$. Let H be the structure obtained by interpreting $R(x;a_i)$ as A_i for any $i < \omega$ and $R(x;b)$ as the empty set for any b in H distinct from the a_i's. Let $\mathbb{K}$ be the category with single object H and single morphism the identity morphism on H.

- Assume $A_{i+1} \subsetneq A_i$ for any $i < \omega$, clearly then $R(x;t)$ is not $\aleph$-equational in $\mathbb{K}$.

- Assume $A_i \subsetneq A_{i+1}$ for any $i < \omega$. Then $R(x;t)$ is $\aleph$-strongly-equational but has infinite height.

b) Choose a family of infinite sets $(|H_i|)_{i<\omega}$ such that $|H_i| \subsetneq |H_{i+1}|$ for any $i < \omega$; a family of sets $(A_i)_{i<\omega}$ and a sequence $(a_i)_{i<\omega}$ such that for any $i < \omega$, $A_i \subset |H_i|$, $a_i \in |H_i|$.

For $i \leq j < \omega$ interpret $R(x;a_i)$ as A_i in $|H_j|$ and $R(x;b)$ as the empty set for $b \in H_j \setminus \{a_i : i \leq j\}$. Let H_i denote the structure thus obtained, and let $\mathbb{K}$ be the category which has the H_i's for objects and inclusion maps for morphisms.

– Assume $A_{i+1} \subsetneq A_i$ for any $i < \omega$ and $\{a_i; i < \omega\}$ is a subset of H_0. Then, $R(x;t)$ is ℓ-strongly-equational in $\mathbb{K}$, has infinite ℓ-height and is not equational in $\mathbb{K}$.

3. **Corollary.** If the objects of $\mathbb{K}$ are the models of a first order theory T then $\varphi(\vec{x};\vec{t})$ is ℓ-equational iff $\varphi(\vec{x};\vec{t})$ has finite ℓ-height.

Proof. Add a countable set C of new individual constants to L, $C = \{\vec{c}_i; i < \omega\}$. Consider the following sets of sentences in $L \cup C$: for $n < \omega$,

$$T_n = \{ \exists\vec{x}\wedge_{i<k}\varphi(\vec{x};\vec{c}_i)\wedge\neg\varphi(\vec{x};\vec{c}_k); 0 < k < n\} \cup \{ \exists\vec{x}\wedge_{i<n}\varphi(\vec{x};c_i)\}$$

$$T_\omega = \{ \exists\vec{x}\wedge_{i<n}\varphi(\vec{x};\vec{c}_i)\wedge\neg\varphi(\vec{x};\vec{c}_n); 0 < n < \omega\}.$$

Clearly $T_\omega \equiv \cup_{n<\omega}T_n$. Hence, by compactness, $T \cup T_\omega$ is consistent iff $T \cup T_n$ is consistent for every $n < \omega$ (note, $T_{n+1} \supset T_n$). But $T \cup T_n$ is consistent iff ℓ-height$(\varphi) \geq n$ while by lemma 1.(i), $T \cup T_\omega$ is consistent iff φ is not ℓ-equational. The assertion immediately follows. ◻

4. **Lemma.** Suppose S closed under finite conjunctions and disjunctions, and $\mathbb{K}$ reflects S. Then, for any morphism

$f: H \longrightarrow F$ in $\mathbb{K}$ and p, a set of formulas in $cl(S^{\vec{x}})$ with parameters in H, p is consistent in $\mathbb{K}$ over H iff fp is consistent in $\mathbb{K}$ over F.

Proof. Clearly, if fp is consistent in $\mathbb{K}$ over F then p is consistent in $\mathbb{K}$ over H.

Suppose p is consistent in $\mathbb{K}$ over H but fp is not consistent in $\mathbb{K}$ over F. We might as well take p finite and therefore a single formula in $cl(S^{\vec{x}})$. Writing p in normal form (up to equivalence in $\mathbb{K}$) we see that we can assume p is a finite conjunction of formulas and negated formulas in $S^{\vec{x}}$ with parameters in H. Say $p = \wedge_{i \in I}\varphi_i \wedge \wedge_{j \in J}\neg\varphi_j$, I and J finite sets.

Since fp is inconsistent in $\mathbb{K}$, we have

$$\wedge_{i \in I} f\varphi_i \models_F \vee_{j \in J} f\varphi_j$$

(we convene $\wedge_{i \in I}\varphi_i = (\vec{x} = \vec{x})$ if I is empty and $\vee_{j \in J} f\varphi_j = (\vec{x} \neq \vec{x})$ if J is empty).

By reflection, we get $\wedge_{i \in I}\varphi_i \models_H \wedge_{j \in J}\varphi_j$ i.e. p is inconsistent in $\mathbb{K}$ H. ◻

Note. If S is closed under finite conjunctions, disjunctions, and S contains the formulas $(x = x)$ for x a type variable then $\mathbb{K}$ reflects S iff $\mathbb{K}$ reflects $cl(S)$. Indeed, it is easy to check that for S as above $cl(S)^{\vec{x}} = cl(S^{\vec{x}})$. Thus, if $\varphi(\vec{x})$ and $\psi(\vec{x})$ are formulas in $cl(S)^{\vec{x}}$

with parameters in H and $f: H \longrightarrow F$ is a morphism such that $f\varphi \models_F f\psi$ but $\varphi \not\models_H \psi$ then, $p = \{\varphi \wedge \neg\psi\}$ would be consistent in $\mathbb{K}$ while fp is inconsistent in $\mathbb{K}$. But that contradicts lemma 4.

Let us say that $\mathbb{K}$ is ω-conservative, if for any sequence of morphisms $(f_\beta : H_\beta \longrightarrow H_{\beta+1})_{\beta<\omega}$ in $\mathbb{K}$ there is a structure H and morphisms $g_\beta : H_\beta \longrightarrow H$ ($\beta < \omega$) such that $g_{\beta+1} \cdot f_\beta = g_\beta$ for any $\beta < \omega$.

ω-conservativeness is similar to closure under unions of chains, but here we do not request H to be a limit of the chain.

(One could define α-conservativeness for arbitrary ordinals α, but we won't need more than ω-conservativeness).

5. Lemma. Suppose in addition to the asumptions in lemma 4., that $\mathbb{K}$ is S-elementary. Let $f_{-1}: H \longrightarrow H_0$ be a morphism in $\mathbb{K}$ and for $i < \alpha$, α a finite ordinal, let p_i be a finite set of formulas in $cl(S^{\vec{x}})$ with parameters in H. If for every $i < \alpha$, p_i is realized in $\mathbb{K}$ then there exists a morphism $g: H_0 \longrightarrow G$ such that for every $i < \alpha$, $g \cdot f_{-1} p_i$ is realized in G.

If in fact $\mathbb{K}$ is ω-conservative, then the claim above holds with $\alpha = \omega$ instead of a finite ordinal.

Proof. First, we construct by induction a sequence of morphisms $(f_i)_{i<\alpha}$ (whether α is finite or $\alpha = \omega$), $f_i : H_i \longrightarrow H_{i+1}$, such that for every $i < \alpha$, $g_i p_i$ is realized in H_{i+1}, where

$g_i = f_i \cdot f_{i-1} \cdot \dots \cdot f_0 \cdot f_{-1}$, (say $g_{-1} = f_{-1}$).

Suppose the construction achieved up to i. By lemma 4, $g_{i-1} p_i$ is consistent in $\mathbb{K}$, hence there exists a morphism $f_i : H_i \longrightarrow H_{i+1}$ such that $f_i \cdot g_{i-1} p_i$ is realized in H_{i+1}. That finishes the inductive step of the construction.

Let H and $(h_\beta : H_\beta \longrightarrow H)_{\beta<\alpha}$ be such that $h_{\beta+1} \cdot f_\beta = h_\beta$: if $\mathbb{K}$ is ω-conservative and $\alpha = \omega$, H and h_β are given by ω-conservativeness; if α is finite, take $H = H_\alpha$ and $h_\beta = f_{\alpha-1} \cdot \dots \cdot f_\beta$.

By S-elementariness, since $g_i p_i$ is realized in H_{i+1} for every $i < \alpha$, $h_i \cdot g_i p_i = h_0 \cdot f_{-1} p_i$ is realized in H for every $i < \alpha$. Thus $g = h_0$ satisfies our conditions. ◻

6. Proposition. Assume S closed under finite conjunctions, $\mathbb{K}$ reflects S and $\mathbb{K}$ is S-elementary. Then,

(i) $\mathbb{K}$-height(S) $= m$ iff height(S) $= m$.

(ii) If in addition $\mathbb{K}$ is ω-conservative, then,

a) S is $\mathbb{K}$-equational iff S is equational.

b) S is $\mathbb{K}$-m-strongly-equational iff S is m-strongly-equational.

Proof.

(i) Clearly, ℓ-height$(S) \geq m$ implies height$(S) \geq m$. Conversely, height$(S) \geq m$ implies ℓ-height$(S) \geq m$: for $m = 0$, the assertion is trivial. Suppose $m > 0$. Then, there are $\vec{x}$, H and $(\varphi_i)_{i<m}$, a sequence of formulas in $S^{\vec{x}}$ with parameters in H, such that the formulas $\wedge_{i<k}\varphi_i \wedge \neg\varphi_k$ for $k < m$, and $\wedge_{i<m}\varphi_i$ are consistent in $\mathbb{K}$.

Following similar arguments to those in lemma 5, (the proof is exactly the same but for the fact that we don't need here S to be closed under disjunctions) we find a morphism $f: H \longrightarrow F$ such that the formulas $f(\wedge_{i<k}\varphi_i \wedge \neg\varphi_k)$ for $k < m$ and $f(\wedge_{i<m}\varphi_i)$ are realized in F. This of course implies that ℓ-height$(S) \geq m$ which is what we wanted.

(ii) Assume $\mathbb{K}$ is ω-conservative

a) If S is not ℓ-equational then clearly S is not equational.

Conversely, suppose S is not equational. Then, by proposition 1.(i), there are $\vec{x}$, H and $(\varphi_i)_{i<\omega}$, a sequence of formulas in $S^{\vec{x}}$ with parameters in H, such that $\wedge_{i\leq n}\varphi_i \wedge \neg\varphi_{n+1}$ is consistent in $\mathbb{K}$ for any $n < \omega$.

Using similar arguments to those in lemma 5 (the proof is exactly the same but for the fact that here we don't need S to be closed under disjunctions) we find a morphism $f:H \longrightarrow F$ such that the formulas $f(\wedge_{i \leq n}\varphi_i \wedge \neg\varphi_{n+1})$ for $n < \omega$ are realized in F. But that clearly implies that S is not ℓ-equational (as $\{f\varphi_i; i < \omega\}$ is not equivalent in F to any finite subset).

b) If S is not ℓ-m-strongly-equational then clearly S is not m-strongly-equational.

Conversely, suppose S is not m-strongly-equational. If S is not equational then by a) S is not ℓ-equational, whence S is not ℓ-stronly-equational and we are done.

If S is equational, then by proposition 1.(i), there are $\vec{x}$, H and $p = \{\varphi_i; i < m + 1\}$, where, for $i < m + 1$, φ_i is a formula in $S^{\vec{x}}$ with parameters in H, and p is not equivalent in $\mathbb{K}$ to any proper subset. Again, following similar arguments to lemma 5 we find a morphism $f:H \longrightarrow F$ such that fp is not logically equivalent *in* F to any proper subset. But that implies S is not ℓ-m-strongly-equational. ◻

Let $\wedge S$ denote the closure of S under finite conjunctions. For n a positive integer, let

$$cl_n(S) = \{\psi = \vee_{i<n}\varphi_i ; \varphi_i \in S\}.$$

7. **Proposition.**

(i) Height $(S^{\vec{x}})$ = height$(\wedge(S^{\vec{x}}))$.

(ii) If S has height less than $m - 1$, then, for $n \geq 2$, $cl_n(S^{\vec{x}})$ has height less than n^m, (for any given $\vec{x}$).

(iii) S is equational iff $cl^+(S)$ is equational.

7.(bis) **Proposition.** The analogue of proposition 7 with the notions of $\mathbb{I}$-height and $\mathbb{I}$-equationality.

Proposition 7.bis follows from proposition 7 in this way: consider for each object H in $\mathbb{K}$ the category $\mathbb{K}_H$ which has for single object H and single morphism id_H; observe that S is $\mathbb{I}$-equational (resp. has $\mathbb{I}$-height less than m) in $\mathbb{K}$ iff S is equational (resp. has height less than m) in $\mathbb{K}_H$ for every H in $\mathbb{K}$. Now apply proposition 7 to $\mathbb{K}_H$.

Proof of proposition 7. For φ and ψ, formulas with parameters in H, we write $\varphi \subset \psi$ if $\varphi \vdash_H \psi$ and $\psi \not\vdash_H \varphi$.

(i) Clearly if height$(S^{\vec{x}}) \geq m$ then height$(\wedge(S^{\vec{x}})) \geq m$. To show the converse we need first the following:

(*) For $i < m$, let $\psi_i = \wedge_{j \in J_i}\varphi_j$ where J_i is a finite set and $\varphi_j\,(j\in J_i)$ is a formula in $S^{\vec{x}}$ with parameters in H; and suppose

$$\wedge_{i\leq k+1}\psi_i \subset \wedge_{i\leq k}\psi_i \text{ for any } k < m-1.$$

Then, we can find a sequence $(j_i)_{i<m}$ such that $j_i \in J_i$ and for any $k < m-1$

$$\wedge_{i\leq k+1}\varphi_{j_i} \subset \wedge_{i\leq k}\varphi_{j_i}.$$

Moreover j_0 can be arbitrarily chosen.

Proof of (*). We choose j_i by induction on $i < m$. Take j_0 to be any element of J_0 and suppose $j_0,\cdots,j_i$ have been chosen. We have

$$\wedge_{k\leq i+1}\psi_k \subset \wedge_{k\leq i}\psi_k \vdash_{\overline{H}} \wedge_{k\leq i}\varphi_{j_k};$$

hence $\wedge_{k\leq i}\varphi_{j_k}\wedge\neg\psi_{i+1}$ is consistent in $\mathbb{K}$. It follows that $\wedge_{k\leq i}\varphi_{j_k}\wedge\neg\varphi_j$ is consistent in $\mathbb{K}$ for some $j \in J_{i+1}$; let then $j_{i+1}=j$. Obviously $\wedge_{k\leq i+1}\varphi_{j_k} \subset \wedge_{k\leq i}\varphi_{j_k}$. That proves (*).

Now, if height$(\wedge(S^{\vec{x}})) \geq m$, then by definition, a sequence $(\psi_i)_{i<m}$ as above does exist with in addition the property that $\wedge_{i<m}\psi_i$ is consistent.

But then the sequence $(\varphi_{j_i})_{i<m}$ given by (*) is such that $\wedge_{i\leq k+1}\varphi_{j_i} \subset \wedge_{i\leq k}\varphi_{j_i}$ and furthermore $\wedge_{i<m}\varphi_{j_i}$ is consistent in $\mathbb{K}$, since $\wedge_{i<m}\psi_i \vdash_{\overline{H}} \wedge_{i<m}\varphi_{j_i}$. That implies height$(S^{\vec{x}}) \geq m$

(ii) We can assume $S = S^{\vec{x}}$ and by (i) we can assume $\wedge S = S$.

We first show the following.

(+) For $i < n^m$, let $\psi_i = \vee_{j \in J_i} \varphi_j$, where $\mathrm{card} J_i = n$ and φ_j is a formula in $S^{\vec{x}}$ with parameters in H. Suppose that

$$\wedge_{i \le k+1} \psi_i \subset \wedge_{i \le k} \psi_i \text{ for any } k < n^m - 1.$$

Then there is a sequence $(j_i)_{i<m}$, such that $j_i \in J_i$ and $\wedge_{i \le k+1} \varphi_{j_i} \subset \wedge_{i \le k} \varphi_{j_i}$ for any $k < m - 1$.

Proof of (+). By induction on m.

For $m = 0$ the assertion is trivial.

Suppose the assertion holds for $m - 1$, and $(\psi_i)_{i<n^m}$ is given as above.

For any k, $0 < k < n^m$, there is $j_k \in J_0$ such that

$$\varphi_{j_k} \wedge \wedge_{i \le k} \psi_i \subset \varphi_{j_k} \wedge \wedge_{i \le k-1} \psi_i \text{: for if}$$

$$\varphi_j \wedge \wedge_{i \le k} \psi_i \ \widetilde{H}\ \varphi_j \wedge \wedge_{i \le k-1} \psi_i$$

for any $j \in J_0$, then

$$(\vee_{j \in J_0} \varphi_j) \wedge \wedge_{i \le k} \psi_k \ \widetilde{H}\ (\vee_{j \in J_0} \varphi_j) \wedge \wedge_{i \le k-1} \psi_i \text{ i.e.}$$

$$\psi_0 \wedge \wedge_{i \le k} \psi_i \ \widetilde{H}\ \psi_0 \wedge \wedge_{i \le k-1} \psi_i;$$

which implies $\wedge_{i \le k} \psi_i \ \widetilde{H}\ \wedge_{i \le k-1} \psi_i$ ※.

For $0 < i, k < n^m$ write $i \equiv k$ if $j_i = j_k$. We partition in this way the set $\{i, 0 < i < n^m\}$ into n subsets. Since $n \geq 2$, one such subset I must have cardinality at least n^{m-1}. Let j_0 denote the common value of the j_i's for $i \in I$.

If $i \in I$ and $k < i$ then $\varphi_{j_0} \wedge \bigwedge_{\ell \leq i} \psi_\ell \subset \varphi_{j_0} \wedge \bigwedge_{\ell \leq k} \psi_\ell$: for $\varphi_{j_0} \wedge \bigwedge_{\ell \leq i} \psi_\ell \subset \varphi_{j_0} \wedge \bigwedge_{\ell \leq i-1} \psi_\ell \vdash_{\overline{H}} \varphi_{j_0} \wedge \bigwedge_{\ell \leq k} \psi_\ell$.

In particular $\varphi_{j_0} \wedge \bigwedge_{\ell \leq i} \psi_\ell \subset \varphi_{j_0}$ (since $\varphi_{j_0} \wedge \psi_0 \, \widetilde{H} \, \varphi_{j_0}$). Let $(i_k)_{0<k\leq n^{m-1}}$ be an increasing sequence of elements in I; let $\chi_0 = \varphi_{j_0}$ and, for $k < n^{m-1}$, $\chi_{k+1} = \bigwedge_{i_k < \ell \leq i_{k+1}} (\varphi_{j_0} \wedge \psi_\ell)$ (put $i_0 = 0$). We have $\bigwedge_{\ell \leq k+1} \chi_\ell \subset \bigwedge_{\ell \leq k} \chi_k$ for any $k < n^{m-1}$.

By (*) (see the proof of (i)) it follows there is a sequence $(h_k)_{k\leq n^{m-1}}$, $i_k < h_{k+1} \leq i_{k+1}$ and $h_0 = 0$ such that

$$\bigwedge_{\ell \leq k+1} (\varphi_{j_0} \wedge \psi_{h_\ell}) \subset \bigwedge_{\ell \leq k} (\varphi_{j_0} \wedge \psi_{h_\ell})$$

for any $k < n^{m-1}$. Clearly now, the formulas $\varphi_{j_0} \wedge \psi_{h_\ell}$, $\ell \leq n^{m-1}$, can be considered as formulas in $cl_n(S)$; thus the induction hypothesis applies to the sequence $(\varphi_{j_0} \wedge \psi_{h_\ell})_{0<\ell\leq n^{m-1}}$. In other words there is a sequence $(j_\ell)_{0<\ell<m}$ such that $j_\ell \in J_{h_\ell}$ and

$$\bigwedge_{i\leq k+1} (\varphi_{j_0} \wedge \varphi_{j_i}) \subset \bigwedge_{i \leq k} (\varphi_{j_0} \wedge \varphi_{j_i}) \text{ for } 0 < k < m-1.$$

Upon observing that

$$\varphi_{j_0} \wedge \varphi_{j_1} \vdash \varphi_{j_0} \wedge \psi_{h_1} \subset \varphi_{j_0} \wedge \psi_{h_0} = \varphi_{j_0},$$

we conclude that

$$\wedge_{i \le k+1} \varphi_{j_i} \subset \wedge_{i \le k} \varphi_{j_i}$$

for any $k < m$. That finishes the proof of (+).

Now if $\mathrm{cl}_n(S^{\vec{x}})$ has height greater or equal to n^m then a sequence $(\psi_i)_{i<n}$. such as given in (+) does exist; let φ_{j_i} be the sequence of formulas in $S^{\vec{x}}$ with parameters in H given by (+).

We have that $\wedge_{i \le k} \varphi_{j_i} \wedge \neg \varphi_{j_{k+1}}$ is consistent for any $k < m - 1$; in particular $\wedge_{i \le k} \varphi_{j_i} \wedge \neg \varphi_{j_{k+1}}$ is consistent for any $k < m - 2$ and $\wedge_{i < m-1} \varphi_{j_i}$ is consistent.

But that implies height(S) $\ge m - 1$, which is what we wanted.

(iii) Clearly S is equational iff $S \cup \{(\vec{x} = \vec{x})\}$, $\vec{x}$ a tuple of variables, is equational.

On the other hand S is equational iff $\wedge S$ is equational: for if p is a set of formulas in $(\wedge S)^{\vec{x}}$ with parameters in H then, writing every formula in p as a conjunct of formulas in S, p is seen to be equivalent to a set of the form $q = \{\varphi(\vec{y}) \wedge (\vec{x} = \vec{x})$; φ a formula in $S^{\vec{y}}$ with parameters in H, $\vec{y}$ a subtuple of $\vec{x}\}$. Applying equationality on the set of formulas $\varphi(\vec{y})$ mentioned in q, for each subtuple $\vec{y}$ separately, we find that p is equivalent in K to a finite subset.

Thus, without loss of generality we can assume S containing the formulas $(\vec{x} = \vec{x})$ and closed under finite conjunctions. It follows that any formula in $(cl^{+}(S))^{\vec{x}}$ is equivalent to a disjunct of formulas in $S^{\vec{x}}$.

Now, obviously, if $cl^{+}(S)$ is equational then S is equational.

Suppose S is equational, but not $cl^{+}(S)$. Then there exists H, $\vec{x}$ and a sequence $(\psi_n)_{n<\omega}$, ψ_n a formula in $cl^{+}(S)^{\vec{x}}$ with parameters in H such that

$$\psi_{n+1} \subset \psi_n \text{ for every } n < \omega.$$

Write $\psi_0' = \vee_{i<m}\varphi_i$, φ_i a formula in $S^{\vec{x}}$ with parameters in H. It is easy to see that for every $n < \omega$ there is an $i < m$ such that $\varphi_i \wedge \psi_{n+1} \subset \varphi_i \wedge \psi_n$. We infer that there is an $\ell_0 < m$ and an infinite subsequence (ψ_{n_j}) of $(\psi_n)_{n<\omega}$, $\psi_{n_0} = \psi_0$, such that

$$\varphi_{\ell_0} \wedge \psi_{n_{j+1}} \subset \varphi_{\ell_0} \wedge \psi_{n_j},$$

for every $j < \omega$.

Let $\chi_j = \varphi_0 \wedge \psi_{n_j}$; χ_j is a formula in $cl^{+}(S)$. Repeat this procedure with the sequence $(\chi_j)_{0<j<\omega}$ instead of $(\psi_n)_{n<\omega}$ to obtain φ_{ℓ_1}, φ_{ℓ_2} etc...

Ultimately we get a sequence $(\varphi_{\ell_n})_{n<\omega}$, φ_{ℓ_n} a formula in $S^{\vec{x}}$ over H such that

$$\varphi_{\ell_{n+1}} \subset \varphi_{\ell_n}$$

for every $n < \omega$.

This contradicts the equationality of S. ∎

Remark. There are strongly equational sets S with $\mathrm{cl}_2(S)$ not strongly equational: let $L = \{R\}$, R a 2-ary relation. Let $|H|$ be an infinite set and $\langle P_n \rangle_{n<\omega}$, $\langle Q_n \rangle_{n<\omega}$ two *sequences of chains* of subsets of $|H|$ such that:

(i) For $n < \omega$, $P_n = (C_i^n)_{i<n}$, $Q_n = (D_i^n)_{i<n}$ with C_i^n, $D_i^n \subset |H|$,

$C_i^n \subsetneq C_{i+1}^n$, $D_i^n \subsetneq D_{i+1}^n$, $(i < n-1)$, $C_0^n, D_0^n \neq \emptyset$.

(ii) If $n \neq m$ then $C_i^n \cap C_j^m = \emptyset$, $D_i^n \cap D_j^m = \emptyset$, for any $i < n$, $j < m$.

Choose an interpretation of R in $|H|$ such that the C_i^n's and D_i^n's, $n < \omega$, $i < n$, are the only interpretations in H of instances of R (an instance of R is here meant to be a formula of the form $R(x;h)$, $h \in H$). Let H be the structure thus obtained, $\mathbb{K}$ the category with single object H and single morphism the identity on H. It is easy to check that $R(x;t)$ is 2-strongly-equational in $\mathbb{K}$.

Fix $n < \omega$. Consider the sequence $(E_i)_{i<m}$ in $\mathrm{cl}_2(R(x;t))_H$ where $E_i = C_i^n \cup D_{n-i-1}^n$ (for simplicity we shall write C_i and D_i for C_i^n and D_i^n respectively).

We shall construct the C_i's and D_i's in such a way that $\bigcap_{i \leq n} E_i \neq \emptyset$ and is not equal to any proper subintersection. Moreover it will be immediate that this can be done for every $n < w$, preserving the conditions (i) and (ii) above. This then clearly implies that $cl_2(R(x;t))$ is not strongly equational.

Construction of C_i and D_i: let (B_{ij}), $0 \leq i$, $j \leq n + 1$, be pairwise disjoint, non-empty subsets of H. Present the B_{ij}'s in a $(n+1)\times(n+1)$-matrix,

$$\begin{matrix} B_{1,1} & B_{1,2} & \cdots & B_{1,n+1} \\ B_{2,1} & B_{2,2} & \cdots & B_{2,n+1} \\ \vdots & \vdots & & \vdots \\ B_{n+1,1} & B_{n+1,2} & \cdots & B_{n+1,n+1} \end{matrix}$$

For $i < n$, let C_i be the union of the $i + 1$ first lines of the matrix, while D_i is the union of the $i + 1$ first columns of the matrix.

$$C_i = \bigcup_{\substack{1 \leq k \leq i+1 \\ 1 \leq \ell \leq n+1}} B_{k\ell} \qquad D_i = \bigcup_{\substack{1 \leq k \leq n+1 \\ 1 \leq \ell \leq i+1}} B_{k\ell}$$

Note that $B_{i+2,n+1-i}$ $(i < n)$ belongs to $E_j = C_j \cup D_{n-j-1}$ iff $j \neq i$ (since $B_{i+2,n+1-i}$ is the meet of the $(i + 2)$-th line with the $(n - i + 1)$-th column).

Thus $\bigcap_{i<n} E_i \neq \bigcap_{i\in I} E_i$ once $I \subsetneq n$, for if $i \in n\setminus I$ then $B_{i+2,n+1-i} \subset \bigcap_{i\in I} E_i$ while $B_{i+2,n+1-i} \not\subset \bigcap_{i<n} E_i$.

Moreover $\bigcap_{i<n} E_i \neq 0$ since $B_{1,1} \subset \bigcap_{i<n} E_i$. ◻

Let $\varphi = \varphi(\vec{x};\vec{t})$. Up to now we have dealt with φ as a formula in $\vec{x}$; instances of φ have been formulas of the kind $\varphi(\vec{x};\vec{a})$ over some structure H.

It is clear however that, given a subtuple $\vec{u}$ of $\vec{x} \cup \vec{t}$ one can consider $\vec{u}$ as the tuple of type variables and the rest of the variables in $\vec{x} \cup \vec{t}$ as parameter variables; instances of φ are then of the form $\varphi(\vec{u};\vec{c})$, $\vec{c} \in H$. Also, all definitions or properties which apply to φ as a formula in $\vec{x}$ apply to φ as a formula in $\vec{u}$. We sometimes write $\varphi^{\vec{u}}$ instead of φ to underline the fact that we consider φ as a formula in $\vec{u}$.

More formally, we say that $\varphi^{\vec{u}}$ satisfies a certain property P or that φ is ······ in $\vec{u}$ if the formula $\hat{\varphi}$, obtained from φ by substituting $\vec{u}$ by type variables and the rest of the variables in $\vec{x} \cup \vec{t}$ by parameter variables, satisfies P or that $\hat{\varphi}$ is ······.

For instance, φ is an equation *in* $\vec{t}$ if $\varphi(\vec{t}_1;\vec{x}_1)$ $(\varphi(\vec{t}_1;\vec{x}_1) = \varphi(\vec{x};\vec{t})[{}^{\vec{x}}_{\vec{t}_1}{}^{\vec{t}}_{\vec{x}_1}])$ is an equation.

8. **Proposition.** Let $\varphi = \varphi(\vec{x};\vec{t})$

(i) ℓ-height$(\varphi^{\vec{x}}) = m$ iff ℓ-height$(\varphi^{\vec{t}}) = m$.

(ii) Assuming $\varphi^{\vec{t}}$ is ℓ-equational, if $\varphi^{\vec{x}}$ is ℓ-m-strongly-equational then $\varphi^{\vec{t}}$ is ℓ-m-strongly-equational.

Proof. We prove (ii); the argumant for (i) is similar and is left to the reader (see 9 for a different proof).

Suppose $\varphi^{\vec{t}}$ is ℓ-equational but not ℓ-m-strongly-equational. By proposition 1.(ii), there is H, $\vec{a}_0, \cdots, \vec{a}_m$ in H such that for any $k \leq m$, the formula

$$\psi_k(\vec{t}) = \bigwedge_{i \in m+1-\{k\}} \varphi(\vec{a}_i;\vec{t}) \wedge \neg\varphi(\vec{a}_k;\vec{t})$$

is consistent in H.

Let b_k realize ψ_k in H. We have the following diagram of true statements in H:

$$
\begin{array}{ccccc}
\neg\varphi(\vec{a}_0;\vec{b}_0) & \varphi(\vec{a}_1;\vec{b}_0) & - - - - - - - - - - - - & & \varphi(\vec{a}_m;\vec{b}_0) \\
\varphi(\vec{a}_0;\vec{b}_1) & \neg\varphi(\vec{a}_1;\vec{b}_1) & - - - - - - - - - - - - & & \varphi(\vec{a}_m;\vec{b}_1) \\
| & & & & | \\
| & & & & | \\
| & & & & | \\
| & & & & | \\
| & & & & | \\
| & & & & | \\
| & & & & | \\
| & & & & | \\
| & & & & | \\
| & & & & | \\
\varphi(\vec{a}_0;\vec{b}_{m-1}) & \varphi(\vec{a}_1;\vec{b}_{m-1}) & - - - & \neg\varphi(\vec{a}_{m-1};\vec{b}_{m-1}) & \varphi(\vec{a}_{m-1};\vec{b}_{m-1}) \\
\varphi(\vec{a}_0;\vec{b}_m) & \varphi(\vec{a}_1;\vec{b}_m) & - - - & \varphi(\vec{a}_{m-1};\vec{b}_m) & \neg\varphi(a_m;ab_m)
\end{array}
$$

Considering the columns of this diagram we obtain that for any ℓ, $0 \le \ell \le m$ the formula

$$\bigwedge_{i \in m+1-\{\ell\}} \varphi(\vec{x};\vec{b}_i) \wedge \neg\varphi(\vec{x};\vec{b}_\ell)$$

is realized in H by $\vec{a}_\ell$. That easily implies $\varphi^{\vec{x}}$ is not ℓ-m-strongly-equational. □

Example. Let R be a noetherian ring with a unit, L the language of rings, $\mathbb{K}$ the category with single object R and single morphism the identity map on R. Consider the formula

$\varphi(x;t) = \exists s(x = st)$; for a in R, $\varphi(a;t)$ defines the set of divisors of a, while $\varphi(x;a)$ defines the set of multiples of a.

φ is an equation in t: for if $(a_i)_{i<\omega}$ is a sequence of elements in R, the ideal $\langle a_i; i < \omega\rangle$ is finitely generated, hence equals $\langle a_i; i < n\rangle$ for some n. It follows that t divides a_i for $i < \omega$ iff t divides a_i for $i < n$.

However, if R contains an infinite sequence $(a_i)_{i<\omega}$ such that a_i strictly divides a_{i+1}, take for instance $R = \mathbb{Z}\times\mathbb{Z}$, and $a_i = (2^i;1)$, then φ is not equational in x.

Note that in $\mathbb{Z}\times\mathbb{Z}$ there is an element $a_\omega \neq 0$ such that a_i divides a_ω for every $i < \omega$, namely $a_\omega = (0,1)$, (in other words $\{\varphi(x;a_i); i < \omega\}$ is realized by a_ω in R).

9. Let $\varphi = \varphi(\vec{x};\vec{t})$, m = length $\vec{x}$, n = length $\vec{t}$ and $H \in \mathbb{K}$. Let $L_{\vec{x}}$ (resp. $L_{\vec{t}}$) be the (obvious) semi-lattice whose underlying set is the class of subsets of H^m (resp. H^n) which are definable by conjuncts of instances of $\varphi^{\vec{x}}$ (resp. $\varphi^{\vec{t}}$) (an instance of $\varphi^{\vec{t}}$ is a formula of the kind $\varphi(\vec{a};\vec{t})$, $\vec{a}$ in H).

a) Assume φ is $\mathfrak{K}$-equational in $\vec{t}$. We define a map

$$*_{\vec{t}} = *^{H}_{\varphi,\vec{t}} : \quad L_{\vec{x}} \longrightarrow L_{\vec{t}}$$

$$X \longmapsto \bigcap \{\varphi(\vec{b}_i;H); \vec{b}_i \in X\}$$

$$(\varphi(\vec{b}_i;H) = \{\vec{a} \in H^n; H \models \varphi(\vec{b}_i;\vec{a})\})$$

The intersection mentioned in the definition of $*_{\vec{t}}$ above is finite because of ℓ-equationality.

Similarly, if φ is ℓ-equational in $\vec{x}$, we define the map $*_{\vec{x}}: L_{\vec{t}} \longrightarrow L_{\vec{x}}$.

b) For $X, Y \in L_{\vec{x}}$, $X \subset Y \Longrightarrow *_{\vec{t}}(X) \supset *_{\vec{t}}(Y)$.

This is immediate, since

$$\{\varphi(\vec{b}_i;H); \vec{b}_i \in X\} \subset \{\varphi(\vec{b}_i;H); \vec{b}_i \in Y\}.$$

c) Note that if $X = \wedge_{i<n}\varphi(H;\vec{a}_i)$ then $\vec{a}_i \in *_{\vec{t}}(X)$ for every $i < n$.

d) Suppose $Y = \bigcap_{j<k}\varphi(H;\vec{a}_j)$ and $*_{\vec{t}}(X) = \bigcap_{i<n}\varphi(\vec{b}_i;H)$, $\vec{b}_i \in X$.

Then, $X \subset Y$ iff $\vec{b}_i \in Y$ for every $i < n$ iff $\vec{a}_j \in *_{\vec{t}}(X)$ for every $j < k$.

Proof. Clearly, $\vec{b}_i \in Y$ for every $i < n$ iff $H \models \wedge_{i<n,\, j<k}\varphi(\vec{b}_i;\vec{a}_j)$ iff $\vec{a}_j \in *_{\vec{t}}(X)$ for every $j < k$.

Obviously, if $X \subset Y$, then $\vec{b}_i$ ($\in X$) belongs to Y for every $i < n$.

Conversely, if $\vec{b}_i \in Y$ for every $i < n$ then $\vec{a}_j \in *_{\vec{t}}(X)$ for $j < k$ and therefore, by definition of $*_{\vec{t}}(X)$, $\vec{a}_j \in \varphi(\vec{b};H)$ for any $\vec{b} \in \vec{X}$ and $j < k$. In other words, $\vec{b} \in \bigcap_{j<k}\varphi(H;\vec{a}_j)$ for any $\vec{b}$ in $\vec{X}$, which means $X \subset Y$. □

e) For $X,Y \in L_{\vec{x}}$, $X \subset Y \Leftrightarrow *_{\vec{t}}(X) \supset *_{\vec{t}}(Y)$.

We already have one direction (a)). Suppose $Y = \bigcap_{j<k}\varphi(H;\vec{a}_j)$ and $*_{\vec{t}}(X) \supset *_{\vec{t}}(Y)$. In c) we have noted that $\vec{a}_j \in *_{\vec{t}}(Y)$ for $j < k$. Hence, $\vec{a}_j \in *_{\vec{t}}(X)$ for $j < k$. By d), we conclude that $X \subset Y$.

As a corollary one gets another proof of lemma 8.(i), namely that ℓ-height$(\varphi^{\vec{x}})$ = ℓ-height$(\varphi^{\vec{t}})$. For it follows from e) that any chain of elements in $L_{\vec{x}}$ of length m gives rise to a chain of elements in $L_{\vec{t}}$ of length m.

10. Proposition. If $\varphi(\vec{x};\vec{t})$ is ℓ-equational in $\vec{x}$ and $\vec{t}$, then $*_{\vec{t}}$ is a dual isomorphism from $L_{\vec{x}}$ onto $L_{\vec{t}}$ with inverse $*_{\vec{x}}$. (By dual we just mean the property stated in 9.e).)

Proof. We already know from 9.e) above that $*_{\vec{t}}$ is a dual isomorphism from $L_{\vec{x}}$ into $L_{\vec{t}}$. Remains to show that $*_{\vec{x}}$ is the inverse of $*_{\vec{t}}$.

We show $*_{\vec{x}} \cdot *_{\vec{t}}$ is the identity on $L_{\vec{t}}$. Let

$X = \bigcap_{i<n} \varphi(H;\vec{a}_i)$

By definition, $*_{\vec{x}} \cdot *_{\vec{t}}(X) = \bigcap \{\varphi(H;\vec{c}_i);\vec{c}_i \in *_{\vec{t}}(X)\}$.

From 9.c), $\vec{a}_i \in *_{\vec{t}}(X)$ for $i < n$; hence $*_{\vec{x}} \cdot *_{\vec{t}}(X) \subset X$.

On the other hand, by definition,

$$\vec{c} \in *_{\vec{t}}(X) \text{ iff } H \models \varphi(\vec{b};\vec{c}) \text{ for all } \vec{b} \in X.$$

Thus, $\vec{c} \in *_{\vec{t}}(X) \Longrightarrow X \subset \varphi(H;\vec{c})$;

hence $*_{\vec{x}} \cdot *_{\vec{t}}(X) \supset X$.

We conclude $*_{\vec{x}} \cdot *_{\vec{t}}(X) = X$. ◻

The proposition below gives a criterion for equationality using complete types.

First, we introduce some new terminology.

11. Terminology. Let $f:A \longrightarrow B$ be a morphism in $\hat{\mathbb{K}}$.

A set p of formulas in L with parameters in A is said *consistent in* $\mathbb{K}$ *over* f if fp is consistent in $\mathbb{K}$ over B.

Given a set Δ of formulas in L, a set p of formulas in $\mathrm{cl}(\Delta^{\vec{x}})$ with parameters in A is called a Δ-*type in* $\vec{x}$ over f

if p is consistent in $\mathbb{K}$ over f;

if f is an inclusion map we say p is a Δ-type in $\vec{x}$ over A.

A Δ-type over f is a Δ-type in some tuple of variables $\vec{x}$ over f.

– Given morphisms $f:A\longrightarrow B$ and $g:B\longrightarrow C$ in $\hat{\mathbb{K}}$, Δ as above and p an L-type over g, we let

$$p\restriction f = \{\varphi(x;\vec{a});\ \vec{a} \text{ in } A,\ \varphi(\vec{x};f\vec{a}) \in p\};$$

$$p\restriction\Delta = \{\varphi(\vec{x};\vec{b});\ \varphi(\vec{x};\vec{t}) \in \mathrm{cl}(\Delta),\ \text{and } \varphi(\vec{x};\vec{b}) \in p\}.$$

Clearly $p\restriction f$ is an L-type over $g\cdot f$ and $p\restriction\Delta$ is a Δ-type over g.

If f is an inclusion map, we write $p\restriction A$ instead of $p\restriction f$.

A Δ-type p in $\vec{x}$ over f ($f:A\longrightarrow B$) is Δ-complete if p is maximal, with respect to inclusion, among Δ-types in $\vec{x}$ over f.

12. **Lemma.** $f:A\longrightarrow B$ a morphism in $\hat{\mathbb{K}}$

a) A Δ-type p in $\vec{x}$ over f is Δ-complete iff for any formula φ in $\mathrm{cl}(\Delta^{\vec{x}})$ over A, either φ or $\neg\varphi$ belongs to p.

b) If p is an L-complete L-type over B then $p\restriction f$ (resp. $p\restriction\Delta$) is L-complete (resp. Δ-complete).

c) An L-type over f can always be extended to an L-complete L-type over f.

Proof.

a) For p and φ as in a), necessarily either $p \cup \{\varphi\}$ or $p \cup \{\neg\varphi\}$ is consistent over f: since $p \cup \{\varphi\}$ and $p \cup \{\neg\varphi\}$ inconsistent over f means there are finite subsets p_1 and p_2 of p such that $p_1 \cup \{\varphi\}$ and $p_2 \cup \{\neg\varphi\}$ are inconsistent over f, whence $p_1 \cup p_2 \cup \{\varphi \vee \neg\varphi\}$ is inconsistent over f, contradicting the fact that p is consistent over f.

Thus if p is Δ-complete then, by maximality, either φ or $\neg\varphi$ must belong to p.

b) Follows immediately from a)

c) Is an immediate application of Zorn's lemma. □

For the remainder of this chapter, by type or complete type we mean an L-type or L-complete L-type;

for p a complete type over H we let

$$p^S = \{\varphi(\vec{x};\vec{a});\ \vec{a} \in H,\ \varphi \in S,\ p \models_H \varphi\}$$

13. **Proposition. The following assertions are equivalent:**

(i) S is equational.

(ii) For any structure H in $\mathbb{K}$, $\vec{x}$ and complete type q in $\vec{x}$ over H, there is a finite subset q_0 of q such that $q_0 \models_H q^S$.

Proof. (i)$\Longrightarrow$(ii) is obvious.

(ii)$\Longrightarrow$(i): Assume (ii) holds. Let $H \in K$, p a set of formulas in $S^{\vec{x}}$ with parameters in H. We want to show the existence of $p_0 \subset p$, p_0 finite such that $p_0 \vdash_{\overline{H}} p$.

If p is inconsistent in $\mathbb{K}$ then clearly such a p_0 exists. Suppose p is consistent. Let P be the set of complete types in $\vec{x}$ over H containing p; by lemma 12.c), P is not empty.

Moreover, by assumption, for any $q \in P$ we can find $q_0 \subset q$, q_0 finite and $q_0 \vdash_{\overline{H}} q^S$. Let

$$p' = p \cup \{\neg(\wedge q_0); q \in P\}.$$

Claim. p' is inconsistent in $\mathbb{K}$. For if p' is consistent, we can extend it to a complete type q over H so that $q \in P$. But then, $\neg(\wedge q_0) \in p' \subset q$ and $q_0 \subset q$ ☒.

So p' is inconsistent. Hence, there are $q^1, \cdots, q^n$ in P and p_0 a finite subset of p such that $p_0 \vdash_{\overline{H}} \vee_{i=1}^{n}(\wedge q_0^i)$. On the other hand, for any i, $1 \leq i \leq n$, $q_0^i \vdash_{\overline{H}} q^{S'} \vdash_{\overline{H}} p$; hence $\vee_{i=1}^{n}(\wedge q_0^i) \vdash_{\overline{H}} p$. We conclude $p_0 \vdash_{\overline{H}} p$, which is what we wanted. ∎

Proposition 13 enables us to give another proof of 7.(ii), namely:

Corollary 1. S is equational iff $R = cl^{+}(S)$ is equational.

