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Capacity and Information Rates for Multiple Antenna Wireless Systems with Multi-dimensional Modulation

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of the degree of Doctor of Philosophy.

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Abstract

Multiple-Input Multiple-Output (MIMO) systems have shown a great potential for increasing the data transmission rates in wireless communications. These systems are characterized by a high design flexibility which allows a wide variety of efficient signaling techniques and resource allocation methods. In this thesis, we consider the capacity and transmission rates for MIMO systems using various multi-dimensional space-time modulation formats. The latter refer to various allocations of orthogonal signal dimensions (or modulating waveforms) to transmit antennas. For a single-user system with a fixed transmit power, we investigate the most efficient allocations of power and signal dimensions to transmit antennas so that transmission rates can be maximized under different cases of channel state information at the transmitter (CSIT). In many situations, we show that using all the signal dimensions on all the transmit antennas yields the largest rates. An analysis of the capacity and transmission rates is also provided at low and high SNR, where the effect of the spatial channel parameters is pointed out. For a broadcast system, we consider the power and signal dimensions allocations not only for transmit antennas, but also among users. As such, the signal dimensions are divided into orthogonal subsets where one of these subsets is shared between all users, and the other subsets are assigned each to a different user. We consider the rate region obtained using dirty paper coding (DPC) under a total transmit power constraint. We derive inner and outer bounds for the DPC rate region. By analyzing the impact of space-time modulation formats on these bounds, we show that higher transmission rates are obtained when the signal dimensions assigned to each user are used on all the transmit antennas. By considering the asymptotical behavior of these bounds at low and high SNR, important insights are obtained as for the optimal power allocation that maximizes the sum rate for each of these bounds in both SNR regimes. We also investigate the spatial multiplexing gains associated with these bounds, and we analyze the impact of increasing the number of shared signal dimensions and the number of time resolvable clusters.

Sommaire

Les systèmes à entrée multiple et sortie multiple permettent d'augmenter le débit de transmission de données dans les réseaux de communication sans fil. Ces systèmes sont caractérisés par une grande flexibilité de conception, ce qui leur permet d'accommoder une grande variété de techniques de signalisation et d'attribution de ressources. Dans cette thèse, nous considérons la capacité et les débits de transmission possibles pour les systèmes à entrée multiple et sortie multiple utilisant une variété de formats de modulation espace-temps multidimensionnelle. Ces formats correspondent à différentes manières d'attribuer les dimensions orthogonales du signal aux antennes de transmission. Pour un système ayant un seul usager et moyennant une puissance de transmission fixe, nous examinons comment attribuer la puissance et les dimensions du signal aux antennes de transmission, de manière à maximiser les débits de transmission en tenant compte de l'information sur l'état du canal dont dispose le transmetteur. Dans plusieurs situations, nous démontrons qu'utiliser toutes les dimensions du signal sur chaque antenne de transmission permet d'atteindre de plus hauts débits de transmission. Nous présentons aussi une analyse de l'effet des paramètres du canal de propagation sur la capacité et les débits de transmission à faible et haut rapport de signal sur bruit. Pour un système de diffusion avec multiples usagers, nous considérons l'attribution de la puissance et des dimensions du signal non seulement aux antennes de transmission mais aussi aux usagers. Les dimensions du signal sont donc divisées en sous-ensembles orthogonaux, où l'un de ces sous-ensembles est partagé entre tous les usagers, et les autres sous-ensembles sont attribués à des usagers distincts. Nous considérons la région de débits obtenue par la technique *Dirty Paper Coding (DPC)* sous une contrainte de puissance de transmission totale. Nous examinons des bornes inférieures et extérieures sur la région de débits obtenue par le DPC. En analysant l'effet des différents formats de modulation espace-temps sur ces bornes, nous démontrons que lorsque toutes les dimensions du signal attribuées à chaque usager sont utilisées sur chaque antenne de transmission, de plus grands débits de transmission peuvent être atteints. En considérant le comportement asymptotique de ces bornes à faible et haut rapport de signal sur bruit, nous pouvons cal-

culer l'attribution de puissance optimale permettant de maximiser le débit cumulatif relatif à chacune de ces bornes dans chaque cas. Nous étudions aussi les gains de multiplexage spatial associés avec ces bornes, et nous analysons l'effet d'augmenter le nombre de dimensions du signal qui sont partagées et le nombre de groupes de diffuseurs qui sont résolus dans le temps.

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List of Abbreviations

3GPP	Third Generation Partnership Project	9
AOA	Angle of Arrival	12
AOD	Angle of Departure	12
ATA	Aggregate Transmit Antenna	17
BC	Broadcast Channel	5
CDMA	Code Division Multiple Access	18
CH	Convex Hull	101
CP	Cyclic Prefix	21
CSI	Channel State Information	2
CSIR	Channel State Information at the Receiver	2
CSIT	Channel State Information at the Transmitter	2
DCT	Discrete Cosine Transform	45
DFT	Discrete Fourier Transform	45
DPC	Dirty Paper Coding	5
DSM	Degraded Same Marginal	7
FT	Fourier Transform	21
i.i.d.	independent and identically distributed	14
IFT	Inverse Fourier Transform	21
ISI	Inter-Symbol Interference	19

KKT	Karush-Kuhn-Tucker	56
KLT	Karhunen-Loeve Transform	45
LOS	Line of Sight	12
MAC	Multiple Access Channel	5
MIMO	Multiple-Input Multiple-Output	1
OFDM	Orthogonal Frequency Division Multiplexing	7
OTA	Orthogonal Transmit Antenna	17
PA	Power Allocation	76
PDF	Probability Density Function	55
POTA	Partially Orthogonal Transmit Antenna	17
RHS	Right Hand Side	193
SNR	Signal-to-Noise Ratio	2
TDMA	Time Division Multiple Access	6
TOG	Transmit Orthogonal Group	17

List of Symbols

$(\cdot)^{(k)}$	A symbol with the superscript (k) refers to user k in the broadcast system. Unless otherwise specified, its description is the same as that used in the single-user system	26
$^\circ$	Angle in degrees	13
$\approx$	Approximation	39
$(\cdot)^*$	Conjugation of a complex number or matrix	16
$\cdot$	Matrix product	15
$(\cdot)^\dagger$	Hermitian conjugation of a matrix or a vector	15
$\doteq$	Exponential equality	53
$\otimes$	Kronecker matrix product	17
$\mathbf{0}_n$	The $n \times 1$ zero column vector	29
$\mathbf{0}_{n \times n}$	The $n \times n$ zero matrix	53
$\mathbf{A} \succeq \mathbf{0}_{n \times n}$	means $\mathbf{A}$ is an $n \times n$ Hermitian positive semidefinite matrix	53
$\mathbf{A} \succeq \mathbf{B}$	means $\mathbf{A} - \mathbf{B} \succeq \mathbf{0}_{n \times n}$	135
$ \mathcal{A} $	Cardinality of a set $\mathcal{A}$	135
$[\mathbf{A}]_{*,q}$	The q^{th} column of a matrix $\mathbf{A}$	59
$[\mathbf{A}]_{q,*}$	The q^{th} row of a matrix $\mathbf{A}$	24
$[\mathbf{A}]_{l,i}$	The (l, i) component of a matrix $\mathbf{A}$	15
$\ \mathbf{a}\ $	The 2-norm of a vector $\mathbf{a}$	15
$\{a\}^+$	$= \max(0, a)$	36
$\Pr$	Probability operator	52
$(\cdot)^T$	Transpose of a matrix or a vector	17
Tr	Trace operator	34

$h(\mathbf{a})$	Differential entropy of a random variable (or vector) $\mathbf{a}$	104
$h(\mathbf{a} \mathbf{b})$	Conditional differential entropy of a random variable (or vector) $\mathbf{a}$ given the random variable (or vector) $\mathbf{b}$	104
$\mathbf{A}$	The $\tilde{D} \times N_T D$ space-time modulation matrix, $\mathbf{A} = \text{diag}_{i=1}^{N_T}[\mathbf{A}_i]$	20
A	$= \sum_{i=1}^{N_G} \sum_{l=1}^{r_i} \log(\lambda_{i;l})$	41
$\mathbf{A}_i$	The $D_G \times D$ space-time modulation matrix mapping $\mathbf{x}_i$ onto $\mathbf{y}_i$, $i = 1, \dots, N_T$	17
A_m	The maximum attenuation for the antenna gain pattern at the base station	14
A_O	$= \sum_{i=1}^{N_G} \sum_{l=1}^{r_{O;i}} \log(\lambda_{O;i;l})$	71
A_{RX}	Mean AOA for cluster 3	78
A_{TX}	Mean AOD for cluster 3	78
$\mathcal{A}_G$	The set of TOG pairs $(a, \tilde{a})$, $a \in \mathcal{A}_{G;c}^{(2)}$ and $\tilde{a} \in \mathcal{A}_{G;c}^{(1)}$, that share signal dimensions	134
$\tilde{\mathcal{A}}_G^{(1)}$	The subset of TOG pairs $(a, \tilde{a}) \in \mathcal{A}_G$ such that $\tilde{a} \in \tilde{\mathcal{A}}_{G;c}^{(1)}$ for spread spectrum like MIMO BCs, or $\tilde{a} \in \tilde{\mathcal{A}}_{O;G;c}^{(1)}$ for MIMO-OFDM BCs	135
$\tilde{\mathcal{A}}_G^{(2)}$	The subset of TOG pairs $(a, \tilde{a}) \in \mathcal{A}_G$ such that $a \in \tilde{\mathcal{A}}_{G;c}^{(2)}$ for spread spectrum like MIMO BCs, or $a \in \tilde{\mathcal{A}}_{O;G;c}^{(2)}$ for MIMO-OFDM BCs	135
$\mathcal{A}_{G;c}^{(k)}$	The subset of indexes of user k 's TOGs that employ shared signal dimensions	130
$\tilde{\mathcal{A}}_{G;c}^{(k)}$	The subset of indexes $j \in \mathcal{A}_{G;c}^{(k)}$ for which $\text{rank}(\mathbf{N}_j^{(p,k)}) < L_t^{(p)} N_R^{(p)}$ for $(p, k) = (2, 1)$ and $(p, k) = (1, 2)$	132
$\tilde{\mathcal{A}}_{O;G;c}^{(k)}$	The subset of indexes $j \in \mathcal{A}_{G;c}^{(k)}$ for which $\min(N_R^{(k)}, n_j^{(k)}) < N_R^{(p)}$ for $(p, k) = (2, 1)$ and $(p, k) = (1, 2)$	167
$\mathcal{A}_x$	The set of matrices Σ with $\text{Tr}(\Sigma) \leq P_T$	190
$\mathcal{A}_y$	The set of matrices Σ_y with $\text{Tr}(\Sigma_y) \leq P_T$	190
$\mathcal{A}_\eta$	The set of TOG power allocation vectors η	58

$\mathcal{A}_{\Sigma}$	The set of input covariance matrices $(\Sigma^{(1)}, \Sigma^{(2)})$ satisfying $Tr(\Sigma^{(1)}) + Tr(\Sigma^{(2)}) \leq P_T$ in the two-user broadcast system	101
$\mathcal{A}_{\varrho}$	$= \{\mathbf{u} \in \mathbb{R}^{m_T} : u_{i;1} > u_{i;2} > \dots > u_{i;m_i} > 0 \forall i, \sum_{i=1}^{N_G} \sum_{l=1}^{m_i} u_{i;l} < W\}$	55
$\mathbf{a}_{k;s}$	The $N_R \times 1$ receive space signature vector of path s in cluster k	15
$\alpha_{k;s}$	Parameter defining the receive space signature vector, $\alpha_{k;s} = 2\pi \frac{d_R}{\lambda} \sin(\theta_{R;k;s})$	15
α_p	$= P^{(1)} / P_T$	102
$\alpha_p^{(k)}$	The fraction of transmit power allocated to user k in the broadcast system	102
$\alpha_{p;s}^{(k)}$	The fraction of power allocated to user k (in the broadcast system) that is associated with the maximum sum rate for $\mathcal{R}_{out}$	106
$\tilde{\alpha}_{p;s}^{(k)}$	The fraction of power allocated to user k (in the broadcast system) that is associated with the maximum sum rate for $\mathcal{R}_{in}$	125
$\alpha_{O;p;s}^{(k)}$	The fraction of power allocated to user k (in the MIMO-OFDM broadcast system) that is associated with the maximum sum rate for $\mathcal{R}_{O;out}$	151
$\tilde{\alpha}_{O;p;s}^{(k)}$	The fraction of power allocated to user k (in the MIMO-OFDM broadcast system) that is associated with the maximum sum rate for $\mathcal{R}_{O;in}$	162
B	$= N_G \sum_{i=1}^{N_G} \sum_{l=1}^{r_{s;i}} \frac{\eta_i \xi_{i;l}}{r_{s;i}}$	58
B_A	Value of $B_{\max}$ using ATA	60
B_O	Value of $B_{\max}$ using OTA	60
B_P	Value of $B_{\max}$ using POTA	60
$\tilde{B}$	$= N_G \sum_{i=1}^{N_G} \sum_{l=1}^{r_{s;i}} \frac{\eta_i \xi_{O;i;l}}{r_{s;i}} D_G$	73
$B_{\max}$	$= N_G \max\{\ [\mathbf{C}_i]_{*,\max}\ ^2\}_{i=1}^{N_G}$ (maximum value of B)	60
$\tilde{B}_{\max}$	$= N_G \check{f}_{\max}$ (maximum value of $\tilde{B}$)	74
B_f	$= N_G \sum_{i=1}^{N_G} \frac{Tr(\mathbf{C}_i^H \mathbf{C}_i)}{n_i} \eta_i$ ($\equiv B$ when $r_{s;i} = n_i$)	61

$B_{f;A}$	Value of $B_{f;\max}$ using ATA	61
$B_{f;O}$	Value of $B_{f;\max}$ using OTA	61
$B_{f;P}$	Value of $B_{f;\max}$ using POTA	61
$B_{f;\max}$	$= N_G \max \left\{ \frac{\text{Tr}(\mathbf{C}_i^\dagger \cdot \mathbf{C}_i)}{n_i} \right\}_{i=1}^{N_G}$	61
$\mathbf{B}^{(k,p)}$	The $D^{(k)} \times D^{(p)}$ matrix indicating the shared dimensions between users k and p in the broadcast system	29
$\mathbf{B}_{O;i}$	$= \sum_{q=1}^{D_G} \check{\mathbf{F}}_{i;q} \otimes (\mathbf{e}_{q;D_G} \cdot \mathbf{e}_{q;D_G}^T)$	69
$\tilde{\mathbf{B}}_{O;i}^{(k)}$	$= \text{diag}[\mathbf{I}_{r_{O,i}^{(k)}}, \mathbf{0}_{(n_i^{(k)} D_G^{(k)} - r_{O,i}^{(k)}) \times (n_i^{(k)} D_G^{(k)} - r_{O,i}^{(k)})}]$	159
$\mathbf{B}_{O;s;i}$	The submatrix formed by the first $r_{s;i} D_G$ columns of $\mathbf{B}_{O;i}$	72
$\mathbf{B}_j^{(k)}$	$= \text{diag}[\mathbf{I}_{r_j^{(k)}}, \mathbf{0}_{(n_j^{(k)} - r_j^{(k)}) \times (n_j^{(k)} - r_j^{(k)})}]$ for $j = 1, \dots, N_G^{(k)}$	127
$\mathbf{B}_j^{(k,p)}$	The j^{th} $D^{(k)} \times D_G^{(p)}$ submatrix of $\mathbf{B}^{(k,p)}$	130
b_i	$= \sum_{p=1}^{i-1} n_p$ for $2 \leq i \leq N_G$, and $b_1 = 0$	21
b_{RX}	Receive cluster angle spread	78
b_{TX}	Transmitter cluster angle spread	78
$\mathbf{b}_{i;k;s}$	The $n_i \times 1$ transmit space signature vector of path s in cluster k associated with TOG i	21
$\mathbf{b}_{k;s}$	The $N_T \times 1$ transmit space signature vector of path s in cluster k	15
$\beta_{k;s}$	Parameter defining the transmit space signature vector, $\beta_{k;s} = 2\pi \frac{d_T}{\lambda} \sin(\theta_{T;k;s})$	15
$\mathbb{C}$	The set of complex numbers	43
C	Shannon capacity in bps/Hz	35
C_O	Shannon capacity in bps/Hz for MIMO-OFDM systems	66
$\mathbf{C}$	The $L_t N_R \times N_T$ channel propagation matrix	21
$\mathbf{C}_i$	The $L_t N_R \times n_i$ channel propagation matrix associated with TOG i	21

$\tilde{\mathbf{C}}_j^{(k)}$	The j^{th} submatrix of size $L_t^{(k)} N_R^{(k)} \times n_j^{(p)}$ in $\mathbf{C}^{(k)} = [\tilde{\mathbf{C}}_1^{(k)} \tilde{\mathbf{C}}_2^{(k)} \dots \tilde{\mathbf{C}}_{N_G^{(p)}}^{(k)}]$, for $(k, p) = (1, 2)$ and $(k, p) = (2, 1)$	130
$\tilde{\mathbf{C}}_{N;j}^{(k,p)}$	The submatrix formed by the first $r_j^{(p)}$ columns of $\tilde{\mathbf{C}}_j^{(k)} \cdot [\mathbf{M}_j^{(p)}]^\dagger$, for $(k, p) = (1, 2)$ and $(k, p) = (2, 1)$	142
$\tilde{\mathbf{C}}_{N;j,q}^{(k,p)}$	The submatrix formed by the first $\min(N_R^{(p)}, n_j^{(p)})$ columns of $FT[\tilde{\mathbf{F}}_j^{(k,p)}(\tau)]_{i^{(k)}(v^{(k)}(j,q))} \cdot [\check{\mathbf{U}}_{O;j;v^{(k,p)}(j,q)}^{(p)}]^\dagger$, for $(k, p) = (1, 2)$ and $(k, p) = (2, 1)$	167
$\check{\mathbf{C}}_i$	The $L_t N_R \times n_i$ matrix $\check{\mathbf{C}}_i = \mathbf{C}_i \cdot \mathbf{U}_{T;i}$	44
$\check{\mathbf{C}}_i^{DCT}$	The matrix $\check{\mathbf{C}}_i$ associated with DCT	45
$\check{\mathbf{C}}_i^{DFT}$	The matrix $\check{\mathbf{C}}_i$ associated with DFT	45
$\ [\mathbf{C}_i]_{*,\max} \ ^2$	$= \max\{\ [\mathbf{C}_i]_{*,q} \ ^2\}_{q=1}^{n_i}$	59
C_{merged}	Shannon capacity after merging TOGs	191
$C_{original}$	Shannon capacity prior to merging TOGs	191
C^{sum}	Sum capacity for the MIMO BC described by (4.1)	106
C_{out}^{sum}	$\triangleq \max_{(R^{(1)}, R^{(2)}) \in \mathcal{R}_{out}} R^{(1)} + R^{(2)}$	106
C_O^{sum}	Sum capacity for the MIMO-OFDM BC described by (4.86)	150
$C_{O,out}^{sum}$	$\triangleq \max_{(R^{(1)}, R^{(2)}) \in \mathcal{R}_{O,out}} R^{(1)} + R^{(2)}$	151
$\mathbf{C}_{s;i}$	The submatrix formed by the first $r_{s;i}$ columns of $\mathbf{C}_i, i = 1, \dots, N_G$	53
$\mathcal{C}$	Capacity region of the MIMO BC described by (4.1)	104
$\mathcal{C}(\mathbf{A})$	The column space of matrix $\mathbf{A}$	146
$\mathcal{C}_O$	Capacity region of the MIMO-OFDM BC described by (4.86)	148
c	$= \frac{1}{r_T} \sum_{i=1}^{N_G} \sum_{l=1}^{r_i} \lambda_{i;l}^{-1}$	41
c_O	$= \frac{1}{r_O} \sum_{i=1}^{N_G} \sum_{l=1}^{r_{O;i}} \lambda_{O;i;l}^{-1}$	71
$\cos(\cdot)$	The cosine function	15

D	Number of orthogonal signal dimensions used in the modulation process for a user's signals	17
$\check{D}^{(k)}$	Normalizing constant for user k 's rate in the broadcast system, $\check{D}^{(k)} = D^{(k)}$ or $\check{D}^{(k)} = D_T$	102
$\tilde{D}$	$= N_T D_G$	20
$\tilde{D}_c$	$= D_c / n_c$	134
$\mathbf{D}^{(l,p)}$	The diagonal matrix defined as $[\mathbf{D}^{(l,p)}]_{q,q} = 1$ if $[\bar{\Lambda}^{(l,p)}]_{q,q} = 0$ and $[\mathbf{D}^{(l,p)}]_{q,q} = 0$ if $[\bar{\Lambda}^{(l,p)}]_{q,q} \neq 0$, for $(l,p) = (1,2)$ and $(l,p) = (2,1)$	127
$\mathbf{D}_C$	$= \mathbf{C}^\dagger \cdot \mathbf{C}$	193
$\mathbf{D}_O^{(l,p)}$	The diagonal matrix defined as $[\mathbf{D}_O^{(l,p)}]_{q,q} = 1$ if $[\bar{\Lambda}_O^{(l,p)}]_{q,q} = 0$ and $[\mathbf{D}_O^{(l,p)}]_{q,q} = 0$ if $[\bar{\Lambda}_O^{(l,p)}]_{q,q} \neq 0$, for $(l,p) = (1,2)$ and $(l,p) = (2,1)$	160
D_c	Number of common signal dimensions that are shared by all users in the broadcast system	26
$D_{c;j}^{(l)}$	Number of shared signal dimensions of user l that are employed by TOG $j \in \mathcal{A}_{G;c}^{(k)}$ of user k for $(k,l) = (1,2)$ and $(k,l) = (2,1)$	130
$D_{c;j;a}^{(k)}$	Number of signal dimensions that are shared between TOG j of user l and TOG a of user k for $(k,l) = (1,2)$ and $(k,l) = (2,1)$	132
D_G	Number of signal dimensions per TOG, $D_G = \frac{D}{N_G}$	17
$D_{G;0}^{(k)}$	Initial value of $D_{p;a}^{(k)}$ for $a = 1, \dots, N_G^{(k)}$	137
$\mathbf{D}_i$	$= \text{stack}_{k=1}^{L_t} [\text{stack}_{m=1}^{N_R} [\mathbf{d}_{i,k;m}^\dagger]], i = 1, \dots, N_G$	42
$\tilde{\mathbf{D}}_i$	$= \text{stack}_{d=1}^{D_G} [\text{stack}_{m=1}^{N_R} [\tilde{\mathbf{d}}_{i,d;m}^\dagger]], i = 1, \dots, N_G$	72
D_T	Total number of signal dimensions in the broadcast system	26
$D_p^{(k)}$	Number of proper signal dimensions assigned to user k in the broadcast system	26
$D_{p;0}^{(k)}$	Initial value of $D_p^{(k)}$	128
$D_{p;a}^{(k)}$	The number of proper signal dimensions allocated to TOG a of user k	132
$\text{diag}[\cdot]$	Operator denoting the direct sum of matrices	20

d_R	Distance between adjacent receive antennas	15
d_T	Distance between adjacent transmit antennas	15
$\mathbf{d}_{i;k;m}$	$= \sum_{s=0}^{S_k-1} G_{k;s}^* e^{-j(m-1)\alpha_{k;s}} \mathbf{b}_{i;k;s}$	42
$\tilde{\mathbf{d}}_{i;d;m}$	$= \sum_{k=1}^{L_t} \sum_{s=0}^{S_k-1} G_{k;s}^* e^{j2\pi \frac{p_i(d)}{T_F} \tau_k} e^{-j(m-1)\alpha_{k;s}} (\mathbf{b}_{i;k;s} \otimes \mathbf{e}_{d;D_G})$	72
$\dim[\mathcal{A}]$	The dimension of space $\mathcal{A}$	146
$\Delta(E_b/N_0)_T$	Asymptotic SNR separation between C and I_u ($= (E_b/N_0)_T - (E_b/N_0)_{T;u}$ dB)	64
Δ_i	$= \frac{ \Lambda_i^{-1} + \eta_i \kappa \sigma^2 \mathbf{C}_{s;i}^\dagger \cdot \mathbf{C}_{s;i} }{ \mathbf{I}_{r;s;i} + \eta_i \kappa \sigma^2 \mathbf{C}_{s;i}^\dagger \cdot \mathbf{C}_{s;i} }$	54
$\Delta^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)})$	$= C^{(k)}(\alpha_p^{(k)}) - \check{R}^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)})$, see (4.29)	114
$\Delta_\infty^{(k)}$	Limiting value for $\Delta^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ when (P_T/N_0) $\rightarrow \infty$, see (4.52) and (4.54)	123
$\Delta_O^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)})$	$= C_O^{(k)}(\alpha_p^{(k)}) - \check{R}_O^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)})$	154
$\Delta_{O;\infty}^{(k)}$	Limiting value for $\Delta_O^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ when (P_T/N_0) $\rightarrow \infty$, see (4.129) and (4.131)	161
$\delta(\cdot)$	Dirac's delta function	15
$\delta(i, l)$	Kronecker delta defined as $\delta(i, l) = 0$ if $i \neq l$ and $\delta(i, l) = 1$ if $i = l$	17
$\check{\delta}_q^{(k)}$	Perturbation in the eigenvalue $\check{\lambda}_q^{(k)}$ of $[\mathbf{H}^{(k)}]^\dagger \cdot [\Psi^{(k)}]^{-1} \cdot \mathbf{H}^{(k)}$ at low SNR, see (4.34) and (4.35)	116
$\check{\delta}_{O;q}^{(k)}$	Perturbation in the eigenvalue $\check{\lambda}_{O;q}^{(k)}$ of $[\mathbf{H}_O^{(k)}]^\dagger \cdot [\Psi_O^{(k)}]^{-1} \cdot \mathbf{H}_O^{(k)}$ at low SNR	155
$\delta D^{(k)}$	The number of signal dimensions that are made shared out of the $D_{p;0}^{(k)}$ proper signal dimensions	128
$\delta D_G^{(k)}$	The number of signal dimensions per TOG that are made shared out of the $D_{G;0}^{(k)}$ proper signal dimensions	137
$E[\cdot]$	Expectation	16
$\mathbf{E}_n^{(k)}$	$= \mathbf{e}_{n;D^{(k)}} \cdot \mathbf{e}_{n;D^{(k)}}^\dagger$ for $n = 1, \dots, D^{(k)}$	131
$\mathbf{E}_{a;n}^{(k)}$	The a^{th} diagonal block of $\mathbf{E}_n^{(k)} = \text{diag}_{j=1}^{N_G^{(k)}} [\mathbf{E}_{j;n}^{(k)}]$	211
$(E_b)_T$	Transmitted energy-per-bit	35
$(E_b/N_0)_R$	Received SNR-per-bit	39
$(E_b/N_0)_T$	Transmitted SNR-per-bit	35

$\left[\frac{E_b}{N_0}\right]_{\min}$	Minimum $(E_b/N_0)_T$ for which $C > 0$	39
$(E_b/N_0)_{T;u}$	Transmitted SNR-per-bit with unknown or partially known channel at the transmitter	58
$\left[\frac{E_b}{N_0}\right]_{u,\min}$	Minimum $(E_b/N_0)_{T;u}$ for which $I_u > 0$	58
$\mathbf{e}_{i;p}$	The $p \times 1$ vector $\mathbf{e}_{i;p} = \text{vec}(\{\delta(i, l)\}_{l=1}^p)$	17
$\boldsymbol{\eta}$	$= \text{vec}(\{\eta_i\}_{i=1}^{N_G})$	58
η_i	The fraction of power assigned to TOG i	53
$\tilde{\mathbf{F}}(t, \tau)$	The channel impulse response matrix, see (2.10)	15
$\tilde{\mathbf{F}}(\tau)$	The time-invariant channel impulse response matrix	24
$\tilde{\mathbf{F}}_j^{(k,p)}(\tau)$	$= \sum_{l=1}^{L_t^{(k)}} \delta(\tau - \tau_l^{(k)}) \mathbf{F}_{j;l}^{(k,p)}$	165
$\mathbf{F}_{i;k}$	The $N_R \times n_i$ propagation matrix of cluster k associated with TOG $i, i = 1, \dots, N_G$	21
$\check{\mathbf{F}}_{i;d}$	$= \sum_{k=1}^{L_t} \mathbf{F}_{i;k} e^{-j2\pi \frac{p_i(d)}{T_F} \tau_k}$ where $d = 1, \dots, D_G$ and $p_i(d) \in \{-N_F/2, \dots, N_F/2 - 1\}$ corresponds to the d^{th} orthogonal subcarrier employed on the antennas of TOG $i, i = 1, \dots, N_G$	67
$\mathbf{F}_{j;l}^{(k,p)}$	The $N_R^{(k)} \times n_j^{(p)}$ submatrix of $\mathbf{F}_l^{(k)}$ $[\mathbf{F}_{1;l}^{(k,p)} \dots \mathbf{F}_{N_G^{(p)};l}^{(k,p)}], j = 1, \dots, N_G^{(p)}$	165
$\check{\mathbf{F}}_{s;i;q}$	The submatrix formed by the first $r_{s;i}$ columns of $\check{\mathbf{F}}_{i;q}$	72
$\mathbf{F}_k$	The $N_R \times N_T$ propagation matrix of cluster k	15
$f_A(\mathbf{X})$	The matrix mapping $f_A(\mathbf{X}) = \mathbf{A}^T \cdot \mathbf{X} \cdot \mathbf{A}$	190
$f_{d;BC}^{(k)}$	The decoding function at receiver k in the broadcast system, $k = 1, 2$	97
$f_{e;BC}$	The encoding function for the broadcast system	97
$\check{f}_{i;l}$	$= \sum_{d=1}^{D_G} \ \check{\mathbf{F}}_{i;d}\ _{*,l}^2$	73
$\check{f}_{\max}$	$= \max\{\check{f}_{i;l_{i;\max}}\}_{i=1}^{N_G}, l_{i;\max} = \arg \max_{l=1, \dots, n_i} \check{f}_{i;l}$	74
g	Number of active TOGs i (for which $\eta_i \neq 0$)	62
$\mathbf{G}^{(2)}$	$= [\mathbf{H}^{(2)} \cdot \Sigma^{(2)} \cdot [\mathbf{H}^{(2)}]^\dagger + N_0 \mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}}]^{-1}$	200
$\mathbf{G}_1^{(2)}$	$= \Sigma^{(2)} \cdot [\mathbf{H}^{(2)}]^\dagger \cdot \mathbf{G}^{(2)} \cdot \mathbf{H}^{(2,1)}$	200
$\check{\mathbf{G}}_p^{(k)}$	$= (FT[\tilde{\mathbf{F}}^{(k)}(\tau)]_{\frac{i^{(k)}(p)}{T_F}})^\dagger \cdot FT[\tilde{\mathbf{F}}^{(k)}(\tau)]_{\frac{i^{(k)}(p)}{T_F}}$	220
$\check{\mathbf{G}}_{p;a}^{(k)}$	The a^{th} diagonal block of $\check{\mathbf{G}}_p^{(k)}$	220

$G_{in}^{(k)}$	Multiplexing gain associated with the rate	123
$\check{R}^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)}), \forall \alpha_p^{(k)} \neq 0$		
$\tilde{G}_{in}^{(k)}$	The value at which $G_{in}^{(k)}$ saturates when $L_t^{(k)}$ be-	144
	comes sufficiently large	
$G_{O;in}^{(k)}$	Multiplexing gain associated with the rate	161
$\check{R}_O^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)}), \forall \alpha_p^{(k)} \neq 0$		
$G_{k;s}$	Complex gain of path s in cluster k	14
$G_{out}^{(k)}$	Multiplexing gain associated with $C^{(k)}(\alpha_p^{(k)})$,	119
	$\forall \alpha_p^{(k)} \neq 0$	
$\tilde{G}_{out}^{(k)}$	$= \frac{N_T D^{(k)}}{\tilde{D}^{(k)}}$	129
$G_{O;out}^{(k)}$	Multiplexing gain associated with $C_O^{(k)}(\alpha_p^{(k)})$,	157
	$\forall \alpha_p^{(k)} \neq 0$	
$G_\tau^{(1)}$	$= \tau G_{out}^{(1)} + (1 - \tau) G_{in}^{(1)}$ for $\tau \in [0, 1]$	139
$G_\tau^{(2)}$	$= \tau G_{in}^{(2)} + (1 - \tau) G_{out}^{(2)}$ for $\tau \in [0, 1]$	139
$g(\theta_{T;k;s}, \theta_{R;k;s})$	Magnitude of the complex gain of path s in clus-	14
	ter k as a function of the AOD and AOA	
$g_R(\theta_{R;k;s})$	Antenna gain pattern at the mobile station in the	14
	direction $\theta_{R;k;s}$	
$g_T(\theta_{T;k;s})$	Antenna gain pattern at the base station in the di-	14
	rection $\theta_{T;k;s}$	
$\mathbf{H}$	The $L_t N_R D \times \tilde{D}$ discrete time channel matrix	20
$\mathbf{H}^{(k,p)}$	$= [\mathbf{C}^{(k)} \otimes \mathbf{I}_{D^{(k)}}] \cdot [\mathbf{I}_{N_T} \otimes \mathbf{B}^{(k,p)}] \cdot [\mathbf{A}^{(p)}]^T, k, p =$	30
	$1, \dots, K$	
$\mathbf{H}_O$	The $N_R D \times \tilde{D}$ discrete time channel matrix for	25
	MIMO-OFDM systems	
$\mathbf{H}_O^{(k,g)}$	$= [\mathbf{O}_{m,q}^{(k)}] \cdot [\mathbf{I}_{N_T} \otimes \mathbf{B}^{(k,g)}] \cdot [\mathbf{A}^{(g)}]^T, k, g = 1, \dots, K$	33
$\mathbf{H}_{O;p}$	The $N_R \times N_T$ discrete time channel matrix asso-	24
	ciated with subcarrier p in the OFDM system	
$h(\mathbf{a}, \mathbf{b})$	Joint differential entropy of a random variables	201
	(or vectors) $\mathbf{a}$ and $\mathbf{b}$	
$I(\mathbf{x}; \mathbf{r})$	Mutual information between $\mathbf{x}$ and $\mathbf{r}$	34
$\tilde{I}_{\mathcal{A}_\theta}$	$= \int_{\mathcal{A}_\theta} \prod_{i=1}^{N_G} \left\{ \frac{K_{m_i, p_i}^{-1} (\ln 2)^{m_i}}{\kappa^{m_i p_i}} \prod_{l=1}^{m_i} [2^{u_{i,l}} (2^{u_{i,l}} - \right.$	55
	$1)^{p_i - m_i}] \prod_{j=l+1}^{m_i} (2^{u_{i,l}} - 2^{u_{i,j}})^2 \} d\mathbf{u}$	

$I_{(i)}$	TOG i 's mutual information in the water-filling scheme	191
$I_{(m+p)}$	Mutual information in the water-filling scheme of the TOG obtained by merging TOGs m and p	191
$\mathbf{I}_n$	The $n \times n$ identity matrix	16
I_u	Information rate with unknown or partially known channel at the transmitter in bps/Hz	57
$I_{u,O}$	Information rate with partially known channel at the transmitter in bps/Hz for MIMO-OFDM systems	72
$I_\sigma(\mathbf{x}; \mathbf{r})$	Input/output mutual information assuming a non-informative a-prior probability distribution on $\mathbf{C}$	52
$\mathbf{i}(t)$	Vector of noise waveforms received on the N_R antennas, $\mathbf{i}(t) = \text{vec}(\{i_l(t)\}_{l=1}^{N_R})$	16
$i_l(t)$	Noise waveform received on antenna l	16
$J(\omega_i \rightarrow \boldsymbol{\varrho}_i)$	Jacobian of the transformation from ω_i to $\boldsymbol{\varrho}_i$	192
K	Number of users in the broadcast system	26
$\mathbf{K}_j^{(k)}$	Diagonal matrix with $[\mathbf{K}_j^{(k)}]_{l,l} = 1$ if $\lambda_{j;l}^{(k)} = \lambda_{\max}^{(k)}$, and $[\mathbf{K}_j^{(k)}]_{l,l} = 0$ else, $l = 1, \dots, n_j^{(k)}$ and $j = 1, \dots, N_G^{(k)}$	115
$\mathbf{K}_{j;a}^{(k,l)}$	$= \frac{\alpha_p^{(k)}}{\tilde{r}^{(k,l)}} (\mathbf{C}_a^{(k)} \cdot \tilde{\mathbf{W}}_{j;a}^{(k,l)} \cdot [\mathbf{C}_a^{(k)}]^\dagger) + \frac{N_G^{(l)} \alpha_p^{(l)}}{r_T^{(l)} D^{(l)}} \mathbf{N}_j^{(k,l)}$	133
$\mathbf{K}_{O;j}^{(k)}$	Diagonal matrix with $[\mathbf{K}_{O;j}^{(k)}]_{l,l} = 1$ if $\lambda_{O;j;l}^{(k)} = \lambda_{O;\max}^{(k)}$, and $[\mathbf{K}_{O;j}^{(k)}]_{l,l} = 0$ else, $l = 1, \dots, n_j^{(k)} D_G^{(k)}$ and $j = 1, \dots, N_G^{(k)}$	155
$\mathbf{K}_{O;j;q;a}^{(k,l)}$	$= \frac{\alpha_p^{(k)}}{\tilde{r}_O^{(k,l)}} \mathbf{Z}_{O;j;q;a}^{(k,l)} + \frac{\alpha_p^{(l)}}{r_O^{(l)}} \mathbf{N}_{O;j;q}^{(k,l)}$	168
$K_{m_i;p_i}$	Normalizing constant	55
κ	$= \frac{P_T}{N_0 D_G}$	53
$\tilde{\kappa}_a^{(2)}(\tau)$	$= \tau r_a^{(2)} + (1 - \tau) \frac{r_T^{(2)}}{N_G^{(2)}}$	139
$\kappa_{j;a}^{(k,p)}$	$= \text{rank}(\mathbf{K}_{j;a}^{(k,p)}) - \text{rank}(\mathbf{N}_j^{(k,p)})$ for $(k,p) = (1,2)$ and $(k,p) = (2,1)$	135
$\kappa_{O;j;a}^{(k,p)}$	$= r_{K;j;a}^{(k,p)} - \min(N_R^{(p)}, n_j^{(p)})$	169
$\tilde{\kappa}^{(2,1)}(\tau)$	$= \sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(1)}} \tau \frac{\kappa_{j;a}^{(2,1)}}{n_c} + (1 - \tau) \frac{r_T^{(2)}}{N_G^{(2)}}$	139

L	Total number of scatterers	40
L_t	Number of time resolvable clusters	12
$\ln$	Natural logarithm	39
$\log$	Base-2 logarithm	34
Λ_i	The $r_{s,i} \times r_{s,i}$ submatrix of the diagonal matrix $\text{diag}[\eta_i \frac{P_T}{D_G} \Lambda_i, \mathbf{0}_{(n_i-r_{s,i}) \times (n_i-r_{s,i})}]$ in the eigenvalue decomposition of $\mathbf{S}_i$	53
$\check{\Lambda}^{(l)}$	The diagonal matrix in the eigenvalue decomposition of $\mathbf{H}^{(l,p)} \cdot (\frac{1}{P_T} \Sigma_{\max}^{(p)}) \cdot [\mathbf{H}^{(l,p)}]^\dagger + \mathbf{H}^{(l)} \cdot (\frac{1}{P_T} \check{\Sigma}^{(l)}) \cdot [\mathbf{H}^{(l)}]^\dagger$, for $(l, p) = (1, 2)$ and $(l, p) = (2, 1)$	122
$\check{\Lambda}_O^{(l)}$	The diagonal matrix in the eigenvalue decomposition of $\mathbf{H}_O^{(l,p)} \cdot (\frac{1}{P_T} \Sigma_{O,\max}^{(p)}) \cdot [\mathbf{H}_O^{(l,p)}]^\dagger + \mathbf{H}_O^{(l)} \cdot (\frac{1}{P_T} \check{\Sigma}_O^{(l)}) \cdot [\mathbf{H}_O^{(l)}]^\dagger$, for $(l, p) = (1, 2)$ and $(l, p) = (2, 1)$	158
$\bar{\Lambda}^{(l,p)}$	The diagonal matrix in the eigenvalue decomposition of the matrix $\mathbf{H}^{(l,p)} \cdot \left\{ \text{diag}_{j=1}^{N_G^{(p)}} \left[[\mathbf{M}_j^{(p)}]^\dagger \cdot \mathbf{B}_j^{(p)} \cdot \mathbf{M}_j^{(p)} \right] \otimes \mathbf{I}_{D_G^{(p)}} \right\} \cdot [\mathbf{H}^{(l,p)}]^\dagger$, for $(l, p) = (1, 2)$ and $(l, p) = (2, 1)$	127
$\bar{\Lambda}_O^{(l,p)}$	The diagonal matrix in the eigenvalue decomposition of the matrix $\mathbf{H}_O^{(l,p)} \cdot \text{diag}_{j=1}^{N_G^{(p)}} \left[[\mathbf{M}_{O,j}^{(p)}]^\dagger \cdot \tilde{\mathbf{B}}_{O,j}^{(p)} \cdot \mathbf{M}_{O,j}^{(p)} \right] \cdot [\mathbf{H}_O^{(l,p)}]^\dagger$, for $(l, p) = (1, 2)$ and $(l, p) = (2, 1)$	159
$\check{\Lambda}^{(l,p)}$	The diagonal matrix in the eigenvalue decomposition of $\mathbf{H}^{(l,p)} \cdot (\frac{1}{P_T} \Sigma_{\max}^{(p)}) \cdot [\mathbf{H}^{(l,p)}]^\dagger$, for $(l, p) = (1, 2)$ and $(l, p) = (2, 1)$	122
$\check{\Lambda}_O^{(l,p)}$	The diagonal matrix in the eigenvalue decomposition of $\mathbf{H}_O^{(l,p)} \cdot (\frac{1}{P_T} \Sigma_{O,\max}^{(p)}) \cdot [\mathbf{H}_O^{(l,p)}]^\dagger$, for $(l, p) = (1, 2)$ and $(l, p) = (2, 1)$	158
λ	Carrier wavelength	15
$\lambda_q^{(k)}$	Eigenvalues of $[\mathbf{H}^{(k)}]^\dagger \cdot \mathbf{H}^{(k)}$, $q = 1, \dots, \tilde{D}^{(k)}$	105
$\check{\lambda}_q^{(k)}$	Eigenvalues of $[\mathbf{H}^{(k)}]^\dagger \cdot [\Psi^{(k)}]^{-1} \cdot \mathbf{H}^{(k)}$, $q = 1, \dots, \tilde{D}^{(k)}$	109

$\check{\lambda}_{O;q}^{(k)}$	Eigenvalues of $[\mathbf{H}_O^{(k)}]^\dagger \cdot [\Psi_O^{(k)}]^{-1} \cdot \mathbf{H}_O^{(k)}$, $q = 1, \dots, \tilde{D}^{(k)}$	151
$\lambda_{i;p}$	Eigenvalues of $\mathbf{C}_i^\dagger \cdot \mathbf{C}_i$, $p = 1, \dots, n_i$, $i = 1, \dots, N_G$	36
$\lambda_{O;i;l}$	Eigenvalues of $\tilde{\mathbf{O}}_i$, $l = 1, \dots, n_i D_G$, $i = 1, \dots, N_G$	68
$\lambda_l(\mathbf{A})$	The l^{th} eigenvalue of an arbitrary matrix $\mathbf{A}$	133
$\lambda_{\max}$	Maximum eigenvalue of $\text{diag}_{i=1}^{N_G}[\mathbf{C}_i^\dagger \cdot \mathbf{C}_i]$ or $\mathbf{H}^\dagger \cdot \mathbf{H}$, see (3.6)	39
$\check{\lambda}_{\max}^{(k)}$	Maximum eigenvalue of $[\mathbf{H}^{(k)}]^\dagger \cdot [\Psi^{(k)}]^{-1} \cdot \mathbf{H}^{(k)}$	117
$\lambda_{O;\max}$	Maximum eigenvalue of $\text{diag}_{i=1}^{N_G}[\tilde{\mathbf{O}}_i]$ or $\mathbf{H}_O^\dagger \cdot \mathbf{H}_O$	69
$\check{\lambda}_{O;\max}^{(k)}$	Maximum eigenvalue of $[\mathbf{H}_O^{(k)}]^\dagger \cdot [\Psi_O^{(k)}]^{-1} \cdot \mathbf{H}_O^{(k)}$	154
$\lambda_{\max;A}$	Value of $\lambda_{\max}$ using ATA	40
$\lambda_{\max;O}$	Value of $\lambda_{\max}$ using OTA	40
$\lambda_{\max;P}$	Value of $\lambda_{\max}$ using POTA	40
$\lambda_{O;\max;A}$	Value of $\lambda_{O;\max}$ using ATA	69
$\lambda_{O;\max;P}$	Value of $\lambda_{O;\max}$ using POTA	69
$\lambda_{O;\max;O}$	Value of $\lambda_{O;\max}$ using OTA	69
$\mathbf{M}$	The unitary matrix in the eigenvalue decomposition of $\mathbf{H}^\dagger \cdot \mathbf{H}$	35
$\mathbf{M}_O$	The unitary matrix in the eigenvalue decomposition of $\mathbf{H}_O^\dagger \cdot \mathbf{H}_O$	66
M	$= \sum_{i=1}^g \sum_{l=1}^{r'_{s;i}} \log(\frac{\eta_i N_G}{r_{s;i}} \xi_{i;l})$	62
M_O	$= \sum_{i=1}^{N_G} \sum_{l=1}^{r_{O;i}} \log\left(\frac{\eta_i N_G}{n_i} \lambda_{O;i;l}\right)$	74
$\check{\mathbf{M}}^{(k)}$	The unitary matrix in the eigenvalue decomposition of $[\mathbf{H}^{(k)}]^\dagger \cdot [\Psi^{(k)}]^{-1} \cdot \mathbf{H}^{(k)}$	110
$\check{\mathbf{M}}_a^{(2)}$	The unitary matrix in the eigenvalue decomposition of $\mathbf{R}_a^{(2,1)}$	212
$\check{\mathbf{M}}^{(l)}$	The unitary matrix in the eigenvalue decomposition of $\mathbf{H}^{(l,p)} \cdot (\frac{1}{P_T} \Sigma_{\max}^{(p)}) \cdot [\mathbf{H}^{(l,p)}]^\dagger + \mathbf{H}^{(l)} \cdot (\frac{1}{P_T} \check{\Sigma}^{(l)}) \cdot [\mathbf{H}^{(l)}]^\dagger$, for $(l, p) = (1, 2)$ and $(l, p) = (2, 1)$	122
$\check{\mathbf{M}}_{O;a}^{(2)}$	The unitary matrix in the eigenvalue decomposition of $\mathbf{R}_{O;a}^{(2,1)}$	221

$\tilde{\mathbf{M}}_O^{(l)}$	The unitary matrix in the eigenvalue decomposition of $\mathbf{H}_O^{(l,p)} \cdot (\frac{1}{P_T} \Sigma_{O;\max}^{(p)}) \cdot [\mathbf{H}_O^{(l,p)}]^\dagger + \mathbf{H}_O^{(l)} \cdot (\frac{1}{P_T} \tilde{\Sigma}_O^{(l)}) \cdot [\mathbf{H}_O^{(l)}]^\dagger$, for $(l, p) = (1, 2)$ and $(l, p) = (2, 1)$	158
$\bar{\mathbf{M}}^{(l,p)}$	The unitary matrix in the eigenvalue decomposition of the matrix $\mathbf{H}^{(l,p)} \cdot \left\{ \text{diag}_{j=1}^{N_G^{(p)}} \left[[\mathbf{M}_j^{(p)}]^\dagger \cdot \mathbf{B}_j^{(p)} \cdot \mathbf{M}_j^{(p)} \right] \otimes \mathbf{I}_{D_G^{(p)}} \right\} \cdot [\mathbf{H}^{(l,p)}]^\dagger$, for $(l, p) = (1, 2)$ and $(l, p) = (2, 1)$	127
$\bar{\mathbf{M}}_O^{(l,p)}$	The unitary matrix in the eigenvalue decomposition of the matrix $\mathbf{H}_O^{(l,p)} \cdot \text{diag}_{j=1}^{N_G^{(p)}} \left[[\mathbf{M}_{O;j}^{(p)}]^\dagger \cdot \tilde{\mathbf{B}}_{O;j}^{(p)} \cdot \mathbf{M}_{O;j}^{(p)} \right] \cdot [\mathbf{H}_O^{(l,p)}]^\dagger$, for $(l, p) = (1, 2)$ and $(l, p) = (2, 1)$	159
$\tilde{\mathbf{M}}^{(l,p)}$	The unitary matrix in the eigenvalue decomposition of $\mathbf{H}^{(l,p)} \cdot (\frac{1}{P_T} \Sigma_{\max}^{(p)}) \cdot [\mathbf{H}^{(l,p)}]^\dagger$, for $(l, p) = (1, 2)$ and $(l, p) = (2, 1)$	122
$\tilde{\mathbf{M}}_O^{(l,p)}$	The unitary matrix in the eigenvalue decomposition of $\mathbf{H}_O^{(l,p)} \cdot (\frac{1}{P_T} \Sigma_{O;\max}^{(p)}) \cdot [\mathbf{H}_O^{(l,p)}]^\dagger$, for $(l, p) = (1, 2)$ and $(l, p) = (2, 1)$	158
$\mathbf{M}_i$	Unitary matrix in the eigenvalue decomposition of $\mathbf{C}_i^\dagger \cdot \mathbf{C}_i$, $i = 1, \dots, N_G$	35
$\mathbf{M}_{O;i}$	Unitary matrix in the eigenvalue decomposition of $\tilde{\mathbf{O}}_i$, $i = 1, \dots, N_G$	68
$\tilde{\mathbf{M}}_{O;j;d}^{(k)}$	See (4.133) for definition	164
$m^{(k)}(\alpha_p^{(k)})$	The number of active eigenmodes associated with the single-user capacity for user k with power allocation $\alpha_p^{(k)}$, $k = 1, 2$	105
$\mathbf{m}_{MS}$	Mobile station velocity vector	15
m_T	$= \sum_{i=1}^{N_G} m_i$	54
m_i	$= \min(r_{s;i}, L_t N_R)$	54
$\mathbf{m}_q^{(k)}$	The q^{th} eigenvector of $[\mathbf{H}^{(k)}]^\dagger \cdot \mathbf{H}^{(k)}$, $q = 1, \dots, \tilde{D}^{(k)}$	114
$\mathbf{m}_{O;q}^{(k)}$	The q^{th} eigenvector of $[\mathbf{H}_O^{(k)}]^\dagger \cdot \mathbf{H}_O^{(k)}$, $q = 1, \dots, \tilde{D}^{(k)}$	155

$\bar{\mathbf{m}}_q^{(k,p)}$	The columns of $[\bar{\mathbf{M}}^{(k,p)}]^\dagger$	207
$\bar{\mathbf{m}}_{O;q}^{(k,p)}$	The columns of $[\bar{\mathbf{M}}_O^{(k,p)}]^\dagger$	218
$\mathbf{m}_{i,l}^*$	The l^{th} eigenvector of $\mathbf{C}_i^\dagger \cdot \mathbf{C}_i$, $l = 1, \dots, n_i$	37
μ	Water-filling constant associated with C	36
μ_O	Water-filling constant associated with C_O	68
$\check{\mu}^{(k)}$	Water-filling constant associated with $\check{R}^{(k)}$	110
$\check{\mu}_O^{(k)}$	Water-filling constant associated with $\check{R}_O^{(k)}$	152
$\mu_{\theta;R}$	$= \sum_{k=1}^{L_t} \frac{P_k}{S_k} \left(\sum_{s=0}^{S_k-1} \theta_{R;k;s} \right)$	79
$\mu_{\theta;T}$	$= \sum_{k=1}^{L_t} \frac{P_k}{S_k} \left(\sum_{s=0}^{S_k-1} \theta_{T;k;s} \right)$	79
N	Code blocklength	97
N_0	Variance of the components of the additive white Gaussian noise	16
N_F	Number of subcarriers in an OFDM system	21
N_G	Number of TOGs	17
$\mathbf{N}_j^{(k,p)}$	$= \tilde{\mathbf{C}}_j^{(k)} \cdot [\mathbf{M}_j^{(p)}]^\dagger \cdot \mathbf{B}_j^{(p)} \cdot \mathbf{M}_j^{(p)} \cdot [\tilde{\mathbf{C}}_j^{(k)}]^\dagger$	131
$\tilde{\mathbf{N}}_{j;l}^{(k,p)}$	$= (\mathbf{C}^{(k)})^\dagger \cdot \mathbf{n}_l(\mathbf{N}_j^{(k,p)}) \cdot \mathbf{n}_l^\dagger(\mathbf{N}_j^{(k,p)}) \cdot \mathbf{C}^{(k)}$	133
$\check{\mathbf{N}}_{j;a}^{(k,p)}$	$= \sum_{l=\text{rank}(\mathbf{N}_j^{(k,p)})+1}^{L_t^{(k)} N_R^{(k)}} \check{\mathbf{N}}_{j;l;a}^{(k,p)} \text{ for } j \in \tilde{\mathcal{A}}_{G;c}^{(p)}$	133
$\tilde{\mathbf{N}}_{j;l;a}^{(k,p)}$	$= (\mathbf{C}_a^{(k)})^\dagger \cdot \mathbf{n}_l(\mathbf{N}_j^{(k,p)}) \cdot \mathbf{n}_l^\dagger(\mathbf{N}_j^{(k,p)}) \cdot \mathbf{C}_a^{(k)}$	133
$\mathbf{N}_{O;j;q}^{(k,p)}$	$= FT[\tilde{\mathbf{F}}_j^{(k,p)}(\tau)]_{\frac{i^{(k)}(v^{(k)}(j,q))}{T_F}} \cdot \check{\mathbf{M}}_{O;j;v^{(k,p)}(j,q)}^{(p)} \cdot (FT[\tilde{\mathbf{F}}_j^{(k,p)}(\tau)]_{\frac{i^{(k)}(v^{(k)}(j,q))}{T_F}})^\dagger$	166
$\check{\mathbf{N}}_{O;j;q;a}^{(k,p)}$	See (4.142) for definition	167
N_R	Number of receive antennas	12
N_T	Number of transmit antennas	12
$N(m, \sigma^2)$	Gaussian or normal distribution with mean m and variance σ^2	14
$\mathcal{N}(\mathbf{A})$	The null space of matrix $\mathbf{A}$	146
$n(\lambda_{\max})$	Multiplicity of $\lambda_{\max}$	39
$n(\lambda_{O;\max})$	Multiplicity of $\lambda_{O;\max}$	69
$n(\check{\lambda}_{\max}^{(k)})$	Multiplicity of $\check{\lambda}_{\max}^{(k)}$	117
$n(\check{\lambda}_{O;\max}^{(k)})$	Multiplicity of $\check{\lambda}_{O;\max}^{(k)}$	154
n_c	$= N_G^{(1)} = N_G^{(2)}$ for broadcast systems with equal number number of TOGs	134

n_i	Number of transmit antennas in TOG i , $i = 1, \dots, N_G$	17
$\mathbf{n}_l(\mathbf{A})$	The l^{th} eigenvector of an arbitrary matrix $\mathbf{A}$	133
$\mathbf{O}_{m,q}$	$\triangleq \text{diag}_{p=-N_F/2}^{N_F/2-1} [[\mathbf{H}_{O;p}]_{m,q}]$, $m = 1, \dots, N_R$ and $q = 1, \dots, N_T$	24
$\tilde{\mathbf{O}}_i$	The i^{th} diagonal block of $\mathbf{H}_O^\dagger \cdot \mathbf{H}_O = \text{diag}_{i=1}^{N_G} [\tilde{\mathbf{O}}_i]$	66
$\tilde{\mathbf{O}}_{r,q}$	$= \sum_{m=1}^{N_R} \mathbf{O}_{m,r}^\dagger \cdot \mathbf{O}_{m,q}$	66
$\tilde{\mathbf{O}}_{r,q;i}$	The i^{th} diagonal block of $\tilde{\mathbf{O}}_{r,q}$	66
$P^{(k)}$	Transmit power allocated to user k in the broadcast system	100
P_T	Total transmit power	34
$P_{T;i}$	The transmit power assigned to TOG i in the water-filling scheme associated with capacity	191
$P_{T;m+p}$	The transmit power assigned to the TOG obtained by merging TOGs m and p in the water-filling scheme associated with capacity	191
$P_{e;BC}$	Average error probability for the broadcast system	97
P_k	Normalized power assigned to cluster k	14
$\mathbf{P}_{k;s}$	The $N_R \times N_T$ propagation matrix of path s in cluster k	15
P_σ	$= \Pr[I_\sigma(\mathbf{x}; \mathbf{r}) < R]$ given rate R	52
p_i	$= \max(r_{s;i}, L_t N_R)$	54
$p_{\mathbf{r}^{(1)}, \mathbf{r}^{(2)} \mathbf{x}_0}(\mathbf{a}_r^{(1)}, \mathbf{a}_r^{(2)} \mathbf{a}_0)$	Transition PDF for the MIMO BC of (4.1), $\mathbf{a}_r^{(1)} \in \mathcal{R}^{(1)}$, $\mathbf{a}_r^{(2)} \in \mathcal{R}^{(2)}$ and $\mathbf{a}_0 \in \mathcal{X}_0$	97
$p_{\mathbf{u}^{(1)}, \mathbf{u}^{(2)}}(\mathbf{a}_u^{(1)}, \mathbf{a}_u^{(2)})$	Joint PDF of $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$, $\mathbf{a}_u^{(k)} \in \mathcal{R}_u^{(k)}$, $k = 1, 2$	199
$p_{\mathbf{u}^{(2)}, \mathbf{x}^{(2)} \mathbf{u}^{(1)}}(\mathbf{a}_u^{(2)}, \mathbf{a}_x^{(2)} \mathbf{a}_u^{(1)})$	Conditional PDF of $\mathbf{u}^{(2)}, \mathbf{x}^{(2)}$ given $\mathbf{u}^{(1)}$ with $\mathbf{a}_x^{(2)} \in \mathbb{C}^{\bar{D}^{(2)}}$ and $\mathbf{a}_u^{(k)} \in \mathcal{R}_u^{(k)}$, $k = 1, 2$	200
$p_{\mathbf{u}^{(2)}, \mathbf{x}^{(2)} \mathbf{x}^{(1)}}(\mathbf{a}_u^{(2)}, \mathbf{a}_x^{(2)} \mathbf{a}_x^{(1)})$	Conditional PDF of $\mathbf{u}^{(2)}, \mathbf{x}^{(2)}$ given $\mathbf{x}^{(1)}$ with $\mathbf{a}_u^{(2)} \in \mathcal{R}_u^{(2)}$ and $\mathbf{a}_x^{(k)} \in \mathbb{C}^{\bar{D}^{(k)}}$, $k = 1, 2$	200
$p_{\mathbf{x}_0 \mathbf{u}^{(1)}, \mathbf{u}^{(2)}}(\mathbf{a}_0 \mathbf{a}_u^{(1)}, \mathbf{a}_u^{(2)})$	Conditional PDF of $\mathbf{x}_0$ given $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}$ with $\mathbf{a}_0 \in \mathcal{X}_0$ and $\mathbf{a}_u^{(k)} \in \mathcal{R}_u^{(k)}$, $k = 1, 2$	199
$p_{\mathbf{g}}(\mathbf{u})$	PDF of $\mathbf{g}$, $\mathbf{u} = \text{vec}(\{\{u_{i;l}\}_{l=1}^{m_i}\}_{i=1}^{N_G})$ with $u_{i,1} \geq u_{i,2} \geq \dots \geq u_{i,m_i} \geq 0, \forall i$	55

$p_{\omega_i}(\mathbf{u}_i)$	PDF of ω_i , $\mathbf{u}_i = \text{vec}(\{u_{i;l}\}_{l=1}^{m_i})$ with $u_{i;1} \geq u_{i;2} \geq \dots \geq u_{i;m_i} \geq 0, \forall i$	191
$\phi(t)$	$= \text{vec}(\{\phi_d(t)\}_{d=1}^D)$	17
$\phi_d(t)$	Orthonormal basis function associated with signal dimension d , $d = 1, \dots, D$	17
$\Psi^{(k)}$	$= \mathbf{I}_{L_t^{(k)} N_R^{(k)} D^{(k)}} + \frac{1}{N_0} \mathbf{H}^{(k,p)} \cdot \Sigma_{\max}^{(p)} \cdot [\mathbf{H}^{(k,p)}]^\dagger$ for $(k, p) = (1, 2)$ and $(k, p) = (2, 1)$	107
$\Psi_O^{(k)}$	$= \mathbf{I}_{N_R^{(k)} D^{(k)}} + \frac{1}{N_0} \mathbf{H}_O^{(k,p)} \cdot \Sigma_{O,\max}^{(p)} \cdot [\mathbf{H}_O^{(k,p)}]^\dagger$ for $(k, p) = (1, 2)$ and $(k, p) = (2, 1)$	151
$Q^{(k)}$	Encoded message for user k in the broadcast system, $k = 1, 2$	97
$\hat{Q}^{(k)}$	Decoded message for user k in the broadcast system, $k = 1, 2$	97
$\mathbf{Q}$	The diagonal matrix in the eigenvalue decomposition of $\mathbf{H}^\dagger \cdot \mathbf{H}$	35
$\mathbf{Q}_O$	The diagonal matrix in the eigenvalue decomposition of $\mathbf{H}_O^\dagger \cdot \mathbf{H}_O$	66
$\mathbf{Q}_i$	Diagonal matrix in the eigenvalue decomposition of $\mathbf{C}_i^\dagger \cdot \mathbf{C}_i$	35
$\mathbf{q}_{i;l}$	Eigenvector of $\mathbf{D}_i \cdot \mathbf{D}_i^\dagger$ associated with $\lambda_{i;l}$	42
$R^{(k)}$	Achievable rate for user k in the broadcast system, $k = 1, 2$	97
$\check{R}^{(k)}$	Maximal DPC rate for user k ($k = 1, 2$) in the broadcast system when the other user transmits at capacity rate, see (4.19)	108
$\check{R}_O^{(k)}$	Maximal DPC rate for user k ($k = 1, 2$) in the MIMO-OFDM broadcast system when the other user transmits at capacity rate, see (4.100)	151
$R_{DPC}^{(k)}$	Achievable rate for user k ($k = 1, 2$) using DPC with user 1 being encoded before user 2, see (4.2) and (4.9)	100
$R_{O,DPC}^{(k)}$	Achievable rate for user k ($k = 1, 2$) using DPC for MIMO-OFDM BCs with user 1 being encoded before user 2, see (4.89)	148

$\tilde{R}_{DPC}^{(k)}$	Achievable rate for user k ($k = 1, 2$) using DPC with user 2 being encoded before user 1, see (4.3) and (4.10)	100
$\tilde{R}_{O;DPC}^{(k)}$	Achievable rate for user k ($k = 1, 2$) using DPC for MIMO-OFDM BCs with user 2 being encoded before user 1, see (4.90)	148
$\mathbf{R}_a^{(2,1)}$	See (B.29) for definition	211
$\mathbf{R}_{O;a}^{(2,1)}$	See (B.45) for definition	221
$\mathcal{R}^{(k)}$	Output alphabet for user k in the broadcast system, $k = 1, 2$	97
$\mathcal{R}_u^{(k)}$	Alphabet of the auxiliary random vector $\mathbf{u}^{(k)}$ in the broadcast system, $k = 1, 2$	199
$\mathcal{R}_{k,p}(\Sigma^{(1)}, \Sigma^{(2)})$	Achievable rate region using DPC, see (4.4)-(4.5)	100
$\mathcal{R}_{O;k,p}(\Sigma^{(1)}, \Sigma^{(2)})$	Achievable rate region using DPC for MIMO-OFDM BCs, see (4.87)-(4.88)	147
$\mathcal{R}_{DPC}$	The DPC rate region, see (4.6)	101
$\mathcal{R}_{O;DPC}$	The DPC rate region for MIMO-OFDM BCs, see (4.91)	148
$\mathcal{R}_{in}$	Inner bound for the DPC rate region	107
$\mathcal{R}_{O;in}$	Inner bound for the DPC rate region of MIMO-OFDM BCs	151
$\mathcal{R}_{out}$	Outer bound for the DPC rate region	103
$\mathcal{R}_{O;out}$	Outer bound for the DPC rate region of MIMO-OFDM BCs	148
r	Rank of $\mathbf{H}$ (or $\mathbf{H}^\dagger \cdot \mathbf{H}$)	35
r_O	Rank of $\mathbf{H}_O$ (or $\mathbf{H}_O^\dagger \cdot \mathbf{H}_O$)	67
$\mathbf{r}$	The $L_t N_R D \times 1$ discrete time channel output vector, $\mathbf{r} = \text{vec}(\{\{\mathbf{r}_{m;u}\}_{m=1}^{N_R}\}_{u=1}^{L_t})$	20
$\tilde{r}^{(k)}$	Rank of $[\mathbf{H}^{(k)}]^\dagger \cdot [\Psi^{(k)}]^{-1} \cdot \mathbf{H}^{(k)}$	109
$\tilde{r}_O^{(k)}$	Rank of $[\mathbf{H}_O^{(k)}]^\dagger \cdot [\Psi_O^{(k)}]^{-1} \cdot \mathbf{H}_O^{(k)}$	151
$\mathbf{r}_O$	The $N_R D \times 1$ discrete time channel output vector for MIMO-OFDM systems, $\mathbf{r}_O = \text{vec}(\{\mathbf{r}_{O;m}\}_{m=1}^{N_R})$	24
$r_{O;i}$	Rank of $\tilde{\mathbf{O}}_i$	67

$\mathbf{r}_{O;m}$	The $D \times 1$ discrete time vector received on antenna m in the MIMO-OFDM system	21
$\check{\mathbf{r}}_{O;m}^{(k)}$	The $D^{(k)} \times 1$ discrete time interference vector received on antenna m of user k in the MIMO-OFDM broadcast system	31
$\tilde{\mathbf{r}}_{O;m}^{(k)}$	The $D^{(k)} \times 1$ discrete time desired output vector received on antenna m of user k in the MIMO-OFDM broadcast system	31
r_T	Rank of $\text{diag}_{i=1}^{N_G}[\mathbf{C}_i^\dagger \cdot \mathbf{C}_i]$ ($= \sum_{i=1}^{N_G} r_i$)	35
$r_{T;A}$	Value of r_T using ATA	41
$r_{T;O}$	Value of r_T using OTA	41
$r_{T;P}$	Value of r_T using POTA	41
r_i	Rank of $\mathbf{C}_i$	35
$\check{r}^{(l,p)}$	Rank of $[\mathbf{H}^{(l)}]^\dagger \cdot [\bar{\mathbf{M}}^{(l,p)}]^\dagger \cdot \mathbf{D}^{(l,p)} \cdot \bar{\mathbf{M}}^{(l,p)} \cdot \mathbf{H}^{(l)}$, for $(l, p) = (1, 2)$ and $(l, p) = (2, 1)$	126
$\check{r}_O^{(l,p)}$	Rank of $[\mathbf{H}_O^{(l)}]^\dagger \cdot [\bar{\mathbf{M}}_O^{(l,p)}]^\dagger \cdot \mathbf{D}_O^{(l,p)} \cdot \bar{\mathbf{M}}_O^{(l,p)} \cdot \mathbf{H}_O^{(l)}$, for $(l, p) = (1, 2)$ and $(l, p) = (2, 1)$	160
$\check{r}_a^{(2,1)}$	Rank of $\mathbf{R}_a^{(2,1)}$	212
$\check{r}_{O;a}^{(2,1)}$	Rank of $\mathbf{R}_{O;a}^{(2,1)}$	221
$\check{r}_{K;j;a}^{(l,p)}$	Rank of $\mathbf{K}_{O;j;a}^{(l,p)}$	168
$r_{\max}^{(l)}$	Rank of the matrix $\mathbf{H}^{(l,p)} \cdot (\frac{1}{P_T} \Sigma_{\max}^{(p)}) \cdot [\mathbf{H}^{(l,p)}]^\dagger + \mathbf{H}^{(l)} \cdot (\frac{1}{P_T} \check{\Sigma}^{(l)}) \cdot [\mathbf{H}^{(l)}]^\dagger$, for $(l, p) = (1, 2)$ and $(l, p) = (2, 1)$	122
$r_{\max;O}^{(l)}$	Rank of the matrix $\mathbf{H}_O^{(l,p)} \cdot (\frac{1}{P_T} \Sigma_{O;\max}^{(p)}) \cdot [\mathbf{H}_O^{(l,p)}]^\dagger + \mathbf{H}_O^{(l)} \cdot (\frac{1}{P_T} \check{\Sigma}_O^{(l)}) \cdot [\mathbf{H}_O^{(l)}]^\dagger$, for $(l, p) = (1, 2)$ and $(l, p) = (2, 1)$	159
$r_{\max}^{(l,p)}$	Rank of $\mathbf{H}^{(l,p)} \cdot (\frac{1}{P_T} \Sigma_{\max}^{(p)}) \cdot [\mathbf{H}^{(l,p)}]^\dagger$, for $(l, p) = (1, 2)$ and $(l, p) = (2, 1)$	122
$r_{\max;O}^{(l,p)}$	Rank of $\mathbf{H}_O^{(l,p)} \cdot (\frac{1}{P_T} \Sigma_{O;\max}^{(p)}) \cdot [\mathbf{H}_O^{(l,p)}]^\dagger$, for $(l, p) = (1, 2)$ and $(l, p) = (2, 1)$	159
$\mathbf{r}_{m;u}$	The $D \times 1$ discrete time vector associated with the delay τ_u and received on antenna m	20

$\mathbf{\check{r}}_{m;u}^{(k)}$	The $D^{(k)} \times 1$ discrete time interference vector associated with the delay $\tau_u^{(k)}$ and received on antenna m of user k in the broadcast system	29
$\tilde{\mathbf{r}}_{m;u}^{(k)}$	The $D^{(k)} \times 1$ discrete time desired output vector associated with the delay $\tau_u^{(k)}$ and received on antenna m of user k in the broadcast system	29
$r'_{s;T}$	$= \sum_{i=1}^g r'_{s;i}$	62
$r_{s;i}$	Rank of $\mathbf{S}_i$	53
$r'_{s;i}$	Rank of $\mathbf{C}_{s;i}$	62
$r'_{O;s;i}$	Rank of $\mathbf{B}_{O;s;i}$	74
$\mathbf{S}_i$	The $n_i \times n_i$ matrix in the representation $\Sigma_i = \mathbf{S}_i \otimes \mathbf{I}_{D_G}$	52
S_k	Number of scatterers or space resolvable paths in cluster k	12
$\mathcal{S}_{P;a}^{(k)}$	The subset of indexes in $\{1, \dots, D_G^{(k)}\}$ that correspond to the proper signal dimensions of TOG a of user k	212
$\mathcal{S}_{j;a}^{(k)}$	The subset of indexes in $\{1, \dots, D_G^{(k)}\}$ that correspond to the signal dimensions that are shared between TOG a of user k and TOG j of user l for $(k, l) = (1, 2)$ and $(k, l) = (2, 1)$	213
$\tilde{\mathbf{s}}(t)$	$= \text{vec}(\{\tilde{s}_i(t)\}_{i=1}^{N_T})$	27
$\mathbf{s}(t)$	$= \text{vec}(\{s_i(t)\}_{i=1}^{N_T})$	16
$s_i(t)$	Transmitted waveform on antenna i in the single-user system, $i = 1, \dots, N_T$	16
$\tilde{s}_i(t)$	Multiplexed waveform transmitted on antenna i in the broadcast system	26
$\sin(\cdot)$	The sine function	15
$\text{stack}[\cdot]$	Operator denoting the vertical stack of matrices having equal number of columns	21
Σ	The $\tilde{D} \times \tilde{D}$ covariance matrix of $\mathbf{x}$	34
$\tilde{\Sigma}$	The matrix given by $\tilde{\Sigma} = \mathbf{M} \cdot \Sigma \cdot \mathbf{M}^\dagger$ for spread spectrum like MIMO systems and by $\tilde{\Sigma} = \mathbf{M}_O \cdot \Sigma \cdot \mathbf{M}_O^\dagger$ for MIMO-OFDM systems	35

Σ_0	The covariance matrix of $\mathbf{x}_0$	100
$\check{\Sigma}^{(k)}$	The input covariance matrix associated with rate $\check{R}^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ for user k in the broadcast system	108
$\check{\Sigma}_O^{(k)}$	The input covariance matrix associated with rate $\check{R}_O^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ for user k in the MIMO-OFDM broadcast system	151
$\check{\Sigma}_a^{(2)}$	$= \sum_{l=1}^{r_a^{(2,1)}} [(\check{\mathbf{M}}_a^{(2)})^\dagger]_{*,l} \cdot [(\check{\mathbf{M}}_a^{(2)})^\dagger]_{*,l}^\dagger$	212
$\check{\Sigma}_{O;a}^{(2)}$	$= \sum_{l=1}^{r_{O;a}^{(2,1)}} [(\check{\mathbf{M}}_{O;a}^{(2)})^\dagger]_{*,l} \cdot [(\check{\mathbf{M}}_{O;a}^{(2)})^\dagger]_{*,l}^\dagger$	221
Σ_i	The covariance matrix of $\mathbf{x}'_i$	36
$\tilde{\Sigma}_i$	The i^{th} diagonal block of $\tilde{\Sigma}$	36
$\Sigma_{\max}^{(k)}$	The input covariance matrix obtained by water-filling over the eigenmodes of $[\mathbf{H}^{(k)}]^\dagger \cdot \mathbf{H}^{(k)}$ for user k in the broadcast system, see (4.15)	105
$\Sigma_{O,\max}^{(k)}$	The input covariance matrix obtained by water-filling over the eigenmodes of $[\mathbf{H}_O^{(k)}]^\dagger \cdot \mathbf{H}_O^{(k)}$ for user k in the MIMO-OFDM broadcast system, see (4.98)	150
$\Sigma_{\mathbf{r}}^{(2)}$	The covariance matrix of $\mathbf{r}^{(2)}$	200
$\Sigma_{\mathbf{r},\mathbf{u}}^{(2)}$	$= E[\mathbf{r}^{(2)} \cdot [\mathbf{u}^{(2)}]^\dagger]$	201
$\Sigma_{\mathbf{u}}^{(2)}$	The covariance matrix of $\mathbf{u}^{(2)}$	200
$\Sigma_{\mathbf{u},\mathbf{r}}^{(2)}$	$= E[\mathbf{u}^{(2)} \cdot [\mathbf{r}^{(2)}]^\dagger]$	201
Σ_y	The $N_T D \times N_T D$ covariance matrix of $\mathbf{y}$	190
σ^2	Variance of the entries of the channel propagation matrix without CSIT	52
$\sigma_{R,k}^2(P_k)$	Variance of cluster k 's mean AOA as a function of the cluster power P_k	15
σ_{RX}	Angle spread at the receiver, see (3.71b)	78
σ_{SF}	Lognormal shadow fading coefficient	14
σ_{TX}	Angle spread at the transmitter, see (3.71a)	78
T_F	The effective duration for an OFDM system	21
T_s	The symbol duration	17
t_i	The TOG to which transmit antenna i belongs	17
τ_k	Propagation delay associated with cluster k	14

θ_{3dB}	The 3 dB beamwidth in degrees for the antenna gain pattern at the base station	14
θ_{BS}	Angle of the LOS direction between the base station and the user with respect to the antenna array normal (or broadside) at the base station	12
θ_{MS}	Angle of the LOS direction between the base station and the user with respect to the antenna array normal (or broadside) at the user equipment	12
θ_m	Angle of the mobile station velocity vector	15
$\bar{\theta}_{R,k}$	Mean AOA of cluster k	12
$\hat{\theta}_{R,k;s}$	Offset AOA of path s with respect to the mean AOA of cluster k	12
$\bar{\theta}_{R,k;s}$	See (3.73) for definition	79
$\theta_{R,k;s}$	AOA of path s in cluster k with respect to the broadside of the receive antenna array	13
$\bar{\theta}_{T,k}$	Mean AOD of cluster k	12
$\hat{\theta}_{T,k;s}$	Offset AOD of path s with respect to the mean AOD of cluster k	12
$\bar{\theta}_{T,k;s}$	See (3.72) for definition	79
$\theta_{T,k;s}$	AOD of path s in cluster k with respect to the broadside of the transmit antenna array	12
$U(a, b)$	Uniform distribution over the interval $[a, b]$	13
$U_{s;i}$	Unitary matrix in the eigenvalue decomposition of S_i	53
$U_{T,i}$	An $n_i \times n_i$ unitary matrix associated with TOG i	45
$\check{U}_{O;j;d}^{(k)}$	The unitary matrix in the eigenvalue decomposition of $\check{M}_{O;j;d}^{(k)}$	167
$U_{T,i}^{DCT}$	The matrix $U_{T,i}$ associated with DCT	45
$U_{T,i}^{DFT}$	The matrix $U_{T,i}$ associated with DFT	45
$u^{(k)}$	Auxiliary random vector used to encode user k in the broadcast system	199
$V_0, \tilde{V}, V_1$	Points on the boundary of the asymptotical outer bound $\mathcal{R}_{out}$ at high SNR, see Table 4.1	119

$\tilde{V}_0, \tilde{V}_1$	Points on the boundary of the asymptotical inner bound $\mathcal{R}_{in}$ at high SNR, see Table 4.2	124
$\mathbf{v}$	The $L_t N_R D \times 1$ discrete time channel noise vector, $\mathbf{v} = \text{vec}(\{\{\mathbf{v}_{m;u}\}_{m=1}^{N_R}\}_{u=1}^{L_t})$	20
$v^{(k)}(j, d)$	The integer in the set $\{1, \dots, D^{(k)}\}$ that denotes the index of the d^{th} shared signal dimension, $d = 1, \dots, D_{c;j}^{(k)}$, employed by TOG $j \in \mathcal{A}_{G;c}^{(l)}$ of user l for $(k, l) = (1, 2)$ and $(k, l) = (2, 1)$	130
$\mathbf{v}_O$	The $N_R D \times 1$ discrete time channel noise vector for MIMO-OFDM systems, $\mathbf{v}_O = \text{vec}(\{\mathbf{v}_{O;m}\}_{m=1}^{N_R})$	24
$v^{(k,p)}(j, d)$	The integer in the set $\{1, \dots, D_G^{(p)}\}$ that denotes the index of the d^{th} shared signal dimension, $d = 1, \dots, D_{c;j}^{(k)}$, employed by TOG $j \in \mathcal{A}_{G;c}^{(p)}$ of user p for $(k, p) = (1, 2)$ and $(k, p) = (2, 1)$	165
$\tilde{v}^{(k)}(d')$	The integer in the set $\{1, \dots, D^{(k)}\}$ that denotes the index of the proper signal dimension $d' = 1, \dots, D_p^{(k)}$ of user k	130
$\mathbf{v}_{O;m}$	The $D \times 1$ discrete time channel noise vector received on antenna m in the MIMO-OFDM system	23
$\mathbf{v}_{m;u}$	The $D \times 1$ discrete time channel noise vector associated with the delay τ_u and received on antenna m	20
$\text{vec}(\cdot)$	Operator denoting the stack of matrix columns, or columns vectors, or scalars in a column vector	15
$\varphi_{k;s}$	Carrier phase of path s in cluster k	14
$\boldsymbol{\varrho}$	$= \text{vec}(\{\boldsymbol{\varrho}_i\}_{i=1}^{N_G})$	55
$\boldsymbol{\varrho}_i$	$= \text{vec}(\{\varrho_{i;l}\}_{l=1}^{m_i}), i = 1, \dots, N_G$	54
$\varrho_{i;l}$	$\equiv f_i(\omega_{i;l}) = \log(1 + \eta_i \kappa \sigma^2 w_{i;l}), l = 1, \dots, m_i$	54
W	$= R/D_G - \sum_{i=1}^{N_G} \log(\Lambda_i)$ given rate R	55
$\mathbf{W}_i$	$= \mathbf{C}_{s;i}^\dagger \cdot \mathbf{C}_{s;i}$ if $r_{s;i} \leq L_t N_R$, and $\mathbf{W}_i = \mathbf{C}_{s;i} \cdot \mathbf{C}_{s;i}^\dagger$ if $L_t N_R < r_{s;i}$	54

$\check{\mathbf{W}}_{j;a}^{(k,p)}$	$= \sum_{q=1}^{\text{rank}(\check{\mathbf{N}}_{j;a}^{(k,p)})} \mathbf{n}_q(\check{\mathbf{N}}_{j;a}^{(k,p)}) \cdot \mathbf{n}_q^\dagger(\check{\mathbf{N}}_{j;a}^{(k,p)})$	133
$\check{\mathbf{W}}_{O;j;q;a}^{(k,p)}$	$= \sum_{l=1}^{\text{rank}(\check{\mathbf{N}}_{O;j;q;a}^{(k,p)})} \mathbf{n}_l(\check{\mathbf{N}}_{O;j;q;a}^{(k,p)}) \cdot \mathbf{n}_l^\dagger(\check{\mathbf{N}}_{O;j;q;a}^{(k,p)})$	168
ω	The angle between the transmit and the receive antenna array broadside directions	13
ω_i	$= \text{vec}(\{\omega_{i,l}\}_{l=1}^{m_i}), i = 1, \dots, N_G$	191
$\omega_{i,l}$	Eigenvalues of $\mathbf{W}_i, l = 1, \dots, m_i$	54
	Also satisfying $\omega_{i,l} \equiv f_i^{-1}(\varrho_{i,l}) = \frac{2\varrho_{i,l}-1}{\eta_i \kappa \sigma^2}$	54
$\mathcal{X}_0$	Input alphabet for the broadcast system	97
$\mathbf{X}_i$	$= [\mathbf{x}_{i,1} \mathbf{x}_{i,2} \dots \mathbf{x}_{i,n_i}]$	37
$\tilde{\mathbf{X}}_i$	$= [\tilde{\mathbf{x}}_{i,1} \tilde{\mathbf{x}}_{i,2} \dots \tilde{\mathbf{x}}_{i,n_i}]$	37
$\mathbf{x}$	The $\tilde{D} \times 1$ discrete time channel input vector, $\mathbf{x} = \text{vec}(\{\mathbf{x}_i\}_{i=1}^{N_T})$	20
$\mathbf{x}_0$	$= \text{vec}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)})$	96
$\mathbf{x}_i$	The $D_G \times 1$ discrete time input vector transmitted on antenna $i, i = 1, \dots, N_T$	17
$\mathbf{x}'_i$	The $n_i D_G \times 1$ discrete time input vector to TOG $i, \mathbf{x}'_i = \text{vec}(\{\mathbf{x}_{i,l}\}_{l=1}^{n_i}), i = 1, \dots, N_G$	36
$\tilde{\mathbf{x}}'_i$	The $n_i D_G \times 1$ vector whose covariance matrix is $\tilde{\Sigma}_i$	37
$\mathbf{x}_{i,l}$	The $D_G \times 1$ discrete time input vector transmitted on antenna l of TOG $i, l = 1, \dots, n_i, i = 1, \dots, N_G$	36
$\tilde{\mathbf{x}}_{i,l}$	The $D_G \times 1$ subvectors of $\tilde{\mathbf{x}}'_i = \text{vec}(\{\tilde{\mathbf{x}}_{i,l}\}_{l=1}^{n_i})$	37
$\xi_{i,l}$	Eigenvalues of $\mathbf{C}_{s,i}^\dagger \cdot \mathbf{C}_{s,i}, l = 1, \dots, r_{s,i}$	57
$\xi_{O;i,l}$	Eigenvalues of $\mathbf{B}_{O;s,i}^\dagger \cdot \mathbf{B}_{O;s,i}, l = 1, \dots, r_{s,i} D_G$	72
$\xi^{(k,p)}$	$= (\sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(p)}} \kappa_{j;a}^{(k,p)}) - r_T^{(k)}$	138
$\mathbf{y}$	The $N_T D \times 1$ discrete time space signal vector, $\mathbf{y} = \text{vec}(\{\mathbf{y}_q\}_{q=1}^{N_T})$	20
$\mathbf{y}_i$	The $D \times 1$ discrete time space signal vector transmitted on antenna $i, i = 1, \dots, N_T$	17
$\check{\mathbf{y}}_q^{(k,p)}$	$= \mathbf{B}^{(k,p)} \cdot \mathbf{y}_q^{(p)}, q = 1, \dots, N_T$	30
$\mathbf{Z}_{O;j;q;a}^{(k,p)}$	See (4.143) for definition	168

$\mathbf{z}(t)$	Vector of received waveforms on the N_R antennas, $\mathbf{z}(t) = \text{vec}(\{z_l(t)\}_{l=1}^{N_R})$	16
z_k	Lognormal randomness for cluster k in a micro-cell environment	14
$z_l(t)$	Received waveform on antenna l	16

Chapter 1

Introduction

1.1 Background and literature review

The increasing demand for wireless communication makes it important to consider the capacity of the underlying channels for these systems. In his seminal work from 1948 [1], Claude E. Shannon defined the channel capacity as the maximum rate at which it is possible to send user information through the channel with as small an error probability as desired by proper encoding. A major challenge for the scientific community has been to design systems and to find practical methods for increasing the data transmission rates. With this goal in mind, a new technology has emerged in the mid 1990s based on the use of multiple antennas at the transmitter and receiver, which creates a Multiple-Input Multiple-Output (MIMO) channel. Being an active area of research, MIMO technology has been shown to significantly increase the data transmission capacity with fixed bandwidth in a rich scattering environment [2–4]. The capacity gain obtained with MIMO systems is due to the fact that the use of multiple transmit and receive antennas provides an additional spatial dimension for communication that can be exploited by spatially multiplexing several data streams onto the MIMO channel. The high design flexibility of MIMO systems is represented by the wide range of resource allocation and signaling methods that can be used with multiple antennas [5, 6]. In this section we take a brief look at the role of allocating resources (mainly power and bandwidth) and signaling techniques in MIMO systems. We also review the recent results on the capacity of MIMO channels in single-user and multi-user (broadcast) environments.

We start by considering single-user systems. When both transmitter and receiver employ multiple antennas, the MIMO channel can be decomposed into parallel independent channels. By multiplexing independent data onto these independent channels, we obtain a significant increase of the data rate compared to channels with a single antenna at the transmitter and receiver. Maximal rate, or capacity, is obtained by optimizing the input signaling structure and the power allocation between these independent channels. This is equivalent to optimizing the input covariance matrix structure subject to a fixed transmit power constraint. A key factor in such optimization is the extent of channel knowledge, or channel state information (CSI), at the transmitter [3, 7]. Hence, channel state information at the transmitter (CSIT) affects the capacity which is also affected by what is known about the channel gain matrix or its distribution at the receiver. In many situations (such as static or slowly varying channels) a good estimate of the channel gain matrix can be easily obtained at the receiver, which leads to assuming that channel state information is known at the receiver. The relation of CSIT to the transmit power allocation strategy that achieves capacity has been investigated in the literature because of its importance to the design and operation of wireless communication systems. With instantaneous CSIT, transmit power allocation by water-filling achieves capacity; otherwise transmit power is uniformly distributed among antennas [3, 8–10]. Reference [11] discusses the capacity-achieving input covariance matrix for a broad class of channels, assuming transmitter knowledge of the channel distribution. A special case where CSIT is a deterministic function of the channel state information at the receiver (CSIR) is investigated in [12], that also presents the water-filling power allocation as capacity achieving policy even with nonperfect CSIT. In a Rayleigh fading environment with both CSIT and CSIR, large capacity gains are possible compared to systems with CSIR only [13]. In a Rician fading environment with CSIR, [14] shows that transmitter knowledge of the fading distribution can achieve significant capacity gains compared to channels with no CSIT at all. When the CSI feedback link is of finite rate, the capacity gain due to CSIT is more pronounced at low Signal-to-Noise Ratio (SNR) than at high SNR for block-fading channels with CSIR [15, 16]. The effect of CSIT has been illustrated in [17] through actual channel measurements considering infinite and finite signal constellation sizes.

In addition to transmit power, another important resource can be subject to various allocations in a MIMO system. This resource is bandwidth. Using the fact that a continuous-time signal with bandwidth W can be represented by W complex signal dimensions per second [6, p. 28], allocating bandwidth becomes equivalent to allocating the orthonormal

basis functions (or signal dimensions) that span the space of transmitted waveforms. For a given signal space with large dimensionality, there exists a multitude of ways in which these signal dimensions can be allocated to transmit antennas. This issue and its impact on the performance of MIMO systems have been considered in [18–20] under the title of multi-dimensional space-time modulation. In these references, an allocation of signal dimensions to transmit antennas is referred to as multi-dimensional space-time modulation format. Considering multi-dimensional space-time modulation results in a more general framework for MIMO systems which can accommodate, as special cases, many practical systems in wireless communications. For transmit correlated block fading channels, a space-time modulation and coding scheme based on transmit beamforming integrated with a combination of space-time coding and orthogonal transmit diversity is shown in [20] to yield a good performance when the correlation is high. For spatially uncorrelated fading channels, [18] suggests that better diversity is obtained by allocating different signal dimensions to different transmit antennas. Assuming both CSIT and CSIR, [19] considers the Shannon capacity using two different space-time modulation formats over a deterministic space dispersive channel with three equal gain propagation paths. However, no extensive capacity analysis is provided in [19] for more general propagation channels and CSIT scenarios.

Furthermore, the use of multiple antennas lays foundations for transmission rate improvement techniques such as antenna selection and beamforming. Benefits of transmit antenna selection for approaching capacity at lower cost and enhancing the system performance are discussed in [21–26]. Beamforming refers to directing energy to an intended user while reducing interference to others [27, 28]. For a Rayleigh fading MIMO channel with CSIR and uncorrelated fades at the receiver, [29] investigates the optimality of beamforming in two cases: transmitter with mean feedback (when the mean values of the channel coefficients are known at the transmitter and their covariance modeled as white), and transmitter with covariance information (when the channel coefficients are zero mean with nonwhite covariance matrix known at the transmitter). In the first case, beamforming achieves capacity if the quality of the channel estimate at the transmitter improves, while in the second case, beamforming becomes optimal if the eigenvalues of the transmit fading covariance matrix become more disparate. The necessary and sufficient conditions for beamforming optimality have been extended to MIMO channels with both transmit and receive fading correlations and with covariance feedback in [30]. The latter reference shows that increasing receive antennas correlation lowers the SNR at which beamforming becomes

capacity achieving. Using a more generic concept known as eigen-beamformer (i.e. beamforming with eigenvectors), signaling on the dominant eigenvectors of the channel covariance matrix has been shown to enhance diversity and interference control capabilities [31]. Over correlated fading MIMO channels, [32, 33] point out that eigen-beamforming can be used to maximize the mutual information of the channel with transmitter assumed to know only the channel correlation matrix.

Apart from resource allocation and signaling techniques, channel capacity is significantly affected by the propagation environment [34] where the presence of scatterers results in multipath signal propagation. A review of spatial propagation models with associated multipath parameter distributions can be found in [35]. In [36] the capacity of a deterministic MIMO channel at high SNR has been shown to grow linearly with SNR in dB, with an asymptotic slope that is proportional to the minimum of the number of multipath components, the number of transmit antennas, or the number of receive antennas. Similar conclusions are drawn in [37], which considers a parametric MIMO fading channel based on that of [36] but with physical parameters modeled as random variables. The capacity of space dispersive multipath channels has been investigated in [38] for different antenna array structures with channel matrix modeled as a deterministic function of random parameters. With perfect CSIR and no CSIT, [39] illustrates the negative impact of space correlation and rank-deficiency (keyhole effect) on the channel capacity. In a realistic propagation environment the fades are often correlated due, for example, to insufficient spacing between antenna elements. Reference [40] shows that as the fading correlation increases more disparity is added to the distribution of the power gains of channel eigenmodes, which leads to a decrease in the number of active eigenmodes for a given transmit power and, hence, to a decrease in channel capacity. By introducing receive spatial correlations to the parametric MIMO fading channel of [37], [41] shows that the ergodic capacity increases as the distribution of the channel eigenvalues (i.e. the squared singular values of the channel matrix) becomes more flat (i.e. less correlated fades), or as the angle spread increases, or as additional multipath components or input/output antennas are used. A possible way to investigate capacity is to consider the statistics of the mutual information such as the characteristic function [42] and the moment generating function [43, 44]. Powerful tools from random matrix theory have also been used to analyze channel capacity [45]. Important insights are obtained by investigating the asymptotic capacity when the numbers of transmit and receive antennas scale indefinitely, with their ratio being kept constant. This approach for capacity analysis has been lately receiving great attention because of its accurate results

for systems with finite (even relatively small) number of antennas [3, 46–50].

So far we have considered single-user systems where one transmitter communicates with one receiver. In multi-user systems, efficient resource allocation and signaling techniques become even more important since multiple users have to be simultaneously accommodated. For the remainder of this section, we focus on broadcast systems. A broadcast channel (BC) models the downlink of a wireless communication system where the transmitter sends information to multiple users, and each user decodes the information it receives without cooperating with the other users. As for any multi-user system, the capacity of a broadcast system cannot be represented by a single number but rather by the closure of all sets of simultaneously achievable rate vectors. The capacity region of the class of scalar degraded BCs has been completely characterized in [51]. Assuming two users, the channel of one user (the worst user) is a noisier version of that of the other user (the best user). In this case, the capacity region is achieved through superposition coding and successive decoding where the best user first decodes and subtracts the signal intended for the worst user before decoding its own signal. An example of scalar degraded BCs is the BC with additive white Gaussian noise considered in [5, Section 14.5.2]. In order to emphasize the importance of resource sharing between users under a total power and bandwidth constraints, [5, Section 14.5.2] presents for this channel the achievable rate regions associated with the following spectrum sharing (or multiple access) techniques: time division, frequency division, and superposition code division with successive interference cancellation. The capacity region of degraded BCs can also be achieved using the dirty paper coding (DPC) technique based on interference cancellation at the sender side using some user ordering. As such, while the signals of the subsequently encoded users are treated as noise, a user sees the signals of the previously encoded users as noncausal interference that can be subtracted without affecting its transmission rate. The DPC technique was introduced by Costa in [52] which used the coding theorem of [53] to prove that capacity is invariant to interference for Gaussian channels with i.i.d. Gaussian interference noncausally known at the transmitter but not at the receiver. Various generalizations of Costa's result can be found in [54–57]. Practical DPC implementations are provided in [57] using nested coding, and in [58] which considers an end-to-end transmission system achieving a significant portion of the theoretical capacity gain. Another approach for constructing practical codes for the binary and Gaussian dirty-paper channels (i.e. channels following the model of [52]) has been proposed in [59] based on superposition coding and a decoding analogous to decoding over a multiple access channel (MAC).

We present now an overview of the recent capacity results associated with MIMO broadcast systems. For a BC with multiple antennas at the transmitter and receivers, the users cannot be ranked by their channel quality since, in general, their channel matrices are different. Therefore, MIMO BCs are not degraded, which complicates the evaluation of their capacity region. Superposition coding cannot be applied to non-degraded BCs since different users may have different rates which makes successive decoding difficult if not impossible. As for DPC, previous work on MIMO BCs shows that it yields an achievable rate region which is a special case of Marton's region [60] for general memoryless BCs. Being investigated in [61], the optimality of DPC in terms of maximizing the sum rate has been proved by direct calculations in [62] for a constant two-user BC with single-antenna receivers and known channel at the transmitter and all receivers. Generalization of results from [62] to systems with arbitrary number of users and multiple transmit and receive antennas has been carried out independently in [63] and [64]. The sum rate optimality of DPC for Gaussian MIMO BCs has also been investigated in [10, 65, 66] using the duality [67] between the DPC rate region of the MIMO BC and the capacity region of a Gaussian MIMO MAC with similar power constraint. Such duality is shown in [68] to hold under arbitrary linear constraints using the more general concept of minimax Lagrangian duality. Comparing the sum capacity obtained with DPC to that obtained with time division multiple access (TDMA), it is shown in [69] that for Gaussian MIMO BCs the DPC gain over TDMA is upper bounded by the minimum of the number of transmit antennas and the number of users. For an uncorrelated Rayleigh fading MIMO BC, [69] also derives a tighter bound for the ratio of the average DPC sum rate to the average TDMA sum rate at high SNR. Scaling laws of the sum rate for block fading Rayleigh MIMO BCs with large number of users are considered in [70] using DPC, TDMA and beamforming. The capacity gains obtained in a cellular system using DPC are illustrated through simulations in [71] for the circuit data and the packet data scenarios. In order to compare the performance of different transmission strategies in the high SNR regime, an affine approximation of the sum capacity is introduced in [72] for the MIMO BC. At high SNR, [72] shows that the sum capacity loss of beamforming relative to DPC decreases as the number of transmit antennas becomes sufficiently larger than the number of receivers in a Rayleigh fading environment. Simple and fast iterative water-filling algorithms are provided in [73, 74] for sum capacity computation based on the BC - MAC duality. Suboptimal algorithms yielding a tight sum capacity approximation have been investigated in [75] based on a zero-forcing allocation strategy. The results of [75] also include an extension to broadcast systems that employ an orthogonal frequency

division multiplexing (OFDM) scheme.

Recently, it was shown that the DPC rate region is in fact the MIMO BC capacity region [76–81]. In [76, 77] an outer bound on the rate region of the Gaussian vector BC is derived by ranking the users and allowing each user to know the channel outputs of the subsequent users, thus, creating a degraded MIMO BC out of the original (non-degraded) channel. The intersection of such degraded BC rate regions over all possible user orderings and over all joint conditional channel distributions that have the same marginals as the original BC is called in [76] the degraded same marginal (DSM) upper bound. It is shown that DPC achieves the rate points inside the DSM region if only Gaussian codebooks are used. However, neither [76] nor [77] proves the optimality of Gaussian inputs. The latter problem has been solved in [78, 79] that use the idea of enhanced degraded channels (for which a converse proof similar to Bergmans' [82] can be conducted) to show that the capacity region of degraded Gaussian MIMO BCs is the achievable region due to Gaussian coding. Hence, the DSM region yields the capacity region of the general MIMO BC and it is achieved using DPC. Reference [80] provides another converse proof for the Gaussian MIMO BC that is different from [78] and mainly based on the duality theory [65] and optimality conditions for a convex optimization problem. A more complete study is provided in [81] which employs enhanced degraded channels to extend the converse proof to different classes of non-degraded MIMO BCs using the notion of supporting hyperplanes. Note that while [78, 79] prove the optimality of DPC under a total transmit power constraint and an average transmit covariance matrix constraint, [81] considers a broader class of input constraints including, for example, the per antenna power constraint.

1.2 Subject of the thesis

1.2.1 Motivation and objectives

In the previous section, we have emphasized the impact of efficient resource allocation and signaling techniques on the performance of MIMO systems both in the single-user and broadcast environments. We have also referred to the more general framework of multi-dimensional space-time modulation introduced in [18–20] which offers the system designer more flexibility for allocating the signal dimensions to transmit antennas. As such, space-time modulation provides a new tool for exploiting the space dimension in wireless

communication systems with multiple antennas. This thesis investigates the capacity and information rates for MIMO systems with multi-dimensional space-time modulation and considers the associated signaling techniques.

For single-user MIMO systems, the available resources are dedicated to one user only. However, one needs to consider the most efficient allocation to transmit antennas not of transmit power alone, or signal dimensions alone, but of these resources together. Therefore, one can ask the following questions. How to allocate the available signal dimensions to transmit antennas so that maximal transmission rates can be obtained under different cases of CSIT ? Also, once a space-time modulation format has been chosen using which transmit power allocation and in which SNR regime does it maximize the transmission rate ? In Section 1.1 we also pointed out another interesting issue which is the effect of the spatial channel parameters on capacity. With multi-dimensional space-time modulation, different channel gain matrices are obtained with different space-time modulation formats for the same underlying propagation medium. Hence, it would be interesting to see how the spatial channel parameters affect transmission capacity for each space-time modulation format. This yields the following question: Are there any channels for which a transmitter that lacks channel information can achieve the same capacity as with CSIT by using an appropriate space-time modulation format and appropriate transmit power allocation ? Compensating for the transmitter lack of channel information is important since acquiring CSIT is not always possible and it requires, in general, a feedback channel from the receiver to the transmitter.

For broadcast MIMO systems, the issue becomes not only allocating the signal dimensions and transmit power to transmit antennas but also sharing these resources among users. Finding the optimal system design in this case is more difficult than in the single-user case because of the mutual interference that users cause into each other when sharing the same communication medium. As seen in Section 1.1, the DPC technique turned out to be optimal (in capacity sense) for MIMO BCs where the issue of multi-dimensional space-time modulation was not considered. A first question to ask is can DPC yield, at least, an achievable rate region for MIMO BCs that employ multi-dimensional space-time modulation ? If so, what is the best approximation for the DPC rate region, knowing that finding it is very difficult since it requires an exhaustive search over the space of input covariance matrices that satisfy the total transmit power constraint [10]. A natural approach is to find inner and outer bounds for this rate region. But then, how to choose appropriate bounds that can pro-

vide insights into the impact of multi-dimensional space-time modulation, as well as into the behavior of the DPC rate region in extreme SNR cases ? We finally come to the issue of sharing signal dimensions between users. A signal dimension allocated to a user is either shared with the other users or proper to this user. On one hand, increasing the number of shared signal dimensions (while fixing the number of proper signal dimensions) increases the number of signal dimensions allocated to each user. This yields an increase in the bandwidth allocated to each user. On the other hand, if more signal dimensions are shared between users the interference that each one of them experiences also increases. Is the tradeoff bandwidth versus interference in favor of sharing more or less signal dimensions between users ?

The objective of this thesis is to provide answers to the questions raised above. We consider the downlink of a wireless or cellular communication system, assuming a single cell scenario where a single base station communicates with one or more users. We employ the Third Generation Partnership Project (3GPP) spatial channel model from [83] which is a standard model for practical wireless channels. This thesis considers deterministic channels or samples of the 3GPP channel model for single-user and broadcast systems. A deterministic model is considered because of the extent of insights it can provide into the effect of multi-dimensional space-time modulation and the effect of the channel parameters on information rates. In many cases a deterministic model serves as a first step in the analysis of capacity for random channels.

1.2.2 Original contributions

Our results are an application of information-theoretical concepts to channel models that are important for standardization. Throughout this thesis, we consider MIMO systems with spread spectrum like and OFDM signaling techniques. A list of the original contributions of this thesis follows:

1. Identification of the space-time modulation formats that maximize the single-user capacity or the information transmission rate for different cases of CSIT and different transmit power allocation criteria. Furthermore, we prove the optimality of uniform power allocation among transmit antennas for single-user MIMO systems with space-time modulation, assuming no CSIT and a particular class of input covariance matrices. The optimality criterion in this case consists in choosing the input covariance

matrix that makes the fraction of channels for which the mutual information is less than any given rate converge fastest to zero, as the transmitter channel uncertainty increases (Chapter 3).

2. Derivation of outer and inner bounds for the DPC achievable rate region for MIMO BCs using multi-dimensional space-time modulation and signal dimensions sharing. The purpose of considering these bounds is to provide an approximation of the DPC rate region and to point out the effect of space-time modulation (Chapter 4).
3. Evaluation of the bandwidth versus interference tradeoff in MIMO BCs by investigating the impact of signal dimensions sharing on the spatial multiplexing gains associated with the inner and outer bounds of the DPC rate region. For this, we fix the number of proper signal dimensions allocated to each user and we analyze the impact of varying the number of shared signal dimensions. The impact of varying the number of shared signal dimensions at the expense of the number of proper signal dimensions is also analyzed, assuming fixed total number of signal dimensions (Chapter 4).
4. Asymptotical characterization of the inner and outer bounds for the DPC rate region in the low and high SNR regimes. This includes the derivation of the optimal power allocation among users that is associated with the maximal sum rate in both regimes (Chapter 4).

1.3 Outline of the thesis

Assuming a microcell environment, we begin by describing the parametric space and time dispersive 3GPP channel model in Chapter 2. Employing multi-dimensional space-time modulation from [18–20], we derive a discrete time channel model for single-user MIMO systems using spread spectrum like and OFDM signaling techniques. Using these techniques, we also derive a discrete time model for broadcast MIMO systems which employ space-time modulation and allow signal dimensions sharing among users. The channel models presented in this chapter and used in the subsequent chapters of this thesis are assumed to be memoryless and time-invariant.

The capacity and information rates for single-user spread spectrum like MIMO systems

and MIMO-OFDM systems with space-time modulation are considered in Chapter 3, assuming perfect CSIR and different cases of CSIT. Expressions and bounds for the channel capacity are derived in terms of the spatial channel parameters. We also investigate the asymptotical behavior of the transmission rates obtained with different transmit power allocations in the low and high SNR regimes. Throughout, the effect of space-time modulation is pointed out.

Dirty paper coding for spread spectrum like MIMO BCs and MIMO-OFDM BCs with space-time modulation and signal dimensions sharing is considered in Chapter 4 for a two-user system. In this chapter, the transmission rates in bps/Hz (or spectral efficiencies) are defined using two possible normalizations. We derive inner and outer bounds for the DPC rate region and we analyze their asymptotical behavior at low and high SNR. We address the tradeoff between bandwidth and interference by investigating the impact of signal dimensions sharing on the spatial multiplexing gains associated with these bounds. The effect of the number of time resolvable clusters on these multiplexing gains is also studied.

Conclusions to this thesis are provided in Chapter 5. Possible extensions of the results and future work are also discussed in this chapter. Appendices A and B provide the mathematical details for Chapters 3 and 4, respectively. Appendix C presents the identities and results from matrix theory that have been used in the mathematical derivations. Appendix D presents a summary of the Matlab routines that are used to generate the numerical results of Chapters 3 and 4. The Matlab routines are provided on a CD-ROM attached to this thesis.

Chapter 2

Channel and system models

2.1 The parametric 3GPP channel model

The presence of scatterers in the propagation medium creates a space and time dispersive multipath channel, where some (or none) of the multipath components could be resolvable in time and/or space. We employ the 3GPP space and time dispersive multipath channel model from [83] with L_t time resolvable clusters, each consisting of several space resolvable propagation paths which have equal propagation delays. Let S_k denote the number of space resolvable paths (or scatterers) in cluster k .

We consider the downlink of a system with N_T transmit antennas at the base station (or transmitter) and N_R receive antennas at the user equipment (or receiver). From [83], the transmitter and receiver employ uniform linear antenna arrays, with N_T and N_R being arbitrary. The angular parameters used in the 3GPP channel model are depicted in Figure 2.1. Let θ_{BS} and θ_{MS} denote the angles of the line of sight (LOS) direction between the base station and the user with respect to the antenna array normal (or broadside) at the base station and at the user equipment, respectively. Cluster k is characterized by the mean angle of departure (AOD) $\bar{\theta}_{T;k}$ and the mean angle of arrival (AOA) $\bar{\theta}_{R;k}$, with respect to the LOS direction. Path s in cluster k is characterized by the offset AOD $\hat{\theta}_{T;k;s}$ with respect to $\bar{\theta}_{T;k}$ and the offset AOA $\hat{\theta}_{R;k;s}$ with respect to $\bar{\theta}_{R;k}$. Hence, the AOD of path s with respect to the broadside of the transmit antenna array is

$$\theta_{T;k;s} = \theta_{BS} + \bar{\theta}_{T;k} + \hat{\theta}_{T;k;s} \quad (2.1)$$

Also, the AOA of path s with respect to the broadside of the receive antenna array is

$$\theta_{R;k;s} = \theta_{MS} + \bar{\theta}_{R;k} + \hat{\theta}_{R;k;s} \quad (2.2)$$

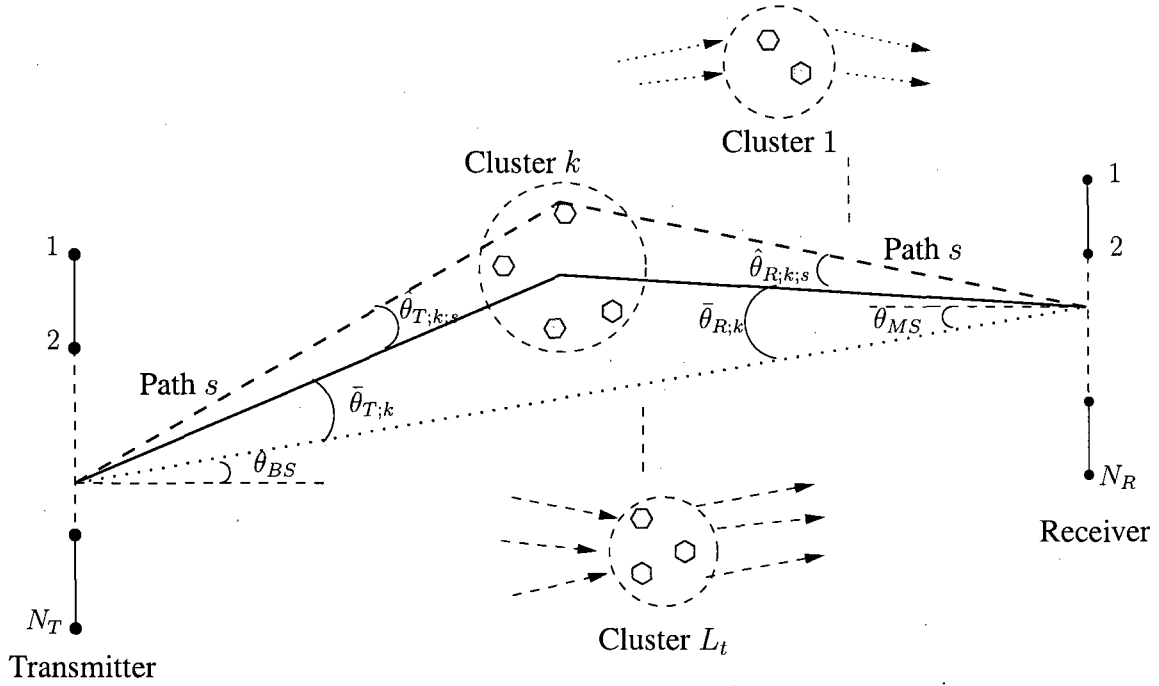


Fig. 2.1 The 3GPP spatial channel model.

All the angles in Figure 2.1 are assumed to be positive in value when measured in counter-clockwise direction. Since no particular specification for the placement of the user with respect to the base station is provided in [83], we choose θ_{BS} to be a random variable with uniform distribution $U(-60^\circ, 60^\circ)$. We also take $\theta_{MS} = \theta_{BS} + \omega$ where ω denotes the angle between the transmit and the receive antenna array broadside directions. We assume ω to be a random variable with distribution $U(0^\circ, 360^\circ)$. From [83, Section 5.3.2], we provide a brief description of the channel parameters in urban microcell environments only since we use such environments for the numerical examples that support our theoretical findings in this thesis. A description of the channel parameters in different environments such as urban and suburban macrocell environments can be found in [83]. Even though [83, Section 5.3.2] considers $L_t = 6$, we restrict the number of time resolvable clusters to $L_t = 3$ for simplicity. From [83], the number of space resolvable paths in each cluster is given by $S_k = 20$ for all $k = 1, \dots, L_t$.

The angles of departure: The mean AODs $\{\bar{\theta}_{T;k}\}_{k=1}^{L_t}$ are independent and identically

distributed (i.i.d.) random variables with uniform distribution $U(-40^\circ, 40^\circ)$. Fixed values for the offset AODs $\{\hat{\theta}_{T;k;s}\}_{s=0}^{S_k-1}$ are provided in [83, Table 5.2].

The path gain coefficients: The gain coefficient of path s in cluster k is given by $G_{k;s} = g(\theta_{T;k;s}, \theta_{R;k;s}) e^{j\varphi_{k;s}}$ with $\varphi_{k;s}$ being the carrier phase. The angles $\{\varphi_{k;s}\}_{s=0}^{S_k-1}$ are i.i.d. random variables with uniform distribution $U(0, 2\pi)$. The gain magnitude $g(\theta_{T;k;s}, \theta_{R;k;s})$ is given by

$$g(\theta_{T;k;s}, \theta_{R;k;s}) = \sqrt{\frac{P_k \sigma_{SF}}{S_k}} \sqrt{g_T(\theta_{T;k;s}) g_R(\theta_{R;k;s})} \quad (2.3)$$

where $g_T(\theta_{T;k;s})$ and $g_R(\theta_{R;k;s})$, respectively, denote the antenna gain patterns at the base station and at the user equipment, and P_k denotes the power assigned to cluster k . The power P_k is equally divided between the space resolvable paths of cluster k . The coefficient σ_{SF} corresponds to the lognormal shadow fading, with $\sigma_{SF} = 10^\gamma$ and γ being a random variable with Gaussian distribution $N(0, 1)$. We consider a base station that employs 3-sector antennas with

$$g_T(\theta_{T;k;s}) = 10^{\left\{-0.1 \min\left(12 \left[\frac{\theta_{T;k;s}}{\theta_{3dB}}\right]^2, A_m\right)\right\}} \quad (2.4)$$

where θ_{3dB} denotes the 3 dB beamwidth in degrees and A_m is the maximum attenuation in dB. From [83, Section 4.5], we have for the 3-sector antennas $\theta_{3dB} = 70^\circ$ and $A_m = 20$ dB. We assume that the receiver employs omnidirectional antennas with $g_R(\theta_{R;k;s}) = 1$ for all $\theta_{R;k;s}$, which leads using (2.3) and (2.4) to

$$g(\theta_{T;k;s}, \theta_{R;k;s}) = \sqrt{\frac{P_k \sigma_{SF}}{S_k}} 10^{\left\{-0.1 \min\left(12 \left[\frac{\theta_{T;k;s}}{\theta_{3dB}}\right]^2, A_m\right)\right\}} \quad (2.5)$$

and hence,

$$G_{k;s} = \sqrt{\frac{P_k \sigma_{SF}}{S_k}} 10^{\left\{-0.1 \min\left(12 \left[\frac{\theta_{T;k;s}}{\theta_{3dB}}\right]^2, A_m\right)\right\}} e^{j\varphi_{k;s}} \quad (2.6)$$

The power P_k is exponentially decaying in time with the addition of a lognormal randomness which is independent of the path delay. From [83, Section 5.3.2], we have

$$P_k = \frac{10^{-(\tau_k + z_k/10)}}{\sum_{l=1}^{L_t} 10^{-(\tau_l + z_l/10)}} \quad (2.7)$$

where the delays τ_k are i.i.d. random variables with uniform distribution $U(0, 1.2\mu s)$ and z_k are i.i.d. random variables with Gaussian distribution $N(0, 9)$, $k = 1, \dots, L_t$.

The angles of arrival: The mean AOAs $\{\bar{\theta}_{R;k}\}_{k=1}^{L_k}$ are independent random variables with respective Gaussian distributions $N(0, \sigma_{R;k}^2(P_k))$ where P_k is given in (2.7) and $\sigma_{R;k}(P_k) = 104.12(1 - e^{-0.265|10 \log_{10}(P_k)|})$. Fixed values for the offset AOAs $\{\hat{\theta}_{R;k;s}\}_{s=0}^{S_k-1}$ are provided in [83, Table 5.2]. The offset AODs and offset AOAs associated with the S_k paths of each cluster are obtained by randomly pairing the fixed values of [83, Table 5.2].

Let $\mathbf{m}_{MS}$ denote the velocity vector associated with the user movement. This velocity vector is characterized by its magnitude $\|\mathbf{m}_{MS}\|$ and by its angle θ_m with respect to the broadside of the receive antenna array. Reference [83] does not determine the velocity distribution according to which $\|\mathbf{m}_{MS}\|$ is drawn. However, θ_m is drawn according to a uniform distribution $U(0^\circ, 360^\circ)$. Let $\mathbf{P}_{k;s}(t)$ denote the $N_R \times N_T$ matrix describing the signal propagation over path s in cluster k from the transmitter to the receiver. Let d_T and d_R denote the spacing between adjacent transmit and receive antennas, respectively. We use λ to denote the carrier wavelength. Define the $N_R \times 1$ receive space signature vector $\mathbf{a}_{k;s} = \text{vec}(\{e^{j(l-1)\alpha_{k;s}}\}_{l=1}^{N_R})$ with $\alpha_{k;s} = 2\pi \frac{d_R}{\lambda} \sin(\theta_{R;k;s})$. Define also the $N_T \times 1$ transmit space signature vector $\mathbf{b}_{k;s} = \text{vec}(\{e^{-j(l-1)\beta_{k;s}}\}_{l=1}^{N_T})$ with $\beta_{k;s} = 2\pi \frac{d_T}{\lambda} \sin(\theta_{T;k;s})$. We can write

$$\mathbf{P}_{k;s}(t) = G_{k;s} e^{j \frac{2\pi}{\lambda} \|\mathbf{m}_{MS}\| \cos(\theta_{R;k;s} - \theta_m) t} \mathbf{a}_{k;s} \cdot \mathbf{b}_{k;s}^\dagger \quad (2.8)$$

where $G_{k;s}$ is given in (2.6). Hence, for cluster k the $N_R \times N_T$ matrix describing the signal propagation from the transmitter to the receiver is given by [83, Section 5.4]

$$\mathbf{F}_k(t) = \sum_{s=0}^{S_k-1} \mathbf{P}_{k;s}(t) = \sum_{s=0}^{S_k-1} G_{k;s} e^{j \frac{2\pi}{\lambda} \|\mathbf{m}_{MS}\| \cos(\theta_{R;k;s} - \theta_m) t} \mathbf{a}_{k;s} \cdot \mathbf{b}_{k;s}^\dagger \quad (2.9)$$

As such, $[\mathbf{F}_k]_{l,i}(t)$ denotes cluster k 's coefficient that describes the signal propagation from transmit antenna i to receive antenna l . Due to user mobility, the location of the scatterers and the propagation delays associated with the time resolvable clusters will change over time. Hence, the channel described by the matrices $\{\mathbf{F}_k(t)\}_{k=1}^{L_t}$ is time-varying [5, 84]. Using t and $t - \tau$ to denote, respectively, the observation time and application time of the input signal, we can write the $N_R \times N_T$ time-varying channel impulse response matrix as

$$\tilde{\mathbf{F}}(t, \tau) = \sum_{k=1}^{L_t} \delta(\tau - \tau_k) \mathbf{F}_k(t) \quad (2.10)$$

In this thesis, we consider a deterministic 3GPP channel model where the spatial channel

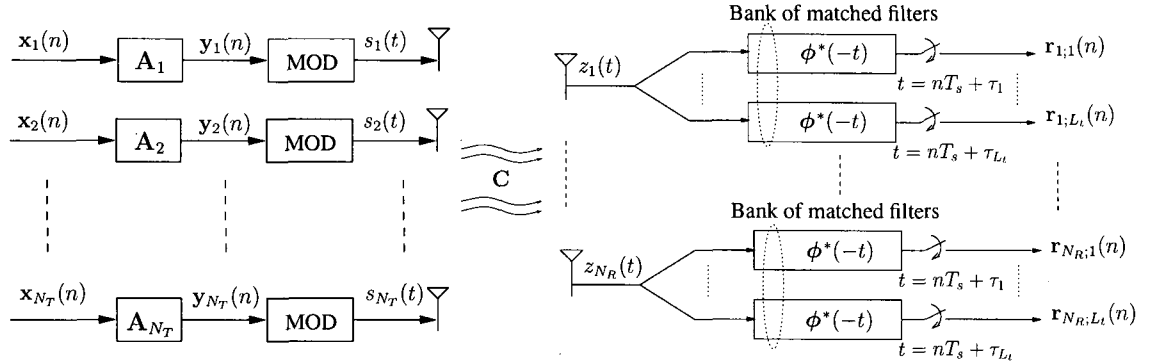


Fig. 2.2 Single-user MIMO system model.

parameters are constant over time. For simplicity, we assume $\|\mathbf{m}_{MS}\| = 0$. Hence, $\mathbf{F}_k(t)$ of (2.9) reduces to the following time-invariant matrix

$$\mathbf{F}_k = \sum_{s=0}^{S_k-1} G_{k;s} \mathbf{a}_{k;s} \cdot \mathbf{b}_{k;s}^\dagger \quad (2.11)$$

Next, we consider single-user and broadcast MIMO systems using space-time modulation over deterministic 3GPP channels. We employ the complex baseband representation.

2.2 Single-user systems with multi-dimensional space-time modulation

We consider the MIMO system illustrated in Figure 2.2. Let $s_i(t)$ denote the transmitted signal on antenna i and $z_l(t)$ denote the received signal on antenna l . Define the vectors $\mathbf{s}(t) = \text{vec}(\{s_i(t)\}_{i=1}^{N_T})$ and $\mathbf{z}(t) = \text{vec}(\{z_l(t)\}_{l=1}^{N_R})$. Denoting the noise waveform corrupting the signal received on antenna l by $i_l(t)$, the noise vector $\mathbf{i}(t) = \text{vec}(\{i_l(t)\}_{l=1}^{N_R})$ is circularly symmetric complex Gaussian with zero mean and autocovariance matrix $E[\mathbf{i}(t) \cdot \mathbf{i}^\dagger(t - \tau)] = N_0 \delta(\tau) \mathbf{I}_{N_R}$. Using (2.10) with $\mathbf{F}_k$ of (2.11), the input/output relation associated with the continuous time channel can be written as

$$\mathbf{z}(t) = \sum_{k=1}^{L_t} \int_{-\infty}^{\infty} \delta(\tau - \tau_k) \mathbf{F}_k \cdot \mathbf{s}(t - \tau) d\tau + \mathbf{i}(t) = \sum_{k=1}^{L_t} \mathbf{F}_k \cdot \mathbf{s}(t - \tau_k) + \mathbf{i}(t) \quad (2.12)$$

The MIMO system of Figure 2.2 employs multi-dimensional space-time modulation

from [18–20]. We consider a modulation process that employs a set of D orthonormal basis functions, $\phi(t) = \text{vec}(\{\phi_d(t)\}_{d=1}^D)$, as signal dimensions. A multi-dimensional space-time modulation format represents an allocation of the signal dimensions to the N_T transmit antennas. We define a Transmit Orthogonal Group (TOG) to be the group of transmit antennas that share a given subset of the signal dimensions. Let N_G denote the number of TOGs ($1 \leq N_G \leq N_T$) and n_i denote the number of antennas in TOG i . For simplicity, we choose the antenna elements in each TOG to be adjacent and we assume an equal number of signal dimensions per TOG, $D_G = \frac{D}{N_G}$. We use the notation $\mathbf{e}_{i,p} = \text{vec}(\{\delta(i,l)\}_{l=1}^p)$, $i = 1, \dots, p$, with the Kronecker symbol $\delta(i,l) = 0$ if $i \neq l$ and $\delta(i,l) = 1$ if $i = l$. Let $\mathbf{x}_i(n)$ denote the $D_G \times 1$ discrete time complex input vector transmitted on antenna i . The corresponding $D \times 1$ discrete time signal space vector is $\mathbf{y}_i(n) = \mathbf{A}_i^T \cdot \mathbf{x}_i(n)$, with $\mathbf{A}_i = \mathbf{e}_{t_i, N_G}^T \otimes \mathbf{I}_{D_G}$ and t_i indicating the TOG to which transmit antenna i belongs ($1 \leq t_i \leq N_G$). The transmitted waveform on antenna i can be written as

$$s_i(t) = \sum_{n'=-\infty}^{\infty} (\mathbf{y}_i(n'))^T \cdot \phi(t - n'T_s) = \sum_{n'=-\infty}^{\infty} \sum_{d=1}^D [\mathbf{y}_i(n')]_d \phi_d(t - n'T_s) \quad (2.13)$$

where $1/T_s$ is the symbol rate. The components of the $D_G \times D$ matrix $\mathbf{A}_i$ are zero or one, with each row having a single nonzero element and no column having more than one nonzero element. The matrix $\mathbf{A}_i$ determines the signal dimensions (indexed by $d \in \{1, \dots, D\}$) which can be used on transmit antenna i by making $[\mathbf{y}_i(n)]_d = 0$ if $\phi_d(t)$ is not used on antenna i , and making $[\mathbf{y}_i(n)]_d = [\mathbf{x}_i(n)]_{d'}$ with $d' \in \{1, \dots, D_G\}$ if $\phi_d(t)$ is used on antenna i . The multi-dimensional space-time modulation format is determined by the matrices $\{\mathbf{A}_i\}_{i=1}^{N_T}$. With $N_G = 1$ and $N_G = N_T$, we have the space-time modulation formats Aggregate Transmit Antenna (ATA) and Orthogonal Transmit Antenna (OTA), respectively. Otherwise, the space-time modulation format is called Partially Orthogonal Transmit Antenna (POTA) which can be viewed as a combination of ATA and OTA with $1 < N_G < N_T$. We consider such space-time modulation formats since they are generalizations of those used in several applications. The space-time coding system [85] is an example of ATA. The Alamouti transmit diversity scheme with two transmit antennas [86] can be seen as ATA since the same orthonormal basis functions are used on both antennas in the following manner: transmit symbol s_1 on antenna 1 and symbol s_2 on antenna 2 during one symbol period then, during the next symbol period, transmit symbol $-s_2^*$ on antenna 1 and symbol s_1^* on antenna 2. The transmitted waveforms (2.13) on antenna 1 and antenna 2 are $s_1(t) = \sum_{n'=-\infty}^{\infty} y_1(n') g(t - n'T_s)$ and

$s_2(t) = \sum_{n'=-\infty}^{\infty} y_2(n') g(t - n'T_s)$, where $g(t)$ is a real-valued signal shaping pulse [84, Section 4.3] with $(y_1(n'), y_1(n'+1)) = (s_1, -s_2^*)$ and $(y_2(n'), y_2(n'+1)) = (s_2, s_1^*)$. Here, the fact that $g(t)$ is used on all transmit antennas makes this scheme ATA. The orthogonal transmit diversity technique of Code Division Multiple Access (CDMA) 2000 [87, 88] is an example of OTA. Such technique is illustrated in Figure 2.3 using two transmit antennas where the input data stream $x(n)$ is split into even and odd streams $x_e(n)$ and $x_o(n)$, each transmitted on a transmit antenna. Here, the two basis functions are represented by the orthogonal Walsh codes $w_e(t)$ and $w_o(t)$ with duration T_s . Furthermore, the transmitted waveforms are $s_1(t) = \{\sum_{n=-\infty}^{\infty} [x_e(n)w_e(t - nT_s) + \sqrt{P_{p;1}}w_{p;1}(t - nT_s)]\}p(t)$ and $s_2(t) = \{\sum_{n=-\infty}^{\infty} [x_o(n)w_o(t - nT_s) + \sqrt{P_{p;2}}w_{p;2}(t - nT_s)]\}p(t)$ where $\{P_{p;j}\}_{j=1}^2$ and $\{w_{p;j}(t)\}_{j=1}^2$ denote the powers and Walsh codes dedicated to the pilots associated with the transmit antennas, and $p(t)$ is the long pseudo-random covering code for the transmitter [88]. With the pseudo-random code satisfying $p(t)p^*(t) = 1$ for all t , we have $\int_0^{T_s} w_e(t)p(t)p^*(t)w_o(t)dt = \int_0^{T_s} w_e(t)w_o(t)dt = 0$ since $w_e(t)$ and $w_o(t)$ are orthogonal. Another example of systems that can be seen as fitting into the ATA category are the MIMO-OFDM systems [89–91]. What we refer to as orthogonal basis functions in our framework correspond in MIMO-OFDM systems to the tones or subcarriers that are associated with the orthogonal flat fading sub-channels into which the original frequency selective channel has been decoupled. At the end of this section, we consider in detail MIMO-OFDM systems with space-time modulation which can be seen as a generalization of those considered in the literature. Examples of ATA, OTA and POTA (with $N_G = 2$) are, respectively, presented in Figures 2.4a, 2.4b and 2.4c assuming $N_T = 4$ and $D = 4$.

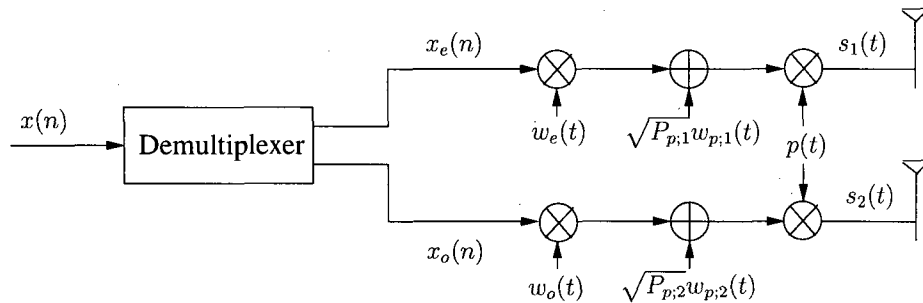


Fig. 2.3 Orthogonal transmit diversity sender with two antennas.

Consider the receiver of Figure 2.2. The discrete time channel output vector is obtained by using banks of sampled matched filters. We consider a receiver operating in perfect synchronism with the transmitter and we assume perfect multipath time resolvability. We

(a) ATA					
Transmit Antennas	4	X	X	X	X
	3	X	X	X	X
	2	X	X	X	X
	1	X	X	X	X
		1	2	3	4
Dimensions					

(b) OTA					
Transmit Antennas	4				X
	3			X	
	2		X		
	1	X			
		1	2	3	4
Dimensions					

(c) POTA ($N_G = 2$)					
Transmit Antennas	4			X	X
	3			X	X
	2	X	X		
	1	X	X		
		1	2	3	4
Dimensions					

Fig. 2.4 Examples of multi-dimensional space-time modulation formats. The assignment of a signal dimension j to transmit antenna i is represented by an X in row i and column j .

can write

$$\int_{-\infty}^{\infty} \phi_d(t - \tau_k - nT_s) \phi_{d'}^*(t - \tau_{k'} - n'T_s) dt = \delta(k, k') \delta(d, d') \delta(n, n') \quad (2.14)$$

for all $k, k' \in \{1, \dots, L_t\}$, all $d, d' \in \{1, \dots, D\}$ and all n, n' . As an example of basis functions that satisfy (2.14), $\{\phi_d(t)\}_{d=1}^D$ can be chosen to be spread spectrum signals to yield negligible inter-symbol interference (ISI) with sufficiently large processing gain [92–94]. We interpret (2.14) as follows. If $d = d'$ then each basis function $\phi_d(t)$ has an auto-correlation function that is a delta function. If $d \neq d'$ then the cross-correlation between the basis functions $\phi_d(t)$ and $\phi_{d'}(t)$ is zero. For spread-spectrum signals with large time-bandwidth product, the auto-correlation function is approximately equal to a delta function [94, Section III.A], [95, p. 47-48]. Furthermore, from [94, Section III.A] spread spectrum signals can be chosen so that their cross-correlation functions are very small compared to the main peak of their auto-correlation functions. According to [96, p. 661], such signals are approximately uniformly orthogonal where the uniform orthogonality of two signals refers to the fact that their cross-correlation is equal to zero for all time shifts [96, (17)]. Note also that the receiver of Figure 2.2 is similar to a RAKE receiver in the sense that it assumes no ISI and perfect multipath resolvability. However, our receiver differs from the RAKE receiver in that we do not add the outputs of the different fingers on each receive antenna and we assume that the vectors $\mathbf{r}_{m;u}(n)$ are applied to an optimum demodulator. Next, we derive the discrete time channel model using (2.14). Since (2.14) does not hold for MIMO-OFDM systems, the derivation of the channel model for these systems requires a different method, as it will be explained at the end of this section.

Let $\mathbf{r}_{m;u}(n)$ and $\mathbf{v}_{m;u}(n)$ respectively denote the $D \times 1$ discrete time channel output and noise vectors associated with the delay τ_u and received on antenna m . From Figure 2.2, $\mathbf{r}_{m;u}(n)$ denotes the stack of the outputs, sampled at time $t = nT_s + \tau_u$, of D matched filters with respective impulse responses $\{\phi_p^*(-t)\}_{p=1}^D$, each having $z_m(t)$ as input. Equivalently, we can write $[\mathbf{r}_{m;u}(n)]_p = \int_{-\infty}^{\infty} z_m(t) \phi_p^*(t - \tau_u - nT_s) dt$. Similarly, we can write $[\mathbf{v}_{m;u}(n)]_p = \int_{-\infty}^{\infty} i_m(t) \phi_p^*(t - \tau_u - nT_s) dt$. Writing $z_m(t) = \sum_{r=1}^{L_t} \sum_{q=1}^{N_T} [\mathbf{F}_r]_{m,q} s_q(t - \tau_r) + i_m(t)$ from (2.12), we have

$$[\mathbf{r}_{m;u}(n)]_p = \sum_{r=1}^{L_t} \sum_{q=1}^{N_T} \int_{-\infty}^{\infty} [\mathbf{F}_r]_{m,q} s_q(t - \tau_r) \phi_p^*(t - \tau_u - nT_s) dt + [\mathbf{v}_{m;u}(n)]_p \quad (2.15)$$

Using (2.13), (2.15) becomes

$$[\mathbf{r}_{m;u}(n)]_p = \sum_{r=1}^{L_t} \sum_{q=1}^{N_T} \sum_{n'=-\infty}^{\infty} \sum_{d=1}^D [\mathbf{F}_r]_{m,q} [\mathbf{y}_q(n')]_d \int_{-\infty}^{\infty} \phi_d(t - \tau_r - n'T_s) \phi_p^*(t - \tau_u - nT_s) dt + [\mathbf{v}_{m;u}(n)]_p \quad (2.16)$$

Using (2.14), (2.16) reduces to $[\mathbf{r}_{m;u}(n)]_p = \sum_{q=1}^{N_T} [\mathbf{F}_u]_{m,q} [\mathbf{y}_q(n)]_p + [\mathbf{v}_{m;u}(n)]_p$, thus, leading to

$$\mathbf{r}_{m;u}(n) = \sum_{q=1}^{N_T} [\mathbf{F}_u]_{m,q} \mathbf{y}_q(n) + \mathbf{v}_{m;u}(n) = \sum_{q=1}^{N_T} ([\mathbf{F}_u]_{m,q} \otimes \mathbf{I}_D) \cdot \mathbf{y}_q(n) + \mathbf{v}_{m;u}(n) \quad (2.17)$$

From (2.17), the output vector associated with cluster u is given by

$$\text{vec}(\{\mathbf{r}_{m;u}(n)\}_{m=1}^{N_R}) = [\mathbf{F}_u \otimes \mathbf{I}_D] \cdot \mathbf{y}(n) + \text{vec}(\{\mathbf{v}_{m;u}(n)\}_{m=1}^{N_R}) \quad (2.18)$$

where $\mathbf{y}(n) = \text{vec}(\{\mathbf{y}_q(n)\}_{q=1}^{N_T})$. Define $\tilde{D} = N_T D_G$. Let $\mathbf{x}(n) = \text{vec}(\{\mathbf{x}_i(n)\}_{i=1}^{N_T})$ and $\mathbf{r}(n) = \text{vec}(\{\{\mathbf{r}_{m;u}(n)\}_{m=1}^{N_R}\}_{u=1}^{L_t})$ respectively denote the $\tilde{D} \times 1$ input vector and $L_t N_R D \times 1$ output vector of the discrete time MIMO channel described by

$$\mathbf{r}(n) = \mathbf{H} \cdot \mathbf{x}(n) + \mathbf{v}(n) \quad (2.19)$$

with noise vector $\mathbf{v}(n) = \text{vec}(\{\{\mathbf{v}_{m;u}(n)\}_{m=1}^{N_R}\}_{u=1}^{L_t})$. Defining $\mathbf{A} = \text{diag}_{i=1}^{N_T} [\mathbf{A}_i]$, we have $\mathbf{y}(n) = \mathbf{A}^T \cdot \mathbf{x}(n)$. Due to the perfect multipath time resolvability assumption, the $L_t N_R D \times \tilde{D}$ discrete time channel matrix $\mathbf{H}$ can be seen as the stack of $N_R D \times \tilde{D}$ L_t submatrices, each associated with a time resolvable cluster. Therefore, we have $\mathbf{H} = \text{stack}_{k=1}^{L_t} [[\mathbf{F}_k \otimes \mathbf{I}_D] \cdot \mathbf{A}^T]$

or, equivalently,

$$\mathbf{H} = (\text{stack}_{k=1}^{L_t} [\mathbf{F}_k] \otimes \mathbf{I}_D) \cdot \mathbf{A}^T = [\mathbf{C} \otimes \mathbf{I}_D] \cdot \mathbf{A}^T \quad (2.20)$$

with an $L_t N_R \times N_T$ channel propagation matrix $\mathbf{C} \triangleq \text{stack}_{k=1}^{L_t} [\mathbf{F}_k]$. We consider the submatrices $\{\mathbf{F}_{i,k}\}_{i=1}^{N_G}$ of $\mathbf{F}_k = [\mathbf{F}_{1,k} \ \mathbf{F}_{2,k} \ \cdots \ \mathbf{F}_{N_G,k}]$ where $\mathbf{F}_{i,k}$ is the $N_R \times n_i$ matrix that describes the signal propagation over cluster k from the antenna elements of TOG i to the receiver. Write $\mathbf{b}_{k;s} = \text{vec}(\{\mathbf{b}_{i,k;s}\}_{i=1}^{N_G})$ where $\mathbf{b}_{i,k;s}$ is the $n_i \times 1$ transmit space signature vector associated with TOG i . We have $\mathbf{b}_{i,k;s} = e^{-j b_i \beta_{k;s}} \text{vec}(\{e^{-j(l-1)\beta_{k;s}}\}_{l=1}^{n_i})$ with $b_i = \sum_{p=1}^{i-1} n_p$ for $2 \leq i \leq N_G$ and $b_1 = 0$. From (2.11), one can easily show that

$$\mathbf{F}_{i,k} = \sum_{s=0}^{S_k-1} G_{k;s} \mathbf{a}_{k;s} \cdot \mathbf{b}_{i,k;s}^\dagger \quad (2.21)$$

The $L_t N_R \times n_i$ matrix that describes the signal propagation over the 3GPP channel from the antenna elements of TOG i to the receiver is given by

$$\mathbf{C}_i = \text{stack}_{k=1}^{L_t} [\mathbf{F}_{i,k}], \quad (2.22)$$

and $\mathbf{C} = [\mathbf{C}_1 \ \mathbf{C}_2 \ \cdots \ \mathbf{C}_{N_G}]$.

Using [91, Section 5.2], we now derive the discrete time channel model for a MIMO-OFDM system with space-time modulation where the available subcarriers are not necessarily used on all transmit antennas. Let N_F denote the number of orthogonal subcarriers (which is a power of 2) and T_F denote the effective duration of an OFDM symbol. The frequency spacing between adjacent subcarriers is given by $1/T_F$. We have $D = N_F$ and the orthonormal basis functions are given by $\phi_p(t) = \frac{1}{\sqrt{T_F}} e^{j2\pi \frac{p}{T_F} t}$ for $-N_F/2 \leq p \leq N_F/2 - 1$ and $0 \leq t \leq T_F$. The system model is illustrated in Figure 2.5. On transmit antenna i , the signal space vector $\mathbf{y}_i(n)$ is first applied to the Inverse Fourier Transform (IFT) presented in Figure 2.6a. Then, the outputs of the IFT are summed and a Cyclic Prefix (CP) is inserted. On each receive antenna, the CP is removed and the received signal is then applied to the Fourier Transform (FT) of Figure 2.6b. From Figure 2.5, $\mathbf{r}_{O,m}(n)$ denotes the stack of the outputs of D integrators, each having an output of the FT as input. For the remainder of this section, we replace the signal dimensions indexes $1, \dots, D$ by the equivalent indexes $-N_F/2, \dots, N_F/2 - 1$. The sequence of transmitted OFDM symbols is represented in Figure 2.7. In this figure, T_{CP} represents the duration of the CP. The waveform associated

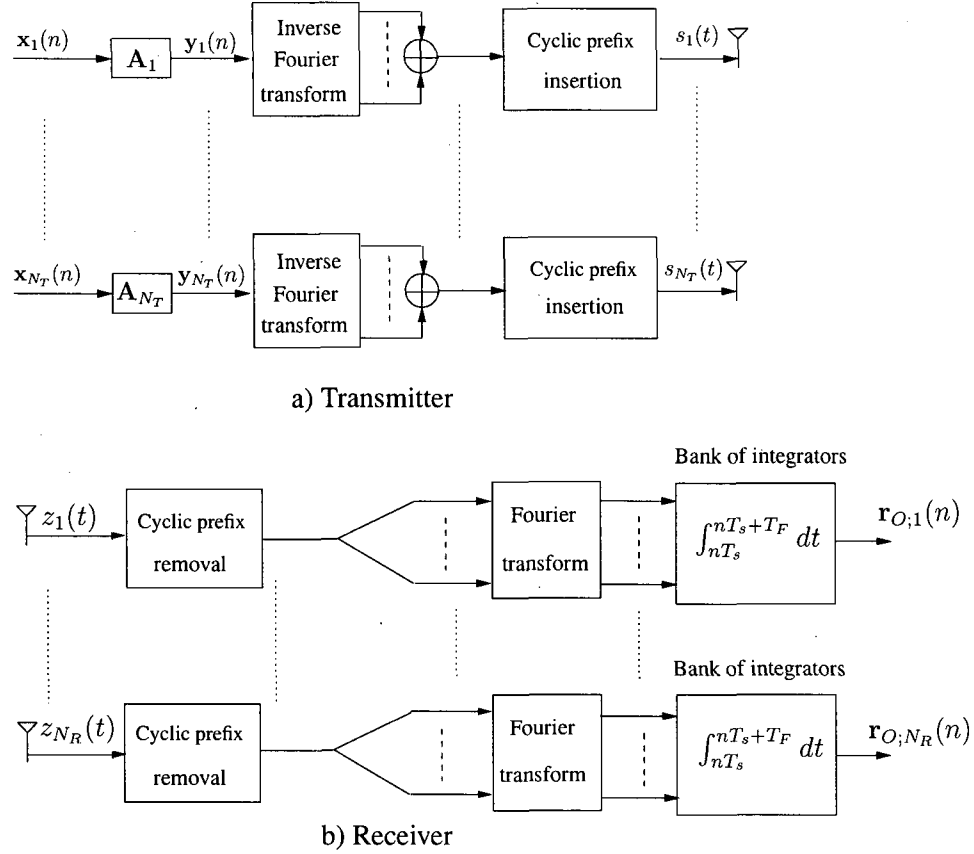


Fig. 2.5 Single-user MIMO-OFDM system model.

with the n^{th} OFDM symbol and transmitted on antenna i is given by

$$s_{i;n}(t - nT_s) = \begin{cases} \sum_{d=-N_F/2}^{N_F/2-1} [y_i(n)]_{d+N_F/2+1} \frac{1}{\sqrt{T_F}} e^{j2\pi \frac{d}{T_F}(t-nT_s)}, & t \in [nT_s - T_{CP}, nT_s + T_F] \\ 0, & \text{else} \end{cases} \quad (2.23)$$

The transmitted waveform on antenna i can be written as $s_i(t) = \sum_{n=-\infty}^{\infty} s_{i;n}(t - nT_s)$. We assume that the maximum excess channel propagation delay is less than T_{CP} , that is, $T_{CP} > \max_{k,u=1,\dots,L_t} |\tau_k - \tau_u|$. Hence, assuming that the transmitter and receiver operate in perfect synchronism, there is no interference of one OFDM symbol on the consecutive symbol and ISI is suppressed. Assuming that the receiver knows the exact time instants nT_s at which the OFDM symbols start, the output component associated with subcarrier p and received on antenna m is given by $[r_{O;m}(n)]_{p+N_F/2+1} = \int_{nT_s}^{nT_s+T_F} z_m(t) \phi_p^*(t - nT_s) dt$.

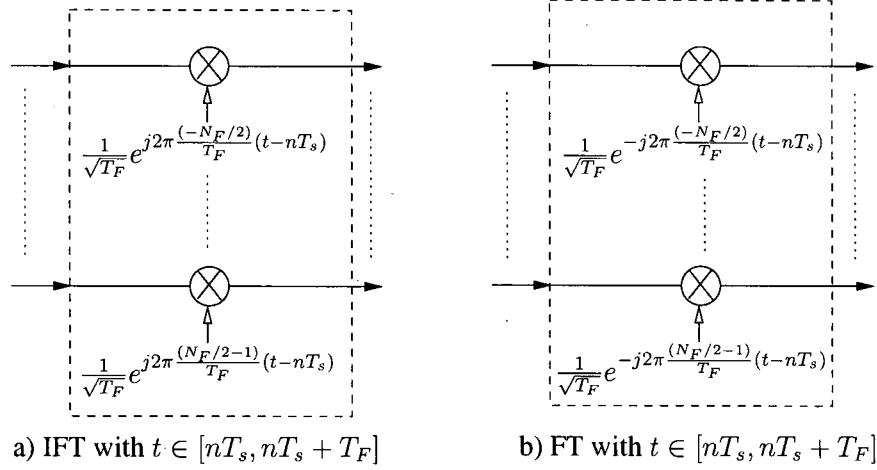


Fig. 2.6 Inverse Fourier transform and Fourier transform.

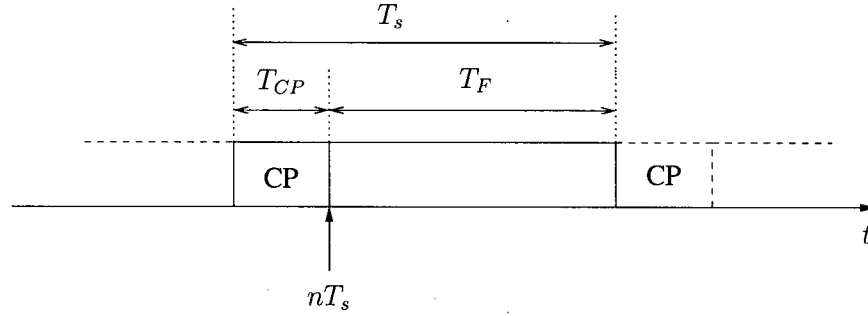


Fig. 2.7 Sequence of transmitted OFDM symbols.

Writing $z_m(t) = \sum_{k=1}^{L_t} \sum_{q=1}^{N_T} [\mathbf{F}_k]_{m,q} s_q(t - \tau_k) + i_m(t)$ from (2.12), we have

$$[\mathbf{r}_{O;m}(n)]_{p+N_F/2+1} = \frac{1}{\sqrt{T_F}} \sum_{k=1}^{L_t} \sum_{q=1}^{N_T} [\mathbf{F}_k]_{m,q} \int_{nT_s}^{nT_s+T_F} s_q(t - \tau_k) e^{-j2\pi \frac{p}{T_F} (t-nT_s)} dt + [\mathbf{v}_{O;m}(n)]_{p+N_F/2+1} \quad (2.24)$$

for $p = -N_F/2, \dots, N_F/2 - 1$ where $\mathbf{v}_{O;m}(n)$ denotes the $D \times 1$ noise vector associated with receive antenna m with components $[\mathbf{v}_{O;m}(n)]_{p+N_F/2+1} = \int_{nT_s}^{nT_s+T_F} i_m(t) \phi_p^*(t - nT_s) dt$. Assuming no ISI, and replacing $s_q(t - \tau_k)$ by $s_{q;n}(t - nT_s - \tau_k)$ from (2.23) and

substituting $u = t - nT_s$, leads to

$$\begin{aligned} [\mathbf{r}_{O;m}(n)]_{p+N_F/2+1} &= \frac{1}{T_F} \sum_{k=1}^{L_t} \sum_{q=1}^{N_T} \sum_{d=-N_F/2}^{N_F/2-1} [\mathbf{F}_k]_{m,q} e^{-j2\pi \frac{d}{T_F} \tau_k} [\mathbf{y}_q(n)]_{d+N_F/2+1} \\ &\quad \times \int_0^{T_F} e^{-j2\pi \frac{(p-d)}{T_F} u} du + [\mathbf{v}_{O;m}(n)]_{p+N_F/2+1} \end{aligned} \quad (2.25)$$

Using the orthogonality of the basis functions $\phi_d(t)$, (2.25) reduces to $[\mathbf{r}_{O;m}(n)]_{p+N_F/2+1} = \sum_{k=1}^{L_t} \sum_{q=1}^{N_T} [\mathbf{F}_k]_{m,q} e^{-j2\pi \frac{p}{T_F} \tau_k} [\mathbf{y}_q(n)]_{p+N_F/2+1} + [\mathbf{v}_{O;m}(n)]_{p+N_F/2+1}$. Equivalently, we have

$$\begin{aligned} [\mathbf{r}_{O;m}(n)]_{p+N_F/2+1} &= \sum_{q=1}^{N_T} [\mathbf{H}_{O;p}]_{m,q} [\mathbf{y}_q(n)]_{p+N_F/2+1} + [\mathbf{v}_{O;m}(n)]_{p+N_F/2+1} \\ &= \{ [\mathbf{H}_{O;p}]_{m,*} \otimes \mathbf{e}_{(p+N_F/2+1);D}^T \} \cdot \mathbf{y}(n) + [\mathbf{v}_{O;m}(n)]_{p+N_F/2+1} \end{aligned} \quad (2.26)$$

where $N_F = D$ and $\mathbf{H}_{O;p}$ is the $N_R \times N_T$ channel matrix associated with subcarrier p with $[\mathbf{H}_{O;p}]_{m,q} = \sum_{k=1}^{L_t} [\mathbf{F}_k]_{m,q} e^{-j2\pi \frac{p}{T_F} \tau_k}$, for $m = 1, \dots, N_R$ and $q = 1, \dots, N_T$. Thus, we have $[\mathbf{H}_{O;p}]_{m,q} = \int_{-\infty}^{\infty} [\sum_{k=1}^{L_t} [\mathbf{F}_k]_{m,q} \delta(\tau - \tau_k)] e^{-j2\pi \frac{p}{T_F} \tau} d\tau = FT [\sum_{k=1}^{L_t} [\mathbf{F}_k]_{m,q} \delta(\tau - \tau_k)]_{\frac{p}{T_F}}$ which denotes the Fourier transform at the frequency $\frac{p}{T_F}$ of $\sum_{k=1}^{L_t} [\mathbf{F}_k]_{m,q} \delta(\tau - \tau_k)$. Equivalently, we use the matrix notation $\mathbf{H}_{O;p} = FT [\tilde{\mathbf{F}}(\tau)]_{\frac{p}{T_F}}$ where $\tilde{\mathbf{F}}(\tau) = \sum_{k=1}^{L_t} \delta(\tau - \tau_k) \mathbf{F}_k$ is the time-invariant channel impulse response matrix (2.10). From (2.26), we obtain

$$\begin{aligned} \mathbf{r}_{O;m}(n) &= \text{stack}_{p=-N_F/2}^{N_F/2-1} \{ [\mathbf{H}_{O;p}]_{m,*} \otimes \mathbf{e}_{(p+N_F/2+1);D}^T \} \cdot \mathbf{y}(n) + \mathbf{v}_{O;m}(n) \\ &= \left\{ \sum_{p=-N_F/2}^{N_F/2-1} [\mathbf{H}_{O;p}]_{m,*} \otimes [\mathbf{e}_{(p+N_F/2+1);D} \cdot \mathbf{e}_{(p+N_F/2+1);D}^T] \right\} \cdot \mathbf{y}(n) + \mathbf{v}_{O;m}(n) \\ &= [\mathbf{O}_{m,1} \mathbf{O}_{m,2} \dots \mathbf{O}_{m,N_T}] \cdot \mathbf{y}(n) + \mathbf{v}_{O;m}(n) \end{aligned} \quad (2.27)$$

with the $D \times D$ diagonal matrices $\mathbf{O}_{m,q} \triangleq \text{diag}_{p=-N_F/2}^{N_F/2-1} [[\mathbf{H}_{O;p}]_{m,q}]$, $m = 1, \dots, N_R$ and $q = 1, \dots, N_T$. The matrix $[\mathbf{e}_{(p+N_F/2+1);D} \cdot \mathbf{e}_{(p+N_F/2+1);D}^T]$ is a $D \times D$ diagonal matrix with diagonal element $(p+N_F/2+1)$ equal to 1 and the remaining diagonal elements equal to 0. Let $\mathbf{r}_O(n) = \text{vec}(\{\mathbf{r}_{O;m}(n)\}_{m=1}^{N_R})$ and $\mathbf{v}_O(n) = \text{vec}(\{\mathbf{v}_{O;m}(n)\}_{m=1}^{N_R})$ denote the $N_R D \times 1$ channel output and noise vectors, respectively. Using $\mathbf{y}(n) = \mathbf{A}^T \cdot \mathbf{x}(n)$, it follows that the discrete time channel model for MIMO-OFDM systems with space-time modulation is

described by the input/output relation

$$\mathbf{r}_O(n) = \mathbf{H}_O \cdot \mathbf{x}(n) + \mathbf{v}_O(n) \quad (2.28)$$

where the $N_R D \times \tilde{D}$ channel matrix $\mathbf{H}_O$ is given by

$$\mathbf{H}_O = \left\{ \sum_{p=-N_F/2}^{N_F/2-1} FT[\tilde{\mathbf{F}}(\tau)]_{\frac{p}{T_F}} \otimes [\mathbf{e}_{(p+N_F/2+1);D} \cdot \mathbf{e}_{(p+N_F/2+1);D}^T] \right\} \cdot \mathbf{A}^T \equiv [\mathbf{O}_{m,q}] \cdot \mathbf{A}^T \quad (2.29)$$

The $(m, q)^{th}$ block of $[\mathbf{O}_{m,q}]$ is the diagonal matrix

$$\mathbf{O}_{m,q} = \text{diag}_{p=-N_F/2}^{N_F/2-1} \left[FT[\tilde{\mathbf{F}}(\tau)]_{m,q} \right]_{\frac{p}{T_F}} \quad (2.30)$$

where diagonal element $(p + N_F/2 + 1)$ represents the Fourier transform of $[\tilde{\mathbf{F}}(\tau)]_{m,q}$ at the frequency p/T_F , for $p = -N_F/2, \dots, N_F/2 + 1$. The matrix $\mathbf{A}^T$ determines (as in (2.20)) the space-time modulation format. For systems with single transmit and receive antennas ($N_R = N_T = 1$ with $N_G = 1$ and $D_G = D$), $\tilde{\mathbf{F}}(\tau)$ reduces to a scalar $\tilde{f}(\tau)$ and (2.29) leads to a diagonal channel matrix $\mathbf{H}_O = \mathbf{O}_{1,1} \cdot \mathbf{A}_1^T \equiv \text{diag}_{p=-N_F/2}^{N_F/2-1} [FT[\tilde{f}(\tau)]_{\frac{p}{T_F}}]$ since $\mathbf{A}_1^T = \mathbf{I}_D$. Furthermore, since OFDM does not resolve the multipath the MIMO-OFDM system (2.28) collapses into a system with one cluster and with impulse response matrix $\tilde{\mathbf{F}}(\tau)$. Let us now compare between the MIMO-OFDM system (2.28) and the MIMO system (2.19) with one time resolvable cluster. For $L_t = 1$ the channel propagation matrix becomes $\mathbf{C} = \mathbf{F}_1$ with a channel impulse response matrix $\tilde{\mathbf{F}}_1(\tau) = \delta(\tau - \tau_1)\mathbf{F}_1$, and (2.20) reduces to $\mathbf{H} = [\mathbf{F}_1 \otimes \mathbf{I}_D] \cdot \mathbf{A}^T$. The structure of this matrix is comparable to that of $\mathbf{H}_O$ of (2.29) in the sense that the $(m, q)^{th}$ block of $[\mathbf{F}_1 \otimes \mathbf{I}_D]$, given by $[\mathbf{F}_1]_{m,q} \mathbf{I}_D$, is a diagonal matrix similarly to $\mathbf{O}_{m,q}$. The matrix $\mathbf{O}_{m,q}$ can be seen as a generalization of $[\mathbf{F}_1]_{m,q} \mathbf{I}_D$ since its diagonal elements are not necessarily equal. Moreover, the matrix $[\mathbf{F}_1]_{m,q} \mathbf{I}_D$ is associated with the impulse response matrix $[\tilde{\mathbf{F}}_1(\tau)]_{m,q} \mathbf{I}_D$ in the time domain. By taking the Fourier transform of diagonal element $(p + N_F/2 + 1)$ of this matrix at frequency $\frac{p}{T_F}$ for $p = -N_F/2, \dots, N_F/2 + 1$, we obtain a matrix similar to $\mathbf{O}_{m,q}$ of (2.30). Hence, we can say that the channel matrix $\mathbf{H}_O$ of a MIMO-OFDM system is the Fourier transform of the channel matrix $\mathbf{H}$ of the MIMO system (2.19) with one time resolvable cluster.

2.3 Broadcast systems with multi-dimensional space-time modulation

In this section, we consider a broadcast channel (BC) where a single transmitter, equipped with N_T transmit antennas, sends independent information to K non-cooperative receivers with multiple antennas. Throughout this thesis, we refer to this channel as MIMO BC since its input and outputs are vector valued. Using the same notations as for the single-user system, we use the superscript (k) to refer to user k in the broadcast system, $k = 1, \dots, K$. We employ for user k the 3GPP spatial channel model from Section 2.1 with $L_t^{(k)}$ time resolvable clusters, each consisting of $S_l^{(k)}$ space resolvable paths which have a propagation delay equal to $\tau_l^{(k)}$, $l = 1, \dots, L_t^{(k)}$. We assume that user k employs $D^{(k)}$ orthonormal real basis functions $(\phi^{(k)}(t) = \text{vec}(\{\phi_d^{(k)}(t)\}_{d=1}^{D^{(k)}}))$ as signal dimensions. Following Section 2.2, a multi-dimensional space-time modulation format describing the allocation of these $D^{(k)}$ signal dimensions to the N_T transmit antennas can be defined with $N_G^{(k)}$ TOGs, each associated with $D_G^{(k)} = D^{(k)}/N_G^{(k)}$ signal dimensions. The space-time modulation format of user k is determined by the matrices $\{\mathbf{A}_i^{(k)}\}_{i=1}^{N_T}$, and the number of antenna elements in his j^{th} TOG is denoted by $n_j^{(k)}$, $j = 1, \dots, N_G^{(k)}$. The multiplexed waveform transmitted on antenna i can be written as $\tilde{s}_i(t) = \sum_{k=1}^K s_i^{(k)}(t)$ where $s_i^{(k)}(t)$ denotes the transmitted waveform on antenna i corresponding to user k . From (2.13), we have $s_i^{(k)}(t) = \sum_{n'=-\infty}^{\infty} [\mathbf{y}_i^{(k)}(n')]^T \cdot \phi^{(k)}(t - n'T_s)$. Figure 2.8 illustrates the transmitted signals associated with user k at the base station. In general, the users can have different space-time modulation formats as well as different spatial channel parameters.

The broadcast system employs a total of D_T signal dimensions that correspond to the orthonormal time functions that span the space of the transmitted waveforms for all users. We divide the D_T dimensions into $K + 1$ subsets as follows. For $k = 1, \dots, K$, subset k with cardinality $D_p^{(k)}$ contains the proper dimensions assigned to user k only. Subset $K + 1$ with cardinality D_c contains the common (or shared) dimensions that are shared by all users. Thus, we have $D^{(k)} = D_p^{(k)} + D_c$ for user k ($k = 1, \dots, K$) and $D_T = \sum_{k=1}^K D_p^{(k)} + D_c$. There are other possibilities for signal dimensions allocations, however, we consider the simplest configuration where all shared signal dimensions are the same for all users. The MIMO-OFDM broadcast systems considered in [97] are examples of ATA with $D_p^{(k)} = 0$ and $D_c = D_T = N_F$ where all the N_F subcarriers are employed on all transmit antennas and used by all users. At the end of this section, we derive a more general model for MIMO-OFDM BCs with space-time modulation and signal dimensions sharing. Figure 2.9 presents an example of signal dimensions allocations in a two-user MIMO BC ($K = 2$).

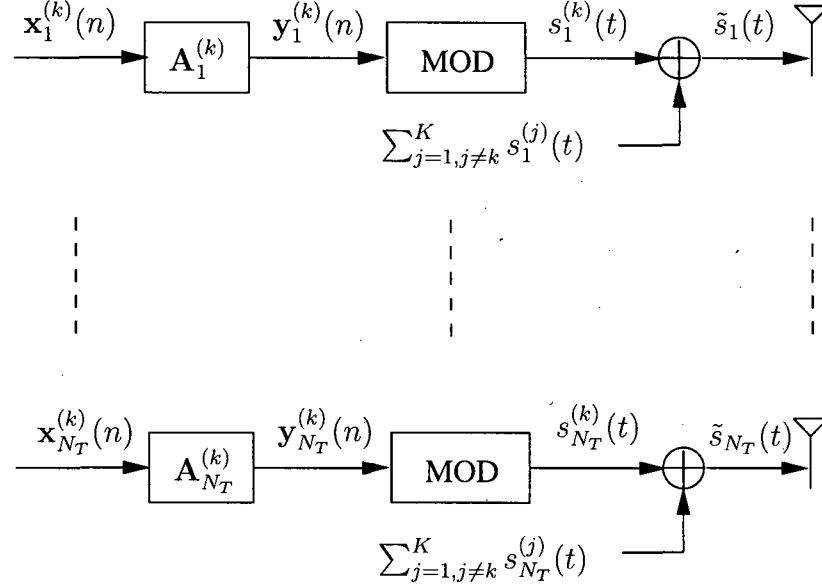


Fig. 2.8 Transmitted signals associated with user k in the broadcast system model.

with $N_T = 4$.

Define the vector of multiplexed waveforms $\tilde{\mathbf{s}}(t) = \text{vec}(\{\tilde{s}_i(t)\}_{i=1}^{N_T})$. As in (2.12), the continuous time channel of user k is described by

$$\mathbf{z}^{(k)}(t) = \sum_{l=1}^{L_t^{(k)}} \mathbf{F}_l^{(k)} \cdot \tilde{\mathbf{s}}(t - \tau_l^{(k)}) + \mathbf{i}^{(k)}(t) \stackrel{(a)}{=} \sum_{l=1}^{L_t^{(k)}} \sum_{p=1}^K \mathbf{F}_l^{(k)} \cdot \mathbf{s}^{(p)}(t - \tau_l^{(k)}) + \mathbf{i}^{(k)}(t) \quad (2.31)$$

where (a) is due to $\tilde{\mathbf{s}}(t) = \sum_{p=1}^K \mathbf{s}^{(p)}(t)$, $\mathbf{s}^{(p)}(t) = \text{vec}(\{s_i^{(p)}(t)\}_{i=1}^{N_T})$ being the vector of waveforms corresponding to user p and transmitted on the N_T antennas. As in the single-user system model, we assume perfect multipath time resolvability and that every receiver is in perfect synchronism with the transmitter. We can write for user k

$$\int_{-\infty}^{\infty} \phi_d^{(k)}(t - \tau_l^{(k)} - nT_s) \phi_{d'}^{(k)}(t - \tau_{l'}^{(k)} - n'T_s) dt = \delta(l, l') \delta(d, d') \delta(n, n') \quad (2.32)$$

a)

4	X				O	O
3		X			O	O
2			O	O	X	
1			O	O		X
	1	2	3	4	5	6

Dimensions

b)

4	X		O	O		
3		X	O	O		
2					~~X~~	O
1					O	~~X~~
	1	2	3	4	5	6

Dimensions

Fig. 2.9 Examples of signal dimensions allocations in a two-user MIMO BC: We consider $D_T = 6$ such that dimensions 1, 2 are proper to user 1 ($D_p^{(1)} = 2$), dimensions 3, 4 are proper to user 2 ($D_p^{(2)} = 2$) and dimensions 5, 6 are shared ($D_c = 2$). We use X and O to denote the dimensions assigned to user 1 and to user 2, respectively. The superposition of the notations X and O is denoted by $\otimes$. User 1 employs OTA while user 2 employs POTA with different configurations in a) and b).

and, for $k \neq k'$,

$$\int_{-\infty}^{\infty} \phi_d^{(k)}(t - \tau_l^{(k)} - nT_s) \phi_{d'}^{(k')}(t - \tau_{l'}^{(k')} - n'T_s) dt = \begin{cases} \delta(l, l')\delta(n, n')\delta(d, d'), & \text{if } \phi_d^{(k)}(t) \text{ is shared with } \phi_{d'}^{(k')}(t) = \phi_v^{(k')}(t) \\ 0, & \text{if } \phi_d^{(k)}(t) \text{ is proper to user } k \end{cases} \quad (2.33)$$

for all $l, l' \in \{1, \dots, L_t^{(k)}\}$, all n, n' and all $d \in \{1, \dots, D^{(k)}\}$ and $v, d' \in \{1, \dots, D^{(k')}\}$. Next, we derive the discrete time channel model using (2.32) and (2.33). Note that since (2.32) and (2.33) do not hold for MIMO-OFDM systems, we use another method to derive the BC model for these systems, as it will be explained at the end of this section.

Assuming that user k has $N_R^{(k)}$ receive antennas, its receiver has the same structure as that depicted in Figure 2.2 for single-user systems. Hence, the $D^{(k)} \times 1$ discrete time channel output vector $\mathbf{r}_{m;u}^{(k)}(n)$ and the noise vector $\mathbf{v}_{m;u}^{(k)}(n)$ associated with the delay $\tau_u^{(k)}$ and received on antenna m of user k are, respectively, defined as $[\mathbf{r}_{m;u}^{(k)}(n)]_d = \int_{-\infty}^{\infty} z_m^{(k)}(t) \phi_d^{(k)}(t - \tau_u^{(k)} - nT_s) dt$ and $[\mathbf{v}_{m;u}^{(k)}(n)]_d = \int_{-\infty}^{\infty} i_m^{(k)}(t) \phi_d^{(k)}(t - \tau_u^{(k)} - nT_s) dt$, $d = 1, \dots, D^{(k)}$. Writing

$z_m^{(k)}(t) = \sum_{l=1}^{L_t^{(k)}} \sum_{p=1}^K \sum_{q=1}^{N_T} [\mathbf{F}_l^{(k)}]_{m,q} s_q^{(p)}(t - \tau_l^{(k)}) + i_m^{(k)}(t)$ from (2.31), we have

$$[\mathbf{r}_{m;u}^{(k)}(n)]_d = [\tilde{\mathbf{r}}_{m;u}^{(k)}(n)]_d + [\check{\mathbf{r}}_{m;u}^{(k)}(n)]_d + [\mathbf{v}_{m;u}^{(k)}(n)]_d \quad (2.34)$$

where $\tilde{\mathbf{r}}_{m;u}^{(k)}(n)$ represents the $D^{(k)} \times 1$ desired output vector of user k and $\check{\mathbf{r}}_{m;u}^{(k)}(n)$ represents the $D^{(k)} \times 1$ interference from the other users, with

$$[\tilde{\mathbf{r}}_{m;u}^{(k)}(n)]_d = \sum_{l=1}^{L_t^{(k)}} \sum_{q=1}^{N_T} \int_{-\infty}^{\infty} [\mathbf{F}_l^{(k)}]_{m,q} s_q^{(k)}(t - \tau_l^{(k)}) \phi_d^{(k)}(t - \tau_u^{(k)} - nT_s) dt \quad (2.35)$$

$$[\check{\mathbf{r}}_{m;u}^{(k)}(n)]_d = \sum_{l=1}^{L_t^{(k)}} \sum_{q=1}^{N_T} \sum_{p=1, p \neq k}^K \int_{-\infty}^{\infty} [\mathbf{F}_l^{(k)}]_{m,q} s_q^{(p)}(t - \tau_l^{(k)}) \phi_d^{(k)}(t - \tau_u^{(k)} - nT_s) dt \quad (2.36)$$

Writing $s_q^{(k)}(t - \tau_l^{(k)}) = \sum_{n'=-\infty}^{\infty} \sum_{d'=1}^{D^{(k)}} [\mathbf{y}_q^{(k)}(n')]_{d'} \phi_{d'}^{(k)}(t - \tau_l^{(k)} - n'T_s)$ from (2.13) and using (2.32), we obtain $[\tilde{\mathbf{r}}_{m;u}^{(k)}(n)]_d = \sum_{q=1}^{N_T} [\mathbf{F}_u^{(k)}]_{m,q} [\mathbf{y}_q^{(k)}(n)]_d$, thus, leading to

$$\tilde{\mathbf{r}}_{m;u}^{(k)}(n) = \sum_{q=1}^{N_T} ([\mathbf{F}_u^{(k)}]_{m,q} \otimes \mathbf{I}_{D^{(k)}}) \cdot \mathbf{y}_q^{(k)}(n) \quad (2.37)$$

For $p \neq k$, let $\mathbf{B}^{(k,p)}$ denote the $D^{(k)} \times D^{(p)}$ matrix indicating the shared dimensions between users k and p in the system. Hence, $\mathbf{B}^{(k,p)}$ is defined as $[\mathbf{B}^{(k,p)}]_{d,*} = \mathbf{0}_{D^{(p)}}^T$ (i.e. the d^{th} row of $\mathbf{B}^{(k,p)}$ is the zero row vector) if $\phi_d^{(k)}(t)$ is not shared, and $[\mathbf{B}^{(k,p)}]_{d,*} = \mathbf{e}_{v;D^{(p)}}^T$ if $\phi_d^{(k)}(t)$ is shared and $\phi_d^{(k)}(t) = \phi_v^{(p)}(t)$ for $v \in \{1, \dots, D^{(p)}\}$. Since there are D_c shared dimensions, $\mathbf{B}^{(k,p)}$ has D_c rows of the form $\mathbf{e}_{v;D^{(p)}}^T$ (each with a different v) and $D^{(k)} - D_c \equiv D_p^{(k)}$ zero rows. An important issue that needs to be considered here is the way the integers $1, \dots, D_T$ are associated with the basis functions $\{\{\phi_d^{(k)}(t)\}_{d=1}^{D^{(k)}}\}_{k=1}^K$. For simplicity, we assign the increasingly ordered elements of the set $\{1, \dots, D_T\}$ that are associated with user k to the basis functions $\phi_1^{(k)}(t), \dots, \phi_{D^{(k)}}^{(k)}(t)$, respectively. For example, consider a two-user MIMO BC with $D_T = 6$ and $D_p^{(1)} = D_p^{(2)} = D_c = 2$ as in Figure 2.9. Here, the elements 1, 2, 5, 6 of the set $\{1, \dots, 6\}$ are associated with user 1 and correspond to $\phi_1^{(1)}(t), \phi_2^{(1)}(t), \phi_3^{(1)}(t), \phi_4^{(1)}(t)$, respectively. The elements 3, 4, 5, 6 of $\{1, \dots, 6\}$ are associated with user 2 and correspond to $\phi_1^{(2)}(t), \phi_2^{(2)}(t), \phi_3^{(2)}(t), \phi_4^{(2)}(t)$, respectively. In this

case, we have $\phi_3^{(1)}(t) = \phi_3^{(2)}(t)$ and $\phi_4^{(1)}(t) = \phi_4^{(2)}(t)$ with

$$\mathbf{B}^{(1,2)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and $\mathbf{B}^{(2,1)} = [\mathbf{B}^{(1,2)}]^T$. Now, for every $q \in \{1, \dots, N_T\}$, we define the $D^{(k)} \times 1$ vector $\check{\mathbf{y}}_q^{(k,p)}(n) = \mathbf{B}^{(k,p)} \cdot \mathbf{y}_q^{(p)}(n)$. As such, we have $[\check{\mathbf{y}}_q^{(k,p)}(n)]_d = [\mathbf{B}^{(k,p)}]_{d,*} \cdot \mathbf{y}_q^{(p)}(n)$, which leads to $[\check{\mathbf{y}}_q^{(k,p)}(n)]_d = \mathbf{0}_{D^{(p)}}^T \cdot \mathbf{y}_q^{(p)}(n) = 0$ if $\phi_d^{(k)}(t)$ is not shared, and $[\check{\mathbf{y}}_q^{(k,p)}(n)]_d = \mathbf{e}_{v;D^{(p)}}^T \cdot \mathbf{y}_q^{(p)}(n) = [\mathbf{y}_q^{(p)}(n)]_v$ if $\phi_d^{(k)}(t)$ is shared and $\phi_d^{(k)}(t) = \phi_v^{(p)}(t)$ for $v \in \{1, \dots, D^{(p)}\}$. Writing $s_q^{(p)}(t - \tau_l^{(k)}) = \sum_{n'=-\infty}^{\infty} \sum_{d'=1}^{D^{(p)}} [\mathbf{y}_q^{(p)}(n')]_{d'} \phi_{d'}^{(p)}(t - \tau_l^{(k)} - n'T_s)$ from (2.13) and using (2.33), we can show that (2.36) is equivalent to

$$[\check{\mathbf{r}}_{m;u}^{(k)}(n)]_d = \sum_{p=1, p \neq k}^K \sum_{q=1}^{N_T} [\mathbf{F}_u^{(k)}]_{m,q} [\check{\mathbf{y}}_q^{(k,p)}(n)]_d$$

which yields

$$\check{\mathbf{r}}_{m;u}^{(k)}(n) = \sum_{p=1, p \neq k}^K \sum_{q=1}^{N_T} ([\mathbf{F}_u^{(k)}]_{m,q} \otimes \mathbf{I}_{D^{(k)}}) \cdot \check{\mathbf{y}}_q^{(k,p)}(n) \quad (2.38)$$

Define $\check{\mathbf{y}}^{(k,p)}(n) = \text{vec}(\{\check{\mathbf{y}}_q^{(k,p)}(n)\}_{q=1}^{N_T})$. Thus, we have

$$\check{\mathbf{y}}^{(k,p)}(n) = [\mathbf{I}_{N_T} \otimes \mathbf{B}^{(k,p)}] \cdot \mathbf{y}^{(p)}(n) \quad (2.39)$$

Using equations (2.34), (2.37), (2.38) and (2.39), we have

$$\begin{aligned} \text{vec}(\{\mathbf{r}_{m;u}^{(k)}(n)\}_{m=1}^{N_R^{(k)}}) &= [\mathbf{F}_u^{(k)} \otimes \mathbf{I}_{D^{(k)}}] \cdot \mathbf{y}^{(k)}(n) + \sum_{p=1, p \neq k}^K [\mathbf{F}_u^{(k)} \otimes \mathbf{I}_{D^{(k)}}] \cdot [\mathbf{I}_{N_T} \otimes \mathbf{B}^{(k,p)}] \cdot \mathbf{y}^{(p)}(n) \\ &\quad + \text{vec}(\{\mathbf{v}_{m;u}^{(k)}(n)\}_{m=1}^{N_R^{(k)}}) \end{aligned}$$

Hence, the discrete time channel for user k is described by

$$\mathbf{r}^{(k)}(n) = \mathbf{H}^{(k)} \cdot \mathbf{x}^{(k)}(n) + \sum_{p=1, p \neq k}^K \mathbf{H}^{(k,p)} \cdot \mathbf{x}^{(p)}(n) + \mathbf{v}^{(k)}(n), \quad (2.40)$$

where the $L_t^{(k)} N_R^{(k)} D^{(k)} \times \tilde{D}^{(k)}$ matrix $\mathbf{H}^{(k)}$ is given by

$$\mathbf{H}^{(k)} = [\mathbf{C}^{(k)} \otimes \mathbf{I}_{D^{(k)}}] \cdot [\mathbf{A}^{(k)}]^T \quad (2.41)$$

similarly to the single-user channel matrix (2.20), and where the $L_t^{(k)} N_R^{(k)} D^{(k)} \times \tilde{D}^{(p)}$ matrix $\mathbf{H}^{(k,p)}$ is given by

$$\mathbf{H}^{(k,p)} = [\mathbf{C}^{(k)} \otimes \mathbf{I}_{D^{(k)}}] \cdot [\mathbf{I}_{N_T} \otimes \mathbf{B}^{(k,p)}] \cdot [\mathbf{A}^{(p)}]^T \quad (2.42)$$

We now derive the discrete time model for MIMO-OFDM BCs with space-time modulation and signal dimensions sharing. We assume a total of $D_T = N_{F,T}$ orthogonal subcarriers indexed by the integers $-N_{F,T}/2, \dots, N_{F,T}/2 - 1$ with frequency spacing $1/T_F$ between adjacent subcarriers. Let $D^{(k)}$ denote the number of orthogonal subcarriers that are employed by user k , for $k = 1, \dots, K$. For every $p \in \{1, \dots, D^{(k)}\}$, we define the mapping $i^{(k)}(p)$ such that $i^{(k)}(p) \in \{-N_{F,T}/2, \dots, N_{F,T}/2 - 1\}$ denotes the subcarrier index associated with integer p of user k , with $i^{(k)}(1) < \dots < i^{(k)}(D^{(k)})$. The transmitted signals associated with user k can be represented as in Figure 2.8 with the only exception that the modulator for each transmit antenna is replaced by an IFT (similar to that of Figure 2.6a with the subcarriers indexes $-N_F/2, \dots, N_F/2 - 1$ replaced by $i^{(k)}(1), \dots, i^{(k)}(D^{(k)})$, respectively) followed by a cyclic prefix insertion. User k 's receiver has a similar structure to that of Figure 2.5b with the only exception that the FT used for each receive antenna is obtained from Figure 2.6b by replacing the subcarriers indexes $-N_F/2, \dots, N_F/2 - 1$ by $i^{(k)}(1), \dots, i^{(k)}(D^{(k)})$, respectively. Hence, the $D^{(k)} \times 1$ output vector $\mathbf{r}_{O;m}^{(k)}(n)$ and noise vector $\mathbf{v}_{O;m}^{(k)}(n)$ received on antenna m of user k are, respectively, defined as $[\mathbf{r}_{O;m}^{(k)}(n)]_p = \frac{1}{\sqrt{T_F}} \int_{nT_s}^{nT_s+T_F} z_m^{(k)}(t) e^{-j2\pi \frac{i^{(k)}(p)}{T_F}(t-nT_s)} dt$ and $[\mathbf{v}_{O;m}^{(k)}(n)]_p = \frac{1}{\sqrt{T_F}} \int_{nT_s}^{nT_s+T_F} i_m^{(k)}(t) e^{-j2\pi \frac{i^{(k)}(p)}{T_F}(t-nT_s)} dt$, with $p = 1, \dots, D^{(k)}$. Now writing $z_m^{(k)}(t) = \sum_{l=1}^{L_t^{(k)}} \sum_{g=1}^K \sum_{q=1}^{N_T} [\mathbf{F}_l^{(k)}]_{m,q} s_q^{(g)}(t - \tau_l^{(k)}) + i_m^{(k)}(t)$ from (2.31), we have

$$[\mathbf{r}_{O;m}^{(k)}(n)]_p = [\tilde{\mathbf{r}}_{O;m}^{(k)}(n)]_p + [\tilde{\mathbf{r}}_{O;m}^{(k)}(n)]_p + [\mathbf{v}_{O;m}^{(k)}(n)]_p \quad (2.43)$$

where $\tilde{\mathbf{r}}_{O;m}^{(k)}(n)$ represents the $D^{(k)} \times 1$ desired output vector of user k and $\tilde{\mathbf{r}}_{O;m}^{(k)}(n)$ represents

the $D^{(k)} \times 1$ interference from the other users, with

$$[\tilde{\mathbf{r}}_{O;m}^{(k)}(n)]_p = \frac{1}{\sqrt{T_F}} \sum_{l=1}^{L_t^{(k)}} \sum_{q=1}^{N_T} \int_{nT_s}^{nT_s+T_F} [\mathbf{F}_l^{(k)}]_{m,q} s_q^{(k)}(t - \tau_l^{(k)}) e^{-j2\pi \frac{i^{(k)}(p)}{T_F}(t-nT_s)} dt \quad (2.44)$$

$$[\tilde{\mathbf{r}}_{O;m}^{(k)}(n)]_p = \frac{1}{\sqrt{T_F}} \sum_{l=1}^{L_t^{(k)}} \sum_{q=1}^{N_T} \sum_{g=1, g \neq k}^K \int_{nT_s}^{nT_s+T_F} [\mathbf{F}_l^{(k)}]_{m,q} s_q^{(g)}(t - \tau_l^{(k)}) e^{-j2\pi \frac{i^{(k)}(p)}{T_F}(t-nT_s)} dt \quad (2.45)$$

Assuming that the CP duration satisfies $T_{CP} > \max_{l,u=1,\dots,L_t^{(k)}} |\tau_l^{(k)} - \tau_u^{(k)}|$ for each $k = 1, \dots, K$, ISI is suppressed for each user. Hence, the integrals in (2.44) and (2.45) can be evaluated by replacing $s_q^{(g)}(t - \tau_l^{(k)})$ by $s_{q;n}^{(g)}(t - \tau_l^{(k)} - nT_s)$, $\forall g$, where $s_{q;n}^{(g)}(t - \tau_l^{(k)} - nT_s) = \sum_{d=1}^{D^{(g)}} [\mathbf{y}_q^{(g)}(n)]_d \frac{1}{\sqrt{T_F}} e^{j2\pi \frac{i^{(g)}(d)}{T_F}(t-\tau_l^{(k)}-nT_s)}$ from (2.23). From (2.44), we obtain as in (2.26) that $[\tilde{\mathbf{r}}_{O;m}^{(k)}(n)]_p = \sum_{q=1}^{N_T} [\mathbf{H}_{O;i^{(k)}(p)}^{(k)}]_{m,q} [\mathbf{y}_q^{(k)}(n)]_p$ where $[\mathbf{H}_{O;i^{(k)}(p)}^{(k)}]_{m,q} = \sum_{l=1}^{L_t^{(k)}} [\mathbf{F}_l^{(k)}]_{m,q} e^{-j2\pi \frac{i^{(k)}(p)}{T_F} \tau_l^{(k)}} = FT \left[\sum_{l=1}^{L_t^{(k)}} [\mathbf{F}_l^{(k)}]_{m,q} \delta(\tau - \tau_l^{(k)}) \right]_{\frac{i^{(k)}(p)}{T_F}}$. Thus, we have as in (2.27)

$$\tilde{\mathbf{r}}_{O;m}^{(k)}(n) = [\mathbf{O}_{m,1}^{(k)} \mathbf{O}_{m,2}^{(k)} \dots \mathbf{O}_{m,N_T}^{(k)}] \cdot \mathbf{y}^{(k)}(n) \quad (2.46)$$

with the $D^{(k)} \times D^{(k)}$ diagonal matrices $\mathbf{O}_{m,q}^{(k)} = \text{diag}_{p=1}^{D^{(k)}} [[\mathbf{H}_{O;i^{(k)}(p)}^{(k)}]_{m,q}]$, $m = 1, \dots, N_R^{(k)}$ and $q = 1, \dots, N_T$. We also have $[\tilde{\mathbf{r}}_{O;m}^{(k)}(n)]_p = \sum_{g=1, g \neq k}^K \sum_{q=1}^{N_T} [\mathbf{H}_{O;i^{(k)}(p)}^{(k)}]_{m,q} [\tilde{\mathbf{y}}_q^{(k,g)}(n)]_p$ from (2.45) with $[\tilde{\mathbf{y}}_q^{(k,g)}(n)]_p = [\mathbf{B}^{(k,g)}]_{p,*} \cdot \mathbf{y}_q^{(g)}(n)$. Here, $[\mathbf{B}^{(k,g)}]_{p,*} = \mathbf{0}_{D^{(g)}}^T$ if subcarrier $i^{(k)}(p)$ is not shared, and $[\mathbf{B}^{(k,g)}]_{p,*} = \mathbf{e}_{d;D^{(g)}}^T$ if subcarrier $i^{(k)}(p)$ is shared and $i^{(k)}(p) = i^{(g)}(d)$ for $d \in \{1, \dots, D^{(g)}\}$. This leads to

$$\tilde{\mathbf{r}}_{O;m}^{(k)}(n) = \sum_{g=1, g \neq k}^K [\mathbf{O}_{m,1}^{(k)} \mathbf{O}_{m,2}^{(k)} \dots \mathbf{O}_{m,N_T}^{(k)}] \cdot \tilde{\mathbf{y}}^{(k,g)}(n) \quad (2.47)$$

We define the $N_R^{(k)} D^{(k)} \times 1$ output vector $\mathbf{r}_O^{(k)}(n) = \text{vec}(\{\mathbf{r}_{O;m}^{(k)}(n)\}_{m=1}^{N_R^{(k)}})$ and $N_R^{(k)} D^{(k)} \times 1$ noise vector $\mathbf{v}_O^{(k)}(n) = \text{vec}(\{\mathbf{v}_{O;m}^{(k)}(n)\}_{m=1}^{N_R^{(k)}})$ for user k . Using (2.43), (2.46), (2.47) and (2.39) with $\mathbf{y}^{(g)}(n) = [\mathbf{A}^{(g)}]^T \cdot \mathbf{x}^{(g)}(n)$, we obtain

$$\mathbf{r}_O^{(k)}(n) = \mathbf{H}_O^{(k)} \cdot \mathbf{x}^{(k)}(n) + \sum_{g=1, g \neq k}^K \mathbf{H}_O^{(k,g)} \cdot \mathbf{x}^{(g)}(n) + \mathbf{v}_O^{(k)}(n) \quad (2.48)$$

where the $N_R^{(k)} D^{(k)} \times \tilde{D}^{(k)}$ matrix $\mathbf{H}_O^{(k)}$ is given by

$$\mathbf{H}_O^{(k)} = [\mathbf{O}_{m,q}^{(k)}] \cdot [\mathbf{A}^{(k)}]^T \quad (2.49)$$

similarly to the single-user channel matrix (2.29), and where the $N_R^{(k)} D^{(k)} \times \tilde{D}^{(g)}$ matrix $\mathbf{H}_O^{(k,g)}$ is given by

$$\mathbf{H}_O^{(k,g)} = [\mathbf{O}_{m,q}^{(k)}] \cdot [\mathbf{I}_{N_T} \otimes \mathbf{B}^{(k,g)}] \cdot [\mathbf{A}^{(g)}]^T \quad (2.50)$$

with $[\mathbf{O}_{m,q}^{(k)}] = \sum_{p=1}^{D^{(k)}} FT[\tilde{\mathbf{F}}^{(k)}(\tau)]_{\frac{i^{(k)}(p)}{T_F}} \otimes [\mathbf{e}_{p,D^{(k)}} \cdot \mathbf{e}_{p,D^{(k)}}^T]$. We can write

$$\mathbf{O}_{m,q}^{(k)} = \text{diag}_{p=1}^{D^{(k)}} \left[FT[\tilde{\mathbf{F}}^{(k)}(\tau)]_{m,q} \right]_{\frac{i^{(k)}(p)}{T_F}} \quad (2.51)$$

where the matrix $\tilde{\mathbf{F}}^{(k)}(\tau) = \sum_{l=1}^{L_t^{(k)}} \delta(\tau - \tau_l^{(k)}) \mathbf{F}_l^{(k)}$ denotes the channel impulse response matrix for user k . As for the single-user case, the matrices (2.49)-(2.50) characterizing the MIMO-OFDM BC (2.48) (for which the multipath is not resolvable in time) have a similar structure as the matrices (2.41)-(2.42) that characterize the MIMO BC (2.40) with one cluster for each user. With $L_t^{(k)} = 1$, we have $\mathbf{C}^{(k)} = \mathbf{F}_1^{(k)}$ in (2.41)-(2.42), and the impulse response matrix in the time domain associated with the $(m, q)^{th}$ block of the matrix $[\mathbf{F}_1^{(k)} \otimes \mathbf{I}_{D^{(k)}}]$ is given by the diagonal matrix $[\tilde{\mathbf{F}}_1^{(k)}(\tau)]_{m,q} \mathbf{I}_{D^{(k)}} = \delta(\tau - \tau_1^{(k)}) [\mathbf{F}_1^{(k)}]_{m,q} \mathbf{I}_{D^{(k)}}$. By taking the Fourier transform of the p^{th} diagonal element of this matrix at frequency $\frac{i^{(k)}(p)}{T_F}$ we obtain a matrix similar to $\mathbf{O}_{m,q}^{(k)}$ of (2.51), for $m = 1, \dots, N_R^{(k)}$ and $q = 1, \dots, N_T$.

Assuming a memoryless channel, we can drop the dependencies on the time index n for the remaining chapters of this thesis. Information rates for the single-user MIMO system of Section 2.2 and rate regions for the MIMO BC of Section 2.3 are considered next.

Chapter 3

Capacity and information rates for single-user MIMO channels

We consider the discrete time MIMO channel described by (2.19). Throughout this chapter we assume perfect CSIR. That is, the receiver has perfect knowledge of the channel propagation matrix (i.e. $\mathbf{C}$) and the multi-dimensional space-time modulation format (i.e. $\mathbf{A}$) that is employed at the transmitter. Depending on CSIT, different transmit power allocation criteria and hence, different information rates are considered in the next sections.

3.1 Capacity with full CSIT

Let $\Sigma = E[\mathbf{x} \cdot \mathbf{x}^\dagger]$ denote the input covariance matrix with power constraint $Tr(\Sigma) \leq P_T$ (or $E[\|\mathbf{x}\|^2] \leq P_T$). The input/output mutual information, maximized by a Gaussian input distribution, is given by [3]

$$I(\mathbf{x}; \mathbf{r}) = \log \left(\left| \mathbf{I}_{L_t N_R D} + \frac{1}{N_0} \mathbf{H} \cdot \Sigma \cdot \mathbf{H}^\dagger \right| \right) \text{ bits/channel-use} \quad (3.1)$$

From [3,51], Shannon capacity is given by $\max_{Tr(\Sigma) \leq P_T} I(\mathbf{x}; \mathbf{r})$ in bits/channel-use. Equivalently, using the fact that a signal with bandwidth W and duration T can be represented by WT complex signal dimensions and since the signal space is spanned by D complex

dimensions, the channel capacity in bps/Hz (or spectral efficiency) can be written as

$$C = \frac{1}{D} \max_{Tr(\Sigma) \leq P_T} I(\mathbf{x}; \mathbf{r}) \text{ bps/Hz} \quad (3.2)$$

The transmitted energy-per-bit is $(E_b)_T = \frac{P_T}{CD}$ when operating at capacity rate and the transmitted SNR per bit $(E_b/N_0)_T$ are related by

$$\frac{P_T}{N_0 D} = C (E_b/N_0)_T \quad (3.3)$$

Consider the eigenvalue decomposition $\mathbf{H}^\dagger \cdot \mathbf{H} = \mathbf{M}^\dagger \cdot \mathbf{Q} \cdot \mathbf{M}$ where the matrices $\mathbf{Q}$ and $\mathbf{M}$ are $\tilde{D} \times \tilde{D}$ diagonal and unitary, respectively. Define $\tilde{\Sigma} = \mathbf{M} \cdot \Sigma \cdot \mathbf{M}^\dagger$. Using (C.1), we can write (3.1) as

$$I(\mathbf{x}; \mathbf{r}) = \log \left(\left| \mathbf{I}_{\tilde{D}} + \frac{1}{N_0} \tilde{\Sigma} \cdot \mathbf{Q} \right| \right) \quad (3.4)$$

and the capacity (3.2) as

$$C = \frac{1}{D} \max_{Tr(\tilde{\Sigma}) \leq P_T} \log \left(\left| \mathbf{I}_{\tilde{D}} + \frac{1}{N_0} \tilde{\Sigma} \cdot \mathbf{Q} \right| \right) \text{ bps/Hz.} \quad (3.5)$$

From (2.20) and (C.3), we can write $\mathbf{H}^\dagger \cdot \mathbf{H} = \mathbf{A} \cdot [(\mathbf{C}^\dagger \cdot \mathbf{C}) \otimes \mathbf{I}_D] \cdot \mathbf{A}^T$. Using $\mathbf{A} = \text{diag}_{i=1}^{N_T} [\mathbf{e}_{t_i; N_G}^T \otimes \mathbf{I}_{D_G}]$ and (C.2)-(C.3), we have

$$\begin{aligned} \mathbf{H}^\dagger \cdot \mathbf{H} &= \text{diag}_{i=1}^{N_T} [\mathbf{e}_{t_i; N_G}^T \otimes \mathbf{I}_{D_G}] \cdot [(\mathbf{C}^\dagger \cdot \mathbf{C}) \otimes (\mathbf{I}_{N_G} \otimes \mathbf{I}_{D_G})] \cdot \text{diag}_{i=1}^{N_T} [\mathbf{e}_{t_i; N_G} \otimes \mathbf{I}_{D_G}] \\ &= \{ \text{diag}_{i=1}^{N_T} [\mathbf{e}_{t_i; N_G}^T] \cdot [(\mathbf{C}^\dagger \cdot \mathbf{C}) \otimes \mathbf{I}_{N_G}] \cdot \text{diag}_{i=1}^{N_T} [\mathbf{e}_{t_i; N_G}] \} \otimes \mathbf{I}_{D_G} \\ &= \{ \text{diag}_{i=1}^{N_G} [\mathbf{I}_{n_i} \otimes \mathbf{e}_{i; N_G}^T] \cdot [(\mathbf{C}^\dagger \cdot \mathbf{C}) \otimes \mathbf{I}_{N_G}] \cdot \text{diag}_{i=1}^{N_G} [\mathbf{I}_{n_i} \otimes \mathbf{e}_{i; N_G}] \} \otimes \mathbf{I}_{D_G} \end{aligned}$$

which reduces to

$$\mathbf{H}^\dagger \cdot \mathbf{H} = \text{diag}_{i=1}^{N_G} [\mathbf{C}_i^\dagger \cdot \mathbf{C}_i] \otimes \mathbf{I}_{D_G} \quad (3.6)$$

Let r denote the rank of $\mathbf{H}$ (or $\mathbf{H}^\dagger \cdot \mathbf{H}$). One can see from (3.6) that $r = r_T D_G$ with $r_T = \sum_{i=1}^{N_G} r_i$ denoting the rank of $\text{diag}_{i=1}^{N_G} [\mathbf{C}_i^\dagger \cdot \mathbf{C}_i]$ and r_i denoting the rank of $\mathbf{C}_i$. Let the $n_i \times n_i$ matrices $\mathbf{M}_i$ and $\mathbf{Q}_i$ denote the unitary and diagonal matrices associated with the eigenvalue decomposition $\mathbf{C}_i^\dagger \cdot \mathbf{C}_i = \mathbf{M}_i^\dagger \cdot \mathbf{Q}_i \cdot \mathbf{M}_i$. Using (3.6) with identity (C.3), one can easily write $\mathbf{Q} = \text{diag}_{i=1}^{N_G} [\mathbf{Q}_i \otimes \mathbf{I}_{D_G}]$ and $\mathbf{M} = \text{diag}_{i=1}^{N_G} [\mathbf{M}_i \otimes \mathbf{I}_{D_G}]$. Hence, the MIMO channel (2.19) can be equivalently seen as a set of N_G parallel subchannels, each

consisting of a TOG and the receiver. The input signal $\mathbf{x}$ can be represented as N_G parallel input streams $\{\mathbf{x}'_i\}_{i=1}^{N_G}$ (since $\mathbf{x} = \text{vec}(\{\mathbf{x}'_i\}_{i=1}^{N_G})$), where $\mathbf{x}'_i = \text{vec}(\{\mathbf{x}_{i,l}\}_{l=1}^{n_i})$ is the $n_i D_G \times 1$ input signal to TOG i and $\mathbf{x}_{i,l}$ denotes the $D_G \times 1$ vector transmitted on antenna l of TOG i . Hence, (3.5) is equivalent to

$$C = \frac{1}{D} \max_{\text{Tr}(\tilde{\Sigma}) \leq P_T} \log \left(\left| \mathbf{I}_{\tilde{D}} + \frac{1}{N_0} \tilde{\Sigma} \cdot \text{diag}_{i=1}^{N_G} [\mathbf{Q}_i \otimes \mathbf{I}_{D_G}] \right| \right) \text{ bps/Hz} \quad (3.7)$$

The term $\log(|\mathbf{I}_{\tilde{D}} + \frac{1}{N_0} \tilde{\Sigma} \cdot \text{diag}_{i=1}^{N_G} [\mathbf{Q}_i \otimes \mathbf{I}_{D_G}]|)$ denotes the input/output mutual information of $\sum_{i=1}^{N_G} n_i D_G$ independent parallel scalar Gaussian channels with a common power constraint. The optimization problem in (3.7) can be solved using the standard water-filling procedure of [51, Section 10.4], and the resulting water level μ is the same for all these parallel channels. The inputs achieving capacity are independent zero mean circularly symmetric Gaussian with covariance matrix $\tilde{\Sigma} = \text{diag}_{i=1}^{N_G} [\tilde{\Sigma}_i]$ such that $\frac{1}{N_0} \tilde{\Sigma}_i = \text{diag}_{p=1}^{n_i} [\{\mu - \lambda_{i,p}^{-1}\}^+] \otimes \mathbf{I}_{D_G}$, and $\{\lambda_{i,p}\}_{p=1}^{n_i}$ denote the (real and non-negative) eigenvalues of $\mathbf{C}_i^\dagger \cdot \mathbf{C}_i$ (i.e. $[\mathbf{Q}_i]_{p,p} = \lambda_{i,p}$) in decreasing order $\lambda_{i,1} \geq \dots \geq \lambda_{i,n_i} \geq 0$. Hence, using $\Sigma = \mathbf{M}^\dagger \cdot \tilde{\Sigma} \cdot \mathbf{M}$, the covariance matrix of the input vector $\mathbf{x}$ to the original channel (2.19) becomes $\Sigma = \text{diag}_{i=1}^{N_G} [\Sigma_i]$ where Σ_i denotes the covariance matrix of $\mathbf{x}'_i$ satisfying $\Sigma_i = (\mathbf{M}_i \otimes \mathbf{I}_{D_G})^\dagger \cdot \tilde{\Sigma}_i \cdot (\mathbf{M}_i \otimes \mathbf{I}_{D_G})$. Equivalently, we can write

$$\begin{aligned} \Sigma_i &= (\mathbf{M}_i \otimes \mathbf{I}_{D_G})^\dagger \cdot (N_0 \text{diag}_{p=1}^{n_i} [\{\mu - \lambda_{i,p}^{-1}\}^+] \otimes \mathbf{I}_{D_G}) \cdot (\mathbf{M}_i \otimes \mathbf{I}_{D_G}) \\ &\stackrel{(a)}{=} [\mathbf{M}_i^\dagger \cdot (N_0 \text{diag}_{p=1}^{n_i} [\{\mu - \lambda_{i,p}^{-1}\}^+]) \cdot \mathbf{M}_i] \otimes \mathbf{I}_{D_G} \end{aligned} \quad (3.8)$$

where (a) is due to (C.3) and (C.14). Using the water-filling matrix $\tilde{\Sigma}$ in (3.7), Shannon capacity can be written as follows

$$\begin{aligned} C &= \frac{1}{D} \sum_{i=1}^{N_G} \sum_{l=1}^{n_i D_G} \log (1 + [\text{diag}_{p=1}^{n_i} [\{\mu - \lambda_{i,p}^{-1}\}^+] \otimes \mathbf{I}_{D_G}]_{l,l} [\mathbf{Q}_i \otimes \mathbf{I}_{D_G}]_{l,l}) \\ &= \frac{1}{D} \sum_{i=1}^{N_G} [D_G \sum_{p=1}^{n_i} \log (1 + \{\mu - \lambda_{i,p}^{-1}\}^+ [\mathbf{Q}_i]_{p,p})] \\ &\stackrel{(a)}{=} \frac{1}{N_G} \sum_{i=1}^{N_G} \sum_{p=1}^{n_i} \{\log (\mu \lambda_{i,p})\}^+ \text{ bps/Hz} \end{aligned} \quad (3.9)$$

with $\sum_{i=1}^{N_G} \sum_{p=1}^{n_i} \{\mu - \lambda_{i,p}^{-1}\}^+ = \frac{N_G P_T}{N_0 D}$, and (a) due to $D = N_G D_G$, $[\mathbf{Q}_i]_{p,p} = \lambda_{i,p}$.

Let $\tilde{\mathbf{x}}'_i = \text{vec}(\{\tilde{\mathbf{x}}_{i,l}\}_{l=1}^{n_i})$ denote the $n_i D_G \times 1$ zero mean circularly symmetric Gaussian vector with covariance matrix $\tilde{\Sigma}_i$. Let $\mathbf{X}_i = [\mathbf{x}_{i,1} \mathbf{x}_{i,2} \cdots \mathbf{x}_{i,n_i}]$ and $\tilde{\mathbf{X}}_i = [\tilde{\mathbf{x}}_{i,1} \tilde{\mathbf{x}}_{i,2} \cdots \tilde{\mathbf{x}}_{i,n_i}]$. We have $\mathbf{x}'_i = \text{vec}(\mathbf{X}_i)$ and $\tilde{\mathbf{x}}'_i = \text{vec}(\tilde{\mathbf{X}}_i)$. Since $\Sigma_i = (\mathbf{M}_i \otimes \mathbf{I}_{D_G})^\dagger \cdot \tilde{\Sigma}_i \cdot (\mathbf{M}_i \otimes \mathbf{I}_{D_G})$, we can write $\mathbf{x}'_i = (\mathbf{M}_i^\dagger \otimes \mathbf{I}_{D_G}) \cdot \tilde{\mathbf{x}}'_i$ or, $\text{vec}(\mathbf{X}_i) = (\mathbf{M}_i^\dagger \otimes \mathbf{I}_{D_G}) \cdot \text{vec}(\tilde{\mathbf{X}}_i)$ which, using (C.4), leads to $\text{vec}(\mathbf{X}_i) = \text{vec}(\mathbf{I}_{D_G} \cdot \tilde{\mathbf{X}}_i \cdot \mathbf{M}_i^*)$, and

$$\mathbf{X}_i = \tilde{\mathbf{X}}_i \cdot \mathbf{M}_i^* = \sum_{l=1}^{n_i} \tilde{\mathbf{x}}_{i,l} \cdot \mathbf{m}_{i,l}^\dagger \quad (3.10)$$

where $\mathbf{m}_{i,l}^\dagger = [\mathbf{M}_i^*]_{l,*}$ (i.e. $\mathbf{M}_i^\dagger = [\mathbf{m}_{i,1}^* \mathbf{m}_{i,2}^* \cdots \mathbf{m}_{i,n_i}^*]$) and $\mathbf{m}_{i,l}^*$ is the l^{th} eigenvector of $\mathbf{C}_i^\dagger \cdot \mathbf{C}_i = \mathbf{M}_i^\dagger \cdot \mathbf{Q}_i \cdot \mathbf{M}_i$ corresponding to $\lambda_{i,l}$. Given l , if $\{\mu - \lambda_{i,l}^{-1}\}^+ = 0$ then $\tilde{\mathbf{x}}_{i,l}$ becomes the zero vector since it reduces to a zero mean Gaussian vector with zero covariance matrix. Otherwise, $\tilde{\mathbf{x}}_{i,l}$ is nonzero and $\tilde{\mathbf{x}}_{i,l} \cdot \mathbf{m}_{i,l}^\dagger$ represents a set of beams using weights that are the components of $\mathbf{m}_{i,l}^*$, subsequently called eigenbeams. The rank of $\mathbf{C}_i$ being r_i , there are at most r_i such eigenbeam sets in the sum (3.10) and the associated eigenvectors $\mathbf{m}_{i,l}^*$ are orthogonal since $\mathbf{M}_i$ is unitary. Thus, within TOG i transmission is carried out along the eigenvectors $\mathbf{m}_{i,l}^*$ for which $\{\mu - \lambda_{i,l}^{-1}\}^+ \neq 0$ (i.e. corresponding to active eigenmodes), and the resulting eigenbeams transmit in orthogonal "directions". Notice that there are two different levels of orthogonality. The first one is the orthogonality associated with the transmission along orthogonal eigenvectors within each TOG. The second one is the signal space orthogonality between different TOGs, due to the fact that the input signals to different TOGs occupy orthogonal signal subspaces. Given N_G , the higher r_i are the more orthogonal eigenbeams may be formed, which could yield a capacity increase. The eigen-beamforming signal structure is illustrated in Figure 3.1.

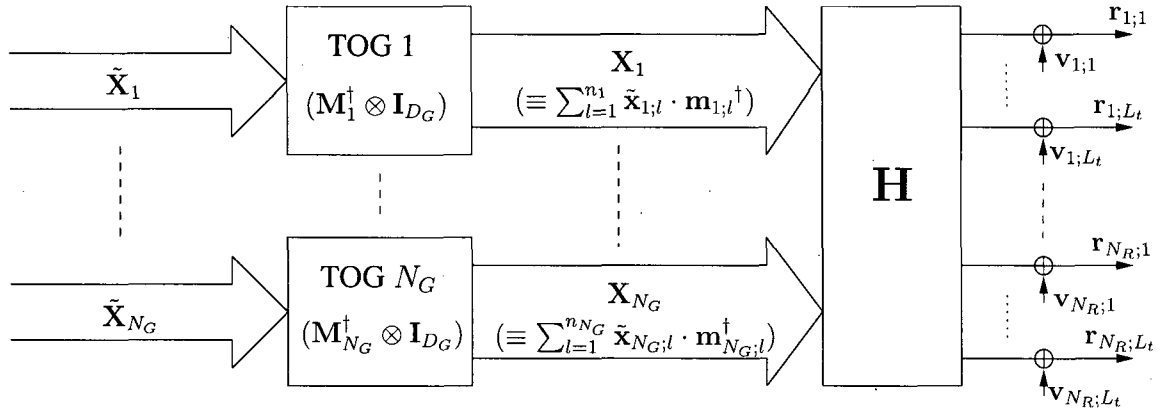


Fig. 3.1 A model for transmission with eigenbeams.

With a given transmit power, different space-time modulation formats yield different capacities (3.9). Next we compare the capacity obtained with ATA, POTA and OTA.

Theorem 3.1.1 *Shannon capacity is maximized by the ATA space-time modulation format.*

The proof of Theorem 3.1.1 is provided in Appendix A. Subsequently, we consider the following definition for merging TOGs.

Definition 3.1.1 (Merging TOGs) *Assuming $N_G \geq 2$, we consider two TOGs i and j having n_i and n_j antenna elements and employing D_i and D_j signal dimensions, respectively. Merging TOGs i and j refers to the process of creating a new TOG with $n_i + n_j$ antenna elements and employing the original $D_i + D_j$ signal dimensions.*

Using Definition 3.1.1, we prove in Appendix A the following corollary.

Corollary 3.1.1 *For MIMO systems with $N_G \geq 2$, merging TOGs cannot decrease capacity.*

The previous theorem and corollary apply to any MIMO channel with arbitrary spatial channel model, not necessarily following the 3GPP specification. By Theorem 3.1.1 and Corollary 3.1.1 maximum, intermediate and minimum values for C are, respectively, obtained using ATA, POTA and OTA, provided that the system using POTA can be obtained from that using OTA by merging TOGs. Note that due to the fact that we restricted the TOGs to have equal number of signal dimensions D_G , we can have POTA systems that do not come from OTA by merging TOGs according to Definition 3.1.1. For example, consider a system with $N_T = 4$ and $D = 4$ that employs OTA with $D_G = 1$. For this system, two POTA schemes with $N_G = 2$ can be obtained as follows. The first scheme has $n_1 = n_2 = 2$ with $D_G = 2$, and it is obtained from OTA by merging the first two antennas into one TOG and the second two antennas into another TOG. The second scheme has $n_1 = 1$ and $n_2 = 3$ with $D_G = 2$, and it is not obtained from OTA by merging TOGs. However, by allowing TOGs to have different number of signal dimensions with the possibility of assigning zero signal dimension to a transmit antenna, any POTA scheme can come from an OTA scheme. Despite its poor performance in terms of Shannon capacity, OTA requires a relatively low implementation complexity. Such advantage has been pointed out in [88] for IS-2000 systems with Orthogonal Transmit Diversity. Furthermore, it has been shown in [18] that

over a spatially uncorrelated block-fading Rayleigh channel it is easier for a code to have full rank error matrices, and hence to achieve full diversity, with OTA than with ATA. The space-time modulation format POTA can also present some advantages over OTA and ATA. For instance, a POTA scheme that employs repetitions and transmit beamforming in each TOG has been shown in [20] to yield the best performance in a highly transmit correlated block-fading Rayleigh channel.

3.1.1 Shannon capacity at low SNR

As in [98, 99], we characterize the low SNR capacity using the minimum $(E_b/N_0)_T$ and the slope of spectral efficiency curve therein. We assume for the low SNR regime that the ratio P_T/N_0 is very small with respect to the inverses of the nonzero eigenvalues $\lambda_{i,l}$, that is, $(P_T/N_0) \ll [\lambda_{i,l}]^{-1}$ or $(P_T/N_0)\lambda_{i,l} \ll 1$ for all $\lambda_{i,l} \neq 0$, $l = 1, \dots, n_i$ and $i = 1, \dots, N_G$. In this case, only the eigenmodes corresponding to the maximum eigenvalue $\lambda_{\max} = \max\{\{\lambda_{i,l}\}_{l=1}^{n_i}\}_{i=1}^{N_G}$ are active. Denoting the multiplicity of $\lambda_{\max}$ by $n(\lambda_{\max})$, $n(\lambda_{\max})D_G$ eigenmodes are active and (3.9) becomes $C = \frac{n(\lambda_{\max})}{N_G} \log(\mu \lambda_{\max})$ with

$$\mu = \lambda_{\max}^{-1} + \frac{N_G P_T}{n(\lambda_{\max}) N_0 D} \quad (3.11)$$

Thus,

$$C = \frac{n(\lambda_{\max})}{N_G} \log \left(1 + \lambda_{\max} \frac{N_G P_T}{n(\lambda_{\max}) N_0 D} \right), \quad (3.12)$$

where $(P_T/N_0)\lambda_{\max} \ll 1$ and hence, $\lambda_{\max} \frac{N_G P_T}{n(\lambda_{\max}) N_0 D} \ll 1$ since $\frac{N_G}{n(\lambda_{\max}) D} = \frac{1}{n(\lambda_{\max}) D_G} \leq 1$. Using (3.3), (3.12) becomes $C = \frac{n(\lambda_{\max})}{N_G} \log(1 + \lambda_{\max} \frac{N_G C}{n(\lambda_{\max})} (E_b/N_0)_T)$, yielding

$$(E_b/N_0)_T = \frac{2^{\frac{N_G C}{n(\lambda_{\max})}} - 1}{\frac{N_G C}{n(\lambda_{\max})} \lambda_{\max}} = \frac{\sum_{k=0}^{\infty} \left(\frac{N_G C}{n(\lambda_{\max})} \ln 2 \right)^k / k! - 1}{\frac{N_G C}{n(\lambda_{\max})} \lambda_{\max}} \approx \frac{N_G (\ln 2)^2}{2 n(\lambda_{\max}) \lambda_{\max}} C + \frac{\ln 2}{\lambda_{\max}} \quad (3.13)$$

where the approximation is of the second order. If $C \rightarrow 0$, then $(E_b/N_0)_T \rightarrow \left[\frac{E_b}{N_0} \right]_{\min} = \frac{\ln 2}{\lambda_{\max}} (or -1.6 - 10 \log_{10}(\lambda_{\max}) \text{ dB})$, that is analogous to [98, Theorem 1] (i.e. $\left[\frac{E_b}{N_0} \right]_{\min} = \frac{\ln 2}{G}$) for deterministic channels with multi-dimensional space-time modulation whose channel gain is given by [98, (18)]

$$G = \frac{(E_b/N_0)_R}{(E_b/N_0)_T} = \frac{E[\|\mathbf{H} \cdot \mathbf{x}\|^2]}{E[\|\mathbf{x}\|^2]} = \frac{E[\text{Tr}(\mathbf{H} \cdot \mathbf{x} \cdot \mathbf{x}^\dagger \cdot \mathbf{H}^\dagger)]}{E[\text{Tr}(\mathbf{x} \cdot \mathbf{x}^\dagger)]} = \frac{\text{Tr}(\mathbf{H} \cdot \Sigma \cdot \mathbf{H}^\dagger)}{\text{Tr}(\Sigma)} \quad (3.14)$$

where $(E_b/N_0)_R$ denotes the received SNR-per-bit. Since $Tr(\mathbf{H} \cdot \Sigma \cdot \mathbf{H}^\dagger) = Tr(\Sigma \cdot \mathbf{H}^\dagger \cdot \mathbf{H}) = Tr(\tilde{\Sigma} \cdot \mathbf{Q}) = n(\lambda_{\max})N_0 D_G(\mu - \lambda_{\max}^{-1})\lambda_{\max}$, we obtain using (3.11) and $Tr(\Sigma) = P_T$ that $G = \lambda_{\max}$ and hence, $(E_b/N_0)_R = \lambda_{\max}(E_b/N_0)_T$. Thus, the minimum received SNR-per-bit required for reliable communication is given by $[\frac{E_b}{N_0}]_{R;\min} = \lambda_{\max}[\frac{E_b}{N_0}]_{\min} = \ln 2$ (or -1.6 dB), which corresponds to the Shannon limit [84, p.262]. Let $\lambda_{\max;A}$, $\lambda_{\max;P}$ and $\lambda_{\max;O}$ denote the values of $\lambda_{\max}$ using ATA, POTA and OTA, respectively. We prove in Appendix A the following theorem.

Theorem 3.1.2 *We have $\lambda_{\max;A} \geq \lambda_{\max;P} \geq \lambda_{\max;O}$.*

Therefore, ATA provides the smallest value for $[\frac{E_b}{N_0}]_{\min}$. From (3.13), we can also write $C \approx \frac{2n(\lambda_{\max})\lambda_{\max}}{N_G(\ln 2)^2}(E_b/N_0)_T - \frac{2n(\lambda_{\max})}{N_G \ln 2}$ which is linear in $(E_b/N_0)_T$ with slope $\frac{2n(\lambda_{\max})\lambda_{\max}}{N_G(\ln 2)^2}$, or [98, (29)]

$$\lim_{(E_b/N_0)_T \rightarrow [\frac{E_b}{N_0}]_{\min}} \frac{10 \log_{10}(2) C}{10 \log_{10}((E_b/N_0)_T) - 10 \log_{10}([\frac{E_b}{N_0}]_{\min})} = \frac{2n(\lambda_{\max})}{N_G} \text{ bps/Hz/(3 dB)}, \quad (3.15)$$

denoting the rate of change of C with $(E_b/N_0)_T$ dB at $[\frac{E_b}{N_0}]_{\min}$. The increase of $(E_b/N_0)_T$ dB in (3.15) is measured in 3 dB units because of the term $10 \log_{10}(2)$. The multiplicity $n(\lambda_{\max})$ depends on the spatial channel and the space-time modulation format. Denoting the total number of scatterers by L , we consider for instance the two extreme cases from [39]: the keyhole channel ($L = 1$) and the i.i.d. channel ($L \rightarrow \infty$). For keyhole channels, (2.21) becomes $\mathbf{C}_i = G_{1;0} \mathbf{a}_{1;0} \cdot \mathbf{b}_{i;1;0}^\dagger$ with a single nonzero eigenvalue per TOG, $\lambda_{i;1} = |G_{1;0}|^2 \|\mathbf{a}_{1;0}\|^2 \|\mathbf{b}_{i;1;0}\|^2 = |G_{1;0}|^2 N_R n_i$. In this case, the slope given by (3.15) depends on the space-time modulation formats as follows. With POTA, this slope reduces to $2(N_{G;\max}/N_G)$ where $N_{G;\max}$ denotes the number of TOGs having the maximum number of antenna elements. For a given N_G , the more disparate $n_1, \dots, n_{N_G}$ are the lower the ratio $N_{G;\max}/N_G$ is. This ratio is maximum (equal to 1) when all TOGs have equal number of antenna elements, $n_1 = \dots = n_{N_G} = N_T/N_G$. It is minimum (equal to $1/N_G$) when only one TOG has the largest number of antenna elements n_i . Hence, the slope (3.15) with POTA is lower bounded by $2/N_G$ bps/Hz/(3 dB) and upper bounded by 2 bps/Hz/(3 dB). For keyhole channels with ATA and OTA we have $n(\lambda_{\max}) = 1$ and $n(\lambda_{\max}) = N_T$, respectively, which reduces the slope of (3.15) to 2 bps/Hz/(3 dB) in both cases. For i.i.d. channels, $\{\mathbf{C}_i^\dagger \cdot \mathbf{C}_i\}_{i=1}^{N_G}$ are independent and the probability that any two eigenvalues $\lambda_{i;l}$ are equal is zero [100, Section 3.1]. It follows that $n(\lambda_{\max}) = 1$ for all space-time modulation

formats, and the slope (3.15) reduces to $2/N_G$ bps/Hz/(3 dB). Although capacity is known to be lower for keyhole than for i.i.d. channels [39], i.i.d. channels cannot yield a capacity slope at $\left[\frac{E_b}{N_0}\right]_{\min}$ greater than that of keyhole channels for space-time modulation formats with $N_G > 1$.

3.1.2 Shannon capacity at high SNR

In the high SNR regime, the ratio P_T/N_0 is assumed to be very large. Hence, all $r_T D_G$ eigenmodes are active, reducing (3.9) to $C = \frac{1}{N_G} \sum_{i=1}^{N_G} \sum_{l=1}^{r_i} \log(\mu \lambda_{i,l})$ with $\mu = \frac{N_G P_T}{r_T N_0 D} + \frac{1}{r_T} \left[\sum_{i=1}^{N_G} \sum_{l=1}^{r_i} \lambda_{i,l}^{-1} \right]$. With $A = \sum_{i=1}^{N_G} \sum_{l=1}^{r_i} \log(\lambda_{i,l})$ and $c = \frac{1}{r_T} \sum_{i=1}^{N_G} \sum_{l=1}^{r_i} \lambda_{i,l}^{-1}$, we have

$$C = \frac{A}{N_G} + \frac{r_T}{N_G} \log \left(\frac{N_G P_T}{r_T N_0 D} + c \right), \quad (3.16)$$

from which we can write

$$\lim_{P_T/N_0 \rightarrow \infty} \frac{C}{\log(P_T/N_0)} = \frac{r_T}{N_G} \quad (3.17)$$

Thus, at high SNR the capacity scales like $\frac{r_T}{N_G} \log(P_T/N_0)$. In [6, 101], the parameter $\frac{r_T}{N_G}$ is referred to as spatial multiplexing gain which determines the number of spatial degrees of freedom for communication over the MIMO channel, or the number of parallel spatial channels that can be created between the transmitter and the receiver. The higher the multiplexing gain is the more parallel spatial channels can be created and hence, the higher is the transmission capacity obtained by multiplexing independent data streams onto these channels. For MIMO systems with space-time modulation, the multiplexing gain is determined by the number of TOGs N_G as well as the rank r_T of the matrix $\text{diag}_{i=1}^{N_G} [\mathbf{C}_i^\dagger \cdot \mathbf{C}_i]$ as seen from (3.17). Let $r_{T;A}$, $r_{T;P}$ and $r_{T;O}$ denote the values of r_T using ATA, POTA and OTA, respectively.

Theorem 3.1.3 *Assuming full rank TOG propagation matrices $\mathbf{C}_i$, we have $r_{T;A} \geq \frac{r_{T;P}}{N_G} \geq \frac{r_{T;O}}{N_T}$.*

The proof of Theorem 3.1.3 can be found in Appendix A. It follows that for channels with $r_i = \min(n_i, L_t N_R)$, $\forall i$, the multiplexing gain (3.17) is largest with ATA, intermediate with

POTA and smallest with OTA. Furthermore, using (3.3) and (3.16), we obtain

$$(E_b/N_0)_T = \frac{r_T}{N_G C} (2^{\lfloor \frac{N_G C - A}{r_T} \rfloor} - c) \approx \frac{r_T}{N_G C} 2^{\lfloor \frac{N_G C - A}{r_T} \rfloor}. \quad (3.18)$$

The approximation is due to $2^{\lfloor \frac{N_G C - A}{r_T} \rfloor} \gg c$ since C becomes large at high SNR. As $C \rightarrow \infty$, $(E_b/N_0)_T$ increases exponentially with C , making C linear in $(E_b/N_0)_T$ dB as in [102] with slope $[10 \log_{10}(2) \frac{N_G}{r_T}]^{-1} \approx \frac{r_T}{3N_G}$ bps/Hz/dB or $\frac{r_T}{N_G}$ bps/Hz/(3 dB). Such slope can be larger or smaller than that of (3.15) depending on the channel. For example, i.i.d. channels using OTA with $N_T \geq 2$ have larger slope at high SNR than at low SNR since $1 \geq 2/N_T$. For keyhole channels using ATA we have $r_T = N_G = 1$, making the high SNR slope equal to 1 bps/Hz/(3 dB) which is half the low SNR slope. By Theorem 3.1.3, the rate of growth of C with $(E_b/N_0)_T$ dB is largest with ATA, intermediate with POTA and smallest with OTA.

3.1.3 Impact of the propagation channel

Capacity can be sometimes expressed in terms of the spatial channel parameters [103]. Based on (3.9), it is sufficient to represent the eigenvalues $\lambda_{i,l}$ in terms of the channel parameters. This issue is addressed in this section, and an illustrative example of MIMO systems where analytical expressions for $\lambda_{i,l}$ can be obtained in terms of the components of the matrices $\mathbf{C}_i^\dagger \cdot \mathbf{C}_i$ is provided. For more general systems, this section investigates some tools that can be used to derive capacity bounds (in terms of the spatial channel parameters) in the low and high SNR regimes where the capacity depends on fewer parameters. The results of this section apply to all space and time dispersive MIMO channels (not only 3GPP channels [83]) whose propagation matrix $\mathbf{C} = [\mathbf{C}_1 \ \mathbf{C}_2 \ \cdots \ \mathbf{C}_{N_G}]$ satisfies (2.21) and (2.22). Using $\mathbf{d}_{i,k;m} = \sum_{s=0}^{S_k-1} G_{k;s}^* e^{-j(m-1)\alpha_{k;s}} \mathbf{b}_{i,k;s}$ and (2.21), we have

$$\mathbf{C}_i^\dagger \cdot \mathbf{C}_i = \sum_{k=1}^{L_t} \mathbf{F}_{i,k}^\dagger \cdot \mathbf{F}_{i,k} = \sum_{k=1}^{L_t} \sum_{m=1}^{N_R} \mathbf{d}_{i,k;m} \cdot \mathbf{d}_{i,k;m}^\dagger \quad (3.19)$$

Defining the $L_t N_R \times n_i$ matrix $\mathbf{D}_i = \text{stack}_{k=1}^{L_t} [\text{stack}_{m=1}^{N_R} [\mathbf{d}_{i,k;m}^\dagger]]$, (3.19) becomes $\mathbf{C}_i^\dagger \cdot \mathbf{C}_i = \mathbf{D}_i^\dagger \cdot \mathbf{D}_i$. If $n_i \leq L_t N_R$, then $\{\lambda_{i,l}\}$ and $\{\mathbf{m}_{i,l}^*\}$ can be found by solving $\mathbf{D}_i^\dagger \cdot \mathbf{D}_i \cdot \mathbf{m}_{i,l}^* = \lambda_{i,l} \mathbf{m}_{i,l}^*$. If $L_t N_R \leq n_i$, then $\{\lambda_{i,l}\}$ can be found by solving $\mathbf{D}_i \cdot \mathbf{D}_i^\dagger \cdot \mathbf{q}_{i,l} = \lambda_{i,l} \mathbf{q}_{i,l}$. Multiplying both sides of the last equality by $\mathbf{D}_i^\dagger$, we have $\mathbf{D}_i^\dagger \cdot \mathbf{D}_i \cdot (\mathbf{D}_i^\dagger \cdot \mathbf{q}_{i,l}) = \lambda_{i,l} (\mathbf{D}_i^\dagger \cdot \mathbf{q}_{i,l})$, which compared to $\mathbf{D}_i^\dagger \cdot \mathbf{D}_i \cdot \mathbf{m}_{i,l}^* = \lambda_{i,l} \mathbf{m}_{i,l}^*$ yields $\mathbf{m}_{i,l}^* = \mathbf{D}_i^\dagger \cdot \mathbf{q}_{i,l}$ and $\mathbf{D}_i \cdot \mathbf{m}_{i,l}^* = \lambda_{i,l} \mathbf{q}_{i,l}$.

Analytical solutions to $|\mathbf{D}_i^\dagger \cdot \mathbf{D}_i - x\mathbf{I}_{n_i}| = 0$ or $|\mathbf{D}_i \cdot \mathbf{D}_i^\dagger - x\mathbf{I}_{L_t N_R}| = 0$ can be obtained up to an order of 4 as in [104].

Example: Consider $L_t = 2$, $N_R = 1$ and $L_t N_R \leq n_i$. We have $\mathbf{D}_i = \text{stack}[\mathbf{d}_{i,1,1}^\dagger, \mathbf{d}_{i,2,1}^\dagger]$. There are at most two nonzero eigenvalues $\lambda_{i,1}$, $\lambda_{i,2}$. If $\mathbf{d}_{i,1,1}$ and $\mathbf{d}_{i,2,1}$ are linearly independent, solving the characteristic equation $|\mathbf{D}_i \cdot \mathbf{D}_i^\dagger - x\mathbf{I}_{L_t N_R}| = 0$ yields

$$\begin{cases} \lambda_{i,1} = \frac{(\mathbf{d}_{i,1,1}^\dagger \cdot \mathbf{d}_{i,1,1} + \mathbf{d}_{i,2,1}^\dagger \cdot \mathbf{d}_{i,2,1}) + \sqrt{(\mathbf{d}_{i,1,1}^\dagger \cdot \mathbf{d}_{i,1,1} - \mathbf{d}_{i,2,1}^\dagger \cdot \mathbf{d}_{i,2,1})^2 + 4|\mathbf{d}_{i,1,1}^\dagger \cdot \mathbf{d}_{i,2,1}|^2}}{2} \\ \lambda_{i,2} = \frac{(\mathbf{d}_{i,1,1}^\dagger \cdot \mathbf{d}_{i,1,1} + \mathbf{d}_{i,2,1}^\dagger \cdot \mathbf{d}_{i,2,1}) - \sqrt{(\mathbf{d}_{i,1,1}^\dagger \cdot \mathbf{d}_{i,1,1} - \mathbf{d}_{i,2,1}^\dagger \cdot \mathbf{d}_{i,2,1})^2 + 4|\mathbf{d}_{i,1,1}^\dagger \cdot \mathbf{d}_{i,2,1}|^2}}{2} \end{cases} \quad (3.20)$$

where we have $\mathbf{d}_{i,1,1}^\dagger \cdot \mathbf{d}_{i,2,1} = \sum_{l=1}^{n_i} \sum_{s=0}^{S_1-1} \sum_{p=0}^{S_2-1} G_{1,s} G_{2,p}^* e^{j(b_i+l-1)(\beta_{1,s}-\beta_{2,p})}$ and, for $k = 1, 2$, $\mathbf{d}_{i,k,1}^\dagger \cdot \mathbf{d}_{i,k,1} = \sum_{l=1}^{n_i} |\sum_{s=0}^{S_k-1} G_{k,s}^* e^{-j(b_i+l-1)\beta_{k,s}}|^2$. The eigenvectors of $\mathbf{D}_i \cdot \mathbf{D}_i^\dagger$ are given by $\mathbf{q}_{i,1} = a_1 \text{vec}(1, -\frac{\mathbf{d}_{i,1,1}^\dagger \cdot \mathbf{d}_{i,1,1} - \lambda_{i,1}}{\mathbf{d}_{i,1,1}^\dagger \cdot \mathbf{d}_{i,2,1}})$ and $\mathbf{q}_{i,2} = a_2 \text{vec}(1, -\frac{\mathbf{d}_{i,1,1}^\dagger \cdot \mathbf{d}_{i,1,1} - \lambda_{i,2}}{\mathbf{d}_{i,1,1}^\dagger \cdot \mathbf{d}_{i,2,1}})$, where a_1 and a_2 denote two normalizing constants. The eigenvectors $\mathbf{m}_{i,l}^*$ of $\mathbf{C}_i^\dagger \cdot \mathbf{C}_i$ (i.e. $\mathbf{D}_i^\dagger \cdot \mathbf{D}_i$) are given by

$$\mathbf{m}_{i,l}^* = [\mathbf{q}_{i,l}]_1 \mathbf{d}_{i,1,1} + [\mathbf{q}_{i,l}]_2 \mathbf{d}_{i,2,1} \quad (3.21)$$

for $l = 1, 2$. If $\mathbf{d}_{i,1,1}$ and $\mathbf{d}_{i,2,1}$ are colinear (i.e. $\mathbf{d}_{i,2,1} = \varrho \mathbf{d}_{i,1,1}$ for some $\varrho \in \mathbb{C}$) we have

$$\lambda_{i,1} = \mathbf{d}_{i,1,1}^\dagger \cdot \mathbf{d}_{i,1,1} + \mathbf{d}_{i,2,1}^\dagger \cdot \mathbf{d}_{i,2,1} \quad \text{and} \quad \lambda_{i,2} = 0 \quad (3.22)$$

with $\mathbf{q}_{i,1} = a_1 \text{vec}(1, \varrho^*)$ and $\mathbf{m}_{i,1}^*$ given by (3.21). The eigenbeams of (3.10) are a weighted sum of beams defined by the space signature vectors $\mathbf{b}_{i,k,s}$ through $\mathbf{d}_{i,k,m}$. If $\lambda_{i,1}$ and $\lambda_{i,2}$ of (3.20) are nonzero more channel eigenmodes can be active, yielding larger capacity especially at high SNR where the scaling depends on r_T . At low SNR, C increases by maximizing $\lambda_{\max} \equiv \max\{\lambda_{i,1}\}_{i=1}^{N_G}$.

As n_i and N_R increase, the analytical evaluation of the ranks r_i and eigenvalues $\lambda_{i,l}$ becomes very complicated. Thus, at high and low SNR we consider bounds on the rank r of $\mathbf{H}$ and on the maximum eigenvalue $\lambda_{\max}$, respectively, which can provide bounds to the water-filling capacity in both regimes.

Bounding $r = \frac{r_T}{N_G} D$ yields bounds to the high SNR slope of C as a function of $(E_b/N_0)_T$ dB and to the multiplexing gain (3.17). Using $L = \sum_{k=1}^{L_t} S_k$, equations (2.21) and (3.19), we have $r_i \leq \min(n_i, L_t N_R, L)$ leading to $r \leq D \min(\frac{N_T}{N_G}, L_t N_R, L)$ since $r = \frac{D}{N_G} (\sum_{i=1}^{N_G} r_i)$. If $\frac{N_T}{N_G} \leq \min(L_t N_R, L)$, r can increase with the average number of transmit antennas per

TOG $\frac{N_T}{N_G}$ which, given N_T , becomes maximum when using ATA. If $L_t N_R \leq \min(\frac{N_T}{N_G}, L)$, r can increase through the product $L_t N_R$ by increasing either L_t or N_R . While L_t is determined by the channel, a system designer can increase N_R yielding an $L_t D$ -fold increase for r . With rich scattering (i.e. large L), we have $r \leq D \min(\frac{N_T}{N_G}, L_t N_R)$ and r can be increased by increasing N_R or $\frac{N_T}{N_G}$. If the scattering is small and $L \leq \min(\frac{N_T}{N_G}, L_t N_R)$, r is determined by LD and it can be increased only by increasing D (i.e. increasing bandwidth) provided that L_t (and consequently L) does not increase. In case of small scattering with limited time dispersion $L_t N_R \leq \min(\frac{N_T}{N_G}, L)$, r is determined by $L_t N_R D$ and can be easily increased by increasing N_R .

The bounds on $\lambda_{\max}$ translate into bounds on $\left[\frac{E_b}{N_0}\right]_{\min}$ which is a key performance metric at low SNR denoting the minimum energy-per-bit required to convey any positive rate reliably. The maximum eigenvalue $\lambda_{\max}$ of $\text{diag}_{i=1}^{N_G}[\mathbf{C}_i^\dagger \cdot \mathbf{C}_i]$ can be bounded as follows [105, Chapter 5]

$$\frac{1}{N_T} \sum_{i=1}^{N_G} \text{Tr}(\mathbf{C}_i^\dagger \cdot \mathbf{C}_i) \leq \lambda_{\max} \quad (3.23a)$$

$$\lambda_{\max} \leq \sum_{i=1}^{N_G} \text{Tr}(\mathbf{C}_i^\dagger \cdot \mathbf{C}_i) \quad (3.23b)$$

Writing $\text{Tr}(\mathbf{C}_i^\dagger \cdot \mathbf{C}_i) = \sum_{k=1}^{L_t} \sum_{m=1}^{N_R} \text{Tr}(\mathbf{d}_{i;k;m} \cdot \mathbf{d}_{i;k;m}^\dagger) = \sum_{k=1}^{L_t} \sum_{m=1}^{N_R} \|\mathbf{d}_{i;k;m}\|^2$ from (3.19), we have

$$\text{Tr}(\mathbf{C}_i^\dagger \cdot \mathbf{C}_i) = \sum_{k=1}^{L_t} \sum_{m=1}^{N_R} \sum_{l=1}^{n_i} \left| \sum_{s=0}^{S_k-1} G_{k;s}^* e^{-j(m-1)\alpha_{k;s}} e^{-j(b_i+l-1)\beta_{k;s}} \right|^2 \quad (3.24)$$

The lower and upper bounds on $\lambda_{\max}$ given in (3.23a) and (3.23b) are often not sufficiently tight. Thus, we seek tighter bounds on $\lambda_{\max}$ based on the elements of the matrix $\text{diag}_{i=1}^{N_G}[\mathbf{C}_i^\dagger \cdot \mathbf{C}_i]$. From [106, 9.B.1], $\lambda_{\max}$ can be lower bounded by the maximum diagonal element of $\text{diag}_{i=1}^{N_G}[\mathbf{C}_i^\dagger \cdot \mathbf{C}_i]$. As for the upper bound, we consider $R_{i,p}$ and $T_{i,p}$ which are the sums of the magnitudes of the entries of $\mathbf{C}_i^\dagger \cdot \mathbf{C}_i$ in the p^{th} row and column, respectively. Since $\mathbf{C}_i^\dagger \cdot \mathbf{C}_i$ is Hermitian, we have $R_{i,p} = T_{i,p}$ for all p , and $\lambda_{\max}$ can be upper bounded by $\max_{i=1, \dots, N_G} \max_{p=1, \dots, n_i} T_{i,p}$ from [106, 9.I.1]. Using (3.13), one can see that at low SNR lower and upper bounds on $\lambda_{\max}$ yield, respectively, lower and upper bounds on C , given $(E_b/N_0)_T$, $n(\lambda_{\max})$ and N_G .

Since the eigenvalues do not change when performing unitary similarity transformations, $\lambda_{\max}$ is also the maximum eigenvalue of $\text{diag}_{i=1}^{N_G}[\check{\mathbf{C}}_i^\dagger \cdot \check{\mathbf{C}}_i]$, where $\check{\mathbf{C}}_i^\dagger \cdot \check{\mathbf{C}}_i = \mathbf{U}_{T,i}^\dagger \cdot \mathbf{C}_i^\dagger \cdot \mathbf{C}_i \cdot \mathbf{U}_{T,i}$

and $\mathbf{U}_{T,i}$ is an $n_i \times n_i$ unitary matrix. An ideal transformation is the diagonalization of $\mathbf{C}_i^\dagger \cdot \mathbf{C}_i$ (using $\mathbf{M}_i$ which is the Karhunen-Loeve Transform (KLT)), where $\lambda_{\max}$ would be the maximum diagonal element of the transformed matrix $\text{diag}_{i=1}^{N_G}[\check{\mathbf{C}}_i^\dagger \cdot \check{\mathbf{C}}_i]$. Subsequently, we consider the Discrete Fourier Transform (DFT) and Discrete Cosine Transform (DCT) since they employ unitary matrices which are fixed and independent of the channel. These transforms have been shown to approximate the KLT [107, 108], which in our case can provide a tighter bounds on $\lambda_{\max}$. Note that the DFT has been used in [109] to provide an imaging interpretation for the scattering environment of space dispersive MIMO channels via the so-called virtual channel representation.

The DFT matrices $\mathbf{U}_{T,i}^{DFT}$ are defined by $[\mathbf{U}_{T,i}^{DFT}]_{m,l} = \frac{1}{\sqrt{n_i}} e^{-j\frac{2\pi}{n_i}(m-1)(l-1)}$ for $m, l = 1, \dots, n_i$. Denoting the transformed matrix product $\mathbf{C}_i^\dagger \cdot \mathbf{C}_i$ using DFT by $(\check{\mathbf{C}}_i^{DFT})^\dagger \cdot \check{\mathbf{C}}_i^{DFT}$, we have

$$\max_{i=1, \dots, N_G} \max_{p=1, \dots, n_i} [(\check{\mathbf{C}}_i^{DFT})^\dagger \cdot \check{\mathbf{C}}_i^{DFT}]_{p,p} \leq \lambda_{\max} \quad (3.25a)$$

$$\lambda_{\max} \leq \max_{i=1, \dots, N_G} \max_{p=1, \dots, n_i} \sum_{l=1}^{n_i} |[(\check{\mathbf{C}}_i^{DFT})^\dagger \cdot \check{\mathbf{C}}_i^{DFT}]_{l,p}| \quad (3.25b)$$

The DCT matrices $\mathbf{U}_{T,i}^{DCT}$ are defined by $[\mathbf{U}_{T,i}^{DCT}]_{m,l} = \frac{b_{T,m}}{\sqrt{n_i}} \cos \left[\frac{m\pi}{2n_i}(2l+1) \right]$ for $m, l = 0, \dots, n_i - 1$, $b_{T,0} = 1$ and $b_{T,m} = \sqrt{2}$ for all $m \geq 1$. The transformed matrix product $\mathbf{C}_i^\dagger \cdot \mathbf{C}_i$ using DCT is denoted by $(\check{\mathbf{C}}_i^{DCT})^\dagger \cdot \check{\mathbf{C}}_i^{DCT}$, and the lower and upper bounds on $\lambda_{\max}$ are as follows

$$\max_{i=1, \dots, N_G} \max_{p=0, \dots, n_i-1} [(\check{\mathbf{C}}_i^{DCT})^\dagger \cdot \check{\mathbf{C}}_i^{DCT}]_{p,p} \leq \lambda_{\max} \quad (3.26a)$$

$$\lambda_{\max} \leq \max_{i=1, \dots, N_G} \max_{p=0, \dots, n_i-1} \sum_{l=0}^{n_i-1} |[(\check{\mathbf{C}}_i^{DCT})^\dagger \cdot \check{\mathbf{C}}_i^{DCT}]_{l,p}| \quad (3.26b)$$

Consequently, $\lambda_{\max}$ can be lower bounded by the maximum of the lower bounds (3.23a), (3.25a) and (3.26a) and upper bounded by the minimum of the upper bounds (3.23b), (3.25b) and (3.26b). We compare these bounds for MIMO channels that are realizations of the 3GPP channel model presented in Chapter 2. For numerical evaluations, we consider four 3GPP channel realizations specified in Tables 3.1 - 3.4, respectively. Our results consider TOG 1 (with $b_1 = 0$). The values of r_i and $\lambda_{\max}$ as well as the bounds (3.23), (3.25) and (3.26) are listed in Table 3.5 for different values of (n_i, N_R) . In all these examples, we have $n(\lambda_{\max}) = 1$, reducing the low SNR capacity slope (3.15) at $\left[\frac{E_b}{N_0} \right]_{\min}$ to $2/N_G$

bps/Hz/(3 dB). As seen from Table 3.5, the bounds (3.23), (3.25) and (3.26) are equal to $\lambda_{\max}$ for $n_i = N_R = 1$, while (3.25a) and (3.26a) (respectively (3.25b) and (3.26b)) present identical lower (respectively upper) bounds for $n_i = N_R = 2$. Moreover, DFT provides a tighter lower bound (3.25a) than DCT (3.26a) for channels A and B. Such observation holds for channels C and D only for $n_i = N_R = 6, 7, 8, 9, 10$ and $n_i = N_R = 3, 4$, respectively. For all channels, (3.25a) and (3.26a) provide tighter lower bounds than that provided by (3.23a). As for the upper bound on $\lambda_{\max}$, (3.23b) provides the best result for channel A (except for $n_i = N_R = 9$) and it is outperformed by (3.25b) and (3.26b) for channels B, C (except for $n_i = N_R = 2, 3, 4$) and D (except for $n_i = N_R = 2$). Table 3.5 shows that the difference between the bounds (3.25b) and (3.26b) is often not very large. For instance, (3.25b) is tighter for Channel D (except for $n_i = N_R = 6$) and (3.26b) is tighter for Channel C (except for $n_i = N_R = 10$). Hence by providing tighter bounds on $\lambda_{\max}$ in many cases, DFT and DCT turn out to be a useful tool to bound C at low SNR. It is also seen that $r_i = n_i = \min(n_i, L_t N_R)$ making $\mathbf{C}_i$ full rank for the randomly generated examples of the 3GPP channels (Tables 3.1 - 3.4).

It was found by generating 10^4 realizations of the 3GPP channel [83] using $n_i = N_R = 2$ that:

Observation 1: For any channel realization, $\{\lambda_{i,l}\}_{l=1}^{n_i}$ were found to be distinct.

Observation 2: No channel realization were found rank-deficient, $r_i = 2 = \min(n_i, L_t N_R)$. Intuitively, this can be explained as follows. From (2.21), the matrix $\mathbf{F}_{i,k}$ is a summation of S_k different rank one matrices $\{G_{k;s} \mathbf{a}_{k;s} \cdot \mathbf{b}_{i;k;s}^\dagger\}_{s=0}^{S_k-1}$, each describing the signal propagation from TOG i to the receiver over a path s in cluster k . These matrices employ realizations of independent random variables ($\varphi_{k;s}$ in $G_{k;s}$ (2.6) for $s = 0, \dots, S_k - 1$) and different parameters (the offset AODs $\{\hat{\theta}_{T;k;s}\}_{s=0}^{S_k-1}$ and AOAs $\{\hat{\theta}_{R;k;s}\}_{s=0}^{S_k-1}$ from [83, Table 5.2]). In this case and since S_k is greater than N_R and n_i , the matrix $\mathbf{F}_{i,k}$ is most likely full rank, $\text{rank}(\mathbf{F}_{i,k}) = \min(N_R, n_i)$ for each k . For different clusters the matrices $\mathbf{F}_{i,k}$ employ also realizations of independent random variables ($\{P_k\}_{k=1}^{L_t}$ (2.7), the mean AODs $\{\bar{\theta}_{T;k}\}_{k=1}^{L_t}$ and mean AOAs $\{\bar{\theta}_{R;k}\}_{k=1}^{L_t}$) and hence, the columns (and rows) of any two different matrices $\mathbf{F}_{i,1}, \dots, \mathbf{F}_{i,L_t}$ are most likely linearly independent. It follows that $\mathbf{C}_i$ (2.22), which is the stack of $\{\mathbf{F}_{i,k}\}_{k=1}^{L_t}$, is most likely full rank and $r_i = \min(n_i, L_t N_R)$. Hence, Theorem 3.1.3's assumption of full rank TOG propagation matrices holds most likely, making the high SNR capacity slope and the multiplexing gain (3.17) largest with ATA for the 3GPP MIMO channels. For the remainder of this chapter we consider full rank channel propagation matrices only.

Channel A with $(\theta_{BS}, \theta_{MS}) = (34.5116, 112.7349)$	Cluster 1 with $(\bar{\theta}_{T,1}, \bar{\theta}_{R,1}) = (-8.386, -4.2713)$ and $(G_{1,s}, \theta_{T,1;s}, \theta_{R,1;s})$: $(-0.0438 - 0.1492i, 26.3491, 183.891), (-0.1341 + 0.0807i, 25.9019, 126.4128),$ $(-0.1465 - 0.0485i, 26.8319, 148.646), (0.048 - 0.1501i, 25.4191, 161.6452),$ $(-0.0668 - 0.1378i, 27.3716, 68.2812), (0.1035 - 0.1205i, 24.8794, 33.0362),$ $(0.1311 - 0.0762i, 27.9833, 84.6737), (-0.0937 - 0.1299i, 24.2677, 139.4174),$ $(-0.1486 + 0.0201i, 28.6897, 103.5189), (-0.141 - 0.0791i, 23.5613, 55.282),$ $(-0.1207 - 0.0855i, 29.5241, 77.5098), (0.0094 + 0.1632i, 22.7269, 117.186),$ $(0.0429 - 0.1389i, 30.5475, 132.2535), (0.0865 + 0.1412i, 21.7035, 99.7412),$ $(0.0619 - 0.1278i, 31.8658, 95.4591), (0.0101 + 0.1679i, 20.3852, 110.0285),$ $(-0.1189 - 0.0684i, 33.7229, 106.8987), (0.0874 - 0.1478i, 18.5281, 113.4083),$ $(0.0789 - 0.1018i, 36.9008, 121.4681), (0.1745 - 0.0291i, 15.3502, 90.5144).$
	Cluster 2 with $(\bar{\theta}_{T,2}, \bar{\theta}_{R,2}) = (-22.4994, -65.6786)$ and $(G_{2,s}, \theta_{T,2;s}, \theta_{R,2;s})$: $(-0.0209 + 0.0781i, 12.2357, 16.1024), (0.0604 + 0.0541i, 11.7885, 38.3338),$ $(-0.0786 + 0.0178i, 12.7185, 100.2378), (-0.0812 - 0.0045i, 11.3057, 70.8461),$ $(-0.0782 + 0.0182i, 13.2582, 122.4836), (0.0434 + 0.0691i, 10.766, -28.3712),$ $(-0.0757 - 0.0254i, 13.8699, 65.0054), (-0.002 - 0.0819i, 10.1543, 23.2663),$ $(0.0162 + 0.0778i, 14.5763, 6.8738), (-0.054 - 0.062i, 9.4479, 42.1115),$ $(-0.0431 + 0.066i, 15.4107, 48.6211), (0.0592 - 0.0576i, 8.6135, 52.0009),$ $(-0.0776 - 0.0095i, 16.4341, 87.2386), (0.0416 - 0.0718i, 7.5901, 45.4913),$ $(0.0476 + 0.0607i, 17.7524, -6.1254), (0.0526 + 0.0648i, 6.2718, 60.0607),$ $(-0.0732 - 0.0193i, 19.6095, 55.7786), (-0.0819 + 0.0179i, 4.4147, 29.107),$ $(0.0143 + 0.0714i, 22.7874, 34.0517), (0.0541 - 0.0647i, 1.2368, 78.01).$
	Cluster 3 with $(\bar{\theta}_{T,3}, \bar{\theta}_{R,3}) = (-6.1398, -72.9047)$ and $(G_{3,s}, \theta_{T,3;s}, \theta_{R,3;s})$: $(0.1166 - 0.0465i, 28.5953, -0.3523), (-0.0909 + 0.0878i, 28.1481, 21.8809),$ $(0.0016 - 0.1245i, 29.0781, 93.0117), (-0.1238 - 0.0301i, 27.6653, 8.8763),$ $(-0.0753 + 0.0978i, 29.6178, 41.395), (0.1284 + 0.0035i, 27.1256, 80.0125),$ $(-0.1168 + 0.0358i, 30.2295, 57.7793), (0.0211 - 0.1279i, 26.5139, 16.0402),$ $(0.1174 + 0.0279i, 30.9359, 34.8854), (0.0219 + 0.1291i, 25.8075, 52.8346),$ $(0.0631 + 0.1007i, 31.7703, -35.5973), (0.0651 + 0.1154i, 24.9731, 70.7839),$ $(0.1099 + 0.0394i, 32.7937, 63.62), (0.0404 - 0.1282i, 23.9497, 26.8256),$ $(0.0518 + 0.1013i, 34.112, 115.2575), (-0.1107 - 0.0803i, 22.6314, 44.7748),$ $(0.0962 + 0.0527i, 35.9691, 38.2652), (-0.0062 - 0.1398i, 20.7743, 31.1077),$ $(-0.091 - 0.0473i, 39.147, 48.5525), (-0.0169 - 0.1438i, 17.5964, -13.3515).$

Table 3.1 Examples of the 3GPP channel in microcell environments: Channel A

Channel B with $(\theta_{BS}, \theta_{MS}) = (8.0962, 173.8729)$	Cluster 1 with $(\bar{\theta}_{T;1}, \bar{\theta}_{R;1}) = (-12.4334, -6.6011)$ and $(G_{1;s}, \theta_{T;1;s}, \theta_{R;1;s})$: $(-0.0724 + 0.1045i, -4.1136, 191.0617), (0.1203 - 0.0406i, -4.5608, 175.9942),$ $(0.0269 - 0.1244i, -3.6308, 172.2165), (0.019 + 0.1254i, -5.0436, 143.4819),$ $(0.0381 + 0.1216i, -3.0911, 149.3226), (-0.0959 + 0.0827i, -5.5833, 158.5494),$ $(-0.0183 + 0.1262i, -2.4794, 180.2763), (-0.073 + 0.1032i, -6.195, 242.6992),$ $(-0.0718 - 0.1055i, -1.773, 168.8367), (0.1075 - 0.0659i, -6.9014, 127.0894),$ $(0.114 - 0.0577i, -0.9386, 185.221), (0.1251 - 0.0115i, -7.7358, 198.2256);$ $(0.1266 - 0.0173i, 0.0848, 165.7069), (-0.0901 - 0.0867i, -8.7592, 162.3271),$ $(0.0077 + 0.1275i, 1.4031, 207.4542), (0.0736 + 0.1i, -10.0775, 91.8444),$ $(0.1182 + 0.0475i, 3.2602, 220.4534), (0.1164 + 0.039i, -11.9346, 154.2673),$ $(0.046 + 0.1176i, 6.4381, 136.318), (-0.1039 - 0.0596i, -15.1125, 114.0902).$
	Cluster 2 with $(\bar{\theta}_{T;2}, \bar{\theta}_{R;2}) = (32.7591, -64.4589)$ and $(G_{2;s}, \theta_{T;2;s}, \theta_{R;2;s})$: $(0.048 + 0.0313i, 41.0789, 96.4094), (0.0151 - 0.0559i, 40.6317, 107.849),$ $(0.0528 - 0.0207i, 41.5617, 140.3677), (-0.0227 - 0.054i, 40.1489, 149.5963),$ $(-0.0425 + 0.0365i, 42.1014, 127.3631), (0.022 + 0.0551i, 39.6092, 184.8413),$ $(0.0496 - 0.0243i, 42.7131, 69.2315), (0.0094 - 0.0594i, 38.9975, 162.5955),$ $(-0.0425 - 0.0337i, 43.4195, 114.3586), (0.0519 + 0.0321i, 38.2911, 100.6915),$ $(0.0523 + 0.0091i, 44.2539, 91.4647), (0.0611 - 0.0114i, 37.4567, 110.9788),$ $(0.0458 - 0.0241i, 45.2773, 133.2038), (-0.0096 - 0.0628i, 36.4333, 78.4601),$ $(-0.0491 - 0.0097i, 46.5956, 56.2323), (0.0524 - 0.0388i, 35.115, 122.4184),$ $(-0.006 + 0.0472i, 48.4527, 33.9865), (0.0658 + 0.0155i, 33.2579, 85.624),$ $(0.0155 - 0.0407i, 51.6306, 118.1363), (0.0333 - 0.0633i, 30.08, 104.4692).$
	Cluster 3 with $(\bar{\theta}_{T;3}, \bar{\theta}_{R;3}) = (36.0489, -43.5504)$ and $(G_{3;s}, \theta_{T;3;s}, \theta_{R;3;s})$: $(-0.0017 + 0.119i, 44.3687, 128.7576), (-0.0984 + 0.0693i, 43.9215, 139.0449),$ $(-0.0127 + 0.1169i, 44.8515, 77.1409), (0.0252 + 0.1192i, 43.4387, 135.2672),$ $(-0.0095 + 0.1156i, 45.3912, 170.5049), (0.0077 + 0.1232i, 42.899, 99.3687),$ $(-0.0764 - 0.0848i, 46.0029, 121.6001), (0.0798 - 0.0965i, 42.2873, 183.5041),$ $(-0.0486 - 0.101i, 46.7093, 148.2717), (0.1057 - 0.071i, 41.5809, 143.327),$ $(-0.0363 - 0.1035i, 47.5437, 161.2763), (-0.1261 + 0.0309i, 40.7465, 54.8951),$ $(-0.0924 + 0.0532i, 48.5671, 125.3778), (-0.1124 + 0.071i, 39.7231, 112.3733),$ $(-0.102 + 0.0128i, 49.8854, 131.8874), (0.1297 - 0.0437i, 38.4048, 205.7499),$ $(-0.0938 - 0.0264i, 51.7425, 154.1124), (-0.1351 - 0.0448i, 36.5477, 90.1401),$ $(0.048 - 0.0745i, 54.9204, 117.318), (0.1499 - 0.0216i, 33.3698, 106.5326).$

Table 3.2 Examples of the 3GPP channel in microcell environments: Channel B

Channel C with $(\theta_{BS}, \theta_{MS}) = (-36.2193, 276.0062)$	Cluster 1 with $(\bar{\theta}_{T;1}, \bar{\theta}_{R;1}) = (18.9707, -25.6825)$ and $(G_{1;s}, \theta_{T;1;s}, \theta_{R;1;s})$:
	$(0.0284 - 0.0118i, -17.025, 245.379), (-0.0041 + 0.0303i, -17.4722, 219.3699),$ $(0.0188 - 0.0245i, -16.5422, 268.2729), (-0.0263 - 0.0154i, -17.955, 237.3192),$ $(0.0309 - 0.0022i, -16.0025, 290.5061), (-0.0271 - 0.0134i, -18.4947, 255.2684),$ $(0.0092 + 0.0298i, -15.3908, 259.0461), (0.005 + 0.0296i, -19.1064, 232.3745),$ $(-0.0285 - 0.0132i, -14.6844, 241.6013), (0.0266 - 0.0136i, -19.8128, 274.1136),$ $(-0.0253 - 0.019i, -13.85, 251.8886), (-0.0211 + 0.0207i, -20.6472, 303.5053),$ $(-0.0159 - 0.0276i, -12.8266, 281.2775), (-0.0287 + 0.0053i, -21.6706, 210.1413),$ $(-0.019 - 0.0259i, -11.5083, 263.3282), (0.028 + 0.0062i, -22.9889, 197.1421),$ $(-0.001 - 0.0324i, -9.6512, 248.7588), (-0.0276 - 0.0049i, -24.846, 325.7511),$ $(-0.0306 - 0.0122i, -6.4733, 174.8963), (-0.0195 + 0.0182i, -28.0239, 226.5338).$
	Cluster 2 with $(\bar{\theta}_{T;2}, \bar{\theta}_{R;2}) = (35.1232, -31.7676)$ and $(G_{2;s}, \theta_{T;2;s}, \theta_{R;2;s})$:
	$(-0.0208 + 0.0268i, -0.8725, 257.2431), (0.0123 - 0.0316i, -1.3197, 275.1924),$ $(0.0125 - 0.0316i, -0.3897, 231.2341), (-0.0339 + 0.0003i, -1.8025, 204.0562),$ $(-0.0311 - 0.0136i, 0.15, 245.8035), (-0.0338 - 0.0024i, -2.3422, 284.421),$ $(-0.0317 + 0.0122i, 0.7617, 220.4487), (-0.0282 - 0.0188i, -2.9539, 226.2894),$ $(0.0026 + 0.0338i, 1.4681, 268.0285), (-0.0165 + 0.0295i, -3.6603, 239.2939),$ $(0.0011 - 0.0339i, 2.3025, 297.4202), (0.0024 + 0.0337i, -4.4947, 252.961),$ $(0.0331 - 0.007i, 3.3259, 168.8112), (0.0226 - 0.0249i, -5.5181, 242.6737),$ $(0.0337 + 0.0014i, 4.6442, 235.5162), (-0.032 - 0.01i, -6.8364, 319.666),$ $(0.0316 + 0.0113i, 6.5013, 262.1878), (-0.0283 + 0.0174i, -8.6935, 191.057),$ $(0.0209 - 0.0256i, 9.6792, 213.2848), (-0.0041 + 0.0324i, -11.8714, 249.1833).$
	Cluster 3 with $(\bar{\theta}_{T;3}, \bar{\theta}_{R;3}) = (-17.7332, 31.7418)$ and $(G_{3;s}, \theta_{T;3;s}, \theta_{R;3;s})$:
	$(0.0113 - 0.0004i, -53.7289, 306.183), (0.0074 - 0.0083i, -54.1761, 289.7987),$ $(-0.0019 - 0.0113i, -53.2461, 347.9303), (-0.0009 + 0.011i, -54.6589, 316.4703),$ $(0.0057 + 0.0102i, -52.7064, 320.7524), (-0.0015 - 0.0107i, -55.1986, 312.6926),$ $(0.0027 + 0.0116i, -52.0947, 299.0255), (-0.0029 - 0.0102i, -55.8103, 267.5655),$ $(0.0057 - 0.0107i, -51.3883, 309.3128), (0.004 - 0.0096i, -56.5167, 383.1753),$ $(-0.0084 - 0.0092i, -50.5539, 254.5663), (0.0084 + 0.0055i, -57.3511, 338.7017),$ $(-0.0019 + 0.0126i, -49.5305, 294.7434), (0.0092 - 0.0032i, -58.3745, 283.958),$ $(-0.0124 - 0.0047i, -48.2122, 325.6971), (0.0087 - 0.0035i, -59.6928, 331.5378),$ $(-0.0099 - 0.0098i, -46.3551, 232.3205), (0.0028 - 0.0083i, -61.5499, 276.7941),$ $(0.0146 - 0.0038i, -43.1772, 302.8032), (-0.003 + 0.0072i, -64.7278, 360.9295).$

Table 3.3 Examples of the 3GPP channel in microcell environments: Channel C

Channel D with $(\theta_{BS}, \theta_{MS}) = (-42.2881, 223.0918)$	Cluster 1 with $(\bar{\theta}_{T,1}, \bar{\theta}_{R,1}) = (39.9062, -73.7786)$ and $(G_{1;s}, \theta_{T,1;s}, \theta_{R,1;s})$:
	$(0.0979 - 0.0246i, -2.1584, 158.0356), (0.0669 - 0.0755i, -2.6056, 173.1031),$ $(0.101 + 0.0015i, -1.6756, 224.7406), (-0.0979 + 0.024i, -3.0884, 131.364),$ $(-0.0575 + 0.0831i, -1.1359, 162.3177), (-0.0247 - 0.0976i, -3.6281, 147.7483),$ $(-0.0754 + 0.0673i, -0.5242, 144.3685), (-0.1004 - 0.0057i, -4.2398, 73.8858),$ $(-0.0109 + 0.1005i, 0.1822, 189.4956), (0.0992 + 0.0153i, -4.9462, 125.5233),$ $(-0.0611 + 0.0805i, 1.0166, 167.2624), (-0.0657 - 0.0756i, -5.7806, 150.8781),$ $(-0.053 - 0.0859i, 2.04, 96.1316), (0.0532 - 0.0844i, -6.804, 154.2579),$ $(0.0665 - 0.0757i, 3.3583, 118.3594), (-0.059 + 0.0798i, -8.1223, 202.4948),$ $(0.0409 + 0.0916i, 5.2154, 140.5908), (0.034 + 0.0922i, -9.9794, 136.3087),$ $(-0.043 - 0.0893i, 8.3933, 109.1308), (0.0962 + 0.0013i, -13.1573, 180.267).$
	Cluster 2 with $(\bar{\theta}_{T,2}, \bar{\theta}_{R,2}) = (31.6724, 18.2899)$ and $(G_{2;s}, \theta_{T,2;s}, \theta_{R,2;s})$:
	$(0.0807 - 0.0466i, -10.3921, 188.2001), (-0.0856 + 0.0364i, -10.8393, 232.6593),$ $(-0.072 - 0.0597i, -9.9093, 201.1993), (0.0891 + 0.0255i, -11.3221, 272.3355),$ $(0.0392 - 0.0852i, -9.3696, 239.8168), (-0.0234 - 0.0894i, -11.8618, 217.5918),$ $(-0.0637 + 0.0693i, -8.7579, 246.3264), (0.0025 - 0.092i, -12.4735, 281.5641),$ $(0.0917 + 0.0223i, -8.0515, 210.4279), (-0.076 - 0.0511i, -13.1799, 316.8091),$ $(0.0758 + 0.0568i, -7.2171, 236.437), (0.0651 + 0.0635i, -14.0143, 165.9543),$ $(0.0948 + 0.0072i, -6.1937, 259.3309), (0.0762 + 0.0483i, -15.0377, 228.3772),$ $(-0.0947 + 0.0122i, -4.8754, 250.1041), (0.0767 - 0.0455i, -16.356, 254.3862),$ $(-0.0241 + 0.0928i, -3.0183, 265.1716), (0.0858 + 0.0172i, -18.2131, 223.4325),$ $(-0.0019 - 0.0961i, 0.1596, 294.5633), (0.0504 - 0.0678i, -21.391, 242.9466).$
	Cluster 3 with $(\bar{\theta}_{T,3}, \bar{\theta}_{R,3}) = (-1.3216, -96.9731)$ and $(G_{3;s}, \theta_{T,3;s}, \theta_{R,3;s})$:
	$(-0.0309 + 0.0453i, -43.3861, 127.6836), (0.0478 - 0.0257i, -43.8333, 85.9363),$ $(0.0517 + 0.0201i, -42.9033, 131.0634), (0.0531 - 0.0077i, -44.3161, 201.5461),$ $(0.0561 + 0.0042i, -42.3636, 50.6913), (-0.0522 - 0.0088i, -44.8558, 149.9086),$ $(-0.0118 + 0.0558i, -41.7519, 113.1142), (-0.0502 - 0.014i, -45.4675, 139.1232),$ $(-0.0078 - 0.0575i, -41.0455, 166.3011), (-0.0325 + 0.0395i, -46.1739, 179.3003),$ $(-0.0587 - 0.0072i, -40.2111, 95.1649), (0.0173 - 0.0469i, -47.0083, 108.1695),$ $(0.0524 - 0.0302i, -39.1877, 102.3288), (-0.0186 + 0.045i, -48.0317, 144.0679),$ $(0.0548 + 0.0296i, -37.8694, 117.3963), (0.0043 + 0.0467i, -49.35, 72.9371),$ $(0.0423 + 0.0489i, -36.0123, 124.5538), (0.0442 - 0.0053i, -51.2071, 134.8411),$ $(-0.0058 - 0.0686i, -32.8344, 121.174), (-0.0288 + 0.0284i, -54.385, 157.0725).$

Table 3.4 Examples of the 3GPP channel in microcell environments: Channel D

	Parameters		Lower bounds on $\lambda_{\max}$			Exact value	Upper bounds on $\lambda_{\max}$		
	(n_i, N_R)	r_i	(3.23a)	(3.25a)	(3.26a)	$\lambda_{\max}$	(3.23b)	(3.25b)	(3.26b)
Channel A	(1,1)	1	1.2311	1.2311	1.2311	1.2311	1.2311	1.2311	1.2311
	(2,2)	2	2.2153	2.5074	2.5074	4.3313	4.4306	4.6031	4.6031
	(3,3)	3	2.8107	5.5469	4.8994	7.8114	8.4322	10.1695	10.0448
	(4,4)	4	3.8556	12.2601	8.1992	13.6279	15.4222	18.479	18.1518
	(5,5)	5	5.3378	21.9471	11.0854	22.2992	26.6888	26.7961	28.4562
	(6,6)	6	6.2354	25.7388	16.1572	29.3979	37.4126	41.6524	40.1776
	(7,7)	7	7.2212	22.9946	17.7175	38.6224	50.5483	51.8013	51.6122
	(8,8)	8	7.7392	38.3137	24.0207	45.4863	61.9133	68.6132	66.3385
	(9,9)	9	9.0354	55.1282	26.8003	56.6091	81.3187	72.5259	81.1333
	(10,10)	10	10.0333	58.193	36.0052	70.0158	100.33	104.2209	105.6716
Channel B	(1,1)	1	0.9627	0.9627	0.9627	0.9627	0.9627	0.9627	0.9627
	(2,2)	2	2.1304	2.2444	2.2444	3.2852	4.2609	3.3936	3.3936
	(3,3)	3	2.8377	5.6498	3.1413	5.6897	8.513	6.1856	6.2889
	(4,4)	4	3.6822	5.9109	5.4227	9.5989	14.7288	12.7234	12.7378
	(5,5)	5	4.0008	9.0512	7.0287	11.3537	20.0041	16.8886	15.588
	(6,6)	6	4.3886	14.9585	8.11	15.5909	26.3315	19.9692	20.9158
	(7,7)	7	4.8157	11.3674	9.6778	19.0357	33.7096	29.1121	27.6824
	(8,8)	8	5.5081	19.6297	12.9033	23.2239	44.0646	38.4992	35.9358
	(9,9)	9	6.4767	28.2351	14.477	29.6554	58.2907	41.3296	43.024
	(10,10)	10	7.5686	21.1306	16.9906	34.0757	75.6859	55.9425	52.5542
Channel C	(1,1)	1	0.0333	0.0333	0.0333	0.0333	0.0333	0.0333	0.0333
	(2,2)	2	0.0646	0.1005	0.1005	0.1223	0.1292	0.1457	0.1457
	(3,3)	3	0.1594	0.3148	0.3246	0.4355	0.4782	0.568	0.5428
	(4,4)	4	0.1905	0.3679	0.4005	0.6578	0.7619	0.8395	0.7963
	(5,5)	5	0.2375	0.5139	0.5591	0.9556	1.1876	1.2294	1.1796
	(6,6)	6	0.3066	0.8901	0.7849	1.2996	1.8397	1.7986	1.6321
	(7,7)	7	0.3836	1.3723	1.0771	1.7039	2.6851	2.3753	2.1393
	(8,8)	8	0.4154	1.6321	1.1698	1.8555	3.3233	2.5408	2.3194
	(9,9)	9	0.4417	1.7649	1.2803	2.0608	3.975	2.8815	2.8722
	(10,10)	10	0.4706	1.8398	1.3284	2.2156	4.7062	3.1128	3.2808
Channel D	(1,1)	1	0.2077	0.2077	0.2077	0.2077	0.2077	0.2077	0.2077
	(2,2)	2	0.4735	0.6244	0.6244	0.8391	0.947	0.9574	0.9574
	(3,3)	3	0.7551	1.1214	1.0341	1.58	2.2652	1.7453	1.7666
	(4,4)	4	1.0845	1.941	1.6863	2.102	4.3378	2.5203	2.6456
	(5,5)	5	1.913	5.4971	5.5809	5.8109	9.5651	7.4434	7.5346
	(6,6)	6	2.702	9.7271	10.3224	10.6297	16.2117	14.6206	13.7869
	(7,7)	7	3.2374	11.6229	12.0031	12.4679	22.6619	16.4649	16.8074
	(8,8)	8	3.8163	13.4373	13.523	14.0732	30.5307	18.0885	19.0721
	(9,9)	9	4.4845	15.9923	16.5186	17.3011	40.3606	23.8095	23.8334
	(10,10)	10	4.7271	17.4729	17.5302	18.5978	47.2707	26.2129	27.1195

Table 3.5 Maximum eigenvalue and bounds for Channels A, B, C and D

3.2 Rates for full rank channels without CSIT or with partial CSIT

In this section, $\mathbf{C}$ has full rank and it is unknown at the transmitter. In our deterministic model, we represent the channel lack of knowledge at the transmitter by a non-informative a-prior probability distribution on the components of the channel propagation matrix. We assume that the entries of this matrix are realizations of i.i.d. circularly symmetric complex Gaussian random variables, that are unknown at the transmitter, and held fixed during all channel uses. These entries are also assumed to have zero mean and independent real and imaginary parts, each with variance $\frac{\sigma^2}{2}$. Thus, we start this section by writing the channel propagation matrix as $\sigma\mathbf{C}$, where $E[|[\mathbf{C}]_{i,j}|^2] = 1$. As $\sigma \rightarrow \infty$, the a-prior distribution becomes non-informative. The mutual information (3.1) becomes

$$I_\sigma(\mathbf{x}; \mathbf{r}) = \log(|\mathbf{I}_{L_t N_R D} + \frac{\sigma^2}{N_0} \mathbf{H} \cdot \Sigma \cdot \mathbf{H}^\dagger|) \quad (3.27)$$

which is a random variable because of its dependency on $\mathbf{C}$ through $\mathbf{H}$, (2.20). Consider a rate $R > 0$. In our "deterministic" context, the probability $P_\sigma \equiv \Pr[I_\sigma(\mathbf{x}; \mathbf{r}) < R]$ represents the fraction of channels for which $I_\sigma(\mathbf{x}; \mathbf{r})$ is less than R , given Σ and σ . When $\sigma \rightarrow \infty$, the transmitter has zero channel knowledge and $P_\sigma \rightarrow 0$ no matter what rate R is chosen, provided that the input covariance matrix Σ is nonzero. We are interested in finding the matrix Σ that satisfies the following optimality criterion:

Definition 3.2.1 (Optimality Criterion 1) *A matrix Σ subject to $\text{Tr}(\Sigma) \leq P_T$ is said to be optimal in sense 1 if it causes fastest convergence of P_σ to 0 for any rate $0 < R < \infty$.*

We start by investigating the manner in which P_σ decreases with σ . Hence, we propose to extend the approach from [101] that considers the manner in which the outage probability for non-ergodic channels decreases when the SNR increases. With informed transmitter, capacity is achieved by an input covariance matrix for TOG i of the form $\Sigma_i = \mathbf{S}_i \otimes \mathbf{I}_{D_G}$, as seen from (3.8). With uninformed transmitter and despite the fact that it can lead to a suboptimal solution when considering Optimality Criterion 1, we choose to restrict attention to an input covariance matrix $\Sigma = \text{diag}_{i=1}^{N_G}[\mathbf{S}_i \otimes \mathbf{I}_{D_G}]$ since it is similar to the input covariance matrix that achieves capacity with full CSIT. As such, the input vectors to different TOGs $\{\mathbf{x}'_i\}_{i=1}^{N_G}$ are uncorrelated and hence, statistically independent because they are Gaussian. Furthermore, the input vectors $\{\mathbf{x}_{i,l}\}_{l=1}^{n_i}$ transmitted on the antennas of

TOG i are such that $E[\mathbf{x}_{i,l} \cdot \mathbf{x}_{i,p}^\dagger] = [\mathbf{S}_i]_{l,p} \mathbf{I}_{D_G}$. That is, the components of $\mathbf{x}_{i,l}$ and $\mathbf{x}_{i,p}$ associated with different signal dimensions are uncorrelated (and hence independent) while those associated with the same signal dimensions have a correlation of $[\mathbf{S}_i]_{l,p}$. Hence, with $\Sigma = \text{diag}_{i=1}^{N_G} [\mathbf{S}_i \otimes \mathbf{I}_{D_G}]$ we need to find the $n_i \times n_i$ Hermitian positive semidefinite matrices $\mathbf{S}_i \succeq \mathbf{0}_{n_i \times n_i}$ (with rank $r_{s;i}$) such that Σ satisfies Optimality Criterion 1. Using (C.1) and (3.6), (3.27) becomes

$$I_\sigma(\mathbf{x}; \mathbf{r}) = D_G \sum_{i=1}^{N_G} \log(|\mathbf{I}_{L_t N_R} + \frac{\sigma^2}{N_0} \mathbf{C}_i \cdot \mathbf{S}_i \cdot \mathbf{C}_i^\dagger|) \quad (3.28)$$

Assume $N_G \geq 2$. Systems employing ATA with $N_G = 1$ are investigated thereafter. Let η_i denote the fraction of power assigned to TOG i . We assume $0 < \eta_i < 1 \forall i$ and $\sum_{i=1}^{N_G} \eta_i = 1$. Under this assumption, if η_i is sufficiently small, then TOG i becomes inactive due to the vanishing small power that is allocated to it. The amount of power assigned to TOG i is given by $\text{Tr}(\Sigma_i) = \text{Tr}(\mathbf{S}_i \otimes \mathbf{I}_{D_G}) = \eta_i P_T$, yielding $\text{Tr}(\mathbf{S}_i) = \eta_i \frac{P_T}{D_G}$. Assuming arbitrary rank $r_{s;i}$ ($1 \leq r_{s;i} \leq n_i$), we consider the eigenvalue decomposition $\mathbf{S}_i = \mathbf{U}_{s;i}^\dagger \cdot \text{diag}[\eta_i \frac{P_T}{D_G} \Lambda_i, \mathbf{0}_{(n_i-r_{s;i}) \times (n_i-r_{s;i})}] \cdot \mathbf{U}_{s;i}$ with $n_i \times n_i$ unitary $\mathbf{U}_{s;i}$ and $r_{s;i} \times r_{s;i}$ (positive definite) diagonal Λ_i with $\text{Tr}(\Lambda_i) = 1$. By the invariance of the distribution of $\mathbf{C}_i$ when multiplied by $\mathbf{U}_{s;i}$ [3, Lemma 5], the distribution of $I_\sigma(\mathbf{x}; \mathbf{r})$ does not change if we choose $\mathbf{S}_i = \text{diag}[\eta_i \frac{P_T}{D_G} \Lambda_i, \mathbf{0}_{(n_i-r_{s;i}) \times (n_i-r_{s;i})}]$. For such $\mathbf{S}_i$, we have $\mathbf{C}_i \cdot \mathbf{S}_i \cdot \mathbf{C}_i^\dagger = \eta_i \frac{P_T}{D_G} \mathbf{C}_{s;i} \cdot \Lambda_i \cdot \mathbf{C}_{s;i}^\dagger$ where $\mathbf{C}_{s;i}$ is the submatrix formed by the first $r_{s;i}$ columns of $\mathbf{C}_i$. Using $\kappa = \frac{P_T}{N_0 D_G}$ and (C.1), (3.28) becomes

$$I_\sigma(\mathbf{x}; \mathbf{r}) = D_G \sum_{i=1}^{N_G} \log(|\mathbf{I}_{L_t N_R} + \eta_i \frac{P_T}{D_G} \frac{\sigma^2}{N_0} \mathbf{C}_{s;i} \cdot \Lambda_i \cdot \mathbf{C}_{s;i}^\dagger|) \quad (3.29)$$

$$= D_G \sum_{i=1}^{N_G} \log(|\Lambda_i|) + D_G \sum_{i=1}^{N_G} \log(|\Lambda_i^{-1} + \eta_i \kappa \sigma^2 \mathbf{C}_{s;i}^\dagger \cdot \mathbf{C}_{s;i}|) \quad (3.30)$$

We use the exponential equality [101] denoted by $\doteq$, defined for any function $f(\sigma)$ by

$$f(\sigma) \doteq \sigma^b \iff \lim_{\sigma \rightarrow \infty} \frac{\log[f(\sigma)]}{\log(\sigma)} = b$$

Consider the ratio $\Delta_i = \frac{|\Lambda_i^{-1} + \eta_i \kappa \sigma^2 \mathbf{C}_{s;i}^\dagger \cdot \mathbf{C}_{s;i}|}{|\mathbf{I}_{r_{s;i}} + \eta_i \kappa \sigma^2 \mathbf{C}_{s;i}^\dagger \cdot \mathbf{C}_{s;i}|} = \frac{|(1/\sigma^2)\Lambda_i^{-1} + \eta_i \kappa \mathbf{C}_{s;i}^\dagger \cdot \mathbf{C}_{s;i}|}{|(1/\sigma^2)\mathbf{I}_{r_{s;i}} + \eta_i \kappa \mathbf{C}_{s;i}^\dagger \cdot \mathbf{C}_{s;i}|}$. If $r_{s;i} \leq L_t N_R$, then as $\sigma \rightarrow \infty$ we have $\Delta_i \rightarrow \frac{|\eta_i \kappa \mathbf{C}_{s;i}^\dagger \cdot \mathbf{C}_{s;i}|}{|\eta_i \kappa \mathbf{C}_{s;i}^\dagger \cdot \mathbf{C}_{s;i}|} \equiv 1$ with $|\eta_i \kappa \mathbf{C}_{s;i}^\dagger \cdot \mathbf{C}_{s;i}| \neq 0$ since $\eta_i \neq 0$ and $\mathbf{C}_{s;i}^\dagger \cdot \mathbf{C}_{s;i}$ is nonsingular with probability one [110, Theorem 3.1.4]. Based on the same reference, if $L_t N_R < r_{s;i}$ then $\mathbf{C}_{s;i} \cdot \mathbf{C}_{s;i}^\dagger$ and $\mathbf{C}_{s;i} \cdot \Lambda_i \cdot \mathbf{C}_{s;i}^\dagger$ are nonsingular with probability one. Identity (C.7) leads to

$$\Delta_i = \frac{|\Lambda_i^{-1}| |(1/\sigma^2)\mathbf{I}_{L_t N_R} + \eta_i \kappa \mathbf{C}_{s;i} \cdot \Lambda_i \cdot \mathbf{C}_{s;i}^\dagger|}{|(1/\sigma^2)\mathbf{I}_{L_t N_R} + \eta_i \kappa \mathbf{C}_{s;i} \cdot \mathbf{C}_{s;i}^\dagger|}$$

which converges to $\frac{|\Lambda_i^{-1}| |\mathbf{C}_{s;i} \cdot \Lambda_i \cdot \mathbf{C}_{s;i}^\dagger|}{|\mathbf{C}_{s;i} \cdot \mathbf{C}_{s;i}^\dagger|}$ as $\sigma \rightarrow \infty$. Therefore, for any $r_{s;i}$ and $L_t N_R$, Δ_i tends to a constant as $\sigma \rightarrow \infty$, which leads to $\Delta_i \doteq \sigma^0 = 1$ and

$$|\Lambda_i^{-1} + \eta_i \kappa \sigma^2 \mathbf{C}_{s;i}^\dagger \cdot \mathbf{C}_{s;i}| \doteq |\mathbf{I}_{r_{s;i}} + \eta_i \kappa \sigma^2 \mathbf{C}_{s;i}^\dagger \cdot \mathbf{C}_{s;i}| \quad (3.31)$$

Define $m_i = \min(r_{s;i}, L_t N_R)$, $p_i = \max(r_{s;i}, L_t N_R)$. Define the Wishart matrix $\mathbf{W}_i = \mathbf{C}_{s;i}^\dagger \cdot \mathbf{C}_{s;i}$ if $r_{s;i} \leq L_t N_R$, and $\mathbf{W}_i = \mathbf{C}_{s;i} \cdot \mathbf{C}_{s;i}^\dagger$ if $L_t N_R < r_{s;i}$. Using $|\mathbf{I}_{r_{s;i}} + \eta_i \kappa \sigma^2 \mathbf{C}_{s;i}^\dagger \cdot \mathbf{C}_{s;i}| = |\mathbf{I}_{m_i} + \eta_i \kappa \sigma^2 \mathbf{W}_i|$ and (3.31), equation (3.30) reduces to

$$I_\sigma(\mathbf{x}; \mathbf{r}) \doteq D_G \sum_{i=1}^{N_G} \log(|\Lambda_i|) + D_G \sum_{i=1}^{N_G} \log(|\mathbf{I}_{m_i} + \eta_i \kappa \sigma^2 \mathbf{W}_i|)$$

Hence, we can write

$$\begin{aligned} P_\sigma &\doteq \Pr \left[D_G \sum_{i=1}^{N_G} \log(|\Lambda_i|) + D_G \sum_{i=1}^{N_G} \log(|\mathbf{I}_{m_i} + \eta_i \kappa \sigma^2 \mathbf{W}_i|) < R \right] \\ &\doteq \Pr \left[\sum_{i=1}^{N_G} \sum_{l=1}^{m_i} \log(1 + \eta_i \kappa \sigma^2 w_{i,l}) < \frac{R}{D_G} - \sum_{i=1}^{N_G} \log(|\Lambda_i|) \right] \end{aligned} \quad (3.32)$$

where $\{\omega_{i,l}\}_{l=1}^{m_i}$ denote the eigenvalues of $\mathbf{W}_i$ satisfying $\omega_{i,1} \geq \omega_{i,2} \geq \dots \geq \omega_{i,m_i} \geq 0$. Since Λ_i is positive definite with $\text{Tr}(\Lambda_i) = 1$, we have $0 < [\Lambda_i]_{l,l} < 1$ for all $1 \leq l \leq r_{s;i}$ yielding $\log(|\Lambda_i|) < 0$. Define $\varrho_{i,l} \equiv f_i(\omega_{i,l}) = \log(1 + \eta_i \kappa \sigma^2 w_{i,l})$, or equivalently, $\omega_{i,l} \equiv f_i^{-1}(\varrho_{i,l}) = \frac{2^{\varrho_{i,l}} - 1}{\eta_i \kappa \sigma^2}$. Since the log function is monotonically increasing, we have $\varrho_{i,1} \geq \varrho_{i,2} \geq \dots \geq \varrho_{i,m_i} \geq 0$. Define $\boldsymbol{\varrho}_i = \text{vec}(\{\varrho_{i,l}\}_{l=1}^{m_i})$ and let $m_T = \sum_{i=1}^{N_G} m_i$.

Lemma 3.2.1 *The probability density function (PDF) of $\mathbf{p} = \text{vec}(\{\mathbf{p}_i\}_{i=1}^{N_G})$ is given by*

$$p_{\mathbf{p}}(\mathbf{u}) = \prod_{i=1}^{N_G} \left\{ \frac{K_{m_i, p_i}^{-1} (\ln 2)^{m_i}}{(\eta_i \kappa \sigma^2)^{m_i p_i}} \prod_{l=1}^{m_i} \left[e^{\left\{ -\frac{2^{u_{i,l}} - 1}{\eta_i \kappa \sigma^2} \right\}} 2^{u_{i,l}} (2^{u_{i,l}} - 1)^{\{p_i - m_i\}} \prod_{j=l+1}^{m_i} (2^{u_{i,l}} - 2^{u_{i,j}})^2 \right] \right\} \quad (3.33)$$

where $\mathbf{u} = \text{vec}(\{\mathbf{u}_i\}_{i=1}^{N_G})$ with $\mathbf{u}_i = \text{vec}(\{u_{i,l}\}_{l=1}^{m_i})$, $u_{i,1} \geq u_{i,2} \geq \dots \geq u_{i,m_i} \geq 0$, and $\{K_{m_i, p_i}\}_{i=1}^{N_G}$ are given constants.

The proof of Lemma 3.2.1 can be found in Appendix A. We define $W = R/D_G - \sum_{i=1}^{N_G} \log(|\Lambda_i|) = R/D_G + \log(1/\prod_{i=1}^{N_G} |\Lambda_i|)$. Equation (3.32) becomes

$$P_{\sigma} \doteq \Pr \left[\sum_{i=1}^{N_G} \sum_{l=1}^{m_i} \varrho_{i,l} < W \right] = \int_{\mathcal{A}_{\varrho}} p_{\mathbf{p}}(\mathbf{u}) d\mathbf{u} \quad (3.34)$$

where $\mathcal{A}_{\varrho} = \{\mathbf{u} \in \mathbb{R}^{m_T}: u_{i,1} > u_{i,2} > \dots > u_{i,m_i} > 0 \forall i, \sum_{i=1}^{N_G} \sum_{l=1}^{m_i} u_{i,l} < W\}$ with strict inequalities $u_{i,l} > u_{i,j}$ for $j > l$ since, otherwise, the integrand in (3.34) reduces to zero.

Theorem 3.2.1 *Given rate $R < \infty$, the fraction of channels P_{σ} for which $I_{\sigma}(\mathbf{x}; \mathbf{r}) < R$ satisfies*

$$P_{\sigma} \doteq \left(\prod_{i=1}^{N_G} \eta_i^{-r_{s,i} L_t N_R} \right) \tilde{I}_{\mathcal{A}_{\varrho}} \sigma^{-2(\sum_{i=1}^{N_G} r_{s,i}) L_t N_R} \quad (3.35)$$

where $\tilde{I}_{\mathcal{A}_{\varrho}} = \int_{\mathcal{A}_{\varrho}} \prod_{i=1}^{N_G} \left\{ \frac{K_{m_i, p_i}^{-1} (\ln 2)^{m_i}}{\kappa^{m_i p_i}} \prod_{l=1}^{m_i} [2^{u_{i,l}} (2^{u_{i,l}} - 1)^{\{p_i - m_i\}} \prod_{j=l+1}^{m_i} (2^{u_{i,l}} - 2^{u_{i,j}})^2] \right\} d\mathbf{u}$.

The proof of Theorem 3.2.1 is provided in Appendix A. Notice that $\tilde{I}_{\mathcal{A}_{\varrho}}$ is independent of σ and $\{\eta_i\}_{i=1}^{N_G}$. From (3.35), the fastest convergence of P_{σ} to zero when $\sigma \rightarrow \infty$ is obtained by: first maximizing the exponent of σ^{-1} (which is dominant), and then minimizing the coefficient $(\prod_{i=1}^{N_G} \eta_i^{-r_{s,i} L_t N_R}) \tilde{I}_{\mathcal{A}_{\varrho}}$.

Problem 1: We consider the maximization of the exponent of σ^{-1} or, equivalently, $\sum_{i=1}^{N_G} r_{s,i}$ subject to $1 \leq r_{s,i} \leq n_i \forall i$. This is obtained when $r_{s,i} = n_i$, yielding full rank matrices $\{\Sigma_i\}_{i=1}^{N_G}$.

Problem 2: Using $r_{s,i} = n_i$ for all TOGs, we consider the minimization of $\prod_{i=1}^{N_G} \eta_i^{-n_i L_t N_R}$ subject to the constraints $0 < \eta_i < 1$ and $\sum_{i=1}^{N_G} \eta_i = 1$. Such problem is equivalent to the

minimization of $-\sum_{i=1}^{N_G} n_i L_t N_R \log(\eta_i)$. Consider the following Lagrangian

$$\mathcal{G}(\eta_1, \dots, \eta_{N_G}) = -\sum_{i=1}^{N_G} n_i L_t N_R \log(\eta_i) + \sum_{i=1}^{N_G} \kappa_i (\eta_i - 1) + \sum_{i=1}^{N_G} \kappa'_i (-\eta_i) + \nu \left(\sum_{i=1}^{N_G} \eta_i - 1 \right) \quad (3.36)$$

The optimal solution $\{\tilde{\eta}_i\}_{i=1}^{N_G}$ to such convex optimization problem is obtained when the following necessary and sufficient Karush-Kuhn-Tucker (KKT) conditions are satisfied [111]:

$$\text{a) } \left[\frac{\partial \mathcal{G}(\eta_1, \dots, \eta_{N_G})}{\partial \eta_i} \right]_{\eta_i = \tilde{\eta}_i} = -\frac{n_i L_t N_R}{\tilde{\eta}_i \ln 2} + \kappa_i - \kappa'_i + \nu = 0 \text{ for all } i = 1, \dots, N_G.$$

$$\text{b) } 0 < \tilde{\eta}_i < 1 \text{ for all } i \text{ and } \sum_{i=1}^{N_G} \tilde{\eta}_i = 1.$$

$$\text{c) } \kappa_i \geq 0, \kappa'_i \geq 0, \kappa_i (\tilde{\eta}_i - 1) = 0 \text{ and } \kappa'_i (-\tilde{\eta}_i) = 0 \text{ for all } i.$$

One can easily verify that the KKT conditions are satisfied for $\tilde{\eta}_i = \frac{n_i}{N_T}$, $\kappa_i = \kappa'_i = 0$ and $\nu = \frac{N_T L_t N_R}{\ln 2}$. Therefore, $\tilde{\eta}_i = \frac{n_i}{N_T}$ is the optimal solution for Problem 2.

Problem 3: Using $r_{s,i} = n_i$ for all TOGs, we consider the minimization of $\tilde{I}_{\mathcal{A}_\theta}$. Since the integrand of $\tilde{I}_{\mathcal{A}_\theta}$ is positive, one can easily show that $\tilde{I}_{\mathcal{A}'_\theta} \leq \tilde{I}_{\mathcal{A}_\theta}$ where $\mathcal{A}'_\theta$ is any area of integration satisfying $\mathcal{A}'_\theta \subseteq \mathcal{A}_\theta$. It follows that $\tilde{I}_{\mathcal{A}_\theta}$ is minimum when $\mathcal{A}_\theta$ is minimum, corresponding to $\mathcal{A}_\theta = \mathcal{A}_{\theta, \min} \equiv \{\mathbf{u} \in \mathbb{R}^{m_T}: u_{i,1} > u_{i,2} > \dots > u_{i,m_i} > 0 \forall i, \sum_{i=1}^{N_G} \sum_{l=1}^{m_i} u_{i,l} < \min(W)\}$. Now for any $0 < R < \infty$, $W \equiv R/D_G + \log(1/\prod_{i=1}^{N_G} |\Lambda_i|)$ is minimum when $\prod_{i=1}^{N_G} |\Lambda_i|$ is maximum. Hence, Problem 3 is solved by maximizing $|\Lambda_i|$ for each TOG i subject to the individual constraints $Tr(\Lambda_i) = 1$, which corresponds to a uniform power distribution among the antennas $\Lambda_i = \frac{1}{n_i} \mathbf{I}_{n_i}$.

So far we have investigated systems with $N_G \geq 2$. The matrix Σ satisfying Optimality Criterion 1 for systems with $N_G = 1$ can be found using the substitution $N_G = 1$ and, hence, $\eta_i = 1$ in (3.29) - (3.35). In this case, (3.35) reduces to $P_\sigma \doteq \tilde{I}_{\mathcal{A}_\theta} \sigma^{-2r_{s,1} L_t N_R}$ and its fastest convergence to zero when $\sigma \rightarrow \infty$ is obtained by: maximizing $r_{s,1}$ as in Problem 1 resulting in $r_{s,1} = N_T$, and then minimizing $\tilde{I}_{\mathcal{A}_\theta}$ as in Problem 3 resulting in $\Lambda_1 = \frac{1}{N_T} \mathbf{I}_{N_T}$. Problem 2 is irrelevant if $N_G = 1$. Thus, $\forall N_G \in \{1, \dots, N_T\}$ Optimality Criterion 1 is satisfied using a full rank diagonal input covariance matrix with uniform power allocation

$$\Sigma = \text{diag}_{i=1}^{N_G} [\mathbf{S}_i \otimes \mathbf{I}_{D_G}] = \frac{N_G P_T}{N_T D} \mathbf{I}_{\tilde{D}} \quad (3.37)$$

The following Lemma considers the behavior of the information rate obtained using the power allocation (3.37) for the particular class of MIMO channels where $\mathbf{C}^\dagger \cdot \mathbf{C}$ is diagonal.

Lemma 3.2.2 *For MIMO channels whose $N_T \times N_T$ matrix $\mathbf{C}^\dagger \cdot \mathbf{C}$ is diagonal, the information rate resulting from the uniform power allocation (3.37) is monotonically decreasing with N_G .*

The proof of Lemma 3.2.2 can be found in Appendix A. Consequently, for MIMO channels with diagonal matrix product $\mathbf{C}^\dagger \cdot \mathbf{C}$ the uniform power allocation (3.37) results in maximum, intermediate and minimum information rates using ATA, POTA and OTA, respectively. For more general channels with arbitrary $\mathbf{C}^\dagger \cdot \mathbf{C}$ (including the 3GPP channels), we put forward a conjecture that will be proved subsequently for low and high SNR.

Conjecture 3.2.1 *With uniform power allocation (3.37), the space-time modulation formats ATA, POTA and OTA result in maximum, intermediate and minimum information rates, respectively.*

Even though no transmitter channel knowledge is needed to satisfy Optimality Criterion 1, the uniform power allocation (3.37) does not necessarily maximize the mutual information $I(\mathbf{x}; \mathbf{r})$. Thus, we also consider other TOG power allocations η_i that more directly impact $I(\mathbf{x}; \mathbf{r})$ while requiring partial transmitter channel knowledge. While we relax the TOG power allocation requirements by letting $0 \leq \eta_i \leq 1$ and by allowing the rank $r_{s;i}$ to vary between 1 and n_i , we keep the uniform power allocation among the active antennas of each TOG, which yields $\Lambda_i = \frac{1}{r_{s;i}} \mathbf{I}_{r_{s;i}}$. The resulting input signal $\mathbf{x}$ is zero mean circularly symmetric Gaussian with independent components and covariance matrix

$$\Sigma = \text{diag}_{i=1}^{N_G} \left[\text{diag} \left[\frac{\eta_i}{r_{s;i}} \frac{P_T}{D_G} \mathbf{I}_{r_{s;i}}, \mathbf{0}_{(n_i-r_{s;i}) \times (n_i-r_{s;i})} \right] \otimes \mathbf{I}_{D_G} \right] \quad (3.38)$$

which can be seen as a generalization of (3.37). Let I_u denote the channel information rate $I(\mathbf{x}; \mathbf{r})$ associated with (3.38). As in (3.29), this information rate can be written as $D_G \sum_{i=1}^{N_G} \log(|\mathbf{I}_{L_t N_R} + \frac{\eta_i}{r_{s;i}} \frac{P_T}{D_G N_0} \mathbf{C}_{s;i} \cdot \mathbf{C}_{s;i}^\dagger|)$ in bits/channel-use or, equivalently, as $I_u = \frac{D_G}{D} \sum_{i=1}^{N_G} \log(|\mathbf{I}_{L_t N_R} + \frac{\eta_i}{r_{s;i}} \frac{P_T}{D_G N_0} \mathbf{C}_{s;i} \cdot \mathbf{C}_{s;i}^\dagger|)$ in bps/Hz. Using (C.1), this becomes $I_u = \frac{D_G}{D} \sum_{i=1}^{N_G} \log(|\mathbf{I}_{r_{s;i}} + \frac{\eta_i}{r_{s;i}} \frac{P_T}{D_G N_0} \mathbf{C}_{s;i}^\dagger \cdot \mathbf{C}_{s;i}|)$ bps/Hz. Let $\{\xi_{i,l}\}_{l=1}^{r_{s;i}}$ denote the (real) eigenvalues of $\mathbf{C}_{s;i}^\dagger \cdot \mathbf{C}_{s;i}$ satisfying $\xi_{i,1} \geq \dots \geq \xi_{i,r_{s;i}} \geq 0$ for all i . We can write

$$I_u = \frac{1}{N_G} \sum_{i=1}^{N_G} \sum_{l=1}^{r_{s;i}} \log \left(1 + \frac{\eta_i}{r_{s;i}} \frac{N_G P_T}{N_0 D} \xi_{i,l} \right) \text{ bps/Hz} \quad (3.39)$$

Subsequently, we use $\boldsymbol{\eta} = \text{vec}(\{\eta_i\}_{i=1}^{N_G})$ to denote a power allocation. The set of all TOG power allocations is defined as $\mathcal{A}_\eta = \{\boldsymbol{\eta} : 0 \leq \eta_i \leq 1 \text{ and } \sum_{i=1}^{N_G} \eta_i = 1\}$. Maximization of I_u over $\mathcal{A}_\eta$ can be done using the KKT technique [111], which requires full transmitter knowledge of the eigenvalues $\xi_{i,l}$. We are interested in the case when much less channel information is available at the transmitter. Thus, we consider the maximization of I_u over all possible ranks $\{r_{s,i}\}_{i=1}^{N_G}$ and TOG power allocations $\boldsymbol{\eta}$ at low and high SNR where only partial CSIT may be required.

3.2.1 Channel information rate at low SNR

We consider the low SNR regime where the ratio P_T/N_0 is very small with respect to the inverses of the nonzero eigenvalues $\xi_{i,l}$, that is, $(P_T/N_0) \ll \xi_{i,l}^{-1}$ or $(P_T/N_0)\xi_{i,l} \ll 1$ for all $\xi_{i,l} \neq 0$, $l = 1, \dots, r_{s,i}$ and $i = 1, \dots, N_G$. From (3.39), we have

$$I_u \approx \frac{1}{N_G} \log \left(1 + \sum_{i=1}^{N_G} \sum_{l=1}^{r_{s,i}} \frac{\eta_i}{r_{s,i}} \frac{N_G P_T}{N_0 D} \xi_{i,l} \right) = \frac{1}{N_G} \log \left(1 + \frac{B P_T}{N_0 D} \right) \quad (3.40)$$

where $B = N_G \sum_{i=1}^{N_G} \sum_{l=1}^{r_{s,i}} \frac{\eta_i \xi_{i,l}}{r_{s,i}}$, and $(1 + \frac{B P_T}{N_0 D})$ is a first order approximation of the product $\prod_{i=1}^{N_G} \prod_{l=1}^{r_{s,i}} (1 + \frac{\eta_i}{r_{s,i}} \frac{N_G P_T}{N_0 D} \xi_{i,l})$. Using (3.3) at rate I_u , we can write $I_u \approx \frac{1}{N_G} \log(1 + B I_u (E_b/N_0)_{T,u})$ where $(E_b/N_0)_{T,u}$ denotes the SNR-per-bit with uninformed transmitter. Thus, we have

$$(E_b/N_0)_{T,u} \approx \frac{2^{N_G I_u} - 1}{B I_u} = \frac{\sum_{k=0}^{\infty} (N_G I_u \ln 2)^k / k! - 1}{B I_u} \approx \frac{N_G^2 (\ln 2)^2}{2B} I_u + \frac{N_G \ln 2}{B} \quad (3.41)$$

The last approximation is of the second order. If $I_u \rightarrow 0$, then $(E_b/N_0)_{T,u} \rightarrow \left[\frac{E_b}{N_0} \right]_{u,\min} = \frac{N_G \ln 2}{B}$ (or $-1.6 + 10 \log_{10}(N_G) - 10 \log_{10}(B)$ dB), that is analogous to [98, Theorem 1] (i.e. $\left[\frac{E_b}{N_0} \right]_{\min} = \frac{\ln 2}{G}$) for deterministic channels with multi-dimensional space-time modulation whose channel gain G is given by (3.14). Writing $\text{Tr}(\mathbf{H} \cdot \boldsymbol{\Sigma} \cdot \mathbf{H}^\dagger) = \text{Tr}(\boldsymbol{\Sigma} \cdot \mathbf{H}^\dagger \cdot \mathbf{H})$ and using (3.6) and (3.38), (3.14) leads to $G = \sum_{i=1}^{N_G} \frac{\text{Tr}(\mathbf{C}_{s,i}^\dagger \cdot \mathbf{C}_{s,i})}{r_{s,i}} \eta_i \equiv \frac{B}{N_G}$. It follows that the minimum received SNR-per-bit required for reliable communication is given by $\left[\frac{E_b}{N_0} \right]_{R,u,\min} = \frac{B}{N_G} \left[\frac{E_b}{N_0} \right]_{u,\min} = \ln 2$ (or -1.6 dB), which corresponds to the Shannon limit [84, p.262]. From (3.41), write $I_u \approx \frac{2B}{N_G^2 (\ln 2)^2} (E_b/N_0)_{T,u} - \frac{2}{N_G \ln 2}$ which is linear

in $(E_b/N_0)_{T;u}$ with slope $\frac{2B}{N_G^2(\ln 2)^2}$, or [98, (29)]

$$\lim_{(E_b/N_0)_{T;u} \rightarrow \left[\frac{E_b}{N_0}\right]_{u;\min}} \frac{10 \log_{10}(2) I_u}{10 \log_{10}((E_b/N_0)_{T;u}) - 10 \log_{10}(\left[\frac{E_b}{N_0}\right]_{u;\min})} = \frac{2}{N_G} \text{ bps/Hz/(3 dB)} \quad (3.42)$$

One can easily see that such slope is independent of the ranks $\{r_{s;i}\}_{i=1}^{N_G}$, the power allocation η and the channel propagation matrix. The impact of these parameters on I_u is captured in $\left[\frac{E_b}{N_0}\right]_{u;\min}$ and, more precisely, in the parameter B which is also dependent on the space-time modulation format.