Proof. Indeed, let $H \in K$ and q a complete type in $\vec{x}$ over H. Then clearly $q^R \underset{H}{\simeq} q^S$. Since S is equational there is a finite subset q_0 of q such that $q^R \underset{H}{\simeq} q^S \underset{H}{\simeq} q_0$. We conclude by proposition 13 that R is equational. ◻

We close this chapter by giving two applications of proposition 13 which are somewhat typical. They both rely on the existence of a "division rule". The first application is almost immediate (see I.4.(i)) if one uses Hilbert's theorem which says that a polynomial ring over a noetherian ring is noetherian. Here, we give direct proofs.

Application 1. Let $\mathbb{K} = \mathbb{F}$ be the category of fields with field embeddings. Then the set S of algebraic equations in the variables $\{x_1,\cdots,x_n\}$ is equational.

Proof. By induction on the number of variables n.

Suppose the assertion holds for $n-1$. Let H be an object in $\mathbb{K}$ and q a complete type in $\vec{x} = \langle x_1,\cdots,x_n\rangle$ over H. q^S is (up to equivalence in $\mathbb{K}$) of the form $\{(P = 0);\ P \in I\}$, where I is an ideal in $H[x_1,\cdots,x_n]$.

Consider each element P of I as an element of $H[x_1,\cdots,x_{n-1}][x_n]$; let U_P be its leading coefficient, U_P

$\in H[x_1,\cdots,x_{n-1}]$. Let $J = \{P \in I\colon U_P \notin I\}$. It is easily seen that

$q^S \underset{H}{\sim} q_0 \cup q_1$, where $q_0 = \{(P = 0);\ P \in J\}$ and $q_1 = \{(U_P = 0);\ P \in I\setminus J\}$. By the induction hypothesis q_1 is equivalent to a finite subset p_1.

As for q_0, choose $Q \in J$ of smallest degree in x_n; denote this degree by $d(Q)$.

Claim. $(Q = 0)\wedge(U_Q \neq 0)\wedge p_1 \vdash_{\overline{H}} q^S$.

Indeed, given $P \in I$, $d(P) \geq d(Q)$, there is $m \in \omega$ and a polynomial R_0, such that

$U_Q\cdot P = U_P\cdot x_n^m\cdot Q + R_0$ with $d(R_0) < d(P)$.

($d(R_0)$ = the degree of R_0 in x_n).

Clearly $R_0 \in I$ and

$$((Q = 0)\wedge(R_0 = 0)\wedge(U_Q \neq 0)) \vdash_{\overline{H}} (P = 0).$$

If $d(R_0) \geq d(Q)$, we repeat the same process with R_0 instead of P. We get $R_1 \in I$, $d(R_1) < d(R_0)$ and

$((Q = 0)\wedge(U_Q \neq 0)\wedge(R_1 = 0)) \vdash_{\overline{H}} (R_0 = 0)$; hence,

$((Q = 0)\wedge(U_Q \neq 0)\wedge(R_1 = 0)) \vdash_{\overline{H}} (P = 0)$.

Ultimately we find R_n, $R_n \in I$, $d(R_n) < d(Q)$, whence $R_n \in J$, and $((Q = 0)\wedge(U_Q \neq 0)\wedge(R_n = 0)) \vdash_{\overline{H}} (P = 0)$. The claim follows immediately.

Now, since the formulas $(Q = 0)$ and $(U_Q \neq 0)$ belong to

q, we conlude by proposition 13 that S is equational. ■

Let $\mathbb{D}F_p$ be the category of differential fields of characteristic p with differential field embeddings. L is then the language of fields plus a unary operation symbol $d(-)$ representing the derivation function.

Application 2. With $\mathbb{K} = \mathbb{D}F_0$ and S the set of atomic formulas in $\vec{x} = \{x_0,\cdots,x_n\}$, S is equational.

Proof. *The proof is similar to that of application 1 via a "division rule" for differential equations.*

Let H be a differential field. Recall, a differential polynomial P in $\vec{x}$ with coefficients in H, is a polynomial in a sequence of variables $\vec{X}$ with coefficients in H where $\vec{X}$ is of the form:

$$\vec{X} = \langle x_0,\cdots,x_n,dx_0,\cdots,dx_n,\cdots,d^m x_0,\cdots,d^m x_n\rangle,$$

for some $m < \omega$. Let ord$P$ (order of P in x_n) be the highest number m such that $d^m x_n$ occurs non-trivially in P; let $u_P = d^m x_n$ for $m = \text{ord}P$.

Thus, we can write the formal equality of polynomials: $P = \sum_{i=0}^{r} I_i u_P^i$ where I_i, for $o \le i \le r$, is a polynomial in the sequence of variables

$$\langle x_0,\cdots,x_n,dx_0,\cdots,dx_n,\cdots,d^{m-1}x_0,\cdots,d^{m-1}x_n,d^m x_0,\ldots,d^m x_{n-1}\rangle$$

$m = \text{ord}P$; let $I_P = I_r$ and $S_P = \sum_{i=1}^{r} i I_i u_P^{i-1}$.

Note first that, since a differential equation over H can be considered as an algebraic equation over H (H as a field) and since algebraic equations are equational in the category of fields it follows immediately that a differential equation is equational in the category of differential fields (of any characteristic).

However, this does not entail that the set S of *all* differential equations is equational. To show that S is equational we need a division rule on differential equations. Such a rule is given by lemma 5 of chapter I.8 in [Kolchin] which we reproduce below:

Lemma. (cf [Kol. I.8]). For any differential polynomial P and m, $0 < m < \omega$, $d^m P - S_P d^m u_P$ has lower order than $d^m u_P$.

Proof of the lemma. Write $P = \sum_{i=0}^{r} I_i u_p^i$. Then,

$$dP = S_P du_P + \sum_{i=0}^{r} d(I_i) u_P^i.$$

Since every derivative of x_n present in I_i is strictly lower than u_P (i.e. I_i has a lower order than u_P) and u_P has a lower order than du_P, we find that $dP - S_P du_P$ has lower order than du_P. This proves the lemma for $m = 1$. The lemma for arbritrary m follows quickly by induction on m.

Back to our proof. Let $H \in \mathbb{D}F_0$. Expanding polynomial expressions and using the porperties of the derivation, every atomic formula $\varphi(\vec{x};\vec{a})$ with coefficients in H, can be written in a natural way in the form:

$$\varphi(\vec{x};\vec{a}) \underset{H}{\simeq} (P_\varphi(\vec{x}) = 0)$$

where $P_\varphi(\vec{x})$ is a differential polynomial in $\vec{x}$ with coefficients in H. (P_φ is uniquely determined up to formal equality of polynomials). Let ordφ, I_φ, S_φ, u_φ denote respectively ordP_φ, I_{P_φ}, S_{P_φ}, and u_{P_φ}.

Let p be a complete type in $\vec{x}$ over H; let

$$p^S = \{\varphi(\vec{x};\vec{a});\ \varphi \in S,\ \vec{a} \in H,\ p \vdash_H \varphi\},$$

$$q = \{\varphi \in p^S;\ (I_\varphi = 0) \notin p,\ u_\varphi \neq 0\},$$

$$p_1 = \{\varphi \in p^S;\ u_\varphi = 0\}.$$

p_1 is the set of atomic formulas in p which do not mention (non-trivially) x_n. By induction hypothesis, we can assume p_1 equivalent in $\mathbb{K}$ to a finite subset. Let $m = \min\{\text{ord}\varphi;\ \varphi \in q\}$; let φ be an element of q with ord$\varphi = m$ and such that P_φ has lowest possible degree say r in $d^m x_n$. Let $P = P_\varphi$.

Claim. $(S_\varphi = 0) \notin p^S$. For, either $r > 1$, in which case ord$S_\varphi = m$, the degree of S_φ in $d^m x_n$ is strictly less than

r, $I_{S_\varphi} = r \cdot I_\varphi$, whence $(I_{S_\varphi} = 0) \notin p$ (since $(I_\varphi = 0) \notin p$), and therefore S_φ cannot belong to p by the minimal choice of φ;

or $r = 1$ in which case $S_\varphi = I_\varphi$ and $(S_\varphi = 0) \notin p$ since $\varphi \in q$.

Consider now an element ψ of $p^S \setminus p_1$. Write

$$P_\psi = Q = \Sigma_{i=0}^{k} I_i u_Q^i, \; u_Q = d^\ell x_n.$$

1st case: $\ell > m$. Then, let

$$R_0 = S_p \cdot Q - I_Q \cdot d^{\ell-m} P \cdot (d^{\ell-m} u_p)^{k-1} =$$

$$= \Sigma_{i=0}^{k-1} S_p I_i u_Q^i + S_p I_Q u_Q^k - I_Q \cdot d^{\ell-m} P \cdot u_Q^{k-1}$$

$$= \Sigma_{i=0}^{k-1} I_i S_p \cdot u_Q^i + I_Q \cdot u_Q^{k-1} \cdot (S_p \cdot u_Q - d^{\ell-m} P).$$

By the lemma above we see that either $\mathrm{ord} R_0 < \ell$ or the degree of R_0 in u_Q is strictly less than k (Recall k is the degree of Q in u_Q).

Moreover, since $(Q = 0) \wedge (P = 0) \vdash_{\overline{H}} (R = 0)$, $(R_0 = 0)$ belongs to p^S. On the other hand

$$(R = 0) \wedge (P = 0) \wedge (S_p \neq 0) \vdash_{\overline{H}} (Q = 0).$$

If $\mathrm{ord} R_0 > m$, we repeat the same process with R_0 instead of Q and obtain R_1. Ultimately we find

$R_0, R_1, \cdots, R_j$, $j < \omega$, $R_j \in p^S$,

$(R_j = 0) \wedge (P = 0) \wedge (S_p \neq 0) \vdash_{\overline{H}}$

$(R_{j-1} = 0) \wedge (P = 0) \wedge (S_p \neq 0) \vdash_{\overline{H}} \cdots \vdash_{\overline{H}} (\hat{Q} = 0)$, and $\mathrm{ord} R_j \leq m$.

Thus we have come down to the case of $\ell \leq m$.

2nd case: $\ell = m$ and $k \geq r$. Clearly then, if $Q_0 = I_p Q - I_Q P \cdot u_p^{k-r}$, the degree of Q_0 in u_p is strictly less than k. Moreover $(Q_0 = 0) \in p^S$ and

$$(Q_0 = 0) \wedge (P = 0) \wedge (I_p \neq 0) \vdash_{\overline{H}} (Q = 0).$$

If the degree of Q_0 is greater or equal to r and $\mathrm{ord} Q_0 = m$ we repeat the same process with Q_0 instead of Q to obtain Q_1. Ultimately we find

$$Q_0, Q_1, \cdots, Q_d,\ d < \omega,\ (Q_d = 0) \in p^S,$$

$$(Q_d = 0) \wedge (P = 0) \wedge (I_p \neq 0) \vdash_{\overline{H}}$$

$$(Q_{d-1} = 0) \wedge (P = 0) \wedge (I_p \neq 0) \vdash_{\overline{H}} \cdots \vdash_{\overline{H}} (Q = 0),$$

and, either $\mathrm{ord} Q_d < m$ or [$\mathrm{ord} Q_d = m$ and the degree of Q_d in u_p is strictly less than r].

3rd case. $\ell < m$ or [$\ell = m$ and $k < r$].

Claim. $p_1 \vdash_{\overline{H}} (Q = 0)$. Indeed, by the minimal choice of P, $(Q = 0)$ cannot belong to q. Hence, either $u_Q = 0$, in which case
$(Q = 0) \in p_1$, or

$$(I_Q = 0) \in p^S, \text{ in which case}$$

$(I_Q = 0) \wedge (Q_0 = 0) \models_{\overline{H}} (Q = 0)$ where $Q = Q_0 + I_Q u_Q^k$ $((Q_0 = 0) \in p^S)$:

by induction on the order of Q and the degree of Q in u_Q we can assume $p_1 \models_{\overline{H}} (Q_0 = 0)$ and since $p_1 \models_{\overline{H}} (I_Q = 0)$, we conclude $p_1 \models_{\overline{H}} (Q = 0)$.

Combining the three cases above we deduce that

$$p_1 \wedge (P = 0) \wedge (S_p \neq 0) \wedge (I_p \neq 0) \models_{\overline{H}} (Q = 0).$$

Since p_1 is equivalent to a finite subset,

$$(P = 0) \in p, \ (S_p \neq 0) \in p \text{ and } (I_p \neq 0) \in p,$$

it follows that there is a finite subset p_0 of p such that $p_0 \vdash p^S$. By proposition 13 we conclude that S is equational. ◻

Chapter II

S_H-Minimal Extensions of Types

We introduce in this chapter the notion of an S_H-minimal extension of a type, for H a structure in $\mathbb{K}$ and S a set of formulas in L. As we said in the introduction, S_H-minimal extensions play in our context the role that non-forking extensions play in stability theory.

Informally speaking, given $A \subset B \subset H$, p a type over A, we are interested in extensions q of p to B which do not satisfy over B any relation with respect to the formulas in S that is not induced by p. We are bound then to consider *all* possible relations with respect to formulas in S that q might imply.

Thus it would be ideal if we are able to find an extension q of p to B such that for any morphism $f:H \longrightarrow F$, fq implies a *minimal* possible set of formulas in S with parameters in F.

We divide the problem into two parts:

First, given $A \subset B \subset H$, p a type over A, we investigate those extensions of p to B which imply a minimal possible set of formulas in S with parameters *in H*. Such extensions are called S_H-minimal extensions of p to B

and will be the object of study of section A.

We show then, that S_H-minimal extensions of p to B always exist (c.f. proposition 2), that, for S an equational set, there are, up to S_H-equivalence, finitely many such extensions (c.f. theorem 13) and that the property of monotonicity-transitivity holds for such extensions when considering types over sets which are S_H-closed in H (c.f. theorem 12).

Second, we consider the types q (over some structure F), for which, intuitively, all the formulas in S (with parameters in some G, when given $f:F\longrightarrow G$) that q implies, are represented in q. More precisely, we call q *S-full* (c.f. B.7) if for any morphism $g:F\longrightarrow G$ and φ a formula in S with parameters in G,

$$gq \vdash_{G} \varphi \Longrightarrow f[q_F^S] \vdash_{G} \varphi,$$

where

$$q_F^S=\{\psi(\vec{x};\vec{a}),\ \psi \in S,\ \vec{a} \in F \text{ and } q \vdash \psi\}.$$

A structure F is S-full if every complete type over F is S-full.

Then, given a type p over $A \subset H$ we investigate the S_H-minimal extensions q of p to H such that $p \cup q_H^S$ is S-full. If q is such an extension, we call q_H^S an S_H-component of p.

In section B, we define S-full types, S_H-components and S-full structures. We show (c.f. 5.(ii)) that if p is over $A \subset H$ and q is an S_H-minimal extension of p to H which is S-full, then for any morphism $f: H \longrightarrow F$ such that fq is consistent in $\mathbb{K}$, fq is an S_F-*minimal* extension of p to fH; furthermore, we show that if S has finite height or that S is equational and $\mathbb{K}$ is closed under unions of countable chains then there is a morphism $f: H \longrightarrow F$ such that for any S_F-minimal extension r of p to f, r_F^S is an S_F-component of p, (c.f. theorem B.10). Finally if $\mathbb{K}$ is closed under unions of chains then for any H in $\mathbb{K}$ there is $f: H \longrightarrow F$ such that F is S-full. (In chapter III, we make the connection between S-full structures and existentially closed structures).

It is apparent all along this chapter that the general theory of sections A and B goes through in a very general abstract setting that has nothing to do with structures or formulas; in section C we describe such a setting.

Section A: S_H-Minimal Extensions

0. Preliminaries.

- **We fix a class Δ of formulas in L closed under boolean combinations and substitution of parameter variables. Unless stated otherwise, all formulas considered are in Δ; a type shall mean a Δ-type and a complete type shall mean a Δ-complete Δ-type.**

The motivation for fixing Δ is that in general, given a category $\mathbb{K}$, we only work with a particular type of formulas e.g. quantifier-free, existential, positive existential etc... with the assumptions or results on $\mathbb{K}$ depending only on such formulas.

We sometimes call $\mathbb{K}$ a Δ-category to underline the choice of Δ.

- S is a fixed subset of Δ which is closed under finite conjunctions, disjunctions and substitutions of parameter variables.

  S usually stands for an equational set or a set of equations.

- Given $A \in \hat{\mathbb{K}}$, we shall consider formulas with parameters in A up to equivalence in $\mathbb{K}$ over A; in other words we do not distinguish between $\varphi(\vec{x};\vec{a})$, $\vec{a}$ in A, and $\varphi(\vec{x};\vec{a})/_{\overline{A}}$.

  Note that if $f: A \longrightarrow B$ is in $\hat{\mathbb{K}}$, $\vec{a}$, $\vec{b}$ in A and $\varphi(\vec{x};\vec{a}) \underset{A}{\sim} \psi(\vec{x};\vec{b})$, then $\varphi(\vec{x};f\vec{a}) \underset{B}{\sim} \psi(\vec{x};f\vec{b})$.

  If R is a set of formulas in Δ, we let $D^R_{\vec{x}}(A)$ denote the set of formulas in $R^{\vec{x}}$ with parameters in A (considered up to equivalence in $\mathbb{K}$ over A), i.e.

$$D^R_{\vec{x}}(A) = \{\varphi(\vec{x};\vec{a})/_{\overline{A}};\ \varphi(\vec{x};\vec{t}) \in R,\ \vec{a} \in A\};$$

$$D^R(A) = \bigcup_{\vec{x}} D^R_{\vec{x}}(A).$$

We let $D_{\vec{x}}(A) = D^{\Delta}_{\vec{x}}(A)$ and $D(A) = D^{\Delta}(A)$.

– Given $A \subset B \in \hat{\mathbb{K}}$, p and q types in $\vec{x}$ over A, we let

$$p^S_B = \{\varphi \in D^S_{\vec{x}}(B);\ p \models_B \varphi\};$$

We say p and q are S_B-equivalent if $p^S_B \underset{B}{\cong} q^S_B$. if $f: A \longrightarrow B$ is a morphism in $\hat{\mathbb{K}}$, p and q types in $\vec{x}$ over f, we let $p^S_f = (fp)^S_B$; we say p and q are S_f-equivalent if fp and fq are S_B-equivalent i.e. if $p^S_f \underset{B}{\cong} q^S_f$.

Example. Let T be the theory of one equivalence relation $E(x;y)$ with two classes, one, infinite, the other, of cardinality n. Let $\mathbb{K}$ be the category of models of T with elementary embeddings, Δ the set of all formulas, $S = cl^+(E(x;t))$.

Let M be a model of T, $a \in M$, and p a type over M. if $p = \{\neg E(x;a)\}$ then

$$p^S_M = \{E(x;b);\ b \in M,\ M \models \neg E(b;a)\}.$$

If p says, the class of x contains more than n elements then

$$p^S_M = \{E(x;b);\ b \in M \text{ the class of } b \text{ is infinite}\}.$$

1. Definition. Given $A \subset B \subset C \in \hat{\mathbb{K}}$, p a type over A and q a

complete type over B extending p, we say q is an S_C-*minimal extension of p to B* if for any complete type q_1 over B extending p,

$$q_C^S \supset (q_1)_C^S \Longrightarrow q_C^S = (q_1)_C^S.$$

(Of course here p, q and q_1 are types in a common tuple of variables $\vec{x}$).

Given $f:A \longrightarrow B$ and $g:B \longrightarrow C$ morphisms in $\hat{\mathbb{K}}$, p a type over $g \cdot f$ and q a complete type over g, (c.f. definition of type in I.11), we say q is an S_g-*minimal extension of p to f* if gq (as a type over $g(B)$) is an S_C-minimal extension of gfp to $g(B)$, i.e. whenever q_1 is a complete type over g extending p to f,

$$q_g^S \supset (q_1)_g^S \Longrightarrow q_g^S = (q_1)_g^S.$$

If g (resp. f) is an inclusion map we say q is an S_C- (resp. S_g-) minimal extension of p to f (resp. to B) instead of q is an S_g-minimal extension of p to f.

2. Proposition. Given $B \subset C \in \hat{\mathbb{K}}$, p a type in $\vec{x}$ over B and q_1 a complete type over B extending p, there is an S_C-minimal extension q of p to B such that $(q_1)_C^S \supset q_C^S$.

Proof. For simplicity write q^S instead of q_C^S for any type q over C.

Let $Q = \{q^S;\ q$ a complete type over B extending p and $q^S \subseteq q_1^S\}$.

We are searching for a minimal element (minimal for inclusion) in Q.

Let $(q_\alpha^S)_{\alpha<\lambda}$ be a decreasing chain of elements in Q (i.e. $q_\alpha^S \subseteq q_\beta^S$ whenever $\beta < \alpha < \lambda$).

Let $\Xi = \{\Theta \in D_{\bar{x}}^{+}(B);$ there is $\varphi \in D_{\bar{x}}^{S}(C)$, $\Theta \vdash_C \varphi$ and $\varphi \notin \bigcap_{\alpha<\lambda} q_\alpha^S\}$.

Consider the set $q = p \cup \{\neg\Theta;\ \Theta \in \Xi\}$.

Claim. q is a type over C.

Indeed, suppose q is inconsistent. Then, there are $\Theta_0,\dots,\Theta_{n-1}$ in Ξ, $p \vdash_C \Theta_0 \vee\dots\vee \Theta_{n-1}$. Let $\varphi_0,\dots,\varphi_{n-1} \in D^S(C)$ such that $\Theta_i \vdash_C \varphi_i$ for $i < n$ and $\varphi_i \notin \bigcap_{\alpha<\lambda} q_\alpha^S$. Since (q_α^S) is decreasing, there is $\beta < \lambda$, $\varphi_i \notin q_\beta^S$ for any $i < n$. But q_β is complete and $q_\beta \supset p$; hence, for some $i < n$, $\Theta_i \in q_\beta$; hence

$q_\beta \vdash \varphi_i$ ✗. That shows q is consistent and proves the claim.

Let q' be a complete type over B extending q (c.f. I.12.c)).

Claim. $(q')^S \subset q^S_\alpha$ for any $\alpha < \lambda$. For if $q' \vdash_C \varphi$, $\varphi \in D^S(C)$, then $\Theta \vdash_C \varphi$ for some $\Theta \in q'$. If $\varphi \notin \bigcap_{\alpha<\lambda} q^S_\alpha$ then $\Theta \in \Xi$ and $\neg\Theta \in q$ which contradicts the consistency of q'. Thus $\varphi \in \bigcap_{\alpha<\lambda} q^S_\alpha$. The claim is proved.

So we have shown that $(q^S)_{\alpha<\lambda}$ has a minorant in Q. We conclude by Zorn's lemma that Q admits a minimal element. ■

Note that it follows from proposition 2 that if $f: A \longrightarrow B$ and $g: B \longrightarrow C$ are morphisms in $\hat{\mathbb{K}}$, p a type over gf and q_1 a complete type over g extending p to then there is an S_g-minimal extension of p to f such that $(q_1)^S_g \supset q^S_g$.

Indeed, it suffices to apply proposition 2 to gfp and gq_1 as types ober gB (note that gq_1 is complete over gB). We find then an S_C-minimal extension q' of gfp to gB such that

$(gq_1)^S_C \supset (q')^S_C$. But that means $q = q' \restriction g$ is an S_g-minimal extension of p to f (note $gq = q'$), and $(q_1)^S_g \supset q^S_g$.

3. Lemma. Let $A \subset B \in \hat{\mathbb{K}}$. Given a complete type p over A and a type q_0 over A with $p^S_A \supset (q_0)^S_A$, there is a complete type q over A extending q_0 such that $p^S_A \supset q^S_A$.

Proof. Let $q_1 = q_0 \cup \{\neg\varphi;\ \varphi \in D^S(A)$ and $\varphi \notin p^S_A\}$.

Claim. q_1 is consistent over A. For if not, there are $\varphi_1, \cdots, \varphi_n \in D^S(A)$,

$$\varphi_i \notin p^S_A \quad (1 \leq i \leq n) \text{ and } q_0 \vdash_{\bar{A}} \vee_{i=1}^n \varphi_i$$

Hence, $(q_0)^S_A \vdash_{\bar{A}} \vee_{i=1}^n \varphi_i$ (recall we had fixed S such that $cl^+(S) = S$).

Therefore $p^S_A \vdash_{\bar{A}} \vee_{i=1}^n \varphi_i$.

By completeness of p, it follows that for some i, $1 \leq i \leq n$, $p^S_A \vdash \varphi_i$ ※. So q_1 is consistent over A.

Now let q be a complete extension of q_1 to A. Clearly $q^S_A \subset p^S_A$. ■

Let $A \subset B \subset C \subset H$, p a type over A (= (A,H)), r a type over C. In view of a monotonicity-transitivity theorem for S_H-minimal extensions, one expects that if r is an S_H-minimal extension of p to C then $r \upharpoonright B$ should be an S_H-minimal extension of p to B. But this is not the case in general as we can have situations of the kind: r and q complete types over C, r and q S_H-equivalent but $r \upharpoonright B$ and $q \upharpoonright B$ not S_H-equivalent.

Subsets B of C in which such situations do not occur are called S_H-closed in C. Lemma 5 justifies this terminology.

First, the exact definition,

4. Definition. Given $A \subset B \in \hat{\mathbb{K}}$, we say *A is S_B-closed in B* if for any $\vec{x}$ and p_1, q_1 complete types in $\vec{x}$ over B,

$$(p_1)^S_B \supset (q_1)^S_B \implies p^S_B \supset q^S_B, \text{ where } p = p_1 \restriction A \text{ and } q = q_1 \restriction A.$$

If $f: A \longrightarrow B$ is a morphism in $\hat{\mathbb{K}}$, we say *A is S_f-closed in B* if fA is S_B-closed in B (i.e. if whenever p_1, q_1 are complete types over B with $(p_1)^S_B \supset (q_1)^S_B$ then $p^S_f \supset q^S_f$ where $p = p_1 \restriction f$ and $q = q_1 \restriction f$).

We will see in chapter V that if $\mathbb{K}$ is the category of models of a first-order theory with elementary embeddings and if S is equational in $\mathbb{K}$ then for any model M in $\mathbb{K}$ and $A \subset M$, A is S_M-closed in M.

5. Lemma. Let $f: A \longrightarrow B$ be a morphism in $\hat{\mathbb{K}}$. Then, A is S_f-closed in B iff for any complete type p in $\vec{x}$ over f and any formula φ in p^S_f *there is* a formula Θ in $D(A)$ such that,

$$p^S_f \vdash_B f\Theta \text{ and } f\Theta \vdash_B \varphi.$$

If in addition S is equational in $\mathbb{K}$ then A is S_f-closed in B iff for any p complete over f, if p^S_f is not empty then p_f^S is equivalent to a single formula $f\Theta$ for some Θ in $\mathbb{D}(A)$.

Proof. Suppose A is S_f-closed in B, p is a complete type in $\vec{x}$ over f and $\varphi \in p_f^S$. Let

$$q_0 = p_f^S \cup \{\neg f\Theta;\ \Theta \in D(A) \text{ and } f\Theta \vdash_{\overline{B}} \varphi\}.$$

Suppose q_0 is consistent in $\mathbb{K}$ over B. Then, we can extend q_0 to a complete type q_1 over B. Since $q_1 \supset p_f^S$, by lemma 3 (since $(q_1)_B^S \supset (fp)_B^S = p_f^S$) there is a complete type p_1 over B, $p_1 \supset fp$ and $(q_1)_B^S \supset (p_1)_B^S$.

By S_f-closure, it follows that $q_f^S \supset p_f^S$, where $q = q_1 \upharpoonright f$. Hence $\varphi \in q_f^S$, which means there is $\Theta \in q$ such that $f\Theta \vdash_{\overline{B}} \varphi$. But then $\neg f\Theta \in q_0 \subset q_1$ ✗.

Thus q_0 must be inconsistent in $\mathbb{K}$. Therefore there are formulas $\Theta_1, \cdots, \Theta_n$ in $D(A)$ such that

$$f\Theta_i \vdash_{\overline{B}} \varphi,\ 1 \le i \le n, \text{ and } p_f^S \vdash_{\overline{B}} f(\vee_{i=1}^n \Theta_i).$$

Let $\Theta = \vee_{i=1}^n \Theta_i$. We have $\Theta \in D(A)$, $p_f^S \vdash_{\overline{B}} f\Theta$ and $f\Theta \vdash_{\overline{B}} \varphi$. That proves one direction of the assertion.

Conversely, suppose f satisifes the right hand side term of the assertion. Let p_1, q_1 be complete types in $\vec{x}$ over B such that $(p_1)_B^S \supset (q_1)_B^S$; let $p = p_1 \upharpoonright f$ and $q = q_1 \upharpoonright f$.

If $\varphi \in q_f^S$, then, by assumption, there is $\Theta \in D(A)$ such that $q_f^S \vdash_{\overline{B}} f\Theta$ and $f\Theta \vdash_{\overline{B}} \varphi$. Since $q_f^S \subset (q_1)_B^S \subset (p_1)_B^S \subset p_1$ we have that $p_1 \vdash_{\overline{B}} f\Theta$. Therefore, by completeness, Θ must belong to p. It follows that $fp \vdash_{\overline{B}} \varphi$ i.e. $\varphi \in p_f^S$. We conclude $p_f^S \supset q_f^S$, which is what we wanted.

Finally, if S is equational, then given a complete type p over A, p_f^S, if not empty, is equivalent in $\mathbb{K}$ to a single formula φ in $D^S(B)$. Hence, A is S_f-closed in B iff for any complete type p over A with p_f^S non-empty, there is a formula Θ in $D(A)$ such that, $p_f^S \vdash_{\overline{B}} f\Theta \vdash_{\overline{B}} p_f^S$ i.e. $f\Theta \underset{\overline{B}}{\sim} p_f^S$. ▩

In short, lemma 5 says that, for S equational, A is S_f-closed in B iff for any p, complete over f with $p_f^S \neq \emptyset$, p_f^S is "definable" over A.

6. **Proposition.** **Let $A \subset C \in \hat{\mathbb{K}}$, S equational in $\mathbb{K}$. Then, there is B, $A \subset B \subset C$, card$(B) \leq$ card(A) + card(Δ) + $\aleph_0$ and B is S_C-closed in C.**

Proof. We first construct by induction a sequence $(A_i)_{i<\omega}$ such that, for $i < \omega$,

$$A_0 = A,\ A_i \subset A_{i+1},\ \text{card}(A_i) \leq \text{card}(A) + \text{card}(\Delta) + \aleph_0$$

and, for p complete over A, $p^S_C \underset{C}{\approx} p^S_{A_{i+1}}$. Suppose A_i has been constructed, we want to construct A_{i+1}. Observe first that there are at most card(Δ) + card(A_i) non-S_C-equivalent complete types over A_i: indeed, to every complete type p over A_i, one can assign a formula Θ_p in $D(A_i)$ such that $\Theta_p \vdash_C p^S_C$. Since there are at most card(A_i) + card(Δ) such formulas, the claim follows.

Now, for every p^S_C, where p is complete over A_i, there is a finite set $\bar{a}$ such that $p^S_C \underset{C}{\approx} p^S_{A_i \cup \bar{a}}$: for p^S_C is equivalent in $\mathbb{K}$ to a single formula φ in $D^S(C)$. Take then $\bar{a}$ to be the set of parameters in φ.

It easily follows now that we can find a set A_{i+1}, $A_i \subset A_{i+1} \subset C$, card$A_{i+1} \leq$ card(A_i) + card(Δ) $\leq$ card(A) + card(Δ) + $\aleph_0$, and such that for any p, complete over A_i, $p^S_C \underset{C}{\approx} p^S_{A_{i+1}}$. This finishes the construction of A_{i+1} and hence of $(A_i)_{i<\omega}$.

Let $B = \bigcup_{i<\omega} A_i$. Then B satisfies the required properties: clearly B has the right cardinality. Furthermore, if p is a complete type over B, then there is $\Theta \in p$, $\Theta \vdash_C p^S_C$; choose $i < \omega$ so that $\Theta \in p \upharpoonright A_i$; then $p^S_C \underset{C}{\approx} (p \upharpoonright A_i)^S_C$ and therefore by the choice of A_{i+1}, $p^S_C \underset{C}{\approx} (p \upharpoonright A_i)^S_{A_{i+1}}$; thus $p^S_C \underset{C}{\approx} p^S_B$. Hence, for any p complete over B, p_C^S is equivalent in $\mathbb{K}$ to a single formula in $D^S(B)$. We conclude by lemma 5 that B is S_C-closed in C. □

Note. If $A \subset C \in \hat{\mathbb{K}}$ and $S = \bigcup_{i \in I} S_i$ where each S_i is equational, then, using a similar argument to that of proposition 6, one can find a set B, $A \subset B \subset C$, card(B) $\leq$ card$(|A \cup \Delta|^I) + \aleph_0$, and B is S_C-closed in C. (The point to observe is that if $A \subset A' \subset C$ and p is complete over A' then, since $p_C^S \sim \bigcup_{i \in I} p_C^{S_i}$, there is a set $p_0 = \{\theta_i ; i \in I\}$ of formulas in $D(A_i)$ such that $p_0 \vdash_C p_C^S$. Now, cardp_0 $\leq$ card$I$; it follows there are at most card$(|A \cup \Delta|^I)$ non-S_C-equivalent complete types over A'. One constructs then a sequence $(A_\alpha)_{\alpha<\lambda}$, $\lambda = (\text{card}I + \aleph_0)^+$ similar to the sequence $(A_i)_{i<\omega}$ in 6 and shows that $B = \bigcup_{\alpha<\lambda} A_\alpha$ satisfies the required conditions. (In fact, if I is countable, we can still find B with card$B \leq$ card$A$ + card$\Delta + \aleph_0$.)

7. Remark. In fact, it will be sufficient for what follows to consider a weaker notion of S-closure than that given in definition 4. Namely, if the property stated in definition 4 holds whenever we assume in addition that p_1 and q_1 are respectively S_B-minimal extensions of p and q to B.

Such a notion could be interesting if one intends, in the study of K to consider only those types which are realized in K (see 0.5).

We will assign the sign (+) in front of each proposition which uses only this weaker notion. The reader may ignore it if he wishes.

8. **Lemma(+).** Let $A \subset B \in \hat{K}$, A S_B-closed in B. Given a complete type p over A and a type q_0 over A with $p_B^S \supset (q_0)_B^S$, there is a complete type q over A extending q_0 and such that $p_B^S \supset q_B^S$. (Compare with lemma 3)

Proof. Let p_1 be an S_B-minimal extension of p to B. We have $(p_1)_B^S \supset p_B^S \supset (q_0)_B^S$. By lemma 3 there is a complete type q' over B extending q_0 and such that $(p_1)_B^S \supset (q')_B^S$. Let $q = q' \upharpoonright A$. By proposition 2 there is an S_B-minimal extension q_1 of q to B such that $(q')_B^S \supset (q_1)_B^S$.

We have now,

p_1 and q_1 are repectively S_B-minimal extensions of p and q and, $(p_1)_B^S \supset (q_1)_B^S$.

By S_B-closure, it follows that $p_B^S \supset q_B^S$. Since $q \supset q_0$, q satisfies the conditions of the claim. ■

9. **Definition.** Let $A \subset B \in \hat{K}$. Given a type p over A and a

complete type p_1 over A extending p, we say that $(p_1)^S_B$ is an S_B-minimal extension of p^S_B to A if $(p_1)^S_B \supset p^S_B$ and for any complete type q_1 over A,

$$[(q_1)^S_B \supset p^S_B \text{ and } (p_1)^S_B \supset (q_1)^S_B] \Longrightarrow (p_1)^S_B = (q_1)^S_B.$$

If $f: A \longrightarrow B$ is a morphism in $\hat{\mathcal{K}}$, p a type over f and p_1 a complete type over f, we say that $(p_1)^S_f$ is an S_f-minimal extension of p^S_f to A if fp_1, as a (complete) type over fA, and fp (as a type over fA) are such that $(fp_1)^S_B$ is an S_B-minimal extension of $(fp)^S_B$ to fA.

10. Lemma(+). Let $A \subset B \in \hat{\mathcal{K}}$, A S_B-closed in B; p a type over A, p_1 a complete type over A. Then p_1 is S_B-equivalent to an S_B-minimal extension of p to A iff $(p_1)^S_B$ is an S_B-minimal extension of p^S_B to A.

Proof. Suppose p_1 is S_B-equivalent to an S_B-minimal extension p' of p to A. In other words $(p_1)^S_B \underset{B}{\simeq} (p')^S_B$, which in fact implies $(p_1)^S_B = (p')^S_B$.

Suppose now $(p_1)^S_B \supset (q_1)^S_B$ and $(q_1)^S_B \supset p^S_B$, where q_1 is a complete type over A.

By lemma 8, there is p_2, complete over A, such that p_2

extends p and $(q_1)^S_B \supset (p_2)^S_B$. We have now $(p')^S_B = (p_1)^S_B$ $\supset (q_1)^S_B \supset (p_2)^S_B$. Since p_2 extends p and p' is an S_B-minimal extension of p to A we deduce that $(p_2)^S_B = (p')^S_B$, whence $(p_1)^S_B = (q_1)^S_B$. That proves one direction of the claim.

Conversely, suppose $(p_1)^S_B$ is an S_B-minimal extension of p^S_B to A. in particular $(p_1)^S_B \supset p^S_B$. By lemma 8, there is a complete type p' over A extending p and such that $(p_1)^S_B$ $\supset (p')^S_B$. By proposition 2 we can assume p' is an S_B-minimal extension of p to A. By minimality of p_1 we deduce $(p_1)^S_B$ $= (p')^S_B$ i.e. p_1 is S_B-equivalent to p', which is what we wanted. ■

11. **Lemma(+).** Let $A \subset B \in \hat{K}$, A S_B-closed in B; p_0 a type over A, p_1 a complete type over A and p_2 a complete type over B with $p_0 \subset p_1 \subset p_2$.

Then, p_2 is an S_B-minimal extension of p_0 to B iff p_2 is an S_B-minimal extension of p_1 to B and p_1 is an S_B-minimal extension of p_0 to A.

Proof. Suppose p_2 is an S_B-minimal extension of p_0 to B. Then, clearly p_2 is an S_B-minimal extension of p_1 to B

since $p_2 \supset p_1 \supset p_0$;

on the other hand, if $(p_1)^S_B \supset (q_1)^S_B$ where q_1 is a complete extension of p_0 to A, by lemma 8 (since $(p_2)^S_B \supset (q_1)^S_B$) there is a complete extension q_2 of q_1 to B such that $(p_2)^S_B \supset (q_2)^S_B$ (By proposition 2 we can assume q_2 is an S_B-minimal extension of q_1 to B).

Since p_2 is an S_B-minimal extension of p_0 to B, it follows that $(p_2)^S_B = (q_2)^S_B$. By S_B-closure of A we conclude $(p_1)^S_B = (q_1)^S_B$. That shows p_1 is an S_B-minimal extension of p_0 to A.

Conversely, suppose p_2 is an S_B-minimal of p_1 to B and p_1 is an S_B-minimal extension of p_0 to A.

If $(p_2)^S_B \supset (q_2)^S_B$ where q_2 is a complete extension of p_0 to B then, by S_B-closure of A, (note that by proposition 2 we can assume q_2 is an S_B-minimal extension of $q_2 \upharpoonright A$), $(p_1)^S_B \supset (q_1)^S_B$ where $q_1 = q_2 \upharpoonright A$.

But p_1 is an S_B-minimal extension of p_0 and $q_1 \supset p_0$, thus $(p_1)^S_B = (q_1)^S_B$.

By lemma 10, $(p_2)^S_B$ is an S_B-minimal extension of $(p_1)^S_B$ to B. Since $(q_2)^S_B \supset (q_1)^S_B = (p_1)^S_B$, we conclude $(p_2)^S_B = (q_2)^S_B$. That shows p_2 is an S_B-minimal extension of p_0 to

B. ■

12. **Theorem** (Monotonicity-transitivity) (+). Let $A \subset B \subset C \in \mathfrak{K}$, p_0 a type over A, p_1 a complete type over A, p_2 a complete type over B; $p_0 \subset p_1 \subset p_2$. Suppose A and B are S_B-closed in C. Then, p_2 is an S_C-minimal extension of p_0 to B iff p_2 is an S_C-minimal extension of p_1 to B and p_1 is an S_C-minimal extension of p_0 to A.

Proof. Let p_3 be an S_C-minimal extension of p_2 to C. Consider the following statements:

1. p_2 is an S_C-minimal extension of p_0 to B.
2. p_3 is an S_C-minimal extension of p_2 to C and p_2 is an S_C-minimal extension of p_0 to B.
3. p_3 is an S_C-minimal extension of p_0 to C.
4. p_3 is an S_C-minimal extension of p_1 to C and p_1 is an S_C-minimal extension of p_0 to A.
5. p_3 is an S_C-minimal extension of p_2 to C, p_2 is an S_C-minimal extension of p_1 to B and p_1 is an S_C-minimal extension of p_0 to A.
6. p_2 is an S_C-minimal extension of p_1 to B and p_1 is an S_C-minimal extension of p_0 to A.

By the choice of p_3, obviously 1⟺2.

By lemma 11 applied to p_3 we have successively, 2⟺3, 3⟺4 and 4⟺5; 5⟺6 is obvious.

We conclude 1⟺6 which is what we wanted. ■

Note. Given morphisms $f: A \longrightarrow B$ and $g: B \longrightarrow C$ in $\hat{K}$ such that A is $S_{g \cdot f}$-closed in C and B is S_g-closed in C; p_0 a type over $g \cdot f$, p_1 a complete type over $g \cdot f$ and p_2 a complete type over g, $p_2 \supset fp_1 \supset fp_0$, it follows from theorem 12 that p_2 is an S_g-minimal extension of p_0 to f iff p_2 is an S_g-minimal extension of p_1 to f and p_1 is an $S_{g \cdot f}$-minimal extension of p_0 to A. Indeed, consider the types $g \cdot fp_0$ over $g \cdot fA$, $g \cdot fp_1$ over $g \cdot fA$ and gp_2 over gB. gfp_1 and gp_2 are then complete and by assumption $g \cdot fA$ and gB are S_C-closed in C. Thus theorem 12 applies: gp_2 is an S_C-minimal extension of $g \cdot fp_0$ to gB iff gp_2 is an S_C-minimal extension of $g \cdot fp_1$ to gB and $g \cdot fp_1$ is an S_C-minimal extension of $g \cdot fp_0$ to $g \cdot fA$. Translating what preceded in terms of $S_{g \cdot f}$-minimal and S_g-minimal extensions (see definition 1) we obtain what we wanted.

13. **Theorem.** Suppose S is equational in **K**. Let $f: A \longrightarrow B$ be a morphism in $\hat{K}$ and p a type in $\vec{x}$ over f. Then, up to S_f-

equivalence, p has finitely many S_f-minimal extensions to A. In other words, there exist $q_0, \cdots, q_{n-1}$, S_f-minimal extensions of p to A such that for any S_f-minimal extension q of p to A, there is $i < n$, $q^S_f \underset{B}{\approx} (q_i)^S_f$.

In fact, with the q_i's as above, we have

$p^S_f \underset{B}{\approx} \vee_{i<n} (q_i)^S_f$.