Given $N_G \in \{1, \dots, N_T\}$, we investigate first the ranks $\{r_{s;i}\}_{i=1}^{N_G}$ that maximize I_u for fixed η . Maximizing I_u of (3.40) is obtained by maximizing B . Thus, we need to maximize $\sum_{i=1}^{N_G} \frac{\text{Tr}(\mathbf{C}_{s;i}^\dagger \cdot \mathbf{C}_{s;i})}{r_{s;i}} \eta_i$ subject to $1 \leq r_{s;i} \leq n_i, \forall i$. This can be done by maximizing the ratios $\left\{ \frac{\text{Tr}(\mathbf{C}_{s;i}^\dagger \cdot \mathbf{C}_{s;i})}{r_{s;i}} \right\}_{i=1}^{N_G}$ as follows. For each TOG, we have

$$\frac{\text{Tr}(\mathbf{C}_{s;i}^\dagger \cdot \mathbf{C}_{s;i})}{r_{s;i}} = \frac{\sum_{l=1}^{r_{s;i}} [\mathbf{C}_{s;i}^\dagger \cdot \mathbf{C}_{s;i}]_{l,l}}{r_{s;i}} \leq \max\{[\mathbf{C}_{s;i}^\dagger \cdot \mathbf{C}_{s;i}]_{l,l}\}_{l=1}^{r_{s;i}} \leq \max\{[\mathbf{C}_i^\dagger \cdot \mathbf{C}_i]_{q,q}\}_{q=1}^{n_i}$$

since $\{[\mathbf{C}_{s;i}^\dagger \cdot \mathbf{C}_{s;i}]_{l,l}\}_{l=1}^{r_{s;i}}$ is a subset of $\{[\mathbf{C}_i^\dagger \cdot \mathbf{C}_i]_{q,q}\}_{q=1}^{n_i}$. Writing $[\mathbf{C}_i^\dagger \cdot \mathbf{C}_i]_{q,q} = \|\mathbf{C}_i\|_{*,q}^2$, we have $\frac{\text{Tr}(\mathbf{C}_{s;i}^\dagger \cdot \mathbf{C}_{s;i})}{r_{s;i}} \leq \max\{\|\mathbf{C}_i\|_{*,q}^2\}_{q=1}^{n_i}$. Equality is ensured if $r_{s;i} = 1$ and $\mathbf{C}_{s;i}$ is the column of $\mathbf{C}_i$ with maximal 2-norm. As such, only the antenna associated with $\max\{\|\mathbf{C}_i\|_{*,q}^2\}_{q=1}^{n_i}$ can be active in TOG i . Moreover, $\Lambda_i = 1$ and $\mathbf{S}_i$ becomes rank one. Denoting $\|\mathbf{C}_i\|_{*,\max}^2 = \max\{\|\mathbf{C}_i\|_{*,q}^2\}_{q=1}^{n_i}$, we have

$$B = N_G \sum_{i=1}^{N_G} \|\mathbf{C}_i\|_{*,\max}^2 \eta_i \quad (3.43)$$

If $N_G \geq 2$, we consider the maximization of (3.43) over $\mathcal{A}_\eta$. This is a linear programming problem [112, Chapter 4] whose geometry shows that the optimal η corresponds to allocating the total power P_T to the antenna associated with $\max\{\|\mathbf{C}_i\|_{*,\max}^2\}_{i=1}^{N_G}$. If such antenna belongs to TOG $i_{\max}$, then $\eta_{i_{\max}} = 1$ and $\eta_i = 0$ for all $i \neq i_{\max}$. We refer to this power allocation as the strongest antenna power allocation. The optimality of the strongest antenna selection in terms of maximizing I_u agrees with that of [21] where it is seen as a tool for maximizing the channel gain (defined in [98, (57)]) as the first order derivative of the spectral efficiency with respect to the SNR at zero SNR) for systems with large number

of transmit antennas. If there are several strongest antennas, B is maximized by allocating P_T to these antennas using an arbitrary power distribution among them. In all cases, the maximum value for B is given by $B_{\max} = N_G \max\{\|[\mathbf{C}_i]_{*,\max}\|^2\}_{i=1}^{N_G}$. Now one can see that $\max\{\|[\mathbf{C}_i]_{*,\max}\|^2\}_{i=1}^{N_G} = \max\{\|[\mathbf{C}]_{*,l}\|^2\}_{l=1}^{N_T}$, thus, yielding

$$B_{\max} = N_G \max\{\|[\mathbf{C}]_{*,l}\|^2\}_{l=1}^{N_T} \quad (3.44)$$

The transmitter needs only knowledge of $\{\|[\mathbf{C}]_{*,l}\|^2\}_{l=1}^{N_T}$ which can be evaluated at the receiver and fed back to the transmitter. If $N_G = 1$, the maximization of (3.43) over $\mathcal{A}_\eta$ is irrelevant, and the strongest antenna power allocation makes (3.43) equal to (3.44). Hence, $B_{\max}$ of (3.44) is the maximum B for all N_G . Let B_A , B_P and B_O denote the values of $B_{\max}$ using ATA, POTA and OTA, respectively. We prove in Appendix A the following result.

Lemma 3.2.3 *Using the strongest antenna power allocation, the largest rate I_u is provided by ATA while POTA and OTA result in intermediate and low rates, respectively.*

Let us investigate the relationship between the strongest antenna power allocation and the opportunistic beamforming technique employed in the downlink of a multi-user system with K users each employing a single receive antenna [113]. For simplicity, we use a setting similar to that of [113], that is, we assume $L_t = 1$ and we employ ATA when comparing the two techniques. In this case, the channel equation (2.19) reduces to $\mathbf{r} = [\mathbf{C} \otimes \mathbf{I}_D] \cdot \mathbf{x} + \mathbf{v}$ with $\mathbf{r}$ and $\mathbf{x}$ being of size $N_R D \times 1$ and $N_T D \times 1$, respectively. As such, the $D \times 1$ output vector received on antenna m is given by

$$\mathbf{r}_m = \sum_{n=1}^{N_T} [\mathbf{C}]_{m,n} \mathbf{x}_n + \mathbf{v}_m \quad (3.45)$$

for $m = 1, \dots, N_R$. In [113], the same $T \times 1$ block of symbols $\mathbf{x}'$ (with the power constraint $E[\|\mathbf{x}'\|^2] = PT$) is transmitted from all the transmit antennas except that it is multiplied by a complex number $\sqrt{\alpha_n} e^{j\theta_n}$ at antenna n , for $n = 1, \dots, N_T$, such that $\sum_{n=1}^{N_T} \alpha_n = 1$. Using $c_{k,n}$ to denote the channel propagation coefficient from transmit antenna n to user k ($k = 1, \dots, K$), the $T \times 1$ output vector received by user k is given by [113, (2)]

$$\mathbf{r}'_k = \sum_{n=1}^{N_T} c_{k,n} \sqrt{\alpha_n} e^{j\theta_n} \mathbf{x}' + \mathbf{v}'_k \equiv c_k \mathbf{x}' + \mathbf{v}'_k \quad (3.46)$$

where $c_k = \sum_{n=1}^{N_T} c_{k,n} \sqrt{\alpha_n} e^{j\theta_n}$ and $\mathbf{v}'_k$ is a $T \times 1$ zero mean circularly symmetric Gaussian noise vector with autocorrelation matrix $N_0 \mathbf{I}_T$. Even though (3.45) and (3.46) are similar, the base station employs different selection criteria for the strongest antenna power allocation and the opportunistic beamforming. In the former case, the selection criterion is based on maximizing the information-theoretic rate at low SNR and the base station selects the transmit antenna that is associated with $\max_{n=1, \dots, N_T} \left(\sum_{m=1}^{N_R} |[\mathbf{C}]_{m,n}|^2 \right)$. In the latter case, an SNR-oriented selection criterion is employed. Here, $\{\alpha_n\}_{n=1}^{N_T}$ and $\{\theta_n\}_{n=1}^{N_T}$ are varied in a pseudorandom manner and at any time the base station selects the user whose instantaneous SNR $\frac{P|c_k|^2}{N_0}$ is closest to its peak value $\frac{P \sum_{n=1}^{N_T} |c_{k,n}|^2}{N_0}$ which is achieved by beamforming using $\alpha_n = \frac{|c_{k,n}|^2}{\sum_{p=1}^{N_T} |c_{k,p}|^2}$ and $\theta_n = -\arg(c_{k,n})$ for all $n = 1, \dots, N_T$. Hence, one can see that the main difference between the strongest antenna power allocation and opportunistic beamforming is that the former involves a maximization over the norms of the column vectors of $\mathbf{C}$ while the latter employs the norms of the row vectors of the matrix $[\mathbf{C}]_{k,n}$ (without any maximization) to find the user whose SNR is closest to its peak beamforming value.

Further assessment of the strongest antenna power allocation can be obtained by comparing it to the power allocation that maximizes I_u when $\{\mathbf{S}_i\}_{i=1}^{N_G}$ are full rank (i.e. $r_{s;i} = n_i$). In this case, B of (3.40) reduces to $B_f = N_G \sum_{i=1}^{N_G} \frac{\text{Tr}(\mathbf{C}_i^\dagger \mathbf{C}_i)}{n_i} \eta_i$. Similarly to (3.43), if $N_G \geq 2$ further maximization of B_f over $\mathcal{A}_\eta$ is obtained by allocating the total power P_T to the TOG associated with $\max\left\{\frac{\text{Tr}(\mathbf{C}_i^\dagger \mathbf{C}_i)}{n_i}\right\}_{i=1}^{N_G}$. We refer to this power allocation as the strongest TOG power allocation for which the transmitter needs only knowledge of $\{\text{Tr}(\mathbf{C}_i^\dagger \mathbf{C}_i)\}_{i=1}^{N_G}$. The resulting maximum value of B_f is $B_{f;\max} = N_G \max\left\{\frac{\text{Tr}(\mathbf{C}_i^\dagger \mathbf{C}_i)}{n_i}\right\}_{i=1}^{N_G}$, which can be easily shown to hold also for ATA ($N_G = 1$ and $\eta_i = 1$). Let $B_{f,A}$, $B_{f,P}$ and $B_{f,O}$ denote the values of $B_{f;\max}$ using ATA, POTA and OTA, respectively. With ATA, we have $B_{f,A} = \frac{\text{Tr}(\mathbf{C}^\dagger \mathbf{C})}{N_T}$ yielding $B_{f,A} \leq B_A$. With POTA, denoting $\left[\frac{\text{Tr}(\mathbf{C}_i^\dagger \mathbf{C}_i)}{n_i}\right]_{\max} = \max\left\{\frac{\text{Tr}(\mathbf{C}_i^\dagger \mathbf{C}_i)}{n_i}\right\}_{i=1}^{N_G}$, we have $B_{f,P} = N_G \left[\frac{\text{Tr}(\mathbf{C}_i^\dagger \mathbf{C}_i)}{n_i}\right]_{\max}$ yielding $B_{f,P} \leq B_P$. With OTA, we have $B_{f,O} = N_T \max\{\|\mathbf{C}\|_{*,l}^2\}_{l=1}^{N_T} = B_O$.

Lemma 3.2.4 *Using the strongest TOG power allocation, the largest rate I_u is provided by OTA while POTA and ATA result in intermediate and low rates, respectively.*

The proof of Lemma 3.2.4 can be found in Appendix A. As $r_{s;i}$ drops from n_i to 1, $\left[\frac{E_b}{N_0}\right]_{u;\min}$ drops from $\frac{\ln 2}{B_{f,A}}$ to $\frac{\ln 2}{B_A}$ for ATA and from $\frac{N_G \ln 2}{B_{f,P}}$ to $\frac{N_G \ln 2}{B_P}$ for POTA. The slope of

I_u with respect to $(E_b/N_0)_{T;u}$ on a linear scale increases from $\frac{2B_{f;A}}{(\ln 2)^2}$ to $\frac{2B_A}{(\ln 2)^2}$ for ATA and from $\frac{2B_{f;P}}{N_G^2(\ln 2)^2}$ to $\frac{2B_P}{N_G^2(\ln 2)^2}$ for POTA. The slope (3.42) at $\left[\frac{E_b}{N_0}\right]_{u;\min}$ remains unchanged.

Using the uniform power allocation, Conjecture 3.2.1 can be proved at low SNR as follows:

Proof of Conjecture 3.2.1 at low SNR: Using Σ of (3.37), we have $B_f = N_G \frac{\text{Tr}(\mathbf{C}^\dagger \cdot \mathbf{C})}{N_T}$ yielding $\left[\frac{E_b}{N_0}\right]_{u;\min} = \frac{N_T \ln 2}{\text{Tr}(\mathbf{C}^\dagger \cdot \mathbf{C})}$. Writing $\left[\frac{E_b}{N_0}\right]_{u;\min} = \frac{\ln 2}{G}$ as in [98, Theorem 1], the maximum channel gain $G \equiv \frac{\text{Tr}(\mathbf{C}^\dagger \cdot \mathbf{C})}{N_T}$ (which agrees with [98, (206)], $G \equiv \frac{\text{Tr}(E[\mathbf{C}^\dagger \cdot \mathbf{C}])}{N_T}$, for random channels) is the same for all space-time modulation formats. Hence, $\left[\frac{E_b}{N_0}\right]_{u;\min}$ is the same for all N_G while the slope of I_u at $\left[\frac{E_b}{N_0}\right]_{u;\min}$ decreases with increasing N_G as seen from (3.42). $\square$

3.2.2 Channel information rate at high SNR

In the high SNR regime, the ratio P_T/N_0 is assumed to be very large. Let $r'_{s;i}$ denote the rank of $\mathbf{C}_{s;i}$, $r'_{s;i} \leq \min(r_{s;i}, L_t N_R)$. Let g denote the number of active TOGs for which $\eta_i \neq 0$. We have $1 \leq g \leq N_G$ for all $N_G = 1, \dots, N_T$. Assuming such g TOGs to be the first ones, we can write $\eta_i = 0$ for $i = g+1, \dots, N_G$. Using $r'_{s;T} = \sum_{i=1}^g r'_{s;i}$ and $M = \sum_{i=1}^g \sum_{l=1}^{r'_{s;i}} \log\left(\frac{\eta_i N_G}{r_{s;i}} \xi_{i;l}\right)$, (3.39) reduces for large P_T/N_0 to

$$I_u \approx \frac{1}{N_G} \sum_{i=1}^g \sum_{l=1}^{r'_{s;i}} \log\left(\frac{\eta_i}{r_{s;i}} \frac{N_G P_T}{N_0 D} \xi_{i;l}\right) = \frac{M}{N_G} + \frac{r'_{s;T}}{N_G} \log\left(\frac{P_T}{N_0 D}\right) \quad (3.47)$$

with a spatial multiplexing gain given by

$$\lim_{P_T/N_0 \rightarrow \infty} \frac{I_u}{\log(P_T/N_0)} = \frac{r'_{s;T}}{N_G} \quad (3.48)$$

Hence, I_u scales like $\frac{r'_{s;T}}{N_G} \log\left(\frac{P_T}{N_0}\right)$ as P_T/N_0 becomes large. Furthermore, using (3.3) at rate I_u , (3.47) leads to

$$(E_b/N_0)_{T;u} = 2^{\left[\frac{N_G I_u - M}{r'_{s;T}}\right]} I_u^{-1} \quad (3.49)$$

As $I_u \rightarrow \infty$, $(E_b/N_0)_{T;u}$ increases exponentially with I_u , making I_u linear in $(E_b/N_0)_{T;u}$ dB with slope $[10 \log_{10}(2) \frac{N_G}{r'_{s;T}}]^{-1} \approx \frac{r'_{s;T}}{3N_G}$ bps/Hz/dB or $\frac{r'_{s;T}}{N_G}$ bps/Hz/(3 dB). This is larger than the low SNR slope (3.42) whenever $r'_{s;T} \geq 2$.

For any $N_G \in \{1, \dots, N_T\}$, we maximize I_u of (3.47) over all possible ranks $\{r_{s;i}\}_{i=1}^{N_G}$. For this, it is sufficient to maximize the multiplexing gain $\frac{r'_{s;T}}{N_G}$, i.e. g and $\{r'_{s;i}\}_{i=1}^g$. Maximum g is obtained when all TOGs are active, $g = N_G$. Secondly, since $\mathbf{C}_{s;i}$ is a submatrix of $\mathbf{C}_i$ we have $r'_{s;i} \leq r_i$ from [114, Theorem 4.4.10], yielding $r'_{s;T} \leq r_T$. Equality is reached if $r_{s;i} = n_i$, making $r'_{s;i} = r_i$ for all i and hence $r'_{s;T} = r_T$. Such optimal solution is similar to [22, Lemma 1] in the sense that it yields a full rank channel matrix $\mathbf{C}_{s;i} \equiv \mathbf{C}_i$ for each TOG. Also, $\{\mathbf{S}_i\}_{i=1}^{N_G}$ are full rank and (3.47) reduces to

$$I_u = \frac{M}{N_G} + \frac{r_T}{N_G} \log \left(\frac{P_T}{N_0 D} \right), \quad (3.50)$$

where M is given by

$$M = \sum_{i=1}^{N_G} \sum_{l=1}^{r_i} \log \left(\frac{N_G \eta_i}{n_i} \lambda_{i,l} \right) \quad (3.51)$$

In this case, the multiplexing gain of (3.48) becomes $\frac{r_T}{N_G}$ which is similar to that of (3.17) and which, by Theorem 3.1.3, is largest, intermediate and smallest with ATA, POTA and OTA, respectively.

If $N_G \geq 2$, I_u of (3.50) can be further maximized over $\mathcal{A}_\eta$. This corresponds to maximizing M of (3.51) or, equivalently, minimizing $-\sum_{i=1}^{N_G} r_i \log(\eta_i)$. Writing the Lagrangian as in (3.36) and solving as in Problem 2, the optimal power allocation is given by $\eta_i = \frac{r_i}{r_T}$. We refer to such power allocation as "high SNR optimal TOG power allocation". The transmitter needs only knowledge of $\{r_i\}_{i=1}^{N_G}$. If all TOGs have equal ranks r_i , then $\eta_i = \frac{r_i}{r_T}$ reduces to a uniform TOG power allocation $\eta_i = \frac{1}{N_G}$. However, if $N_G = 1$ (i.e. ATA), we have $\eta_i = 1$ in (3.51) and the high SNR optimal power allocation reduces to a uniform power allocation among all transmit antennas. We prove in Appendix A the following result.

Lemma 3.2.5 *For any power allocation such that $\eta_i \neq 0 \forall i$ ($g = N_G$) and if $r_{s;i} = n_i \forall i$, the largest rate I_u is provided by ATA while POTA and OTA result in intermediate and low rates, respectively.*

In particular, Lemma 3.2.5 holds for $\eta_i = \frac{r_i}{r_T}$, $\eta_i = \frac{1}{N_G}$ and $\eta_i = \frac{n_i}{N_T}$. For the uniform power allocation $\eta_i = \frac{n_i}{N_T}$, the proof of Lemma 3.2.5 is a proof for Conjecture 3.2.1 in the high SNR regime.

3.2.3 Comparison between the information rate I_u and Shannon capacity C

Regardless of the TOG power allocation η and ranks $\{r_{s;i}\}_{i=1}^{N_G}$, I_u (3.39) cannot exceed Shannon capacity C (3.9). However, equality can be achieved for certain channels.

Theorem 3.2.2 *The Shannon capacity for informed transmitter is equal to the information rate for uninformed transmitter using full rank TOG input covariance matrices (i.e. $r_{s;i} = n_i$) if and only if all TOGs have full column rank propagation matrices $\mathbf{C}_i$ (i.e. $r_i = n_i$) with all eigenmodes active and equal eigenvalues $\lambda_{i;l} = \lambda_i$ for all $1 \leq l \leq n_i$, $1 \leq i \leq N_G$, such that*

$$\lambda_i^{-1} + \frac{\eta_i N_G P_T}{n_i N_0 D} = \text{Constant} \quad (3.52)$$

The proof of Theorem 3.2.2 is provided in Appendix A. We assume that P_T/N_0 is sufficient to make all channel eigenmodes active. For example, Theorem 3.2.2 applies for MIMO channels using OTA with propagation matrix $\mathbf{C}$ having all columns of equal norm $\tilde{c}^2$, yielding $\lambda_i = \|\mathbf{C}_{*,i}\|^2 = \tilde{c}^2$ and $\eta_i = \frac{1}{N_T}$ from (3.52). The capacity C and the rate I_u with uniform power allocation are equal to $\log(1 + \tilde{c}^2 \frac{P_T}{N_0 D})$ bps/Hz. However, according to Observation 1 at the end of Section 3.1, randomly generated realizations of the 3GPP [83] channels have distinct eigenvalues $\lambda_{i;l}$ (for each TOG) almost surely. Thus, most likely Theorem 3.2.2 cannot be satisfied for the 3GPP channels.

Low SNR: Consider (3.13) and (3.41) where B takes the value $B_{\max}$ of (3.44) associated with the strongest antenna power allocation. We have $\frac{B_{\max}}{N_G} \leq \lambda_{\max}$ since $\lambda_{\max}$ is lower bounded by the maximum diagonal element of the matrix $\text{diag}_{i=1}^{N_G}[\mathbf{C}_i^\dagger \cdot \mathbf{C}_i]$ that is given by $\max\{\|\mathbf{C}_{*,l}\|^2\}_{l=1}^{N_T}$. Therefore, as C and $I_u \rightarrow 0$ we have $\left[\frac{E_b}{N_0}\right]_{u;\min} \geq \left[\frac{E_b}{N_0}\right]_{\min}$. Since $n(\lambda_{\max}) \geq 1$, C grows at a steeper slope than I_u as seen from (3.15) and (3.42). As the SNR goes to zero, it can be easily shown that

$$\frac{C}{I_u} \approx \frac{\frac{n(\lambda_{\max})}{N_G} \log(1 + \lambda_{\max} \frac{N_G P_T}{n(\lambda_{\max}) N_0 D})}{\frac{1}{N_G} \log(1 + \frac{B_{\max} P_T}{N_0 D})} \rightarrow \frac{\lambda_{\max}}{(B_{\max}/N_G)}$$

With OTA, we have $\frac{C}{I_u} \rightarrow 1$ since $\lambda_{\max} = \frac{B_{\max}}{N_G} \equiv \frac{B_O}{N_T}$, yielding also $\left[\frac{E_b}{N_0}\right]_{u;\min} = \left[\frac{E_b}{N_0}\right]_{\min}$.

High SNR: Consider I_u of (3.50) assuming $\eta_i \neq 0$ for all i . Using the high SNR results from Sections 3.1.2 and 3.2.2, the systems with and without CSIT have similar high SNR slope (equal to $\frac{r_T}{N_G}$ bps/Hz/(3 dB)). Let $\Delta(E_b/N_0)_T$ denote the power offset

[102] or, equivalently, the asymptotic SNR separation in dB between C and I_u . Using (3.18) and (3.49) with $r'_{s,T} = r_T$, we have $\Delta(E_b/N_0)_T = (E_b/N_0)_T - (E_b/N_0)_{T,u} = 10^{\frac{(M-A)}{r_T}} \log_{10}(2) - 10 \log_{10}(\frac{N_G}{r_T})$ dB which, using M of (3.51), reduces to

$$\Delta(E_b/N_0)_T = \frac{10}{r_T} \sum_{i=1}^{N_G} r_i \log_{10} \left(\frac{N_G \eta_i}{n_i} \right) - 10 \log_{10} \left(\frac{N_G}{r_T} \right) \text{ dB} \quad (3.53)$$

Theorem 3.2.3 *The asymptotic SNR separation in dB is zero if and only if TOG i has a full column rank propagation matrix $\mathbf{C}_i$ ($r_i = n_i$) with $\eta_i = \frac{n_i}{N_T}$, for all $i = 1, \dots, N_G$.*

The proof of Theorem 3.2.3 is provided in Appendix A. It follows that for MIMO channels having $r_i = n_i \forall i$, the high SNR optimal TOG power allocation (which reduces to the uniform power allocation since $\eta_i = \frac{r_i}{r_T} = \frac{n_i}{N_T}$) not only maximizes I_u but also compensates the lack of transmitter channel knowledge by making I_u equal to C . With ATA, (3.53) becomes $\Delta(E_b/N_0)_T = 10 \log_{10}(\frac{r_T}{N_T})$ dB reducing to 0 dB if $\mathbf{C}$ is full column rank. With OTA, the uniform power allocation $\eta_i = \frac{1}{N_T}$ ensures $\Delta(E_b/N_0)_T = 0$ dB. Moreover, the result provided for the equality $C = I_u$ in Theorem 3.2.3 reinforces that of Theorem 3.2.2 at any SNR. The conditions of Theorem 3.2.2 reduce to those of Theorem 3.2.3 for large P_T/N_0 since, in addition to $r_i = n_i$, (3.52) becomes $\frac{\eta_i N_G P_T}{n_i N_0 D} = \text{Constant}$ regardless of $\{\lambda_i\}$, yielding $\eta_i = \frac{n_i}{N_T} \forall i$.

3.3 Capacity and rate results for MIMO-OFDM systems

So far we have considered spread spectrum like MIMO systems whose discrete time channel model has been obtained based on assumption (2.14). As seen from Section 3.1, the Shannon capacity of these systems is obtained with full CSIT using a water-filling power allocation and eigen-beamforming input signaling structure. This capacity is shown to be maximal with ATA. It scales like $\frac{r_T}{N_G} \log(P_T/N_0)$ at high SNR, and it is linear with $(E_b/N_0)_T$ dB at low SNR with a slope at $\left[\frac{E_b}{N_0}\right]_{\min}$ that is determined by the number of TOGs N_G and the multiplicity of the maximal eigenvalue of $\text{diag}_{i=1}^{N_G}[\mathbf{C}_i^\dagger \cdot \mathbf{C}_i]$. Considering these extreme SNR cases with partial CSIT, Section 3.2 shows that the information rate associated with an input covariance matrix of the form (3.38) is maximized by transmit antenna selection at low SNR, and by making all the antennas active at high SNR with an

optimal power allocation determined by the ranks of the TOG propagation matrices. The objective of this section is to conduct a similar capacity and information rate analysis for the MIMO-OFDM systems described by equations (2.28)-(2.30).

Assuming full CSIT, this section considers first the capacity C_O of the MIMO-OFDM system (2.28). Using the eigenvalue decomposition $\mathbf{H}_O^\dagger \cdot \mathbf{H}_O = \mathbf{M}_O^\dagger \cdot \mathbf{Q}_O \cdot \mathbf{M}_O$ and $\tilde{\Sigma} = \mathbf{M}_O \cdot \Sigma \cdot \mathbf{M}_O^\dagger$, we can write as in (3.5)

$$C_O = \frac{1}{D} \max_{\text{Tr}(\tilde{\Sigma}) \leq P_T} \log \left(\left| \mathbf{I}_{\tilde{D}} + \frac{1}{N_0} \tilde{\Sigma} \cdot \mathbf{Q}_O \right| \right) \text{ bps/Hz.} \quad (3.54)$$

Using (2.29), we have $\mathbf{H}_O^\dagger \cdot \mathbf{H}_O = \mathbf{A} \cdot [\mathbf{O}_{m,q}]^\dagger \cdot [\mathbf{O}_{m,q}] \cdot \mathbf{A}^T = \mathbf{A} \cdot [\tilde{\mathbf{O}}_{r,q}] \cdot \mathbf{A}^T$ where the $(r, q)^{\text{th}}$ block of $[\tilde{\mathbf{O}}_{r,q}]$ is given by $\tilde{\mathbf{O}}_{r,q} = \sum_{m=1}^{N_R} \mathbf{O}_{m,r}^\dagger \cdot \mathbf{O}_{m,q}$ for $r, q = 1, \dots, N_T$. From (2.30), we can write

$$\begin{aligned} \tilde{\mathbf{O}}_{r,q} &= \text{diag}_{p=-N_F/2}^{N_F/2-1} \left[\sum_{m=1}^{N_R} \left(FT[\tilde{\mathbf{F}}(\tau)]_{m,r}^* \frac{p}{T_F} FT[\tilde{\mathbf{F}}(\tau)]_{m,q} \right) \right] \\ &= \text{diag}_{p=-N_F/2}^{N_F/2-1} \left[\sum_{m=1}^{N_R} \left(\sum_{k=1}^{L_t} \sum_{l=1}^{L_t} [\mathbf{F}_k]_{m,r}^* [\mathbf{F}_l]_{m,q} e^{j2\pi \frac{p}{T_F} (\tau_k - \tau_l)} \right) \right] \\ &= \text{diag}_{p=-N_F/2}^{N_F/2-1} \left[[\mathbf{H}_{O;p}^\dagger \cdot \mathbf{H}_{O;p}]_{r,q} \right] = \text{diag}_{i=1}^{N_G} [\tilde{\mathbf{O}}_{r,q;i}] \end{aligned} \quad (3.55)$$

where $\mathbf{H}_{O;p} = \sum_{k=1}^{L_t} \mathbf{F}_k e^{-j2\pi \frac{p}{T_F} \tau_k}$ as defined in Chapter 2, and $\tilde{\mathbf{O}}_{r,q;i}$ is a $D_G \times D_G$ diagonal matrix whose diagonal elements $[\mathbf{H}_{O;p}^\dagger \cdot \mathbf{H}_{O;p}]_{r,q}$ have their index p associated with the D_G orthogonal subcarriers that are employed on the antennas of TOG i . Using $\mathbf{A} = \text{diag}_{i=1}^{N_T} [\mathbf{e}_{i;N_G}^T \otimes \mathbf{I}_{D_G}] = \text{diag}_{i=1}^{N_G} [\mathbf{I}_{n_i} \otimes (\mathbf{e}_{i;N_G}^T \otimes \mathbf{I}_{D_G})]$ and since

$$(\mathbf{e}_{i;N_G}^T \otimes \mathbf{I}_{D_G}) \cdot \tilde{\mathbf{O}}_{r,q} \cdot (\mathbf{e}_{j;N_G} \otimes \mathbf{I}_{D_G}) = \begin{cases} \mathbf{0}_{D_G \times D_G}, & \text{if } i \neq j \\ \tilde{\mathbf{O}}_{r,q;i}, & \text{if } i = j \end{cases}$$

for all $i, j = 1, \dots, N_G$, it follows that

$$\mathbf{H}_O^\dagger \cdot \mathbf{H}_O = \text{diag}_{i=1}^{N_G} [\tilde{\mathbf{O}}_i] \quad (3.56)$$

where $\tilde{\mathbf{O}}_i$ is an $n_i D_G \times n_i D_G$ matrix defined as $\tilde{\mathbf{O}}_i = [\tilde{\mathbf{O}}_{r',q';i}]$ with r', q' denoting the indexes of the transmit antennas that belong to TOG i , that is, $r', q' = 1, \dots, n_1$ if $i = 1$ and $r', q' = \sum_{j=1}^{i-1} n_j + 1, \dots, \sum_{j=1}^i n_j$ if $i > 1$. For $i = 1, \dots, N_G$ and $d = 1, \dots, D_G$, we

define the $N_R \times n_i$ matrix $\check{\mathbf{F}}_{i,d} = \sum_{k=1}^{L_t} \mathbf{F}_{i,k} e^{-j2\pi \frac{p_i(d)}{T_F} \tau_k}$ where $p_i(d)$ denotes the element of the set $\{-N_F/2, \dots, N_F/2-1\}$ that corresponds to the d^{th} orthogonal subcarrier employed on the antennas of TOG i . Let $(\check{r}, \check{q}) = (r', q')$ if $i = 1$, and $(\check{r}, \check{q}) = (r' - \sum_{j=1}^{i-1} n_j, q' - \sum_{j=1}^{i-1} n_j)$ if $i > 1$, hence $\check{r}, \check{q} = 1, \dots, n_i$. The d^{th} diagonal element of $\tilde{\mathbf{O}}_{r',q';i}$ is given by $[\tilde{\mathbf{O}}_{r',q';i}]_{d,d} = [\mathbf{H}_{O;p_i(d)}^\dagger \cdot \mathbf{H}_{O;p_i(d)}]_{r',q'} = \sum_{m=1}^{N_R} \left(\sum_{k=1}^{L_t} \sum_{l=1}^{L_t} [\mathbf{F}_k]_{m,r'}^* [\mathbf{F}_l]_{m,q'} e^{j2\pi \frac{p_i(d)}{T_F} (\tau_k - \tau_l)} \right)$, that is,

$$[\tilde{\mathbf{O}}_{r',q';i}]_{d,d} = \sum_{m=1}^{N_R} \left(\sum_{k=1}^{L_t} \sum_{l=1}^{L_t} [\mathbf{F}_{i,k}]_{m,\check{r}}^* [\mathbf{F}_{i,l}]_{m,\check{q}} e^{j2\pi \frac{p_i(d)}{T_F} (\tau_k - \tau_l)} \right) = [\check{\mathbf{F}}_{i,d}^\dagger \cdot \check{\mathbf{F}}_{i,d}]_{\check{r},\check{q}}$$

It follows that the $(\check{r}, \check{q})^{th}$ block of $\tilde{\mathbf{O}}_i$ is given by $\tilde{\mathbf{O}}_{r',q';i} \equiv \text{diag}_{d=1}^{D_G} \left[[\check{\mathbf{F}}_{i,d}^\dagger \cdot \check{\mathbf{F}}_{i,d}]_{\check{r},\check{q}} \right]$, yielding

$$\tilde{\mathbf{O}}_i = \sum_{d=1}^{D_G} \left[(\check{\mathbf{F}}_{i,d}^\dagger \cdot \check{\mathbf{F}}_{i,d}) \otimes (\mathbf{e}_{d,D_G} \cdot \mathbf{e}_{d,D_G}^T) \right] \quad (3.57)$$

which is a summation of Hermitian positive semidefinite matrices. Moreover, for different d the matrices $(\check{\mathbf{F}}_{i,d}^\dagger \cdot \check{\mathbf{F}}_{i,d}) \otimes (\mathbf{e}_{d,D_G} \cdot \mathbf{e}_{d,D_G}^T)$ have their column spaces (and also their row spaces) essentially disjoint. Hence, we have from [114, Theorem 18.5.7] that $r_{O;i} \triangleq \text{rank}(\tilde{\mathbf{O}}_i) = \sum_{d=1}^{D_G} \text{rank}((\check{\mathbf{F}}_{i,d}^\dagger \cdot \check{\mathbf{F}}_{i,d}) \otimes (\mathbf{e}_{d,D_G} \cdot \mathbf{e}_{d,D_G}^T))$ which, using (C.16), leads to

$$r_{O;i} = \sum_{d=1}^{D_G} \text{rank}(\check{\mathbf{F}}_{i,d}^\dagger \cdot \check{\mathbf{F}}_{i,d}) \stackrel{(a)}{=} \sum_{d=1}^{D_G} \text{rank}(\check{\mathbf{F}}_{i,d}) \equiv \sum_{d=1}^{D_G} \text{rank} \left(\sum_{k=1}^{L_t} \mathbf{F}_{i,k} e^{-j2\pi \frac{p_i(d)}{T_F} \tau_k} \right)$$

with (a) due to (C.15a). As for the TOG propagation matrix $\mathbf{C}_i$, the matrix $\check{\mathbf{F}}_{i,d}$ (which is the summation of different and most likely full rank matrices) is most likely full rank for 3GPP channels, that is, $\text{rank}(\check{\mathbf{F}}_{i,d}) = \min(N_R, n_i)$. With this, we have $r_{O;i} = D_G \min(N_R, n_i)$. Using r_O to denote the rank of $\mathbf{H}_O$ (or $\mathbf{H}_O^\dagger \cdot \mathbf{H}_O$) we can write from (3.56) that $r_O = \sum_{i=1}^{N_G} r_{O;i} = D_G \sum_{i=1}^{N_G} \min(N_R, n_i)$. This rank is different from the rank r of the channel matrix (2.20) of spread spectrum like MIMO systems over 3GPP channels where $r = r_T D_G = D_G \sum_{i=1}^{N_G} \min(L_t N_R, n_i)$. Since $L_t \geq 1$, we can easily show that $r \geq r_O$. Furthermore, the spatial channel parameter L_t does not affect r_O and the MIMO-OFDM system operates as if there is only one cluster (i.e. without multipath time resolvability). For spread spectrum like MIMO systems, if L_t increases the rank r increases if there is at least one TOG with $n_i > L_t N_R$. Both r and r_O can be increased by increasing D_G , however, the impact of increasing the number of transmit and receive antennas on these ranks can be differ-

ent. If $n_i > N_R$ for some or all i , then adding more receive antennas increases r_O while it increases r only if $n_i > L_t N_R$. Moreover, for the TOGs i with $n_i > N_R$ increasing n_i does not increase r_O while it increases r only if $L_t N_R > n_i$. If $n_i \leq N_R$ for all TOGs i (as with OTA for example), increasing N_R does not change r and r_O , however, increasing n_i for some or all i can increase both ranks. Now for every $i = 1, \dots, N_G$, let $\{\lambda_{O;i,l}\}_{l=1}^{n_i D_G}$ denote the (real and non-negative) eigenvalues of $\tilde{\mathbf{O}}_i$ in decreasing order $\lambda_{O;i,1} \geq \dots \geq \lambda_{O;i,n_i D_G} \geq 0$. We consider the eigenvalue decomposition $\tilde{\mathbf{O}}_i = \mathbf{M}_{O;i}^\dagger \cdot \text{diag}_{l=1}^{n_i D_G} [\lambda_{O;i,l}] \cdot \mathbf{M}_{O;i}$ where $\mathbf{M}_{O;i}$ is an $n_i D_G \times n_i D_G$ unitary matrix. Using (3.56), the eigenvalue decomposition of $\mathbf{H}_O^\dagger \cdot \mathbf{H}_O$ employs the matrices $\mathbf{M}_O = \text{diag}_{i=1}^{N_G} [\mathbf{M}_{O;i}]$ and $\mathbf{Q}_O = \text{diag}[\{\{\lambda_{O;i,l}\}_{l=1}^{n_i D_G}\}_{i=1}^{N_G}]$.

Following similar steps to those used in Section 3.1 to evaluate the capacity C of spread spectrum like MIMO systems, the capacity C_O is obtained using a water-filling power allocation over the eigenmodes of $\mathbf{H}_O^\dagger \cdot \mathbf{H}_O$. As in (3.9), we can write

$$C_O = \frac{1}{D} \sum_{i=1}^{N_G} \sum_{l=1}^{n_i D_G} \{\log(\mu_O \lambda_{O;i,l})\}^+ \text{ bps/Hz} \quad (3.58)$$

with $\sum_{i=1}^{N_G} \sum_{l=1}^{n_i D_G} \{\mu_O - \lambda_{O;i,l}^{-1}\}^+ = \frac{P_T}{N_0}$. This capacity is achieved by a zero mean Gaussian input vector $\mathbf{x}$ with covariance matrix $\Sigma = \text{diag}_{i=1}^{N_G} [\Sigma_i]$ where $\Sigma_i = \mathbf{M}_{O;i}^\dagger \cdot \tilde{\Sigma}_i \cdot \mathbf{M}_{O;i}$ and $\tilde{\Sigma}_i = N_0 \text{diag}_{l=1}^{n_i D_G} [\{\mu_O - \lambda_{O;i,l}^{-1}\}^+]$. With $\tilde{\mathbf{x}}'_i$ denoting (as in Section 3.1) the $n_i D_G \times 1$ zero mean Gaussian vector whose covariance matrix is $\tilde{\Sigma}_i$, the input vector to TOG i satisfies

$$\mathbf{x}'_i = \text{vec}(\{\mathbf{x}_{i,l}\}_{l=1}^{n_i}) = \mathbf{M}_{O;i}^\dagger \cdot \tilde{\mathbf{x}}'_i = \sum_{l=1}^{n_i D_G} [\tilde{\mathbf{x}}'_i]_l [\mathbf{M}_{O;i}^\dagger]_{*,l}$$

where the column vectors $[\mathbf{M}_{O;i}^\dagger]_{*,l}$ of $\mathbf{M}_{O;i}^\dagger$ denote the eigenvectors of $\tilde{\mathbf{O}}_i$. The input signaling structure corresponds to eigen-beamforming with the eigenbeams being formed using weights that are the components of the vectors $[\mathbf{M}_{O;i}^\dagger]_{*,l}$.

For spread spectrum like MIMO systems, Theorem 3.1.1 shows that ATA provides the largest value for C among all space-time modulation formats. This result holds also for the capacity C_O of MIMO-OFDM systems and can be shown following a similar proof to that of Theorem 3.1.1 since, using (C.1) and (2.29), the input/output mutual information can be written as

$$I(\mathbf{x}; \mathbf{r}) = \log(|\mathbf{I}_{\tilde{D}} + \frac{1}{N_0} \Sigma \cdot \mathbf{H}_O^\dagger \cdot \mathbf{H}_O|) = \log\left(\left|\mathbf{I}_{N_T D} + \frac{1}{N_0} \mathbf{A}^T \cdot \Sigma \cdot \mathbf{A} \cdot [\mathbf{O}_{m,q}]^\dagger \cdot [\mathbf{O}_{m,q}]\right|\right)$$

which is similar to (A.1) with $[(C^\dagger \cdot C) \otimes \mathbf{I}_D]$ of (A.1) being replaced by $[\mathbf{O}_{m,q}]^\dagger \cdot [\mathbf{O}_{m,q}]$. Also, one can easily see that Corollary 3.1.1 holds for MIMO-OFDM systems. Hence, similarly to C , C_O is maximal, intermediate and minimal when using ATA, POTA and OTA, respectively.

Next, we consider the asymptotic behavior of C_O in the low and high SNR regimes. At low SNR, the ratio P_T/N_0 is assumed to satisfy $(P_T/N_0) \ll [\lambda_{O;i;l}]^{-1}$ for all $\lambda_{O;i;l} \neq 0$, $l = 1, \dots, n_i D_G$ and $i = 1, \dots, N_G$. In this case, only the strongest eigenmodes corresponding to $\lambda_{O;\max} = \max\{\{\lambda_{O;i;l}\}_{l=1}^{n_i D_G}\}_{i=1}^{N_G}$ are active, which reduces (3.58) to

$$C_O = \frac{n(\lambda_{O;\max})}{D} \log \left(1 + \lambda_{O;\max} \frac{P_T}{N_0 n(\lambda_{O;\max})} \right)$$

where $n(\lambda_{O;\max})$ denotes the multiplicity of $\lambda_{O;\max}$. Following similar steps as in Section 3.1.1, we can show that the minimum transmitted SNR-per-bit for MIMO-OFDM systems is given by $\left[\frac{E_b}{N_0}\right]_{\min} = \frac{\ln 2}{\lambda_{O;\max}}$. Let $\lambda_{O;\max;A}$, $\lambda_{O;\max;P}$ and $\lambda_{O;\max;O}$ denote the values of $\lambda_{O;\max}$ using ATA, POTA and OTA, respectively. We define the $N_R D_G \times n_i D_G$ matrix $\mathbf{B}_{O;i} \triangleq \sum_{q=1}^{D_G} \check{\mathbf{F}}_{i,q} \otimes (\mathbf{e}_{q;D_G} \cdot \mathbf{e}_{q;D_G}^T)$. Using (C.3) with $\mathbf{e}_{d;D_G}^T \cdot \mathbf{e}_{q;D_G} = \delta(d, q)$, we can show that (3.57) is equivalent to

$$\tilde{\mathbf{O}}_i = \mathbf{B}_{O;i}^\dagger \cdot \mathbf{B}_{O;i} \quad (3.59)$$

or

$$\tilde{\mathbf{O}}_i = \left[\sum_{d=1}^{D_G} \left(\sum_{k=1}^{L_t} \mathbf{F}_{i,k} e^{-j2\pi \frac{p_i(d)}{T_F} \tau_k} \right) \otimes (\mathbf{e}_{d;D_G} \cdot \mathbf{e}_{d;D_G}^T) \right]^\dagger \cdot \left[\sum_{q=1}^{D_G} \left(\sum_{k=1}^{L_t} \mathbf{F}_{i,k} e^{-j2\pi \frac{p_i(q)}{T_F} \tau_k} \right) \otimes (\mathbf{e}_{q;D_G} \cdot \mathbf{e}_{q;D_G}^T) \right] \quad (3.60)$$

With $1 < N_G < N_T$, the $N_R D_G \times n_i D_G$ matrix $\left[\sum_{d=1}^{D_G} \left(\sum_{k=1}^{L_t} \mathbf{F}_{i,k} e^{-j2\pi \frac{p_i(d)}{T_F} \tau_k} \right) \otimes (\mathbf{e}_{d;D_G} \cdot \mathbf{e}_{d;D_G}^T) \right]$ can be obtained from the $N_R D \times N_T D$ matrix $\left[\sum_{d=1}^D \left(\sum_{k=1}^{L_t} \mathbf{F}_k e^{-j2\pi \frac{p_1(d)}{T_F} \tau_k} \right) \otimes (\mathbf{e}_{d;D} \cdot \mathbf{e}_{d;D}^T) \right]$ (associated with $N_G = 1$) by deleting some columns and rows, for every $i = 1, \dots, N_G$. Hence, using the interlacing property for singular values [105, Theorem 7.3.9], we can show as in Theorem 3.1.2 that $\lambda_{O;\max;A} \geq \lambda_{O;\max;P}$. Similarly, since for every $i = 1, \dots, N_G$ and every $l = 1, \dots, n_i$, the matrix $\left[\sum_{d=1}^{D'_G} \left(\sum_{k=1}^{L_t} [\mathbf{F}_{i,k}]_{*,l} e^{-j2\pi \frac{p_i(d)}{T_F} \tau_k} \right) \otimes (\mathbf{e}_{d;D'_G} \cdot \mathbf{e}_{d;D'_G}^T) \right]$ (with D'_G denoting the number of signal dimensions per TOG or antenna when using OTA, $D'_G = \frac{D}{N_T} < \frac{D}{N_G} = D_G$) can be obtained from $\left[\sum_{d=1}^{D_G} \left(\sum_{k=1}^{L_t} \mathbf{F}_{i,k} e^{-j2\pi \frac{p_i(d)}{T_F} \tau_k} \right) \otimes (\mathbf{e}_{d;D_G} \cdot \mathbf{e}_{d;D_G}^T) \right]$ by deleting some columns and rows, we can show that $\lambda_{O;\max;P} \geq \lambda_{O;\max;O}$. It follows that for MIMO-OFDM systems ATA provides the smallest value for $\left[\frac{E_b}{N_0}\right]_{\min}$

while POTA and OTA result in intermediate and largest values, respectively. Furthermore, it can be shown as in Section 3.1.1 that $C_O \approx \frac{2n(\lambda_{O;\max})\lambda_{O;\max}}{D(\ln 2)^2} (E_b/N_0)_T - \frac{2n(\lambda_{O;\max})}{D \ln 2}$. Thus, the rate of change of C_O with $(E_b/N_0)_T$ dB at $\left[\frac{E_b}{N_0}\right]_{\min}$ is [98, (29)]

$$\lim_{(E_b/N_0)_T \rightarrow \left[\frac{E_b}{N_0}\right]_{\min}} \frac{10 \log_{10}(2) C_O}{10 \log_{10}((E_b/N_0)_T) - 10 \log_{10}\left(\left[\frac{E_b}{N_0}\right]_{\min}\right)} = \frac{2 n(\lambda_{O;\max})}{D} \text{ bps/Hz/(3 dB)}, \quad (3.61)$$

As in Section 3.1.1, we investigate this slope for keyhole channels (i.e. $L = 1$) and for i.i.d. channels (i.e. $L \rightarrow \infty$). For keyhole channels, we have $L_t = 1$ and $\mathbf{C}_i = \mathbf{F}_{i,1} = G_{1,0} \mathbf{a}_{1,0} \cdot \mathbf{b}_{i,1,0}^\dagger$, thus, making $\tilde{\mathbf{F}}_{i,d} = G_{1,0} e^{-j2\pi \frac{n_i(d)}{T_F} \tau_1} \mathbf{a}_{1,0} \cdot \mathbf{b}_{i,1,0}^\dagger$. From (3.57), we obtain

$$\tilde{\mathbf{O}}_i = \sum_{d=1}^{D_G} \left[\left(\|G_{1,0}\|^2 \|\mathbf{a}_{1,0}\|^2 \mathbf{b}_{i,1,0} \mathbf{b}_{i,1,0}^\dagger \right) \otimes (\mathbf{e}_{d,D_G} \mathbf{e}_{d,D_G}^T) \right] \stackrel{(a)}{=} \|G_{1,0}\|^2 N_R \left[(\mathbf{b}_{i,1,0} \mathbf{b}_{i,1,0}^\dagger) \otimes \mathbf{I}_{D_G} \right]$$

where (a) is due to $\|\mathbf{a}_{1,0}\|^2 = N_R$ and to (C.5b) with $\sum_{d=1}^{D_G} \mathbf{e}_{d,D_G} \mathbf{e}_{d,D_G}^T = \mathbf{I}_{D_G}$. Hence, for every i , $\tilde{\mathbf{O}}_i$ has one nonzero eigenvalue $\lambda_{O;i,l} = \|G_{1,0}\|^2 N_R n_i$ with multiplicity D_G according to Theorem C.0.1. In this case, the slope (3.61) for MIMO-OFDM systems with an underlying keyhole channel becomes $\frac{2 N_{G;\max} D_G}{D} = \frac{2 N_{G;\max}}{N_G} \text{ bps/Hz/(3 dB)}$ when using POTA (where $N_{G;\max}$ is the number of TOGs having the maximum number of antenna elements), and 2 bps/Hz/(3 dB) when using ATA or OTA. This slope is equal to the slope (3.15) for spread spectrum like MIMO systems with an underlying keyhole channel. For i.i.d. channels, the matrices $\{\{\mathbf{F}_{i,k}\}_{k=1}^{L_t}\}_{i=1}^{N_G}$ are independent and the probability that two eigenvalues $\lambda_{O;i,l}$ of $\tilde{\mathbf{O}}_i$ (3.60) are equal is zero [100, Section 3.1]. It follows that $n(\lambda_{O;\max}) = 1$ for all space-time modulation formats, which reduces (3.61) to $\frac{2}{D} \text{ bps/Hz/(3 dB)}$. This slope cannot be greater than that of keyhole channels. Also, from Section 3.1.1 we have that for spread spectrum like MIMO systems over i.i.d. channels the capacity slope at $\left[\frac{E_b}{N_0}\right]_{\min}$ is equal to $\frac{2}{N_G} \text{ bps/Hz/(3 dB)}$, which is larger than what we obtained for MIMO-OFDM systems since $D = N_G D_G$ and $D_G \geq 1$. The reason behind this lies in the fact that for i.i.d. channels the maximal eigenvalue of the block diagonal matrix $\mathbf{H}_O^\dagger \cdot \mathbf{H}_O$ (3.56) has a multiplicity of $n(\lambda_{O;\max}) \equiv 1$ while the maximal eigenvalue of the block diagonal matrix $\mathbf{H}^\dagger \cdot \mathbf{H}$ (3.6) has a multiplicity of $D_G n(\lambda_{\max}) \equiv D_G$. The diagonal blocks of these matrices that are associated with TOG i are given by $\tilde{\mathbf{O}}_i$ of (3.57) and by $(\mathbf{C}_i^\dagger \cdot \mathbf{C}_i) \otimes \mathbf{I}_{D_G} = \sum_{d=1}^{D_G} [(\mathbf{C}_i^\dagger \cdot \mathbf{C}_i) \otimes (\mathbf{e}_{d,D_G} \mathbf{e}_{d,D_G}^T)]$, respectively. Hence, for spread spectrum like MIMO systems each signal dimension d allocated to TOG i is associated with the matrix $(\mathbf{C}_i^\dagger \cdot \mathbf{C}_i) \otimes (\mathbf{e}_{d,D_G} \mathbf{e}_{d,D_G}^T)$ where $\mathbf{C}_i$ is independent of d . Using (3.57) for MIMO-

OFDM systems, signal dimension d is associated with the matrix $(\check{\mathbf{F}}_{i,d}^\dagger \cdot \check{\mathbf{F}}_{i,d}) \otimes (\mathbf{e}_{d,D_G} \cdot \mathbf{e}_{d,D_G}^T)$ where $\check{\mathbf{F}}_{i,d}$ represents the Fourier transform at frequency $\frac{p_i(d)}{T_F}$ of $\sum_{k=1}^{L_t} \delta(\tau - \tau_k) \mathbf{F}_{i,k}$. Consequently, the transition from the time domain to the frequency domain leads to decreasing the multiplicity of the maximal eigenvalue from D_G (for $\mathbf{H}^\dagger \cdot \mathbf{H}$) to 1 (for $\mathbf{H}_O^\dagger \cdot \mathbf{H}_O$) as well as to decreasing the capacity slope at $\left[\frac{E_b}{N_0}\right]_{\min}$ from $\frac{2}{N_G}$ bps/Hz/(3 dB) for spread spectrum like MIMO systems to $\frac{2}{D}$ bps/Hz/(3 dB) for MIMO-OFDM systems.

Consider now the high SNR regime where P_T/N_0 is assumed to be very large. In this case, all $r_O \equiv \sum_{i=1}^{N_G} r_{O,i} = D_G \sum_{i=1}^{N_G} \min(N_R, n_i)$ eigenmodes are active which reduces (3.58) to

$$C_O = \frac{r_O}{D} \log \left(\frac{P_T}{r_O N_0} + c_O \right) + \frac{A_O}{D} \quad (3.62)$$

with $A_O = \sum_{i=1}^{N_G} \sum_{l=1}^{r_{O,i}} \log(\lambda_{O,i,l})$ and $c_O = \frac{1}{r_O} \sum_{i=1}^{N_G} \sum_{l=1}^{r_{O,i}} \lambda_{O,i,l}^{-1}$. The associated spatial multiplexing gain is given by $\lim_{P_T/N_0 \rightarrow \infty} \frac{C_O}{\log(P_T/N_0)} = \frac{r_O}{D} = \sum_{i=1}^{N_G} \frac{\min(N_R, n_i)}{N_G}$. Using a proof similar to that of Theorem 3.1.3, we can easily show that maximal, intermediate and minimal values for $\frac{r_O}{D}$ are provided by ATA, POTA and OTA, respectively. From (3.17) the spatial multiplexing gain for spread spectrum like MIMO systems is given by $\frac{r_T}{N_G}$, which satisfies $\frac{r_T}{N_G} = \sum_{i=1}^{N_G} \frac{\min(L_t N_R, n_i)}{N_G}$ for 3GPP channels. Since $L_t \geq 1$, it follows that

$$\frac{r_T}{N_G} \geq \frac{r_O}{D} \quad (3.63)$$

Therefore, the multiplexing gain and capacity of spread spectrum like MIMO systems are larger or equal than those of MIMO-OFDM systems. Equality is achieved when $L_t = 1$, in which case both spread spectrum like MIMO systems and MIMO-OFDM systems have no multipath time resolvability. Also, (3.63) is satisfied with equality when $L_t > 1$ and $\min(N_R, n_i) = \min(L_t N_R, n_i)$ for all $i = 1, \dots, N_G$. This occurs when $n_i \leq N_R$ for all i (as with OTA for example), in which case the system's performance is determined by the number of transmit antennas only regardless of the multipath time resolvability and the number of receive antennas. Moreover, an increase in the number of time resolvable clusters L_t can lead to an increase in both $\frac{r_T}{N_G}$ and C while it does not affect neither $\frac{r_O}{D}$ nor C_O . As in (3.18), we can show using (3.62) that $(E_b/N_0)_T \approx \frac{r_O}{D C_O} 2^{\left[\frac{D C_O - A_O}{r_O}\right]}$. Hence, as $C_O \rightarrow \infty$, $(E_b/N_0)_T$ increases exponentially with C_O , making C_O linear in $(E_b/N_0)_T$ dB with slope $\frac{r_O}{D}$ bps/Hz/(3 dB). This slope can be larger or smaller than that of (3.61). For example, for i.i.d. channels the slope at high SNR is larger than at low SNR if $r_O \geq 2$, and smaller if $r_O \leq 2$. For keyhole channels using ATA with one receive antenna, the slope at

high SNR reduces to $\min(N_R, N_T) = 1$ bps/Hz/(3 dB) which is half the low SNR slope.

From Section 3.1.3, the capacity of spread spectrum like MIMO systems can be expressed in terms of the spatial channel parameters. This also holds for MIMO-OFDM systems. Using (2.21), we can write $\check{\mathbf{F}}_{i;d} = \sum_{k=1}^{L_t} \sum_{s=0}^{S_k-1} G_{k;s} e^{-j2\pi \frac{p_i(d)}{T_F} \tau_k} \mathbf{a}_{k;s} \cdot \mathbf{b}_{i;k;s}^\dagger$. Defining the $n_i D_G \times 1$ vectors $\tilde{\mathbf{d}}_{i;d;m} = \sum_{k=1}^{L_t} \sum_{s=0}^{S_k-1} G_{k;s}^* e^{j2\pi \frac{p_i(d)}{T_F} \tau_k} e^{-j(m-1)\alpha_{k;s}} (\mathbf{b}_{i;k;s} \otimes \mathbf{e}_{d;D_G})$ for $d = 1, \dots, D_G$ and $m = 1, \dots, N_R$ and using (C.3), we can easily show that (3.57) is equivalent to

$$\tilde{\mathbf{O}}_i = \sum_{d=1}^{D_G} \sum_{m=1}^{N_R} \tilde{\mathbf{d}}_{i;d;m} \cdot \tilde{\mathbf{d}}_{i;d;m}^\dagger \quad (3.64)$$

which, similarly to $\mathbf{C}_i^\dagger \cdot \mathbf{C}_i$ of (3.19), is a summation of rank one matrices. Now using $\tilde{\mathbf{D}}_i = \text{stack}_{d=1}^{D_G} [\text{stack}_{m=1}^{N_R} [\tilde{\mathbf{d}}_{i;d;m}^\dagger]]$ and following a similar method as in Section 3.1.3, analytical expressions for the eigenvalues and eigenvectors of $\tilde{\mathbf{O}}_i$ can be found by solving the equations $|\tilde{\mathbf{D}}_i^\dagger \cdot \tilde{\mathbf{D}}_i - x \mathbf{I}_{n_i D_G}| = 0$ or $|\tilde{\mathbf{D}}_i \cdot \tilde{\mathbf{D}}_i^\dagger - x \mathbf{I}_{N_R D_G}| = 0$ up to an order of 4 (i.e. with $n_i D_G \leq 4$ or $N_R D_G \leq 4$) as in [104]. For larger values of $n_i D_G$ and $N_R D_G$, bounds for the maximal eigenvalue $\lambda_{O,\max}$ (and hence, for $[\frac{E_b}{N_0}]_{\min}$) can be found as in (3.23) using $\text{Tr}(\tilde{\mathbf{O}}_i) = \sum_{d=1}^{D_G} \sum_{m=1}^{N_R} \|\tilde{\mathbf{d}}_{i;d;m}\|^2$, and as in (3.25) and (3.26) using the components of the matrices $[\tilde{\mathbf{U}}_{T;i}^{DFT}]^\dagger \cdot \tilde{\mathbf{O}}_i \cdot \tilde{\mathbf{U}}_{T;i}^{DFT}$ and $[\tilde{\mathbf{U}}_{T;i}^{DCT}]^\dagger \cdot \tilde{\mathbf{O}}_i \cdot \tilde{\mathbf{U}}_{T;i}^{DCT}$, respectively, where $\tilde{\mathbf{U}}_{T;i}^{DFT}$ and $\tilde{\mathbf{U}}_{T;i}^{DCT}$ denote the $n_i D_G \times n_i D_G$ DFT and DCT matrices.

Assuming partial CSIT, we consider the low and high SNR regimes and we investigate the maximal information rate $I_{u;O}$ that can be obtained for MIMO-OFDM systems with an input covariance matrix of the form (3.38). Using (3.56) and writing (3.38) as $\Sigma = \text{diag}_{i=1}^{N_G} [\Sigma_i]$ with $\Sigma_i = \text{diag}[\frac{\eta_i}{r_{s;i}} \frac{P_T}{D_G} \mathbf{I}_{r_{s;i}}, \mathbf{0}_{(n_i-r_{s;i}) \times (n_i-r_{s;i})}] \otimes \mathbf{I}_{D_G}$, the corresponding information rate (in bps/Hz) is $I_{u;O} = \frac{1}{D} \log |\mathbf{I}_{\tilde{D}} + \frac{1}{N_0} \Sigma \cdot \mathbf{H}_O^\dagger \cdot \mathbf{H}_O|$, or

$$I_{u;O} \stackrel{(a)}{=} \frac{1}{D} \sum_{i=1}^{N_G} \log |\mathbf{I}_{n_i D_G} + \frac{1}{N_0} \Sigma_i \cdot \mathbf{B}_{O;i}^\dagger \cdot \mathbf{B}_{O;i}| \stackrel{(b)}{=} \frac{1}{D} \sum_{i=1}^{N_G} \log |\mathbf{I}_{N_R D_G} + \frac{1}{N_0} \mathbf{B}_{O;i} \cdot \Sigma_i \cdot \mathbf{B}_{O;i}^\dagger|$$

where (a) is due to (3.56) and (3.59), and (b) is due to (C.1). Since Σ_i is a diagonal matrix with only the first $r_{s;i} D_G$ diagonal elements nonzero, one can easily show that $\mathbf{B}_{O;i} \cdot \Sigma_i \cdot \mathbf{B}_{O;i}^\dagger = \mathbf{B}_{O;s;i} \cdot \frac{\eta_i}{r_{s;i}} \frac{P_T}{D_G} \mathbf{I}_{r_{s;i} D_G} \cdot \mathbf{B}_{O;s;i}^\dagger = \frac{\eta_i}{r_{s;i}} \frac{P_T}{D_G} \mathbf{B}_{O;s;i} \cdot \mathbf{B}_{O;s;i}^\dagger$ where $\mathbf{B}_{O;s;i}$ is the submatrix formed by the first $r_{s;i} D_G$ columns of $\mathbf{B}_{O;i}$, that is, $\mathbf{B}_{O;s;i} = \sum_{q=1}^{D_G} \check{\mathbf{F}}_{s;i;q} \otimes (\mathbf{e}_{q;D_G} \cdot \mathbf{e}_{q;D_G}^T)$ where $\check{\mathbf{F}}_{s;i;q}$ is the submatrix formed by the first $r_{s;i}$ columns of $\check{\mathbf{F}}_{i;q}$. It follows, using (C.1), that $I_{u;O} = \frac{1}{D} \sum_{i=1}^{N_G} \log |\mathbf{I}_{r_{s;i} D_G} + \frac{\eta_i}{r_{s;i}} \frac{N_G P_T}{N_0 D} \mathbf{B}_{O;s;i}^\dagger \cdot \mathbf{B}_{O;s;i}|$. Let $\{\xi_{O;i;l}\}_{l=1}^{r_{s;i} D_G}$ denote the (real

and non-negative) eigenvalues of $\mathbf{B}_{O;s;i}^\dagger \cdot \mathbf{B}_{O;s;i}$ satisfying $\xi_{O;i;1} \geq \dots \geq \xi_{O;i;r_{s,i}D_G} \geq 0$. We can write

$$I_{u;O} = \frac{1}{D} \sum_{i=1}^{N_G} \sum_{l=1}^{r_{s,i}D_G} \log \left(1 + \frac{\eta_i}{r_{s,i}} \frac{N_G P_T}{N_0 D} \xi_{O;i;l} \right) \text{ bps/Hz} \quad (3.65)$$

We investigate the rate $I_{u;O}$ (3.65) in the low SNR regime where $(P_T/N_0)\xi_{O;i;l} \ll 1$ for all $\xi_{O;i;l} \neq 0$, $l = 1, \dots, r_{s,i}D_G$ and $i = 1, \dots, N_G$. Using a first order approximation as in (3.40), we can write $I_{u;O} \approx \frac{1}{D} \log \left(1 + \frac{\tilde{B}P_T}{N_0 D} \right)$ where $\tilde{B} = N_G \sum_{i=1}^{N_G} \sum_{l=1}^{r_{s,i}D_G} \frac{\eta_i \xi_{O;i;l}}{r_{s,i}}$. Similarly to (3.41), one can easily show that $(E_b/N_0)_{T;u} \approx \frac{D^2(\ln 2)^2}{2\tilde{B}} I_{u;O} + \frac{D \ln 2}{\tilde{B}}$ and hence, $\left[\frac{E_b}{N_0} \right]_{u;\min} = \frac{D \ln 2}{\tilde{B}}$. Also, we have $I_{u;O} \approx \frac{2\tilde{B}}{D^2(\ln 2)^2} (E_b/N_0)_{T;u} - \frac{2}{D \ln 2}$ which is linear in $(E_b/N_0)_{T;u}$ with slope [98, (29)]

$$\lim_{(E_b/N_0)_{T;u} \rightarrow \left[\frac{E_b}{N_0} \right]_{u;\min}} \frac{10 \log_{10}(2) I_{u;O}}{10 \log_{10}((E_b/N_0)_{T;u}) - 10 \log_{10}(\left[\frac{E_b}{N_0} \right]_{u;\min})} = \frac{2}{D} = \frac{2}{N_G D_G} \text{ bps/Hz/(3 dB)} \quad (3.66)$$

This slope is the same for all space-time modulation formats, which is not the case for the slope (3.42) for spread spectrum like MIMO systems without CSIT. Moreover, the slope of (3.66) is smaller than that of (3.42) since $D_G \geq 1$. Following the same approach as in Section 3.2.1, let us consider the optimal power allocation that maximizes $I_{u;O}$ at low SNR. For fixed power allocation $\boldsymbol{\eta}$, maximizing $I_{u;O}$ is equivalent to maximizing $\tilde{B} \equiv N_G \sum_{i=1}^{N_G} \frac{\eta_i}{r_{s,i}} \text{Tr}(\mathbf{B}_{O;s;i}^\dagger \cdot \mathbf{B}_{O;s;i})$, which can be done by maximizing the ratios $\frac{\text{Tr}(\mathbf{B}_{O;s;i}^\dagger \cdot \mathbf{B}_{O;s;i})}{r_{s,i}}$ as follows. For each TOG, we can write $\mathbf{B}_{O;s;i}^\dagger \cdot \mathbf{B}_{O;s;i} = \left[\sum_{d=1}^{D_G} \check{\mathbf{F}}_{s;i;d}^\dagger \otimes (\mathbf{e}_{d;D_G} \cdot \mathbf{e}_{d;D_G}^T) \right] \cdot \left[\sum_{q=1}^{D_G} \check{\mathbf{F}}_{s;i;q} \otimes (\mathbf{e}_{q;D_G} \cdot \mathbf{e}_{q;D_G}^T) \right]$ which, using (C.3) and $\mathbf{e}_{d;D_G}^T \cdot \mathbf{e}_{q;D_G} = \delta(d, q)$, leads to $\mathbf{B}_{O;s;i}^\dagger \cdot \mathbf{B}_{O;s;i} = \sum_{d=1}^{D_G} (\check{\mathbf{F}}_{s;i;d}^\dagger \cdot \check{\mathbf{F}}_{s;i;d}) \otimes (\mathbf{e}_{d;D_G} \cdot \mathbf{e}_{d;D_G}^T)$. Thus, we have $\text{Tr}(\mathbf{B}_{O;s;i}^\dagger \cdot \mathbf{B}_{O;s;i}) = \sum_{d=1}^{D_G} \text{Tr}[(\check{\mathbf{F}}_{s;i;d}^\dagger \cdot \check{\mathbf{F}}_{s;i;d}) \otimes (\mathbf{e}_{d;D_G} \cdot \mathbf{e}_{d;D_G}^T)] = \sum_{d=1}^{D_G} \text{Tr}[\check{\mathbf{F}}_{s;i;d}^\dagger \cdot \check{\mathbf{F}}_{s;i;d}]$ with the last equality due to the property $\text{Tr}(\mathbf{A} \otimes \mathbf{B}) = \text{Tr}(\mathbf{A})\text{Tr}(\mathbf{B})$ from [114, p. 338] and to $\text{Tr}(\mathbf{e}_{d;D_G} \cdot \mathbf{e}_{d;D_G}^T) = 1$. By writing $\check{\mathbf{F}}_{s;i;d} = [\check{\mathbf{F}}_{i;d,*1} \dots \check{\mathbf{F}}_{i;d,*r_{s,i}}]$, it follows that $\text{Tr}(\mathbf{B}_{O;s;i}^\dagger \cdot \mathbf{B}_{O;s;i}) = \sum_{d=1}^{D_G} \sum_{l=1}^{r_{s,i}} \|\check{\mathbf{F}}_{i;d,*l}\|^2 = \sum_{l=1}^{r_{s,i}} \check{f}_{i;l}$ where $\check{f}_{i;l} \triangleq \sum_{d=1}^{D_G} \|\check{\mathbf{F}}_{i;d,*l}\|^2$ or, equivalently, $\check{f}_{i;l} \triangleq \sum_{d=1}^{D_G} \left\| \sum_{k=1}^{L_t} [\mathbf{F}_{i;k}]_{*,l} e^{-j2\pi \frac{p_i(d)}{T_F} \tau_k} \right\|^2$. As such, we obtain

$$\frac{\text{Tr}(\mathbf{B}_{O;s;i}^\dagger \cdot \mathbf{B}_{O;s;i})}{r_{s,i}} = \frac{\sum_{l=1}^{r_{s,i}} \check{f}_{i;l}}{r_{s,i}} \leq \max\{\check{f}_{i;l}\}_{l=1}^{r_{s,i}} \leq \max\{\check{f}_{i;l}\}_{l=1}^{n_i}$$

Denoting $l_{i;\max} = \arg \max_{l=1, \dots, n_i} \check{f}_{i;l}$, we have $\frac{\text{Tr}(\mathbf{B}_{O;s;i}^\dagger \cdot \mathbf{B}_{O;s;i})}{r_{s,i}} \leq \check{f}_{i;l_{i;\max}}$. Equality is ensured if $r_{s,i} = 1$ and $\check{\mathbf{F}}_{s;i;d}$ is column $l_{i;\max}$ of $\check{\mathbf{F}}_{i;d}$ for every $d = 1, \dots, D_G$. In this case, we

have $\tilde{B} = N_G \sum_{i=1}^{N_G} \check{f}_{i,l_{i,\max}} \eta_i$. If $N_G \geq 2$, this $\tilde{B}$ is maximized over $\mathcal{A}_\eta$ by allocating the total power P_T to the (strongest) antenna associated with $\check{f}_{\max} \triangleq \max\{\check{f}_{i,l_{i,\max}}\}_{i=1}^{N_G}$. With ATA ($N_G = 1$), the maximization over $\mathcal{A}_\eta$ is irrelevant and $\check{f}_{\max} = \check{f}_{1,l_{1,\max}} = \max\{\check{f}_{1,l}\}_{l=1}^{N_T}$ where $\check{f}_{1,l} = \sum_{d=1}^D \left\| \sum_{k=1}^{L_t} [\mathbf{F}_k]_{*,l} e^{-j2\pi \frac{p_l(d)}{T_F} \tau_k} \right\|^2 \equiv \sum_{p=-N_F/2}^{N_F/2-1} \left\| \sum_{k=1}^{L_t} [\mathbf{F}_k]_{*,l} e^{-j2\pi \frac{p}{T_F} \tau_k} \right\|^2 = \sum_{p=-N_F/2}^{N_F/2-1} \left\| [\mathbf{H}_{O;p}]_{*,l} \right\|^2$. Hence, for any space-time modulation format the maximal value for $\tilde{B}$ is given by

$$\tilde{B}_{\max} = N_G \check{f}_{\max} \quad (3.67)$$

We conclude that for MIMO-OFDM systems the information rate $I_{u,O}$ is maximized at low SNR using also a strongest antenna power allocation, where the selected antenna is the one associated with $\check{f}_{\max}$. This strongest antenna selection criterion is different from the one used for spread spectrum like MIMO systems where the strongest antenna is defined in Section 3.2.1 as the antenna associated with the column of $\mathbf{C}$ that has the maximal norm.

We now consider the information rate $I_{u,O}$ in the high SNR regime where P_T/N_0 is assumed to be very large. With g denoting the number of active TOGs and $r'_{O;s;i}$ denoting the rank of $\mathbf{B}_{O;s;i}$, (3.65) can be written as $I_{u,O} \approx \frac{1}{D} \sum_{i=1}^g \sum_{l=1}^{r'_{O;s;i}} \log \left(\frac{\eta_i}{r_{s;i}} \frac{N_G P_T}{N_0 D} \xi_{O;i,l} \right)$, or

$$I_{u,O} \approx \frac{1}{D} \sum_{i=1}^g \sum_{l=1}^{r'_{O;s;i}} \log \left(\frac{\eta_i N_G}{r_{s;i}} \xi_{O;i,l} \right) + \frac{(\sum_{i=1}^g r'_{O;s;i})}{D} \log \left(\frac{P_T}{N_0 D} \right)$$

Since the second term on the right hand side is dominant, maximizing $I_{u,O}$ reduces to maximizing $\sum_{i=1}^g r'_{O;s;i}$, i.e. g and $\{r'_{O;s;i}\}_{i=1}^g$. As in Section 3.2.2, this corresponds to making all the TOGs active ($g = N_G$), and to making $r'_{O;s;i} = r_{O;i}$ which happens when $\mathbf{B}_{O;s;i} = \mathbf{B}_{O;i}$ and $r_{s,i} = n_i$ for all i . The resulting information rate $I_{u,O}$ is

$$I_{u,O} = \frac{M_O}{D} + \frac{r_O}{D} \log \left(\frac{P_T}{N_0 D} \right) \quad (3.68)$$

where M_O is given by

$$M_O = \sum_{i=1}^{N_G} \sum_{l=1}^{r_{O;i}} \log \left(\frac{\eta_i N_G}{n_i} \lambda_{O;i,l} \right) \quad (3.69)$$

If $N_G \geq 2$, further maximization of $I_{u,O}$ (3.68) can be obtained by maximizing M_O over the set of power allocations $\mathcal{A}_\eta$. This is equivalent to minimizing $-\sum_{i=1}^{N_G} r_{O;i} \log(\eta_i)$ which, as in Section 3.2.2, leads to the optimal power allocation $\eta_i = \frac{r_{O;i}}{r_O}$. The latter is similar to the high SNR optimal power allocation for spread spectrum like MIMO systems in the

sense that it is the ratio of the rank of $\tilde{\mathbf{O}}_i$ to that of $\mathbf{H}_O^\dagger \cdot \mathbf{H}_O$. If $N_G = 1$, we have $\eta_i = 1$ in (3.69) and the optimal power allocation reduces to a uniform power allocation among all transmit antennas. Note that the multiplexing gain associated with $I_{u,O}$ (3.68) is equal to $\frac{r_O}{D}$, which is similar to that of C_O (3.62) and is maximum, intermediate and minimum when using ATA, POTA and OTA, respectively. It follows, similarly to the result of Lemma 3.2.5 for spread spectrum like MIMO systems, that the rate $I_{u,O}$ (3.68) is also maximum, intermediate and minimum when using ATA, POTA and OTA, respectively. Moreover, for a given space-time modulation format and for the same $\frac{P_T}{N_0 D}$, the rate $I_{u,O}$ (3.68) is smaller than the rate I_u (3.50) because of the inequality (3.63).

We conclude this section by considering the necessary and sufficient conditions for the capacity C_O (3.58) to be equal to the rate $I_{u,O}$ that is obtained using full rank input covariance matrices (i.e. $r_{s,i} = n_i$ for all i) and given from (3.65) by

$$I_{u,O} = \frac{1}{D} \sum_{i=1}^{N_G} \sum_{l=1}^{n_i D_G} \log \left(1 + \frac{\eta_i}{n_i} \frac{N_G P_T}{N_0 D} \lambda_{O,i,l} \right) \text{ bps/Hz} \quad (3.70)$$

Following a proof similar to that of Theorem 3.2.2, these conditions are that all matrices $\mathbf{B}_{O,i}$ are full column rank (i.e. $r_{O,i} = n_i D_G$) with all eigenmodes active and equal eigenvalues $\lambda_{O,i,l} = \lambda_{O,i}$ for all $1 \leq l \leq n_i D_G$, $1 \leq i \leq N_G$, such that

$$\lambda_{O,i}^{-1} + \frac{\eta_i}{n_i} \frac{N_G P_T}{N_0 D} = \text{Constant}$$

In the high SNR regime with very large P_T/N_0 , the last expression reduces to $\frac{\eta_i}{n_i} \frac{N_G P_T}{N_0 D} = \text{Constant}$ regardless of the eigenvalues $\lambda_{O,i}$ and hence, the necessary and sufficient conditions reduce to having $\eta_i = \frac{n_i}{N_T}$ (corresponding to the uniform power allocation) in addition to $r_{O,i} = n_i D_G$ for all $i = 1, \dots, N_G$. The latter result is similar to that of Theorem 3.2.3 for spread spectrum like MIMO systems and can be proved in a similar manner.

3.4 Numerical results for spread spectrum like systems

This section presents some numerical examples of the results obtained in Sections 3.1 and 3.2. For the numerical results illustrated in Figures 3.2-3.7, we employ Channel A with the parameters of Table 3.1 and we consider a system with $N_T = 6$, $N_R = 1, 2$ using ATA,

POTA ($N_G = 2$, $n_1 = 2$, $n_2 = 4$) and OTA. Note that the values of N_R and N_T used in this section are arbitrary. Unless otherwise specified, we consider $L_t = 3$. As such, we have a total of $L = 60$ multipath components which is larger than n_i and $L_t N_R$, yielding $r_i = \min(n_i, L_t N_R)$ for all i . Our results confirm the following theoretical findings:

1) *Effect of transmit power allocation:* Assuming $N_R = 1$, Figure 3.2 compares ATA with POTA while Figure 3.3 compares ATA with OTA. We calculated the capacity C using the water-filling Power Allocation (PA) as well as the information rate I_u using the following PAs: strongest antenna PA, uniform PA ($\eta_i = \frac{n_i}{N_T}$), uniform TOG PA ($\eta_i = \frac{1}{N_G}$), high SNR optimal TOG PA ($\eta_i = \frac{r_i}{r_T}$). Plots of I_u are also provided using the one-TOG PAs $\eta_i = \delta(i, i_b)$ and $\eta_i = \delta(i, i_w)$, where $i_b = \arg \max_{i=1, \dots, N_G} \frac{\text{Tr}(\mathbf{C}_i^\dagger \cdot \mathbf{C}_i)}{n_i}$ and $i_w = \arg \min_{i=1, \dots, N_G} \frac{\text{Tr}(\mathbf{C}_i^\dagger \cdot \mathbf{C}_i)}{n_i}$. Hence, $\eta_i = \delta(i, i_b)$ corresponds to the strongest TOG power allocation. With ATA we plotted I_u using two PAs: $\eta_1 = 1$ and the strongest antenna PA. With OTA we plotted I_u using three PAs: $\eta_i = \frac{1}{N_T}$, $\eta_i = \delta(i, i_w)$ and the strongest antenna PA.

Regardless of the PA used for I_u , we have $I_u \leq C$ at any SNR. For any space-time modulation format, inflection points are observed in the capacity curves. These inflection points occur since the number of active eigenmodes in the water-filling scheme increases as $(E_b/N_0)_T$ increases. Such phenomenon is absent in the I_u curves since for any PA the number of active eigenmodes does not change with the SNR.

Low SNR: Using the strongest antenna PA, the highest rate is provided by ATA (Lemma 3.2.3) and all space-time modulation formats yield $\left[\frac{E_b}{N_0}\right]_{u;\min} = -3.85$ dB in Figures 3.2, 3.3. If the strongest TOG PA is used, OTA provides the highest rate I_u in the SNR interval $[-3.85, -3.5]$ dB (Lemma 3.2.4) with $\left[\frac{E_b}{N_0}\right]_{\min} = \left[\frac{E_b}{N_0}\right]_{u;\min} = -3.85$ dB for informed and uninformed transmitters in Figure 3.3. Using the uniform PA $\eta_i = \frac{n_i}{N_T}$, we have $\left[\frac{E_b}{N_0}\right]_{u;\min} = -3.31$ dB in Figures 3.2, 3.3 and $\left[\frac{E_b}{N_0}\right]_{u;\min} = -5.32$ dB in Figure 3.4 for all space-time modulation formats.

High SNR: One can see from Figures 3.2 - 3.4 that Lemma 3.2.5 holds for each of the PAs $\eta_i = \frac{n_i}{N_T}$, $\eta_i = \frac{r_i}{r_T}$ and $\eta_i = \frac{1}{N_G}$. With POTA, Figure 3.2 shows that the PAs $\eta_i = \frac{n_i}{N_T}$ and $\eta_i = \frac{1}{N_G}$ result in information rates that are very close to the maximal rate provided by the PA $\eta_i = \frac{r_i}{r_T}$. With OTA and $\eta_i = \frac{1}{N_T}$ we have $\Delta(E_b/N_0)_T = 0$ dB starting from relatively low SNR in Figures 3.3, 3.4, 3.7.

2) *Effect of the number of receive antennas:* From Figures 3.2 - 3.4, one can see that regardless of space-time modulation format both C and I_u increase with N_R . For I_u , we considered $\eta_i = \frac{n_i}{N_T}$ only since it satisfies Optimality Criterion 1 without requiring any transmitter channel knowledge. By adding more receive antennas the ranks $r_i = \min(n_i, L_t N_R)$ can increase (when $n_i \geq L_t N_R$), which can lead to a larger number of channel eigenmodes and a higher slope $\frac{r_T}{N_G}$ at high SNR.

3) *Effect of the multipath propagation:* Figures 3.5 (ATA), 3.6 (POTA) and 3.7 (OTA) illustrate the capacity C and rate I_u for each of the following propagation scenarios:

- i) $(L_t, L) = (1, 20)$: We consider cluster 1 with 20 space resolvable paths ($L = S_1 = 20$).
- ii) $(L_t, L) = (1, 40)$: There are 40 paths resolvable in space and not in time ($L = S_1 + S_2 = 40$).
- iii) $(L_t, L) = (1, 60)$: There are 60 paths resolvable in space and not in time ($L = S_1 + S_2 + S_3 = 60$).
- iv) $(L_t, L) = (2, 40)$: We consider clusters 1, 2 with different time delays and 20 space resolvable paths per cluster ($S_1 = S_2 = 20$), $L = S_1 + S_2 = 40$.

The case $(L_t, L) = (3, 60)$ where all three clusters are considered with different time delays and 20 space resolvable paths per cluster is presented in Figure 3.4. In all these examples, the powers (2.7) associated with time dispersive clusters are normalized such that their sum is 1, and equal amounts of power are assigned to the space resolvable paths in a cluster. In all propagation scenarios, $L \geq \min(n_i, L_t N_R)$, $\forall i$, hence we always have $r_i = \min(n_i, L_t N_R)$. Consider scenarios i)-iii). Figures 3.5 - 3.7 show that adding more space resolvable paths may not always increase transmission rates mainly because it does not increase the ranks r_i (which are determined only by n_i and $L_t N_R$) while leading to arbitrary variations in the eigenvalues $\lambda_{i,l}$ as well as in C and I_u . When comparing scenarios ii) and iv), one can see that increasing L_t can increase r_i for TOGs with $n_i \geq L_t N_R$, which can increase C and I_u especially at high SNR where the slope is given by $\frac{r_T}{N_G}$ bps/Hz/(3 dB). This is observed in Figures 3.5 and 3.6 but not in Figure 3.7 where $r_i = n_i = 1$ for OTA regardless of L_t . These conclusions are valid not only for the 3GPP channels but for all channels with full rank TOG propagation matrices (i.e. $r_i = \min(n_i, L_t N_R)$ for all $i = 1, \dots, N_G$). Next, we consider the effect of varying the angle spread at the transmitter and receiver for the 3GPP channels using the angle spread definition from [83].

4) *Effect of the angle spread:* We now consider the effect of angle spread on the capacity and the transmission rate using uniform PA. For the numerical results illustrated in Figures

3.9, 3.10 and 3.11, we consider a system with $(N_T, N_R) = (6, 2)$ using ATA, POTA ($N_G = 2, n_1 = 2, n_2 = 4$) and OTA, respectively. For the numerical results illustrated in Figures 3.12, 3.13 and 3.14, we consider a system with $(N_T, N_R) = (4, 4)$ using ATA, POTA ($N_G = 2, n_1 = 2, n_2 = 2$) and OTA, respectively. For Figures 3.15 and 3.16, we employ ATA for systems with $(N_T, N_R) = (6, 2)$ and $(N_T, N_R) = (4, 4)$, respectively. For simplicity, we assume that the AODs $\theta_{T;k;s}$ associated with clusters 1, 2 and 3 are equally spaced in the intervals $[-A_{TX} - b_{TX}, -A_{TX} + b_{TX}]$, $[-b_{TX}, b_{TX}]$ and $[A_{TX} - b_{TX}, A_{TX} + b_{TX}]$, respectively, where b_{TX} determines the transmitter cluster angle spread and $-A_{TX}$ and A_{TX} denote the mean AODs for clusters 1 and 3, respectively. Similarly, the AOA $\theta_{R;k;s}$ associated with clusters 1, 2 and 3 are equally spaced in the intervals $[-A_{RX} - b_{RX}, -A_{RX} + b_{RX}]$, $[-b_{RX}, b_{RX}]$ and $[A_{RX} - b_{RX}, A_{RX} + b_{RX}]$, respectively, where b_{RX} determines the receive cluster angle spread and $-A_{RX}$ and A_{RX} denote the mean AOAs for clusters 1 and 3, respectively. Cluster 2 has zero mean AOD and zero mean AOA. We consider the following realizations of the remaining channel parameters: $\sigma_{SF} = 0.0714$, $P_1 = 0.3650$, $P_2 = 0.2254$, $P_3 = 0.4096$, and the carrier phases (in radians)

$$\begin{aligned} \{\varphi_{1;s}\}_{s=0}^{S_1-1} &= \{6.2477, 5.9914, 2.3861, 1.8615, 6.1815, 4.6323, 3.2250, 3.0169, 0.7559, 5.3451, \\ &\quad 6.1664, 4.2965, 3.5908, 2.0910, 0.9543, 2.3924, 3.0198, 1.4370, 2.7083, 3.0542\} \\ \{\varphi_{2;s}\}_{s=0}^{S_2-1} &= \{0.4007, 5.2148, 0.5838, 0.7081, 3.6872, 2.6063, 0.5312, 0.5535, 1.3317, 1.5721, \\ &\quad 2.8203, 0.0644, 0.2854, 4.0472, 3.2529, 4.6819, 3.3210, 5.2014, 4.6053, 5.9652\} \\ \{\varphi_{3;s}\}_{s=0}^{S_3-1} &= \{0.5114, 5.8802, 6.2335, 0.1885, 0.7996, 2.9448, 1.8821, 2.0198, 6.0766, 0.9274, \\ &\quad 3.5040, 3.7861, 3.6942, 0.7761, 2.3987, 1.6952, 5.8212, 1.3055, 1.2956, 5.2335\} \end{aligned}$$

From the 3GPP specification [83, Annex A] and using the fact that for each cluster k the power P_k is equally divided between its S_k space resolvable paths with $\sum_{k=1}^{L_t} P_k = 1$ as seen from (2.7), the angle spreads at the transmitter and receiver are given by

$$\sigma_{TX} = \sqrt{\sum_{k=1}^{L_t} \frac{P_k}{S_k} \left(\sum_{s=0}^{S_k-1} \bar{\theta}_{T;k;s}^2 \right)} \quad (3.71a)$$

$$\sigma_{RX} = \sqrt{\sum_{k=1}^{L_t} \frac{P_k}{S_k} \left(\sum_{s=0}^{S_k-1} \bar{\theta}_{R;k;s}^2 \right)}, \quad (3.71b)$$

$$\bar{\theta}_{T;k;s} = \begin{cases} 360^\circ + (\theta_{T;k;s} - \mu_{\theta;T}), & \text{if } \theta_{T;k;s} - \mu_{\theta;T} < -180^\circ \\ (\theta_{T;k;s} - \mu_{\theta;T}), & \text{if } -180^\circ \leq \theta_{T;k;s} - \mu_{\theta;T} \leq 180^\circ \\ 360^\circ - (\theta_{T;k;s} - \mu_{\theta;T}), & \text{if } 180^\circ < \theta_{T;k;s} - \mu_{\theta;T} \end{cases} \quad (3.72)$$

$$\bar{\theta}_{R;k;s} = \begin{cases} 360^\circ + (\theta_{R;k;s} - \mu_{\theta;R}), & \text{if } \theta_{R;k;s} - \mu_{\theta;R} < -180^\circ \\ (\theta_{R;k;s} - \mu_{\theta;R}), & \text{if } -180^\circ \leq \theta_{R;k;s} - \mu_{\theta;R} \leq 180^\circ \\ 360^\circ - (\theta_{R;k;s} - \mu_{\theta;R}), & \text{if } 180^\circ < \theta_{R;k;s} - \mu_{\theta;R} \end{cases} \quad (3.73)$$

with $\mu_{\theta;T} = \sum_{k=1}^{L_t} \frac{P_k}{S_k} (\sum_{s=0}^{S_k-1} \theta_{T;k;s})$ and $\mu_{\theta;R} = \sum_{k=1}^{L_t} \frac{P_k}{S_k} (\sum_{s=0}^{S_k-1} \theta_{R;k;s})$. The angle spreads σ_{TX} and σ_{RX} are represented in Figures 3.8a and 3.8b as functions of the angles (A_{TX}, b_{TX}) and (A_{RX}, b_{RX}) , respectively. One can see that for a given b_{TX} and b_{RX} , σ_{TX} and σ_{RX} increase with increasing A_{TX} and A_{RX} , respectively. Furthermore, for a given A_{TX} (respectively, A_{RX}) the angle spread σ_{TX} (respectively, σ_{RX}) varies slowly with b_{TX} (respectively, b_{RX}). The larger A_{TX} and A_{RX} are, the smaller these variations become. Hence, we conclude that the angle spreads σ_{TX} and σ_{RX} are mainly determined by the mean AOD A_{TX} and mean AOA A_{RX} , respectively. A large angle spread σ_{TX} (respectively, σ_{RX}) is obtained when A_{TX} (respectively, A_{RX}) is large, regardless of the cluster angle spread b_{TX} or b_{RX} . Using $\frac{P_T}{N_0 D} = 15$ and different space-time modulation formats, plots for the channel capacity C as a function of (A_{TX}, b_{TX}) and (A_{RX}, b_{RX}) are provided in Figures 3.9, 3.10 and 3.11 with $(N_T, N_R) = (6, 2)$ and in Figures 3.12, 3.13 and 3.14 with $(N_T, N_R) = (4, 4)$. For the same $\frac{P_T}{N_0 D}$ and using ATA with uniform PA, plots of the transmission rate I_u as a function of (A_{TX}, b_{TX}) and (A_{RX}, b_{RX}) are provided in Figure 3.15 with $(N_T, N_R) = (6, 2)$ and in Figure 3.16 with $(N_T, N_R) = (4, 4)$. In all these figures, we take $(A_{RX}, b_{RX}) = (50^\circ, 10^\circ)$ when varying (A_{TX}, b_{TX}) and $(A_{TX}, b_{TX}) = (20^\circ, 5^\circ)$ when varying (A_{RX}, b_{RX}) . Considering a similar range for (A_{RX}, b_{RX}) as that of (A_{TX}, b_{TX}) (i.e. $A_{RX} \in [1^\circ, 45^\circ]$ and $b_{RX} \in [1^\circ, 15^\circ]$), we observe in each of Figures 3.9-3.16 an asymmetric behavior of C (or I_u) when taken as a function of (A_{TX}, b_{TX}) and as a function of (A_{RX}, b_{RX}) . Moreover, we have the following observations:

a) Impact of the mean AOD A_{TX} and mean AOA A_{RX} : Figures 3.9a-3.16a show that for a given b_{TX} , C and I_u become (in general) smaller when A_{TX} becomes larger, that is, when the mean AODs for clusters 1 and 3 move away from the broadside of the transmit antenna array. If b_{RX} is fixed and small (i.e. $b_{RX} < 10^\circ$), Figures 3.9b-3.16b show that the capacity C and information rate I_u vary slowly when A_{RX} increases in the interval $[1^\circ, 90^\circ]$, and they increase when A_{RX} increases in the interval $[90^\circ, 150^\circ]$. Such an increase is more pronounced for $(N_T, N_R) = (4, 4)$ than for $(N_T, N_R) = (6, 2)$. If b_{RX} is fixed and large,

Figures 3.9b-3.16b show that when A_{RX} increases C and I_u decrease and then increase. The lowest values for C and I_u are reached for a mean AOA close to 90° , which corresponds to receiving the waveforms associated with clusters 1 and 3 inline with the antenna array. We conclude that, for a fixed cluster angle spread at the transmitter and/or receiver, the closer are the clusters mean AODs and mean AOAs to the antenna array broadside the higher are the channel capacity C and information rate I_u . In terms of angle spread and if we restrict the mean AOD A_{TX} (respectively, mean AOA A_{RX}) to vary in the interval $[1^\circ, 90^\circ]$, we can claim that the smaller σ_{TX} (respectively, σ_{RX}) is the larger C and I_u are, since σ_{TX} (respectively, σ_{RX}) is a monotonically increasing function of A_{TX} (respectively, A_{RX}) as seen from Figure 3.8a (respectively, Figure 3.8b). Our results on the impact of mean AOA agree with those of [115] which considers a MIMO system with infinite number of transmit antennas, with CSIR only and with correlated fades at the receiver. In this reference, it is shown that as the mean AOA moves away from the broadside angle the spatial correlation increases, thus, yielding a decrease in the ergodic capacity. Let us now compare our results to those obtained in [41, 116] for the ergodic capacity of the uplink of an OFDM-based MIMO system with infinite number of transmit antennas, and using uniform power allocation across all space-frequency subchannels. Assuming a Rayleigh fading channel with spatially correlated fades at the receiver, [41, 116] show that the ergodic capacity increases by increasing the total angle spread which denotes the spread of the clusters mean AOA taken with respect to the inline direction of the receive antenna array. This conclusion has been extended to Rician fading MIMO channels in [117]. By noticing that in our case the angle spread σ_{RX} is taken with respect to the broadside of the antenna array whereas in [41, 116, 117] it is taken with respect to the inline direction (i.e. a large angle spread in [41, 116, 117] means a small σ_{RX} and vice versa), our results also agree with those of [41, 116, 117].

b) Impact of the cluster angle spread at the transmitter b_{TX} and at the receiver b_{RX} : From Figures 3.9a-3.11a and 3.12a-3.14a we can see that for a given A_{TX} the capacity C does not present significant variations when the cluster angle spread b_{TX} varies. This holds also for I_u in Figures 3.15a and 3.16a except for small values of the mean AOD A_{TX} (i.e. $A_{TX} \in [1^\circ, 10^\circ]$) for which the rate increases as b_{TX} increases. We can also see from Figures 3.9a-3.16a that the impact of the transmitter cluster angle spread b_{TX} on C and I_u is dominated by that of the mean AOD A_{TX} . Figures 3.9b-3.11b and 3.15b show that for a mean AOA A_{RX} that is closer to the antenna array broadside than to the inline direction, C and I_u increase in general when b_{RX} increases. In Figures 3.12b-3.14b and 3.16b, C and I_u

present much more variations as functions of b_{RX} . In all these figures and for a mean AOA A_{RX} that is close to the inline direction (i.e. to 90°), C and I_u decrease with increasing b_{RX} . Assuming one cluster, [41, 115, 116, 118] show that the ergodic capacity in the limit of large number of transmit antennas increases when the receive cluster angle spread increases since the spatial correlation decreases in this case. A propagation scenario with one cluster as in [41, 115, 116, 118] can be obtained based on our channel model by choosing $A_{RX} = 1^\circ$ since in this case all three clusters almost coincide and can be regarded as one cluster. We compare the results of [41, 115, 116, 118] with what we obtained for $A_{RX} = 1^\circ$ using $N_T = 6$ which is larger than the other value $N_T = 4$ that we also considered for the numerical results. With $A_{RX} = 1^\circ$, Figures 3.9b-3.11b and 3.15b show that C and I_u monotonically increase when b_{RX} increases in the interval $[10^\circ, 30^\circ]$ (in accordance with the findings of [41, 115, 116, 118] for an infinite number of transmit antennas) but can be slightly decreasing when b_{RX} increases in the interval $[1^\circ, 10^\circ]$.

5) *Effect of the space-time modulation format:* Without CSIT, Conjecture 3.2.1 can be easily seen to hold throughout Figures 3.2 - 3.7. With CSIT, the results of Theorem 3.1.1 and Corollary 3.1.1 hold for the system with $N_T = 6$ and $N_R = 1, 2$ considered in Figures 3.2-3.7 and 3.9-3.11, as well as for the system with $N_T = N_R = 4$ considered in Figures 3.12-3.14. In Figures 3.2-3.7, one can see that ATA provides the highest value for C at any given $(E_b/N_0)_T$ while OTA and POTA provide poor and intermediate performance, respectively. Figures 3.9-3.11 and Figures 3.12-3.14 show that for a given angles (A_{TX}, b_{TX}) or (A_{RX}, b_{RX}) , largest, intermediate and smallest values for C are provided by ATA, POTA and OTA, respectively.

We now briefly discuss how the results provided in this chapter can be used for random channels. Based on the results of Section 3.1, the impact of space-time modulation on the ergodic capacity for fast fading channels with both CSIT and CSIR can be analyzed. For these channels, the transmitter employs a water-filling power allocation and eigenbeamforming signaling structure for each fading channel realization. The ergodic capacity is the average (over all channel propagation matrix realizations) of capacities associated with each channel realization. The consequences of this definition are two-fold. First, the ergodic capacity is maximum, intermediate and minimum with ATA, POTA and OTA, respectively, since each of these capacities is so by Theorem 3.1.1 and Corollary 3.1.1. Second, assuming that L_t is constant the ergodic capacity scales like $\frac{r_T}{N_G} \log(P_T/N_0) \equiv \frac{\sum_{i=1}^{N_G} \min(n_i, L_t N_R)}{N_G} \log(P_T/N_0)$ at high SNR as seen from (3.17). The results of Section 3.2

can be used for random channels with CSIR and without CSIT as follows. We consider non-ergodic fading channels where the entries of the channel propagation matrix $\mathbf{C}$ are i.i.d. circularly symmetric Gaussian random variables with zero mean and independent real and imaginary parts, each with variance $1/2$. In this case, $\mathbf{C}$ is chosen randomly at the beginning and held fixed during all channel uses. Using the normalization $\Sigma' = \frac{1}{P_T} \Sigma = \text{diag}_{i=1}^{N_G} [\mathbf{S}'_i \otimes \mathbf{I}_{D_G}]$ with $\mathbf{S}'_i = \frac{1}{P_T} \mathbf{S}_i$, the mutual information (3.1) reduces as in (3.28) to

$$I(\mathbf{x}; \mathbf{r}) = \log \left(\left| \mathbf{I}_{L_t N_R D} + \frac{P_T}{N_0} \mathbf{H} \cdot \Sigma' \cdot \mathbf{H}^\dagger \right| \right) = D_G \sum_{i=1}^{N_G} \log \left(\left| \mathbf{I}_{L_t N_R} + \frac{P_T}{N_0} \mathbf{C}_i \cdot \mathbf{S}'_i \cdot \mathbf{C}_i^\dagger \right| \right)$$

Using the same notations as in Section 3.2, this mutual information has the same distribution as if $\mathbf{S}'_i$ is a diagonal matrix of the form $\mathbf{S}'_i = \text{diag}[\frac{\eta_i}{D_G} \Lambda_i, \mathbf{0}_{(n_i - r_{s,i}) \times (n_i - r_{s,i})}]$. Equivalently, using $\kappa = 1/D_G$, $I(\mathbf{x}; \mathbf{r})$ has a similar distribution as

$$I(\mathbf{x}; \mathbf{r}) = D_G \sum_{i=1}^{N_G} \log(|\Lambda_i|) + D_G \sum_{i=1}^{N_G} \log \left(|\Lambda_i^{-1} + \eta_i \kappa \frac{P_T}{N_0} \mathbf{C}_{s,i}^\dagger \cdot \mathbf{C}_{s,i}| \right)$$

which has a similar form as $I_\sigma(\mathbf{x}; \mathbf{r})$ of (3.30). Hence, following similar steps to those used in Section 3.2 with P_T/N_0 now playing the role of σ^2 , our proof for the optimality of the uniform power allocation in terms of satisfying Optimality Criterion 1 can be used to show that at high SNR and over all (nonzero) input covariance matrices $\Sigma = \text{diag}_{i=1}^{N_G} [\mathbf{S}_i \otimes \mathbf{I}_{D_G}]$ with $\text{Tr}(\Sigma) \leq P_T$, the uniform power allocation (3.37) causes the fastest convergence to zero of $\Pr[I(\mathbf{x}; \mathbf{r}) < R]$ as $P_T/N_0 \rightarrow \infty$, for any given rate $0 < R < \infty$. If the channel is changing rapidly over time and the input covariance matrix $\Sigma = \text{diag}_{i=1}^{N_G} [\mathbf{S}_i \otimes \mathbf{I}_{D_G}]$ is full rank, the information rate averaged over all channel realizations scales at high SNR like $\frac{r_T}{N_G} \log(P_T/N_0) \equiv \frac{\sum_{i=1}^{N_G} \min(n_i, L_t N_R)}{N_G} \log(P_T/N_0)$ (assuming that L_t is constant) similarly to the information rate (3.50) associated with each channel realization. In this case, ATA achieves the largest average rate followed by POTA then by OTA (Theorem 3.1.3). Furthermore, the necessary and sufficient conditions of Theorem 3.2.3 hold for the equality of the average rate and the ergodic capacity at high SNR.

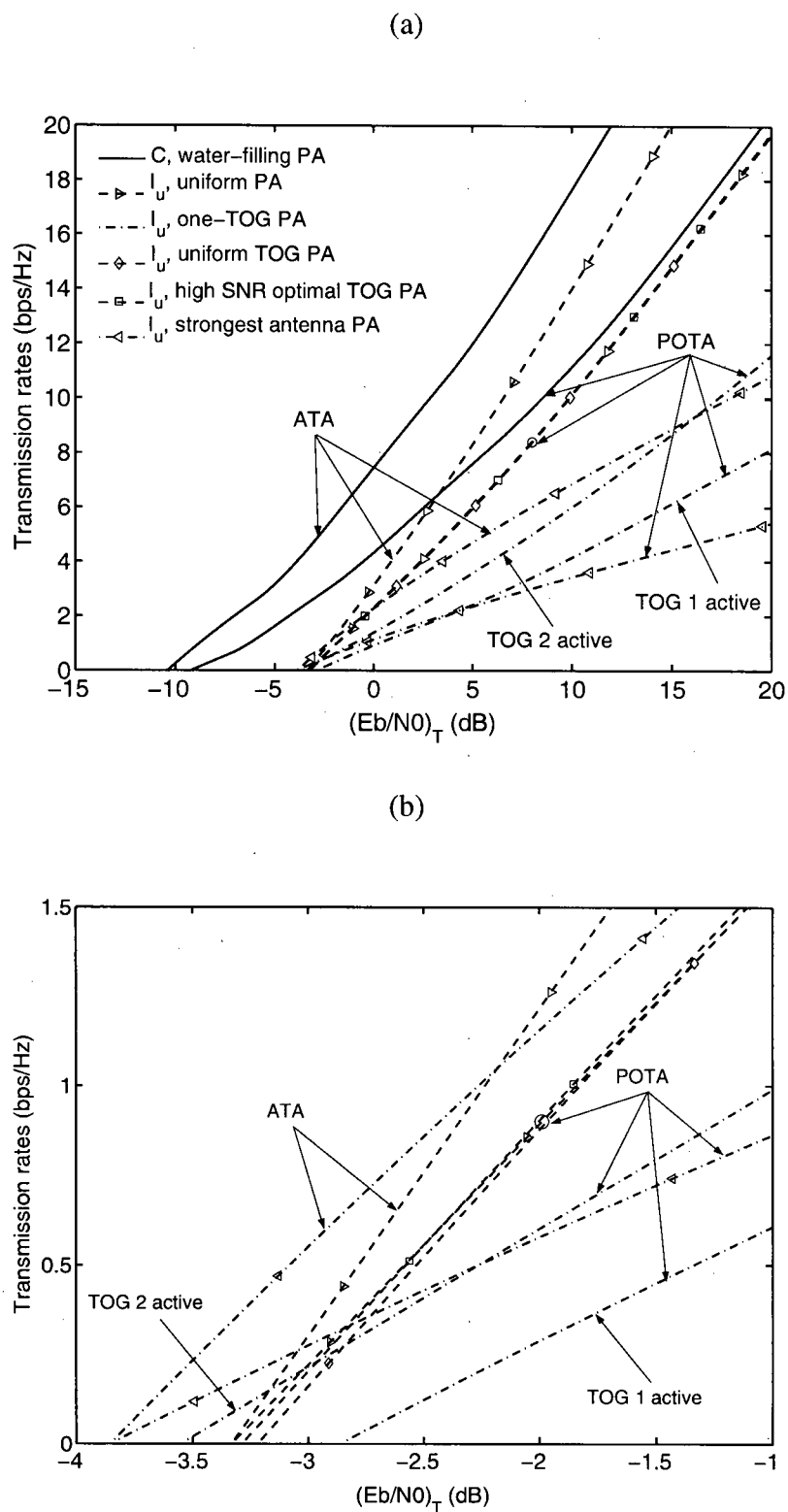


Fig. 3.2 a) Comparison of ATA and POTA for Channel A with one receive antenna $N_R = 1$, b) Zoom in the SNR interval $[-4, -1]$ dB with the same legend as in a).

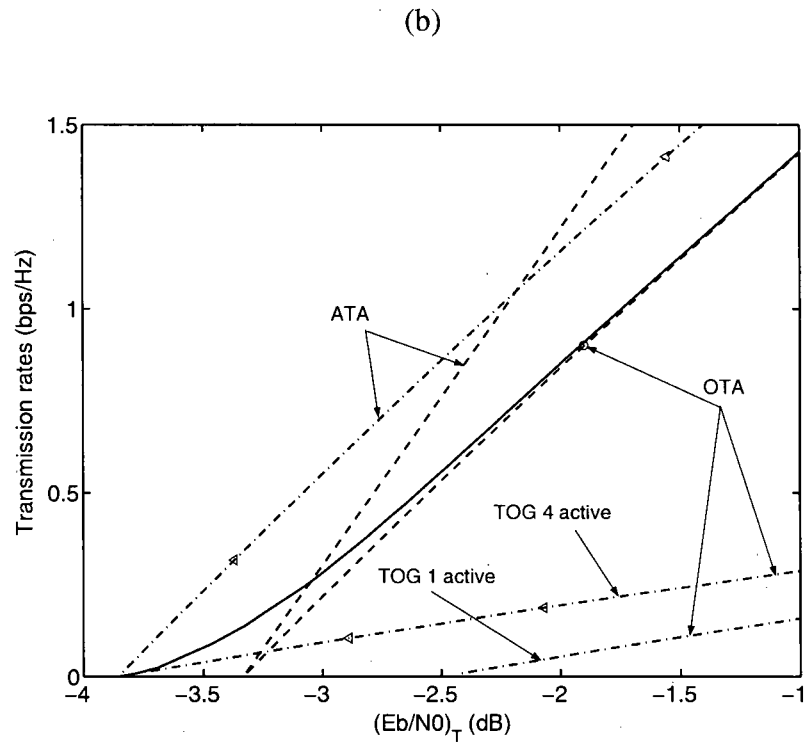
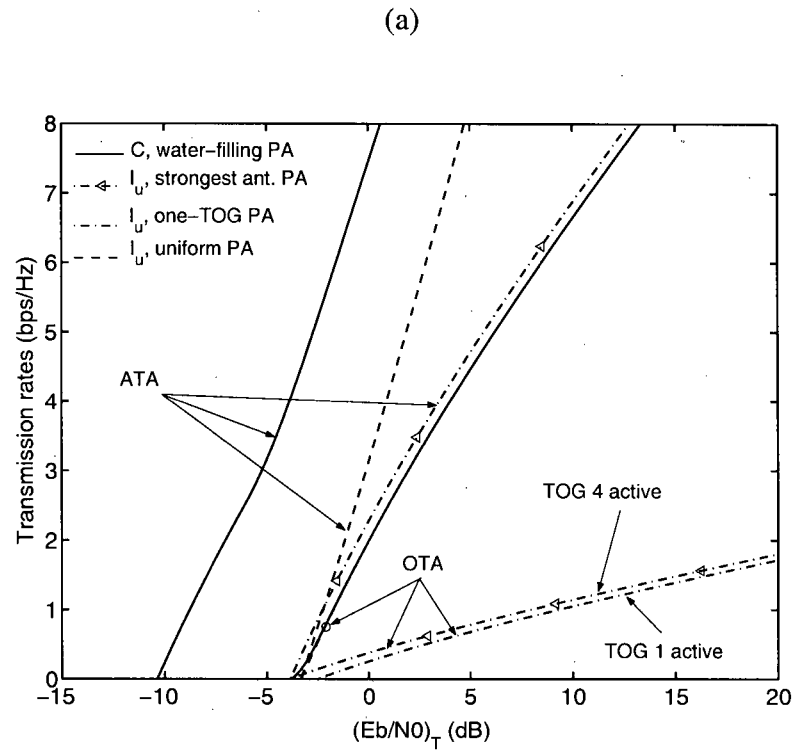


Fig. 3.3 a) Comparison of ATA and OTA for Channel A with one receive antenna $N_R = 1$, b) Zoom in the SNR interval $[-4, -1]$ dB with the same legend as in a).

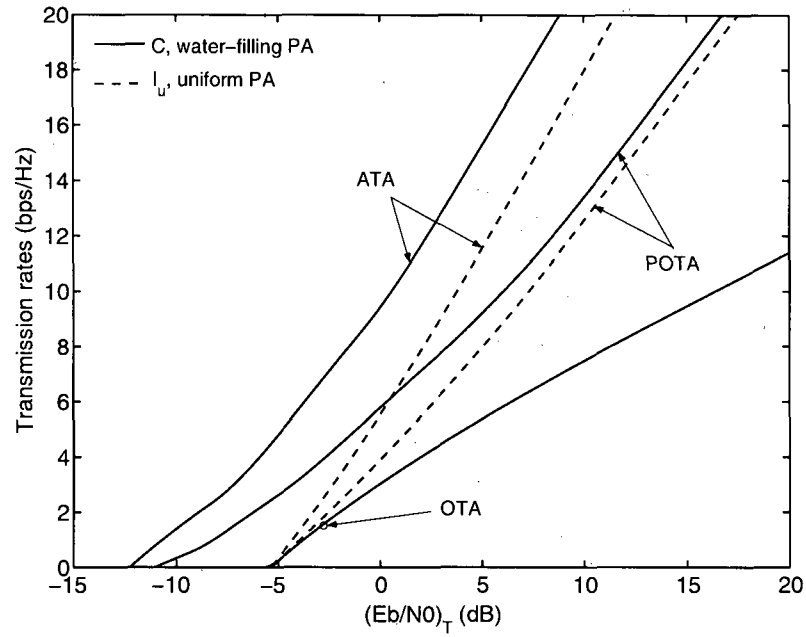


Fig. 3.4 Comparison of ATA, POTA and OTA for Channel A with two receive antennas $N_R = 2$.

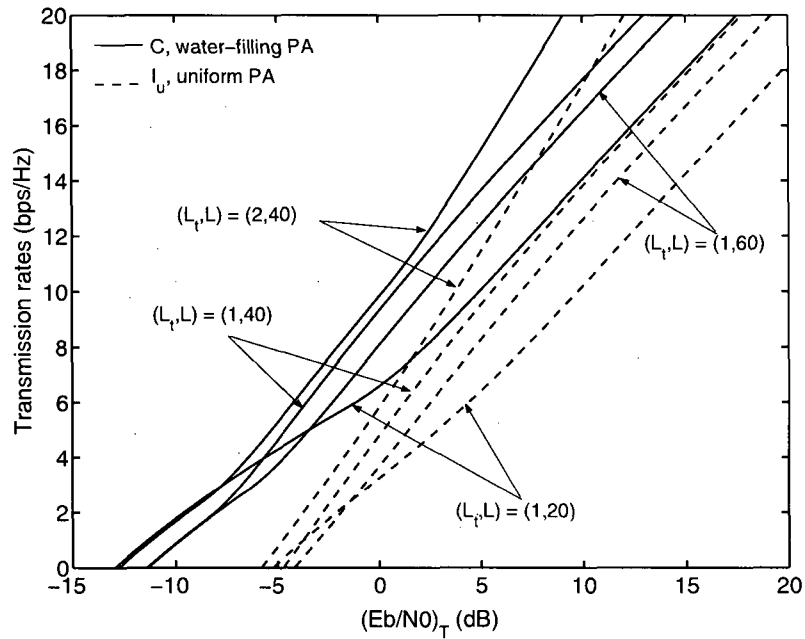


Fig. 3.5 Transmission rates for Channel A with different propagation scenarios using ATA and $N_R = 2$.

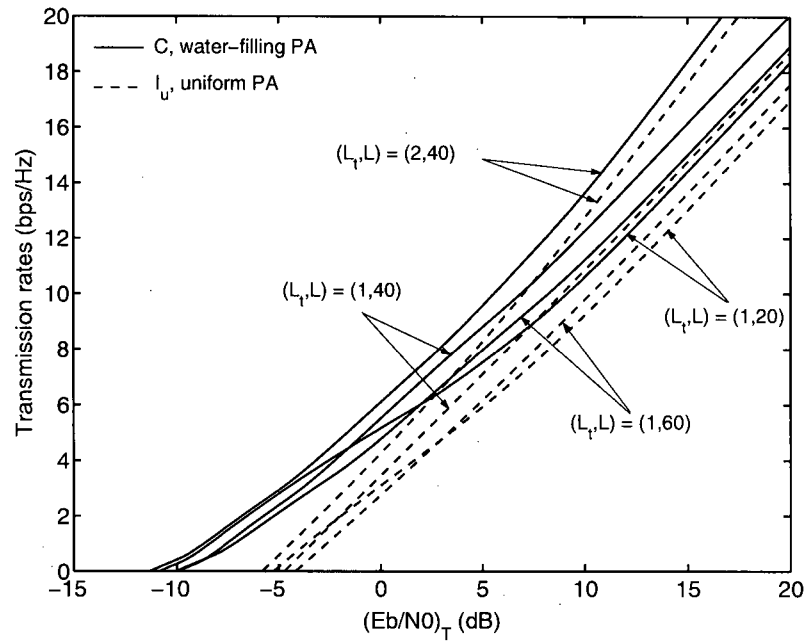


Fig. 3.6 Transmission rates for Channel A with different propagation scenarios using POTA and $N_R = 2$.

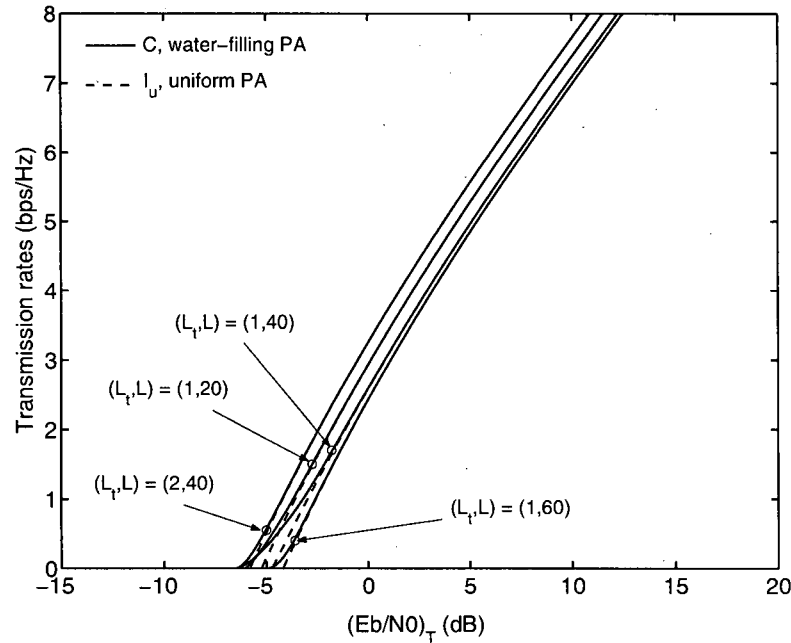


Fig. 3.7 Transmission rates for Channel A with different propagation scenarios using OTA and $N_R = 2$.

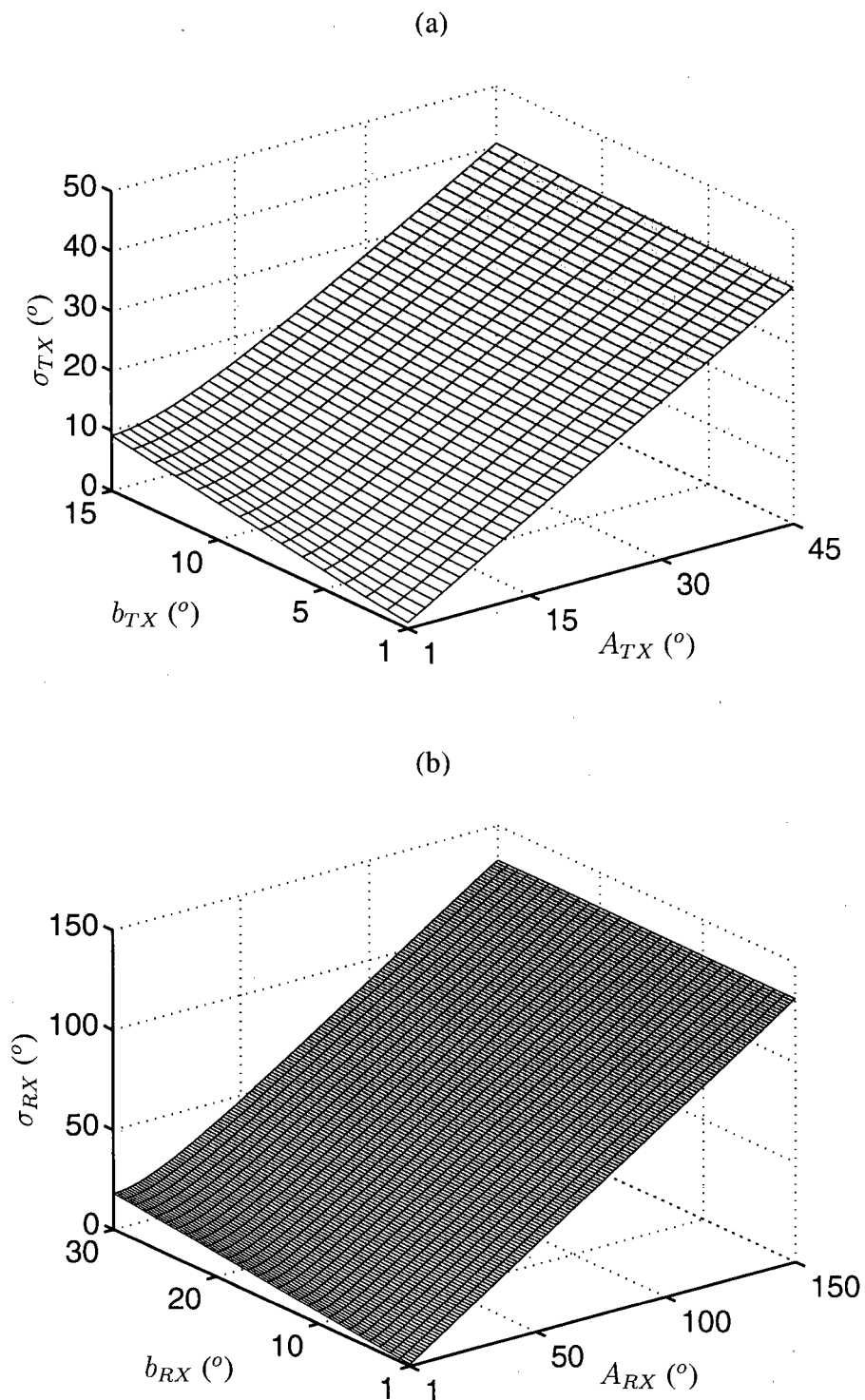


Fig. 3.8 a) Angle spread at the transmitter σ_{TX} as a function of A_{TX} and b_{TX} ,
 b) Angle spread at the receiver σ_{RX} as a function of A_{RX} and b_{RX} .

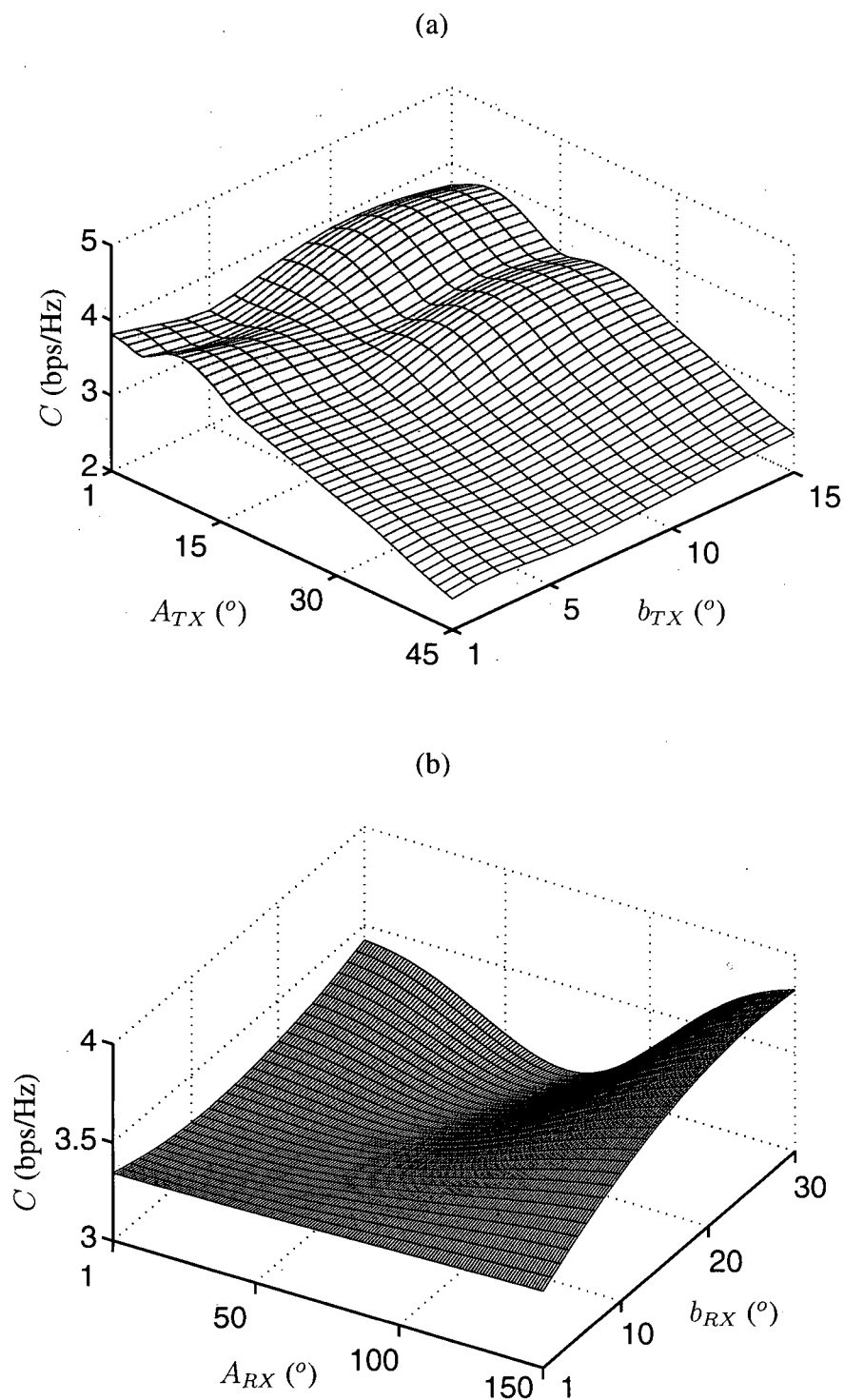


Fig. 3.9 Capacity using ATA with $N_T = 6$, $N_R = 2$ and $\frac{P_T}{N_0 D} = 15$. a) Capacity as a function of A_{TX} and b_{TX} with $(A_{RX}, b_{RX}) = (50^\circ, 10^\circ)$, b) Capacity as a function of A_{RX} and b_{RX} with $(A_{TX}, b_{TX}) = (20^\circ, 5^\circ)$.

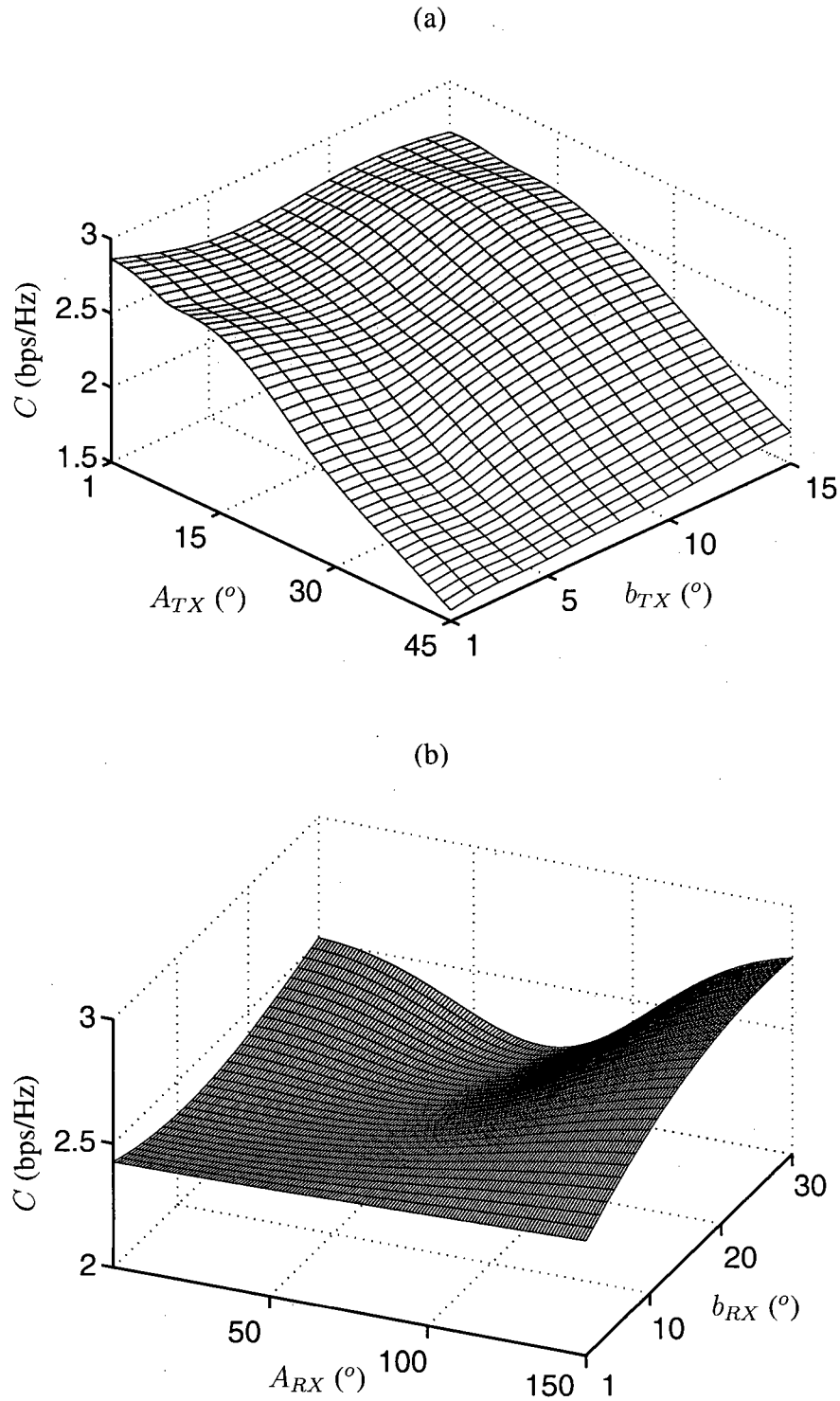


Fig. 3.10 Capacity using POTA with $N_T = 6$, $N_R = 2$ and $\frac{P_T}{N_0 D} = 15$. a) Capacity as a function of A_{TX} and b_{TX} with $(A_{RX}, b_{RX}) = (50^\circ, 10^\circ)$, b) Capacity as a function of A_{RX} and b_{RX} with $(A_{TX}, b_{TX}) = (20^\circ, 5^\circ)$.

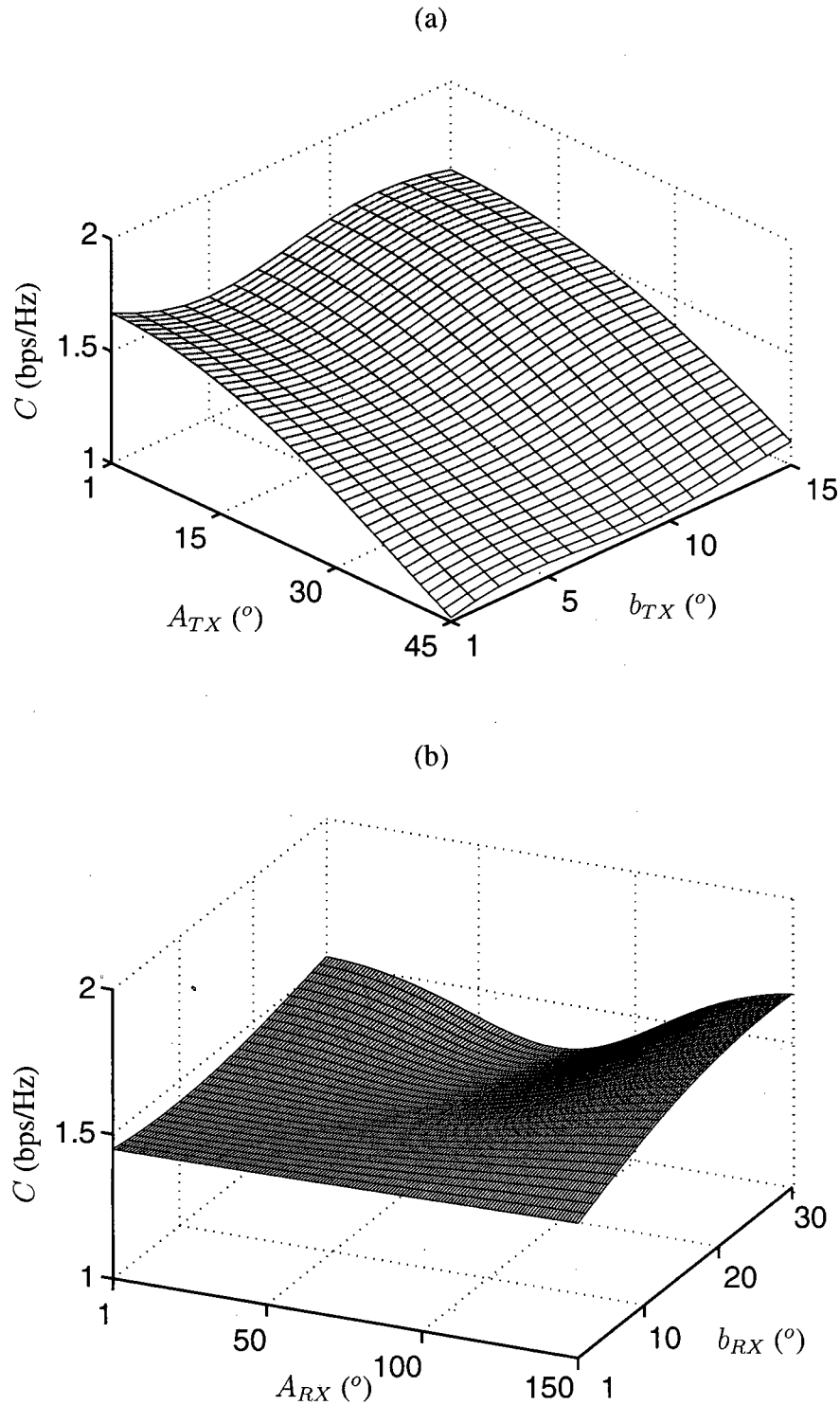


Fig. 3.11 Capacity using OTA with $N_T = 6$, $N_R = 2$ and $\frac{P_T}{N_0 D} = 15$. a) Capacity as a function of A_{TX} and b_{TX} with $(A_{RX}, b_{RX}) = (50^\circ, 10^\circ)$, b) Capacity as a function of A_{RX} and b_{RX} with $(A_{TX}, b_{TX}) = (20^\circ, 5^\circ)$.

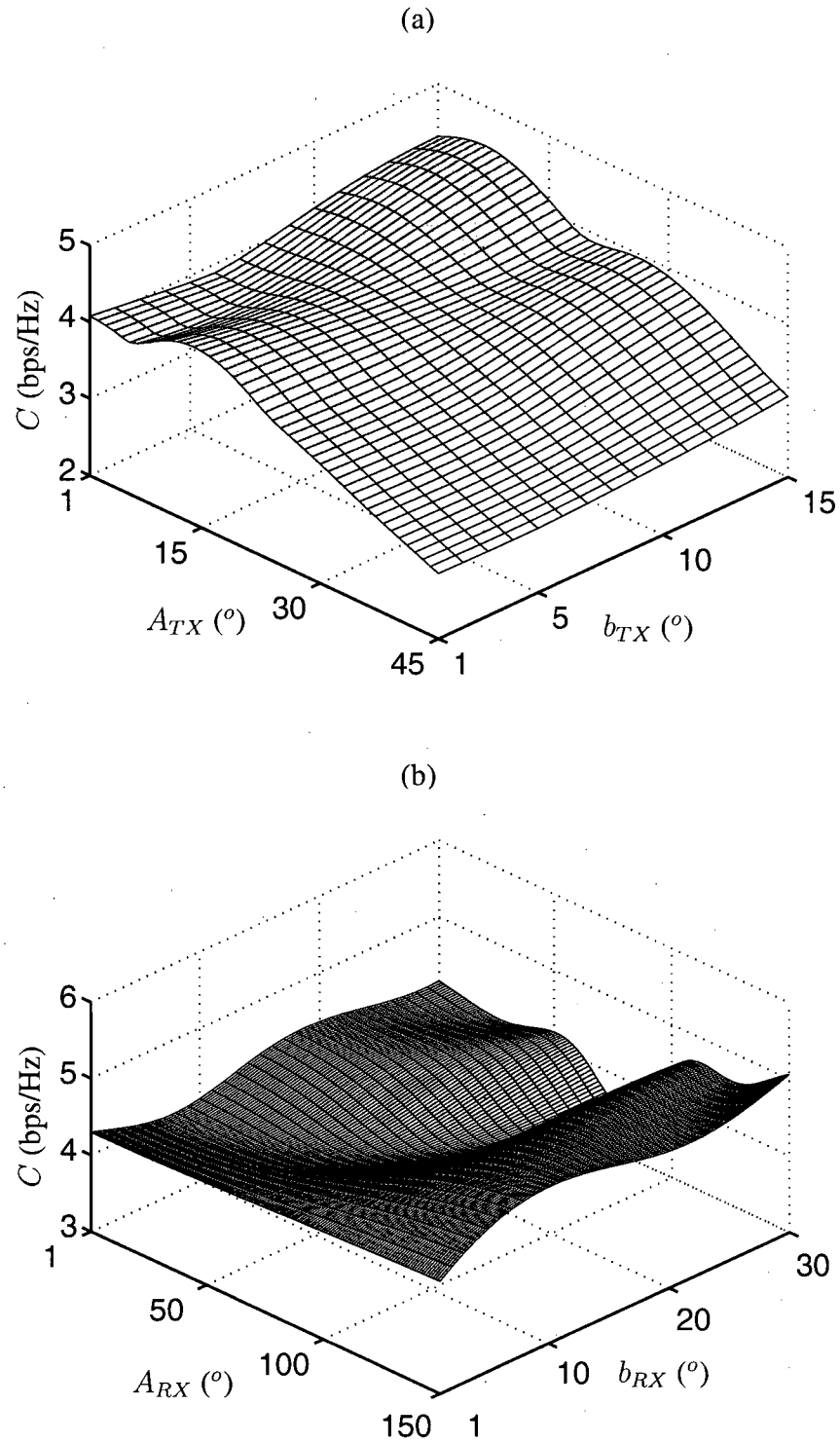


Fig. 3.12 Capacity using ATA with $N_T = N_R = 4$ and $\frac{P_T}{N_0 D} = 15$. a) Capacity as a function of A_{TX} and b_{TX} with $(A_{RX}, b_{RX}) = (50^\circ, 10^\circ)$, b) Capacity as a function of A_{RX} and b_{RX} with $(A_{TX}, b_{TX}) = (20^\circ, 5^\circ)$.

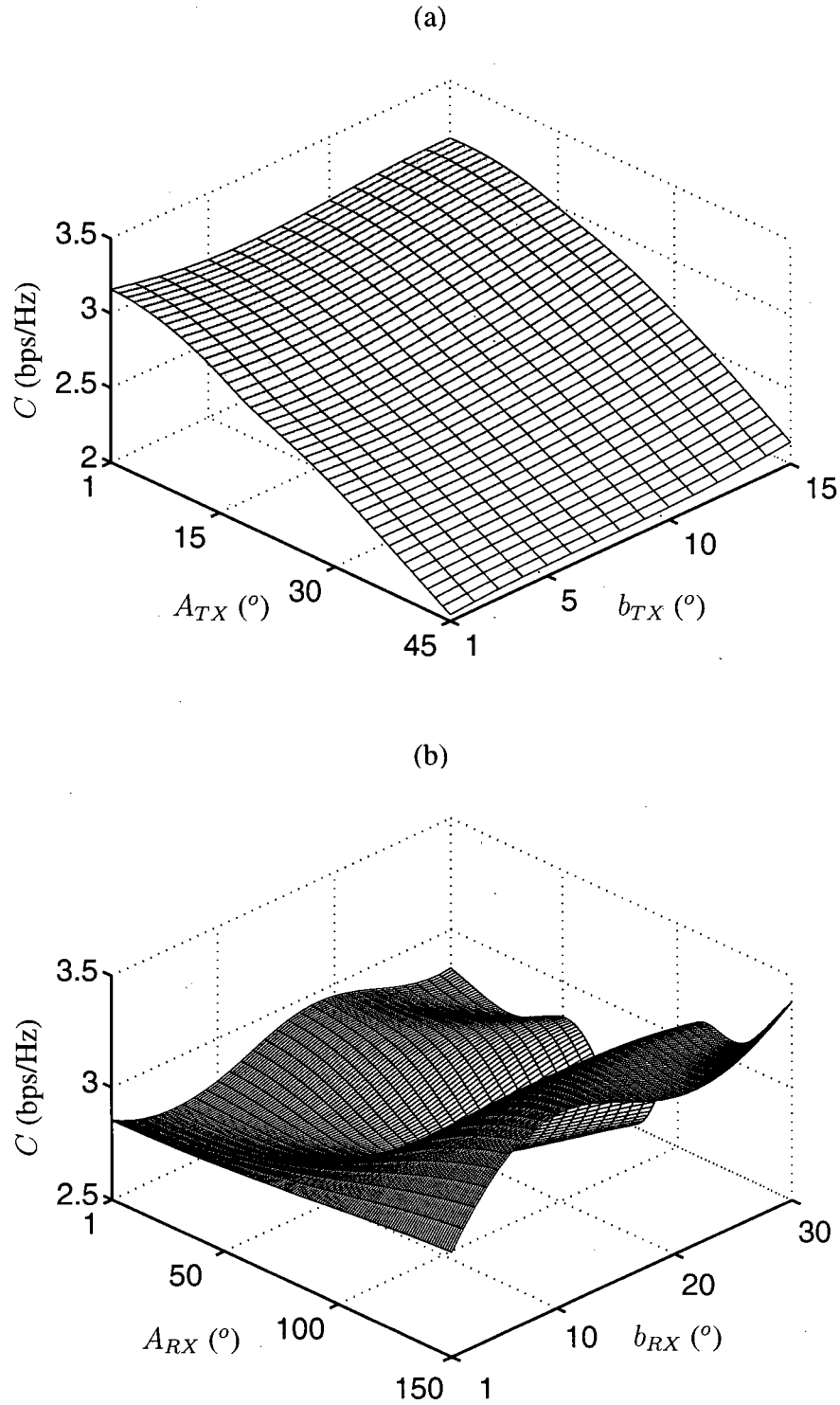


Fig. 3.13 Capacity using POTA with $N_T = N_R = 4$ and $\frac{P_T}{N_0 D} = 15$. a) Capacity as a function of A_{TX} and b_{TX} with $(A_{RX}, b_{RX}) = (50^\circ, 10^\circ)$, b) Capacity as a function of A_{RX} and b_{RX} with $(A_{TX}, b_{TX}) = (20^\circ, 5^\circ)$.

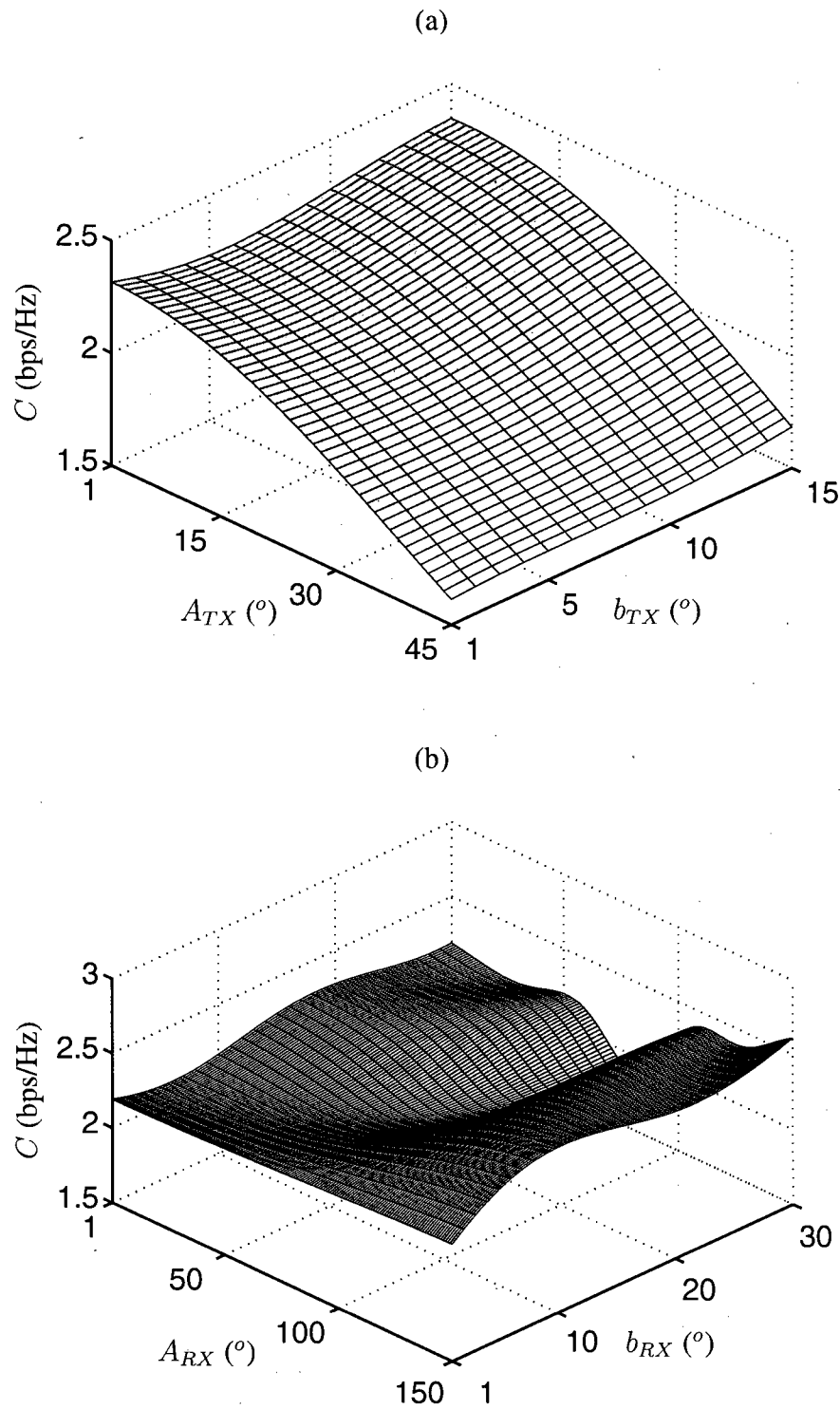


Fig. 3.14 Capacity using OTA with $N_T = N_R = 4$ and $\frac{P_T}{N_0 D} = 15$. a) Capacity as a function of A_{TX} and b_{TX} with $(A_{RX}, b_{RX}) = (50^\circ, 10^\circ)$, b) Capacity as a function of A_{RX} and b_{RX} with $(A_{TX}, b_{TX}) = (20^\circ, 5^\circ)$.

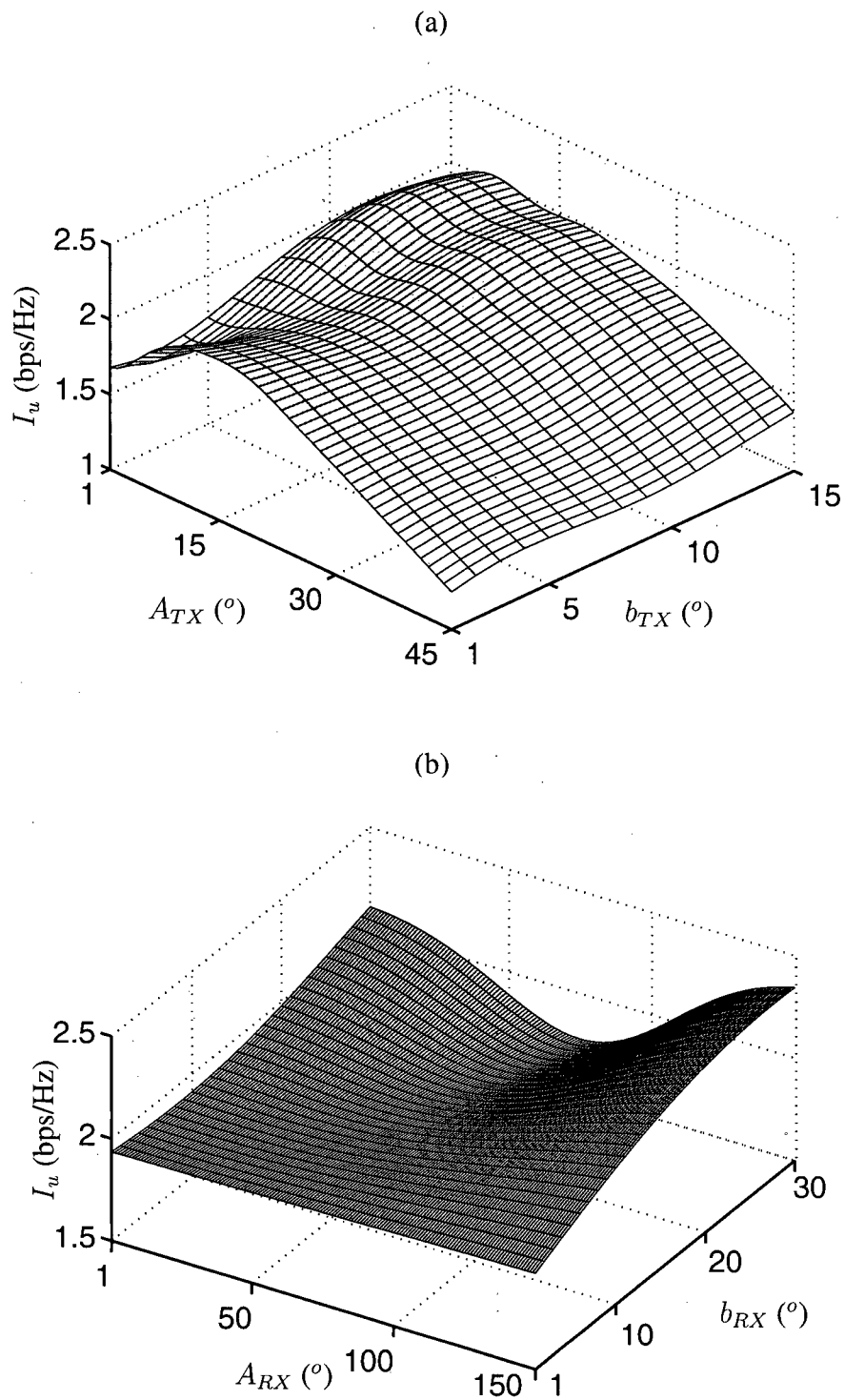


Fig. 3.15 Transmission rate I_u using ATA with uniform PA, $N_T = 6$, $N_R = 2$ and $\frac{P_T}{N_0 D} = 15$. a) I_u as a function of A_{TX} and b_{TX} with $(A_{RX}, b_{RX}) = (50^\circ, 10^\circ)$, b) I_u as a function of A_{RX} and b_{RX} with $(A_{TX}, b_{TX}) = (20^\circ, 5^\circ)$.

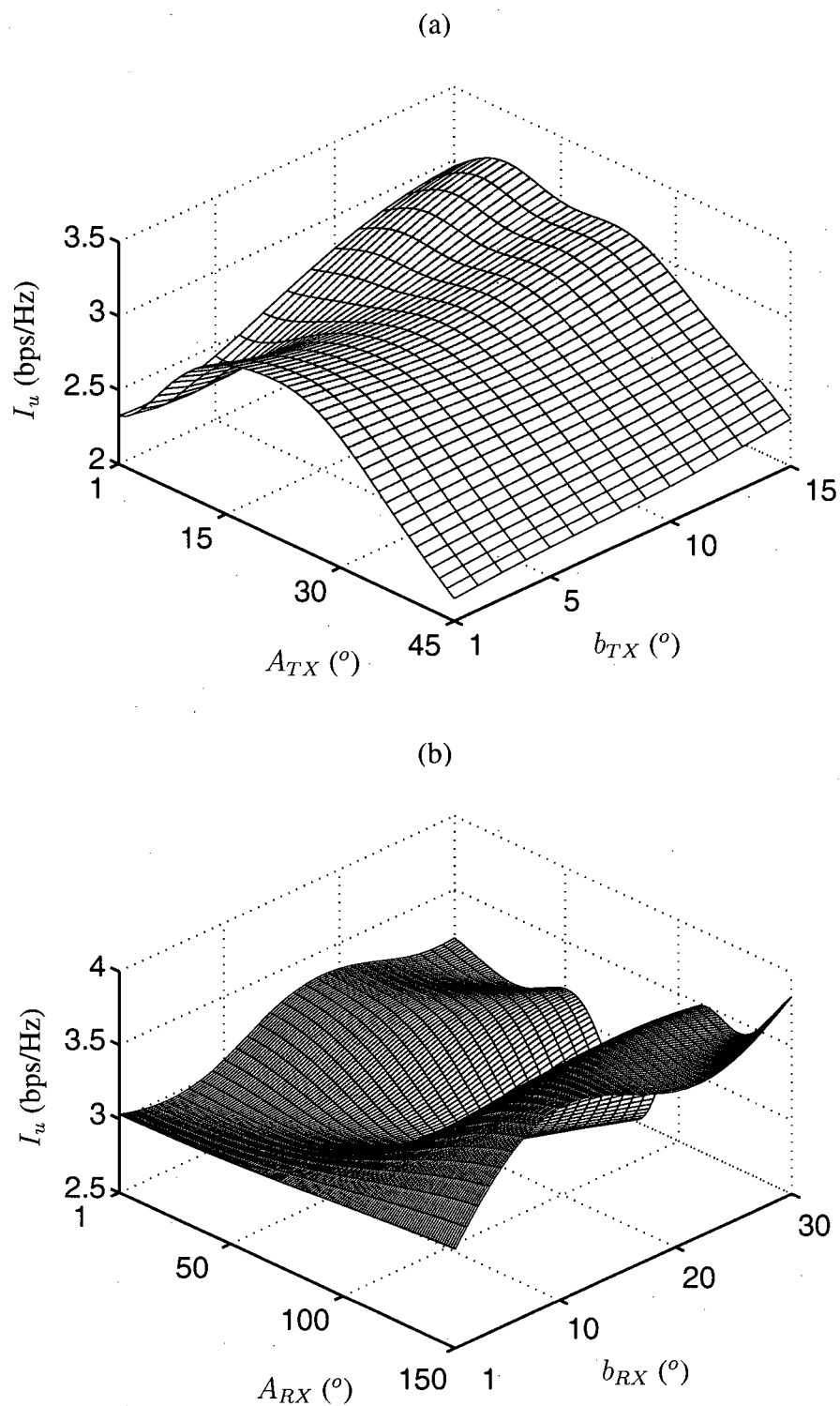


Fig. 3.16 Transmission rate I_u using ATA with uniform PA, $N_T = N_R = 4$ and $\frac{P_T}{N_0 D} = 15$. a) I_u as a function of A_{TX} and b_{TX} with $(A_{RX}, b_{RX}) = (50^\circ, 10^\circ)$, b) I_u as a function of A_{RX} and b_{RX} with $(A_{TX}, b_{TX}) = (20^\circ, 5^\circ)$.

Chapter 4

Rate regions for broadcast MIMO channels

We consider the MIMO BCs introduced in Section 2.3, assuming two users in the system ($K = 2$). The results can be easily extended to systems with a larger number of users. Throughout this chapter we assume perfect CSIT and CSIR. That is, the transmitter and both receivers have perfect knowledge of: the channel propagation matrices (i.e. $\{\mathbf{C}^{(k)}\}_{k=1}^2$) for spread spectrum like systems and the matrices $\{[\mathbf{O}_{m,q}^{(k)}]\}_{k=1}^2$ for MIMO-OFDM systems, the matrices $\mathbf{B}^{(1,2)}$ and $\mathbf{B}^{(2,1)}$ (with $\mathbf{B}^{(2,1)} = [\mathbf{B}^{(1,2)}]^T$) that indicate the shared signal dimensions between the two users, and the multi-dimensional space-time modulation formats (i.e. $\{\mathbf{A}^{(k)}\}_{k=1}^2$) that are employed at the transmitter.

4.1 The dirty paper rate region

Let $\mathbf{x}_0$ denote the channel input (or transmitted) vector, defined as $\mathbf{x}_0 \triangleq \text{vec}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)})$ where $\mathbf{x}^{(k)}$ is the $\tilde{D}^{(k)} \times 1$ input vector intended for user k , $k = 1, 2$. Using (2.40), the two-user MIMO BC with multi-dimensional space-time modulation is described by

$$\mathbf{r}^{(1)} = \mathbf{H}^{(1)} \cdot \mathbf{x}^{(1)} + \mathbf{H}^{(1,2)} \cdot \mathbf{x}^{(2)} + \mathbf{v}^{(1)} = [\mathbf{H}^{(1)} \quad \mathbf{H}^{(1,2)}] \cdot \mathbf{x}_0 + \mathbf{v}^{(1)} \quad (4.1a)$$

$$\mathbf{r}^{(2)} = \mathbf{H}^{(2,1)} \cdot \mathbf{x}^{(1)} + \mathbf{H}^{(2)} \cdot \mathbf{x}^{(2)} + \mathbf{v}^{(2)} = [\mathbf{H}^{(2,1)} \quad \mathbf{H}^{(2)}] \cdot \mathbf{x}_0 + \mathbf{v}^{(2)} \quad (4.1b)$$

where $\mathbf{H}^{(1)}, \mathbf{H}^{(2)}$ are given by (2.41), and $\mathbf{H}^{(1,2)}, \mathbf{H}^{(2,1)}$ are given by (2.42). We represent

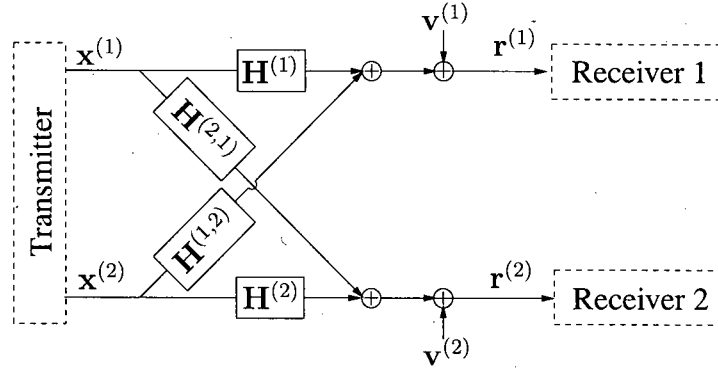


Fig. 4.1 Two-user BC with space-time modulation and signal dimensions sharing.

this channel in Figure 4.1. The model (4.1) represents a memoryless BC where a single transmitter sends independent information to two non-cooperative receivers. We briefly review the information-theoretic definitions associated with this channel from [51, Section 14.6]. The BC described by (4.1) consists of an input alphabet $\mathcal{X}_0$ (with $(\tilde{D}^{(1)} + \tilde{D}^{(2)}) \times 1$ vector valued elements), two output alphabets $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$ (with $L_t^{(1)} N_R^{(1)} D^{(1)} \times 1$ and $L_t^{(2)} N_R^{(2)} D^{(2)} \times 1$ vector valued elements, respectively), and a transition probability function $p_{\mathbf{r}^{(1)}, \mathbf{r}^{(2)} | \mathbf{x}_0}(\mathbf{a}_r^{(1)}, \mathbf{a}_r^{(2)} | \mathbf{a}_0)$, $\mathbf{a}_r^{(1)} \in \mathcal{R}^{(1)}$, $\mathbf{a}_r^{(2)} \in \mathcal{R}^{(2)}$ and $\mathbf{a}_0 \in \mathcal{X}_0$. Assuming block codes of length N , two independent messages $Q^{(1)} \in \{1, \dots, 2^{NR^{(1)}}\}$ (associated with user 1) and $Q^{(2)} \in \{1, \dots, 2^{NR^{(2)}}\}$ (associated with user 2) are jointly encoded at the transmitter. The receivers decode their signals independently. A $((2^{NR^{(1)}}, 2^{NR^{(2)}}), N)$ code for this BC consists of an encoding function $f_{e,BC} : \{1, \dots, 2^{NR^{(1)}}\} \times \{1, \dots, 2^{NR^{(2)}}\} \rightarrow \mathcal{X}_0^N$, and two decoding functions $f_{d,BC}^{(1)} : [\mathcal{R}^{(1)}]^N \rightarrow \{1, \dots, 2^{NR^{(1)}}\}$ and $f_{d,BC}^{(2)} : [\mathcal{R}^{(2)}]^N \rightarrow \{1, \dots, 2^{NR^{(2)}}\}$. Using $\hat{Q}^{(1)}$ and $\hat{Q}^{(2)}$ to denote the decoded messages for user 1 and user 2, respectively, the average probability of error is $P_{e,BC} = \Pr[\hat{Q}^{(1)} \neq Q^{(1)} \text{ or } \hat{Q}^{(2)} \neq Q^{(2)}]$. A rate pair $(R^{(1)}, R^{(2)})$ is said to be achievable if there exists a sequence of $((2^{NR^{(1)}}, 2^{NR^{(2)}}), N)$ codes for which $P_{e,BC} \rightarrow 0$ when $N \rightarrow \infty$. The BC capacity region is the closure of the set of achievable rate pairs. In this chapter we will also use time-sharing between two codes. This is done as follows [51, Chapter 14]. Consider two sequences of codes $((2^{NR_1^{(1)}}, 2^{NR_1^{(2)}}), N)$ and $((2^{NR_2^{(1)}}, 2^{NR_2^{(2)}}), N)$ with rate pairs $(R_1^{(1)}, R_1^{(2)})$ and $(R_2^{(1)}, R_2^{(2)})$, respectively. Time-sharing between these two codes is obtained by using the first codebook for a fraction of time τ and the second codebook for the remaining fraction of time $(1 - \tau)$ as follows. Let p and q denote two positive integers such that $p \leq q$ and τ is equal to, or approximated by, the ratio p/q . For user k , time sharing consists of transmitting p codewords

from the first codebook followed by $(q - p)$ codewords from the second codebook, which results in a new codebook with

$$M_{new}^{(k)} \triangleq (2^{NR_1^{(k)}})^p (2^{NR_2^{(k)}})^{(q-p)} = 2^{N[pR_1^{(k)} + (q-p)R_2^{(k)}]}$$

messages. The rate of the new code for user k is given by $R_{new}^{(k)} = \frac{\log M_{new}^{(k)}}{qN} = \frac{p}{q}R_1^{(k)} + \frac{(q-p)}{q}R_2^{(k)} \equiv \tau R_1^{(k)} + (1 - \tau)R_2^{(k)}$, $k = 1, 2$. Since the overall probability of error cannot be larger than the maximum of the probabilities of error for each code, the probability of error of the new code goes to zero as $N \rightarrow \infty$. Hence, the new code is achievable.

A MIMO BC is referred to as Gaussian MIMO BC if the transition probability function $p_{\mathbf{r}^{(1)}, \mathbf{r}^{(2)} | \mathbf{x}_0}$ is Gaussian. Hence, since the noise vectors $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ in (4.1) are independent Gaussian and since they are independent of $\mathbf{x}_0$, the model (4.1) represents a Gaussian MIMO BC, with $\mathcal{X}_0 = \mathbb{C}^{(\tilde{D}^{(1)} + \tilde{D}^{(2)})}$, $\mathcal{R}^{(1)} = \mathbb{C}^{L_t^{(1)} N_R^{(1)} D^{(1)}}$ and $\mathcal{R}^{(2)} = \mathbb{C}^{L_t^{(2)} N_R^{(2)} D^{(2)}}$. In this section, we use dirty paper coding (DPC) to establish an achievable rate region for this channel. The concept of DPC was introduced by Costa in [52] for a scalar memoryless Gaussian channel with noncausally known interference at the transmitter. Costa's channel is modeled as

$$Y = X + S + Z$$

where X and Y are the transmitted and received signals, respectively, S is an i.i.d. Gaussian interfering signal whose realizations are noncausally known at the transmitter but not at the receiver, and Z is a Gaussian noise independent of S . Coding for Costa's channel employs the random binning technique presented in [53] for memoryless channels with i.i.d. channel state process noncausally known at the transmitter, but not at the receiver. Using the capacity results from [53], Costa showed that under a fixed power constraint, the capacity of the channel with interference is the same as if the interference does not exist. Furthermore, the capacity achieving input signal X is Gaussian and statistically independent of S . Referred to as DPC, Costa's technique can be seen as a tool to "presubtract" the interference at the transmitter without increasing the transmit power. An extension of Costa's DPC result to the case of vector memoryless Gaussian channels with noncausally known interference has been proved in [119, 120]. Based on this extension, it was shown in [10, 63, 65, 66] that using independent Gaussian inputs and applying DPC at the transmitter yield an achievable rate region for the Gaussian MIMO BC under a total transmit power constraint. For the two-user BC, the encoding procedure associated with this DPC rate region is as follows. Assuming that user 1 is encoded first and user 2 encoded second, the transmitter picks a

codeword for user 1. Then, the transmitter picks a codeword for user 2 while treating the codeword of user 1 as noncausally known interference. Hence, the codeword of user 1 can be "presubtracted" such that it won't be seen by user 2 as interference. User 1, however, sees the signals intended for user 2 as interference. Note that for BCs with more than two users ($K > 2$) the DPC procedure generalizes as follows. For a given users ordering, the transmitter picks a codeword for the k^{th} user while treating the codewords of the previously encoded users (if any) as noncausally known interference that can be "presubtracted" such that it won't be seen by the k^{th} user as interference. However, the signals intended for the subsequently encoded users are seen by the k^{th} user as interference. The transmitted signal is the sum of the users signals. By changing the users encoding order and following a similar encoding procedure, another set of achievable rate vectors can be obtained. The DPC rate region is defined in [10, 65] as the convex hull of the union of all such achievable rate vectors over all positive semidefinite input covariance matrices that satisfy the total power constraint and over all user orderings. Recently, the DPC rate region was shown to coincide with the capacity region of the Gaussian MIMO BC [81].

The model (4.1) for MIMO BCs with multi-dimensional space-time modulation and signal dimensions sharing differs from the MIMO BC model considered in the references cited above in that the user signals $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are multiplied by different matrices in equations (4.1a) and (4.1b). Using (2.41) and (2.42), one can see that, in general, $\mathbf{H}^{(1,2)}$ is different from $\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2,1)}$ is different from $\mathbf{H}^{(2)}$. The equalities $\mathbf{H}^{(1,2)} = \mathbf{H}^{(1)}$ and $\mathbf{H}^{(2,1)} = \mathbf{H}^{(2)}$ are obtained only when all signal dimensions are used by all users ($D^{(1)} = D^{(2)} = D_T = D_c$, $\mathbf{B}^{(1,2)} = \mathbf{B}^{(2,1)} = \mathbf{I}_{D_T}$) and when the users have similar space-time modulation formats ($\mathbf{A}^{(1)} = \mathbf{A}^{(2)}$). Equations (4.1) reduce to $\mathbf{r}^{(k)} = \mathbf{H}^{(k)} \cdot (\mathbf{x}^{(1)} + \mathbf{x}^{(2)}) + \mathbf{v}^{(k)}$, $k = 1, 2$, if $\mathbf{H}^{(1,2)} = \mathbf{H}^{(1)}$ and $\mathbf{H}^{(2,1)} = \mathbf{H}^{(2)}$. Apart from this particular case, the transmitted vector cannot be taken as $\mathbf{x}^{(1)} + \mathbf{x}^{(2)}$ as in [10, 63, 65, 66] but rather it is given by $\mathbf{x}_0$ which is the stack of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$. We keep the assumption of independent zero mean circularly symmetric Gaussian input vectors $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ since it will be shown to yield an achievable rate region for the MIMO BC of (4.1). More precisely, based on this assumption, we will apply in Theorem 4.1.1 below the extension of Costa's DPC result to the case of vector Gaussian channels from [119, 120] which can be used to show that if $\mathbf{x}^{(1)}$ (respectively, $\mathbf{x}^{(2)}$) is noncausally known at the transmitter, user 2 (respectively, user 1) can achieve a rate equal to its input/output mutual information as if the interfering term $\mathbf{H}^{(2,1)} \cdot \mathbf{x}^{(1)}$ (respectively, $\mathbf{H}^{(1,2)} \cdot \mathbf{x}^{(2)}$) is not present in (4.1b) (respectively, (4.1a)). Using this result, we will follow the encoding procedure explained above in order to obtain achievable rates for the BC of

(4.1). Let $\Sigma^{(1)}$ and $\Sigma^{(2)}$ denote the covariance matrices of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$, respectively. The broadcast system has a total transmit power of $P^{(1)} + P^{(2)} = P_T$ where, for $k = 1, 2$, $P^{(k)}$ denotes the power allocated to user k , and we have $\text{Tr}(\Sigma^{(k)}) \leq P^{(k)}$. The covariance matrix of the transmitted vector $\mathbf{x}_0$ is $\Sigma_0 = E[\mathbf{x}_0 \cdot \mathbf{x}_0^\dagger] = \text{diag}[\Sigma^{(1)}, \Sigma^{(2)}]$ due to the independence of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$, with the transmit power constraint $\text{Tr}(\Sigma_0) = \text{Tr}(\Sigma^{(1)}) + \text{Tr}(\Sigma^{(2)}) \leq P_T$. Applying DPC at the transmitter, the following rate pair (in bits/channel-use) is achievable when encoding user 1 first and user 2 second

$$R_{DPC}^{(1)}(\Sigma^{(1)}, \Sigma^{(2)}) = \log \frac{\left| \mathbf{I}_{L_t^{(1)} N_R^{(1)} D^{(1)}} + \frac{1}{N_0} \mathbf{H}^{(1,2)} \cdot \Sigma^{(2)} \cdot [\mathbf{H}^{(1,2)}]^\dagger + \frac{1}{N_0} \mathbf{H}^{(1)} \cdot \Sigma^{(1)} \cdot [\mathbf{H}^{(1)}]^\dagger \right|}{\left| \mathbf{I}_{L_t^{(1)} N_R^{(1)} D^{(1)}} + \frac{1}{N_0} \mathbf{H}^{(1,2)} \cdot \Sigma^{(2)} \cdot [\mathbf{H}^{(1,2)}]^\dagger \right|} \quad (4.2a)$$

$$R_{DPC}^{(2)}(\Sigma^{(2)}) = \log \left| \mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}} + \frac{1}{N_0} \mathbf{H}^{(2)} \cdot \Sigma^{(2)} \cdot [\mathbf{H}^{(2)}]^\dagger \right| \quad (4.2b)$$

Also, the following rate pair (in bits/channel-use) is achievable when encoding user 2 first and user 1 second

$$\tilde{R}_{DPC}^{(1)}(\Sigma^{(1)}) = \log \left| \mathbf{I}_{L_t^{(1)} N_R^{(1)} D^{(1)}} + \frac{1}{N_0} \mathbf{H}^{(1)} \cdot \Sigma^{(1)} \cdot [\mathbf{H}^{(1)}]^\dagger \right| \quad (4.3a)$$

$$\tilde{R}_{DPC}^{(2)}(\Sigma^{(1)}, \Sigma^{(2)}) = \log \frac{\left| \mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}} + \frac{1}{N_0} \mathbf{H}^{(2,1)} \cdot \Sigma^{(1)} \cdot [\mathbf{H}^{(2,1)}]^\dagger + \frac{1}{N_0} \mathbf{H}^{(2)} \cdot \Sigma^{(2)} \cdot [\mathbf{H}^{(2)}]^\dagger \right|}{\left| \mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}} + \frac{1}{N_0} \mathbf{H}^{(2,1)} \cdot \Sigma^{(1)} \cdot [\mathbf{H}^{(2,1)}]^\dagger \right|} \quad (4.3b)$$

The next theorem provides achievable rate regions for the Gaussian MIMO BC of (4.1) using rate pairs of the form of (4.2) and (4.3).

Theorem 4.1.1 (Rate Regions Achievability) *Assuming independent Gaussian input vectors $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$, the following rate regions $\mathcal{R}_{1,2}(\Sigma^{(1)}, \Sigma^{(2)})$ and $\mathcal{R}_{2,1}(\Sigma^{(1)}, \Sigma^{(2)})$ are achievable*

$$\mathcal{R}_{1,2}(\Sigma^{(1)}, \Sigma^{(2)}) = \left\{ (R^{(1)}, R^{(2)}) : 0 \leq R^{(1)} \leq R_{DPC}^{(1)}(\Sigma^{(1)}, \Sigma^{(2)}), 0 \leq R^{(2)} \leq R_{DPC}^{(2)}(\Sigma^{(2)}) \right\} \quad (4.4)$$

$$\mathcal{R}_{2,1}(\Sigma^{(1)}, \Sigma^{(2)}) = \left\{ (R^{(1)}, R^{(2)}) : 0 \leq R^{(1)} \leq \tilde{R}_{DPC}^{(1)}(\Sigma^{(1)}), 0 \leq R^{(2)} \leq \tilde{R}_{DPC}^{(2)}(\Sigma^{(1)}, \Sigma^{(2)}) \right\} \quad (4.5)$$

where $(\Sigma^{(1)}, \Sigma^{(2)}) \in \mathcal{A}_\Sigma$ and

$$\mathcal{A}_\Sigma = \{(\Sigma^{(1)}, \Sigma^{(2)}) : \Sigma^{(1)} \succeq \mathbf{0}_{\tilde{D}^{(1)} \times \tilde{D}^{(1)}}, \Sigma^{(2)} \succeq \mathbf{0}_{\tilde{D}^{(2)} \times \tilde{D}^{(2)}} \text{ with } \text{Tr}(\Sigma^{(1)}) + \text{Tr}(\Sigma^{(2)}) \leq P_T\}.$$

The proof of Theorem 4.1.1 can be found in Appendix B. As in [10, 65], the DPC rate region is the convex hull (CH) of the union of the regions $\mathcal{R}_{1,2}(\Sigma^{(1)}, \Sigma^{(2)})$ and $\mathcal{R}_{2,1}(\Sigma^{(1)}, \Sigma^{(2)})$ over all input covariance matrices $\Sigma^{(1)}$ and $\Sigma^{(2)}$ that satisfy the total power constraint,

$$\mathcal{R}_{DPC} = \text{CH} \left(\bigcup_{(\Sigma^{(1)}, \Sigma^{(2)}) \in \mathcal{A}_\Sigma} \left[\mathcal{R}_{1,2}(\Sigma^{(1)}, \Sigma^{(2)}) \cup \mathcal{R}_{2,1}(\Sigma^{(1)}, \Sigma^{(2)}) \right] \right) \quad (4.6)$$

Points on the boundary of $\mathcal{R}_{DPC}$ are either of the form $(R_{DPC}^{(1)}(\Sigma^{(1)}, \Sigma^{(2)}), R_{DPC}^{(2)}(\Sigma^{(1)}, \Sigma^{(2)}))$ or $(\tilde{R}_{DPC}^{(1)}(\Sigma^{(1)}, \Sigma^{(2)}), \tilde{R}_{DPC}^{(2)}(\Sigma^{(1)}, \Sigma^{(2)}))$, or obtained by time-sharing between points that correspond to such rate pairs.

If all signal dimensions are used by all users ($D_c = D_T$) and if the users employ similar space-time modulation formats (i.e. $\mathbf{H}^{(1,2)} = \mathbf{H}^{(1)}$ and $\mathbf{H}^{(2,1)} = \mathbf{H}^{(2)}$), then the results of [81] can be used to show that $\mathcal{R}_{DPC}$ is the capacity region. If $0 < D_c < D_T$, DPC yields achievable rate pairs where the interference associated with the D_c shared signal dimensions (and determined by the terms $\mathbf{H}^{(1,2)} \cdot \mathbf{x}^{(2)}$ in (4.1a) and $\mathbf{H}^{(2,1)} \cdot \mathbf{x}^{(1)}$ in (4.1b)) is canceled out for one user only. Furthermore, the rate expressions in (4.2) and (4.3) involve non-convex functions of the input covariance matrices $\Sigma^{(1)}$ and $\Sigma^{(2)}$. This makes finding the region $\mathcal{R}_{DPC}$ very difficult, because generally the entire space of input covariance matrices satisfying $\text{Tr}(\Sigma^{(1)}) + \text{Tr}(\Sigma^{(2)}) \leq P_T$ must be searched over. In the particular case where the users share no signal dimensions ($D_c = 0$), the MIMO BC of (4.1) reduces to the orthogonally multiplexed MIMO BC [121, 122] described by the following discrete time input/output relations

$$\mathbf{r}^{(k)} = \mathbf{H}^{(k)} \cdot \mathbf{x}^{(k)} + \mathbf{v}^{(k)} \text{ for } k = 1, 2 \quad (4.7)$$

which are obtained from (4.1) using $\mathbf{H}^{(1,2)} = \mathbf{0}_{L_t^{(1)} N_R^{(1)} D^{(1)} \times \tilde{D}^{(2)}}$ and $\mathbf{H}^{(2,1)} = \mathbf{0}_{L_t^{(2)} N_R^{(2)} D^{(2)} \times \tilde{D}^{(1)}}$ since $\mathbf{B}^{(1,2)} = \mathbf{0}_{D^{(1)} \times D^{(2)}}$ and $\mathbf{B}^{(2,1)} = \mathbf{0}_{D^{(2)} \times D^{(1)}}$. For orthogonally multiplexed MIMO BCs, DPC is not needed because each user experiences no interference from the other user. In this case, the users can be encoded independently and the capacity region can be easily characterized (as explained in Section 4.2). In the more general case of a MIMO BC with $D_c \neq 0$, subsequent sections consider inner and outer bounds for $\mathcal{R}_{DPC}$.

The rates considered above are in bits/channel-use. We now consider two possible normalizations in order to express these rates in bps/Hz. Let $\check{D}^{(k)}$ denote a normalizing constant defined as $\check{D}^{(k)} = D^{(k)}$ if we only consider the number of signal dimensions allocated to user k , and as $\check{D}^{(k)} = D_T$ if we consider the total number of signal dimensions, $k = 1, 2$. Throughout this chapter the rate in bps/Hz of user k is obtained by dividing its rate in bits/channel-use by $\check{D}^{(k)}$. Although both normalizations can be used for any BC, one normalization may be more convenient than the other, depending on the system or on the metric that is used to assess the system's performance. For instance, for orthogonally multiplexed BCs ($D_c = 0$), where the channel can be seen as a set of parallel and decoupled single-user channels, normalizing by $\check{D}^{(k)} = D^{(k)}$ seems to be more appropriate. On the other hand, when assessing the BC performance using metrics that involve the rates of both users such as the rate sum or the sum capacity, it is more convenient to use $\check{D}^{(k)} = D_T$.

Denote $\alpha_p = P^{(1)}/P_T$, thus $\alpha_p \in [0, 1]$ and $(1 - \alpha_p) = P^{(2)}/P_T$. The fraction of transmit power allocated to user k is given by

$$\alpha_p^{(k)} = \begin{cases} \alpha_p, & \text{if } k = 1 \\ (1 - \alpha_p), & \text{if } k = 2 \end{cases} \quad (4.8)$$

Normalizing by $\check{D}^{(k)}$, we rewrite the rates of (4.2) and (4.3) in bps/Hz as functions of $\alpha_p^{(k)}$, $k = 1, 2$, as follows

$$R_{DPC}^{(1)}(\Sigma^{(1)}, \Sigma^{(2)}) \equiv R_{DPC}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}) = \frac{1}{\check{D}^{(1)}} \log \frac{\left| \mathbf{I}_{L_t^{(1)} N_R^{(1)} D^{(1)}} + \frac{1}{N_0} \mathbf{H}^{(1,2)} \cdot \Sigma^{(2)} \cdot [\mathbf{H}^{(1,2)}]^\dagger + \frac{1}{N_0} \mathbf{H}^{(1)} \cdot \Sigma^{(1)} \cdot [\mathbf{H}^{(1)}]^\dagger \right|}{\left| \mathbf{I}_{L_t^{(1)} N_R^{(1)} D^{(1)}} + \frac{1}{N_0} \mathbf{H}^{(1,2)} \cdot \Sigma^{(2)} \cdot [\mathbf{H}^{(1,2)}]^\dagger \right|}} \text{ bps/Hz} \quad (4.9a)$$

$$R_{DPC}^{(2)}(\Sigma^{(2)}) \equiv R_{DPC}^{(2)}(\alpha_p^{(2)}) = \frac{1}{\check{D}^{(2)}} \log \left| \mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}} + \frac{1}{N_0} \mathbf{H}^{(2)} \cdot \Sigma^{(2)} \cdot [\mathbf{H}^{(2)}]^\dagger \right| \text{ bps/Hz} \quad (4.9b)$$

and

$$\begin{aligned}\tilde{R}_{DPC}^{(1)}(\Sigma^{(1)}) &\equiv \tilde{R}_{DPC}^{(1)}(\alpha_p^{(1)}) = \\ &\frac{1}{\tilde{D}^{(1)}} \log \left| \mathbf{I}_{L_t^{(1)} N_R^{(1)} D^{(1)}} + \frac{1}{N_0} \mathbf{H}^{(1)} \cdot \Sigma^{(1)} \cdot [\mathbf{H}^{(1)}]^\dagger \right| \text{ bps/Hz} \quad (4.10a)\end{aligned}$$

$$\begin{aligned}\tilde{R}_{DPC}^{(2)}(\Sigma^{(1)}, \Sigma^{(2)}) &\equiv \tilde{R}_{DPC}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}) = \\ &\frac{1}{\tilde{D}^{(2)}} \log \frac{\left| \mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}} + \frac{1}{N_0} \mathbf{H}^{(2,1)} \cdot \Sigma^{(1)} \cdot [\mathbf{H}^{(2,1)}]^\dagger + \frac{1}{N_0} \mathbf{H}^{(2)} \cdot \Sigma^{(2)} \cdot [\mathbf{H}^{(2)}]^\dagger \right|}{\left| \mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}} + \frac{1}{N_0} \mathbf{H}^{(2,1)} \cdot \Sigma^{(1)} \cdot [\mathbf{H}^{(2,1)}]^\dagger \right|} \text{ bps/Hz} \quad (4.10b)\end{aligned}$$

where $Tr(\Sigma^{(1)}) \leq \alpha_p^{(1)} P_T$ and $Tr(\Sigma^{(2)}) \leq \alpha_p^{(2)} P_T$. The next section considers an outer bound for the DPC rate region $\mathcal{R}_{DPC}$ given by (4.6).

4.2 Outer bound based on orthogonally multiplexed MIMO BCs

From Section 4.1, applying DPC at the transmitter consists in ordering the users and encoding them successively. That is, the transmitter picks a codeword for the first user, then uses the noncausal knowledge of this codeword to encode the second user such that the second user does not see the signals intended for the first user as interference. The signals intended for the second user are seen by the first user as interference, however. Hence, for the two-user MIMO BC, DPC ensures that interference is canceled for one user only. An outer bound for the DPC rate region $\mathcal{R}_{DPC}$ can be obtained if the interference is canceled for both users. Let $\mathcal{R}_{out}$ denote such outer bound. Consider a coding scheme for $\mathcal{R}_{out}$ that cancels for each user the interference from the other user without any power penalty. Obviously, such coding scheme is not realizable in practice since it is impossible for the transmitter to pick two codewords (one for each user) and, then use the noncausal knowledge of each user's codeword to cancel the interference this user creates for the other user. If all interference is canceled, the MIMO BC of (4.1) becomes equivalent to two non-interfering single-user MIMO channels with a total power constraint. In this case, each user can be encoded at the same rate as if the interfering terms $\mathbf{H}^{(1,2)} \cdot \mathbf{x}^{(2)}$ and $\mathbf{H}^{(2,1)} \cdot \mathbf{x}^{(1)}$ are not present in (4.1a) and (4.1b), respectively, i.e. as if $\mathbf{r}^{(k)} = \mathbf{H}^{(k)} \cdot \mathbf{x}^{(k)} + \mathbf{v}^{(k)}$ for $k = 1, 2$.

Hence, all rate pairs $(R^{(1)}, R^{(2)})$ such that

$$R^{(1)} \leq I(\mathbf{x}^{(1)}; \mathbf{r}^{(1)}) \quad (4.11a)$$

$$R^{(2)} \leq I(\mathbf{x}^{(2)}; \mathbf{r}^{(2)}) \quad (4.11b)$$

are achievable. Furthermore, we can write

$$\begin{aligned} I(\mathbf{x}^{(k)}; \mathbf{r}^{(k)}) &= h(\mathbf{r}^{(k)}) - h(\mathbf{r}^{(k)} | \mathbf{x}^{(k)}) \\ &\stackrel{(a)}{=} h(\mathbf{H}^{(k)} \cdot \mathbf{x}^{(k)} + \mathbf{v}^{(k)}) - h(\mathbf{v}^{(k)} | \mathbf{x}^{(k)}) \\ &\stackrel{(b)}{=} h(\mathbf{H}^{(k)} \cdot \mathbf{x}^{(k)} + \mathbf{v}^{(k)}) - h(\mathbf{v}^{(k)}) \\ &\stackrel{(c)}{=} \log(|\mathbf{I}_{L_t^{(k)} N_R^{(k)} D^{(k)}} + \frac{1}{N_0} \mathbf{H}^{(k)} \cdot \Sigma^{(k)} \cdot [\mathbf{H}^{(k)}]^\dagger|) \end{aligned}$$

where (a) is due to the invariance of the differential entropy with translation [51, Theorem 9.6.3], (b) is due to the independence of $\mathbf{v}^{(k)}$ and $\mathbf{x}^{(k)}$, and (c) is due to the Gaussianity of $\mathbf{x}^{(k)}$ and $\mathbf{v}^{(k)}$. This is the mutual information (3.1) of user k in the single-user system with power constraint $\text{Tr}(\Sigma^{(k)}) \leq \alpha_p^{(k)} P_T$, $k = 1, 2$. The outer bound $\mathcal{R}_{out}$ is the closure of all the rate pairs $(R^{(1)}, R^{(2)})$ that satisfy (4.11), that is,

$$\begin{aligned} \mathcal{R}_{out} = \{ (R^{(1)}, R^{(2)}) : 0 \leq R^{(1)} \leq C^{(1)}(\alpha_p^{(1)}), 0 \leq R^{(2)} \leq C^{(2)}(\alpha_p^{(2)}), \\ \text{for all } (\alpha_p^{(1)}, \alpha_p^{(2)}) \text{ satisfying } 0 \leq \alpha_p^{(1)}, \alpha_p^{(2)} \leq 1, \alpha_p^{(1)} + \alpha_p^{(2)} = 1 \} \end{aligned} \quad (4.12)$$

where $C^{(k)}(\alpha_p^{(k)}) = \max_{\text{Tr}(\Sigma^{(k)}) \leq \alpha_p^{(k)} P_T} \log(|\mathbf{I}_{L_t^{(k)} N_R^{(k)} D^{(k)}} + \frac{1}{N_0} \mathbf{H}^{(k)} \cdot \Sigma^{(k)} \cdot [\mathbf{H}^{(k)}]^\dagger|)$ is the single-user (Shannon) capacity of user k in bits/channel-use, $k = 1, 2$. Moreover, $\mathcal{R}_{out}$ is also an outer bound for the capacity region $\mathcal{C}$ of the MIMO BC described by (4.1) since it is the largest rate region that can be achieved assuming that each user experiences no interference from the other user. An outer bound for the DPC and capacity regions of MIMO BCs with $K > 2$ can be also obtained similarly to $\mathcal{R}_{out}$ by assuming that the interference is canceled for all users, in which case each user k ($k = 1, \dots, K$) transmits at a rate equal to the single-user capacity $C^{(k)}(\alpha_p^{(k)})$ obtained subject to its power constraint. The outer bound $\mathcal{R}_{out}$ of (4.12) is similar to the capacity region [121, 122] of orthogonally multiplexed MIMO BCs (4.7). The main difference between $\mathcal{R}_{out}$ and the capacity region of orthogonally multiplexed MIMO BCs is that for the same number of signal dimensions allocated to user k , $D^{(k)} = D_p^{(k)} + D_c$, we have $D_c = 0$ when considering the capacity region of orthogonally multiplexed MIMO BCs while $D_c \neq 0$ when considering $\mathcal{R}_{out}$. For

single-user systems, the capacity in bps/Hz is given by (3.2) or (3.9) where it is normalized by the number of signal dimensions that span the user signal space. For broadcast systems, since we choose two possible normalizations depending on $\check{D}^{(k)}$ we can write

$$C^{(k)}(\alpha_p^{(k)}) = \frac{1}{\check{D}^{(k)}} \max_{Tr(\Sigma^{(k)}) \leq \alpha_p^{(k)} P_T} \log(|\mathbf{I}_{L_t^{(k)} N_R^{(k)} D^{(k)}} + \frac{1}{N_0} \mathbf{H}^{(k)} \cdot \Sigma^{(k)} \cdot [\mathbf{H}^{(k)}]^\dagger|) \text{ bps/Hz} \quad (4.13)$$

for $k = 1, 2$. With $\{\lambda_{j;l}^{(k)}\}_{l=1}^{n_j^{(k)}}$ being the decreasingly ordered eigenvalues of $[\mathbf{C}_j^{(k)}]^\dagger \cdot \mathbf{C}_j^{(k)}$ and using the water-filling solution of Section 3.1, (4.13) reduces similarly to (3.9) to

$$C^{(k)}(\alpha_p^{(k)}) = \frac{D_G^{(k)}}{\check{D}^{(k)}} \sum_{j=1}^{N_G^{(k)}} \sum_{l=1}^{n_j^{(k)}} \{ \log(\mu^{(k)}(\alpha_p^{(k)}) \lambda_{j;l}^{(k)}) \}^+ \text{ bps/Hz} \quad (4.14)$$

with a water-filling constant $\mu^{(k)}(\alpha_p^{(k)})$ that satisfies $\sum_{j=1}^{N_G^{(k)}} \sum_{l=1}^{n_j^{(k)}} \{ \mu^{(k)}(\alpha_p^{(k)}) - (\lambda_{j;l}^{(k)})^{-1} \}^+ = \frac{N_G^{(k)} \alpha_p^{(k)} P_T}{N_0 D^{(k)}}$. From Section 3.1, $C^{(k)}(\alpha_p^{(k)})$ is achieved by a zero mean circularly symmetric Gaussian input vector $\mathbf{x}^{(k)}$ with covariance matrix

$$\Sigma_{\max}^{(k)} \triangleq \text{diag}_{j=1}^{N_G^{(k)}} \left[(\mathbf{M}_j^{(k)})^\dagger \cdot (N_0 \text{diag}_{p=1}^{n_j^{(k)}} [\{ \mu^{(k)}(\alpha_p^{(k)}) - (\lambda_{j;p}^{(k)})^{-1} \}^+]) \cdot \mathbf{M}_j^{(k)} \right] \otimes \mathbf{I}_{D_G^{(k)}} \quad (4.15)$$

from (3.8) where, for every j , $\mathbf{M}_j^{(k)}$ denotes the unitary matrix in the eigenvalue decomposition $[\mathbf{C}_j^{(k)}]^\dagger \cdot \mathbf{C}_j^{(k)} = [\mathbf{M}_j^{(k)}]^\dagger \cdot \text{diag}_{l=1}^{n_j^{(k)}} [\lambda_{j;l}^{(k)}] \cdot \mathbf{M}_j^{(k)}$. The corresponding transmission scheme is eigen-beamforming. We also provide an equivalent expression for $C^{(k)}(\alpha_p^{(k)})$ in terms of the eigenvalues of $[\mathbf{H}^{(k)}]^\dagger \cdot \mathbf{H}^{(k)}$ that will be useful for subsequent derivations. Using (3.6) and Theorem C.0.1, each eigenvalue $\lambda_q^{(k)}$ of $[\mathbf{H}^{(k)}]^\dagger \cdot \mathbf{H}^{(k)}$ is an eigenvalue of $\text{diag}_{j=1}^{N_G^{(k)}} [[\mathbf{C}_j^{(k)}]^\dagger \cdot \mathbf{C}_j^{(k)}]$ with multiplicity $D_G^{(k)}$. Let $m^{(k)}(\alpha_p^{(k)})$ denote the number of active eigenmodes associated with $C^{(k)}(\alpha_p^{(k)})$ for user k , $m^{(k)}(\alpha_p^{(k)}) \leq r^{(k)}$ for all $\alpha_p^{(k)} \in [0, 1]$. Assuming that the eigenvalues $\{\lambda_q^{(k)}\}_{q=1}^{\check{D}^{(k)}}$ are arranged in decreasing order, (4.14) is equivalent to

$$C^{(k)}(\alpha_p^{(k)}) = \frac{1}{\check{D}^{(k)}} \sum_{q=1}^{m^{(k)}(\alpha_p^{(k)})} \log(\mu^{(k)}(\alpha_p^{(k)}) \lambda_q^{(k)}) \text{ bps/Hz} \quad (4.16)$$

where $\mu^{(k)}(\alpha_p^{(k)}) = \frac{\alpha_p^{(k)} P_T}{N_0 m^{(k)}(\alpha_p^{(k)})} + \frac{1}{m^{(k)}(\alpha_p^{(k)})} \sum_{q=1}^{m^{(k)}(\alpha_p^{(k)})} (\lambda_q^{(k)})^{-1}$.

The outer bound $\mathcal{R}_{out}$ becomes tight whenever the interference becomes small since, in this case, canceling the interference for one user only or canceling the interference for both

users does not significantly affect the transmission rates. This occurs in the low SNR regime where P_T/N_0 is sufficiently small and hence, the interference caused by each user's signals is small compared to the noise. Also, $\mathcal{R}_{out}$ becomes tight when $\alpha_p \rightarrow 0$ with user 2 being encoded before user 1 since, in this case, the interference that user 1 experiences due to user 2's signals is canceled by DPC whereas the interference that user 2 experiences due to user 1's signals is small because the power allocated to user 1 is small ($\alpha_p^{(1)} \rightarrow 0$). Similarly, $\mathcal{R}_{out}$ becomes tight when $\alpha_p \rightarrow 1$ with user 1 being encoded before user 2 since, in this case, the interference that user 2 experiences due to user 1's signals is canceled by DPC whereas the interference that user 1 experiences due to user 2's signals is small because the power allocated to user 2 is small ($\alpha_p^{(2)} \rightarrow 0$). Using a time-sharing argument as in [51, Theorem 14.3.2], one can easily prove that $\mathcal{R}_{out}$ is convex. Moreover, since the boundary of $\mathcal{R}_{out}$ is determined by the single-user capacity pairs $(C^{(1)}(\alpha_p^{(1)}), C^{(2)}(\alpha_p^{(2)}))$ as seen from (4.12), the impact of the users space-time modulation formats on $\mathcal{R}_{out}$ can be easily investigated using Theorem 3.1.1 and Corollary 3.1.1 from Chapter 3.

Further characterization of $\mathcal{R}_{out}$ can be obtained by considering its maximum sum rate C_{out}^{sum} . Using the rate pairs $(R^{(1)}, R^{(2)})$ specified by $\mathcal{R}_{out}$ (4.12), we have

$$C_{out}^{sum} = \max_{(R^{(1)}, R^{(2)}) \in \mathcal{R}_{out}} \{R^{(1)} + R^{(2)}\} \stackrel{(a)}{=} \max_{\alpha_p^{(1)}, \alpha_p^{(2)} \in [0,1], \alpha_p^{(1)} + \alpha_p^{(2)} = 1} \{C^{(1)}(\alpha_p^{(1)}) + C^{(2)}(\alpha_p^{(2)})\}$$

with (a) due to the fact that the maximum sum rate over all the rate pairs $(R^{(1)}, R^{(2)}) \in \mathcal{R}_{out}$ is obtained at a boundary point of $\mathcal{R}_{out}$ where these boundary points are specified by $(C^{(1)}(\alpha_p^{(1)}), C^{(2)}(\alpha_p^{(2)}))$, with $C^{(k)}(\alpha_p^{(k)})$ given by (4.14) or (4.16) for $k = 1, 2$ and $\alpha_p^{(1)}, \alpha_p^{(2)} \in [0, 1]$ with $\alpha_p^{(1)} + \alpha_p^{(2)} = 1$. We use $(\alpha_{p,s}^{(1)}, \alpha_{p,s}^{(2)})$ to denote the (not necessarily unique) solution to the last optimization problem. Note that since $\mathcal{R}_{out}$ is an outer bound for the capacity region $\mathcal{C}$ of the MIMO BC described by (4.1), C_{out}^{sum} is an upper bound to the sum capacity, denoted by C^{sum} , of this channel. The sum capacity C^{sum} is defined as the maximum sum rate over all achievable rate pairs $(R^{(1)}, R^{(2)})$ for the channel (4.1), i.e. $C^{sum} = \max_{(R^{(1)}, R^{(2)})} \{R^{(1)} + R^{(2)}\}$. We have for $\mathcal{R}_{out}$ the following result from convexity theory.

Theorem 4.2.1 *The sum rate $R^{(1)} + R^{(2)}$, $(R^{(1)}, R^{(2)}) \in \mathcal{R}_{out}$, is maximized at a boundary point $(C^{(1)}(\alpha_p^{(1)}), C^{(2)}(\alpha_p^{(2)}))$ if and only if the support line to the boundary of $\mathcal{R}_{out}$ at $(C^{(1)}(\alpha_p^{(1)}), C^{(2)}(\alpha_p^{(2)}))$ has a slope equal to -1 .*

Proof of Theorem 4.2.1: This statement follows easily from the convexity properties [123] of $\mathcal{R}_{out}$. $\square$

Prior to investigating the behavior of $\mathcal{R}_{out}$ and C_{out}^{sum} in the low and high SNR regimes, the next section introduces an inner bound for $\mathcal{R}_{DPC}$ which, together with $\mathcal{R}_{out}$, provide an approximation of $\mathcal{R}_{DPC}$.

4.3 Inner bound

This section considers an inner bound for the rate region $\mathcal{R}_{DPC}$. As seen from equations (4.4)-(4.6), $\mathcal{R}_{DPC}$ is defined using the rate pairs $(R_{DPC}^{(1)}(\Sigma^{(1)}, \Sigma^{(2)}), R_{DPC}^{(2)}(\Sigma^{(1)}, \Sigma^{(2)}))$ of (4.9) and $(\tilde{R}_{DPC}^{(1)}(\Sigma^{(1)}, \Sigma^{(2)}), \tilde{R}_{DPC}^{(2)}(\Sigma^{(1)}, \Sigma^{(2)}))$ of (4.10) which are defined over the set of positive semidefinite input covariance matrices that satisfy $Tr(\Sigma^{(1)}) + Tr(\Sigma^{(2)}) \leq P_T$. An inner bound for $\mathcal{R}_{DPC}$ can be obtained by considering only a subset of these input covariance matrices and by using the associated subset of DPC rate pairs of (4.9) and (4.10). We build this subset of DPC rate pairs by following the method proposed in [124] for the downlink of a wireless system where, based on a first-come first-served scheduling, newly entering users are accommodated without affecting the existing users. Next, we explain the method from [124] and we use $\mathcal{R}_{in}$ to denote the resulting inner bound for $\mathcal{R}_{DPC}$.

Throughout, we use the expressions of the DPC rate pairs (4.9) and (4.10) in terms of the fractions of power $\alpha_p^{(k)}$, $k = 1, 2$. For a given $(\alpha_p^{(1)}, \alpha_p^{(2)})$, we define for user 1 and user 2 the following Hermitian positive definite matrices

$$\Psi^{(1)} \triangleq \mathbf{I}_{L_t^{(1)} N_R^{(1)} D^{(1)}} + \frac{1}{N_0} \mathbf{H}^{(1,2)} \cdot \Sigma_{\max}^{(2)} \cdot [\mathbf{H}^{(1,2)}]^\dagger \quad (4.17a)$$

$$\Psi^{(2)} \triangleq \mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}} + \frac{1}{N_0} \mathbf{H}^{(2,1)} \cdot \Sigma_{\max}^{(1)} \cdot [\mathbf{H}^{(2,1)}]^\dagger \quad (4.17b)$$

where $\Sigma_{\max}^{(k)}$ is given by (4.15) and it denotes the input covariance matrix of user k obtained by water-filling over the eigenmodes of $[\mathbf{H}^{(k)}]^\dagger \cdot \mathbf{H}^{(k)}$, $Tr(\Sigma_{\max}^{(k)}) = \alpha_p^{(k)} P_T$, $k = 1, 2$. Considering (4.9b) with $\Sigma^{(2)} = \Sigma_{\max}^{(2)}$, we have for user 2

$$R_{DPC}^{(2)}(\alpha_p^{(2)}) = \frac{1}{D^{(2)}} \log \left| \mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}} + \frac{1}{N_0} \mathbf{H}^{(2)} \cdot \Sigma_{\max}^{(2)} \cdot [\mathbf{H}^{(2)}]^\dagger \right| = C^{(2)}(\alpha_p^{(2)})$$

where $C^{(2)}(\alpha_p^{(2)})$ is the single-user capacity given by (4.14) or (4.16) with $k = 2$. The second equality follows from the fact that $C^{(2)}(\alpha_p^{(2)})$ is achieved by a zero mean Gaussian input vector $\mathbf{x}^{(2)}$ with covariance matrix $\Sigma_{\max}^{(2)}$ as shown in Section 3.1. Using $\Sigma^{(2)} = \Sigma_{\max}^{(2)}$ in (4.9a), we obtain

$$\begin{aligned} R_{DPC}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}) &\stackrel{(a)}{=} \frac{1}{\check{D}^{(1)}} \log \left| \mathbf{I}_{L_t^{(1)} N_R^{(1)} D^{(1)}} + \frac{1}{N_0} \mathbf{H}^{(1)} \cdot \Sigma^{(1)} \cdot [\mathbf{H}^{(1)}]^\dagger \cdot \left[\mathbf{I}_{L_t^{(1)} N_R^{(1)} D^{(1)}} + \frac{1}{N_0} \mathbf{H}^{(1,2)} \cdot \Sigma_{\max}^{(2)} \cdot [\mathbf{H}^{(1,2)}]^\dagger \right]^{-1} \right| \\ &\stackrel{(b)}{=} \frac{1}{\check{D}^{(1)}} \log \left| \mathbf{I}_{L_t^{(1)} N_R^{(1)} D^{(1)}} + \frac{1}{N_0} \mathbf{H}^{(1)} \cdot \Sigma^{(1)} \cdot [\mathbf{H}^{(1)}]^\dagger \cdot [\Psi^{(1)}]^{-1} \right| \end{aligned} \quad (4.18)$$

where (a) is due to (C.17) and (C.10), and (b) is due to (4.17a). Following [124], we choose the input covariance matrix $\Sigma^{(1)}$ for user 1 so that it maximizes $R_{DPC}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ of (4.18) under the constraint $Tr(\Sigma^{(1)}) \leq \alpha_p^{(1)} P_T$. Denoting the maximum rate $R_{DPC}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ by $\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)})$, we have

$$\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}) = \frac{1}{\check{D}^{(1)}} \max_{Tr(\Sigma^{(1)}) \leq \alpha_p^{(1)} P_T} \log \left| \mathbf{I}_{L_t^{(1)} N_R^{(1)} D^{(1)}} + \frac{1}{N_0} \mathbf{H}^{(1)} \cdot \Sigma^{(1)} \cdot [\mathbf{H}^{(1)}]^\dagger \cdot [\Psi^{(1)}]^{-1} \right| \quad (4.19)$$

in bps/Hz. Let $\check{\Sigma}^{(1)}$ denote the optimal input covariance matrix that achieves $\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)})$. The optimization problem in (4.19) is similar to that formulated in (3.2) with $I(\mathbf{x}; \mathbf{r})$ of (3.1), and such that the matrices $\mathbf{H}$ and $\mathbf{H}^\dagger$ in (3.1) are replaced by $\mathbf{H}^{(1)}$ and $[\mathbf{H}^{(1)}]^\dagger \cdot [\Psi^{(1)}]^{-1}$, respectively. Therefore, following a method similar to that used for solving the optimization problem (3.2) we can show that $\check{\Sigma}^{(1)}$ is obtained by water-filling over the eigenmodes of $[\mathbf{H}^{(1)}]^\dagger \cdot [\Psi^{(1)}]^{-1} \cdot \mathbf{H}^{(1)}$ with $Tr(\check{\Sigma}^{(1)}) = \alpha_p^{(1)} P_T$. By choosing $(\Sigma^{(1)}, \Sigma^{(2)}) = (\check{\Sigma}^{(1)}, \Sigma_{\max}^{(2)})$, the DPC rate pair of (4.9) reduces to $(\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}), C^{(2)}(\alpha_p^{(2)}))$. Since for a given $(\Sigma^{(1)}, \Sigma^{(2)})$ the DPC rate pair of (4.9) is an element of $\mathcal{R}_{1,2}(\Sigma^{(1)}, \Sigma^{(2)})$, as seen from (4.4), $(\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}), C^{(2)}(\alpha_p^{(2)})) \in \mathcal{R}_{1,2}(\check{\Sigma}^{(1)}, \Sigma_{\max}^{(2)})$ and it is an achievable rate pair by Theorem 4.1.1. Following a similar approach, we consider now particular DPC rate pairs of the form of (4.10). Considering (4.10a) with $\Sigma^{(1)} = \Sigma_{\max}^{(1)}$, we have for user 1

$$\tilde{R}_{DPC}^{(1)}(\alpha_p^{(1)}) = \frac{1}{\check{D}^{(1)}} \log \left| \mathbf{I}_{L_t^{(1)} N_R^{(1)} D^{(1)}} + \frac{1}{N_0} \mathbf{H}^{(1)} \cdot \Sigma_{\max}^{(1)} \cdot [\mathbf{H}^{(1)}]^\dagger \right| = C^{(1)}(\alpha_p^{(1)})$$

where $C^{(1)}(\alpha_p^{(1)})$ is the single-user capacity given by (4.14) or (4.16) with $k = 1$. The second equality follows from the fact that $C^{(1)}(\alpha_p^{(1)})$ is achieved by a zero mean Gaussian

input vector $\mathbf{x}^{(1)}$ with covariance matrix $\Sigma_{\max}^{(1)}$ as shown in Section 3.1. Using $\Sigma^{(1)} = \Sigma_{\max}^{(1)}$ in (4.10b), we obtain

$$\begin{aligned} \tilde{R}_{DPC}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}) &\stackrel{(a)}{=} \frac{1}{\check{D}^{(2)}} \log \left| \mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}} + \frac{1}{N_0} \mathbf{H}^{(2)} \cdot \Sigma^{(2)} \cdot [\mathbf{H}^{(2)}]^\dagger \cdot \left[\mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}} + \frac{1}{N_0} \mathbf{H}^{(2,1)} \cdot \Sigma_{\max}^{(1)} \cdot [\mathbf{H}^{(2,1)}]^\dagger \right]^{-1} \right| \\ &\stackrel{(b)}{=} \frac{1}{\check{D}^{(2)}} \log \left| \mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}} + \frac{1}{N_0} \mathbf{H}^{(2)} \cdot \Sigma^{(2)} \cdot [\mathbf{H}^{(2)}]^\dagger \cdot [\Psi^{(2)}]^{-1} \right| \end{aligned} \quad (4.20)$$

where (a) is due to (C.17) and (C.10), and (b) is due to (4.17b). Following [124], we choose the input covariance matrix $\Sigma^{(2)}$ for user 2 so that it maximizes $\tilde{R}_{DPC}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ of (4.20) under the constraint $Tr(\Sigma^{(2)}) \leq \alpha_p^{(2)} P_T$. Denoting the maximum rate $\tilde{R}_{DPC}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ by $\check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})$, we have

$$\check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}) = \frac{1}{\check{D}^{(2)}} \max_{Tr(\Sigma^{(2)}) \leq \alpha_p^{(2)} P_T} \log \left| \mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}} + \frac{1}{N_0} \mathbf{H}^{(2)} \cdot \Sigma^{(2)} \cdot [\mathbf{H}^{(2)}]^\dagger \cdot [\Psi^{(2)}]^{-1} \right| \quad (4.21)$$

in bps/Hz. Let $\check{\Sigma}^{(2)}$ denote the optimal input covariance matrix that achieves $\check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})$. The optimization problem in (4.21) is similar to those of (4.19) and (3.2) and hence, it can be solved using a similar method as for (3.2). It follows that $\check{\Sigma}^{(2)}$ is obtained by water-filling over the eigenmodes of $[\mathbf{H}^{(2)}]^\dagger \cdot [\Psi^{(2)}]^{-1} \cdot \mathbf{H}^{(2)}$ with $Tr(\check{\Sigma}^{(2)}) = \alpha_p^{(2)} P_T$. By choosing $(\Sigma^{(1)}, \Sigma^{(2)}) = (\Sigma_{\max}^{(1)}, \check{\Sigma}^{(2)})$, the DPC rate pair of (4.10) reduces to $(C^{(1)}(\alpha_p^{(1)}), \check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}))$. Since for a given $(\Sigma^{(1)}, \Sigma^{(2)})$ the DPC rate pair of (4.10) is an element of $\mathcal{R}_{2,1}(\Sigma^{(1)}, \Sigma^{(2)})$, as seen from (4.5), $(C^{(1)}(\alpha_p^{(1)}), \check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})) \in \mathcal{R}_{2,1}(\Sigma_{\max}^{(1)}, \check{\Sigma}^{(2)})$ and it is an achievable rate pair by Theorem 4.1.1.

Before introducing the inner bound $\mathcal{R}_{in}$, let us consider the water-filling schemes associated with the rates $\check{R}^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)})$, $k = 1, 2$, for a given pair $(\alpha_p^{(1)}, \alpha_p^{(2)})$. Let $\check{r}^{(k)}$ and $\{\check{\lambda}_q^{(k)}\}_{q=1}^{\check{D}^{(k)}}$ denote the rank and eigenvalues of $[\mathbf{H}^{(k)}]^\dagger \cdot [\Psi^{(k)}]^{-1} \cdot \mathbf{H}^{(k)}$, respectively, such that $\check{\lambda}_1^{(k)} \geq \dots \geq \check{\lambda}_{\check{D}^{(k)}}^{(k)} \geq 0$. The eigenvalues $\check{\lambda}_q^{(k)}$ depend on $(\alpha_p^{(1)}, \alpha_p^{(2)})$ and on the total transmit power P_T because of the dependency of $\Psi^{(k)}$ on the input covariance matrix of the other user as seen from (4.17). Furthermore, since $\Psi^{(k)}$ is Hermitian positive definite, $[\Psi^{(k)}]^{-1}$ is also Hermitian positive definite from [105, p. 397]. By [105, Theorem 7.2.6] there exists a unique Hermitian positive definite matrix $[\Psi^{(k)}]^{-1/2}$ such that

$[\Psi^{(k)}]^{-1/2} \cdot [\Psi^{(k)}]^{-1/2} = [\Psi^{(k)}]^{-1}$. We can write

$$\begin{aligned}
 \check{r}^{(k)} &= \text{rank}([\mathbf{H}^{(k)}]^\dagger \cdot [\Psi^{(k)}]^{-1/2} \cdot [\Psi^{(k)}]^{-1/2} \cdot \mathbf{H}^{(k)}) \\
 &\stackrel{(a)}{=} \text{rank}([\Psi^{(k)}]^{-1/2} \cdot \mathbf{H}^{(k)} \cdot [\mathbf{H}^{(k)}]^\dagger \cdot [\Psi^{(k)}]^{-1/2}) \\
 &\stackrel{(b)}{=} \text{rank}(\mathbf{H}^{(k)} \cdot [\mathbf{H}^{(k)}]^\dagger) \\
 &\stackrel{(c)}{=} \text{rank}([\mathbf{H}^{(k)}]^\dagger \cdot \mathbf{H}^{(k)}) = r_T^{(k)} D_G^{(k)}
 \end{aligned} \tag{4.22}$$

where (a) is due to (C.15c), (b) is due to the fact that the rank of $\mathbf{H}^{(k)} \cdot [\mathbf{H}^{(k)}]^\dagger$ is unchanged upon left and right multiplication by the nonsingular matrix $[\Psi^{(k)}]^{-1/2}$ from [105, p. 13], and (c) is also due to (C.15c). With the water-filling solution, equations (4.19) and (4.21) can be put in the following equivalent form

$$\check{R}^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)}) = \frac{1}{\check{D}^{(k)}} \sum_{q=1}^{\check{D}^{(k)}} \{\log(\check{\mu}^{(k)}(\alpha_p^{(k)}) \check{\lambda}_q^{(k)})\}^+ \text{ bps/Hz} \tag{4.23}$$

where the water-filling constant $\check{\mu}^{(k)}(\alpha_p^{(k)})$ satisfies $\sum_{q=1}^{\check{D}^{(k)}} \{\check{\mu}^{(k)}(\alpha_p^{(k)}) - (\check{\lambda}_q^{(k)})^{-1}\}^+ = \frac{\alpha_p^{(k)} P_T}{N_0}$ and where the number of active eigenmodes is less or equal than $r_T^{(k)} D_G^{(k)}$ from (4.22), $k = 1, 2$. The associated input covariance matrix for user k is given by

$$\check{\Sigma}^{(k)} = [\check{\mathbf{M}}^{(k)}]^\dagger \cdot (N_0 \text{diag}_{q=1}^{\check{D}^{(k)}} [\{\check{\mu}^{(k)}(\alpha_p^{(k)}) - (\check{\lambda}_q^{(k)})^{-1}\}^+]) \cdot \check{\mathbf{M}}^{(k)} \tag{4.24}$$

where $\check{\mathbf{M}}^{(k)}$ denotes the $\check{D}^{(k)} \times \check{D}^{(k)}$ unitary matrix in the eigenvalue decomposition $[\mathbf{H}^{(k)}]^\dagger \cdot [\Psi^{(k)}]^{-1} \cdot \mathbf{H}^{(k)} = [\check{\mathbf{M}}^{(k)}]^\dagger \cdot \text{diag}_{q=1}^{\check{D}^{(k)}} [\check{\lambda}_q^{(k)}] \cdot \check{\mathbf{M}}^{(k)}$. Hence, considering the subset of DPC rate pairs $(\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}), C^{(2)}(\alpha_p^{(2)}))$ and $(C^{(1)}(\alpha_p^{(1)}), \check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}))$, we define the inner bound $\mathcal{R}_{in}$ as follows

$$\begin{aligned}
 \mathcal{R}_{in} = \text{CH} \Big\{ (0, 0) \cup \Big\{ (\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}), C^{(2)}(\alpha_p^{(2)})), (C^{(1)}(\alpha_p^{(1)}), \check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})) : \\
 0 \leq \alpha_p^{(1)}, \alpha_p^{(2)} \leq 1, \alpha_p^{(1)} + \alpha_p^{(2)} = 1 \Big\} \Big\}
 \end{aligned} \tag{4.25}$$

As such, $\mathcal{R}_{in}$ is an inner bound for the DPC rate region $\mathcal{R}_{DPC}$ as well as for the capacity region $\mathcal{C}$ of the MIMO BC (4.1). The inner bound $\mathcal{R}_{in}$ and the outer bound $\mathcal{R}_{out}$ of Section 4.2 can be used to provide an approximation for $\mathcal{R}_{DPC}$ and $\mathcal{C}$. Note that for BCs with more than two users ($K > 2$), an inner bound to the capacity and DPC rate regions can be obtained similarly to $\mathcal{R}_{in}$, that is, by maximizing the transmission rate of the k^{th} user

subject to its power constraint and provided that the subsequently encoded users (if any) are transmitting at maximal rates. As an example we consider the MIMO BC with 3 users, described by (2.40) as follows

$$\begin{aligned} \mathbf{r}^{(1)} &= \mathbf{H}^{(1)} \cdot \mathbf{x}^{(1)} + \mathbf{H}^{(1,2)} \cdot \mathbf{x}^{(2)} + \mathbf{H}^{(1,3)} \cdot \mathbf{x}^{(3)} + \mathbf{v}^{(1)} \\ \mathbf{r}^{(2)} &= \mathbf{H}^{(2,1)} \cdot \mathbf{x}^{(1)} + \mathbf{H}^{(2)} \cdot \mathbf{x}^{(2)} + \mathbf{H}^{(2,3)} \cdot \mathbf{x}^{(3)} + \mathbf{v}^{(2)} \\ \mathbf{r}^{(3)} &= \mathbf{H}^{(3,1)} \cdot \mathbf{x}^{(1)} + \mathbf{H}^{(3,2)} \cdot \mathbf{x}^{(2)} + \mathbf{H}^{(3)} \cdot \mathbf{x}^{(3)} + \mathbf{v}^{(3)} \end{aligned}$$

where $\{\mathbf{H}^{(k)}\}_{k=1}^3$ are given by (2.41), and $\mathbf{H}^{(k,p)}$ for $(k,p) = (1,2), (1,3), (2,1), (2,3), (3,1), (3,2)$ are given by (2.42). Assuming that user k_1 is encoded first, user k_2 second and user k_3 third, $k_1, k_2, k_3 \in \{1, 2, 3\}$, the DPC procedure leads to the following rates (in bps/Hz)

$$\begin{aligned} R_{DPC}^{(k_1)}(\alpha_p^{(k_1)}, \alpha_p^{(k_2)}, \alpha_p^{(k_3)}) &= \\ \frac{1}{\check{D}^{(k_1)}} \log &\frac{\left| \mathbf{I}_{L_t^{(k_1)} N_R^{(k_1)} D^{(k_1)}} + \frac{1}{N_0} \sum_{l=k_2}^{k_3} \mathbf{H}^{(k_1,l)} \cdot \Sigma^{(l)} \cdot [\mathbf{H}^{(k_1,l)}]^\dagger + \frac{1}{N_0} \mathbf{H}^{(k_1)} \cdot \Sigma^{(k_1)} \cdot [\mathbf{H}^{(k_1)}]^\dagger \right|}{\left| \mathbf{I}_{L_t^{(k_1)} N_R^{(k_1)} D^{(k_1)}} + \frac{1}{N_0} \sum_{l=k_2}^{k_3} \mathbf{H}^{(k_1,l)} \cdot \Sigma^{(l)} \cdot [\mathbf{H}^{(k_1,l)}]^\dagger \right|}} \\ R_{DPC}^{(k_2)}(\alpha_p^{(k_2)}, \alpha_p^{(k_3)}) &= \\ \frac{1}{\check{D}^{(k_2)}} \log &\frac{\left| \mathbf{I}_{L_t^{(k_2)} N_R^{(k_2)} D^{(k_2)}} + \frac{1}{N_0} \mathbf{H}^{(k_2,k_3)} \cdot \Sigma^{(k_3)} \cdot [\mathbf{H}^{(k_2,k_3)}]^\dagger + \frac{1}{N_0} \mathbf{H}^{(k_2)} \cdot \Sigma^{(k_2)} \cdot [\mathbf{H}^{(k_2)}]^\dagger \right|}{\left| \mathbf{I}_{L_t^{(k_2)} N_R^{(k_2)} D^{(k_2)}} + \frac{1}{N_0} \mathbf{H}^{(k_2,k_3)} \cdot \Sigma^{(k_3)} \cdot [\mathbf{H}^{(k_2,k_3)}]^\dagger \right|}} \\ R_{DPC}^{(k_3)}(\alpha_p^{(k_3)}) &= \frac{1}{\check{D}^{(k_3)}} \log \left| \mathbf{I}_{L_t^{(k_3)} N_R^{(k_3)} D^{(k_3)}} + \frac{1}{N_0} \mathbf{H}^{(k_3)} \cdot \Sigma^{(k_3)} \cdot [\mathbf{H}^{(k_3)}]^\dagger \right| \end{aligned}$$

where $\sum_{i=1}^3 \alpha_p^{(k_i)} = 1$ and $\alpha_p^{(k_i)} \in [0, 1]$ for $i = 1, 2, 3$. In order to determine $\mathcal{R}_{in}$ for the 3 users BC, we consider the triples $(\check{R}^{(k_1)}(\alpha_p^{(k_1)}, \alpha_p^{(k_2)}, \alpha_p^{(k_3)}), \check{R}^{(k_2)}(\alpha_p^{(k_2)}, \alpha_p^{(k_3)}), C^{(k_3)}(\alpha_p^{(k_3)}))$ where

$$C^{(k_3)}(\alpha_p^{(k_3)}) = \frac{1}{\check{D}^{(k_3)}} \log \left| \mathbf{I}_{L_t^{(k_3)} N_R^{(k_3)} D^{(k_3)}} + \frac{1}{N_0} \mathbf{H}^{(k_3)} \cdot \Sigma_{\max}^{(k_3)} \cdot [\mathbf{H}^{(k_3)}]^\dagger \right|$$

denotes the single-user capacity for user k_3 obtained using $\Sigma^{(k_3)} = \Sigma_{\max}^{(k_3)}$ with $Tr(\Sigma_{\max}^{(k_3)}) = \alpha_p^{(k_3)} P_T$, the rate $\check{R}^{(k_2)}(\alpha_p^{(k_2)}, \alpha_p^{(k_3)})$ is given by

$$\check{R}^{(k_2)}(\alpha_p^{(k_2)}, \alpha_p^{(k_3)}) = \max_{Tr(\Sigma^{(k_2)}) \leq \alpha_p^{(k_2)} P_T} \left[R_{DPC}^{(k_2)}(\alpha_p^{(k_2)}, \alpha_p^{(k_3)}) \right]_{\Sigma^{(k_3)} = \Sigma_{\max}^{(k_3)}}$$

which denotes the maximal rate $R_{DPC}^{(k_2)}(\alpha_p^{(k_2)}, \alpha_p^{(k_3)})$ provided that user k_3 transmits at ca-

capacity rate, and the rate $\check{R}^{(k_1)}(\alpha_p^{(k_1)}, \alpha_p^{(k_2)}, \alpha_p^{(k_3)})$ is given by

$$\check{R}^{(k_1)}(\alpha_p^{(k_1)}, \alpha_p^{(k_2)}, \alpha_p^{(k_3)}) = \max_{Tr(\Sigma^{(k_1)}) \leq \alpha_p^{(k_1)} P_T} \left[R_{DPC}^{(k_1)}(\alpha_p^{(k_1)}, \alpha_p^{(k_2)}, \alpha_p^{(k_3)}) \right]_{\Sigma^{(k_2)} = \check{\Sigma}^{(k_2)}, \Sigma^{(k_3)} = \Sigma_{\max}^{(k_3)}}$$

with $\check{\Sigma}^{(k_2)}$ denoting the input covariance matrix for user k_2 associated with $\check{R}^{(k_2)}(\alpha_p^{(k_2)}, \alpha_p^{(k_3)})$ and obtained by water-filling over the eigenmodes of $[\mathbf{H}^{(k_2)}]^\dagger \cdot [\mathbf{I}_{L_t^{(k_2)} N_R^{(k_2)} D^{(k_2)}} + \frac{1}{N_0} \mathbf{H}^{(k_2, k_3)} \cdot \Sigma_{\max}^{(k_3)} \cdot [\mathbf{H}^{(k_2, k_3)}]^\dagger]^{-1} \cdot \mathbf{H}^{(k_2)}$. Thus, the rate $\check{R}^{(k_1)}(\alpha_p^{(k_1)}, \alpha_p^{(k_2)}, \alpha_p^{(k_3)})$ denotes the maximal rate for user k_1 provided that user k_3 operates at capacity rate and user k_2 operates at rate $\check{R}^{(k_2)}(\alpha_p^{(k_2)}, \alpha_p^{(k_3)})$. Similarly to (4.25), $\mathcal{R}_{in}$ can be defined for $K = 3$ as

$$\mathcal{R}_{in} = \text{CH} \left\{ (0, 0, 0) \bigcup \left\{ \bigcup_{(k_1, k_2, k_3) \in \Pi_3} (\check{R}^{(k_1)}(\alpha_p^{(k_1)}, \alpha_p^{(k_2)}, \alpha_p^{(k_3)}), \check{R}^{(k_2)}(\alpha_p^{(k_2)}, \alpha_p^{(k_3)}), C^{(k_3)}(\alpha_p^{(k_3)})) : \right. \right. \\ \left. \left. 0 \leq \alpha_p^{(k_1)}, \alpha_p^{(k_2)}, \alpha_p^{(k_3)} \leq 1, \alpha_p^{(k_1)} + \alpha_p^{(k_2)} + \alpha_p^{(k_3)} = 1 \right\} \right\}$$

with $\Pi_3 = \{(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)\}$ denoting the set of all possible users encoding orders. Note that the more users are added the larger becomes the gap between the inner bound $\mathcal{R}_{in}$ and outer bound $\mathcal{R}_{out}$. This is due to the fact that by adding a new user to the system the interference experienced by the previously encoded users increases, which yields an increase in the gap between the rate $\check{R}^{(k)}(\alpha_p^{(1)}, \dots, \alpha_p^{(K)})$ and the capacity $C^{(k)}(\alpha_p^{(k)})$ for these users.

In Chapter 3 we have investigated the impact of space-time modulation on the single-user capacity $C^{(k)}(\alpha_p^{(k)})$. Using Theorem 3.1.1 and Corollary 3.1.1, we have shown that $C^{(k)}(\alpha_p^{(k)})$ is maximum, intermediate and minimum when user k employs ATA, POTA and OTA, respectively, provided that the POTA system for user k is obtained from OTA by merging TOGs, $k = 1, 2$. For a given $(\alpha_p^{(1)}, \alpha_p^{(2)})$, the rate $\check{R}^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ denotes the maximum transmission rate for user k provided that the other user in the system is operating at capacity rate. Thus, we are interested in investigating the impact of space-time modulation on $\check{R}^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ assuming a fixed space-time modulation format for the other user in the system. We prove in Appendix B the following theorem.

Theorem 4.3.1 *Assume that for each user the POTA system is obtained from OTA by merging TOGs. If the space-time modulation format for user 2 (respectively user 1) is fixed then maximum, intermediate and minimum values for $\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ (respectively $\check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})$) are obtained when user 1 (respectively user 2) employs ATA, POTA and OTA, respectively.*

Hence, similarly to Theorem 3.1.1, Theorem 4.3.1 illustrates the fact that with water-filling power allocation ATA provides higher transmission rates than those achieved by other space-time modulation formats. The next section considers the behavior of the inner bound $\mathcal{R}_{in}$ and outer bound $\mathcal{R}_{out}$ in the low and high SNR regimes.

4.4 Asymptotical behavior in extreme SNR cases

4.4.1 The low SNR regime

In this section we consider the low SNR regime where the ratio P_T/N_0 is assumed to be very small with respect to the inverses of the nonzero eigenvalues of $[\mathbf{H}^{(k)}]^\dagger \cdot \mathbf{H}^{(k)}$ (or, equivalently, of $\text{diag}_{i=1}^{N_G^{(k)}} [[\mathbf{C}_i^{(k)}]^\dagger \cdot \mathbf{C}_i^{(k)}]$), that is, $(P_T/N_0) \ll [\lambda_q^{(k)}]^{-1}$ or $(P_T/N_0)\lambda_q^{(k)} \ll 1$ for all $\lambda_q^{(k)} \neq 0$, $q = 1, \dots, \tilde{D}^{(k)}$ and $k = 1, 2$. We start by investigating the behavior of the outer bound $\mathcal{R}_{out}$. Considering user k , only the $n(\lambda_{\max}^{(k)})D_G^{(k)}$ strongest eigenmodes corresponding to $\lambda_{\max}^{(k)} = \max\{\{\lambda_{i,l}^{(k)}\}_{l=1}^{n_i^{(k)}}\}_{i=1}^{N_G^{(k)}} = \max\{\lambda_q^{(k)}\}_{q=1}^{\tilde{D}^{(k)}}$ are active. As in (3.12), the single-user capacity given by (4.14) or (4.16) can be written at low SNR as

$$C^{(k)}(\alpha_p^{(k)}) = \frac{n(\lambda_{\max}^{(k)})D_G^{(k)}}{\tilde{D}^{(k)}} \log \left(1 + \lambda_{\max}^{(k)} \frac{N_G^{(k)} \alpha_p^{(k)} P_T}{n(\lambda_{\max}^{(k)}) N_0 D^{(k)}} \right) \text{ bps/Hz} \quad (4.26)$$

where $(P_T/N_0)\lambda_{\max}^{(k)} \ll 1$, yielding $\lambda_{\max}^{(k)} \frac{N_G^{(k)} \alpha_p^{(k)} P_T}{n(\lambda_{\max}^{(k)}) N_0 D^{(k)}} \ll 1$ since $\frac{N_G^{(k)} \alpha_p^{(k)}}{n(\lambda_{\max}^{(k)}) D^{(k)}} = \frac{\alpha_p^{(k)}}{n(\lambda_{\max}^{(k)}) D_G^{(k)}} \leq 1$. Using the approximation $\ln(1+x) \approx x$, (4.26) leads to

$$C^{(k)}(\alpha_p^{(k)}) \approx \alpha_p^{(k)} \frac{\lambda_{\max}^{(k)}}{\ln 2} \frac{P_T}{N_0 \tilde{D}^{(k)}} \approx \alpha_p^{(k)} C^{(k)}(1) \quad (4.27)$$

From (4.27) and (4.8), we can write $C^{(1)}(\alpha_p^{(1)}) \approx \frac{\lambda_{\max}^{(1)}}{\ln 2} \frac{\alpha_p P_T}{N_0 \tilde{D}^{(1)}}$ and $C^{(2)}(\alpha_p^{(2)}) \approx \frac{\lambda_{\max}^{(2)}}{\ln 2} \frac{(1-\alpha_p) P_T}{N_0 \tilde{D}^{(2)}}$. Hence, the outer bound $\mathcal{R}_{out}$ reduces in the two-dimensional plane defined by the axes $R^{(1)}$ and $R^{(2)}$ to a triangle bounded by the line

$$R^{(2)} = -\frac{\lambda_{\max}^{(2)}}{\lambda_{\max}^{(1)}} \frac{\tilde{D}^{(1)}}{\tilde{D}^{(2)}} R^{(1)} + \frac{\lambda_{\max}^{(2)}}{\ln 2} \frac{P_T}{N_0 \tilde{D}^{(2)}} \quad (4.28)$$

in the positive quadrant.

We now find the maximum sum rate C_{out}^{sum} for this triangular region $\mathcal{R}_{out}$ and the associated power allocation $(\alpha_{p;s}^{(1)}, \alpha_{p;s}^{(2)})$. Theorem 4.2.1 leads to the following results. If $-\frac{\lambda_{max}^{(2)} \check{D}^{(1)}}{\lambda_{max}^{(1)} \check{D}^{(2)}} < -1$ then C_{out}^{sum} is achieved by allocating the total power to user 2 only, that is, $(\alpha_{p;s}^{(1)}, \alpha_{p;s}^{(2)}) = (0, 1)$ and $C_{out}^{sum} = C^{(2)}(1)$. If $-\frac{\lambda_{max}^{(2)} \check{D}^{(1)}}{\lambda_{max}^{(1)} \check{D}^{(2)}} > -1$ then C_{out}^{sum} is achieved by allocating the total power to user 1 only, that is, $(\alpha_{p;s}^{(1)}, \alpha_{p;s}^{(2)}) = (1, 0)$ and $C_{out}^{sum} = C^{(1)}(1)$. Otherwise, if $-\frac{\lambda_{max}^{(2)} \check{D}^{(1)}}{\lambda_{max}^{(1)} \check{D}^{(2)}} = -1$ then C_{out}^{sum} is achieved by an arbitrary power allocation $(\alpha_{p;s}^{(1)}, \alpha_{p;s}^{(2)})$ among users and $C_{out}^{sum} = C^{(1)}(1) = C^{(2)}(1)$. Hence, the power allocation $(\alpha_{p;s}^{(1)}, \alpha_{p;s}^{(2)})$ associated with C_{out}^{sum} depends on the users spatial channel parameters and space-time modulation formats through $\frac{\lambda_{max}^{(2)}}{\lambda_{max}^{(1)}}$ and on the rate normalization constants through $\frac{\check{D}^{(1)}}{\check{D}^{(2)}}$. If $\check{D}^{(1)} = \check{D}^{(2)}$ (which occurs if $\check{D}^{(1)} = \check{D}^{(2)} = D_T$ or if $\check{D}^{(k)} = D^{(k)}$ for $k = 1, 2$ with $D_p^{(1)} = D_p^{(2)}$), then $(\alpha_{p;s}^{(1)}, \alpha_{p;s}^{(2)})$ depends only on $\frac{\lambda_{max}^{(2)}}{\lambda_{max}^{(1)}}$ and C_{out}^{sum} is obtained by allocating the total transmit power to the user that has the larger $\lambda_{max}^{(k)}$.

The boundary of $\mathcal{R}_{out}$ coincides with that of the inner bound $\mathcal{R}_{in}$ at the extreme points $(0, C^{(2)}(1))$ and $(C^{(1)}(1), 0)$ associated with $(\alpha_p^{(1)}, \alpha_p^{(2)}) = (0, 1)$ and $(\alpha_p^{(1)}, \alpha_p^{(2)}) = (1, 0)$, respectively. For all other pairs $(\alpha_p^{(1)}, \alpha_p^{(2)})$, we consider the separation between the corresponding boundary points of $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$ and we use the following metric to measure this separation

$$\Delta^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)}) = C^{(k)}(\alpha_p^{(k)}) - \check{R}^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)}) \text{ bps/Hz} \quad (4.29)$$

for $k = 1, 2$. As such, for a given power allocation $(\alpha_p^{(1)}, \alpha_p^{(2)})$, $\Delta^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ represents the difference in rates for user 2 when considering the rate pairs $(C^{(1)}(\alpha_p^{(1)}), \check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}))$ of $\mathcal{R}_{in}$ and $(C^{(1)}(\alpha_p^{(1)}), C^{(2)}(\alpha_p^{(2)}))$ of $\mathcal{R}_{out}$. Similarly, $\Delta^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ represents the difference in rates for user 1 when considering the rate pairs $(\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}), C^{(2)}(\alpha_p^{(2)}))$ of $\mathcal{R}_{in}$ and $(C^{(1)}(\alpha_p^{(1)}), C^{(2)}(\alpha_p^{(2)}))$ of $\mathcal{R}_{out}$.

Let us evaluate the separation (4.29) at low SNR. Consider the rate $\check{R}^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ for user k , $k = 1, 2$, and fix $(\alpha_p^{(1)}, \alpha_p^{(2)})$. From Section 4.4, we know that $\check{R}^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ is achieved by water-filling over the eigenmodes of $[\mathbf{H}^{(k)}]^\dagger \cdot [\Psi^{(k)}]^{-1} \cdot \mathbf{H}^{(k)}$ where $\Psi^{(k)}$ is given by (4.17) for $k = 1, 2$. Let us first provide an approximation for the eigenvalues of $[\mathbf{H}^{(k)}]^\dagger \cdot [\Psi^{(k)}]^{-1} \cdot \mathbf{H}^{(k)}$ in the low SNR regime. Let $\{\mathbf{m}_q^{(k)}\}_{q=1}^{\check{D}^{(k)}}$ denote the eigenvectors of $[\mathbf{H}^{(k)}]^\dagger \cdot \mathbf{H}^{(k)}$ associated with the eigenvalues $\{\lambda_q^{(k)}\}_{q=1}^{\check{D}^{(k)}}$ for user k such that $\|\mathbf{m}_q^{(k)}\| = 1$, $k = 1, 2$. We start by assuming $k = 1$ and we consider $\Psi^{(1)}$ of (4.17a). Since only the $n(\lambda_{max}^{(2)})D_G^{(2)}$ strongest eigenmodes associated with $\lambda_{max}^{(2)}$ are active in the water-filling scheme associated with $C^{(2)}(\alpha_p^{(2)})$, the water-filling constant for user 2 can be written as in

(3.11), i.e.

$$\mu^{(2)}(\alpha_p^{(2)}) = (\lambda_{\max}^{(2)})^{-1} + \frac{N_G^{(2)} \alpha_p^{(2)} P_T}{n(\lambda_{\max}^{(2)}) N_0 D^{(2)}}$$

which, using (4.15), results in

$$\Sigma_{\max}^{(2)} = \left(\frac{N_G^{(2)} \alpha_p^{(2)} P_T}{n(\lambda_{\max}^{(2)}) D^{(2)}} \right) \text{diag}_{j=1}^{N_G^{(2)}} \left[\{ [\mathbf{M}_j^{(2)}]^\dagger \cdot \mathbf{K}_j^{(2)} \cdot \mathbf{M}_j^{(2)} \} \otimes \mathbf{I}_{D_G^{(2)}} \right] \quad (4.30)$$

where $\mathbf{K}_j^{(2)}$ is an $n_j^{(2)} \times n_j^{(2)}$ diagonal matrix defined as $[\mathbf{K}_j^{(2)}]_{l,l} = 1$ if $\lambda_{j,l}^{(2)} = \lambda_{\max}^{(2)}$, and $[\mathbf{K}_j^{(2)}]_{l,l} = 0$ else. Hence, $\mathbf{K}_j^{(2)}$ is Hermitian and $\mathbf{K}_j^{(2)} \cdot \mathbf{K}_j^{(2)} = \mathbf{K}_j^{(2)}$. Consider the spectral norm of the matrix $\frac{1}{N_0} \mathbf{H}^{(1,2)} \cdot \Sigma_{\max}^{(2)} \cdot [\mathbf{H}^{(1,2)}]^\dagger$. We have

$$\begin{aligned} \left\| \frac{1}{N_0} \mathbf{H}^{(1,2)} \cdot \Sigma_{\max}^{(2)} \cdot [\mathbf{H}^{(1,2)}]^\dagger \right\| &\stackrel{(a)}{\leq} \frac{1}{N_0} \|\mathbf{H}^{(1,2)}\| \|\Sigma_{\max}^{(2)}\| \|\mathbf{H}^{(1,2)}\|^\dagger \\ &\stackrel{(b)}{=} \left(\frac{N_G^{(2)} \alpha_p^{(2)} P_T}{n(\lambda_{\max}^{(2)}) N_0 D^{(2)}} \right) \|\mathbf{H}^{(1,2)}\| \|\text{diag}_{j=1}^{N_G^{(2)}} [\{ [\mathbf{M}_j^{(2)}]^\dagger \cdot \mathbf{K}_j^{(2)} \cdot \mathbf{M}_j^{(2)} \} \otimes \mathbf{I}_{D_G^{(2)}}]\| \|\mathbf{H}^{(1,2)}\|^\dagger \end{aligned}$$

where (a) is due to the fact that the spectral norm is submultiplicative and homogeneous [105, p. 290] and (b) is due to (4.30). From (2.42), we have $\mathbf{H}^{(1,2)} = [\mathbf{C}^{(1)} \otimes \mathbf{I}_{D^{(1)}}] \cdot [\mathbf{I}_{N_T} \otimes \mathbf{B}^{(1,2)}] \cdot [\mathbf{A}^{(2)}]^T$ where $\mathbf{B}^{(1,2)}$ is defined in Section 2.3. Thus, using the fact that for an arbitrary $n \times n$ matrix $\mathbf{U}$ we have $\|\mathbf{U}\| = \max_{l=1,\dots,n} \sqrt{\lambda_l(\mathbf{U}^\dagger \cdot \mathbf{U})}$ from [105, p. 295], we can write

$$\|\mathbf{H}^{(1,2)}\|^\dagger = \|\mathbf{H}^{(1,2)}\| \leq \|[\mathbf{C}^{(1)} \otimes \mathbf{I}_{D^{(1)}}]\| \|\mathbf{I}_{N_T} \otimes \mathbf{B}^{(1,2)}\| \|\mathbf{A}^{(2)}\|^T \stackrel{(a)}{=} \sqrt{\lambda_{\max,A}^{(1)}} \quad (4.31)$$

where, using (C.3), (C.20) and Theorem C.0.1, (a) is due to $\|[\mathbf{C}^{(1)} \otimes \mathbf{I}_{D^{(1)}}]\| = \sqrt{\lambda_{\max,A}^{(1)}}$ (where $\lambda_{\max,A}^{(1)}$ is the maximum eigenvalue of $[\mathbf{C}^{(1)}]^\dagger \cdot \mathbf{C}^{(1)}$), $\|\mathbf{A}^{(2)}\|^T = 1$ since $\mathbf{A}^{(2)} \cdot [\mathbf{A}^{(2)}]^T = \mathbf{I}_{\bar{D}^{(2)}}$, and $\|\mathbf{I}_{N_T} \otimes \mathbf{B}^{(1,2)}\| = 1$ since $[\mathbf{B}^{(1,2)}]^T \cdot \mathbf{B}^{(1,2)}$ is $D^{(2)} \times D^{(2)}$ diagonal matrix with diagonal elements equal to 0 or 1. Since $\mathbf{M}_j^{(2)}$ is unitary and $\mathbf{K}_j^{(2)}$ is diagonal with $\mathbf{K}_j^{(2)} \cdot \mathbf{K}_j^{(2)} = \mathbf{K}_j^{(2)}$ for all j , one can show that $\|\text{diag}_{j=1}^{N_G^{(2)}} [\{ [\mathbf{M}_j^{(2)}]^\dagger \cdot \mathbf{K}_j^{(2)} \cdot \mathbf{M}_j^{(2)} \} \otimes \mathbf{I}_{D_G^{(2)}}]\| = \max_{l=1,\dots,\bar{D}^{(2)}} \sqrt{\lambda_l(\text{diag}_{j=1}^{N_G^{(2)}} [\{ [\mathbf{M}_j^{(2)}]^\dagger \cdot \mathbf{K}_j^{(2)} \cdot \mathbf{M}_j^{(2)} \} \otimes \mathbf{I}_{D_G^{(2)}}])} = 1$ with the last equality due to (C.3), (C.20) and Theorem C.0.1. It follows that

$$\left\| \frac{1}{N_0} \mathbf{H}^{(1,2)} \cdot \Sigma_{\max}^{(2)} \cdot [\mathbf{H}^{(1,2)}]^\dagger \right\| \leq \left(\frac{N_G^{(2)} \alpha_p^{(2)} P_T}{n(\lambda_{\max}^{(2)}) N_0 D^{(2)}} \right) \lambda_{\max,A}^{(1)} \stackrel{(a)}{<} 1 \quad (4.32)$$

with (a) due to $(P_T/N_0)\lambda_{\max;A}^{(1)} \ll 1$ at low SNR and to $\frac{N_G^{(2)}\alpha_p^{(2)}}{n(\lambda_{\max}^{(2)})D^{(2)}} = \frac{\alpha_p^{(2)}}{n(\lambda_{\max}^{(2)})D_G^{(2)}} \leq 1$. With (4.32) and using a first order approximation in the expansion

$$\left(\mathbf{I}_{L_t^{(1)}N_R^{(1)}D^{(1)}} + \frac{1}{N_0} \mathbf{H}^{(1,2)} \cdot \Sigma_{\max}^{(2)} \cdot [\mathbf{H}^{(1,2)}]^\dagger \right)^{-1} = \sum_{p=0}^{\infty} \left(-\frac{1}{N_0} \mathbf{H}^{(1,2)} \cdot \Sigma_{\max}^{(2)} \cdot [\mathbf{H}^{(1,2)}]^\dagger \right)^p$$

from [105, p. 301], where $\left(-\frac{1}{N_0} \mathbf{H}^{(1,2)} \cdot \Sigma_{\max}^{(2)} \cdot [\mathbf{H}^{(1,2)}]^\dagger \right)^p = \mathbf{I}_{L_t^{(1)}N_R^{(1)}D^{(1)}}$ for $p = 0$, equation (4.17a) leads to $[\Psi^{(1)}]^{-1} = \mathbf{I}_{L_t^{(1)}N_R^{(1)}D^{(1)}} - \frac{1}{N_0} \mathbf{H}^{(1,2)} \cdot \Sigma_{\max}^{(2)} \cdot [\mathbf{H}^{(1,2)}]^\dagger$. Using (4.30), this yields

$$\begin{aligned} [\mathbf{H}^{(1)}]^\dagger \cdot [\Psi^{(1)}]^{-1} \cdot \mathbf{H}^{(1)} &= [\mathbf{H}^{(1)}]^\dagger \cdot \mathbf{H}^{(1)} - \left(\frac{N_G^{(2)}\alpha_p^{(2)}P_T}{n(\lambda_{\max}^{(2)})N_0D^{(2)}} \right) \\ &\times [\mathbf{H}^{(1)}]^\dagger \cdot \mathbf{H}^{(1,2)} \cdot \text{diag}_{j=1}^{N_G^{(2)}} \left[\{ [\mathbf{M}_j^{(2)}]^\dagger \cdot \mathbf{K}_j^{(2)} \cdot \mathbf{M}_j^{(2)} \} \otimes \mathbf{I}_{D_G^{(2)}} \right] \cdot [\mathbf{H}^{(1,2)}]^\dagger \cdot \mathbf{H}^{(1)} \end{aligned} \quad (4.33)$$

By Lemma C.0.2 and for sufficiently small P_T/N_0 , the eigenvalues of $[\mathbf{H}^{(1)}]^\dagger \cdot [\Psi^{(1)}]^{-1} \cdot \mathbf{H}^{(1)}$ can be approximated as

$$\check{\lambda}_q^{(1)} = \lambda_q^{(1)} - \frac{P_T}{N_0} \check{\delta}_q^{(1)} \quad (4.34)$$

where, for $q = 1, \dots, \tilde{D}^{(1)}$,

$$\begin{aligned} \check{\delta}_q^{(1)} &= \left(\frac{N_G^{(2)}\alpha_p^{(2)}}{n(\lambda_{\max}^{(2)})D^{(2)}} \right) [\mathbf{m}_q^{(1)}]^\dagger \cdot [\mathbf{H}^{(1)}]^\dagger \cdot \mathbf{H}^{(1,2)} \\ &\cdot \text{diag}_{j=1}^{N_G^{(2)}} \left[\{ [\mathbf{M}_j^{(2)}]^\dagger \cdot \mathbf{K}_j^{(2)} \cdot \mathbf{M}_j^{(2)} \} \otimes \mathbf{I}_{D_G^{(2)}} \right] \cdot [\mathbf{H}^{(1,2)}]^\dagger \cdot \mathbf{H}^{(1)} \cdot \mathbf{m}_q^{(1)} \end{aligned} \quad (4.35)$$

This is equivalent to $\check{\delta}_q^{(1)} = \left(\frac{N_G^{(2)}\alpha_p^{(2)}}{n(\lambda_{\max}^{(2)})D^{(2)}} \right) \| \text{diag}_{j=1}^{N_G^{(2)}} [(\mathbf{K}_j^{(2)} \cdot \mathbf{M}_j^{(2)}) \otimes \mathbf{I}_{D_G^{(2)}}] \cdot [\mathbf{H}^{(1,2)}]^\dagger \cdot \mathbf{H}^{(1)} \cdot \mathbf{m}_q^{(1)} \|^2$, using (C.3) and $\mathbf{K}_j^{(2)} \cdot \mathbf{K}_j^{(2)} = \mathbf{K}_j^{(2)}$. Thus, $\check{\delta}_q^{(1)} \geq 0$ and $(P_T/N_0)\check{\lambda}_q^{(1)} \leq (P_T/N_0)\lambda_q^{(1)} \ll 1$ at low SNR. Furthermore, since the matrix spectral norm is the norm induced by the vector 2-norm [105, p. 296], we can write using [105, Theorem 5.6.2] and the submultiplicativity property of the spectral norm [105, p. 290]

$$\check{\delta}_q^{(1)} \leq \left(\frac{N_G^{(2)}\alpha_p^{(2)}}{n(\lambda_{\max}^{(2)})D^{(2)}} \right) \| \text{diag}_{j=1}^{N_G^{(2)}} [(\mathbf{K}_j^{(2)} \cdot \mathbf{M}_j^{(2)}) \otimes \mathbf{I}_{D_G^{(2)}}] \|^2 \| [\mathbf{H}^{(1,2)}]^\dagger \|^2 \| \mathbf{H}^{(1)} \|^2 \| \mathbf{m}_q^{(1)} \|^2$$

Since $\mathbf{M}_j^{(2)}$ is unitary and since $\mathbf{K}_j^{(2)}$ is diagonal with $[\mathbf{K}_j^{(2)}]_{l,l} = 0$ or $[\mathbf{K}_j^{(2)}]_{l,l} = 1$ such that $\mathbf{K}_j^{(2)} \cdot \mathbf{K}_j^{(2)} = \mathbf{K}_j^{(2)}$, one can easily show that $\| \text{diag}_{j=1}^{N_G^{(2)}} [(\mathbf{K}_j^{(2)} \cdot \mathbf{M}_j^{(2)}) \otimes \mathbf{I}_{D_G^{(2)}}] \| =$

$\max_{l=1, \dots, \tilde{D}^{(2)}} \sqrt{\lambda_l(\text{diag}_{j=1}^{N_G^{(2)}} [\{[\mathbf{M}_j^{(2)}]^\dagger \cdot \mathbf{K}_j^{(2)} \cdot \mathbf{M}_j^{(2)}\} \otimes \mathbf{I}_{D_G^{(2)}}])} = 1$ with the last equality due to (C.3), (C.20) and Theorem C.0.1. Moreover, from equation (2.41), we can write $\|\mathbf{H}^{(1)}\| \leq \|[\mathbf{C}^{(1)} \otimes \mathbf{I}_{D^{(1)}}]\| \|\mathbf{A}^{(1)}\|^T$, becoming $\|\mathbf{H}^{(1)}\| \leq \sqrt{\lambda_{\max;A}^{(1)}}$ since $\|[\mathbf{C}^{(1)} \otimes \mathbf{I}_{D^{(1)}}]\| = \sqrt{\lambda_{\max;A}^{(1)}}$ and $\|\mathbf{A}^{(1)}\|^T = 1$ since $\mathbf{A}^{(1)} \cdot [\mathbf{A}^{(1)}]^T = \mathbf{I}_{\tilde{D}^{(1)}}$. With (4.31) and $\|\mathbf{m}_q^{(k)}\| = 1$, we finally obtain

$$\check{\delta}_q^{(1)} \leq \left(\frac{N_G^{(2)} \alpha_p^{(2)}}{n(\lambda_{\max}^{(2)}) D^{(2)}} \right) [\lambda_{\max;A}^{(1)}]^2 \quad (4.36)$$

for all $q = 1, \dots, \tilde{D}^{(1)}$. Following the same steps as above, an approximation similar to that of (4.34) can be obtained for the eigenvalues $\check{\lambda}_q^{(2)}$ of $[\mathbf{H}^{(2)}]^\dagger \cdot [\Psi^{(2)}]^{-1} \cdot \mathbf{H}^{(2)}$, that is, $\check{\lambda}_q^{(2)} = \lambda_q^{(2)} - \frac{P_T}{N_0} \check{\delta}_q^{(2)}$ where, for $q = 1, \dots, \tilde{D}^{(2)}$, $\check{\delta}_q^{(2)}$ satisfies (4.35) and (4.36) with the user indexes 1 and 2 switched. Furthermore, $\check{\delta}_q^{(2)} \geq 0$ and $(P_T/N_0) \check{\lambda}_q^{(2)} \leq (P_T/N_0) \lambda_q^{(2)} \ll 1$ at low SNR. For $k = 1, 2$, we define $\check{\lambda}_{\max}^{(k)} = \max\{\check{\lambda}_q^{(k)}\}_{q=1}^{\tilde{D}^{(k)}}$ with multiplicity $n(\check{\lambda}_{\max}^{(k)})$. For sufficiently small P_T/N_0 and with decreasingly ordered eigenvalues $\lambda_q^{(1)}$ and $\check{\lambda}_q^{(1)}$, we can assume that $\lambda_{\max}^{(k)} = \lambda_1^{(k)}$ and $\check{\lambda}_{\max}^{(k)} = \check{\lambda}_1^{(k)}$, $k = 1, 2$. It follows that $\check{\lambda}_{\max}^{(k)} = \lambda_{\max}^{(k)} - \frac{P_T}{N_0} \check{\delta}_1^{(k)}$ and,

$$\lambda_{\max}^{(1)} - \check{\lambda}_{\max}^{(1)} \leq \frac{P_T}{N_0} \left(\frac{N_G^{(2)} \alpha_p^{(2)}}{n(\lambda_{\max}^{(2)}) D^{(2)}} \right) [\lambda_{\max;A}^{(1)}]^2 \quad (4.37a)$$

$$\lambda_{\max}^{(2)} - \check{\lambda}_{\max}^{(2)} \leq \frac{P_T}{N_0} \left(\frac{N_G^{(1)} \alpha_p^{(1)}}{n(\lambda_{\max}^{(1)}) D^{(1)}} \right) [\lambda_{\max;A}^{(2)}]^2 \quad (4.37b)$$

with (4.37a) due to (4.36) for user 1, and (4.37b) similarly obtained for user 2 by switching the user indexes 1 and 2. With only the $n(\check{\lambda}_{\max}^{(k)})$ strongest eigenmodes active at low SNR, (4.23) reduces to $\check{R}^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)}) = \frac{n(\check{\lambda}_{\max}^{(k)})}{\check{D}^{(k)}} \log(\check{\mu}^{(k)}(\alpha_p^{(k)}) \check{\lambda}_{\max}^{(k)})$ with $\check{\mu}^{(k)}(\alpha_p^{(k)}) = (\check{\lambda}_{\max}^{(k)})^{-1} + \frac{\alpha_p^{(k)} P_T}{n(\check{\lambda}_{\max}^{(k)}) N_0}$, $k = 1, 2$. Equivalently, we have

$$\check{R}^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)}) = \frac{n(\check{\lambda}_{\max}^{(k)})}{\check{D}^{(k)}} \log \left(1 + \check{\lambda}_{\max}^{(k)} \frac{\alpha_p^{(k)} P_T}{n(\check{\lambda}_{\max}^{(k)}) N_0} \right) \quad (4.38)$$

with $(P_T/N_0) \check{\lambda}_{\max}^{(k)} \ll 1$ and hence, $\check{\lambda}_{\max}^{(k)} \frac{\alpha_p^{(k)} P_T}{n(\check{\lambda}_{\max}^{(k)}) N_0} \ll 1$ since $\frac{\alpha_p^{(k)}}{n(\check{\lambda}_{\max}^{(k)})} \leq 1$. Now using (4.26), (4.38) and the approximation $\ln(1+x) \approx x$, (4.29) reduces at low SNR to

$$\Delta^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)}) \approx (\lambda_{\max}^{(k)} - \check{\lambda}_{\max}^{(k)}) \frac{\alpha_p^{(k)} P_T}{N_0 \check{D}^{(k)} \ln 2} \text{ bps/Hz} \quad (4.39)$$

As $(P_T/N_0) \rightarrow 0$, (4.37) and (4.39) show that $\Delta^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)}) \rightarrow 0$ for $k = 1, 2$. Thus, the rate pairs $(C^{(1)}(\alpha_p^{(1)}), \check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}))$ and $(\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}), C^{(2)}(\alpha_p^{(2)}))$ converge to $(C^{(1)}(\alpha_p^{(1)}), C^{(2)}(\alpha_p^{(2)}))$ as $(P_T/N_0) \rightarrow 0$, for all $(\alpha_p^{(1)}, \alpha_p^{(2)})$ with $\alpha_p^{(1)} \in (0, 1)$ and $\alpha_p^{(2)} = 1 - \alpha_p^{(1)}$. Since $\mathcal{R}_{out}$ is convex, one can easily see using (4.25) that $\mathcal{R}_{in} \rightarrow \mathcal{R}_{out}$ as $(P_T/N_0) \rightarrow 0$. Furthermore, since $\mathcal{R}_{out}$ is an outer bound and $\mathcal{R}_{in}$ is an inner bound for $\mathcal{R}_{DPC}$ and for $\mathcal{C}$, their convergence implies that as $(P_T/N_0) \rightarrow 0$ both $\mathcal{R}_{DPC}$ and $\mathcal{C}$ converge to the triangle bounded by the line (4.28). Consequently, in the low SNR regime $\mathcal{R}_{DPC}$ converges towards $\mathcal{C}$, thus, making DPC a capacity achieving strategy in this regime. By observing that the limiting triangular region $\mathcal{R}_{DPC}$ is similar to the TDMA rate region considered in [69], our result extends [69, Theorem 5] in the sense that it makes DPC and TDMA asymptotically equivalent at low SNR not only in terms of the sum capacity (as shown in [69, Theorem 5]) but rather in terms of the whole capacity region. The reason behind such asymptotical equivalence is the following. Since $\mathcal{R}_{DPC}$ converges to $\mathcal{R}_{out}$ any point on the boundary of $\mathcal{R}_{DPC}$ with power allocation $(\alpha_p^{(1)}, \alpha_p^{(2)})$ converges to $(C^{(1)}(\alpha_p^{(1)}), C^{(2)}(\alpha_p^{(2)})) = (\alpha_p^{(1)}C^{(1)}(1), (1 - \alpha_p^{(1)})C^{(2)}(1))$ with the last equality due to (4.27). The latter rate pair can be obtained by time-sharing between two codes with rate pairs $(C^{(1)}(1), 0)$ and $(0, C^{(2)}(1))$, respectively, and such that the first code is used for a fraction of time $\alpha_p^{(1)}$ while the second code is used for the remaining fraction of time $(1 - \alpha_p^{(1)})$. Hence, implementing DPC or using this time-sharing method leads to the same limiting triangular capacity region. Note that the coding used for $\mathcal{R}_{out}$ cannot be considered as capacity achieving because it is not realizable in practice. Moreover, $\mathcal{C}$ exhibits similar properties to those of $\mathcal{R}_{out}$. Hence, similarly to $\mathcal{R}_{out}$, $\mathcal{C}$ is largest, intermediate and smallest when the users employ ATA, POTA and OTA, respectively. Furthermore, the power allocation $(\alpha_p^{(1)}, \alpha_p^{(2)})$ associated with the sum capacity C^{sum} (with C^{sum} converging towards its upper bound C_{out}^{sum}) is determined by $\frac{\lambda_{\max}^{(2)} \check{D}^{(1)}}{\lambda_{\max}^{(1)} \check{D}^{(2)}}$ (see the discussion presented above for C_{out}^{sum}). Also, the effect of varying the number of proper and shared signal dimensions on the capacity region $\mathcal{C}$ can be investigated at low SNR using $\mathcal{R}_{out}$ for which all the interference is canceled. When D_c increases or when $D_p^{(1)}$ and $D_p^{(2)}$ increase $\check{D}^{(k)}$ increases (in both cases $\check{D}^{(k)} = D^{(k)}$ or $\check{D}^{(k)} = D_T$), and the points with coordinates $(0, \frac{\lambda_{\max}^{(2)}}{\ln 2} \frac{P_T}{N_0 \check{D}^{(2)}})$ and $(\frac{\lambda_{\max}^{(1)}}{\ln 2} \frac{P_T}{N_0 \check{D}^{(1)}}, 0)$ become closer to the origin in the two-dimensional plane defined by the axes $R^{(1)}$ and $R^{(2)}$. Therefore, the limiting triangular region $\mathcal{R}_{out}$ bounded by the line (4.28) becomes smaller and hence, $\mathcal{C}$ becomes smaller when more proper and/or shared signal dimensions are added. This behavior of $\mathcal{C}$ holds with all the transmission rates taken in bps/Hz. In bits/channel-use units, the limiting asymptotical capacity region is the triangle

bounded by the line joining the points $(0, \frac{\lambda_{\max}^{(2)} P_T}{\ln 2 N_0})$ and $(\frac{\lambda_{\max}^{(1)} P_T}{\ln 2 N_0}, 0)$. The latter region is independent of D_c , which can be explained by the fact that at low SNR the effect of the interference associated with the D_c shared signal dimensions becomes insignificant compared to the effect of the noise.

4.4.2 The high SNR regime

In this section, the high SNR regime refers to the situation where P_T/N_0 is assumed to be very large. We start by investigating the behavior of the outer bound $\mathcal{R}_{out}$. If $\alpha_p^{(k)} \rightarrow 0$, then $C^{(k)}(\alpha_p^{(k)}) \rightarrow 0$ because the power allocated to user k converges to 0. Otherwise, assuming $0 < \alpha_p^{(k)}$ with $(\alpha_p^{(k)} P_T)/N_0$ very large, all channel eigenmodes are active for user k , and as in (3.16) the single-user capacity (4.14) can be written as

$$C^{(k)}(\alpha_p^{(k)}) = \frac{A^{(k)} D_G^{(k)}}{\check{D}^{(k)}} + \frac{r_T^{(k)} D_G^{(k)}}{\check{D}^{(k)}} \log \left(\frac{N_G^{(k)} \alpha_p^{(k)} P_T}{r_T^{(k)} N_0 D^{(k)}} + c^{(k)} \right) \text{ bps/Hz} \quad (4.40)$$

where $A^{(k)} = \sum_{i=1}^{N_G^{(k)}} \sum_{l=1}^{r_i^{(k)}} \log(\lambda_{i,l}^{(k)})$ and $c^{(k)} = \frac{1}{r_T^{(k)}} \sum_{i=1}^{N_G^{(k)}} \sum_{l=1}^{r_i^{(k)}} (\lambda_{i,l}^{(k)})^{-1}$. The associated water-filling constant is given by

$$\mu^{(k)}(\alpha_p^{(k)}) = \frac{N_G^{(k)} \alpha_p^{(k)} P_T}{r_T^{(k)} N_0 D^{(k)}} + c^{(k)} \stackrel{(a)}{\approx} \frac{N_G^{(k)} \alpha_p^{(k)} P_T}{r_T^{(k)} N_0 D^{(k)}} \quad (4.41)$$

where (a) is due to $\frac{N_G^{(k)} \alpha_p^{(k)} P_T}{r_T^{(k)} N_0 D^{(k)}} \gg c^{(k)}$. From (4.40), the spatial multiplexing gain corresponding to $C^{(k)}(\alpha_p^{(k)})$ with $\alpha_p^{(k)} \neq 0$ is given by

$$G_{out}^{(k)} \triangleq \lim_{(P_T/N_0) \rightarrow \infty} \frac{C^{(k)}(\alpha_p^{(k)})}{\log(P_T/N_0)} = \frac{r_T^{(k)} D_G^{(k)}}{\check{D}^{(k)}} \quad (4.42)$$

Using the points V_0 , $\tilde{V}$, and V_1 with the coordinates of Table 4.1, the following theorem considers the behavior of $\mathcal{R}_{out}$ in the high SNR regime.

Theorem 4.4.1 *As $(P_T/N_0) \rightarrow \infty$, $\mathcal{R}_{out}$ converges towards the rectangle defined by the origin, V_0 , $\tilde{V}$ and V_1 .*

Point	$R^{(1)}$ -coordinate	$R^{(2)}$ -coordinate
V_0	0	$\frac{r_T^{(2)} D_G^{(2)}}{\check{D}^{(2)}} \log(P_T/N_0)$
$\tilde{V}$	$\frac{r_T^{(1)} D_G^{(1)}}{\check{D}^{(1)}} \log(P_T/N_0)$	$\frac{r_T^{(2)} D_G^{(2)}}{\check{D}^{(2)}} \log(P_T/N_0)$
V_1	$\frac{r_T^{(1)} D_G^{(1)}}{\check{D}^{(1)}} \log(P_T/N_0)$	0

Table 4.1 Corner points of the limiting outer bound $\mathcal{R}_{out}$ for spread spectrum like BCs

We prove Theorem 4.4.1 in Appendix B. From Section 4.2, the outer bound $\mathcal{R}_{out}$ (similarly to the capacity region of orthogonally multiplexed MIMO BCs) is associated with a broadcast system consisting of two non-interfering single-user MIMO channels that are only coupled through a total power constraint. As $P_T/N_0 \rightarrow \infty$, $(\alpha_p^{(k)} P_T)/N_0 \rightarrow \infty$ for $\alpha_p^{(k)} > 0$, $k = 1, 2$, and this total power constraint does not hold anymore. This makes the asymptotical outer bound of Theorem 4.4.1 similar to the rectangular capacity region of the orthogonal broadcast channels of [125]. By Theorem 4.2.1, C_{out}^{sum} is asymptotically achieved only at the upper right corner $\tilde{V}$ of the limiting rectangle, which is the limiting point of all the boundary points $(C^{(1)}(\alpha_p^{(1)}), C^{(2)}(\alpha_p^{(2)}))$ of $\mathcal{R}_{out}$ whose power allocation satisfies $0 < \alpha_p^{(1)}, \alpha_p^{(2)} < 1$ and $\alpha_p^{(1)} + \alpha_p^{(2)} = 1$. Thus, the power allocation $(\alpha_{p;s}^{(1)}, \alpha_{p;s}^{(2)})$ associated with C_{out}^{sum} satisfies $0 < \alpha_{p;s}^{(1)}, \alpha_{p;s}^{(2)} < 1$ and $\alpha_{p;s}^{(1)} + \alpha_{p;s}^{(2)} = 1$. We evaluate $(\alpha_{p;s}^{(1)}, \alpha_{p;s}^{(2)})$ as follows. Using $\frac{N_G^{(k)} \alpha_p^{(k)} P_T}{r_T^{(k)} N_0 D^{(k)}} \gg c^{(k)}$, (4.40) can be written as

$$C^{(k)}(\alpha_p^{(k)}) = \frac{A^{(k)} D_G^{(k)}}{\check{D}^{(k)}} + \frac{r_T^{(k)} D_G^{(k)}}{\check{D}^{(k)}} \log \left(\frac{N_G^{(k)} \alpha_p^{(k)} P_T}{r_T^{(k)} N_0 D^{(k)}} \right) \text{ bps/Hz} \quad (4.43)$$

Based on this equation, we now evaluate the derivative $\frac{dC^{(2)}(\alpha_p^{(2)})}{dC^{(1)}(\alpha_p^{(1)})} = \frac{dC^{(2)}(\alpha_p^{(2)})}{d\alpha_p} \frac{d\alpha_p}{dC^{(1)}(\alpha_p^{(1)})}$ using $\alpha_p^{(1)} = \alpha_p$ and $\alpha_p^{(2)} = 1 - \alpha_p$ from (4.8). Hence, we obtain

$$\frac{dC^{(2)}(\alpha_p^{(2)})}{dC^{(1)}(\alpha_p^{(1)})} = -\frac{\check{D}^{(1)}}{r_T^{(1)} D_G^{(1)}} \frac{r_T^{(2)} D_G^{(2)}}{\check{D}^{(2)}} \frac{\alpha_p}{(1 - \alpha_p)} \quad (4.44)$$

which exists for all $\alpha_p \in (0, 1)$. It follows from [123, p. 199] that for any $\alpha_p^{(1)}, \alpha_p^{(2)} \in (0, 1)$ with $\alpha_p^{(1)} + \alpha_p^{(2)} = 1$, the support line to the boundary of $\mathcal{R}_{out}$ at $(C^{(1)}(\alpha_p^{(1)}), C^{(2)}(\alpha_p^{(2)}))$ is unique with a slope given by (4.44). By Theorem 4.2.1, $\alpha_{p;s}^{(1)}$ can be obtained at high SNR

by solving $-\frac{\check{D}^{(1)}}{r_T^{(1)} D_G^{(1)}} \frac{r_T^{(2)} D_G^{(2)}}{\check{D}^{(2)}} \frac{\alpha_{p;s}^{(1)}}{1-\alpha_{p;s}^{(1)}} = -1$. This leads to

$$\alpha_{p;s}^{(1)} = \frac{\check{D}^{(2)}}{r_T^{(2)} D_G^{(2)}} \frac{r_T^{(1)} D_G^{(1)}}{\check{D}^{(1)}} \left[1 + \frac{\check{D}^{(2)}}{r_T^{(2)} D_G^{(2)}} \frac{r_T^{(1)} D_G^{(1)}}{\check{D}^{(1)}} \right]^{-1} = \left[1 + \frac{r_T^{(2)} D_G^{(2)}}{\check{D}^{(2)}} \frac{\check{D}^{(1)}}{r_T^{(1)} D_G^{(1)}} \right]^{-1} \quad (4.45)$$

The corresponding power allocation for user 2 can be obtained using $\alpha_{p;s}^{(2)} = 1 - \alpha_{p;s}^{(1)}$. Writing $\frac{r_T^{(2)} D_G^{(2)}}{\check{D}^{(2)}} \frac{\check{D}^{(1)}}{r_T^{(1)} D_G^{(1)}} = \frac{r_T^{(2)} N_G^{(1)}}{r_T^{(1)} N_G^{(2)}} \frac{D^{(2)} \check{D}^{(1)}}{\check{D}^{(2)} D^{(1)}}$ since $D_G^{(k)} = D^{(k)} / N_G^{(k)}$, the power allocation $(\alpha_{p;s}^{(1)}, \alpha_{p;s}^{(2)})$ associated with C_{out}^{sum} depends on the users spatial channel parameters and space-time modulation formats through $\frac{r_T^{(2)} N_G^{(1)}}{r_T^{(1)} N_G^{(2)}}$, and on the number of signal dimensions and rate normalization constants through $\frac{D^{(2)} \check{D}^{(1)}}{\check{D}^{(2)} D^{(1)}}$. If $\check{D}^{(k)} = D^{(k)}$, $k = 1, 2$, or if $\check{D}^{(1)} = \check{D}^{(2)} = D_T$ with $D_p^{(1)} = D_p^{(2)}$ then (4.45) becomes $\alpha_{p;s}^{(1)} = \left[1 + \frac{r_T^{(2)} N_G^{(1)}}{r_T^{(1)} N_G^{(2)}} \right]^{-1}$ in which case $(\alpha_{p;s}^{(1)}, \alpha_{p;s}^{(2)})$ is independent of the number of signal dimensions allocated to the users. If $\check{D}^{(1)} = \check{D}^{(2)} = D_T$ with $D_p^{(1)} \neq D_p^{(2)}$, then (4.45) becomes $\alpha_{p;s}^{(1)} = \left[1 + \frac{r_T^{(2)} N_G^{(1)}}{r_T^{(1)} N_G^{(2)}} \frac{(D_p^{(2)} + D_c)}{(D_p^{(1)} + D_c)} \right]^{-1}$ which is a function of D_c . From Appendix B.2, the ratio $\frac{(D_p^{(2)} + D_c)}{(D_p^{(1)} + D_c)}$ is strictly increasing with D_c (as in Figure B.1a) if $D_p^{(1)} > D_p^{(2)}$ in which case $\alpha_{p;s}^{(1)}$ decreases strictly while $\alpha_{p;s}^{(2)}$ increases strictly with increasing D_c . However, if $D_p^{(1)} < D_p^{(2)}$ then $\frac{(D_p^{(2)} + D_c)}{(D_p^{(1)} + D_c)}$ is strictly decreasing with D_c (as in Figure B.1b) in which case $\alpha_{p;s}^{(1)}$ increases strictly while $\alpha_{p;s}^{(2)}$ decreases strictly with increasing D_c . Thus, if more shared signal dimensions are added the sum capacity upper bound C_{out}^{sum} is obtained by allocating more power to the user that has less proper signal dimensions.

We investigate now the behavior of the inner bound $\mathcal{R}_{in}$ and the separation between $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$ at high SNR. If $\alpha_p^{(k)} \rightarrow 0$, then $\check{R}^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)}) \rightarrow 0$ because the power allocated to user k converges to 0, $k = 1, 2$. Otherwise, if $0 < \alpha_p^{(k)}$ with $(\alpha_p^{(k)} P_T) / N_0$ assumed to be very large, $k = 1, 2$, we consider the pair $(C^{(1)}(\alpha_p^{(1)}), \check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}))$ with $\check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ given by (4.21). The pair $(\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}), C^{(2)}(\alpha_p^{(2)}))$ can be similarly investigated by switching the users indexes 1 and 2. The rate (4.21) is associated with the input covariance matrix $\check{\Sigma}^{(2)}$ obtained from (4.24) with $k = 2$. Thus, using (4.17b), (4.21) is equivalent to

$$\begin{aligned} \check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}) = & \frac{1}{\check{D}^{(2)}} \left\{ -\log \left| \frac{N_0}{P_T} \mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}} + \frac{1}{N_0} \mathbf{H}^{(2,1)} \cdot \left(\frac{N_0}{P_T} \Sigma_{\max}^{(1)} \right) \cdot [\mathbf{H}^{(2,1)}]^\dagger \right| \right. \\ & \left. + \log \left| \frac{N_0}{P_T} \mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}} + \frac{1}{N_0} \mathbf{H}^{(2,1)} \cdot \left(\frac{N_0}{P_T} \Sigma_{\max}^{(1)} \right) \cdot [\mathbf{H}^{(2,1)}]^\dagger + \frac{1}{N_0} \mathbf{H}^{(2)} \cdot \left(\frac{N_0}{P_T} \check{\Sigma}^{(2)} \right) \cdot [\mathbf{H}^{(2)}]^\dagger \right| \right\} \end{aligned}$$

which becomes

$$\check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}) = \frac{1}{\check{D}^{(2)}} \left\{ \log \left| \frac{N_0}{P_T} \mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}} + \check{\Lambda}^{(2)} \right| - \log \left| \frac{N_0}{P_T} \mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}} + \check{\Lambda}^{(2,1)} \right| \right\} \quad (4.46)$$

using the eigenvalue decompositions

$$\mathbf{H}^{(2,1)} \cdot \left(\frac{1}{P_T} \Sigma_{\max}^{(1)} \right) \cdot [\mathbf{H}^{(2,1)}]^\dagger + \mathbf{H}^{(2)} \cdot \left(\frac{1}{P_T} \check{\Sigma}^{(2)} \right) \cdot [\mathbf{H}^{(2)}]^\dagger = [\check{\mathbf{M}}^{(2)}]^\dagger \cdot \check{\Lambda}^{(2)} \cdot \check{\mathbf{M}}^{(2)} \quad (4.47)$$

$$\mathbf{H}^{(2,1)} \cdot \left(\frac{1}{P_T} \Sigma_{\max}^{(1)} \right) \cdot [\mathbf{H}^{(2,1)}]^\dagger = [\check{\mathbf{M}}^{(2,1)}]^\dagger \cdot \check{\Lambda}^{(2,1)} \cdot \check{\mathbf{M}}^{(2,1)} \quad (4.48)$$

where the matrices $\check{\mathbf{M}}^{(2,1)}, \check{\mathbf{M}}^{(2)}$ are unitary and $\check{\Lambda}^{(2,1)}, \check{\Lambda}^{(2)}$ are diagonal, and using identities (C.10) and (C.1). Since $Tr(\frac{1}{P_T} \Sigma_{\max}^{(1)}) = \alpha_p^{(1)}$ and $Tr(\frac{1}{P_T} \check{\Sigma}^{(2)}) = \alpha_p^{(2)}$, the matrices $\alpha_p^{(1)} \mathbf{I}_{\check{D}^{(1)}} - \frac{1}{P_T} \Sigma_{\max}^{(1)}$ and $\alpha_p^{(2)} \mathbf{I}_{\check{D}^{(2)}} - \frac{1}{P_T} \check{\Sigma}^{(2)}$ are Hermitian positive semidefinite by Lemma C.0.1. Thus, $\mathbf{H}^{(2,1)} \cdot [\alpha_p^{(1)} \mathbf{I}_{\check{D}^{(1)}} - \frac{1}{P_T} \Sigma_{\max}^{(1)}] \cdot [\mathbf{H}^{(2,1)}]^\dagger + \mathbf{H}^{(2)} \cdot [\alpha_p^{(2)} \mathbf{I}_{\check{D}^{(2)}} - \frac{1}{P_T} \check{\Sigma}^{(2)}] \cdot [\mathbf{H}^{(2)}]^\dagger$ as well as $\mathbf{H}^{(2,1)} \cdot [\alpha_p^{(1)} \mathbf{I}_{\check{D}^{(1)}} - \frac{1}{P_T} \Sigma_{\max}^{(1)}] \cdot [\mathbf{H}^{(2,1)}]^\dagger$ are also positive semidefinite from [105, p. 399]. Since the trace of a positive semidefinite matrix is non-negative [105, p. 398], it follows using (4.47)-(4.48) and (C.11) that

$$0 \leq Tr(\check{\Lambda}^{(2,1)}) \leq Tr\left(\alpha_p^{(1)} \mathbf{H}^{(2,1)} \cdot [\mathbf{H}^{(2,1)}]^\dagger\right) \quad (4.49a)$$

$$0 \leq Tr(\check{\Lambda}^{(2)}) \leq Tr\left(\alpha_p^{(1)} \mathbf{H}^{(2,1)} \cdot [\mathbf{H}^{(2,1)}]^\dagger + \alpha_p^{(2)} \mathbf{H}^{(2)} \cdot [\mathbf{H}^{(2)}]^\dagger\right) \quad (4.49b)$$

Let $r_{\max}^{(2)}$ and $r_{\max}^{(2,1)}$ be the ranks of $\mathbf{H}^{(2,1)} \cdot (\frac{1}{P_T} \Sigma_{\max}^{(1)}) \cdot [\mathbf{H}^{(2,1)}]^\dagger + \mathbf{H}^{(2)} \cdot (\frac{1}{P_T} \check{\Sigma}^{(2)}) \cdot [\mathbf{H}^{(2)}]^\dagger$ and $\mathbf{H}^{(2,1)} \cdot (\frac{1}{P_T} \Sigma_{\max}^{(1)}) \cdot [\mathbf{H}^{(2,1)}]^\dagger$, respectively. Since these matrices are Hermitian, their ranks are equal to the numbers of nonzero eigenvalues, respectively [105, p. 175]. Hence, by Corollary C.0.1, we must have $r_{\max}^{(2)} \geq r_{\max}^{(2,1)}$. The following lemma considers the impact of $(\alpha_p^{(1)}, \alpha_p^{(2)})$ and P_T/N_0 on $r_{\max}^{(2)}$ and $r_{\max}^{(2,1)}$.

Lemma 4.4.1 Assume $0 < \alpha_p^{(1)}$ and $0 < \alpha_p^{(2)}$. When $(\alpha_p^{(k)} P_T)/N_0 \rightarrow \infty$, $k = 1, 2$, the ranks $r_{\max}^{(2,1)}$ and $r_{\max}^{(2)}$ become independent of $(\alpha_p^{(1)}, \alpha_p^{(2)})$ and P_T/N_0 .

The proof of Lemma 4.4.1 can be found in Appendix B. Writing (4.46) as

$$\begin{aligned} \check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}) &= \frac{1}{\check{D}^{(2)}} \left\{ \sum_{p=1}^{r_{\max}^{(2)}} \log \left(\frac{N_0}{P_T} + [\check{\Lambda}^{(2)}]_{p,p} \right) - \sum_{q=1}^{r_{\max}^{(2,1)}} \log \left(\frac{N_0}{P_T} + [\check{\Lambda}^{(2,1)}]_{q,q} \right) \right. \\ &\quad \left. + (r_{\max}^{(2)} - r_{\max}^{(2,1)}) \log \left(\frac{P_T}{N_0} \right) \right\}, \end{aligned} \quad (4.50)$$

the term $\frac{(r_{\max}^{(2)} - r_{\max}^{(2,1)})}{\check{D}^{(2)}} \log(P_T/N_0)$ becomes dominant when P_T/N_0 becomes very large since $0 < [\check{\Lambda}^{(2)}]_{p,p} < \infty$ and $0 < [\check{\Lambda}^{(2,1)}]_{q,q} < \infty$ as seen from (4.49). Thus, we can write

$$G_{in}^{(2)} \triangleq \lim_{(P_T/N_0) \rightarrow \infty} \frac{\check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})}{\log(P_T/N_0)} = \frac{(r_{\max}^{(2)} - r_{\max}^{(2,1)})}{\check{D}^{(2)}} \quad (4.51)$$

which denotes the spatial multiplexing gain associated with $\check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})$. Therefore, for $0 < \alpha_p^{(k)}$ and for very large $(\alpha_p^{(k)} P_T)/N_0$, $k = 1, 2$, $(C^{(1)}(\alpha_p^{(1)}), \check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})) \rightarrow (r_{\check{D}^{(1)}}^{(1)} \log(\frac{P_T}{N_0}), \frac{(r_{\max}^{(2)} - r_{\max}^{(2,1)})}{\check{D}^{(2)}} \log(\frac{P_T}{N_0}))$ from (4.42) and (4.51). Also, using (4.42) with $k = 2$, (4.29) leads at high SNR to

$$\Delta^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}) \rightarrow \Delta_{\infty}^{(2)} \triangleq \frac{(r_T^{(2)} D_G^{(2)} + r_{\max}^{(2,1)} - r_{\max}^{(2)})}{\check{D}^{(2)}} \log \left(\frac{P_T}{N_0} \right) \text{ bps/Hz} \quad (4.52)$$

as $(P_T/N_0) \rightarrow \infty$. Let $r_{\max}^{(1)}$ and $r_{\max}^{(1,2)}$ denote the ranks of $\mathbf{H}^{(1,2)} \cdot (\frac{1}{P_T} \Sigma_{\max}^{(2)}) \cdot [\mathbf{H}^{(1,2)}]^\dagger + \mathbf{H}^{(1)} \cdot (\frac{1}{P_T} \check{\Sigma}^{(1)}) \cdot [\mathbf{H}^{(1)}]^\dagger$ and $\mathbf{H}^{(1,2)} \cdot (\frac{1}{P_T} \Sigma_{\max}^{(2)}) \cdot [\mathbf{H}^{(1,2)}]^\dagger$, respectively. Similarly to Lemma 4.4.1, $r_{\max}^{(1,2)}$ and $r_{\max}^{(1)}$ become independent of $(\alpha_p^{(1)}, \alpha_p^{(2)})$ and P_T/N_0 for $0 < \alpha_p^{(k)}$ and for very large $(\alpha_p^{(k)} P_T)/N_0$, $k = 1, 2$. By switching the users indexes and following a similar method, we can show as in (4.51) that the spatial multiplexing gain associated with $\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ is given by

$$G_{in}^{(1)} \triangleq \lim_{(P_T/N_0) \rightarrow \infty} \frac{\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)})}{\log(P_T/N_0)} = \frac{(r_{\max}^{(1)} - r_{\max}^{(1,2)})}{\check{D}^{(1)}} \quad (4.53)$$

and hence, $(\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}), C^{(2)}(\alpha_p^{(2)})) \rightarrow (\frac{(r_{\max}^{(1)} - r_{\max}^{(1,2)})}{\check{D}^{(1)}} \log(\frac{P_T}{N_0}), \frac{r_T^{(2)} D_G^{(2)}}{\check{D}^{(2)}} \log(\frac{P_T}{N_0}))$ whenever $0 < \alpha_p^{(k)}$ and $(\alpha_p^{(k)} P_T)/N_0$ very large for $k = 1, 2$. Similarly to (4.52), we have

$$\Delta^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}) \rightarrow \Delta_{\infty}^{(1)} \triangleq \frac{(r_T^{(1)} D_G^{(1)} + r_{\max}^{(1,2)} - r_{\max}^{(1)})}{\check{D}^{(1)}} \log \left(\frac{P_T}{N_0} \right) \text{ bps/Hz} \quad (4.54)$$

Furthermore, if $\alpha_p^{(1)} \rightarrow 0$ (and $\alpha_p^{(2)} \rightarrow 1$) then $(\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}), C^{(2)}(\alpha_p^{(2)})) \rightarrow (0, \frac{r_T^{(2)} D_G^{(2)}}{\check{D}^{(2)}} \log(\frac{P_T}{N_0}))$ and $(C^{(1)}(\alpha_p^{(1)}), \check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})) \rightarrow (0, \frac{(r_{\max}^{(2)} - r_{\max}^{(2,1)})}{\check{D}^{(2)}} \log(\frac{P_T}{N_0}))$. Also, if $\alpha_p^{(2)} \rightarrow 0$ (and $\alpha_p^{(1)} \rightarrow 1$) then $(C^{(1)}(\alpha_p^{(1)}), \check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})) \rightarrow (\frac{r_T^{(1)} D_G^{(1)}}{\check{D}^{(1)}} \log(\frac{P_T}{N_0}), 0)$ and $(\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}), C^{(2)}(\alpha_p^{(2)})) \rightarrow (\frac{(r_{\max}^{(1)} - r_{\max}^{(1,2)})}{\check{D}^{(1)}} \log(\frac{P_T}{N_0}), 0)$. Using (4.25) and the points $\tilde{V}_0$ and $\tilde{V}_1$ with the coordinates of Table 4.2, we can draw the following conclusion.

Point	$R^{(1)}$ -coordinate	$R^{(2)}$ -coordinate
$\tilde{V}_0$	$\frac{(r_{\max}^{(1)} - r_{\max}^{(1,2)})}{\check{D}^{(1)}} \log(P_T/N_0)$	$\frac{r_T^{(2)} D_G^{(2)}}{\check{D}^{(2)}} \log(P_T/N_0)$
$\tilde{V}_1$	$\frac{r_T^{(1)} D_G^{(1)}}{\check{D}^{(1)}} \log(P_T/N_0)$	$\frac{(r_{\max}^{(2)} - r_{\max}^{(2,1)})}{\check{D}^{(2)}} \log(P_T/N_0)$

Table 4.2 Coordinates of the points $\tilde{V}_0, \tilde{V}_1$ for spread spectrum like BCs

Theorem 4.4.2 As $(P_T/N_0) \rightarrow \infty$, the inner bound $\mathcal{R}_{in}$ converges towards the pentagon defined by the origin, $V_0, \tilde{V}_0, \tilde{V}_1$ and V_1 .

Hence, the asymptotical inner bound of Theorem 4.4.2 is characterized by the points $V_0, \tilde{V}_0, \tilde{V}_1$ and V_1 . The asymptotical outer and inner bounds defined in Theorems 4.4.1 and 4.4.2 are depicted in Figure 4.2. The points V_0 and V_1 correspond to the situations where the total transmit power is allocated to user 2 ($(\alpha_p^{(1)}, \alpha_p^{(2)}) = (0, 1)$ and $P^{(2)} = P_T$) and to user 1 ($(\alpha_p^{(1)}, \alpha_p^{(2)}) = (1, 0)$ and $P^{(1)} = P_T$), respectively, with the users operating at capacity rate. For the points $\tilde{V}_0$ and $\tilde{V}_1$ we have $P^{(1)} = \alpha_p^{(1)} P_T$ and $P^{(2)} = \alpha_p^{(2)} P_T$ with $\alpha_p^{(1)} > 0$ and $\alpha_p^{(2)} > 0$. The point $\tilde{V}_0$ corresponds to user 2 being encoded first and operating at capacity rate. The point $\tilde{V}_1$ corresponds to user 1 being encoded first and operating at capacity rate. The points on the segments $V_0 \tilde{V}_0, \tilde{V}_0 \tilde{V}_1$ and $\tilde{V}_1 V_1$ can be achieved by time-sharing. The points on the segments $V_0 \tilde{V}_0$ and $\tilde{V}_1 V_1$ are on the boundaries of $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$ and hence, they are on the boundaries of the DPC rate region $\mathcal{R}_{DPC}$ and the capacity region $\mathcal{C}$ of the MIMO BC (4.1). We conclude that $\mathcal{C}$ and the capacity region of the orthogonal broadcast channels of [125] coincide on the segments $V_0 \tilde{V}_0$ and $\tilde{V}_1 V_1$.

As in Theorem 4.2.1, the sum rate for $\mathcal{R}_{in}$ is maximized at the boundary point at which the support line to the boundary of $\mathcal{R}_{in}$ has a slope equal to -1 . Thus, in the limit when $(P_T/N_0) \rightarrow \infty$, Figure 4.2 shows that the sum rate for the limiting inner bound is maximized at $\tilde{V}_1$ if $\Delta_\infty^{(2)} < \Delta_\infty^{(1)}$, or at $\tilde{V}_0$ if $\Delta_\infty^{(2)} > \Delta_\infty^{(1)}$, or at any point on the line segment joining $\tilde{V}_1$ and $\tilde{V}_0$ if $\Delta_\infty^{(2)} = \Delta_\infty^{(1)}$. Equivalently, the sum rate for the limiting inner bound

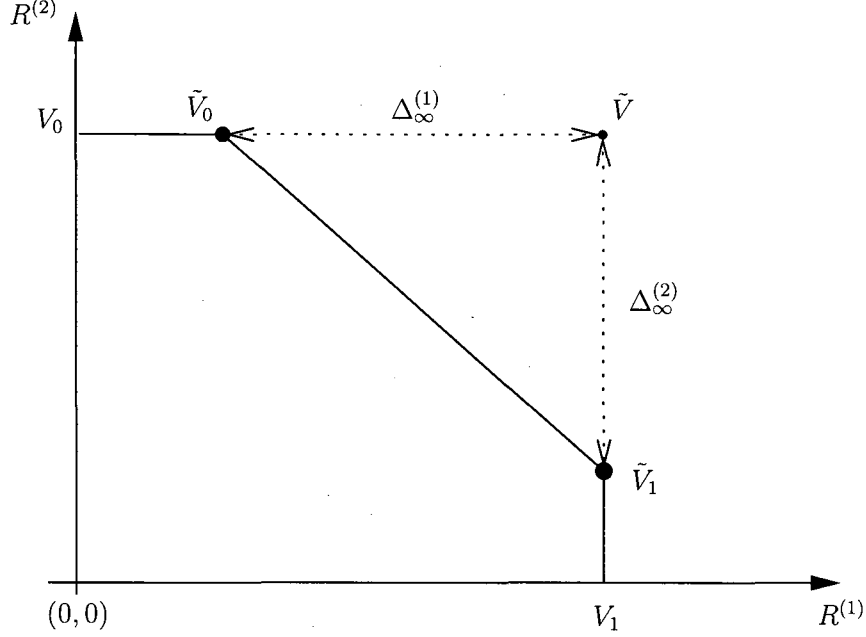


Fig. 4.2 The asymptotical bounds at high SNR. The outer bound is the rectangle defined by $(0, 0)$, V_0 , $\tilde{V}$ and V_1 . The inner bound is the pentagon defined by $(0, 0)$, V_0 , $\tilde{V}_0$, $\tilde{V}_1$ and V_1 .

is maximized at the corner point of the boundary that is closest to $\tilde{V}$. As seen from (4.52) and (4.54), $\Delta_\infty^{(k)} / \log(\frac{P_T}{N_0})$ represents for user k the difference between the multiplexing gains $G_{out}^{(k)}$ and $G_{in}^{(k)}$ (with $k, k' = 1, 2$ and $k \neq k'$) that are associated with the single-user capacity $C^{(k)}(\alpha_p^{(k)})$ and the rate $\check{R}^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)})$, respectively. Note that, similarly to (3.17), $G_{out}^{(k)}$ represents the maximal multiplexing gain for user k . By transmitting at capacity rate, users 1 and 2 achieve their maximal multiplexing gains at $\tilde{V}_1$ and $\tilde{V}_0$, respectively. Thus, for $k, k' = 1, 2$ with $k \neq k'$, if $\Delta_\infty^{(k)} < \Delta_\infty^{(k')}$ this means that by transmitting at rate $\frac{r_{\max}^{(k)} - r_{\max}^{(k,k')}}{\bar{D}^{(k)}} \log(\frac{P_T}{N_0})$ user k is able to achieve a larger fraction of its maximal multiplexing gain than that achieved by user k' when transmitting at rate $\frac{r_{\max}^{(k')} - r_{\max}^{(k',k)}}{\bar{D}^{(k')}} \log(\frac{P_T}{N_0})$. In this case, the sum rate for the asymptotical inner bound is maximized with user k transmitting at rate $\frac{r_{\max}^{(k)} - r_{\max}^{(k,k')}}{\bar{D}^{(k)}} \log(\frac{P_T}{N_0})$ while user k' is operating at capacity rate. Furthermore, the power allocation $(\tilde{\alpha}_{p;s}^{(1)}, \tilde{\alpha}_{p;s}^{(2)})$ associated with the maximum sum rate for $\mathcal{R}_{in}$ in the high SNR regime is such that $0 < \tilde{\alpha}_{p;s}^{(1)}$ and $0 < \tilde{\alpha}_{p;s}^{(2)}$. Let us now evaluate $(\tilde{\alpha}_{p;s}^{(1)}, \tilde{\alpha}_{p;s}^{(2)})$. If $\Delta_\infty^{(2)} \leq \Delta_\infty^{(1)}$, the sum rate is maximized at the boundary point $(C^{(1)}(\tilde{\alpha}_{p;s}^{(1)}), \check{R}^{(2)}(\tilde{\alpha}_{p;s}^{(1)}, \tilde{\alpha}_{p;s}^{(2)}))$ at which the support line to the boundary of $\mathcal{R}_{in}$ has a slope equal to -1 . Here the maximum sum rate for $\mathcal{R}_{in}$ is given by $C^{(1)}(\tilde{\alpha}_{p;s}^{(1)}) + \check{R}^{(2)}(\tilde{\alpha}_{p;s}^{(1)}, \tilde{\alpha}_{p;s}^{(2)})$ which is a lower bound for the sum capacity

C^{sum} . We consider the derivative $\frac{d\check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})}{dC^{(1)}(\alpha_p^{(1)})} = \frac{d\check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})}{d\alpha_p} \frac{d\alpha_p}{dC^{(1)}(\alpha_p^{(1)})}$. Using (4.43) with $k = 1$ and $\alpha_p^{(1)} = \alpha_p$, we obtain

$$\frac{dC^{(1)}(\alpha_p^{(1)})}{d\alpha_p} = \frac{r_T^{(1)} D_G^{(1)}}{\ln 2 \check{D}^{(1)} \alpha_p} \quad (4.55)$$

which exists for all $\alpha_p \in (0, 1)$. An approximation for the rate $\check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ of (4.46) can be obtained as follows. Putting this rate in the form of (4.23) and using the approximation (B.16) for $[\mathbf{H}^{(2)}]^\dagger \cdot [\Psi^{(2)}]^{-1} \cdot \mathbf{H}^{(2)}$, we obtain

$$\check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}) = \frac{1}{\check{D}^{(2)}} \sum_{q=1}^{\check{r}^{(2,1)}} \log(\check{\mu}^{(2)}(\alpha_p^{(2)}) \check{\lambda}_q^{(2)}) \text{ bps/Hz}$$

where the rank $\check{r}^{(2,1)}$ and eigenvalues $\check{\lambda}_q^{(2)}$ are independent of $(\alpha_p^{(1)}, \alpha_p^{(2)})$ (equivalently α_p) and P_T/N_0 as shown in the proof of Lemma 4.4.1 in Appendix B. Using the approximation given by (B.17) for the water-filling constant $\check{\mu}^{(2)}(\alpha_p^{(2)})$ with $\alpha_p^{(2)} = 1 - \alpha_p$, we can write

$$\check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}) = \frac{\check{r}^{(2,1)}}{\check{D}^{(2)}} \log\left(\frac{(1 - \alpha_p) P_T}{N_0 \check{r}^{(2,1)}}\right) + \frac{1}{\check{D}^{(2)}} \sum_{q=1}^{\check{r}^{(2,1)}} \log(\check{\lambda}_q^{(2)}) \text{ bps/Hz}$$

resulting in

$$\frac{d\check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})}{d\alpha_p} = -\frac{\check{r}^{(2,1)}}{\ln 2 \check{D}^{(2)}} \frac{N_0 \check{r}^{(2,1)}}{(1 - \alpha_p) P_T} \frac{P_T}{N_0 \check{r}^{(2,1)}} = -\frac{\check{r}^{(2,1)}}{\ln 2 \check{D}^{(2)} (1 - \alpha_p)} \quad (4.56)$$

which exists for all $\alpha_p \in (0, 1)$. Hence, using (4.55) and (4.56), we obtain

$$\frac{d\check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})}{dC^{(1)}(\alpha_p^{(1)})} = -\frac{\check{D}^{(1)}}{\check{D}^{(2)}} \frac{\check{r}^{(2,1)}}{r_T^{(1)} D_G^{(1)}} \frac{\alpha_p}{(1 - \alpha_p)} \quad (4.57)$$

The support line to the boundary of $\mathcal{R}_{in}$ at $(C^{(1)}(\tilde{\alpha}_{p;s}^{(1)}), \check{R}^{(2)}(\tilde{\alpha}_{p;s}^{(1)}, \tilde{\alpha}_{p;s}^{(2)}))$ is unique and has a slope given by (4.57) with $\alpha_p = \tilde{\alpha}_{p;s}^{(1)}$, [123, p. 199]. Solving $\left[\frac{d\check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})}{dC^{(1)}(\alpha_p^{(1)})}\right]_{\alpha_p=\tilde{\alpha}_{p;s}^{(1)}} = -1$ leads, using (4.57), to

$$\tilde{\alpha}_{p;s}^{(1)} = \frac{\check{D}^{(2)} r_T^{(1)} D_G^{(1)}}{\check{D}^{(1)} \check{r}^{(2,1)}} \left[1 + \frac{\check{D}^{(2)} r_T^{(1)} D_G^{(1)}}{\check{D}^{(1)} \check{r}^{(2,1)}}\right]^{-1} = \left[1 + \frac{\check{D}^{(1)}}{\check{D}^{(2)}} \frac{\check{r}^{(2,1)}}{r_T^{(1)} D_G^{(1)}}\right]^{-1} \quad (4.58)$$

If $\Delta_\infty^{(2)} \geq \Delta_\infty^{(1)}$, the sum rate is maximized at the boundary point $(\check{R}^{(1)}(\tilde{\alpha}_{p;s}^{(1)}, \tilde{\alpha}_{p;s}^{(2)}), C^{(2)}(\tilde{\alpha}_{p;s}^{(2)}))$ at which the support line to the boundary of $\mathcal{R}_{in}$ has a slope equal to -1 . Here the maximum sum rate for $\mathcal{R}_{in}$ is given by $\check{R}^{(1)}(\tilde{\alpha}_{p;s}^{(1)}, \tilde{\alpha}_{p;s}^{(2)}) + C^{(2)}(\tilde{\alpha}_{p;s}^{(2)})$ which is a lower bound for the sum capacity C^{sum} . Following similar steps, we can show that $\frac{dC^{(2)}(\alpha_p^{(2)})}{d\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)})} = -\frac{\check{D}^{(1)} r_T^{(2)} D_G^{(2)}}{\check{D}^{(2)} \check{r}^{(1,2)} (1-\alpha_p)}$ where $\check{r}^{(1,2)}$ denotes the rank of the matrix $[\mathbf{H}^{(1)}]^\dagger \cdot [\bar{\mathbf{M}}^{(1,2)}]^\dagger \cdot \mathbf{D}^{(1,2)} \cdot \bar{\mathbf{M}}^{(1,2)} \cdot \mathbf{H}^{(1)}$ which, similarly to (B.16), can be shown to satisfy

$$[\mathbf{H}^{(1)}]^\dagger \cdot [\Psi^{(1)}]^{-1} \cdot \mathbf{H}^{(1)} \approx [\mathbf{H}^{(1)}]^\dagger \cdot [\bar{\mathbf{M}}^{(1,2)}]^\dagger \cdot \mathbf{D}^{(1,2)} \cdot \bar{\mathbf{M}}^{(1,2)} \cdot \mathbf{H}^{(1)}$$

for very large P_T/N_0 , with $\bar{\mathbf{M}}^{(1,2)}$ denoting the unitary matrix in the eigenvalue decomposition (similarly to (B.13))

$$\mathbf{H}^{(1,2)} \cdot \left\{ \text{diag}_{j=1}^{N_G^{(2)}} \left[[\mathbf{M}_j^{(2)}]^\dagger \cdot \mathbf{B}_j^{(2)} \cdot \mathbf{M}_j^{(2)} \right] \otimes \mathbf{I}_{D_G^{(2)}} \right\} \cdot [\mathbf{H}^{(1,2)}]^\dagger = [\bar{\mathbf{M}}^{(1,2)}]^\dagger \cdot \bar{\Lambda}^{(1,2)} \cdot \bar{\mathbf{M}}^{(1,2)},$$

$\mathbf{B}_j^{(2)} = \text{diag}[\mathbf{I}_{r_j^{(2)}}, \mathbf{0}_{(n_j^{(2)} - r_j^{(2)}) \times (n_j^{(2)} - r_j^{(2)})}]$, and with $\mathbf{D}^{(1,2)}$ being a diagonal matrix defined as $[\mathbf{D}^{(1,2)}]_{p,p} = 1$ if $\frac{N_G^{(2)}}{r_T^{(2)} D^{(2)}} [\bar{\Lambda}^{(1,2)}]_{p,p} = 0$ and $[\mathbf{D}^{(1,2)}]_{p,p} = 0$ if $\frac{N_G^{(2)}}{r_T^{(2)} D^{(2)}} [\bar{\Lambda}^{(1,2)}]_{p,p} \neq 0$ for $p = 1, \dots, L_t^{(1)} N_R^{(1)} D^{(1)}$. Now solving $\left[\frac{dC^{(2)}(\alpha_p^{(2)})}{d\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)})} \right]_{\alpha_p = \tilde{\alpha}_{p;s}^{(1)}} = -1$ leads to

$$\tilde{\alpha}_{p;s}^{(1)} = \frac{\check{D}^{(2)} \check{r}^{(1,2)}}{\check{D}^{(1)} r_T^{(2)} D_G^{(2)}} \left[1 + \frac{\check{D}^{(2)} \check{r}^{(1,2)}}{\check{D}^{(1)} r_T^{(2)} D_G^{(2)}} \right]^{-1} = \left[1 + \frac{\check{D}^{(1)} r_T^{(2)} D_G^{(2)}}{\check{D}^{(2)} \check{r}^{(1,2)}} \right]^{-1} \quad (4.59)$$

Obviously, $\tilde{\alpha}_{p;s}^{(1)}$ of (4.58) and (4.59) reduce to $\alpha_{p;s}^{(1)}$ of (4.45) when $D_c = 0$ since in this case $\mathcal{R}_{in}$ is equal to $\mathcal{R}_{out}$. Next, we investigate the impact of signal dimensions sharing on the multiplexing gains (4.42), (4.51) and (4.53), and we analyze the tightness of the inner and outer bounds at high SNR.