Proof. Let Q be the set of S_f-minimal extensions of p to A. Consider the set

$q_0 = p \cup \{\neg\Theta,\ \Theta \in D(A)$ and $f\Theta \vdash_B q^S_f$ for some $q \in Q\}$.

Suppose q_0 is consistent over f. Extend q_0 to a complete type q_1 over f. By proposition 2 (applied to fq_1 over fA) there is an S_B-minimal extension q' of fq_0 to fA such that

$(fq_1)^S_B \supset (q')^S_B$. In other terms, there is an S_f-minimal extension q of q_0 to A, (take $q = q' \upharpoonright f$), such that $(q_1)^S_f \supset q^S_f$.

Since S is equational, $(q_1)^S_f$ is equivalent to a single formula in $D(B)$. Therefore, by definition of $(q_1)^S_f$, there is a formula Θ in q_1 such that $f\Theta \vdash_B (q_1)^S_f$; hence $f\Theta \vdash_B q^S_f$.

But $q \in Q$, it follows, by definition of q_0, that $\neg\Theta \in q_0$, whence $\neg\Theta \in q_1$ ✗.

Thus q_O is inconsistent over f. That means there are $\Theta_0, \cdots, \Theta_{n-1}$, formulas in $D(A)$ such that,

$$f\Theta_i \models_B (q_i)^S_f \text{ for some } q_i \in {}^\circ Q \ (i < n) \text{ and } p \models_f \vee_{i<n}\Theta_i.$$

It follows that for any q, complete over f, q extending p, there is $i < n$ such that $q \models_f \Theta_i$, and therefore $fq \models_B (q_i)^S_f$.

In particular, if q is an S_f-minimal extension of p to A, we find that for some $i < n$, $(q)^S_f = (q_i)^S_f$. We conclude that, up to S_f-equivalence, $q_O, \cdots, q_{n-1}$ enumerate all the S_f-minimal extensions of p to A.

We have now,

$$fp \models_B \vee_{i<n}\Theta_i \models_B \vee_{i<n}(q_i)^S_f,$$

i.e. $p^S_f \models_B \vee_{i<n}(q_i)^S_f$; on the other hand, since $q_i \supset p$,

$$(q_i)^S_f \vdash p^S_f \quad (i < n);$$

therefore, $p^S_f \underset{B}{\sim} \vee_{i<n}(q_i)^S_f$. ■

Remark. If we are to deal only with those types which are *realized* in **K** and define equationality of S only with respect to such types (see 0.5) then, for a similar result to theorem 13 to go through we need the following conditions:

$A \in \hat{\mathbf{K}}$ and p is a type over A such that for any S_A-minimal extension q of p to A, $q{\restriction}S$ is realized in **K** over A (recall $q{\restriction}S = \{\varphi(\vec{x};\vec{a}) \in q;\ \varphi(\vec{x};\vec{t}) \in cl(S)\}$.

The following is then true: p has, up to S_A-equivalence, finitely many S_A-minimal extensions to A which are realized in **K** over A.

Examples.

a) Let L consist of one binary relation $<$; $S = cl^+((x > t))$. Define the usual order on ${}^{<\omega}2$: $\eta < \nu$ if η is an initial segment of ν. Let $M = ({}^{<\omega}2, <)$ be the structure thus obtained.

Let Δ be the set of all formulas in L and let **K** be the Δ-category of models of $Th(M)$ with elementary embeddings.

Clearly S is not equational in **K**.

Consider the type over $(\emptyset, M)$,

$$p = \{\exists t[(h(t) > n) \wedge (x > t)];\ n < \omega\}$$

where $(h(t) > n)$ means $\exists t_0 \cdots t_n\ t_0 < t_1 < \cdots < t_n < t$.

It is easy to see that for each $\eta \in {}^{\omega}2$, the type

$$p_\eta = \{x > \eta \upharpoonright n;\ n < \omega\}$$

determines an S_M-minimal extension of p to M.

Furthermore, for $\eta, \nu \in {}^{\omega}2$, $\eta \neq \nu$ implies $p_\eta \cup p_\nu$ is inconsistent in **K**. It follows that p has 2^{ω} S_M-minimal extensions to M.

b) Let T be the theory of linear orders, Δ the set of all quantifier-free formulas (in $L = \{<\}$) and **K** the Δ-

category of models of T with embeddings for morphisms.

Let $S = cl^{+}\{(x > t)\}$, $S' = cl^{+}\{(x > t), (x < t)\}$.

Clearly, neither S nor S' is equational in **K**. Let $M \in$ **K** and p a $(\Delta-)$-complete type in x over M which is not realized in M.

p is then equivalent in **K** to a type p_0 of the form

$$p_0 = \{(x > a);\ a \in I\} \cup \{(x < b);\ b \in J\}$$

where I and J are subsets of M such that $a < b$ whenever $a \in I$ and $b \in J$.

Now let $f:M \longrightarrow N$ be a morphism in **K**. p has a unique S_M-minimal extension to f, namely the type

$$p_1 = \{(x > fa);\ a \in I\} \cup \{(x < b);\ b \in N,\ b > fa \text{ for all } a \text{ in } I\}.$$

However, any complete type over N extending fp is an S'_f-minimal extension of p to f.

c) Let $L = \{<,+,\cdot,0,1\}$, Δ the set of quantifier-free formulas in L and **K** the Δ-category of ordered fields with ordered field embeddings for morphisms.

Let S be the set of algebraic equations in L. S is equational (for the same reasons as for **K** = **F**). Let $f:H \longrightarrow F$ be a morphism in **K**. Consider the type p over H,

$$p = \{(x > o),\ (P(x) = 0)\}$$

where $P(x)$ is some polynomial with coefficients in H, such that $P(x)$ has a positive root in some extension of

H.

First, note that p is a type over f (for p has a root in the real closure of H, whence p has a root in the real closure of F).

Now, the S_F-minimal extensions of p to f are the types of the form

$$p_1 \underset{F}{\simeq} \{(x > 0), (Q(x) = 0)\}$$

where $Q(x)$ is an irreducible component of $P(x)$ in F which has a positive root in some extension of F.

One observes here that there is a morphism $g:F \longrightarrow G$, for instance the embedding of F into its real closure, such that the S_G-minimal extensions of p to $g \cdot f$ are the types of the form,

$$p_{a,} \underset{G}{\simeq} \{(x = a);\ a \in G,\ a > 0 \text{ and } a \text{ is a root of } P\}.$$

Of course then, for any morphism $e:G \longrightarrow E$, p_a has a unique S_E-minimal extension to e.

The observation above will be seen in section B to follow from a general statement (see theorem B.8).■

Suppose $S = S_\omega = \bigcup_{n<\omega} S_n$ where, for every $n < \omega$, S_n is an equational set of formulas, $S_n = cl^+(S_n)$, and $S_n \subset S_{n+1}$. Let $f:A \longrightarrow B$ be a morphism in $\hat{K}$ and p a type in $\vec{x}$ over f. Consider the following partially-ordered set $(T, \leq)$:

$T = \{(q,n);\ n \le \omega,\ q$ is an $(S_n)_f$-minimal extension of p to $A\} \cup \{(p,-1)\}$;

for $0 \le n,m \le \omega$, $(q,n) \le (q',m)$ if $n \le m$ and $(q)_f^{S_\bullet} \subseteq (q')_f^{S_\bullet}$;

$(p,-1) \le (q,n)$ for any $0 \le n \le \omega$.

(So $(q,n) \approx (q',m)$ $0 \le n,m \le \omega$, if $n = m$ and $(q)_f^{S_\bullet} = (q')_f^{S_\bullet}$).

$(T,\le)$ satisfies the following properties:

1. Given $-1 \le n < m \le \omega$ and $(q,n) \in T$ there is $(q',m) \in T$ such that $(q,n) \le (q',m)$.
2. Given $-1 \le n < m \le \omega$, and $(q',m) \in T$ there is $(q,n) \in T$ such that $(q,n) \le (q',m)$.
3. $(T,\le)$ has finitely many elements at every finite level.

Indeed, given $-1 \le n < m \le \omega$ and $(q,n) \in T$, by proposition 2 there is an $(S_m)_f$-minimal extension q' of p to A such that $(q)_f^{S_\bullet} \supset (q')_f^{S_\bullet}$. But then $(q)_f^{S_\bullet} \supset (q')_f^{S_\bullet}$ and, since q is assumed an $(S_n)_f$-minimal extension of p, we deduce that $(q)_f^{S_\bullet} = (q')_f^{S_\bullet}$. Thus $(q',m) \in T$ and $(q,n) \le (q',m)$. That proves 1.

Given $-1 \le n < m \le \omega$ and $(q',m) \in T$, by proposition 2 there is an $(S_n)_f$-minimal extension of p to A such that $(q')_f^{S_\bullet} \supset (q)_f^{S_\bullet}$. So $(q,n) \in T$ and $(q,n) \le (q',m)$. That proves 2.

It follows from property 2 that if $(q,n) \in T$ then (q,n) is at the $n+1$-th level, i.e. the $(n+1)$-th level in T is constituted of the $(S_n)_f$-minimal extensions of p to A. By theorem 13 we deduce property 3.

Remark. We will see in chapter III that for K the category of models of a complete theory with elementary embeddings, if p is a complete type over a subset of A then $(T,\leq)$ has more precise properties as for instance the fact that $(T,\lesssim)$ is a tree (c.f. III.14) (i.e. $(q,n) \not\lesssim (q_1,n)$ implies there is no (q',m) in T such that $(q,n) \leq (q',m)$ and $(q_1,n) \leq (q',m)$). This will follow from the fact that, in that case, if (q,n) and (q_1,n) belong to T then (q,n) and (q_1,n) are in some sense conjugate of each other.

Also, lemma 14 below will have more precise versions (c.f. III.17).

14. **Lemma.** Let $S = \cup_{n<\omega} S_n$, $f: A \longrightarrow B$ and p be given as above. Suppose q_ω is an S_f-minimal extension of p to A. Then there is a sequence $(q_n)_{n<\omega}$ of types over f such that,

for $n<\omega$, q_n is an $(S_n)_f$-minimal extension of p to A,

$(q_n)_f^{S_n} \subset (q_{n+1})_f^{S_{n+1}}$ and $(q_\omega)_f^S = \cup_{n<\omega} (q_n)_f^{S_n}$.

Proof. Let $(T,\leq)$ be defined as above; clearly $(q_\omega,\omega) \in T$.

Let

$$(T_0,\le) = \{(q,n) \in T;\ (q,n) \le (q_\omega,\omega)\}.$$

By property 2 of $(T,\le)$, T_0 is infinite.

From property 3 of $(T,\le)$ (see above) we deduce that $(T_0,\le)$ has finite branching at finite levels. It follows by König's lemma that $(T_0,\le)$ admits an infinite branch $\mathcal{E}$.

Write q_n' for q if $(q,n) \in \mathcal{E}$. Let

$$q' = p \cup \{\neg\Theta;\ \Theta \in D_x^{+}(A) \text{ and } f\Theta \models_B \varphi \text{ for some } \varphi \text{ in } D^S(B) \setminus \bigcup_{n<\omega}(q_n)_f^{S}\}.$$

Claim. q' is consistent over f. For if not, there are $\Theta_0,\cdots,\Theta_{k-1}$ in $D(A)$ and $\varphi_0,\cdots,\varphi_{k-1}$ in $D^S(B)\setminus \bigcup_{n<\omega}(q_n)_f^{S}$ such that

$$f\Theta_i \models_B \varphi_i \quad (i < k) \text{ and } p \models_f \vee_{i<k}\Theta_i.$$

Let $m < \omega$ be such that $\varphi_i \in D^{S_m}(B)$ for all $i < k$. Since $q_m \supset p$, we have that $q_m \models_f \vee_{i<k}\Theta_i$; by completeness and since q_m is consistent over f, it follows that $q_m \models_f \Theta_i$ for some $i < k$; hence $fq_m \models_B \varphi_i$ i.e. $\varphi_i \in (q_m)_f^{S_m}$ ✗.

Let q'' be a complete type over f extending q'. We clearly have $(q'')_f^S \subset \bigcup_{n<\omega}(q_n)_f^{S}$.

So $(q'')_f^S \subset \bigcup_{n<\omega}(q_n)_f^{S} \subset q_f^S$. By minimality of q we conclude that $(q'')_f^S = q_f^S$, hence $q_f^S = \bigcup_{n<\omega}(q_n)_f^{S}$. ■

15. **Proposition.** Suppose $S = cl^{+}(\cup_{i<\omega} T_i)$ where each T_i is an equational set of formulas. Given a morphism $f: A \longrightarrow B$ in $\mathfrak{K}$ and a type p in $\vec{x}$ over f there are at most 2^{ω} non-S_f-equivalent S_f-minimal extensions of p to A.

Proof. Let $S_n = cl^{+}(\cup_{i<n} T_i)$; then, by proposition I.7.(iii), S_n is equational, $S_n \subset S_{n+1}$ and $S = \cup_{n<\omega} S_n$. By lemma 14, if q is an S_f-minimal extension of p to A then $q_f^S = \cup_{n<\omega} (q_n)_f^{S_n}$ where each q_n is an $(S_n)_f$-minimal extension of p to A. By proposition 13, for each $n < \omega$, p has finitely many non-$(S_n)_f$-equivalent $(S_n)_f$-minimal extensions to A. We conclude that p has at most 2^{ω} non-S_f-equivalent S_f-minimal extensions to A. ■

[Note. In fact, with p and f as in the proposition above but with $S = cl^{+}(\cup_{i \in I} T_i)$, where I is an arbitrary infinite index set and T_i is equational $(i \in I)$, one can show there are at most $2^{|I|}$ non-S_f-equivalent S_f-minimal extensions of p to A].

16. **Theorem.** Suppose $S = cl^{+}(\cup_{i \in I} S_i)$ where for each $i \in I$, S_i

is an equational set of formulas. Given a morphism $f: B \longrightarrow C$ in $\mathbb{K}$ and a complete type p in $\vec{x}$ over f, there is a set $A \subset B$, of cardinality at most card(I) if I is infinite and finite if I is finite, such that p is, up to S_f-equivalence, the unique S_f-minimal extension of $p \upharpoonright A$ to B.

Proof. Let I^* be the set of finite subsets of I. Clearly $S = \bigcup_{J \in I^*} S_J$, where $S_J = \mathrm{cl}^+ (\bigcup_{i \in J} S_i)$, and $p_f^S = \bigcup_{J \in I^*} p_f^{S_J}$. Now, by proposition I.7.(iii), S_J is equational for $J \in I^*$; thus, there is a single formula Θ_J in $D(B)$ such that $f\Theta_J \vdash_C p_f^{S_J}$.

Let A be the set of parameters occuring in the formulas Θ_J, $J \in I^*$. A is clearly finite if I is finite and of cardinality at most card$I$ if I is infinite.

On the other hand, $f(p \upharpoonright A) \vdash_C p_f^S$. It easily follows that, up to S_f-equivalence, p is the unique S_f-minimal extension of $p \upharpoonright A$ to B. ■

Example. $L = \{<\}$, $<$ a binary relation, Δ the set of quantifier-free formulas in L, $\mathbb{K}$ the Δ-category of linearly-ordered structures with embeddings for morphisms;

$S = \mathrm{cl}^+(\{(x > t)\})$.

Clearly S is not equational in $\mathbb{K}$.

Let $H \in \mathbb{K}$. The set $\{(x > a);\ a \in H\}$ determines a unique (Δ–) complete type p over H.

One easily checks that p is an S_H-minimal extension of $p \upharpoonright A$, for $A \subset H$, iff A is cofinal in H (i.e. $\forall b \in H\ \exists a \in A\ (a \geq b)$).

Thus, if H admits no cofinal subsets of strictly less cardinality, then there is no set $A \subset H$ such that card(A) < card(H) and p is an S_H-minimal extension of $p \upharpoonright A$ to H.

Section B: Irreducibility

0. **Definition.** Let $A \in \hat{K}$ and Σ a set of formulas in $D(A)$ which is closed under finite conjunctions and disjunctions. Let p be a type or just a single formula over A. We say p is Σ-*irreducible* if for any φ_1, φ_2 in Σ,

$$(p \vdash_{\bar{A}} \varphi_1 \vee \varphi_2) \Longrightarrow (p \vdash_{\bar{A}} \varphi_1, \text{ or } p \vdash_{\bar{A}} \varphi_2);$$

p is said Σ-full if p is Σ-irreducible and for any φ in Σ,

$$(p \vdash_{\bar{A}} \varphi) \Longrightarrow (p \cap \Sigma \vdash_{\bar{A}} \varphi).$$

Example. Let $\mathbf{K} = \mathbf{F}$ (the category of fields), $F \in \mathbf{K}$ and p a type in x over F. Let Σ be the set of algebraic equations with coefficients in F. Then, with $Q_p(x)$ the polynomial over F

of smallest degree such that $p \vdash_F (Q_p(x) = 0)$, p is Φ-irreducible iff $Q_p(x)$ is an irreducible polynomial in $F[x]$; p is Φ-full if Q_p' is irreducible and $(a.Q_p(x) = 0) \in p$ for some non-zero element in F.

For A, Φ, p as in definition 0 let us say that p is Φ-complete if for any φ in Φ, either φ or $\neg\varphi$ belongs to p.

Let $p^{\Phi} = \{\varphi \in \Phi;\ p \vdash_A \varphi\}$.

1. **Lemma.** A and Φ as in definition 0. Let p be a type in $\vec{x}$ over A. Then p is Φ-irreducible iff there is a complete type q over A extending p and such that $q^{\Phi} \underset{A}{\sim} p^{\Phi}$ iff there is a Φ-complete type q over A extending p and such that $q^{\Phi} \underset{A}{\sim} p^{\Phi}$.

Proof. Let $q_1 = p \cup p^{\Phi} \cup \{\neg\varphi;\ \varphi \in \Phi^{\vec{x}},\ \varphi \notin p^{\Phi}\}$.

q_1 is consistent in K iff for any formulas $\varphi_0, \cdots, \varphi_{n-1}$ in $\Phi^{\vec{x}}$, if $\varphi_i \notin p^{\Phi}$ $(i < n)$ then $p \cup p^{\Phi} \nvdash_A \vee_{i<n}\varphi_i$. In other words, since $p \vdash_A p^{\Phi}$, q_1 is consistent in K iff for any formulas $\varphi_0, \cdots, \varphi_{n-1}$ in $\Phi^{\vec{x}}$, if $p \vdash_A \vee_{i<n}\varphi_i$ then $p \vdash_A \varphi_i$ for some $i < n$ i.e. iff p is Φ-irreductible.

Now, any complete (resp. Φ-complete) type q over A extending p and such that $q^{\Phi} \underset{A}{\sim} p^{\Phi}$ must extend q_1. On the other hand, any complete (resp. Φ-complete) type q over A extending q_1 is such that $q^{\Phi} \underset{A}{\sim} q_1^{\Phi} \underset{A}{\sim} p^{\Phi}$. The claim has now become clear. ■

Note. With the notation of lemma 1 p is Φ-irreducible iff p^{Φ} is Φ-irreducible: since, Φ being closed under disjunctions, for any φ_1, φ_2 in Φ, $p \vdash_{\overline{A}} \varphi_1 \vee \varphi_2$ iff $p^{\Phi} \vdash_{\overline{A}} \varphi_1 \vee \varphi_2$.

2. **Definition.** Given a morphism $f: A \longrightarrow B$ in $\hat{\mathfrak{K}}$ and Θ, a formula in $D(B)$, we say:

Θ is S-definable if there is a formula φ in $D^S(B)$ such that $\Theta \underset{B}{\sim} \varphi$;

Θ is f-definable if there is a formula ψ in $D(A)$ such that $\Theta \underset{B}{\sim} f\psi$;

Θ is (S,f)-definable if Θ is S-definable and f-definable.

If f is an inclusion map, we say Θ is A-definable. (resp. (S,A)-definable) instead of Θ is f-definable (resp. (S,f)-definable).

3. **Lemma.** If $f: A \longrightarrow B$ is in $\hat{\mathfrak{K}}$, A is S_B-closed in B, S is equational and p is a type over A then p_f^S is (S,f)-definable.

Proof. Let f, A and p satisfy the hypothesis of the lemma. By theorem A.13, p has, up to S_f-equivalence, finitely many S_f-minimal extensions, say $p_0, \cdots, p_{n-1}$, and $p_f^S \underset{B}{\cong} \vee_{i<n} (p_i)_f^S$. By lemma A.5, since p_i $(i < n)$ is a complete type and A is S_f-closed in B, there is a formula Θ_i in $D(A)$ such that $(p_i)_f^S \underset{B}{\cong} f\Theta$. Thus $p_f^S \underset{B}{\cong} f(\vee_{i<n}\Theta_i)$, which implies that p_f^S is (S,f)-definable. ■

4. **Definition.** Let $f: A \longrightarrow B$ be a morphism in $\hat{K}$, p a type over f.

(i) We say p is (S,f)-*irreducible* if fp is Σ-irreducible for Σ the set of (S,f)-definable formulas in $D(B)$.

(ii) p is S_f-*irreducible* (resp. S_f-*full*) if fp is Σ-irreducible (resp. Σ-full) for $\Sigma = D^S(B)$.

If f is an inclusion map, we say p is (S,A)-irreducible (resp. S_B-irreducible, S_B-full) instead of p is (S,f)-irreducible (resp. S_f-irreducible, S_f-full).

Note. With the notations above, p is S_f-irreducible iff fp is (S,B)-irreducible.

Also, p is S_f-irreducible $\Longrightarrow$ p is (S,f)-irreducible.

5. Proposition. $f: A \longrightarrow B$ a morphism in $\mathcal{K}$, p a type over f.

(i) Assuming A is S_f-closed in B and S is equational, p is (S,f)-irreducible iff p has, up to S_f-equivalence, a unique S_f-minimal extension q to A, and in fact q is such that $q_f^S \cong_B p_f^S$, iff there is a complete type q over f extending p with $q_f^S \cong_B p_f^S$.

(ii) p is S_f-irreducible iff p has, up to S_f-equivalence, a unique S_B-minimal extension q to f; and q is such that $q_f^S \cong_B p_f^S$. If in addition p is S_f-full then q is such that $q_B^S \cong_B f(p \cap D^S(A))$.

Proof.

(i) Let Φ be the set of (S,f)-definable formulas in $D(B)$.

Suppose p is (S,f)-irreducible. By definition, it means that fp is Φ-irreducible. Let $q_0, \cdots, q_{n-1}$ be the S_f-minimal extensions of p to A; by theorem A.13 we have $p_f^S \cong_B \vee_{i<n} (q_i)_f^S$. Hence $fp \vdash_B \vee_{i<n} (q_i)_f^S$. By lemma 3 each $(q_i)_f^S$ is equivalent to a formula in Φ; it follows by Φ-irreducibility that $fp \vdash_B (q_i)_f^S$ for some $i < n$. Clearly then, q_i is, up to S_f-equivalence, the unique S_f-minimal extension of p to A; and since $(q_i)_f^S \vdash_B p_f^S$, we have $(q_i)^S_f \underset{B}{\simeq} p^S_f$.

Suppose p has, up to S_f-equivalence, a unique S_f-minimal extension q to A then, by theorem A.13, $p^S_f \underset{B}{\simeq} q^S_f$. Thus, there is a complete type q over A, $q \supset p$, such that $p^S_f \underset{B}{\simeq} q^S_f$.

Finally, suppose there is a complete type q over f extending p such that $p^S_f \underset{B}{\simeq} q^S_f$.

If $fp \models_{\overline{B}} \varphi_1 \vee \varphi_2$, where $\varphi_1, \varphi_2 \in \Phi$ then $fq \models_{\overline{B}} \varphi_1 \vee \varphi_2$.

φ_1, φ_2 being in Φ, they are equivalent to formulas in $D(A)$; it follows by completeness that $fq \models_{\overline{B}} \varphi_i$ for $i = 1$ or 2.

φ_1, φ_2 being in Φ, they are equivalent to formulas in $D^S(B)$; hence, $(fq)^S_B \models_{\overline{B}} \varphi_i$ i.e. $q^S_f \models_{\overline{B}} \varphi_i$. Therefore $p^S_f \models_{\overline{B}} \varphi_i$. We conclude that p is (S,f)-irreducible.

(ii) We have already noted above that p is S_f-irreducible iff fp is (S,B)-irreducible i.e. iff fp is (S,id_B)-irreducible. By (i) (applied to fp and id_B) fp is (S,id_B)-irreducible iff fp has a unique S_B-minimal extension of q to B, and q is such that $q^S_B \underset{B}{\simeq} (fp)^S_B$, i.e. iff p has a unique S_B-minimal extension to f, and $q^S_B \underset{B}{\simeq} p^S_f$.

We conclude that p is S_f-irreducible iff p has a unique S_B-minimal extension q to f, and in fact $q_B^S \underset{B}{\simeq} p_f^S$.

If p is actually S_f-full then $p_f^S \underset{B}{\simeq} f(p \cap D^S(A))$; so the S_B-minimal extension of p to f is such that $q_B^S \underset{B}{\simeq} f(p \cap D^S(A))$.■

6. **Proposition.** Let $f: A \longrightarrow B$ be in $\hat{K}$, Σ the set of (S,f)-definable formulas in $D(B)$, p a type over f. Assume S is equational and A is S_f-closed in B. Let $\Theta_0, \cdots, \Theta_{n-1}$ be Σ-irreducible formulas in Σ and let $p_0, \cdots, p_{n-1}$ be complete types over f such that $(p_i)_f^S \underset{B}{\simeq} \Theta_i$, $i < n$.

Then, $p_0, \cdots, p_{n-1}$ are, up to S_f-equivalence (all) the distinct S_f-minimal extensions of p to A iff $p_f^S \underset{B}{\simeq} \bigvee_{i<n}\Theta_i$ and $\Theta_i \not\models_B \Theta_j$ $(i \neq j)$.

Proof. By theorem A.13, if $p_0, \cdots, p_{n-1}$ are the distinct S_f-minimal extensions of p to A then $p_f^S \underset{B}{\simeq} \bigvee_{i<n}(P_i)_f^S$ i.e. $p_f^S \underset{B}{\simeq} \bigvee_{i<n}\Theta_i$; and of course, since $(p_i)_f^S \not\models_B (p_j)_f^S$ $(i \neq j)$, we have $\Theta_i \not\models_B \Theta_j$ $(i \neq j)$.

Conversely, suppose $p_f^S \underset{B}{\simeq} \bigvee_{i<n}\Theta_i$ and $\Theta_i \not\models_B \Theta_j$ $(i \neq j)$. Let q be an S_f-minimal extension of p to A; we have that

$fq \vdash_B \vee_{i<n}\Theta_i$. Now Θ_i, being in Ξ, is equivalent to a formula over fA; hence, fq being complete over fA, $fq \vdash_B \Theta_i$ for some $i < n$ i.e. $q_f^S \supset (p_i)_f^S$. It follows, by minimality, that $q_f^S = (p_i)_f^S$.

Thus, the S_f-minimal extensions of p to A are among the p_i's ($i < n$). Suppose one of the p_i's, say p_0, is not an S_f-minimal extension of p; then, by proposition A.2 there is an S_f-minimal extension q of p to A such that $(p_0)_f^S \supset q_f^S$. By what preceded, there is i, $0 < i < n$, such that $q_f^S \underset{B}{\simeq} (p_i)_f^S$. We have then $(p_o)_f^S \supset (p_i)_f^S$ i.e. $\Theta_0 \vdash_B \Theta_i$ $\times$, ∎

Note. Let f and Ξ be as in proposition 6; S equational and A S_f-closed in B.

Given Θ in Ξ there is Ψ in $D(A)$ such that $\Theta \underset{B}{\simeq} f\Psi$. Thus, Θ is Ξ-irreducible iff Ψ is (S,f)-irreducible iff (by proposition 5.(i)) there is a complete type p over f such that $p_f^S \underset{B}{\simeq} \Psi_f^S \underset{B}{\simeq} (f\Psi)_B^S \underset{B}{\simeq} \Theta_B^S$. But Θ being S-definable $\Theta_B^S \underset{B}{\simeq} \Theta$.

We conclude that Θ is Ξ-irreducible iff there is a complete type p over f such that $p_f^S \underset{B}{\simeq} \Theta$. So, in proposition 6 above, the existence of the types p_i was already ensured by the assumptions on Θ_i.

7. **Definition.** Let $A \in \hat{\mathcal{K}}$, p a type over A.

(i) We say p is *S-irreducible* (resp. *S-full*) if p is S_f-irreducible (resp. S_f-full) for any morphism $f: A \longrightarrow B$ in $\hat{\mathcal{K}}$ such that p is consistent over f.

(ii) Let $f: A \longrightarrow B$ be a morphism in $\hat{\mathcal{K}}$ such that p is over f; q a complete type over B. We say q_B^S is an *S_f-component* of p, or that q_B^S is an S_B-component of p in case f is an inclusion map, if $q_B^S \supset p_f^S$, q_B^S is an S_B-minimal extension of $p_f^S (= (fp)_B^S)$ to B (see definition A.2) *and* $fp \cup q_B^S$ is S-full.

Example. Let $L = \{R_i(x;t);\ i < \omega\}$, Δ the set of all formulas in L and T the theory such that, M is a model of T if:

1. For any $a \in M$, $R_0(M;a) = M$.
2. For $a, b \in M$, $j \geq i$,

$$R_j(M;a) \cap R_i(M;b) \neq \emptyset \Longrightarrow R_j(M;a) \subset R_i(M;b).$$

3. For any a, $b \in M$, $j \neq i$, $R_j(M;a) \neq R_i(M;b)$.
4. For any a, b, $c \in M$, if $R_{i+1}(M;a) \subset R_i(M;c)$, $R_{i+1}(M;b) \subset R_i(M;c)$ and $R_{i+1}(M;a) \neq R_{i+1}(M;b)$ then $R_i(M;c) = R_{i+1}(M;a) \cup R_{i+1}(M;b)$.
5. For any $a \in M$, $j > i$, there is $b \in M$ such that

$R_j(M;a) \subset R_i(M;b)$.

Let **K** be the Δ-category of models of T with embeddings for morphisms; $S = cl^+(\{R_i(x;t);\ i < \omega\})$, $S_i = cl^+(R_i)$. Clearly S is not equational, while R_i and hence S_i is equational for every $i < \omega$ (for it follows from 2 that for any a, b in M, either

$R_i(M;a) = R_i(M;b)$ or $R_i(M;a) \cap R_i(M;b) = \emptyset$).

Let M be a structure in **K** in which there is a sequence $(a_i)_{i<\omega}$ such that $p = \{R_i(x;a_i);\ i < \omega\}$ is consistent; let $p_i = \{R_i(x;a_i)\}$.

Given $f:M \longrightarrow N$ in **K**, we see that p_i is $(S_j)_f$-irreducible $(j > i)$ iff $R_i(N;fa_i)$ cannot be written in N as a disjunction of 2^{j-i} distinct R_j-definable sets iff $R_i(N;fa_i)$ does not contain 2^{j-i} distinct R_j-definable sets.

It follows that p_i is not S_j-irreducible, for we can always find a morphism $f:M \longrightarrow N$ in **K** such that $R_i(N;fa_i)$ contains 2^{j-i} distinct R_j-definable sets.

p however is S-full; for if $f:M \longrightarrow N$ is a morphism in **K** with p consistent over f, and a, b are elements in N such that $fp \vdash_N R_j(x;a) \vee R_i(x;b)$ for some i, j then either $fp \cup \{R_j(x;a)\}$ is consistent in **K**, in which case $R_j(x;fa_j) \underset{N}{\sim} R_j(x;a)$, whence $f(p \cap D^S(M)) \vdash_N R_j(x;a)$, or

$fp \cup R_i(x;b)$ is consistent in which case

$f(p \cap D^S(H)) \vdash_H R_i(x;b)$.

Now, one can check that p^S_H is an S_H-minimal extension of p_i to H if for any $j > i$, $R_i(H;a_i)$ contains 2^{j-i} R_j-definable subsets, and in that case p^S_H is an S_H-component of p.

8. **Remarks.**

a) In definition 7.(ii) above, we can assume without loss of generality that q extends p. For we know (by lemma A.10 applied to fp and q as types over B, and since B is S_B-closed in B) that q^S_B is an S_B-minimal extension of p^S_f to f iff q is S_B-equivalent to an S_B-minimal extension of p to f.

b) If $f:A \longrightarrow B$ is in $\hat{K}$ and p is a type over f which is S_f-full, it is easy to extend p to a complete type q over f such that q is S_f-full and $q^S_f \cong_B p^S_f$: indeed, take q to be an S_f-minimal extension of p to A; by proposition 5 (i) q is such that $q^S_f \cong_B p^S_f$.

However, unless K has the amalgamation property and K is Δ-elementary, if p is a type over A which is S-full, it does not necessarily follow that p can be extended to a complete type q over A such that q is S-full and $q^S_A \underset{A}{\approx} p^S_A$. This explains why in definition 7.(ii) we only request $fp \cup q^S_B$ to be S-full.

In case $\mathbb{K}$ has the A.P., $\mathbb{K}$ is Δ-elementary and p is an S-full type over A. We let Φ be the set of formulas Θ in $D(A)$ for which there is a morphism $g:A \longrightarrow C$ in $\hat{\mathbb{K}}$ and a formula φ in $D(C)$ such that $g\Theta \vdash_{\overline{C}} \varphi$ and $gp \not\vdash_{\overline{C}} \varphi$.

Let $q_0 = p \cup \{\neg\Theta;\ \Theta \in \Phi\}$.

Claim. q_0 is consistent in $\mathbb{K}$ over A. For suppose $p \vdash_{\overline{A}} \bigvee_{i<n}\Theta_i$; $\Theta_i \in \Phi$ $(i < n)$. Let $g:A \longrightarrow C_i$ be morphisms and φ_i be formulas in $D(C_i)$ such that $g\Theta_i \vdash_{\overline{C_i}} \varphi_i$ and $g_i p \not\vdash_{\overline{C_i}} \varphi_i$ $(i < n)$.

By A.P. we can find morphisms $d_i:C_i \longrightarrow D$ such that $d_i \cdot g_i = d_j \cdot g_j$ $(i, j < n)$. It follows that

$$d_i \cdot g_i p \vdash_{\overline{D}} \bigvee_{i<n} d_i \cdot g_i \Theta_i \vdash_{\overline{D}} \bigvee_{i<n} d_i \varphi_i.$$

Therefore, by S-irreducibility, $d_i g_i p \vdash_{\overline{D}} d_i \varphi_i$ for some $i < n$; hence, by Δ-reflection (c.f. 0.2), $g_i p \vdash_{\overline{C_i}} \varphi_i$ ✗. (Note that we have used all along the fact (c.f. I.4) that if r is a type over H and $f:H \longrightarrow F$ is in $\mathbb{K}$ then r is consistent over f). That proves the claim.

Now, let q be a complete type over A extending q_0. it is easy to see that q is S-full and that in fact, for any morphism $g:A \longrightarrow C$,

$$q_f^S \underset{C}{\simeq} p_f^S \underset{C}{\simeq} f(p \cap D^S(A)).$$

So, if K has the A.P. and K is Δ-elementary, we can take the following as a definition of S_f-components: given $f:A \longrightarrow B$, p a type over f and q a complete type over B, we say q is an S_f-component of p if q is an S_B-minimal extension of p to f which is S-full.

c) With the notations of 7.(ii), if q_B^S is an S_f-component of p then for any morphism $g:B \longrightarrow C$ such that $fp \cup q_B^S$ is consistent over g we have

$$(fp \cup q_B^S)_g^S \underset{C}{\simeq} g(q_B^S).$$

Indeed, since $fp \cup q_B^S$ is S-full,

$$(fp \cup q_B^S)_g^S \underset{C}{\simeq} g[(fp \cup q_B^S) \cap D^S(B)];$$

but $(fp \cup q_B^S) \cap D^S(B) = (fp \cap D^S(B)) \cup q_B^S = q_B^S$ (since $fp \cap D^S(B) \subset p_f^S \subset q_B^S$); we conclude,

$$(fp \cup q_B^S)_g^S \underset{C}{\simeq} g(q_B^S).$$

In particular $(fp \cup q_B^S)_B^S = q_B^S$.

9. **Proposition.** Given morphisms $f:A \longrightarrow B$ and $g:B \longrightarrow C$ in $\mathbb{K}$, p a type over $g \cdot f$, p_1 a complete type over g extending p, and p_2 a complete type over C such that, $(p_1)_B^S$ is an S_f-

component of p and $(p_2)^S_C$ is an S_C-minimal extension of $(p_1)^S_B$ (or equivalently (see the proof) p_2 is an S_C-minimal extension of $fp \cup (p_1)^S_B$) to g *then* $(p_2)^S_C$ is an $S_{g\cdot f}$-component of p.

Proof. Suppose p, p_1, p_2 satisfy the hypothesis of the proposition.

Note first that by remark 8.c),

$$(fp \cup (p_1)^S_B)^S_g \underset{C}{\tilde{=}} g[(p_1)^S_B]$$

$$\text{i.e. } [g(fp \cup (p_1)^S_B)]^S_C \underset{C}{\tilde{=}} g[(p_1)^S_B];$$

so $(p_2)^S_C$ is an S_C-minimal extension of $[g(fp \cup (p_1)^S_B)]^S_C$ to C; by lemma A.10 (since C is S_C-closed in C) it follows that p_2 is S_C-equivalent to an S_C-minimal extension of $fp \cup (p_1)^S_B$ to g.

Therefore, without loss of generality, we can assume p_2 is an S_C-minimal extension of $fp \cup (p_1)^S_B$ to g.

By proposition 3.(ii), since $fp \cup (p_1)^S_B$ is S-full, we must have

$$(p_2)^S_C \underset{C}{\tilde{=}} (fp \cup (p_1)^S_B)^S_g \underset{C}{\tilde{=}} g[(p_1)^S_B].$$

It follows:

1. $(p_2)^S_C \supset p^S_{g\cdot f}$: since

$$(p_2)^S_C \underset{C}{\tilde{=}} (fp \cup (p_1)^S_B)^S_g \supset (fp)^S_g.$$

2. $g \cdot fp \cup (p_2)^S_C$ is S–full: since

$$g \cdot fp \cup (p_2)^S_C \underset{C}{\simeq} g \cdot fp \cup g[(p_1)^S_C] \underset{C}{\simeq} g[fp \cup (p_1)^S_C]$$

and $fp \cup (p_1)^S_C$ is S–full.

3. $(p_2)^S_C$ is an S_C–minimal extension of $p^S_{g \cdot f}$: for suppose $(p_2)^S_C \supset (q_2)^S_C$, q_2 a complete type over C such that $(q_2)^S_C \supset p^S_{g \cdot f}$; since $(p_2)^S_C \underset{C}{\simeq} g[(p_1)^S_B]$ we get $(p_1)^S_B \supset (q_1)^S_B \supset p^S_f$ where $q_1 = q_2 \restriction g$; but $(p_1)^S_B$ is an S_B–minimal extension of p^S_f to f; hence $(p_1)^S_B \underset{B}{\simeq} (q_1)^S_B$, and $(p_2)^S_C \supset (q_2)^S_C \supset g[(p_1)^S_B]$; since $(p_2)^S_C$ in an S_C–minimal extension of $(p_1)^S_B$, we infer that $(p_2)^S_C = (q_2)^S_C$.

We conclude from 1,2 and 3 that $(p_2)^S_C$ is an $S_{g \cdot f}$–component of p. ■

10. **Theorem.** Let $H \in$ **K**, p a type in $\vec{x}$ over H. Suppose S is equational, **K** reflects S and **K** satisfies one of the following conditions:

a) S is the closure under finite disjunctions of a set S_1 which has finite height; say height$(S_1) < n$.

b) **K** is ω–conservative (see chapter I before lemma 5 for a definition of ω–conservativeness).

Then, there is a morphism $f: H \longrightarrow F$ in **K** such that p is consistent over f and, for any S_F-minimal extension q of p to f, q_F^S is an S_f-component of p.

Proof. Assume S is equational and **K** reflects S. For q a type over G ($\in$ **K**) let $*_q$ denote the following assertion,

$*_q$: "there is a morphism $g:G \longrightarrow E$ in **K** such that q is consistent over g and, for any S_E-minimal extension r of q to g, r_E^S is an S_g-component of q".

Claim 1. With $g:G \longrightarrow E$ and q a type over g, let $q_0, \cdots, q_{n-1}$ be, up to S_E-euivalence, the S_E-minimal extensions of q to g, (by theorem A.13 applied to gq, there are finitely many such extensions). Let $r_i = gq \cup (q_i)_E^S$ $(i < n)$. *Then*, if $*_q$ does not hold, there is a morphism $e:E \longrightarrow D$ in **K** and $i < n$ such that $r = e(r_i)$ is consistent in **K** and $*_r$ *does not hold.*

Proof of claim 1. Suppose the assertion of the claim is false. We construct by induction on $i < n$ a sequence of morphisms $e_i:E_i \longrightarrow E_{i+1}$ $(i < n)$, $E_0 = E$, such that:

If $f_i = e_i \cdot e_{i-1} \cdots e_0$ $(i < n)$, $f_{-1} = \mathrm{id}_E$, and r' is an $S_{E_{i+1}}$-minimal extension of $f_{i-1}r_i$ to e_i $(i < n)$ then $(r')_{E_{i+1}}^S$ is an S_{e_i}-component of $f_{i-1}r_i$;

and furthermore, q is consistent over $f_i \cdot g$ $(i < n)$.

Suppose the construction achieved up to $i-1$ $(i < n)$; let $r = f_{i-1}r_i$.

If r is consistent then, by assumption $*_r$ must hold; i.e. there is a morphism $d{:}E_i \longrightarrow D$ such that r is consistent over d and, for any S_D-minimal extension r' of r to d, $(r')^S_D$ is an S_d-component of r. in that case take $e_i = d$. Since r is consistent over $d = e_i$ and $r = f_{i-1}r_i \supset f_{i-1} \cdot gq$ we see that q is consistent over $e_i \cdot f_{i-1} \cdot g$ i.e. over $f_i \cdot g$.

If r is inconsistent take $e_i = \mathrm{id}_{E_i}$, then q is consistent over $f_i \cdot g = f_{i-1} \cdot g$ by the induction hypothesis.

This finishes the inductive step of the construction. Now, let $f = f_{n-1} \cdot g$; $f{:}G \longrightarrow E_n$. q is consistent over f.

Let p_n be an S_{E_n}-minimal extension of q to f; let $p_i = p_n \restriction h_i$ $(i < n)$ where $h_i = e_{n-1} \cdot e_{n-2} \cdots e_i$. p_i is a type over E_i extending q.

By proposition A.2 there is $i < n$ such that $(p_0)^S_{E_0} \supset (q_i)^S_{E_0}$, whence $p_0 \supset r_i$. It follows that $p_i \supset f_{i-1}r_i$.

Again, by proposition A.2 there is an $S_{E_{i+1}}$-minimal extension r' of $f_{i-1}r_i$ to e_i such that $(p_{i+1})^S_{E_{i+1}} \supset (r')^S_{E_{i+1}}$; by the choice of the sequence $(e_i)_{i<n}$, we have

(+) $(r')^S_{E_{i+1}}$ is an S_{e_i}-component of $f_{i-1}r_i$.

On the other hand, since p_n is an S_{E_*}-minimal extension of q to f and

$p_n \supset h_{i+1}[f_i r_i \cup (r')^S_{E_{i+1}}] \supset fq$,

(++) p_n is an S_{E_*}-minimal extension of $f_i r_i \cup (r')^S_{E_{i+1}}$ to h_{i+1}.

Since $f_i r_i \cup (r')^S_{E_{i+1}}$ is S-full it follows that $(p_n)^S_{E_*} \cong_{E_*} h_{i+1}[(r')^S_{E_{i+1}}]$; by reflection we deduce that $(p_{i+1})^S_{E_{i+1}} \cong_{E_{i+1}} (r')^S_{E_{i+1}}$. So we can assume $r' = p_{i+1}$.