4.5 The spatial multiplexing gains

This section considers the multiplexing gains introduced in Section 4.4.2 associated with the outer bound $\mathcal{R}_{out}$ (defined in Theorem 4.4.1) and inner bound $\mathcal{R}_{in}$ (defined in Theorem 4.4.2) of Figure 4.2. We investigate the following two scenarios. First, we fix the number of proper signal dimensions $D_p^{(k)}$ for each user ($k = 1, 2$) and we vary the number of shared signal dimensions D_c . Second, we fix the total number of signal dimensions D_T and we

increase the number of shared signal dimensions D_c at the expense of the number of proper signal dimensions $D_p^{(1)}$ and $D_p^{(2)}$. In this case, we consider two initial values $D_{p;0}^{(1)}$ and $D_{p;0}^{(2)}$ for $D_p^{(1)}$ and $D_p^{(2)}$, respectively, such that $D_{p;0}^{(1)} + D_{p;0}^{(2)} = D_T$. Then we start to increase D_c by making $\delta D^{(1)}$ (with $0 < \delta D^{(1)} \leq D_{p;0}^{(1)}$) signal dimensions shared out of the initial $D_{p;0}^{(1)}$ proper signal dimensions for user 1 and also by making $\delta D^{(2)}$ (with $0 < \delta D^{(2)} \leq D_{p;0}^{(2)}$) signal dimensions shared out of the initial $D_{p;0}^{(2)}$ proper signal dimensions for user 2. Hence, we obtain $D_p^{(1)} = D_{p;0}^{(1)} - \delta D^{(1)}$, $D_p^{(2)} = D_{p;0}^{(2)} - \delta D^{(2)}$, and $D_c = \delta D^{(1)} + \delta D^{(2)}$ which we increase by increasing $\delta D^{(1)}$ with $\delta D^{(2)}$ held fixed or vice versa. The bandwidth assigned to any given user increases with increasing D_c , which could yield an increase in its transmission rate. However, the interference that this user experiences when encoded first also increases with increasing D_c , which could yield a decrease in its transmission rate. The fact that increasing D_c leads to opposite effects on the transmission rates creates a tradeoff between bandwidth and interference. In this section, we analyze the behavior of the multiplexing gains (4.42) for $\mathcal{R}_{out}$ as well as (4.51) and (4.53) for $\mathcal{R}_{in}$ as a functions of D_c in the two scenarios previously introduced. Furthermore, we analyze the impact of the number of time resolvable clusters on these multiplexing gains and we consider the tightness of the inner and outer bounds at high SNR.

4.5.1 Multiplexing gains associated with $\mathcal{R}_{out}$

Using $G_{out}^{(k)}$ of (4.42), the multiplexing gains associated with the corner points V_0 , $\tilde{V}$ and V_1 of $\mathcal{R}_{out}$ in Figure 4.2 are given by $(0, G_{out}^{(2)})$, $(G_{out}^{(1)}, G_{out}^{(2)})$ and $(G_{out}^{(1)}, 0)$, respectively. The boundary points of $\mathcal{R}_{out}$ that belong to the segment $V_0\tilde{V}$ (respectively, $\tilde{V}V_1$) can be achieved by time-sharing between the codes associated with V_0 and $\tilde{V}$ (respectively, $\tilde{V}$ and V_1). Recall that time-sharing between two codes with rate pairs $(R_1^{(1)}, R_1^{(2)})$ and $(R_2^{(1)}, R_2^{(2)})$ (with multiplexing gains $(g_{mux;1}^{(1)}, g_{mux;1}^{(2)})$ and $(g_{mux;2}^{(1)}, g_{mux;2}^{(2)})$, respectively) results in a new code with rates $(\tau R_1^{(1)} + (1 - \tau)R_2^{(1)}, \tau R_1^{(2)} + (1 - \tau)R_2^{(2)})$, $\tau \in [0, 1]$. One can easily show that the multiplexing gains associated with this new rate pair are: $\tau g_{mux;1}^{(1)} + (1 - \tau)g_{mux;2}^{(1)}$ for user 1 and $\tau g_{mux;1}^{(2)} + (1 - \tau)g_{mux;2}^{(2)}$ for user 2. Considering a point on the segment $V_0\tilde{V}$, the multiplexing gains are given by $\tilde{\tau}_0 G_{out}^{(1)}$, $\tilde{\tau}_0 \in [0, 1]$, for user 1 and by $G_{out}^{(2)}$ for user 2. Considering a point on the segment $\tilde{V}V_1$, the multiplexing gains are given by $G_{out}^{(1)}$ for user 1 and by $\tilde{\tau}_1 G_{out}^{(2)}$, $\tilde{\tau}_1 \in [0, 1]$, for user 2. Investigating the behavior of these multiplexing gains (for a given $\tilde{\tau}_0$ or $\tilde{\tau}_1$) is equivalent to investigating that of $G_{out}^{(k)}$. If $\check{D}^{(k)} = D^{(k)}$, then

$[G_{out}^{(k)}]_{\check{D}^{(k)}=D^{(k)}} = \frac{r_T^{(k)} D_G^{(k)}}{D^{(k)}} = \frac{r_T^{(k)}}{N_G^{(k)}}$ which is independent of D_c . If $\check{D}^{(k)} = D_T$, then

$$[G_{out}^{(k)}]_{\check{D}^{(k)}=D_T} = \frac{r_T^{(k)} (D_p^{(k)} + D_c)}{N_G^{(k)} (D_p^{(1)} + D_p^{(2)} + D_c)} \quad (4.60)$$

In the first scenario where $D_p^{(1)}$ and $D_p^{(2)}$ are fixed, $[G_{out}^{(k)}]_{\check{D}^{(k)}=D_T}$ is a monotonically increasing function of D_c from Appendix B.2. Considering the second scenario, we have from (4.60) that $[G_{out}^{(1)}]_{\check{D}^{(1)}=D_T} = \frac{r_T^{(1)} (D_{p,0}^{(1)} + \delta D^{(2)})}{N_G^{(1)} D_T}$ and $[G_{out}^{(2)}]_{\check{D}^{(2)}=D_T} = \frac{r_T^{(2)} (D_{p,0}^{(2)} + \delta D^{(1)})}{N_G^{(2)} D_T}$. Thus, $[G_{out}^{(1)}]_{\check{D}^{(1)}=D_T}$ (respectively, $[G_{out}^{(2)}]_{\check{D}^{(2)}=D_T}$) increases whenever $\delta D^{(2)}$ (respectively, $\delta D^{(1)}$) increases, that is, when the number of signal dimensions allocated to user 1 (respectively, user 2) increases at the expense of the number of proper signal dimensions of the other user. However, for $k = 1, 2$, the multiplexing gain $[G_{out}^{(k)}]_{\check{D}^{(k)}=D_T}$ remains constant when $\delta D^{(k)}$ increases. We conclude that when D_c increases in both scenarios the outer bound $\mathcal{R}_{out}$ becomes larger. This is due to the fact that for $\mathcal{R}_{out}$ all the interference is canceled out so that when D_c increases there is no increase in interference while the bandwidth of each user increases, which increases the transmission rates in bits/channel-use.

Let us consider the behavior of $G_{out}^{(k)}$ and $\mathcal{R}_{out}$ when $D_c \rightarrow \infty$. If $\check{D}^{(k)} = D_T$, then one can see from (4.60) that $\lim_{D_c \rightarrow \infty} [G_{out}^{(k)}]_{\check{D}^{(k)}=D_T} = \frac{r_T^{(k)}}{N_G^{(k)}}$ which is equal to $[G_{out}^{(k)}]_{\check{D}^{(k)}=D^{(k)}}$. Thus, in the limit when $D_c \rightarrow \infty$ the outer bound $\mathcal{R}_{out}$ associated with $\check{D}^{(k)} = D_T$ converges to that associated with $\check{D}^{(k)} = D^{(k)}$ which is a rectangle with lower left corner $(0, 0)$ and upper right corner $(\frac{r_T^{(1)}}{N_G^{(1)}} \log(P_T/N_0), \frac{r_T^{(2)}}{N_G^{(2)}} \log(P_T/N_0))$. Also, we can show that $[G_{out}^{(k)}]_{\check{D}^{(k)}=D^{(k)}} - [G_{out}^{(k)}]_{\check{D}^{(k)}=D_T}$ and hence, the separation between $[\mathcal{R}_{out}]_{\check{D}^{(k)}=D^{(k)}}$ and $[\mathcal{R}_{out}]_{\check{D}^{(k)}=D_T}$ are monotonically decreasing functions of D_c .

We finally investigate the effect of the number of time resolvable clusters $L_t^{(k)}$ on $G_{out}^{(k)}$, $k = 1, 2$. The multiplexing gain $G_{out}^{(k)}$ depends on $L_t^{(k)}$ through the rank $r_T^{(k)}$ which is given by $r_T^{(k)} = \sum_{i=1}^{N_G^{(k)}} r_i^{(k)} = \sum_{i=1}^{N_G^{(k)}} \min(n_i^{(k)}, L_t^{(k)} N_R^{(k)})$ for 3GPP channels. If there is at least one TOG i of user k such that $L_t^{(k)} N_R^{(k)} < n_i^{(k)}$, an increase in $L_t^{(k)}$ leads to an increase in $r_T^{(k)}$ and hence, in $G_{out}^{(k)}$. However, if all the TOGs of user k have less antenna elements than the product $L_t^{(k)} N_R^{(k)}$, then the multiplexing gain $G_{out}^{(k)}$ remains unchanged when $L_t^{(k)}$ increases, yielding $r_T^{(k)} = N_T$ and $G_{out}^{(k)} = \tilde{G}_{out}^{(k)}$ with $\tilde{G}_{out}^{(k)} \triangleq \frac{N_T D_G^{(k)}}{\check{D}^{(k)}}$. Moreover, since $r_T^{(k)}$ is independent of user k 's angle spread at the transmitter and receiver, $G_{out}^{(k)}$ remains unaffected when the angle spread changes.

4.5.2 Multiplexing gains associated with $\mathcal{R}_{in}$

This section considers the multiplexing gains $G_{in}^{(k)}$ of (4.51) and (4.53) that are associated with the boundary points of $\mathcal{R}_{in}$ for which $\alpha_p^{(k)} \in (0, 1)$ for $k = 1, 2$. For the extreme cases $\alpha_p^{(k)} = 0$ and $\alpha_p^{(k)} = 1$ the multiplexing gain of user k is equal to zero and to $G_{out}^{(k)}$, respectively. We start by evaluating $G_{in}^{(2)}$ as a function of the number of shared and proper signal dimensions of user 2. The evaluation of $G_{in}^{(1)}$ as a function of the number of shared and proper signal dimensions of user 1 can be similarly done by switching the users indexes. For $k = 1, 2$, let $\mathcal{A}_{G;c}^{(k)}$ denote the subset of integers $j \in \{1, \dots, N_G^{(k)}\}$ which correspond to the TOGs of user k that employ shared signal dimensions. Note that each shared signal dimension can be employed by only one TOG of $\mathcal{A}_{G;c}^{(k)}$ since, otherwise, the orthogonality between the subsets of signal dimensions assigned to different TOGs of user k does not hold anymore. For $j \in \mathcal{A}_{G;c}^{(1)}$, let $D_{c;j}^{(2)}$ be the number of shared signal dimensions of user 2 that are employed by TOG j of user 1, $D_{c;j}^{(2)} \leq D_G^{(1)}$ and $\sum_{j \in \mathcal{A}_{G;c}^{(1)}} D_{c;j}^{(2)} = D_c$. For $j \in \mathcal{A}_{G;c}^{(1)}$ and $d = 1, \dots, D_{c;j}^{(2)}$ we define for user 2 the mapping $v^{(2)}(j, d)$ such that $v^{(2)}(j, d) \in \{1, \dots, D^{(2)}\}$ denotes the index of the shared signal dimension associated with integer d and TOG j . Also, for every proper signal dimension $d' = 1, \dots, D_p^{(2)}$ of user 2 we define the mapping $\tilde{v}^{(2)}(d')$ such that $\tilde{v}^{(2)}(d') \in \{1, \dots, D^{(2)}\}$ denotes the index of the proper signal dimension associated with integer d' . Clearly, $\tilde{v}^{(2)}(d') \neq v^{(2)}(j, d)$. Let us evaluate the rank $r_{\max}^{(2,1)}$ of $\mathbf{H}^{(2,1)} \cdot \left(\frac{1}{P_T} \Sigma_{\max}^{(1)}\right) \cdot [\mathbf{H}^{(2,1)}]^\dagger$ first. For $j = 1, \dots, N_G^{(1)}$, let $\mathbf{B}_j^{(2,1)}$ denote the $D^{(2)} \times D_G^{(1)}$ submatrix of $\mathbf{B}^{(2,1)}$ such that $\mathbf{B}^{(2,1)} = [\mathbf{B}_1^{(2,1)} \ \mathbf{B}_2^{(2,1)} \ \dots \ \mathbf{B}_{N_G^{(1)}}^{(2,1)}]$. Hence, $\mathbf{B}_j^{(2,1)}$ describes the signal dimensions sharing scheme between user 2 and TOG j of user 1. Using $[\mathbf{A}^{(1)}]^T = \text{diag}_{i=1}^{N_T} [\mathbf{e}_{t_i^{(1)}, N_G^{(1)}} \otimes \mathbf{I}_{D_G^{(1)}}]$, we can show that $[\mathbf{I}_{N_T} \otimes \mathbf{B}^{(2,1)}] \cdot [\mathbf{A}^{(1)}]^T = \text{diag}_{i=1}^{N_T} [\mathbf{B}^{(2,1)} \cdot (\mathbf{e}_{t_i^{(1)}, N_G^{(1)}} \otimes \mathbf{I}_{D_G^{(1)}})] = \text{diag}_{i=1}^{N_T} [\mathbf{B}_{t_i^{(1)}}^{(2,1)}]$, yielding

$$[\mathbf{I}_{N_T} \otimes \mathbf{B}^{(2,1)}] \cdot [\mathbf{A}^{(1)}]^T = \text{diag}_{j=1}^{N_G^{(1)}} [\mathbf{I}_{n_j^{(1)}} \otimes \mathbf{B}_j^{(2,1)}] \quad (4.61)$$

Write $\mathbf{C}^{(2)} = [\tilde{\mathbf{C}}_1^{(2)} \ \tilde{\mathbf{C}}_2^{(2)} \ \dots \ \tilde{\mathbf{C}}_{N_G^{(1)}}^{(2)}]$ where the submatrix $\tilde{\mathbf{C}}_j^{(2)}$ is of size $L_t^{(2)} N_R^{(2)} \times n_j^{(1)}$, $j = 1, \dots, N_G^{(1)}$. Equation (B.12) in Appendix B yields

$$\begin{aligned} & \mathbf{H}^{(2,1)} \cdot \left(\frac{1}{P_T} \Sigma_{\max}^{(1)}\right) \cdot [\mathbf{H}^{(2,1)}]^\dagger \approx \\ & \frac{N_G^{(1)} \alpha_p^{(1)}}{r_T^{(1)} D^{(1)}} \mathbf{H}^{(2,1)} \cdot \left\{ \text{diag}_{j=1}^{N_G^{(1)}} \left[[\mathbf{M}_j^{(1)}]^\dagger \cdot \mathbf{B}_j^{(1)} \cdot \mathbf{M}_j^{(1)} \right] \otimes \mathbf{I}_{D_G^{(1)}} \right\} \cdot [\mathbf{H}^{(2,1)}]^\dagger \end{aligned} \quad (4.62)$$

with $\mathbf{B}_j^{(1)} = \text{diag}[\mathbf{I}_{r_j^{(1)}}, \mathbf{0}_{(n_j^{(1)} - r_j^{(1)}) \times (n_j^{(1)} - r_j^{(1)})}]$. Hence, using (2.42), (4.61), (4.62) and (C.3) we can write

$$\begin{aligned} \mathbf{H}^{(2,1)} \cdot \left(\frac{1}{P_T} \Sigma_{\max}^{(1)} \right) \cdot [\mathbf{H}^{(2,1)}]^\dagger &= \frac{N_G^{(1)} \alpha_p^{(1)}}{r_T^{(1)} D^{(1)}} [\mathbf{C}^{(2)} \otimes \mathbf{I}_{D^{(2)}}] \cdot \text{diag}_{j=1}^{N_G^{(1)}} [\mathbf{I}_{n_j^{(1)}} \otimes \mathbf{B}_j^{(2,1)}] \\ &\cdot \text{diag}_{j=1}^{N_G^{(1)}} \left[([\mathbf{M}_j^{(1)}]^\dagger \cdot \mathbf{B}_j^{(1)} \cdot \mathbf{M}_j^{(1)}) \otimes \mathbf{I}_{D_G^{(1)}} \right] \cdot \text{diag}_{j=1}^{N_G^{(1)}} [\mathbf{I}_{n_j^{(1)}} \otimes (\mathbf{B}_j^{(2,1)})^\dagger] \cdot [\mathbf{C}^{(2)} \otimes \mathbf{I}_{D^{(2)}}]^\dagger \\ &= \frac{N_G^{(1)} \alpha_p^{(1)}}{r_T^{(1)} D^{(1)}} [(\tilde{\mathbf{C}}_1^{(2)} \otimes \mathbf{I}_{D^{(2)}}) \cdots (\tilde{\mathbf{C}}_{N_G^{(1)}}^{(2)} \otimes \mathbf{I}_{D^{(2)}})] \\ &\cdot \text{diag}_{j=1}^{N_G^{(1)}} [([\mathbf{M}_j^{(1)}]^\dagger \cdot \mathbf{B}_j^{(1)} \cdot \mathbf{M}_j^{(1)}) \otimes \{\mathbf{B}_j^{(2,1)} \cdot (\mathbf{B}_j^{(2,1)})^\dagger\}] \cdot \text{stack}_{j=1}^{N_G^{(1)}} [(\tilde{\mathbf{C}}_j^{(2)})^\dagger \otimes \mathbf{I}_{D^{(2)}}] \end{aligned}$$

Using the $L_t^{(2)} N_R^{(2)} \times L_t^{(2)} N_R^{(2)}$ matrix $\mathbf{N}_j^{(2,1)} \triangleq \tilde{\mathbf{C}}_j^{(2)} \cdot [\mathbf{M}_j^{(1)}]^\dagger \cdot \mathbf{B}_j^{(1)} \cdot \mathbf{M}_j^{(1)} \cdot [\tilde{\mathbf{C}}_j^{(2)}]^\dagger$, this yields

$$\mathbf{H}^{(2,1)} \cdot \left(\frac{1}{P_T} \Sigma_{\max}^{(1)} \right) \cdot [\mathbf{H}^{(2,1)}]^\dagger = \frac{N_G^{(1)} \alpha_p^{(1)}}{r_T^{(1)} D^{(1)}} \sum_{j=1}^{N_G^{(1)}} [\mathbf{N}_j^{(2,1)} \otimes \{\mathbf{B}_j^{(2,1)} \cdot [\mathbf{B}_j^{(2,1)}]^\dagger\}] \quad (4.63)$$

By definition, the p^{th} column of $\mathbf{B}^{(2,1)}$ is the $D^{(2)} \times 1$ zero vector if the p^{th} signal dimension of user 1 is not shared; otherwise, it is $\mathbf{e}_{q,D^{(2)}}$ where q denotes the integer in the set $\{1, \dots, D^{(2)}\}$ for which $\phi_p^{(1)}(t) = \phi_q^{(2)}(t)$. Hence, if TOG j of user 1 employs no shared signal dimensions (i.e. $j \notin \mathcal{A}_{G;c}^{(1)}$) then $\mathbf{B}_j^{(2,1)} = \mathbf{0}_{D^{(2)} \times D_G^{(1)}}$. Thus, (4.63) reduces to

$$\mathbf{H}^{(2,1)} \cdot \left(\frac{1}{P_T} \Sigma_{\max}^{(1)} \right) \cdot [\mathbf{H}^{(2,1)}]^\dagger = \frac{N_G^{(1)} \alpha_p^{(1)}}{r_T^{(1)} D^{(1)}} \sum_{j \in \mathcal{A}_{G;c}^{(1)}} [\mathbf{N}_j^{(2,1)} \otimes \{\mathbf{B}_j^{(2,1)} \cdot [\mathbf{B}_j^{(2,1)}]^\dagger\}] \quad (4.64)$$

Now with $j \in \mathcal{A}_{G;c}^{(1)}$, $\mathbf{B}_j^{(2,1)}$ has $D_{c;j}^{(2)}$ columns of the form $\mathbf{e}_{v^{(2)}(j,d);D^{(2)}}$ with $d = 1, \dots, D_{c;j}^{(2)}$ and the remaining $D_G^{(1)} - D_{c;j}^{(2)}$ columns are zero, which leads to

$$\mathbf{B}_j^{(2,1)} \cdot [\mathbf{B}_j^{(2,1)}]^\dagger = \sum_{d=1}^{D_{c;j}^{(2)}} \mathbf{E}_{v^{(2)}(j,d)}^{(2)} \quad (4.65)$$

using the notation $\mathbf{E}_n^{(2)} \triangleq \mathbf{e}_{n,D^{(2)}} \cdot \mathbf{e}_{n,D^{(2)}}^\dagger$ for $n = 1, \dots, D^{(2)}$. Thus, $\mathbf{B}_j^{(2,1)} \cdot [\mathbf{B}_j^{(2,1)}]^\dagger$ is a diagonal matrix with diagonal elements $\{v^{(2)}(j,d)\}_{d=1}^{D_{c;j}^{(2)}}$ equal to one and the remaining $D^{(2)} - D_{c;j}^{(2)}$ diagonal elements equal to zero, therefore, making $\text{rank}(\mathbf{B}_j^{(2,1)} \cdot [\mathbf{B}_j^{(2,1)}]^\dagger) = D_{c;j}^{(2)}$. Furthermore, since every row in the matrix $\mathbf{B}^{(2,1)}$ has at most one nonzero element

(equal to 1), we have $v^{(2)}(j_1, d_1) \neq v^{(2)}(j_2, d_2)$ for $j_1 = j_2, d_1 \neq d_2$ as well as for $j_1 \neq j_2$ with arbitrary d_1, d_2 . It follows that for different values of j in $\mathcal{A}_{G;c}^{(1)}$ the matrices $\mathbf{B}_j^{(2,1)} \cdot [\mathbf{B}_j^{(2,1)}]^\dagger$ have their column (and row) spaces essentially disjoint. Consequently, different matrices in the summation of (4.64) have their column (and row) spaces essentially disjoint, which satisfies the conditions for the rank additivity from [114, Theorem 18.5.7]. Thus, the rank $r_{\max}^{(2,1)}$ of the matrix $\mathbf{H}^{(2,1)} \cdot (\frac{1}{P_T} \Sigma_{\max}^{(1)}) \cdot [\mathbf{H}^{(2,1)}]^\dagger$ can be written as

$$r_{\max}^{(2,1)} = \sum_{j \in \mathcal{A}_{G;c}^{(1)}} \text{rank}(\mathbf{N}_j^{(2,1)} \otimes \{\mathbf{B}_j^{(2,1)} \cdot [\mathbf{B}_j^{(2,1)}]^\dagger\}) \stackrel{(a)}{=} \sum_{j \in \mathcal{A}_{G;c}^{(1)}} \text{rank}(\mathbf{N}_j^{(2,1)}) \text{rank}(\mathbf{B}_j^{(2,1)} \cdot [\mathbf{B}_j^{(2,1)}]^\dagger)$$

with (a) due to (C.16). We finally obtain

$$r_{\max}^{(2,1)} = \sum_{j \in \mathcal{A}_{G;c}^{(1)}} D_{c;j}^{(2)} \text{rank}(\mathbf{N}_j^{(2,1)}) \quad (4.66)$$

We now consider the rank $r_{\max}^{(2)}$ of $\mathbf{H}^{(2,1)} \cdot (\frac{1}{P_T} \Sigma_{\max}^{(1)}) \cdot [\mathbf{H}^{(2,1)}]^\dagger + \mathbf{H}^{(2)} \cdot (\frac{1}{P_T} \check{\Sigma}^{(2)}) \cdot [\mathbf{H}^{(2)}]^\dagger$. If $r_{\max}^{(2,1)} = L_t^{(2)} N_R^{(2)} D^{(2)}$ then $r_{\max}^{(2)} = L_t^{(2)} N_R^{(2)} D^{(2)}$ as mentioned in the proof of Lemma 4.4.1 in Appendix B, which from (4.51) yields $G_{in}^{(2)} = 0$ and makes it independent of the number of shared and proper signal dimensions. Thus, we subsequently assume that $r_{\max}^{(2,1)} < L_t^{(2)} N_R^{(2)} D^{(2)}$. Prior to evaluating $r_{\max}^{(2)}$, we introduce some notations and definitions. Let $\tilde{\mathcal{A}}_{G;c}^{(1)}$ denote the subset of indexes $j \in \mathcal{A}_{G;c}^{(1)}$ for which $\mathbf{N}_j^{(2,1)}$ is not full rank, i.e. $\text{rank}(\mathbf{N}_j^{(2,1)}) < L_t^{(2)} N_R^{(2)}$. Using $\mathcal{A}_{G;c}^{(1)} - \tilde{\mathcal{A}}_{G;c}^{(1)}$ to denote the subset of indexes $j \in \mathcal{A}_{G;c}^{(1)}$ and $j \notin \tilde{\mathcal{A}}_{G;c}^{(1)}$, we have $\text{rank}(\mathbf{N}_j^{(2,1)}) = L_t^{(2)} N_R^{(2)}$ for all $j \in \mathcal{A}_{G;c}^{(1)} - \tilde{\mathcal{A}}_{G;c}^{(1)}$. Consider the following allocations of proper and shared signal dimensions for user 2. For $a = 1, \dots, N_G^{(2)}$, let $D_{p;a}^{(2)}$ be the number of proper signal dimensions allocated to TOG a of user 2. Also, we denote the number of shared signal dimensions that are employed by TOG a of user 2 and TOG j of user 1 by $D_{c;j;a}^{(2)}$. Similar allocations of the proper and shared signal dimensions can be obtained for user 1 by switching the users indexes. Thus, for $(k, l) = (1, 2), (2, 1)$, we have the following relationships

$$\sum_{a=1}^{N_G^{(k)}} D_{p;a}^{(k)} = D_p^{(k)} \quad (4.67a)$$

$$\sum_{a=1}^{N_G^{(k)}} D_{c;j;a}^{(k)} = D_{c;j}^{(k)} \quad (4.67b)$$

$$\sum_{a=1}^{N_G^{(k)}} \sum_{j \in \mathcal{A}_{G;c}^{(l)}} D_{c;j;a}^{(k)} = D_c \quad (4.67c)$$

$$D_{p;a}^{(k)} + \sum_{j \in \mathcal{A}_{G;c}^{(l)}} D_{c;j;a}^{(k)} = D_G^{(k)} \equiv \frac{D_p^{(k)} + D_c}{N_G^{(k)}}, \quad \forall a \quad (4.67d)$$

$$\sum_{a=1}^{N_G^{(k)}} D_{p;a}^{(k)} + \sum_{\tilde{a}=1}^{N_G^{(l)}} D_{p;\tilde{a}}^{(l)} + \sum_{a=1}^{N_G^{(k)}} \sum_{j \in \mathcal{A}_{G;c}^{(l)}} D_{c;j;a}^{(k)} = D_T \quad (4.67e)$$

For an arbitrary $n \times n$ Hermitian matrix $\mathbf{A}$, we subsequently use the notation $\mathbf{n}_l(\mathbf{A})$ to denote the l^{th} eigenvector of $\mathbf{A}$ associated with the eigenvalue $\lambda_l(\mathbf{A})$ for $l = 1, \dots, n$. For $j \in \tilde{\mathcal{A}}_{G;c}^{(1)}$, $a = 1, \dots, N_G^{(2)}$ and $l = \text{rank}(\mathbf{N}_j^{(2,1)}) + 1, \dots, L_t^{(2)} N_R^{(2)}$, we define the $N_T \times N_T$ matrix $\tilde{\mathbf{N}}_{j;l}^{(2,1)} \triangleq (\mathbf{C}^{(2)})^\dagger \cdot \mathbf{n}_l(\mathbf{N}_j^{(2,1)}) \cdot \mathbf{n}_l^\dagger(\mathbf{N}_j^{(2,1)}) \cdot \mathbf{C}^{(2)}$ and the $n_a^{(2)} \times n_a^{(2)}$ submatrix $\tilde{\mathbf{N}}_{j;l;a}^{(2,1)} \triangleq (\mathbf{C}_a^{(2)})^\dagger \cdot \mathbf{n}_l(\mathbf{N}_j^{(2,1)}) \cdot \mathbf{n}_l^\dagger(\mathbf{N}_j^{(2,1)}) \cdot \mathbf{C}_a^{(2)}$ of $\tilde{\mathbf{N}}_{j;l}^{(2,1)}$. Denoting $\check{\mathbf{N}}_{j;a}^{(2,1)} = \sum_{l=\text{rank}(\mathbf{N}_j^{(2,1)})+1}^{L_t^{(2)} N_R^{(2)}} \tilde{\mathbf{N}}_{j;l;a}^{(2,1)}$, we define the $n_a^{(2)} \times n_a^{(2)}$ matrix $\check{\mathbf{W}}_{j;a}^{(2,1)} \triangleq \sum_{q=1}^{\text{rank}(\check{\mathbf{N}}_{j;a}^{(2,1)})} \mathbf{n}_q(\check{\mathbf{N}}_{j;a}^{(2,1)}) \cdot \mathbf{n}_q^\dagger(\check{\mathbf{N}}_{j;a}^{(2,1)})$. Finally, we define the $L_t^{(2)} N_R^{(2)} \times L_t^{(2)} N_R^{(2)}$ matrix $\mathbf{K}_{j;a}^{(2,1)} = \frac{\alpha_p^{(2)}}{\check{r}^{(2,1)}} (\mathbf{C}_a^{(2)} \cdot \check{\mathbf{W}}_{j;a}^{(2,1)} \cdot [\mathbf{C}_a^{(2)}]^\dagger) + \frac{N_G^{(1)} \alpha_p^{(1)}}{r_T^{(1)} D^{(1)}} \mathbf{N}_j^{(2,1)}$, where (as mentioned in the proof of Lemma 4.4.1 in Appendix B) $\check{r}^{(2,1)}$ denotes the rank of the matrix $[\mathbf{H}^{(2)}]^\dagger \cdot [\bar{\mathbf{M}}^{(2,1)}]^\dagger \cdot \mathbf{D}^{(2,1)} \cdot \bar{\mathbf{M}}^{(2,1)} \cdot \mathbf{H}^{(2)}$ with $\bar{\mathbf{M}}^{(2,1)}$ and $\mathbf{D}^{(2,1)}$ being also defined in Appendix B. Using the previous notations and definitions, we have the following lemma.

Lemma 4.5.1 *The rank $r_{\max}^{(2)}$ of $\mathbf{H}^{(2,1)} \cdot (\frac{1}{P_T} \Sigma_{\max}^{(1)}) \cdot [\mathbf{H}^{(2,1)}]^\dagger + \mathbf{H}^{(2)} \cdot (\frac{1}{P_T} \check{\Sigma}^{(2)}) \cdot [\mathbf{H}^{(2)}]^\dagger$ is given by*

$$r_{\max}^{(2)} = \sum_{a=1}^{N_G^{(2)}} r_a^{(2)} D_{p;a}^{(2)} + \sum_{a=1}^{N_G^{(2)}} \sum_{j \in \tilde{\mathcal{A}}_{G;c}^{(1)}} D_{c;j;a}^{(2)} \text{rank}(\mathbf{K}_{j;a}^{(2,1)}) + \sum_{a=1}^{N_G^{(2)}} \sum_{j \in \mathcal{A}_{G;c}^{(1)} - \tilde{\mathcal{A}}_{G;c}^{(1)}} D_{c;j;a}^{(2)} \text{rank}(\mathbf{N}_j^{(2,1)}) \quad (4.68)$$

The proof of Lemma 4.5.1 can be found in Appendix B. Hence, using (4.66), (4.67b) with $k = 2$ and (4.68), the multiplexing gain $G_{in}^{(2)}$ of (4.51) becomes

$$G_{in}^{(2)} = \frac{\sum_{a=1}^{N_G^{(2)}} [r_a^{(2)} D_{p;a}^{(2)} + \sum_{j \in \tilde{\mathcal{A}}_{G;c}^{(1)}} D_{c;j;a}^{(2)} \{rank(\mathbf{K}_{j;a}^{(2,1)}) - rank(\mathbf{N}_j^{(2,1)})\}]}{\tilde{D}^{(2)}} \quad (4.69)$$

By Corollary C.0.1, the number of nonzero eigenvalues of the (Hermitian) matrix $\mathbf{K}_{j;a}^{(2,1)}$ cannot be less than that of the (Hermitian) matrix $\mathbf{N}_j^{(2,1)}$. Since the number of nonzero eigenvalues of a Hermitian matrix is equal to its rank [105, p. 175], we obtain $rank(\mathbf{K}_{j;a}^{(2,1)}) - rank(\mathbf{N}_j^{(2,1)}) \geq 0$. An expression similar to (4.69) can be obtained for the spatial multiplexing gain $G_{in}^{(1)}$ of (4.53) by switching the user indexes 1 and 2 in (4.69) and in all the definitions of the matrices and parameters used in this equation, that is,

$$G_{in}^{(1)} = \frac{\sum_{\tilde{a}=1}^{N_G^{(1)}} [r_{\tilde{a}}^{(1)} D_{p;\tilde{a}}^{(1)} + \sum_{\tilde{j} \in \tilde{\mathcal{A}}_{G;c}^{(2)}} D_{c;\tilde{j};\tilde{a}}^{(1)} \{rank(\mathbf{K}_{\tilde{j};\tilde{a}}^{(1,2)}) - rank(\mathbf{N}_{\tilde{j}}^{(1,2)})\}]}{\tilde{D}^{(1)}} \quad (4.70)$$

Thus, the multiplexing gains associated with the corner points $V_0, \tilde{V}_0, \tilde{V}_1$ and V_1 (Figure 4.2) of the limiting inner bound $\mathcal{R}_{in}$ (4.25) are, respectively, given by $(0, G_{out}^{(2)})$, $(G_{in}^{(1)}, G_{out}^{(2)})$, $(G_{out}^{(1)}, G_{in}^{(2)})$ and $(G_{out}^{(1)}, 0)$ with $G_{out}^{(k)}$ given by (4.42) for $k = 1, 2$, $G_{in}^{(2)}$ given by (4.69) and $G_{in}^{(1)}$ given by (4.70). Note that if $D_c = 0$ then for $k = 1, 2$ we have $D_{p;a}^{(k)} = D_G^{(k)}$ from (4.67d), yielding $G_{in}^{(k)} = \frac{\sum_{a=1}^{N_G^{(k)}} r_a^{(k)} D_G^{(k)}}{\tilde{D}^{(k)}} \equiv \frac{r_T^{(k)} D_G^{(k)}}{\tilde{D}^{(k)}} = G_{out}^{(k)}$. Next, we investigate the impact of signal dimensions sharing on the spatial multiplexing gains associated with the boundary points of $\mathcal{R}_{in}$ for which $\alpha_p^{(1)}, \alpha_p^{(2)} \in (0, 1)$.

Impact of signal dimensions sharing

In this subsection we assume that the users have equal number of TOGs ($n_c = N_G^{(1)} = N_G^{(2)}$) and that for each user each TOG employs an equal number of shared signal dimensions $\tilde{D}_c \triangleq D_c/n_c$. Hence, $\mathcal{A}_{G;c}^{(k)} = \{1, \dots, n_c\}$ for $k = 1, 2$. For such broadcast systems, each TOG of user k has also an equal number of proper signal dimensions since the total number of signal dimensions per TOG is equal to $D_G^{(k)}$, $k = 1, 2$. As such, when D_c varies $D_G^{(k)}$ varies but the assumption of equal number of signal dimensions per TOG remains satisfied for each user. For more simplicity, we also assume that each TOG $a \in \mathcal{A}_{G;c}^{(2)}$ shares signal dimensions only with one TOG $\tilde{a} \in \mathcal{A}_{G;c}^{(1)}$ and we let $\mathcal{A}_G$ denote the set of these pairs $(a, \tilde{a})$,

$|\mathcal{A}_G| = n_c$. Hence, we have $D_{c;\tilde{a};a}^{(2)} = D_{c;\tilde{a};a}^{(1)} = \tilde{D}_c$ if $(a, \tilde{a}) \in \mathcal{A}_G$ and $D_{c;\tilde{a};a}^{(2)} = D_{c;\tilde{a};a}^{(1)} = 0$ if $(a, \tilde{a}) \notin \mathcal{A}_G$. Now let $\tilde{\mathcal{A}}_G^{(1)} \subseteq \mathcal{A}_G$ denote the subset of pairs $(a, \tilde{a})$ such that $\tilde{a} \in \tilde{\mathcal{A}}_{G;c}^{(1)}$, and let $\tilde{\mathcal{A}}_G^{(2)} \subseteq \mathcal{A}_G$ denote the subset of pairs $(a, \tilde{a})$ such that $a \in \tilde{\mathcal{A}}_{G;c}^{(2)}$. Based on the definition of $\tilde{\mathcal{A}}_{G;c}^{(1)}$ (respectively, $\tilde{\mathcal{A}}_{G;c}^{(2)}$), for each $\tilde{a} \in \tilde{\mathcal{A}}_{G;c}^{(1)}$ (respectively, $a \in \tilde{\mathcal{A}}_{G;c}^{(2)}$) we have $\text{rank}(\mathbf{N}_{\tilde{a}}^{(2,1)}) < L_t^{(2)} N_R^{(2)}$ (respectively, $\text{rank}(\mathbf{N}_a^{(1,2)}) < L_t^{(1)} N_R^{(1)}$). We have $|\tilde{\mathcal{A}}_G^{(k)}| \leq n_c$ for $k = 1, 2$. The multiplexing gains (4.69) and (4.70) become

$$G_{in}^{(2)} = \frac{\sum_{a=1}^{N_G^{(2)}} r_a^{(2)} D_{p;a}^{(2)} + D_c \sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(1)}} (\kappa_{j;a}^{(2,1)} / n_c)}{\check{D}^{(2)}} \quad (4.71)$$

with $\kappa_{j;a}^{(2,1)} \triangleq \text{rank}(\mathbf{K}_{j;a}^{(2,1)}) - \text{rank}(\mathbf{N}_j^{(2,1)})$, and

$$G_{in}^{(1)} = \frac{\sum_{\tilde{a}=1}^{N_G^{(1)}} r_{\tilde{a}}^{(1)} D_{p;\tilde{a}}^{(1)} + D_c \sum_{(\tilde{j}, \tilde{a}) \in \tilde{\mathcal{A}}_G^{(2)}} (\kappa_{\tilde{j};\tilde{a}}^{(1,2)} / n_c)}{\check{D}^{(1)}} \quad (4.72)$$

with $\kappa_{\tilde{j};\tilde{a}}^{(1,2)} \triangleq \text{rank}(\mathbf{K}_{\tilde{j};\tilde{a}}^{(1,2)}) - \text{rank}(\mathbf{N}_{\tilde{j}}^{(1,2)})$.

Considering $G_{in}^{(2)}$, we start by examining the dependency on the number of proper and shared signal dimensions of the terms in (4.71). As it can be seen from their definitions, the Hermitian positive semidefinite matrices $\mathbf{N}_j^{(2,1)}$, $\tilde{\mathbf{N}}_{j;\tilde{a}}^{(2,1)}$, $\check{\mathbf{N}}_{j;\tilde{a}}^{(2,1)}$ and $\check{\mathbf{W}}_{j;\tilde{a}}^{(2,1)}$ are determined by the users spatial channel parameters regardless of the number of proper and shared signal dimensions. Hence, $\text{rank}(\mathbf{N}_j^{(2,1)})$ and $r_a^{(2)}$ are independent of D_c , $D_p^{(1)}$ and $D_p^{(2)}$. It remains to show that $\text{rank}(\mathbf{K}_{j;a}^{(2,1)})$ (and hence, $\kappa_{j;a}^{(2,1)}$) is independent of D_c , $D_p^{(1)}$ and $D_p^{(2)}$. For arbitrary $n \times n$ Hermitian matrices $\mathbf{A}$ and $\mathbf{B}$, we use the partial order relation $\mathbf{A} \succeq \mathbf{B}$ from [105] to indicate that the $n \times n$ matrix $\mathbf{A} - \mathbf{B}$ (obtained by subtracting $\mathbf{B}$ from $\mathbf{A}$) is positive semidefinite, $\mathbf{A} - \mathbf{B} \succeq \mathbf{0}_{n \times n}$. Here a partial order relation between the matrices $\frac{\alpha_p^{(2)}}{\check{r}^{(2,1)}} \{ \mathbf{C}_a^{(2)} \cdot \check{\mathbf{W}}_{j;a}^{(2,1)} \cdot [\mathbf{C}_a^{(2)}]^\dagger + \mathbf{N}_j^{(2,1)} \}$, $\mathbf{K}_{j;a}^{(2,1)}$ and $\frac{N_G^{(1)} \alpha_p^{(1)}}{r_T^{(1)} D^{(1)}} \{ \mathbf{C}_a^{(2)} \cdot \check{\mathbf{W}}_{j;a}^{(2,1)} \cdot [\mathbf{C}_a^{(2)}]^\dagger + \mathbf{N}_j^{(2,1)} \}$ can be obtained by distinguishing two cases: $\frac{\alpha_p^{(2)}}{\check{r}^{(2,1)}} \leq \frac{N_G^{(1)} \alpha_p^{(1)}}{r_T^{(1)} D^{(1)}}$ and $\frac{\alpha_p^{(2)}}{\check{r}^{(2,1)}} > \frac{N_G^{(1)} \alpha_p^{(1)}}{r_T^{(1)} D^{(1)}}$. If $\frac{\alpha_p^{(2)}}{\check{r}^{(2,1)}} \leq \frac{N_G^{(1)} \alpha_p^{(1)}}{r_T^{(1)} D^{(1)}}$ then $\frac{N_G^{(1)} \alpha_p^{(1)}}{r_T^{(1)} D^{(1)}} \{ \mathbf{C}_a^{(2)} \cdot \check{\mathbf{W}}_{j;a}^{(2,1)} \cdot [\mathbf{C}_a^{(2)}]^\dagger + \mathbf{N}_j^{(2,1)} \} \succeq \mathbf{K}_{j;a}^{(2,1)} \succeq \frac{\alpha_p^{(2)}}{\check{r}^{(2,1)}} \{ \mathbf{C}_a^{(2)} \cdot \check{\mathbf{W}}_{j;a}^{(2,1)} \cdot [\mathbf{C}_a^{(2)}]^\dagger + \mathbf{N}_j^{(2,1)} \}$. Let $\lambda_l(\mathbf{C}_a^{(2)} \cdot \check{\mathbf{W}}_{j;a}^{(2,1)} \cdot [\mathbf{C}_a^{(2)}]^\dagger + \mathbf{N}_j^{(2,1)})$ and $\lambda_l(\mathbf{K}_{j;a}^{(2,1)})$ denote the l^{th} eigenvalues (in decreasing order) of the matrices $\mathbf{C}_a^{(2)} \cdot \check{\mathbf{W}}_{j;a}^{(2,1)} \cdot [\mathbf{C}_a^{(2)}]^\dagger + \mathbf{N}_j^{(2,1)}$ and

$\mathbf{K}_{j;a}^{(2,1)}$, respectively. From [106, Theorem F.1, p. 475], we have that

$$\begin{aligned} \frac{N_G^{(1)} \alpha_p^{(1)}}{r_T^{(1)} D^{(1)}} \lambda_l(\mathbf{C}_a^{(2)} \cdot \check{\mathbf{W}}_{j;a}^{(2,1)} \cdot [\mathbf{C}_a^{(2)}]^\dagger + \mathbf{N}_j^{(2,1)}) &\geq \lambda_l(\mathbf{K}_{j;a}^{(2,1)}) \\ &\geq \frac{\alpha_p^{(2)}}{\check{r}^{(2,1)}} \lambda_l(\mathbf{C}_a^{(2)} \cdot \check{\mathbf{W}}_{j;a}^{(2,1)} \cdot [\mathbf{C}_a^{(2)}]^\dagger + \mathbf{N}_j^{(2,1)}) \geq 0. \end{aligned}$$

One can easily show that $\lambda_l(\mathbf{C}_a^{(2)} \cdot \check{\mathbf{W}}_{j;a}^{(2,1)} \cdot [\mathbf{C}_a^{(2)}]^\dagger + \mathbf{N}_j^{(2,1)}) = 0$ if and only if $\lambda_l(\mathbf{K}_{j;a}^{(2,1)}) = 0$. Since the rank of a Hermitian matrix is equal to the number of its nonzero eigenvalues [105, p. 175], we have $\text{rank}(\mathbf{K}_{j;a}^{(2,1)}) = \text{rank}(\mathbf{C}_a^{(2)} \cdot \check{\mathbf{W}}_{j;a}^{(2,1)} \cdot [\mathbf{C}_a^{(2)}]^\dagger + \mathbf{N}_j^{(2,1)})$ which is independent of D_c , $D_p^{(1)}$ and $D_p^{(2)}$. This result can be obtained using a similar approach if $\frac{\alpha_p^{(2)}}{\check{r}^{(2,1)}} > \frac{N_G^{(1)} \alpha_p^{(1)}}{r_T^{(1)} D^{(1)}}$. We conclude that the dependency of $G_{in}^{(2)}$ on the number of proper and shared signal dimensions is only through the terms $D_{p;a}^{(2)}$, D_c and $\check{D}^{(2)}$. Similarly, $G_{in}^{(1)}$ of (4.72) depends on the number of proper and shared signal dimensions only through the terms $D_{p;\tilde{a}}^{(1)}$, D_c and $\check{D}^{(1)}$. The coefficients $\kappa_{j;a}^{(2,1)}$ and $\kappa_{\tilde{j};\tilde{a}}^{(1,2)}$ are determined by the spatial channels of both users, while the ranks $r_a^{(2)}$ and $r_{\tilde{a}}^{(1)}$ depend only on the TOG propagation matrices of user 2 and user 1, respectively. Furthermore, each shared signal dimension contributes to the multiplexing gain $G_{in}^{(2)}$ (respectively, $G_{in}^{(1)}$) by an amount $\sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(1)}} (\kappa_{j;a}^{(2,1)}/n_c)$ (respectively, $\sum_{(\tilde{j},\tilde{a}) \in \tilde{\mathcal{A}}_G^{(2)}} (\kappa_{\tilde{j};\tilde{a}}^{(1,2)}/n_c)$). The term $r_a^{(k)} D_{p;a}^{(k)}$ (with $k = 2$ in (4.71) and $k = 1$ in (4.72)) represents the contribution to the multiplexing gain of user k by the proper signal dimensions of its a^{th} TOG. Furthermore, for any pair (a, j) we have

$$\begin{aligned} \kappa_{j;a}^{(2,1)} &\equiv \text{rank}(\mathbf{C}_a^{(2)} \cdot \check{\mathbf{W}}_{j;a}^{(2,1)} \cdot [\mathbf{C}_a^{(2)}]^\dagger + \mathbf{N}_j^{(2,1)}) - \text{rank}(\mathbf{N}_j^{(2,1)}) \\ &\stackrel{(a)}{\leq} \text{rank}(\mathbf{C}_a^{(2)} \cdot \check{\mathbf{W}}_{j;a}^{(2,1)} \cdot [\mathbf{C}_a^{(2)}]^\dagger) \stackrel{(b)}{\leq} r_a^{(2)} \end{aligned} \quad (4.73)$$

with (a) due to (C.24) and (b) due to (C.23). Similarly, we can show that $\kappa_{\tilde{j};\tilde{a}}^{(1,2)} \leq r_{\tilde{a}}^{(1)}$ for all pairs $(\tilde{j}, \tilde{a})$.

The multiplexing gains associated with the corner points V_0 , $\tilde{V}_0$, $\tilde{V}_1$ and V_1 of $\mathcal{R}_{in}$ in Figure 4.2 are given by $(0, G_{out}^{(2)})$, $(G_{in}^{(1)}, G_{out}^{(2)})$, $(G_{out}^{(1)}, G_{in}^{(2)})$ and $(G_{out}^{(1)}, 0)$, respectively. We analyze the behavior of the multiplexing gains associated with the different boundary points of $\mathcal{R}_{in}$ using the normalizations $\check{D}^{(k)} = D^{(k)}$ and $\check{D}^{(k)} = D_T$, $k = 1, 2$. Recall that the segments $V_1 \tilde{V}_1$ and $V_0 \tilde{V}_0$ are also part of the boundary of the capacity region since $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$ coincide on these segments. We consider the following cases.

a) *Points on the segment $V_1 \tilde{V}_1$:* Using time-sharing between the codes associated with the

corner points V_1 and $\tilde{V}_1$, any point on the segment $V_1\tilde{V}_1$ corresponds to achievable rates, with spatial multiplexing gains given by $G_{out}^{(1)}$ for user 1 and by $\tau_{11}G_{in}^{(2)}$ for user 2 where $\tau_{11} \in [0, 1]$. The behavior of $G_{out}^{(1)}$ as a function of the number of shared signal dimensions has been analyzed in Section 4.5.1. We now analyze the behavior of $\tau_{11}G_{in}^{(2)}$ which, for a given τ_{11} , is equivalent to the behavior of $G_{in}^{(2)}$. If $\check{D}^{(2)} = D^{(2)}$, (4.71) can be written as

$$[G_{in}^{(2)}]_{\check{D}^{(2)}=D^{(2)}} = \frac{\sum_{a=1}^{N_G^{(2)}} r_a^{(2)} D_{p;a}^{(2)} + D_c \sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(1)}} (\kappa_{j;a}^{(2,1)} / n_c)}{D_p^{(2)} + D_c} \quad (4.74)$$

In the first scenario, we increase D_c by increasing $\tilde{D}_c$ with $D_p^{(1)}$, $D_p^{(2)}$ held fixed. In this case, the multiplexing gain $[G_{in}^{(2)}]_{\check{D}^{(2)}=D^{(2)}}$ increases monotonically if $D_p^{(2)} \sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(1)}} (\kappa_{j;a}^{(2,1)} / n_c) \geq \sum_{a=1}^{N_G^{(2)}} r_a^{(2)} D_{p;a}^{(2)}$, and decreases monotonically if $D_p^{(2)} \sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(1)}} (\kappa_{j;a}^{(2,1)} / n_c) \leq \sum_{a=1}^{N_G^{(2)}} r_a^{(2)} D_{p;a}^{(2)}$ from Appendix B.2. For MIMO BCs where all the signal dimensions of user 2 are shared ($D_p^{(2)} = 0$), (4.74) becomes $[G_{in}^{(2)}]_{\check{D}^{(2)}=D^{(2)}} = \sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(1)}} (\kappa_{j;a}^{(2,1)} / n_c)$ which is independent of D_c . Note that in this case the set $\tilde{\mathcal{A}}_G^{(1)}$ cannot be empty since, otherwise, the set $\tilde{\mathcal{A}}_{G;c}^{(1)}$ is empty and $\text{rank}(\mathbf{N}_j^{(2,1)}) = L_t^{(2)} N_R^{(2)}$ for all $j \in \mathcal{A}_{G;c}^{(1)}$ which, using (4.66) and $\sum_{j \in \mathcal{A}_{G;c}^{(1)}} D_{c;j}^{(2)} = D_c = D^{(2)}$, results in $r_{\max}^{(2,1)} = L_t^{(2)} N_R^{(2)} D^{(2)} = r_{\max}^{(2)}$ and $[G_{in}^{(2)}]_{\check{D}^{(2)}=D^{(2)}} = 0$. We can interpret the behavior of $[G_{in}^{(2)}]_{\check{D}^{(2)}=D^{(2)}}$ as follows. If $D_p^{(2)} = 0$ and since each shared signal dimension contributes to $[G_{in}^{(2)}]_{\check{D}^{(2)}=D^{(2)}}$ by an equal amount $\sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(1)}} (\kappa_{j;a}^{(2,1)} / n_c)$, $[G_{in}^{(2)}]_{\check{D}^{(2)}=D^{(2)}}$ becomes independent of D_c when adding all these contributions and normalizing by $D^{(2)} \equiv D_c$. When $D_p^{(2)} > 0$, the contribution to $[G_{in}^{(2)}]_{\check{D}^{(2)}=D^{(2)}}$ of the proper signal dimensions allocated to a given TOG a is different from that of the other proper signal dimensions allocated to the other TOGs as well as from the contribution of each shared signal dimension. In this case, adding the contributions of all user 2's signal dimensions and normalizing by $D^{(2)}$ does not yield a multiplexing gain that remains constant when D_c varies. We now consider the second scenario where D_T is held fixed and D_c increases at the expense of $D_p^{(1)}$ and $D_p^{(2)}$ by making $\delta D^{(1)}$ and $\delta D^{(2)}$ signal dimensions shared out of some initial values $D_{p;0}^{(1)}$ and $D_{p;0}^{(2)}$ of $D_p^{(1)}$ and $D_p^{(2)}$, respectively. For $k = 1, 2$ and $a = 1, \dots, N_G^{(k)}$, let $D_{G;0}^{(k)}$ denote the initial value of $D_{p;a}^{(k)}$, that is, all the TOGs have equal number of signal dimensions that are all proper initially. Thus, we have $D_{p;0}^{(k)} = N_G^{(k)} D_{G;0}^{(k)} \equiv n_c D_{G;0}^{(k)}$. For each TOG a , we make $\delta D_G^{(k)}$ signal dimensions shared (with $\delta D_G^{(k)} < D_{G;0}^{(k)}$) out of its initial $D_{G;0}^{(k)}$ proper signal dimensions. This leads to $D_{p;a}^{(k)} = D_{G;0}^{(k)} - \delta D_G^{(k)}$ for each a and $\delta D^{(k)} = n_c \delta D_G^{(k)}$ for $k = 1, 2$. The number of shared signal dimensions for a pair of TOGs $(a, \tilde{a}) \in \mathcal{A}_G$ is $\tilde{D}_c = \delta D_G^{(k)} + \delta D_G^{(l)}$, for $(k, l) = (1, 2), (2, 1)$,

yielding $D_G^{(k)} = D_{p;a}^{(k)} + \tilde{D}_c = D_{G;0}^{(k)} + \delta D_G^{(l)}$ and $D_c = n_c \tilde{D}_c = \delta D^{(1)} + \delta D^{(2)}$. Using $\sum_{a=1}^{N_G^{(2)}} r_a^{(2)} = \sum_{a=1}^{n_c} r_a^{(2)} = r_T^{(2)}$ and $D_p^{(2)} = D_{p;0}^{(2)} - \delta D^{(2)}$, we can write $[G_{in}^{(2)}]_{\check{D}^{(2)}=D^{(2)}}$ of (4.74) as

$$[G_{in}^{(2)}]_{\check{D}^{(2)}=D^{(2)}} = \frac{r_T^{(2)} D_{G;0}^{(2)} + (\delta D^{(1)} + \delta D^{(2)}) \left[\sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(1)}} \frac{\kappa_{j;a}^{(2,1)}}{n_c} \right] - \delta D^{(2)} \frac{r_T^{(2)}}{n_c}}{D_{p;0}^{(2)} + \delta D^{(1)}} \quad (4.75)$$

Assuming fixed $\delta D^{(1)}$, $[G_{in}^{(2)}]_{\check{D}^{(2)}=D^{(2)}}$ is a linear function of $\delta D^{(2)}$. One can see that incrementing $\delta D^{(2)}$ by one increases the numerator in (4.75) by $\sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(1)}} \frac{\kappa_{j;a}^{(2,1)}}{n_c}$ and decreases it by $\frac{r_T^{(2)}}{n_c}$. Let $\xi^{(2,1)} \triangleq \left(\sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(1)}} \kappa_{j;a}^{(2,1)} \right) - r_T^{(2)}$. Since $\kappa_{j;a}^{(2,1)} \leq r_a^{(2)}$ from (4.73), we can write

$$\sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(1)}} \kappa_{j;a}^{(2,1)} \leq \sum_{a: (a,j) \in \tilde{\mathcal{A}}_G^{(1)}} r_a^{(2)} \leq \sum_{a=1}^{N_G^{(2)}} r_a^{(2)} = r_T^{(2)}$$

Thus, $\xi^{(2,1)} \leq 0$ and $[G_{in}^{(2)}]_{\check{D}^{(2)}=D^{(2)}}$ decreases linearly with increasing $\delta D^{(2)}$. We now fix $\delta D^{(2)}$ and increase $\delta D^{(1)}$. In this case, we have from Appendix B.2 that $[G_{in}^{(2)}]_{\check{D}^{(2)}=D^{(2)}}$ increases monotonically if $(D_{p;0}^{(2)} - \delta D^{(2)}) \left[\sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(1)}} (\kappa_{j;a}^{(2,1)} / n_c) \right] - r_T^{(2)} D_{G;0}^{(2)} + \delta D^{(2)} \frac{r_T^{(2)}}{n_c} \geq 0$, and decreases monotonically if $(D_{p;0}^{(2)} - \delta D^{(2)}) \left[\sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(1)}} (\kappa_{j;a}^{(2,1)} / n_c) \right] - r_T^{(2)} D_{G;0}^{(2)} + \delta D^{(2)} \frac{r_T^{(2)}}{n_c} \leq 0$. Using $D_{p;0}^{(2)} = n_c D_{G;0}^{(2)}$ and $\delta D^{(2)} = n_c \delta D_G^{(2)}$, we have

$$(D_{p;0}^{(2)} - \delta D^{(2)}) \left[\sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(1)}} (\kappa_{j;a}^{(2,1)} / n_c) \right] - r_T^{(2)} D_{G;0}^{(2)} + \delta D^{(2)} \frac{r_T^{(2)}}{n_c} = (D_{G;0}^{(2)} - \delta D_G^{(2)}) \xi^{(2,1)} \leq 0$$

since $\xi^{(2,1)} \leq 0$ and $\delta D_G^{(2)} < D_{G;0}^{(2)}$. Hence, $[G_{in}^{(2)}]_{\check{D}^{(2)}=D^{(2)}}$ decreases monotonically with increasing $\delta D^{(1)}$. We now consider the normalization $\check{D}^{(2)} = D_T$. In this case, (4.71) can be written as

$$[G_{in}^{(2)}]_{\check{D}^{(2)}=D_T} = \frac{\sum_{a=1}^{N_G^{(2)}} r_a^{(2)} D_{p;a}^{(2)} + D_c \sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(1)}} (\kappa_{j;a}^{(2,1)} / n_c)}{D_p^{(1)} + D_p^{(2)} + D_c} \quad (4.76)$$

In the first scenario where D_c increases with $D_p^{(1)}$, $D_p^{(2)}$ held fixed, the multiplexing gain $[G_{in}^{(2)}]_{\check{D}^{(2)}=D_T}$ increases monotonically if $[D_p^{(1)} + D_p^{(2)}] \sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(1)}} (\kappa_{j;a}^{(2,1)} / n_c) \geq \sum_{a=1}^{N_G^{(2)}} r_a^{(2)} D_{p;a}^{(2)}$, and it decreases monotonically if $[D_p^{(1)} + D_p^{(2)}] \sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(1)}} (\kappa_{j;a}^{(2,1)} / n_c) \leq \sum_{a=1}^{N_G^{(2)}} r_a^{(2)} D_{p;a}^{(2)}$.

from Appendix B.2. If $D_p^{(2)} = 0$, $[G_{in}^{(2)}]_{\check{D}^{(2)}=D_T}$ becomes independent of D_c only when all the signal dimensions are shared (i.e. $D_p^{(1)} = D_p^{(2)} = 0$). If $D_p^{(2)} = 0$ and $D_p^{(1)} \neq 0$, then $[G_{in}^{(2)}]_{\check{D}^{(2)}=D_T}$ increases with increasing D_c . In the second scenario where D_T is held fixed and D_c increases at the expense of $D_p^{(1)}$ and $D_p^{(2)}$, we can write $[G_{in}^{(2)}]_{\check{D}^{(2)}=D_T}$ using the previous notations as

$$[G_{in}^{(2)}]_{\check{D}^{(2)}=D_T} = \frac{r_T^{(2)} D_{G;0}^{(2)} + (\delta D^{(1)} + \delta D^{(2)}) \left[\sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(1)}} \frac{\kappa_{j;a}^{(2,1)}}{n_c} \right] - \delta D^{(2)} \frac{r_T^{(2)}}{n_c}}{D_T} \quad (4.77)$$

If $\delta D^{(1)}$ is fixed and $\delta D^{(2)}$ increases then, similarly to $[G_{in}^{(2)}]_{\check{D}^{(2)}=D^{(2)}}$ of (4.75), the multiplexing gain $[G_{in}^{(2)}]_{\check{D}^{(2)}=D_T}$ decreases linearly since $\xi^{(2,1)} \leq 0$. If $\delta D^{(2)}$ is fixed, increasing $\delta D^{(1)}$ leads always to a linear increase in $[G_{in}^{(2)}]_{\check{D}^{(2)}=D_T}$ since $\sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(1)}} (\kappa_{j;a}^{(2,1)} / n_c) \geq 0$.

b) Points on the segment $\tilde{V}_1 \tilde{V}_0$: Using time-sharing between the codes associated with the corner points $\tilde{V}_1$ and $\tilde{V}_0$, any point on the segment $\tilde{V}_1 \tilde{V}_0$ corresponds to achievable rates, with spatial multiplexing gains given by $G_\tau^{(1)} \triangleq \tau G_{out}^{(1)} + (1 - \tau) G_{in}^{(1)}$ for user 1 (where $\tau \in [0, 1]$) and by $G_\tau^{(2)} \triangleq \tau G_{in}^{(2)} + (1 - \tau) G_{out}^{(2)}$ for user 2. We start by considering user 2's multiplexing gain with $\check{D}^{(2)} = D^{(2)}$ and for a given $\tau \in (0, 1)$. Using $[G_{out}^{(2)}]_{\check{D}^{(2)}=D^{(2)}} = \frac{r_T^{(2)}}{N_G^{(2)}}$ and (4.74), we can write

$$[G_\tau^{(2)}]_{\check{D}^{(2)}=D^{(2)}} = \tau \frac{\sum_{a=1}^{N_G^{(2)}} r_a^{(2)} D_{p;a}^{(2)} + D_c \sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(1)}} (\kappa_{j;a}^{(2,1)} / n_c)}{D_p^{(2)} + D_c} + (1 - \tau) \frac{r_T^{(2)}}{N_G^{(2)}} \quad (4.78)$$

With the second term on the right hand side of (4.78) being constant, the behavior of $[G_\tau^{(2)}]_{\check{D}^{(2)}=D^{(2)}}$ as a function of D_c is similar to that of $G_{in}^{(2)}$ of (4.74) in the two scenarios that we considered for increasing D_c . If $\check{D}^{(2)} = D_T$, $G_\tau^{(2)}$ can be written using (4.60) with (4.67a) and (4.76) as

$$[G_\tau^{(2)}]_{\check{D}^{(2)}=D_T} = \frac{\sum_{a=1}^{N_G^{(2)}} [\tau r_a^{(2)} + (1 - \tau) \frac{r_T^{(2)}}{N_G^{(2)}}] D_{p;a}^{(2)} + D_c \left[\sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(1)}} \tau \frac{\kappa_{j;a}^{(2,1)}}{n_c} + (1 - \tau) \frac{r_T^{(2)}}{N_G^{(2)}} \right]}{D_p^{(1)} + D_p^{(2)} + D_c}$$

Using $\tilde{\kappa}_a^{(2)}(\tau) \triangleq \tau r_a^{(2)} + (1 - \tau) \frac{r_T^{(2)}}{N_G^{(2)}}$ and $\tilde{\kappa}^{(2,1)}(\tau) \triangleq \sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(1)}} \tau \frac{\kappa_{j;a}^{(2,1)}}{n_c} + (1 - \tau) \frac{r_T^{(2)}}{N_G^{(2)}}$, we have

$$[G_\tau^{(2)}]_{\check{D}^{(2)}=D_T} = \frac{\sum_{a=1}^{N_G^{(2)}} \tilde{\kappa}_a^{(2)}(\tau) D_{p;a}^{(2)} + D_c \tilde{\kappa}^{(2,1)}(\tau)}{D_p^{(1)} + D_p^{(2)} + D_c} \quad (4.79)$$

In the first scenario where D_c increases with $D_p^{(1)}, D_p^{(2)}$ held fixed, $[G_\tau^{(2)}]_{\check{D}^{(2)}=D_T}$ increases monotonically if $[D_p^{(1)} + D_p^{(2)}]\tilde{\kappa}^{(2,1)}(\tau) \geq \sum_{a=1}^{N_G^{(2)}} \tilde{\kappa}_a^{(2)}(\tau) D_{p,a}^{(2)}$ and it decreases monotonically if $[D_p^{(1)} + D_p^{(2)}]\tilde{\kappa}^{(2,1)}(\tau) \leq \sum_{a=1}^{N_G^{(2)}} \tilde{\kappa}_a^{(2)}(\tau) D_{p,a}^{(2)}$ from Appendix B.2. For MIMO BCs where all the signal dimensions are shared ($D_p^{(1)} = D_p^{(2)} = 0$), (4.79) reduces to $[G_\tau^{(2)}]_{\check{D}^{(2)}=D_T} = \tilde{\kappa}^{(2,1)}(\tau)$ which is a function of τ and independent of D_c . If $D_p^{(2)} = 0$ and $D_p^{(1)} \neq 0$ then $[G_\tau^{(2)}]_{\check{D}^{(2)}=D_T}$ increases with increasing D_c for any τ . In the second scenario where D_T is held fixed and D_c increases at the expense of $D_p^{(1)}$ and $D_p^{(2)}$, we can write $[G_\tau^{(2)}]_{\check{D}^{(2)}=D_T}$, using the previous notations with $n_c = N_G^{(2)}$ and $\sum_{a=1}^{N_G^{(2)}} \tilde{\kappa}_a^{(2)}(\tau) = r_T^{(2)}$, as

$$[G_\tau^{(2)}]_{\check{D}^{(2)}=D_T} = \frac{\sum_{a=1}^{N_G^{(2)}} \tilde{\kappa}_a^{(2)}(\tau) D_{G,0}^{(2)} + (\delta D^{(1)} + \delta D^{(2)})\tilde{\kappa}^{(2,1)}(\tau) - \delta D^{(2)} \frac{r_T^{(2)}}{N_G^{(2)}}}{D_T} \quad (4.80)$$

If $\delta D^{(1)}$ is fixed and $\delta D^{(2)}$ increases then $[G_\tau^{(2)}]_{\check{D}^{(2)}=D_T}$ decreases linearly since $\tilde{\kappa}^{(2,1)}(\tau) - \frac{r_T^{(2)}}{N_G^{(2)}} = \frac{\tau}{n_c} \xi^{(2,1)} \leq 0$. If $\delta D^{(2)}$ is fixed, increasing $\delta D^{(1)}$ leads always to an increase in the multiplexing gain $[G_\tau^{(2)}]_{\check{D}^{(2)}=D_T}$ since $\tilde{\kappa}^{(2,1)}(\tau) \geq 0$. The behavior of the multiplexing gain $G_\tau^{(1)}$ of user 1 can be investigated similarly to $G_\tau^{(2)}$ by switching the user indexes and by replacing τ by $1 - \tau$.

c) *Points on the segment $V_0\tilde{V}_0$* : Using time-sharing between the codes associated with the corner points V_0 and $\tilde{V}_0$, any point on the segment $V_0\tilde{V}_0$ corresponds to achievable rates, with spatial multiplexing gains given by $\tau_{00}G_{in}^{(1)}$ for user 1 (where $\tau_{00} \in [0, 1]$) and by $G_{out}^{(2)}$ for user 2. Here, the analysis of the behavior of $\tau_{00}G_{in}^{(1)}$ or, equivalently, $G_{in}^{(1)}$ for a given τ_{00} is similar to the analysis of $G_{in}^{(2)}$ in case a) above. Hence, similar conclusions as for case a) can be drawn with the users indexes switched. The behavior of $G_{out}^{(2)}$ as a function of the number of shared signal dimension has been analyzed in Section 4.5.1.

An important conclusion from the previous analysis is the following. When D_T is fixed and D_c is increasing at the expense of $D_p^{(1)}$ and $D_p^{(2)}$, the behavior of the multiplexing gains as functions of D_c is determined in most of the cases by the users spatial channels through the parameters $\xi^{(2,1)}$ and $\xi^{(1,2)} \triangleq (\sum_{(\tilde{j}, \tilde{a}) \in \tilde{\mathcal{A}}_G^{(2)}} \kappa_{\tilde{j}; \tilde{a}}^{(1,2)}) - r_T^{(1)}$. Since $\xi^{(2,1)} \leq 0$ (respectively, $\xi^{(1,2)} \leq 0$), maximal multiplexing gain for user 2 (respectively, user 1) is obtained when all the signal dimensions allocated to this user are proper. This observation can be interpreted as follows. Considering user 2, the parameter $\xi^{(2,1)} \equiv (\sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(1)}} \kappa_{j;a}^{(2,1)}) - r_T^{(2)}$ represents the difference between $(\sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(1)}} \kappa_{j;a}^{(2,1)})$ which depends on the quality of both users' channels and $r_T^{(2)}$ which is determined only by the channel of user 2. With $\xi^{(2,1)} \leq 0$,

the more our signaling scheme exploits the channels of both users (by sharing signal dimensions), the smaller is the multiplexing gain that can be obtained for user 2. Thus, the channel of user 1 can be seen as destructively contributing through these shared signal dimensions to the decrease of the multiplexing gain of user 2. By switching the users indexes, a similar analysis can be carried out for the multiplexing gain of user 1 based on the quality parameter $\xi^{(1,2)}$.

We finally consider the behavior of the multiplexing gains $G_{in}^{(2)}$ (4.71), $G_{in}^{(1)}$ (4.72) and $\mathcal{R}_{in}$ when $D_c \rightarrow \infty$ assuming fixed $D_p^{(1)}$, $D_p^{(2)}$ and fixed (nonzero) SNR per dimension, i.e. $\frac{(P_T/N_0)}{D_T} = \text{Constant}$. We denote this constant by c_T , i.e. $\frac{(P_T/N_0)}{D_T} = c_T > 0$. From (4.71) and since $\{r_a^{(2)}\}_{a=1}^{N_G^{(2)}}$ and $\sum_{(a,j) \in \check{\mathcal{A}}_G^{(1)}} (\kappa_{j;a}^{(2,1)}/n_c)$ are independent of D_c , one can easily show that

$$\lim_{D_c \rightarrow \infty} G_{in}^{(2)} = \lim_{D_c \rightarrow \infty} \frac{D_c \sum_{(a,j) \in \check{\mathcal{A}}_G^{(1)}} (\kappa_{j;a}^{(2,1)}/n_c)}{\check{D}^{(2)}} \stackrel{(a)}{=} \sum_{(a,j) \in \check{\mathcal{A}}_G^{(1)}} (\kappa_{j;a}^{(2,1)}/n_c)$$

where (a) is due to $\frac{D_c}{\check{D}^{(2)}} = \frac{D_c}{D_p^{(2)} + D_c} \rightarrow 1$ if $\check{D}^{(2)} = D^{(2)}$ and to $\frac{D_c}{\check{D}^{(2)}} = \frac{D_c}{D_p^{(1)} + D_p^{(2)} + D_c} \rightarrow 1$ if $\check{D}^{(2)} = D_T$. It follows that, when $D_c \rightarrow \infty$,

$$G_{in}^{(2)} \log(P_T/N_0) \equiv G_{in}^{(2)} \log[c_T(D_p^{(1)} + D_p^{(2)} + D_c)] \rightarrow \sum_{(a,j) \in \check{\mathcal{A}}_G^{(1)}} (\kappa_{j;a}^{(2,1)}/n_c) \log(D_c)$$

since $\log[c_T(D_p^{(1)} + D_p^{(2)} + D_c)] \approx \log[c_T D_c] \approx \log(D_c)$ for sufficiently large D_c . Similarly, we can show from (4.72) that when $D_c \rightarrow \infty$

$$G_{in}^{(1)} \log(P_T/N_0) \rightarrow \sum_{(\tilde{j}, \tilde{a}) \in \check{\mathcal{A}}_G^{(2)}} (\kappa_{\tilde{j}; \tilde{a}}^{(1,2)}/n_c) \log(D_c)$$

for $\check{D}^{(1)} = D^{(1)}$ or $\check{D}^{(1)} = D_T$. Moreover, we can show using Section 4.5.1 that, when $D_c \rightarrow \infty$, $G_{out}^{(1)} \log(P_T/N_0) \rightarrow \frac{r_T^{(1)}}{N_G^{(1)}} \log(D_c)$ and $G_{out}^{(2)} \log(P_T/N_0) \rightarrow \frac{r_T^{(2)}}{N_G^{(2)}} \log(D_c)$, regardless of $\check{D}^{(1)}$ and $\check{D}^{(2)}$. It follows, using Tables 4.1 and 4.2 with (4.42) for $k = 1, 2$, (4.51) and (4.53), that the coordinates of the corner points $V_0, \tilde{V}_0, \tilde{V}_1$ and V_1 of $\mathcal{R}_{in}$ in Figure 4.2 converge to $(0, \frac{r_T^{(2)}}{N_G^{(2)}} \log(D_c))$, $(\sum_{(\tilde{j}, \tilde{a}) \in \check{\mathcal{A}}_G^{(2)}} (\kappa_{\tilde{j}; \tilde{a}}^{(1,2)}/n_c) \log(D_c), \frac{r_T^{(2)}}{N_G^{(2)}} \log(D_c))$, $(\frac{r_T^{(1)}}{N_G^{(1)}} \log(D_c), \sum_{(a,j) \in \check{\mathcal{A}}_G^{(1)}} (\kappa_{j;a}^{(2,1)}/n_c) \log(D_c))$ and $(\frac{r_T^{(1)}}{N_G^{(1)}} \log(D_c), 0)$, respectively. Hence, the boundary of $\mathcal{R}_{in}$ scales logarithmically with D_c , regardless of the constant c_T .

The results presented in this subsection for the impact of signal dimensions sharing are mainly based on the fact that the coefficients $r_a^{(2)}$ and $\sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(1)}} (\kappa_{j;a}^{(2,1)} / n_c)$ in (4.71) and $r_{\tilde{a}}^{(1)}$ and $\sum_{(\tilde{j}, \tilde{a}) \in \tilde{\mathcal{A}}_G^{(2)}} (\kappa_{\tilde{j}; \tilde{a}}^{(1,2)} / n_c)$ in (4.72) are independent of the number of shared and proper signal dimensions. These coefficients are determined by the users' spatial channels as we show in the following two examples. We consider keyhole channels for the first example. In this case, we have $r_a^{(2)} = 1, \forall a$, and $r_{\tilde{a}}^{(1)} = 1, \forall \tilde{a}$, from Section 3.1.1. Using (4.73), it follows that $0 \leq \kappa_{j;a}^{(2,1)} \leq 1$ for keyhole channels, or $\kappa_{j;a}^{(2,1)} \in \{0, 1\}$ since $\kappa_{j;a}^{(2,1)}$ is an integer. Similarly, we can show that $\kappa_{\tilde{j}; \tilde{a}}^{(1,2)}$ is either 0 or 1 for keyhole channels. We now consider 3GPP channels as a second example. In this case, we have $r_a^{(2)} = \min(n_a^{(2)}, L_t^{(2)} N_R^{(2)}), \forall a$, and $r_{\tilde{a}}^{(1)} = \min(n_{\tilde{a}}^{(1)}, L_t^{(1)} N_R^{(1)}), \forall \tilde{a}$. Using (4.73), it follows that $0 \leq \kappa_{j;a}^{(2,1)} \leq \min(n_a^{(2)}, L_t^{(2)} N_R^{(2)})$, or $\kappa_{j;a}^{(2,1)} \in \{0, 1, \dots, \min(n_a^{(2)}, L_t^{(2)} N_R^{(2)})\}$. Similarly, we can show that $\kappa_{\tilde{j}; \tilde{a}}^{(1,2)} \in \{0, 1, \dots, \min(n_{\tilde{a}}^{(1)}, L_t^{(1)} N_R^{(1)})\}$ for 3GPP channels. Thus, for different spatial channels the coefficients $r_a^{(2)}$ and $\sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(1)}} (\kappa_{j;a}^{(2,1)} / n_c)$ in (4.71) and $r_{\tilde{a}}^{(1)}$ and $\sum_{(\tilde{j}, \tilde{a}) \in \tilde{\mathcal{A}}_G^{(2)}} (\kappa_{\tilde{j}; \tilde{a}}^{(1,2)} / n_c)$ in (4.72) can take different values, resulting eventually in different behavior of the multiplexing gains $G_{in}^{(1)}$ and $G_{in}^{(2)}$ as functions of D_c . The next subsection investigates the multiplexing gain impact of the parameter $L_t^{(k)}$ for 3GPP channels, using the multiplexing gains of (4.69) and (4.70) which are more general than those of (4.71) and (4.72).

Impact of the number of time resolvable clusters

Assuming a fixed number of time resolvable clusters for the other user in the system, we now investigate the effect of the parameter $L_t^{(k)}$ on the multiplexing gain of user k ($k = 1, 2$) for 3GPP channels. We consider the boundary points of $\mathcal{R}_{in}$ as follows.

a) Points on the segment $V_1 \tilde{V}_1$: Each of these points is associated with the pair of multiplexing gains $(G_{out}^{(1)}, \tau_{11} G_{in}^{(2)})$, $\tau_{11} \in [0, 1]$. For user 1, the effect of $L_t^{(1)}$ on $G_{out}^{(1)}$ has been discussed in Section 4.5.1 and it is independent of $L_t^{(2)}$. Assuming $L_t^{(1)}$ fixed, we investigate the effect of $L_t^{(2)}$ on $\tau_{11} G_{in}^{(2)}$ or, equivalently, on $G_{in}^{(2)}$. First we start by evaluating the rank of the matrices $\mathbf{N}_j^{(2,1)}$ for 3GPP channels. For these channels, $\tilde{\mathbf{C}}_j^{(2)}$ is most likely full rank, i.e. $\text{rank}(\tilde{\mathbf{C}}_j^{(2)}) = \min(L_t^{(2)} N_R^{(2)}, n_j^{(1)})$. Since $\mathbf{M}_j^{(1)}$ is unitary, this leads to $\text{rank}(\tilde{\mathbf{C}}_j^{(2)} \cdot [\mathbf{M}_j^{(1)}]^\dagger) = \min(L_t^{(2)} N_R^{(2)}, n_j^{(1)})$ from [105, p. 13]. Let $\tilde{\mathbf{C}}_{N;j}^{(2,1)}$ denote the $L_t^{(2)} N_R^{(2)} \times r_j^{(1)}$ submatrix formed by the first $r_j^{(1)} \equiv \min(L_t^{(1)} N_R^{(1)}, n_j^{(1)})$ columns of the product matrix $\tilde{\mathbf{C}}_j^{(2)} \cdot [\mathbf{M}_j^{(1)}]^\dagger$. Since the matrices $\tilde{\mathbf{C}}_j^{(2)}$ and $\mathbf{C}_j^{(1)}$ (and, hence, $\mathbf{M}_j^{(1)}$) are randomly generated,

the matrix $\tilde{\mathbf{C}}_{N;j}^{(2,1)}$ is most likely full rank, that is, $\text{rank}(\tilde{\mathbf{C}}_{N;j}^{(2,1)}) = \min(L_t^{(2)} N_R^{(2)}, r_j^{(1)})$. With $\mathbf{B}_j^{(1)} = \text{diag}[\mathbf{I}_{r_j^{(1)}}, \mathbf{0}_{(n_j^{(1)} - r_j^{(1)}) \times (n_j^{(1)} - r_j^{(1)})}]$, one can easily show that $\mathbf{N}_j^{(2,1)} = \tilde{\mathbf{C}}_{N;j}^{(2,1)} \cdot [\tilde{\mathbf{C}}_{N;j}^{(2,1)}]^\dagger$. It follows using (C.15) that $\text{rank}(\mathbf{N}_j^{(2,1)}) = \text{rank}(\tilde{\mathbf{C}}_{N;j}^{(2,1)}) = \min(L_t^{(2)} N_R^{(2)}, r_j^{(1)})$. This result shows that we have $\text{rank}(\mathbf{N}_j^{(2,1)}) < L_t^{(2)} N_R^{(2)}$ only if $r_j^{(1)} < L_t^{(2)} N_R^{(2)}$, in which case $\text{rank}(\mathbf{N}_j^{(2,1)}) = r_j^{(1)}$. Hence, for every $j \in \tilde{\mathcal{A}}_{G;c}^{(1)}$ we have $\text{rank}(\mathbf{N}_j^{(2,1)}) = r_j^{(1)}$. It follows, using the substitution $r_a^{(2)} = \min(n_a^{(2)}, L_t^{(2)} N_R^{(2)})$ in (4.69), that

$$G_{in}^{(2)} = \frac{\sum_{a=1}^{N_G^{(2)}} [D_{p;a}^{(2)} \min(n_a^{(2)}, L_t^{(2)} N_R^{(2)}) + \sum_{j \in \tilde{\mathcal{A}}_{G;c}^{(1)}} D_{c;j;a}^{(2)} \{\text{rank}(\mathbf{K}_{j;a}^{(2,1)}) - r_j^{(1)}\}]}{\tilde{D}^{(2)}} \quad (4.81)$$

with $r_j^{(1)}$ independent of $L_t^{(2)}$. Investigating the effect of $L_t^{(2)}$ on $G_{in}^{(2)}$ can be done by considering each term $D_{p;a}^{(2)} \min(n_a^{(2)}, L_t^{(2)} N_R^{(2)}) + \sum_{j \in \tilde{\mathcal{A}}_{G;c}^{(1)}} D_{c;j;a}^{(2)} \{\text{rank}(\mathbf{K}_{j;a}^{(2,1)}) - r_j^{(1)}\}$ in the numerator on the right hand side of (4.81). If TOG a of user 2 employs proper signal dimensions (i.e. $D_{p;a}^{(2)} \neq 0$), an increase in $L_t^{(2)}$ leads to an increase in the term $D_{p;a}^{(2)} \min(n_a^{(2)}, L_t^{(2)} N_R^{(2)})$ (and, hence, in $G_{in}^{(2)}$) if and only if $n_a^{(2)} > L_t^{(2)} N_R^{(2)}$. Otherwise, we have $D_{p;a}^{(2)} \min(n_a^{(2)}, L_t^{(2)} N_R^{(2)}) \equiv D_{p;a}^{(2)} n_a^{(2)}$, which remains constant when $L_t^{(2)}$ increases. If TOG a of user 2 employs shared signal dimensions, then there exists at least one TOG $j \in \mathcal{A}_{G;c}^{(1)}$ of user 1 for which $D_{c;j;a}^{(2)} \neq 0$. For each such TOG j of user 1, we consider the following two cases. If $r_j^{(1)} < L_t^{(2)} N_R^{(2)}$ then $\text{rank}(\mathbf{N}_j^{(2,1)}) = r_j^{(1)}$ and $j \in \tilde{\mathcal{A}}_{G;c}^{(1)}$, in which case increasing $L_t^{(2)}$ can lead to an increase in the term $D_{c;j;a}^{(2)} \{\text{rank}(\mathbf{K}_{j;a}^{(2,1)}) - r_j^{(1)}\}$ only if $\text{rank}(\mathbf{K}_{j;a}^{(2,1)})$ increases (since $r_j^{(1)}$ is not affected). Note that $\text{rank}(\mathbf{K}_{j;a}^{(2,1)})$ does not always increase when the size of the $(L_t^{(2)} N_R^{(2)} \times L_t^{(2)} N_R^{(2)})$ matrix $\mathbf{K}_{j;a}^{(2,1)}$ increases because it has an upper bound that is independent of $L_t^{(2)}$,

$$\text{rank}(\mathbf{K}_{j;a}^{(2,1)}) \stackrel{(a)}{\leq} \text{rank}(\mathbf{C}_a^{(2)} \cdot \check{\mathbf{W}}_{j;a}^{(2,1)} \cdot [\mathbf{C}_a^{(2)}]^\dagger) + r_j^{(1)} \stackrel{(b)}{\leq} n_a^{(2)} + r_j^{(1)}$$

where (a) is due to (C.24) and (b) is due to (C.23) with $\text{rank}(\mathbf{C}_a^{(2)} \cdot \check{\mathbf{W}}_{j;a}^{(2,1)} \cdot [\mathbf{C}_a^{(2)}]^\dagger) \leq r_a^{(2)} \leq n_a^{(2)}$. If $r_j^{(1)} \geq L_t^{(2)} N_R^{(2)}$ then $\text{rank}(\mathbf{N}_j^{(2,1)}) = L_t^{(2)} N_R^{(2)}$ and $j \in \mathcal{A}_{G;c}^{(1)} - \tilde{\mathcal{A}}_{G;c}^{(1)}$. In this case, increasing $L_t^{(2)}$ can make $r_j^{(1)}$ less or equal than $L_t^{(2)} N_R^{(2)}$ and make j an element of the subset $\tilde{\mathcal{A}}_{G;c}^{(1)}$ of $\mathcal{A}_{G;c}^{(1)}$. This yields an increase in the cardinality of $\tilde{\mathcal{A}}_{G;c}^{(1)}$ and, thus, leads to an increase in the numerator of (4.81) by adding a new (non-negative) term of the form $D_{c;j;a}^{(2)} \{\text{rank}(\mathbf{K}_{j;a}^{(2,1)}) - r_j^{(1)}\}$. Using this analysis for every TOG a of user 2, we conclude that with $L_t^{(1)}$ held fixed $G_{in}^{(2)}$ increases in general when $L_t^{(2)}$ increases. The multiplexing gain $G_{in}^{(2)}$ remains constant when $L_t^{(2)}$ increases only if the following three conditions are satisfied: $n_a^{(2)} < L_t^{(2)} N_R^{(2)}$ for all $a \in \{1, \dots, N_G^{(2)}\}$, $\tilde{\mathcal{A}}_{G;c}^{(1)} = \mathcal{A}_{G;c}^{(1)}$ (that

is, $\text{rank}(\mathbf{N}_j^{(2,1)}) = r_j^{(1)} < L_t^{(2)} N_R^{(2)}$ for all j), and the ranks of the matrices $\mathbf{K}_{j;a}^{(2,1)}$ remain constant when $L_t^{(2)}$ increases for all $j \in \mathcal{A}_{G;c}^{(1)}$ and all $a \in \{1, \dots, N_G^{(2)}\}$. One can easily see that the first two conditions are satisfied when $L_t^{(2)}$ becomes sufficiently large. Let us examine the ranks of the matrices $\mathbf{K}_{j;a}^{(2,1)}$ when $L_t^{(2)}$ is sufficiently large. Using $\text{rank}(\mathbf{N}_j^{(2,1)}) = r_j^{(1)}$, we can write $\check{\mathbf{N}}_{j;a}^{(2,1)} = \sum_{l=r_j^{(1)}+1}^{L_t^{(2)} N_R^{(2)}} (\mathbf{C}_a^{(2)})^\dagger \cdot \mathbf{n}_l(\mathbf{N}_j^{(2,1)}) \cdot \mathbf{n}_l^\dagger(\mathbf{N}_j^{(2,1)}) \cdot \mathbf{C}_a^{(2)}$, which is a summation of $L_t^{(2)} N_R^{(2)} - r_j^{(1)}$ different rank one matrices of size $n_a^{(2)} \times n_a^{(2)}$ and is most likely to satisfy $\text{rank}(\check{\mathbf{N}}_{j;a}^{(2,1)}) = n_a^{(2)}$ when $L_t^{(2)}$ is sufficiently large. With this, we have $\check{\mathbf{W}}_{j;a}^{(2,1)} = \sum_{q=1}^{n_a^{(2)}} \mathbf{n}_q(\check{\mathbf{N}}_{j;a}^{(2,1)}) \cdot [\mathbf{n}_q(\check{\mathbf{N}}_{j;a}^{(2,1)})]^\dagger = \mathbf{I}_{n_a^{(2)}}$ where the last equality is due to the fact that $\check{\mathbf{N}}_{j;a}^{(2,1)}$ is an $n_a^{(2)} \times n_a^{(2)}$ Hermitian matrix whose eigenvectors $\{\mathbf{n}_q(\check{\mathbf{N}}_{j;a}^{(2,1)})\}_{q=1}^{n_a^{(2)}}$ form an orthonormal basis. It follows that $\mathbf{K}_{j;a}^{(2,1)} = \frac{\alpha_p^{(2)}}{\check{r}^{(2,1)}} \mathbf{C}_a^{(2)} \cdot [\mathbf{C}_a^{(2)}]^\dagger + \frac{N_G^{(1)} \alpha_p^{(1)}}{r_T^{(1)} D^{(1)}} \mathbf{N}_j^{(2,1)} \equiv \frac{\alpha_p^{(2)}}{\check{r}^{(2,1)}} \mathbf{C}_a^{(2)} \cdot [\mathbf{C}_a^{(2)}]^\dagger + \frac{N_G^{(1)} \alpha_p^{(1)}}{r_T^{(1)} D^{(1)}} \check{\mathbf{C}}_{N;j}^{(2,1)} \cdot [\check{\mathbf{C}}_{N;j}^{(2,1)}]^\dagger$ and $\text{rank}(\mathbf{K}_{j;a}^{(2,1)}) = \text{rank}(\mathbf{C}_a^{(2)} \cdot [\mathbf{C}_a^{(2)}]^\dagger + \check{\mathbf{C}}_{N;j}^{(2,1)} \cdot [\check{\mathbf{C}}_{N;j}^{(2,1)}]^\dagger)$. We conclude that when $L_t^{(2)}$ becomes sufficiently large, the multiplexing gain (4.81) saturates at

$$\tilde{G}_{in}^{(2)} = \frac{\sum_{a=1}^{N_G^{(2)}} [D_{p;a}^{(2)} n_a^{(2)} + \sum_{j \in \mathcal{A}_{G;c}^{(1)}} D_{c;j;a}^{(2)} \{ \text{rank}(\mathbf{C}_a^{(2)} \cdot [\mathbf{C}_a^{(2)}]^\dagger + \check{\mathbf{C}}_{N;j}^{(2,1)} \cdot [\check{\mathbf{C}}_{N;j}^{(2,1)}]^\dagger) - r_j^{(1)} \}]}{\check{D}^{(2)}} \quad (4.82)$$

By using (4.42) with $k = 2$ and (4.51), (4.52) becomes $\Delta_\infty^{(2)} = (G_{out}^{(2)} - G_{in}^{(2)}) \log(P_T/N_0)$. When $L_t^{(2)}$ becomes sufficiently large, this reduces to $\Delta_\infty^{(2)} = (\tilde{G}_{out}^{(2)} - \tilde{G}_{in}^{(2)}) \log(P_T/N_0)$ with $\tilde{G}_{in}^{(2)}$ of (4.82) and $\tilde{G}_{out}^{(2)} = \frac{\sum_{a=1}^{N_G^{(2)}} n_a^{(2)} D_G^{(2)}}{\check{D}^{(2)}}$. Using (4.67d), we can finally write

$$\Delta_\infty^{(2)} = \frac{\sum_{a=1}^{N_G^{(2)}} \sum_{j \in \mathcal{A}_{G;c}^{(1)}} D_{c;j;a}^{(2)} \{ n_a^{(2)} - \text{rank}(\mathbf{C}_a^{(2)} \cdot [\mathbf{C}_a^{(2)}]^\dagger + \check{\mathbf{C}}_{N;j}^{(2,1)} \cdot [\check{\mathbf{C}}_{N;j}^{(2,1)}]^\dagger) + r_j^{(1)} \}}{\check{D}^{(2)}} \log\left(\frac{P_T}{N_0}\right) \quad (4.83)$$

b) Points on the segment $\tilde{V}_1 \tilde{V}_0$: Each of these points is associated with the pair of multiplexing gains $(G_\tau^{(1)}, G_\tau^{(2)})$ with $G_\tau^{(1)} = \tau G_{out}^{(1)} + (1-\tau) G_{in}^{(1)}$ for user 1, $G_\tau^{(2)} = \tau G_{in}^{(2)} + (1-\tau) G_{out}^{(2)}$ for user 2, $\tau \in (0, 1)$, $G_{in}^{(2)}$ given by (4.81) and $G_{in}^{(1)}$ given by

$$G_{in}^{(1)} = \frac{\sum_{\tilde{a}=1}^{N_G^{(1)}} [D_{p;\tilde{a}}^{(1)} \min(n_{\tilde{a}}^{(1)}, L_t^{(1)} N_R^{(1)}) + \sum_{j \in \mathcal{A}_{G;c}^{(2)}} D_{c;j;\tilde{a}}^{(1)} \{ \text{rank}(\mathbf{K}_{j;\tilde{a}}^{(1,2)}) - r_j^{(2)} \}]}{\check{D}^{(1)}} \quad (4.84)$$

similarly to (4.81). We investigate the effect of $L_t^{(2)}$ on $G_\tau^{(2)}$ with fixed $L_t^{(1)}$. Writing

$$G_{out}^{(2)} = \frac{\tau_T^{(2)} D_G^{(2)}}{\check{D}^{(2)}} \equiv \frac{\sum_{a=1}^{N_G^{(2)}} D_G^{(2)} \min(n_a^{(2)}, L_t^{(2)} N_R^{(2)})}{\check{D}^{(2)}} \text{ and using (4.81), we obtain}$$

$$G_\tau^{(2)} = \frac{\sum_{a=1}^{N_G^{(2)}} [\{\tau D_{p;a}^{(2)} + (1-\tau) D_G^{(2)}\} \min(n_a^{(2)}, L_t^{(2)} N_R^{(2)}) + \sum_{j \in \tilde{\mathcal{A}}_{G;c}^{(1)}} \tau D_{c;j;a}^{(2)} \{rank(\mathbf{K}_{j;a}^{(2,1)}) - r_j^{(1)}\}]}{\check{D}^{(2)}}$$

Following a similar approach to that used for analyzing the effect of $L_t^{(2)}$ on $G_{in}^{(2)}$ of (4.81), we conclude that $G_\tau^{(2)}$ increases in general when $L_t^{(2)}$ increases. However, $G_\tau^{(2)}$ remains constant if $n_a^{(2)} < L_t^{(2)} N_R^{(2)}$ for all a and $\tilde{\mathcal{A}}_{G;c}^{(1)} = \mathcal{A}_{G;c}^{(1)}$ with $rank(\mathbf{K}_{j;a}^{(2,1)})$ remaining constant independently of $L_t^{(2)}$ for all $j \in \mathcal{A}_{G;c}^{(1)}$ and all $a \in \{1, \dots, N_G^{(2)}\}$. Now assuming $L_t^{(1)}$ fixed, the effect of $L_t^{(1)}$ on $G_\tau^{(1)}$ can be investigated similarly to the effect of $L_t^{(2)}$ on $G_\tau^{(2)}$ by switching the users indexes and by replacing τ by $1 - \tau$.

c) Points on the segment $V_0 \tilde{V}_0$: Each of these points is associated with the pair of multiplexing gains $(\tau_{00} G_{in}^{(1)}, G_{out}^{(2)})$, $\tau_{00} \in [0, 1]$. Assuming $L_t^{(2)}$ fixed, the effect of $L_t^{(1)}$ on $\tau_{00} G_{in}^{(1)}$ or, equivalently, on $G_{in}^{(1)}$ of (4.84) can be investigated similarly to the effect of $L_t^{(2)}$ on $G_{in}^{(2)}$ of (4.81) in case a) above with the users indexes switched. For user 2, the effect of $L_t^{(2)}$ on $G_{out}^{(2)}$ has been discussed in Section 4.5.1 and it is independent of $L_t^{(1)}$.

The next section investigates the tightness of the bounds $\mathcal{R}_{out}$ and $\mathcal{R}_{in}$ at high SNR.

4.5.3 Tightness of the inner and outer bounds

We now assume arbitrary spatial users' channels, not necessarily following the 3GPP model. The multiplexing gains $G_{out}^{(k)}$ (4.42) and $G_{in}^{(k)}$ (4.69)-(4.70) can be used to evaluate the separation $\Delta_\infty^{(k)} = (G_{out}^{(k)} - G_{in}^{(k)}) \log(P_T/N_0)$ for $k = 1, 2$. We start by evaluating $\Delta_\infty^{(2)}$. By writing $G_{out}^{(2)} = \frac{\sum_{a=1}^{N_G^{(2)}} r_a^{(2)} D_G^{(2)}}{\check{D}^{(2)}}$ with $D_G^{(2)} = D_{p;a}^{(2)} + \sum_{j \in \mathcal{A}_{G;c}^{(1)}} D_{c;j;a}^{(2)}$ for all a from (4.67d) and by using (4.69), we obtain

$$\Delta_\infty^{(2)} = \frac{\sum_{a=1}^{N_G^{(2)}} [\sum_{j \in \mathcal{A}_{G;c}^{(1)}} D_{c;j;a}^{(2)} r_a^{(2)} - \sum_{j \in \tilde{\mathcal{A}}_{G;c}^{(1)}} D_{c;j;a}^{(2)} \kappa_{j;a}^{(2,1)}]}{\check{D}^{(2)}} \log\left(\frac{P_T}{N_0}\right) \quad (4.85)$$

Clearly, $\Delta_\infty^{(2)}$ depends only on the number of shared signal dimensions and it is independent of the number of proper signal dimensions. Moreover, $\Delta_\infty^{(2)}$ reduces to zero if $D_c = 0$ in

which case $D_{c;j,a}^{(2)} = 0$ for all a and all j . For a given signal dimensions sharing scheme, the tightness of the bounds is determined by the spatial channels of both users. We can see from (4.85) that the closer the terms $\sum_{j \in \mathcal{A}_{G;c}^{(1)}} D_{c;j,a}^{(2)} r_a^{(2)}$ and $\sum_{j \in \tilde{\mathcal{A}}_{G;c}^{(1)}} D_{c;j,a}^{(2)} \kappa_{j;a}^{(2,1)}$ are for each a the smaller is $\Delta_\infty^{(2)}$ (i.e. the closer are the points $\tilde{V}_1$ and $\tilde{V}$ in Figure 4.2) and hence, the more accurate is the approximation of the capacity region by the inner and outer bounds. For a given a , the difference between the terms $\sum_{j \in \mathcal{A}_{G;c}^{(1)}} D_{c;j,a}^{(2)} r_a^{(2)}$ and $\sum_{j \in \tilde{\mathcal{A}}_{G;c}^{(1)}} D_{c;j,a}^{(2)} \kappa_{j;a}^{(2,1)}$ becomes small if $\tilde{\mathcal{A}}_{G;c}^{(1)} = \mathcal{A}_{G;c}^{(1)}$ and if $\kappa_{j;a}^{(2,1)}$ is close to $r_a^{(2)}$ for every $j \in \tilde{\mathcal{A}}_{G;c}^{(1)}$. The first condition is satisfied if $\text{rank}(\mathbf{N}_j^{(2,1)}) < L_t^{(2)} N_R^{(2)}$ for all $j \in \mathcal{A}_{G;c}^{(1)}$, which happens if the rank $r_j^{(1)}$ of the propagation matrix for each TOG $j \in \mathcal{A}_{G;c}^{(1)}$ is strictly less than $L_t^{(2)} N_R^{(2)}$ since $\text{rank}(\mathbf{N}_j^{(2,1)}) \equiv \text{rank}(\tilde{\mathbf{C}}_j^{(2)} \cdot [\mathbf{M}_j^{(1)}]^\dagger \cdot \mathbf{B}_j^{(1)} \cdot \mathbf{M}_j^{(1)} \cdot [\tilde{\mathbf{C}}_j^{(2)}]^\dagger) \leq \min(r_j^{(1)}, n_j^{(1)}, L_t^{(2)} N_R^{(2)}) = \min(r_j^{(1)}, L_t^{(2)} N_R^{(2)})$ from (C.23). For the second condition and since $\kappa_{j;a}^{(2,1)} \leq r_a^{(2)}$ from (4.73), we examine the conditions on the spatial channel parameters for which equality is reached in inequalities (a) and (b) of (4.73) for a given a and j . From [114, Theorem 18.5.7], inequality (a) is satisfied with equality if and only if the column spaces (and the row spaces) of $\mathbf{C}_a^{(2)} \cdot \check{\mathbf{W}}_{j;a}^{(2,1)} \cdot [\mathbf{C}_a^{(2)}]^\dagger$ and $\mathbf{N}_j^{(2,1)}$ are essentially disjoint. Consider now inequality (b). Since $\check{\mathbf{W}}_{j;a}^{(2,1)}$ is Hermitian positive semidefinite, we can write $\check{\mathbf{W}}_{j;a}^{(2,1)} = [\check{\mathbf{W}}_{j;a}^{(2,1)}]^{1/2} \cdot [\check{\mathbf{W}}_{j;a}^{(2,1)}]^{1/2}$ where $[\check{\mathbf{W}}_{j;a}^{(2,1)}]^{1/2}$ is the unique $n_a^{(2)} \times n_a^{(2)}$ Hermitian positive semidefinite square root of $\check{\mathbf{W}}_{j;a}^{(2,1)}$ [105, Theorem 7.2.6]. Let $\dim[\mathcal{A}]$ denote the dimension of a space $\mathcal{A}$, and let $\mathcal{N}(\mathbf{A})$ and $\mathcal{C}(\mathbf{A})$ denote the null space and the column space of matrix $\mathbf{A}$, respectively. Using (C.15c), we can write $\text{rank}(\mathbf{C}_a^{(2)} \cdot \check{\mathbf{W}}_{j;a}^{(2,1)} \cdot [\mathbf{C}_a^{(2)}]^\dagger) = \text{rank}([\check{\mathbf{W}}_{j;a}^{(2,1)}]^{1/2} \cdot [\mathbf{C}_a^{(2)}]^\dagger) = \text{rank}([\mathbf{C}_a^{(2)}]^\dagger) - \dim[\mathcal{C}([\mathbf{C}_a^{(2)}]^\dagger) \cap \mathcal{N}([\check{\mathbf{W}}_{j;a}^{(2,1)}]^{1/2})] \equiv r_a^{(2)} - \dim[\mathcal{C}([\mathbf{C}_a^{(2)}]^\dagger) \cap \mathcal{N}([\check{\mathbf{W}}_{j;a}^{(2,1)}]^{1/2})]$ with the second equality due to [97, Corollary 6.2] or [114, Theorem 17.5.4]. It follows that inequality (b) is satisfied with equality if and only if $\mathcal{C}([\mathbf{C}_a^{(2)}]^\dagger)$ and $\mathcal{N}([\check{\mathbf{W}}_{j;a}^{(2,1)}]^{1/2})$ are essentially disjoint. Hence, if $\tilde{\mathcal{A}}_{G;c}^{(1)} = \mathcal{A}_{G;c}^{(1)}$ then $\Delta_\infty^{(2)} = 0$ if the users spatial channels are such that inequalities (a) and (b) in (4.73) are satisfied with equality for all $a = 1, \dots, N_G^{(2)}$ and all $j \in \mathcal{A}_{G;c}^{(1)}$. Let us examine the physical interpretation of these mathematical conditions. Since $\text{rank}(\mathbf{N}_j^{(2,1)}) \leq \min(r_j^{(1)}, L_t^{(2)} N_R^{(2)})$, the equality $\tilde{\mathcal{A}}_{G;c}^{(1)} = \mathcal{A}_{G;c}^{(1)}$ is obtained if $L_t^{(2)} N_R^{(2)}$ is strictly larger than the ranks $r_j^{(1)}$ for all $j \in \mathcal{A}_{G;c}^{(1)}$. This happens if the channel of user 2 is highly time dispersive (i.e. $L_t^{(2)}$ is large), or if user 2 employs more receive antennas than the transmitter ($N_R^{(2)} > N_T$), or both. Moreover, the parameter $\kappa_{j;a}^{(2,1)}$ is determined by the spatial channels of both users through the propagation matrices associated with TOG j of user 1 and TOG a of user 2, while $r_a^{(2)}$ is only determined by the propagation matrix of TOG a of user 2. Equality in (a) and (b) of (4.73) means that the channel of user 1 can be constructively exploited to

cancel the interference associated with the $D_{c;j;a}^{(2)}$ signal dimensions that are shared between TOG j of user 1 and TOG a of user 2. Consequently, if $\Delta_\infty^{(2)} = 0$ then $\mathcal{R}_{in} = \mathcal{R}_{out}$ and the users spatial channels compensate the interference experienced by user 2 (when encoded first) by making the capacity region equal to $\mathcal{R}_{out}$. By following a similar approach and by switching the users indexes, we can evaluate the separation $\Delta_\infty^{(1)}$ between the points $\tilde{V}_0$ and $\tilde{V}$ in Figure 4.2 (which is due to the interference experienced by user 1 when encoded first) and investigate the tightness of the bounds $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$.

The results presented so far in this chapter correspond to spread spectrum like MIMO broadcast systems whose discrete time channel model (2.40)-(2.42) is derived based on assumptions (2.32) and (2.33). The next section considers the DPC rate region and its bounds for the MIMO-OFDM BCs described by equations (2.48)-(2.51).

4.6 Inner and outer bounds for MIMO-OFDM BCs

Considering MIMO-OFDM BCs, the objective of this section is to derive inner and outer bounds for the DPC rate region that are similar to $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$. We analyze the behavior of these bounds at low and high SNR and we investigate their multiplexing gains. Using (2.48), a two-user MIMO-OFDM BC is described by

$$\mathbf{r}_O^{(1)} = \mathbf{H}_O^{(1)} \cdot \mathbf{x}^{(1)} + \mathbf{H}_O^{(1,2)} \cdot \mathbf{x}^{(2)} + \mathbf{v}_O^{(1)} = [\mathbf{H}_O^{(1)} \quad \mathbf{H}_O^{(1,2)}] \cdot \mathbf{x}_0 + \mathbf{v}_O^{(1)} \quad (4.86a)$$

$$\mathbf{r}_O^{(2)} = \mathbf{H}_O^{(2,1)} \cdot \mathbf{x}^{(1)} + \mathbf{H}_O^{(2)} \cdot \mathbf{x}^{(2)} + \mathbf{v}_O^{(2)} = [\mathbf{H}_O^{(2,1)} \quad \mathbf{H}_O^{(2)}] \cdot \mathbf{x}_0 + \mathbf{v}_O^{(2)} \quad (4.86b)$$

where $\mathbf{H}_O^{(1)}, \mathbf{H}_O^{(2)}$ are given by (2.49), and $\mathbf{H}_O^{(1,2)}, \mathbf{H}_O^{(2,1)}$ are given by (2.50). The only difference between this BC model and that of the spread spectrum like MIMO BC (4.1) is that the matrices $\mathbf{H}^{(1)}, \mathbf{H}^{(2)}, \mathbf{H}^{(1,2)}$ and $\mathbf{H}^{(2,1)}$ of (4.1) are replaced here by $\mathbf{H}_O^{(1)}, \mathbf{H}_O^{(2)}, \mathbf{H}_O^{(1,2)}$ and $\mathbf{H}_O^{(2,1)}$, respectively. Hence, as in Theorem 4.1.1, DPC can be used to obtain achievable rate regions $\mathcal{R}_{O;1,2}(\Sigma^{(1)}, \Sigma^{(2)})$ and $\mathcal{R}_{O;2,1}(\Sigma^{(1)}, \Sigma^{(2)})$ that are similar to $\mathcal{R}_{1,2}(\Sigma^{(1)}, \Sigma^{(2)})$ and $\mathcal{R}_{2,1}(\Sigma^{(1)}, \Sigma^{(2)})$ of (4.4)-(4.5), i.e.

$$\mathcal{R}_{O;1,2}(\Sigma^{(1)}, \Sigma^{(2)}) = \left\{ (R^{(1)}, R^{(2)}) : 0 \leq R^{(1)} \leq R_{O,DPC}^{(1)}(\Sigma^{(1)}, \Sigma^{(2)}), 0 \leq R^{(2)} \leq R_{O,DPC}^{(2)}(\Sigma^{(2)}) \right\} \quad (4.87)$$

$$\mathcal{R}_{O;2,1}(\Sigma^{(1)}, \Sigma^{(2)}) = \left\{ (R^{(1)}, R^{(2)}) : 0 \leq R^{(1)} \leq \tilde{R}_{O,DPC}^{(1)}(\Sigma^{(1)}), 0 \leq R^{(2)} \leq \tilde{R}_{O,DPC}^{(2)}(\Sigma^{(1)}, \Sigma^{(2)}) \right\} \quad (4.88)$$

where $(\Sigma^{(1)}, \Sigma^{(2)}) \in \mathcal{A}_\Sigma$ and the DPC rates are defined in bps/Hz as in (4.9)-(4.10), i.e.

$$R_{O,DPC}^{(1)}(\Sigma^{(1)}, \Sigma^{(2)}) \equiv R_{O,DPC}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}) = \frac{1}{\check{D}^{(1)}} \log \frac{\left| \mathbf{I}_{N_R^{(1)}D^{(1)}} + \frac{1}{N_0} \mathbf{H}_O^{(1,2)} \cdot \Sigma^{(2)} \cdot [\mathbf{H}_O^{(1,2)}]^\dagger + \frac{1}{N_0} \mathbf{H}_O^{(1)} \cdot \Sigma^{(1)} \cdot [\mathbf{H}_O^{(1)}]^\dagger \right|}{\left| \mathbf{I}_{N_R^{(1)}D^{(1)}} + \frac{1}{N_0} \mathbf{H}_O^{(1,2)} \cdot \Sigma^{(2)} \cdot [\mathbf{H}_O^{(1,2)}]^\dagger \right|}} \quad (4.89a)$$

$$R_{O,DPC}^{(2)}(\Sigma^{(1)}, \Sigma^{(2)}) \equiv R_{O,DPC}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}) = \frac{1}{\check{D}^{(2)}} \log \left| \mathbf{I}_{N_R^{(2)}D^{(2)}} + \frac{1}{N_0} \mathbf{H}_O^{(2)} \cdot \Sigma^{(2)} \cdot [\mathbf{H}_O^{(2)}]^\dagger \right| \quad (4.89b)$$

and

$$\tilde{R}_{O,DPC}^{(1)}(\Sigma^{(1)}) \equiv \tilde{R}_{O,DPC}^{(1)}(\alpha_p^{(1)}) = \frac{1}{\check{D}^{(1)}} \log \left| \mathbf{I}_{N_R^{(1)}D^{(1)}} + \frac{1}{N_0} \mathbf{H}_O^{(1)} \cdot \Sigma^{(1)} \cdot [\mathbf{H}_O^{(1)}]^\dagger \right| \quad (4.90a)$$

$$\tilde{R}_{O,DPC}^{(2)}(\Sigma^{(1)}, \Sigma^{(2)}) \equiv \tilde{R}_{O,DPC}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}) = \frac{1}{\check{D}^{(2)}} \log \frac{\left| \mathbf{I}_{N_R^{(2)}D^{(2)}} + \frac{1}{N_0} \mathbf{H}_O^{(2,1)} \cdot \Sigma^{(1)} \cdot [\mathbf{H}_O^{(2,1)}]^\dagger + \frac{1}{N_0} \mathbf{H}_O^{(2)} \cdot \Sigma^{(2)} \cdot [\mathbf{H}_O^{(2)}]^\dagger \right|}{\left| \mathbf{I}_{N_R^{(2)}D^{(2)}} + \frac{1}{N_0} \mathbf{H}_O^{(2,1)} \cdot \Sigma^{(1)} \cdot [\mathbf{H}_O^{(2,1)}]^\dagger \right|}} \quad (4.90b)$$

where $Tr(\Sigma^{(1)}) \leq \alpha_p^{(1)} P_T$ and $Tr(\Sigma^{(2)}) \leq \alpha_p^{(2)} P_T$. The DPC rate region for MIMO-OFDM BCs can be defined similarly to (4.6) as

$$\mathcal{R}_{O,DPC} = \text{CH} \left(\bigcup_{(\Sigma^{(1)}, \Sigma^{(2)}) \in \mathcal{A}_\Sigma} \left[\mathcal{R}_{O,1,2}(\Sigma^{(1)}, \Sigma^{(2)}) \cup \mathcal{R}_{O,2,1}(\Sigma^{(1)}, \Sigma^{(2)}) \right] \right) \quad (4.91)$$

Following a similar approach as for deriving $\mathcal{R}_{out}$ in Section 4.2, an outer bound for $\mathcal{R}_{O,DPC}$ and also for the capacity region $\mathcal{C}_O$ of the MIMO-OFDM BC (4.86) can be obtained by assuming that the interference is canceled for both users. Similarly to (4.12), this outer bound is given by

$$\mathcal{R}_{O,out} = \{ (R^{(1)}, R^{(2)}) : 0 \leq R^{(1)} \leq C_O^{(1)}(\alpha_p^{(1)}), 0 \leq R^{(2)} \leq C_O^{(2)}(\alpha_p^{(2)}), \text{ for all } (\alpha_p^{(1)}, \alpha_p^{(2)}) \text{ satisfying } 0 \leq \alpha_p^{(1)}, \alpha_p^{(2)} \leq 1, \alpha_p^{(1)} + \alpha_p^{(2)} = 1 \} \quad (4.92)$$

where $C_O^{(k)}(\alpha_p^{(k)})$ denotes the single-user capacity of user k which, similarly to (4.13), can

be written in bps/Hz as

$$C_O^{(k)}(\alpha_p^{(k)}) = \frac{1}{\tilde{D}^{(k)}} \max_{\text{Tr}(\Sigma^{(k)}) \leq \alpha_p^{(k)} P_T} \log(|\mathbf{I}_{N_R^{(k)} D^{(k)}} + \frac{1}{N_0} \mathbf{H}_O^{(k)} \cdot \Sigma^{(k)} \cdot [\mathbf{H}_O^{(k)}]^\dagger|) \quad (4.93)$$

for $k = 1, 2$. Clearly, $C_O^{(k)}(\alpha_p^{(k)})$ is achieved by water-filling over the eigenmodes of $[\mathbf{H}_O^{(k)}]^\dagger \cdot [\mathbf{H}_O^{(k)}]$. This matrix product can be evaluated similarly to the matrix product $\mathbf{H}_O^\dagger \cdot \mathbf{H}_O$ for the MIMO-OFDM single-user system in Section 3.3. Thus, we have $[\mathbf{H}_O^{(k)}]^\dagger \cdot [\mathbf{H}_O^{(k)}] = \mathbf{A}^{(k)} \cdot [\mathbf{O}_{m,q}^{(k)}]^\dagger \cdot [\mathbf{O}_{m,q}^{(k)}] \cdot [\mathbf{A}^{(k)}]^T = \mathbf{A}^{(k)} \cdot [\tilde{\mathbf{O}}_{r,q}^{(k)}] \cdot (\mathbf{A}^{(k)})^T$ where $\tilde{\mathbf{O}}_{r,q}^{(k)} = \sum_{m=1}^{N_R^{(k)}} (\mathbf{O}_{m,r}^{(k)})^\dagger \cdot \mathbf{O}_{m,q}^{(k)}$ for $r, q = 1, \dots, N_T$. Using (2.51) and following steps similar to those that led to (3.55), we can show that

$$\tilde{\mathbf{O}}_{r,q}^{(k)} = \text{diag}_{p=1}^{D^{(k)}} \left[[(\mathbf{H}_{O;i^{(k)}(p)}^{(k)})^\dagger \cdot \mathbf{H}_{O;i^{(k)}(p)}^{(k)}]_{r,q} \right] = \text{diag}_{i=1}^{N_G^{(k)}} [\tilde{\mathbf{O}}_{r,q;i}^{(k)}] \quad (4.94)$$

where $\mathbf{H}_{O;i^{(k)}(p)}^{(k)} = \sum_{l=1}^{L_t^{(k)}} \mathbf{F}_l^{(k)} e^{-j2\pi \frac{i^{(k)}(p)}{T_F} \tau_l^{(k)}} = FT \left[\sum_{l=1}^{L_t^{(k)}} \mathbf{F}_l^{(k)} \delta(\tau - \tau_l^{(k)}) \right]_{\frac{i^{(k)}(p)}{T_F}}$ as defined in Chapter 2, and $\tilde{\mathbf{O}}_{r,q;i}^{(k)}$ is a $D_G^{(k)} \times D_G^{(k)}$ diagonal matrix whose diagonal elements $[(\mathbf{H}_{O;i^{(k)}(p)}^{(k)})^\dagger \cdot \mathbf{H}_{O;i^{(k)}(p)}^{(k)}]_{r,q}$ have their index $i^{(k)}(p) \in \{-N_{F,T}/2, \dots, N_{F,T}/2 - 1\}$ associated with the $D_G^{(k)}$ orthogonal subcarriers that are employed by TOG i of user k . Using $\mathbf{A}^{(k)} = \text{diag}_{i=1}^{N_T} [\mathbf{e}_{t_i^{(k)}; N_G^{(k)}}^T \otimes \mathbf{I}_{D_G^{(k)}}] = \text{diag}_{i=1}^{N_G^{(k)}} [\mathbf{I}_{n_i^{(k)}} \otimes (\mathbf{e}_{i; N_G^{(k)}}^T \otimes \mathbf{I}_{D_G^{(k)}})]$ and since

$$(\mathbf{e}_{i; N_G^{(k)}}^T \otimes \mathbf{I}_{D_G^{(k)}}) \cdot \tilde{\mathbf{O}}_{r,q}^{(k)} \cdot (\mathbf{e}_{j; N_G^{(k)}} \otimes \mathbf{I}_{D_G^{(k)}}) = \begin{cases} \mathbf{0}_{D_G^{(k)} \times D_G^{(k)}}, & \text{if } i \neq j \\ \tilde{\mathbf{O}}_{r,q;i}^{(k)}, & \text{if } i = j \end{cases}$$

for all $i, j = 1, \dots, N_G^{(k)}$, it follows as in (3.56) that

$$[\mathbf{H}_O^{(k)}]^\dagger \cdot \mathbf{H}_O^{(k)} = \text{diag}_{i=1}^{N_G^{(k)}} [\tilde{\mathbf{O}}_i^{(k)}] \quad (4.95)$$

where $\tilde{\mathbf{O}}_i^{(k)}$ is an $n_i^{(k)} D_G^{(k)} \times n_i^{(k)} D_G^{(k)}$ matrix defined as $\tilde{\mathbf{O}}_i^{(k)} = [\tilde{\mathbf{O}}_{r',q';i}^{(k)}]$ with r', q' denoting the indexes of the transmit antennas that belong to TOG i of user k , that is, $r', q' = 1, \dots, n_1^{(k)}$ if $i = 1$ and $r', q' = \sum_{j=1}^{i-1} n_j^{(k)} + 1, \dots, \sum_{j=1}^i n_j^{(k)}$ if $i > 1$. For $i = 1, \dots, N_G^{(k)}$ and $d = 1, \dots, D_G^{(k)}$, we define the $N_R^{(k)} \times n_i^{(k)}$ matrix $\tilde{\mathbf{F}}_{i,d}^{(k)} = \sum_{l=1}^{L_t^{(k)}} \mathbf{F}_{i,l}^{(k)} e^{-j2\pi \frac{i^{(k)}(i,d)}{T_F} \tau_l^{(k)}}$ where $i^{(k)}(i,d)$ denotes the element of the set $\{-N_{F,T}/2, \dots, N_{F,T}/2 - 1\}$ that corresponds to the d^{th} orthogonal subcarrier employed on the antennas of TOG i of user k . Let $(\check{r}, \check{q}) = (r', q')$ if $i = 1$, and $(\check{r}, \check{q}) = (r' - \sum_{j=1}^{i-1} n_j^{(k)}, q' - \sum_{j=1}^{i-1} n_j^{(k)})$ if $i > 1$,

hence $\check{r}, \check{q} = 1, \dots, n_i^{(k)}$. The d^{th} diagonal element of $\tilde{\mathbf{O}}_{r',q';i}^{(k)}$ is given by $[\tilde{\mathbf{O}}_{r',q';i}^{(k)}]_{d,d} = [\mathbf{H}_{O;i^{(k)}(i,d)}^\dagger \cdot \mathbf{H}_{O;i^{(k)}(i,d)}]_{r',q'} = \sum_{m=1}^{N_R^{(k)}} \left(\sum_{v=1}^{L_t^{(k)}} \sum_{l=1}^{L_t^{(k)}} [\mathbf{F}_v^{(k)}]_{m,r'}^* [\mathbf{F}_l^{(k)}]_{m,q'} e^{j2\pi \frac{i^{(k)}(i,d)}{T_F} (\tau_v^{(k)} - \tau_l^{(k)})} \right)$, that is,

$$[\tilde{\mathbf{O}}_{r',q';i}^{(k)}]_{d,d} = \sum_{m=1}^{N_R^{(k)}} \left(\sum_{v=1}^{L_t^{(k)}} \sum_{l=1}^{L_t^{(k)}} [\mathbf{F}_{i,v}^{(k)}]_{m,\check{r}}^* [\mathbf{F}_{i,l}^{(k)}]_{m,\check{q}} e^{j2\pi \frac{i^{(k)}(i,d)}{T_F} (\tau_v^{(k)} - \tau_l^{(k)})} \right) = [(\check{\mathbf{F}}_{i,d}^{(k)})^\dagger \cdot \check{\mathbf{F}}_{i,d}^{(k)}]_{\check{r},\check{q}}$$

It follows that the $(\check{r}, \check{q})^{th}$ block of $\tilde{\mathbf{O}}_i^{(k)}$ is given by $\tilde{\mathbf{O}}_{r',q';i}^{(k)} \equiv \text{diag}_{d=1}^{D_G^{(k)}} \left[[(\check{\mathbf{F}}_{i,d}^{(k)})^\dagger \cdot \check{\mathbf{F}}_{i,d}^{(k)}]_{\check{r},\check{q}} \right]$, leading as in (3.57) to

$$\tilde{\mathbf{O}}_i^{(k)} = \sum_{d=1}^{D_G^{(k)}} \left[((\check{\mathbf{F}}_{i,d}^{(k)})^\dagger \cdot \check{\mathbf{F}}_{i,d}^{(k)}) \otimes (\mathbf{e}_{d;D_G^{(k)}} \cdot \mathbf{e}_{d;D_G^{(k)}}^T) \right] \quad (4.96)$$

For 3GPP channels, we use from Section 3.3 the fact that the rank of this matrix is given by $r_{O,i}^{(k)} = D_G^{(k)} \min(N_R^{(k)}, n_i^{(k)})$ for all i , thus, making the rank of $[\mathbf{H}_O^{(k)}]^\dagger \cdot \mathbf{H}_O^{(k)}$ (4.95) (or, equivalently, of $\mathbf{H}_O^{(k)}$) equal to $r_O^{(k)} = D_G^{(k)} \sum_{i=1}^{N_G^{(k)}} \min(N_R^{(k)}, n_i^{(k)})$. Let $\{\lambda_{O;i,l}^{(k)}\}_{l=1}^{n_i^{(k)} D_G^{(k)}}$ denote the (real and non-negative) eigenvalues of $\tilde{\mathbf{O}}_i^{(k)}$ in decreasing order $\lambda_{O;i,1}^{(k)} \geq \dots \geq \lambda_{O;i,n_i^{(k)} D_G^{(k)}}^{(k)} \geq 0$. Consider the eigenvalue decomposition $\tilde{\mathbf{O}}_i^{(k)} = (\mathbf{M}_{O,i}^{(k)})^\dagger \cdot \text{diag}_{l=1}^{n_i^{(k)} D_G^{(k)}} [\lambda_{O;i,l}^{(k)}] \cdot \mathbf{M}_{O,i}^{(k)}$ where $\mathbf{M}_{O,i}^{(k)}$ is an $n_i^{(k)} D_G^{(k)} \times n_i^{(k)} D_G^{(k)}$ unitary matrix. From (3.58), $C_O^{(k)}(\alpha_p^{(k)})$ of (4.93) can be written as

$$C_O^{(k)}(\alpha_p^{(k)}) = \frac{1}{D^{(k)}} \sum_{i=1}^{N_G^{(k)}} \sum_{l=1}^{n_i^{(k)} D_G^{(k)}} \{ \log(\mu_O^{(k)}(\alpha_p^{(k)}) \lambda_{O;i,l}^{(k)}) \}^+ \text{ bps/Hz} \quad (4.97)$$

with $\sum_{i=1}^{N_G^{(k)}} \sum_{l=1}^{n_i^{(k)} D_G^{(k)}} \{ \mu_O^{(k)}(\alpha_p^{(k)}) - (\lambda_{O;i,l}^{(k)})^{-1} \}^+ = \frac{\alpha_p^{(k)} P_T}{N_0}$. This capacity is achieved by a zero mean Gaussian input vector $\mathbf{x}^{(k)}$ with covariance matrix

$$\Sigma_{O,\max}^{(k)} \triangleq \text{diag}_{i=1}^{N_G^{(k)}} \left[(\mathbf{M}_{O,i}^{(k)})^\dagger \cdot (N_0 \text{diag}_{l=1}^{n_i^{(k)} D_G^{(k)}} [\{ \mu_O^{(k)}(\alpha_p^{(k)}) - (\lambda_{O;i,l}^{(k)})^{-1} \}^+]) \cdot \mathbf{M}_{O,i}^{(k)} \right] \quad (4.98)$$

Similarly to $\mathcal{R}_{out}$, the outer bound $\mathcal{R}_{O,out}$ can be shown to be convex by using a time-sharing argument as in [51, Theorem 14.3.2]. From Section 3.3 also and based on the impact of space-time modulation on the single-user capacity, one can easily show that $\mathcal{R}_{O,out}$ is largest, intermediate and smallest when the users employ ATA, POTA and OTA, respectively. Finally, an upper bound to the sum capacity C_O^{sum} of the MIMO-OFDM BC (4.86)

is obtained by considering the maximum sum rate $C_{O;out}^{sum}$ of $\mathcal{R}_{O;out}$, where

$$C_{O;out}^{sum} = \max_{(R^{(1)}, R^{(2)}) \in \mathcal{R}_{O;out}} \{R^{(1)} + R^{(2)}\} \equiv \max_{\alpha_p^{(1)}, \alpha_p^{(2)} \in [0,1], \alpha_p^{(1)} + \alpha_p^{(2)} = 1} \{C_O^{(1)}(\alpha_p^{(1)}) + C_O^{(2)}(\alpha_p^{(2)})\}$$

The solution to this optimization problem can be found by using Theorem 4.2.1 which holds also for $\mathcal{R}_{O;out}$ since it is convex. Subsequently, we use $(\alpha_{O;p;s}^{(1)}, \alpha_{O;p;s}^{(2)})$ to denote the (not necessarily unique) solution to this problem.

We now derive an inner bound $\mathcal{R}_{O;in}$ for the DPC rate region $\mathcal{R}_{O;DPC}$ following a similar approach to that used for deriving $\mathcal{R}_{in}$ in Section 4.3. For a given $(\alpha_p^{(1)}, \alpha_p^{(2)})$, we start by defining the Hermitian matrices

$$\Psi_O^{(1)} \triangleq \mathbf{I}_{N_R^{(1)} D^{(1)}} + \frac{1}{N_0} \mathbf{H}_O^{(1,2)} \cdot \Sigma_{O;max}^{(2)} \cdot [\mathbf{H}_O^{(1,2)}]^\dagger \quad (4.99a)$$

$$\Psi_O^{(2)} \triangleq \mathbf{I}_{N_R^{(2)} D^{(2)}} + \frac{1}{N_0} \mathbf{H}_O^{(2,1)} \cdot \Sigma_{O;max}^{(1)} \cdot [\mathbf{H}_O^{(2,1)}]^\dagger \quad (4.99b)$$

where $\Sigma_{O;max}^{(k)}$ is given by (4.98) for $k = 1, 2$. For $k = 1, 2$, let $\check{r}_O^{(k)}$ and $\{\check{\lambda}_{O;q}^{(k)}\}_{q=1}^{\check{D}^{(k)}}$ denote the rank and eigenvalues of $[\mathbf{H}_O^{(k)}]^\dagger \cdot [\Psi_O^{(k)}]^{-1} \cdot \mathbf{H}_O^{(k)}$, respectively, such that $\check{\lambda}_{O;1}^{(k)} \geq \dots \geq \check{\lambda}_{O;\check{D}^{(k)}}^{(k)} \geq 0$. Furthermore, let $\check{\mathbf{M}}_O^{(k)}$ denote the $\check{D}^{(k)} \times \check{D}^{(k)}$ unitary matrix in the eigenvalue decomposition $[\mathbf{H}_O^{(k)}]^\dagger \cdot [\Psi_O^{(k)}]^{-1} \cdot \mathbf{H}_O^{(k)} = [\check{\mathbf{M}}_O^{(k)}]^\dagger \cdot \text{diag}_{q=1}^{\check{D}^{(k)}} [\check{\lambda}_{O;q}^{(k)}] \cdot \check{\mathbf{M}}_O^{(k)}$. Considering the DPC rate pair (4.89) with $\Sigma^{(2)} = \Sigma_{O;max}^{(2)}$, we obtain $R_{O;DPC}^{(2)}(\alpha_p^{(2)}) = C_O^{(2)}(\alpha_p^{(2)})$ from (4.89b) where $C_O^{(2)}(\alpha_p^{(2)})$ denotes the single-user capacity (4.97) for user 2. Also, from (4.89a) the rate for user 1 can be written, similarly to (4.18), as

$$R_{O;DPC}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}) = \frac{1}{\check{D}^{(1)}} \log \left| \mathbf{I}_{N_R^{(1)} D^{(1)}} + \frac{1}{N_0} \mathbf{H}_O^{(1)} \cdot \Sigma^{(1)} \cdot [\mathbf{H}_O^{(1)}]^\dagger \cdot [\Psi_O^{(1)}]^{-1} \right|$$

We choose the input covariance matrix $\Sigma^{(1)}$ so that this rate is maximized under the constraint $Tr(\Sigma^{(1)}) \leq \alpha_p^{(1)} P_T$. The resulting maximal rate (in bps/Hz) for user 1 becomes

$$\check{R}_O^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}) = \frac{1}{\check{D}^{(1)}} \max_{Tr(\Sigma^{(1)}) \leq \alpha_p^{(1)} P_T} \log \left| \mathbf{I}_{N_R^{(1)} D^{(1)}} + \frac{1}{N_0} \mathbf{H}_O^{(1)} \cdot \Sigma^{(1)} \cdot [\mathbf{H}_O^{(1)}]^\dagger \cdot [\Psi_O^{(1)}]^{-1} \right| \quad (4.100)$$

Let $\check{\Sigma}_O^{(1)}$ denote the optimal matrix that achieves this rate. Hence, $\check{\Sigma}_O^{(1)}$ is obtained by water-filling against the eigenvalues $\check{\lambda}_{O;q}^{(1)}$ of $[\mathbf{H}_O^{(1)}]^\dagger \cdot [\Psi_O^{(1)}]^{-1} \cdot \mathbf{H}_O^{(1)}$ with $Tr(\Sigma^{(1)}) = \alpha_p^{(1)} P_T$. The eigenvalues $\check{\lambda}_{O;q}^{(1)}$ depend on the total transmit power P_T because of the dependency of $\Psi_O^{(1)}$ on $\Sigma_{O;max}^{(2)}$. Moreover, following steps that are similar to those that led to (4.22), we

can show that $\check{r}_O^{(1)} = \text{rank}([\mathbf{H}_O^{(1)}]^\dagger \cdot \mathbf{H}_O^{(1)})$, thus, yielding $\check{r}_O^{(1)} = r_O^{(1)}$. With the water-filling solution, (4.100) can be written as

$$\check{R}_O^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}) = \frac{1}{\check{D}^{(1)}} \sum_{q=1}^{\check{D}^{(1)}} \{\log(\check{\mu}_O^{(1)}(\alpha_p^{(1)}) \check{\lambda}_{O;q}^{(1)})\}^+ \text{ bps/Hz} \quad (4.101)$$

where $\sum_{q=1}^{\check{D}^{(1)}} \{\check{\mu}_O^{(1)}(\alpha_p^{(1)}) - (\check{\lambda}_{O;q}^{(1)})^{-1}\}^+ = \frac{\alpha_p^{(1)} P_T}{N_0}$ and where the number of active eigenmodes is less or equal than $r_O^{(1)}$. The associated input covariance matrix for user 1 is given by

$$\check{\Sigma}_O^{(1)} = [\check{\mathbf{M}}_O^{(1)}]^\dagger \cdot (N_0 \text{diag}_{q=1}^{\check{D}^{(1)}} [\{\check{\mu}_O^{(1)}(\alpha_p^{(1)}) - (\check{\lambda}_{O;q}^{(1)})^{-1}\}^+]) \cdot \check{\mathbf{M}}_O^{(1)} \quad (4.102)$$

Hence, by choosing $(\Sigma^{(1)}, \Sigma^{(2)}) = (\check{\Sigma}_O^{(1)}, \Sigma_{O;\max}^{(2)})$, the DPC rate pair of (4.89) reduces to $(\check{R}_O^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}), C_O^{(2)}(\alpha_p^{(2)}))$ which is achievable since it is an element of $\mathcal{R}_{O;1,2}(\check{\Sigma}_O^{(1)}, \Sigma_{O;\max}^{(2)})$. Similarly, by considering the rate pair (4.90) with $\Sigma^{(1)} = \Sigma_{O;\max}^{(1)}$, we obtain $\check{R}_{O;DPC}^{(1)}(\alpha_p^{(1)}) = C_O^{(1)}(\alpha_p^{(1)})$ from (4.90a) and

$$\check{R}_{O;DPC}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}) = \frac{1}{\check{D}^{(2)}} \log \left| \mathbf{I}_{N_R^{(2)} D^{(2)}} + \frac{1}{N_0} \mathbf{H}_O^{(2)} \cdot \Sigma^{(2)} \cdot [\mathbf{H}_O^{(2)}]^\dagger \cdot [\Psi_O^{(2)}]^{-1} \right|$$

from (4.90b). The maximal value for this rate is obtained by water-filling over the eigenmodes of $[\mathbf{H}_O^{(2)}]^\dagger \cdot [\Psi_O^{(2)}]^{-1} \cdot \mathbf{H}_O^{(2)}$ and it is given by

$$\check{R}_O^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}) = \frac{1}{\check{D}^{(2)}} \sum_{q=1}^{\check{D}^{(2)}} \{\log(\check{\mu}_O^{(2)}(\alpha_p^{(2)}) \check{\lambda}_{O;q}^{(2)})\}^+ \text{ bps/Hz} \quad (4.103)$$

where $\sum_{q=1}^{\check{D}^{(2)}} \{\check{\mu}_O^{(2)}(\alpha_p^{(2)}) - (\check{\lambda}_{O;q}^{(2)})^{-1}\}^+ = \frac{\alpha_p^{(2)} P_T}{N_0}$ and where the number of active eigenmodes is less or equal than $r_O^{(2)}$. The associated input covariance matrix for user 2 is given by

$$\check{\Sigma}_O^{(2)} = [\check{\mathbf{M}}_O^{(2)}]^\dagger \cdot (N_0 \text{diag}_{q=1}^{\check{D}^{(2)}} [\{\check{\mu}_O^{(2)}(\alpha_p^{(2)}) - (\check{\lambda}_{O;q}^{(2)})^{-1}\}^+]) \cdot \check{\mathbf{M}}_O^{(2)} \quad (4.104)$$

By choosing $(\Sigma^{(1)}, \Sigma^{(2)}) = (\Sigma_{O;\max}^{(1)}, \check{\Sigma}_O^{(2)})$, the DPC rate pair of (4.90) becomes $(C_O^{(1)}(\alpha_p^{(1)}), \check{R}_O^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}))$ which is achievable since it is an element of $\mathcal{R}_{O;2,1}(\Sigma_{O;\max}^{(1)}, \check{\Sigma}_O^{(2)})$. Considering the subset of rate pairs $(\check{R}_O^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}), C_O^{(2)}(\alpha_p^{(2)}))$ and $(C_O^{(1)}(\alpha_p^{(1)}), \check{R}_O^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}))$, an inner bound for both the capacity region $\mathcal{C}_O$ of the MIMO-OFDM BC (4.86) and the DPC

rate region $\mathcal{R}_{O,DPC}$ is defined similarly to (4.25) as follows

$$\mathcal{R}_{O,in} = \text{CH} \left\{ (0, 0) \cup \left\{ (\check{R}_O^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}), C_O^{(2)}(\alpha_p^{(2)})), (C_O^{(1)}(\alpha_p^{(1)}), \check{R}_O^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})) : \right. \right. \\ \left. \left. 0 \leq \alpha_p^{(1)}, \alpha_p^{(2)} \leq 1, \alpha_p^{(1)} + \alpha_p^{(2)} = 1 \right\} \right\} \quad (4.105)$$

For spread spectrum like MIMO BCs and assuming that for each user the POTA system is obtained from OTA by merging TOGs, Theorem 4.3.1 shows that if the space-time modulation format for user k is fixed then maximum, intermediate and minimum values for $\check{R}_O^{(l)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ are obtained when user l employs ATA, POTA and OTA, respectively, for $(k, l) = (1, 2), (2, 1)$. This result holds also for the rates $\check{R}_O^{(l)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ of MIMO-OFDM BCs and can be shown following a similar proof to that of Theorem 4.3.1 with the matrix $[\mathbf{C}^{(l)} \otimes \mathbf{I}_{D^{(l)}}]$ being simply replaced by the matrix $[\mathbf{O}_{m,q}^{(l)}]$ throughout this proof.

Let us investigate the behavior of the inner bound $\mathcal{R}_{O,in}$ and outer bound $\mathcal{R}_{O,out}$ in the low SNR regime where P_T/N_0 is assumed to be very small compared to the inverses of the nonzero eigenvalues of $(\mathbf{H}_O^{(k)})^\dagger \cdot \mathbf{H}_O^{(k)}$, that is, $P_T/N_0 \ll [\lambda_{O;i,l}^{(k)}]^{-1}$ for all $\lambda_{O;i,l}^{(k)} \neq 0$, $l = 1, \dots, n_i^{(k)} D_G^{(k)}$, $i = 1, \dots, N_G^{(k)}$ and $k = 1, 2$. Consider a boundary point $(C_O^{(1)}(\alpha_p^{(1)}), C_O^{(2)}(\alpha_p^{(2)}))$ of $\mathcal{R}_{O,out}$. From Section 3.3, the capacity $C_O^{(k)}(\alpha_p^{(k)})$ for user k with $\alpha_p^{(k)} \neq 0$ can be written (in bps/Hz) at low SNR as

$$C_O^{(k)}(\alpha_p^{(k)}) = \frac{n(\lambda_{O;\max}^{(k)})}{\check{D}^{(k)}} \log \left(1 + \lambda_{O;\max}^{(k)} \frac{\alpha_p^{(k)} P_T}{N_0 n(\lambda_{O;\max}^{(k)})} \right) \quad (4.106)$$

where $n(\lambda_{O;\max}^{(k)})$ denotes the multiplicity of $\lambda_{O;\max}^{(k)} = \max \{ \{ \lambda_{O;i,l}^{(k)} \}_{l=1}^{n_i^{(k)} D_G^{(k)}} \}_{i=1}^{N_G^{(k)}} \}$. Using the approximation $\ln(1+x) \approx x$, we can write $C_O^{(k)}(\alpha_p^{(k)}) \approx \alpha_p^{(k)} \frac{\lambda_{O;\max}^{(k)}}{\ln 2} \frac{P_T}{N_0 \check{D}^{(k)}}$ which, using (4.8), leads to $C_O^{(1)}(\alpha_p^{(1)}) \approx \frac{\lambda_{O;\max}^{(1)}}{\ln 2} \frac{\alpha_p^{(1)} P_T}{N_0 \check{D}^{(1)}}$ and $C_O^{(2)}(\alpha_p^{(2)}) \approx \frac{\lambda_{O;\max}^{(2)}}{\ln 2} \frac{(1-\alpha_p^{(1)}) P_T}{N_0 \check{D}^{(2)}}$. Hence, similarly to $\mathcal{R}_{out}$, $\mathcal{R}_{O,out}$ reduces in the two-dimensional plane defined by the axes $R^{(1)}$ and $R^{(2)}$ to a triangle bounded by the line

$$R^{(2)} = - \frac{\lambda_{O;\max}^{(2)}}{\lambda_{O;\max}^{(1)}} \frac{\check{D}^{(1)}}{\check{D}^{(2)}} R^{(1)} + \frac{\lambda_{O;\max}^{(2)}}{\ln 2} \frac{P_T}{N_0 \check{D}^{(2)}} \quad (4.107)$$

in the positive quadrant. Since this equation is similar to (4.28), the maximal sum rate $C_{O,out}^{sum}$ can be investigated similarly to C_{out}^{sum} in Section 4.4.1. As such, $C_{O,out}^{sum}$ is achieved

by allocating the total power to user 2 (respectively, user 1) only, or $(\alpha_{O;p;s}^{(1)}, \alpha_{O;p;s}^{(2)}) = (0, 1)$ (respectively, $(\alpha_{O;p;s}^{(1)}, \alpha_{O;p;s}^{(2)}) = (1, 0)$), if $-\frac{\lambda_{O;\max}^{(2)}}{\lambda_{O;\max}^{(1)}} \frac{\check{D}^{(1)}}{\check{D}^{(2)}} < -1$ (respectively, $-\frac{\lambda_{O;\max}^{(2)}}{\lambda_{O;\max}^{(1)}} \frac{\check{D}^{(1)}}{\check{D}^{(2)}} > -1$). Otherwise, if $-\frac{\lambda_{O;\max}^{(2)}}{\lambda_{O;\max}^{(1)}} \frac{\check{D}^{(1)}}{\check{D}^{(2)}} = -1$ then $C_{O;\text{out}}^{\text{sum}}$ is achieved by an arbitrary power allocation $(\alpha_{O;p;s}^{(1)}, \alpha_{O;p;s}^{(2)})$ among users. Consider now user k 's rate $\check{R}_O^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ at low SNR. This rate is given by (4.101) for $k = 1$ and by (4.103) for $k = 2$. Using $n(\check{\lambda}_{O;\max}^{(k)})$ to denote the multiplicity of the maximal eigenvalue $\check{\lambda}_{O;\max}^{(k)} = \max\{\check{\lambda}_{O;q}^{(k)}\}_{q=1}^{\check{D}^{(k)}}$ and since only the $n(\check{\lambda}_{O;\max}^{(k)})$ strongest eigenmodes are active at low SNR, the rate $\check{R}_O^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ of user k reduces to

$$\check{R}_O^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)}) = \frac{n(\check{\lambda}_{O;\max}^{(k)})}{\check{D}^{(k)}} \log \left(1 + \check{\lambda}_{O;\max}^{(k)} \frac{\alpha_p^{(k)} P_T}{n(\check{\lambda}_{O;\max}^{(k)}) N_0} \right) \quad (4.108)$$

which is similar to (4.38). For a given pair $(\alpha_p^{(1)}, \alpha_p^{(2)})$ with $\alpha_p^{(1)} \in (0, 1)$ and $\alpha_p^{(2)} = 1 - \alpha_p^{(1)}$, let $\Delta_O^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)}) \triangleq C_O^{(k)}(\alpha_p^{(k)}) - \check{R}_O^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ denote the difference in rates for user k , $k = 1, 2$, when considering the rate pairs $(C_O^{(1)}(\alpha_p^{(1)}), C_O^{(2)}(\alpha_p^{(2)}))$ of $\mathcal{R}_{O;\text{out}}$ and each of the rate pairs $(\check{R}_O^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}), C_O^{(2)}(\alpha_p^{(2)}))$ and $(C_O^{(1)}(\alpha_p^{(1)}), \check{R}_O^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}))$ of $\mathcal{R}_{O;\text{in}}$. The boundaries of $\mathcal{R}_{O;\text{out}}$ and $\mathcal{R}_{O;\text{in}}$ coincide at the extreme points $(C_O^{(1)}(1), 0)$ and $(0, C_O^{(2)}(1))$ associated with $(\alpha_p^{(1)}, \alpha_p^{(2)}) = (1, 0)$ and $(\alpha_p^{(1)}, \alpha_p^{(2)}) = (0, 1)$, respectively. First let us investigate the difference between the eigenvalues $\lambda_{O;\max}^{(k)}$ and $\check{\lambda}_{O;\max}^{(k)}$ for user k . An upper bound to $(\lambda_{O;\max}^{(k)} - \check{\lambda}_{O;\max}^{(k)})$ can be obtained as in (4.37) following similar steps. Assuming $k = 1$ and similarly to $[\Psi^{(1)}]^{-1}$ in Section 4.4.1, $[\Psi_O^{(1)}]^{-1}$ can be written as $[\Psi_O^{(1)}]^{-1} = \mathbf{I}_{N_R^{(1)} D^{(1)}} - \frac{1}{N_0} \mathbf{H}_O^{(1,2)} \cdot \Sigma_{O;\max}^{(2)} \cdot [\mathbf{H}_O^{(1,2)}]^\dagger$ based on a first order approximation in the expansion [105, p. 301]

$$\left(\mathbf{I}_{N_R^{(1)} D^{(1)}} + \frac{1}{N_0} \mathbf{H}_O^{(1,2)} \cdot \Sigma_{O;\max}^{(2)} \cdot [\mathbf{H}_O^{(1,2)}]^\dagger \right)^{-1} = \sum_{p=0}^{\infty} \left(-\frac{1}{N_0} \mathbf{H}_O^{(1,2)} \cdot \Sigma_{O;\max}^{(2)} \cdot [\mathbf{H}_O^{(1,2)}]^\dagger \right)^p$$

with $\left(-\frac{1}{N_0} \mathbf{H}_O^{(1,2)} \cdot \Sigma_{O;\max}^{(2)} \cdot [\mathbf{H}_O^{(1,2)}]^\dagger \right)^p = \mathbf{I}_{N_R^{(1)} D^{(1)}}$ for $p = 0$, and using the fact that $\left\| \frac{1}{N_0} \mathbf{H}_O^{(1,2)} \cdot \Sigma_{O;\max}^{(2)} \cdot [\mathbf{H}_O^{(1,2)}]^\dagger \right\| \ll 1$. This last inequality can be shown by writing

$$\left\| \frac{1}{N_0} \mathbf{H}_O^{(1,2)} \cdot \Sigma_{O;\max}^{(2)} \cdot [\mathbf{H}_O^{(1,2)}]^\dagger \right\| \leq \frac{1}{N_0} \left\| \mathbf{H}_O^{(1,2)} \right\| \left\| \Sigma_{O;\max}^{(2)} \right\| \left\| [\mathbf{H}_O^{(1,2)}]^\dagger \right\|$$

where $\left\| \mathbf{H}_O^{(1,2)} \right\| = \left\| [\mathbf{H}_O^{(1,2)}]^\dagger \right\| \leq \left\| [\mathbf{O}_{m,q}^{(1)}] \right\| \left\| [\mathbf{I}_{N_T} \otimes \mathbf{B}^{(1,2)}] \right\| \left\| [\mathbf{A}^{(2)}]^T \right\|$, becoming $\left\| \mathbf{H}_O^{(1,2)} \right\| = \left\| [\mathbf{H}_O^{(1,2)}]^\dagger \right\| \leq \left\| [\mathbf{O}_{m,q}^{(1)}] \right\|$ since $\left\| [\mathbf{I}_{N_T} \otimes \mathbf{B}^{(1,2)}] \right\| = \left\| [\mathbf{A}^{(2)}]^T \right\| = 1$ as shown in

Section 4.4.1. At low SNR, the matrix $\Sigma_{O;\max}^{(2)}$ (4.98) becomes

$$\Sigma_{O;\max}^{(2)} = \left(\frac{\alpha_p^{(2)} P_T}{n(\lambda_{O;\max}^{(2)})} \right) \text{diag}_{i=1}^{N_G^{(2)}} [(\mathbf{M}_{O;i}^{(2)})^\dagger \cdot \mathbf{K}_{O;i}^{(2)} \cdot \mathbf{M}_{O;i}^{(2)}] \quad (4.109)$$

with $\mathbf{K}_{O;i}^{(2)}$ being an $n_i^{(2)} D_G^{(2)} \times n_i^{(2)} D_G^{(2)}$ diagonal matrix defined as $[\mathbf{K}_{O;i}^{(2)}]_{q,q} = 1$ if $\lambda_{O;i,q}^{(2)} = \lambda_{O;\max}^{(2)}$ and $[\mathbf{K}_{O;i}^{(2)}]_{q,q} = 0$ else. One can easily show that $\| \text{diag}_{i=1}^{N_G^{(2)}} [(\mathbf{M}_{O;i}^{(2)})^\dagger \cdot \mathbf{K}_{O;i}^{(2)} \cdot \mathbf{M}_{O;i}^{(2)}] \| = \max_{i=1,\dots,N_G^{(2)}} \max_{l=1,\dots,n_i^{(2)} D_G^{(2)}} \sqrt{\lambda_l((\mathbf{M}_{O;i}^{(2)})^\dagger \cdot \mathbf{K}_{O;i}^{(2)} \cdot \mathbf{M}_{O;i}^{(2)})} = 1$. It follows that

$$\left\| \frac{1}{N_0} \mathbf{H}_O^{(1,2)} \cdot \Sigma_{O;\max}^{(2)} \cdot [\mathbf{H}_O^{(1,2)}]^\dagger \right\| \leq \frac{\alpha_p^{(2)} P_T}{N_0 n(\lambda_{O;\max}^{(2)})} \|\mathbf{O}_{m,q}^{(1)}\|^2 \ll 1$$

since $\frac{P_T}{N_0} \|\mathbf{O}_{m,q}^{(1)}\|^2 \equiv \frac{P_T}{N_0} (\max_{l=1,\dots,N_T D^{(1)}} \lambda_l([\mathbf{O}_{m,q}^{(1)}]^\dagger \cdot [\mathbf{O}_{m,q}^{(1)}])) \ll 1$ at low SNR and $\frac{\alpha_p^{(2)}}{n(\lambda_{O;\max}^{(2)})} \leq 1$. Thus, using (4.109), a first order approximation for the matrix product $[\mathbf{H}_O^{(1)}]^\dagger \cdot [\Psi_O^{(1)}]^{-1} \cdot \mathbf{H}_O^{(1)} \equiv [\mathbf{H}_O^{(1)}]^\dagger \cdot (\mathbf{I}_{N_R^{(1)} D^{(1)}} - \frac{1}{N_0} \mathbf{H}_O^{(1,2)} \cdot \Sigma_{O;\max}^{(2)} \cdot [\mathbf{H}_O^{(1,2)}]^\dagger) \cdot \mathbf{H}_O^{(1)}$ can be written as

$$\begin{aligned} [\mathbf{H}_O^{(1)}]^\dagger \cdot [\Psi_O^{(1)}]^{-1} \cdot \mathbf{H}_O^{(1)} &= [\mathbf{H}_O^{(1)}]^\dagger \cdot \mathbf{H}_O^{(1)} \\ &\quad - \left(\frac{\alpha_p^{(2)} P_T}{N_0 n(\lambda_{O;\max}^{(2)})} \right) [\mathbf{H}_O^{(1)}]^\dagger \cdot \mathbf{H}_O^{(1,2)} \cdot \text{diag}_{i=1}^{N_G^{(2)}} [(\mathbf{M}_{O;i}^{(2)})^\dagger \cdot \mathbf{K}_{O;i}^{(2)} \cdot \mathbf{M}_{O;i}^{(2)}] \cdot [\mathbf{H}_O^{(1,2)}]^\dagger \cdot \mathbf{H}_O^{(1)} \end{aligned}$$

By Lemma C.0.2 and for sufficiently small P_T/N_0 , we can write $\check{\lambda}_{O,l}^{(1)} = \lambda_{O,l}^{(1)} - \frac{P_T}{N_0} \check{\delta}_{O,l}^{(1)}$ where $\{\lambda_{O,l}^{(1)}\}_{l=1}^{\bar{D}^{(1)}}$ denote the eigenvalues of $[\mathbf{H}_O^{(1)}]^\dagger \cdot \mathbf{H}_O^{(1)}$, and

$$\check{\delta}_{O,l}^{(1)} = \frac{\alpha_p^{(2)}}{n(\lambda_{O;\max}^{(2)})} [\mathbf{m}_{O,l}^{(1)}]^\dagger \cdot [\mathbf{H}_O^{(1)}]^\dagger \cdot \mathbf{H}_O^{(1,2)} \cdot \text{diag}_{i=1}^{N_G^{(2)}} [(\mathbf{M}_{O;i}^{(2)})^\dagger \cdot \mathbf{K}_{O;i}^{(2)} \cdot \mathbf{M}_{O;i}^{(2)}] \cdot [\mathbf{H}_O^{(1,2)}]^\dagger \cdot \mathbf{H}_O^{(1)} \cdot \mathbf{m}_{O,l}^{(1)}$$

with $\mathbf{m}_{O,l}^{(1)}$ being the eigenvector of $[\mathbf{H}_O^{(1)}]^\dagger \cdot \mathbf{H}_O^{(1)}$ associated with the eigenvalue $\lambda_{O,l}^{(1)}$ such that $\|\mathbf{m}_{O,l}^{(1)}\| = 1$. Since $\mathbf{K}_{O;i}^{(2)} \cdot \mathbf{K}_{O;i}^{(2)} = \mathbf{K}_{O;i}^{(2)}$, we can write $\check{\delta}_{O,l}^{(1)} = \frac{\alpha_p^{(2)}}{n(\lambda_{O;\max}^{(2)})} \|\text{diag}_{i=1}^{N_G^{(2)}} [\mathbf{K}_{O;i}^{(2)} \cdot \mathbf{M}_{O;i}^{(2)}] \cdot [\mathbf{H}_O^{(1,2)}]^\dagger \cdot \mathbf{H}_O^{(1)} \cdot \mathbf{m}_{O,l}^{(1)}\|^2$. Since the matrix spectral norm is the norm induced by the vector 2-norm [105, p. 296], we have using [105, Theorem 5.6.2] and the submultiplicativ-

ity property of the spectral norm [105, p. 290] that

$$\check{\delta}_{O;l}^{(1)} \leq \frac{\alpha_p^{(2)}}{n(\lambda_{O;\max}^{(2)})} \| \text{diag}_{i=1}^{N_G^{(2)}} [\mathbf{K}_{O;i}^{(2)} \cdot \mathbf{M}_{O;i}^{(2)}] \|^2 \| [\mathbf{H}_O^{(1,2)}]^\dagger \|^2 \| \mathbf{H}_O^{(1)} \|^2 \| \mathbf{m}_{O;l}^{(1)} \|^2 \leq \frac{\alpha_p^{(2)}}{n(\lambda_{O;\max}^{(2)})} \| [\mathbf{O}_{m,q}^{(1)}] \|^4$$

since $\| \mathbf{H}_O^{(1)} \| \leq \| [\mathbf{O}_{m,q}^{(1)}] \| \| [\mathbf{A}^{(1)}]^T \| = \| [\mathbf{O}_{m,q}^{(1)}] \|$ using $\| [\mathbf{A}^{(1)}]^T \| = 1$, $\| \text{diag}_{i=1}^{N_G^{(2)}} [\mathbf{K}_{O;i}^{(2)} \cdot \mathbf{M}_{O;i}^{(2)}] \| = \max_{i=1, \dots, N_G^{(2)}} \max_{l=1, \dots, n_i^{(2)}} D_G^{(2)} \sqrt{\lambda_l((\mathbf{M}_{O;i}^{(2)})^\dagger \cdot \mathbf{K}_{O;i}^{(2)} \cdot \mathbf{M}_{O;i}^{(2)})} = 1$, and $\| [\mathbf{H}_O^{(1,2)}]^\dagger \| \leq \| [\mathbf{O}_{m,q}^{(1)}] \|$. From this, we can write

$$\lambda_{O;l}^{(1)} - \check{\lambda}_{O;l}^{(1)} = \frac{P_T}{N_0} \check{\delta}_{O;l}^{(1)} \leq \frac{P_T}{N_0} \frac{\alpha_p^{(2)}}{n(\lambda_{O;\max}^{(2)})} \| [\mathbf{O}_{m,q}^{(1)}] \|^4$$

Moreover, since $\check{\delta}_{O;l}^{(1)} \geq 0$, we have $(P_T/N_0) \check{\lambda}_{O;l}^{(1)} \leq (P_T/N_0) \lambda_{O;l}^{(1)} \ll 1$ at low SNR. By following a similar approach and by switching the users indexes, we can also write $\lambda_{O;l}^{(2)} - \check{\lambda}_{O;l}^{(2)} \leq \frac{P_T}{N_0} \frac{\alpha_p^{(1)}}{n(\lambda_{O;\max}^{(1)})} \| [\mathbf{O}_{m,q}^{(2)}] \|^4$. Assuming that $\lambda_{O;\max}^{(k)} = \lambda_{O;1}^{(k)}$ and $\check{\lambda}_{O;\max}^{(k)} = \check{\lambda}_{O;1}^{(k)}$, it follows for $(k, l) = (1, 2), (2, 1)$ that

$$\lambda_{O;\max}^{(k)} - \check{\lambda}_{O;\max}^{(k)} \leq \frac{P_T}{N_0} \frac{\alpha_p^{(l)}}{n(\lambda_{O;\max}^{(l)})} \| [\mathbf{O}_{m,q}^{(k)}] \|^4 \quad (4.110)$$

which is similar to (4.37) for spread spectrum like MIMO BCs. Using (4.106) and (4.108) with the approximation $\ln(1+x) \approx x$, we obtain

$$\Delta_O^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)}) \approx (\lambda_{O;\max}^{(k)} - \check{\lambda}_{O;\max}^{(k)}) \frac{\alpha_p^{(k)} P_T}{N_0 \check{D}^{(k)} \ln 2} \text{ bps/Hz} \quad (4.111)$$

similarly to (4.39). As $P_T/N_0 \rightarrow 0$, (4.110) and (4.111) show that $\Delta_O^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)}) \rightarrow 0$ for $k = 1, 2$. Thus, the rate pairs $(\check{R}_O^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}), C_O^{(2)}(\alpha_p^{(2)}))$ and $(C_O^{(1)}(\alpha_p^{(1)}), \check{R}_O^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}))$ converge to $(C_O^{(1)}(\alpha_p^{(1)}), C_O^{(2)}(\alpha_p^{(2)}))$ for all $(\alpha_p^{(1)}, \alpha_p^{(2)})$ with $\alpha_p^{(1)} \in (0, 1)$ and $\alpha_p^{(2)} = 1 - \alpha_p^{(1)}$. It follows that $\mathcal{R}_{O;\text{in}} \rightarrow \mathcal{R}_{O;\text{out}}$ as $P_T/N_0 \rightarrow 0$, and both the DPC rate region $\mathcal{R}_{O;\text{DPC}}$ and the capacity region $\mathcal{C}_O$ converge to the triangle bounded by the line (4.107). Similarly to spread spectrum like MIMO BCs, DPC achieves the capacity region for MIMO-OFDM BCs at low SNR and it is asymptotically equivalent to TDMA (as explained in Section 4.4.1). Furthermore, the sum capacity as well as the impact of signal dimensions sharing and space-time modulation on the capacity region can be investigated based on the outer bound and following the approach of Section 4.4.1 for spread spectrum like MIMO BCs.

Consider now the behavior of the outer bound $\mathcal{R}_{O;out}$ in the high SNR regime where P_T/N_0 is assumed to be very large. From Section 3.3, the single-user capacity with nonzero power allocation $\alpha_p^{(k)} > 0$ can be written at high SNR as in (3.62), i.e.

$$C_O^{(k)}(\alpha_p^{(k)}) = \frac{r_O^{(k)}}{\check{D}^{(k)}} \log \left(\frac{\alpha_p^{(k)} P_T}{r_O^{(k)} N_0} + c_O^{(k)} \right) + \frac{A_O^{(k)}}{\check{D}^{(k)}} \quad (4.112)$$

where $A_O^{(k)} = \sum_{i=1}^{N_G^{(k)}} \sum_{l=1}^{r_{O;i}^{(k)}} \log(\lambda_{O;i;l}^{(k)})$ and $c_O^{(k)} = \frac{1}{r_O^{(k)}} \sum_{i=1}^{N_G^{(k)}} \sum_{l=1}^{r_{O;i}^{(k)}} (\lambda_{O;i;l}^{(k)})^{-1}$. The associated water-filling constant is given by

$$\mu_O^{(k)}(\alpha_p^{(k)}) = \frac{\alpha_p^{(k)} P_T}{r_O^{(k)} N_0} + c_O^{(k)} \stackrel{(a)}{\approx} \frac{\alpha_p^{(k)} P_T}{r_O^{(k)} N_0} \quad (4.113)$$

with (a) due to $\frac{\alpha_p^{(k)} P_T}{r_O^{(k)} N_0} \gg c_O^{(k)}$. From (4.112), the spatial multiplexing gain corresponding to $C_O^{(k)}(\alpha_p^{(k)})$ with $\alpha_p^{(k)} \neq 0$ is given by

$$G_{O;out}^{(k)} \triangleq \lim_{(P_T/N_0) \rightarrow \infty} \frac{C_O^{(k)}(\alpha_p^{(k)})}{\log(P_T/N_0)} = \frac{r_O^{(k)}}{\check{D}^{(k)}} \quad (4.114)$$

Following a similar proof to that of Theorem 4.4.1 with $C_O^{(k)}(\alpha_p^{(k)})$ and $r_O^{(k)}$ replacing $C^{(k)}(\alpha_p^{(k)})$ and $r_T^{(k)} D_G^{(k)}$, respectively, we can show that $\mathcal{R}_{O;out}$ converges towards the rectangle defined by the origin and the points V_0 , $\tilde{V}$ and V_1 whose coordinates are now provided in Table 4.3.

Point	$R^{(1)}$ -coordinate	$R^{(2)}$ -coordinate
V_0	0	$\frac{r_O^{(2)}}{\check{D}^{(2)}} \log(P_T/N_0)$
$\tilde{V}$	$\frac{r_O^{(1)}}{\check{D}^{(1)}} \log(P_T/N_0)$	$\frac{r_O^{(2)}}{\check{D}^{(2)}} \log(P_T/N_0)$
V_1	$\frac{r_O^{(1)}}{\check{D}^{(1)}} \log(P_T/N_0)$	0

Table 4.3 Corner points of the limiting outer bound $\mathcal{R}_{O;out}$ for MIMO-OFDM BCs

Consider now the maximum sum rate for $\mathcal{R}_{O;out}$. As for C_{out}^{sum} , $C_{O;out}^{sum}$ is asymptotically achieved only at the upper right corner $\tilde{V}$ of the limiting rectangle, which is the limiting point of all the boundary points $(C_O^{(1)}(\alpha_p^{(1)}), C_O^{(2)}(\alpha_p^{(2)}))$ of $\mathcal{R}_{O;out}$ whose power allocation satisfies $0 < \alpha_p^{(1)}, \alpha_p^{(2)} < 1$ and $\alpha_p^{(1)} + \alpha_p^{(2)} = 1$. Thus, the power allocation $(\alpha_{O;p;s}^{(1)}, \alpha_{O;p;s}^{(2)})$

associated with $C_{O;out}^{sum}$ satisfies $0 < \alpha_{O;p;s}^{(1)}, \alpha_{O;p;s}^{(2)} < 1$ and $\alpha_{O;p;s}^{(1)} + \alpha_{O;p;s}^{(2)} = 1$. We evaluate this power allocation following a similar method to that used in Section 4.4.2 and based on Theorem 4.2.1 for evaluating the power allocation $(\alpha_{p;s}^{(1)}, \alpha_{p;s}^{(2)})$ associated with C_{out}^{sum} . As such, $\alpha_{O;p;s}^{(1)}$ is the solution to $\frac{dC_O^{(2)}(\alpha_p^{(2)})}{dC_O^{(1)}(\alpha_p^{(1)})} = -1$. Using $\frac{\alpha_p^{(k)} P_T}{r_O^{(k)} N_0} \gg c_O^{(k)}$, (4.112) can be written for $k = 1, 2$ as

$$C_O^{(k)}(\alpha_p^{(k)}) = \frac{r_O^{(k)}}{\check{D}^{(k)}} \log \left(\frac{\alpha_p^{(k)} P_T}{r_O^{(k)} N_0} \right) + \frac{A_O^{(k)}}{\check{D}^{(k)}} \quad (4.115)$$

which, using (4.8), leads to $\frac{dC_O^{(2)}(\alpha_p^{(2)})}{dC_O^{(1)}(\alpha_p^{(1)})} = \frac{dC_O^{(2)}(\alpha_p^{(2)})}{d\alpha_p} \frac{d\alpha_p}{dC_O^{(1)}(\alpha_p^{(1)})} = -\frac{\check{D}^{(1)}}{r_O^{(1)}} \frac{r_O^{(2)}}{\check{D}^{(2)}} \frac{\alpha_p}{(1-\alpha_p)}$ similarly to (4.44). Solving $-\frac{\check{D}^{(1)}}{r_O^{(1)}} \frac{r_O^{(2)}}{\check{D}^{(2)}} \frac{\alpha_p}{(1-\alpha_p)} = -1$ results in

$$\alpha_{O;p;s}^{(1)} = \frac{\check{D}^{(2)}}{r_O^{(2)}} \frac{r_O^{(1)}}{\check{D}^{(1)}} \left[1 + \frac{\check{D}^{(2)}}{r_O^{(2)}} \frac{r_O^{(1)}}{\check{D}^{(1)}} \right]^{-1} = \left[1 + \frac{r_O^{(2)}}{\check{D}^{(2)}} \frac{\check{D}^{(1)}}{r_O^{(1)}} \right]^{-1} \quad (4.116)$$

The corresponding power allocation for user 2 is given by $\alpha_{O;p;s}^{(2)} = 1 - \alpha_{O;p;s}^{(1)}$. Equation (4.116) is similar to (4.45) with $r_T^{(k)} D_G^{(k)}$ being replaced by $r_O^{(k)} \equiv D_G^{(k)} \sum_{i=1}^{N_G^{(k)}} \min(N_R^{(k)}, n_i^{(k)})$ for $k = 1, 2$. Therefore, the impact of increasing D_c on $(\alpha_{O;p;s}^{(1)}, \alpha_{O;p;s}^{(2)})$ is similar to the impact of increasing D_c on $(\alpha_{p;s}^{(1)}, \alpha_{p;s}^{(2)})$ (see the discussion presented in Section 4.4.2).

We now investigate the behavior of the inner bound $\mathcal{R}_{O;in}$ and the separation between $\mathcal{R}_{O;in}$ and $\mathcal{R}_{O;out}$ in the high SNR regime. We will follow a similar approach as that used to investigate the high SNR behavior of $\mathcal{R}_{in}$ in Section 4.4.2. Assuming $0 < \alpha_p^{(k)}$ with $(\alpha_p^{(k)} P_T)/N_0$ very large for $k = 1, 2$, we consider the rate $\check{R}_O^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ of (4.103) for user 2 which can be obtained from (4.90b) using the substitution $(\Sigma^{(1)}, \Sigma^{(2)}) = (\Sigma_{O;\max}^{(1)}, \check{\Sigma}_O^{(2)})$ with $\check{\Sigma}_O^{(2)}$ of (4.104). As such, using the eigenvalue decompositions

$$\mathbf{H}_O^{(2,1)} \cdot \left(\frac{1}{P_T} \Sigma_{O;\max}^{(1)} \right) \cdot [\mathbf{H}_O^{(2,1)}]^\dagger + \mathbf{H}_O^{(2)} \cdot \left(\frac{1}{P_T} \check{\Sigma}_O^{(2)} \right) \cdot [\mathbf{H}_O^{(2)}]^\dagger = [\check{\mathbf{M}}_O^{(2)}]^\dagger \cdot \check{\Lambda}_O^{(2)} \cdot \check{\mathbf{M}}_O^{(2)} \quad (4.117)$$

$$\mathbf{H}_O^{(2,1)} \cdot \left(\frac{1}{P_T} \Sigma_{O;\max}^{(1)} \right) \cdot [\mathbf{H}_O^{(2,1)}]^\dagger = [\check{\mathbf{M}}_O^{(2,1)}]^\dagger \cdot \check{\Lambda}_O^{(2,1)} \cdot \check{\mathbf{M}}_O^{(2,1)} \quad (4.118)$$

where the matrices $\check{\mathbf{M}}_O^{(2,1)}, \check{\mathbf{M}}_O^{(2)}$ are unitary and $\check{\Lambda}_O^{(2,1)}, \check{\Lambda}_O^{(2)}$ are diagonal, the rate $\check{R}_O^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ can be put in a form similar to that of $\check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ (4.46) for spread spectrum like MIMO BCs, i.e.

$$\check{R}_O^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}) = \frac{1}{\check{D}^{(2)}} \left\{ \log \left| \frac{N_0}{P_T} \mathbf{I}_{N_R^{(2)} D^{(2)}} + \check{\Lambda}_O^{(2)} \right| - \log \left| \frac{N_0}{P_T} \mathbf{I}_{N_R^{(2)} D^{(2)}} + \check{\Lambda}_O^{(2,1)} \right| \right\} \quad (4.119)$$

Similarly to (4.49), we can show that

$$0 \leq \text{Tr}(\check{\Lambda}_O^{(2,1)}) \leq \text{Tr}(\alpha_p^{(1)} \mathbf{H}_O^{(2,1)} \cdot [\mathbf{H}_O^{(2,1)}]^\dagger) \quad (4.120a)$$

$$0 \leq \text{Tr}(\check{\Lambda}_O^{(2)}) \leq \text{Tr}(\alpha_p^{(1)} \mathbf{H}_O^{(2,1)} \cdot [\mathbf{H}_O^{(2,1)}]^\dagger + \alpha_p^{(2)} \mathbf{H}_O^{(2)} \cdot [\mathbf{H}_O^{(2)}]^\dagger) \quad (4.120b)$$

Let $r_{\max;O}^{(2)}$ and $r_{\max;O}^{(2,1)}$ be the ranks of $\mathbf{H}_O^{(2,1)} \cdot (\frac{1}{P_T} \Sigma_{O;\max}^{(1)}) \cdot [\mathbf{H}_O^{(2,1)}]^\dagger + \mathbf{H}_O^{(2)} \cdot (\frac{1}{P_T} \check{\Sigma}_O^{(2)}) \cdot [\mathbf{H}_O^{(2)}]^\dagger$ and $\mathbf{H}_O^{(2,1)} \cdot (\frac{1}{P_T} \Sigma_{O;\max}^{(1)}) \cdot [\mathbf{H}_O^{(2,1)}]^\dagger$, respectively. As for $r_{\max}^{(2)}$ and $r_{\max}^{(2,1)}$, the result of Lemma 4.4.1 holds also for the ranks $r_{\max;O}^{(2)}$ and $r_{\max;O}^{(2,1)}$. We briefly outline the proof which is similar to that of Lemma 4.4.1. Using (4.98) and writing $\mu_O^{(1)}(\alpha_p^{(1)}) - (\lambda_{O;i,l}^{(1)})^{-1} \approx \mu_O^{(1)}(\alpha_p^{(1)}) \approx \frac{\alpha_p^{(1)} P_T}{r_O^{(1)} N_0}$ from (4.113), for all $l = 1, \dots, r_{O;i}^{(1)}$ and $i = 1, \dots, N_G^{(1)}$, we can write similarly to (B.11) that

$$\begin{aligned} \Sigma_{O;\max}^{(1)} &\approx \text{diag}_{i=1}^{N_G^{(1)}} \left[[\mathbf{M}_{O;i}^{(1)}]^\dagger \cdot (N_0 \text{diag}[\mu_O^{(1)}(\alpha_p^{(1)}) \mathbf{I}_{r_{O;i}^{(1)}}, \mathbf{0}_{(n_i^{(1)} D_G^{(1)} - r_{O;i}^{(1)}) \times (n_i^{(1)} D_G^{(1)} - r_{O;i}^{(1)})}]) \cdot \mathbf{M}_{O;i}^{(1)} \right] \\ &\approx \frac{\alpha_p^{(1)} P_T}{r_O^{(1)}} \text{diag}_{i=1}^{N_G^{(1)}} \left[[\mathbf{M}_{O;i}^{(1)}]^\dagger \cdot \tilde{\mathbf{B}}_{O;i}^{(1)} \cdot \mathbf{M}_{O;i}^{(1)} \right] \end{aligned} \quad (4.121)$$

where $\tilde{\mathbf{B}}_{O;i}^{(1)} = \text{diag}[\mathbf{I}_{r_{O;i}^{(1)}}, \mathbf{0}_{(n_i^{(1)} D_G^{(1)} - r_{O;i}^{(1)}) \times (n_i^{(1)} D_G^{(1)} - r_{O;i}^{(1)})}]$. Hence, (4.121) yields

$$\mathbf{H}_O^{(2,1)} \cdot (\frac{1}{P_T} \Sigma_{O;\max}^{(1)}) \cdot [\mathbf{H}_O^{(2,1)}]^\dagger \approx \frac{\alpha_p^{(1)}}{r_O^{(1)}} \mathbf{H}_O^{(2,1)} \cdot \text{diag}_{i=1}^{N_G^{(1)}} \left[[\mathbf{M}_{O;i}^{(1)}]^\dagger \cdot \tilde{\mathbf{B}}_{O;i}^{(1)} \cdot \mathbf{M}_{O;i}^{(1)} \right] \cdot [\mathbf{H}_O^{(2,1)}]^\dagger \quad (4.122)$$

From this equation, one can see that $r_{\max;O}^{(2,1)}$ is equal to the rank of $\mathbf{H}_O^{(2,1)} \cdot \text{diag}_{i=1}^{N_G^{(1)}} \left[[\mathbf{M}_{O;i}^{(1)}]^\dagger \cdot \tilde{\mathbf{B}}_{O;i}^{(1)} \cdot \mathbf{M}_{O;i}^{(1)} \right] \cdot [\mathbf{H}_O^{(2,1)}]^\dagger$ and hence, it is independent of $(\alpha_p^{(1)}, \alpha_p^{(2)})$ and P_T/N_0 since $\mathbf{M}_{O;i}^{(1)}, r_{O;i}^{(1)}$ ($\forall i$) and $\mathbf{H}_O^{(2,1)}$ are independent of $(\alpha_p^{(1)}, \alpha_p^{(2)})$ and P_T/N_0 . If $r_{\max;O}^{(2,1)} = N_R^{(2)} D^{(2)}$ then $r_{\max;O}^{(2)} = N_R^{(2)} D^{(2)}$ since $N_R^{(2)} D^{(2)} \geq r_{\max;O}^{(2)} \geq r_{\max;O}^{(2,1)}$ where the last inequality is due to the fact that the number of nonzero eigenvalues (and hence, the rank) of $\mathbf{H}_O^{(2,1)} \cdot (\frac{1}{P_T} \Sigma_{O;\max}^{(1)}) \cdot [\mathbf{H}_O^{(2,1)}]^\dagger + \mathbf{H}_O^{(2)} \cdot (\frac{1}{P_T} \check{\Sigma}_O^{(2)}) \cdot [\mathbf{H}_O^{(2)}]^\dagger$ cannot be less than that of $\mathbf{H}_O^{(2,1)} \cdot (\frac{1}{P_T} \Sigma_{O;\max}^{(1)}) \cdot [\mathbf{H}_O^{(2,1)}]^\dagger$ by Corollary C.0.1. In this case, $r_{\max;O}^{(2)}$ is independent of $(\alpha_p^{(1)}, \alpha_p^{(2)})$ and P_T/N_0 . Assume that $r_{\max;O}^{(2,1)} < N_R^{(2)} D^{(2)}$. As in (B.13), we consider the eigenvalue decomposition

$$\mathbf{H}_O^{(2,1)} \cdot \text{diag}_{i=1}^{N_G^{(1)}} \left[[\mathbf{M}_{O;i}^{(1)}]^\dagger \cdot \tilde{\mathbf{B}}_{O;i}^{(1)} \cdot \mathbf{M}_{O;i}^{(1)} \right] \cdot [\mathbf{H}_O^{(2,1)}]^\dagger = [\bar{\mathbf{M}}_O^{(2,1)}]^\dagger \cdot \bar{\Lambda}_O^{(2,1)} \cdot \bar{\mathbf{M}}_O^{(2,1)}$$

where the $N_R^{(2)} D^{(2)} \times N_R^{(2)} D^{(2)}$ matrices $\bar{\mathbf{M}}_O^{(2,1)}$ and $\bar{\Lambda}_O^{(2,1)}$ are unitary and diagonal, respectively. From this and using (4.99b) and (4.122), we obtain $\Psi_O^{(2)} = \mathbf{I}_{N_R^{(2)} D^{(2)}} + \frac{\alpha_p^{(1)} P_T}{r_O^{(1)} N_0} [\bar{\mathbf{M}}_O^{(2,1)}]^\dagger$.

$\bar{\Lambda}_O^{(2,1)} \cdot \bar{\mathbf{M}}_O^{(2,1)} \equiv [\bar{\mathbf{M}}_O^{(2,1)}]^\dagger \cdot [\mathbf{I}_{N_R^{(2)}D^{(2)}} + \frac{\alpha_p^{(1)}P_T}{r_O^{(1)}N_0}\bar{\Lambda}_O^{(2,1)}] \cdot \bar{\mathbf{M}}_O^{(2,1)}$. Similarly to (B.15) and for $\alpha_p^{(1)} > 0$ with $(\alpha_p^{(1)}P_T)/N_0$ very large, we can write $[\mathbf{I}_{N_R^{(2)}D^{(2)}} + \frac{\alpha_p^{(1)}P_T}{r_O^{(1)}N_0}\bar{\Lambda}_O^{(2,1)}]^{-1} \approx \mathbf{D}_O^{(2,1)}$ where $\mathbf{D}_O^{(2,1)}$ is an $N_R^{(2)}D^{(2)} \times N_R^{(2)}D^{(2)}$ diagonal matrix defined as $[\mathbf{D}_O^{(2,1)}]_{p,p} = 1$ if $\frac{1}{r_O^{(1)}}[\bar{\Lambda}_O^{(2,1)}]_{p,p} = 0$ and $[\mathbf{D}_O^{(2,1)}]_{p,p} = 0$ if $\frac{1}{r_O^{(1)}}[\bar{\Lambda}_O^{(2,1)}]_{p,p} \neq 0$. Hence, we have

$$[\mathbf{H}_O^{(2)}]^\dagger \cdot [\Psi_O^{(2)}]^{-1} \cdot \mathbf{H}_O^{(2)} \approx [\mathbf{H}_O^{(2)}]^\dagger \cdot [\bar{\mathbf{M}}_O^{(2,1)}]^\dagger \cdot \mathbf{D}_O^{(2,1)} \cdot \bar{\mathbf{M}}_O^{(2,1)} \cdot \mathbf{H}_O^{(2)} \quad (4.123)$$

as in (B.16). The matrix on the right hand side of (4.123) as well as its rank $\check{r}_O^{(2,1)}$ and eigenvalues $\{\check{\lambda}_{O,q}^{(2)}\}_{q=1}^{\check{D}^{(2)}}$ are independent of $(\alpha_p^{(1)}, \alpha_p^{(2)})$ and P_T/N_0 . Using water-filling against the eigenvalues $\check{\lambda}_{O,q}^{(2)}$ with $\alpha_p^{(2)} > 0$ and $(\alpha_p^{(2)}P_T)/N_0$ very large, the water-filling constant satisfies $\sum_{q=1}^{\check{r}_O^{(2,1)}} \{\check{\mu}_O^{(2)}(\alpha_p^{(2)}) - (\check{\lambda}_{O,q}^{(2)})^{-1}\} = \frac{\alpha_p^{(2)}P_T}{N_0}$ or, equivalently,

$$\check{\mu}_O^{(2)}(\alpha_p^{(2)}) = \frac{\alpha_p^{(2)}P_T}{N_0\check{r}_O^{(2,1)}} + \frac{1}{\check{r}_O^{(2,1)}} \sum_{q=1}^{\check{r}_O^{(2,1)}} (\check{\lambda}_{O,q}^{(2)})^{-1} \stackrel{(a)}{\approx} \frac{\alpha_p^{(2)}P_T}{N_0\check{r}_O^{(2,1)}} \quad (4.124)$$

where (a) is due to $\frac{\alpha_p^{(2)}P_T}{N_0\check{r}_O^{(2,1)}} \gg \frac{1}{\check{r}_O^{(2,1)}} \sum_{q=1}^{\check{r}_O^{(2,1)}} (\check{\lambda}_{O,q}^{(2)})^{-1}$. The resulting transmission rate for user 2 is given by

$$\check{R}_O^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}) = \frac{\check{r}_O^{(2,1)}}{\check{D}^{(2)}} \log \left(\frac{\alpha_p^{(2)}P_T}{N_0\check{r}_O^{(2,1)}} \right) + \frac{1}{\check{D}^{(2)}} \sum_{q=1}^{\check{r}_O^{(2,1)}} \log(\check{\lambda}_{O,q}^{(2)}) \quad (4.125)$$

Moreover, using $\check{\mu}_O^{(2)}(\alpha_p^{(2)}) - (\check{\lambda}_{O,q}^{(2)})^{-1} \approx \frac{\alpha_p^{(2)}P_T}{N_0\check{r}_O^{(2,1)}}$, the input covariance matrix of user 2 is given, as in (4.104), by

$$\check{\Sigma}_O^{(2)} \approx [\check{\mathbf{M}}_O^{(2)}]^\dagger \cdot \left(\frac{\alpha_p^{(2)}P_T}{\check{r}_O^{(2,1)}} \text{diag}[\mathbf{I}_{\check{r}_O^{(2,1)}}, \mathbf{0}_{(\check{D}^{(2)}-\check{r}_O^{(2,1)}) \times (\check{D}^{(2)}-\check{r}_O^{(2,1)})}] \right) \cdot \check{\mathbf{M}}_O^{(2)} \quad (4.126)$$

with $\check{\mathbf{M}}_O^{(2)}$ being the $\check{D}^{(2)} \times \check{D}^{(2)}$ unitary matrix in the eigenvalue decomposition of $[\mathbf{H}_O^{(2)}]^\dagger \cdot [\bar{\mathbf{M}}_O^{(2,1)}]^\dagger \cdot \mathbf{D}_O^{(2,1)} \cdot \bar{\mathbf{M}}_O^{(2,1)} \cdot \mathbf{H}_O^{(2)}$ and is independent of $(\alpha_p^{(1)}, \alpha_p^{(2)})$ and P_T/N_0 . Using (4.122) and (4.126), it follows that

$$\mathbf{H}_O^{(2,1)} \cdot \left(\frac{1}{P_T} \Sigma_{O,\max}^{(1)} \right) \cdot [\mathbf{H}_O^{(2,1)}]^\dagger + \mathbf{H}_O^{(2)} \cdot \left(\frac{1}{P_T} \check{\Sigma}_O^{(2)} \right) \cdot [\mathbf{H}_O^{(2)}]^\dagger = \alpha_p^{(1)} \Pi_{O,0}^{(2)} + \alpha_p^{(2)} \Pi_{O,1}^{(2)}$$

with the matrices $\Pi_{O;0}^{(2)} = \frac{1}{r_O^{(1)}} \mathbf{H}_O^{(2,1)} \cdot \text{diag}_{i=1}^{N_G^{(1)}} [\mathbf{M}_{O;i}^{(1)\dagger} \cdot \tilde{\mathbf{B}}_{O;i}^{(1)} \cdot \mathbf{M}_{O;i}^{(1)}] \cdot [\mathbf{H}_O^{(2,1)}]^\dagger$ and $\Pi_{O;1}^{(2)} = \frac{1}{\check{r}_O^{(2,1)}} \check{\mathbf{H}}_O^{(2)} \cdot [\check{\mathbf{M}}_O^{(2)}]^\dagger \cdot (\text{diag}[\mathbf{I}_{\check{r}_O^{(2,1)}}, \mathbf{0}_{(\check{D}^{(2)} - \check{r}_O^{(2,1)}) \times (\check{D}^{(2)} - \check{r}_O^{(2,1)})}] \cdot \check{\mathbf{M}}_O^{(2)} \cdot [\mathbf{H}_O^{(2)}]^\dagger$ being independent of $(\alpha_p^{(1)}, \alpha_p^{(2)})$ and P_T/N_0 . As for the proof of Lemma 4.4.1, we can show that $r_{\max;O}^{(2)} = \text{rank}(\Pi_{O;0}^{(2)} + \Pi_{O;1}^{(2)})$ which is independent of $(\alpha_p^{(1)}, \alpha_p^{(2)})$ and P_T/N_0 . An equivalent expression for $\check{R}_O^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ of (4.119) is given, similarly to (4.50), by

$$\begin{aligned} \check{R}_O^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}) = \frac{1}{\check{D}^{(2)}} \left\{ \sum_{p=1}^{r_{\max;O}^{(2)}} \log \left(\frac{N_0}{P_T} + [\check{\Lambda}_O^{(2)}]_{p,p} \right) - \sum_{q=1}^{r_{\max;O}^{(2,1)}} \log \left(\frac{N_0}{P_T} + [\check{\Lambda}_O^{(2,1)}]_{q,q} \right) \right. \\ \left. + (r_{\max;O}^{(2)} - r_{\max;O}^{(2,1)}) \log \left(\frac{P_T}{N_0} \right) \right\}, \end{aligned} \quad (4.127)$$

Using $[\check{\Lambda}_O^{(2)}]_{p,p} \ll (P_T/N_0)$ and $[\check{\Lambda}_O^{(2,1)}]_{q,q} \ll (P_T/N_0)$ from (4.120), we obtain the spatial multiplexing gain

$$G_{O;in}^{(2)} \triangleq \lim_{(P_T/N_0) \rightarrow \infty} \frac{\check{R}_O^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})}{\log(P_T/N_0)} = \frac{(r_{\max;O}^{(2)} - r_{\max;O}^{(2,1)})}{\check{D}^{(2)}} \quad (4.128)$$

Considering the boundary points of $\mathcal{R}_{O;in}$ and $\mathcal{R}_{O;out}$, the difference in rates (in bps/Hz) for user 2 at high SNR satisfies

$$\Delta_O^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}) \longrightarrow \Delta_{O;\infty}^{(2)} \triangleq \frac{(r_O^{(2)} + r_{\max;O}^{(2,1)} - r_{\max;O}^{(2)})}{\check{D}^{(2)}} \log \left(\frac{P_T}{N_0} \right) \equiv (G_{O;out}^{(2)} - G_{O;in}^{(2)}) \log \left(\frac{P_T}{N_0} \right) \quad (4.129)$$

The multiplexing gain and difference in rates (in bps/Hz) for user 1 can be similarly obtained with the users indexes switched, that is

$$G_{O;in}^{(1)} \triangleq \lim_{(P_T/N_0) \rightarrow \infty} \frac{\check{R}_O^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)})}{\log(P_T/N_0)} = \frac{(r_{\max;O}^{(1)} - r_{\max;O}^{(1,2)})}{\check{D}^{(1)}} \quad (4.130)$$

and

$$\Delta_O^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}) \longrightarrow \Delta_{O;\infty}^{(1)} \triangleq \frac{(r_O^{(1)} + r_{\max;O}^{(1,2)} - r_{\max;O}^{(1)})}{\check{D}^{(1)}} \log \left(\frac{P_T}{N_0} \right) \equiv (G_{O;out}^{(1)} - G_{O;in}^{(1)}) \log \left(\frac{P_T}{N_0} \right) \quad (4.131)$$

where $r_{\max;O}^{(1)}$ and $r_{\max;O}^{(1,2)}$ denote the ranks of the matrices $\mathbf{H}_O^{(1,2)} \cdot (\frac{1}{P_T} \Sigma_{O;\max}^{(2)}) \cdot [\mathbf{H}_O^{(1,2)}]^\dagger + \mathbf{H}_O^{(1)} \cdot (\frac{1}{P_T} \Sigma_O^{(1)}) \cdot [\mathbf{H}_O^{(1)}]^\dagger$ and $\mathbf{H}_O^{(1,2)} \cdot (\frac{1}{P_T} \Sigma_{O;\max}^{(2)}) \cdot [\mathbf{H}_O^{(1,2)}]^\dagger$, respectively, and are independent

of $(\alpha_p^{(1)}, \alpha_p^{(2)})$ and P_T/N_0 . Hence, similarly to $\mathcal{R}_{in}$, one can show that Theorem 4.4.2 holds for MIMO-OFDM BCs and, as $(P_T/N_0) \rightarrow \infty$, $\mathcal{R}_{O;in}$ converges towards the pentagon defined by the origin, V_0 , $\tilde{V}_0$, $\tilde{V}_1$ and V_1 . The coordinates of $\tilde{V}_0$ and $\tilde{V}_1$ are now provided in Table 4.4, while those of V_0 and V_1 are provided in Table 4.3. The asymptotical regions $\mathcal{R}_{O;in}$ and $\mathcal{R}_{O;out}$ can be represented as in Figure 4.2, with $\Delta_\infty^{(k)}$ being replaced by $\Delta_{O;\infty}^{(k)}$ for $k = 1, 2$. For MIMO-OFDM BCs also, the capacity region $\mathcal{C}_O$ and the DPC rate region $\mathcal{R}_{O,DPC}$ coincide on the segments $V_0\tilde{V}_0$ and $\tilde{V}_1V_1$ since these segments are part of the boundaries of $\mathcal{R}_{O;in}$ and $\mathcal{R}_{O;out}$.

Point	$R^{(1)}$ -coordinate	$R^{(2)}$ -coordinate
$\tilde{V}_0$	$\frac{(r_{\max;O}^{(1)} - r_{\max;O}^{(1,2)})}{\check{D}^{(1)}} \log(P_T/N_0)$	$\frac{r_O^{(2)}}{\check{D}^{(2)}} \log(P_T/N_0)$
$\tilde{V}_1$	$\frac{r_O^{(1)}}{\check{D}^{(1)}} \log(P_T/N_0)$	$\frac{(r_{\max;O}^{(2)} - r_{\max;O}^{(2,1)})}{\check{D}^{(2)}} \log(P_T/N_0)$

Table 4.4 Coordinates of the points $\tilde{V}_0, \tilde{V}_1$ for MIMO-OFDM BCs

As for $\mathcal{R}_{in}$ and based on Theorem 4.2.1, the sum rate for $\mathcal{R}_{O;in}$ is maximized at the boundary point ($\tilde{V}_0$ or $\tilde{V}_1$) that is closest to $\tilde{V}$. If $\Delta_{O;\infty}^{(2)} \leq \Delta_{O;\infty}^{(1)}$, then the sum rate is maximized at the boundary point $(C_O^{(1)}(\tilde{\alpha}_{O;p;s}^{(1)}, \check{R}_O^{(2)}(\tilde{\alpha}_{O;p;s}^{(1)}, \tilde{\alpha}_{O;p;s}^{(2)}))$ at which the support line to the boundary of $\mathcal{R}_{O;in}$ has a slope equal to -1 . Using (4.115) with $k = 1$ and $\alpha_p^{(1)} = \alpha_p$, we can write $\frac{dC_O^{(1)}(\alpha_p^{(1)})}{d\alpha_p} = \frac{r_O^{(1)}}{\ln 2 \check{D}^{(1)}\alpha_p}$. Also, using (4.125) with $\alpha_p^{(2)} = 1 - \alpha_p$ we obtain $\frac{d\check{R}_O^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})}{d\alpha_p} = -\frac{\check{r}_O^{(2,1)}}{\ln 2 \check{D}^{(2)}(1-\alpha_p)}$, yielding

$$\frac{d\check{R}_O^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})}{dC_O^{(1)}(\alpha_p^{(1)})} = \frac{d\check{R}_O^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})}{d\alpha_p} \frac{d\alpha_p}{dC_O^{(1)}(\alpha_p^{(1)})} = -\frac{\check{r}_O^{(2,1)} \check{D}^{(1)}\alpha_p}{\check{D}^{(2)}(1-\alpha_p) r_O^{(1)}}$$

Solving $\frac{d\check{R}_O^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})}{dC_O^{(1)}(\alpha_p^{(1)})} = -1$ leads to $\tilde{\alpha}_{O;p;s}^{(1)} = [1 + \frac{\check{r}_O^{(2,1)} \check{D}^{(1)}}{r_O^{(1)} \check{D}^{(2)}}]^{-1}$ and $\tilde{\alpha}_{O;p;s}^{(2)} = 1 - \tilde{\alpha}_{O;p;s}^{(1)}$. Following similar steps, if $\Delta_{O;\infty}^{(1)} \leq \Delta_{O;\infty}^{(2)}$, then the sum rate is maximized at the boundary point $(\check{R}_O^{(1)}(\tilde{\alpha}_{O;p;s}^{(1)}, \tilde{\alpha}_{O;p;s}^{(2)}, C_O^{(2)}(\tilde{\alpha}_{O;p;s}^{(2)}))$ with $\tilde{\alpha}_{O;p;s}^{(1)} = [1 + \frac{r_O^{(2)} \check{D}^{(1)}}{\check{r}_O^{(1,2)} \check{D}^{(2)}}]^{-1}$ and $\tilde{\alpha}_{O;p;s}^{(2)} = 1 - \tilde{\alpha}_{O;p;s}^{(1)}$. Here, $\check{r}_O^{(1,2)}$ is the rank of $[\mathbf{H}_O^{(1)}]^\dagger \cdot [\bar{\mathbf{M}}_O^{(1,2)}]^\dagger \cdot \mathbf{D}_O^{(1,2)} \cdot \bar{\mathbf{M}}_O^{(1,2)} \cdot \mathbf{H}_O^{(1)}$ where $\bar{\mathbf{M}}_O^{(1,2)}$ is the unitary matrix in the eigenvalue decomposition

$$\mathbf{H}_O^{(1,2)} \cdot \text{diag}_{i=1}^{N_G^{(2)}} [\mathbf{M}_{O;i}^{(2)}]^\dagger \cdot \tilde{\mathbf{B}}_{O;i}^{(2)} \cdot \mathbf{M}_{O;i}^{(2)} \cdot [\mathbf{H}_O^{(1,2)}]^\dagger = [\bar{\mathbf{M}}_O^{(1,2)}]^\dagger \cdot \bar{\Lambda}_O^{(1,2)} \cdot \bar{\mathbf{M}}_O^{(1,2)}$$

$\tilde{\mathbf{B}}_{O;i}^{(2)} = \text{diag}[\mathbf{I}_{r_{O,i}^{(2)}}, \mathbf{0}_{(n_i^{(2)} D_G^{(2)} - r_{O,i}^{(2)}) \times (n_i^{(2)} D_G^{(2)} - r_{O,i}^{(2)})}]$, and $\mathbf{D}_O^{(1,2)}$ is the $N_R^{(1)} D^{(1)} \times N_R^{(1)} D^{(1)}$

diagonal matrix defined as $[D_O^{(1,2)}]_{p,p} = 1$ if $\frac{1}{r_O^{(2)}}[\bar{\Lambda}_O^{(1,2)}]_{p,p} = 0$ and $[D_O^{(1,2)}]_{p,p} = 0$ if $\frac{1}{r_O^{(2)}}[\bar{\Lambda}_O^{(1,2)}]_{p,p} \neq 0$.

In Section 4.5, we studied the multiplexing gains associated with $\mathcal{R}_{out}$ and $\mathcal{R}_{in}$, and we analyzed the impact of signal dimensions sharing and the number of time resolvable clusters on these multiplexing gains. Considering the high SNR regime with $(P_T/N_0) \rightarrow \infty$, we conduct a similar analysis for the multiplexing gains associated with $\mathcal{R}_{O,out}$ and $\mathcal{R}_{O,in}$. For the remainder of this section, we consider 3GPP channels for which the rank of $\mathbf{H}_O^{(k)}$ is given by $r_O^{(k)} = D_G^{(k)} \sum_{i=1}^{N_G^{(k)}} \min(N_R^{(k)}, n_i^{(k)})$ for $k = 1, 2$. We start by considering $\mathcal{R}_{O,out}$. For MIMO-OFDM BCs, a point on the segment $V_0\tilde{V}$ (respectively, $\tilde{V}V_1$) in Figure 4.2 is associated with the pair of multiplexing gains $(\tilde{\tau}_0 G_{O,out}^{(1)}, G_{O,out}^{(2)})$ with $\tilde{\tau}_0 \in [0, 1]$ (respectively, $(G_{O,out}^{(1)}, \tilde{\tau}_1 G_{O,out}^{(2)})$ with $\tilde{\tau}_1 \in [0, 1]$) where $\{G_{O,out}^{(k)}\}_{k=1}^2$ are given by (4.114), or

$$G_{O,out}^{(k)} = \frac{D^{(k)} \sum_{i=1}^{N_G^{(k)}} \min(N_R^{(k)}, n_i^{(k)})}{N_G^{(k)} \check{D}^{(k)}} \quad (4.132)$$

for $k = 1, 2$. From (4.132), $G_{O,out}^{(k)}$ is independent of the number of time resolvable clusters $L_t^{(k)}$. Writing (4.42) as $G_{out}^{(k)} = \frac{D^{(k)} \sum_{i=1}^{N_G^{(k)}} \min(L_t^{(k)} N_R^{(k)}, n_i^{(k)})}{N_G^{(k)} \check{D}^{(k)}}$ for 3GPP channels, one can see that the only difference between $G_{O,out}^{(k)}$ and $G_{out}^{(k)}$ lies in the presence of the parameter $L_t^{(k)}$. Since $L_t^{(k)} \geq 1$, it can be shown similarly to (3.63) that $G_{out}^{(k)} \geq G_{O,out}^{(k)}$, yielding $\mathcal{R}_{O,out} \subseteq \mathcal{R}_{out}$. The two regions are equal if $G_{out}^{(k)} = G_{O,out}^{(k)}$ for $k = 1$ and $k = 2$. Based on the results of Section 3.3, $G_{out}^{(k)}$ and $G_{O,out}^{(k)}$ are equal if $L_t^{(k)} = 1$ or if $n_i^{(k)} \leq N_R^{(k)}$ for all $i = 1, \dots, N_G^{(k)}$. The behavior of $G_{O,out}^{(k)}$ as a function of D_c is similar to that of $G_{out}^{(k)}$. Thus, for $k = 1, 2$, $[G_{O,out}^{(k)}]_{\check{D}^{(k)}=D^{(k)}} \equiv \frac{\sum_{i=1}^{N_G^{(k)}} \min(N_R^{(k)}, n_i^{(k)})}{N_G^{(k)}}$ is independent of D_c . Moreover, $[G_{O,out}^{(k)}]_{\check{D}^{(k)}=D_T}$ increases monotonically when D_c increases, either with $D_p^{(1)}, D_p^{(2)}$ held fixed, or at the expense of the number of proper signal dimensions of the other user in the system with D_T held fixed. When $D_c \rightarrow \infty$, $[G_{O,out}^{(k)}]_{\check{D}^{(k)}=D_T}$ converges to $[G_{O,out}^{(k)}]_{\check{D}^{(k)}=D^{(k)}}$ and hence, $[\mathcal{R}_{O,out}]_{\check{D}^{(k)}=D_T} \rightarrow [\mathcal{R}_{O,out}]_{\check{D}^{(k)}=D^{(k)}}$. We can also show that as D_c increases the separation between $[\mathcal{R}_{O,out}]_{\check{D}^{(k)}=D_T}$ and $[\mathcal{R}_{O,out}]_{\check{D}^{(k)}=D^{(k)}}$ decreases monotonically. Note finally that $G_{O,out}^{(k)}$ is independent of user k 's angle spread at the transmitter and receiver.

We now consider $\mathcal{R}_{O,in}$ and we start by evaluating $G_{O,in}^{(2)}$ of (4.128). The evaluation of $G_{O,in}^{(1)}$ of (4.130) can be done by switching the users indexes and following similar steps.

Consider the rank $r_{\max;O}^{(2,1)}$ of the matrix $\mathbf{H}_O^{(2,1)} \cdot (\frac{1}{P_T} \Sigma_{O;\max}^{(1)}) \cdot [\mathbf{H}_O^{(2,1)}]^\dagger$ given by (4.122). For every $j = 1, \dots, N_G^{(1)}$, we can write $(\mathbf{M}_{O;j}^{(1)})^\dagger \cdot \tilde{\mathbf{B}}_{O;j}^{(1)} \cdot \mathbf{M}_{O;j}^{(1)} = \sum_{l=1}^{r_{O;j}^{(1)}} [(\mathbf{M}_{O;j}^{(1)})^\dagger]_{*,l} \cdot [(\mathbf{M}_{O;j}^{(1)})^\dagger]_{*,l}^\dagger$. The vectors $[(\mathbf{M}_{O;j}^{(1)})^\dagger]_{*,l}$ are the eigenvectors of $\tilde{\mathbf{O}}_j^{(1)}$ that are associated with the (nonzero) eigenvalues $\lambda_{O;j;l}^{(1)}$, $l = 1, \dots, r_{O;j}^{(1)}$ with $r_{O;j}^{(1)} = D_G^{(1)} \min(N_R^{(1)}, n_j^{(1)})$. From (4.96) we can show, using (C.3) and $\mathbf{e}_{i;n}^T \cdot \mathbf{e}_{p;n} = \delta(i, p)$ for all $i, p = 1, \dots, n$, that the eigenvectors of $\tilde{\mathbf{O}}_j^{(1)}$ are given by

$$[(\mathbf{M}_{O;j}^{(1)})^\dagger]_{*,l} \equiv \mathbf{n}_{j;l}(\tilde{\mathbf{O}}_j^{(1)}) = \mathbf{n}_q((\check{\mathbf{F}}_{j;d}^{(1)})^\dagger \cdot \check{\mathbf{F}}_{j;d}^{(1)}) \otimes \mathbf{e}_{d;D_G^{(1)}}$$

where $\mathbf{n}_q((\check{\mathbf{F}}_{j;d}^{(1)})^\dagger \cdot \check{\mathbf{F}}_{j;d}^{(1)})$ denotes the q^{th} eigenvector of $(\check{\mathbf{F}}_{j;d}^{(1)})^\dagger \cdot \check{\mathbf{F}}_{j;d}^{(1)}$ that is associated with a nonzero eigenvalue, $q = 1, \dots, \min(N_R^{(1)}, n_j^{(1)})$ and $d = 1, \dots, D_G^{(1)}$. Thus, we have

$$[(\mathbf{M}_{O;j}^{(1)})^\dagger \cdot \tilde{\mathbf{B}}_{O;j}^{(1)} \cdot \mathbf{M}_{O;j}^{(1)}] \stackrel{(a)}{=} \sum_{d=1}^{D_G^{(1)} \min(N_R^{(1)}, n_j^{(1)})} \sum_{q=1}^{\min(N_R^{(1)}, n_j^{(1)})} \{ \mathbf{n}_q((\check{\mathbf{F}}_{j;d}^{(1)})^\dagger \cdot \check{\mathbf{F}}_{j;d}^{(1)}) \cdot \mathbf{n}_q^\dagger((\check{\mathbf{F}}_{j;d}^{(1)})^\dagger \cdot \check{\mathbf{F}}_{j;d}^{(1)}) \} \otimes \{ \mathbf{e}_{d;D_G^{(1)}} \cdot \mathbf{e}_{d;D_G^{(1)}}^T \}$$

where (a) is due to (C.3). Define the $n_j^{(1)} \times n_j^{(1)}$ matrix

$$\check{\mathbf{M}}_{O;j;d}^{(1)} \triangleq \sum_{q=1}^{\min(N_R^{(1)}, n_j^{(1)})} \{ \mathbf{n}_q((\check{\mathbf{F}}_{j;d}^{(1)})^\dagger \cdot \check{\mathbf{F}}_{j;d}^{(1)}) \cdot \mathbf{n}_q^\dagger((\check{\mathbf{F}}_{j;d}^{(1)})^\dagger \cdot \check{\mathbf{F}}_{j;d}^{(1)}) \} \quad (4.133)$$

Using (C.5a), we obtain

$$[(\mathbf{M}_{O;j}^{(1)})^\dagger \cdot \tilde{\mathbf{B}}_{O;j}^{(1)} \cdot \mathbf{M}_{O;j}^{(1)}] = \sum_{d=1}^{D_G^{(1)}} \check{\mathbf{M}}_{O;j;d}^{(1)} \otimes \{ \mathbf{e}_{d;D_G^{(1)}} \cdot \mathbf{e}_{d;D_G^{(1)}}^T \} \quad (4.134)$$

Moreover, using (2.50) with $(k, g) = (2, 1)$ and (4.61), we can write

$$\begin{aligned} & \mathbf{H}_O^{(2,1)} \cdot \text{diag}_{i=1}^{N_G^{(1)}} \left[[(\mathbf{M}_{O;i}^{(1)})^\dagger \cdot \tilde{\mathbf{B}}_{O;i}^{(1)} \cdot \mathbf{M}_{O;i}^{(1)}] \cdot [\mathbf{H}_O^{(2,1)}]^\dagger \right] \\ &= [\mathbf{O}_{m,q}^{(2)}] \cdot \text{diag}_{j=1}^{N_G^{(1)}} \left[\{ \mathbf{I}_{n_j^{(1)}} \otimes \mathbf{B}_j^{(2,1)} \} \cdot [(\mathbf{M}_{O;j}^{(1)})^\dagger \cdot \tilde{\mathbf{B}}_{O;j}^{(1)} \cdot \mathbf{M}_{O;j}^{(1)}] \cdot \{ \mathbf{I}_{n_j^{(1)}} \otimes (\mathbf{B}_j^{(2,1)})^\dagger \} \right] \cdot [\mathbf{O}_{m,q}^{(2)}]^\dagger \\ &\stackrel{(a)}{=} [\mathbf{O}_{m,q}^{(2)}] \cdot \text{diag}_{j=1}^{N_G^{(1)}} \left[\sum_{d=1}^{D_G^{(1)}} \check{\mathbf{M}}_{O;j;d}^{(1)} \otimes \{ \mathbf{B}_j^{(2,1)} \cdot \mathbf{e}_{d;D_G^{(1)}} \cdot \mathbf{e}_{d;D_G^{(1)}}^\dagger \cdot (\mathbf{B}_j^{(2,1)})^\dagger \} \right] \cdot [\mathbf{O}_{m,q}^{(2)}]^\dagger \end{aligned}$$

where (a) is due to (4.134) and (C.3). Since $\mathbf{B}_j^{(2,1)} \cdot \mathbf{e}_{d;D_G^{(1)}} = [\mathbf{B}_j^{(2,1)}]_{*,d}$, it follows that

$$\begin{aligned} & \mathbf{H}_O^{(2,1)} \cdot \text{diag}_{i=1}^{N_G^{(1)}} \left[[\mathbf{M}_{O;i}^{(1)}]^\dagger \cdot \tilde{\mathbf{B}}_{O;i}^{(1)} \cdot \mathbf{M}_{O;i}^{(1)} \right] \cdot [\mathbf{H}_O^{(2,1)}]^\dagger \\ &= [\mathbf{O}_{m,q}^{(2)}] \cdot \text{diag}_{j=1}^{N_G^{(1)}} \left[\sum_{d=1}^{D_G^{(1)}} \tilde{\mathbf{M}}_{O;j;d}^{(1)} \otimes \{ [\mathbf{B}_j^{(2,1)}]_{*,d} \cdot [\mathbf{B}_j^{(2,1)}]_{*,d}^\dagger \} \right] \cdot [\mathbf{O}_{m,q}^{(2)}]^\dagger \end{aligned} \quad (4.135)$$

For each $l \in \{1, \dots, L_t^{(2)}\}$, we define the $N_R^{(2)} \times n_j^{(1)}$ submatrices $\mathbf{F}_{j;l}^{(2,1)}$, $j = 1, \dots, N_G^{(1)}$, such that $\mathbf{F}_l^{(2)} = [\mathbf{F}_{1;l}^{(2,1)} \dots \mathbf{F}_{N_G^{(1)};l}^{(2,1)}]$. Also, define the $N_R^{(2)} \times n_j^{(1)}$ matrix $\tilde{\mathbf{F}}_j^{(2,1)}(\tau) \triangleq \sum_{l=1}^{L_t^{(2)}} \delta(\tau - \tau_l^{(2)}) \mathbf{F}_{j;l}^{(2,1)}$. Hence, user 2's channel impulse response matrix $\tilde{\mathbf{F}}^{(2)}(\tau)$ (defined in Section 2.3) can be written as $\tilde{\mathbf{F}}^{(2)}(\tau) = [\tilde{\mathbf{F}}_1^{(2,1)}(\tau) \dots \tilde{\mathbf{F}}_{N_G^{(1)}}^{(2,1)}(\tau)]$. Using the notation $\mathbf{E}_p^{(2)} = \mathbf{e}_{p;D^{(2)}} \cdot \mathbf{e}_{p;D^{(2)}}^T \equiv \mathbf{e}_{p;D^{(2)}} \cdot \mathbf{e}_{p;D^{(2)}}^\dagger$ from Section 4.5.2, we have

$$[\mathbf{O}_{m,q}^{(2)}] = \sum_{p=1}^{D^{(2)}} FT[\tilde{\mathbf{F}}^{(2)}(\tau)]_{\frac{i^{(2)}(p)}{T_F}} \otimes \mathbf{E}_p^{(2)} \quad (4.136a)$$

$$= \sum_{p=1}^{D^{(2)}} \left[FT[\tilde{\mathbf{F}}_1^{(2,1)}(\tau)]_{\frac{i^{(2)}(p)}{T_F}} \dots FT[\tilde{\mathbf{F}}_{N_G^{(1)}}^{(2,1)}(\tau)]_{\frac{i^{(2)}(p)}{T_F}} \right] \otimes \mathbf{E}_p^{(2)} \quad (4.136b)$$

from Section 2.3. Using (4.136b), (4.135) and identity (C.3), we obtain

$$\begin{aligned} & \mathbf{H}_O^{(2,1)} \cdot \text{diag}_{i=1}^{N_G^{(1)}} \left[[\mathbf{M}_{O;i}^{(1)}]^\dagger \cdot \tilde{\mathbf{B}}_{O;i}^{(1)} \cdot \mathbf{M}_{O;i}^{(1)} \right] \cdot [\mathbf{H}_O^{(2,1)}]^\dagger \\ &= \sum_{j=1}^{N_G^{(1)}} \sum_{p=1}^{D^{(2)}} \sum_{p'=1}^{D^{(2)}} \sum_{d=1}^{D_G^{(1)}} \left\{ FT[\tilde{\mathbf{F}}_j^{(2,1)}(\tau)]_{\frac{i^{(2)}(p)}{T_F}} \otimes \mathbf{E}_p^{(2)} \right\} \\ & \quad \cdot \left\{ \tilde{\mathbf{M}}_{O;j;d}^{(1)} \otimes [[\mathbf{B}_j^{(2,1)}]_{*,d} \cdot [\mathbf{B}_j^{(2,1)}]_{*,d}^\dagger] \right\} \cdot \left\{ \left(FT[\tilde{\mathbf{F}}_j^{(2,1)}(\tau)]_{\frac{i^{(2)}(p')}{T_F}} \right)^\dagger \otimes \mathbf{E}_{p'}^{(2)} \right\} \\ &\stackrel{(a)}{=} \sum_{j \in \mathcal{A}_{G;c}^{(1)}} \sum_{p=1}^{D^{(2)}} \sum_{p'=1}^{D^{(2)}} \sum_{d=1}^{D_G^{(1)}} \left\{ FT[\tilde{\mathbf{F}}_j^{(2,1)}(\tau)]_{\frac{i^{(2)}(p)}{T_F}} \otimes \mathbf{E}_p^{(2)} \right\} \\ & \quad \cdot \left\{ \tilde{\mathbf{M}}_{O;j;d}^{(1)} \otimes [[\mathbf{B}_j^{(2,1)}]_{*,d} \cdot [\mathbf{B}_j^{(2,1)}]_{*,d}^\dagger] \right\} \cdot \left\{ \left(FT[\tilde{\mathbf{F}}_j^{(2,1)}(\tau)]_{\frac{i^{(2)}(p')}{T_F}} \right)^\dagger \otimes \mathbf{E}_{p'}^{(2)} \right\} \end{aligned}$$

with (a) due to $\mathbf{B}_j^{(2,1)} = \mathbf{0}_{D^{(2)} \times D_G^{(1)}}$ for all $j \notin \mathcal{A}_{G;c}^{(1)}$. As in Section 4.5.2, each TOG $j \in \mathcal{A}_{G;c}^{(1)}$ employs $D_{c;j}^{(2)}$ shared signal dimensions, $D_{c;j}^{(2)} \leq D_G^{(1)}$. For $q = 1, \dots, D_{c;j}^{(2)}$, let $v^{(2,1)}(j, q) \in \{1, \dots, D_G^{(1)}\}$ denote the index of the shared signal dimension that is

employed by TOG j and associated with integer q . If $d \in \{1, \dots, D_G^{(1)}\}$ is equal to an index $v^{(2,1)}(j, q)$ then $[\mathbf{B}_j^{(2,1)}]_{*,d} \equiv [\mathbf{B}_j^{(2,1)}]_{*,v^{(2,1)}(j,q)} = \mathbf{e}_{v^{(2)}(j,q);D^{(2)}}$ and $[\mathbf{B}_j^{(2,1)}]_{*,d} \cdot [\mathbf{B}_j^{(2,1)}]_{*,d}^\dagger = \mathbf{E}_{v^{(2)}(j,q)}^{(2)}$ where, as in Section 4.5.2, $v^{(2)}(j, q) \in \{1, \dots, D^{(2)}\}$ denotes for user 2 the index of the shared signal dimension associated with integer q and TOG j of user 1. If the d^{th} signal dimension of TOG j of user 1 is not shared (i.e. does not correspond to any of the indexes $v^{(2,1)}(j, q)$) then $[\mathbf{B}_j^{(2,1)}]_{*,d} = \mathbf{0}_{D^{(2)}}$. It follows that

$$\begin{aligned} & \mathbf{H}_O^{(2,1)} \cdot \text{diag}_{i=1}^{N_G^{(1)}} \left[[\mathbf{M}_{O,i}^{(1)}]^\dagger \cdot \tilde{\mathbf{B}}_{O,i}^{(1)} \cdot \mathbf{M}_{O,i}^{(1)} \right] \cdot [\mathbf{H}_O^{(2,1)}]^\dagger \\ &= \sum_{j \in \mathcal{A}_{G;c}^{(1)}} \sum_{p=1}^{D^{(2)}} \sum_{p'=1}^{D^{(2)}} \sum_{q=1}^{D_{c;j}^{(2)}} \left\{ FT[\tilde{\mathbf{F}}_j^{(2,1)}(\tau)]_{\frac{i^{(2)}(p)}{T_F}} \otimes \mathbf{E}_p^{(2)} \right\} \\ & \cdot \left\{ \tilde{\mathbf{M}}_{O;j;v^{(2,1)}(j,q)}^{(1)} \otimes \mathbf{E}_{v^{(2)}(j,q)}^{(2)} \right\} \cdot \left\{ \left(FT[\tilde{\mathbf{F}}_j^{(2,1)}(\tau)]_{\frac{i^{(2)}(p')}{T_F}} \right)^\dagger \otimes \mathbf{E}_{p'}^{(2)} \right\} \end{aligned}$$

Define $\mathbf{N}_{O;j;q}^{(2,1)} \triangleq FT[\tilde{\mathbf{F}}_j^{(2,1)}(\tau)]_{\frac{i^{(2)}(v^{(2)}(j,q))}{T_F}} \cdot \tilde{\mathbf{M}}_{O;j;v^{(2,1)}(j,q)}^{(1)} \cdot (FT[\tilde{\mathbf{F}}_j^{(2,1)}(\tau)]_{\frac{i^{(2)}(v^{(2)}(j,q))}{T_F}})^\dagger$ for $q = 1, \dots, D_{c;j}^{(2)}$ and $j \in \mathcal{A}_{G;c}^{(1)}$. The matrix $\mathbf{N}_{O;j;q}^{(2,1)}$ is Hermitian positive semidefinite of size $N_R^{(2)} \times N_R^{(2)}$. Using (C.3) with $\mathbf{E}_n^{(2)} \cdot \mathbf{E}_i^{(2)} = \mathbf{e}_{n;D^{(2)}} \cdot \mathbf{e}_{n;D^{(2)}}^T \cdot \mathbf{e}_{i;D^{(2)}} \cdot \mathbf{e}_{i;D^{(2)}}^T = \delta(i, n) \mathbf{E}_n^{(2)}$, we finally obtain

$$\mathbf{H}_O^{(2,1)} \cdot \text{diag}_{i=1}^{N_G^{(1)}} \left[[\mathbf{M}_{O,i}^{(1)}]^\dagger \cdot \tilde{\mathbf{B}}_{O,i}^{(1)} \cdot \mathbf{M}_{O,i}^{(1)} \right] \cdot [\mathbf{H}_O^{(2,1)}]^\dagger = \sum_{j \in \mathcal{A}_{G;c}^{(1)}} \sum_{q=1}^{D_{c;j}^{(2)}} \mathbf{N}_{O;j;q}^{(2,1)} \otimes \mathbf{E}_{v^{(2)}(j,q)}^{(2)} \quad (4.137)$$

Using (4.122), this yields

$$\mathbf{H}_O^{(2,1)} \cdot \left(\frac{1}{P_T} \Sigma_{O;\max}^{(1)} \right) \cdot [\mathbf{H}_O^{(2,1)}]^\dagger = \frac{\alpha_p^{(1)}}{r_O^{(1)}} \sum_{j \in \mathcal{A}_{G;c}^{(1)}} \sum_{q=1}^{D_{c;j}^{(2)}} \mathbf{N}_{O;j;q}^{(2,1)} \otimes \mathbf{E}_{v^{(2)}(j,q)}^{(2)} \quad (4.138)$$

Since $v^{(2)}(j, q) \neq v^{(2)}(j', q')$ for $(j, q) \neq (j', q')$ from Section 4.5.2, the matrices $\mathbf{E}_{v^{(2)}(j,q)}^{(2)}$ and, hence, $\mathbf{N}_{O;j;q}^{(2,1)} \otimes \mathbf{E}_{v^{(2)}(j,q)}^{(2)}$ have their column (and row) spaces essentially disjoint. Thus, using [114, Theorem 18.5.7] for rank additivity, the rank $r_{\max;O}^{(2,1)}$ of $\mathbf{H}_O^{(2,1)} \cdot \left(\frac{1}{P_T} \Sigma_{O;\max}^{(1)} \right) \cdot [\mathbf{H}_O^{(2,1)}]^\dagger$ is equal to

$$r_{\max;O}^{(2,1)} = \sum_{j \in \mathcal{A}_{G;c}^{(1)}} \sum_{q=1}^{D_{c;j}^{(2)}} \text{rank}(\mathbf{N}_{O;j;q}^{(2,1)} \otimes \mathbf{E}_{v^{(2)}(j,q)}^{(2)}) \stackrel{(a)}{=} \sum_{j \in \mathcal{A}_{G;c}^{(1)}} \sum_{q=1}^{D_{c;j}^{(2)}} \text{rank}(\mathbf{N}_{O;j;q}^{(2,1)}) \quad (4.139)$$

with (a) due to (C.16) and to $\text{rank}(\mathbf{E}_{v^{(2)}(j,q)}^{(2)}) = 1$. Let us evaluate $\text{rank}(\mathbf{N}_{O;j,q}^{(2,1)})$ for 3GPP channels. For this, we write the matrices $\check{\mathbf{M}}_{O;j,d}^{(1)}$ of (4.133) as $\check{\mathbf{M}}_{O;j,d}^{(1)} = [\check{\mathbf{U}}_{O;j,d}^{(1)}]^\dagger \cdot \mathbf{B}_{O;j}^{(1)} \cdot \check{\mathbf{U}}_{O;j,d}^{(1)}$ where $\mathbf{B}_{O;j}^{(1)} = \text{diag}[\mathbf{I}_{\min(N_R^{(1)}, n_j^{(1)})}, \mathbf{0}_{(n_j^{(1)} - \min(N_R^{(1)}, n_j^{(1)})) \times (n_j^{(1)} - \min(N_R^{(1)}, n_j^{(1)})}]$ and $\check{\mathbf{U}}_{O;j,d}^{(1)}$ denotes the $n_j^{(1)} \times n_j^{(1)}$ unitary matrix in the eigenvalue decomposition of $(\check{\mathbf{F}}_{j;d}^{(1)})^\dagger \cdot \check{\mathbf{F}}_{j;d}^{(1)}$, i.e. the columns of the matrix $[\check{\mathbf{U}}_{O;j,d}^{(1)}]^\dagger$ are the eigenvectors $\{\mathbf{n}_q((\check{\mathbf{F}}_{j;d}^{(1)})^\dagger \cdot \check{\mathbf{F}}_{j;d}^{(1)})\}_{q=1}^{n_j^{(1)}}$. It follows that

$$\mathbf{N}_{O;j,q}^{(2,1)} = FT[\tilde{\mathbf{F}}_j^{(2,1)}(\tau)]_{\frac{i^{(2)}(v^{(2)}(j,q))}{T_F}} \cdot [\check{\mathbf{U}}_{O;j,v^{(2,1)}(j,q)}^{(1)}]^\dagger \cdot \mathbf{B}_{O;j}^{(1)} \cdot \check{\mathbf{U}}_{O;j,v^{(2,1)}(j,q)}^{(1)} \cdot (FT[\tilde{\mathbf{F}}_j^{(2,1)}(\tau)]_{\frac{i^{(2)}(v^{(2)}(j,q))}{T_F}})^\dagger \quad (4.140)$$

From this, we can write $\mathbf{N}_{O;j,q}^{(2,1)} = \tilde{\mathbf{C}}_{N;j,q}^{(2,1)} \cdot [\tilde{\mathbf{C}}_{N;j,q}^{(2,1)}]^\dagger$ where $\tilde{\mathbf{C}}_{N;j,q}^{(2,1)}$ denotes the $N_R^{(2)} \times \min(N_R^{(1)}, n_j^{(1)})$ matrix formed by the first $\min(N_R^{(1)}, n_j^{(1)})$ columns of the $N_R^{(2)} \times n_j^{(1)}$ product matrix $FT[\tilde{\mathbf{F}}_j^{(2,1)}(\tau)]_{\frac{i^{(2)}(v^{(2)}(j,q))}{T_F}} \cdot [\check{\mathbf{U}}_{O;j,v^{(2,1)}(j,q)}^{(1)}]^\dagger$. Since the parameters of the users spatial channels are randomly generated, $\tilde{\mathbf{C}}_{N;j,q}^{(2,1)}$ is most likely full rank, thus, yielding $\text{rank}(\mathbf{N}_{O;j,q}^{(2,1)}) = \min(N_R^{(2)}, \min(N_R^{(1)}, n_j^{(1)})) = \min(N_R^{(2)}, N_R^{(1)}, n_j^{(1)})$ from (C.15) for all q . Equation (4.139) reduces to

$$r_{\max;O}^{(2,1)} = \sum_{j \in \mathcal{A}_{G;c}^{(1)}} D_{c;j}^{(2)} \min(N_R^{(2)}, N_R^{(1)}, n_j^{(1)}) \quad (4.141)$$

We now consider the rank $r_{\max;O}^{(2)}$ of $\mathbf{H}_O^{(2,1)} \cdot (\frac{1}{P_T} \Sigma_{O;\max}^{(1)}) \cdot [\mathbf{H}_O^{(2,1)}]^\dagger + \mathbf{H}_O^{(2)} \cdot (\frac{1}{P_T} \check{\Sigma}_O^{(2)}) \cdot [\mathbf{H}_O^{(2)}]^\dagger$. We use an approach similar to the one used in Section 4.5.2 to evaluate the rank $r_{\max}^{(2)}$. Since $r_{\max;O}^{(2,1)} \leq r_{\max;O}^{(2)} \leq N_R^{(2)} D^{(2)}$, we have $r_{\max;O}^{(2)} = r_{\max;O}^{(2,1)} = N_R^{(2)} D^{(2)}$ if $r_{\max;O}^{(2,1)} = N_R^{(2)} D^{(2)}$, in which case $G_{O;\max}^{(2)} = 0$ from (4.128). We subsequently assume that $r_{\max;O}^{(2,1)} < N_R^{(2)} D^{(2)}$. Prior to evaluating $r_{\max;O}^{(2)}$, we introduce some notations and definitions. We use $\tilde{\mathcal{A}}_{O;G;c}^{(1)} \subseteq \mathcal{A}_{G;c}^{(1)}$ to denote the set of $j \in \mathcal{A}_{G;c}^{(1)}$ such that $\min(N_R^{(1)}, n_j^{(1)}) < N_R^{(2)}$. Hence, for $j \in \tilde{\mathcal{A}}_{O;G;c}^{(1)}$ and for all $q = 1, \dots, D_{c;j}^{(2)}$, the matrices $\mathbf{N}_{O;j,q}^{(2,1)}$ are not full rank with $\text{rank}(\mathbf{N}_{O;j,q}^{(2,1)}) = \min(N_R^{(1)}, n_j^{(1)}) < N_R^{(2)}$. Using $\mathcal{A}_{G;c}^{(1)} - \tilde{\mathcal{A}}_{O;G;c}^{(1)}$ to denote the subset of indexes $j \in \mathcal{A}_{G;c}^{(1)}$ and $j \notin \tilde{\mathcal{A}}_{O;G;c}^{(1)}$, we have $\text{rank}(\mathbf{N}_{O;j,q}^{(2,1)}) = N_R^{(2)} \leq \min(N_R^{(1)}, n_j^{(1)})$ for all $q = 1, \dots, D_{c;j}^{(2)}$ and all $j \in \mathcal{A}_{G;c}^{(1)} - \tilde{\mathcal{A}}_{O;G;c}^{(1)}$. Furthermore, we define for $j \in \tilde{\mathcal{A}}_{O;G;c}^{(1)}$ the $n_a^{(2)} \times n_a^{(2)}$ matrices

$$\check{\mathbf{N}}_{O;j,q;a}^{(2,1)} \triangleq \sum_{l=\min(N_R^{(1)}, n_j^{(1)})+1}^{N_R^{(2)}} (FT[\tilde{\mathbf{F}}_a^{(2)}(\tau)]_{\frac{i^{(2)}(v^{(2)}(j,q))}{T_F}})^\dagger \cdot \mathbf{n}_l(\mathbf{N}_{O;j,q}^{(2,1)}) \cdot \mathbf{n}_l^\dagger(\mathbf{N}_{O;j,q}^{(2,1)}) \cdot FT[\tilde{\mathbf{F}}_a^{(2)}(\tau)]_{\frac{i^{(2)}(v^{(2)}(j,q))}{T_F}} \quad (4.142)$$

and $\check{\mathbf{W}}_{O;j;q;a}^{(2,1)} \triangleq \sum_{l=1}^{\text{rank}(\check{\mathbf{N}}_{O;j;q;a}^{(2,1)})} \mathbf{n}_l(\check{\mathbf{N}}_{O;j;q;a}^{(2,1)}) \cdot \mathbf{n}_l^\dagger(\check{\mathbf{N}}_{O;j;q;a}^{(2,1)})$. Also, we define the $N_R^{(2)} \times N_R^{(2)}$ matrices

$$\mathbf{Z}_{O;j;q;a}^{(2,1)} \triangleq FT[\tilde{\mathbf{F}}_a^{(2)}(\tau)]_{\frac{i^{(2)}(v^{(2)}(j,q))}{T_F}} \cdot \check{\mathbf{W}}_{O;j;q;a}^{(2,1)} \cdot (FT[\tilde{\mathbf{F}}_a^{(2)}(\tau)]_{\frac{i^{(2)}(v^{(2)}(j,q))}{T_F}})^\dagger \quad (4.143)$$

and $\mathbf{K}_{O;j;q;a}^{(2,1)} \triangleq \frac{\alpha_p^{(2)}}{r_O^{(2,1)}} \mathbf{Z}_{O;j;q;a}^{(2,1)} + \frac{\alpha_p^{(1)}}{r_O^{(1)}} \mathbf{N}_{O;j;q}^{(2,1)}$. Using the previous notations and definitions, we have the following lemma.

Lemma 4.6.1 *The rank $r_{\max;O}^{(2)}$ of $\mathbf{H}_O^{(2,1)} \cdot (\frac{1}{P_T} \Sigma_{O;\max}^{(1)}) \cdot [\mathbf{H}_O^{(2,1)}]^\dagger + \mathbf{H}_O^{(2)} \cdot (\frac{1}{P_T} \check{\Sigma}_O^{(2)}) \cdot [\mathbf{H}_O^{(2)}]^\dagger$ is given by*

$$r_{\max;O}^{(2)} = \sum_{a=1}^{N_G^{(2)}} D_{p;a}^{(2)} \min(N_R^{(2)}, n_a^{(2)}) + \sum_{a=1}^{N_G^{(2)}} \sum_{j \in \tilde{\mathcal{A}}_{O;G;c}^{(1)}} D_{c;j;a}^{(2)} r_{K;j;a}^{(2,1)} + \sum_{a=1}^{N_G^{(2)}} \sum_{j \in \mathcal{A}_{G;c}^{(1)} - \tilde{\mathcal{A}}_{O;G;c}^{(1)}} D_{c;j;a}^{(2)} N_R^{(2)} \quad (4.144)$$

where $r_{K;j;a}^{(2,1)}$ denotes the rank of $\mathbf{K}_{O;j;q;a}^{(2,1)}$ for all q and it is independent of $L_t^{(1)}, L_t^{(2)}$.

The proof of Lemma 4.6.1 can be found in Appendix B. Writing

$$r_{\max;O}^{(2,1)} = \sum_{a=1}^{N_G^{(2)}} \sum_{j \in \mathcal{A}_{G;c}^{(1)} - \tilde{\mathcal{A}}_{O;G;c}^{(1)}} D_{c;j;a}^{(2)} N_R^{(2)} + \sum_{a=1}^{N_G^{(2)}} \sum_{j \in \tilde{\mathcal{A}}_{O;G;c}^{(1)}} D_{c;j;a}^{(2)} \min(N_R^{(1)}, n_j^{(1)})$$

from (4.141) and (4.67b) with $k = 2$, the multiplexing gain $G_{O;in}^{(2)}$ of (4.128) becomes

$$G_{O;in}^{(2)} = \frac{\sum_{a=1}^{N_G^{(2)}} D_{p;a}^{(2)} \min(N_R^{(2)}, n_a^{(2)}) + \sum_{a=1}^{N_G^{(2)}} \sum_{j \in \tilde{\mathcal{A}}_{O;G;c}^{(1)}} D_{c;j;a}^{(2)} [r_{K;j;a}^{(2,1)} - \min(N_R^{(1)}, n_j^{(1)})]}{\check{D}^{(2)}} \quad (4.145)$$

For user 1, the multiplexing gain $G_{O;in}^{(1)}$ can be obtained by following a similar method and by switching the users indexes throughout the derivations,

$$G_{O;in}^{(1)} = \frac{\sum_{\tilde{a}=1}^{N_G^{(1)}} D_{p;\tilde{a}}^{(1)} \min(N_R^{(1)}, n_{\tilde{a}}^{(1)}) + \sum_{\tilde{a}=1}^{N_G^{(1)}} \sum_{\tilde{j} \in \tilde{\mathcal{A}}_{O;G;c}^{(2)}} D_{c;\tilde{j};\tilde{a}}^{(1)} [r_{K;\tilde{j};\tilde{a}}^{(1,2)} - \min(N_R^{(2)}, n_{\tilde{j}}^{(2)})]}{\check{D}^{(1)}} \quad (4.146)$$

Thus, the multiplexing gains associated with the corner points $V_0, \tilde{V}_0, \tilde{V}_1$ and V_1 in Figure 4.2 of the limiting inner bound $\mathcal{R}_{O;in}$ (4.105) are, respectively, given by $(0, G_{O;out}^{(2)})$,

$(G_{O;in}^{(1)}, G_{O;out}^{(2)})$, $(G_{O;out}^{(1)}, G_{O;in}^{(2)})$ and $(G_{O;out}^{(1)}, 0)$ with $G_{O;out}^{(k)}$ given by (4.132) for $k = 1, 2$, $G_{O;in}^{(2)}$ given by (4.145) and $G_{O;in}^{(1)}$ given by (4.146).

For 3GPP channels, the main difference between the multiplexing gains (4.145), (4.146) associated with MIMO-OFDM BCs and the multiplexing gains (4.81), (4.84) associated with spread spectrum like MIMO BCs is the fact that the formers are independent of $\{L_t^{(k)}\}_{k=1}^2$ while the latters can increase as the number of time resolvable clusters increases (as seen from Section 4.5.2). Furthermore, let $\kappa_{O;j;a}^{(2,1)} \triangleq r_{K;j;a}^{(2,1)} - \min(N_R^{(1)}, n_j^{(1)})$ and $\kappa_{O;\tilde{j};\tilde{a}}^{(1,2)} \triangleq r_{K;\tilde{j};\tilde{a}}^{(1,2)} - \min(N_R^{(2)}, n_{\tilde{j}}^{(2)})$. Assuming as in Section 4.5.2 that $N_G^{(1)} = N_G^{(2)} = n_c$ with equal number of shared signal dimensions per TOG $\tilde{D}_c$ and that each TOG $a \in \mathcal{A}_{G;c}^{(2)}$ shares signal dimensions with only one TOG $\tilde{a} \in \mathcal{A}_{G;c}^{(1)}$, $(a, \tilde{a}) \in \mathcal{A}_G$, the multiplexing gains (4.145) and (4.146) reduce to

$$G_{O;in}^{(2)} = \frac{\sum_{a=1}^{N_G^{(2)}} D_{p;a}^{(2)} \min(N_R^{(2)}, n_a^{(2)}) + D_c \sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(1)}} (\kappa_{O;j;a}^{(2,1)}/n_c)}{\tilde{D}^{(2)}} \quad (4.147a)$$

$$G_{O;in}^{(1)} = \frac{\sum_{\tilde{a}=1}^{N_G^{(1)}} D_{p;\tilde{a}}^{(1)} \min(N_R^{(1)}, n_{\tilde{a}}^{(1)}) + D_c \sum_{(\tilde{j},\tilde{a}) \in \tilde{\mathcal{A}}_G^{(2)}} (\kappa_{O;\tilde{j};\tilde{a}}^{(1,2)}/n_c)}{\tilde{D}^{(1)}} \quad (4.147b)$$

Here, $\tilde{\mathcal{A}}_G^{(1)} \subseteq \mathcal{A}_G$ (respectively, $\tilde{\mathcal{A}}_G^{(2)} \subseteq \mathcal{A}_G$) is the set of pairs $(a, j) \in \mathcal{A}_G$ (respectively, $(\tilde{j}, \tilde{a}) \in \mathcal{A}_G$) such that $j \in \tilde{\mathcal{A}}_{O;G;c}^{(1)}$ (respectively, $\tilde{j} \in \tilde{\mathcal{A}}_{O;G;c}^{(2)}$). Using (4.147), the impact of signal dimensions sharing on the multiplexing gains associated with $\mathcal{R}_{O;in}$ can be investigated as in Section 4.5.2 with $\min(N_R^{(2)}, n_a^{(2)})$ and $\kappa_{O;j;a}^{(2,1)}$ in (4.147a) respectively replacing $r_a^{(2)}$ and $\kappa_{j;a}^{(2,1)}$ in (4.71), and with $\min(N_R^{(1)}, n_{\tilde{a}}^{(1)})$ and $\kappa_{O;\tilde{j};\tilde{a}}^{(1,2)}$ in (4.147b) respectively replacing $r_{\tilde{a}}^{(1)}$ and $\kappa_{\tilde{j};\tilde{a}}^{(1,2)}$ in (4.72). Moreover, by using (4.67d) we can write (4.132) for $k = 2$ as

$$G_{O;out}^{(2)} = \frac{\sum_{a=1}^{N_G^{(2)}} \min(N_R^{(2)}, n_a^{(2)}) [D_{p;a}^{(2)} + \sum_{j \in \mathcal{A}_{G;c}^{(1)}} D_{c;j;a}^{(2)}]}{\tilde{D}^{(2)}} \quad (4.148)$$

Also, we have

$$D_c \sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(1)}} (\kappa_{O;j;a}^{(2,1)}/n_c) = \sum_{(a,j) \in \tilde{\mathcal{A}}_G^{(1)}} \tilde{D}_c \kappa_{O;j;a}^{(2,1)} \stackrel{(a)}{=} \sum_{a=1}^{N_G^{(2)}} \sum_{j \in \tilde{\mathcal{A}}_{O;G;c}^{(1)}} D_{c;j;a}^{(2)} \kappa_{O;j;a}^{(2,1)}$$

where (a) is due to $D_{c;j;a}^{(2)} = \tilde{D}_c$ if $(a, j) \in \tilde{\mathcal{A}}_G^{(1)}$ and $D_{c;j;a}^{(2)} = 0$ otherwise. Using this result

in (4.147a) and using (4.148), $\Delta_{O;\infty}^{(2)}$ of (4.129) reduces to

$$\Delta_{O;\infty}^{(2)} = \frac{\sum_{a=1}^{N_G^{(2)}} \left[\sum_{j \in \mathcal{A}_{G;c}^{(1)}} D_{c;j;a}^{(2)} \min(N_R^{(2)}, n_a^{(2)}) - \sum_{j \in \tilde{\mathcal{A}}_{O;G;c}^{(1)}} D_{c;j;a}^{(2)} \kappa_{O;j;a}^{(2,1)} \right]}{\check{D}^{(2)}} \log \left(\frac{P_T}{N_0} \right) \quad (4.149)$$

which is similar to $\Delta_{\infty}^{(2)}$ of (4.85) with $\min(N_R^{(2)}, n_a^{(2)})$, $\tilde{\mathcal{A}}_{O;G;c}^{(1)}$ and $\kappa_{O;j;a}^{(2,1)}$ replacing $r_a^{(2)}$, $\tilde{\mathcal{A}}_{G;c}^{(1)}$ and $\kappa_{j;a}^{(2,1)}$, respectively. Therefore, (4.149) can be used to analyze the tightness of the bounds $\mathcal{R}_{O;out}$ and $\mathcal{R}_{O;in}$ by following similar steps as in Section 4.5.3 for analyzing the tightness of the bounds $\mathcal{R}_{out}$ and $\mathcal{R}_{in}$ for spread spectrum like MIMO BCs.

4.7 Numerical results for spread spectrum like systems

In this section we provide some numerical examples that illustrate the theoretical results obtained in the previous sections of this chapter. We consider a two-user MIMO BC (4.1) where the propagation channels of users 1 and 2 correspond to Channel A and Channel B with the parameters of Tables 3.1 and 3.2, respectively. Unless otherwise specified, we consider a system with $D_T = 6$ such that dimensions 1 and 4 are shared ($D_c = 2$), dimensions 2 and 5 are proper to user 1 ($D_p^{(1)} = 2$ and $D^{(1)} = 4$) and dimensions 3 and 6 are proper to user 2 ($D_p^{(2)} = 2$ and $D^{(2)} = 4$). From Section 2.3, the basis functions $\{\phi_d^{(1)}(t)\}_{d=1}^4$ of user 1 are associated with the signal dimensions 1, 2, 4, 5, respectively. Also, the basis functions $\{\phi_d^{(2)}(t)\}_{d=1}^4$ of user 2 are associated with the signal dimensions 1, 3, 4, 6, respectively. The associated matrix $\mathbf{B}^{(1,2)}$ that describes the signal dimensions sharing among users is therefore given by

$$\mathbf{B}^{(1,2)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (4.150)$$

with $\mathbf{B}^{(2,1)} = [\mathbf{B}^{(1,2)}]^T$. We consider a system with $N_T = 4$, $N_R^{(k)} = 1, 2$ using ATA ($N_G^{(k)} = 1$), POTA ($N_G^{(k)} = 2$, $n_1^{(k)} = 2$, $n_2^{(k)} = 2$) and OTA ($N_G^{(k)} = 4$), $k = 1, 2$. For each user, the POTA system can be obtained from OTA by merging TOGs. The assignments of signal dimensions to transmit antennas for ATA, POTA and OTA are provided in Table 4.5.

Our results confirm the following theoretical findings:

	User 1	User 2
ATA	Dimensions 1 (shared), 2 (proper), 4 (shared) and 5 (proper) are used on all antennas.	Dimensions 1 (shared), 3 (proper), 4 (shared) and 6 (proper) are used on all antennas.
POTA	TOG 1 consists of antennas 1 and 2, and employs dimensions 1 (shared) and 2 (proper).	TOG 1 consists of antennas 1 and 2, and employs dimensions 1 (shared) and 3 (proper).
	TOG 2 consists of antennas 3 and 4, and employs dimensions 4 (shared) and 5 (proper).	TOG 2 consists of antennas 3 and 4, and employs dimensions 4 (shared) and 6 (proper).
OTA	TOG 1 consists of antenna 1, and employs dimension 1 (shared).	TOG 1 consists of antenna 1, and employs dimension 1 (shared).
	TOG 2 consists of antenna 2, and employs dimension 2 (proper).	TOG 2 consists of antenna 2, and employs dimension 3 (proper).
	TOG 3 consists of antenna 3, and employs dimension 4 (shared).	TOG 3 consists of antenna 3, and employs dimension 4 (shared).
	TOG 4 consists of antenna 4, and employs dimension 5 (proper).	TOG 4 consists of antenna 4, and employs dimension 6 (proper).

Table 4.5 Assignments of signal dimensions to transmit antennas with $D_c = 2$

1) *Effect of space-time modulation:* With $P_T/N_0 = 10$ (or, equivalently, 10 dB), Figure 4.3 (with $\check{D}^{(k)} = D^{(k)}$ for $k = 1, 2$) and Figure 4.4 (with $\check{D}^{(k)} = D_T$ for $k = 1, 2$) illustrate the effect of space-time modulation on $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$ using two receive antennas for user 1 (i.e. $N_R^{(1)} = 2$) and one receive antenna for user 2 (i.e. $N_R^{(2)} = 1$). Assuming that the space-time modulation format of user 1 is ATA, Figure 4.3 shows that the inner bound $\mathcal{R}_{in}$ and outer bound $\mathcal{R}_{out}$ (and, hence, the DPC rate region $\mathcal{R}_{DPC}$ and the capacity region $\mathcal{C}$) are largest, intermediate and smallest when user 2 employs ATA, POTA and OTA, respectively. Similarly, assuming that the space-time modulation format of user 2 is POTA, Figure 4.4 shows that $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$ (and, hence, $\mathcal{R}_{DPC}$ and $\mathcal{C}$) are largest, intermediate and smallest when user 1 employs ATA, POTA and OTA, respectively. The behavior of $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$ can be explained as follows. For a given $\alpha_p^{(k)}$ and regardless of the other user in the system, Theorem 3.1.1 and Corollary 3.1.1 show that for user k the capacity $C^{(k)}(\alpha_p^{(k)})$ is maximum, intermediate and minimum when user k employs ATA, POTA and OTA, respectively, $k = 1, 2$. Even though this result has been proved in Chapter 3 with the transmission rate being normalized by the user number of signal dimensions (equivalently, $\check{D}^{(k)} = D^{(k)}$) it holds also when normalizing by the total number of signal dimensions in the BC since $[C^{(k)}(\alpha_p^{(k)})]_{\check{D}^{(k)}=D_T}$ can be written as $\frac{D^{(k)}}{D_T}[C^{(k)}(\alpha_p^{(k)})]_{\check{D}^{(k)}=D^{(k)}}$

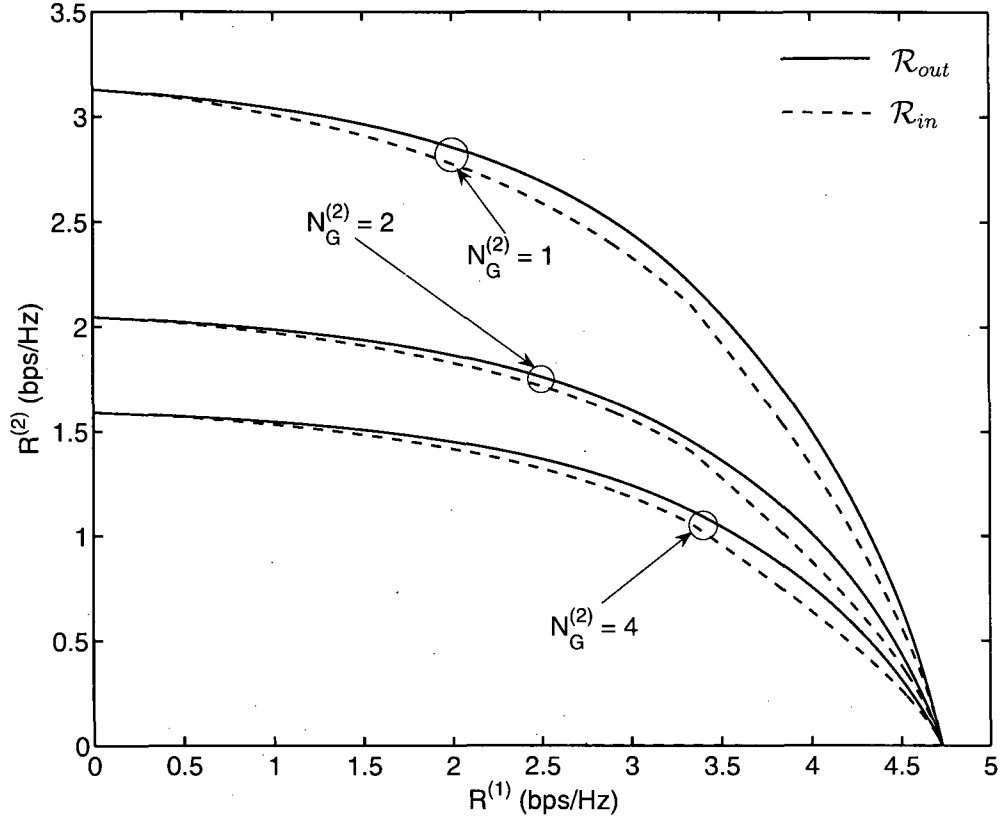


Fig. 4.3 The bounds $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$ with $\check{D}^{(k)} = D^{(k)}$ for $k = 1, 2$, $D_c = 2$ and $P_T/N_0 = 10$. User 1 employs ATA with $N_R^{(1)} = 2$. User 2 employs ATA ($N_G^{(2)} = 1$), POTA ($N_G^{(2)} = 2$) and OTA ($N_G^{(2)} = 4$) with $N_R^{(2)} = 1$.

and hence, it has the same behavior as a function of $N_G^{(k)}$ as $[C^{(k)}(\alpha_p^{(k)})]_{\check{D}^{(k)}=D^{(k)}}$. It follows that for every $(\alpha_p^{(1)}, \alpha_p^{(2)})$, the k^{th} coordinate in the pair $(C^{(1)}(\alpha_p^{(1)}), C^{(2)}(\alpha_p^{(2)}))$ increases when $N_G^{(k)}$ decreases, $k = 1, 2$. Hence, by varying $\alpha_p^{(1)}, \alpha_p^{(2)}$ in the interval $[0, 1]$ with $\alpha_p^{(2)} = 1 - \alpha_p^{(1)}$, one can see why $\mathcal{R}_{out}$ expands in Figures 4.3 and 4.4. We now consider $\mathcal{R}_{in}$ with the scenario of Figure 4.3. For this, we investigate the behavior of user 2's rate in the pairs $(C^{(1)}(\alpha_p^{(1)}), \check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}))$ and $(\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}), C^{(2)}(\alpha_p^{(2)}))$ when user 1's rate is fixed. For a given $(\alpha_p^{(1)}, \alpha_p^{(2)})$ and for the same capacity $C^{(1)}(\alpha_p^{(1)})$, Theorem 4.3.1 shows that the rate $\check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ is maximum, intermediate and minimum when user 2 employs ATA, POTA and OTA, respectively. Furthermore, for a given $\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)})$, the rate pair $(\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}), C^{(2)}(\alpha_p^{(2)}))$ is such that $C^{(2)}(\alpha_p^{(2)})$ increases when $N_G^{(2)}$ decreases by Theorem 3.1.1 and Corollary 3.1.1. Hence, by varying $\alpha_p^{(1)}, \alpha_p^{(2)}$ in the interval $[0, 1]$ with

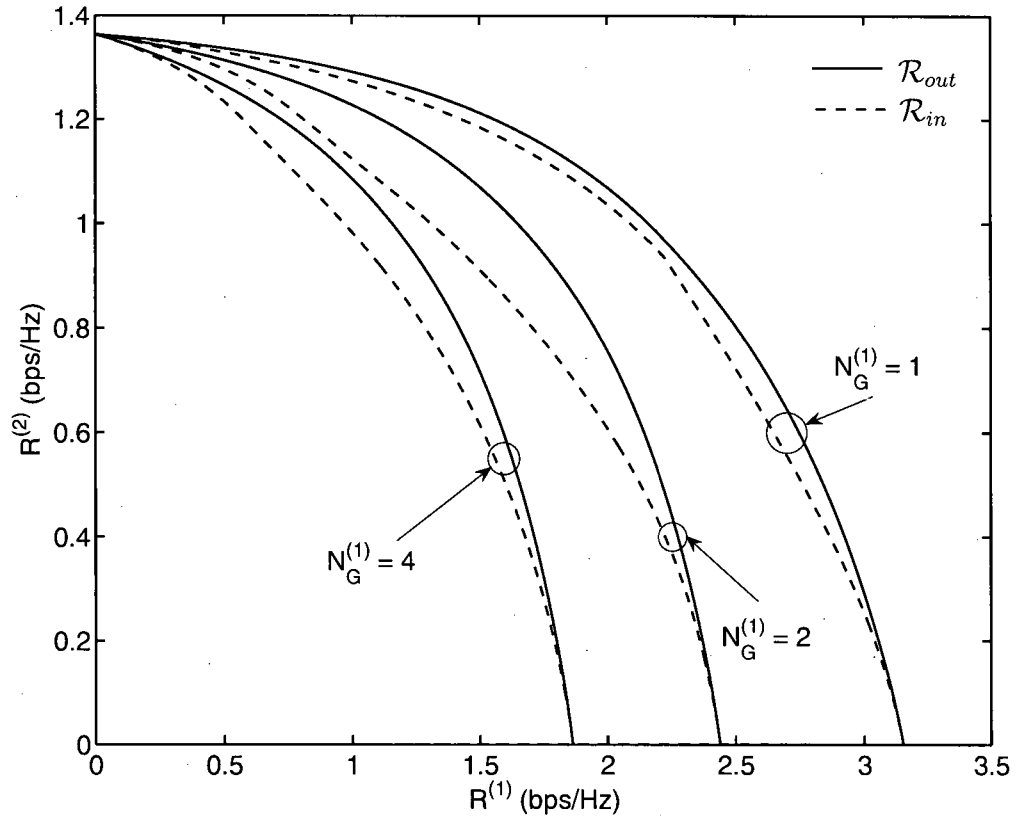


Fig. 4.4 The bounds $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$ with $\check{D}^{(k)} = D_T$ for $k = 1, 2$, $D_c = 2$ and $P_T/N_0 = 10$. User 1 employs ATA ($N_G^{(1)} = 1$), POTA ($N_G^{(1)} = 2$) and OTA ($N_G^{(1)} = 4$) with $N_R^{(1)} = 2$. User 2 employs POTA with $N_R^{(2)} = 1$.

$\alpha_p^{(2)} = 1 - \alpha_p^{(1)}$, we conclude that for a fixed $N_G^{(1)}$ the inner bound $\mathcal{R}_{in}$ expands when $N_G^{(2)}$ decreases. Following a similar approach and switching the user indexes 1 and 2, we can explain why $\mathcal{R}_{in}$ expands in Figure 4.4 when $N_G^{(1)}$ decreases with fixed $N_G^{(2)}$.

2) *Effect of the number of receive antennas:* With $P_T/N_0 = 10$ (or, equivalently, 10 dB), Figure 4.5 illustrates the effect of increasing the number of receive antennas, assuming that both users employ POTA and using $\check{D}^{(k)} = D^{(k)}$ for $k = 1, 2$. One can see that both $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$ expand when $N_R^{(1)}$ or $N_R^{(2)}$ increases. This can be explained as follows. If $N_R^{(2)}$ increases, the rank $r^{(2)} \equiv r_T^{(2)} D_G^{(2)}$ of $[\mathbf{H}^{(2)}]^\dagger \cdot \mathbf{H}^{(2)}$ can only increase since $r_T^{(2)} = \sum_{j=1}^{N_G^{(2)}} r_j^{(2)}$ with $r_j^{(2)} = \min(n_j^{(2)}, L_t^{(2)} N_R^{(2)})$ almost surely for 3GPP channels. It follows that the number of active eigenmodes in the water-filling scheme associated with $C^{(2)}(\alpha_p^{(2)})$ can only in-

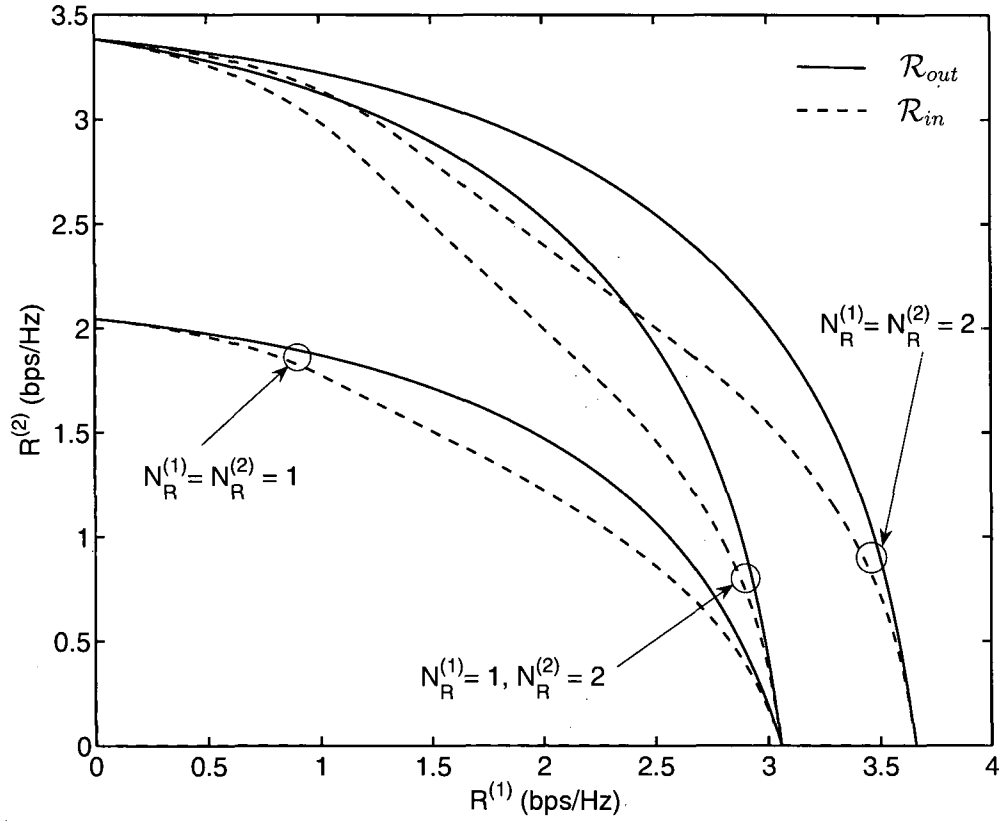
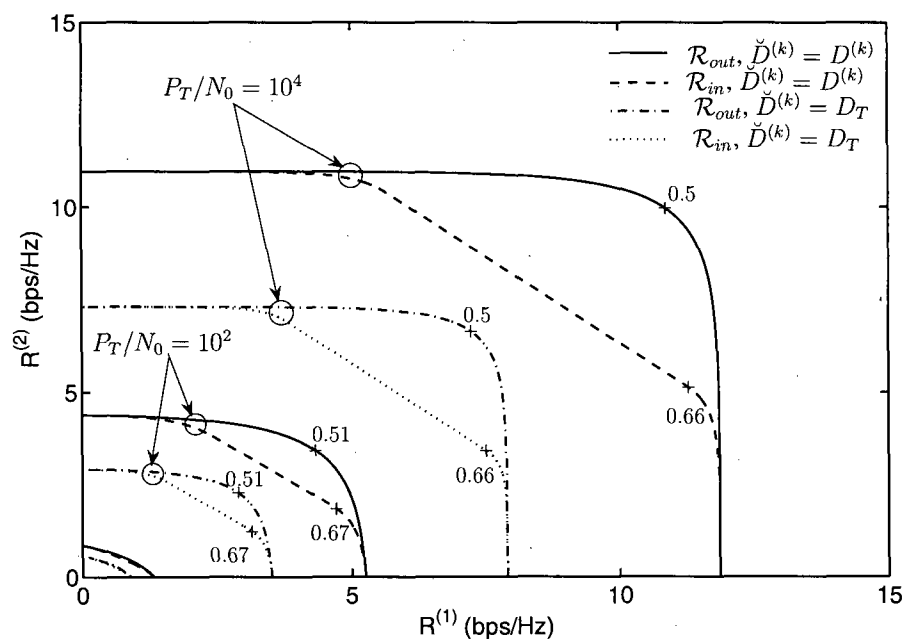


Fig. 4.5 The bounds $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$ with $\check{D}^{(k)} = D^{(k)}$ for $k = 1, 2$, $D_c = 2$ and $P_T/N_0 = 10$. User 1 and user 2 employ POTA ($N_G^{(1)} = N_G^{(2)} = 2$) and have different number of receive antennas.

crease in which case $C^{(2)}(\alpha_p^{(2)})$ increases for every $\alpha_p^{(2)}$. Furthermore, since $\check{r}^{(2)} = r_T^{(2)} D_G^{(2)}$ from (4.22), the number of active eigenmodes in the water-filling scheme associated with $\check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ can also increase if $N_R^{(2)}$ increases in which case the rate $\check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ increases for every $(\alpha_p^{(1)}, \alpha_p^{(2)})$. Similarly, the single-user capacity $C^{(1)}(\alpha_p^{(1)})$ and the rate $\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ can only increase when $N_R^{(1)}$ increases. We conclude that an increase in the number of receive antennas for one or both users makes $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$ expand.

3) *Effect of the number of shared signal dimensions:* Figures 4.6-4.11 illustrate the behavior of $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$ when D_c increases from 2 to 6 while keeping $D_p^{(k)} = 2$ for $k = 1, 2$. The rate regions presented in each of these figures are associated with $\check{D}^{(k)} = D^{(k)}$ and $\check{D}^{(k)} = D_T$, $k = 1, 2$. Obviously, a rate region associated with $\check{D}^{(k)} = D_T$ cannot be larger than that associated with $\check{D}^{(k)} = D^{(k)}$ since $D^{(k)} \leq D_T$ for each k , which makes user k 's

(a)



(b)

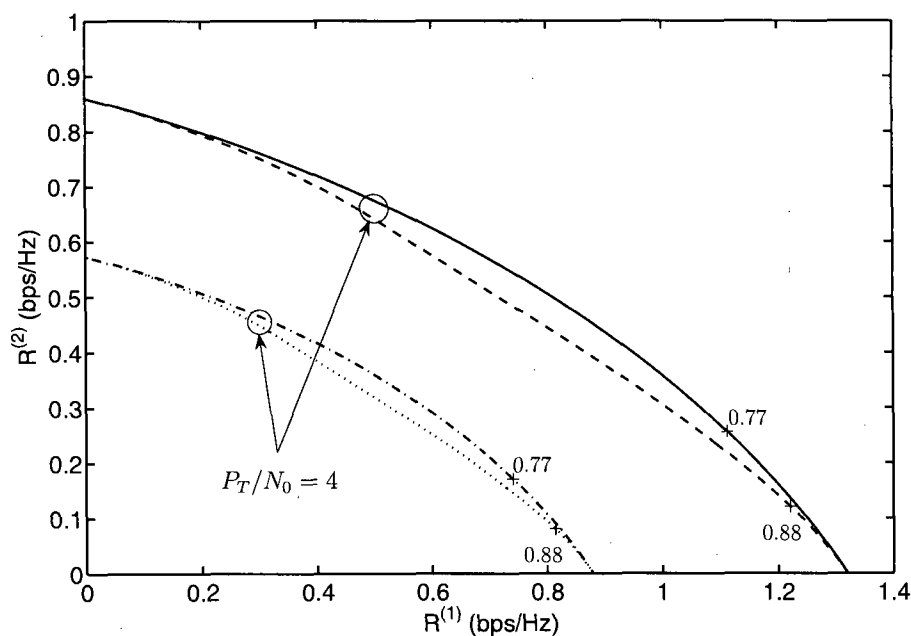


Fig. 4.6 a) The bounds $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$ with $D_c = 2$. User 1 and user 2 employ OTA ($N_G^{(1)} = N_G^{(2)} = 4$) with $N_R^{(1)} = N_R^{(2)} = 1$. The maximum sum rate points are marked by '+' with the associated $\alpha_{p,s}^{(1)}$. b) Zoom for $P_T/N_0 = 4$.

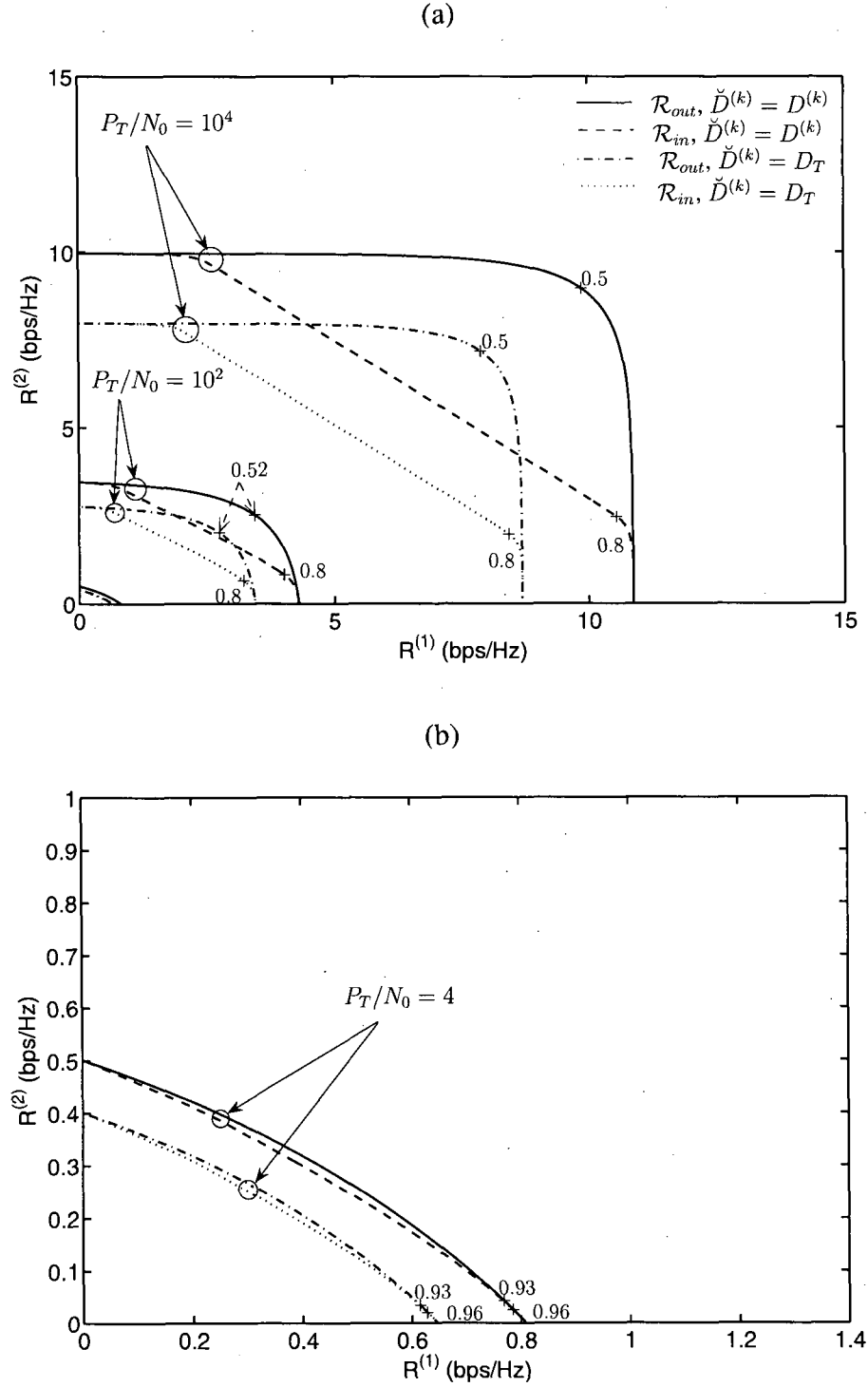


Fig. 4.7 a) The bounds $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$ with $D_c = 6$. User 1 and user 2 employ OTA ($N_G^{(1)} = N_G^{(2)} = 4$) with $N_R^{(1)} = N_R^{(2)} = 1$. The maximum sum rate points are marked by '+' with the associated $\alpha_{p;s}^{(1)}$. b) Zoom for $P_T/N_0 = 4$.

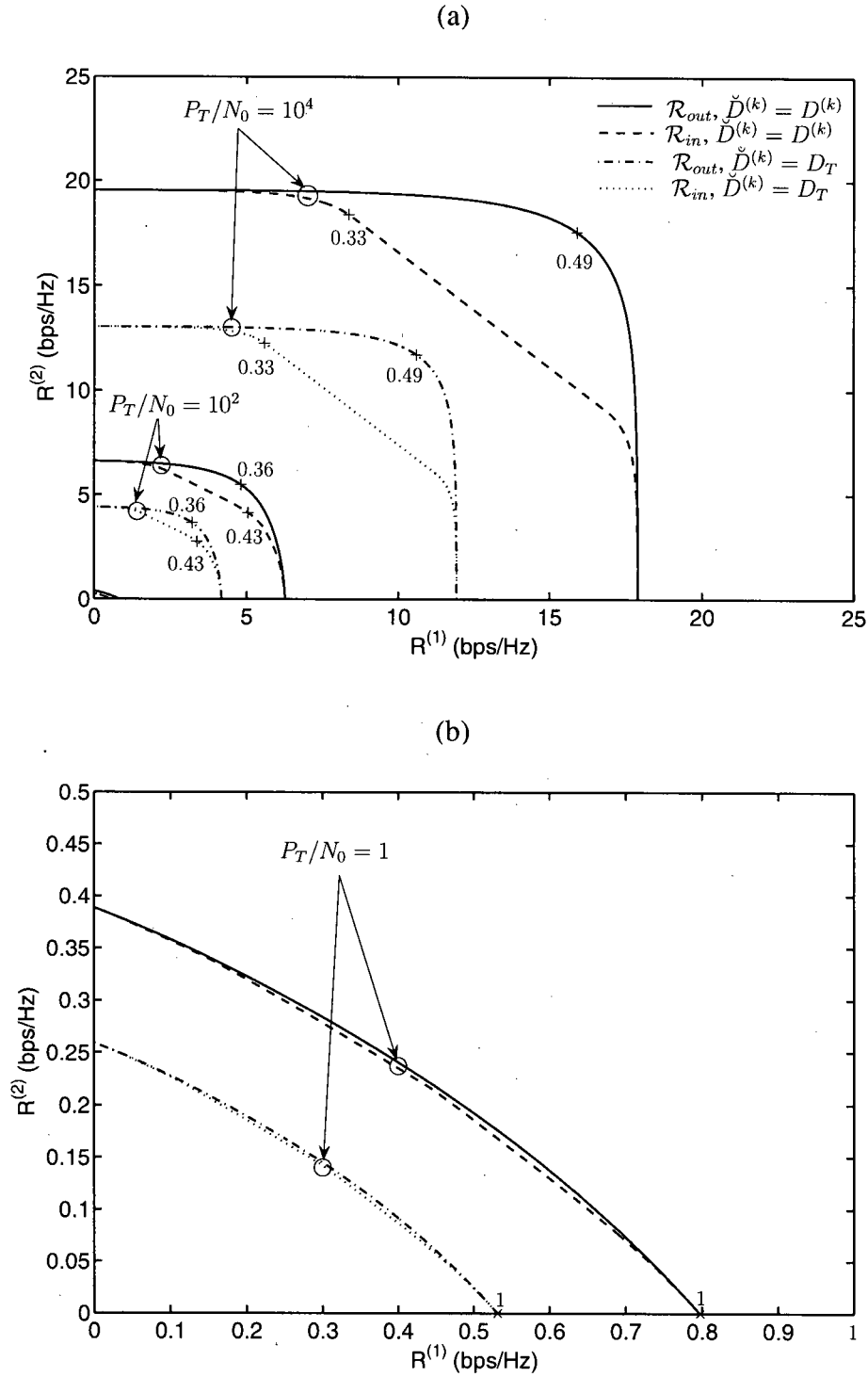


Fig. 4.8 a) The bounds $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$ with $D_c = 2$. User 1 and user 2 employ POTA ($N_G^{(1)} = N_G^{(2)} = 2$) with $N_R^{(1)} = N_R^{(2)} = 1$. The maximum sum rate points are marked by '+' or 'x' with the associated $\alpha_{p,s}^{(1)}$. b) Zoom for $P_T/N_0 = 1$.

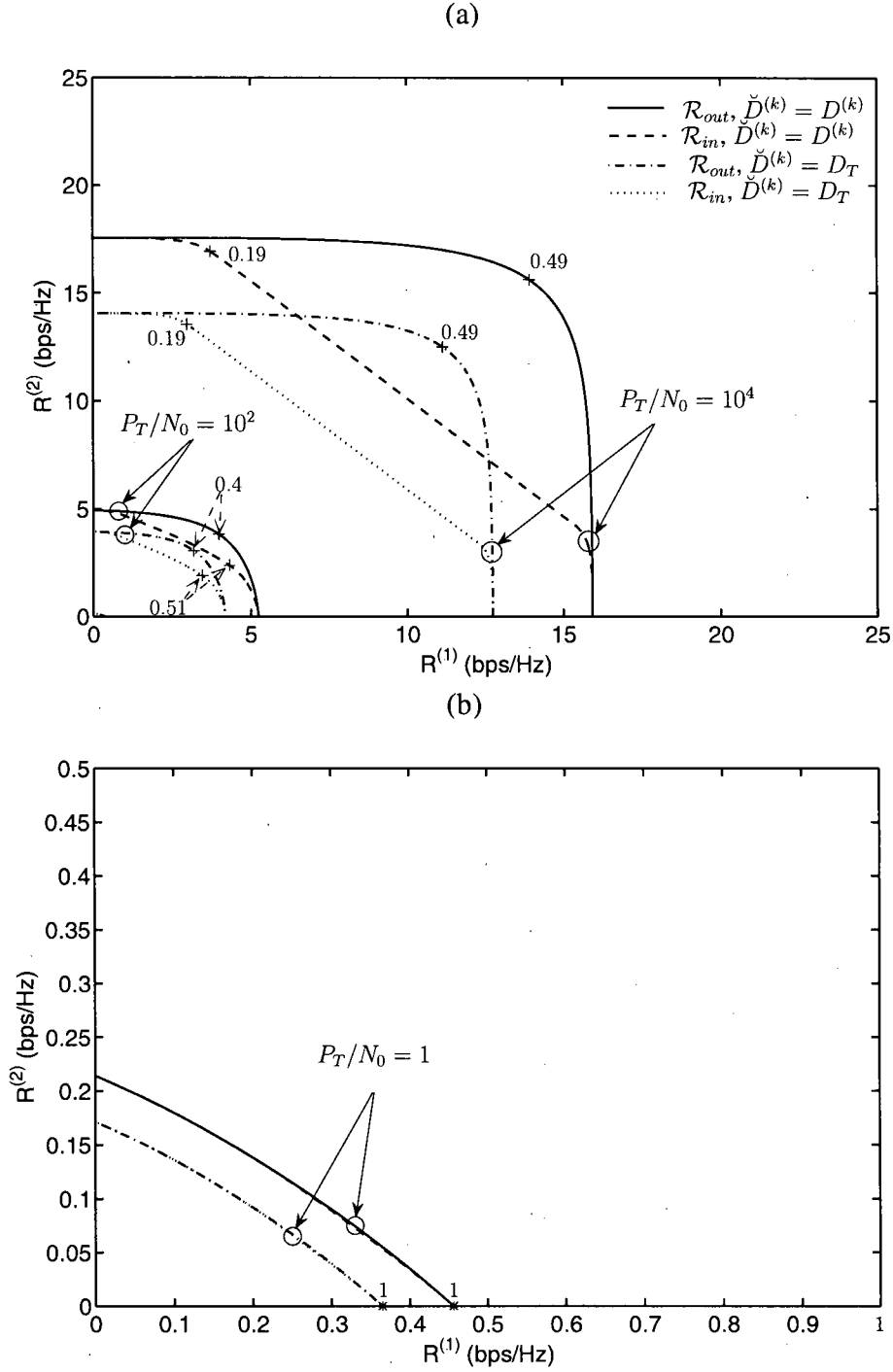


Fig. 4.9 a) The bounds $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$ with $D_c = 6$. User 1 and user 2 employ POTAs ($N_G^{(1)} = N_G^{(2)} = 2$) with $N_R^{(1)} = N_R^{(2)} = 1$. The maximum sum rate points are marked by '+' or 'x' with the associated $\alpha_{p;s}^{(1)}$. b) Zoom for $P_T/N_0 = 1$.

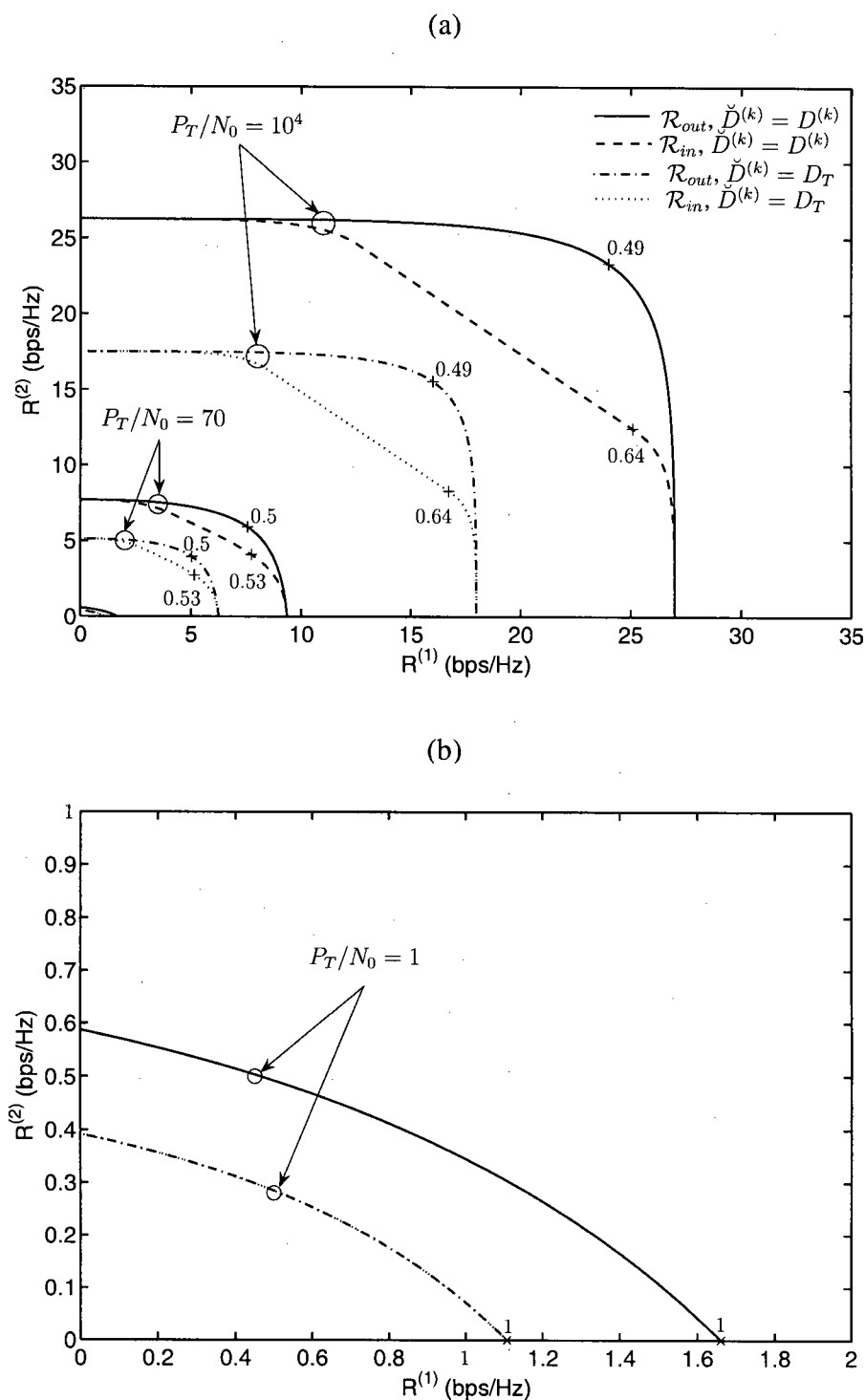


Fig. 4.10 a) The bounds $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$ with $D_c = 2$. User 1 and user 2 employ ATA ($N_G^{(1)} = N_G^{(2)} = 1$) with $N_R^{(1)} = 2$ and $N_R^{(2)} = 1$. The maximum sum rate points are marked by '+' or 'x' with the associated $\alpha_{p;s}^{(1)}$. b) Zoom for $P_T/N_0 = 1$.

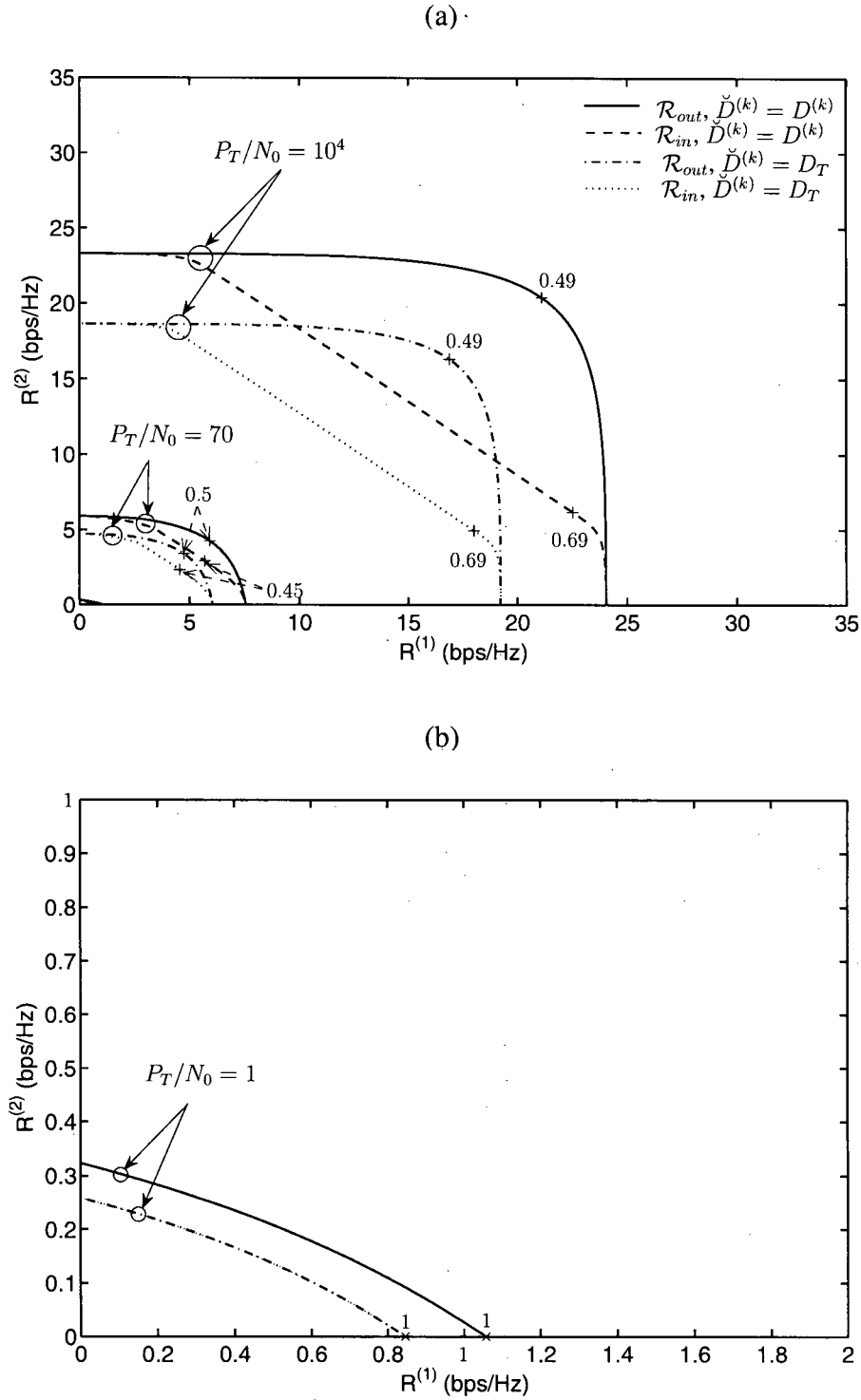


Fig. 4.11 a) The bounds $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$ with $D_c = 6$. User 1 and user 2 employ ATA ($N_G^{(1)} = N_G^{(2)} = 1$) with $N_R^{(1)} = 2$ and $N_R^{(2)} = 1$. The maximum sum rate points are marked by '+' or 'x' with the associated $\alpha_{p;s}^{(1)}$. b) Zoom for $P_T/N_0 = 1$.

rate associated with $\check{D}^{(k)} = D_T$ less or equal than that associated with $\check{D}^{(k)} = D^{(k)}$. When $D_c = 6$, we keep dimensions 2 and 5 proper to user 1 and dimensions 3 and 6 proper to user 2 and we add four new shared signal dimensions numbered as 7, 8, 9, 10. Hence, dimensions 1, 4, 7, 8, 9, 10 denote the shared signal dimensions. We have $D_T = 10$ and $D^{(k)} = 8$ for $k = 1, 2$. Following Section 2.3, the basis functions $\{\phi_d^{(1)}(t)\}_{d=1}^8$ of user 1 are associated with the signal dimensions 1, 2, 4, 5, 7, 8, 9, 10, respectively. Also, the basis functions $\{\phi_d^{(2)}(t)\}_{d=1}^8$ of user 2 are associated with the signal dimensions 1, 3, 4, 6, 7, 8, 9, 10, respectively. The matrix $\mathbf{B}^{(1,2)}$ becomes

$$\mathbf{B}^{(1,2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (4.151)$$

with $\mathbf{B}^{(2,1)} = [\mathbf{B}^{(1,2)}]^T$. For the sake of clarity, we provide the assignments of the signal dimensions to transmit antennas in Table 4.6 below for different space-time modulation formats.

In Figures 4.6 and 4.7, we consider a MIMO BC where the two users employ OTA with $N_R^{(1)} = N_R^{(2)} = 1$ and $P_T/N_0 = 4, 10^2, 10^4$ (or, equivalently, $P_T/N_0 = 6.02, 20, 40$ dB). In Figures 4.8 and 4.9, we consider a MIMO BC where the two users employ POTA with $N_R^{(1)} = N_R^{(2)} = 1$ and $P_T/N_0 = 1, 10^2, 10^4$ (or, equivalently, $P_T/N_0 = 0, 20, 40$ dB). In Figures 4.10 and 4.11, we consider a MIMO BC where the two users employ ATA with $N_R^{(1)} = 2, N_R^{(2)} = 1$ and $P_T/N_0 = 1, 70, 10^4$ (or, equivalently, $P_T/N_0 = 0, 18.45, 40$ dB). Figures 4.6-4.11 show that for any space-time modulation format and except for the curves associated with $P_T/N_0 = 10^4$ and $\check{D}^{(k)} = D_T$, both $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$ become smaller when D_c increases. If $P_T/N_0 = 10^4$ and $\check{D}^{(k)} = D_T$, then Figures 4.6a-4.11a show that as D_c increases the outer bound $\mathcal{R}_{out}$ increases while $\mathcal{R}_{in}$ can be increasing or decreasing depending on the values of α_p . Furthermore, one can see from Figures 4.6a-4.11a that, for $P_T/N_0 = 10^4$ and for a given space-time modulation format, the separation between the region $\mathcal{R}_{out}$ associated with $\check{D}^{(k)} = D^{(k)}$ and the region $\mathcal{R}_{out}$ associated with $\check{D}^{(k)} = D_T$

	User 1	User 2
ATA	Dimensions 1 (shared), 2 (proper), 4 (shared), 5 (proper) and 7, 8, 9, 10 (shared) are used on all antennas.	Dimensions 1 (shared), 3 (proper), 4 (shared), 6 (proper) and 7, 8, 9, 10 (shared) are used on all antennas.
POTA	TOG 1 consists of antennas 1 and 2, and employs dimensions 1 (shared), 2 (proper), 4 (shared) and 5 (proper).	TOG 1 consists of antennas 1 and 2, and employs dimensions 1 (shared), 3 (proper), 4 (shared) and 6 (proper).
	TOG 2 consists of antennas 3 and 4, and employs dimensions 7, 8, 9, 10 (shared).	TOG 2 consists of antennas 3 and 4, and employs dimensions 7, 8, 9, 10 (shared).
OTA	TOG 1 consists of antenna 1, and employs dimensions 1 (shared) and 2 (proper).	TOG 1 consists of antenna 1 and, employs dimension 1 (shared) and 3 (proper).
	TOG 2 consists of antenna 2, and employs dimensions 4 (shared) and 5 (proper).	TOG 2 consists of antenna 2, and employs dimensions 4 (shared) and 6 (proper).
	TOG 3 consists of antenna 3, and employs dimensions 7, 8 (shared).	TOG 3 consists of antenna 3, and employs dimensions 7, 8 (shared).
	TOG 4 consists of antenna 4, and employs dimensions 9, 10 (shared).	TOG 4 consists of antenna 4, and employs dimensions 9, 10 (shared).

Table 4.6 Assignments of signal dimensions to transmit antennas with $D_c = 6$

decreases when D_c increases from 2 to 6. Whether $\check{D}^{(k)} = D_T$ or $\check{D}^{(k)} = D^{(k)}$, one can also see that, for $P_T/N_0 = 10^4$, the separation between the outer bound $\mathcal{R}_{out}$ and inner bound $\mathcal{R}_{in}$ increases when D_c increases from 2 to 6.

4) *Extreme SNR cases*: Figures 4.6-4.11 also illustrate the low and high SNR behavior of the bounds $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$.

Low SNR: For a given normalization constant (i.e $\check{D}^{(k)} = D^{(k)}$ or $\check{D}^{(k)} = D_T$) and for small P_T/N_0 , Figures 4.6b - 4.11b illustrate the near triangular shape of $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$. Furthermore, the separation between $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$ is small in Figures 4.6b-4.7b, very small in Figures 4.8b-4.9b, and zero in Figures 4.10b-4.11b. On the other hand, we have $(\lambda_{\max}^{(1)}, \lambda_{\max}^{(2)}) = (1.684, 0.9627)$ when both users employ OTA, $(\lambda_{\max}^{(1)}, \lambda_{\max}^{(2)}) = (3.2571, 1.3832)$ when they employ POTA and $(\lambda_{\max}^{(1)}, \lambda_{\max}^{(2)}) = (8.6493, 2.0105)$ when they employ ATA. In all three cases, $\lambda_{\max}^{(1)} > \lambda_{\max}^{(2)}$. If $\check{D}^{(k)} = D_T$ the maximum sum rate for $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$, which is also the sum capacity C^{sum} , is achieved by allocating the total transmit power to user 1, i.e $(\alpha_{p;s}^{(1)}, \alpha_{p;s}^{(2)}) = (1, 0)$, regardless of D_c . This also holds

when the rate normalization constant is $\check{D}^{(k)} = D^{(k)}$ since $D_p^{(1)} = D_p^{(2)}$, thus, yielding $\frac{\lambda_{\max}^{(2)} \check{D}^{(1)}}{\lambda_{\max}^{(1)} \check{D}^{(2)}} = \frac{\lambda_{\max}^{(2)}}{\lambda_{\max}^{(1)}}$ and making $(\alpha_{p;s}^{(1)}, \alpha_{p;s}^{(2)})$ independent of D_c as explained in Section 4.5.1. Recall from Section 4.5.1 that since the capacity region $\mathcal{C}$ becomes triangular at low SNR, C^{sum} corresponds to the maximum of the single-user capacities $C^{(1)}(1)$ and $C^{(2)}(1)$. Thus, having $(\alpha_{p;s}^{(1)}, \alpha_{p;s}^{(2)}) = (1, 0)$ can be intuitively explained by the fact that user 1 has a better channel than user 2 (since $\lambda_{\max}^{(1)} > \lambda_{\max}^{(2)}$) and hence, for the same transmit power it can achieve a larger capacity than user 2.

High SNR: For a given normalization constant (i.e $\check{D}^{(k)} = D^{(k)}$ or $\check{D}^{(k)} = D_T$) and for large P_T/N_0 , Figures 4.6a - 4.11a illustrate the pentagonal shape of $\mathcal{R}_{in}$ and the rectangular shape of $\mathcal{R}_{out}$. Furthermore, the separation between $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$ is large for any space-time modulation format and any rate normalization constant $\check{D}^{(k)}$. Now using $r_T^{(k)} = \sum_{j=1}^{N_G^{(k)}} r_j^{(k)}$ with $r_j^{(k)} = \min(n_j^{(k)}, L_t^{(k)} N_R^{(k)})$, the power allocation (4.45) associated with the maximum sum rate C_{out}^{sum} for $\mathcal{R}_{out}$ reduces to $\alpha_{p;s}^{(1)} = 0.5$ when the users employ OTA and POTA and to $\alpha_{p;s}^{(1)} = 0.5714$ when the users employ ATA. The theoretical value for $\alpha_{p;s}^{(1)}$ is reached with OTA in Figures 4.6a and 4.7a for $P_T/N_0 = 10^4$, but not with POTA and ATA in Figures 4.8a-4.11a for which $P_T/N_0 = 10^4$ is not large enough. Consider now the maximum sum rate points on the boundary of $\mathcal{R}_{in}$. When both users employ OTA, one can see from Figures 4.6a and 4.7a that the sum rate for $\mathcal{R}_{in}$ with $\check{D}^{(k)} = D^{(k)}$ and $\check{D}^{(k)} = D_T$ is maximized (for $P_T/N_0 = 10^4$) at boundary points of the form $(C^{(1)}(\tilde{\alpha}_{p;s}^{(1)}), \check{R}^{(2)}(\tilde{\alpha}_{p;s}^{(1)}, \tilde{\alpha}_{p;s}^{(2)}))$. With OTA, the theoretical power allocation $\tilde{\alpha}_{p;s}^{(1)}$, calculated using (4.58), is given for both values of $\check{D}^{(k)}$ by $\tilde{\alpha}_{p;s}^{(1)} = 0.66$ when $D_c = 2$ and by $\tilde{\alpha}_{p;s}^{(1)} = 0.8$ when $D_c = 6$. The theoretical values for $\tilde{\alpha}_{p;s}^{(1)}$ match those provided in Figures 4.6a (for $D_c = 2$) and 4.7a (for $D_c = 6$) with $P_T/N_0 = 10^4$. When both users employ POTA, one can see from Figures 4.8a and 4.9a that the sum rate for $\mathcal{R}_{in}$ with $\check{D}^{(k)} = D^{(k)}$ and $\check{D}^{(k)} = D_T$ is maximized (for $P_T/N_0 = 10^4$) at boundary points of the form $(\check{R}^{(1)}(\tilde{\alpha}_{p;s}^{(1)}, \tilde{\alpha}_{p;s}^{(2)}), C^{(2)}(\tilde{\alpha}_{p;s}^{(2)}))$. With POTA, the theoretical power allocation $\tilde{\alpha}_{p;s}^{(1)}$, calculated using (4.59), is given for both values of $\check{D}^{(k)}$ by $\tilde{\alpha}_{p;s}^{(1)} = 0.33$ when $D_c = 2$ and by $\tilde{\alpha}_{p;s}^{(1)} = 0.2$ when $D_c = 6$. With $P_T/N_0 = 10^4$, the theoretical values for $\tilde{\alpha}_{p;s}^{(1)}$ match those provided in Figure 4.8a (for $D_c = 2$), but do not correspond exactly to those provided in Figure 4.9a (for $D_c = 6$). When both users employ ATA, one can see from Figures 4.10a and 4.11a that the sum rate for $\mathcal{R}_{in}$ with $\check{D}^{(k)} = D^{(k)}$ and $\check{D}^{(k)} = D_T$ is maximized (for $P_T/N_0 = 10^4$) at boundary points of the form $(C^{(1)}(\tilde{\alpha}_{p;s}^{(1)}), \check{R}^{(2)}(\tilde{\alpha}_{p;s}^{(1)}, \tilde{\alpha}_{p;s}^{(2)}))$. With ATA, the theoretical power allocation $\tilde{\alpha}_{p;s}^{(1)}$, calculated using (4.58), is given by $\tilde{\alpha}_{p;s}^{(1)} = 0.72$ when $D_c = 2$ and by $\tilde{\alpha}_{p;s}^{(1)} = 0.84$ when $D_c = 6$. The theoretical values for $\tilde{\alpha}_{p;s}^{(1)}$ do not match those provided in Figures 4.10a (for $D_c = 2$)

and 4.11a (for $D_c = 6$) for which $P_T/N_0 = 10^4$ is not large enough.

Finally, Figures 4.6-4.11 show that as P_T/N_0 increases the separation between the bounds $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$ also increases. Therefore, the lower P_T/N_0 is the tighter these bounds are, and the more accurate $\mathcal{R}_{DPC}$ can be approximated.

Chapter 5

Conclusions and future work

5.1 Summary and conclusions

This thesis considered Multiple-Input Multiple-Output (MIMO) systems that employ various multi-dimensional space-time modulation formats. The latter refer to various allocations of orthogonal signal dimensions to transmit antennas. For single-user systems, our objective was to study the information rates that can be achieved with such space-time modulation schemes. Assuming perfect receiver channel knowledge, we considered different power allocation criteria depending on the channel state information at the transmitter (CSIT). An analysis of the different transmission rates is provided at low and high SNR. We also examined the possibility for a transmitter without CSIT to achieve a capacity similar to that with CSIT by using appropriate space-time modulation format and transmit power allocation. For broadcast systems, our model allowed signal dimensions sharing among users. Here, our objective was to investigate the impact of space-time modulation and signal dimensions sharing on the achievable rates. Assuming perfect channel knowledge at the transmitter and all receivers, we started by showing that an achievable rate region can be obtained by using dirty paper coding (DPC). By deriving inner and outer bounds for the DPC rate region we obtained important insights into the effect of space-time modulation. A detailed analysis of these bounds is presented in the low and high SNR regimes. Furthermore, we investigated the impact of varying the number of shared signal dimensions on the spatial multiplexing gains associated with these bounds. The main contributions of this work are provided by Chapters 3 and 4.

We considered three space-time modulation formats: all dimensions are used by all antennas (ATA), each subset of signal dimensions is used by a single group of antennas (POTA) or by a single antenna (OTA). A Transmit Orthogonal Group (TOG) denotes a set of transmit antennas that share a given subset of the signal dimensions. Throughout, we used the 3GPP spatial channel model. For MIMO systems using spread spectrum like signaling, we have the following conclusions from Chapter 3. With CSIT, we considered the Shannon capacity, achieved using water-filling power allocation and an eigen-beamforming signaling structure. We showed that this capacity is maximum, intermediate and minimum with ATA, POTA and OTA, respectively, provided that the POTA system can be obtained from OTA by merging TOGs. Capacity calculations in terms of the spatial channel parameters have been provided for systems with small number of antennas and time resolvable clusters. Without CSIT, we used an a-prior probability distribution on the components of the channel propagation matrix to represent the lack of channel knowledge. Considering a particular class of input covariance matrices, we proved the optimality of the uniform power allocation in the sense that it makes the fraction of channels for which the mutual information is less than any given rate converge fastest to zero, as the a-prior distribution becomes non-informative. Using the uniform power allocation, ATA provided the best rates at low and high SNR. With partial CSIT, the information rate is maximized at low SNR through transmit antenna selection, while at high SNR it is maximized when all antennas are active with the fraction of power allocated to each TOG being determined by the ranks of the TOG propagation matrices. In both regimes, ATA provided the largest rate followed by POTA then by OTA. At high SNR, we showed that transmitters without CSI can achieve rates equal to capacity if and only if they employ uniform power allocation and all their TOGs have full column rank propagation matrices. The previous conclusions hold for spread spectrum like MIMO systems. As shown in Chapter 3, MIMO-OFDM systems present many similarities to spread spectrum like MIMO systems with one time resolvable cluster. We showed that Shannon capacity for MIMO-OFDM systems is also maximum, intermediate and minimum with ATA, POTA and OTA, respectively. With partial CSIT, the information rate is also maximized for MIMO-OFDM systems by transmit antenna selection at low SNR and by making all the antennas active at high SNR. However, the strongest antenna selection criterion at low SNR and the optimal power allocation at high SNR are different from those of spread spectrum like MIMO systems. Also, the spatial multiplexing gain associated with the Shannon capacity is larger for spread spectrum like MIMO systems than for MIMO-OFDM systems.