By (+), (++), and proposition 9 (applied to p = our $f_{i-1}r_i$, p_1 = our p_{i+1} and p_2 = our p_n) it follows that p_n is an S_{h_i}-component of $f_{i-1}r_i$. In particular $h_i \cdot f_{i-1}r_i \cup (p_n)^S_{E_*}$ is S-full.

But $h_i \cdot f_{i-1}r_i = f_{n-1}(gq \cup (q_i)^S_E)$ and $f_{n-1}((q_i)^S_E) \subset (p_n)^S_{E_*}$. So we actually have that $fq \cup (p_n)^S_{E_*}$ is S-full. We deduce that $(P_n)^S_{E_*}$ is an S_f-component of q.

We have shown that the morphism $f:G \longrightarrow E_n$ is such that q is consistent over f and, for any S_{E_*}-minimal extension p_n of q to f, $(p_n)^S_{E_*}$ is an S_f-component of q. In other words $*_q$ holds. ✖.

This ends the proof of claim 1.

Claim 2. If q is a type over G ($\in$ **K**) and q is not S-full, then there is a morphism $g:G \longrightarrow E$ such that q is consistent over g and for any complete extension r of gq to E,

$$r^S_E \models_E g(q \cap D^S(G)) \text{ and } g(q \cap D^S(G)) \not\models_E r^S_E.$$

Proof of claim 2. Suppose q is not S-full; then there is a morphism $g:G \longrightarrow E$ such that q is consistent over g and q is not S_g-full. In other terms, there are formulas φ_1 and φ_2 in $D^S(E)$ such that

$$gq \models_E \varphi_1 \vee \varphi_2,\ g(q \cap D^S(G)) \not\models_E \varphi_1 \text{ and } g(q \cap D^S(G)) \not\models_E \varphi_2.$$

Now, if r is a complete type over E extending q then $r \models_E \varphi_1 \vee \varphi_2$; hence, by completeness, either $r \models_E \varphi_1$ or $r \models_E \varphi_2$. In any case, we have

$$r^S_E \models_E g(q \cap D^S(G)) \text{ and } g(q \cap D^S(G)) \not\models_E r^S_E.$$

Thus g satisfies our conditions.

Claim 3. Suppose p is a type over H such that $*_p$ does not hold. Then there is a sequence of morphisms

$(f_{i+1}:H_i \longrightarrow H_{i+1})_{i<\omega}$,

$H_0 = H$, $f_0 = \mathrm{id}_{H_0}$, and a sequence of types $(p_i)_{i<\omega}$, $p_0 = p$ such that:

p_i is a type over H_i such that $*_{p_i}$ does not hold and, if $S = \mathrm{cl}^+(S_1)$, then

$$p_{i+1} \cap D^{S_1}(H_{i+1}) \models_{H_{i+1}} f_{i+1}(p_i \cap D^{S_1}(H_i)) \text{ and}$$

$$f_{i+1}(p_i \cap D^{S_1}(H_i)) \not\vdash_{H_{i+1}} p_{i+1} \cap D^{S_1}(H_{i+1}).$$

Proof of claim 3. We construct $(f_i)_{i<\omega}$ and $(p_i)_{i<\omega}$ by induction on $i < \omega$. Suppose the construction of (f_i) and (p_i) with its required properties achieved up to $i < \omega$.

Since $*_{p_i}$ does not hold, p_i is not S-full. Therefore, by claim 2, there is a morphism $g: H_i \longrightarrow E$ such that p_i is consistent over g and for any complete extension q of p_i to g,

$$q_E^S \vdash_E g(p_i \cap D^S(H_i)) \text{ and } g(p_i \cap D^S(H_i)) \not\vdash_E q_E^S.$$

By claim 1, there is an S_E-minimal extension q_i of p_i to g and a morphism $e: E \longrightarrow D$ such that, for $r_i = gp_i \cup (q_i)_E^{S_1}$ and $r = e(r_i)$, r is consistent in K and $*_r$ does not hold.

Take $f_{i+1} = e \cdot g$ and $p_{i+1} = r$. Suppose $S = cl^+(S_1)$.

Since r extends p_i to $e \cdot g$,

$$r \cap D^{S_1}(D) \vdash_D e \cdot g(p_i \cap D^{S_1}(H_i)).$$

On the other hand,

$$e \cdot g(p_i \cap D^{S_1}(H_i)) \not\vdash_D r \cap D^{S_1}(D);$$

for if not, then,

$$e \cdot g(p_i \cap D^{S_1}(H_i)) \vdash_D r \cap D^{S_1}(D) \vdash_D e((q_i)_E^{S_1});$$

q_i being complete over S, $(q_i)_E^{S_1} \simeq_E (q_i)_E^S$; thus

$e \cdot g(p_i \cap D^S(H_i)) \not\vdash_D e((q_i)^S_E)$; by reflection, it follows $g(p_i \cap D^S(H_i)) \not\vdash_E (q_i)^S_E$ which contradicts the choice of the morphism g.

Thus, p_{i+1} and f_{i+1} as chosen satisfy the required conditions.

Proof of the theorem. Suppose condition a) (resp. condition b)) holds while $*_p$ does not hold. We want to show a contradiction.

Consider the sequences $(f_i)_{i<\omega}$ and $(p_i)_{i<\omega}$, constructed in claim 3, up to $n+1$ (resp. up to ω), (recall height$(S_1) < h$).

Let $G \in \mathbf{K}$ and let $g_i : H_i \longrightarrow G$ be morphisms in $\mathbf{K}$ such that $g_{i+1} \cdot f_{i+1} = g_i$ for $i < n+1$ (resp. for $i < \omega$): take for instance $G = H_{n+2}$ and $g_i = f_{n+1} \cdot \dots \cdot f_i$ (resp. G and the morphisms g_i are given by ω-conservativeness).

We have,

$$g_{i+1}(p_{i+1} \cap D^{S_1}(H_{i+1})) \vdash_G g_i(p_i \cap D^{S_1}(H_i)),$$

and (by reflection),

$$g_i(p_i \cap D^{S_1}(H_i)) \not\vdash_G g_{i+1}(p_{i+1} \cap D^{S_1}(H_{i+1})),$$

for $i < n+1$ (resp. $i < \omega$).

The implications above imply that S_1 (c.f. I.7.(i)) has height greater or equal to n (resp. taking $S = S_1$, S is not equational) ✗.

We conclude that $*_p$ must hold, which is what we wanted. ■

Let $f:A\longrightarrow F$ be in $\hat{K}$ and p a type over f such that for any S_F-minimal extension q of p to f, q_F^S is an S_f-component of p. The proposition below investigates in what measure are the S_f-components of p intresically defined.

11. **Proposition.** Let $f:A\longrightarrow F$ be in $\hat{K}$, p a type over A such that for any S_F-minimal extension q of p to f, q_F^S is an S_f-component of p. Then, given $g:A\longrightarrow G$ in K, p_1, a complete type over G extending p such that $(p_1)_G^S$ is an S_g-component of p and, given $e_1:F\longrightarrow E$ and $e_2:G\longrightarrow E$ such that $e_1\cdot f\restriction A = e_2\cdot g\restriction A$ and p_1 is consistent over e_2, *there is* an S_F-minimal extension q of p to f such that $e_1(q_F^S) \not\geq_E e_2((p_1)_G^S)$.

Proof. (Recall that if $A=(A,H)$, then the morphism $f:A\longrightarrow F$ is formally defined as a morphism $f:H\longrightarrow F$ and similarly for $g:A\longrightarrow G$, so that the equality $e_1\cdot f\restriction A = e_2\cdot g\restriction A$ is strictly weaker than the equality $e_1\cdot f = e_2\cdot g$). Let f, g, e_1, e_2, p and p_1 satisfy the hypothesis of the proposition. Since $(p_1)_G^S$ is an S_g-component of p,

$gp \cup (p_1)^S_G$ is S-full; hence, by proposition A.5.(ii) applied to $e_2(gp \cup (p_1)^S_G)$, $gp \cup (p_1)^S_G$ has an S_E-minimal extension p_2 to e_2 such that

$$(p_2)^S_E \underset{E}{\simeq} e_2[(gp \cup (p_1)^S_G) \cap D^S(G)] \underset{E}{\simeq} e_2((p_1)^S_G).$$

By proposition 2, $(p_2)^S_E$ is an $S_{e_2 \cdot g}$-component of p; in particular, $(p_2)^S_E$ is an S_E-minimal extension of $p^S_{e_2 \cdot g}$ to $e_2 \cdot g$. Since p_2 extends p, p_2 is actually an S_E-minimal extension of p to $e_2 \cdot g$ or equivalently, to $e_1 \cdot f$. By proposition A.2 there is an S_F-minimal extension q of p to f such that $(p_2 \restriction e_1)^S_F \supset q^S_F$. So we have

$$p_2 \supset e_1 \cdot (fp \cup q^S_F) \supset e_1 \cdot fp.$$

It easily follows that p_2 is an S_E-minimal extension of $fp \cup q^S_F$ to e_1. But q^S_F is an S_f-component of p; hence $fp \cup q^S_F$ is S-full; hence, by proposition 5.(ii) $fp \cup q^S_F$ has, up to S_E-equivalence, a unique S_E-minimal extension q' to e_1 and $(q')^S_E \underset{E}{\simeq} e_1(q^S_F)$.

Thus, $(p_2)^S_E \underset{E}{\simeq} (q')^S_E \underset{E}{\simeq} e_1(q^S_F)$.

We conclude, $e_2((p_1)^S_G) \underset{E}{\simeq} e_1(q^S_F)$. ■

12. **Definition.** $H \in$ **K**. **We say** H **is** S**-irreducible if every complete type over** H **is** S**-irreducible;**

H is S-full if for every complete type p over H and subset p_0 of p, $p_0 \cup p_H^S$ is S-full.

In particular if H is S-full then every complete type over H is S-full. We have an immediate converse in case **K** reflects Δ.

13. **Lemma.** If **K** reflects Δ then H is S-full iff every complete type over H is S-full.

Proof. One direction has already been proved.

Suppose every complete type over H is S-full. Let p be a complete type over H, $p_0 \subset p$ and $f: H \longrightarrow F$ an arbitrary morphism in **K**. Since **K** reflects Δ, p is consistent over f (c.f. I.4). Since p is S_f-full, we easily deduce that $p_0 \cup p_H^S$ is S_f-full. Thus $p_0 \cup p_H^S$ is S_f-full for any morphism $f: H \longrightarrow F$ i.e.

$$p_0 \cup p_H^S \text{ is } S\text{-full}.$$

We conclude that H is S-full. ■

Example. In **K** = **F** (the category of fields with field embeddings, c.f. 0.4.(i)) with S the set of algebraic equations, Δ the set of quantifier-free formulas, any algebraically-closed field H is S-full: for if p is a complete type over H, $H \subset F$ and φ_1, φ_2 are algebraic equations with coefficients in F such that $p \models_F \varphi_1 \vee \varphi_2$, then $p_H^S \models_F \varphi_1 \vee \varphi_2 \vee \bigvee_{i<n} \psi_i$ where ψ_i are algebraic equations with coefficients in H and $\psi_i \notin p$. Now $\wedge p_H^S$ defines an irreducible variety in H, hence $\wedge p_H^S$ defines an irreducible variety in F. It follows that $p_H^S \models_F \varphi_j$ for $j = 1$ or 2 (since $p_H^S \not\models_F \psi_i$ for any $i < n$).■

In fact we will see in chapter III that there is in general a strong relation between S-full structures and existentially-closed structures. For instance if **K** is the category of models of a first-order theory with elementary embeddings for morphisms then any structure in **K** is S-full (c.f. III.3).

14. Properties of S-full structures. $H \in$ **K**.

(i) If H is S-irreducible (resp. S-full) then for any morphism $f: H \longrightarrow F$ and complete type p over f, p has, up to S_F-equivalences, a unique S_F-minimal extension q to f; moreover q is such that

$q_F^S \underset{F}{\approx} p_f^S$ (resp. $q_F^S \underset{F}{\approx} f(p_H^S) \underset{F}{\approx} p_f^S$).

(ii) If H is S-full then for any morphism $f: H \longrightarrow F$, H is S_f-closed in F.

(iii) If H is S-full, p is a type over H and q is an S_H-

minimal extension of p to H then q_H^S is an S_H-component of p.

Proof.

(i) Follows immediately from proposition 5 and the fact that any complete type over H is S-irreducible (resp. S-full).

(ii) From (i) we have that for any morphism $f:H \longrightarrow F$ and any complete type p over f, $p_F^S \cong f(p_H^S) \supseteq f(D(H))$. The claim now follows from lemma A.5

(iii) Immediate. ■

Let us say **K** is inductive if for any sequence of morphisms $(f_{ij}:H_i \longrightarrow H_j)_{i<j<\alpha}$ in **K** (α an ordinal) with $f_{jk} \cdot f_{ij} = f_{ik}$ there is a structure H_α in **K** and morphisms $g_i:H_i \longrightarrow H_\alpha$ $(i < \alpha)$ such that $g_j \cdot f_{ij} = g_i$ $(i < j < \alpha)$ and $|H_\alpha| = \bigcup_{i<\alpha} g_i(H_i)$; we write

$$(H_\alpha, g_i; i < \alpha) = \text{colim}(H_i, f_{ij}; i < \alpha).$$

(Inductiveness is just the analogue of closure under unions of chains).

The theorem below proves the existence of S-full structures under general assumptions.

15. **Theorem.** **Assume S is equational, $\mathbf{K}$ reflects S and $\mathbf{K}$ is inductive. Then, given $H_0 \in \mathbf{K}$, there is a morphism $f: H_0 \longrightarrow H$ such that H is S-full.**

Proof. We construct by induction a sequence $(h_i: H_i \longrightarrow H_{i+1})_{i<\omega}$ such that for any type p over h_i and $S_{H_{i+1}}$-minimal extension q of p to h_i, $q^S_{H_{i+1}}$ is an S_{h_i}-component of p $(i < \omega)$.

Suppose for a moment the construction done. Let then

$$(H,\ g_i;\ i < \omega) = \operatorname{colim}(H_i,\ h_i;\ i < \omega).$$

Claim. H is S-full.

Indeed, given a complete type p over H and $p_0 \subset p$, $p_0 \cup p^S_H$ is S-full: for let $f: H \longrightarrow F$ be in $\mathbf{K}$, φ_1, φ_2 in $D^S(F)$ such that $f(p_0 \cup p^S_H) \models_F \varphi_1 \vee \varphi_2$ and $p_0 \cup p^S_H$ is consistent over f.

Since S is equational, p^S_H is equivalent to a single formula in $D^S(H)$; so there is a *finite* subset q of $p_0 \cup p^S_H$ such that $q^S_H \underset{H}{\approx} p^S_H$ and $fq \models_F \varphi_1 \vee \varphi_2$; moreover, since $H = \bigcup_{i<\omega} g_i(H_i)$, there is $i < \omega$ such that $q = q \restriction g_i(H_i)$ and $q^S_H \underset{H}{\approx} q \cap D^S(g_i(H_i))$. Let $q_j = q \restriction g_j$ $(j < \omega)$.

We have

$$g_i[(q_i)^S_{H_i}] \underset{H}{\approx} q^S_H \underset{H}{\approx} (q_i)^S_{g_i} \underset{H}{\approx} g_{i+1}((q_{i+1})^S_{H_{i+1}});$$

by reflection it follows that

$$(q_i)^S_{h_i} \ \widetilde{H}_{i+1} \ h_i((q_i')^S_{H_i}) \ \widetilde{H}_{i+1} \ (q_{i+1})^S_{H_{i+1}}.$$

Thus, q_{i+1} is an $S_{H_{i+1}}$-minimal extension of q_i to h_i; hence $(q_{i+1})^S_{H_{i+1}}$ is an S_{h_i}-component of q_i, which in particular implies that $h_i q_i \cup (q_{i+1})^S_{H_{i+1}}$ is S-full.

Now

$$f \cdot g_{i+1}(h_i q_i \ \cup \ (q_{i+1})^S_{H_{i+1}}) \Vdash_F fq \Vdash_F \varphi_1 \vee \varphi_2;$$

since $h_i q_i \ \cup \ (q_{i+1})^S_{H_{i+1}}$ is S-full, it follows that $f \cdot g_{i+1}((q_{i+1})^S_{H_{i+1}}) \Vdash_F \varphi_j$ for $j = 1$ or 2. Hence $f(q^S_H) \Vdash_F \varphi_j$ for $j = 1$ or 2 i.e. $f(p^S_H) \Vdash_F \varphi_j$ for $j = 1$ or 2 which is what we wanted.

Construction of $(h_i)_{i<\omega}$. Suppose the construction achieved up to $i - 1 < \omega$, $(h_{-1} = \mathrm{id}_{H_0}, H_{-1} = H_0)$. Let $(p_\alpha)_{\alpha<\lambda}$ enumerate the types over H_i. We construct, by induction on $\alpha < \lambda$, the sequences

$(f_\alpha : F_\alpha \longrightarrow F_{\alpha+1})_{\alpha<\lambda}$ and $(d_{\beta\alpha} : F_\beta \longrightarrow F_\alpha)_{\beta<\alpha<\lambda}$ such that:

- $F_0 = H_i$, $d_0 = \mathrm{id}_{H_i}$ and $d_{\beta\alpha} \cdot d_{\gamma\beta} = d_{\gamma\alpha}$, $f_\alpha = d_{\alpha\alpha+1}$.
- For α limit,

  $(F_\alpha, d_{\beta\alpha}, \beta < \alpha) = \mathrm{colim}(F_\beta; d_{\beta\gamma}; \beta<\gamma< \alpha)$.
- If $d_{0\alpha} p_\alpha$ is inconsistent in K over F_α, $f_\alpha = \mathrm{id}_{F_\alpha}$

- If $d_{o\alpha}p_\alpha$ is consistent in K over F_α, f_α is such that $d_{o\alpha}p_\alpha$ is consistent over f_α and for any $S_{F_{\alpha+1}}$-minimal extension q of $d_{o\alpha}p_\alpha$ to f_α, $q^S_{F_{\alpha+1}}$ is an S_{f_α}-component od $d_{o\alpha}p_\alpha$.

f_α is given by theorem 8.

Let $(H_{i+1}, e_\alpha; \alpha < \lambda) = \mathrm{colim}(F_\alpha, d_{\beta\alpha}; \beta<\alpha< \lambda)$; $h_i = e_O$. We have to show that $h_i : H_i \longrightarrow H_{i+1}$ satisfy the required property of the sequence $(h_i)_{i<\omega}$: so let p be a type over h_i, say $p = p_\alpha$ $(\alpha < \lambda)$, and let q be an $S_{H_{i+1}}$-minimal extension of p to h_i.

Let $q_\alpha = q \upharpoonright e_\alpha$ (recall $e_\alpha : F_\alpha \longrightarrow H_{i+1}$).

By proposition A.2, there is an $S_{F_{\alpha+1}}$-minimal extension r of $d_{o\alpha}p_\alpha$ to f_α $(f_\alpha : F_\alpha \longrightarrow F_{\alpha+1})$ such that $(q_{\alpha+1})^S_{F_{\alpha+1}} \supset r^S_{F_{\alpha+1}}$. Clearly then, q is an $S_{H_{i+1}}$-minimal extension of $d_{o\alpha+1}p_\alpha \cup r^S_{F_{\alpha+1}}$ to $e_{\alpha+1}$.

On the other hand, by construction, $r^S_{F_{\alpha+1}}$ is an S_{f_α}-component of $d_{o\alpha}p_\alpha$; in particular $d_{o\alpha+1}p_\alpha \cup r^S_{F_{\alpha+1}}$ is S-full.

Whence, by proposition A.5.(ii),

$$q^S_{H_{i+1}} \widetilde{H}_{i+1}\ e_{\alpha+1}[(d_{o\alpha+1}p_\alpha \cup r^S_{F_{\alpha+1}})^S_{F_{\alpha+1}}]\ \widetilde{H}_{i+1}\ e_{\alpha+1}[r^S_{F_{\alpha+1}}].$$

By reflection, we deduce,

$(q_{\alpha+1})^S_{F_{\alpha+1}} \cong_{F_{\alpha+1}} r^S_{F_{\alpha+1}}$. So, we might as well assume $r = q_{\alpha+1}$ and therefore r is consistent over $e_{\alpha+1}$.

By proposition 9 it follows that $q^{S_i}_{H_{i+1}}$ is an S_{e_i}-component of $d_{\alpha\alpha}p_\alpha$; hence $q^S_{H_{i+1}}$ is an S_{H_i}-component of $e_\alpha \cdot d_{\alpha\alpha}p_\alpha$ i.e. of $p_\alpha = p$, which is what we wanted. ■

Section C: The Abstract Context

As we observed in the introduction to this chapter, the general theory of sections A and B goes through in a very general abstract context that has nothing to do with either structures or formulas.

0. The abstract context can be described as follows:

We have $\mathcal{A}$, an abstract category ($\mathcal{A}$ will stand for a category of L-structures in the case we are really interested in);

two functors

$D:\mathcal{A} \longrightarrow$ Boole (= category of boolean algebras)

$D:\mathcal{A} \longrightarrow$ Dist. latt. (= category of distributive lattices) and a natural transformation

$$i : D^S \longrightarrow D$$

such that for each $A \in \mathrm{Ob}(\mathcal{A})$ ($\mathrm{Ob}(\mathcal{A})$ the class of objects in

$\mathcal{A}$), i_A is a (1–1)-distributive lattice homomorphism (of $D^S(A)$ into $D(A)$). //

In other words, we have an assignment:

$$A \longmapsto D(A).$$

which to every object A in $\mathcal{A}$ assigns a boolean algebra $D(A)$, (do not confuse yet the notation $D(A)$ with that of the set of formulas with parameters in A; A here is not necessarily a structure)

and an assignment

$$\begin{array}{c} A \\ f\downarrow \\ B \end{array} \longmapsto \begin{array}{c} D(A) \\ \downarrow D(f) \\ D(B) \end{array}$$

which to every morphism $f: A \longrightarrow B$ in $\mathcal{A}$ assigns a boolean algebra homomorphism $D(f): D(A) \longrightarrow D(B)$, so that $D(\mathrm{id}) = \mathrm{id}$ and $D(f\cdot g) = D(f)\cdot D(g)$.

The assignments above define the functor D.

Similarly for D^S: now $D^S(A)$ is a distributive lattice and $D^S(f)$ is a distributive lattice homomorphism.

Finally the assumptions on i mean we have an assignment which to every $A \in \mathrm{Ob}(\mathcal{A})$ assigns a (1–1)-distributive lattice homomorphism

$$D^S(A) \xrightarrow{i_A} D(A)$$

such that, for all $A \xrightarrow{f} B$ in $\mathcal{A}$, the diagram

$$\begin{array}{ccc} D^S(A) & \xrightarrow{i_A} & D(A) \\ {\scriptstyle D^S(f)}\downarrow & & \downarrow{\scriptstyle D(f)} \\ D^S(B) & \xrightarrow{i_B} & D(B) \end{array}$$

commutes.

1. Now, in the context of sections A and B i.e., if **K** is a category of structures, Δ is a boolean-closed set of formulas and S is a set of formulas in Δ with $S = cl^+(S)$ then, for a fixed tuple of variable $\vec{x}$,

   we let $\mathcal{A}$ stand for $\hat{\mathbf{K}}$;

   for $A \in \mathrm{Ob}(\mathcal{A})$, $D(\mathcal{A})$ (in the abstract context) stands for $D_{\vec{x}}(A)$ (in the context of sections A and B, i.e. the set of formulas in $\vec{x}$ with parameters in A considered up to equivalence in **K**; $D_{\vec{x}}(A)$ is considered as a boolean algebra in the obvious way). Note for instance that for φ, ψ in $D(A)$, $\varphi \leq \psi$ iff $\varphi \vdash_{\overline{A}} \psi$, (where $\leq$ denotes the usual partial order in the boolean algebra $D(A)$);

   for $f: A \longrightarrow B$ a morphism in $\mathcal{A}$, $D(f)$ is such that

$$D(f)(\varphi(\vec{x};\vec{a})) = \varphi(\vec{x};f\vec{a}) \quad (\vec{a} \in A).$$

Similarly, for $A \in \mathcal{A}$, $D^S(A)$ (in the abstract context) $= D^S_{\vec{x}}(A)$, ($D^S_{\vec{x}}(A)$ considered as a distributive lattice in the obvious way), and $D^S(f) = D(f)\restriction D^S(A)$;

finally for $A \in \mathrm{Ob}(\mathcal{A})$, i_A is the obvious embedding of $D^S_{\vec{x}}(A)$ into $D_{\vec{x}}(A)$.

We shall refer to the context just described above, when no confusion arises, by the symbol **K** and call it the "standard" context.

2. **Back to the general context.** We define,

D^S is *equational* if for all $A \in \mathcal{A}$, $D^S(A)$ (as a distributive lattice) has the descending chain property (i.e. there is no sequence $(a_i)_{i<\omega}$ of elements in $D^S(A)$ such that $a_{i+1} < a_i$ for any $i < \omega$).

$\mathcal{A}$ *reflects* D^S if for all $f:A \longrightarrow B$ in $\mathcal{A}$, $D^S(f)$ is injective.

A *type* over A ($\in \mathcal{A}$) is a "filter-base" in $D(A)$: i.e. a set of elements in $D(A)$ such that all finite intersections of elements of it are $\neq \emptyset$. [Note that this is the same as consistency in the standard case, under the above translation in 1];

a *complete type* is an ultrafilter in $D(A)$.

One uses the notation $p \vdash \varphi$ in the general context;

here p is a type over A, φ is a "formula over A" i.e. an element of $D(A)$. $p \vdash \varphi$ means that for some finite $p' \subset p$,

$$\wedge p' \leq \varphi,$$

where $\wedge p'$ is the intersection of the elements of p' and $\leq$ is the usual partial order in the boolean algebra $D(A)$.

Given $f: A \longrightarrow B$, p a type over A and q a type over B, let

$$fp = \{f\varphi;\ \varphi \in p\},\ q \upharpoonright f = \{\varphi \in D(A);\ f\varphi \in q\};$$

we say p is a *type over* f or p is *consistent over* f if fp is a type over B;

$$p^S_f = \{\varphi \in D^S(B);\ fp \vdash \varphi\},\ p^S_A = p^S_{\mathrm{id}_A}.$$

For f as above, A is S_f-closed in B if for any complete types p' and q' over B,

$$(p')^S_B \supset (q')^S_B \Longrightarrow p^S_f \supset q^S_f$$

where $p = p' \upharpoonright f$ and $q = q' \upharpoonright f$.

Given $f: A \longrightarrow B$ and $g: B \longrightarrow C$ in $\mathcal{a}$, p a type over f and q a complete type over g, extending p to f, we say that q is an S_g-minimal extension of p to f iff for any complete type q_1 over g extending p to f, if $q^S_g \supset (q_1)^S_g$ then $q^S_g = (q_1)^S_g$.

Of course, one easily checks that, under the translation given above, (when $\mathcal{a}$ stands for $\hat{K}$, etc...), the definitions that preceded coincide with those given in the standard case.

We reproduce below the analogues of theorems A.2, A.12 and A.13 in the abstract context; their proofs are exactly the same as those given in the standard context.

3. **Theorem.** Given $f: B \longrightarrow C$, p a type over f and q_1 a complete type over f extending p, there is an S_f-minimal extension q of p to B such that $(q_1)^S_f \supset q^S_f$.

4. **Theorem.** Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be morphisms in $\mathcal{Q}$ such that A is $S_{g \cdot f}$-closed in C and B is S_g-closed in C, p_0 a type over $g \cdot f$, p_1 a complete type over $g \cdot f$, p_2 a complete type over g, $p_2 \supset fp_1 \supset fp_0$. Then p_2 is an S_g-minimal extension of p_0 to f iff p_2 is an S_g-minimal extension of p_1 to f and p_1 is an $S_{g \cdot f}$-minimal extension of p_0 to A.

5. **Theorem.** Suppose D^S is equational. Let $f: A \longrightarrow B$ be in $\mathcal{Q}$, p a type over f. Then, there exist $q_0, \cdots, q_{n-1}$, S_f-minimal extensions of p to A such that for any S_f-minimal extension of p to A, there is $i < n$, $q^S_f \underset{B}{\cong} (q_i)^S_f$. In fact, with the q_i's as above, $p^S_f \underset{B}{\cong} \bigvee_{i<n} (q_i)^S_f$.

In a similar fashion one defines in the abstract context S-full and S-irreducible types as well as S-full structures etc...

For instance, given $f:A \longrightarrow B$ in $\mathcal{A}$, p a type over f, p is S-full if for any φ_1, φ_2 in $D^S(B)$,

$$fp \vdash \varphi_1 \vee \varphi_2 \Longrightarrow f[p \cap D^S(A)] \vdash \varphi_i \text{ for } i = 1 \text{ or } 2;$$

and similarly, the results of section B have their analogues in the general context.

6. **Note.** In the statements of the definitions and theorems above there is no mention of any tuples of variables. Thus, we can apply theorems 3, 4 and 5 to the following situation:

Let K, Δ and S be given as in section A and suppose Δ and S contain the formulas $(y = y)$, y any variable.

Fix $\vec{x}$ a, *possible infinite*, tuple of variables.

For $A \in \hat{K}$, $\varphi(\vec{y};\vec{a})$ and $\psi(\vec{z};\vec{b})$ formulas with parameters in A, write

$$\varphi(\vec{y};\vec{a}) \approx \psi(\vec{z};\vec{b})$$

if

$$\varphi(\vec{y};\vec{a}) \wedge (\vec{z} = \vec{z}) \underset{A}{\sim} \psi(\vec{z};\vec{b}) \wedge (\vec{y} = \vec{y});$$

Clearly $\approx$ is an equivalence relation.

Now, let $\mathcal{A}$ stand for $\hat{K}$;

for $A \in \mathcal{A}$, let $D(A)$ (in the abstract context) be the (obvious) boolean algebra whose underlying set is.

$$\{\varphi(\vec{y};\vec{a})/\approx;\ \vec{a} \in A \text{ and } \vec{y} \text{ a subtuple of } \vec{x}\},$$

let $D^S(A)$ be the distributive lattice whose underlying set is

$$\{\varphi(\vec{y};\vec{a})/\approx;\ \vec{a} \in A,\ \varphi \in S \text{ and } \vec{y} \text{ a subtuple of } \vec{x}\}.$$

For $f\colon A \longrightarrow B$, let

$$D(f)(\varphi(\vec{y};\vec{a})/\approx) = \varphi(\vec{y};f\vec{a})/\approx.$$

Note that if $\varphi(\vec{y};\vec{a}) \approx \psi(\vec{z};\vec{b})$ then $\varphi(\vec{y};f\vec{a}) \approx \psi(\vec{z};f\vec{b})$ so that $D(f)$ is a well defined homomorphism of boolean algebras. Let $D^S(f) = D(f){\restriction}D^S(A)$. We refer to the context just described above by the symbol $\mathbf{K}/\approx$.

With the setting above, for $A \in \mathcal{A}$, a type $p/\approx$ (the notation $p/\approx$ will be explained later on), over A becomes a set of "formulas" $\varphi(\vec{y};\vec{a})/\approx$ with parameters in A and with $\vec{y}$ a subtuple of $\vec{x}$ such that any finite subset of $p/\approx$ is consistent over A;

$p/\approx$ is complete over A if for any $\varphi(\vec{y};\vec{a})/\approx$ in $D(A)$ either $\varphi(\vec{y};\vec{a})/\approx$ belongs to $p/\approx$ or $\neg\varphi(\vec{y};\vec{a})/\approx$ belongs to $p/\approx$;

Given $f\colon A \longrightarrow B$ in $\hat{\mathbf{K}}$,

$$(p/\approx)^S_f = \{\varphi \in D^S(B),\ p \vdash \varphi\},$$

i.e.

$$(p/\approx)^S_f = \{\varphi(\vec{y};\vec{b})/\approx;\ \vec{b} \in B,\ \varphi \in S,\ \vec{y} \text{ a subtuple of } \vec{x}$$
$$\text{and for some } p_0 \subset p/\approx,\ p_0 \text{ finite},\ fp_0 \vdash \varphi\}.$$

[Note that, for $\theta(\vec{z};\vec{c})/\approx$ and $\varphi(\vec{y};\vec{b})/\approx$ in $D(B)$,

$$\theta(\vec{z};\vec{c})/\approx \vdash \varphi(\vec{y};\vec{b})/\approx \text{ in } \mathcal{A} \text{ if}$$

$$\theta(\vec{z};\vec{c})\wedge(\vec{y} = \vec{y}) \vdash_{\overline{B}} \varphi(\vec{y};\vec{b})\wedge(\vec{z} = \vec{z}) \text{ (in the standard context)]}$$

Now, we can apply to $\mathbf{K}/\approx$, the definitions and theorems given above in the general context.

For instance, given $A \subset N \in \mathbf{K}$, and $p/\approx$ a type over A (with $\vec{x}$ a possibly infinite tuple of variables), we can speak of S_H-minimal extension of $p/\approx$ to H, and, say, the monotonocity-transitivity theorem (c.f. theorem 4) applies to such types.

Note that if $\vec{x}$ is finite, then for all practical uses, a type $p/\approx$ over A ($\in \mathcal{A}$) as defined in $\mathbf{K}/\approx$ can be considered as a type in $\vec{x}$ over A as defined in the standard context $\mathbf{K}$. In fact the map which to a type $p/\approx$ over A, corresponds the set $p = \{\varphi(\vec{x};\vec{a});\varphi(\vec{x};\vec{a})/\approx \in p/\approx\}$ is one-to-one from the set of types over A (in $\mathbf{K}/\approx$) onto the set of types in $\vec{x}$ over A (in $\mathbf{K}$).

Furthermore, one can easily check the following:

p is complete (in the standard sense) iff $p/\approx$ is complete; if $f: A \longrightarrow B$ is in $\hat{\mathbf{K}}$, and q is a complete type in $\vec{x}$ over A, then q is an S_f-minimal extension of p to A (in $\mathbf{K}$) iff $q/\approx$ is an S_f-minimal extension of $p/\approx$ to A (in $\mathbf{K}/\approx$).

More generally, if $\vec{x}$ is infinite, then a type $p/\approx$ over A ($\in \mathfrak{A}$) in $\mathbf{K}/\approx$ can be considered as a consistent set of formulas (in $\mathbf{K}$) in subtuples of $\vec{x}$ with parameters in A; to $p/\approx$ we correspond the set

$$p = \{\varphi(\vec{y};\vec{a});\ \vec{y} \text{ a subtuple of } \vec{x},\ \varphi(\vec{y};\vec{a})/\approx\ \in p\}.$$

This leads us to generalize the notion of a type, in the standard context $\mathbf{K}$, to include types in infinite tuples of variables.

If $\vec{x}$ is an infinite tuple of variables and $A \in \hat{\mathbf{K}}$, we define a type p in $\vec{x}$ over A to be a consistent set of formulas in subtuples of $\vec{x}$ with parameters in A; and we let

$$p/\approx = \{\varphi(\vec{y};\vec{a})/\approx;\ \varphi(\vec{y};\vec{a}) \in p\},$$

(clearly $p/\approx$ is a type in $\mathbf{K}/\approx$).

We then apply on such types p the terminology applied on the types $p/\approx$ in $\mathbf{K}/\approx$.

Thus, for instance, (for p as above) we say, p is complete if $p/\approx$ is complete (in $\mathbf{K}/\approx$), i.e. p is complete if for any formula $\varphi(\vec{y};\vec{a})$, $\vec{y}$ a subtuple of $\vec{x}$ and $\vec{a}$ in A, either $\varphi(\vec{y};\vec{a})$ or $\neg\varphi(\vec{y};\vec{a})$ belong to p;

given $\varphi(\vec{y};\vec{a})$, $\vec{y}$ a subtuple of $\vec{x}$, $\vec{a} \in A$, $p \vdash_{\overline{A}} \varphi(\vec{y};\vec{a})$ if $p/\approx\ \vdash \varphi(\vec{y};\vec{a})/\approx$ in $\mathbf{K}/\approx$. it follows that $p \vdash_{\overline{A}}$

$\varphi(\vec{y};\vec{a})$ if there is a formula $\Theta(\vec{z};\vec{b})$ in p such that

$$\Theta(\vec{z};\vec{b})\wedge(\vec{y} = \vec{y}) \vdash_{\overline{A}} \varphi(\vec{y};\vec{a})\wedge(\vec{z} = \vec{z});$$

given $f:A \longrightarrow B$, q a complete type over f, q is an S_f-minimal extension of p to A if $q/\approx$ is an S_f-minimal extension of $p/\approx$ to A, i.e. q is an S_f-minimal extension of p to A if for any complete type q' over f, extending p,

$$q^S_f \supset (q')^S_f \Longrightarrow q^S_f = (q')^S_f,$$

etc...

Also, the theorems which apply to types in $\mathbf{K}/\approx$, apply to types in $\mathbf{K}$.

For instance, given $f:B \longrightarrow C$ in $\hat{\mathbf{K}}$, p a type in $\vec{x}$ over f, ($\vec{x}$ possibly infinite), q_1 a complete type over f extending p, there is an S_f-minimal extension q of p to A such that $(q_1)^S_f \supset q^S_f$: indeed, by theorem 3 applied in $\mathbf{K}/\approx$, there is an S_f-minimal extension $q/\approx$ of $p/\approx$ to A such that $(q_1/\approx)^S_f \supset (q/\approx)^S_f$; under the translation described above, this means, q is an S_f-minimal extension of p to A and $(q_1)^S_f \supset q^S_f$, which is what we wanted.

Remark.

(even if $\vec{x}$ is infinite), given $f:A \longrightarrow B$ in $\hat{\mathbf{K}}$, if A is S_f-closed in B (in the standard sense) then A is S_f-closed in B (in the new setting).

Indeed, lemma A.5 still holds in $K/\approx$. Suppose $p/\approx$ is a complete type (in $K/\approx$) over A, $\varphi \in D^S(B)$ and $\varphi \in p^S_f$ i.e. $fp/\approx \vdash \varphi$. By definition, this means there is a finite subset p_0 of $p/\approx$ such that $fp_0 \vdash \varphi$. Say $\varphi = \varphi(\vec{y};\vec{b})/\approx$, $\vec{y}$ a subtuple of $\vec{x}$. it follows easily that there is a finite tuple of variables $\vec{z}$ and a complete type q', in the standard sense, in $\vec{z}$ over A such that

$q'/\approx \subset p'/\approx$, and

$$q' \vdash_{\overline{B}} \varphi(\vec{y};\vec{b}) \wedge (\vec{z} = \vec{z}), \text{ in the standard sense.}$$

Since A is S_f-closed in B, in the standard sense, there is $\Theta(\vec{z};\vec{a})$ in q' such that

$$(q')^S_f \vdash_{\overline{B}} \Theta(\vec{z};\vec{a}) \vdash \varphi(\vec{y};\vec{b}) \wedge (\vec{z} = \vec{z}),$$

in the standard sense.

Thus, $p/\approx \vdash \Theta/\approx \vdash \varphi/\approx$ (in $K/\approx$).

By lemma 5, we conclude that A is S_f-closed in B (in the new setting).

CHAPTER III

The First-Order Case

Let Δ be the set of all formulas in L;

Γ denotes a fixed set of formulas in L, $\Gamma = cl(\Gamma)$, contains the formulas $(x = x)$, x a variable, and Γ is closed under substitution of (any) variable (so if $\varphi(\vec{x}) \in \Gamma$ then $\varphi(\vec{t}) \in \Gamma$ etc...);

S is a fixed set of formulas in Γ, $S = cl^{+}(S)$, (S will stand for a set of equations).

We study in this chapter the case when **K** is the Δ-category of models of a first-order theory T with the Γ-elementary embeddings (c.f. 0.1.(i)) for morphisms and **K** reflects S.

Mainly, we investigate S_H-minimal extensions of types for H a $\Sigma_1(\Gamma)$-closed structure (i.e. a structure for which any formula in Γ with parameters in H is realized in **K** iff it is realized in H).

We show then, that H is S-full (c.f. 3) and that any subset of H is S_H-closed in H, whenever S is a set of equations (c.f. 11). Furthermore, for S a set of equations, S_H-minimal extensions have the local-character property (c.f. 17).

Let us give a sketch of the arguments used to prove the results stated above:

first we show (c.f. lemma 7) that if $A \subset H$ and Θ is a formula in $D^{\Gamma}(H)$ which is A-invariant (i.e. for any $f_1, f_2 : H \longrightarrow F$ with $f_1 \upharpoonright A = f_2 \upharpoonright A$, $f_1\Theta \underset{F}{\approx} f_2\Theta$), then Θ is A-definable (i.e. there is ψ in $D(A)$ such that $\Theta \underset{H}{\approx} \psi$).

Next, given a complete type p over $A \subset H$ and p_0 an S_H-minimal extension of p to H for S an equational set of formulas we let $p_1, \cdots, p_{n-1}$ be the S_H-minimal extension of p to H such that $(p_i)_H^S$ is an A-conjugate of $(p_0)_H^S$ (i.e. there are $f_0, f_1 : H \longrightarrow F$, $f_0 \upharpoonright A = f_1 \upharpoonright A$ and $f_0[(p_0)_H^S] \underset{F}{\approx} f_i[(p_i)_H^S]$). We show then that $\vee_{i<n}(p_i)_H^S$ is A-invariant, whence A-definable, whence $p \vdash_H \vee_{i<n}(p_i)_H^S$.

It follows that $p_H^S \underset{H}{\approx} \vee_{i<n}(p_i)_H^S$ and A is S_H-closed in H. It follows also that the p_i's $(i < n)$, are (all) the S_H-minimal extensions of p to H.

Conversely, we show (c.f. 12) that if p is as above, and p_0 is a complete type over H extending p such that $(p_0)_H^S$ has finitely many A-conjugates then p_0 is an S_H-minimal extension of p.

Finally, if p is a complete type over A and p_0 is an S_H-minimal extension of p to H for S an equational set then, from what preceded, $(p_0)_H^S$ has finitely many A-

conjugates; hence, if $R \subset S$, $(p_0)^R_H$ has finitely many A-conjugates; hence p_0 is an R_H-minimal extension of p to H.

This sketches the proof of the local-character property of S_H-minimal extensions for S an equational set. For S an arbitrary set of equations, the argument is slightly more sophisticated. (c.f. 15 and 17).

0. Preliminaries.

- Δ, Γ, S, K and T are fixed as above.
- $\Sigma_1(\Gamma)$ denotes the closure of Γ under existential quantifiers (i.e. $\Sigma_1(\Gamma)$ is the set of formulas of the form $\exists \vec{w}\Theta(\vec{w};\vec{v})$ where $\Theta \in \Gamma$);

  a formula in $\Sigma_1(\Gamma)$ is also called a $\Sigma_1(\Gamma)$-formula.
- A structure H in K is $\Sigma_1(\Gamma)$-*closed* if for any formula Θ in $\Sigma_1(\Gamma)$ with parameters in H, Θ is realized in K iff Θ is realized in H.
- Given H in K we let $L(H) = L \cup \{c_a;\ a \in H\}$, where c_a is an individual constant not occuring in L and $c_a \neq c_b$ for $a \neq b \in H$; if $\vec{a} = \langle a_1, \cdots, a_n \rangle$ is a tuple of elements in H, we let $\vec{c}_{\vec{a}} = \langle c_{a_1}, \cdots, c_{a_n} \rangle$.

$$\mathrm{diag}(H) = \{\Theta(\vec{c}_{\vec{a}});\ \Theta(\vec{x}) \in \Gamma;\ H \models \Theta(\vec{a})\}.$$

Clearly, for $F \in$ K, F is a model of diag(H) iff there is a Γ-elementary embedding (whence a morphism in

K) from H into F.