For MIMO broadcast channels (BCs) with space-time modulation, the signal dimensions are divided into orthogonal subsets where only one of these subsets is shared between all users, and the other subsets are assigned each to a different user. We have the following conclusions from Chapter 4 where we considered spread spectrum like MIMO BCs and MIMO-OFDM BCs with two users for simplicity. Assuming perfect channel knowledge at the transmitter and all receivers, we used DPC to obtain an achievable rate region. In order to obtain rates in bps/Hz units, we considered two possible normalizations. An outer bound for the capacity and DPC rate regions was obtained by assuming interference cancellation for all users. This outer bound exhibits similar properties to the capacity region of orthogonally multiplexed MIMO BCs and, thus, it is largest, intermediate and smallest when the users employ ATA, POTA and OTA, respectively. We also derived an inner bound for the capacity and DPC rate regions by considering the subset of DPC rate pairs where the user that is encoded first transmits at maximal rate, assuming that the user that is encoded second is operating at capacity rate. For this inner bound, we have shown that if the space-time modulation format of the latter user is fixed, the rate of the former user is maximum, intermediate and minimum when this user employs ATA, POTA and OTA, respectively. At low SNR, the inner and outer bounds converge. The limiting region in this case is the asymptotical capacity region at low SNR, and it is equivalent to the rate region obtained by time-sharing. Also, in this regime the sum capacity is achieved by allocating power to the user that provides the largest capacity when assigned the total transmit power. At high SNR, we calculated the power allocations that correspond to maximum sum rate for the inner and outer bounds, and we evaluated the spatial multiplexing gains associated with these bounds. The analysis of these multiplexing gains as a functions of the number of shared signal dimensions leads to the following results. Considering the outer bound, the multiplexing gain of a user is independent of the number of shared signal dimensions if its rate is normalized by its number of signal dimensions. However, if the user rate is normalized by the total number of signal dimensions, this multiplexing gain increases when more shared signal dimensions are added with fixed (nonzero) number of proper signal dimensions. Considering the inner bound, we investigated the impact of signal dimensions sharing on the multiplexing gains when the users have the same number of TOGs and when each TOG of a user shares an equal number of signal dimensions with only one TOG of the other user. For the points on the inner bound's boundary, the multiplexing gain of the user that is encoded last is similar to that associated with the outer bound's boundary. Analyzing the multiplexing gain of the user that is encoded first leads to the following conclusions. If

the number of proper signal dimensions for each user is fixed then, depending on the users spatial channel and regardless of the rate normalization method, this multiplexing gain can increase or decrease when more shared signal dimensions are added. In case the total number of signal dimensions is fixed and the user's rate is normalized by the number of its signal dimensions, this multiplexing gain decreases when the number of shared signal dimensions increases at the expense of the number of proper signal dimensions of any user in the system. However, if the user's rate is normalized by the total number of signal dimensions, then its multiplexing gain decreases (respectively, increases) if the number of shared signal dimensions increases at the expense of the number of proper signal dimensions of this user (respectively, of the other user). When taken as functions of the number of time resolvable clusters, the spatial multiplexing gains present different behaviors for spread spectrum like MIMO BCs and for MIMO-OFDM BCs. For spread spectrum like MIMO BCs and for a given user, we have shown that the multiplexing gains associated with both the inner and outer bounds increase in general when the number of time resolvable clusters of this user increases. For MIMO-OFDM BCs, the multiplexing gains associated with both the inner and outer bounds are independent of the number of time resolvable clusters in the users propagation medium.

Our results can be applied to many wireless communication systems in the literature which can be accommodated as special cases of the more general framework of multi-dimensional space-time modulation. Furthermore, since we considered the 3GPP spatial channel model our results can be also applied to channel models that can be seen as particular cases of this model such as space or time dispersive channels. In a single-user environment and for a given CSIT case and SNR regime, our analysis from Chapter 3 can be used to design better MIMO systems by determining the most efficient power allocation and space-time modulation format so that maximal information rates can be obtained. For MIMO BCs, the analysis of the DPC rate region's bounds in Chapter 4 can be used to design MIMO broadcast systems with larger achievable rate regions using appropriate coding techniques and signal dimensions allocations. For instance, at low SNR our results show that we can use the simpler time-sharing technique instead of DPC for achieving the BC capacity region. At high SNR, our results can be used to determine the optimal signal dimensions sharing schemes so that the multiplexing gains associated with the inner bound can be maximized, thus, yielding a larger achievable rate region.

5.2 Future extensions

Because of its large number of parameters, the 3GPP channel model can be used to describe many practical wireless communication channels. In this thesis, we discussed the transmission rates impact of the number of time resolvable clusters and of the angle spread for single-user MIMO systems. For MIMO BCs, we also considered the impact of the number of time resolvable clusters of a user on its spatial multiplexing gains that are associated with the inner and outer bounds of the DPC rate region. An important extension here is to study the effect of the user angle spread on these multiplexing gains. More generally, one needs to investigate the effect of the users spatial channel parameters on the achievable DPC rate region. This is interesting because the parameters of each user's channel affect not only the rates of this user but also those of the other users in the broadcast system.

For the MIMO BC with multi-dimensional space-time modulation, another interesting extension would be to show whether the achievable DPC rate region is in fact the capacity region of this channel. Note that the converse proof provided in the literature for MIMO BCs without space-time modulation does not readily apply to our framework. Furthermore, it is also important to investigate the implementation of DPC in our framework.

For broadcast systems also, a deeper investigation of the bandwidth versus interference tradeoff is very important for the system design. In this thesis, we only considered the impact of varying the number of shared signal dimensions on the spatial multiplexing gains associated with inner and outer bounds for the DPC rate region. A more complete analysis requires the investigation of the impact of varying the number of shared signal dimensions on the whole DPC rate region. For this, one also needs to consider broadcast systems where the users have different numbers of TOGs and such that for each user each TOG shares an arbitrary number of signal dimensions with an arbitrary number of TOGs of the other user.

Appendix A

Mathematical details and proofs for Chapter 3

We provide in this appendix the proofs of the theorems, lemmas and corollaries stated in Chapter 3.

Proof of Theorem 3.1.1: From (3.1) and (C.1), we have $I(\mathbf{x}; \mathbf{r}) = \log(|\mathbf{I}_{\tilde{D}} + \frac{1}{N_0} \Sigma \cdot \mathbf{H}^\dagger \cdot \mathbf{H}|)$ which, using (2.20) and (C.1), reduces to

$$I(\mathbf{x}; \mathbf{r}) = \log \left(\left| \mathbf{I}_{N_T D} + \frac{1}{N_0} \mathbf{A}^T \cdot \Sigma \cdot \mathbf{A} \cdot [(\mathbf{C}^\dagger \cdot \mathbf{C}) \otimes \mathbf{I}_D] \right| \right) \quad (\text{A.1})$$

The $N_T D \times N_T D$ matrix $\Sigma_y = \mathbf{A}^T \cdot \Sigma \cdot \mathbf{A}$ is the covariance matrix of $\mathbf{y} = \text{vec}(\{\mathbf{y}_i\}_{i=1}^{N_T})$. Thus, using the mapping $f_{\mathbf{A}}(\mathbf{X}) = \mathbf{A}^T \cdot \mathbf{X} \cdot \mathbf{A}$, $\Sigma_y = f_{\mathbf{A}}(\Sigma)$ is the image of Σ and (A.1) becomes

$$I(\mathbf{x}; \mathbf{r}) = \log \left(\left| \mathbf{I}_{N_T D} + \frac{1}{N_0} \Sigma_y \cdot [(\mathbf{C}^\dagger \cdot \mathbf{C}) \otimes \mathbf{I}_D] \right| \right) \quad (\text{A.2})$$

Using $\mathbf{A} = \text{diag}_{i=1}^{N_T} [\mathbf{e}_{t_i, N_G}^T \otimes \mathbf{I}_{D_G}]$ and (C.3), it is easily seen that $\mathbf{A} \cdot \mathbf{A}^T = \mathbf{I}_{\tilde{D}}$. Thus, using (C.11), the power constraint becomes $\text{Tr}(\Sigma_y) = \text{Tr}(\Sigma) \leq P_T$.

Let $\mathcal{A}_x$ and $\mathcal{A}_y$ denote respectively the sets of $\tilde{D} \times \tilde{D}$ and $N_T D \times N_T D$ covariance matrices Σ and Σ_y that satisfy the transmit power constraint. It can be shown that the mapping $f_{\mathbf{A}}$ is a mapping of $\mathcal{A}_x$ into $\mathcal{A}_y$, i.e. $f_{\mathbf{A}}(\mathcal{A}_x) \subseteq \mathcal{A}_y$. Equality is achieved only when $\mathbf{A}$ is invertible, corresponding to the ATA space-time modulation format for which

we have $\mathbf{A} = \mathbf{I}_{N_T D}$. Therefore, the maximization of (A.1) over $\Sigma \in \mathcal{A}_x$ cannot lead to a capacity greater than that obtained from the maximization of (A.2) over $\Sigma_y \in \mathcal{A}_y$. We conclude that the capacity achieved by ATA cannot be less than that achieved by any other multi-dimensional space-time modulation format. $\square$

Proof of Corollary 3.1.1: Let $I_{(i)}$ and $P_{T;i}$ respectively denote the mutual information and power associated with TOG i in the water-filling scheme achieving capacity. Such capacity is given by $C_{\text{original}} = \frac{1}{D} \sum_{i=1}^{N_G} I_{(i)}$. Now, merging TOGs m and p results in a new TOG with mutual information $I_{(m+p)}$ and power $P_{T;m+p}$. By Theorem 3.1.1, the maximum value $I_{(m+p)}$ obtained under the constraint $P_{T;m+p} \leq P_{T;m} + P_{T;p}$ cannot be less than $I_{(m)} + I_{(p)}$. Hence, we can write $C_{\text{original}} \leq \frac{1}{D} \left[\sum_{i=1, i \neq m, p}^{N_G} I_{(i)} + I_{(m+p)} \right] \leq C_{\text{merged}}$ where C_{merged} denotes the capacity of the new system. $\square$

Proof of Theorem 3.1.2: Based on the interlacing property for singular values [105, Theorem 7.3.9], the maximum singular value of $\mathbf{C}$ is larger or equal than the maximum singular value of any submatrix $\mathbf{C}_i$ obtained by deleting one or more columns of $\mathbf{C}$. Writing the singular values of $\mathbf{C}_i$ as square roots of the eigenvalues of $\mathbf{C}_i^\dagger \cdot \mathbf{C}_i$ yields $\sqrt{\lambda_{\max}(\mathbf{C}_i^\dagger \cdot \mathbf{C}_i)} \geq \max\{\sqrt{\lambda_{i;l}}\}_{l=1}^{n_i}, \forall i$, and $\lambda_{\max;A} \geq \lambda_{\max;P}$ since $\lambda_{\max;A} = \lambda_{\max}(\mathbf{C}^\dagger \cdot \mathbf{C})$ and $\lambda_{\max;P} = \max\{\{\lambda_{i;l}\}_{l=1}^{n_i}\}_{i=1}^{N_G}$. Similarly, using [105, Theorem 7.3.9] within each $\mathbf{C}_i$, we can write $\max\{\sqrt{\lambda_{i;l}}\}_{l=1}^{n_i} \geq \max\{\|\mathbf{C}_i\|_{*,q}\}_{q=1}^{n_i}$ where $\|\mathbf{C}_i\|_{*,q}$ is the singular value of column $[\mathbf{C}_i]_{*,q}$, thus, leading to $\lambda_{\max;P} \geq \lambda_{\max;O}$ since $\lambda_{\max;O} = \max\{\{\|\mathbf{C}_i\|_{*,q}\}^2\}_{q=1}^{n_i}\}_{i=1}^{N_G}$. $\square$

Proof of Theorem 3.1.3: Since $r_i = \min(n_i, L_t N_R) \forall i$, the ratio $\frac{r_T}{N_G}$ can be written as: $r_{T;A} = \min(N_T, L_t N_R)$ with ATA, $\frac{r_{T;P}}{N_G} = \sum_{i=1}^{N_G} \min(\frac{n_i}{N_G}, \frac{L_t N_R}{N_G})$ with POTA, and $\frac{r_{T;O}}{N_T} = \sum_{i=1}^{N_T} \min(\frac{1}{N_T}, \frac{L_t N_R}{N_T}) = 1$ with OTA. Consider POTA. We have $\min(\frac{n_i}{N_G}, \frac{L_t N_R}{N_G}) \leq \frac{n_i}{N_G}$, yielding $\frac{r_{T;P}}{N_G} \leq \sum_{i=1}^{N_G} \frac{n_i}{N_G} = \frac{N_T}{N_G}$. Similarly, using $\min(\frac{n_i}{N_G}, \frac{L_t N_R}{N_G}) \leq \frac{L_t N_R}{N_G}$ we have $\frac{r_{T;P}}{N_G} \leq \sum_{i=1}^{N_G} \frac{L_t N_R}{N_G} = L_t N_R$. Thus, we can write

$$\frac{r_{T;P}}{N_G} \leq \min\left(\frac{N_T}{N_G}, L_t N_R\right) \leq \min(N_T, L_t N_R) = r_{T;A}$$

Using $\min(\frac{n_i}{N_G}, \frac{L_t N_R}{N_G}) \geq \frac{1}{N_G}$ one can easily see that $\frac{r_{T;P}}{N_G} \geq \sum_{i=1}^{N_G} \frac{1}{N_G} = 1 = \frac{r_{T;O}}{N_T}$. $\square$

Proof of Lemma 3.2.1: From [3], the PDF of $\boldsymbol{\omega}_i = \text{vec}(\{\omega_{i;l}\}_{l=1}^{m_i})$ is given by

$$p_{\boldsymbol{\omega}_i}(\mathbf{u}_i) = K_{m_i; p_i}^{-1} \prod_{l=1}^{m_i} \left[e^{-u_{i;l}} u_{i;l}^{\{p_i - m_i\}} \prod_{j=l+1}^{m_i} (u_{i;l} - u_{i;j})^2 \right] \quad (\text{A.3})$$

where $K_{m_i;p_i}$ are normalizing constants. By [126, Theorem 3.1.1], we have

$$p_{\boldsymbol{\varrho}_i}(\mathbf{u}_i) = p_{\boldsymbol{\omega}_i}(f_i^{-1}(u_{i,1}), \dots, f_i^{-1}(u_{i,m_i})) J(\boldsymbol{\omega}_i \longrightarrow \boldsymbol{\varrho}_i) \quad (\text{A.4})$$

where $J(\boldsymbol{\omega}_i \longrightarrow \boldsymbol{\varrho}_i)$ is the Jacobian of the transformation from $\boldsymbol{\omega}_i$ to $\boldsymbol{\varrho}_i$. Using (A.3), we have

$$\begin{aligned} p_{\boldsymbol{\omega}_i}(f_i^{-1}(u_{i,1}), \dots, f_i^{-1}(u_{i,m_i})) \\ &= K_{m_i;p_i}^{-1} \prod_{l=1}^{m_i} \left[e^{\{-\frac{2^{u_{i,l}}-1}{\eta_i \kappa \sigma^2}\}} \left(\frac{2^{u_{i,l}}-1}{\eta_i \kappa \sigma^2} \right)^{\{p_i-m_i\}} \prod_{j=l+1}^{m_i} \left(\frac{2^{u_{i,l}}-2^{u_{i,j}}}{\eta_i \kappa \sigma^2} \right)^2 \right] \\ &= \frac{K_{m_i;p_i}^{-1} \prod_{l=1}^{m_i} \left[e^{\{-\frac{2^{u_{i,l}}-1}{\eta_i \kappa \sigma^2}\}} (2^{u_{i,l}}-1)^{\{p_i-m_i\}} \prod_{j=l+1}^{m_i} (2^{u_{i,l}}-2^{u_{i,j}})^2 \right]}{(\eta_i \kappa \sigma^2)^{\sum_{l=1}^{m_i} \{p_i-m_i+2(m_i-l)\}}} \end{aligned}$$

Furthermore, we have $\sum_{l=1}^{m_i} \{p_i-m_i+2(m_i-l)\} = m_i(p_i+m_i) - 2 \sum_{l=1}^{m_i} l = m_i(p_i-1)$ since $\sum_{l=1}^{m_i} l = \frac{m_i(m_i+1)}{2}$, yielding

$$\begin{aligned} p_{\boldsymbol{\omega}_i}(f_i^{-1}(u_{i,1}), \dots, f_i^{-1}(u_{i,m_i})) \\ &= \frac{K_{m_i;p_i}^{-1} \prod_{l=1}^{m_i} \left[e^{\{-\frac{2^{u_{i,l}}-1}{\eta_i \kappa \sigma^2}\}} (2^{u_{i,l}}-1)^{\{p_i-m_i\}} \prod_{j=l+1}^{m_i} (2^{u_{i,l}}-2^{u_{i,j}})^2 \right]}{(\eta_i \kappa \sigma^2)^{\{m_i(p_i-1)\}}} \quad (\text{A.5}) \end{aligned}$$

The Jacobian $J(\boldsymbol{\omega}_i \longrightarrow \boldsymbol{\varrho}_i)$ is given by [126, Section 2.15]

$$J(\boldsymbol{\omega}_i \longrightarrow \boldsymbol{\varrho}_i) = \left| |\mathbf{J}_i| \right|, \quad [\mathbf{J}_i]_{p,q} = \frac{\partial \omega_{i,p}}{\partial \varrho_{i,q}}, \quad p, q = 1, \dots, m_i$$

where $\left| |\mathbf{J}_i| \right|$ is the absolute value of the determinant $|\mathbf{J}_i|$. One can see that $[\mathbf{J}_i]_{p,q} = 0$ if $p \neq q$ and, otherwise, $[\mathbf{J}_i]_{q,q} \equiv \frac{\partial}{\partial \varrho_{i,q}} \left(\frac{2^{\varrho_{i,q}}-1}{\eta_i \kappa \sigma^2} \right) = \frac{\ln 2}{\eta_i \kappa \sigma^2} 2^{\varrho_{i,q}}$. Therefore, we have $|\mathbf{J}_i| = \prod_{q=1}^{m_i} \left(\frac{\ln 2}{\eta_i \kappa \sigma^2} 2^{\varrho_{i,q}} \right)$ and

$$J(\boldsymbol{\omega}_i \longrightarrow \boldsymbol{\varrho}_i) = \left(\frac{\ln 2}{\eta_i \kappa \sigma^2} \right)^{m_i} \prod_{l=1}^{m_i} 2^{\varrho_{i,l}} \quad (\text{A.6})$$

From (A.4)-(A.6) we have

$$p_{\boldsymbol{\varrho}_i}(\mathbf{u}_i) = \frac{K_{m_i;p_i}^{-1} (\ln 2)^{m_i}}{(\eta_i \kappa \sigma^2)^{m_i p_i}} \prod_{l=1}^{m_i} \left[e^{\{-\frac{2^{u_{i,l}}-1}{\eta_i \kappa \sigma^2}\}} 2^{u_{i,l}} (2^{u_{i,l}}-1)^{\{p_i-m_i\}} \prod_{j=l+1}^{m_i} (2^{u_{i,l}}-2^{u_{i,j}})^2 \right] \quad (\text{A.7})$$

Since the entries of the channel propagation matrix $\mathbf{C}$ are independent, the matrices $\mathbf{C}_{s,i}$

(which are submatrices of $\mathbf{C}$) and, thus, $\mathbf{W}_i$ are independent for different TOGs i . Hence, the vectors $\boldsymbol{\omega}_i$ are independent for different i , and hence the vectors $\boldsymbol{\rho}_i$ are independent for different i . It follows that $p_{\boldsymbol{\rho}}(\mathbf{u}) = \prod_{i=1}^{N_G} p_{\boldsymbol{\rho}_i}(\mathbf{u}_i)$ which, using (A.7), leads to (3.33). $\square$

Proof of Theorem 3.2.1: Using (3.33), (3.34) becomes equivalent to

$$P_{\sigma} \doteq \left\{ \prod_{i=1}^{N_G} (\eta_i \sigma^2)^{-m_i p_i} \right\} \int_{\mathcal{A}_{\theta}} \prod_{i=1}^{N_G} \left\{ \frac{K_{m_i; p_i}^{-1} (\ln 2)^{m_i}}{\kappa^{m_i p_i}} \right. \\ \left. \times \prod_{l=1}^{m_i} \left[e^{\{-\frac{2^{u_{i,l}}-1}{\eta_i \kappa \sigma^2}\}} 2^{u_{i,l}} (2^{u_{i,l}} - 1)^{p_i - m_i} \prod_{j=l+1}^{m_i} (2^{u_{i,l}} - 2^{u_{i,j}})^2 \right] \right\} d\mathbf{u} \quad (\text{A.8})$$

For all $\mathbf{u} \in \mathcal{A}_{\theta}$, we have $u_{i,1} > u_{i,2} > \dots > u_{i,m_i} > 0$ and $\sum_{i=1}^{N_G} \sum_{l=1}^{m_i} u_{i,l} < W$, thus, leading to $0 < u_{i,l} < W$. We bound the exponential term $e^{\{-\frac{2^{u_{i,l}}-1}{\eta_i \kappa \sigma^2}\}}$ as follows: $e^{\{-\frac{2^W-1}{\eta_i \kappa \sigma^2}\}} < e^{\{-\frac{2^{u_{i,l}}-1}{\eta_i \kappa \sigma^2}\}} < 1$. Hence, using $\tilde{I}_{\sigma; \mathcal{A}_{\theta}}$ to denote the integral on the Right Hand Side (RHS) of (A.8), we can write

$$\left\{ \prod_{i=1}^{N_G} \prod_{l=1}^{m_i} e^{\{-\frac{2^{u_{i,l}}-1}{\eta_i \kappa \sigma^2}\}} \right\} \tilde{I}_{\mathcal{A}_{\theta}} \leq \tilde{I}_{\sigma; \mathcal{A}_{\theta}} \leq \tilde{I}_{\mathcal{A}_{\theta}} \quad (\text{A.9})$$

Moreover, we have

$$\prod_{i=1}^{N_G} \prod_{l=1}^{m_i} e^{\{-\frac{2^{u_{i,l}}-1}{\eta_i \kappa \sigma^2}\}} = e^{\{-(\frac{2^W-1}{\kappa \sigma^2})(\sum_{i=1}^{N_G} m_i / \eta_i)\}} = [e^{\{-\frac{2^W-1}{\kappa \sigma^2}\}}]^{(\sum_{i=1}^{N_G} m_i / \eta_i)} \xrightarrow{\sigma \rightarrow \infty} 1$$

which leads to $\prod_{i=1}^{N_G} \prod_{l=1}^{m_i} e^{\{-\frac{2^{u_{i,l}}-1}{\eta_i \kappa \sigma^2}\}} \doteq 1$ and, consequently, $\tilde{I}_{\sigma; \mathcal{A}_{\theta}} \doteq \tilde{I}_{\mathcal{A}_{\theta}}$ as seen from (A.9). Hence, (A.8) reduces to $P_{\sigma} \doteq \left\{ \prod_{i=1}^{N_G} (\eta_i \sigma^2)^{-m_i p_i} \right\} \tilde{I}_{\mathcal{A}_{\theta}}$. Since $m_i = \min(r_{s;i}, L_t N_R)$ and $p_i = \max(r_{s;i}, L_t N_R)$, we have $m_i p_i = r_{s;i} L_t N_R$, and the proof of Theorem 3.2.1 is concluded. $\square$

Proof of Lemma 3.2.2: Denote $\mathbf{D}_C = \mathbf{C}^{\dagger} \cdot \mathbf{C}$. Writing $I(\mathbf{x}; \mathbf{r})$ as in (A.1) and using Σ of (3.37), we have $I(\mathbf{x}; \mathbf{r}) = \log(|\mathbf{I}_{N_T D} + N_G \frac{P_T}{N_0 N_T D} \mathbf{A}^T \cdot \mathbf{A} \cdot [\mathbf{D}_C \otimes \mathbf{I}_D]|)$. With $\mathbf{A} =$

$diag_{i=1}^{N_T}[\mathbf{e}_{t_i;N_G}^T \otimes \mathbf{I}_{D_G}]$, this becomes

$$\begin{aligned} I(\mathbf{x}; \mathbf{r}) &= \log \left(\left| \mathbf{I}_{N_T D} \right. \right. \\ &\quad \left. \left. + N_G \frac{P_T}{N_0 N_T D} diag_{i=1}^{N_T}[(\mathbf{e}_{t_i;N_G} \cdot \mathbf{e}_{t_i;N_G}^T) \otimes \mathbf{I}_{D_G}] \cdot [\mathbf{D}_C \otimes (\mathbf{I}_{N_G} \otimes \mathbf{I}_{D_G})] \right| \right) \\ &= \log \left(\left| \mathbf{I}_{N_T N_G} \otimes \mathbf{I}_{D_G} \right. \right. \\ &\quad \left. \left. + \left\{ N_G \frac{P_T}{N_0 N_T D} diag_{i=1}^{N_T}[\mathbf{e}_{t_i;N_G} \cdot \mathbf{e}_{t_i;N_G}^T] \cdot [\mathbf{D}_C \otimes \mathbf{I}_{N_G}] \right\} \otimes \mathbf{I}_{D_G} \right| \right), \end{aligned}$$

yielding $I(\mathbf{x}; \mathbf{r}) = D_G \log \left(\left| \mathbf{I}_{N_T N_G} + N_G \frac{P_T}{N_0 N_T D} diag_{i=1}^{N_T}[\mathbf{e}_{t_i;N_G} \cdot \mathbf{e}_{t_i;N_G}^T] \cdot [\mathbf{D}_C \otimes \mathbf{I}_{N_G}] \right| \right)$ by (C.5a) and (C.6). One can easily show that $diag_{i=1}^{N_T}[\mathbf{e}_{t_i;N_G} \cdot \mathbf{e}_{t_i;N_G}^T] \cdot [\mathbf{D}_C \otimes \mathbf{I}_{N_G}] = diag_{i=1}^{N_T}[[\mathbf{D}_C]_{i,i}(\mathbf{e}_{t_i;N_G} \cdot \mathbf{e}_{t_i;N_G}^T)]$. Hence, we have

$$\begin{aligned} I(\mathbf{x}; \mathbf{r}) &= D_G \log \left(\left| \mathbf{I}_{N_T N_G} + N_G \frac{P_T}{N_0 N_T D} diag_{i=1}^{N_T}[(\mathbf{e}_{t_i;N_G} \cdot \mathbf{e}_{t_i;N_G}^T) \cdot ([\mathbf{D}_C]_{i,i} \mathbf{I}_{N_G})] \right| \right) \\ &= D_G \sum_{i=1}^{N_T} \log \left(\left| \mathbf{I}_{N_G} + N_G [\mathbf{D}_C]_{i,i} \frac{P_T}{N_0 N_T D} [\mathbf{e}_{t_i;N_G} \cdot \mathbf{e}_{t_i;N_G}^T] \right| \right) \\ &\stackrel{(a)}{=} \frac{D}{N_G} \sum_{i=1}^{N_T} \log \left(1 + N_G [\mathbf{D}_C]_{i,i} \frac{P_T}{N_0 N_T D} \right) \end{aligned} \tag{A.10}$$

with (a) due to (C.1), $D_G = \frac{D}{N_G}$. The RHS of (A.10) being the sum of monotonically decreasing non-negative functions of N_G , $I(\mathbf{x}; \mathbf{r})$ is therefore monotonically decreasing with N_G . $\square$

Proof of Lemma 3.2.3: We have $\left[\frac{E_b}{N_0} \right]_{u;\min}$ equal to $\frac{\ln 2}{B_A}$, $\frac{N_G \ln 2}{B_P}$ and $\frac{N_T \ln 2}{B_O}$ when using ATA, POTA and OTA, respectively. Since $\frac{B_{\max}}{N_G} = B_A = \frac{B_P}{N_G} = \frac{B_O}{N_T} = \max\{\|[\mathbf{C}]_{*,l}\|^2\}_{l=1}^{N_T}$ from (3.44), all space-time modulation formats result in similar $\left[\frac{E_b}{N_0} \right]_{u;\min}$. Moreover, (3.42) shows that the slope of I_u at $\left[\frac{E_b}{N_0} \right]_{u;\min}$ is largest, intermediate and smallest with ATA, POTA and OTA, respectively. We conclude that I_u decreases with increasing N_G . $\square$

Proof of Lemma 3.2.4: Since I_u has respectively largest, intermediate and smallest slope with ATA, POTA and OTA from (3.42), it is sufficient to show that the respective $\left[\frac{E_b}{N_0} \right]_{u;\min}$ are in decreasing order, i.e. $B_{f;A} \leq \frac{B_{f;P}}{N_G} \leq \frac{B_{f;O}}{N_T}$. By definition, the strongest TOG power allocation maximizes $\sum_{i=1}^{N_G} \frac{Tr(\mathbf{C}_i^\dagger \cdot \mathbf{C}_i)}{n_i} \eta_i$ over all possible power allocations η and this maximum is given by $\frac{B_{f;P}}{N_G} \equiv \left[\frac{Tr(\mathbf{C}_i^\dagger \cdot \mathbf{C}_i)}{n_i} \right]_{\max}$ for POTA ($N_G \geq 2$). Particularly,

with $\eta_i = \frac{n_i}{N_T}$ we have $\frac{B_{f;P}}{N_G} \geq \sum_{i=1}^{N_G} \frac{\text{Tr}(\mathbf{C}_i^\dagger \cdot \mathbf{C}_i)}{N_T} = \frac{\text{Tr}(\mathbf{C}^\dagger \cdot \mathbf{C})}{N_T}$, yielding $\frac{B_{f;P}}{N_G} \geq B_{f;A}$. With OTA, the maximum of $\sum_{l=1}^{N_T} \|\mathbf{C}_{*,l}\|^2 \eta_l$ over $\mathcal{A}_\eta$ is given by $\frac{B_{f;O}}{N_T} \equiv \max\{\|\mathbf{C}_{*,l}\|^2\}_{l=1}^{N_T}$. Thus, we can write $\frac{B_{f;O}}{N_T} \geq \sum_{l=1}^{N_T} \|\mathbf{C}_{*,l}\|^2 \eta_l$ for all power allocations η_l . Particularly, chose η_l equal for all antennas l belonging to one TOG (as defined by POTA), i.e. all the n_i antennas in TOG i have their power allocations η_l equal to η_{n_i} , $1 \leq l \leq n_i$. With such power allocation, we have $\sum_{l=1}^{N_T} \|\mathbf{C}_{*,l}\|^2 \eta_l = \sum_{i=1}^{N_G} \text{Tr}(\mathbf{C}_i^\dagger \cdot \mathbf{C}_i) \eta_{n_i}$ with $\sum_{i=1}^{N_G} n_i \eta_{n_i} = 1$, and

$$\frac{B_{f;O}}{N_T} \geq \sum_{l=1}^{N_T} \|\mathbf{C}_{*,l}\|^2 \eta_l = \sum_{i=1}^{N_G} \text{Tr}(\mathbf{C}_i^\dagger \cdot \mathbf{C}_i) \eta_{n_i}$$

Choosing $\eta_{n_{\max}} = \frac{1}{n_{\max}}$ for the TOG associated with $\left[\frac{\text{Tr}(\mathbf{C}_i^\dagger \cdot \mathbf{C}_i)}{n_i}\right]_{\max}$ and allocating no power $\eta_{n_i} = 0$ to all other TOGs, we have $\frac{B_{f;O}}{N_T} \geq \left[\frac{\text{Tr}(\mathbf{C}_i^\dagger \cdot \mathbf{C}_i)}{n_i}\right]_{\max}$ and, therefore, $\frac{B_{f;O}}{N_T} \geq \frac{B_{f;P}}{N_G}$.

Proof of Lemma 3.2.5: Consider I_u of (3.50). Regardless of η , $\frac{r_T}{N_G} \log\left(\frac{P_T}{N_0 D}\right)$ becomes dominant for large P_T/N_0 . Thus, the impact of space-time modulation on I_u is determined through $\frac{r_T}{N_G}$ which, according to Theorem 3.1.3, is maximum, intermediate and minimum with ATA, POTA and OTA, respectively. $\square$

Proof of Theorem 3.2.2: We prove Theorem 3.2.2 as follows.

1) Suppose that the capacity C from (3.9) using water-filling power allocation (3.8) is equal to the rate I_u obtained from (3.1) using the power allocation $\Sigma = \text{diag}_{i=1}^{N_G} \left[\frac{\eta_i}{n_i} \frac{P_T}{D_G} \mathbf{I}_{n_i D_G} \right]$ (obtained from (3.38) for $r_{s,i} = n_i$). Due to the uniqueness of the input covariance matrix that maximizes the mutual information $I(\mathbf{x}; \mathbf{r})$ [73, Appendix II], the input covariance matrices associated with capacity C and rate I_u must be equal when $C = I_u$. Therefore, we can write $\{\mathbf{M}_i^\dagger \cdot (N_0 \text{diag}_{l=1}^{n_i} [\{\mu - \lambda_{i,l}^{-1}\}^+]) \cdot \mathbf{M}_i\} \otimes \mathbf{I}_{D_G} = \frac{\eta_i}{n_i} \frac{N_G P_T}{D} \mathbf{I}_{n_i D_G}$ for all i , leading to

$$\text{diag}_{l=1}^{n_i} [\{\mu - \lambda_{i,l}^{-1}\}^+] = \frac{\eta_i}{n_i} \frac{N_G P_T}{N_0 D} \mathbf{I}_{n_i} \quad (\text{A.11})$$

It follows that $\{\mu - \lambda_{i,l}^{-1}\}^+ = \frac{\eta_i}{n_i} \frac{N_G P_T}{N_0 D}$ for all $l = 1, \dots, n_i$, yielding $r_i = n_i$ with all eigenmodes active in TOG i . Equivalently, we have $\mu - \lambda_{i,l}^{-1} = \frac{\eta_i}{n_i} \frac{N_G P_T}{N_0 D}$ or, $\lambda_{i,l}^{-1} + \frac{\eta_i}{n_i} \frac{N_G P_T}{N_0 D} = \mu = \text{Constant}$ for all i . Thus, all eigenvalues in TOG i are equal ($\lambda_{i,l} = \lambda_i$) and the water level μ in (3.9) is given by

$$\mu = \lambda_i^{-1} + \frac{\eta_i}{n_i} \frac{N_G P_T}{N_0 D}, \quad (\text{A.12})$$

which proves (3.52).

2) Consider N_G TOGs with $r_i = n_i$ and $\lambda_{i,l} = \lambda_i$ for all l , such that (3.52) is satisfied for all i . Using (3.1), (C.1) and $\Sigma = \text{diag}_{i=1}^{N_G} [\frac{\eta_i P_T}{n_i D_G} \mathbf{I}_{n_i D_G}]$, the information rate for uninformed transmitter is

$$\begin{aligned} I_u &= \frac{1}{N_G} \sum_{i=1}^{N_G} \sum_{l=1}^{n_i} \log \left(\lambda_i^{-1} + \frac{\eta_i N_G P_T}{n_i N_0 D} \right) + \frac{1}{N_G} \sum_{i=1}^{N_G} n_i \log(\lambda_i) \\ &= \frac{N_T}{N_G} \log \left[\prod_{i=1}^{N_G} \prod_{l=1}^{n_i} \left(\lambda_i^{-1} + \frac{\eta_i N_G P_T}{n_i N_0 D} \right)^{\frac{1}{N_T}} \right] + \frac{1}{N_G} \sum_{i=1}^{N_G} n_i \log(\lambda_i) \\ &\stackrel{(a)}{\leq} \frac{N_T}{N_G} \log \left[\frac{1}{N_T} \sum_{i=1}^{N_G} \sum_{l=1}^{n_i} \left(\lambda_i^{-1} + \frac{\eta_i N_G P_T}{n_i N_0 D} \right) \right] + \frac{1}{N_G} \sum_{i=1}^{N_G} n_i \log(\lambda_i) \text{ bps/Hz} \end{aligned}$$

where (a) is due to the arithmetic-geometric inequality [127, p. 1094], for which equality is achieved since $\lambda_i^{-1} + \frac{\eta_i N_G P_T}{n_i N_0 D}$ are equal according to (3.52). Hence, we have

$$I_u = \frac{N_T}{N_G} \log \left[\frac{1}{N_T} \left(\sum_{i=1}^{N_G} n_i \lambda_i^{-1} \right) + \frac{N_G P_T}{N_0 N_T D} \right] + \frac{1}{N_G} \sum_{i=1}^{N_G} n_i \log(\lambda_i) \text{ bps/Hz} \quad (\text{A.13})$$

For informed transmitter, (3.52) shows that there exists a water-filling scheme where all eigenmodes in all TOGs are active and TOG i is assigned a fraction η_i of the total power P_T . The water level in such scheme is given by $\mu = \lambda_i^{-1} + \frac{\eta_i N_G P_T}{n_i N_0 D}$. Summing for $1 \leq l \leq n_i$ and $1 \leq i \leq N_G$, we also have $\mu = \frac{1}{N_T} \left(\sum_{i=1}^{N_G} n_i \lambda_i^{-1} \right) + \frac{N_G P_T}{N_0 N_T D}$, reducing (3.9) to

$$C = \frac{N_T}{N_G} \log \left[\frac{1}{N_T} \left(\sum_{i=1}^{N_G} n_i \lambda_i^{-1} \right) + \frac{N_G P_T}{N_0 N_T D} \right] + \frac{1}{N_G} \sum_{i=1}^{N_G} n_i \log(\lambda_i) \text{ bps/Hz} \quad (\text{A.14})$$

which is similar to (A.13). The proof of Theorem 3.2.2 is therefore concluded. $\square$

Proof of Theorem 3.2.3: We prove Theorem 3.2.3 as follows.

1) If $r_i = n_i$ and $\eta_i = \frac{n_i}{N_T} \forall i$, it is straightforward to show from (3.53) that $\Delta(E_b/N_0)_T = 0$ dB.

2) Assume $\Delta(E_b/N_0)_T = 0$ dB. Thus, (3.53) yields $\log_{10}(\frac{r_T}{N_G}) = \frac{1}{r_T} \sum_{i=1}^{N_G} r_i \log_{10}(\frac{n_i}{N_G \eta_i})$ which, by writing $\frac{n_i}{N_G \eta_i} = \frac{r_i}{r_i(N_G \eta_i/n_i)}$ and using the log sum inequality [51, p. 483], leads to

$$\log_{10} \left(\frac{r_T}{N_G} \right) \stackrel{(a)}{\geq} \frac{1}{r_T} \left(\sum_{i=1}^{N_G} r_i \right) \log_{10} \left[\frac{\sum_{i=1}^{N_G} r_i}{\sum_{i=1}^{N_G} r_i (N_G \eta_i/n_i)} \right] = \log_{10} \left[\frac{r_T}{\sum_{i=1}^{N_G} r_i (N_G \eta_i/n_i)} \right]$$

Equality in (a) is reached if and only if η_i/n_i is constant $\forall i$. Such constant can be easily shown to be $1/N_T$ using $\sum_{i=1}^{N_G} \eta_i = 1$ and $\sum_{i=1}^{N_G} n_i = N_T$. Hence, equality in (a) is reached if and only if $\eta_i = \frac{n_i}{N_T} \forall i$. Furthermore, since $r_i \leq n_i$ we have $\sum_{i=1}^{N_G} r_i(N_G \eta_i/n_i) \leq \sum_{i=1}^{N_G} (N_G \eta_i) = N_G$, and

$$\log_{10} \left[\frac{r_T}{\sum_{i=1}^{N_G} r_i(N_G \eta_i/n_i)} \right] \stackrel{(b)}{\geq} \log_{10} \left(\frac{r_T}{N_G} \right)$$

Equality in (b) holds only if $r_i = n_i \forall i$. Hence, $\Delta(E_b/N_0)_T = 0$ dB implies that (a) and (b) are both satisfied with equality, yielding $r_i = n_i$ and $\eta_i = \frac{n_i}{N_T}$ for all $i = 1, \dots, N_G$. $\square$

Appendix B

Mathematical details and proofs for Chapter 4

B.1 Proofs of the theorems and lemmas of Chapter 4

We provide in this section the proofs of the theorems and lemmas stated in Chapter 4.

Proof of Theorem 4.1.1: For this theorem, $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are independent zero mean circularly symmetric Gaussian input vectors with covariance matrices $\Sigma^{(1)}$ and $\Sigma^{(2)}$, respectively, such that $Tr(\Sigma^{(1)}) + Tr(\Sigma^{(2)}) \leq P_T$. The proof of achievability of the rate regions (4.4) and (4.5) follows closely that of [63, Theorem 1] and it employs Marton's achievable rate region from [60, 128] for a general discrete memoryless BC. Although the achievability of Marton's region has been originally proved for discrete BCs it can be shown to hold for BCs with continuous alphabets based on a quantization argument [129, Chapter 7]. For instance, such a quantization method has been used in the context of extending the rate results for channels with noncausal CSIT from the discrete case to the continuous case, assuming memoryless channels [52]. Roughly speaking, the continuous alphabet is first discretized by using a sequence of increasingly fine quantizers, then the rate achievability results for discrete channels are applied considering the discretized alphabet. In the limit of fine quantization, the rate achievability is finally established for the continuous alphabet. Marton's region has been used in [62, 63] for Gaussian BCs, with its definition from [60] as

the set of rate pairs $(R^{(1)}, R^{(2)})$ such that

$$R^{(1)} \leq I(\mathbf{u}^{(1)}; \mathbf{r}^{(1)}) \quad (\text{B.1a})$$

$$R^{(2)} \leq I(\mathbf{u}^{(2)}; \mathbf{r}^{(2)}) \quad (\text{B.1b})$$

$$R^{(1)} + R^{(2)} \leq I(\mathbf{u}^{(1)}; \mathbf{r}^{(1)}) + I(\mathbf{u}^{(2)}; \mathbf{r}^{(2)}) - I(\mathbf{u}^{(1)}; \mathbf{u}^{(2)}) \quad (\text{B.1c})$$

where $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ are auxiliary random vectors taking values in an alphabets $\mathcal{R}_u^{(1)}$ and $\mathcal{R}_u^{(2)}$, respectively, and where the mutual information is evaluated under a joint distribution $p_{\mathbf{r}^{(1)}, \mathbf{r}^{(2)} | \mathbf{x}_0}(\mathbf{a}_r^{(1)}, \mathbf{a}_r^{(2)} | \mathbf{a}_0) p_{\mathbf{x}_0 | \mathbf{u}^{(1)}, \mathbf{u}^{(2)}}(\mathbf{a}_0 | \mathbf{a}_u^{(1)}, \mathbf{a}_u^{(2)}) p_{\mathbf{u}^{(1)}, \mathbf{u}^{(2)}}(\mathbf{a}_u^{(1)}, \mathbf{a}_u^{(2)})$ where $\mathbf{a}_0 \in \mathcal{X}_0$, $\mathbf{a}_u^{(k)} \in \mathcal{R}_u^{(k)}$ and $\mathbf{a}_r^{(k)} \in \mathcal{R}^{(k)}$, $k = 1, 2$.

Let us first prove the achievability of the rate region $\mathcal{R}_{1,2}(\Sigma^{(1)}, \Sigma^{(2)})$ of (4.4) using (B.1). For this, we need to find a pair of auxiliary vectors $(\mathbf{u}^{(1)}, \mathbf{u}^{(2)})$ for which $I(\mathbf{u}^{(1)}; \mathbf{r}^{(1)})$ and $I(\mathbf{u}^{(2)}; \mathbf{r}^{(2)}) - I(\mathbf{u}^{(1)}; \mathbf{u}^{(2)})$ correspond to the rates $R_{DPC}^{(1)}(\Sigma^{(1)}, \Sigma^{(2)})$ and $R_{DPC}^{(2)}(\Sigma^{(2)})$ of (4.2a) and (4.2b), respectively. We choose $\mathbf{u}^{(1)}$ to be zero mean circularly symmetric Gaussian with $\mathbf{u}^{(1)} = \mathbf{x}^{(1)}$ (and $\mathcal{R}_u^{(1)} = \mathbb{C}^{\tilde{D}^{(1)}}$) since, as shown below, this makes $I(\mathbf{u}^{(1)}; \mathbf{r}^{(1)})$ equal to the rate of (4.2a). With this $\mathbf{u}^{(1)}$, user 1 can be first encoded at a rate $I(\mathbf{u}^{(1)}; \mathbf{r}^{(1)})$ (bits/channel-use) which, using (4.1a), can be written as

$$\begin{aligned} I(\mathbf{u}^{(1)}; \mathbf{r}^{(1)}) &\equiv I(\mathbf{x}^{(1)}; \mathbf{r}^{(1)}) = h(\mathbf{r}^{(1)}) - h(\mathbf{r}^{(1)} | \mathbf{x}^{(1)}) \\ &= h(\mathbf{H}^{(1)} \cdot \mathbf{x}^{(1)} + \mathbf{H}^{(1,2)} \cdot \mathbf{x}^{(2)} + \mathbf{v}^{(1)}) - h(\mathbf{H}^{(1)} \cdot \mathbf{x}^{(1)} + \mathbf{H}^{(1,2)} \cdot \mathbf{x}^{(2)} + \mathbf{v}^{(1)} | \mathbf{x}^{(1)}) \\ &\stackrel{(a)}{=} h(\mathbf{H}^{(1)} \cdot \mathbf{x}^{(1)} + \mathbf{H}^{(1,2)} \cdot \mathbf{x}^{(2)} + \mathbf{v}^{(1)}) - h(\mathbf{H}^{(1,2)} \cdot \mathbf{x}^{(2)} + \mathbf{v}^{(1)} | \mathbf{x}^{(1)}) \\ &\stackrel{(b)}{=} h(\mathbf{H}^{(1)} \cdot \mathbf{x}^{(1)} + \mathbf{H}^{(1,2)} \cdot \mathbf{x}^{(2)} + \mathbf{v}^{(1)}) - h(\mathbf{H}^{(1,2)} \cdot \mathbf{x}^{(2)} + \mathbf{v}^{(1)}) \\ &\stackrel{(c)}{=} \log \frac{\left| \mathbf{I}_{L_t^{(1)} N_R^{(1)} D^{(1)}} + \frac{1}{N_0} \mathbf{H}^{(1,2)} \cdot \Sigma^{(2)} \cdot [\mathbf{H}^{(1,2)}]^\dagger + \frac{1}{N_0} \mathbf{H}^{(1)} \cdot \Sigma^{(1)} \cdot [\mathbf{H}^{(1)}]^\dagger \right|}{\left| \mathbf{I}_{L_t^{(1)} N_R^{(1)} D^{(1)}} + \frac{1}{N_0} \mathbf{H}^{(1,2)} \cdot \Sigma^{(2)} \cdot [\mathbf{H}^{(1,2)}]^\dagger \right|} \\ &= R_{DPC}^{(1)}(\Sigma^{(1)}, \Sigma^{(2)}) \end{aligned}$$

where (a) is due to the fact that translation does not change the differential entropy [51, Theorem 9.6.3], (b) due to the fact that $\mathbf{x}^{(2)}$ and $\mathbf{v}^{(1)}$ are independent of $\mathbf{x}^{(1)}$, and (c) due to $\mathbf{H}^{(1)} \cdot \mathbf{x}^{(1)} + \mathbf{H}^{(1,2)} \cdot \mathbf{x}^{(2)} + \mathbf{v}^{(1)}$ and $\mathbf{H}^{(1,2)} \cdot \mathbf{x}^{(2)} + \mathbf{v}^{(1)}$ being Gaussian since they are sums of independent Gaussian vectors. Here, the term $\mathbf{H}^{(1,2)} \cdot \mathbf{x}^{(2)}$ in (4.1a) is treated as additional noise. Now we need to find an auxiliary vector $\mathbf{u}^{(2)}$ such that $I(\mathbf{u}^{(2)}; \mathbf{r}^{(2)}) - I(\mathbf{u}^{(1)}; \mathbf{u}^{(2)})$ is equal to the rate of (4.2b). User 2 can be encoded using the noncausal knowledge of user 1's codeword. Hence, due to the transmitter knowledge of $\mathbf{H}^{(2,1)}$ and since $\mathbf{x}^{(1)}$ is Gaussian, the

term $\mathbf{H}^{(2,1)} \cdot \mathbf{x}^{(1)}$ in (4.1b) can be considered as Gaussian interference noncausally known at the transmitter. From [53], the maximal rate for user 2 is given by

$$\begin{aligned} R_{DPC}^{(2)}(\Sigma^{(1)}, \Sigma^{(2)}) &\triangleq \max_{p_{\mathbf{u}^{(2)}, \mathbf{x}^{(2)} | \mathbf{u}^{(1)}}(\mathbf{a}_u^{(2)}, \mathbf{a}_x^{(2)} | \mathbf{a}_u^{(1)})} [I(\mathbf{u}^{(2)}; \mathbf{r}^{(2)}) - I(\mathbf{u}^{(1)}; \mathbf{u}^{(2)})] \\ &\equiv \max_{p_{\mathbf{u}^{(2)}, \mathbf{x}^{(2)} | \mathbf{x}^{(1)}}(\mathbf{a}_u^{(2)}, \mathbf{a}_x^{(2)} | \mathbf{a}_x^{(1)})} [I(\mathbf{u}^{(2)}; \mathbf{r}^{(2)}) - I(\mathbf{x}^{(1)}; \mathbf{u}^{(2)})] \end{aligned} \quad (\text{B.2})$$

It is known by [119, Lemma 3.1] (or [120, Theorem 3]), which extends Costa's DPC result from [52] to the vector case, that $R_{DPC}^{(2)}(\Sigma^{(1)}, \Sigma^{(2)})$ is equal to the mutual information of user 2 as if the interfering term $\mathbf{H}^{(2,1)} \cdot \mathbf{x}^{(1)}$ is not present in (4.1b), i.e. $R_{DPC}^{(2)}(\Sigma^{(1)}, \Sigma^{(2)}) = I(\mathbf{x}^{(2)}; \mathbf{r}^{(2)})$ with $\mathbf{r}^{(2)} = \mathbf{H}^{(2)} \cdot \mathbf{x}^{(2)} + \mathbf{v}^{(2)}$. For the sake of completeness, we prove this claim using an auxiliary random vector $\mathbf{u}^{(2)}$ similar to that used in the proof of [119, Lemma 3.1]. Denoting $\mathbf{G}^{(2)} = [\mathbf{H}^{(2)} \cdot \Sigma^{(2)} \cdot [\mathbf{H}^{(2)}]^\dagger + N_0 \mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}}]^{-1}$, we define

$$\mathbf{u}^{(2)} = \mathbf{x}^{(2)} + \mathbf{G}_1^{(2)} \cdot \mathbf{x}^{(1)} \quad (\text{B.3a})$$

$$\mathbf{G}_1^{(2)} = \Sigma^{(2)} \cdot [\mathbf{H}^{(2)}]^\dagger \cdot \mathbf{G}^{(2)} \cdot \mathbf{H}^{(2,1)} \quad (\text{B.3b})$$

with $\mathcal{R}_u^{(2)} = \mathbb{C}^{\bar{D}^{(2)}}$. Since $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are zero mean Gaussian and statistically independent, $\mathbf{u}^{(2)}$ is also zero mean Gaussian. Let $\Sigma_u^{(2)}$ denote the covariance matrix of $\mathbf{u}^{(2)}$. From (B.3a), we have $\Sigma_u^{(2)} = \Sigma^{(2)} + \mathbf{G}_1^{(2)} \cdot \Sigma^{(1)} \cdot [\mathbf{G}_1^{(2)}]^\dagger$. We can write

$$\begin{aligned} I(\mathbf{x}^{(1)}; \mathbf{u}^{(2)}) &= h(\mathbf{u}^{(2)}) - h(\mathbf{u}^{(2)} | \mathbf{x}^{(1)}) \\ &= h(\mathbf{u}^{(2)}) - h(\mathbf{x}^{(2)} + \mathbf{G}_1^{(2)} \cdot \mathbf{x}^{(1)} | \mathbf{x}^{(1)}) \\ &\stackrel{(a)}{=} h(\mathbf{u}^{(2)}) - h(\mathbf{x}^{(2)} | \mathbf{x}^{(1)}) \\ &\stackrel{(b)}{=} h(\mathbf{u}^{(2)}) - h(\mathbf{x}^{(2)}) = \log \frac{|\Sigma_u^{(2)}|}{|\Sigma^{(2)}|} \end{aligned}$$

where (a) is due to the invariance of the differential entropy with translation [51, Theorem 9.6.3], and (b) is due to the independence of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$. From (4.1b), $\mathbf{r}^{(2)}$ is zero mean Gaussian (being the sum of independent zero mean Gaussian vectors) with covariance matrix

$$\Sigma_r^{(2)} = \mathbf{H}^{(2,1)} \cdot \Sigma^{(1)} \cdot [\mathbf{H}^{(2,1)}]^\dagger + \mathbf{H}^{(2)} \cdot \Sigma^{(2)} \cdot [\mathbf{H}^{(2)}]^\dagger + N_0 \mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}} \quad (\text{B.4})$$

which is Hermitian positive definite and hence, nonsingular. Define $\Sigma_{u,r}^{(2)} = E[\mathbf{u}^{(2)} \cdot [\mathbf{r}^{(2)}]^\dagger]$.

Using (4.1b) and (B.3a), we can write

$$\Sigma_{\mathbf{u},\mathbf{r}}^{(2)} = \mathbf{G}_1^{(2)} \cdot \Sigma^{(1)} \cdot [\mathbf{H}^{(2,1)}]^\dagger + \Sigma^{(2)} \cdot [\mathbf{H}^{(2)}]^\dagger \stackrel{(a)}{=} \Sigma^{(2)} \cdot [\mathbf{H}^{(2)}]^\dagger \cdot \mathbf{G}^{(2)} \cdot \Sigma_{\mathbf{r}}^{(2)} \quad (\text{B.5})$$

where (a) is due to (B.3b) and (B.4). Also, let $\Sigma_{\mathbf{r},\mathbf{u}}^{(2)} = E[\mathbf{r}^{(2)} \cdot [\mathbf{u}^{(2)}]^\dagger] = [\Sigma_{\mathbf{u},\mathbf{r}}^{(2)}]^\dagger$. Writing $I(\mathbf{u}^{(2)}; \mathbf{r}^{(2)}) = h(\mathbf{u}^{(2)}) + h(\mathbf{r}^{(2)}) - h(\mathbf{u}^{(2)}, \mathbf{r}^{(2)})$, we obtain

$$I(\mathbf{u}^{(2)}; \mathbf{r}^{(2)}) = \log \frac{|\Sigma_{\mathbf{r}}^{(2)}| |\Sigma_{\mathbf{u}}^{(2)}|}{\begin{vmatrix} \Sigma_{\mathbf{u}}^{(2)} & \Sigma_{\mathbf{u},\mathbf{r}}^{(2)} \\ \Sigma_{\mathbf{r},\mathbf{u}}^{(2)} & \Sigma_{\mathbf{r}}^{(2)} \end{vmatrix}} \stackrel{(a)}{=} \log \frac{|\Sigma_{\mathbf{u}}^{(2)}|}{|\Sigma_{\mathbf{u}}^{(2)} - \Sigma_{\mathbf{u},\mathbf{r}}^{(2)} \cdot [\Sigma_{\mathbf{r}}^{(2)}]^{-1} \cdot \Sigma_{\mathbf{r},\mathbf{u}}^{(2)}|}$$

where (a) is due to (C.8). Consequently, we can write

$$I(\mathbf{u}^{(2)}; \mathbf{r}^{(2)}) - I(\mathbf{x}^{(1)}; \mathbf{u}^{(2)}) = \log \frac{|\Sigma^{(2)}|}{|\Sigma_{\mathbf{u}}^{(2)} - \Sigma_{\mathbf{u},\mathbf{r}}^{(2)} \cdot [\Sigma_{\mathbf{r}}^{(2)}]^{-1} \cdot \Sigma_{\mathbf{r},\mathbf{u}}^{(2)}|} \quad (\text{B.6})$$

Using (B.4), (B.5) and $\Sigma_{\mathbf{r},\mathbf{u}}^{(2)} = [\Sigma_{\mathbf{u},\mathbf{r}}^{(2)}]^\dagger$, one can easily show that

$$\begin{aligned} \Sigma_{\mathbf{u},\mathbf{r}}^{(2)} \cdot [\Sigma_{\mathbf{r}}^{(2)}]^{-1} \cdot \Sigma_{\mathbf{r},\mathbf{u}}^{(2)} &= \Sigma^{(2)} \cdot [\mathbf{H}^{(2)}]^\dagger \cdot \mathbf{G}^{(2)} \cdot \mathbf{H}^{(2)} \cdot \Sigma^{(2)} \\ &+ \Sigma^{(2)} \cdot [\mathbf{H}^{(2)}]^\dagger \cdot \mathbf{G}^{(2)} \cdot \mathbf{H}^{(2,1)} \cdot \Sigma^{(1)} \cdot [\mathbf{H}^{(2,1)}]^\dagger \cdot \mathbf{G}^{(2)} \cdot \mathbf{H}^{(2)} \cdot \Sigma^{(2)} \end{aligned} \quad (\text{B.7})$$

Also, using (B.3), we have

$$\Sigma_{\mathbf{u}}^{(2)} = \Sigma^{(2)} + \Sigma^{(2)} \cdot [\mathbf{H}^{(2)}]^\dagger \cdot \mathbf{G}^{(2)} \cdot \mathbf{H}^{(2,1)} \cdot \Sigma^{(1)} \cdot [\mathbf{H}^{(2,1)}]^\dagger \cdot \mathbf{G}^{(2)} \cdot \mathbf{H}^{(2)} \cdot \Sigma^{(2)} \quad (\text{B.8})$$

Hence, using (B.7) and (B.8), we can write

$$\begin{aligned} |\Sigma_{\mathbf{u}}^{(2)} - \Sigma_{\mathbf{u},\mathbf{r}}^{(2)} \cdot [\Sigma_{\mathbf{r}}^{(2)}]^{-1} \cdot \Sigma_{\mathbf{r},\mathbf{u}}^{(2)}| &= |\Sigma^{(2)} - \Sigma^{(2)} \cdot [\mathbf{H}^{(2)}]^\dagger \cdot \mathbf{G}^{(2)} \cdot \mathbf{H}^{(2)} \cdot \Sigma^{(2)}| \\ &\stackrel{(a)}{=} |\Sigma^{(2)}| |\mathbf{I}_{\tilde{D}^{(2)}} - [\mathbf{H}^{(2)}]^\dagger \cdot \mathbf{G}^{(2)} \cdot \mathbf{H}^{(2)} \cdot \Sigma^{(2)}| \\ &\stackrel{(b)}{=} \frac{|\Sigma^{(2)}| |N_0 \mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}}|}{|\mathbf{H}^{(2)} \cdot \Sigma^{(2)} \cdot [\mathbf{H}^{(2)}]^\dagger + N_0 \mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}}|} \end{aligned}$$

where (a) is due to (C.10) and (b) is obtained using (C.9). It follows from (B.6) that

$$I(\mathbf{u}^{(2)}; \mathbf{r}^{(2)}) - I(\mathbf{x}^{(1)}; \mathbf{u}^{(2)}) = \log \left| \mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}} + \frac{1}{N_0} \mathbf{H}^{(2)} \cdot \Sigma^{(2)} \cdot [\mathbf{H}^{(2)}]^\dagger \right|$$

which is the mutual information $I(\mathbf{x}^{(2)}; \mathbf{r}^{(2)})$ for user 2 (given $\Sigma^{(2)}$) as if the interfering term

$\mathbf{H}^{(2,1)} \cdot \mathbf{x}^{(1)}$ is not present in (4.1b), i.e. as if $\mathbf{r}^{(2)} = \mathbf{H}^{(2)} \cdot \mathbf{x}^{(2)} + \mathbf{v}^{(2)}$. Thus, the assumption that $\mathbf{u}^{(2)}$ is of the form of (B.3a) where $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are Gaussian and statistically independent turns out to be without loss of generality. Using the notation $R_{DPC}^{(2)}(\Sigma^{(2)})$ instead of $R_{DPC}^{(2)}(\Sigma^{(1)}, \Sigma^{(2)})$, (B.2) becomes

$$R_{DPC}^{(2)}(\Sigma^{(2)}) = \log |\mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}} + \frac{1}{N_0} \mathbf{H}^{(2)} \cdot \Sigma^{(2)} \cdot [\mathbf{H}^{(2)}]^\dagger|$$

which is (4.2b). Based on the achievability of Marton's region (B.1) associated with $\mathbf{u}^{(1)} = \mathbf{x}^{(1)}$ and with $\mathbf{u}^{(2)}$ of (B.3), all the rate pairs $(R^{(1)}, R^{(2)})$ satisfying $R^{(1)} \leq R_{DPC}^{(1)}(\Sigma^{(1)}, \Sigma^{(2)})$ and $R^{(2)} \leq R_{DPC}^{(2)}(\Sigma^{(2)})$ are achievable. We conclude that the rate region $\mathcal{R}_{1,2}(\Sigma^{(1)}, \Sigma^{(2)})$ of (4.4) is achievable. Similarly, we can prove the achievability of the rate region $\mathcal{R}_{2,1}(\Sigma^{(1)}, \Sigma^{(2)})$ of (4.5) by reversing the users encoding order. $\square$

Proof of Theorem 4.3.1: The proof follows closely that of Theorem 3.1.1 and Corollary 3.1.1. Fix the space-time modulation format for user 2. We start by showing that $\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ of (4.19) is maximum when user 1 employs ATA. Taking $\Sigma^{(2)} = \Sigma_{\max}^{(2)}$ and using (2.41) and (C.1), the input/output mutual information for user 1 associated with (4.1a) can be written as

$$I(\mathbf{x}^{(1)}; \mathbf{r}^{(1)}) = \log |\mathbf{I}_{N_T D^{(1)}} + \frac{1}{N_0} \{[\mathbf{A}^{(1)}]^T \cdot \Sigma^{(1)} \cdot \mathbf{A}^{(1)}\} \cdot ([\mathbf{C}^{(1)}]^\dagger \otimes \mathbf{I}_{D^{(1)}}) \cdot [\Psi^{(1)}]^{-1} \cdot (\mathbf{C}^{(1)} \otimes \mathbf{I}_{D^{(1)}})| \quad (\text{B.9})$$

Since the space-time modulation format for user 2 is fixed, the matrix $\Psi^{(1)}$ (4.17a) is also fixed. The $N_T D^{(1)} \times N_T D^{(1)}$ matrix $\Sigma_y^{(1)} = [\mathbf{A}^{(1)}]^T \cdot \Sigma^{(1)} \cdot \mathbf{A}^{(1)}$ is the covariance matrix of $\mathbf{y}^{(1)} = [\mathbf{A}^{(1)}]^T \cdot \mathbf{x}^{(1)}$. Thus, using the mapping $f_{\mathbf{A}^{(1)}}(\mathbf{X}) = [\mathbf{A}^{(1)}]^T \cdot \mathbf{X} \cdot \mathbf{A}^{(1)}$, $\Sigma_y^{(1)} = f_{\mathbf{A}^{(1)}}(\Sigma^{(1)})$ is the image of $\Sigma^{(1)}$ and (B.9) becomes

$$I(\mathbf{x}^{(1)}; \mathbf{r}^{(1)}) = \log |\mathbf{I}_{N_T D^{(1)}} + \frac{1}{N_0} \Sigma_y^{(1)} \cdot ([\mathbf{C}^{(1)}]^\dagger \otimes \mathbf{I}_{D^{(1)}}) \cdot [\Psi^{(1)}]^{-1} \cdot (\mathbf{C}^{(1)} \otimes \mathbf{I}_{D^{(1)}})| \quad (\text{B.10})$$

Since $\mathbf{A}^{(1)} = \text{diag}_{i=1}^{N_T} [\mathbf{e}_{t_i^{(1)}; N_G^{(1)}}^T \otimes \mathbf{I}_{D_G^{(1)}}]$, we can show using identity (C.3) that $\mathbf{A}^{(1)} \cdot [\mathbf{A}^{(1)}]^T = \mathbf{I}_{\tilde{D}^{(1)}}$. Using (C.11), the power constraint for user 1 becomes $\text{Tr}(\Sigma_y^{(1)}) = \text{Tr}(\Sigma^{(1)}) \leq \alpha_p^{(1)} P_T$. Let $\mathcal{A}_x^{(1)}$ and $\mathcal{A}_y^{(1)}$ denote respectively the sets of $\tilde{D}^{(1)} \times \tilde{D}^{(1)}$ and $N_T D^{(1)} \times N_T D^{(1)}$ covariance matrices $\Sigma^{(1)}$ and $\Sigma_y^{(1)}$ that satisfy this power constraint. It can be shown that the mapping $f_{\mathbf{A}^{(1)}}$ is a mapping of $\mathcal{A}_x^{(1)}$ into $\mathcal{A}_y^{(1)}$, i.e. $f_{\mathbf{A}^{(1)}}(\mathcal{A}_x^{(1)}) \subseteq \mathcal{A}_y^{(1)}$. Equality is achieved only when $\mathbf{A}^{(1)}$ is invertible, corresponding to the ATA space-time modulation format for which we have $\mathbf{A}^{(1)} = \mathbf{I}_{N_T D^{(1)}}$. Therefore, the maximization of

(B.9) over $\Sigma_x^{(1)} \in \mathcal{A}_x^{(1)}$ cannot lead to a rate $\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ greater than that obtained from the maximization of (B.10) over $\Sigma_y^{(1)} \in \mathcal{A}_y^{(1)}$. We conclude that the rate $\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ achieved by ATA cannot be less than that achieved by any other multi-dimensional space-time modulation format.

We show now that $\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ of (4.19) cannot decrease by merging TOGs. Let $I_{(i)}^{(1)}$ and $P_{T;i}^{(1)}$ respectively denote the mutual information and power associated with TOG i of user 1 in the water-filling scheme corresponding to $\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)})$. This rate can be written (in bps/Hz) as $\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}) = \frac{1}{\bar{D}^{(1)}} \sum_{i=1}^{N_G^{(1)}} I_{(i)}^{(1)}$. Now, merging the m^{th} and p^{th} TOGs associated with user 1 results in a new TOG with mutual information $I_{(m+p)}^{(1)}$ and power $P_{T;m+p}^{(1)}$. Using the earlier statement that maximum mutual information is obtained with ATA, the maximum value $I_{(m+p)}^{(1)}$ obtained under the constraint $P_{T;m+p}^{(1)} \leq P_{T;m}^{(1)} + P_{T;p}^{(1)}$ cannot be less than $I_{(m)}^{(1)} + I_{(p)}^{(1)}$. Hence, we can write $\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}) \leq \frac{1}{\bar{D}^{(1)}} [\sum_{i=1, i \neq m, p}^{N_G^{(1)}} I_{(i)}^{(1)} + I_{(m+p)}^{(1)}] \leq \check{R}_{merged}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ where $\check{R}_{merged}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ denotes the rate (4.19) of the new system.

We conclude that $\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ is maximum, intermediate and minimum when user 1 employs ATA, POTA and OTA, respectively. Following similar steps, we can show that when the space-time modulation format of user 1 is fixed the rate $\check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ of (4.21) is maximum, intermediate and minimum when user 2 employs ATA, POTA and OTA, respectively. $\square$

Proof of Theorem 4.4.1: Take $(P_T/N_0) \rightarrow \infty$. We distinguish the following three cases:

- i) If $\alpha_p^{(1)} \rightarrow 0$ (and $\alpha_p^{(2)} \rightarrow 1$): Then $C^{(1)}(\alpha_p^{(1)}) \rightarrow 0$, and $C^{(2)}(\alpha_p^{(2)}) \rightarrow \frac{r_T^{(2)} D_G^{(2)}}{\bar{D}^{(2)}} \log\left(\frac{P_T}{N_0}\right)$ from (4.42), thus, yielding $(C^{(1)}(\alpha_p^{(1)}), C^{(2)}(\alpha_p^{(2)})) \rightarrow (0, \frac{r_T^{(2)} D_G^{(2)}}{\bar{D}^{(2)}} \log\left(\frac{P_T}{N_0}\right))$.
- ii) If $\alpha_p^{(2)} \rightarrow 0$ (and $\alpha_p^{(1)} \rightarrow 1$): Similarly to case i) above, $(C^{(1)}(\alpha_p^{(1)}), C^{(2)}(\alpha_p^{(2)})) \rightarrow (\frac{r_T^{(1)} D_G^{(1)}}{\bar{D}^{(1)}} \log\left(\frac{P_T}{N_0}\right), 0)$.
- iii) If $0 < \alpha_p^{(k)}$ with $(\alpha_p^{(k)} P_T)/N_0$ assumed to be very large for $k = 1, 2$: Then, from (4.42), $(C^{(1)}(\alpha_p^{(1)}), C^{(2)}(\alpha_p^{(2)})) \rightarrow (\frac{r_T^{(1)} D_G^{(1)}}{\bar{D}^{(1)}} \log\left(\frac{P_T}{N_0}\right), \frac{r_T^{(2)} D_G^{(2)}}{\bar{D}^{(2)}} \log\left(\frac{P_T}{N_0}\right)) = \tilde{V}$.

Hence, as $(P_T/N_0) \rightarrow \infty$ the boundary of $\mathcal{R}_{out}$ becomes characterized by the rate pairs $(0, \frac{r_T^{(2)} D_G^{(2)}}{\bar{D}^{(2)}} \log(P_T/N_0))$, $(\frac{r_T^{(1)} D_G^{(1)}}{\bar{D}^{(1)}} \log(P_T/N_0), 0)$ and $(\frac{r_T^{(1)} D_G^{(1)}}{\bar{D}^{(1)}} \log(P_T/N_0), \frac{r_T^{(2)} D_G^{(2)}}{\bar{D}^{(2)}} \log(P_T/N_0))$. Furthermore, from (4.12), $\mathcal{R}_{out}$ reduces to the set of rate pairs $(R^{(1)}, R^{(2)})$ such that $0 \leq$

$R^{(1)} \leq \frac{r_T^{(1)} D_G^{(1)}}{D^{(1)}} \log(P_T/N_0)$ and $0 \leq R^{(2)} \leq \frac{r_T^{(2)} D_G^{(2)}}{D^{(2)}} \log(P_T/N_0)$. Thus, the limiting outer bound $\mathcal{R}_{out}$ is a rectangle with upper right corner $\tilde{V}$. $\square$

Proof of Lemma 4.4.1: This lemma assumes $\alpha_p^{(k)} > 0$ and $(\alpha_p^{(k)} P_T)/N_0$ very large for $k = 1, 2$. We want to show that as long as these assumptions are satisfied, the ranks $r_{\max}^{(2,1)}$ and $r_{\max}^{(2)}$ are independent of the values of $\alpha_p^{(k)}$, $k = 1, 2$, and P_T/N_0 . Consider first the rank $r_{\max}^{(2,1)}$. Using $\Sigma_{\max}^{(1)}$ of (4.15) with $k = 1$, all the channel eigenmodes are active, that is, $\mu^{(1)}(\alpha_p^{(1)}) - (\lambda_{j,l}^{(1)})^{-1} > 0$ for all $l = 1, \dots, r_j^{(1)}$ and $j = 1, \dots, N_G^{(1)}$. Furthermore, with $\mu^{(1)}(\alpha_p^{(1)})$ satisfying (4.41), one can see that $\mu^{(1)}(\alpha_p^{(1)}) \gg (\lambda_{j,l}^{(1)})^{-1}$ and hence, we can write $\mu^{(1)}(\alpha_p^{(1)}) - (\lambda_{j,l}^{(1)})^{-1} \approx \mu^{(1)}(\alpha_p^{(1)})$, yielding

$$\begin{aligned} \Sigma_{\max}^{(1)} &\approx \text{diag}_{j=1}^{N_G^{(1)}} \left[[\mathbf{M}_j^{(1)}]^\dagger \cdot (N_0 \text{diag}[\mu^{(1)}(\alpha_p^{(1)}) \mathbf{I}_{r_j^{(1)}}, \mathbf{0}_{(n_j^{(1)}-r_j^{(1)}) \times (n_j^{(1)}-r_j^{(1)})}]) \cdot \mathbf{M}_j^{(1)} \right] \otimes \mathbf{I}_{D_G^{(1)}} \\ &\stackrel{(a)}{\approx} \frac{N_G^{(1)} \alpha_p^{(1)} P_T}{r_T^{(1)} D^{(1)}} \text{diag}_{j=1}^{N_G^{(1)}} \left[[\mathbf{M}_j^{(1)}]^\dagger \cdot \mathbf{B}_j^{(1)} \cdot \mathbf{M}_j^{(1)} \right] \otimes \mathbf{I}_{D_G^{(1)}} \end{aligned} \quad (\text{B.11})$$

where (a) is due to (4.41) and where $\mathbf{B}_j^{(1)} = \text{diag}[\mathbf{I}_{r_j^{(1)}}, \mathbf{0}_{(n_j^{(1)}-r_j^{(1)}) \times (n_j^{(1)}-r_j^{(1)})}]$. Hence, (B.11) yields

$$\begin{aligned} \mathbf{H}^{(2,1)} \cdot \left(\frac{1}{P_T} \Sigma_{\max}^{(1)} \right) \cdot [\mathbf{H}^{(2,1)}]^\dagger &\approx \\ \frac{N_G^{(1)} \alpha_p^{(1)}}{r_T^{(1)} D^{(1)}} \mathbf{H}^{(2,1)} \cdot \left\{ \text{diag}_{j=1}^{N_G^{(1)}} \left[[\mathbf{M}_j^{(1)}]^\dagger \cdot \mathbf{B}_j^{(1)} \cdot \mathbf{M}_j^{(1)} \right] \otimes \mathbf{I}_{D_G^{(1)}} \right\} \cdot [\mathbf{H}^{(2,1)}]^\dagger \end{aligned} \quad (\text{B.12})$$

Thus, using (C.13), $r_{\max}^{(2,1)}$ is equal to the rank of the matrix $\mathbf{H}^{(2,1)} \cdot \left\{ \text{diag}_{j=1}^{N_G^{(1)}} \left[[\mathbf{M}_j^{(1)}]^\dagger \cdot \mathbf{B}_j^{(1)} \cdot \mathbf{M}_j^{(1)} \right] \otimes \mathbf{I}_{D_G^{(1)}} \right\} \cdot [\mathbf{H}^{(2,1)}]^\dagger$ which is independent of $(\alpha_p^{(1)}, \alpha_p^{(2)})$ and P_T/N_0 since $\mathbf{M}_j^{(1)}, r_j^{(1)}$ (for $j = 1, \dots, N_G^{(1)}$) and $\mathbf{H}^{(2,1)}$ are independent of $(\alpha_p^{(1)}, \alpha_p^{(2)})$ and P_T/N_0 by definition. We conclude that $r_{\max}^{(2,1)}$ is the same for all the values of $(\alpha_p^{(1)}, \alpha_p^{(2)})$ and P_T/N_0 that satisfy $\alpha_p^{(k)} > 0$ with $(\alpha_p^{(k)} P_T)/N_0$ being very large for $k = 1, 2$.

Consider now the rank $r_{\max}^{(2)}$. If $r_{\max}^{(2,1)} = L_t^{(2)} N_R^{(2)} D^{(2)}$, then $r_{\max}^{(2)} = L_t^{(2)} N_R^{(2)} D^{(2)}$ since $L_t^{(2)} N_R^{(2)} D^{(2)} \geq r_{\max}^{(2)} \geq r_{\max}^{(2,1)}$ and hence, $r_{\max}^{(2)}$ is independent of $(\alpha_p^{(1)}, \alpha_p^{(2)})$ and P_T/N_0 . We assume $r_{\max}^{(2,1)} < L_t^{(2)} N_R^{(2)} D^{(2)}$ for the remainder of this proof. The rank $r_{\max}^{(2)}$ is the rank of the matrix $\mathbf{H}^{(2,1)} \cdot \left(\frac{1}{P_T} \Sigma_{\max}^{(1)} \right) \cdot [\mathbf{H}^{(2,1)}]^\dagger + \mathbf{H}^{(2)} \cdot \left(\frac{1}{P_T} \tilde{\Sigma}^{(2)} \right) \cdot [\mathbf{H}^{(2)}]^\dagger$. With $\mathbf{H}^{(2,1)} \cdot \left(\frac{1}{P_T} \Sigma_{\max}^{(1)} \right) \cdot [\mathbf{H}^{(2,1)}]^\dagger$ given by (B.12), it remains to find an expression for $\mathbf{H}^{(2)} \cdot \left(\frac{1}{P_T} \tilde{\Sigma}^{(2)} \right) \cdot [\mathbf{H}^{(2)}]^\dagger$ under the assumptions that $\alpha_p^{(k)} > 0$ and $(\alpha_p^{(k)} P_T)/N_0$ very large for $k = 1, 2$. Hence, we need to

find an expression for $\check{\Sigma}^{(2)}$ under these assumptions, using (4.24). For this, we first derive an expression for $\Psi^{(2)}$, and then we consider the eigenvalue decomposition of $[\mathbf{H}^{(2)}]^\dagger \cdot [\Psi^{(2)}]^{-1}$. $\mathbf{H}^{(2)}$ in order to evaluate $\check{\mu}^{(2)}(\alpha_p^{(2)})$, $\{\check{\lambda}_q^{(2)}\}_{q=1}^{\check{D}^{(2)}}$ and $\check{\mathbf{M}}^{(2)}$ of (4.24).

We start by deriving an expression for $\Psi^{(2)}$ under the assumptions that $\alpha_p^{(k)} > 0$ and $(\alpha_p^{(k)} P_T)/N_0$ very large for $k = 1, 2$. Consider the eigenvalue decomposition

$$\mathbf{H}^{(2,1)} \cdot \left\{ \text{diag}_{j=1}^{N_G^{(1)}} \left[[\mathbf{M}_j^{(1)}]^\dagger \cdot \mathbf{B}_j^{(1)} \cdot \mathbf{M}_j^{(1)} \right] \otimes \mathbf{I}_{D_G^{(1)}} \right\} \cdot [\mathbf{H}^{(2,1)}]^\dagger = [\bar{\mathbf{M}}^{(2,1)}]^\dagger \cdot \bar{\Lambda}^{(2,1)} \cdot \bar{\mathbf{M}}^{(2,1)} \quad (\text{B.13})$$

where $\bar{\mathbf{M}}^{(2,1)}$ is $L_t^{(2)} N_R^{(2)} D^{(2)} \times L_t^{(2)} N_R^{(2)} D^{(2)}$ unitary and $\bar{\Lambda}^{(2,1)}$ is $L_t^{(2)} N_R^{(2)} D^{(2)} \times L_t^{(2)} N_R^{(2)} D^{(2)}$ diagonal. Since the matrix on the left hand side of (B.13) is independent of $(\alpha_p^{(1)}, \alpha_p^{(2)})$ and P_T/N_0 , so are $\bar{\mathbf{M}}^{(2,1)}$ and $\bar{\Lambda}^{(2,1)}$. Using (B.12), (B.13) and (C.13), one can see that the rank of $\bar{\Lambda}^{(2,1)}$ is equal to $r_{\max}^{(2,1)}$ and hence, $\bar{\Lambda}^{(2,1)}$ has only $r_{\max}^{(2,1)}$ nonzero diagonal elements. Hence, using (B.12) and (B.13), (4.17b) becomes

$$\Psi^{(2)} = \mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}} + \frac{N_G^{(1)} \alpha_p^{(1)} P_T}{N_0 r_T^{(1)} D^{(1)}} [\bar{\mathbf{M}}^{(2,1)}]^\dagger \cdot \bar{\Lambda}^{(2,1)} \cdot \bar{\mathbf{M}}^{(2,1)} \quad (\text{B.14})$$

Based on (B.14), we now consider $[\mathbf{H}^{(2)}]^\dagger \cdot [\Psi^{(2)}]^{-1} \cdot \mathbf{H}^{(2)}$. Using (B.14) with $[\bar{\mathbf{M}}^{(2,1)}]^\dagger \cdot \bar{\mathbf{M}}^{(2,1)} = \mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}}$, we can write

$$[\Psi^{(2)}]^{-1} = [\bar{\mathbf{M}}^{(2,1)}]^\dagger \cdot \left[\mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}} + \frac{N_G^{(1)} \alpha_p^{(1)} P_T}{N_0 r_T^{(1)} D^{(1)}} \bar{\Lambda}^{(2,1)} \right]^{-1} \cdot \bar{\mathbf{M}}^{(2,1)}$$

where the dependence on $(\alpha_p^{(1)}, \alpha_p^{(2)})$ and P_T/N_0 is only through $\left[\mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}} + \frac{N_G^{(1)} \alpha_p^{(1)} P_T}{N_0 r_T^{(1)} D^{(1)}} \bar{\Lambda}^{(2,1)} \right]^{-1}$.

Since $\bar{\Lambda}^{(2,1)}$ is diagonal, the matrix $\left[\mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}} + \frac{N_G^{(1)} \alpha_p^{(1)} P_T}{N_0 r_T^{(1)} D^{(1)}} \bar{\Lambda}^{(2,1)} \right]$ is diagonal and so is its inverse, with $\left[\mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}} + \frac{N_G^{(1)} \alpha_p^{(1)} P_T}{N_0 r_T^{(1)} D^{(1)}} \bar{\Lambda}^{(2,1)} \right]_{p,p}^{-1} = \left[1 + \frac{N_G^{(1)} \alpha_p^{(1)} P_T}{N_0 r_T^{(1)} D^{(1)}} [\bar{\Lambda}^{(2,1)}]_{p,p} \right]^{-1}$. Thus, with $0 < \alpha_p^{(1)}$ and $(\alpha_p^{(1)} P_T)/N_0$ assumed to be very large, we can write $\left[\mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}} + \frac{N_G^{(1)} \alpha_p^{(1)} P_T}{N_0 r_T^{(1)} D^{(1)}} \bar{\Lambda}^{(2,1)} \right]_{p,p}^{-1} = 1$ if $[\bar{\Lambda}^{(2,1)}]_{p,p} = 0$ and $\left[\mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}} + \frac{N_G^{(1)} \alpha_p^{(1)} P_T}{N_0 r_T^{(1)} D^{(1)}} \bar{\Lambda}^{(2,1)} \right]_{p,p}^{-1} \approx 0$ else. Equivalently, we can write

$$\left[\mathbf{I}_{L_t^{(2)} N_R^{(2)} D^{(2)}} + \frac{N_G^{(1)} \alpha_p^{(1)} P_T}{N_0 r_T^{(1)} D^{(1)}} \bar{\Lambda}^{(2,1)} \right]^{-1} \approx \mathbf{D}^{(2,1)} \quad (\text{B.15})$$

where $\mathbf{D}^{(2,1)}$ is an $L_t^{(2)} N_R^{(2)} D^{(2)} \times L_t^{(2)} N_R^{(2)} D^{(2)}$ diagonal matrix defined as $[\mathbf{D}^{(2,1)}]_{p,p} = 1$

if $\frac{N_G^{(1)}}{r_T^{(1)} D^{(1)}} [\bar{\Lambda}^{(2,1)}]_{p,p} = 0$ and $[\mathbf{D}^{(2,1)}]_{p,p} = 0$ if $\frac{N_G^{(1)}}{r_T^{(1)} D^{(1)}} [\bar{\Lambda}^{(2,1)}]_{p,p} \neq 0$. From this, we have $[\Psi^{(2)}]^{-1} \approx [\bar{\mathbf{M}}^{(2,1)}]^\dagger \cdot \mathbf{D}^{(2,1)} \cdot \bar{\mathbf{M}}^{(2,1)}$. Note that this matrix is singular since $\mathbf{D}^{(2,1)}$ can not be full rank (because $\mathbf{D}^{(2,1)}$ is full rank only when $\bar{\Lambda}^{(2,1)}$ is zero) and hence, expression (4.22) for the rank of $[\mathbf{H}^{(k)}]^\dagger \cdot [\Psi^{(k)}]^{-1} \cdot \mathbf{H}^{(k)}$ does not hold in the limit when $(P_T/N_0) \rightarrow \infty$. We can write

$$[\mathbf{H}^{(2)}]^\dagger \cdot [\Psi^{(2)}]^{-1} \cdot \mathbf{H}^{(2)} \approx [\mathbf{H}^{(2)}]^\dagger \cdot [\bar{\mathbf{M}}^{(2,1)}]^\dagger \cdot \mathbf{D}^{(2,1)} \cdot \bar{\mathbf{M}}^{(2,1)} \cdot \mathbf{H}^{(2)} \quad (\text{B.16})$$

Now since $\bar{\Lambda}^{(2,1)}$ is independent of $(\alpha_p^{(1)}, \alpha_p^{(2)})$ and P_T/N_0 , the matrix $\mathbf{D}^{(2,1)}$ and, hence, $[\mathbf{H}^{(2)}]^\dagger \cdot [\bar{\mathbf{M}}^{(2,1)}]^\dagger \cdot \mathbf{D}^{(2,1)} \cdot \bar{\mathbf{M}}^{(2,1)} \cdot \mathbf{H}^{(2)}$ are also independent of $(\alpha_p^{(1)}, \alpha_p^{(2)})$ and P_T/N_0 . Let $\check{r}^{(2,1)}$ and $\{\check{\lambda}_q^{(2)}\}_{q=1}^{\check{D}^{(2)}}$ denote the rank and eigenvalues of $[\mathbf{H}^{(2)}]^\dagger \cdot [\bar{\mathbf{M}}^{(2,1)}]^\dagger \cdot \mathbf{D}^{(2,1)} \cdot \bar{\mathbf{M}}^{(2,1)} \cdot \mathbf{H}^{(2)}$, respectively. Also, let $\check{\mathbf{M}}^{(2)}$ denote the unitary matrix associated with the eigenvalue decomposition of this matrix. The matrix $\check{\mathbf{M}}^{(2)}$ and eigenvalues $\check{\lambda}_q^{(2)}$ are independent of $(\alpha_p^{(1)}, \alpha_p^{(2)})$ and P_T/N_0 . Furthermore, only $\check{r}^{(2,1)}$ eigenvalues $\check{\lambda}_q^{(2)}$ are nonzero. Using water-filling against the eigenvalues $\{\check{\lambda}_q^{(2)}\}_{q=1}^{\check{D}^{(2)}}$ with $0 < \alpha_p^{(2)}$ and $(\alpha_p^{(2)} P_T)/N_0$ assumed to be very large, the water-filling constant satisfies $\sum_{q=1}^{\check{r}^{(2,1)}} \{\check{\mu}^{(2)}(\alpha_p^{(2)}) - (\check{\lambda}_q^{(2)})^{-1}\} = \frac{\alpha_p^{(2)} P_T}{N_0}$ or, equivalently,

$$\check{\mu}^{(2)}(\alpha_p^{(2)}) = \frac{\alpha_p^{(2)} P_T}{N_0 \check{r}^{(2,1)}} + \frac{1}{\check{r}^{(2,1)}} \sum_{q=1}^{\check{r}^{(2,1)}} (\check{\lambda}_q^{(2)})^{-1} \stackrel{(a)}{\approx} \frac{\alpha_p^{(2)} P_T}{N_0 \check{r}^{(2,1)}} \quad (\text{B.17})$$

where (a) is due to $\frac{\alpha_p^{(2)} P_T}{N_0 \check{r}^{(2,1)}} \gg \frac{1}{\check{r}^{(2,1)}} \sum_{q=1}^{\check{r}^{(2,1)}} (\check{\lambda}_q^{(2)})^{-1}$. Using $\check{\mu}^{(2)}(\alpha_p^{(2)}) - (\check{\lambda}_q^{(2)})^{-1} \approx \frac{\alpha_p^{(2)} P_T}{N_0 \check{r}^{(2,1)}}$, the input covariance matrix of user 2 is given, as in (4.24), by

$$\check{\Sigma}^{(2)} \approx [\check{\mathbf{M}}^{(2)}]^\dagger \cdot \left(\frac{\alpha_p^{(2)} P_T}{\check{r}^{(2,1)}} \text{diag}[\mathbf{I}_{\check{r}^{(2,1)}}, \mathbf{0}_{(\check{D}^{(2)} - \check{r}^{(2,1)}) \times (\check{D}^{(2)} - \check{r}^{(2,1)})}] \right) \cdot \check{\mathbf{M}}^{(2)} \quad (\text{B.18})$$

Now that we have $\check{\Sigma}^{(2)}$, an expression for $\mathbf{H}^{(2)} \cdot \left(\frac{1}{P_T} \check{\Sigma}^{(2)} \right) \cdot [\mathbf{H}^{(2)}]^\dagger$ under the assumptions that $\alpha_p^{(k)} > 0$ and $(\alpha_p^{(k)} P_T)/N_0$ very large for $k = 1, 2$ can be written as follows

$$\begin{aligned} \mathbf{H}^{(2)} \cdot \left(\frac{1}{P_T} \check{\Sigma}^{(2)} \right) \cdot [\mathbf{H}^{(2)}]^\dagger &\approx \\ \frac{\alpha_p^{(2)}}{\check{r}^{(2,1)}} \mathbf{H}^{(2)} \cdot [\check{\mathbf{M}}^{(2)}]^\dagger \cdot \text{diag}[\mathbf{I}_{\check{r}^{(2,1)}}, \mathbf{0}_{(\check{D}^{(2)} - \check{r}^{(2,1)}) \times (\check{D}^{(2)} - \check{r}^{(2,1)})}] \cdot \check{\mathbf{M}}^{(2)} \cdot [\mathbf{H}^{(2)}]^\dagger \end{aligned} \quad (\text{B.19})$$

As such, using (B.12) and (B.19), we have

$$\mathbf{H}^{(2,1)} \cdot \left(\frac{1}{P_T} \Sigma_{\max}^{(1)} \right) \cdot [\mathbf{H}^{(2,1)}]^\dagger + \mathbf{H}^{(2)} \cdot \left(\frac{1}{P_T} \check{\Sigma}^{(2)} \right) \cdot [\mathbf{H}^{(2)}]^\dagger = \alpha_p^{(1)} \Pi_0^{(2)} + \alpha_p^{(2)} \Pi_1^{(2)} \quad (\text{B.20})$$

where

$$\begin{cases} \Pi_0^{(2)} = \frac{N_G^{(1)}}{r_T^{(1)} D^{(1)}} \mathbf{H}^{(2,1)} \cdot \left\{ \text{diag}_{j=1}^{N_G^{(1)}} \left[[\mathbf{M}_j^{(1)}]^\dagger \cdot \mathbf{B}_j^{(1)} \cdot \mathbf{M}_j^{(1)} \right] \otimes \mathbf{I}_{D_G^{(1)}} \right\} \cdot [\mathbf{H}^{(2,1)}]^\dagger \\ \Pi_1^{(2)} = \frac{1}{\check{r}^{(2,1)}} \mathbf{H}^{(2)} \cdot [\check{\mathbf{M}}^{(2)}]^\dagger \cdot \text{diag}[\mathbf{I}_{\check{r}^{(2,1)}}, \mathbf{0}_{(\check{D}^{(2)} - \check{r}^{(2,1)}) \times (\check{D}^{(2)} - \check{r}^{(2,1)})}] \cdot \check{\mathbf{M}}^{(2)} \cdot [\mathbf{H}^{(2)}]^\dagger \end{cases}$$

Both $\Pi_0^{(2)}$ and $\Pi_1^{(2)}$ are Hermitian positive semidefinite, and independent of $(\alpha_p^{(1)}, \alpha_p^{(2)})$ and P_T/N_0 . Using (B.20), we are now able to investigate the rank $r_{\max}^{(2)}$. A partial order relation between the matrices $\alpha_p^{(1)} \Pi_0^{(2)} + \alpha_p^{(2)} \Pi_1^{(2)}$, $\alpha_p^{(1)} [\Pi_0^{(2)} + \Pi_1^{(2)}]$ and $\alpha_p^{(2)} [\Pi_0^{(2)} + \Pi_1^{(2)}]$ can be obtained by distinguishing two cases: $\alpha_p^{(1)} \leq \alpha_p^{(2)}$ and $\alpha_p^{(1)} > \alpha_p^{(2)}$. If $\alpha_p^{(1)} \leq \alpha_p^{(2)}$, then $\alpha_p^{(2)} [\Pi_0^{(2)} + \Pi_1^{(2)}] \succeq \alpha_p^{(1)} \Pi_0^{(2)} + \alpha_p^{(2)} \Pi_1^{(2)} \succeq \alpha_p^{(1)} [\Pi_0^{(2)} + \Pi_1^{(2)}]$ and hence,

$$\alpha_p^{(2)} \lambda_q(\Pi_0^{(2)} + \Pi_1^{(2)}) \geq \lambda_q(\alpha_p^{(1)} \Pi_0^{(2)} + \alpha_p^{(2)} \Pi_1^{(2)}) \geq \alpha_p^{(1)} \lambda_q(\Pi_0^{(2)} + \Pi_1^{(2)}) \geq 0$$

from [106, Theorem F.1, p. 475] where $\lambda_q(\Pi_0^{(2)} + \Pi_1^{(2)})$ and $\lambda_q(\alpha_p^{(1)} \Pi_0^{(2)} + \alpha_p^{(2)} \Pi_1^{(2)})$ denote the q^{th} eigenvalues (in decreasing order) of the matrices $\Pi_0^{(2)} + \Pi_1^{(2)}$ and $\alpha_p^{(1)} \Pi_0^{(2)} + \alpha_p^{(2)} \Pi_1^{(2)}$, respectively. One can easily show that $\lambda_q(\alpha_p^{(1)} \Pi_0^{(2)} + \alpha_p^{(2)} \Pi_1^{(2)}) = 0$ if and only if $\lambda_q(\Pi_0^{(2)} + \Pi_1^{(2)}) = 0$. Since the rank of a Hermitian matrix is equal to the number of its nonzero eigenvalues [105, p. 175], we must have $r_{\max}^{(2)} \equiv \text{rank}(\alpha_p^{(1)} \Pi_0^{(2)} + \alpha_p^{(2)} \Pi_1^{(2)}) = \text{rank}(\Pi_0^{(2)} + \Pi_1^{(2)})$, which makes $r_{\max}^{(2)}$ independent of $(\alpha_p^{(1)}, \alpha_p^{(2)})$ and P_T/N_0 since $\Pi_0^{(2)}$ and $\Pi_1^{(2)}$ are independent of $(\alpha_p^{(1)}, \alpha_p^{(2)})$ and P_T/N_0 . This result can be obtained using a similar approach if $\alpha_p^{(1)} > \alpha_p^{(2)}$. We conclude that $r_{\max}^{(2)}$ is the same for all the values of $(\alpha_p^{(1)}, \alpha_p^{(2)})$ and P_T/N_0 that satisfy $\alpha_p^{(k)} > 0$ with $(\alpha_p^{(k)} P_T)/N_0$ being very large for $k = 1, 2$. $\square$

Proof of Lemma 4.5.1: In order to evaluate the rank $r_{\max}^{(2)}$, we start by finding an expression for the matrix $\mathbf{H}^{(2)} \cdot (\frac{1}{P_T} \check{\Sigma}^{(2)}) \cdot [\mathbf{H}^{(2)}]^\dagger$ using $\check{\Sigma}^{(2)}$ of (B.18). For this, we need to evaluate $[\check{\mathbf{M}}^{(2)}]^\dagger$ in (B.18) by evaluating the eigenvectors of the matrix $[\mathbf{H}^{(2)}]^\dagger \cdot [\bar{\mathbf{M}}^{(2,1)}]^\dagger \cdot \mathbf{D}^{(2,1)} \cdot \bar{\mathbf{M}}^{(2,1)} \cdot \mathbf{H}^{(2)}$ which are the columns of $[\check{\mathbf{M}}^{(2)}]^\dagger$. Based on the definition of $\mathbf{D}^{(2,1)}$ that we previously introduced in this appendix, the matrix product $[\bar{\mathbf{M}}^{(2,1)}]^\dagger \cdot \mathbf{D}^{(2,1)} \cdot \bar{\mathbf{M}}^{(2,1)}$ can be written as

$$[\bar{\mathbf{M}}^{(2,1)}]^\dagger \cdot \mathbf{D}^{(2,1)} \cdot \bar{\mathbf{M}}^{(2,1)} = \sum_{q=1}^{L_t^{(2)} N_R^{(2)} D^{(2)} - r_{\max}^{(2,1)}} \bar{\mathbf{m}}_q^{(2,1)} \cdot [\bar{\mathbf{m}}_q^{(2,1)}]^\dagger \quad (\text{B.21})$$

where $\bar{\mathbf{m}}_q^{(2,1)}$ denote the columns of $[\bar{\mathbf{M}}^{(2,1)}]^\dagger$, or the eigenvectors of $\mathbf{H}^{(2,1)} \cdot \left\{ \text{diag}_{j=1}^{N_G^{(1)}} \left[[\mathbf{M}_j^{(1)}]^\dagger \cdot \mathbf{B}_j^{(1)} \cdot \mathbf{M}_j^{(1)} \right] \otimes \mathbf{I}_{D_G^{(1)}} \right\} \cdot [\mathbf{H}^{(2,1)}]^\dagger$ (B.13), that are associated with the zero eigenvalues. Let us

now evaluate these vectors $\bar{\mathbf{m}}_q^{(2,1)}$. Using equations (B.12) and (4.64), we can write

$$\begin{aligned} & \mathbf{H}^{(2,1)} \cdot \left\{ \text{diag}_{j=1}^{N_G^{(1)}} \left[[\mathbf{M}_j^{(1)}]^\dagger \cdot \mathbf{B}_j^{(1)} \cdot \mathbf{M}_j^{(1)} \right] \otimes \mathbf{I}_{D_G^{(1)}} \right\} \cdot [\mathbf{H}^{(2,1)}]^\dagger \\ &= \sum_{j \in \mathcal{A}_{G;c}^{(1)}} \left[\mathbf{N}_j^{(2,1)} \otimes \{ \mathbf{B}_j^{(2,1)} \cdot [\mathbf{B}_j^{(2,1)}]^\dagger \} \right] \stackrel{(a)}{=} \sum_{j \in \mathcal{A}_{G;c}^{(1)}} \sum_{d=1}^{D_{c;j}^{(2)}} \mathbf{N}_j^{(2,1)} \otimes \mathbf{E}_{v^{(2)}(j,d)}^{(2)} \end{aligned} \quad (\text{B.22})$$

with (a) due to (4.65). We will show that some of the vectors $\bar{\mathbf{m}}_q^{(2,1)}$ can be determined by using the eigenvectors of $\mathbf{N}_j^{(2,1)}$ that are associated with zero eigenvalues. Since $\mathbf{N}_j^{(2,1)}$ is Hermitian positive semidefinite, then $\lambda_l(\mathbf{N}_j^{(2,1)}) > 0$ for $l = 1, \dots, \text{rank}(\mathbf{N}_j^{(2,1)})$ and $\lambda_l(\mathbf{N}_j^{(2,1)}) = 0$ for $l = \text{rank}(\mathbf{N}_j^{(2,1)}) + 1, \dots, L_t^{(2)} N_R^{(2)}$. Therefore, only the matrices $\mathbf{N}_j^{(2,1)}$ with $\text{rank}(\mathbf{N}_j^{(2,1)}) < L_t^{(2)} N_R^{(2)}$ or, equivalently with $j \in \tilde{\mathcal{A}}_{G;c}^{(1)}$, have zero eigenvalues. For every $\tilde{j} \in \tilde{\mathcal{A}}_{G;c}^{(1)}$ and every $\tilde{l} = \text{rank}(\mathbf{N}_{\tilde{j}}^{(2,1)}) + 1, \dots, L_t^{(2)} N_R^{(2)}$, the eigenvectors $\mathbf{n}_{\tilde{l}}(\mathbf{N}_{\tilde{j}}^{(2,1)})$ are associated with the zero eigenvalues of $\mathbf{N}_{\tilde{j}}^{(2,1)}$. Consequently, we can choose the first $\sum_{j \in \tilde{\mathcal{A}}_{G;c}^{(1)}} D_{c;j}^{(2)} (L_t^{(2)} N_R^{(2)} - \text{rank}(\mathbf{N}_j^{(2,1)}))$ vectors $\bar{\mathbf{m}}_q^{(2,1)}$ to be of the form $\mathbf{n}_{\tilde{l}}(\mathbf{N}_{\tilde{j}}^{(2,1)}) \otimes \mathbf{e}_{v^{(2)}(\tilde{j}, \tilde{d}); D^{(2)}}$ where $\tilde{l} = \text{rank}(\mathbf{N}_{\tilde{j}}^{(2,1)}) + 1, \dots, L_t^{(2)} N_R^{(2)}$, $\tilde{d} = 1, \dots, D_{c;\tilde{j}}^{(2)}$ and $\tilde{j} \in \tilde{\mathcal{A}}_{G;c}^{(1)}$. This is due to the fact that

$$\begin{aligned} & \mathbf{H}^{(2,1)} \cdot \left\{ \text{diag}_{j=1}^{N_G^{(1)}} \left[[\mathbf{M}_j^{(1)}]^\dagger \cdot \mathbf{B}_j^{(1)} \cdot \mathbf{M}_j^{(1)} \right] \otimes \mathbf{I}_{D_G^{(1)}} \right\} \cdot [\mathbf{H}^{(2,1)}]^\dagger \cdot [\mathbf{n}_{\tilde{l}}(\mathbf{N}_{\tilde{j}}^{(2,1)}) \otimes \mathbf{e}_{v^{(2)}(\tilde{j}, \tilde{d}); D^{(2)}}] \\ & \stackrel{(a)}{=} \left[\sum_{j \in \mathcal{A}_{G;c}^{(1)}} \sum_{d=1}^{D_{c;j}^{(2)}} \mathbf{N}_j^{(2,1)} \otimes \mathbf{E}_{v^{(2)}(j,d)}^{(2)} \right] \cdot [\mathbf{n}_{\tilde{l}}(\mathbf{N}_{\tilde{j}}^{(2,1)}) \otimes \mathbf{e}_{v^{(2)}(\tilde{j}, \tilde{d}); D^{(2)}}] \\ & \stackrel{(b)}{=} \sum_{j \in \mathcal{A}_{G;c}^{(1)}} \sum_{d=1}^{D_{c;j}^{(2)}} [\mathbf{N}_j^{(2,1)} \cdot \mathbf{n}_{\tilde{l}}(\mathbf{N}_{\tilde{j}}^{(2,1)})] \otimes [\mathbf{E}_{v^{(2)}(j,d)}^{(2)} \cdot \mathbf{e}_{v^{(2)}(\tilde{j}, \tilde{d}); D^{(2)}}] \\ & \stackrel{(c)}{=} [\mathbf{N}_{\tilde{j}}^{(2,1)} \cdot \mathbf{n}_{\tilde{l}}(\mathbf{N}_{\tilde{j}}^{(2,1)})] \otimes \mathbf{e}_{v^{(2)}(\tilde{j}, \tilde{d}); D^{(2)}} \stackrel{(d)}{=} \mathbf{0}_{L_t^{(2)} N_R^{(2)}} \otimes \mathbf{e}_{v^{(2)}(\tilde{j}, \tilde{d}); D^{(2)}} = \mathbf{0}_{L_t^{(2)} N_R^{(2)} D^{(2)}} \end{aligned}$$

where (a) is due to (B.22), (b) is due to (C.3), (c) is due to

$$\mathbf{E}_{v^{(2)}(j,d)}^{(2)} \cdot \mathbf{e}_{v^{(2)}(\tilde{j}, \tilde{d}); D^{(2)}} = \mathbf{e}_{v^{(2)}(j,d); D^{(2)}} \cdot [\mathbf{e}_{v^{(2)}(j,d); D^{(2)}}]^\dagger \cdot \mathbf{e}_{v^{(2)}(\tilde{j}, \tilde{d}); D^{(2)}} = \delta(j, \tilde{j}) \delta(d, \tilde{d}) \mathbf{e}_{v^{(2)}(j,d); D^{(2)}} \quad (\text{B.23})$$

and (d) is due to $\mathbf{N}_{\tilde{j}}^{(2,1)} \cdot \mathbf{n}_{\tilde{l}}(\mathbf{N}_{\tilde{j}}^{(2,1)}) = \mathbf{0}_{\mathbf{n}_{\tilde{l}}(\mathbf{N}_{\tilde{j}}^{(2,1)})} = \mathbf{0}_{L_t^{(2)} N_R^{(2)}}$. Hence, $\mathbf{n}_{\tilde{l}}(\mathbf{N}_{\tilde{j}}^{(2,1)}) \otimes \mathbf{e}_{v^{(2)}(\tilde{j}, \tilde{d}); D^{(2)}}$ is an eigenvector associated with a zero eigenvalue of the matrix $\mathbf{H}^{(2,1)} \cdot \left\{ \text{diag}_{j=1}^{N_G^{(1)}} \left[[\mathbf{M}_j^{(1)}]^\dagger \cdot \mathbf{B}_j^{(1)} \cdot \mathbf{M}_j^{(1)} \right] \otimes \mathbf{I}_{D_G^{(1)}} \right\} \cdot [\mathbf{H}^{(2,1)}]^\dagger$. Following similar steps, one can easily

show that the eigenvectors of this matrix which are associated with nonzero eigenvalues are of the form $\mathbf{n}_l(\mathbf{N}_j^{(2,1)}) \otimes \mathbf{e}_{v^{(2)}(j,d);D^{(2)}}$ with $l = 1, \dots, \text{rank}(\mathbf{N}_j^{(2,1)})$, $d = 1, \dots, D_{c;j}^{(2)}$ and $j \in \mathcal{A}_{G;c}^{(1)}$, and that these eigenvectors satisfy

$$\begin{aligned} & [\mathbf{n}_l(\mathbf{N}_j^{(2,1)}) \otimes \mathbf{e}_{v^{(2)}(\tilde{j},\tilde{d});D^{(2)}}]^\dagger \cdot [\mathbf{n}_l(\mathbf{N}_j^{(2,1)}) \otimes \mathbf{e}_{v^{(2)}(j,d);D^{(2)}}] \stackrel{(a)}{=} \\ & [\mathbf{n}_l^\dagger(\mathbf{N}_j^{(2,1)}) \cdot \mathbf{n}_l(\mathbf{N}_j^{(2,1)})] \otimes [\mathbf{e}_{v^{(2)}(\tilde{j},\tilde{d});D^{(2)}}^\dagger \cdot \mathbf{e}_{v^{(2)}(j,d);D^{(2)}}] \stackrel{(b)}{=} 0 \end{aligned}$$

with (a) due to (C.3), and (b) due to $\mathbf{e}_{v^{(2)}(\tilde{j},\tilde{d});D^{(2)}}^\dagger \cdot \mathbf{e}_{v^{(2)}(j,d);D^{(2)}} = 0$ if $(\tilde{j}, \tilde{d}) \neq (j, d)$ and to $\mathbf{n}_l^\dagger(\mathbf{N}_j^{(2,1)}) \cdot \mathbf{n}_l(\mathbf{N}_j^{(2,1)}) = 0$ if $(\tilde{j}, \tilde{d}) = (j, d)$. We now determine the remaining $L_t^{(2)} N_R^{(2)} D_p^{(2)}$ vectors $\bar{\mathbf{m}}_q^{(2,1)}$. We choose these vectors to be of the form $\mathbf{e}_{r;L_t^{(2)} N_R^{(2)}} \otimes \mathbf{e}_{\tilde{v}^{(2)}(s);D^{(2)}}$ for $r = 1, \dots, L_t^{(2)} N_R^{(2)}$ and $s = 1, \dots, D_p^{(2)}$. Let us now show that these are eigenvectors of $\mathbf{H}^{(2,1)} \cdot \left\{ \text{diag}_{j=1}^{N_G^{(1)}} \left[[\mathbf{M}_j^{(1)}]^\dagger \cdot \mathbf{B}_j^{(1)} \cdot \mathbf{M}_j^{(1)} \right] \otimes \mathbf{I}_{D_G^{(1)}} \right\} \cdot [\mathbf{H}^{(2,1)}]^\dagger$ associated with zero eigenvalues. Using (B.22) and (C.3), we can write

$$\begin{aligned} & \mathbf{H}^{(2,1)} \cdot \left\{ \text{diag}_{j=1}^{N_G^{(1)}} \left[[\mathbf{M}_j^{(1)}]^\dagger \cdot \mathbf{B}_j^{(1)} \cdot \mathbf{M}_j^{(1)} \right] \otimes \mathbf{I}_{D_G^{(1)}} \right\} \cdot [\mathbf{H}^{(2,1)}]^\dagger \cdot [\mathbf{e}_{r;L_t^{(2)} N_R^{(2)}} \otimes \mathbf{e}_{\tilde{v}^{(2)}(s);D^{(2)}}] = \\ & \sum_{j \in \mathcal{A}_{G;c}^{(1)}} \sum_{d=1}^{D_{c;j}^{(2)}} (\mathbf{N}_j^{(2,1)} \cdot \mathbf{e}_{r;L_t^{(2)} N_R^{(2)}}) \otimes (\mathbf{E}_{v^{(2)}(j,d)}^{(2)} \cdot \mathbf{e}_{\tilde{v}^{(2)}(s);D^{(2)}}) = \mathbf{0}_{L_t^{(2)} N_R^{(2)} D^{(2)}} \end{aligned}$$

where the last equality is due to

$$\mathbf{E}_{v^{(2)}(j,d)}^{(2)} \cdot \mathbf{e}_{\tilde{v}^{(2)}(s);D^{(2)}} = \mathbf{e}_{v^{(2)}(j,d);D^{(2)}} \cdot [\mathbf{e}_{v^{(2)}(j,d);D^{(2)}}]^\dagger \cdot \mathbf{e}_{\tilde{v}^{(2)}(s);D^{(2)}} = \mathbf{0}_{D^{(2)}} \quad (\text{B.24})$$

for all $j \in \mathcal{A}_{G;c}^{(1)}$, $d = 1, \dots, D_{c;j}^{(2)}$ and $s = 1, \dots, D_p^{(2)}$ since proper and shared signal dimensions have different indexes in the set $\{1, \dots, D^{(2)}\}$ (i.e. $v^{(2)}(j, d) \neq \tilde{v}^{(2)}(s)$). We finally show that the vectors $\mathbf{e}_{r;L_t^{(2)} N_R^{(2)}} \otimes \mathbf{e}_{\tilde{v}^{(2)}(s);D^{(2)}}$ are orthogonal to the remaining eigenvectors $\mathbf{n}_l(\mathbf{N}_j^{(2,1)}) \otimes \mathbf{e}_{v^{(2)}(j,d);D^{(2)}}$ where $l = 1, \dots, L_t^{(2)} N_R^{(2)}$, $d = 1, \dots, D_{c;j}^{(2)}$ and $j \in \mathcal{A}_{G;c}^{(1)}$. Using (C.3), we have

$$\begin{aligned} & [\mathbf{e}_{r;L_t^{(2)} N_R^{(2)}} \otimes \mathbf{e}_{\tilde{v}^{(2)}(s);D^{(2)}}]^\dagger \cdot [\mathbf{n}_l(\mathbf{N}_j^{(2,1)}) \otimes \mathbf{e}_{v^{(2)}(j,d);D^{(2)}}] = \\ & [\mathbf{e}_{r;L_t^{(2)} N_R^{(2)}}^\dagger \cdot \mathbf{n}_l(\mathbf{N}_j^{(2,1)})] \otimes [\mathbf{e}_{\tilde{v}^{(2)}(s);D^{(2)}}^\dagger \cdot \mathbf{e}_{v^{(2)}(j,d);D^{(2)}}] = 0 \end{aligned}$$

with the last equality due to $\tilde{v}^{(2)}(s) \neq v^{(2)}(j, d)$. Hence, we conclude that the vectors $\bar{\mathbf{m}}_q^{(2,1)}$ are given by $\{ \{ \mathbf{n}_l(\mathbf{N}_j^{(2,1)}) \otimes \mathbf{e}_{v^{(2)}(j,d);D^{(2)}} \}_{l=\text{rank}(\mathbf{N}_j^{(2,1)})+1}^{L_t^{(2)} N_R^{(2)}} \}_{d=1}^{D_{c;j}^{(2)}}, j \in \tilde{\mathcal{A}}_{G;c}^{(1)}$, and $\{ \{ \mathbf{e}_{r;L_t^{(2)} N_R^{(2)}} \otimes \mathbf{e}_{\tilde{v}^{(2)}(s);D^{(2)}} \}_{s=1}^{D_p^{(2)}} \}_{r=1}^{L_t^{(2)} N_R^{(2)}}$.

$\mathbf{e}_{\tilde{v}^{(2)}(s); D^{(2)}}\}_{r=1}^{L_t^{(2)} N_R^{(2)}}\}_{s=1}^{D_p^{(2)}} \cdot$ As such, using (C.3), (C.5) and $\sum_{r=1}^{L_t^{(2)} N_R^{(2)}} \mathbf{e}_{r; L_t^{(2)} N_R^{(2)}} \cdot \mathbf{e}_{r; L_t^{(2)} N_R^{(2)}}^\dagger = \mathbf{I}_{L_t^{(2)} N_R^{(2)}}$, (B.21) reduces to

$$\begin{aligned} [\bar{\mathbf{M}}^{(2,1)}]^\dagger \cdot \mathbf{D}^{(2,1)} \cdot \bar{\mathbf{M}}^{(2,1)} &= \mathbf{I}_{L_t^{(2)} N_R^{(2)}} \otimes \left[\sum_{s=1}^{D_p^{(2)}} \mathbf{E}_{\tilde{v}^{(2)}(s)}^{(2)} \right] \\ &+ \sum_{j \in \bar{\mathcal{A}}_{G;c}^{(1)}} \sum_{d=1}^{D_{c;j}^{(2)}} \sum_{l=\text{rank}(\mathbf{N}_j^{(2,1)})+1}^{L_t^{(2)} N_R^{(2)}} [\mathbf{n}_l(\mathbf{N}_j^{(2,1)}) \cdot \mathbf{n}_l^\dagger(\mathbf{N}_j^{(2,1)})] \otimes \mathbf{E}_{v^{(2)}(j,d)}^{(2)} \end{aligned} \quad (\text{B.25})$$

Equation (B.25) is now used to obtain an expression for the matrix $[\mathbf{H}^{(2)}]^\dagger \cdot [\bar{\mathbf{M}}^{(2,1)}]^\dagger \cdot \mathbf{D}^{(2,1)} \cdot \bar{\mathbf{M}}^{(2,1)} \cdot \mathbf{H}^{(2)}$ which can be used to evaluate the eigenvectors of this matrix and hence, the matrix $[\tilde{\mathbf{M}}^{(2)}]^\dagger$. Using (2.41) with $k = 2$, we have

$$\begin{aligned} &[\mathbf{H}^{(2)}]^\dagger \cdot [\bar{\mathbf{M}}^{(2,1)}]^\dagger \cdot \mathbf{D}^{(2,1)} \cdot \bar{\mathbf{M}}^{(2,1)} \cdot \mathbf{H}^{(2)} \\ &= \mathbf{A}^{(2)} \cdot [(\mathbf{C}^{(2)})^\dagger \otimes \mathbf{I}_{D^{(2)}}] \cdot [\bar{\mathbf{M}}^{(2,1)}]^\dagger \cdot \mathbf{D}^{(2,1)} \cdot \bar{\mathbf{M}}^{(2,1)} \cdot [\mathbf{C}^{(2)} \otimes \mathbf{I}_{D^{(2)}}] \cdot [\mathbf{A}^{(2)}]^T \end{aligned}$$

which, using (B.25) as well as identities (C.3) and (C.5b), leads to

$$\begin{aligned} &[\mathbf{H}^{(2)}]^\dagger \cdot [\bar{\mathbf{M}}^{(2,1)}]^\dagger \cdot \mathbf{D}^{(2,1)} \cdot \bar{\mathbf{M}}^{(2,1)} \cdot \mathbf{H}^{(2)} = \\ &\sum_{s=1}^{D_p^{(2)}} \mathbf{A}^{(2)} \cdot \left\{ [(\mathbf{C}^{(2)})^\dagger \cdot \mathbf{C}^{(2)}] \otimes \mathbf{E}_{\tilde{v}^{(2)}(s)}^{(2)} \right\} \cdot [\mathbf{A}^{(2)}]^T \\ &+ \sum_{j \in \bar{\mathcal{A}}_{G;c}^{(1)}} \sum_{d=1}^{D_{c;j}^{(2)}} \sum_{l=\text{rank}(\mathbf{N}_j^{(2,1)})+1}^{L_t^{(2)} N_R^{(2)}} \mathbf{A}^{(2)} \cdot \left\{ \tilde{\mathbf{N}}_{j;l}^{(2,1)} \otimes \mathbf{E}_{v^{(2)}(j,d)}^{(2)} \right\} \cdot [\mathbf{A}^{(2)}]^T \end{aligned} \quad (\text{B.26})$$

where $\tilde{\mathbf{N}}_{j;l}^{(2,1)}$ is defined in Section 4.5.2. Each term on the right hand side of (B.26) is of the form $\mathbf{A}^{(2)} \cdot \{\mathbf{X} \otimes \mathbf{E}_q^{(2)}\} \cdot [\mathbf{A}^{(2)}]^T$ where $\mathbf{X} = [\mathbf{X}_{a,b}]$ is an $N_T \times N_T$ matrix composed of $(N_G^{(2)})^2$ submatrices $\mathbf{X}_{a,b}$ of size $n_a^{(2)} \times n_b^{(2)}$, $a, b = 1, \dots, N_G^{(2)}$. That is, the submatrix $\mathbf{X}_{a,b}$ contains the components of $\mathbf{X}$ that belong to rows $l = \sum_{i=1}^{a-1} n_i^{(2)} + 1, \dots, \sum_{i=1}^a n_i^{(2)}$ if $a \neq 1$ (or $l = 1, \dots, n_a^{(2)}$ if $a = 1$) and to columns $p = \sum_{i=1}^{b-1} n_i^{(2)} + 1, \dots, \sum_{i=1}^b n_i^{(2)}$ if $b \neq 1$ (or $p = 1, \dots, n_b^{(2)}$ if $b = 1$). By writing $\mathbf{A}^{(2)} = \text{diag}_{i=1}^{N_T} [\mathbf{e}_{t_i^{(2)}; N_G^{(2)}}^T \otimes \mathbf{I}_{D_G^{(2)}}] = \text{diag}_{j=1}^{N_G^{(2)}} [\mathbf{I}_{n_j^{(2)}} \otimes (\mathbf{e}_{j; N_G^{(2)}}^T \otimes \mathbf{I}_{D_G^{(2)}})]$, we can show that $\mathbf{A}^{(2)} \cdot \{\mathbf{X} \otimes \mathbf{E}_q^{(2)}\} \cdot [\mathbf{A}^{(2)}]^T = [\tilde{\mathbf{X}}_{a,b}]$

with the $n_a^{(2)} D_G^{(2)} \times n_b^{(2)} D_G^{(2)}$ matrix defined by

$$\tilde{\mathbf{X}}_{a,b} = \mathbf{X}_{a,b} \otimes [(\mathbf{e}_{a;N_G^{(2)}}^T \otimes \mathbf{I}_{D_G^{(2)}}) \cdot \mathbf{E}_q^{(2)} \cdot (\mathbf{e}_{b;N_G^{(2)}} \otimes \mathbf{I}_{D_G^{(2)}})] = \mathbf{X}_{a,b} \otimes \delta(a,b) \mathbf{E}_{a;q}^{(2)}$$

where the $D_G^{(2)} \times D_G^{(2)}$ matrix $\mathbf{E}_{a;q}^{(2)}$ denotes the a^{th} diagonal block of $\mathbf{E}_q^{(2)} = \text{diag}_{j=1}^{N_G^{(2)}} [\mathbf{E}_{j;q}^{(2)}]$, and $(\mathbf{e}_{a;N_G^{(2)}}^T \otimes \mathbf{I}_{D_G^{(2)}}) \cdot \mathbf{E}_q^{(2)} \cdot (\mathbf{e}_{a;N_G^{(2)}} \otimes \mathbf{I}_{D_G^{(2)}}) = \mathbf{E}_{a;q}^{(2)}$. We obtain

$$\mathbf{A}^{(2)} \cdot \{\mathbf{X} \otimes \mathbf{E}_q^{(2)}\} \cdot [\mathbf{A}^{(2)}]^T = \text{diag}_{a=1}^{N_G^{(2)}} [\tilde{\mathbf{X}}_{a,a}] = \text{diag}_{a=1}^{N_G^{(2)}} [\mathbf{X}_{a,a} \otimes \mathbf{E}_{a;q}^{(2)}] \quad (\text{B.27})$$

Using the fact that the diagonal blocks of the matrices $(\mathbf{C}^{(2)})^\dagger \cdot \mathbf{C}^{(2)}$ and $\tilde{\mathbf{N}}_{j;l}^{(2,1)}$ are given by $(\mathbf{C}_a^{(2)})^\dagger \cdot \mathbf{C}_a^{(2)}$ and $\tilde{\mathbf{N}}_{j;l;a}^{(2,1)}$ (which is defined in Section 4.5.2), respectively, (B.27) can be used to write (B.26) as follows

$$\begin{aligned} [\mathbf{H}^{(2)}]^\dagger \cdot [\tilde{\mathbf{M}}^{(2,1)}]^\dagger \cdot \mathbf{D}^{(2,1)} \cdot \tilde{\mathbf{M}}^{(2,1)} \cdot \mathbf{H}^{(2)} &= \sum_{s=1}^{D_p^{(2)}} \text{diag}_{a=1}^{N_G^{(2)}} \left[[(\mathbf{C}_a^{(2)})^\dagger \cdot \mathbf{C}_a^{(2)}] \otimes \mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)} \right] \\ &+ \sum_{j \in \tilde{\mathcal{A}}_{G;c}^{(1)}} \sum_{d=1}^{D_{c;j}^{(2)}} \sum_{l=\text{rank}(\mathbf{N}_j^{(2,1)})+1}^{L_t^{(2)} N_R^{(2)}} \text{diag}_{a=1}^{N_G^{(2)}} \left[\tilde{\mathbf{N}}_{j;l;a}^{(2,1)} \otimes \mathbf{E}_{a;v^{(2)}(j,d)}^{(2)} \right] \end{aligned}$$

which, using (C.5a) and the matrix $\check{\mathbf{N}}_{j;a}^{(2,1)}$ defined in Section 4.5.2, is equivalent to

$$\begin{aligned} &[\mathbf{H}^{(2)}]^\dagger \cdot [\tilde{\mathbf{M}}^{(2,1)}]^\dagger \cdot \mathbf{D}^{(2,1)} \cdot \tilde{\mathbf{M}}^{(2,1)} \cdot \mathbf{H}^{(2)} = \\ &\text{diag}_{a=1}^{N_G^{(2)}} \left[\sum_{s=1}^{D_p^{(2)}} [(\mathbf{C}_a^{(2)})^\dagger \cdot \mathbf{C}_a^{(2)}] \otimes \mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)} + \sum_{j \in \tilde{\mathcal{A}}_{G;c}^{(1)}} \sum_{d=1}^{D_{c;j}^{(2)}} \check{\mathbf{N}}_{j;a}^{(2,1)} \otimes \mathbf{E}_{a;v^{(2)}(j,d)}^{(2)} \right] \quad (\text{B.28}) \end{aligned}$$

From (B.28), the matrix $[\mathbf{H}^{(2)}]^\dagger \cdot [\tilde{\mathbf{M}}^{(2,1)}]^\dagger \cdot \mathbf{D}^{(2,1)} \cdot \tilde{\mathbf{M}}^{(2,1)} \cdot \mathbf{H}^{(2)}$ is block diagonal, with diagonal block a ($a = 1, \dots, N_G^{(2)}$) being given by

$$\mathbf{R}_a^{(2,1)} \triangleq \sum_{s=1}^{D_p^{(2)}} [(\mathbf{C}_a^{(2)})^\dagger \cdot \mathbf{C}_a^{(2)}] \otimes \mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)} + \sum_{j \in \tilde{\mathcal{A}}_{G;c}^{(1)}} \sum_{d=1}^{D_{c;j}^{(2)}} \check{\mathbf{N}}_{j;a}^{(2,1)} \otimes \mathbf{E}_{a;v^{(2)}(j,d)}^{(2)} \quad (\text{B.29})$$

Based on their definitions, $\mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)}$ and $\mathbf{E}_{a;v^{(2)}(j,d)}^{(2)}$ are diagonal matrices with at most one diagonal element equal to one while the remaining diagonal elements are equal to zero.

Using the fact that the Kronecker product of two Hermitian positive semidefinite matrices is also Hermitian positive semidefinite from [106, 19.G.2.a], each term on the right hand side of (B.29) is Hermitian positive semidefinite. Therefore, the $n_a^{(2)} D_G^{(2)} \times n_a^{(2)} D_G^{(2)}$ matrix $\mathbf{R}_a^{(2,1)}$ is also Hermitian positive semidefinite. As such, the unitary matrix $\check{\mathbf{M}}^{(2)}$ in the eigenvalue decomposition of $[\mathbf{H}^{(2)}]^\dagger \cdot [\bar{\mathbf{M}}^{(2,1)}]^\dagger \cdot \mathbf{D}^{(2,1)} \cdot \bar{\mathbf{M}}^{(2,1)} \cdot \mathbf{H}^{(2)} \equiv \text{diag}_{a=1}^{N_G^{(2)}} [\mathbf{R}_a^{(2,1)}]$ can be easily shown to be also block diagonal, $\check{\mathbf{M}}^{(2)} = \text{diag}_{a=1}^{N_G^{(2)}} [\check{\mathbf{M}}_a^{(2)}]$ where $\check{\mathbf{M}}_a^{(2)}$ is the $n_a^{(2)} D_G^{(2)} \times n_a^{(2)} D_G^{(2)}$ unitary matrix in the eigenvalue decomposition $\mathbf{R}_a^{(2,1)} = [\check{\mathbf{M}}_a^{(2)}]^\dagger \cdot \text{diag}_{l=1}^{n_a^{(2)} D_G^{(2)}} [\lambda_l(\mathbf{R}_a^{(2,1)})] \cdot \check{\mathbf{M}}_a^{(2)}$. Moreover, the rank $\check{r}^{(2,1)}$ of $[\mathbf{H}^{(2)}]^\dagger \cdot [\bar{\mathbf{M}}^{(2,1)}]^\dagger \cdot \mathbf{D}^{(2,1)} \cdot \bar{\mathbf{M}}^{(2,1)} \cdot \mathbf{H}^{(2)}$ satisfies $\check{r}^{(2,1)} = \sum_{a=1}^{N_G^{(2)}} \check{r}_a^{(2,1)}$ where $\check{r}_a^{(2,1)}$ denotes the rank of $\mathbf{R}_a^{(2,1)}$. We use the matrix $\check{\mathbf{M}}^{(2)}$ to determine the input covariance matrix of user 2 given by (B.18) or, equivalently,

$$\check{\Sigma}^{(2)} \approx \frac{\alpha_p^{(2)} P_T}{\check{r}^{(2,1)}} \sum_{l=1}^{\check{r}^{(2,1)}} [(\check{\mathbf{M}}^{(2)})^\dagger]_{*,l} \cdot [(\check{\mathbf{M}}^{(2)})^\dagger]_{*,l}^\dagger \quad (\text{B.30})$$

where $[(\check{\mathbf{M}}^{(2)})^\dagger]_{*,l}$ denote the column vectors of $(\check{\mathbf{M}}^{(2)})^\dagger$ that are associated with nonzero eigenvalues. Since $\check{\mathbf{M}}^{(2)} = \text{diag}_{a=1}^{N_G^{(2)}} [\check{\mathbf{M}}_a^{(2)}]$, (B.30) can be put in the form

$$\check{\Sigma}^{(2)} \approx \frac{\alpha_p^{(2)} P_T}{\check{r}^{(2,1)}} \text{diag}_{a=1}^{N_G^{(2)}} \left[\sum_{l=1}^{\check{r}_a^{(2,1)}} [(\check{\mathbf{M}}_a^{(2)})^\dagger]_{*,l} \cdot [(\check{\mathbf{M}}_a^{(2)})^\dagger]_{*,l}^\dagger \right] \equiv \frac{\alpha_p^{(2)} P_T}{\check{r}^{(2,1)}} \text{diag}_{a=1}^{N_G^{(2)}} [\check{\Sigma}_a^{(2)}] \quad (\text{B.31})$$

where $\check{\Sigma}_a^{(2)} \triangleq \sum_{l=1}^{\check{r}_a^{(2,1)}} [(\check{\mathbf{M}}_a^{(2)})^\dagger]_{*,l} \cdot [(\check{\mathbf{M}}_a^{(2)})^\dagger]_{*,l}^\dagger$ and where the columns $[(\check{\mathbf{M}}_a^{(2)})^\dagger]_{*,l}$ are the eigenvectors of $\mathbf{R}_a^{(2,1)}$ associated with nonzero eigenvalues.