In II.C.6 we generalized the notion of a type to include types in infinite tuples of variables: a type in an infinite tuple $\vec{x}$ over A is a consistent set of formulas in subtuples of $\vec{x}$ with parameters in A. Furthermore, we have shown how the terminology and results of the preceding chapters apply to such types; in particular, the monotonicity-transivity property.

We recall (c.f. II.C.6) that for p a type in an infinite tuple $\vec{x}$ over f, when $f{:}A \longrightarrow B$ is in $\hat{\mathbf{K}}$,

$p_f^S = \{\varphi(\vec{y};\vec{a});\ \vec{y}$ a subtuple of $\vec{x};\ \vec{a} \in A,\ \varphi \in S,\ fp \vdash_B \varphi(\vec{y};\vec{a})\}$,

where $fp \vdash_B \varphi(\vec{y};\vec{a})$ means there is a formula $\Theta(\vec{z};\vec{a})$ ($\vec{a} \in A$, $\vec{z}$ a subtuple of $\vec{x}$), which is a finite conjunction of formulas in p and such that

$$\Theta(\vec{z};f\vec{a}) \wedge (\vec{y} = \vec{y}) \vdash_B \varphi(\vec{y};\vec{a}) \wedge (\vec{z} = \vec{z});$$

a complete type q in $\vec{x}$ over f extending p is then an S_f-minimal extension of p to A iff for any complete type q' in $\vec{x}$ over f, extending p

$$q_f^S \supset (q')_f^S \Longrightarrow q_f^S = (q')_f^S.$$

If p is a type in $\vec{x}$ over A ($\in \hat{\mathbf{K}}$) and $\vec{y}$ is a finite subtuple of $\vec{x}$ we let

$p\restriction\vec{y} = \{\varphi(\vec{y};\vec{a}) \in p\}$;

clearly, if p is complete over A then so is $p\restriction\vec{y}$.

1. **Lemma.** Let $H \in \mathbf{K}$, $\Theta(\vec{x};\vec{a})$, $\varphi(\vec{x};\vec{b})$ formulas in $D(H)$. Suppose $\Theta(\vec{x};\vec{a}) \models_H \varphi(\vec{x};\vec{b})$, then there is a $\Sigma_1(\Gamma)$-formula $\psi(\vec{t};\vec{u})$ such that $H \models \psi(\vec{a};\vec{b})$ and

$T \models \forall\vec{t}\forall\vec{u}[\psi(\vec{t};\vec{u}) \longrightarrow \forall\vec{x}(\Theta(\vec{x};\vec{t}) \longrightarrow \varphi(\vec{x};\vec{u}))]$.

Proof. Clearly,

$$T \cup \mathrm{diag}(H) \models \forall\vec{x}(\Theta(\vec{x};\vec{c}_a) \longrightarrow \varphi(\vec{x};\vec{c}_b)).$$

By compactness, there is a Γ-sentence $\chi(\vec{c};\vec{c}_a;\vec{c}_b)$ in $L(H)$ ($\vec{c}$ a tuple of individual constants in $L(H)$ disjoint from $\vec{c}_a$ and $\vec{c}_b$) such that

$$T \cup \chi(\vec{c};\vec{c}_a;\vec{c}_b) \models \forall\vec{x}(\Theta(\vec{x};\vec{c}_a) \longrightarrow \varphi(\vec{x};\vec{c}_b)).$$

Hence,

$$T \models \chi(\vec{c};\vec{c}_a;\vec{c}_b) \longrightarrow \forall\vec{x}(\Theta(\vec{x};\vec{c}_a) \longrightarrow \varphi(\vec{x};\vec{c}_b)),$$

i.e.

$$T \models \forall\vec{t}\vec{u}[\exists\vec{v}\chi(\vec{v};\vec{t};\vec{u}) \longrightarrow \forall\vec{x}(\Theta(\vec{x};\vec{t}) \longrightarrow \varphi(\vec{x};\vec{u}))].$$

Thus $\psi(\vec{t};\vec{u}) = \exists\vec{v}\chi(\vec{v};\vec{t};\vec{u})$ satisfies our conditions. ■

2. **Corollary.** $f{:}H \longrightarrow F$ a morphism in $\mathbf{K}$; H a $\Sigma_1(\Gamma)$-closed structure.

(i) Then any type over H is a type over f (i.e. is consistent over f).

(ii) f reflects Δ.

Proof.

(i) It suffices to show that if $\Theta(\vec{x};\vec{a})$ is a formula in $D(H)$ such that $f\Theta(\vec{x};\vec{a}) \vdash_F (\vec{x} \neq \vec{x})$ then $\Theta(\vec{x};\vec{a}) \vdash_H \vec{x} \neq \vec{x}$.

Suppose $\Theta(\vec{x};f\vec{a}) \vdash_F \vec{x} \neq \vec{x}$. By lemma 1, there is a $\Sigma_1(\Gamma)$ formula $\psi(\vec{t})$ such that $F \models \psi(f\vec{a})$ and

$$T \models \psi(\vec{t}) \longrightarrow \forall\vec{x}(\Theta(\vec{x};\vec{t}) \longrightarrow (\vec{x} \neq \vec{x})).$$

Since H is $\Sigma_1(\Gamma)$-closed and $F \models \psi(f\vec{a})$, $H \models \psi(\vec{a})$. Hence, whenever $g:H \longrightarrow G$ is a morphism in **K**, $G \models \psi(g\vec{a})$ and therefore $G \models \forall\vec{x}(\Theta(\vec{x};g\vec{a}) \longrightarrow (\vec{x} \neq \vec{x}))$. We conclude

$$\Theta(\vec{x};\vec{a}) \vdash_H (\vec{x} \neq \vec{x}).$$

(ii) Suppose Θ and φ are formulas in $D(H)$ such that $f\Theta \vdash_F f\varphi$. Then $f(\Theta\wedge\neg\varphi)$ is inconsistent in **K** over F. It follows from (i) that $\Theta\wedge\neg\varphi$ is inconsistent in **K** over H i.e. $\Theta \vdash_H \varphi$.

Thus, f reflects Δ. ■

3. **Proposition.** Suppose every formula in S is equational. Then any $\Sigma_1(\Gamma)$-closed structure H in **K** is S-full.

Proof. Let H be a $\Sigma_1(\Gamma)$-closed structure in **K**, p a complete type in $\vec{x}$ over H, $p_0 \subset p$. We want to show $p_0 \cup p_H^S$ is S-full.

So let $f:H \longrightarrow F$ be a morphism in $\mathbb{K}$ with $p_0 \cup p_H^S$ consistent over f, and $\varphi_1(\vec{x};\vec{a})$, $\varphi_2(\vec{x};\vec{b})$ formulas in $D^S(F)$ such that

$$f(p_0 \cup p_H^S) \vdash_F \varphi_1(\vec{x};\vec{a}) \vee \varphi_2(\vec{x};\vec{b}).$$

We need to prove that $f(p_H^S) \vdash_F \varphi_i$ for $i = 1$ or 2. let $S_1 = \mathrm{cl}^+(\{\varphi_1(\vec{x};\vec{t}), \varphi_2(\vec{x};\vec{u})\})$; since φ_1 and φ_2 are equations, by proposition I.7.(ii), S_1 is equational. Hence there is a formula Φ in $D^{S_1}(H)$ such that $p_H^{S_1} \underset{H}{\sim} \Phi$.

Suppose $f\Phi \nvdash_F \varphi_1$ and $f\Phi \nvdash_F \varphi_2$; we will show a contradiction.

Since $\mathbb{K}$ reflects S there is a morphism $e:F \longrightarrow E$ such that $ef\Phi \wedge \neg e\varphi_1$ and $ef\Phi \wedge \neg e\varphi_2$ are realized in E (c.f. I.A.5). Now let

$\Theta \in p_0 \cup p_H^S$ such that

$$f\Theta \vdash_F \varphi_1(\vec{x};\vec{a}) \vee \varphi_2(\vec{x};\vec{b}).$$

By lemma 1 there is a $\Sigma_1(\Gamma)$-formula $\psi(\vec{t};\vec{u})$ in L such that $F \models \psi(\vec{a};\vec{b})$ and

$$T \models \psi(\vec{t};\vec{u}) \longrightarrow \forall \vec{x}(\Theta(\vec{x}) \longrightarrow \varphi_1(\vec{x};\vec{t}) \vee \varphi_2(\vec{x};\vec{u})).$$

Since $\psi \in \Sigma_1(\Gamma)$, $E \models \psi(e\vec{a};e\vec{b})$. We have

$\mathcal{E} \models \exists \vec{t}\vec{u}\vec{x}_1\vec{x}_2 \psi(\vec{t};\vec{u}) \wedge e\cdot f\Phi(\vec{x}_1) \wedge \neg\varphi_1(\vec{x}_1;\vec{t}) \wedge e\cdot f\Phi(\vec{x}_2) \wedge \neg\varphi_2(\vec{x}_2;\vec{u})$.

Since H is $\Sigma_1(\Gamma)$-closed, it follows that the $\Sigma_1(\Gamma)$-formula

$$\psi(\vec{t};\vec{u}) \wedge \Phi(\vec{x}_1) \wedge \neg\varphi_1(\vec{x}_1;\vec{t}) \wedge \Phi(\vec{x}_2) \wedge \neg\varphi_2(\vec{x}_2;\vec{u})$$

is realized in H.

In other words there are tuples $\vec{c}$ and $\vec{d}$ in H such that $H \models \psi(\vec{c};\vec{d})$, $H \models \exists\vec{x}\Phi(\vec{x}) \wedge \neg\varphi_1(\vec{x};\vec{c})$ and $H \models \exists\vec{x}\Phi(\vec{x}) \wedge \neg\varphi_2(\vec{x};\vec{d})$.

But then

$$T \models \forall\vec{x}(\Theta(\vec{x}) \longrightarrow \varphi_1(\vec{x};\vec{c}) \vee \varphi_2(\vec{x};\vec{d}))$$

i.e.

$$\Theta \models_{H} \varphi_1(\vec{x};\vec{c}) \vee \varphi_2(\vec{x};\vec{d})$$

hence

$$\rho \models_{H} \varphi_1(\vec{x};\vec{c}) \vee \varphi_2(\vec{x};\vec{d})$$

and therefore, by completeness, either $\rho \models_{H} \varphi_1(\vec{x};\vec{c})$ or $\rho \models_{H} \varphi_2(\vec{x};\vec{c})$;

i.e. either $\Phi \models_{H} \rho_H^{S_1} \models_{H} \varphi_1(\vec{x};\vec{c})$ or $\Phi \models_{H} \rho_H^{S_1} \vdash \varphi_2(\vec{x};\vec{d})$, contradicting the fact that $\Phi \wedge \neg\varphi_1(\vec{x};\vec{c})$ and $\Phi \wedge \neg\varphi_2(\vec{x};\vec{d})$ are realized in H. ■

4. **Corollary.** If **K** has elementary embeddings for morphisms, (i.e. if Γ is the set of all formulas) and S is a set of equations then any structure in **K** i.e. any model of T is S-full.

Proof. Follows immediately from proposition 3 and the fact that in this case any model of T is $\Sigma_1(\Gamma)$-closed. ■

5. **Proposition.** Let $h: A \longrightarrow H$ and $f: A \longrightarrow F$ be morphisms in $\hat{K}$, $A \in \hat{K}$, H and F $\Sigma_1(\Gamma)$-closed structures in K which are S-full; p a type over A which is consistent over f and g. Then, given $g_1: H \longrightarrow G$ and $g_2: F \longrightarrow G$ such that $g_1 \cdot h \upharpoonright A = g_2 \cdot f \upharpoonright A$ and an S_H-minimal extension p_1 of p to h, there is an S_F-minimal extension p_2 of p to f such that

$$g_1[(p_1)^S_H] \underset{G}{\approx} g_2[(p_2)^S_F].$$

Proof. Since H and F are assumed S-full, for any S_H- (resp. S_F-) minimal extension q of p to h (resp. to f), q^S_H (resp. q^S_F) is an S_h-component (resp. S_f-component) of p. In particular, $(p_1)^S_H$ is an S_h-component of p.

By corollary 2, p is consistent over g_1.

It follows by proposition II.B.11 that there exists an S_F-minimal extension p_2 of p to f such that $g_2[(p_2)^S_F] \underset{G}{\approx} g_1[(p_1)^S_H]$. ■

Note. In proposition 5 above, if S is a set of equations, then by proposition 3, H and F are S-full once H and F are $\Sigma_1(\Gamma)$-

closed.

6. **Definition.** Let $A \subset H \in \mathbf{K}$,

Θ a formula in $D(H)$. We say Θ is A-invariant if for any morphisms $f_1 : H \longrightarrow F$ and $f_2 : H \longrightarrow F$ such that $f_1 \upharpoonright A = f_2 \upharpoonright A$ we have $f_1\Theta \underset{F}{\approx} f_2\Theta$.

Equivalently, Θ is A-invariant if for any *isomorphism* $\sigma : H \longrightarrow F$ and morphisms $g_1 : F \longrightarrow G$ and $g_2 : H \longrightarrow G$ such that $g_1 \cdot \sigma \upharpoonright A = g_2 \upharpoonright A$, we have $g_1\sigma\Theta \underset{G}{\approx} g_2\Theta$.

Indeed, to obtain the second version from the first, apply the first version to $f_1 = g_1\sigma$ and $f_2 = g_2$; to obtain the first version from the second, apply the second version to $\sigma = \mathrm{id}_H$, $g_1 = f_1$, $g_2 = f_2$.

If $h : A \longrightarrow H$ is in $\hat{\mathbf{K}}$ we say $\Theta(\in D(H))$ is h-invariant if Θ is $h(A)$-invariant.

7. **Lemma.** Let $A \subset H \in \mathbf{K}$.

$\Theta(\vec{x};\vec{b})$ a formula in $D(H)$. Then Θ is A-invariant iff Θ is A-definable (i,e, there is ψ in $D(A)$, $\Theta \underset{H}{\approx} \psi$). If $\Theta \in D^{\Gamma}(H)$ and Θ is A-invariant then in fact there is a formula ψ in $\Sigma_1(\Gamma)$ with parameters in A such that $\Theta \underset{H}{\approx} \psi$.

Proof. It is easy to see that if Θ is A-definable then Θ is A-invariant.

Conversely, suppose Θ is A-invariant. Let σH be an isomorphic copy of H; σ, the isomorphism from H onto σH. Then, by (the second version) definition of A-invariance:

$$T \cup \mathrm{diag}(H) \cup \mathrm{diag}(\sigma H) \cup \{(c_a = c_{\sigma a});\ a \in A\} \models$$
$$\Theta(\vec{x};\vec{c}_b) \longleftrightarrow \Theta(\vec{x};\vec{c}_{\sigma\vec{b}}).$$

By compactness, it follows there is a sentence $\varphi(\vec{c}_h;\vec{c}_a)$ in $\mathrm{diag}(H)$, where $\vec{a} \in A$ and $\vec{h} = \vec{b}\frown\vec{b}_1$ a tuple of elements in $H \setminus A$, such that

$$(*) T \cup \{\varphi(\vec{c}_h;\vec{c}_a),\ \varphi(\vec{c}_{\sigma h};\vec{c}_{\sigma a})\} \cup \{(\vec{c}_a = \vec{c}_{\sigma a})\} \models$$
$$\Theta(\vec{x};\vec{c}_b) \longleftrightarrow \Theta(\vec{x};\vec{c}_{\sigma b}).$$

Consider the formula

$$\psi(\vec{x};\vec{a}) = \exists\vec{t}\vec{t}_1\Theta(\vec{x};\vec{t}) \wedge \varphi(\vec{t}\frown\vec{t}_1;\vec{a}),\quad (\psi \in D(A));$$

length$\vec{t}$ = length$\vec{b}$, length$\vec{t}_1$ = length$\vec{b}_1$, (recall $\vec{h} = \vec{b}\frown\vec{b}_1$).

Claim. $\Theta(\vec{x};\vec{b}) \underset{H}{\sim} \psi(\vec{x};\vec{a})$.

For, given a morphism $f:H \longrightarrow F$ in **K** and $\vec{d}$ in F such that $F \models \Theta(\vec{d};f\vec{b})$, we have that

$F \models \varphi(f\vec{h};f\vec{a})$ (since $\varphi(\vec{c}_h;\vec{c}_a) \in \mathrm{diag}(H)$), and therefore

$$F \models \exists\vec{t}\vec{t}_1\Theta(\vec{d};\vec{t}) \wedge \varphi(\vec{t}\frown\vec{t}_1;f\vec{a})$$

i.e. $F \models \psi(\vec{d}; f\vec{a})$.

Thus $F \models \forall\vec{x}(f\Theta(\vec{x}) \longrightarrow f\psi(\vec{x}))$. Since f was arbitrary, it follows that $\Theta \models_{\overline{H}} \psi$.

Conversely, given a morphism $f: H \longrightarrow F$ and $\vec{d}$ in F, such that $F \models \psi(\vec{d}; f\vec{a})$, there exist (by definition of ψ) $\vec{b}'$ and $\vec{h}'$, tuples in F such that

$$F \models \Theta(\vec{d}; \vec{b}') \wedge \varphi(\vec{h}'; f\vec{a}).$$

So now we have

$$F \models \varphi(f\vec{h}; f\vec{a}) \wedge \varphi(\vec{h}'; f\vec{a});$$

from (*) above it follows that

$$F \models \forall\vec{x}(\Theta(\vec{x}; f\vec{b}) \longleftrightarrow \Theta(\vec{x}; \vec{b}')).$$

Since $F \models \Theta(\vec{d}; \vec{b}')$ we get $F \models \Theta(\vec{d}; f\vec{b})$. Thus

$$F \models \forall\vec{x}(f\psi(\vec{x}) \longrightarrow f\Theta(\vec{x})).$$

Since f was arbitrary, we deduce that $\psi \models_{\overline{H}} \Theta$ and conclude, $\Theta \underset{H}{\simeq} \psi$.

Finally, it is clear that if Θ is in $D^{\Gamma}(H)$ then ψ as defined is in $\Sigma_1(\Gamma)$. ∎

8. **Definition.** Let $A \subset H \in \mathbf{K}$; p_1 and p_2 types over H. We say p_2 is an A-conjugate of p_1 if there are morphisms $f_1: H \longrightarrow F$ and $f_2: H \longrightarrow F$ such that p_1 is consistent over f_1, $f_1 \upharpoonright A = f_2 \upharpoonright A$ and $f_1 p_1 \underset{F}{\simeq} f_2 p_2$.

Equivalently, p_2 is an A-conjugate of p_1 if there is an isomorphism $\sigma : H \longrightarrow F$ and morphisms $g_1 : F \longrightarrow G$ and $g_2 : H \longrightarrow G$ such that σp_1 is consistent over g_1, $g_1 \cdot \sigma \restriction A = g_2 \restriction A$ and $g_2 p_2 \underset{G}{\approx} g_1 \sigma p_1$.

Indeed to obtain the second version from the first, take $\sigma = \mathrm{id}_H$, $g_1 = f_1$ and $g_2 = f_2$; to obtain the first version from the second, take $f_1 = g_1 \sigma$ and $f_2 = g_2$.

If $h : A \longrightarrow H$ is in $\hat{K}$, p_1, p_2 as above, we say p_2 is an h-conjugate of p_1, if p_2 is an $h(A)$-conjugate of p_1.

9. Lemma. Given $A \subset H \in \hat{K}$, with H $\Sigma_1(\Gamma)$-closed, A-conjugation is an equivalence relation on the set of types over H.

Proof. Reflexivity and symmetry are immediate.

Suppose p_1, p_2, p_3 are types over H such that p_2 is an A-conjugate of p_1 and p_3 is an A-conjugate of p_2. Hence, by definition, there are morphisms $f_1, f_2 : H \longrightarrow F$ and $g_2, g_3 : H \longrightarrow G$ such that $f_1 \restriction A = f_2 \restriction A$, $g_2 \restriction A = g_3 \restriction A$ and $f_1 p_1 \underset{F}{\approx} f_2 p_2$, $g_2 p_2 \underset{G}{\approx} g_3 p_3$.

Since H is $\Sigma_1(\Gamma)$-closed, we can find morphisms $h_1 : F \longrightarrow E$ and $h_3 : G \longrightarrow E$ in $\mathbb{K}$ such that $h_1 \cdot f_2 = h_3 \cdot g_2$. Let $e_1 = h_1 \cdot f_1$ and $e_3 = h_3 \cdot g_3$; we have

$$e_1{\restriction}A = h_1\cdot f_1{\restriction}A = h_1\cdot f_2{\restriction}A = h_3\cdot g_2{\restriction}A = e_3{\restriction}A, \text{ and}$$

$$e_1p_1 \underset{E}{\cong} h_1\cdot f_2p_2 \underset{E}{\cong} h_3\cdot g_2p_2 \underset{E}{\cong} e_3p_3.$$

We conclude p_3 is an A-conjugate of p_1. ■

10. **Theorem. Assume S is equational. Let $h:A\longrightarrow H$ be in $\hat{K}$ with H $\Sigma_1(\Gamma)$-closed; p a *complete* type over h, p_1 an S_H-minimal extension of p to h. Then a complete type p_2 over H, extending p to h, is an S_H-minimal extension of p to h iff $(p_2)^S_H$ is an h-conjugate of $(p_1)^S_H$. Moreover $(p)^S_H$ is h-definable.**

Proof. Suppose p_2 is complete over H and $(p_2)^S_H$ is an h-conjugate of $(p_1)^S_H$. Then there are morphisms $f_1:H\longrightarrow F$ and $f_2:H\longrightarrow F$ such that $f_1\cdot h{\restriction}A = f_2\cdot h{\restriction}A$ and $f_1((p_1)^S_H) \underset{F}{\cong} f_2((p_2)^S_H)$.

Let q_1 (resp. q_2) be an S_F-minimal extension of p_1 (resp. p_2) to f_1 (resp. f_2). Since H is $\Sigma_1(\Gamma)$-closed, H is S-full (by proposition 3); hence (c.f. II.B.5.(ii))

$$(q_1)^S_F \underset{F}{\cong} f_1((p_1)^S_H) \underset{F}{\cong} f_2((p_2)^S_H) \underset{F}{\cong} (q_2)^S_F.$$

Now H is S_{f_1}-closed and S_{f_2}-closed in F (c.f. II.B.14.(ii)); hence, by transitivity (II.A.12) q_1 is an S_F-minimal extension of p to $f_1\cdot h$. Since $f_1h{\restriction}A = f_2h{\restriction}A$ and

$(q_2)^S_F \underset{F}{\approx} (q_1)^S_F$, q_2 is an S_F-minimal extension of p to $f_2 \cdot h$.

By monotonicity (II.A.12) it follows that p_2 is an S_H-minimal extension of p to h. That proves one direction of the claim.

Suppose now p_2 is an S_H-minimal extension of p to h.

Let $q_0, \cdots, q_{n-1}$ be, up to S_H-equivalence, all the S_H-minimal extensions of p to h such that $(q_i)^S_H$ $(i < n)$ is an h-conjugate of $(p_1)^S_H$, (by II.A.13 there are finitely many such extensions);

let $\Theta = \vee_{i<n} (q_i)^S_H$.

Claim. Θ is h-invariant.

Proof of the claim. Let $f_1, f_2 : H \longrightarrow F$ be morphisms in K, such that $f_1 \cdot h \restriction A = f_2 \cdot h \restriction A$; we have to show $f_1 \Theta \underset{F}{\approx} f_2 \Theta$.

By proposition 5, for each $i < n$ there is an S_H-minimal extension q_i' of p to h such that

$$f_1((q_i)^S_H) \underset{F}{\approx} f_2((q_i')^S_H).$$

But then, by definition, $(q_i)^S_H$ is an h-conjugate of $(q_i')^S_H$, whence (by lemma 9) an h-conjugate of $(p_1)^S_H$. Hence there is $j < n$ such that

$$(q_i')^S_H \underset{H}{\approx} (q_j)^S_H.$$

It easily follows that there is a permutation τ of n such that

$$f_1((q_i)^S_H) \underset{F}{\simeq} f_2(q_{\tau i})^S_H) \quad (i < n).$$

Clearly then

$$f_1(\vee_{i<n}(q_i)^S_H) \underset{F}{\simeq} f_2(\vee_{i<n}(q_{\tau i})^S_H) \underset{F}{\simeq} f_2(\vee_{i<n}(q_i)^S_H).$$

We conclude $f_1\Theta \underset{F}{\simeq} f_2\Theta$, which is what we wanted.

By lemma 7 it follows that Θ is h-definable i.e. there is ψ in $D(A)$ such that $\Theta \underset{H}{\simeq} h\psi$.

Obviously, $(q_i)^S_H \Vdash_H \Theta$ (any $i < n$); hence $q_i \Vdash_H h\psi$. Therefore

$$\psi \in q_i \upharpoonright h = p \text{ and } p^S_h \vdash h\psi \vdash \vee_{i<n}(q_i)^S_H.$$

On the other hand,

$$h\psi \underset{H}{\simeq} \vee_{i<n}(q_i)^S_H \Vdash_H p^S_h \text{ (since } q_i \text{ extends } p \text{ to } h).$$

We conclude that

$$p^S_h \underset{H}{\simeq} h\psi \underset{H}{\simeq} \vee_{i<n}(q_i)^S_H.$$

It follows easily that, up to S_H-equivalence, the types q_i $(i < n)$ enumerate all the S_H-minimal extensions of p to h (c.f. II.B.6). Hence $(p_2)^S_H \sim (q_i)^S_H$ for some $i < n$, which means that $(p_2)^S_H$ is an h-conjugate of $(p_1)^S_H$. Finally, since $p^S_h \underset{H}{\simeq} h\psi$, p^S_h is h-definable. ■

11. **Corollary.** Let $h: A \longrightarrow H$ be a morphism in $\mathcal{K}$ with H a $\Sigma_1^*(\Gamma)$-

closed structure in K. If S is a set of equations, then A is S_h-closed in H.

Proof. By II.A.5, A is S_h-closed in H iff for any complete type p over h and φ in $D^S(H)$ such that $hp \vdash_{\overline{H}} \varphi$ there is Θ in $D(A)$ such that $p^S_H \vdash_{\overline{H}} h\Theta \vdash_{\overline{H}} \varphi$.

But, if $hp \vdash_{\overline{H}} \varphi$ then $p^{S_1}_h \vdash_{\overline{H}} \varphi$ where $S_1 = cl^+(\{\varphi\})$, and by theorem 10, $p^{S_1}_h$ is h-definable i.e. $p^{S_1}_h \underset{\overline{H}}{\sim} h\Theta$ for some Θ in $D(A)$. The conclusion is now clear. ∎

So, if S is a set of equations and H is $\Sigma_1(\Gamma)$-closed, any subset of H is S_H-closed in H. thus, the monotonicity-transitivity theorem (II.A.12) applies to S_H-minimal extensions of types over subsets of H.

Theorem 12 below intuitively says that, (given $h:A \longrightarrow H$, p over h, p_1 complete over H), p_1 is an S_H-minimal extension of p to h iff $(p_1)^S_H$ is "almost over h". The exact definition of "almost over" will be given in chapter V.

12. **Theorem.** Let S, h and p be as in theorem 10; p_1 a complete type over H, extending p to h. Then p_1 is an S_H-minimal

extension of p to h iff there are $p_2, \cdots, p_n$, complete types over H extending p such that $(p_i)_H^S$ is an h-conjugate of $(p_1)_H^S$ $(1 < i < n)$ and for any morphisms $f_1, f_2 : H \longrightarrow F$ with $f_1 \cdot h \restriction A = f_2 \cdot h \restriction A$ and any i, $1 \leq i \leq n$, there is j, $1 \leq j \leq n$ such that $f_1((p_i)_H^S) \underset{F}{\approx} f_2((p_j)_H^S)$.

Proof. suppose p_1 is an S_H-minimal extension of p to h. Let $p_1, \cdots, p_n$ be, up to S_H-equivalence, all the S_H-minimal extensions of p to h. By theorem 10, $(p_i)_H^S$ is an h-conjugate of $(p_1)_H^S$, $(1 \leq i \leq n)$. By proposition 5, for any morphisms $f_1 : f_2 : H \longrightarrow F$ and any i, $1 \leq i \leq n$, there is j, $1 \leq j \leq n$, such that

$$f_1((p_i)_H^S) \underset{F}{\approx} f_2((p_j)_H^S).$$

That proves one direction of the claim.

Conversely, suppose there are $p_2, \cdots, p_n$ such as they satisfy the right hand side term of the claim. Let $\Theta = \vee_{1 \leq i \leq n} (p_1)_H^S$.

Θ is h-invariant: for if $f_1, f_2 : H \longrightarrow F$ are such that $f_1 \cdot h \restriction A = f_2 \cdot h \restriction A$ then it is easily seen that $f_1 \Theta \underset{F}{\approx} f_2 \Theta$.

By lemma 7 it follows that Θ is h-definable i.e. there is ψ in $D(A)$ such that $\Theta \underset{H}{\approx} h\psi$. Now of course $\Theta \in p_1$; hence $\psi \in p$, for $hp \subset p_1$. Thus $hp \vdash_H \Theta$ i.e.

$$hp \vdash_H \bigvee_{1 \le i \le n} (p_i)^S_H.$$

It follows that at least one of the p_i's is an S_H-minimal extension of p to h (c.f. II.B.6). By theorem 10, since the $(p_i)^S_H$ are assumed h-conjugate of each other , it follows that all the p_i's and in particular p_1, are S_H-minimal extensions of p to h.

13. **Corollary.** Let S, h and p be as in theorem 10 and suppose S contains the formulas $(x = x)$, x any variable; let p_1 be an S_H-minimal extension of p to h. Say p is a type in $\vec{x}$; let $\vec{y}$ be a subtuple of $\vec{x}$ and $R \subset S$, $R = \mathrm{cl}^+(R)$. Then $(p_1 \restriction \vec{y})$ is an $(R)_H$-minimal extension of $p \restriction \vec{y}$ to h.

Proof. Let us show first that if q_1 and q_2 are complete types over H and $f_1, f_2 : H \longrightarrow F$ are morphisms in **K** such that

$$f_1[(q_1)^S_H] \underset{F}{\approx} f_2[(q_2)^S_H]$$

then

$$f_1[(r_1)^R_H] \underset{F}{\approx} f_2[(r_2)^R_H],$$

where $r_1 = q_1 \restriction \vec{y}$ and $r_2 = q_2 \restriction \vec{y}$.

Indeed, suppose q_1, q_2, f_1, f_2 are given as above.

We have that

$$f_1(q_1) \vdash_F f_2[(q_2)^S_H];$$

since S contains the formulas $(x = x)$, it easily follows that

$$f_1(r_1) \models_F f_2[(r_2)^R_H].$$

Now, $R \subset S$ implies R is equational; therefore, by proposition 3, H is R-full. In particular r_1 is R_{f_1}-full.

Since

$$f_2[(r_2)^R_H] \subset D^R(H) \text{ and } f_1(r_1) \models_F f_2[(r_2)^R_H],$$

we deduce that

$$f_1[(r_1)^R_H] \models_H f_2[(r_2)^R_H].$$

By symmetry, we get

$$f_2[(r_2)^R_h] \models_F f_1[(r_1)^R_H].$$

We conclude

$$f_1[(r_1)^R_H] \cong_F f_2[(r_2)^R_H],$$

which is what we wanted.

Back to our proof of the corollary: by proposition 12, since p_1 is an S_H-minimal extension of p to h, there are $p_2, \cdots, p_n$, complete types over H extending p, such that $(p_i)^S_H$ is an h-conjugate of $(p_1)^S_H$ and, for any $f_1, f_2 : H \longrightarrow F$ with $f_1 \cdot h \restriction A = f_2 \cdot h \restriction A$ and $1 \le i \le n$, there is j, $1 \le j \le n$, such that

$$f_1[(p_i)^S_H] \cong_F f_2[(p_j)^S_H].$$

Let $r_i = p_i \restriction \vec{y}$. Then, using the claim above, we see that $(r_i)^R_H$ is an h-conjugate of $(r_1)^R_H$ and for any

$f_1, f_2: H \longrightarrow F$ with $f_1 \cdot h \upharpoonright A = f_2 \cdot h \upharpoonright A$ and $1 \le i \le n$ there is $1 \le j \le n$ such that

$$f_1[(r_i)^R_H] \underset{F}{\simeq} f_2[(r_j)^R_H].$$

We conclude by proposition 12 that r_1 is an R_H-minimal extension of p to h. ■

14. Proposition. Let S be equational, $h: A \longrightarrow H$ in $\hat{\mathbf{K}}$ with H $\Sigma_1(\Gamma)$-closed, p complete over h, p_1 and p_2 non S_H-equivalent S_H-minimal extensions of p to h. Then $hp \cup \{(p_1)^S_H, (p_2)^S_H\}$ is inconsistent in $\mathbf{K}$.

Proof. Let $q_0, \cdots, q_{n-1}$ be, up to S_H-equivalence, all the S_H-minimal extensions of p to h.

Let Θ be a single formula in $D(H)$ such that

$$\Theta \underset{H}{\simeq} \vee\{(q_i)^S_H \wedge (q_j)^S_H;\ i, j < n,\ i \ne j\}$$

(just consider $(q_i)^S_H$ and $(q_j)^S_H$ as single formulas).

Claim. Θ is h-invariant.

Indeed, let $f_1, f_2: H \longrightarrow F$ be such that $f_1 \cdot h \upharpoonright A = f_2 \cdot h \upharpoonright A$; we want to show $f_1\Theta \underset{F}{\simeq} f_2\Theta$.

By proposition 5, for any $i < n$ there is $\mathbf{1}_i < n$ such that

$$f_1[(q_i)^S_H] \underset{F}{\simeq} f_2[(q_{\mathbf{1}_i})^S_H].$$

Since K reflects S, if $i \neq j$ then $\ell_i \neq \ell_j$.

The map $i \longmapsto \ell_i$ is therefore a permutation of n. It clearly follows that $f_1\Theta \underset{H}{\sim} f_2\Theta$. That proves the claim.

By lemma 7 we deduce that $\Theta \underset{H}{\sim} h\psi$ for some ψ in $D(A)$.

Suppose $hp \cup \{\Theta\}$ is consistent. Then, by completeness, ψ belongs to p; hence $hp \vdash_H \Theta$. Since p_1 extends p, $p_1 \vdash_H \Theta$ i.e.

$$p_1 \vdash_H \vee\{(q_i)^S_H \wedge (q_j)^S_H;\ i,j < n,\ i \neq j\}.$$

By completeness it follows that $p_1 \vdash_H (q_i)^S_H \wedge (q_j)^S_H$ for some $i,j < n$, $i \neq j$.

Thus

$$p_1 \vdash_H (q_i)^S_H \text{ and } p_1 \vdash_H (q_j)^S_H;$$

by minimality of p_1 we must have

$$(p_1)^S_H \underset{H}{\sim} (q_i)^S_H \text{ and } (p_1)^S_H \underset{H}{\sim} (q_j)^S_H;$$

whence $(q_i)^S_H \underset{H}{\sim} (q_j)^S_H$ ✗.

We conclude that $hp \cup \{\Theta\}$ must be inconsistent in K, which means that for any $i,j < n$, $i \neq j$ $hp \cup \{(q_i)^S_H \wedge (q_j)^S_H\}$ is inconsistent in K. In particular $hp \cup \{(p_1)^S_H \wedge (p_2)^S_H\}$ is inconsistent in K. ■

15. Proposition. Let $S = cl^+(\bigcup_{i \in I} S_i)$ where for each $i \in I$,

$S_i = cl^+(S_i)$ is equational, and S contains the formulas $(x = x)$, x any variable; $h: A \longrightarrow H$ in $\hat{K}$, H $\Sigma_1(\Gamma)$-closed; p a complete type in a (possibly infinite) tuple of variables $\vec{x}$; $(\vec{x}_i)_{i \in I}$ a family of finite subtuples of $\vec{x}$ and q_i, for $i \in I$, an $(S_i)_H$-minimal extension of $p \restriction \vec{x}_i$ to h (q_i is a type in $\vec{x}_i$). Suppose $q_0 = hp \cup \{(q_i)^{S_i}_H;\ i \in I\}$ is consistent in K. Then $q = hp \cup \{q_i \restriction S_i;\ i \in I\}$ is consistent in K.

Proof. To prove the consistency of q it suffices to assume $\vec{x}$ is a finite tuple and I is finite. S is then equational.

Let r be a complete extension of q_0 to H. By II.A.2 there is an S_H-minimal extension p_1 of p to h such that $r^S_H \supset (p_1)^S_H$.

By corollary 13, $p_1 \restriction \vec{x}_i$ is an S_i-minimal extension of $p \restriction \vec{x}_i$ to h. Now, $hp \cup \{(p_1 \restriction \vec{x}_i)^{S_i}_H,\ (q_i)^{S_i}_H\}$ is consistent since it is contained in r. By proposition 14, it follows that

$(p_1 \restriction \vec{x}_i)^{S_i}_H \underset{H}{\sim} (q_i)^{S_i}_H$ and therefore $(p_1 \restriction \vec{x}_i) \restriction S_i = q_i \restriction S_i$.

We conclude that $p_1 \supset q$, which implies that q is consistent. ■

Note. With the notations of proposition 15, clearly $(q \restriction \vec{x}_i) \restriction S_i = q_i \restriction S_i$. Thus for any $i \in I$, $q \restriction \vec{x}_i$ is an $(S_i)_H$-

minimal extension of $p \restriction \vec{x}_i$ to h.

16. **Lemma.** Let S, h and p be as in proposition 15; let p_1 be a complete type over H extending p to h. Then p_1 is an S_H-minimal extension of p to h iff for any $i \in I$ and finite subtuple $\vec{y}$ of $\vec{x}$, $p_1 \restriction \vec{y}$ is an $(S_i)_H$-minimal extension of p to h.

Proof. Suppose $p_1 \restriction \vec{y}$ is an $(S_i)_H$-minimal extension of p to h for any $i \in I$ and finite subtuple $\vec{y}$ of $\vec{x}$.

Now, if q is a complete type over H extending p to h with $(p_1)_H^S \supset q_H^S$ then for any $\vec{y}$ and $i \in I$,

$$(p_1 \restriction \vec{y})_H^{S_i} \supset (q \restriction \vec{y})_H^{S_i};$$

hence by $(S_i)_H$-minimality,

$$(p_1 \restriction \vec{y})_H^{S_i} = (q \restriction \vec{y})_H^{S_i}$$

for any $\vec{y}$ and $i \in I$.

But clearly

$$(p_1)_H^S \underset{H}{\sim} \bigcup \{(p_1 \restriction \vec{y})_H^{S_i};\ \vec{y} \text{ a finite subtuple of } \vec{x},\ i \in I\}$$

and

$$q_H^S \underset{H}{\sim} \bigcup \{(q \restriction \vec{y})_H^{S_i};\ \vec{y} \text{ a subtuple of } \vec{x},\ i \in I\}.$$

It follows that $(p_1)_H^S \underset{H}{\sim} q_H^S$, which shows that p_1 is an S_H-minimal extension of p to h.

Conversely, suppose p_1 is an S_H-minimal extension of p to h. By II.A.2, for any finite subtuple $\vec{y}$ of $\vec{x}$ and any $i \in I$, there is an $(S_i)_H$-minimal extension $q_{i,\vec{y}}$ of $p \restriction \vec{y}$ to h such that

$$(p_1 \restriction \vec{y})_H^{S_i} \supset (q_{i,\vec{y}})_H^{S_i}.$$

Let $q_0 = hp \cup \{(q_{i,\vec{y}})_H^{S_i};\ i \in I,\ \vec{y}$ a finite subtuple of $\vec{x}\}$

and $q = hp \cup \{q_{i,\vec{y}} \restriction S_i;\ i \in I,\ \vec{y}$ a finite subtuple of $\vec{x}\}$.

Since $q_0 \subset p_1$, q_0 is consistent in K; hence by proposition 15, q is consistent, (write $S = cl^+(\bigcup_{i,\vec{y}} S_{i,\vec{y}})$ if necessary, where $S_{i,\vec{y}} = S_i$).

Let q_1 be a complete extension of q over H; clearly

$$(q_1)_H^S \underset{H}{\cong} \bigcup_{i,\vec{y}} (q_{i,\vec{y}})_H^{S_i}.$$

Since for any i, $\vec{y}$,

$$(q_{i,\vec{y}})_H^{S_i} \subset (p_1 \restriction \vec{y})_H^{S_i} \subset (p_1)_H^S,$$

we have that $(q_1)_H^S \subset (p_1)_H^S$. By S_H-minimality, we deduce that $(p_1)_H^S = (q_1)_H^S$; hence

$$(p_1 \restriction \vec{y})_H^{S_i} = (q_1 \restriction \vec{y})_H^{S_i} = q_{i,\vec{y}} \restriction S_i.$$

We conclude that $p_1 \restriction \vec{y}$ is an S_i-minimal extension of p to h for any i in I and subtuple $\vec{y}$ of $\vec{x}$. ■

17. Theorem (local character). Let $S = cl^{+}(\bigcup_{i\in I}S_i)$ where for any $i \in I$, $S_i = cl^{+}(S_i)$ and S_i is equational and suppose S contains the formulas $(x = x)$, x any variable; $A \subset B \subset H$, H a $\Sigma_1(\Gamma)$-closed structure in K; p a complete type in $\vec{x}$ over A, $\vec{x}$ possibly infinite; p_1 a complete type over B extending p.

Then, p_1 is an S_H-minimal extension of p to B iff for any i in I and finite subtuple $\vec{y}$ of $\vec{x}$, $p\restriction\vec{y}$ is an $(S_i)_H$-minimal extension of $p\restriction\vec{y}$ to B.

Proof. Let p_2 be an S_H-minimal extension of p_1 to H. Consider the following assertions:

1. p_1 is an S_H-minimal extension of p to B.
2. p_2 is an S_H-minimal extension of p to H.
3. $(p_2\restriction\vec{y})$ is an $(S_i)_H$-minimal extension of $p\restriction\vec{y}$ to H for any i in I and finite subtuple $\vec{y}$ of $\vec{x}$.
4. $p_1\restriction\vec{y}$ is an $(S_i)_H$-minimal extension of $p\restriction\vec{y}$ to B for any i in I and finite subtuple $\vec{y}$ of $\vec{x}$.

Now, by corollary 11, B is S_H-closed in H; hence by the monotonicity-transitivity theorem (c.f. II.A.12 and II.C.6)) we find that 1. $\Leftrightarrow$ 2.

From lemma 15 we have that 2. $\Leftrightarrow$ 3.

By monotonicity (II.A.12) applied to the types $p_2\restriction\vec{y}$

and the equational sets S_i we find 3. ⟺ 4.

We conclude 1. ⟺ 4. which is what we wanted. ■

We close this chapter on a nullstellensatz type of result on $\Sigma_1(\Gamma)$-closed structures.

18. **Proposition.** Let H be a $\Sigma_1(\Gamma)$-closed structure in $\mathbb{K}$; $\varphi(\vec{x};\vec{a})$ a consistent formula in $\mathcal{D}^{\Gamma}(H)$ and $\psi(\vec{x};\vec{t})$ a formula in S. Assume ψ is an equation. Then there is a sequence $(\vec{c}_i)_{i<n}$ of parameters in H, such that for any morphism $f:H \longrightarrow F$ and tuple $\vec{b}$ in F,
$\varphi(\vec{x};\vec{a}) \models_F \psi(\vec{x};\vec{b}) \iff F \models \wedge_{i<n}\psi(f\vec{c}_i;\vec{b})$.