For $a = 1, \dots, N_G^{(2)}$, we now evaluate the rank $\check{r}_a^{(2,1)}$ of $\mathbf{R}_a^{(2,1)}$ and the eigenvectors $[(\check{\mathbf{M}}_a^{(2)})^\dagger]_{*,l}$. For $s = 1, \dots, D_p^{(2)}$, since $\mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)}$ is the a^{th} diagonal block of the matrix $\mathbf{E}_{\tilde{v}^{(2)}(s)}^{(2)} \equiv \mathbf{e}_{\tilde{v}^{(2)}(s);D^{(2)}} \cdot \mathbf{e}_{\tilde{v}^{(2)}(s);D^{(2)}}^\dagger$, we have $\mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)} = \mathbf{0}_{D_G^{(2)} \times D_G^{(2)}}$ if signal dimension $\tilde{v}^{(2)}(s)$ is not allocated to TOG a of user 2. Otherwise, $\tilde{v}^{(2)}(s)$ corresponds to a proper signal dimension that is allocated to TOG a of user 2, and there exists an integer $s' \in \{1, \dots, D_G^{(2)}\}$ such that $\mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)} = \mathbf{e}_{s';D_G^{(2)}} \cdot \mathbf{e}_{s';D_G^{(2)}}^\dagger$ with

$$\begin{cases} \mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)} \cdot \mathbf{e}_{s';D_G^{(2)}} = \mathbf{e}_{s';D_G^{(2)}} \\ \mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)} \cdot \mathbf{e}_{n;D_G^{(2)}} = \mathbf{0}_{D_G^{(2)}}, \quad \text{for all } n \in \{1, \dots, D_G^{(2)}\} \text{ with } n \neq s' \end{cases} \quad (\text{B.32})$$

Let $S_{P,a}^{(2)} \subseteq \{1, \dots, D_G^{(2)}\}$ denote the subset of such integers s' . Since there are $D_{p,a}^{(2)}$ proper

signal dimensions allocated to TOG a of user 2, the cardinality of $\mathcal{S}_{P;a}^{(2)}$ is given by $|\mathcal{S}_{P;a}^{(2)}| = D_{p;a}^{(2)}$. Similarly, we have $\mathbf{E}_{a;v^{(2)}(j,d)}^{(2)} = \mathbf{0}_{D_G^{(2)} \times D_G^{(2)}}$ if the shared signal dimension $v^{(2)}(j,d)$ is not allocated to TOG a of user 2. Otherwise, $v^{(2)}(j,d)$ corresponds to a signal dimension that is shared between TOG j ($j \in \mathcal{A}_{G;c}^{(1)}$) of user 1 and TOG a of user 2, and there exists an integer $d' \in \{1, \dots, D_G^{(2)}\}$ such that $\mathbf{E}_{a;v^{(2)}(j,d)}^{(2)} = \mathbf{e}_{d';D_G^{(2)}} \cdot \mathbf{e}_{d';D_G^{(2)}}^\dagger$ with

$$\begin{cases} \mathbf{E}_{a;v^{(2)}(j,d)}^{(2)} \cdot \mathbf{e}_{d';D_G^{(2)}} = \mathbf{e}_{d';D_G^{(2)}} \\ \mathbf{E}_{a;v^{(2)}(j,d)}^{(2)} \cdot \mathbf{e}_{n;D_G^{(2)}} = \mathbf{0}_{D_G^{(2)}}, \quad \text{for all } n \in \{1, \dots, D_G^{(2)}\} \text{ with } n \neq d' \end{cases} \quad (\text{B.33})$$

Let $\mathcal{S}_{j;a}^{(2)} \subseteq \{1, \dots, D_G^{(2)}\}$ denote the subset of such integers d' . Since there are $D_{c;j;a}^{(2)}$ signal dimensions shared between TOG j of user 1 and TOG a of user 2, the cardinality of $\mathcal{S}_{j;a}^{(2)}$ is given by $|\mathcal{S}_{j;a}^{(2)}| = D_{c;j;a}^{(2)}$. One can easily show that for each a we have $\mathcal{S}_{P;a}^{(2)} \cap (\bigcap_{j \in \mathcal{A}_{G;c}^{(1)}} \mathcal{S}_{j;a}^{(2)}) = \emptyset$. Moreover, the matrices $\mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)}$ and $\mathbf{E}_{a;v^{(2)}(j,d)}^{(2)}$ that are nonzero (i.e. satisfying $\mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)} = \mathbf{e}_{s';D_G^{(2)}} \cdot \mathbf{e}_{s';D_G^{(2)}}^\dagger$ for some $s' \in \mathcal{S}_{P;a}^{(2)}$ and $\mathbf{E}_{a;v^{(2)}(j,d)}^{(2)} = \mathbf{e}_{d';D_G^{(2)}} \cdot \mathbf{e}_{d';D_G^{(2)}}^\dagger$ for some $d' \in \mathcal{S}_{j;a}^{(2)}$) have their column (and row) spaces essentially disjoint since different s' and different d' have different values in the set $\{1, \dots, D_G^{(2)}\} \equiv \mathcal{S}_{P;a}^{(2)} \cup (\bigcup_{j \in \mathcal{A}_{G;c}^{(1)}} \mathcal{S}_{j;a}^{(2)})$. It follows that $\mathbf{R}_a^{(2,1)}$ (B.29) is a summation of matrices with essentially disjoint column (and row) spaces, for which the rank additivity results from [114, Theorem 18.5.7] yield

$$\begin{aligned} \check{r}_a^{(2,1)} &= \sum_{s=1}^{D_p^{(2)}} \text{rank} \left([(\mathbf{C}_a^{(2)})^\dagger \cdot \mathbf{C}_a^{(2)}] \otimes \mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)} \right) + \sum_{j \in \tilde{\mathcal{A}}_{G;c}^{(1)}} \sum_{d=1}^{D_{c;j}^{(2)}} \text{rank} \left(\check{\mathbf{N}}_{j;a}^{(2,1)} \otimes \mathbf{E}_{a;v^{(2)}(j,d)}^{(2)} \right) \\ &\stackrel{(a)}{=} \sum_{s=1}^{D_p^{(2)}} \text{rank}((\mathbf{C}_a^{(2)})^\dagger \cdot \mathbf{C}_a^{(2)}) \text{rank}(\mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)}) + \sum_{j \in \tilde{\mathcal{A}}_{G;c}^{(1)}} \sum_{d=1}^{D_{c;j}^{(2)}} \text{rank}(\check{\mathbf{N}}_{j;a}^{(2,1)}) \text{rank}(\mathbf{E}_{a;v^{(2)}(j,d)}^{(2)}) \\ &\stackrel{(b)}{=} D_{p;a}^{(2)} r_a^{(2)} + \sum_{j \in \tilde{\mathcal{A}}_{G;c}^{(1)}} D_{c;j;a}^{(2)} \text{rank}(\check{\mathbf{N}}_{j;a}^{(2,1)}) \end{aligned}$$

where (a) is due to (C.16), and (b) is due to $\text{rank}((\mathbf{C}_a^{(2)})^\dagger \cdot \mathbf{C}_a^{(2)}) = r_a^{(2)}$ and to the fact that only $D_{p;a}^{(2)}$ out of the $D_p^{(2)}$ matrices $\mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)}$ are nonzero and with rank one, and only $D_{c;j;a}^{(2)}$ out of the $D_{c;j}^{(2)}$ matrices $\mathbf{E}_{a;v^{(2)}(j,d)}^{(2)}$ are nonzero and with rank one for each $j \in \tilde{\mathcal{A}}_{G;c}^{(1)}$. Now let $\mathbf{n}_l((\mathbf{C}_a^{(2)})^\dagger \cdot \mathbf{C}_a^{(2)})$ (for $l = 1, \dots, r_a^{(2)}$) and $\mathbf{n}_q(\check{\mathbf{N}}_{j;a}^{(2,1)})$ (for $q = 1, \dots, \text{rank}(\check{\mathbf{N}}_{j;a}^{(2,1)})$) denote, respectively, the eigenvectors of $[(\mathbf{C}_a^{(2)})^\dagger \cdot \mathbf{C}_a^{(2)}]$ and $\check{\mathbf{N}}_{j;a}^{(2,1)}$ that are associated with nonzero

eigenvalues. Consider the $r_a^{(2)} D_{p;a}^{(2)}$ vectors $\mathbf{n}_l([\mathbf{C}_a^{(2)}]^\dagger \cdot \mathbf{C}_a^{(2)}) \otimes \mathbf{e}_{s'; D_G^{(2)}}$ where $l = 1, \dots, r_a^{(2)}$ and $s' \in \mathcal{S}_{P;a}^{(2)}$. Using (B.29), we have

$$\begin{aligned}
& \mathbf{R}_a^{(2,1)} \cdot [\mathbf{n}_l([\mathbf{C}_a^{(2)}]^\dagger \cdot \mathbf{C}_a^{(2)}) \otimes \mathbf{e}_{s'; D_G^{(2)}}] \\
& \stackrel{(a)}{=} \sum_{s=1}^{D_p^{(2)}} \left\{ [(\mathbf{C}_a^{(2)})^\dagger \cdot \mathbf{C}_a^{(2)}] \cdot [\mathbf{n}_l([\mathbf{C}_a^{(2)}]^\dagger \cdot \mathbf{C}_a^{(2)})] \right\} \otimes \{ \mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)} \cdot \mathbf{e}_{s'; D_G^{(2)}} \} \\
& \quad + \sum_{j \in \tilde{\mathcal{A}}_{G;c}^{(1)}} \sum_{d=1}^{D_{c;j}^{(2)}} \left\{ \check{\mathbf{N}}_{j;a}^{(2,1)} \cdot \mathbf{n}_l([\mathbf{C}_a^{(2)}]^\dagger \cdot \mathbf{C}_a^{(2)}) \right\} \otimes \{ \mathbf{E}_{a;\tilde{v}^{(2)}(j,d)}^{(2)} \cdot \mathbf{e}_{s'; D_G^{(2)}} \} \\
& \stackrel{(b)}{=} \lambda_{a;l}^{(2)} \mathbf{n}_l([\mathbf{C}_a^{(2)}]^\dagger \cdot \mathbf{C}_a^{(2)}) \otimes \mathbf{e}_{s'; D_G^{(2)}} \\
& \quad + \sum_{j \in \tilde{\mathcal{A}}_{G;c}^{(1)}} \sum_{d=1}^{D_{c;j}^{(2)}} \left\{ \check{\mathbf{N}}_{j;a}^{(2,1)} \cdot \mathbf{n}_l([\mathbf{C}_a^{(2)}]^\dagger \cdot \mathbf{C}_a^{(2)}) \right\} \otimes \{ \mathbf{E}_{a;\tilde{v}^{(2)}(j,d)}^{(2)} \cdot \mathbf{e}_{s'; D_G^{(2)}} \} \\
& \stackrel{(c)}{=} \lambda_{a;l}^{(2)} \mathbf{n}_l([\mathbf{C}_a^{(2)}]^\dagger \cdot \mathbf{C}_a^{(2)}) \otimes \mathbf{e}_{s'; D_G^{(2)}}
\end{aligned}$$

where (a) is due to (C.3), (b) is due to $\mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)} \cdot \mathbf{e}_{s'; D_G^{(2)}} = \mathbf{e}_{s'; D_G^{(2)}}$ if $\tilde{v}^{(2)}(s)$ and s' correspond to the same proper signal dimension and, otherwise, to $\mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)} \cdot \mathbf{e}_{s'; D_G^{(2)}} = \mathbf{0}_{D_G^{(2)}}$ from (B.32), and (c) is due to $\mathbf{E}_{a;\tilde{v}^{(2)}(j,d)}^{(2)} \cdot \mathbf{e}_{s'; D_G^{(2)}} = \mathbf{0}_{D_G^{(2)}}$ based on (B.33) with the fact that $s' \notin \mathcal{S}_{j;a}^{(2)}$ for all j . Hence, the first $r_a^{(2)} D_{p;a}^{(2)}$ vectors $[(\mathbf{M}_a^{(2)})^\dagger]_{*,l}$ are of the form $\mathbf{n}_l([\mathbf{C}_a^{(2)}]^\dagger \cdot \mathbf{C}_a^{(2)}) \otimes \mathbf{e}_{s'; D_G^{(2)}}$ and are associated with the nonzero eigenvalues $\lambda_{a;l}^{(2)}$, each with a multiplicity of $D_{p;a}^{(2)}$. Moreover, we have

$$\begin{aligned}
& \sum_{l=1}^{r_a^{(2)}} \sum_{s' \in \mathcal{S}_{P;a}^{(2)}} [\mathbf{n}_l([\mathbf{C}_a^{(2)}]^\dagger \cdot \mathbf{C}_a^{(2)}) \cdot \mathbf{n}_l^\dagger([\mathbf{C}_a^{(2)}]^\dagger \cdot \mathbf{C}_a^{(2)})] \otimes [\mathbf{e}_{s'; D_G^{(2)}} \cdot \mathbf{e}_{s'; D_G^{(2)}}^\dagger] \\
& \stackrel{(a)}{=} \left[\sum_{l=1}^{r_a^{(2)}} \mathbf{n}_l([\mathbf{C}_a^{(2)}]^\dagger \cdot \mathbf{C}_a^{(2)}) \cdot \mathbf{n}_l^\dagger([\mathbf{C}_a^{(2)}]^\dagger \cdot \mathbf{C}_a^{(2)}) \right] \otimes \left[\sum_{s' \in \mathcal{S}_{P;a}^{(2)}} \mathbf{e}_{s'; D_G^{(2)}} \cdot \mathbf{e}_{s'; D_G^{(2)}}^\dagger \right] \quad (\text{B.34}) \\
& \stackrel{(b)}{=} [\mathbf{M}_a^{(2)}]^\dagger \cdot \mathbf{B}_a^{(2)} \cdot \mathbf{M}_a^{(2)} \otimes \left[\sum_{s=1}^{D_p^{(2)}} \mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)} \right]
\end{aligned}$$

where (a) is due to (C.5), and (b) is due to $\sum_{l=1}^{r_a^{(2)}} \mathbf{n}_l([\mathbf{C}_a^{(2)}]^\dagger \cdot \mathbf{C}_a^{(2)}) \cdot \mathbf{n}_l^\dagger([\mathbf{C}_a^{(2)}]^\dagger \cdot \mathbf{C}_a^{(2)}) \equiv$

$[\mathbf{M}_a^{(2)}]^\dagger \cdot \mathbf{B}_a^{(2)} \cdot \mathbf{M}_a^{(2)}$ and to the fact that $\sum_{s=1}^{D_p^{(2)}} \mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)} = \sum_{s' \in \mathcal{S}_{P,a}^{(2)}} \mathbf{e}_{s';D_G^{(2)}} \cdot \mathbf{e}_{s';D_G^{(2)}}^\dagger$ since $\mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)} = \mathbf{e}_{s';D_G^{(2)}} \cdot \mathbf{e}_{s';D_G^{(2)}}^\dagger$ if $\tilde{v}^{(2)}(s)$ corresponds to the proper signal dimension $s' \in \mathcal{S}_{P,a}^{(2)}$ and $\mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)} = \mathbf{0}_{D_G^{(2)} \times D_G^{(2)}}$ otherwise. Using (B.29), (B.32)-(B.33) and following similar steps, we can show that the remaining $\sum_{j \in \tilde{\mathcal{A}}_{G;c}^{(1)}} D_{c;j,a}^{(2)} \text{rank}(\check{\mathbf{N}}_{j;a}^{(2,1)})$ vectors $[(\check{\mathbf{M}}_a^{(2)})^\dagger]_{*,l}$ are of the form $\mathbf{n}_q(\check{\mathbf{N}}_{j;a}^{(2,1)}) \otimes \mathbf{e}_{d';D_G^{(2)}}$ (for $q = 1, \dots, \text{rank}(\check{\mathbf{N}}_{j;a}^{(2,1)})$, $d' \in \mathcal{S}_{j;a}^{(2)}$ and $j \in \tilde{\mathcal{A}}_{G;c}^{(1)}$) and are associated with the eigenvalues $\lambda_q(\check{\mathbf{N}}_{j;a}^{(2,1)})$, each with a multiplicity of $D_{c;j,a}^{(2)}$. Moreover, using the matrix $\check{\mathbf{W}}_{j;a}^{(2,1)}$ defined in Section 4.5.2, we have

$$\begin{aligned}
 & \sum_{j \in \tilde{\mathcal{A}}_{G;c}^{(1)}} \sum_{q=1}^{\text{rank}(\check{\mathbf{N}}_{j;a}^{(2,1)})} \sum_{d' \in \mathcal{S}_{j;a}^{(2)}} [\mathbf{n}_q(\check{\mathbf{N}}_{j;a}^{(2,1)}) \cdot \mathbf{n}_q^\dagger(\check{\mathbf{N}}_{j;a}^{(2,1)})] \otimes [\mathbf{e}_{d';D_G^{(2)}} \cdot \mathbf{e}_{d';D_G^{(2)}}^\dagger] \\
 & \stackrel{(a)}{=} \sum_{j \in \tilde{\mathcal{A}}_{G;c}^{(1)}} \left[\sum_{q=1}^{\text{rank}(\check{\mathbf{N}}_{j;a}^{(2,1)})} \mathbf{n}_q(\check{\mathbf{N}}_{j;a}^{(2,1)}) \cdot \mathbf{n}_q^\dagger(\check{\mathbf{N}}_{j;a}^{(2,1)}) \right] \otimes \left[\sum_{d' \in \mathcal{S}_{j;a}^{(2)}} \mathbf{e}_{d';D_G^{(2)}} \cdot \mathbf{e}_{d';D_G^{(2)}}^\dagger \right] \quad (\text{B.35}) \\
 & \stackrel{(b)}{=} \sum_{j \in \tilde{\mathcal{A}}_{G;c}^{(1)}} \check{\mathbf{W}}_{j;a}^{(2,1)} \otimes \left[\sum_{d=1}^{D_{c;j}^{(2)}} \mathbf{E}_{a;v^{(2)}(j,d)}^{(2)} \right]
 \end{aligned}$$

where (a) is due to (C.5), and (b) is due to the fact that $\sum_{d=1}^{D_{c;j}^{(2)}} \mathbf{E}_{a;v^{(2)}(j,d)}^{(2)} = \sum_{d' \in \mathcal{S}_{j;a}^{(2)}} \mathbf{e}_{d';D_G^{(2)}} \cdot \mathbf{e}_{d';D_G^{(2)}}^\dagger$ since $\mathbf{E}_{a;v^{(2)}(j,d)}^{(2)} = \mathbf{e}_{d';D_G^{(2)}} \cdot \mathbf{e}_{d';D_G^{(2)}}^\dagger$ if $v^{(2)}(j,d)$ corresponds to the shared signal dimension $d' \in \mathcal{S}_{j;a}^{(2)}$ and $\mathbf{E}_{a;v^{(2)}(j,d)}^{(2)} = \mathbf{0}_{D_G^{(2)} \times D_G^{(2)}}$ otherwise. Hence, the matrix $\check{\Sigma}_a^{(2)} \triangleq \sum_{l=1}^{\check{r}_a^{(2,1)}} [(\check{\mathbf{M}}_a^{(2)})^\dagger]_{*,l} \cdot [(\check{\mathbf{M}}_a^{(2)})^\dagger]_{*,l}^\dagger$ can be written using (B.34) and (B.35) as

$$\check{\Sigma}_a^{(2)} = ([\mathbf{M}_a^{(2)}]^\dagger \cdot \mathbf{B}_a^{(2)} \cdot \mathbf{M}_a^{(2)}) \otimes \left[\sum_{s=1}^{D_p^{(2)}} \mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)} \right] + \sum_{j \in \tilde{\mathcal{A}}_{G;c}^{(1)}} \check{\mathbf{W}}_{j;a}^{(2,1)} \otimes \left[\sum_{d=1}^{D_{c;j}^{(2)}} \mathbf{E}_{a;v^{(2)}(j,d)}^{(2)} \right] \quad (\text{B.36})$$

Now using (2.41) with $\mathbf{A}^{(2)} = \text{diag}_{a=1}^{N_G^{(2)}} [\mathbf{I}_{n_a^{(2)}} \otimes (\mathbf{e}_{a;N_G^{(2)}}^T \otimes \mathbf{I}_{D_G^{(2)}})]$ and (B.31), we obtain

$$\begin{aligned}
 \mathbf{H}^{(2)} \cdot \left(\frac{1}{P_T} \check{\Sigma}^{(2)} \right) \cdot [\mathbf{H}^{(2)}]^\dagger &= \frac{\alpha_p^{(2)}}{\check{r}^{(2,1)}} [\mathbf{C}^{(2)} \otimes \mathbf{I}_{D^{(2)}}] \cdot \text{diag}_{a=1}^{N_G^{(2)}} [\tilde{\mathbf{A}}_a^{(2)}] \cdot [(\mathbf{C}^{(2)})^\dagger \otimes \mathbf{I}_{D^{(2)}}] \\
 &\stackrel{(a)}{=} \frac{\alpha_p^{(2)}}{\check{r}^{(2,1)}} \sum_{a=1}^{N_G^{(2)}} (\mathbf{C}_a^{(2)} \otimes \mathbf{I}_{D^{(2)}}) \cdot \tilde{\mathbf{A}}_a^{(2)} \cdot [(\mathbf{C}_a^{(2)})^\dagger \otimes \mathbf{I}_{D^{(2)}}] \quad (\text{B.37})
 \end{aligned}$$

with $\tilde{\Lambda}_a^{(2)} \triangleq \{\mathbf{I}_{n_a^{(2)}} \otimes (\mathbf{e}_{a;N_G^{(2)}} \otimes \mathbf{I}_{D_G^{(2)}})\} \cdot \check{\Sigma}_a^{(2)} \cdot \{\mathbf{I}_{n_a^{(2)}} \otimes (\mathbf{e}_{a;N_G^{(2)}}^T \otimes \mathbf{I}_{D_G^{(2)}})\}$ and (a) due to $\mathbf{C}^{(2)} \otimes \mathbf{I}_{D^{(2)}} = [(\mathbf{C}_1^{(2)} \otimes \mathbf{I}_{D^{(2)}}) \cdots (\mathbf{C}_{N_G^{(2)}}^{(2)} \otimes \mathbf{I}_{D^{(2)}})]$. For any $q = 1, \dots, D^{(2)}$ and for any $a = 1, \dots, N_G^{(2)}$, we can show that

$$(\mathbf{e}_{a;N_G^{(2)}} \otimes \mathbf{I}_{D_G^{(2)}}) \cdot \mathbf{E}_{a;q}^{(2)} \cdot (\mathbf{e}_{a;N_G^{(2)}}^T \otimes \mathbf{I}_{D_G^{(2)}}) = \text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a;q}^{(2)}] \quad (\text{B.38})$$

The matrix on the right hand side of (B.38) is a $D^{(2)} \times D^{(2)}$ block diagonal matrix where the a^{th} diagonal block is equal to $\mathbf{E}_{a;q}^{(2)}$ and the other diagonal blocks are equal to $\mathbf{0}_{D_G^{(2)} \times D_G^{(2)}}$. Equations (B.36), (B.38) and identity (C.3) can be used to put $\tilde{\Lambda}_a^{(2)}$ in the following form

$$\begin{aligned} \tilde{\Lambda}_a^{(2)} &= \sum_{s=1}^{D_p^{(2)}} ([\mathbf{M}_a^{(2)}]^\dagger \cdot \mathbf{B}_a^{(2)} \cdot \mathbf{M}_a^{(2)}) \otimes \text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a;\check{v}^{(2)}(s)}^{(2)}] \\ &\quad + \sum_{j \in \tilde{\mathcal{A}}_{G;c}^{(1)}} \sum_{d=1}^{D_{c;j}^{(2)}} \check{\mathbf{W}}_{j;a}^{(2,1)} \otimes \text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a;\check{v}^{(2)}(j,d)}^{(2)}] \end{aligned}$$

Using this expression and identity (C.3), (B.37) becomes

$$\begin{aligned} &\mathbf{H}^{(2)} \cdot \left(\frac{1}{P_T} \check{\Sigma}^{(2)} \right) \cdot [\mathbf{H}^{(2)}]^\dagger \\ &= \frac{\alpha_p^{(2)}}{\check{r}^{(2,1)}} \sum_{a=1}^{N_G^{(2)}} \sum_{s=1}^{D_p^{(2)}} (\mathbf{C}_a^{(2)} \cdot [\mathbf{M}_a^{(2)}]^\dagger \cdot \mathbf{B}_a^{(2)} \cdot \mathbf{M}_a^{(2)} \cdot [\mathbf{C}_a^{(2)}]^\dagger) \otimes \text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a;\check{v}^{(2)}(s)}^{(2)}] \\ &\quad + \frac{\alpha_p^{(2)}}{\check{r}^{(2,1)}} \sum_{a=1}^{N_G^{(2)}} \sum_{j \in \tilde{\mathcal{A}}_{G;c}^{(1)}} \sum_{d=1}^{D_{c;j}^{(2)}} (\mathbf{C}_a^{(2)} \cdot \check{\mathbf{W}}_{j;a}^{(2,1)} \cdot [\mathbf{C}_a^{(2)}]^\dagger) \otimes \text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a;\check{v}^{(2)}(j,d)}^{(2)}] \end{aligned} \quad (\text{B.39})$$

As a last step before evaluating the rank $r_{\max}^{(2)}$, we rewrite (4.64) using an equivalent form for $\mathbf{B}_j^{(2,1)} \cdot [\mathbf{B}_j^{(2,1)}]^\dagger$ of (4.65). Since $\mathbf{E}_{a;\check{v}^{(2)}(j,d)}^{(2)} = \text{diag}_{m=1}^{N_G^{(2)}} [\mathbf{E}_{a;\check{v}^{(2)}(j,d)}^{(2)}]$, (4.65) is equivalent to

$$\mathbf{B}_j^{(2,1)} \cdot [\mathbf{B}_j^{(2,1)}]^\dagger = \sum_{d=1}^{D_{c;j}^{(2)}} \text{diag}_{a=1}^{N_G^{(2)}} [\mathbf{E}_{a;\check{v}^{(2)}(j,d)}^{(2)}] = \sum_{d=1}^{D_{c;j}^{(2)}} \sum_{a=1}^{N_G^{(2)}} \text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a;\check{v}^{(2)}(j,d)}^{(2)}]$$

Using this expression for $\mathbf{B}_j^{(2,1)} \cdot [\mathbf{B}_j^{(2,1)}]^\dagger$ and identity (C.5b), (4.64) becomes

$$\mathbf{H}^{(2,1)} \cdot \left(\frac{1}{P_T} \Sigma_{\max}^{(1)} \right) \cdot [\mathbf{H}^{(2,1)}]^\dagger = \frac{N_G^{(1)} \alpha_p^{(1)}}{r_T^{(1)} D^{(1)}} \sum_{a=1}^{N_G^{(2)}} \sum_{j \in \mathcal{A}_{G;c}^{(1)}} \sum_{d=1}^{D_{c;j}^{(2)}} \mathbf{N}_j^{(2,1)} \otimes \text{diag}_{m=1}^{N_G^{(2)}} \left[\delta(a, m) \mathbf{E}_{a;v^{(2)}(j,d)}^{(2)} \right]$$

which, using (B.39), leads to

$$\begin{aligned} & \mathbf{H}^{(2)} \cdot \left(\frac{1}{P_T} \check{\Sigma}^{(2)} \right) \cdot [\mathbf{H}^{(2)}]^\dagger + \mathbf{H}^{(2,1)} \cdot \left(\frac{1}{P_T} \Sigma_{\max}^{(1)} \right) \cdot [\mathbf{H}^{(2,1)}]^\dagger = \\ & \frac{\alpha_p^{(2)}}{r^{(2,1)}} \sum_{a=1}^{N_G^{(2)}} \sum_{s=1}^{D_p^{(2)}} \left(\mathbf{C}_a^{(2)} \cdot [\mathbf{M}_a^{(2)}]^\dagger \cdot \mathbf{B}_a^{(2)} \cdot \mathbf{M}_a^{(2)} \cdot [\mathbf{C}_a^{(2)}]^\dagger \right) \otimes \text{diag}_{m=1}^{N_G^{(2)}} \left[\delta(a, m) \mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)} \right] \\ & + \sum_{a=1}^{N_G^{(2)}} \sum_{j \in \tilde{\mathcal{A}}_{G;c}^{(1)}} \sum_{d=1}^{D_{c;j}^{(2)}} \mathbf{K}_{j;a}^{(2,1)} \otimes \text{diag}_{m=1}^{N_G^{(2)}} \left[\delta(a, m) \mathbf{E}_{a;v^{(2)}(j,d)}^{(2)} \right] \\ & + \frac{N_G^{(1)} \alpha_p^{(1)}}{r_T^{(1)} D^{(1)}} \sum_{a=1}^{N_G^{(2)}} \sum_{j \in \mathcal{A}_{G;c}^{(1)} - \tilde{\mathcal{A}}_{G;c}^{(1)}} \sum_{d=1}^{D_{c;j}^{(2)}} \mathbf{N}_j^{(2,1)} \otimes \text{diag}_{m=1}^{N_G^{(2)}} \left[\delta(a, m) \mathbf{E}_{a;v^{(2)}(j,d)}^{(2)} \right] \end{aligned} \quad (\text{B.40})$$

For different values of a , the matrices $\text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a;q}^{(2)}]$ have their column (and row) spaces essentially disjoint, regardless of the index q . For the same a and for different integers $p, q \in \{1, \dots, D^{(2)}\}$, the matrices $\text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a;p}^{(2)}]$ and $\text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a;q}^{(2)}]$ have their column (and row) spaces essentially disjoint. Since $v^{(2)}(j, d)$ and $\tilde{v}^{(2)}(s)$ take different values in $\{1, \dots, D^{(2)}\}$ for different pairs (j, d) and different values of s , one can easily show that all the matrices on the right hand side of (B.40) have their column (and row) spaces essentially disjoint. Hence, using the rank additivity from [114, Theorem 18.5.7], $r_{\max}^{(2)}$ is the sum of the ranks of the matrices on the right hand side of (B.40). Moreover, using identity (C.16) with the fact that for each a only $D_{p;a}^{(2)}$ out of the $D_p^{(2)}$ (rank one) matrices $\{\text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)}]\}_{s=1}^{D_p^{(2)}}$ are nonzero, and that for each pair (a, j) only $D_{c;j,a}^{(2)}$ out of the $D_{c;j}^{(2)}$ (rank one) matrices $\{\text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a;v^{(2)}(j,d)}^{(2)}]\}_{d=1}^{D_{c;j}^{(2)}}$ are nonzero, we finally obtain

$$\begin{aligned} r_{\max}^{(2)} &= \sum_{a=1}^{N_G^{(2)}} D_{p;a}^{(2)} \text{rank}(\mathbf{C}_a^{(2)} \cdot [\mathbf{M}_a^{(2)}]^\dagger \cdot \mathbf{B}_a^{(2)} \cdot \mathbf{M}_a^{(2)} \cdot [\mathbf{C}_a^{(2)}]^\dagger) + \sum_{a=1}^{N_G^{(2)}} \sum_{j \in \tilde{\mathcal{A}}_{G;c}^{(1)}} D_{c;j,a}^{(2)} \text{rank}(\mathbf{K}_{j;a}^{(2,1)}) \\ &+ \sum_{a=1}^{N_G^{(2)}} \sum_{j \in \mathcal{A}_{G;c}^{(1)} - \tilde{\mathcal{A}}_{G;c}^{(1)}} D_{c;j,a}^{(2)} \text{rank}(\mathbf{N}_j^{(2,1)}) \end{aligned}$$

Since $\mathbf{B}_a^{(2)} = \text{diag}[\mathbf{I}_{r_a^{(2)}}, \mathbf{0}_{(n_a^{(2)} - r_a^{(2)}) \times (n_a^{(2)} - r_a^{(2)})}] = \mathbf{B}_a^{(2)} \cdot \mathbf{B}_a^{(2)}$, (C.15c) can be used to write $\text{rank}(\mathbf{C}_a^{(2)} \cdot [\mathbf{M}_a^{(2)}]^\dagger \cdot \mathbf{B}_a^{(2)} \cdot \mathbf{M}_a^{(2)} \cdot [\mathbf{C}_a^{(2)}]^\dagger) = \text{rank}(\mathbf{B}_a^{(2)} \cdot \mathbf{M}_a^{(2)} \cdot [\mathbf{C}_a^{(2)}]^\dagger \cdot \mathbf{C}_a^{(2)} \cdot [\mathbf{M}_a^{(2)}]^\dagger \cdot \mathbf{B}_a^{(2)})$. Substituting $[\mathbf{C}_a^{(2)}]^\dagger \cdot \mathbf{C}_a^{(2)} = [\mathbf{M}_a^{(2)}]^\dagger \cdot \text{diag}_{l=1}^{n_a^{(2)}} [\lambda_{a;l}^{(2)}] \cdot \mathbf{M}_a^{(2)}$, we obtain $\text{rank}(\mathbf{C}_a^{(2)} \cdot [\mathbf{M}_a^{(2)}]^\dagger \cdot \mathbf{B}_a^{(2)} \cdot \mathbf{M}_a^{(2)} \cdot [\mathbf{C}_a^{(2)}]^\dagger) = \text{rank}(\mathbf{B}_a^{(2)} \cdot \text{diag}_{l=1}^{n_a^{(2)}} [\lambda_{a;l}^{(2)}] \cdot \mathbf{B}_a^{(2)}) = r_a^{(2)}$. The last equality results from $\mathbf{B}_a^{(2)}$ and $\text{diag}_{l=1}^{n_a^{(2)}} [\lambda_{a;l}^{(2)}]$ being diagonal with only the first $r_a^{(2)}$ diagonal elements nonzero. With this result, the previous expression for $r_{\max}^{(2)}$ reduces to (4.68). $\square$

Proof of Lemma 4.6.1: In order to evaluate the rank $r_{\max;O}^{(2)}$, we start by evaluating the matrix $[\mathbf{H}_O^{(2)}]^\dagger \cdot [\Psi_O^{(2)}]^{-1} \cdot \mathbf{H}_O^{(2)}$ given by (4.123). Similarly to (B.21) and using the matrices $\bar{\mathbf{M}}_O^{(2,1)}$ and $\mathbf{D}_O^{(2,1)}$ defined in Section 4.6, the matrix product $[\bar{\mathbf{M}}_O^{(2,1)}]^\dagger \cdot \mathbf{D}_O^{(2,1)} \cdot \bar{\mathbf{M}}_O^{(2,1)}$ on the right hand side of (4.123) can be written as

$$[\bar{\mathbf{M}}_O^{(2,1)}]^\dagger \cdot \mathbf{D}_O^{(2,1)} \cdot \bar{\mathbf{M}}_O^{(2,1)} = \sum_{q=1}^{N_R^{(2)} D^{(2)} - r_{\max;O}^{(2,1)}} \bar{\mathbf{m}}_{O;q}^{(2,1)} \cdot [\bar{\mathbf{m}}_{O;q}^{(2,1)}]^\dagger \quad (\text{B.41})$$

where $\bar{\mathbf{m}}_{O;q}^{(2,1)}$ denote the eigenvectors of $\mathbf{H}_O^{(2,1)} \cdot \text{diag}_{i=1}^{N_G^{(1)}} [[\mathbf{M}_{O;i}^{(1)}]^\dagger \cdot \tilde{\mathbf{B}}_{O;i}^{(1)} \cdot \mathbf{M}_{O;i}^{(1)}] \cdot [\mathbf{H}_O^{(2,1)}]^\dagger$ that are associated with the zero eigenvalues. Let us now evaluate these vectors $\bar{\mathbf{m}}_{O;q}^{(2,1)}$. For every $j \in \tilde{\mathcal{A}}_{O;G;c}^{(1)}$, we have $\text{rank}(\mathbf{N}_{O;j;q}^{(2,1)}) = \min(N_R^{(1)}, n_j^{(1)}) < N_R^{(2)}$ and hence, the eigenvectors $\mathbf{n}_l(\mathbf{N}_{O;j;q}^{(2,1)})$ for $l = \min(N_R^{(1)}, n_j^{(1)}) + 1, \dots, N_R^{(2)}$ are associated with zero eigenvalues. Thus, the first $\sum_{j \in \tilde{\mathcal{A}}_{O;G;c}^{(1)}} D_{c;j}^{(2)} [N_R^{(2)} - \min(N_R^{(1)}, n_j^{(1)})]$ vectors $\bar{\mathbf{m}}_{O;q}^{(2,1)}$ are of the form $\mathbf{n}_l(\mathbf{N}_{O;j;q}^{(2,1)}) \otimes \mathbf{e}_{v^{(2)}(j,q);D^{(2)}}$ with $l = \min(N_R^{(1)}, n_j^{(1)}) + 1, \dots, N_R^{(2)}$, $q = 1, \dots, D_{c;j}^{(2)}$ and $j \in \tilde{\mathcal{A}}_{O;G;c}^{(1)}$. This can be shown by using (C.3) and (4.137) and by writing

$$\begin{aligned} & \mathbf{H}_O^{(2,1)} \cdot \text{diag}_{i=1}^{N_G^{(1)}} [[\mathbf{M}_{O;i}^{(1)}]^\dagger \cdot \tilde{\mathbf{B}}_{O;i}^{(1)} \cdot \mathbf{M}_{O;i}^{(1)}] \cdot [\mathbf{H}_O^{(2,1)}]^\dagger \cdot [\mathbf{n}_l(\mathbf{N}_{O;j;q}^{(2,1)}) \otimes \mathbf{e}_{v^{(2)}(j,q);D^{(2)}}] \\ &= \sum_{j' \in \mathcal{A}_{G;c}^{(1)}} \sum_{q'=1}^{D_{c;j'}^{(2)}} [\mathbf{N}_{O;j';q'}^{(2,1)} \cdot \mathbf{n}_l(\mathbf{N}_{O;j;q}^{(2,1)})] \otimes [\mathbf{E}_{v^{(2)}(j',q')}^{(2)} \cdot \mathbf{e}_{v^{(2)}(j,q);D^{(2)}}] \\ &\stackrel{(a)}{=} [\mathbf{N}_{O;j;q}^{(2,1)} \cdot \mathbf{n}_l(\mathbf{N}_{O;j;q}^{(2,1)})] \otimes \mathbf{e}_{v^{(2)}(j,q);D^{(2)}} \stackrel{(b)}{=} \mathbf{0}_{N_R^{(2)} D^{(2)}} \end{aligned}$$

where (a) is due to (B.23) and (b) is due to the fact that $\mathbf{N}_{O;j;q}^{(2,1)} \cdot \mathbf{n}_l(\mathbf{N}_{O;j;q}^{(2,1)}) = \mathbf{0}_{N_R^{(2)}}$. Following similar steps, we can show that the eigenvectors associated with nonzero eigenvalues are of the form $\mathbf{n}_{\tilde{l}}(\mathbf{N}_{O;\tilde{j};\tilde{q}}^{(2,1)}) \otimes \mathbf{e}_{v^{(2)}(\tilde{j},\tilde{q});D^{(2)}}$ with $\tilde{l} = 1, \dots, \min(N_R^{(1)}, n_{\tilde{j}}^{(1)})$, $\tilde{q} = 1, \dots, D_{c;\tilde{j}}^{(2)}$ and $\tilde{j} \in \tilde{\mathcal{A}}_{O;G;c}^{(1)}$ and with $\tilde{l} = 1, \dots, N_R^{(2)}$, $\tilde{q} = 1, \dots, D_{c;\tilde{j}}^{(2)}$ and $\tilde{j} \in \mathcal{A}_{G;c}^{(1)} - \tilde{\mathcal{A}}_{O;G;c}^{(1)}$. Moreover,

different eigenvectors $\mathbf{n}_l(\mathbf{N}_{O;j;q}^{(2,1)}) \otimes \mathbf{e}_{v^{(2)}(j,q);D^{(2)}}$ are orthogonal since

$$\begin{aligned} & [\mathbf{n}_l^\dagger(\mathbf{N}_{O;\tilde{j};\tilde{q}}^{(2,1)}) \otimes \mathbf{e}_{v^{(2)}(\tilde{j},\tilde{q});D^{(2)}}^T] \cdot [\mathbf{n}_l(\mathbf{N}_{O;j;q}^{(2,1)}) \otimes \mathbf{e}_{v^{(2)}(j,q);D^{(2)}}] \\ &= [\mathbf{n}_l^\dagger(\mathbf{N}_{O;\tilde{j};\tilde{q}}^{(2,1)}) \cdot \mathbf{n}_l(\mathbf{N}_{O;j;q}^{(2,1)})] \otimes [\mathbf{e}_{v^{(2)}(\tilde{j},\tilde{q});D^{(2)}}^T \cdot \mathbf{e}_{v^{(2)}(j,q);D^{(2)}}] \stackrel{(a)}{=} 0 \end{aligned}$$

where (a) is due to $\mathbf{e}_{v^{(2)}(\tilde{j},\tilde{q});D^{(2)}}^T \cdot \mathbf{e}_{v^{(2)}(j,q);D^{(2)}} = 0$ if $(\tilde{j}, \tilde{q}) \neq (j, q)$ and to $\mathbf{n}_l^\dagger(\mathbf{N}_{O;\tilde{j};\tilde{q}}^{(2,1)}) \cdot \mathbf{n}_l(\mathbf{N}_{O;j;q}^{(2,1)}) = 0$ if $(\tilde{j}, \tilde{q}) = (j, q)$ and $l \neq \tilde{l}$. We now determine the remaining $N_R^{(2)} D_p^{(2)}$ vectors $\bar{\mathbf{m}}_{O;q}^{(2,1)}$. We choose these vectors to be of the form $\mathbf{e}_{r;N_R^{(2)}} \otimes \mathbf{e}_{\tilde{v}^{(2)}(s);D^{(2)}}$ where $r = 1, \dots, N_R^{(2)}$, $s = 1, \dots, D_p^{(2)}$ and, as in Section 4.5.2, $\tilde{v}^{(2)}(s) \in \{1, \dots, D^{(2)}\}$ denotes the index of user 2's proper signal dimension associated with integer s . One can easily show that these vectors are orthonormal since different values of s yield different values of $\tilde{v}^{(2)}(s)$ and $\mathbf{e}_{r;N_R^{(2)}}^T \cdot \mathbf{e}_{r';N_R^{(2)}} = \delta(r, r')$. Now using (C.3) and the fact that $\mathbf{e}_{v^{(2)}(j,q);D^{(2)}}^T \cdot \mathbf{e}_{\tilde{v}^{(2)}(s);D^{(2)}} = 0$ (since $v^{(2)}(j, q) \neq \tilde{v}^{(2)}(s)$ for all (j, q) and all s), we can show that $\mathbf{e}_{r;N_R^{(2)}} \otimes \mathbf{e}_{\tilde{v}^{(2)}(s);D^{(2)}}$ and $\mathbf{n}_l(\mathbf{N}_{O;j;q}^{(2,1)}) \otimes \mathbf{e}_{v^{(2)}(j,q);D^{(2)}}$ are orthogonal for all l and (j, q) . Moreover, we have from (4.137) and (C.3) that

$$\begin{aligned} & \mathbf{H}_O^{(2,1)} \cdot \text{diag}_{i=1}^{N_G^{(1)}} \left[[\mathbf{M}_{O;i}^{(1)}]^\dagger \cdot \tilde{\mathbf{B}}_{O;i}^{(1)} \cdot \mathbf{M}_{O;i}^{(1)} \right] \cdot [\mathbf{H}_O^{(2,1)}]^\dagger \cdot [\mathbf{e}_{r;N_R^{(2)}} \otimes \mathbf{e}_{\tilde{v}^{(2)}(s);D^{(2)}}] \\ &= \sum_{j \in \mathcal{A}_{G;c}^{(1)}} \sum_{q=1}^{D_{c;j}^{(2)}} [\mathbf{N}_{O;j;q}^{(2,1)} \cdot \mathbf{e}_{r;N_R^{(2)}}] \otimes [\mathbf{E}_{v^{(2)}(j,q)}^{(2)} \cdot \mathbf{e}_{\tilde{v}^{(2)}(s);D^{(2)}}] \stackrel{(a)}{=} \mathbf{0}_{N_R^{(2)} D^{(2)}} \end{aligned}$$

where (a) is due to (B.24). Thus, $\mathbf{e}_{r;N_R^{(2)}} \otimes \mathbf{e}_{\tilde{v}^{(2)}(s);D^{(2)}}$ are eigenvectors of $\mathbf{H}_O^{(2,1)} \cdot \text{diag}_{i=1}^{N_G^{(1)}} [[\mathbf{M}_{O;i}^{(1)}]^\dagger \cdot \tilde{\mathbf{B}}_{O;i}^{(1)} \cdot \mathbf{M}_{O;i}^{(1)}] \cdot [\mathbf{H}_O^{(2,1)}]^\dagger$ associated with zero eigenvalues. Hence, the vectors $\bar{\mathbf{m}}_{O;q}^{(2,1)}$ are given by $\mathbf{n}_l(\mathbf{N}_{O;j;q}^{(2,1)}) \otimes \mathbf{e}_{v^{(2)}(j,q);D^{(2)}}$ for $l = \min(N_R^{(1)}, n_j^{(1)}) + 1, \dots, N_R^{(2)}$, $q = 1, \dots, D_{c;j}^{(2)}$ and $j \in \tilde{\mathcal{A}}_{O;G;c}^{(1)}$, and by $\mathbf{e}_{r;N_R^{(2)}} \otimes \mathbf{e}_{\tilde{v}^{(2)}(s);D^{(2)}}$ for $r = 1, \dots, N_R^{(2)}$ and $s = 1, \dots, D_p^{(2)}$. Using (C.3), (C.5) with $\sum_{r=1}^{N_R^{(2)}} \mathbf{e}_{r;N_R^{(2)}} \cdot \mathbf{e}_{r;N_R^{(2)}}^T = \mathbf{I}_{N_R^{(2)}}$, (B.41) reduces to

$$\begin{aligned} & [\bar{\mathbf{M}}_O^{(2,1)}]^\dagger \cdot \mathbf{D}_O^{(2,1)} \cdot \bar{\mathbf{M}}_O^{(2,1)} = \mathbf{I}_{N_R^{(2)}} \otimes \left[\sum_{s=1}^{D_p^{(2)}} \mathbf{E}_{\tilde{v}^{(2)}(s)}^{(2)} \right] \\ &+ \sum_{j \in \tilde{\mathcal{A}}_{O;G;c}^{(1)}} \sum_{q=1}^{D_{c;j}^{(2)}} \sum_{l=\min(N_R^{(1)}, n_j^{(1)})+1}^{N_R^{(2)}} [\mathbf{n}_l(\mathbf{N}_{O;j;q}^{(2,1)}) \cdot \mathbf{n}_l^\dagger(\mathbf{N}_{O;j;q}^{(2,1)})] \otimes \mathbf{E}_{v^{(2)}(j,q)}^{(2)} \end{aligned} \tag{B.42}$$

For $p = 1, \dots, D^{(2)}$, we define $\check{\mathbf{G}}_p^{(2)} \triangleq (FT[\tilde{\mathbf{F}}^{(2)}(\tau)]_{\frac{i^{(2)}(p)}{T_F}})^{\dagger} \cdot FT[\tilde{\mathbf{F}}^{(2)}(\tau)]_{\frac{i^{(2)}(p)}{T_F}}$ where $\tilde{\mathbf{F}}^{(2)}(\tau) = \sum_{l=1}^{L_t^{(2)}} \delta(\tau - \tau_l^{(2)}) \mathbf{F}_l^{(2)}$ from Section 2.3. The matrices $\check{\mathbf{G}}_p^{(2)}$ are Hermitian positive semidefinite of size $N_T \times N_T$. By using (C.3), (C.5) with $\mathbf{E}_p^{(2)} \cdot \mathbf{E}_{p'}^{(2)} = \mathbf{e}_{p;D^{(2)}} \cdot \mathbf{e}_{p';D^{(2)}}^T \cdot \mathbf{e}_{p';D^{(2)}} \cdot \mathbf{e}_{p;D^{(2)}}^T = \delta(p, p') \mathbf{E}_p^{(2)}$ and by writing $\mathbf{H}_O^{(2)}$ as in (2.49) with $[\mathbf{O}_{m,q}^{(2)}]$ of (4.136a), (B.42) leads to

$$\begin{aligned} & [\mathbf{H}_O^{(2)}]^{\dagger} \cdot [\bar{\mathbf{M}}_O^{(2,1)}]^{\dagger} \cdot \mathbf{D}_O^{(2,1)} \cdot \bar{\mathbf{M}}_O^{(2,1)} \cdot \mathbf{H}_O^{(2)} = \\ & \sum_{s=1}^{D_p^{(2)}} \mathbf{A}^{(2)} \cdot \left\{ \check{\mathbf{G}}_{\check{v}^{(2)}(s)}^{(2)} \otimes \mathbf{E}_{\check{v}^{(2)}(s)}^{(2)} \right\} \cdot [\mathbf{A}^{(2)}]^T \\ & + \sum_{j \in \check{\mathcal{A}}_{O;G;c}^{(1)}} \sum_{q=1}^{D_{c;j}^{(2)}} \mathbf{A}^{(2)} \cdot \left\{ \left[\sum_{l=\min(N_R^{(1)}, n_j^{(1)})+1}^{N_R^{(2)}} (FT[\tilde{\mathbf{F}}^{(2)}(\tau)]_{\frac{i^{(2)}(v^{(2)}(j,q))}{T_F}})^{\dagger} \cdot \mathbf{n}_l(\mathbf{N}_{O;j;q}^{(2,1)}) \cdot \mathbf{n}_l^{\dagger}(\mathbf{N}_{O;j;q}^{(2,1)}) \right. \right. \\ & \quad \left. \left. \cdot FT[\tilde{\mathbf{F}}^{(2)}(\tau)]_{\frac{i^{(2)}(v^{(2)}(j,q))}{T_F}} \right] \otimes \mathbf{E}_{v^{(2)}(j,q)}^{(2)} \right\} \cdot [\mathbf{A}^{(2)}]^T \end{aligned} \quad (\text{B.43})$$

For $a = 1, \dots, N_G^{(2)}$, the a^{th} diagonal block of $\check{\mathbf{G}}_{\check{v}^{(2)}(s)}^{(2)}$ is the $n_a^{(2)} \times n_a^{(2)}$ matrix $\check{\mathbf{G}}_{\check{v}^{(2)}(s);a}^{(2)} \triangleq (FT[\tilde{\mathbf{F}}_a^{(2)}(\tau)]_{\frac{i^{(2)}(\check{v}^{(2)}(s))}{T_F}})^{\dagger} \cdot FT[\tilde{\mathbf{F}}_a^{(2)}(\tau)]_{\frac{i^{(2)}(\check{v}^{(2)}(s))}{T_F}}$ where $\tilde{\mathbf{F}}_a^{(2)}(\tau) = \sum_{l=1}^{L_t^{(2)}} \delta(\tau - \tau_l^{(2)}) \mathbf{F}_{a,l}^{(2)}$ and $\tilde{\mathbf{F}}^{(2)}(\tau) = [\tilde{\mathbf{F}}_1^{(2)}(\tau) \dots \tilde{\mathbf{F}}_{N_G^{(2)}}^{(2)}(\tau)]$. Moreover, the $n_a^{(2)} \times n_a^{(2)}$ diagonal block of the matrix $\sum_{l=\min(N_R^{(1)}, n_j^{(1)})+1}^{N_R^{(2)}} (FT[\tilde{\mathbf{F}}^{(2)}(\tau)]_{\frac{i^{(2)}(v^{(2)}(j,q))}{T_F}})^{\dagger} \cdot \mathbf{n}_l(\mathbf{N}_{O;j;q}^{(2,1)}) \cdot \mathbf{n}_l^{\dagger}(\mathbf{N}_{O;j;q}^{(2,1)}) \cdot FT[\tilde{\mathbf{F}}^{(2)}(\tau)]_{\frac{i^{(2)}(v^{(2)}(j,q))}{T_F}}$ is given by the matrix $\check{\mathbf{N}}_{O;j;q;a}^{(2,1)}$ which is defined in Section 4.6. Each term on the right hand side of (B.43) is of the form $\mathbf{A}^{(2)} \cdot \{\mathbf{X} \otimes \mathbf{E}_q^{(2)}\} \cdot [\mathbf{A}^{(2)}]^T$ where $\mathbf{X}$ is an $N_T \times N_T$ matrix. Hence, we can use (B.27) to write (B.43) as follows

$$\begin{aligned} & [\mathbf{H}_O^{(2)}]^{\dagger} \cdot [\bar{\mathbf{M}}_O^{(2,1)}]^{\dagger} \cdot \mathbf{D}_O^{(2,1)} \cdot \bar{\mathbf{M}}_O^{(2,1)} \cdot \mathbf{H}_O^{(2)} = \\ & \text{diag}_{a=1}^{N_G^{(2)}} \left[\sum_{s=1}^{D_p^{(2)}} \check{\mathbf{G}}_{\check{v}^{(2)}(s);a}^{(2)} \otimes \mathbf{E}_{a;\check{v}^{(2)}(s)}^{(2)} + \sum_{j \in \check{\mathcal{A}}_{O;G;c}^{(1)}} \sum_{q=1}^{D_{c;j}^{(2)}} \check{\mathbf{N}}_{O;j;q;a}^{(2,1)} \otimes \mathbf{E}_{a;v^{(2)}(j,q)}^{(2)} \right] \end{aligned} \quad (\text{B.44})$$

where, for $q = 1, \dots, D^{(2)}$ and $a = 1, \dots, N_G^{(2)}$, $\mathbf{E}_{a;q}^{(2)}$ denotes the a^{th} diagonal block of $\mathbf{E}_q^{(2)} = \text{diag}_{j=1}^{N_G^{(2)}} [\mathbf{E}_{j;q}^{(2)}]$ as in the proof of Lemma 4.5.1. From (B.44), the matrix $[\mathbf{H}_O^{(2)}]^{\dagger} \cdot [\bar{\mathbf{M}}_O^{(2,1)}]^{\dagger} \cdot \mathbf{D}_O^{(2,1)} \cdot \bar{\mathbf{M}}_O^{(2,1)} \cdot \mathbf{H}_O^{(2)}$ is block diagonal with diagonal block a ($a = 1, \dots, N_G^{(2)}$)

being of size $n_a^{(2)} D_G^{(2)} \times n_a^{(2)} D_G^{(2)}$ and given by

$$\mathbf{R}_{O;a}^{(2,1)} \triangleq \sum_{s=1}^{D_p^{(2)}} \check{\mathbf{G}}_{\tilde{v}^{(2)}(s);a}^{(2)} \otimes \mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)} + \sum_{j \in \check{\mathcal{A}}_{O;G;c}^{(1)}} \sum_{q=1}^{D_{c;j}^{(2)}} \check{\mathbf{N}}_{O;j;q;a}^{(2,1)} \otimes \mathbf{E}_{a;v^{(2)}(j,q)}^{(2)} \quad (\text{B.45})$$

As for $\mathbf{R}_a^{(2,1)}$ of (B.29), we can show that $\mathbf{R}_{O;a}^{(2,1)}$ is Hermitian positive semidefinite. Moreover, the unitary matrix $\check{\mathbf{M}}_O^{(2)}$ in the eigenvalue decomposition of $[\mathbf{H}_O^{(2)}]^\dagger \cdot [\bar{\mathbf{M}}_O^{(2,1)}]^\dagger \cdot \mathbf{D}_O^{(2,1)} \cdot \bar{\mathbf{M}}_O^{(2,1)} \cdot \mathbf{H}_O^{(2)} \equiv \text{diag}_{a=1}^{N_G^{(2)}} [\mathbf{R}_{O;a}^{(2,1)}]$ can be easily shown to be block diagonal $\check{\mathbf{M}}_O^{(2)} = \text{diag}_{a=1}^{N_G^{(2)}} [\check{\mathbf{M}}_{O;a}^{(2)}]$ where $\check{\mathbf{M}}_{O;a}^{(2)}$ is the unitary matrix in the eigenvalue decomposition of $\mathbf{R}_{O;a}^{(2,1)}$. The rank $\check{r}_O^{(2,1)}$ of $[\mathbf{H}_O^{(2)}]^\dagger \cdot [\bar{\mathbf{M}}_O^{(2,1)}]^\dagger \cdot \mathbf{D}_O^{(2,1)} \cdot \bar{\mathbf{M}}_O^{(2,1)} \cdot \mathbf{H}_O^{(2)}$ satisfies $\check{r}_O^{(2,1)} = \sum_{a=1}^{N_G^{(2)}} \check{r}_{O;a}^{(2,1)}$ with $\check{r}_{O;a}^{(2,1)} = \text{rank}(\mathbf{R}_{O;a}^{(2,1)})$. We use the matrix $\check{\mathbf{M}}_O^{(2)}$ to determine the input covariance matrix $\check{\Sigma}_O^{(2)}$ given by (4.126). Similarly to $\check{\Sigma}^{(2)}$ of (B.30) - (B.31), $\check{\Sigma}_O^{(2)}$ of (4.126) can be written as $\check{\Sigma}_O^{(2)} = \frac{\alpha_p^{(2)} P_T}{\check{r}_O^{(2,1)}} \text{diag}_{a=1}^{N_G^{(2)}} [\check{\Sigma}_{O;a}^{(2)}]$ where $\check{\Sigma}_{O;a}^{(2)} \triangleq \sum_{l=1}^{\check{r}_{O;a}^{(2,1)}} [(\check{\mathbf{M}}_{O;a}^{(2)})^\dagger]_{*,l} \cdot [(\check{\mathbf{M}}_{O;a}^{(2)})^\dagger]_{*,l}^\dagger$ and $[(\check{\mathbf{M}}_{O;a}^{(2)})^\dagger]_{*,l}$ are the eigenvectors of $\mathbf{R}_{O;a}^{(2,1)}$ associated with nonzero eigenvalues. For $a = 1, \dots, N_G^{(2)}$, we now evaluate the rank $\check{r}_{O;a}^{(2,1)}$ and the eigenvectors $[(\check{\mathbf{M}}_{O;a}^{(2)})^\dagger]_{*,l}$ of $\mathbf{R}_{O;a}^{(2,1)}$. As explained in the proof of Lemma 4.5.1, for every $s = 1, \dots, D_p^{(2)}$, $\mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)} = \mathbf{0}_{D_G^{(2)} \times D_G^{(2)}}$ if the proper signal dimension $\tilde{v}^{(2)}(s)$ is not allocated to TOG a of user 2. Otherwise, $\tilde{v}^{(2)}(s)$ corresponds to an element s' of the subset $\mathcal{S}_{P;a}^{(2)} \subseteq \{1, \dots, D_G^{(2)}\}$ of proper signal dimensions assigned to TOG a of user 2, and $\mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)} = \mathbf{e}_{s';D_G^{(2)}} \cdot \mathbf{e}_{s';D_G^{(2)}}^\dagger$ satisfying (B.32). Also, for $d = 1, \dots, D_{c;j}^{(2)}$, $\mathbf{E}_{a;v^{(2)}(j,d)}^{(2)} = \mathbf{0}_{D_G^{(2)} \times D_G^{(2)}}$ if the shared signal dimension $v^{(2)}(j,d)$ is not allocated to TOG a of user 2. Otherwise, $v^{(2)}(j,d)$ corresponds to an element d' of the subset $\mathcal{S}_{j;a}^{(2)} \subseteq \{1, \dots, D_G^{(2)}\}$ of shared signal dimensions between TOG j of user 1 and TOG a of user 2, and $\mathbf{E}_{a;v^{(2)}(j,d)}^{(2)} = \mathbf{e}_{d';D_G^{(2)}} \cdot \mathbf{e}_{d';D_G^{(2)}}^\dagger$ satisfying (B.33). As in the proof of Lemma 4.5.1, the matrices $\mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)}$ and $\mathbf{E}_{a;v^{(2)}(j,d)}^{(2)}$ have their column (and row) spaces essentially disjoint, and so are the matrices on the right hand side of (B.45). In this case, [114, Theorem 18.5.7] and identity (C.16) lead to

$$\check{r}_{O;a}^{(2,1)} = \sum_{s=1}^{D_p^{(2)}} \text{rank}(\check{\mathbf{G}}_{\tilde{v}^{(2)}(s);a}^{(2)}) \text{rank}(\mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)}) + \sum_{j \in \check{\mathcal{A}}_{O;G;c}^{(1)}} \sum_{q=1}^{D_{c;j}^{(2)}} \text{rank}(\check{\mathbf{N}}_{O;j;q;a}^{(2,1)}) \text{rank}(\mathbf{E}_{a;v^{(2)}(j,q)}^{(2)})$$

Here, $\text{rank}(\mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)}) = 1$ for the $D_{p;a}^{(2)}$ values of s for which $\tilde{v}^{(2)}(s)$ corresponds to an element of $\mathcal{S}_{P;a}^{(2)}$ and $\text{rank}(\mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)}) = 0$ for the other values of s . Also, $\text{rank}(\mathbf{E}_{a;v^{(2)}(j,q)}^{(2)}) =$

1 for the $D_{c;j;a}^{(2)}$ values of q for which $v^{(2)}(j, q)$ corresponds to an element of $\mathcal{S}_{j;a}^{(2)}$ and $\text{rank}(\mathbf{E}_{a;v^{(2)}(j,q)}^{(2)}) = 0$ for the other values of q . The eigenvectors $[(\check{\mathbf{M}}_{O;a}^{(2)})^\dagger]_{*,l}$ of $\mathbf{R}_{O;a}^{(2,1)}$ that are associated with nonzero eigenvalues and the matrix $\check{\Sigma}_{O;a}^{(2)}$ can be determined using the same method as in the proof of Lemma 4.5.1 for determining the eigenvectors of $\mathbf{R}_a^{(2,1)}$ (B.29) and the matrix $\check{\Sigma}_a^{(2)}$ of (B.36). That is, we only consider the proper signal dimensions $\tilde{v}^{(2)}(s)$ which are associated (on a one-to-one basis) with the $D_{p;a}^{(2)}$ elements $s' \in \mathcal{S}_{P;a}^{(2)}$ and the shared signal dimensions $v^{(2)}(j, q)$ that are associated (on a one-to-one basis) with the $D_{c;j;a}^{(2)}$ elements $q' \in \mathcal{S}_{j;a}^{(2)}$. Then, based on (B.32)-(B.33), we can show that the $\check{r}_{O;a}^{(2,1)}$ eigenvectors $[(\check{\mathbf{M}}_{O;a}^{(2)})^\dagger]_{*,l}$ are given by $\check{\mathbf{g}}_{a;l;s'}^{(2)} \otimes \mathbf{e}_{s';D_G^{(2)}}$ where $\check{\mathbf{g}}_{a;l;s'}^{(2)} \triangleq \mathbf{n}_l(\check{\mathbf{G}}_{\tilde{v}^{(2)}(s);a}^{(2)})$ (for $l = 1, \dots, \check{r}_{G;a;s'}^{(2)}$) and $\check{r}_{G;a;s'}^{(2)} \triangleq \text{rank}(\check{\mathbf{G}}_{\tilde{v}^{(2)}(s);a}^{(2)})$ denote the eigenvectors of $\check{\mathbf{G}}_{\tilde{v}^{(2)}(s);a}^{(2)}$ that are associated with nonzero eigenvalues, and by $\check{\mathbf{n}}_{j;a;l';q'}^{(2,1)} \otimes \mathbf{e}_{q';D_G^{(2)}}$ where $\check{\mathbf{n}}_{j;a;l';q'}^{(2,1)} \triangleq \mathbf{n}_{l'}(\check{\mathbf{N}}_{O;j;q;a}^{(2,1)})$ (for $l' = 1, \dots, \check{r}_{N;j;a;q'}^{(2,1)}$ and $\check{r}_{N;j;a;q'}^{(2,1)} \triangleq \text{rank}(\check{\mathbf{N}}_{O;j;q;a}^{(2,1)})$) denote the eigenvectors of $\check{\mathbf{N}}_{O;j;q;a}^{(2,1)}$ that are associated with nonzero eigenvalues. With these eigenvectors $[(\check{\mathbf{M}}_{O;a}^{(2)})^\dagger]_{*,l}$, we obtain

$$\begin{aligned}
\check{\Sigma}_{O;a}^{(2)} &\equiv \sum_{l=1}^{\check{r}_{O;a}^{(2,1)}} [(\check{\mathbf{M}}_{O;a}^{(2)})^\dagger]_{*,l} \cdot [(\check{\mathbf{M}}_{O;a}^{(2)})^\dagger]_{*,l}^\dagger \\
&= \sum_{s' \in \mathcal{S}_{P;a}^{(2)}} \sum_{l=1}^{\check{r}_{G;a;s'}^{(2)}} [\check{\mathbf{g}}_{a;l;s'}^{(2)} \cdot (\check{\mathbf{g}}_{a;l;s'}^{(2)})^\dagger] \otimes [\mathbf{e}_{s';D_G^{(2)}} \cdot \mathbf{e}_{s';D_G^{(2)}}^\dagger] \\
&\quad + \sum_{j \in \tilde{\mathcal{A}}_{O;G;c}^{(1)}} \sum_{q' \in \mathcal{S}_{j;a}^{(2)}} \sum_{l'=1}^{\check{r}_{N;j;a;q'}^{(2,1)}} [\check{\mathbf{n}}_{j;a;l';q'}^{(2,1)} \cdot (\check{\mathbf{n}}_{j;a;l';q'}^{(2,1)})^\dagger] \otimes [\mathbf{e}_{q';D_G^{(2)}} \cdot \mathbf{e}_{q';D_G^{(2)}}^\dagger] \\
&\stackrel{(a)}{=} \sum_{s=1}^{D_p^{(2)}} \sum_{l=1}^{\text{rank}(\check{\mathbf{G}}_{\tilde{v}^{(2)}(s);a}^{(2)})} [\mathbf{n}_l(\check{\mathbf{G}}_{\tilde{v}^{(2)}(s);a}^{(2)}) \cdot \mathbf{n}_l^\dagger(\check{\mathbf{G}}_{\tilde{v}^{(2)}(s);a}^{(2)})] \otimes \mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)} \\
&\quad + \sum_{j \in \tilde{\mathcal{A}}_{O;G;c}^{(1)}} \sum_{q=1}^{D_{c;j}^{(2)}} \sum_{l'=1}^{\text{rank}(\check{\mathbf{N}}_{O;j;q;a}^{(2,1)})} [\mathbf{n}_{l'}(\check{\mathbf{N}}_{O;j;q;a}^{(2,1)}) \cdot \mathbf{n}_{l'}^\dagger(\check{\mathbf{N}}_{O;j;q;a}^{(2,1)})] \otimes \mathbf{E}_{a;v^{(2)}(j,q)}^{(2)}
\end{aligned}$$

where (a) is due to $\mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)} = \mathbf{e}_{s';D_G^{(2)}} \cdot \mathbf{e}_{s';D_G^{(2)}}^\dagger$ if $\tilde{v}^{(2)}(s)$ corresponds to the proper signal dimensions $s' \in \mathcal{S}_{P;a}^{(2)}$ and $\mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)} = \mathbf{0}_{D_G^{(2)} \times D_G^{(2)}}$ otherwise, and to $\mathbf{E}_{a;v^{(2)}(j,q)}^{(2)} = \mathbf{e}_{q';D_G^{(2)}} \cdot \mathbf{e}_{q';D_G^{(2)}}^\dagger$ if $v^{(2)}(j, q)$ corresponds to the shared signal dimensions $q' \in \mathcal{S}_{j;a}^{(2)}$ and $\mathbf{E}_{a;v^{(2)}(j,q)}^{(2)} = \mathbf{0}_{D_G^{(2)} \times D_G^{(2)}}$ otherwise. Defining $\check{\mathbf{W}}_{O;a;s}^{(2)} \triangleq \sum_{l=1}^{\text{rank}(\check{\mathbf{G}}_{\tilde{v}^{(2)}(s);a}^{(2)})} \mathbf{n}_l(\check{\mathbf{G}}_{\tilde{v}^{(2)}(s);a}^{(2)}) \cdot \mathbf{n}_l^\dagger(\check{\mathbf{G}}_{\tilde{v}^{(2)}(s);a}^{(2)})$ and

using the matrix $\check{\mathbf{W}}_{O;j;q;a}^{(2,1)}$ defined in Section 4.6. it follows that

$$\check{\Sigma}_{O;a}^{(2)} = \sum_{s=1}^{D_p^{(2)}} \check{\mathbf{W}}_{O;a;s}^{(2)} \otimes \mathbf{E}_{a;\check{v}^{(2)}(s)}^{(2)} + \sum_{j \in \check{\mathcal{A}}_{O;G;c}^{(1)}} \sum_{q=1}^{D_{c;j}^{(2)}} \check{\mathbf{W}}_{O;j;q;a}^{(2,1)} \otimes \mathbf{E}_{a;v^{(2)}(j,q)}^{(2)} \quad (\text{B.46})$$

Now using $\check{\Sigma}_O^{(2)} = \frac{\alpha_p^{(2)} P_T}{r_O^{(2,1)}} \text{diag}_{a=1}^{N_G^{(2)}} [\check{\Sigma}_{O;a}^{(2)}]$ and (2.49) with $\mathbf{A}^{(2)} = \text{diag}_{a=1}^{N_G^{(2)}} [\mathbf{I}_{n_a^{(2)}} \otimes (\mathbf{e}_{a;N_G^{(2)}}^T \otimes \mathbf{I}_{D_G^{(2)}})]$, we obtain $\mathbf{H}_O^{(2)} \cdot \left(\frac{1}{P_T} \check{\Sigma}_O^{(2)}\right) \cdot [\mathbf{H}_O^{(2)}]^\dagger = \frac{\alpha_p^{(2)}}{r_O^{(2,1)}} [\mathbf{O}_{m,q}^{(2)}] \cdot \text{diag}_{a=1}^{N_G^{(2)}} [\tilde{\Lambda}_{O;a}^{(2)}] \cdot [\mathbf{O}_{m,q}^{(2)}]^\dagger$ where

$$\tilde{\Lambda}_{O;a}^{(2)} \triangleq \{\mathbf{I}_{n_a^{(2)}} \otimes (\mathbf{e}_{a;N_G^{(2)}} \otimes \mathbf{I}_{D_G^{(2)}})\} \cdot \check{\Sigma}_{O;a}^{(2)} \cdot \{\mathbf{I}_{n_a^{(2)}} \otimes (\mathbf{e}_{a;N_G^{(2)}}^T \otimes \mathbf{I}_{D_G^{(2)}})\}$$

Writing (4.136a) as $[\mathbf{O}_{m,q}^{(2)}] = \sum_{p=1}^{D^{(2)}} [FT[\tilde{\mathbf{F}}_1^{(2)}(\tau)]_{\frac{i^{(2)}(p)}{T_F}} \cdots FT[\tilde{\mathbf{F}}_{N_G^{(2)}}^{(2)}(\tau)]_{\frac{i^{(2)}(p)}{T_F}}] \otimes \mathbf{E}_p^{(2)}$, we have

$$\begin{aligned} \mathbf{H}_O^{(2)} \cdot \left(\frac{1}{P_T} \check{\Sigma}_O^{(2)}\right) \cdot [\mathbf{H}_O^{(2)}]^\dagger &= \frac{\alpha_p^{(2)}}{r_O^{(2,1)}} \sum_{a=1}^{N_G^{(2)}} \sum_{p=1}^{D^{(2)}} \sum_{p'=1}^{D^{(2)}} \left\{ FT[\tilde{\mathbf{F}}_a^{(2)}(\tau)]_{\frac{i^{(2)}(p)}{T_F}} \otimes \mathbf{E}_p^{(2)} \right\} \\ &\quad \cdot \tilde{\Lambda}_{O;a}^{(2)} \cdot \left\{ (FT[\tilde{\mathbf{F}}_a^{(2)}(\tau)]_{\frac{i^{(2)}(p')}{T_F}})^\dagger \otimes \mathbf{E}_{p'}^{(2)} \right\} \end{aligned} \quad (\text{B.47})$$

Using (B.46), (B.38) as well as (C.3), we can write

$$\begin{aligned} \tilde{\Lambda}_{O;a}^{(2)} &= \sum_{s=1}^{D_p^{(2)}} \check{\mathbf{W}}_{O;a;s}^{(2)} \otimes \text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a;\check{v}^{(2)}(s)}^{(2)}] \\ &\quad + \sum_{j \in \check{\mathcal{A}}_{O;G;c}^{(1)}} \sum_{q=1}^{D_{c;j}^{(2)}} \check{\mathbf{W}}_{O;j;q;a}^{(2,1)} \otimes \text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a;v^{(2)}(j,q)}^{(2)}] \end{aligned} \quad (\text{B.48})$$

For any $p, v = 1, \dots, D^{(2)}$, we have

$$\begin{aligned} \mathbf{E}_p^{(2)} \cdot \text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a;v}^{(2)}] &= \text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a;v}^{(2)}] \cdot \mathbf{E}_p^{(2)} \\ &= \text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a;p}^{(2)} \cdot \mathbf{E}_{a;v}^{(2)}] = \delta(p, v) \text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a;v}^{(2)}] \end{aligned}$$

Using this result as well as equation (B.48) and identity (C.3), (B.47) reduces to

$$\begin{aligned} \mathbf{H}_O^{(2)} \cdot \left(\frac{1}{P_T} \check{\Sigma}_O^{(2)} \right) \cdot [\mathbf{H}_O^{(2)}]^\dagger &= \frac{\alpha_p^{(2)}}{\check{r}_O^{(2,1)}} \sum_{a=1}^{N_G^{(2)}} \left\{ \sum_{s=1}^{D_p^{(2)}} \mathbf{Z}_{O;a;s}^{(2)} \otimes \text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a;\check{v}^{(2)}(s)}^{(2)}] \right. \\ &\quad \left. + \sum_{j \in \check{\mathcal{A}}_{O;G;c}^{(1)}} \sum_{q=1}^{D_{c;j}^{(2)}} \mathbf{Z}_{O;j;q;a}^{(2,1)} \otimes \text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a;v^{(2)}(j,q)}^{(2)}] \right\} \end{aligned} \quad (\text{B.49})$$

with $\mathbf{Z}_{O;j;q;a}^{(2,1)}$ being defined in Section 4.6 and with the $N_R^{(2)} \times N_R^{(2)}$ matrix

$$\mathbf{Z}_{O;a;s}^{(2)} \triangleq FT[\tilde{\mathbf{F}}_a^{(2)}(\tau)]_{i^{(2)}(\check{v}^{(2)}(s))} \cdot \check{\mathbf{W}}_{O;a;s}^{(2)} \cdot (FT[\tilde{\mathbf{F}}_a^{(2)}(\tau)]_{i^{(2)}(\check{v}^{(2)}(s))})^\dagger$$

By writing $\mathbf{E}_{v^{(2)}(j,q)}^{(2)}$ of (4.138) as

$$\mathbf{E}_{v^{(2)}(j,q)}^{(2)} = \text{diag}_{a=1}^{N_G^{(2)}} [\mathbf{E}_{a;v^{(2)}(j,q)}^{(2)}] = \sum_{a=1}^{N_G^{(2)}} \text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a;v^{(2)}(j,q)}^{(2)}]$$

equations (4.138) and (B.49) lead to

$$\begin{aligned} \mathbf{H}_O^{(2,1)} \cdot \left(\frac{1}{P_T} \Sigma_{O;\max}^{(1)} \right) \cdot [\mathbf{H}_O^{(2,1)}]^\dagger &+ \mathbf{H}_O^{(2)} \cdot \left(\frac{1}{P_T} \check{\Sigma}_O^{(2)} \right) \cdot [\mathbf{H}_O^{(2)}]^\dagger = \\ &\frac{\alpha_p^{(1)}}{r_O^{(1)}} \sum_{a=1}^{N_G^{(2)}} \sum_{j \in \mathcal{A}_{G;c}^{(1)} - \check{\mathcal{A}}_{O;G;c}^{(1)}} \sum_{q=1}^{D_{c;j}^{(2)}} \mathbf{N}_{O;j;q}^{(2,1)} \otimes \text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a;v^{(2)}(j,q)}^{(2)}] \\ &+ \frac{\alpha_p^{(2)}}{\check{r}_O^{(2,1)}} \sum_{a=1}^{N_G^{(2)}} \sum_{s=1}^{D_p^{(2)}} \mathbf{Z}_{O;a;s}^{(2)} \otimes \text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a;\check{v}^{(2)}(s)}^{(2)}] \\ &+ \sum_{a=1}^{N_G^{(2)}} \sum_{j \in \check{\mathcal{A}}_{O;G;c}^{(1)}} \sum_{q=1}^{D_{c;j}^{(2)}} \mathbf{K}_{O;j;q;a}^{(2,1)} \otimes \text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a;v^{(2)}(j,q)}^{(2)}] \end{aligned} \quad (\text{B.50})$$

As for (B.40), we can show that the matrices on the right hand side of (B.50) have their column (and row) spaces essentially disjoint. Therefore, by [114, Theorem 18.5.7] and

using (C.16), we can write

$$\begin{aligned}
 r_{\max;O}^{(2)} &= \sum_{a=1}^{N_G^{(2)}} \sum_{j \in \mathcal{A}_{G;c}^{(1)} - \tilde{\mathcal{A}}_{O;G;c}^{(1)}} \sum_{q=1}^{D_{c;j}^{(2)}} \text{rank}(\mathbf{N}_{O;j;q}^{(2,1)}) \text{rank}(\text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a;\tilde{v}^{(2)}(j,q)}^{(2)}]) \\
 &+ \sum_{a=1}^{N_G^{(2)}} \sum_{s=1}^{D_p^{(2)}} \text{rank}(\mathbf{Z}_{O;a;s}^{(2)}) \text{rank}(\text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)}]) \\
 &+ \sum_{a=1}^{N_G^{(2)}} \sum_{j \in \tilde{\mathcal{A}}_{O;G;c}^{(1)}} \sum_{q=1}^{D_{c;j}^{(2)}} \text{rank}(\mathbf{K}_{O;j;q;a}^{(2,1)}) \text{rank}(\text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a;\tilde{v}^{(2)}(j,q)}^{(2)}])
 \end{aligned} \tag{B.51}$$

From Section 4.6, we have that $\text{rank}(\mathbf{N}_{O;j;q}^{(2,1)}) = N_R^{(2)}$ for all $j \in \mathcal{A}_{G;c}^{(1)} - \tilde{\mathcal{A}}_{O;G;c}^{(1)}$. Moreover, only $D_{p;a}^{(2)}$ (respectively, $D_{c;j;a}^{(2)}$) rank one matrices $\{\text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a;\tilde{v}^{(2)}(s)}^{(2)}]\}_{s=1}^{D_p^{(2)}}$ (respectively, $\{\text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a;\tilde{v}^{(2)}(j,q)}^{(2)}]\}_{q=1}^{D_{c;j}^{(2)}}$) are nonzero. It follows that the first term on the right hand side of (B.51) satisfies

$$\begin{aligned}
 &\sum_{a=1}^{N_G^{(2)}} \sum_{j \in \mathcal{A}_{G;c}^{(1)} - \tilde{\mathcal{A}}_{O;G;c}^{(1)}} \sum_{q=1}^{D_{c;j}^{(2)}} \text{rank}(\mathbf{N}_{O;j;q}^{(2,1)}) \text{rank}(\text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a;\tilde{v}^{(2)}(j,q)}^{(2)}]) \\
 &= \sum_{a=1}^{N_G^{(2)}} \sum_{j \in \mathcal{A}_{G;c}^{(1)} - \tilde{\mathcal{A}}_{O;G;c}^{(1)}} D_{c;j;a}^{(2)} N_R^{(2)}
 \end{aligned}$$

Let us now evaluate $\text{rank}(\mathbf{Z}_{O;a;s}^{(2)})$. For this, we start by evaluating $\text{rank}(\check{\mathbf{G}}_{\tilde{v}^{(2)}(s);a}^{(2)})$. Using identity (C.15a), one can easily see that

$$\text{rank}(\check{\mathbf{G}}_{\tilde{v}^{(2)}(s);a}^{(2)}) = \text{rank}\left(FT[\tilde{\mathbf{F}}_a^{(2)}(\tau)]_{\frac{i^{(2)}(\tilde{v}^{(2)}(s))}{T_F}}\right) \equiv \text{rank}\left(\sum_{l=1}^{L_t^{(2)}} e^{-j2\pi \frac{i^{(2)}(\tilde{v}^{(2)}(s))}{T_F} \tau_l^{(2)}} \mathbf{F}_{a;l}^{(2)}\right)$$

For 3GPP channels, the matrices $\{e^{-j2\pi \frac{i^{(2)}(\tilde{v}^{(2)}(s))}{T_F} \tau_l^{(2)}} \mathbf{F}_{a;l}^{(2)}\}$ are different (full rank) matrices whose components are randomly generated as explained in Chapter 2. Consequently, $\sum_{l=1}^{L_t^{(2)}} e^{-j2\pi \frac{i^{(2)}(\tilde{v}^{(2)}(s))}{T_F} \tau_l^{(2)}} \mathbf{F}_{a;l}^{(2)}$ is an $N_R^{(2)} \times n_a^{(2)}$ matrix with randomly generated components and, therefore, it is most likely full rank. It follows that $\text{rank}(\check{\mathbf{G}}_{\tilde{v}^{(2)}(s);a}^{(2)}) = \min(N_R^{(2)}, n_a^{(2)})$. Let $\mathbf{M}(\check{\mathbf{G}}_{\tilde{v}^{(2)}(s);a}^{(2)})$ and $\mathbf{\Lambda}(\check{\mathbf{G}}_{\tilde{v}^{(2)}(s);a}^{(2)})$ denote the $n_a^{(2)} \times n_a^{(2)}$ unitary and diagonal matrices in

the eigenvalue decomposition $\check{\mathbf{G}}_{\check{v}^{(2)}(s);a}^{(2)} = \mathbf{M}^\dagger(\check{\mathbf{G}}_{\check{v}^{(2)}(s);a}^{(2)}) \cdot \Lambda(\check{\mathbf{G}}_{\check{v}^{(2)}(s);a}^{(2)}) \cdot \mathbf{M}(\check{\mathbf{G}}_{\check{v}^{(2)}(s);a}^{(2)})$. We can write $\check{\mathbf{W}}_{O;a;s}^{(2)} = \mathbf{M}^\dagger(\check{\mathbf{G}}_{\check{v}^{(2)}(s);a}^{(2)}) \cdot \check{\mathbf{B}}_{F;a}^{(2)} \cdot \mathbf{M}(\check{\mathbf{G}}_{\check{v}^{(2)}(s);a}^{(2)})$ with the matrix $\check{\mathbf{B}}_{F;a}^{(2)} = \text{diag}[\mathbf{I}_{\min(N_R^{(2)}, n_a^{(2)})}, \mathbf{0}_{(n_a^{(2)} - \min(N_R^{(2)}, n_a^{(2)})) \times (n_a^{(2)} - \min(N_R^{(2)}, n_a^{(2)}))}] = \check{\mathbf{B}}_{F;a}^{(2)} \cdot \check{\mathbf{B}}_{F;a}^{(2)}$. Using (C.15c), we have for all s

$$\begin{aligned} \text{rank}(\mathbf{Z}_{O;a;s}^{(2)}) &= \text{rank}(\check{\mathbf{B}}_{F;a}^{(2)} \cdot \mathbf{M}(\check{\mathbf{G}}_{\check{v}^{(2)}(s);a}^{(2)}) \cdot \check{\mathbf{G}}_{\check{v}^{(2)}(s);a}^{(2)} \cdot \mathbf{M}^\dagger(\check{\mathbf{G}}_{\check{v}^{(2)}(s);a}^{(2)}) \cdot \check{\mathbf{B}}_{F;a}^{(2)}) \\ &= \text{rank}(\check{\mathbf{B}}_{F;a}^{(2)} \cdot \Lambda(\check{\mathbf{G}}_{\check{v}^{(2)}(s);a}^{(2)}) \cdot \check{\mathbf{B}}_{F;a}^{(2)}) \\ &\stackrel{(a)}{=} \text{rank}(\check{\mathbf{G}}_{\check{v}^{(2)}(s);a}^{(2)}) \equiv \min(N_R^{(2)}, n_a^{(2)}) \end{aligned}$$

with (a) due to the fact that $\check{\mathbf{B}}_{F;a}^{(2)}$ and $\Lambda(\check{\mathbf{G}}_{\check{v}^{(2)}(s);a}^{(2)})$ are diagonal whose first $\min(N_R^{(2)}, n_a^{(2)})$ diagonal elements only are nonzero. The second term on the right hand side of (B.51) becomes

$$\sum_{a=1}^{N_G^{(2)}} \sum_{s=1}^{D_p^{(2)}} \text{rank}(\mathbf{Z}_{O;a;s}^{(2)}) \text{rank}(\text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a;\check{v}^{(2)}(s)}^{(2)}]) = \sum_{a=1}^{N_G^{(2)}} D_{p;a}^{(2)} \min(N_R^{(2)}, n_a^{(2)})$$

We finally consider $\text{rank}(\mathbf{K}_{O;j;q;a}^{(2,1)})$. As for $\text{rank}(\mathbf{K}_{j;a}^{(2,1)})$ in Section 4.5.2, we can show that $\text{rank}(\mathbf{K}_{O;j;q;a}^{(2,1)}) \geq \text{rank}(\mathbf{N}_{O;j;q}^{(2,1)})$ based on Corollary C.0.1 and that $\text{rank}(\mathbf{K}_{O;j;q;a}^{(2,1)}) = \text{rank}(\mathbf{Z}_{O;j;q;a}^{(2,1)} + \mathbf{N}_{O;j;q}^{(2,1)})$ (which is independent of the power allocation $(\alpha_p^{(1)}, \alpha_p^{(2)})$). Consider the eigenvalue decomposition $\mathbf{N}_{O;j;q}^{(2,1)} = \mathbf{M}^\dagger(\mathbf{N}_{O;j;q}^{(2,1)}) \cdot \Lambda(\mathbf{N}_{O;j;q}^{(2,1)}) \cdot \mathbf{M}(\mathbf{N}_{O;j;q}^{(2,1)})$ where the $N_R^{(2)} \times N_R^{(2)}$ matrices $\mathbf{M}(\mathbf{N}_{O;j;q}^{(2,1)})$ and $\Lambda(\mathbf{N}_{O;j;q}^{(2,1)})$ are unitary and diagonal, respectively. Using $\text{rank}(\mathbf{N}_{O;j;q}^{(2,1)}) = \min(N_R^{(1)}, n_j^{(1)})$ for $j \in \tilde{\mathcal{A}}_{O;G;c}^{(1)}$, we can write

$$\check{\mathbf{N}}_{O;j;q;a}^{(2,1)} = (FT[\tilde{\mathbf{F}}_a^{(2)}(\tau)]_{\frac{i^{(2)}(v^{(2)}(j,q))}{T_F}})^\dagger \cdot \mathbf{M}^\dagger(\mathbf{N}_{O;j;q}^{(2,1)}) \cdot \check{\mathbf{B}}_{O;j}^{(2,1)} \cdot \mathbf{M}(\mathbf{N}_{O;j;q}^{(2,1)}) \cdot FT[\tilde{\mathbf{F}}_a^{(2)}(\tau)]_{\frac{i^{(2)}(v^{(2)}(j,q))}{T_F}} \quad (\text{B.52})$$

with $\check{\mathbf{B}}_{O;j}^{(2,1)} = \text{diag}[\mathbf{0}_{(\min(N_R^{(1)}, n_j^{(1)}) \times \min(N_R^{(1)}, n_j^{(1)}))}, \mathbf{I}_{N_R^{(2)} - \min(N_R^{(1)}, n_j^{(1)})}]$. Equation (B.52) is equivalent to $\check{\mathbf{N}}_{O;j;q;a}^{(2,1)} = \check{\mathbf{C}}_{N;j;q;a}^{(2,1)} \cdot (\check{\mathbf{C}}_{N;j;q;a}^{(2,1)})^\dagger$ where $\check{\mathbf{C}}_{N;j;q;a}^{(2,1)}$ denotes the $n_a^{(2)} \times (N_R^{(2)} - \min(N_R^{(1)}, n_j^{(1)}))$ matrix formed by the last $N_R^{(2)} - \min(N_R^{(1)}, n_j^{(1)})$ columns of the matrix $(FT[\tilde{\mathbf{F}}_a^{(2)}(\tau)]_{\frac{i^{(2)}(v^{(2)}(j,q))}{T_F}})^\dagger \cdot \mathbf{M}^\dagger(\mathbf{N}_{O;j;q}^{(2,1)})$. For 3GPP channels, $\check{\mathbf{C}}_{N;j;q;a}^{(2,1)}$ is most likely full rank, yielding $r_{O;j;a}^{(2,1)} \triangleq \text{rank}(\check{\mathbf{N}}_{O;j;q;a}^{(2,1)}) = \text{rank}(\check{\mathbf{C}}_{N;j;q;a}^{(2,1)}) = \min(n_a^{(2)}, N_R^{(2)} - \min(N_R^{(1)}, n_j^{(1)}))$ from (C.15). Consider the eigenvalue decomposition $\check{\mathbf{N}}_{O;j;q;a}^{(2,1)} = \mathbf{M}^\dagger(\check{\mathbf{N}}_{O;j;q;a}^{(2,1)}) \cdot \Lambda(\check{\mathbf{N}}_{O;j;q;a}^{(2,1)}) \cdot \mathbf{M}(\check{\mathbf{N}}_{O;j;q;a}^{(2,1)})$ where the $n_a^{(2)} \times n_a^{(2)}$ matrices $\mathbf{M}(\check{\mathbf{N}}_{O;j;q;a}^{(2,1)})$ and $\Lambda(\check{\mathbf{N}}_{O;j;q;a}^{(2,1)})$ are unitary and di-

agonal, respectively. We can write

$$\mathbf{Z}_{O;j;q;a}^{(2,1)} = FT[\tilde{\mathbf{F}}_a^{(2)}(\tau)]_{\frac{i^{(2)}(v^{(2)}(j,q))}{T_F}} \cdot \mathbf{M}^\dagger(\check{\mathbf{N}}_{O;j;q;a}^{(2,1)}) \cdot \check{\mathbf{B}}_{O;j;a}^{(2,1)} \cdot \mathbf{M}(\check{\mathbf{N}}_{O;j;q;a}^{(2,1)}) \cdot (FT[\tilde{\mathbf{F}}_a^{(2)}(\tau)]_{\frac{i^{(2)}(v^{(2)}(j,q))}{T_F}})^\dagger \quad (\text{B.53})$$

where $\check{\mathbf{B}}_{O;j;a}^{(2,1)} = \text{diag}[\mathbf{I}_{r_{O;j;a}^{(2,1)}}, \mathbf{0}_{(n_a^{(2)} - r_{O;j;a}^{(2,1)}) \times (n_a^{(2)} - r_{O;j;a}^{(2,1)})}]$. Given a and j , different values of q lead to different values of the subcarrier indexes $i^{(2)}(v^{(2)}(j, q))$ and $i^{(1)}(j, v^{(2,1)}(j, q))$, which affects $\mathbf{N}_{O;j;q}^{(2,1)}$ and $\mathbf{Z}_{O;j;q;a}^{(2,1)}$ through $\check{\mathbf{F}}_{j;v^{(2,1)}(j,q)}^{(1)} \equiv \sum_{l=1}^{L_t^{(1)}} \mathbf{F}_{j;l}^{(1)} e^{-j2\pi \frac{i^{(1)}(j, v^{(2,1)}(j,q))}{T_F} \tau_l^{(1)}}$, $FT[\tilde{\mathbf{F}}_j^{(2,1)}(\tau)]_{\frac{i^{(2)}(v^{(2)}(j,q))}{T_F}} \equiv \sum_{l=1}^{L_t^{(2)}} \mathbf{F}_{j;l}^{(2,1)} e^{-j2\pi \frac{i^{(2)}(v^{(2)}(j,q))}{T_F} \tau_l^{(2)}}$ and $FT[\tilde{\mathbf{F}}_a^{(2)}(\tau)]_{\frac{i^{(2)}(v^{(2)}(j,q))}{T_F}} \equiv \sum_{l=1}^{L_t^{(2)}} \mathbf{F}_{a;l}^{(2)} e^{-j2\pi \frac{i^{(2)}(v^{(2)}(j,q))}{T_F} \tau_l^{(2)}}$, as seen from (4.140) and (B.53). Since the delays $\tau_l^{(1)}, \tau_l^{(2)}$ and the matrices $\mathbf{F}_{j;l}^{(1)}, \mathbf{F}_{j;l}^{(2,1)}, \mathbf{F}_{a;l}^{(2)}$ are randomly generated, varying q has most likely no impact on $\text{rank}(\mathbf{Z}_{O;j;q;a}^{(2,1)} + \mathbf{N}_{O;j;q}^{(2,1)})$. Hence, varying q does not affect $\text{rank}(\mathbf{K}_{O;j;q;a}^{(2,1)})$ which we denote by $r_{K;j;a}^{(2,1)}$. Similarly, it can be shown that $r_{K;j;a}^{(2,1)}$ is most likely independent of the number of clusters $L_t^{(1)}$ and $L_t^{(2)}$. The third term on the right hand side of (B.51) becomes

$$\sum_{a=1}^{N_G^{(2)}} \sum_{j \in \tilde{\mathcal{A}}_{O,G;c}^{(1)}} \sum_{q=1}^{D_{c;j}^{(2)}} \text{rank}(\mathbf{K}_{O;j;q;a}^{(2,1)}) \text{rank}(\text{diag}_{m=1}^{N_G^{(2)}} [\delta(a, m) \mathbf{E}_{a,v^{(2)}(j,q)}^{(2)}]) = \sum_{a=1}^{N_G^{(2)}} \sum_{j \in \tilde{\mathcal{A}}_{O,G;c}^{(1)}} D_{c;j;a}^{(2)} r_{K;j;a}^{(2,1)}$$

Based on these results, (B.51) reduces to (4.144). $\square$

B.2 Behavior of the function $f(x) = \frac{a_1 + b_1 x}{a_2 + b_2 x}$

In many situations in Chapter 4, we encounter the function $f(x) = \frac{a_1 + b_1 x}{a_2 + b_2 x}$ of a non-negative variable x with $a_2, b_1, b_2 \neq 0$. For example, this occurs when expressing the multiplexing gains in Section 4.2 as functions of the number of shared and proper signal dimensions. Hence, we briefly analyze the behavior of $f(x)$ in this appendix and we provide graphs to illustrate this behavior. We can write

$$\begin{aligned} f(x) &= \frac{b_1}{b_2} \frac{(a_1/b_1 + x)}{(a_2/b_2 - a_1/b_1 + a_1/b_1 + x)} = \frac{b_1}{b_2} \left[1 + \frac{a_2/b_2 - a_1/b_1}{a_1/b_1 + x} \right]^{-1} \\ &= \left[\frac{b_2}{b_1} + \frac{a_2/b_1 - a_1 b_2/b_1^2}{a_1/b_1 + x} \right]^{-1} \end{aligned}$$

We distinguish two cases. If $(a_2/b_1 - a_1 b_2/b_1^2) \geq 0$ or, equivalently, $(a_2 b_1 - a_1 b_2) \geq 0$,

then $f(x)$ is monotonically increasing with increasing x . In this case, the behavior of $f(x)$ is illustrated in Figure B.1a. If $(a_2/b_1 - a_1b_2/b_1^2) \leq 0$ or, equivalently, $(a_2b_1 - a_1b_2) \leq 0$, then $f(x)$ is monotonically decreasing with increasing x . In this case, the behavior of $f(x)$ is illustrated in Figure B.1b.

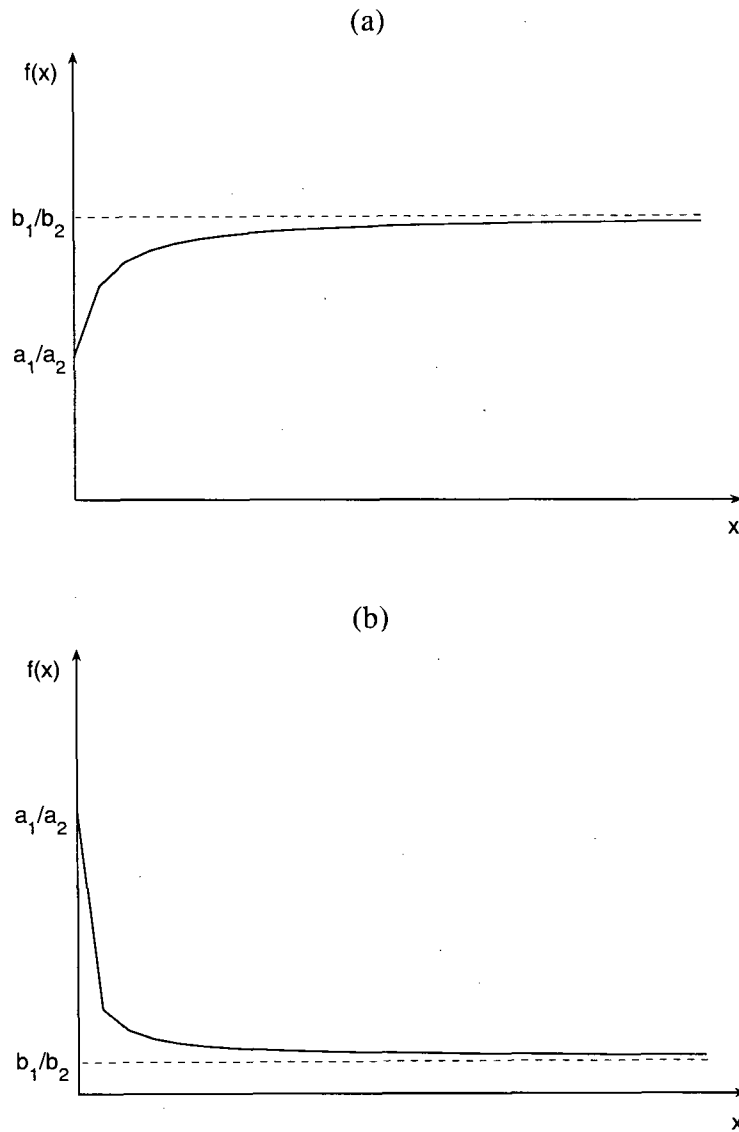


Fig. B.1 a) Behavior of $f(x)$ if $(a_2b_1 - a_1b_2) \geq 0$. b) Behavior of $f(x)$ if $(a_2b_1 - a_1b_2) \leq 0$.

Appendix C

Identities and results from matrix theory

This appendix presents the identities and results from matrix theory that are used in the thesis. Detailed treatment of a broad range of matrix topics can be found in [105, 114].

We start by providing a list of the relevant matrix identities.

1. For any $n \times m$ matrix $\mathbf{U}$ and $m \times n$ matrix $\mathbf{V}$ [114, Corollary 18.1.2]

$$|\mathbf{I}_n + \mathbf{U} \cdot \mathbf{V}| = |\mathbf{I}_m + \mathbf{V} \cdot \mathbf{U}| \quad (\text{C.1})$$

2. For any $n \times m$ matrix $\mathbf{U}$, $p \times q$ matrix $\mathbf{V}$ and $s \times t$ matrix $\mathbf{R}$ [114, Lemma 16.1.1], [130, Theorem T2.1]

$$(\mathbf{U} \otimes \mathbf{V}) \otimes \mathbf{R} = \mathbf{U} \otimes (\mathbf{V} \otimes \mathbf{R}) \quad (\text{C.2})$$

3. For any $n \times m$ matrix $\mathbf{U}$, $p \times q$ matrix $\mathbf{V}$, $m \times t$ matrix $\mathbf{R}$ and $q \times s$ matrix $\mathbf{S}$ [114, Lemma 16.1.2], [130, Theorem T2.4]

$$(\mathbf{U} \otimes \mathbf{V}) \cdot (\mathbf{R} \otimes \mathbf{S}) = (\mathbf{U} \cdot \mathbf{R}) \otimes (\mathbf{V} \cdot \mathbf{S}) \quad (\text{C.3})$$

4. For any $n \times m$ matrix $\mathbf{U}$, $m \times p$ matrix $\mathbf{V}$ and $p \times t$ matrix $\mathbf{R}$ [114, Theorem 16.2.1], [130, Theorem T2.13]

$$\text{vec}(\mathbf{U} \cdot \mathbf{V} \cdot \mathbf{R}) = (\mathbf{R}^T \otimes \mathbf{U}) \cdot \text{vec}(\mathbf{V}) \quad (\text{C.4})$$

5. For any $n \times m$ matrices U and V and any $p \times q$ matrix T [114, p.335]

$$(U + V) \otimes T = (U \otimes T) + (V \otimes T) \quad (C.5a)$$

$$T \otimes (U + V) = (T \otimes U) + (T \otimes V) \quad (C.5b)$$

6. For any $n \times n$ matrix U and $m \times m$ matrix V [114, eq. (3.18) p. 350], [130, Theorem T2.7]

$$|U \otimes V| = |U|^m |V|^n \quad (C.6)$$

7. Let R denote an $n \times n$ matrix, S an $n \times m$ matrix, T an $m \times m$ matrix, and U an $m \times n$ matrix. If R and T are nonsingular, then [114, Theorem 18.1.1]

$$|R + S \cdot T \cdot U| = |R| |T| |T^{-1} + U \cdot R^{-1} \cdot S| \quad (C.7)$$

8. Let U denote an $m \times m$ matrix, R an $m \times n$ matrix, S an $n \times m$ matrix, and T an $n \times n$ matrix. If T is nonsingular, then [114, Theorem 13.3.8]

$$\begin{vmatrix} T & S \\ R & U \end{vmatrix} = \begin{vmatrix} U & R \\ S & T \end{vmatrix} = |T| |U - R \cdot T^{-1} \cdot S| \quad (C.8)$$

If U is also nonsingular, one can easily obtain from (C.8)

$$|T| |U - R \cdot T^{-1} \cdot S| = |U| |T - S \cdot U^{-1} \cdot R| \quad (C.9)$$

9. For $n \times n$ matrices U and T , [114, Theorem 13.3.4], [105, p. 11]

$$|U \cdot T| = |U| |T| \quad (C.10)$$

10. For $n \times m$ matrix U and $m \times n$ matrix T , [114, Lemma 5.2.1]

$$Tr(U \cdot T) = Tr(T \cdot U) \quad (C.11)$$

11. For r scalars $\{k_i\}_{i=1}^r$ and for any $n \times n$ matrices $\{U_i\}_{i=1}^r$, [114, eq. (1.6) p. 50]

$$Tr\left(\sum_{i=1}^r k_i U_i\right) = \sum_{i=1}^r k_i Tr(U_i) \quad (C.12)$$

12. For an arbitrary matrix $\mathbf{T}$ and any nonzero scalar a [114, eq. (4.2) p. 40]

$$\text{rank}(a\mathbf{T}) = \text{rank}(\mathbf{T}) \quad (\text{C.13})$$

13. For any $m \times n$ matrix $\mathbf{U}$ and $p \times q$ matrix $\mathbf{V}$ [114, eq. (1.15) p. 336], [130, Theorem T2.3]

$$(\mathbf{U} \otimes \mathbf{V})^T = (\mathbf{U}^T \otimes \mathbf{V}^T) \quad (\text{C.14})$$

14. For any $m \times n$ matrix $\mathbf{U}$ [105, p. 13]

$$\text{rank}(\mathbf{U}^\dagger \cdot \mathbf{U}) = \text{rank}(\mathbf{U}) \quad (\text{C.15a})$$

$$= \text{rank}(\mathbf{U}^\dagger) \quad (\text{C.15b})$$

$$= \text{rank}(\mathbf{U} \cdot \mathbf{U}^\dagger) \quad (\text{C.15c})$$

15. For any $m \times n$ matrix $\mathbf{U}$ and $p \times q$ matrix $\mathbf{V}$ [114, eq. (1.26) p. 338]

$$\text{rank}(\mathbf{U} \otimes \mathbf{V}) = \text{rank}(\mathbf{U}) \text{rank}(\mathbf{V}) \quad (\text{C.16})$$

16. Let $\mathbf{U}$ represent an $n \times n$ matrix. Then, $\mathbf{U}$ is nonsingular if and only if $|\mathbf{U}| \neq 0$, in which case [114, Theorem 13.3.7]

$$|\mathbf{U}^{-1}| = |\mathbf{U}|^{-1} \quad (\text{C.17})$$

17. For any $n \times n$ matrix $\mathbf{U}$ with eigenvalues $\{\lambda_i(\mathbf{U})\}_{i=1}^n$

$$|\mathbf{I}_n + \mathbf{U}| = \prod_{i=1}^n \lambda_i(\mathbf{I}_n + \mathbf{U}) \stackrel{(a)}{=} \prod_{i=1}^n (1 + \lambda_i(\mathbf{U})) \quad (\text{C.18})$$

where (a) is due to $\lambda_i(\mathbf{I}_n + \mathbf{U})$ being the roots of the characteristic equation [105, p. 87]

$$|\mathbf{I}_n + \mathbf{U} - \lambda \mathbf{I}_n| = |\mathbf{U} - (\lambda - 1)\mathbf{I}_n| = 0,$$

or, equivalently, to

$$\lambda_i(\mathbf{I}_n + \mathbf{U}) = 1 + \lambda_i(\mathbf{U}) \quad (\text{C.19})$$

for all $i = 1, \dots, n$.

18. For any $m \times m$ matrices U and V [106, 9.A.1.a]

$$\{\lambda_1(U \cdot V), \dots, \lambda_m(U \cdot V)\} = \{\lambda_1(V \cdot U), \dots, \lambda_m(V \cdot U)\} \quad (C.20)$$

19. For arbitrary $n \times n$ matrices A and B such that $I_n + A$ is nonsingular, we have

$$|I_n + [I_n + A]^{-1} \cdot B| = |[I_n + A]^{-1} \cdot [I_n + A + B]| \stackrel{(a)}{=} \frac{|I_n + A + B|}{|I_n + A|} \quad (C.21)$$

where (a) is due to (C.10) and (C.17).

20. For any $n \times n$ Hermitian matrices U and V such that U is positive semidefinite (or definite) and V is negative semidefinite, we have

$$\text{Tr}(U \cdot V) \stackrel{(a)}{=} \text{Tr}(U^{1/2} \cdot V \cdot U^{1/2}) \stackrel{(b)}{\leq} 0 \quad (C.22)$$

where $U^{1/2}$ is the Hermitian positive semidefinite square root of U from [105, Theorem 7.2.6] (i.e. $U = U^{1/2} \cdot U^{1/2}$), (a) is due to (C.11), and (b) is due to V (and, hence, $U^{1/2} \cdot V \cdot U^{1/2}$) being negative semidefinite.

21. For arbitrary $m \times k$ matrix A and $k \times n$ matrix B , we have [105, p. 13]

$$\text{rank}(A \cdot B) \leq \min(\text{rank}(A), \text{rank}(B)) \quad (C.23)$$

22. For arbitrary $m \times n$ matrices A and B , we have [105, p. 13]

$$\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B) \quad (C.24)$$

We also use the following results.

Theorem C.0.1 [114, Theorem 21.11.1] *Let θ represent an eigenvalue of an $m \times m$ matrix U , and let τ represent an eigenvalue of a $p \times p$ matrix V . Then, $\theta\tau$ is an eigenvalue of $U \otimes V$.*

Corollary C.0.1 [105, Corollary 4.3.3] *Let T and L be $n \times n$ Hermitian matrices. Assume that $L \succeq 0_{n \times n}$ and that the eigenvalues of T and $T + L$ are arranged in increasing order. Then*

$$\lambda_k(T) \leq \lambda_k(T + L) \text{ for all } k = 1, \dots, n$$

Lemma C.0.1 *Let $\mathbf{A}$ denote an $n \times n$ Hermitian positive semidefinite matrix, and let k denote a non-negative scalar such that $\text{Tr}(\mathbf{A}) = k$. Then, $k\mathbf{I}_n - \mathbf{A}$ is Hermitian positive semidefinite.*

Proof of Lemma C.0.1: Clearly, $k\mathbf{I}_n - \mathbf{A}$ is Hermitian. For all nonzero vectors $\mathbf{x} \in \mathbb{C}^n$, we have

$$\mathbf{x}^\dagger \cdot [k\mathbf{I}_n - \mathbf{A}] \cdot \mathbf{x} = k \|\mathbf{x}\|^2 - \mathbf{x}^\dagger \cdot \mathbf{A} \cdot \mathbf{x} = \|\mathbf{x}\|^2 \left[k - \frac{\mathbf{x}^\dagger \cdot \mathbf{A} \cdot \mathbf{x}}{\|\mathbf{x}\|^2} \right]$$

By [105, Theorem 4.2.2], $\frac{\mathbf{x}^\dagger \cdot \mathbf{A} \cdot \mathbf{x}}{\|\mathbf{x}\|^2}$ is less or equal than the maximum eigenvalue of $\mathbf{A}$ and hence, $\frac{\mathbf{x}^\dagger \cdot \mathbf{A} \cdot \mathbf{x}}{\|\mathbf{x}\|^2} \leq \text{Tr}(\mathbf{A}) = k$. It follows that $\left[k - \frac{\mathbf{x}^\dagger \cdot \mathbf{A} \cdot \mathbf{x}}{\|\mathbf{x}\|^2} \right] \geq 0$, yielding $\mathbf{x}^\dagger \cdot [k\mathbf{I}_n - \mathbf{A}] \cdot \mathbf{x} \geq 0$ for all nonzero $\mathbf{x}$. Hence, $k\mathbf{I}_n - \mathbf{A}$ is positive semidefinite. $\square$

The next lemma shows approximately how the eigenvalues of a Hermitian matrix $\mathbf{L}$ change when a Hermitian perturbation matrix is added to $\mathbf{L}$.

Lemma C.0.2 *Let $\mathbf{L}$ denote an $n \times n$ Hermitian matrix with (not necessarily distinct) eigenvalues $\{\theta_t\}_{t=1}^n$ and with eigenvectors $\{\mathbf{l}_t\}_{t=1}^n$. Let $\tilde{\mathbf{L}}$ denote an $n \times n$ Hermitian perturbation of $\mathbf{L}$. The eigenvalues of $\mathbf{L} + \tilde{\mathbf{L}}$ can be written using a first order approximation as $\theta_t + \mathbf{l}_t^\dagger \cdot \tilde{\mathbf{L}} \cdot \mathbf{l}_t$ for all $t = 1, \dots, n$.*

Proof of Lemma C.0.2: The proof follows similar steps to those provided in [131, Section 6.12]. Since $\mathbf{L}$ is Hermitian, $\{\mathbf{l}_t\}_{t=1}^n$ form an orthonormal set, yielding

$$\mathbf{l}_t^\dagger \cdot \mathbf{l}_k = \delta(t, k) \quad (\text{C.25})$$

Let $\theta_t + \tilde{\theta}_t$ and $\mathbf{l}_t + \tilde{\mathbf{l}}_t$ respectively denote the eigenvalues and eigenvectors of the perturbed matrix $\mathbf{L} + \tilde{\mathbf{L}}$. Since $\mathbf{L}$ and $\tilde{\mathbf{L}}$ are Hermitian, $\mathbf{L} + \tilde{\mathbf{L}}$ is also Hermitian with real eigenvalues $\theta_t + \tilde{\theta}_t$. For all $t = 1, \dots, n$, we have

$$(\mathbf{L} + \tilde{\mathbf{L}}) \cdot (\mathbf{l}_t + \tilde{\mathbf{l}}_t) = (\theta_t + \tilde{\theta}_t)(\mathbf{l}_t + \tilde{\mathbf{l}}_t) \quad (\text{C.26})$$

and

$$\mathbf{L} \cdot \mathbf{l}_t = \theta_t \mathbf{l}_t \quad (\text{C.27})$$

The perturbed eigenvector $\mathbf{l}_t + \tilde{\mathbf{l}}_t$ can be written in the space spanned by $\{\mathbf{l}_k\}_{k=1}^n$ as $\mathbf{l}_t + \tilde{\mathbf{l}}_t = \sum_{k=1}^n a'_{t;k} \mathbf{l}_k$, $a'_{t;k} \in \mathbb{C}$. If $\tilde{\mathbf{L}} = \mathbf{0}_{n \times n}$, then $\tilde{\theta}_t = 0$ for all t but (C.26) holds for any vector $\tilde{\mathbf{l}}_t$ which is a multiple of $\mathbf{l}_t$. To ensure $\tilde{\mathbf{l}}_t = \mathbf{0}_n$ if $\tilde{\mathbf{L}} = \mathbf{0}_{n \times n}$, we normalize $\mathbf{l}_t + \tilde{\mathbf{l}}_t$ by assuming that $a'_{t;t} = 1$ in the expansion $\mathbf{l}_t + \tilde{\mathbf{l}}_t = \sum_{k=1}^n a'_{t;k} \mathbf{l}_k$. Equivalently, we assume that

$$\tilde{\mathbf{l}}_t = \sum_{k=1}^n a_{t;k} \mathbf{l}_k \quad (\text{C.28})$$

with $a_{t,t} = 0$, and $a_{t;k} = a'_{t;k}$ for $k \neq t$. Hence, $\tilde{\mathbf{l}}_t$ is a linear combination of vectors that are orthogonal to $\mathbf{l}_t$, and

$$\tilde{\mathbf{l}}_t^\dagger \cdot \mathbf{l}_k = a_{t;k}^* \quad (\text{C.29})$$

for $k = 1, \dots, n$. Furthermore, for arbitrary $n \times 1$ vectors $\mathbf{a}$ and $\mathbf{b}$, we have

$$(\mathbf{L} \cdot \mathbf{a})^\dagger \cdot \mathbf{b} = \mathbf{a}^\dagger \cdot \mathbf{L}^\dagger \cdot \mathbf{b} = \mathbf{a}^\dagger \cdot (\mathbf{L} \cdot \mathbf{b}) \quad (\text{C.30})$$

since $\mathbf{L}$ is Hermitian. Using (C.26) and (C.27), we have $\mathbf{L} \cdot \tilde{\mathbf{l}}_t + \tilde{\mathbf{L}} \cdot \mathbf{l}_t + \tilde{\mathbf{L}} \cdot \tilde{\mathbf{l}}_t = (\theta_t + \tilde{\theta}_t) \tilde{\mathbf{l}}_t + \tilde{\theta}_t \mathbf{l}_t$ and hence,

$$(\mathbf{L} \cdot \tilde{\mathbf{l}}_t)^\dagger \cdot \mathbf{l}_k + (\tilde{\mathbf{L}} \cdot \mathbf{l}_t)^\dagger \cdot \mathbf{l}_k + (\tilde{\mathbf{L}} \cdot \tilde{\mathbf{l}}_t)^\dagger \cdot \mathbf{l}_k = (\theta_t + \tilde{\theta}_t) \tilde{\mathbf{l}}_t^\dagger \cdot \mathbf{l}_k + \tilde{\theta}_t \mathbf{l}_t^\dagger \cdot \mathbf{l}_k \quad (\text{C.31})$$

for $k = 1, \dots, n$. From (C.30), (C.27) and (C.29) one can easily show that

$$(\mathbf{L} \cdot \tilde{\mathbf{l}}_t)^\dagger \cdot \mathbf{l}_k = \tilde{\mathbf{l}}_t^\dagger \cdot (\mathbf{L} \cdot \mathbf{l}_k) = \theta_k a_{t;k}^* \quad (\text{C.32})$$

Using (C.25), (C.28)-(C.30) and (C.32), equation (C.31) can be written as

$$\theta_k a_{t;k}^* + \mathbf{l}_t^\dagger \cdot \tilde{\mathbf{L}} \cdot \mathbf{l}_k + \left(\sum_{p=1}^n a_{t;p} \mathbf{l}_p \right)^\dagger \cdot \tilde{\mathbf{L}} \cdot \mathbf{l}_k = (\theta_t + \tilde{\theta}_t) a_{t;k}^* + \tilde{\theta}_t \delta(t, k) \quad (\text{C.33})$$

Since $k \in \{1, \dots, n\}$, (C.33) results for each t in a system of n equations with unknowns $\tilde{\theta}_t$ and $\{a_{t;k}\}_{k=1, k \neq t}^n$ (since $a_{t,t} = 0$). An approximation of $\tilde{\theta}_t$ can be obtained as in [131, Section 6.12] by writing (C.31) to the first order, i.e.

$$(\mathbf{L} \cdot \tilde{\mathbf{l}}_t)^\dagger \cdot \mathbf{l}_k + (\tilde{\mathbf{L}} \cdot \mathbf{l}_t)^\dagger \cdot \mathbf{l}_k = \theta_t \tilde{\mathbf{l}}_t^\dagger \cdot \mathbf{l}_k + \tilde{\theta}_t \mathbf{l}_t^\dagger \cdot \mathbf{l}_k$$

Now taking $k = t$, we obtain $(\mathbf{L} \cdot \tilde{\mathbf{l}}_t)^\dagger \cdot \mathbf{l}_t + (\tilde{\mathbf{L}} \cdot \mathbf{l}_t)^\dagger \cdot \mathbf{l}_t = \theta_t \tilde{\mathbf{l}}_t^\dagger \cdot \mathbf{l}_t + \tilde{\theta}_t \mathbf{l}_t^\dagger \cdot \mathbf{l}_t$ which, using

(C.25), (C.27) and (C.29) with $a_{t,t} = 0$, reduces to $\tilde{\theta}_t = \mathbf{l}_t^\dagger \cdot \tilde{\mathbf{L}} \cdot \mathbf{l}_t$. Thus, the eigenvalues of $\mathbf{L} + \tilde{\mathbf{L}}$ can be approximated to the first order as $\theta_t + \mathbf{l}_t^\dagger \cdot \tilde{\mathbf{L}} \cdot \mathbf{l}_t$ for all $t = 1, \dots, n$. $\square$

Appendix D

Matlab routines for the numerical results

This appendix presents a summary of the Matlab routines used to generate the numerical results in Chapters 3 and 4. The Matlab routines are provided on a CD-ROM attached to this thesis.

D.1 Numerical results for Chapter 3

We provide in Table D.1 the list and description of the Matlab routines corresponding to the numerical results of Chapter 3. On the CD-ROM, these routines are stored in the directory called Chapter3. This directory contains three sub-directories. The first one is called Samples-3GPP-ABCD and it contains the Matlab routines that are used to generate Channels A, B, C and D of Tables 3.1-3.4 as well as the maximum eigenvalue results of Table 3.5. The files cinfo-ChAPrime.mat, cinfo-ChB.mat, cinfo-ChC.mat and cinfo-ChD.mat in this sub-directory are used to store Channels A, B, C and D, respectively. The textual description of these channels is provided in the .txt files. The second sub-directory is called ChannelA-Graphs and it contains the Matlab routines associated with the graphs of Figures 3.2-3.7. The third sub-directory is called 3GPP-angle-spread and it contains the Matlab routines associated with the graphs of Figures 3.8-3.14. The file minicinfo.mat in this sub-directory is used to store the values of the parameters σ_{SF} , $\{P_k\}_{k=1}^{L_t}$ and $\{\{\varphi_{k;s}\}_{s=0}^{S_k-1}\}_{k=1}^{L_t}$ that are used while investigating the effect of angle spread in Section 3.4. Table D.1 presents a list of the Matlab files associated with the numerical results of Chapter 3. Detailed description of the corresponding algorithms follows in subsections D.1.1 through D.1.9.

File Name	Description
channel.m	Generates samples of the 3GPP channel following [83]. See Section D.1.1.
Randeig.m	Generates the maximum eigenvalue and its bounds (Table 3.5). See Section D.1.2.
CvsSNRwf.m	It is used to calculate the single-user capacity C and associated $(E_b/N_0)_T$ using water-filling power allocation. See Section D.1.3.
CvsSun.m	It is used to calculate the information rate I_u and associated $(E_b/N_0)_{T,u}$ using different power allocations described by different $\{\eta_i\}_{i=1}^{N_G}$. See Section D.1.4.
CvsSunAs.m	It is used to calculate the information rate I_u and associated $(E_b/N_0)_{T,u}$ using the strongest antenna power allocation. See Section D.1.5.
ATAvsPOTA.m	Generates the graphs of Figure 3.2. See Section D.1.6.
ATAvsOTA.m	Generates the graphs of Figure 3.3. See Section D.1.6.
POATANR2.m	Generates the graphs of Figure 3.4. See Section D.1.6.
ATAMC.m	Generates the graphs of Figure 3.5. See Section D.1.6.
POTAMC.m	Generates the graphs of Figure 3.6. See Section D.1.6.
OTAMC.m	Generates the graphs of Figure 3.7. See Section D.1.6.
AngSpread.m	Calculates the angle spread at the transmitter σ_{TX} and receiver σ_{RX} as well as the AODs $\{\theta_{T,k;s}\}$ and AOAs $\{\theta_{R,k;s}\}$. See Section D.1.7.
WFinfo.m	It is used to calculate the single-user capacity C . See Section D.1.8.
AngSpSTMF.m	Generates the graphs of Figures 3.8-3.14. See Section D.1.9.

Table D.1 Matlab files associated with the numerical results in Chapter 3

D.1.1 Algorithm to generate samples of the 3GPP channel model

We describe the algorithm associated with the Matlab file channel.m and used to generate samples of the 3GPP spatial channel model for urban microcell environments. The algorithm can be found in [83, Section 5.3.2]. For the sake of completeness, we briefly provide the steps of this algorithm.

Input: None.

Output: The path gains $\{\{G_{k;s}\}_{s=0}^{S_k-1}\}_{k=1}^{L_t}$ and the coefficients $\{\{-\alpha_{k;s}\}_{s=0}^{S_k-1}\}_{k=1}^{L_t}$ and $\{\{-\beta_{k;s}\}_{s=0}^{S_k-1}\}_{k=1}^{L_t}$.

Algorithm: All parameters follow the specification of Section 2.1. We choose $L_t = 3$ and $S_k = 20$ for all $k = 1, \dots, L_t$.

Step 1: Generate the location angles θ_{BS} , ω and θ_{MS} .

Step 2: Generate the parameter γ and the lognormal shadow fading coefficient σ_{SF} .

Step 3: Generate the propagation delays τ_k , $k = 1, \dots, L_t$. The minimum of these delays is subtracted from all the delays so that the first delay is equal to zero.

Step 4: Generate the coefficients z_k and the powers P_k given by (2.7), $k = 1, \dots, L_t$.

Step 5: Generate the mean AODs $\bar{\theta}_{T;k}$, $k = 1, \dots, L_t$, and associate each $\bar{\theta}_{T;k}$ with the offset AODs $\hat{\theta}_{T;k;s}$ provided in [83, Table 5.2].

Step 6: Generate the carrier phases $\{\{\varphi_{k;s}\}_{s=0}^{S_k-1}\}_{k=1}^{L_t}$.

Step 7: Generate the mean AOAs $\bar{\theta}_{R;k}$, $k = 1, \dots, L_t$. For each k , we perform a random permutation of the offset AOAs $\hat{\theta}_{R;k;s}$ provided in [83, Table 5.2] and we associate the permuted values with $\bar{\theta}_{R;k}$.

Step 8: Compute the AODs $\theta_{T;k;s}$ using (2.1) and AOAs $\theta_{R;k;s}$ using (2.2), for $s = 0, \dots, S_k - 1$ and $k = 1, \dots, L_t$.

Step 9: Compute the path gains $G_{k;s}$ using (2.6) for $s = 0, \dots, S_k - 1$ and $k = 1, \dots, L_t$.

Step 10: Store the path gains $\{\{G_{k;s}\}_{s=0}^{S_k-1}\}_{k=1}^{L_t}$ and the coefficients $\{\{-\alpha_{k;s}\}_{s=0}^{S_k-1}\}_{k=1}^{L_t}$ and $\{\{-\beta_{k;s}\}_{s=0}^{S_k-1}\}_{k=1}^{L_t}$ in an output file. Terminate.

D.1.2 Algorithm to compute the lower and upper bounds on $\lambda_{\max}$

We describe the algorithm associated with the Matlab file Randeig.m and used to compute the maximum eigenvalue $\lambda_{\max}$ and its lower and upper bounds (3.23) and (3.25)-(3.26).

Input: The spatial channel parameters generated by channel.m.

Output: The eigenvalue $\lambda_{\max}$ and its bounds (3.23) and (3.25)-(3.26).

Algorithm: The results presented in Table 3.5 correspond to different values of (n_i, N_R) and to one TOG. The number of time resolvable clusters is $L_t = 3$.

Step 1: Load the channel parameters that have been generated by channel.m.

For each (n_i, N_R) , we perform the following procedure.

Step 2: Compute the matrices $\mathbf{F}_{1;k}$ for $k = 1, \dots, L_t$ according to (2.21).

Step 3: Compute the matrix product $\mathbf{C}_1^\dagger \cdot \mathbf{C}_1$ using (2.22). Compute $\lambda_{\max}$ and its lower and upper bounds given by (3.23).

Step 4: Compute the DFT matrix $\mathbf{U}_{T,1}^{DFT}$ using the function "dftmtx" in Matlab. Compute the transformed matrix product $(\check{\mathbf{C}}_1^{DFT})^\dagger \cdot \check{\mathbf{C}}_1^{DFT}$ and the lower and upper bounds given by (3.25).

Step 5: Compute the DCT matrix $U_{T;1}^{DCT}$ using the function "dctmtx" in Matlab. Compute the transformed matrix product $(\check{C}_1^{DCT})^\dagger \cdot \check{C}_1^{DCT}$ and the lower and upper bounds given by (3.26).

Step 6: If more (n_i, N_R) are considered then go to Step 2. Else, terminate.

D.1.3 Algorithm to calculate C

We describe the algorithm associated with the Matlab file CvsSNRwf.m and used to calculate the single-user capacity C (3.9) and the associated SNR-per-bit $(E_b/N_0)_T$ using water-filling power allocation. The algorithm consists in varying C within a range $[C_{\min}, C_{\max}]$, using increments of step size δC , and computing the $(E_b/N_0)_T$ associated with each value of C . The minimal value $C_{\min}$ is very small but nonzero in order to avoid division by zero. The algorithm works as follows. Let $\{\lambda_q\}_{q=1}^{N_T}$ denote the eigenvalues $\{\{\lambda_{i,l}\}_{l=1}^{n_i}\}_{i=1}^{N_G}$ arranged in decreasing order, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{r_T} > \lambda_{r_T+1} = \dots = \lambda_{N_T} = 0$. We consider the water-filling scheme associated with the eigenvalues $\{\lambda_q\}_{q=1}^{r_T}$ and depicted in Figure D.1. Let C_m denote the capacity obtained with the water-filling constant $\mu_m = \lambda_{m+1}^{-1}$, for

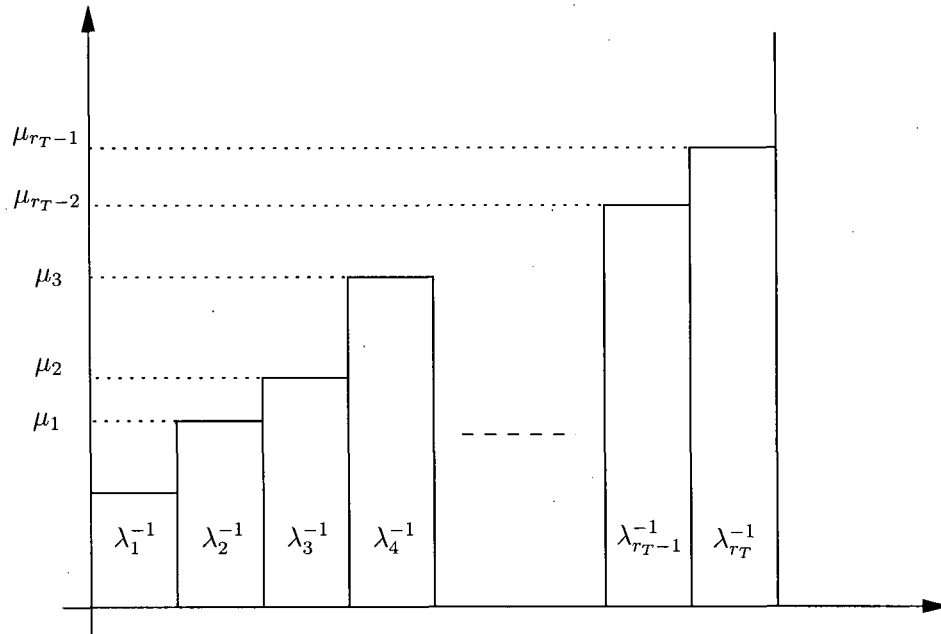


Fig. D.1 The water-filling scheme associated with the eigenvalues $\{\lambda_q\}_{q=1}^{r_T}$.

$m = 1, \dots, r_T - 1$, and $C_0 = C_{\min}$. For each $m = 1, \dots, r_T - 1$, we vary C in the

interval $(C_{m-1}, C_m]$ for which m eigenmodes are active. As such, we have from (3.9) that $C = \frac{m}{N_G} \log(\mu) + A$ with $A = \frac{1}{N_G} \sum_{l=1}^m \log(\lambda_l)$, $\mu = \frac{N_G P_T}{m N_0 D} + d$ and $d = \frac{1}{m} \sum_{l=1}^m \lambda_l^{-1}$. Using these equations and (3.3), the algorithm calculates the SNR-per-bit associated with C as follows

$$(E_b/N_0)_T = \frac{2^{\frac{N_G(C-A)}{m}} - d}{(N_G C/m)}$$

For $C \geq C_{r_T-1}$, all the r_T eigenmodes in Figure D.1 are active, yielding $(E_b/N_0)_T = \frac{2^{\frac{N_G(C-A)}{r_T}} - d}{(N_G C/r_T)}$ with $A = \frac{1}{N_G} \sum_{l=1}^{r_T} \log(\lambda_l)$ and $d = \frac{1}{r_T} \sum_{l=1}^{r_T} \lambda_l^{-1}$. If at any step C exceeds the maximum value $C_{\max}$, then the algorithm stops.

Input: The set of eigenvalues $\{\{\lambda_{i,l}\}_{l=1}^{n_i}\}_{i=1}^{N_G}$, the number of TOGs N_G , the minimum and maximum values for capacity $C_{\min}$ and $C_{\max}$ and the step size δC . Note that the eigenvalues $\lambda_{i,l}$ of the matrix $\text{diag}_{i=1}^{N_G}[\mathbf{C}_i^\dagger \cdot \mathbf{C}_i]$ are obtained using the Matlab function *eig* in the main program prior to calling the routine *CvsSNRwf.m*.

Output: The vectors of values of C and associated $(E_b/N_0)_T$ in dB.

Algorithm: We use the following water-filling procedure.

Step 1: Arrange the nonzero eigenvalues $\{\{\lambda_{i,l}\}_{l=1}^{n_i}\}_{i=1}^{N_G}$ and sort them in decreasing order. We label the newly obtained eigenvalues as $\{\lambda_q\}_{q=1}^{r_T}$. The eigenvalues $\lambda_{i,l}$ that are equal to zero are not used in the water-filling procedure.

Step 2: Initialize $C_0 = C_{\min}$. Let m denote the number of active eigenmodes. Initialize $m = 1$. If $r_T = 1$ then go to Step 5.

Step 3: Calculate the capacity corresponding to a water level equal to λ_{m+1}^{-1} ,

$$C_m = \frac{1}{N_G} \sum_{l=1}^m \log(\lambda_{m+1}^{-1} \lambda_l)$$

If $C_m > C_{\max}$ then $C_m = C_{\max}$. Calculate

$$A = \frac{1}{N_G} \sum_{l=1}^m \log(\lambda_l), \quad d = \frac{1}{m} \sum_{l=1}^m \lambda_l^{-1}$$

Take $\tilde{C} \in [C_0, C_m]$ with increments of step size δC . For each $\tilde{C}$, calculate the associated SNR in dB

$$\tilde{B} = 10 \log_{10} \left(\frac{2^{\frac{N_G(\tilde{C}-A)}{m}} - d}{(N_G \tilde{C}/m)} \right)$$

Update the output vectors C and $(E_b/N_0)_T$. Take $C_0 = C_m + \delta C$.

Step 4: If $C_m = C_{\max}$, then stop. Else, if $m < r_T - 1$ then take $m = m + 1$ and go to Step 3.

Step 5: We consider now the case where all the eigenmodes are active. Calculate

$$A = \frac{1}{N_G} \sum_{l=1}^{r_T} \log(\lambda_l), \quad d = \frac{1}{r_T} \sum_{l=1}^{r_T} \lambda_l^{-1}$$

Take $\tilde{C} \in [C_0, C_{\max}]$ with increments of step size δC . For each $\tilde{C}$, calculate the associated SNR in dB

$$\tilde{B} = 10 \log_{10} \left(\frac{2^{\frac{N_G(\tilde{C}-A)}{r_T}} - d}{(N_G \tilde{C}/r_T)} \right)$$

Update the output vectors C and $(E_b/N_0)_T$. Terminate.

D.1.4 Algorithm to calculate I_u for different $\{\eta_i\}_{i=1}^{N_G}$

We describe the algorithm associated with the Matlab file CvsSun.m. This algorithm is used to calculate the rate I_u and the associated SNR-per-bit $(E_b/N_0)_{T,u}$ where I_u is obtained from (3.39) using $r_{s,i} = n_i$ and $\xi_{i,l} = \lambda_{i,l}$ for all $l = 1, \dots, n_i$ and $i = 1, \dots, N_G$. We used this algorithm with the uniform PA $\eta_i = \frac{n_i}{N_T}$, the uniform TOG PA $\eta_i = \frac{1}{N_G}$, the high SNR optimal TOG PA $\eta_i = \frac{r_i}{r_T}$ and one TOG PA $\eta_i = \delta(i, p)$ where $p = 1, \dots, N_G$. The algorithm consists in varying the ratio $\frac{P_T}{N_0 D}$ (i.e. the product $I_u(E_b/N_0)_{T,u}$) in the range $[0, [\frac{P_T}{N_0 D}]_{\max}]$ where $[\frac{P_T}{N_0 D}]_{\max}$ is an arbitrary high value for $\frac{P_T}{N_0 D}$, and then for each value of $\frac{P_T}{N_0 D}$ we calculate the rate I_u and the associated $(E_b/N_0)_{T,u}$. The algorithm works as follows. We start by arbitrarily fixing a maximum value $[\frac{P_T}{N_0 D}]_{\max}$, and then we choose 100 values for $\frac{P_T}{N_0 D}$ in $[0, [\frac{P_T}{N_0 D}]_{\max}]$ using increments with variable step size. With $\{\eta_i\}_{i=1}^{N_G}$ and the eigenvalues $\lambda_{i,l}$ being given as inputs to the algorithm, we calculate for each value of $\frac{P_T}{N_0 D}$ the information rate I_u obtained from (3.39) by taking $r_{s,i} = n_i$ and given by

$$I_u = \frac{1}{N_G} \sum_{i=1}^{N_G} \sum_{l=1}^{n_i} \log \left(1 + \frac{\eta_i}{n_i} N_G \frac{P_T}{N_0 D} \lambda_{i,l} \right)$$

Once this rate I_u is calculated, we evaluate the associated SNR-per-bit $(E_b/N_0)_{T,u} = \frac{1}{I_u} \frac{P_T}{N_0 D}$.

Input: The set of eigenvalues $\{\{\lambda_{i,l}\}_{l=1}^{n_i}\}_{i=1}^{N_G}$, the number of antenna elements in each TOG $\{n_i\}_{i=1}^{N_G}$, the power allocation $\{\eta_i\}_{i=1}^{N_G}$, and a maximum value $I_{\max}$ for I_u . Note that the eigenvalues $\lambda_{i,l}$ of the matrix $\text{diag}_{i=1}^{N_G}[\mathbf{C}_i^\dagger \cdot \mathbf{C}_i]$ are obtained using the Matlab function *eig* in the main program prior to calling the routine CvsSun.m.

Output: The vectors of values of I_u and associated $(E_b/N_0)_{T;u}$ in dB.

Algorithm: We use the following procedure.

Step 1: Initialize $\left[\frac{P_T}{N_0 D}\right]_{\max} = 10^5 I_{\max}$ where the constant 10^5 is an arbitrary high value of $(E_b/N_0)_{T;u}$. Define the parameter $x = \frac{0.22[\frac{P_T}{N_0 D}]_{\max}}{1.22^{100}-1}$. Initialize $\tilde{P} = 0$ and $p = 0$.

Step 2: Take $\tilde{P} = \tilde{P} + (1.22)^p x$. Calculate the rate

$$\tilde{I} = \frac{1}{N_G} \sum_{i=1}^{N_G} \sum_{l=1}^{n_i} \log \left(1 + \frac{\eta_i}{n_i} N_G \tilde{P} \lambda_{i,l} \right)$$

Calculate the associated SNR in dB

$$\tilde{B} = 10 \log_{10}(\tilde{P}/\tilde{I})$$

Update the output vectors I_u and $(E_b/N_0)_{T;u}$. Take $p = p + 1$.

Step 3: If $p \leq 99$, go to Step 2. Else, terminate.

D.1.5 Algorithm to calculate I_u using the strongest antenna PA

We describe the algorithm associated with the Matlab file CvsSunAs.m. This algorithm is similar to that of the previous section except that its input consists of the element $\text{maxel} \equiv \max\{\|\mathbf{C}_{*,l}\|^2\}_{l=1}^{N_T}$, the number of TOGs N_G , and a maximum value $I_{\max}$ for I_u . Also, we calculate the rate in Step 2 using (3.40) where B is replaced by $B_{\max}$ of (3.44), that is,

$$\tilde{I} = \frac{1}{N_G} \log \left(1 + N_G \tilde{P} \text{maxel} \right)$$

D.1.6 Algorithm to generate the graphs of Figures 3.2-3.7

We present now the algorithm that we used to generate the graphs of Figures 3.2-3.7. The Matlab code can be found in the files specified in Table D.1.

Algorithm:

- Step 1:** Define the number of transmit antennas N_T , and the parameters $C_{\min}$, $C_{\max}$, δC and $I_{\max}$ that are used for capacity and rate calculation.
- Step 2:** Load the spatial channel parameters that were generated by the routine channel.m. The data is stored in the file cinfo-ChAPrime.mat for Channel A.
- Step 3:** Define the space-time modulation format by N_G and the number of antenna elements in each TOG $\{n_i\}_{i=1}^{N_G}$. Define the number of receive antennas N_R .
- Step 4:** Build the matrices $\mathbf{F}_{i;k}$ using (2.21) and $\mathbf{C}_i^\dagger \cdot \mathbf{C}_i$ using (2.22). Evaluate the eigenvalues $\{\{\lambda_{i;l}\}_{l=1}^{n_i}\}_{i=1}^{N_G}$.
- Step 5:** Calculate the capacity C and the rate I_u along with the associated SNRs using the algorithms of Sections D.1.3 - D.1.5.
- Step 6:** If evaluation of the capacity and transmission rate is required for different number of receive antennas or different space-time modulation format, then go to Step 3. If evaluation of the capacity and transmission rate is required for different propagation scenarios (i.e. different (L_t, L_r)), then update the coefficient $\sqrt{\frac{P_k}{S_k}}$ in (2.6) such that the powers $\{P_k\}_{k=1}^{L_t}$ (2.7) sum to one whenever L_t , $\{S_k\}_{k=1}^{L_t}$ change, and go to Step 4. Else, go to Step 7.
- Step 7:** Plot the curves of C and I_u as functions of $(E_b/N_0)_T$. Terminate.

D.1.7 Algorithm to calculate the angle spread and the AODs and AOAs

We describe the algorithm associated with the Matlab file AngSpread.m. This algorithm computes the angle spread at the transmitter σ_{TX} and at the receiver σ_{RX} as well as the AODs and AOAs, following the simplified model used in Section 3.4 while investigating the effect of the angle spread. For this model, the AODs associated with clusters 1, 2 and 3 are equally spaced in the intervals $[-A_{TX} - b_{TX}, -A_{TX} + b_{TX}]$, $[-b_{TX}, b_{TX}]$ and $[A_{TX} - b_{TX}, A_{TX} + b_{TX}]$, respectively, and the AOAs associated with clusters 1, 2 and 3 are equally spaced in the intervals $[-A_{RX} - b_{RX}, -A_{RX} + b_{RX}]$, $[-b_{RX}, b_{RX}]$ and $[A_{RX} - b_{RX}, A_{RX} + b_{RX}]$, respectively. We have equal number of space resolvable paths per cluster $S_t = S_1 = S_2 = S_3 = 20$. The algorithm works as follows. Given A_{TX} and b_{TX} , we start by calculating the AODs $\{\{\theta_{T;k;s}\}_{s=0}^{S_t-1}\}_{k=1}^3$. Given A_{RX} and b_{RX} , we also calculate the AOAs $\{\{\theta_{R;k;s}\}_{s=0}^{S_t-1}\}_{k=1}^3$. Once the AODs and AOAs are calculated, we compute σ_{TX} and σ_{RX} using (3.71). The algorithm is the following.

Input: The angles $A_{TX}, b_{TX}, A_{RX}, b_{RX}$, the cluster powers $\{P_k\}_{k=1}^3$, and the number of space resolvable paths per cluster S_t .

Output: The AODs $\{\{\theta_{T;k;s}\}_{s=0}^{S_t-1}\}_{k=1}^3$, AOA $\{\{\theta_{R;k;s}\}_{s=0}^{S_t-1}\}_{k=1}^3$ and the angle spread values σ_{TX}, σ_{RX} .

Algorithm: We use the following procedure.

Step 1: Calculate the AODs as follows: $\theta_{T;1;s} = -A_{TX} - b_{TX} + s \frac{2b_{TX}}{(S_t-1)}$, $\theta_{T;2;s} = -b_{TX} + s \frac{2b_{TX}}{(S_t-1)}$ and $\theta_{T;3;s} = A_{TX} - b_{TX} + s \frac{2b_{TX}}{(S_t-1)}$ for $s = 0, \dots, S_t - 1$.

Step 2: Calculate the AOAs as follows: $\theta_{R;1;s} = -A_{RX} - b_{RX} + s \frac{2b_{RX}}{(S_t-1)}$, $\theta_{R;2;s} = -b_{RX} + s \frac{2b_{RX}}{(S_t-1)}$ and $\theta_{R;3;s} = A_{RX} - b_{RX} + s \frac{2b_{RX}}{(S_t-1)}$ for $s = 0, \dots, S_t - 1$.

Step 3: Evaluate the angle spread σ_{TX} using (3.71a) and σ_{RX} using (3.71b). Terminate.

D.1.8 The water-filling scheme associated with C

We describe the algorithm associated with the Matlab file WFinfo.m. This algorithm evaluates the water-filling scheme over the eigenmodes $\lambda_{i,l}$. For simplicity of notation, we denote the eigenvalues $\{\{\lambda_{i,l}\}_{l=1}^{n_i}\}_{i=1}^{N_G}$ as $\{\tilde{\lambda}_q\}_{q=1}^{N_T}$. If the number of active eigenmodes is m , the channel capacity is

$$C = \frac{1}{N_G} \sum_{l=1}^m \log(\mu \tilde{\lambda}_l) = A_m + \frac{m}{N_G} \log\left(\frac{N_G P_T}{m N_0 D} + d_m\right) \quad (D.1)$$

where μ satisfies $\sum_{l=1}^m \{\mu - \tilde{\lambda}_l^{-1}\} = \frac{N_G P_T}{N_0 D}$ (and hence $\mu = \frac{N_G P_T}{m N_0 D} + \frac{1}{m} \sum_{l=1}^m \tilde{\lambda}_l^{-1}$), $A_m = \frac{1}{N_G} \sum_{l=1}^m \log(\tilde{\lambda}_l)$ and $d_m = \frac{1}{m} \sum_{l=1}^m \tilde{\lambda}_l^{-1}$. Assuming that the eigenvalues $\tilde{\lambda}_l$ are given, the purpose of this algorithm is evaluating the parameters A_m, d_m and the values of $\frac{P_T}{N_0 D} \equiv C\left(\frac{E_b}{N_0}\right)_T$ when the water-filling constant μ is equal to $\tilde{\lambda}_l^{-1}$ for $l = 2, \dots, r_T$ (as illustrated in Figure D.1), as well as evaluating A_m, d_m when all the eigenmodes are active (i.e. $m = r_T$). The algorithm works as follows. Assuming that the nonzero eigenvalues $\tilde{\lambda}_l$ are decreasingly ordered, we vary m in the set $\{1, \dots, r_T - 1\}$. For each m , we evaluate A_m and d_m and we take $\mu = \tilde{\lambda}_{m+1}^{-1}$. Using this μ and the first equality in (D.1), we evaluate the associated capacity C_m . Then, denoting the SNR-per-bit associated with m by B_m and substituting $\frac{P_T}{N_0 D} = C_m B_m$ in the last expression of (D.1), we obtain

$$B_m = \left[2 \frac{N_G}{m} (C_m - A_m) - d_m\right] \left[\frac{N_G C_m}{m}\right]^{-1} \quad (D.2)$$

which can be used to calculate the ratio $\frac{P_T}{N_0 D}$ corresponding to $\mu = \tilde{\lambda}_{m+1}^{-1}$, $\frac{P_T}{N_0 D} = C_m B_m$. As a last step in this algorithm, we take $m = r_T$ (which corresponds to all the eigenmodes being active) and we evaluate the associated constants A_{r_T} and d_{r_T} . The algorithm can be presented as follows.

Input: The eigenvalues $\tilde{\lambda}_q$ and the number of TOGs N_G .

Output: The vectors m_0, A, d, C_0 and $\frac{P_T}{N_0 D}$ that contain the number of active eigenmodes m , the constants A_m, d_m , the capacities C_m and the products $C_m B_m$, respectively.

Algorithm: We use the following procedure.

Step 1: Initialize the first element in each output vector to zero.

Step 2: Arrange the nonzero eigenvalues $\tilde{\lambda}_q$ and sort them in decreasing order. We label the newly obtained eigenvalues as $\{\tilde{\lambda}_{ORD;q}\}_{q=1}^{r_T}$. The eigenvalues $\tilde{\lambda}_q$ that are equal to zero are not used in the water-filling procedure.

Step 3: Take $m = 1$. If $r_T = 1$ then go to Step 7.

Step 4: Calculate the capacity corresponding to a water level equal to $\lambda_{ORD;m+1}^{-1}$,

$$C_m = \frac{1}{N_G} \sum_{l=1}^m \log(\tilde{\lambda}_{ORD;m+1}^{-1} \tilde{\lambda}_{ORD;l})$$

and the constants $A_m = \frac{1}{N_G} \sum_{l=1}^m \log(\tilde{\lambda}_{ORD;l})$ and $d_m = \frac{1}{m} \sum_{l=1}^m \tilde{\lambda}_{ORD;l}^{-1}$. Update the output vectors A, d, m_0 and C_0 by adding A_m, d_m, m and C_m , respectively.

Step 5: Calculate the SNR-per-bit (D.2) associated with the capacity C_m . Then, update the output vector $\frac{P_T}{N_0 D}$ by adding $C_m B_m$.

Step 6: If $m < r_T - 1$ then take $m = m + 1$ and go to Step 4.

Step 7: We consider now the case where all the eigenmodes are active. Update the output vector m_0 by adding the element r_T . Calculate the constants $A_{r_T} = \frac{1}{N_G} \sum_{l=1}^{r_T} \log(\tilde{\lambda}_{ORD;l})$ and $d_{r_T} = \frac{1}{r_T} \sum_{l=1}^{r_T} \tilde{\lambda}_{ORD;l}^{-1}$. Update the output vectors A and d by adding A_{r_T} and d_{r_T} , respectively. Terminate.

D.1.9 Algorithm to generate the graphs of Figures 3.8 - 3.14

We present now the algorithm that we used to generate the graphs of Figures 3.8 - 3.14. The Matlab code can be found in the file AngSpSTMF.m. The algorithm works as follows.

We fix $\frac{P_T}{N_0 D}$, N_T , N_R and the user space-time modulation format. Following the 3GPP specification, we start by generating the parameters σ_{SF} , $\{P_k\}_{k=1}^3$ and the carrier phases $\varphi_{k;s}$. We have equal number of space resolvable paths per cluster $S_t = S_1 = S_2 = S_3 = 20$. This algorithm calculates the angle spread σ_{TX} , the rate I_u with uniform PA and the capacity C as a functions of (A_{TX}, b_{TX}) for fixed (A_{RX}, b_{RX}) , or it calculates σ_{RX} , I_u and C as a functions of (A_{RX}, b_{RX}) for fixed (A_{TX}, b_{TX}) . For each pair (A_{TX}, b_{TX}) in the first case (or (A_{RX}, b_{RX}) in the second case), the algorithm starts by computing the angle spread, AODs and AOAs using the routine AngSpread.m. Using (2.6) and (2.21)-(2.22), the AODs and AOAs are then used to compute the TOG propagation matrices $\mathbf{C}_i$ as well as the eigenvalues $\lambda_{i;l}$. Using (3.39) with $r_{s;i} = n_i$, $\eta_i = \frac{n_i}{N_T}$ and $\xi_{i;l} = \lambda_{i;l}$, we compute

$$I_u = \frac{1}{N_G} \sum_{i=1}^{N_G} \sum_{l=1}^{n_i} \log \left(1 + \frac{N_G}{N_T} \frac{P_T}{N_0 D} \lambda_{i;l} \right) \quad (\text{D.3})$$

Also, using the output of the routine WFinfo.m we determine the number of active eigenmodes m and the constants A_m and d_m that are associated with the given value of $\frac{P_T}{N_0 D}$ and then we evaluate the capacity C using the second equality in (D.1). The algorithm can be presented as follows.

Algorithm:

- Step 1:** Define the number of transmit and receive antennas, the ratio $\frac{P_T}{N_0 D}$ and the parameters N_G and $\{n_i\}_{i=1}^{N_G}$ that determine the space-time modulation format.
- Step 2:** Generate the 3GPP channel parameters γ , σ_{SF} , $\{\tau_k, P_k\}_{k=1}^3$ and $\{\{\varphi_{k;s}\}_{s=0}^{S_t-1}\}_{k=1}^3$. These parameters are generated once and stored in the file minicinfo.mat.
- Step 3:** Define (A_{TX}, b_{TX}) if the transmission rates are computed as functions of (A_{RX}, b_{RX}) , or (A_{RX}, b_{RX}) if the transmission rates are computed as functions of (A_{TX}, b_{TX}) . In the first case, define the range $[1^\circ, 150^\circ] \times [1^\circ, 30^\circ]$ for (A_{RX}, b_{RX}) and initialize $(A_{RX}, b_{RX}) = (1^\circ, 1^\circ)$. In the second case, define the range $[1^\circ, 45^\circ] \times [1^\circ, 15^\circ]$ for (A_{TX}, b_{TX}) and initialize $(A_{TX}, b_{TX}) = (1^\circ, 1^\circ)$.
- Step 4:** Call the routine AngSpread.m to evaluate σ_{RX} and σ_{TX} as well as $\{\{\theta_{T;k;s}\}_{s=0}^{S_t-1}\}_{k=1}^3$ and $\{\{\theta_{R;k;s}\}_{s=0}^{S_t-1}\}_{k=1}^3$.
- Step 5:** Compute the TOG propagation matrices $\{\mathbf{C}_i\}_{i=1}^{N_G}$ and the eigenvalues $\{\{\lambda_{i;l}\}_{l=1}^{n_i}\}_{i=1}^{N_G}$.
- Step 6:** Compute the rate I_u using (D.3). Call the routine WFinfo.m and use its output to determine the number of active eigenmodes m and the constants A_m and d_m that are associated with the value of $\frac{P_T}{N_0 D}$ defined in Step 1. Compute C using (D.1).

Step 7: If $A_{RX} \leq 150^\circ$ and $b_{RX} \leq 30^\circ$ (or $A_{TX} \leq 45^\circ$ and $b_{TX} \leq 15^\circ$) then increment (A_{RX}, b_{RX}) (or (A_{TX}, b_{TX})) and go to Step 4. Else, plot the curves of the angle spread as well as C and I_u as functions of (A_{RX}, b_{RX}) (or (A_{TX}, b_{TX})) and terminate.

D.2 Numerical results for Chapter 4

We provide in Table D.2 the list and description of the Matlab routines associated with the graphs of Figures 4.3-4.11 in Chapter 4. On the CD-ROM, these routines are stored in the directory called Chapter4. Detailed description of the corresponding algorithms follows in Sections D.2.1 and D.2.2.

File Name	Description
Datawf.m	It is used to calculate the capacity $C^{(k)}(\alpha_p^{(k)})$ or the rate $\check{R}^{(k)}(\alpha_p^{(1)}, \alpha_p^{(2)})$, $k = 1, 2$, that are associated with a given power allocation $(\alpha_p^{(1)}, \alpha_p^{(2)})$. See Section D.2.1.
BCSTMFU1ATA.m	Generates the graphs of Figure 4.3. See Section D.2.2.
BCSTMFU2POTA.m	Generates the graphs of Figure 4.4. See Section D.2.2.
BCNRPOTA.m	Generates the graphs of Figure 4.5. See Section D.2.2.
BCOTALHSNR.m	Generates the graphs of Figure 4.6-4.7. The user can select the value of D_c and associated matrix $\mathbf{B}^{(1,2)}$ in the header of this file. See Section D.2.2.
BCPOTALHSNR.m	Generates the graphs of Figure 4.8-4.9. The user can select the value of D_c and associated matrix $\mathbf{B}^{(1,2)}$ in the header of this file. See Section D.2.2.
BCATALHSNR.m	Generates the graphs of Figure 4.10-4.11. The user can select the value of D_c and associated matrix $\mathbf{B}^{(1,2)}$ in the header of this file. See Section D.2.2.

Table D.2 Matlab routines used for the numerical results in Chapter 4

D.2.1 The water-filling schemes associated with $C^{(k)}$ and $\check{R}^{(k)}$

We describe the algorithm associated with the Matlab file Datawf.m. This algorithm evaluates the water-filling schemes over the eigenmodes $\{\{\lambda_{i,l}^{(k)}\}_{l=1}^{n_i^{(k)}}\}_{i=1}^{N_G^{(k)}}$ and $\{\check{\lambda}_q^{(k)}\}_{q=1}^{\check{D}^{(k)}}$ that are associated with the single-user capacity $C^{(k)}$ (4.14) and the rate $\check{R}^{(k)}$ (4.23), respectively.

Since the procedure is the same for $C^{(k)}$ and $\check{R}^{(k)}$, we present the algorithm using the notation $\check{R}^{(k)}$ to denote $C^{(k)}$ or $\check{R}^{(k)}$ with $\check{\lambda}_q^{(k)}$ referring to $\lambda_{i;l}^{(k)}$ or $\check{\lambda}_q^{(k)}$, respectively. Also, let $\check{r}^{(k)}$ denote the rank $r_T^{(k)}$ if we consider $C^{(k)}$ and $\check{r}^{(k)}$ if we consider $\check{R}^{(k)}$. We use a constant K_0 that satisfies $K_0 = D_G^{(k)}$ if we consider (4.14) and $K_0 = 1$ if we consider (4.23). If the number of active eigenmodes is $m^{(k)}$, the water-filling solution leads to

$$\check{R}^{(k)} = \frac{K_0}{\check{D}^{(k)}} \sum_{l=1}^{m^{(k)}} \log(\check{\mu}^{(k)} \check{\lambda}_l^{(k)}) = A^{(k)} + \frac{K_0}{\check{D}^{(k)}} m^{(k)} \log\left(\frac{\alpha_p^{(k)}}{m^{(k)}} \frac{P_T}{K_0 N_0} + d^{(k)}\right) \quad (\text{D.4})$$

where $\check{\mu}^{(k)}$ satisfies $\sum_{l=1}^{m^{(k)}} \{\check{\mu}^{(k)} - (\check{\lambda}_l^{(k)})^{-1}\} = \frac{\alpha_p^{(k)} P_T}{K_0 N_0}$, $A^{(k)} = \frac{K_0}{\check{D}^{(k)}} \sum_{l=1}^{m^{(k)}} \log(\check{\lambda}_l^{(k)})$ and $d^{(k)} = \frac{1}{m^{(k)}} \sum_{l=1}^{m^{(k)}} (\check{\lambda}_l^{(k)})^{-1}$. Assuming that P_T/N_0 and the eigenvalues $\check{\lambda}_l^{(k)}$ are given, the purpose of this algorithm is evaluating the parameters $A^{(k)}$, $d^{(k)}$, $\alpha_p^{(k)}$ when the water-filling constant $\check{\mu}^{(k)}$ is equal to $(\check{\lambda}_l^{(k)})^{-1}$ for $l = 2, \dots, \check{r}^{(k)}$ (as illustrated in Figure D.1 for the single-user capacity), as well as evaluating $A^{(k)}$, $d^{(k)}$ when all the eigenmodes are active (i.e. $m^{(k)} = \check{r}^{(k)}$). The algorithm works as follows. Assuming that the nonzero eigenvalues $\check{\lambda}_l^{(k)}$ are decreasingly ordered, we vary $m^{(k)}$ in the set $\{1, \dots, \check{r}^{(k)} - 1\}$. For each $m^{(k)}$, we evaluate $A^{(k)}$ and $d^{(k)}$ and we take $\mu^{(k)} = (\check{\lambda}_{m^{(k)}+1}^{(k)})^{-1}$. Using this $\mu^{(k)}$ and the first equality in (D.4), we evaluate $\check{R}^{(k)}$. Then, substituting $\frac{\alpha_p^{(k)} P_T}{N_0} = \check{D}^{(k)} \check{R}^{(k)} \left(\frac{E_b^{(k)}}{N_0}\right)_T$ in the last expression of (D.4), we obtain

$$\left(\frac{E_b^{(k)}}{N_0}\right)_T = \left[2^{\frac{\check{D}^{(k)}}{K_0 m^{(k)}} (\check{R}^{(k)} - A^{(k)})} - d^{(k)}\right] \left[\frac{\check{D}^{(k)} \check{R}^{(k)}}{K_0 m^{(k)}}\right]^{-1}$$

which can be used thereafter to calculate the associated fraction of power assigned to user k , $\alpha_p^{(k)} = \frac{\check{D}^{(k)} \check{R}^{(k)} (E_b^{(k)}/N_0)_T}{P_T/N_0}$. As a last step in this algorithm, we take $m^{(k)} = \check{r}^{(k)}$ (which corresponds to all the eigenmodes being active) and we evaluate the associated constants $A^{(k)}$ and $d^{(k)}$. The algorithm can be presented as follows.

Input: The SNR P_T/N_0 , the eigenvalues $\check{\lambda}_q^{(k)}$, the rate normalization constant $\check{D}^{(k)}$, the number of signal dimensions per TOG $D_G^{(k)}$, and a constant *flag* that takes the value 0 if we consider $C^{(k)}$ or the value 1 if we consider $\check{R}^{(k)}$.

Output: The vectors of values of $\alpha_p^{(k)}$, $A^{(k)}$, $d^{(k)}$ and $m^{(k)}$.

Algorithm: We use the following procedure.

Step 1: If *flag* = 0, set $K_0 = D_G^{(k)}$. If *flag* = 1, set $K_0 = 1$. Initialize the first element in each output vector to zero. This corresponds to the situation where no power is allocated to

user k ($\alpha_p^{(k)} = 0$).

Step 2: Arrange the nonzero eigenvalues $\check{\lambda}_q^{(k)}$ and sort them in decreasing order. We label the newly obtained eigenvalues as $\{\check{\lambda}_{ORD;q}^{(k)}\}_{q=1}^{\check{r}^{(k)}}$. The eigenvalues $\check{\lambda}_q^{(k)}$ that are equal to zero are not used in the water-filling procedure.

Step 3: Take $m = 1$. If $\check{r}^{(k)} = 1$ then go to Step 7.

Step 4: Calculate the rate corresponding to a water level equal to $(\check{\lambda}_{ORD;m+1}^{(k)})^{-1}$,

$$\check{R}_m^{(k)} = \frac{K_0}{\check{D}^{(k)}} \sum_{l=1}^m \log((\check{\lambda}_{ORD;m+1}^{(k)})^{-1} \check{\lambda}_{ORD;l}^{(k)})$$

and the constants $A_m = \frac{K_0}{\check{D}^{(k)}} \sum_{l=1}^m \log(\check{\lambda}_{ORD;l}^{(k)})$ and $d_m = \frac{1}{m} \sum_{l=1}^m (\check{\lambda}_{ORD;l}^{(k)})^{-1}$. Update the output vectors $A^{(k)}$, $d^{(k)}$ and $m^{(k)}$ by adding A_m , d_m and m , respectively.

Step 5: Calculate the SNR-per-bit associated with the rate $\check{R}_m^{(k)}$

$$\check{B}_m^{(k)} = \left[2^{\frac{\check{D}^{(k)}}{K_0 m} (\check{R}_m^{(k)} - A_m)} - d_m \right] \left[\frac{\check{D}^{(k)} \check{R}_m^{(k)}}{K_0 m} \right]^{-1}$$

The corresponding power allocation of user k is given by $\frac{\check{D}^{(k)} \check{R}_m^{(k)} \check{B}_m^{(k)}}{(P_T/N_0)}$. Update the output vectors $\alpha_p^{(k)}$.

Step 6: If $m < \check{r}^{(k)} - 1$ then take $m = m + 1$ and go to Step 4.

Step 7: We consider now the case where all the eigenmodes are active. Update the output vectors $m^{(k)}$ and $\alpha_p^{(k)}$ by adding the elements $\check{r}^{(k)}$ and 1, respectively. Calculate the constants $A_{\check{r}^{(k)}} = \frac{K_0}{\check{D}^{(k)}} \sum_{l=1}^{\check{r}^{(k)}} \log(\check{\lambda}_{ORD;l}^{(k)})$ and $d_{\check{r}^{(k)}} = \frac{1}{\check{r}^{(k)}} \sum_{l=1}^{\check{r}^{(k)}} (\check{\lambda}_{ORD;l}^{(k)})^{-1}$. Update the output vectors $A^{(k)}$ and $d^{(k)}$ by adding $A_{\check{r}^{(k)}}$ and $d_{\check{r}^{(k)}}$, respectively. Terminate.

D.2.2 Algorithm to generate the graphs of Figures 4.3-4.11

We present now the algorithm that we used to generate the inner and outer bounds $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$ in Figures 4.3-4.11. The Matlab code can be found in the files specified in Table D.2. For simplicity, we present this algorithm for a given P_T/N_0 , for given space-time modulation formats for users 1 and 2, for a given normalization method (i.e. $\check{D}^{(k)} = D^{(k)}$ or $\check{D}^{(k)} = D_T$), and for a given number of receive antennas for each user. The algorithm works as follows. We start by defining the system parameters and signal dimensions sharing scheme (that are determined by the matrices $\mathbf{B}^{(k,l)}$), and we calculate the matrices $\mathbf{H}^{(k)}$, $\mathbf{H}^{(k,l)}$ as well as the eigenvalues $\lambda_{i,p}^{(k)}$ of the matrices $\text{diag}_{i=1}^{N_G^{(k)}} [(\mathbf{C}_i^{(k)})^\dagger \cdot \mathbf{C}_i^{(k)}]$ for

$(k, l) = (1, 2), (2, 1)$. In order to calculate the inner and outer bounds $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$ it is sufficient to calculate the rate pairs $(C^{(1)}(\alpha_p^{(1)}), \check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}))$, $(\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}), C^{(2)}(\alpha_p^{(2)}))$ and the capacity pairs $(C^{(1)}(\alpha_p^{(1)}), C^{(2)}(\alpha_p^{(2)}))$ for $\alpha_p^{(1)} \in [0, 1]$ and $\alpha_p^{(2)} = 1 - \alpha_p^{(1)}$. Hence, we vary $\alpha_p^{(1)}$ in the interval $[0, 1]$ and for each value of $\alpha_p^{(1)}$ the algorithm performs the following steps:

- Calculate the capacity $C^{(1)}(\alpha_p^{(1)})$: With $\{\{\lambda_{i,l}^{(1)}\}_{l=1}^{n_i^{(1)}}\}_{i=1}^{N_G^{(1)}}$ as input and $flag = 0$, the outputs of the routine Datawf.m can be used to determine the number of active eigenmodes $m^{(1)}(\alpha_p^{(1)})$ and the constants $A^{(1)}(\alpha_p^{(1)})$, $d^{(1)}(\alpha_p^{(1)})$ that are associated with this value of $\alpha_p^{(1)}$. Then, the capacity $C^{(1)}(\alpha_p^{(1)})$ of (4.14) can be calculated as

$$C^{(1)}(\alpha_p^{(1)}) = A^{(1)}(\alpha_p^{(1)}) + \frac{D_G^{(1)}}{\check{D}^{(1)}} m^{(1)}(\alpha_p^{(1)}) \log \left[\frac{\alpha_p^{(1)} P_T}{m^{(1)}(\alpha_p^{(1)}) D_G^{(1)} N_0} + d^{(1)}(\alpha_p^{(1)}) \right]$$

- Calculate the rate $\check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})$: We start by computing the matrix $\Sigma_{\max}^{(1)}$ using (4.15) and then we evaluate the matrix $\Psi^{(2)}$ using (4.17). We compute the eigenvalues $\check{\lambda}_p^{(2)}$ of $[\mathbf{H}^{(2)}]^\dagger \cdot [\Psi^{(2)}]^{-1} \cdot \mathbf{H}^{(2)}$ and we use them as inputs to the routine Datawf.m with $flag = 1$. The outputs of this routine can be used to determine the number of active eigenmodes $m^{(2)}(\alpha_p^{(2)})$ and the constants $A^{(2)}(\alpha_p^{(2)})$, $d^{(2)}(\alpha_p^{(2)})$ that are associated with the current value of $\alpha_p^{(2)} = 1 - \alpha_p^{(1)}$. Then, the rate $\check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ of (4.23) can be calculated as

$$\check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}) = \frac{m^{(2)}(\alpha_p^{(2)})}{\check{D}^{(2)}} \log \left[\frac{\alpha_p^{(2)} P_T}{m^{(2)}(\alpha_p^{(2)}) N_0} + d^{(2)}(\alpha_p^{(2)}) \right] + A^{(2)}(\alpha_p^{(2)})$$

- Calculate the capacity $C^{(2)}(\alpha_p^{(2)})$: We follow a similar approach to that used for calculating $C^{(1)}(\alpha_p^{(1)})$ with the user indexes 1 and 2 switched.
- Calculate the rate $\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)})$: We follow a similar approach to that used for calculating $\check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ with the user indexes 1 and 2 switched.

Finally, the algorithm computes the inner bound $\mathcal{R}_{in}$ by taking the convex hull of all the calculated rate pairs $(C^{(1)}(\alpha_p^{(1)}), \check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}))$ and $(\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}), C^{(2)}(\alpha_p^{(2)}))$ for different $(\alpha_p^{(1)}, \alpha_p^{(2)})$, and computes $\mathcal{R}_{out}$ by taking the set of the pairs $(C^{(1)}(\alpha_p^{(1)}), C^{(2)}(\alpha_p^{(2)}))$. We also find the points of $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$ that correspond to the maximum sum rate with the associated power allocations $\tilde{\alpha}_{p;s}^{(1)}$ and $\alpha_{p;s}^{(1)}$. The algorithm can be presented as follows.

Algorithm:

Step 1: Define the number of transmit antennas N_T and the numbers of receive antennas $(N_R^{(1)}, N_R^{(2)})$. Define the parameters $\check{D}^{(k)}, D_p^{(k)}$ ($k = 1, 2$), D_c and the matrices $\mathbf{B}^{(1,2)}$ and $\mathbf{B}^{(2,1)}$. Define the space-time modulation formats by $N_G^{(k)}$, the number of antenna elements in each TOG $\{n_i^{(k)}\}_{i=1}^{N_G^{(k)}}$ and the matrices $\mathbf{A}^{(k)}$, $k = 1, 2$. Define P_T/N_0 .

Step 2: Load the spatial channel parameters for the two users. The data is stored in the files cinfo-ChAPrime.mat for Channel A of user 1 and cinfo-ChB.mat for Channel B of user 2.

Step 3: For each $k = 1, 2$: Calculate the matrices $\mathbf{F}_{i,l}^{(k)}$ using (2.21) and $(\mathbf{C}_i^{(k)})^\dagger \cdot \mathbf{C}_i^{(k)}$ using (2.22). Calculate the matrix $\mathbf{H}^{(k)}$ using (2.41). Evaluate the eigenvalues $\{\{\lambda_{i,l}^{(k)}\}_{l=1}^{n_i^{(k)}}\}_{i=1}^{N_G^{(k)}}$ and the matrices $\{\mathbf{M}_i^{(k)}\}_{i=1}^{N_G^{(k)}}$. Use the routine Datawf.m to specify the water-filling scheme that is associated with the eigenvalues $\{\{\lambda_{i,l}^{(k)}\}_{l=1}^{n_i^{(k)}}\}_{i=1}^{N_G^{(k)}}$.

Step 4: Calculate the matrices $\mathbf{H}^{(1,2)}$ and $\mathbf{H}^{(2,1)}$ using (2.42).

Step 5: We vary $\alpha_p^{(1)} \equiv \alpha_p$ in the interval $[0, 1]$ using increments of step size $\delta\alpha_p$. Take $a = 0$. Take $(l, v) = (1, 2)$.

Step 6: Evaluate the rate pairs $(C^{(l)}(\alpha_p^{(l)}), \check{R}^{(v)}(\alpha_p^{(1)}, \alpha_p^{(2)}))$

1. Find the values from the output vectors $\alpha_p^{(l)}, A^{(l)}, d^{(l)}$ and $m^{(l)}$ of Datawf.m (obtained from Step 3 with $k = l$) that correspond to the power allocation $\alpha_p^{(l)} = a$ if $l = 1$ or $\alpha_p^{(l)} = 1 - a$ if $l = 2$. Use these values to calculate $C^{(l)}(\alpha_p^{(l)})$ of (4.14) and to evaluate the matrix $\Sigma_{\max}^{(l)}$ using (4.15).
2. Evaluate the matrix $\Psi^{(v)}$ using (4.17) and the matrix $[\mathbf{H}^{(v)}]^\dagger \cdot [\Psi^{(v)}]^{-1} \cdot \mathbf{H}^{(v)}$. Evaluate the eigenvalues $\{\check{\lambda}_q^{(v)}\}_{l=1}^{\check{D}^{(v)}}$ of this matrix.
3. Use the routine Datawf.m to specify the water-filling scheme that is associated with $\{\check{\lambda}_q^{(v)}\}_{l=1}^{\check{D}^{(v)}}$. Find the values from the resulting output vectors $\alpha_p^{(v)}, A^{(v)}, d^{(v)}$ and $m^{(v)}$ that correspond to the power allocation $\alpha_p^{(v)} = a$ if $v = 1$ or $\alpha_p^{(v)} = 1 - a$ if $v = 2$. Use these values to calculate $\check{R}^{(v)}(\alpha_p^{(1)}, \alpha_p^{(2)})$ of (4.23).

Step 7: If $(l, v) = (1, 2)$, then take $(l, v) = (2, 1)$ and go to Step 6. Else, go to Step 8.

Step 8: Take $a = a + \delta\alpha_p$ and $(l, v) = (1, 2)$. If $a > 1$, then go to Step 9. Else, go to Step 6.

Step 9: Using the function "convhull" in Matlab, compute the convex hull $\mathcal{R}_{in}$ of the rate pairs $(C^{(1)}(\alpha_p^{(1)}), \check{R}^{(2)}(\alpha_p^{(1)}, \alpha_p^{(2)}))$ and $(\check{R}^{(1)}(\alpha_p^{(1)}, \alpha_p^{(2)}), C^{(2)}(\alpha_p^{(2)}))$. The outer bound $\mathcal{R}_{out}$ is simply the set of pairs $(C^{(1)}(\alpha_p^{(1)}), C^{(2)}(\alpha_p^{(2)}))$.

Step 10: Find the points of $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$ that correspond to the maximum sum rate.

Step 11: Evaluate the power allocations $\alpha_{p;s}^{(1)}$ of (4.45) and $\tilde{\alpha}_{p;s}^{(1)}$ of (4.58) and (4.59).

Step 12: Plot the inner and outer bounds $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$. Terminate.

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