Proof. Choose $\vec{c}_0,\cdots,\vec{c}_{n-1}$ in $\varphi(H;\vec{a})$ such that

$$\wedge_{i<n}\psi(\vec{c}_i;\vec{t}) \underset{H}{\approx} \wedge\{\psi(\vec{c};\vec{t}),\ \vec{c} \in \varphi(H;\vec{a})\};$$

$\vec{c}_0,\cdots,\vec{c}_{n-1}$ can be so chosen because $\psi(\vec{x};\vec{t})$ is an equation in $\vec{t}$ (c.f. I.8).

We show that $(c_i)_{i<n}$ satisfies the conditions of the proposition.

Let $f:H \longrightarrow F$ be in $\mathbb{K}$ and $\vec{b}$ in F.

If $\varphi(\vec{x};f\vec{a}) \models_F \psi(\vec{x};\vec{b})$ then $F \models \wedge_{i<n}\psi(f\vec{c}_i;\vec{b})$: for clearly $F \models \wedge_{i<n}\varphi(f\vec{c}_i;f\vec{a})$.

Conversely, if $F \models \bigwedge_{i<n}\psi(f\vec{c}_i;\vec{b})$ then $\varphi(\vec{x};f\vec{a}) \models_F \psi(\vec{x};\vec{b})$: for suppose the contrary holds, i.e. $F \models \bigwedge_{i<n}\psi(f\vec{c}_i;\vec{b})$ and $\varphi(\vec{x};f\vec{a}) \not\models_F \psi(\vec{x};\vec{b})$; then there is a morphism $g:F \longrightarrow G$ such that

$$G \models \exists\vec{x}\varphi(\vec{x};gf\vec{a}) \wedge \neg\psi(\vec{x};g\vec{b})$$

and of course $G \models \bigwedge_{i<n}\psi(gf\vec{c}_i;g\vec{b})$. We have

$$G \models \exists\vec{t}\exists\vec{x}\bigwedge_{i<n}\psi(gf\vec{c}_i;\vec{t}) \wedge \varphi(\vec{x};gf\vec{a}) \wedge \neg\psi(\vec{x};\vec{t}).$$

Since H is $\Sigma_1(\Gamma)$ closed, it follows there exists a tuple $\vec{d}$ of elements in H such that,

$$H \models \bigwedge_{i<n}\psi(\vec{c}_i;\vec{d}) \text{ and } H \models \exists\vec{x}\varphi(\vec{x};\vec{a}) \wedge \neg\psi(\vec{x};\vec{d}).$$

But clearly, if $H \models \bigwedge_{i<n}\psi(\vec{c}_i;\vec{d})$ then $H \models \forall\vec{x}(\varphi(\vec{x};\vec{a}) \longrightarrow \psi(\vec{x};\vec{d}))$, contradiction. ■

19. **Corollary.** With φ, ψ, H as in proposition 5 above, there is a sequence $(\vec{c}_i)_{i<n}$ of parameters in H such that

$$\forall\vec{x}(\varphi(\vec{x};\vec{a}) \longrightarrow \psi(\vec{x};\vec{t})) \underset{H}{\approx} \bigwedge_{i<n}\psi(\vec{c}_i;\vec{t}).$$

Proof. Choose $\vec{c}_0, \cdots, \vec{c}_{n-1}$ in $\varphi(H;\vec{a})$ as in theorem 18 above. Then,

$$\forall x(\varphi(\vec{x};\vec{a}) \longrightarrow \psi(\vec{x};\vec{t})) \models_H \bigwedge_{i<n}\psi(\vec{c}_i;\vec{t}),$$

since for any morphism $f:H \longrightarrow F$, $F \models \bigwedge_{i<n}\varphi(f\vec{c}_i;f\vec{a})$.

The converse follows immediately from theorem 18. ■

Chapter IV

The Symmetry Property

Let us quickly give an idea of how the symmetry property will be proved for R_H-minimal extensions of types ($H \in$ **K**), when for instance **K** is the category of models of a complete theory T with elementary embeddings, and where R is a set of equations such that 'any' formula in L is equivalent in T to a boolean combination of formulas in R.

Step 1. We show that if S is a symmetric set of formulas (i.e. if $\varphi(\vec{x};\vec{t}) \in S$ then $\varphi(\vec{u};\vec{y}) \in S$) and $A \subset B, C \subset H$ then $tp(B;C)$ is an S_C-*minimal* extension of $tp(B;A)$ to C iff $tp(C;B)$ is an S_B-minimal extension of $tp(C;A)$ to B;

Step 2. Since we are in fact interested in S_H-*minimal* extensions, we consider sets of formulas S such that for any $B \subset$ H, and any complete type p over A, $p_H^S \underset{H}{\sim} p_B^S$; we say then that S is full in H. We note that if S is full, $A \subset B \subset H$ and p is a type over A then the S_B-*minimal* extensions of p to B coincide with the S_H-*minimal* extensions of p to B.

Step 3. For S symmetric and full, and $A \subset B, C \subset H$ we deduce from step 1 and step 2 that $tp(B;C)$ is an S_H-*minimal* extension of $tp(B;A)$ to C iff $tp(C;B)$ is an S_H-*minimal* extension of $tp(C;A)$ to B.

Step 4. We show that the set of all equations in K is symmetric and full.

Step 5. We show that if S is the set of all equations in K $A \subset B \subset H$, p a type over A and R is as above then the R_H-minimal extension of p to B coincide with the S_H-minimal extensions of p to B.

Step 6. We deduce the symmetry property for R_H-minimal extensions from step 3, 4 and 5. ■

In this chapter, we prove step 1, 2 and 3 in arbitrary Δ-categories. Then we discuss the existence of full sets of formulas in general.

Preliminaries. Δ is a boolean-closed set of formulas which contains the formulas $(x = x)$, x any variable; S is said to be symmetric if for any formula $\varphi(\vec{x};\vec{t})$, $\varphi^{\vec{x}}$ belongs to S iff $\varphi^{\vec{t}}$ belongs to S (i.e. $\varphi(\vec{x};\vec{t})$ belongs to S iff $\varphi(\vec{t}_1;\vec{x}_1)$

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belongs to S).

S is said completely symmetric if for any formula $\varphi(\vec{x};\vec{t})$, if $\varphi^{\vec{x}}$ belongs to S then $\varphi^{\vec{u}}$ belongs to $\vec{x}$ for any subtuple $\vec{u}$ of $\vec{x}\frown\vec{t}$ (i.e. if $\varphi(\vec{x};\vec{t})$ belongs to S then the formula obtained from φ by substituting $\vec{u}$ by type variables and the rest of the variables in $\vec{x}\frown\vec{t}$ by parameter variables is in S).

Let $\vec{x}_1$ and $\vec{x}_2$ be, possibly infinite, disjoint tuples of variables and assume S is symmetric. We write

$$S(\vec{x}_1;\vec{x}_2) = \{\varphi(\vec{y}_1;\vec{y}_2);\ \vec{y}_1,\ \vec{y}_2 \text{ subtuples of } \vec{x}_1,\ \vec{x}_2 \text{ respectively, and } \varphi(\vec{y}_1;\vec{t}_2) \text{ in } S\}.$$

Thus given $A \in \hat{\mathfrak{K}}$, p a type in $\vec{x}_1\frown\vec{x}_2$ over A, (by definition)

$$p \cap S(\vec{x}_1;\vec{x}_2) = \{\varphi(\vec{y}_1;\vec{y}_2) \in p;\ \varphi(\vec{y}_1;\vec{y}_2) \in S(\vec{x}_1;\vec{x}_2)\}.$$

($p \cap S(\vec{x}_1;\vec{x}_2)$ is a set of formulas without parameters).

0. Definitions. Let $A \in \hat{\mathfrak{K}}$, $\vec{x}_1$, $\vec{x}_2$, possibly infinite, disjoint tuples of variables, p_1 a type in $\vec{x}_1$ over A and p_2 a type in $\vec{x}_2$ over A; q a complete type in $\vec{x}_1\frown\vec{x}_2$ over A extending $p_1 \cup p_2$.

We say q is an *S-minimal amalgam* of p_1 and p_2 over A

if, whenever r is a complete type in $\vec{x}_1{}^\frown\vec{x}_2$ over h extending $p_1 \cup p_2$, such that

$$q \cap S(\vec{x}_1;\vec{x}_2) \supset r \cap S(\vec{x}_1;\vec{x}_2)$$

then

$$q \cap S(\vec{x}_1;\vec{x}_2) = r \cap S(\vec{x}_1;\vec{x}_2).$$

We say q is an *S-free amalgam* of p_1 and p_2 over A if for any complete type r in $\vec{x}_1{}^\frown\vec{x}_2$ over A extending $p_1 \cup p_2$, $q \cap S(\vec{x}_1;\vec{x}_2) \lhd r \cap S(\vec{x}_1;\vec{x}_2)$.

Let $A,B \subset H \in \mathbf{K}$. We denote by $tp(B,A;H)$ the class of types obtained in the following manner:

to every element b in B, assign a variable x_b ($x_b \neq x_c$ if $b \neq c$) and let $\vec{x} = \langle x_b,\ b \in B\rangle$. Then consider the type (in $\vec{x}$)

$$p_{\vec{x}} = \{\varphi(\vec{x}_b;\vec{a});\ \varphi \in \Delta,\ \vec{a} \in A,\ \vec{b} \in B \text{ and } H \models \varphi(\vec{b};\vec{a})\}.$$

(Note that $p_{\vec{x}}$ is complete).

So $tp(B,A;H)$ is the class of types $p_{\vec{x}}$, as described above, when we vary the choice of the tuple $\vec{x}$.

However, when there is no ambiguity, we will confuse $tp(B,A;H)$ with any one of its representatives and consequently apply to $tp(B,A;H)$ the terminology and notations applied on types.

For instance, we will write $tp_H^S(B,A;H)$, meaning that we have beforehand identified $tp(B,A;H)$ with one of its representatives p and let $tp_H^S(B,A;H) = p_H^S$. Similarly, if $A \subset B \subset H$ and $C \subset H$, we say that $tp(C,B;H)$ is an S_H-minimal extension of $tp(C,A;H)$ to B if, once chosen representatives q and p of $tp(C,B;H)$ and $tp(C,A;H)$ (in the same tuple of variables $\vec{x} = \langle x_c,\ c \in C\rangle$), q is an S_H-minimal extension of p to B.

When it is well-understood which structure (H) is considered, we write $tp(B;A)$ instead of $tp(B,A;H)$

If $A \subset B, C \subset H$, $tp(B \mathbin{\dot\cup} C,A;H)$ denotes the following type: let $\vec{x}_1 = \langle x_b;\ b \in B\rangle$ and $\vec{x}_2 = \langle x_c;\ c \in C\rangle$ be disjoint tuples of variables which are in one-to-one correspondence with B and C respectively. Then

$$tp(B \mathbin{\dot\cup} C,A;H) = \{\Theta(x_{\vec{b}},x_{\vec{c}},\vec{a}) \in D(A);\ H \models \Theta(\vec{b},\vec{c},\vec{a})\}.$$

1) **Theorem.** Assume S is symmetric. Given $A \subset B \subset H$ and $A \subset C \subset H$,
the following assertions are equivalent:

a) $tp(B,C;H)$ is an S_C-minimal extension of $tp(B,A;H)$ to C.

b) $tp(C,B;H)$ is an S_B-minimal extension of $tp(C,A;H)$ to B.

c) $tp(B \mathbin{\dot\cup} C,A;H)$ is an S-minimal amalgam of $tp(B,A;H)$ and $tp(C,A;H)$ over A.

If in addition K is as in chapter III, S is a set of equations, Ξ is the set of (S,A)-definable formulas and H is $\Sigma_1(\Gamma)$-closed.

d) $tp(B \mathring{\cup} C, A; H)$ is an S-minimal amalgam of $tp^{\Xi}(B;A)$ and $tp^{\Xi}(C;A)$ over A. (Recall, for p over A, $p^{\Xi} = \{\varphi \in \Xi;\ p \vdash_H \Xi\}$ and note that with the assumptions of d), for p a type over A, $p^{\Xi} \underset{H}{\approx} p^S_H$.

Proof. Let $\vec{x}_1 = \langle x_b;\ b \in B\rangle$, $\vec{x}_2 = x_c;\ c \in C\rangle$; $\vec{x}_1$ and $\vec{x}_2$ disjoint.

– c) ⟶ a)

Suppose c) holds and $tp^S_C(B;C) \supset q^S_C$ where q is a complete type in $\vec{x}_1$ over C extending $tp(B;A)$. Let r be the following type in $\vec{x}_1{}^\frown\vec{x}_2$ over A:

$$r = \{\Theta(\vec{x}_b, \vec{x}_c, \vec{a}) \in D(A);\ \Theta(x_{\vec{b}}, \vec{c}, \vec{a}) \in q\}.$$

It is easy to check that r is complete and that r extends $tp(B;A) \cup tp(C;A)$.

Furthermore, if $\varphi(x_{\vec{b}}, \vec{t})$ is in S and $\varphi(x_{\vec{b}}, x_{\vec{c}})$ belongs to r, then by definition of r, $\varphi(x_{\vec{b}}; \vec{c}) \in q$ i.e. $\varphi(x_{\vec{b}}; \vec{c}) \in q^S_C$; hence $\varphi(x_{\vec{b}}; \vec{c}) \in tp^S_C(B;C)$ and therefore $\varphi(x_{\vec{b}}; x_{\vec{c}}) \in tp(B \mathring{\cup} C; A)$.

Thus $tp(B \mathring{\cup} C; A) \cap S(\vec{x}_1; \vec{x}_2) \supset r \cap S(\vec{x}_1; \vec{x}_2)$; by definition of S-minimal amalgam it follows that

$$tp(B \cup C; A) \cap S(\vec{x}_1;\vec{x}_2) = r \cap S(\vec{x}_1;\vec{x}_2),$$

which easily implies that $tp^S_C(B;C) = q^S_C$.

That shows a) holds.

- a) $\longrightarrow$ c)

Suppose a) holds and r is a complete type in $\vec{x}_1 \frown \vec{x}_2$ over A extending $tp(B:A) \cup tp(C;A)$ (resp. $tp^S_H(B;A) \cup tp^S_H(C;A)$) such that $tp(B \cup C;A) \cap S(\vec{x}_1;\vec{x}_2) \supset r \cap S(\vec{x}_1;\vec{x}_2)$. Consider the set

$q_0 = tp(B;A) \cup \{\neg\varphi(\vec{x}_{\vec{b}};\vec{c});\ \varphi(\vec{x}_{\vec{b}};\vec{c}) \in D^S(C)$ and $\varphi(\vec{x}_{\vec{b}};\vec{x}_{\vec{c}}) \notin r\}$.

Claim. q is consistent. For if not, there are formulas $\varphi_i(\vec{x}_{\vec{b}};\vec{c})$ in $D^S(C)$, $i < n$, such that $\varphi_i(\vec{x}_{\vec{b}};\vec{c}) \notin r$ and $tp(B;A) \vdash_H \vee_{i<n}\varphi_i(x_{\vec{b}};\vec{c})$ (note, we should have taken formulas $\varphi_i(x_{\vec{b}_i};\vec{c}_i)$ with different tuple $\vec{b}_i$ and $\vec{c}_i$ but we can always come back to a single tuple $\vec{b}$ and a single tuple $\vec{c}$ by just adding dummy variables if necessary to φ_i); thus

$$(*)\quad tp^R_H(\vec{b};A) \vdash_H \vee_{i<n}\varphi_i(\vec{x}_{\vec{b}};\vec{c}),$$

where $R = cl^+(\{\varphi_i(x_{\vec{b}};\vec{t});\ i < n\})$.

Hence there is $\theta(\vec{x}_{\vec{b}};\vec{a})$ in $tp(\vec{b};A)$ such that

$\theta(x_{\vec{b}};\vec{a}) \vdash_H \vee_{i<n}\varphi_i(x_{\vec{b}};\vec{c})$;

so, if $X(\vec{a};\vec{x_c}) = \forall x_{\vec{b}}(\Theta(\vec{x_b};\vec{a}) \longrightarrow \vee_{i<n}\varphi_i(\vec{x_b};\vec{x_c}))$, we have $H \models X(\vec{a};\vec{c})$; i.e. $X(\vec{a};\vec{x_c}) \in tp(C;A)$. Now r extends $tp(B;A) \cup tp(C;A)$; hence r contains the formulas $\Theta(x_{\vec{b}};\vec{a})$ and $X(\vec{a};x_{\vec{c}})$. But clearly

$$\Theta(x_{\vec{b}};\vec{a}) \wedge X(\vec{a};x_{\vec{c}}) \models_H \vee_{i<n}\varphi_i(x_{\vec{b}};x_{\vec{c}});$$

by completeness of r it follows that $\varphi_i(x_{\vec{b}};x_{\vec{c}}) \in r$ for some $i < n$ ✖.

That proves the claim.

So q_0 is consistent; let q be a complete extension of q_0 over C. Then, $q_C^S \subset tp_C^S(B;C)$: for if $\varphi(x_{\vec{b}};\vec{c}) \in q_C^S$ then necessarily $\varphi(x_{\vec{b}};x_{\vec{c}}) \in r \cap S(\vec{x}_1;\vec{x}_2)$; hence $\varphi(x_{\vec{b}};x_{\vec{c}}) \in tp(B \mathring{\cup} C;A) \cap S(\vec{x}_1;\vec{x}_2)$ which implies $\varphi(x_{\vec{b}};\vec{c}) \in tp_C^S(B;C)$.

Therefore, by S_C-minimality of $tp(B;C)$, we must have $q_C^S = tp_C^S(B;C)$, which in its turn implies that

$$tp(B \mathring{\cup} C;A) \cap S(\vec{x}_1;\vec{x}_2) = r \cap S(\vec{x}_1;\vec{x}_2).$$

We conclude, $tp(B \mathring{\cup} C;A)$ is an S-minimal amalgam of $tp(B;A)$ and $tp(C;A)$.

- a) $\longrightarrow$ d) (assuming $\mathbf{K}$ is as in section A, S is a set of equations and H is $\Sigma_1(\Gamma)$-closed).

The argument is exactly similar to the argument used in the proof of a) $\longrightarrow$ c) up to the point (*) in the claim where we had $tp^R_H(\vec{b};A) \vdash_H \vee_{i<n}\varphi_i(x_{\vec{b}};\vec{c})$. We continue from that point in the following manner: since for every $i < n$ φ_i is an equation, $R = cl^+(\{\varphi_i; i < n\})$ is equational; by III.A.11, A is R_H-closed in H, so that (c.f. II.A.5) there is a single formula $\varphi(x_{\vec{b}};\vec{a})$ in $D^S(H)$ such that $tp^R_H(\vec{b};\vec{a}) \underset{H}{\sim} \varphi(x_{\vec{b}};\vec{a})$. By corollary III.18, there is a sequence $(\vec{d}_i)_{i<m}$ of parameters in H such that, if $\psi(x_{\vec{b}};\vec{t}) = \vee_{i<n}\varphi_i(x_{\vec{b}};\vec{t})$;

$$\forall x_{\vec{b}}(\varphi(x_{\vec{b}};\vec{a}) \longrightarrow \psi(x_{\vec{b}};\vec{t})) \underset{H}{\sim} \wedge_{i<n}\psi(\vec{d}_i;\vec{t}).$$

So, if $X(\vec{a};x_{\vec{c}}) = \forall x_{\vec{b}}(\varphi(x_{\vec{b}};\vec{a}) \longrightarrow \psi(x_{\vec{b}};x_{\vec{c}}))$, we have $H \models X(\vec{a};\vec{c})$ i.e. $X(\vec{a};x_{\vec{c}}) \in tp(C;A)$ and

$X(\vec{a};x_{\vec{c}}) \underset{H}{\sim} \forall x_b(\varphi(x_{\vec{b}};\vec{a}) \longrightarrow \vee_{i<n}\varphi_i(x_{\vec{b}};x_{\vec{c}})) \underset{H}{\sim} \wedge_{i<n}\psi(\vec{d}_i;x_{\vec{c}})$.

Therefore, $X(\vec{a};x_{\vec{c}}) \in tp^{\Phi}_H(C;B)$. Now r extends $tp^{\Phi}_H(B;A) \cup tp^{\Phi}_H(C;A)$; hence r contains the formulas $\varphi(x_{\vec{b}};\vec{a})$ and $X(\vec{a};x_{\vec{c}})$.

But clearly, $\varphi(x_{\vec{b}};\vec{a}) \wedge X(\vec{a};x_{\vec{c}}) \vdash_H \vee_{i<n}\varphi_i(x_{\vec{b}};x_{\vec{c}})$; by completeness of r it follows that $\varphi_i(x_{\vec{b}};x_{\vec{c}}) \in r$ for some $i < n$. ✖. That proves the claim.

The rest of the proof is the same as that of a) $\longrightarrow$ c) (after the claim).

- By symmetry we get b) ⟶ c), c) ⟶ b) and b) ⟶ d).
- Finally, d) ⟶ c): follows immediately from the definition of S-minimal amalgam and the facts that $tp(B;A) \supset tp^{\Phi}_{H}(B;A)$ and $tp(C;A) \supset tp^{\Phi}_{H}(C;A)$. ■

2. **Remark.** Theorem 1 above states a partial result on the symmetry of S-minimal extensions (see Introduction). The theorem lacks in the fact that is deals only with the case $tp(B;C)$ is an S_C-*minimal* extension of $tp(B;A)$ and not with the case $tp(B;C)$ is an S_H-*minimal* extension of $tp(B;A)$. Of course this lack disappears if it so happens that S_H-*minimal* extensions and S_C-*minimal* extensions are identical, or even if S_H-*minimal* extensions are identical to R_C-*minimal* extensions for some well-chosen set of formulas R. But let us be more explicit: a generalization of the notion of S-closure (say, for S equational) would be the following: given $A \subset B \subset H \in \mathbf{K}$, $\Gamma \subset \Delta$, we say A is (Γ_B, S_H)-closed in H if for any complete type p over A, p^S_H is equivalent to a single formula in $D^{\Gamma}(B)$.

Thus A is S_H-closed in H iff A is (Δ_A, S_H)-closed in H.

Most interesting is the case A is (S_A, S_H)-closed in H, which implies that for any complete type p over A, $p^S_A \underset{H}{\sim} p^S_H$. It is clear then that, for p a type over A and q a complete type over A, q is an S_H-minimal minimal extension of p to A iff q is an S_A-minimal extension of p to A.

We can ask the following questions: suppose A is S_H-closed in H and S is equational, is there a set of formula R, $R \supset S$, such that R is equational and A is (R_A, R_H)-closed in H? Moreover, is it possible to choose R so that $p_A^R \sim_H p_H^S$ for any complete type p over A?

Since A is S_H-closed, p_H^S is equivalent to an (S,A)-definable formula. Thus, the latter question amounts to the ability of uniformly defining over A the (S,A)-definable formulas; i.e. we ask whether there is a set of formulas R such that any formula in $D^R(A)$ is S-definable and any (S,A)-definable formula is equivalent to a formula in $D^R(A)$.

It is therefore important, whenever given a Δ-category K, $A \subset H$ and a set of equations S, to investigate the (S,A)-definable formulas in $D(H)$.

3. **Example.** Let **K** be the category of models of T with elementary embeddings, Δ the set of all formulas and S a set of equations. Consider the following kind of formulas

$$X(\vec{x};\vec{t}) = \Theta(\vec{x};\vec{t}) \wedge \exists\vec{u}\forall\vec{x}(\Theta(\vec{x};\vec{t}) \longleftrightarrow \varphi(\vec{x};\vec{u})),$$

where Θ is an arbitrary formula and φ is in S; let R be the set of such formulas. (Note that $R \supset S$).

Clearly, given $A \subset H \in$ **K**, $\vec{a}$ in A and X as above, if

$X(\vec{x};\vec{a})$ is consistent then there is $\vec{b}$ in M such that $M \models \forall\vec{x}(X(\vec{x};\vec{a}) \longleftrightarrow \varphi(\vec{x};\vec{b}))$; thus any instance of X over A is S-definable, and therefore R is a set of equations.

Conversely, if $\psi(\vec{x};\vec{c})$ is (S,A)-definable, then by definition there is $\Theta(\vec{x};\vec{a})$ in $D(A)$ and $\varphi(\vec{x};\vec{b})$ in $D^S(M)$ such that $\psi(\vec{x};\vec{c}) \underset{M}{\approx} \Theta(\vec{x};\vec{a}) \underset{M}{\approx} \varphi(\vec{x};\vec{b})$, it follows easily that $\psi(\vec{x};\vec{c}) \underset{M}{\approx} X(\vec{x};\vec{a})$ where

$$X(\vec{x};\vec{t}) = \Theta(\vec{x};\vec{t}) \wedge \exists\vec{u}\forall\vec{x}(\Theta(\vec{x};\vec{t}) \longleftrightarrow \varphi(\vec{x};\vec{u})).$$

Thus any (S,A)-definable formula in $D(M)$ is equivalent to a formula in $D^R(A)$.

Now, by corollary III.11, A is S_M-closed in M; it follows that if p is a complete type over A, p^S_M is (S,A)-definable. We conclude from what preceded that $p^S_M \underset{M}{\approx} p^R_A$. ■

4. Let **K** be as in chapter III, $M \in$ **K**, S a set of equations. Consider the formulas X of the kind

$$X(\vec{x};\vec{t}) = \exists\vec{u}\varphi(\vec{x};\vec{u}) \wedge \psi(\vec{u};\vec{t}),$$

where $\varphi(\vec{x};\vec{u})$ is in S and $\psi(\vec{u};\vec{t})$ is a formula in $\Sigma_1(\Gamma)$ such that (*) $T \models \psi(\vec{u};\vec{t}) \wedge \psi(\vec{v};\vec{t}) \longrightarrow \forall\vec{x}(\varphi(\vec{x};\vec{u}) \longleftrightarrow \varphi(\vec{x};\vec{v}))$.

Let R be the set of such formulas.

(i) **Claim.** If X is as above, $\vec{a}$ is in M and $X(\vec{x};\vec{a})$ is consistent *in* M then $X(\vec{x};\vec{a})$ is S-definable in M: for let $\vec{b}$ in M such that $M \models X(\vec{b};\vec{a})$, then in fact

$X(\vec{x};\vec{a}) \underset{H}{\simeq} \varphi(\vec{x};\vec{b})$.

Indeed, let $f:H \longrightarrow F$ be in **K**, since $\psi \in \Sigma_1(\Gamma)$, $F \models \psi(f\vec{b};f\vec{a})$. We easily deduce that

$$F \models \forall\vec{x}(\varphi(\vec{x};f\vec{b}) \longrightarrow X(\vec{x};f\vec{a})).$$

The converse, i.e. the fact that $F \models \forall\vec{x}(X(\vec{x};f\vec{a}) \longrightarrow \varphi(\vec{x};f\vec{b}))$ follows immediately from the property (*) of ψ.

We conclude that $F \models \forall\vec{x}(X(\vec{x};\vec{a}) \longleftrightarrow \varphi(\vec{x};\vec{b}))$.

(ii) **Claim.** If H is $\Sigma_1(\Gamma)$-closed, X is as above, $\vec{a}$ is in H and $X(\vec{x};\vec{a})$ is consistent *in* **K** then $X(\vec{x};\vec{a})$ is S-definable *in* H.

Indeed, since $X(\vec{x};\vec{t})$ is a formula in $\Sigma_1(\Gamma)$ and H is $\Sigma_1(\Gamma)$-closed, $X(\vec{x};\vec{a})$ is consistent in **K** iff $X(\vec{x};\vec{a})$ is consistent in H. The claim now follows from (i). Note that for H arbitrary, if $X(\vec{x};\vec{a})$ is consistent in **K** then there is a morphism $f:H \longrightarrow F$ such that fX is consistent in F, whence by (i), fX is S-definable in F.

(iii) **Claim.** R is a set of $\mathfrak{k}$-equations: follows immediately from (i).

(iv) **Claim.** Given $A \subset H$ and $\varphi(\vec{x};\vec{c})$ a formula in $D(H)$ which is (S,A)-definable, there is a formula $X(\vec{x};\vec{a})$ in $D^R(A)$ such that $\varphi(\vec{x};\vec{c}) \underset{H}{\simeq} X(\vec{x};\vec{a})$.

Indeed, since $\varphi(\vec{x};\vec{c})$ is S-definable, we can assume $\varphi \in S$; since $\varphi(\vec{x};\vec{c})$ is A-definable, there is a formula $\Theta(\vec{x};\vec{a})$ in $D(A)$ such that $\Theta(\vec{x};\vec{a}) \underset{H}{\sim} \varphi(\vec{x};\vec{c})$.

By lemma III.1 there is $\psi(\vec{u};\vec{t})$ in $\Sigma_1(\Gamma)$ such that $H \models \psi(\vec{c};\vec{a})$ and

$$T \models \psi(\vec{u};\vec{t}) \longrightarrow \forall\vec{x}(\Theta(\vec{x};\vec{t}) \longleftrightarrow \varphi(\vec{x};\vec{u})).$$

It follows easily that

$$T \models \psi(\vec{u};\vec{t}) \wedge \psi(\vec{v};\vec{t}) \longrightarrow \forall\vec{x}(\varphi(\vec{x};\vec{u}) \longleftrightarrow \varphi(\vec{x};\vec{v})).$$

and

$$\varphi(\vec{x};\vec{c}) \underset{H}{\sim} X(\vec{x};\vec{a}) \text{ where } X(\vec{x};\vec{t}) = \exists\vec{u}(\varphi(\vec{x};\vec{u}) \wedge \psi(\vec{u};\vec{t})),$$

Clearly $X(\vec{x};\vec{t}) \in R$.

(v) **Claim.** If $A \subset H$ and H is $\Sigma_1(\Gamma)$-closed then for any complete type p over A, $p_H^S \underset{H}{\sim} p_A^R \underset{H}{\sim} p_H^R$.

By claim (ii), any formula in p_H^R is S-definable; hence $p_H^S \vdash_{\overline{H}} p_H^R \vdash_{\overline{H}} p_A^R$.

By corollary III.11, A is S_H-closed in H; hence p_H^S is (S,A)-definable; by claim (iv), it follows that $p_A^R \vdash_{\overline{H}} p_H^S$.

Conclusion. Given $A \subset B \subset H$, H $\Sigma_1(\Gamma)$-closed, p a type over A, q a type over B extending p, q is an S_H-minimal extension of p to B iff q is an R_B-minimal extension of p

to B. (For, by claim (v), given complete extensions q_1 and q_2 of p to B

$$(q_1)^S_H \supset (q_2)^S_H \Leftrightarrow (q_1)^R_B \supset (q_2)^R_B).$$ ■

In view of what preceded, we make the following definition.

5. **Definition.** Given $H \in \mathbf{K}$, a set of formulas R is said to be full in H if for any $A \subset H$ and complete type p over A, $p^R_H \simeq_H p^R_A$.

6. **Theorem (symmetry).** Assume S is symmetric and full in H. Given $A \subset B \subset H$ and $A \subset C \subset H$, the following assertions are equivalent.

   a) $tp(B;C)$ is an S_H-minimal extension of $tp(B;A)$ to C.

   b) $tp(C;B)$ is an S_H-minimal extension of $tp(C;A)$ to B.

   c) $tp(B \mathring{\cup} C;A)$ is an S-minimal amalgam of $tp(B;A)$ and $tp(C;A)$ over A.

Proof. Follows immediately from proposition 1 and the fact that, S being full, $tp(B;C)$ is an S_H-minimal extension of $tp(B;A)$ to C iff $tp(B;C)$ is an S_C-minimal extension of $tp(B;A)$ to C; similarly with $tp(C;B)$. ■

Note. In 4 above, (given that S is a set of equations, $\mathbf{K}$ is as in chapter III and H is $\Sigma_1(\Gamma)$-closed), we have constructed from S in a natural way a set of ℓ-equations R which is full in H. However, given that S is symmetric, R, a priori, is not necessarily symmetric, and therefore theorem 6 does not apply to R.

If $\mathbf{K}$ has elementary embeddings for morphisms, then we can still "close" S to a set of formulas R_ω which is symmetric and full in H in the following manner:

Construct $(R_i)_{i<\omega}$ by induction such that: $R_0 = S$, and for $i < \omega$ R_i is a set of equations,

- R_{2i+1} is full in H: R_{2i+1} is obtained by applying the argument in 4 where we replace S by R_{2i} and let $R_{2i+1} = R$. R_{2i+1} is then a set of ℓ-equations; hence, the morphisms in $\mathbf{K}$ being elementary, R_{2i+1} is a set of equations.
- R_{2i} is symmetric: let $R_{2i} = cl^+(S_{2i})$ where $S_{2i} = \{\varphi(\vec{t}_1,\vec{x}_1);\ \varphi(\vec{x};\vec{t}) \in R_{2i-1}\} \cup R_{2i-1}$.

  (By I.8.(i)) R_{2i} is a set of ℓ-equations; hence, the morphisms in $\mathbf{K}$ being elementary, R_{2i} is a set of equations. Now let $R_\omega = \cup_{i<\omega} R_i$. It is easy to check that R_ω is a set of equations which is symmetric and full in H.

Note that we used above the elementariness of the morphisms in K to identify ℓ-equationality and equationality in K. (The same argument does not immediately apply in case the morphisms in K are just Γ-elementary as in chapter III; however, a similar argument in that case can still be worked out, but needs the direct study of the essential properties of S used in 4 to construct R, (c.f.[S.2])).

7. **Proposition. If K is the category of models of a complete theory with elementary embeddings and S is the set of *all* equations in K, then S is symmetric and full in any given structure in K.**

Proof. Suppose $\varphi(\vec{x};\vec{t})$ is an equation; by corollary I.3 φ has finite ℓ-height; by proposition I.8.(i) it follows that $\varphi^{\vec{t}}$ has finite ℓ-height; by elementariness $\varphi^{\vec{t}}$ has finite height; hence $\varphi^{\vec{t}}$ is an equation.

Thus, if $\varphi^{\vec{x}} \in S$ then $\varphi^{\vec{t}} \in S$; i.e. S is symmetric. We have shown in 3 that there is a set of equations R , $R \supset S$ (up to logical equivalence), and R is full in any given structure H. Since S is the set of all equations, it follows that $R \equiv S$ (up to logical equivalence of formulas) and therefore S is full in any given structure of K. ■

CHAPTER V

The Case of a Complete Theory

In this chapter, we assume **K** is the Δ-category of models of a complete first-order theory T with the elementary embeddings for morphisms and where Δ is the set of all formulas in L.

S is a fixed set of equations, $S = c\ell^{+}(S)$ and S contains the formulas $(x = x)$, x any variable.

Theorem C.6 and corollary C.7 in this chapter, have been proved jointly by the author and A. Pillay (c.f. [P,S]).

Section A: Preliminaries and Summary

We fix a large saturated model $\bar{M}$; without loss of generality we restrict ourselves, in the study of **K**, to the category of elementary submodels of $\bar{M}$ which are of cardinality strictly less than that of $\bar{M}$.

All sets and models considered shall be subsets and elementary submodels of $\bar{M}$ of cardinality strictly less than card$\bar{M}$.

By a type P, we mean a type over $\overline{M}$ in possibly an infinite tuple of variables; $tp(A;B) = tp(A,B,\overline{M})$. If p is a type, we let $p^S = p^S_{\overline{M}}$; and we talk of S-minimal extensions instead of $S_{\overline{M}}$-minimal extensions.

We write $\varphi \vdash \psi$, (resp. $\varphi \sim \psi$), instead of $\varphi \vdash_{\overline{M}} \psi$, (resp. $\varphi \underset{\overline{M}}{\sim} \psi$), and $\models \varphi$, instead of $\overline{M} \models \varphi$; equivalence means logical equivalence in $\overline{M}$ and consistency means consistency in $\overline{M}$.

Let $X \subset \overline{M}$; we say that X is a definable set if there is a formula $\varphi(\vec{x};\vec{a})$ such that, $X = \{\vec{c};\ \models \varphi(\vec{c};\vec{a})\}$ (In principle we should say X is a definable class, since card$X$ can equal card$\overline{M}$; but in that instance, it will be clear what is meant). We do not distinguish between formulas and the sets they define.

By an automorphism we mean an automorphism of $\overline{M}$; if $A \subset \overline{M}$, an automorphism over A is an automorphism which keeps A invariant.

If p and q are types, (or just single formulas), we say that q is an A-conjugate of p if there is an automorphism σ over A such that $\sigma p \sim q$ (where $\sigma p = \{\varphi(\vec{x};\sigma\vec{a});\ \varphi(\vec{x};\vec{a}) \in p\}$); p is almost over A if p has, up to equivalence, finitely many A-conjugates.

[Note that the notion of A-conjugation defined above and the notion given in chapter III (c.f. III.8) coincide when dealing with types over submodels of $\overline{M}$, (this follows immediately from the saturation property of $\overline{M}$), but we should be careful of identifying the two notions when dealing with arbitrary types over $\overline{M}$, (unless the theory is stable).]

Remark. If p is a type and σ is an automorphism, then $p^S \sim \sigma p^S$ iff for any $\varphi \in S$, $p^\Phi \sim \sigma p^\Phi$ where $\Phi = \mathrm{cl}^+(\varphi)$; since Φ is equational, p^Φ is equivalent to a single formula in $D^S(\overline{M})$ so that the statement $p^S \sim \sigma p^S$ is equivalent to an infinite conjunction of first-order statements. Later on we will use the observation above in compactness arguments.

Let us now summarize in this context some of the definitions and results given in the preceding chapters.

0. **Definitions** (c.f. 0.3). A set of formulas R is equational if for any family $(X_i)_{i\in I}$ of R-definable subsets of $\overline{M}$, there is a finite set $I_0 \subset I$, such that

$$\cap_{i\in I_0}\{X_i;\ i \in I\} = \cap_{i\in I}\{X_i;\ i \in I\}.$$

(Note that the definition above coincideswith the regular definition of an equational set in K because of $\overline{M}$ being saturated and lemma I.1.(i)).

A formula $\varphi(\vec{x};\vec{t})$ is an equation if $\{\varphi(\vec{x};\vec{t})\}$ is equational. Let us call a definable set X a closed set if X is definable by an instance of an equation.

Note. It is easy to check, by compactness, that a definable set X is closed if for any family $(X_i)_{i\in I}$ of conjugates of X (i.e. of $\emptyset$-conjugates of X), there is a finite subset I_0 of I, such that,

$$\cap \{X_i ; i \in I_0\} = \cap \{X_i ; i \in I\}.$$

1. Definitions (c.f. II,A.0 and III.0).

(i) Given a type p and $A \subset \bar{H}$, we let

$$p_A^S = \{\varphi(\vec{x};\vec{a}) ; \vec{a} \in A, \varphi \in S \text{ and } p \vdash \varphi(\vec{x};\vec{a})\};$$

let $p^S = p_{\bar{H}}^S$.

Given types p and q we say p and q are S-equivalent if p and q are $S_{\bar{H}}$-equivalent i.e. if $p^S \sim q^S$.

(ii) Given $A \subset B \subset C$, p a type over A and q a complete type over B extending p, we say q is an S_C-minimal extension of p to B if for any extension r of p to B,

$$q_C^S \supset r_C^S \Rightarrow q_C^S = r_C^S;$$

q is an S-minimal extension of p to B if q is an $S_{\bar{H}}$-minimal extension of p to B.

2. **Proposition.** For any $A \subset \bar{M}$ and type p over A, if S is equational then p^S is equivalent to an A-definable formula.

Proof. By III.11 A is $S_{\bar{M}}$-closed in $\bar{M}$; the rest follows by II.A.3. ■

3. **Definitions.**

(i) (c.f. II.B.4) Given a type p and $A \subset \bar{M}$, we say p is A-irreducible if p is (S,A)-irreducible i.e. if, for any (S,A)-definable formulas φ_1 and φ_2 in $D(\bar{M})$

$$p \vdash \varphi_1 \vee \varphi_2 \Longrightarrow p \vdash \varphi_1 \text{ or } p \vdash \varphi_2;$$

p is irreducible if p is $\bar{M}$-irreducible.

p is full if p is $S_{\bar{M}}$-full i.e. if p is irreducible and $p^S \sim p \cap D^S(\bar{M})$ ($= \{\varphi(\vec{x};\vec{a}) \in p;\ \varphi \in S\}$).

(ii) (c.f. II.B.7) A set $A \subset \bar{M}$ is irreducible (resp. full) if every complete type over A is irreducible (resp. full).

4. **Proposition.**

(i) Let p be a type over A. Then p is A-irreducible iff p admits, up to S-equivalence, a unique S-minimal extension q to A (q is then such that $q^S \sim p^S$) iff there is a complete type q over A, extending p and such that $q^S \sim p^S$.

(i) Every model in $\overline{H}$ is full.

Proof. (i) is a restatement of II.B.5.(ii) and (ii) follows from III.3. ■

5. Theorem. Let $A \subset \overline{H}$, p a type in a finite tuple $\vec{x}$ of variables; suppose S is equational. Then,

(i) p has, up to S-equivalence, finitely many S-minimal extensions to A.

(ii) Let $\Theta_0, \cdots, \Theta_{n-1}$, be A-irreducible (S,A)-definable formulas in $\vec{x}$, and let $p_0, \cdots, p_{n-1}$ be complete types over A such that $p_i^S \sim \Theta_i$, $(i < n)$. (Note that, by proposition 4.(i) the p_i's always exist).

Then, up to S-equivalence, $p_0, \cdots, p_{n-1}$ are (all) the distinct S-minimal extensions of p to A iff

$$p^S \sim \vee_{i<n}\Theta_i, \text{ and } \Theta_i \nvdash \Theta_j \quad (i \neq j).$$

Thus it is suggestive to think of the S-minimal extensions of p to A as the A-irreducible components of p to A.

Proof. For (i) c.f. II.A.13 and for (ii) c.f. II.B.6. ■

6. **Theorem.**

(i) *Monotonocity-transitivity.* Given $A \subset B \subset C$, p_0 a type over A, p_1 a complete type over B and p_2 a complete type over C, $p_2 \supset p_1 \supset p_0$, then p_2 is an S-minimal extension of p_0 to C iff p_2 is an S-minimal extension of p_1 to C and p_1 is an S-minimal extension of p_0 to B.

(ii) *Local-character.* Given $A \subset B$ and $C \subset \bar{M}$, $tp(C;B)$ is an S-minimal extension of $tp(C;A)$ to B iff for any formula φ in S and any finite tuple $\vec{c}$ of elements in C, $tp(\vec{c};B)$ is a Σ-minimal extension of $tp(\vec{c};A)$ to B, where $\Sigma = cl^+(\varphi)$.

(iii) *Symmetry.* Given that S is the set of all equations, $A \subset B$ and $A \subset C$, $tp(B;C)$ is an S-minimal extension of $tp(B;A)$ to C iff $tp(C;B)$ is an S-minimal extension of $tp(C;A)$ to B.

Proof. (i) follows from II.A.12 (see also II.B.16); (ii) follows from III.17 and (iii) follows from IV.6 and IV.7. ■

Note. Later on we shall give different statements of the symmetry property than that given above.

7. **Theorem.** Let $A \subset B$, p a complete type over A.

(i) Let q be a complete type over B extending p. Then, q is an S-minimal extension of p to B iff for any formula φ in S, q^{Σ} is almost over A, where $\Sigma = cl^{+}(\varphi)$.

(ii) Suppose B is a model, and q_1 and q_2 are S-minimal extensions of p to B, then q_1^S and q_2^S are A-conjugates of each other.

Proof.

(i) By the local-character property (c.f. 6.(ii)), we can assume without loss of generality $S = cl^{+}(\varphi)$, for some formula φ; and consequently we can assume S is equational.

1st case. B is a model. Then, the claim follows from III.12.

2nd case. B is arbitrary. Let $q_0, \cdots, q_{n-1}$ be the S-minimal extensions of q to some model $M \supset B$; by theorem 5.(ii), $q^S \sim \vee_{i<n} (q_i)^S$.

Suppose q is an S-minimal extension of p to B. Then, by transitivity, q_i is an S-minimal extension of p to M, $(i < n)$; it follows from the first case that $(q_i)^S$ is almost over A, for any $i < n$. Hence $q^S \sim \vee_{i<n} (q_i)^S$ is almost over A.

Conversely, suppose q^S is almost over A; let

$\sigma_j q^S$, $j < m$, be all the distinct A-conjugates of q^S (σ_j an automorphism over A). Then $\vee_{j<m}(\sigma_j q^S)$ is A-invariant, whence A-definable. It follows that $p \vdash \vee_{j<m}(\sigma_j q^S)$. Assume $M = \bar{M}$. Since $q^S \sim \vee_{i<n}(q_i)^S$, we have that

$$p \vdash \bigvee_{\substack{j<m \\ i<n}} (\sigma_j q_i)^S.$$

By theorem 5, it follows that at least one of the $\sigma_j q_i$'s ($i < n$, $j < m$) is an S-minimal extension of p to M. Since, σ_j is an automorphism over A, we deduce that at least one of the q_i's, ($i < n$), is an S-minimal extension of p to $\bar{M}$. We conclude by monotonicity that q is an S-minimal extension of p to B.

(ii) By the local-character property, q_1 and q_2 are Φ-minimal extensions of p to B whenever $\Phi = \mathrm{cl}^+(\varphi)$ and $\varphi \in S$.

By III.12, for any φ in S, q_1^Φ and q_2^Φ are A-conjugates of each other. It follows, by an easy compactness argument, that q_1^S and q_2^S are A-conjugates of each other. (Note that $q_1^S = \cup \{q_1^\Phi;\ \varphi \in S\}$). ■

Section B: Stable and Equational Theories

We recall that T is λ-stable, (λ a cardinal), if for any set A of cardinality λ, there are λ complete types (in a single variable) over A; T is stable if it is λ-stable for some cardinal λ.

We recall also that if T is stable then there is the notion of non-forking extension of a type which satisfy properties 0.– 5. mentioned in the introduction to this thesis. We shall state again, as facts of stability theory, these and further properties when needed.

0. **Definition.** Let $\vec{x}$ be a finite tuple of variables. We say T is S-equational in $\vec{x}$ if any formula in $\vec{x}$ is equivalent in T to a boolean combination of formulas in $S^{\vec{x}}$, T is S-equational if T is S-equational in every tuple $\vec{x}$.

   T is equational (resp. equational in $\vec{x}$) if there is a set of equations R such that T is R-equational (resp. R-equational in $\vec{x}$).

Examples.

a) The theory ACF_p (of algebraically closed fields of characteristic p) is S-equational with S the set of atomic formulas. (see I.4.(i)).

b) The theory DCF_0 (of differentially closed fields of characterisitic 0) is S-equational with S the set of atomic formulas: indeed, it is a fact that DCF_0 has elimination of quantifiers (c.f. [Sa] 40.3) and we know (see application 2 at the end of chapter I) that any differential equation is an equation in DCF_0.

c) Any complete theory of modules is S-equational with S the set of positivie primitive formulas: it is a fact that any complete theory of modules has elimination of quantifiers up to positive primitive formulas (c.f. [Z]) and we know (see 0.4.(ii)) that positive primitive formulas are equations.

Let us finally say without proof that the theory of separably closed fields of finite ershov invariant is equational in a single variable x.

There are many possible variations on definition 1. For instance we could define T is S-equational if T is stable and for any $A \subset B$ and complete type p over B, p does not fork over A iff $p \restriction S$ does not fork over A; or that T is S-equational in a given model M if any definable subset of M is equivalent to a boolean combination of S-definable subsets of M. Also, some of the results below do hold with such definitions.

1. Proposition. T is equational in x iff every definable subset of $\bar{M}$ equals a boolean combination of closed sets.

Proof. Obviously, if T is equational then every definable set equals a boolean combination of closed sets.

Conversely, suppose any definable set equals a boolean combination of closed sets. Let $\Theta(x;\vec{t})$ be an arbitrary formula and let

$$\Phi = \{\varphi(x;\vec{t}) \in \mathrm{cl}(E);\ \varphi(x;\vec{t}) \vdash^{\mathfrak{T}} \Theta(x;\vec{t})\},$$

where E is the set of all equations.

Consider

$$p(x;\vec{t}) = \{\Theta(x;\vec{t})\} \cup \{\neg\varphi(x;\vec{t});\ \varphi \in \Phi\}.$$

Suppose $p(x;\vec{t})$ is consistent; let a, $\vec{b}$ realize p.

Claim. $tp(a;\vec{b})\restriction E \vdash tp(a;\vec{b})$.

More generally we show that for p_1 and p_2 complete types in x over $\vec{b}$, if $p_1\restriction E = p_2\restriction E$ then $p_1 = p_2$.

Indeed, suppose $p_1\restriction E = p_2\restriction E$. Then $(p_1)^E_B = (p_2)^E_B$; but, (by proposition IV.7) E is full in $\bar{M}$; hence $(p_1)^E \sim (p_1)^E_B$, and $(p_2)^E \sim (p_2)^E_B$. So $(p_1)^E = (p_2)^E$.

Now let q_1 be an E-minimal extension of p_1 to $\bar{M}$. Since $q_1 \supset (p_2)^E$, (by II.A.3) there is a complete extension q_2 of p_2 to $\bar{M}$ such that $(q_1)^E \supset (q_2)^E$. Since $q_2 \supset (p_2)^E = (p_1)^E$, (by II.A.3) there is a complete extension q_3 of p_1 to $\bar{M}$ such that $(q_2)^E \supset (q_3)^E$. So $(q_1)^E \supset (q_3)^E$; by minimality it follows that $(q_1)^E = (q_3)^E$, whence $(q_1)^E = (q_2)^E = (q_3)^E$.

Therefore $q_1 \restriction E = q_2 \restriction E$. Since by assumption every definable subset of $\bar{M}$ is equivalent to a boolean combination of instances of formulas in E (i.e. of closed sets) we deduce that $q_1 = q_2$. Hence $p_1 = p_2$, which proves the claim.

Let $\psi(x;\vec{b}) \in tp(a;\vec{b}) \restriction E$, and $\psi(x;\vec{b}) \vdash \Theta(x;\vec{b})$. Write $\psi(x;\vec{t}) = \vee_{j<m} \wedge_{i<n} \varphi_{ij}(x;\vec{t})^{t_{ij}}$, $t_{ij} = 0,1, \varphi_{ij} \in E$.

Let $\delta(\vec{t}) = \forall x(\psi(x;\vec{t}) \longrightarrow \Theta(x;\vec{t}))$; we have $\models \delta(\vec{b})$.

Let $X(x;\vec{t}) = \psi(x;\vec{t}) \wedge \delta(\vec{t})$; clearly $X(x;\vec{b}) \in tp(a;\vec{b})$, and $X(x;\vec{t}) \vdash \Theta(x;\vec{t})$.

Moreover we can write $X(x;\vec{t}) = \vee_{j<m} \wedge_{i<n} \psi_{ij}(x;\vec{t})^{t_{ij}}$, where $\psi_{ij} = \varphi_{ij}(x;\vec{t}) \wedge \delta(\vec{t})$ if $t_{ij} = 0$,

and $\psi_{ij}(x;\vec{t}) = \varphi_{ij}(x;\vec{t}) \vee \neg\delta(\vec{t})$ if $t_{ij} = 1$.

It is easy to check that $\psi_{ij}(x;\vec{t})$ is an equation. Thus $X \in cl(E)$; hence $X(x;\vec{t}) \in \Phi$, and therefore, $\neg X(x;\vec{b}) \in tp(a;\vec{b})$ ✗.

Therefore $p(x;\vec{t})$ is inconsistent i.e. there are $\varphi_0, \cdots, \varphi_{n-1}$ in Φ such that $\varphi_i(x;\vec{t}) \vdash \Theta(x;\vec{t})$, $(i < n)$, and $\Theta(x;\vec{t}) \vdash \vee_{i<n} \varphi_i(x;\vec{t})$.

Thus, $\Theta(x;\vec{t}) \sim \vee_{i<n} \varphi_i(x;\vec{t})$.

We conlude T is E-equational in x. ■

Of course a similar result to proposition 2 holds if, instead of a single variable x, we consider a tuple of variables.

Question. Does T equational in x imply T equational in any tuple of variables?

Proposition. If T is S-equational in $\vec{x}$ then T is stable.

Proof. Suppose T is S-equational in $\vec{x}$. Let $A \subset \bar{M}$ and p a complete type in $\vec{x}$ over A. We have $p \sim p \upharpoonright S$ and $p \upharpoonright S$ is completely determined by the set $\{p \upharpoonright \varphi;\ \varphi \in S\}$; hence p is completely determined by $\{p_A^{\varphi};\ \varphi \in S\}$.

Now, for $\varphi \in S$, p_A^{φ} is equivalent to a single formula in $D^S(A)$.

Since $\mathrm{card} D^S(A) \leq \mathrm{card} S + \mathrm{card} A$, we deduce that there are less than or equal to $(\mathrm{card} S + \mathrm{card} A)^{|S|}$ complete types in $\vec{x}$ over A. In particular, if $\mathrm{card} A = \lambda \geq \mathrm{card} S$, and $\lambda^{|S|} = \lambda$, we find that there are λ complete types in $\vec{x}$ over A i.e. T is λ-stable. ∎

Thus if T is equational in $\vec{x}$, we can speak of non-forking extensions of types.

We recall the following properties of non-forking extensions of types (see properties 0 and 4 in the introduction).

Given $A \subset \overline{M}$, p complete type over A in the finite tuple $\vec{x}$, q a complete extension of p to $\overline{M}$, then

(i) q is a non-forking extension of p iff q has less or equal than $2^{|T|+\aleph_0}$ A-conjugates;

(ii) If q is a non-forking extension of p then for any formula $\Theta(\vec{x};\vec{t})$, $q\restriction\Theta$ has finitely many A-conjugates.

Furthermore, non-forking extensions of types satsify the monotonicity-transitivity property. (see property 1 in the introduction).

3. **Theorem.** **let T be stable, $A \subset B$, p a complete type over B and R a set of equations. If p does not fork over A, then p is an R-minimal extension of $p\restriction A$ to B. If T is actually R-equational in $\vec{x}$ and p is a type in $\vec{x}$ then in fact, p does not fork over A iff p is an R-minimal extension of $p\restriction A$ to B.**

Proof. By monotonicity-transitivity applied to non-forking and R-minimal extensions, we can assume without loss of generality that $B = \overline{M}$.

If p is a non-forking extension of $p\restriction A$ to B, then by property (ii) given above, for any formula φ in R, $p\restriction\varphi$ has finitely many A-conjugates. Since φ is an equation, $p\restriction\varphi$ is completely determined by p^{Φ} where $\Phi = cl^{+}(\varphi)$; it follows that p^{Φ} is almost over A. We deduce from theorem A.7.(i) that p is an R-minimal extension of $p\restriction A$.

Suppose T is R-equational in $\vec{x}$ and p is a type in $\vec{x}$. We have already shown that if p is a non-forking extension of $p\restriction A$ then p is an R-minimal extension of $p\restriction A$ to B.

Conversely, if p is an R-minimal extension of $p\restriction A$ to B, then by theorem A.7.(i), for any φ in R, p^{Φ} is almost over A, where $\Phi = cl^{+}(\varphi)$. It follows that $p\restriction\varphi$ has finitely many A-conjugates. Since T is R-equational, $p \sim \cup \{p\restriction\varphi;\ \varphi \in R\}$; whence P has at most $2^{|T|+\aleph_0}$ A-conjugates. We conclude that p is a non-forking extension of $p\restriction A$. ■

Remark. (for stable theories), one can prove the theorem obove directly, without first investigating the properties of R-minimal extensions (c.f. [P.S]). Then, one deduces the properties of R-minimal extensions from those of non-forking extensions. For instance, the following theorem,

4. **Theorem (symmetry). Suppose T is S-equational, $A \subset \bar{M}$, $\vec{a}$, $\vec{b}$ in $\bar{M}$. Then, $tp(\vec{a};A \cup \vec{b})$ is an S-minimal extension of $tp(\vec{a};A)$ to $A \cup \vec{b}$ iff $tp(\vec{b};A \cup \vec{a})$ is an S-minimal extension of $tp(\vec{b};A)$ to $A \cup \vec{a}$.**

Can be seen as a corollary of theorem 3 and the symmetry property of non-forking extensions.

For completeness sake however, we will prove below theorem 4 without the use of any pre-given result of stability theory.

First, some lemmas,

5. **Lemma.** Suppose T is S-equational in $\vec{x}$, $A \subset \bar{M}$, p a complete type in $\vec{x}$ over A and q an S-minimal extension of p to $\bar{M}$. Then, for any formula $\Theta(\vec{x};\vec{t})$, $q \upharpoonright \Theta$ has finitely many A-conjugates.

Proof. By theorem A.7.(i), for any formula φ in S, q^{Φ} is almost over A, where $\Phi = cl^{+}(\varphi)$; hence $q \upharpoonright \varphi$ has finitely many A-conjugates. it follows that for any finite set S_0 of formulas in S, $q \upharpoonright S_0$ has finitely many A-conjugates.

Now, given a formula $\Theta(\vec{x};\vec{t})$, since T is S-equational in $\vec{x}$, there is a finite set S_0 of formulas in S, such that $\Theta \in cl(S_0)$. But $q \upharpoonright S_0$ i.e. $q \upharpoonright cl(S_0)$ has finitely many A-conjugates; we easily conclude that $q \upharpoonright \Theta$ has finitely many A-conjugates. ■

6. **Lemma.** Suppose T is S-equational in $\vec{x}$, $A \subset B$, $\vec{a}$ in $\bar{M}$, length$\vec{a}$ = length$\vec{x}$. Let E be the set of all equations in $\vec{x}$. Then the following assertions are equivalent:

1. $tp(\vec{a};B)$ is an S-minimal extension of $tp(\vec{a};A)$ to B.

2. $tp(\vec{a} \cup A;B)$ is an S-minimal extension of $tp(\vec{a} \cup A;A)$ to B.

3. $tp(\vec{a} \cup A;B)$ is an E-minimal extension of $tp(\vec{a} \cup A;A)$ to B.

Proof. By A.6.(ii), since $S \subset E$, if $tp(\vec{a} \cup A;B)$ is an E-minimal extension of $tp(\vec{a} \cup A;A)$ to B then $tp(\vec{a} \cup A;B)$ and $tp(\vec{a};B)$ are respectively S-minimal extensions of $tp(\vec{a} \cup A;A)$ and $tp(\vec{a};A)$ to B.

In other words we have 3. $\longrightarrow$ 1. and 3. $\longrightarrow$ 2..

Similarly, 2. $\longrightarrow$ 1.

Remains to show 1. $\longrightarrow$ 3., Suppose 1 holds.

By monotonicity-transitivity we can assume without loss of generality $B = \bar{M}$.

Let $\vec{c}$ be a tuple in A and let $\varphi(\vec{x}\frown\vec{y};\vec{t}) \in E$ length$\vec{y}$ = length$\vec{c}$. By lemma 5, $tp(\vec{a};B) \vdash \varphi(\vec{x};\vec{y}\frown\vec{t})$ has finitely many A-conjugates. It follows easily that $tp(\vec{a}\frown\vec{c};B) \vdash \varphi(\vec{x}\frown\vec{y};\vec{t})$ has finitely many A-conjugates. We deduce, by A.7.(i). that $tp(\vec{a}\frown\vec{c};B)$ is an E-minimal extension of $tp(\vec{a}\frown\vec{c};A)$ to B. Since $\vec{c}$ was arbitrarily chosen in A we conclude by A.6.(ii) that $tp(\vec{a} \cup A;B)$ is an S-minimal extension of $tp(\vec{a} \cup A;A)$ to B. ■

Proof of theorem 4. By lemma 6, if $tp(\vec{a};A \cup \vec{b})$ is an S-minimal extension of $tp(\vec{a};A)$ then $tp(\vec{a} \cup A;A \cup \vec{b})$ is an E-minimal extension of $tp(\vec{a} \cup A;A)$, where E is the set of all equations; since E is full, we deduce, by A.6.(i), that $tp(\vec{b} \cup A,\vec{a} \cup A)$ is an E-minimal extension of $tp(\vec{b} \cup A;A)$. By lemma 6 again we conclude, $tp(\vec{b};\vec{a} \cup A)$ is an S-minimal extension of $tp(\vec{b};A)$.

The converse is given by symmetry of the argument above. ■

7. **Lemma. Suppose T is S-equational, $A \subset B$, $C \subset \bar{M}$ such that $tp(C;B)$ is an S_B-minimal extension of $tp(C;A)$. Then $tp(C;B)$ is an S-minimal extension of $tp(C;A)$.**

Proof. Suppose $tp^S(C;B) \supset p^S$, where p is a complete type over B extending $tp(C;A)$.

Then of course $tp_B^S(C;B) \supset p_B^S$. By S_B-minimality it follows that $tp_B^S(C;B) = p_B^S$; hence, T being S-equational, $tp(C;B) = p$, which implies $tp^S(C;B) = p^S$. Thus $tp(C;B)$ is an S-minimal extension of $tp(C;A)$. ■

8. **Corollary. T S-equational, $A \subset B$, $A \subset C$, such that $tp(C;A)$ is S-irreducible (resp. S-full). Then, the following assertions are equivalent:**

a. $tp(C;B)$ is an S_B-minimal extension of $tp(C;A)$.

b. $tp(C;B)$ is an S-minimal extension of $tp(C;A)$.

c. $tp(C;A) \vdash tp^S_B(C;A)$ (resp. $tp^S_A(C;A) \vdash tp^S_A(C;B)$).

Proof. By lemma 7, a$\longrightarrow$b. By proposition A.4.(i), b$\longrightarrow$c. Finally, c$\longrightarrow$a is clear. ■

9. Theorem. Suppose T is S-equational, $A \subset B$ and $A \subset C$. Assume moreover S is symmetric.

Consider the following assertions:

a. $tp(B \mathring{\cup} C;A)$ is an S-free amalgam of $tp^S_A(B;A)$ and $tp^S_A(C;A)$ over A.

b. $tp(B \mathring{\cup} C;A)$ is an S-free amalgam of $tp^S(B;A)$ and $tp^S(C;A)$ over A.

c. $tp(B \mathring{\cup} C;A)$ is an S-free amalgam of $tp(B;A)$ and $tp(C;A)$ over A.

d. $tp(C;B)$ is an S_B-minimal extension of $tp(C;A)$.

e. $tp(C;B)$ is an S-minimal extension of $tp(C;A)$.

Then, a$\longrightarrow$b$\longrightarrow$c$\longrightarrow$d$\longrightarrow$e.

If in addition $tp(C;A)$ is S-irreducible then b, c, d, and e are equivalent; if $tp(B;A)$ and $tp(C;A)$ are S-full then all the assertions are equivalent.

Proof. a$\longrightarrow$b$\longrightarrow$c is immediate, for $tp(C;A) \supset tp^S(C;A) \supset tp^S_A(C;A)$, and $tp(B;A) \supset tp^S(B;A) \supset tp^S_A(B;A)$.

c$\longrightarrow$d follows from proposition IV.1 (since S-free amalgams are a fortiori S-minimal amalgams.

d$\longrightarrow$e follows from lemma 7.

So we have a$\longrightarrow$b$\longrightarrow$c$\longrightarrow$d$\longrightarrow$e.

Suppose $tp(C;A)$ is S-irreducible. Then,

e$\longrightarrow$b. Suppose e holds. Since $tp(C;A)$ is assumed S-irreducible, by proposition II.5.(ii), we have that $tp(C;A) \vdash tp^S(C;B)$; in particular $tp(C;A) \vdash tp^S_B(C;B)$.

Let $\vec{x}_1 = \langle x_b;\ b \in B\rangle$ and $\vec{x}_2 = \langle x_c;\ c \in C\rangle$ and r a complete type in $\vec{x}_1 \frown \vec{x}_2$ over A extending $tp^S(B;A) \cup tp^S(C;A)$; we want to show

$$tp(B \mathring{\cup} C;A) \cap S(\vec{x}_1;\vec{x}_2) \subset r \cap S(\vec{x}_1;\vec{x}_2).$$

So let $\varphi(\vec{x}_b;\vec{x}_c) \in tp(B \mathring{\cup} C;A) \cap S(\vec{x}_1;\vec{x}_2)$ i.e. $\models \varphi(\vec{b};\vec{c})$ and $\varphi(\vec{x}_b;\vec{t}) \in S$;

we already noted that $tp(C;A) \vdash tp^S_B(C;B)$; hence

$$tp(\vec{c};A) \vdash tp^S_B(\vec{c};B) \text{ and therefore}$$

$$tp^S(\vec{c};A) \vdash tp^S_B(\vec{c};B).$$

It follows there is a subset R of S which is equational and such that $tp^R(\vec{c};A) \vdash \varphi(\vec{b};\vec{x}_c)$.

Let $\Theta(\vec{a};\vec{x_c})$, $\vec{a} \in A$, be such that $\Theta \sim tp^R(\vec{c};A)$; thus Θ is S-definable and the formula

$$X(\vec{x_b};a) = \forall \vec{x_c}(\Theta(\vec{a};\vec{x_c}) \longrightarrow \varphi(\vec{x_b};\vec{x_c})) \quad ,$$

is satisfied by $\vec{b}$.

By III.19, $X(\vec{x_b};\vec{a})$ is S-definable;

thus $tp^S(B;A) \vdash X(\vec{x_b};\vec{a})$. Since also $tp^S(C;A) \vdash tp^R(\vec{c};A) \vdash \Theta(\vec{a};\vec{x_c})$, we deduce that $r \vdash X(\vec{x_b};\vec{a}) \wedge \Theta(\vec{a};\vec{x_c})$.

Now, clearly $X(\vec{x_b};\vec{a}) \wedge \Theta(\vec{a};\vec{x_c}) \vdash \varphi(\vec{x_b};\vec{x_c})$.

Thus $\varphi(\vec{x_b};\vec{x_c}) \in r$.

Since we already have b$\longrightarrow$c$\longrightarrow$d$\longrightarrow$e (from the first part of the proof), we conclude that if $tp(C;A)$ is S-irreducible then b, c, d and e are equivalent.

Finally, suppose $tp(B;A)$ and $tp(C;A)$ are S-full; then $tp^S(B;A) \sim tp^S_A(B;A)$ and $tp^S(C;A) \sim tp^S_A(C;A)$.

b$\longrightarrow$a has now become immediate.

Since, S-full $\Longrightarrow$ S-irreducible and we already know a$\longrightarrow$b, we deduce from what preceded that in that case all the assertions are equivalent. ■

Examples.

1. Suppose T is a universal theory which is S-equational with $S = \mathrm{cl}^+(At)$. Let M be a submodel of $\bar{M}$, $a,b \in \bar{M}$. For $\vec{c} \in \bar{M}$, let $M[\vec{c}]$ denote the substructure of $\bar{M}$ generated

by $M \cup \{\vec{c}\}$; $M[\vec{c}]$ is a model of T.

Given $N_1 \supset M$ and $N_2 \supset M$ N_1, N_2 models of T, a map $f:N_1 \longrightarrow N_2$ is a "homomorphism" over M if $f\restriction M = id_M$ and for any atomic formula $\varphi(\vec{x})$ and $\vec{c}$ in N_1,

$$\models \varphi(\vec{c}) \implies \models \varphi(f\vec{c}).$$

Then, $tp(a;M \cup b)$ is an S-minimal extension of $tp(a;M)$ iff for any $N \supset M$, N a model of T, $f_1:M[a] \longrightarrow N$ and $f_2:M[b] \longrightarrow N$ homomorphisms over M, there is a homomorphism $f:M[a,b] \longrightarrow N$ over M such that $f\restriction M[a] = f_1$ and $f\restriction M[b] = f_2$.

Indeed, since M is a model, $tp(a;M)$ and $tp(b;M)$ are S-full. Now $tp(a;M \cup b)$ is an S-minimal extension of $tp(a;M)$ iff $tp(M \cup a;M \cup b)$ is an S-minimal extension of $tp(M \cup a;M)$ iff (by theorem 7) $tp(M \cup a \,\dot{\cup}\, M \cup b;M)$ is an S-free amalgam of $tp^S_M(M \cup a;M)$ and $tp^S_M(M \cup b;M)$.

To conclude the proof of the claim, it suffices to translate what S-free amalgam means in terms of homomorphisms, (noting that for $N_1,N_2 \supset M$, there is a homomorphism $f:N_1 \longrightarrow N_2$ over M iff $tp^S_M(N_1;M) \subset tp^S_M(N_2;M)$.

2. Suppose T is a complete theory of modules and S is the closure under conjunctions and disjunctions of the set of positive primitive formulas; we already noted T is S-

equational.

Now suppose that M and N are pure submodels of $\overline{M}$ such that the sum $M + N$ is direct (i.e. $M \cap N = \{0\}$) and $M + N$ is pure in $\overline{M}$; then $tp(M;N)$ is an S-minimal extension of $tp(M;\emptyset)$ to N (i.e., M and N are independent over $\emptyset$).

Indeed, it is easily seen that, since $M \cap N = \{0\}$ and $M + N$ is pure in $\overline{M}$, $tp(M \dot{\cup} N;\emptyset)$ is an S-free amalgam of $tp(M;\emptyset)$ and $tp(N;\emptyset)$. It follows from theorem 7 that $tp(M;N)$ is an S-minimal extension of $tp(M;\emptyset)$ to N.

10. **Note.** With the notations of theorem 9 above, but with S an arbitrary set of equations (T not necessarily S-equational), and $tp(C;A)$ S-irreducible, the assertion c$\longrightarrow$b still holds. Furthermore, if $tp(C;A)$ and $tp(B;A)$ are S-full then the assertion e$\longrightarrow$a still holds. Indeed, in the proof of e$\longrightarrow$b (resp. e$\longrightarrow$a when $tp(C;A)$ and $tp(B;A)$ are S-full) above we only used the fact that S is a set of equations.

11. **Corollary.** Let T be stable, M a model, $M \subset B$ and $M \subset C$, R a symmetric set of equations. If $tp(B;C)$ is a non-forking extension of $tp(B;M)$ then $tp(B \dot{\cup} C;M)$ is an R-free amalgam of $tp^{S}_{M}(B;M)$ and $tp^{S}_{M}(C;M)$ over M.

Proof. Since R is a set of equations and M is a model, $tp(B;M)$ and $tp(C;M)$ are R-full. By theorem 3, if $tp(B;C)$ does not fork over M, then $tp(B;C)$ is an R-minimal extension of $tp(B;M)$; hence, by note 10, $tp(B \mathbin{\dot{\cup}} C;M)$ is an R-free amalgam of $tp^S_M(B;M)$ and $tp^S_M(C;M)$. ■

In fact, corollary 11 characterizes a set of equations in a stable theory.

To explain what we mean let us first define a formula $\varphi(\vec{x};\vec{t})$ to be *fundamental* in T (T stable) if;

whenever M is a model and $\vec{a}$, $\vec{b}$ are in $\bar{M}$ such that $tp(\vec{a};M \cup \vec{b})$ does not fork over M and $\models \varphi(\vec{a};\vec{b})$ then, for any $\vec{a}'$, $\vec{b}'$ in $\bar{M}$ such that $tp(\vec{a};M) = tp(\vec{a}';M)$ and $tp(\vec{b};M) = tp(\vec{b}';M)$, we have $\models \varphi(\vec{a}';\vec{b}')$.

It is easy to check that, for R a symmetric set of formulas, R is a set of fundamental formulas iff whenever M is a model and $\vec{a}$, $\vec{b}$ are in $\bar{M}$ such that $tp(\vec{a};M \cup \vec{b})$ does not fork over M then $tp(M \cup \vec{a} \mathbin{\dot{\cup}} M \cup \vec{b};M)$ is an R-free amalgam of $tp(M \cup \vec{a};M)$ and $tp(M \cup \vec{b};M)$.

[Proof. Suppose R is a set of fundamental formulas, and r a complete extension of $tp(M \cup \vec{a};M) \cup tp(M \cup \vec{b};M)$ over M.

Suppose $\varphi(\vec{x};\vec{y}) \in tp(M \cup \vec{a} \mathbin{\dot{\cup}} M \cup \vec{b};M)$ and $\varphi(\vec{x};\vec{t}) \in R$; let $\vec{c}$ and $\vec{d}$ be tuples in $M \cup \vec{a}$ and $M \cup \vec{b}$ respectively, such that $\models \varphi(\vec{c};\vec{d})$. If $tp(\vec{a};M \cup \vec{b})$ does not fork over M then $tp(M \cup \vec{a};M \cup \vec{b})$ does not fork over M; hence $tp(\vec{c};M \cup \vec{b})$ does not fork over M. Since φ is fundamental and $\models \varphi(\vec{c};\vec{d})$, it follows that $\varphi(\vec{x};\vec{y}) \in r$. Thus $tp(M \cup \vec{a} \cup M \cup \vec{b};M)$ is an R-free amalgam of $tp(M \cup \vec{a};M)$ and $tp(M \cup \vec{b};M)$.

The converse follows immediately from the definition of R-free amalgam.]

We will use in the theorem below the following fact of stability theory: if T is stable, $M \subset B \subset \bar{M}$, p a complete type over M and q a non-forking extension of p over B, then, for any formula $\varphi(\vec{x};\vec{t})$ there is a formula $d\varphi(\vec{t};\vec{c})$, $\vec{c}$ in M, such that for $\vec{n}$ in B, $\varphi(\vec{x};\vec{n}) \in q$ iff $\models d\varphi(\vec{n};\vec{c})$; we say that q is an heir of p.

12. **Theorem.** *T* **stable. A formula $\varphi(\vec{x};\vec{t})$ is an equation iff $\varphi(\vec{x};\vec{t})$ is fundamental.**

Proof. By corollary 11, if φ is an equation then whenever M is a model, $\vec{a}$, $\vec{b}$ are in $\bar{M}$ such that $tp(\vec{a};M \cup \vec{b})$ does not fork over M,

$tp(M \cup \vec{a} \cup M \cup \vec{b};M)$ is a Ξ-free amalgam of $tp(M \cup \vec{a};M)$ and $tp(M \cup \vec{b};M)$, where $\Xi = cl^{+}(\{\varphi^{\vec{x}},\varphi^{\vec{t}}\})$. It follows that φ is fundamental.

Conversely, suppose φ is fundamental. Let M be a model and p a complete type in $\vec{x}$ over M.

Since T is stable there is a formula $d\varphi(\vec{t};\vec{c})$, $\vec{c}$ in M such that for $\vec{m}$ in M

$$\varphi(\vec{x};\vec{m}) \in p \iff \models d\varphi(\vec{m};\vec{c}).$$

Claim. $p \cup d\varphi(\vec{t};\vec{c}) \vdash \varphi(\vec{x};\vec{t})$.

Indeed, let $\vec{a}'\frown\vec{b}$ realize $p(\vec{x}) \cup \{d\varphi(\vec{t};\vec{c})\}$. Let $\vec{a}$ be such that $tp(\vec{a};M \cup \vec{b})$ is a non-forking extension of $p(\vec{x})$; then $tp(\vec{a};M \cup \vec{b})$ is an heir of $tp(\vec{a};M)$ which implies, for $\vec{n}$ in $M \cup \vec{b}$,

$$\varphi(\vec{x};\vec{n}) \in tp(\vec{a};M \cup \vec{b}) \iff \models d\varphi(\vec{n};\vec{c}).$$

Since $\models d\varphi(\vec{b};\vec{c})$, $\varphi(\vec{x};\vec{b}) \in tp(\vec{a};M \cup \vec{b})$. We have, $\models \varphi(\vec{a};\vec{b})$ and $tp(\vec{a};M \cup \vec{b})$ is a non-forking extension of $tp(\vec{a};M)$. It follows from the definition of a fundamental formula that $\models \varphi(\vec{a}';\vec{b})$, proving the claim.

By compactness, we deduce there is a finite subset p_0 of p such that, $p_0 \cup d\varphi(\vec{t};\vec{c}) \vdash \varphi(\vec{x};\vec{t})$.

Furthermore, since for any $\vec{m}$ in M such that $\varphi(\vec{x};\vec{m}) \in p$, we have that $\models d\varphi(\vec{m};\vec{c})$, we get

$$p_0 \vdash p_M^{\varphi}.$$

We have shown that for any model M and complete type p over M there is a finite subset p_0 of p such that $p_0 \vdash p_M^{\varphi}$.

We conclude by proposition I.12 that φ is an equation. ■

Section C: Rank and Height

Preliminaries. For simplicity we assume in this section $\text{card}T \leq \aleph_0$. Let us recall some facts and definitions of stability theory. (For more details c.f. [M] and [P.1]).

- T is said *ω-stable* if for any infinite cardinal λ and any set A of cardinality λ, there are λ-many complete types over A.
- T is said superstable if there is a cardinal μ such that for any cardinal $\lambda \geq \mu$ and any set A of cardinality λ, there are λ-many complete types over A.
- The Morley-rank is defined as follows: (by induction on α).
  - for any type p, $MR(p) \geq 0$.
  - for α limit, $MR(p) \geq \alpha$ iff $MR(p) \geq \beta$ for any $\beta < \alpha$.
  - for p a finite type, $MR(p) \geq \alpha + 1$ if there is a sequence $(p_i)_{i<\omega}$ of contradictory types extending p and such that $MR(p_i) \geq \alpha$.
  - for p arbitrary, $MR(p) \geq \alpha + 1$ if $MR(p_0) \geq \alpha+1$ for any finite subset of p.

We write $MR(p) = \alpha$ if $MR(p) \geq \alpha$ and $MR(p) \not\geq \alpha + 1$; $MR(p) = \infty$ if $MR(p) \geq \alpha$ for every α. If $MR(p) = \alpha$, the

Morley-degree of p, which we denote by $\mathrm{Md}(p)$, is the maximum (finite) number of contradictory types extending p and of Morley-rank α.

Fact 1. If p is a complete type over A and q is a non-forking extension of p to $B \supset A$, then $\mathrm{MR}(q) = \mathrm{MR}(p)$.

Fact 2. T is ω-stable iff $\mathrm{MR}(-)$ takes its value in On.

- The Lascar-rank is defined on complete types over subsets of $\bar{M}$, as follows: (by induction on α).
- $U(p) \geq 0$ for any complete type p over a set.
- for α limit, $U(p) \geq \alpha$ if $U(p) \geq \beta$ for any $\beta < \alpha$.
- $U(p) \geq \alpha + 1$ if there is a forking extension of p (over some set) such that $U(q) \geq \alpha$.

Write $U(q) = \alpha$ if $U(q) \geq \alpha$ and $U(q) \not\geq \alpha + 1$; $U(q) = \infty$ if $U(q) \geq \alpha$ for every ordinal α.

Fact 3. If p is complete over A and q is a non-forking extension of p over $B \supset A$, then $U(p) = U(q)$.

Fact 4. T is superstable iff $U(-)$ takes its values in On.

- The fundamental order ($\geq$) on complete types over models is defined as follows:

given p and q, complete types over the models M and N respectively, we write $p \geq q$ if for any formula $\varphi(\vec{x};\vec{m})$ in p there is $\vec{n}$ such that $\varphi(\vec{x};\vec{n})$ is in q.

Fact 5. If q is a non-forking extensions of p then $q \geq p$ (and clearly $p \geq q$).

Fact 6. if $p \geq q$, where q is a complete type over $\bar{M}$ then there is an automorphism σ such that $p \subset \sigma q$.

A type p is said positive if $p \subset D^S(\overline{M})$.

0. **Definitions.**

(i) We say (T,S) has the d.c.c. in $\vec{x}$ if $S^{\vec{x}}$ is equational (recall $S^{\vec{x}} = \{\varphi(\vec{x};\vec{t}) \in S\}$).

(ii) (T,S) has the d.c.c. on irreducible types in $\vec{x}$ if there is no infinite descending chain of irreducible positive types in $\vec{x}$ i.e. if there is no sequence $(p_i)_{i<\omega}$ of types in $\vec{x}$ over $\overline{M}$ such that $p_i \subset D^S(\overline{M})$, p_i is S-irreducible, $p_{i+1} \vdash p_i$ and $p_i \nvdash p_{i+1}$, for $i < \omega$.

Say (T,S) has the d.c.c. (resp. d.c.c. on irreducible types) if (T,S) has the d.c.c. (resp. d.c.c. on irreducible types) in any $\vec{x}$.

(d.c.c. stands for "descending chain condition").

Note. Since a type q over $\overline{M}$ is S-irreducible iff there is a complete type q' over $\overline{M}$ extending q and such that $(q')^S = q^S$, (T,S) has the d.c.c. on irreducible types iff there is no sequence $(p_i)_{i<\omega}$ of complete types over $\overline{M}$ such that

$$p_{i+1}^S \supsetneq p_i^S.$$

Examples.

a) (ACF_0,S) and (DCF_0,S) have the d.c.c., for S the set of quantifier-free positive formulas (see the end of chapter I).

b) Let T be a complete theory of modules, S the set of positive primitive formulas. Then (T,S) has the d.c.c. on irreducible types iff there is no descending chain $(G_i)_{i<\omega}$ of S-definable subgroups (of $\bar{H}$) such that G_{i+1} has infinite index in G_i.

Indeed, suppose p and q are irreducible positive types such that $p \vdash q$ and $q \nvdash p$. Then for any φ in q there is ψ in p such that $\psi \vdash \varphi$ and $q \nvdash \psi$. Now φ and ψ are instances of positive primitive formulas so that φ and ψ define cosets of S-definable groups G_φ and G_ψ respectively (see 0.4.(ii)).

Moreover $G_\psi \subset G_\varphi$ (for $\psi \vdash \varphi$) and G_ψ has infinite index in G_φ: for if G_ψ has finite index in G_φ, then φ can be written as a finite union of cosets of G_ψ; since q is S-irreducible and $\varphi \in q$ it would follow that one of these cosets belong to q, whence to p; so there is a coset X of G_ψ which belong to p and q; but $\psi \in p$ and ψ is a coset of G_ψ, hence necessarily $X = \psi$ (if not $X \cap \psi = \emptyset$ and p is inconsistent); therefore $\psi \in q$ ✗.

It follows easily that if $(p_i)_{i<\omega}$ is a sequence of positive irreducible types with $p_{i+1} \vdash p_i$ and $p_i \nvdash p_{i+1}$

$(i < \omega)$ we can build a chain $(G_i)_{i<\omega}$ of S-definable groups with $G_{i+1} \subset G_i$ and G_{i+1} has infinite index in G_i.

Conversely if $(G_i)_{i<\omega}$ is a chain of S-definable groups as above, let

$$p_i = G_i \cup \{G;\ G \text{ an } S\text{-definable subgroup of } G_i \text{ of finite index in } G_i\}.$$

Then $(p_i)_{i<\omega}$ is a sequence of positive types such that:

- p_i is S-irreducible: for if $p_i \vdash \bigcup_{i<n} X_i$, X_i S-definable, then for some G in p_i, $G \vdash \bigcup_{i<n} X_i$ i.e. $G \subset \bigcup_{i<n} X_i$; also without loss of generality we can assume $X_i \subset G$. It follows (by Von Neumann's lemma) that $G \subset \bigcup_{j\in J} X_j$, $J \subset n$, where for each $j \in J$, X_j is a coset of a subgroup $G(X_j)$ of G_j of finite index in G_i; hence for $j \in J$, $G(X_j) \in p_i$ and therefore $G' = G \cap \bigcap_{j\in J} G(X_j) \in p_i$; now clearly $G' \cap \bigcup_{i<n} X_i$ is empty unless $G' \subset X_i$ for some $i < n$; we conclude that $p_i \vdash X_i$ for some $i < n$.
- $p_{i+1} \vdash p_i$: for if G is a subgroup of G_i of finite index in G_i then $G \cap G_{i+1}$ is a subgroup of G_{i+1} of finite index in G_{i+1}.
- $p_i \not\vdash p_{i+1}$: for $p_i \not\vdash G_{i+1}$ (G_{i+1} having infinite index in G_i).

Thus $(p_i)_{i<\omega}$ is a descending chain of irreducible positive types.

1. **Theorem.** Let T be S-equational in $\vec{x}$.

(i) if (T,S) has the d.c.c. in $\vec{x}$ then T is totally transcendental.

(ii) if (T,S) has the d.c.c on irreducible types in $\vec{x}$ then T is superstable.

Proof.

(i) Suppose (T,S) has the d.c.c. in $\vec{x}$. Let A be a set of cardinality $\lambda \geq \aleph_0$. A complete type p in $\vec{x}$ over A is completely determined by p_A^S; since S is equational p_A^S is equivalent to a single formula over A.

It follows there are at most λ ($=$ card$D(A)$)-many complete types in $\vec{x}$ over A. Thus T is totally transcendental.

(ii) Suppose (T,S) has the d.c.c. on irreducible types in $\vec{x}$.

Claim. For any set C and complete type q in $\vec{x}$ over C there is a finite subset A of C such that q is an S-minimal extension of $q \upharpoonright A$ to C.

Indeed, suppose the claim is false; we first construct by induction (on $i < \omega$) a sequence of sets $(A_i)_{i<\omega}$ and a sequence of types $(q_i)_{i<\omega}$ such that:

A_i is finite, $A_i \subset A_{i+1} \subset B$;

$q_i = q \upharpoonright A_i$, $q^S_{i+1} \vdash q^S_i$ and $q^S_i \nvdash q^S_{i+1}$;

q_{i+1} is not an S-minimal extension of q_i to A_{i+1}.

Take A_0 any finite subset of M and $q_0 = q \upharpoonright A_0$.

Suppose the construction of (A_i) and (q_i) achieved up to i. Since the claim is supposed false, q is not an S-minimal extension of $q \upharpoonright A_i = q_i$.

By theorem A.6.(ii), it follows there is a formula φ in S such that q is not a Φ-minimal extension of q_i, where $\Phi = cl^+(\varphi)$. Hence, there is a finite subset B of M, $B \supset A_i$ such that $q \upharpoonright B$ is not a Φ-minimal extension of q_i to B; therefore (by A.6.(ii)) $q \upharpoonright B$ is not an S-minimal extension of q_i to B. Thus $(q \upharpoonright B)^S \vdash q^S_i$ (since $q \upharpoonright B \supset q_i$) and $q^S_i \nvdash (q \upharpoonright B)^S$ (for if $q^S_i \vdash (q \upharpoonright B)^S$, $q \upharpoonright B$ would be an S-minimal extension of q_i to B).

Let $A_{i+1} = B$ and $q_{i+1} = q \upharpoonright B$. This finishes the inductive step of the construction.

Now, we construct by induction a sequence $(p_i)_{i<\omega}$ of complete types over $\overline{M}$ such that $p^S_{i+1} \supsetneq p^S_i$, and p_i is an S-minimal extension of $\sigma_i q_i$ to $\overline{M}$ for some automorphism σ_i of $\overline{M}$.

Let p_0 be an arbitrary S-minimal extension of q_0 to $\bar{M}$ and suppose the construction of p_i achieved up to i. Let q' be an S-minimal extension of $\sigma_i q_{i+1}$ to $\bar{M}$; since $q_{i+1}^S \vdash q_i^S$,

$$(\sigma_i q_{i+1})^S \vdash (\sigma_i q_i)^S \text{ and } (q')^S \supset (\sigma_i q_i)^S.$$

Hence, by II.A.3, there is an S-minimal extension q_i' of $\sigma_i q_i$ to $\bar{M}$ such that $(q')^S \supset (q_i')^S$. Furthermore $(q')^S \neq (q_i')^S$, for if $(q')^S = (q_i')^S$ then q' would be an S-minimal extension of $\sigma_i q_i$ and by monotonicity $\sigma_i q_{i+1}$ would be an S-minimal extension of $\sigma_i q_i$ to $\sigma_i A_{i+1}$; it would follows that q_{i+1} is an S-minimal extension of q_i to A_{i+1} ✖.

Now, by induction hypothesis p_i is an S-minimal extension of $\sigma_i q_i$ to $\bar{M}$. Thus, by A.7.(ii), there is an automorphism τ over A_i such that $\tau q_i' = p_i$.

Then, $\tau q'$ is an S-minimal extension of $\tau\sigma_i q_{i+1}$ (for q' was chosen an S-minimal extension of $\sigma_i q_{i+1}$), and

$$(\tau q')^S \underset{\neq}{\supset} (\tau q_i')^S = p_i^S, \text{ for } (q')^S \underset{\neq}{\supset} (q_i')^S.$$

Let $\sigma_{i+1} = \tau\sigma_i$ and $p_{i+1} = \tau q'$. This finishes the inductive step of the construction of $(p_i)_{i<\omega}$.

But of course, the sequence $(p_i^S)_{i<\omega}$ as constructed above contradicts the d.c.c. on irreducible types. The claim is now proven.

From the claim we deduce that the number of complete types in $\vec{x}$ over a set C of cardinality λ is at most the number of triples $\langle A,p,q\rangle$ where A is a finite subset of C, p is a complete type over A and q is an S-minimal extension of p to C.

Clearly there are λ-many finite subsets of C, and over a finite set there are at most $2^{\aleph_0}$-many complete types; by II.A.15, a type over C has at most $2^{\aleph_0}$-many S-minimal extensions to C.

We conclude there are at most $2^{\aleph_0} + \lambda$-many complete types in $\vec{x}$ over C, which proves T is superstable. ■

Naturally, one is interested in the converse of theorem (i), (ii).

For instance, if T is a complete theory of modules and S is the set of positive primitive formulas then it is a fact (c.f. [Z]) that T is totally transcendental iff there is no infinite descending chain of S-definable subgroups, and T is superstable iff there is no infinite descending chain $(G_i)_{i<\omega}$ of S-definable subgroups with G_{i+1} of infinite index in G_i. Thus, in that case, T is totally transcendental iff (T,S) has the d.c.c and T is superstable iff (T,S) has the d.c.c on irreducible types (see example b) above).

However, in general, the converses of 1.(i) and 1.(ii) are false as the following example shows:

Let $L = \{P_i ; i < \omega\}$, P_i a unary predicate symbol, and let T be the complete theory which says:

$$P_{i+1} \subset P_i \text{ and } P_i \setminus P_{i+1} \text{ is infinite.}$$

T is clearly S-equational where $S = \mathrm{cl}^+(\{P_i; i < \omega\})$ and T is also totally-transcendental. But (T,S) does not have the d.c.c. on irreducible types, for the sequence $(p_i)_{i<\omega}$, where $p_i = \{P_i(x)\}$, is clearly a descending chain of positive S-irreducible types.

On the other hand, it is obvious that T is R-equational, where $R = \mathrm{cl}^+(\{\neg P_i(x)\})$, and (T,R) has the d.c.c..

The following question then arises:

Question. Given a totally-transcendental (resp. superstable) equational theory T, can we find a set of equations R such that T is R-equational and (T,R) has the d.c.c. (resp. d.c.c. on irreducible types) ?

We do not know the answer to this question; we shall give below criterions for an S-equational totally transcendental theory (resp. superstable) to have the d.c.c. on x (resp. the d.c.c. on irreducible types).

2. **Definitions.**

(i) Given definable sets X and Y, we say X fixes Y if any automorphism σ which fixes *setwise* X fixes setwise Y (i.e. $\sigma X = X \Longrightarrow \sigma Y = Y$).

(ii) A family $(X_i)_{i \in I}$ of definable sets is an invariant family if for any $i, j \in I$, X_i fixes X_j.

Thus, an infinite invariant descending chain of S-definable sets is a sequence $(X_i)_{i<\omega}$ of S-definable sets such that $(X_i)_{i<\omega}$ is an invariant family and $X_{i+1} \subsetneq X_i$, $(i < \omega)$.

3. **Lemma.** Suppose $\varphi(\vec{x};\vec{t})$ is an equation and $\varphi(\vec{x};\vec{a}) \vdash \varphi(\vec{x};\sigma\vec{a})$, where σ is an automorphism, then $\varphi(\vec{x};\vec{a}) \sim \varphi(\vec{x};\sigma\vec{a})$.

Proof. Suppose $\varphi(\vec{x};\vec{a}) \not\sim \varphi(\vec{x};\sigma\vec{a})$. Then we can easily construct a sequence $(\vec{a}_i)_{i<\omega}$ such that $\varphi(\vec{x};\vec{a}_{i+1}) \vdash \varphi(x;\vec{a})$ and $\varphi(\vec{x};\vec{a}_{i+1}) \not\sim \varphi(x;a_i)$; but that contradicts equationality.

4. **Lemma.** Let X and Y be non-empty definable sets; say $X = \varphi(x;\vec{a})$ and $Y = \psi(x;\vec{b})$. Suppose X fixes Y.

Then, if X is A-definable ($A \subset \bar{M}$), so is Y.

If in addition φ is an equation then there is a formula $\chi(x,y_0,\cdots,y_{n-1})$ such that $\psi(x;\vec{b}) \sim \exists y_0\cdots y_{n-1}\chi(x,y_0,\cdots,y_{n-1})\wedge\bigwedge_{i<n}\varphi(y_i;\vec{a})$.

Proof. Since X fixes Y, it is clear that if X is A-definable then Y is invariant over A; hence Y is A-definable.

Suppose in addition that φ is an equation. By I.9.d) there is a finite sequence $c_0,\cdots,c_{n-1}$ of elements in $\varphi(x;\vec{a})$ such that

$$\models \bigwedge_{i<n}\varphi(c_i;\vec{t}) \longrightarrow (\varphi(x;\vec{a}) \longrightarrow \varphi(x;\vec{t}));$$

in other words, $c_0,\cdots,c_{n-1}$ are such that $\varphi(x;\vec{a})$ is the smallest φ-definable set containing $c_0,\cdots,c_{n-1}$.

Claim. For any element b in Y, we have

$\{\theta(x,y_0,\cdots,y_{n-1}) \in tp(b\frown c_0\frown\cdots\frown c_{n-1})\}\wedge\bigwedge_{i<n}\varphi(y_i;\vec{a}) \vdash \psi(x;\vec{b})$.

For, suppose $b',d_0,\cdots,d_{n-1}$ are elements such that $tp(b\frown c_0\frown\cdots\frown c_{n-1}) = tp(b'\frown d_0\frown\cdots\frown d_{n-1})$ and $\models \bigwedge_{i<n}\varphi(d_i;\vec{a})$;

let σ be an automorphism such that $\sigma b = b'$ and $\sigma c_i = d_i$ $(i < n)$. Then, $d_0,\cdots,d_{n-1}$ are such that $\varphi(x;\sigma\vec{a})$ is the smallest φ-definable set containing $d_0, \cdots, d_{n-1}$. Since $\models \wedge_{i<n}\varphi(d_i;\vec{a})$, it follows $\varphi(x;\sigma\vec{a}) \vdash \varphi(x;\vec{a})$; by 3, we infer that $\varphi(x;\sigma\vec{a}) \sim \varphi(x;\vec{a})$ i.e. σ fixes $\varphi(x;\vec{a})$.

By assumption it follows σ fixes $Y = \psi(x;\vec{b})$. Since $\models \psi(b;\vec{b})$, we have $\models \psi(b';\sigma\vec{b})$. We conclude $\models \psi(\vec{b}';\vec{b})$, which proves the claim.

Therefore, for any element b in Y there is a formula $\chi_b(x\frown y_0\frown\cdots\frown y_{n-1})$ in $tp(b\frown c_1\frown\cdots\frown c_{n-1})$ such that

$$\chi_b(x,y_0,\cdots,y_{n-1}) \wedge\wedge_{i<n}\varphi(y_i;\vec{a}) \vdash \psi(x;\vec{b})$$

i.e.

$$\exists y_0\cdots y_{n-1}\chi_b(x,y_0\cdots,y_{n-1}) \wedge\wedge_{i<n}\varphi(y_i;\vec{a}) \vdash \psi(x;\vec{b});$$

and clearly

$$\models \exists y_0\cdots y_{n-1}\chi_b(b,y_0,\cdots,y_{n-1}) \wedge\wedge_{i<n}\varphi(y_i;\vec{a}).$$

By compactness, we deduce that for some $b_0,\cdots,b_{m-1}$

$$\psi(x;\vec{b}) \sim \vee_{i<m}\exists y_0\cdots y_{n-1}\chi_{b_i}(x,y_0,\cdots,y_{n-1}) \wedge\wedge_{i<n}\varphi(y_i;\vec{a}).$$

We conclude,

$$\psi(x,\vec{b}) \sim \exists y_0\cdots y_{n-1}\chi(x,y_0,\cdots,y_{n-1} \wedge\wedge\varphi(y_i;\vec{a}),$$

where $\chi = \vee_{i<m}\chi_{b_i}$. ■

Terminology. We say that a definable set X contains an infinite descending chain of S-definable sets if there is an infinite descending chain $(X_i)_{i<\omega}$ of S-definable sets such that $X \supset X_0$.

A descending chain $(X_i)_{i<\omega}$ of S-definable sets is said minimal if there is a sequence $(\varphi_i)_{i<\omega}$ of formulas in S such that X_i is Φ_i-definable, where $\Phi_i = cl^+(\varphi_i)$, and X_i is minimal (for inclusion) among Φ_i-definable sets which contain an infinite descending chain of S-definable sets.

5. **Lemma.** **If there are infinite descending chains of S-definable sets, there are minimal infinite descending chains of S-definable sets.**

Proof. Suppose there are infinite descending chains of S-definable sets. We construct by induction a descending chain $(X_i)_{i<\omega}$ of S-definable sets such that, for $i < \omega$, there is $\varphi_i \in S$, X_i is Φ_i-definable ($\Phi_i = cl^+(\varphi_i)$), X_i contains an infinite descending chain of S-definable sets and X_i is minimal (for inclusion) among Φ_i-definable sets which contain an infinite descending chain of S-definable sets:

take $\varphi_0 = (x = x)$ and $X_0 = (x = x)$ and suppose the construction done up to i.

Clearly there is $Y \subsetneq X_i$ such that Y contains an infinite descending chain of S-definable sets and Y is φ-definable, for some $\varphi \in S$.

Since $\Phi = cl^+(\varphi)$ is equational, it follows there is a

Φ-definable set $X \subsetneq X_i$, X contains an infinite descending chain of S-definable sets and X is minimal as such. Let $X_{i+1} = X$.

Obviously $(X_i)_{i<\omega}$ is minimal. ■

Note. If $(X_i)_{i<\omega}$ is a minimal descending chain of S-definable sets then, for $i < \omega$, X_{i+1} fixes X_i. For if σ is an automorphism which fixes setwise X_{i+1} then $\sigma X_i \cap X_i \supset X_{i+1}$ which implies that $\sigma X_i \cap X_i$ contains an infinite descending chain of S-definable sets. Now, if X_i is Φ_i-definable, $\Phi_i = cl^+(\varphi_i)$, then $\sigma X_i \cap X_i$ is also Φ_i-definable; it follows by the minimal choice of X_i that $X_i \cap \sigma X_i = X_i$. Hence $X_i \subset \sigma X_i$ and similarly $X_i \subset \sigma^{-1} X_i$. Thus $\sigma X_i = X_i$.

Theorem 6 and corollary 7 below have been proved jointly by the author and A. Pillay (c.f. [P.S]).

6. Theorem. Let T be totally transcendental and S-equational. Then, (T,S) has the d.c.c. iff there are no infinite invariant descending chains of S-definable sets in $\bar{M}$.

Proof. Suppose there are no invariant descending chains of S-definable sets and $(X_i)_{i<\omega}$ is a descending chain of S-

definable sets; we will show a contradiction.

By lemma 5 we can assume $(X_i)_{i<\omega}$ is minimal. By the note above, for any $i < j < \omega$, X_j fixes X_i; thus, if for $j,k > i$, X_i fixes X_j and X_k, then X_j fixes X_k and X_k fixes X_j.

It follows that, given $i < \omega$, X_i does not fix X_j for almost all j, $i < j < \omega$, (for if there is an infinite sequence $(j_n)_{n<\omega}$, $1 < j_n < j_{n+1} < \omega$, such that X_i fixes X_{j_n} for any $n < \omega$, then $(X_{j_n})_{n<\omega}$ is an invariant descending chain of S-definable sets ✗).

We can now easily find a subsequence $(X_{i_n})_{n<\omega}$ of $(X_i)_{i<\omega}$ such that for any $n < \omega$, X_{i_n} does not fix $X_{i_{n+1}}$.

Let $Y_n = X_{i_n}$, and let σ_n^1 be an automorphism which fixes Y_n but not Y_{n+1}, $(n < \omega)$; let $\sigma_n^0 = \text{id}$ (on $\bar{M}$). For $\eta \in {}^n2$, let $Y_\eta = \sigma_\eta Y_n$ where

$$\sigma_\eta = \sigma_0^{\eta(0)} \cdot \sigma_1^{\eta(1)} \cdot \dots \cdot \sigma_n^{\eta(n-1)}, \; \sigma_\emptyset = \text{id}.$$

Claim. For $\nu \in {}^\omega 2$, the set

$p_\nu = \{Y_{\nu\restriction n};\ n < \omega\} \cup \{\neg Y_\eta;\ \eta \in {}^{<\omega}2,\ \eta \text{ not an initial segment of } \nu\}$

is consistent.

Proof of the claim. Note first that, for $n < \omega$, $Y_{\nu\restriction n+1} \subsetneq Y_{\nu\restriction n}$. Suppose p_ν is inconsistent. Then, for some $n < \omega$ and $\eta_0, \dots, \eta_{\ell-1} \in {}^{<\omega}2$, η_k not initial segments of ν

$(k < \ell)$,

$$Y_{\nu \upharpoonright n} \subset \cup_{k<\ell} Y_{\eta_k}.$$

Clearly, $Y_{\nu \upharpoonright n}$ contains an infinite descending chain of S-definable sets; it follows easily that for some $k < \ell$ $Y_{\nu \upharpoonright n} \cap Y_{\eta_k}$ contains an infinite descending chain of S-definable sets.

Write $\eta = \eta_k$. Let m be the smallest possible, such that $\eta(m) \neq \nu(m)$; then $Y_{\nu \upharpoonright m+1} \cap Y_{\eta \upharpoonright m+1} \supset Y_{\nu \upharpoonright n} \cap Y_{\eta}$, whence $Y_{\nu \upharpoonright m+1} \cap Y_{\eta \upharpoonright m+1}$ contains an infinite descending chain of S-definable sets.

But $Y_{\nu \upharpoonright m+1}$ and $Y_{\eta \upharpoonright m+1}$ are distinct conjugates of Y_m; it follows that for some automorphism σ, $\sigma Y_m \neq Y_m$ and $\sigma Y_m \cap Y_m$ contains an infinite descending chain of S-definable sets.

By the minimal choice of Y_m it follows that $\sigma Y_m \cap Y_m = Y_m$ i.e. $Y_m \subset \sigma Y_m$; but of course σY_m satisfies the same minimal property as Y_m so that $\sigma Y_m \subset Y_m$. Thus $\sigma Y_m = Y_m$ ✗. p_ν is therefore consistent.

Now, clearly, for $\nu \neq \eta \in {}^{\omega}2$, p_ν and p_η are contradictory; so we have 2^ω-many types over a countable set (the set of parameters in the Y_η's), thus contradicting the fact that T is totally transcendental. That proves one direction of the theorem.

The converse is obvious. ■

Note. In fact, in the proof of theorem 6, we have shown the following; if T is as in theorem 6 and $(X_i)_{i<\omega}$ is a minimal descending chain of definable sets then there is a subsequence of $(X_i)_{i<\omega}$ which is an invariant descending chain.

7. **Corollary.** Assume in addition to the assumptions of theorem 6 that T is $\aleph_0$-categorical. Then (T,S) has the d.c.c.

Proof. if $(X_i)_{i<\omega}$ is an invariant descending chain of definable sets, we can assume all the sets X_i $(i < \omega)$ defined over the same finite set of parameters A; but then, by $\aleph_0$-categoricity, there are at most finitely many distinct sets definable over A so that the sets X_i $(i < \omega)$ are almost all equal ✗. Thus, there are no invariant descending chains of definable sets.

We conlude by theorem 6. ∎

From now on, we assume T is S-equational in a single variable x, and all types considered shall be types in x. We make such a convention just for notational simplicity; all the results below have analogues in case T is S-equational in $\vec{x}$ and the types considered are types in $\vec{x}$.

Consider the following height function on irreducible types (defined by induction on ordinals).

- $h(0) \geq 0$ for all irreducible types.
- $h(p) \geq \delta$, δ a limit, if $h(p) \geq \alpha$ for all $\alpha < \delta$.
- $h(p) \geq \alpha + 1$ if there is an irreducible type q such that $q^S \vdash p^S$, $p^S \nvdash q^S$ and $h(q) \geq \alpha$.

Write $h(p) = \alpha$ if $h(p) \geq \alpha$ and $h(p) \ngeq \alpha+1$; $h(p) = \infty$ if $h(p) \geq \alpha$ for all $\alpha \in$ On.

Clearly, $h(-)$ takes all its values in On iff (T,S) has the d.c.c. on irreducible types.

Naturally, we would like to compare the function h to the Morley rank or the U-rank. We are lead then to investigate in what measure is the increase of height of irreducible types related to non-minimal extensions of types.

8. **Definitions.**

(i) **Similarly to definition 2, given types p and q, or just single formulas, we say that p fixes q if whenever σ is an automorphism, with $\sigma p \sim p$, then $\sigma q \sim q$.**

(ii) **If p is an irreducible type we let**

$$\mathrm{Inv}(p) = p^S \cup \{\neg X;\ X \text{ is } S\text{-definable},\ p^S \nvdash X, \text{ and } p^S \text{ fixes } X\}.$$

Note. If p is a type over A, X a definable set and p fixes X, then X is A-definable, for obviously X is A-invariant.

One can also show, using compactness, that p fixes X iff there is a formula in p which fixes X.

In particular if p is an irreducible type which is definable over A then every set in Inv(p) is definable over A. It easily follows that if p is complete over A then $p \vdash$ Inv(p). (Note that p^S fixes Inv(p)).

Recall that a type p is irreducible iff there is a complete type p' over $\bar{M}$ such that $(p')^S \sim p^S$; moreover p' is the unique S-minimal extension of p to $\bar{M}$.

Notation. For p an irreducible type, we denote by $\mathbf{p}$ the S-minimal extension of p to $\bar{M}$; we have $\mathbf{p}^S \sim p^S$.

9. **Proposition. Let p and q be irreducible complete types over A, such that $q^S \vdash p^S$. Then, there is an automorphism σ such that $\sigma p^S \sim p^S$ and σq *extends* p iff $q^S \cup$ Inv(p) is consistent.**

Observe that $q^S \cup$ Inv(p) is consistent if there is no S-definable set X such that $p^S \nvdash X$, p^S fixes X and $q^S \vdash X$.

Proof. Let p' be an S-minimal extension of p to some model M (M containing the domain over which p is defined). Then $(p')^S \sim p^S$ and Inv(p') = Inv(p). Thus, without loss of generality, we can assume p is a complete type over a model

M.

Suppose $q^S \cup \mathrm{Inv}(p)$ is consistent.

Then there is an automorphism σ such that $\sigma p^S \sim p^S$ and $\sigma q^S \cup p$ is consistent. For suppose this is not the case; then, by compactness, there are S-definable subsets $X_0, \cdots, X_{n-1}$ of M such that $X_i \notin p$ $(i < \omega)$ and $\sigma q^S \vdash \bigcup_{i<n} X_i$ for any automorphism σ such that $\sigma p^S \sim p^S$.

Let $X = \bigcup_{i<n} X_i$ and

$$Y = \bigcap \{\sigma X,\ \sigma \text{ an automorphism which fixes } p^S\}.$$

Since X is S-definable, Y can be written as a finite intersection of conjugates of X, whence Y is S-definable. Clearly p^S fixes Y, moreover $p^S \nvdash Y$, for $Y \vdash \bigcup_{i<n} X_i$ and $p^S \nvdash \bigcup_{i<n} X_i$. Thus $\neg Y \in \mathrm{Inv}(p)$.

On the other hand, $q^S \vdash Y$, since $q^S \vdash \sigma X$ for any automorphism σ which fixes p^S.

Hence $q^S \cup \mathrm{Inv}(p)$ is inconsistent ✖.

So there is an automorphism σ which fixes p^S and such that $\sigma q^S \cup p$ is consistent.

It follows that σq extends p: for if not, there are S-definable sets $Y_0, \cdots, Y_{n-1}$, $Y_i \notin \sigma q$ i.e. $\sigma q^S \nvdash Y_i$ and $\sigma q^S \cup p \vdash \bigcup_{i<n} Y_i$. Hence, there are S-definable subsets $X_0, \cdots, X_{m-1}$ of M such that, $X_i \notin p$ i.e. $p^S \nvdash X_i$ $(i < m)$, and $\sigma q^S \cup p^S \vdash \bigcup_{i<n} Y_i \vee \bigcup_{i<m} X_i$.

But $\sigma q^S \vdash p^S$ (since $\sigma q^S \vdash \sigma p^S$ and $\sigma p^S \sim p^S$);

so $\sigma q^S \vdash \bigcup_{i<n} Y_i \vee \bigcup_{i<m} X_i$.

By irreducibility of σq^S (which follows from the irreducibility of q^S) we deduce that, either $\sigma q^S \vdash Y_i$ for some $i < n$ and that contradicts the choice of Y_i, or $\sigma q^S \vdash X_i$ for some $i < m$ and that contradicts the consistency of $\sigma q^S \cup p$. So σq does extend p

This proves one direction of the claim.

Conversely, suppose there is an automorphism σ which fixes p^S and such that σq extends p.

As we noted above, $p \vdash \text{Inv}(p)$.

Now, since σ fixes p^S and p^S fixes $\text{Inv}(p)$, σ fixes $\text{Inv}(p)$; hence σ^{-1} fixes $\text{Inv}(p)$.

Finally, since σq extends p, $\sigma q \cup \text{Inv}(p)$ is consistent; hence

$$\sigma^{-1}(\sigma q^S \cup \text{Inv}(p)) \sim q \cup \text{Inv}(p)$$

is consistent. ■

Note. With p and q as in proposition, one should compare the condition $q^S \cup \text{Inv}(p)$ being consistent, to the property of one group having infinite index in another group. Here, $q^S \cup \text{Inv}(p)$ consistent intuitively means q^S has infinite index in p^S.

In fact the following is true: given that $q^S \cup \mathrm{Inv}(p)$ is consistent, then for any S-definable set X, with $p^S \nvdash X$, there is an automorphism σ such that $\sigma p^S \sim p^S$ and $\sigma q^S \cup \{\neg X\}$ is consistent. (For if $\sigma q^S \vdash X$ for all σ as above, then $q^S \vdash \bigcap \sigma X$. $\bigcap \sigma X$ is S-definable and $p^S \nvdash \bigcap \sigma X$; thus $\neg \bigcap \sigma X \in \mathrm{Inv}(p)$, contradicting the fact that $q^S \cup \mathrm{Inv}(p)$ is consistent).

To give an even better approximation of the notion of "infinite index" one needs to speak of an arbitrary S-definable set (not necessarily irreducible) having "infinite index" in another S-definable set; this is possible but requires defining $\mathrm{Inv}(p)$ for an arbitrary type p (c.f. [S.2]). We shall not deal here with such notions.

10. **Theorem.** **Let p and q be irreducible types over A such that $q^S \vdash p^S$ and $p^S \nvdash q^S$. Then, there is an automorphism σ which fixes p^S and such that σq is a forking extension of $p{\restriction}A$ iff $q^S \cup \mathrm{Inv}(p)$ is consistent.**

Proof. We can assume without loss of generality that p is a complete type over a model M (see beginning of the proof of proposition 9).

Suppose $q^S \cup \mathrm{Inv}(p)$ is consistent. By proposition 9 there is an automorphism σ which fixes p^S and such that σq extends p. Clearly then $(\sigma q)^S \vdash p^S$ and, since $p^S \nvdash q^S$, $p^S \nvdash \sigma q^S$. We conclude that σq is a non-minimal extension whence, by proposition 8.3, a forking extension, of $p \upharpoonright A$.

The converse follows immediately from proposition 9.

11. **Theorem.** Let T be superstable. Then, (T,S) has the d.c.c. on irreducible types iff there is no infinite descending chain $(p_i)_{i<\omega}$ of irreducible positive types such that $p_{i+1} \cup \mathrm{Inv}(p_i)$ is inconsistent for any $i < \omega$, i.e. iff there is no infinite sequence $(p_i)_{i<\omega}$ of irreducible positive types such that

$$p_{i+1} \vdash p_i,\ p_i \nvdash p_{i+1} \text{ and,}$$

for $i < \omega$, there is X_i, $p_i \nvdash X_i$, p_i fixes X_i and $p_{i+1} \vdash X_i$.

Proof. Suppose there is no infinite descending chain $(p_i)_{i<\omega}$ such as above and suppose $(q_i)_{i<\omega}$ is an infinite descending chain of irreducible positive types; we will show a contradiction.

Note first that for any $i < j < k < \omega$, if $q_j \cup \mathrm{Inv}(q_i)$ is inconsistent then $q_k \cup \mathrm{Inv}(q_i)$ is inconsistent, (since $q_k \cup \mathrm{Inv}(q_i) \vdash q_j \cup \mathrm{Inv}(q_i)$).

It follows that there is $i < \omega$ such that for any j, i

$< j < \omega$, $q_{j+1} \cup \mathrm{Inv}(q_j)$ is consistent: for if not we easily construct an infinite (increasing) sequence $(i_j)_{j<\omega}$ such that $q_{i_j+1} \cup \mathrm{Inv}(q_{i_j})$ is inconsistent for any $j < \omega$; whence, from the note above, $q_{i_{j+1}} \cup \mathrm{Inv}(q_{i_j})$ is inconsistent for any $j < \omega$. But that means the sequence $(p_j)_{j<\omega} = (q_{i_j})_{j<\omega}$ is an infinite descending chain of irreducible positive types with $p_{j+1} \cup \mathrm{Inv}(p_j)$ inconsistent for all $j < \omega$, thus contradicting our assumption.

So we might as well assume $(q_i)_{i<\omega}$ is such that $q_{i+1} \cup \mathrm{Inv}(q_i)$ is consistent for all $i < \omega$.

We construct now by induction a sequence of types $(p_i)_{i<\omega}$ such that p_i is a complete type over a model M_i; $p_i = \sigma_i q_i \restriction M_i$ for some automorphism σ_i, $(p_i)^S \sim \sigma_i q_i$, and p_{i+1} is a forking extension of p_i.

For $i = 0$ take M_0 an arbitrary model and $p_0 = q_0 \restriction M_0$; suppose the construction of (p_i) done up to i.

Note that, since $p_i^S \sim \sigma_i q_i$, $\mathrm{Inv}(p_i) \sim \mathrm{Inv}(\sigma_i q_i)$; so $q_{i+1} \cup \mathrm{Inv}(\sigma_i^{-1} p_i)$ is consistent. By theorem 10, there is an automorphism τ such that τq_{i+1} is a forking extension of $\sigma_i^{-1} p_i$. Thus $\sigma_i \tau q_{i+1}$ is a forking extension of p_i.

Let $\sigma_{i+1} = \sigma_i \tau$; choose $M_{i+1} \supset M_i$ such that $(\sigma_{i+1} q_{i+1} \restriction M_{i+1})^S \sim \sigma_{i+1} q_{i+1}^S$ and $\sigma_{i+1} q_{i+1} \restriction M_{i+1}$ is a forking extension of p_i. Let $p_{i+1} = \sigma_{i+1} q_{i+1} \restriction M_{i+1}$; p_{i+1} thus chosen satisfies the required properties, and so ends the inductive step of the construction.

But, the sequence $(p_i)_{i<\omega}$ constructed above, contradicts superstability (see Fact 4).

This shows one direction of the claim. The converse is obvious. ■

12. **Theorem. Let p, q be complete types over the models M and N respectively. Then, $p \geq q$ ($\geq$, the fundamental order, see preliminaries) iff there is an automorphism σ such that $\sigma q^S \vdash p^S$ and $\sigma q^S \cup \mathrm{Inv}(p)$ is consistent.**

Proof. Suppose $p \geq q$. Then $p \geq \mathbf{q}$, and therefore (see Fact 6) there is an automorphism σ such that $p \subset \sigma\mathbf{q}$. It is clear then that $\sigma\mathbf{q}^S \vdash p^S$ and $\sigma\mathbf{q}^S \cup \mathrm{Inv}(p)$ is consistent (for $p \vdash \mathrm{Inv}(p)$).

Since $q^S \sim \mathbf{q}^S$, it follows that $\sigma q^S \vdash p^S$ and $\sigma q^S \cup \mathrm{Inv}(p)$ is consistent.

Conversely, suppose $\sigma q^S \vdash p^S$ and $\sigma q^S \cup \mathrm{Inv}(p)$ is consistent for some automorphism σ. Then, by proposition 9 applied to σq, there is an automorphism τ such that $\tau\sigma\mathbf{q}$ extends p.

It follows that $p \geq \tau\sigma\mathbf{q}$, and since $\tau\sigma\mathbf{q} \geq \mathbf{q} \geq q$ (for $\mathbf{q}$ is a non-forking extension of q, see Fact 5), we conclude that $p \geq q$. ■

13. **Proposition.** Let (T,S) have the d.c.c. The following assertions are equivalent

1. For any irreducible positive type p, $MR(p) = h(p)$.
2. For any irreducible type p, $MR(p) = MR(p^S)$ and $Md(p) = Md(p^S) = 1$.
3. For any type p, $MR(p) = MR(p^S)$ and $Md(p) = Md(p^S)$.
4. For any irreducible positive types p and q, if $q \vdash p$ and $p \nvdash q$ then $MR(q) < MR(p)$.

Proof. Note first that for any irreducible type $MR(p) \leq h(p)$ (for if $MR(p) \geq \alpha + 1$, then p^S, which is equivalent to a single formula, contains infinitely many distinct complete types over $\bar{M}$ of Morley rank greater or equal to α. Hence, there is necessarily an irreducible type q such that $q^S \vdash p^S$ and $p^S \nvdash q^S$ i.e. $h(q) < h(p)$, and $MR(q) \geq \alpha$. The claim now follows immediately by induction on $MR(p)$).

$1 \longleftrightarrow 4$: immediate

$4 \longrightarrow 2$: Suppose 4. holds; let p be an irreducible type. Clearly $MR(p) \leq MR(p^S)$.

On the other hand, if $\varphi \in p$ then $p^S \wedge \varphi$ can be written as a disjunction of formulas of the form $\psi = p^S \wedge \bigwedge_{i<n} \neg\varphi_i$ where φ_i is S-definable, $p^S \nvdash \varphi_i$ and $\varphi_i \vdash p^S$ $(i < n)$.

Now, φ_i can be written as a finite disjunction of irreducible positive types (c.f. A.5); say $\varphi_i = \bigvee_{j<m_i} q^i_j$. Thus $q^i_j \vdash \varphi_i \vdash p^S$ and $p^S \nvdash q^i_j$; hence, by 4., $MR(q^i_j) < MR(p^S)$ for any $i < n$ and $j < m_i$; hence $MR(\varphi_i) < MR(p^S)$ for any $i < n$;

hence $MR(\psi) = MR(p^S \wedge \bigwedge_{i<n} \neg\varphi_i) = MR(p^S)$. Thus $MR(p^S \wedge \varphi) = MR(p^S)$ for any φ in p. We conclude $MR(p) = MR(p^S)$.

Finally, it is clear that $Md(p) \leq Md(p^S)$. Suppose $Md(p^S) > 1$; then there is an irreducible type q such that $q^S \vdash p^S$, $p^S \nvdash q^S$ and $MR(q) = MR(p^S)$. From what preceded $MR(q^S) = MR(q) = MR(p^S)$, which contradicts 4.

Hence $Md(p) = Md(p^S) = 1$.

2 $\longrightarrow$ 4. Suppose 2 holds; let p and q be irreducible positive types such that $q \vdash p$ and $p \nvdash q$.

Clearly $MR(q) \leq MR(p)$.

If $MR(q) = MR(p)$ then (considering p and q as single formulas) $MR(p \wedge \neg q) < MR(p)$, for $Md(p) = 1$. But $p \vdash p \wedge \neg q$, since $p^S \sim p$; thus $MR(p) < MR(p)$ i.e. $MR(p) < MR(p^S)$

contradicting 2.

We conclude that $MR(q) < MR(p)$.

$2 \wedge 4 \longrightarrow 3$: Suppose 2 and 4 hold ; let p be an arbitrary type and let $p_0, \cdots, p_{n-1}$ be its minimal extensions to $\bar{M}$. By A.5, $p^S \sim \vee_{i<n} (p_i)^S$.

So there is $i < n$ such that $MR(p^S) = MR(p_i^S)$. By 2 $MR(p_i^S) = MR(p_i)$; since $p_i \supset p$, $MR(p_i) \leq MR(p)$. So $MR(p^S) \leq MR(p)$. Since $p \vdash p^S$ we conclude $MR(p) = MR(p^S)$.

Clearly $Md(p) \leq Md(p^S)$.

Let q be a complete type over $\bar{M}$ extending p^S and such that $MR(q) = MR(p^S)$. Since $q \supset p^S$ there is an S-minimal extension p_i of p to $\bar{M}$ such that $q^S \supset p_i^S$ (c.f. II.A.3).

If $p_i^S \not\vdash q^S$, then, by 4, $MR(q^S) < MR(p_i^S) \leq MR(p^S)$; by 2, $MR(q) = MR(q^S)$, it follows $MR(q) < MR(p^S)$, contradicting the choice of q. hence $q^S \sim p_i^S$. In particular q extends p; moreover,

$$MR(q) = MR(p^S) = MR(p).$$

We showed that if q is a complete extension of p^S to $\bar{M}$ of Morley rank equal the Morley rank of p^S then q extends p and $MR(q) = MR(p)$.

It follows that $Md(p^S) \leq Md(p)$, whence $Md(p^S) = Md(p)$.

$3 \longrightarrow 2$: Suppose 3 holds; let p be an irreducible type. We have immediately that $MR(p) = MR(p^S)$. Since $p^S \sim p^S$, we also have $Md(p) = Md(p^S)$; since $Md(p) = 1$ we conclude that $Md(p^S) = 1$ and $Md(p) = 1$ (for $Md(p) \leq Md(p^S)$). ■

Application. The simplest kind of equations are the equations which have height at most 1 (c.f. I.0, for definition of height) i.e. those formulas $\varphi(\vec{x};\vec{t})$ such that any two instances of φ are either equivalent or contradictory.

Such formulas have been called normal by A. Pillay, and consequently a theory which is S-equational in $\vec{x}$ with S a set of normal formulas is called S-normal in $\vec{x}$ (c.f. [P.2]).

For example, any complete theory of modules is S-normal with S the set of p.p.f.

A normal set is a set which is definable by a normal formula. By compactness, one shows that a definable set X is normal iff any conjugate of X either equals X or is disjoint from X.

Lemma. **Let $\varphi(x;\vec{a})$ and $\psi(x;\vec{b})$ define normal sets and $\varphi(x;\vec{a}) \vdash \psi(x;\vec{b})$. Then $\psi(x;\vec{b})$ fixes $\varphi(x;\vec{a})$ iff there is a formula $\chi(x)$, without parameters, such that**

$\varphi(x;\vec{a}) \sim \psi(x;\vec{b}) \wedge \chi(x).$

Proof. Suppose $\psi(x;\vec{b})$ fixes $\varphi(x;\vec{a})$. Let c be an element satisfying $\varphi(x;\vec{a})$. Then,

$$\{\Theta(x) \in tp(c;\emptyset)\} \cup \{\psi(x;\vec{b})\} \vdash \varphi(x;\vec{a}):$$

for if d is an element such that $tp(d;\emptyset) = tp(c;\emptyset)$ and $\models \psi(d;\vec{b})$, then there is an automorphism σ with $\sigma c = d$. From $\models \psi(\vec{c};\vec{b})$, we get $\models \psi(\vec{d};\sigma\vec{b})$; also by assumption we have $\models \psi(\vec{d};\vec{b})$.

$\psi(x;\vec{b})$ and $\psi(x;\sigma\vec{b})$ being non-contradictory, we deduce (by normality) that $\psi(x;\vec{b})$ and $\psi(x;\sigma\vec{b})$ are equivalent; hence σ fixes $\psi(x;\vec{b})$, and therefore σ fixes $\varphi(x;\vec{a})$, i.e. $\varphi(x;\vec{a}) \sim \varphi(x;\sigma\vec{a})$. Now $\models \varphi(\vec{c};\vec{a})$; hence $\models \varphi(\vec{d};\sigma\vec{a})$ and finally $\models \varphi(\vec{d};\vec{a})$, which is what we wanted.

So there is a formula $\Theta_c(x)$ in $tp(c;\emptyset)$ such that

$$\Theta_c(x) \wedge \psi(x;\vec{b}) \vdash \varphi(x;\vec{a})$$

and of course we have $\models \Theta_c(c) \wedge \psi(c;\vec{b})$.

By compactness we deduce that

$$\varphi(x;\vec{a}) \sim \vee_{c \in A} \Theta_c(x) \wedge \psi(x;\vec{b})$$

where A is a finite set of elements realizing $\varphi(x;\vec{a})$.

We conclude that $\varphi(x;\vec{a}) \sim \chi(x) \wedge \psi(x;\vec{b})$, where

$$\chi(x) = \vee_{c \in A} \Theta_c(x).$$

The converse is immediate. ■

Corollary. Let T be S-normal in x.

(i) Let p and q be irreducible types with $q^S \vdash p^S$, and $p\restriction\emptyset$ complete over $\emptyset$. Then $q^S \cup \mathrm{Inv}(p)$ is consistent iff $q^S \cup p\restriction\emptyset$ is consistent.

(ii) Let p be a complete type over $\emptyset$; consider the following height function h_p on irreducible types *extending* p:

$h_p(q) \geq 0$ for all irreducible types $q \supset p$.

$h_p(q) \geq \lambda$, λ limit, if $h_p(q) \geq \alpha$ for all $\alpha < \lambda$.

$h_p(q) \geq \alpha + 1$ if there is an irreducible type r extending p such that $r^S \vdash q^S$ and $q^S \nvdash r^S$.

Then, for any complete type q over some model, extending p, $U(q) = h_p(q)$ (U the Lascar-rank).

(iii) Let p and q be complete types over the models M and N respectively. Then $p \geq q$ iff there is an automorphism σ such that $\sigma q^S \vdash p^S$ and $p\restriction\emptyset = q\restriction\emptyset$.

Proof.

(i) Clearly $\mathrm{Inv}(p) \vdash p\restriction\emptyset$; thus if $q^S \cup \mathrm{Inv}(p)$ is consistent then so is $q^S \cup p\restriction\emptyset$.

Now if $q^S \cup \mathrm{Inv}(p)$ is inconsistent then there is an S-definable set X such that $p^S \nvdash X$, p^S fixes X and $q^S \vdash X$. It follows (by compactness) that a certain S-

definable set Y in p^S fixes X; hence Y fixes $X \cap Y$. By the lemma above we deduce that $X \cap Y = Y \cap Z$ for some set Z definable over $\emptyset$. We have that $q^S \vdash X \cap Y$ and $p^S \nvdash X \cap Y$; hence $q^S \vdash Z$ and $p \nvdash Z$ (for $p^S \nvdash Y \cap Z = X \cap Y$ and $p^S \vdash Y$) which implies $\neg Z \in p \restriction \emptyset$. We conclude that $q^S \cup p \restriction \emptyset$ is inconsistent.

(ii) Clearly $h_p(q) \geq U(q)$; we show by induction on the ordinal α that $h_p(q) \geq \alpha$ implies $U(q) \geq \alpha$. It will follow $U(q) \geq h_p(q)$ and therefore $U(q) = h_p(q)$.

Suppose the assertion true for all $\beta < \alpha$.

For $\alpha = 0$ or α limit the assertion is obvious; suppose $h_p(q) \geq \alpha + 1$. Then there is by definition of h_p an irreducible type q' extending p such that $(q')^S \vdash q^S$, $q^S \nvdash (q')^S$ and $h_p(q') \geq \alpha$. By taking a minimal extension to a model if necessary we can assume q' is a complete type over a model. By the induction hypothesis, $U(q') \geq \alpha$.

By (i). $(q')^S \cup \mathrm{Inv}(q)$ is consistent; hence, by theorem 10, there is an automorphism σ such that $\sigma q'$ is a forking extension of q, and since $U(\sigma q') = U(q') = U(q') \geq \alpha$ we conclude that $U(q) \geq \alpha + 1$. ¶

(iii) If σ is an automorphism such that $\sigma q^S \vdash p^S$, and $p \restriction \emptyset = q \restriction \emptyset$ then $\sigma q^S \cup p \restriction \emptyset$ is consistent; hence, by (i),

$\sigma q^S \cup \mathrm{Inv}(p)$ is consistent. By theorem 12 it follows that $p \geq q$.

If $p \geq q$ then clearly $p{\restriction}\emptyset = q{\restriction}\emptyset$ and by theorem 12 there is an automorphism σ such that $\sigma q^S \vdash p^S$. ■

Remark. For modules, the translations of formulas and types play a similar role to conjugates of formulas and types.

More explicitly, if T is a complete theory of modules, then, given a definable set X and an element a we let $aX = \{ab;\ b \in X\}$. Note that if X is definable by $\varphi(x;\vec{a})$ then aX is definable by $\varphi(a^{-1}x;\vec{a})$.

If p is a type we let $ap = \{aX;\ X \in p\}$.

Now, similarly to definition 8, one says that p fixes X by translations if whenever a is an element such that $ap \sim p$ then $aX = X$. Also, for S the set of positive primitive formulas, and for p an (S-)irreducible type, one defines

$$\mathrm{Invtr}(p) = p^S \cup \{\neg X;\ X\ S\text{-definable},\ p^S \nvdash X \text{ and } p^S \text{ fixes } X \text{ by translations}\}.$$

And similarly to proposition 9 one shows that, for p and q irreducible types with $q^S \vdash p^S$, $q^S \cup \mathrm{Invtr}(p)$ is consistent iff there is an element a such that aq extends p; we get also analogues of theorems 10, 11 and 12.

But, if X and Y are S-definable, i.e. cosets of groups, and $Y \subset X$ then X fixes Y by translations iff $X = Y$.

It follows that for p an irreducible type Invtr(p) $= p^S$, and therefore, if $q^S \vdash p^S$, then there is always an element a such that aq extends p. We deduce for instance (from the analogue of theorem 10) that if $q^S \vdash p^S$ and $p^S \nvdash q^S$ then there is an element a such that aq is a forking extension of p; it follows easily that $U(p) = h(p)$ for any complete type over a model M.

The important point to underline is that, in general algebraic equational theories one should investigate algebraic transformations (e.g. the translations for modules) which could play a similar role to automorphisms (c.f. [S.2]).

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